Stability Conditions and Applications

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- Want a space (algebraic variety, scheme, algebraic space, stack) that parametrizes equivalence classes of objects.
- Want to be able to deform objects (taking limits): a compact moduli space.

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Once again, if we fix the discrete invariants then the classification problem has finitely many solutions.

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The Zariski topology on $\mathbb{CP}^n$ is the topology whose closed subsets are the zero sets of complex homogeneous polynomials in $n+1$ variables. Each of those closed subsets is called a **projective variety**.

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$$\mathbb{CP}^5 = \{[a_0, a_1, a_3, a_4, a_5]\}$$

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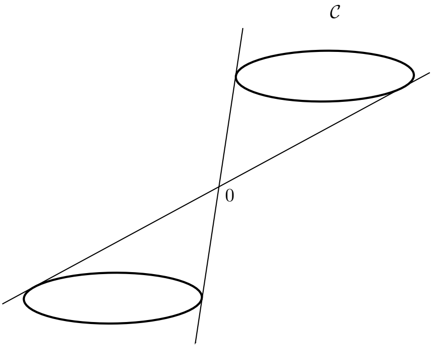

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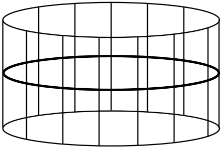

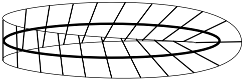

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- Resulting moduli space is “too big”

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- Still not good enough! We need to add an extra condition: a "stability condition"

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Note: μ_ω can be defined for all coherent sheaves by declaring $\mu_\omega(E) = +\infty$ if $ch_0(E) = 0$.

Classical semistabilities

Definition

Let $E \in \mathrm{Coh}(X)$ be a torsion-free sheaf. We say that E is **Mumford semistable** if for all subsheaves $A \hookrightarrow E$ we have

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Note: These semistabilities depend on the ample class ω.

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- (Harder-Narasimhan) Every nonzero $E \in \mathrm{Coh}(X)$ has a unique filtration

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such that E_0 is the torsion subsheaf of E, and the factors $F_i = E_i/E_{i-1}$ are Mumford semistable of decreasing slopes.
- For fixed Chern character v, there is a projective coarse moduli space $M_\omega(v)$ parametrizing S-equivalence classes of Gieseker semistable sheaves of type v.

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- Is this space irreducible or smooth? Can we understand its singular locus? What is its dimension?
- How can we produce varieties that are birational to $M_\omega(v)$?

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Note: For most surfaces $N^1(X) \cong \mathbb{Z}$ and so $M_\omega(v)$ and $M_{\omega'}(v)$ are actually isomorphic. Is there a way to produce interesting birational models in this case?

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Classical	Bridgeland
$Coh(X)$	Abelian subcategory $\mathcal{A} \subset D^b(X)$
$Z_M = -ch_1\omega^{n-1} + ich_0\omega^n$	$Z\colon K(\mathcal{A}) \to \mathbb{C}$
$ch_0(\mathcal{E}) = 0$ implies $ch_1(\mathcal{E})\omega^{n-1} \geq 0$	$\mathfrak{Im}(Z(E)) = 0$ implies $\mathfrak{Re}(Z(E)) > 0$
μ_M	$\mu_Z = \frac{-\mathfrak{Re}(Z)}{\mathfrak{Im}(Z)}$
HN filtrations	Impose this condition
$\Delta(E) \geq 0$ for semistables	Support property

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Theorem (Bridgeland, 2002)

There is a complex manifold $Stab(X)$ parametrizing numerical stability conditions on X. Moreover, for a fixed Chern character v, $Stab(X)$ admits a locally finite wall and chamber decomposition such that σ and σ' are in the same chamber if and only if the sets of σ-semistable and σ'-semistable objects of type v are the same.

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- (Bridgeland) If X is a surface then for every Chern character v there is a distinguished chamber in the wall and chamber decomposition of $Stab(X)$ such that

$$E \text{ is } \sigma\text{-semistable} \iff E \text{ is } \omega\text{-Gieseker semistable}$$

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- (Bertram) If X is a surface then for every fixed Chern character v there is a family of geometric stability conditions $\sigma_t = (Z_t, \mathcal{A}_t)$ with

$$\frac{\mathfrak{Re}(Z_t(A))}{ch_0(A)} = -\left(\frac{\chi(A)}{ch_0(A)\omega^2} - \frac{\chi(v)}{ch_0(v)\omega^2}\right)\omega^2 - t(\mu_\omega(A) - \mu_\omega(v)).$$

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- (Bertram, M.) Moreover, the path $\{\sigma_t\}_{t>0} \subset Stab(X)$ enters the ω-Gieseker chamber for $t \gg 0$ and never leaves.

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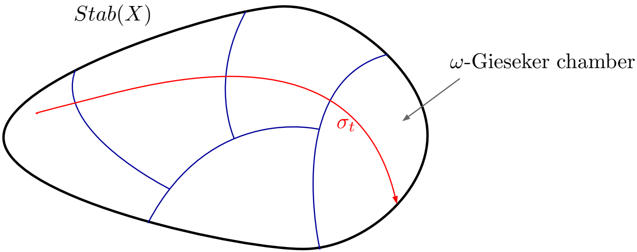

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Theorem (Bertram, M.)

Let X be an arbitrary smooth surface. Given two ample classes ω and ω' and a Chern character v there is a sequence of GIT flips

$$\begin{array}{ccccccccc} M_\omega(v) & & & & M_1 & & & & M_2 & & \cdots & & M_{\omega'}(v) \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & N_1 & & & & N_2 & & & & \cdots & & \end{array}$$

where M_i and N_i are projective moduli spaces of Bridgeland semistable **sheaves** *of type v.*

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- (Bridgeland; Arcara, Bertram) If X is a surface and σ is geometric with $Z_\sigma(\mathbb{C}_x) = -1$ then there are classes $B, \omega \in N^1(X)_\mathbb{Q}$ with ω ample such that

$$Z_\sigma(E) = -\int e^{-B-i\omega} ch(E).$$

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From now on we denote this tilted slope by $\nu_{B,\omega}$ and refer to it as **the tilt**.

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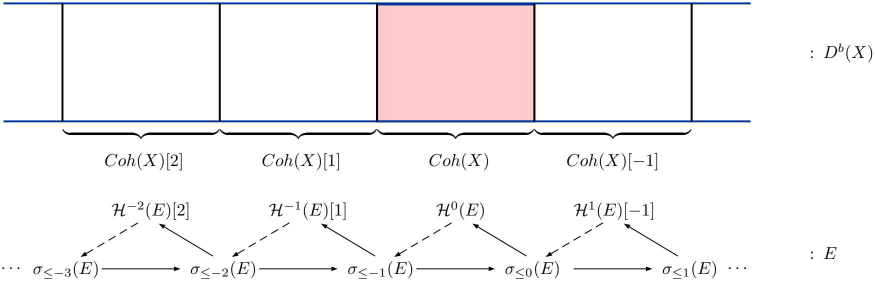

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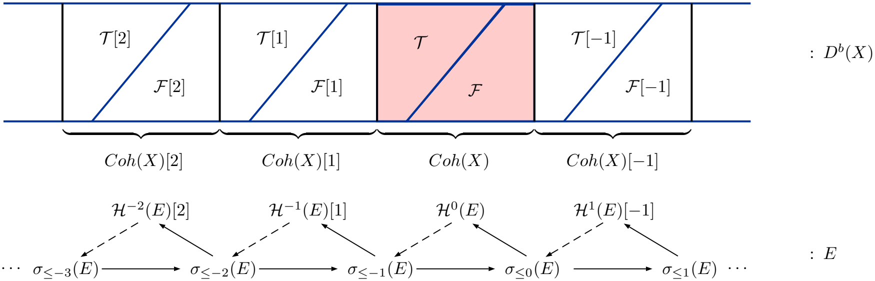

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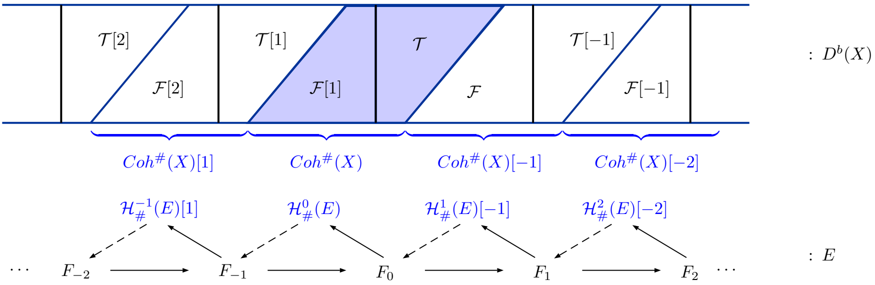

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- Since $\mathcal{A}_\sigma$ is obtained as a tilt of $Coh(X)$, from now on we will denote it by $Coh^{B,\omega}(X)$.

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Conjecture (Bayer, Bertram, Macrì, Toda; 2011)

On a threefold X the linear map

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Equivalently

Theorem (Bayer, Macrì,Toda)

The pair $(Z_{B,\omega}, \mathcal{A}_{B,\omega})$ *defines a geometric stability condition on* X *if and only if every* $\nu_{B,\omega}$*-semistable object* $E \in Coh^{B,\omega}(X)$ *with* $\nu_{B,\omega}(E) = 0$ *also satisfies*

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Generalized Bogomolov-Gieseker inequality

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- Conjecture fails on $Bl_p\mathbb{P}^3$. Schmidt proves the existence of classes β, ω and a line bundle so that the GBG inequality fails.
- Does this mean that there should be corrections to the central charge?

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- Can we find divisors D on a threefold X with special intersection properties so that $\mathcal{O}_D$ violates the GBG inequality?

Using Bertram's estimates for walls...

Theorem (M., Schmidt)

Let X be a smooth projective threefold. Suppose that there is an effective divisor D and an ample divisor H such that

$$D^3 > \frac{(D \cdot H^2)^3}{4(H^3)^2} + \frac{3}{4}\frac{(D^2 \cdot H)^2}{D \cdot H^2}.$$

Then there exists a pair of numbers $\alpha_0 > 0, \beta_0$ such that $\mathcal{O}_D$ violates the GBG inequality for $\omega = \alpha_0 H$ and $B = \beta_0 H$.

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In particular, the GBG inequality fails on any blow-up of a point on a smooth threefold.

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Note: ALL known counterexamples to the GBG inequality are constructed this way.

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Theorem (M., Schmidt)

There is a heart of a t-structure $\tilde{\mathcal{A}}$ on $D^b(\tilde{X})$ such that the linear map

$$Z_{\alpha,\beta,s} = -\left(ch_3^{\tilde{B}} - s\alpha^2\tilde{H}^2 ch_1^{\tilde{B}} - \frac{E^2}{6} ch_1^{\tilde{B}}\right) + i\left(\tilde{H} ch_2^{\tilde{B}} - \frac{\alpha^3}{2}\tilde{H}^3 ch_0^{\tilde{B}}\right)$$

is the central charge of a stability condition on $\tilde{\mathcal{A}}$ for every $s > 1/6$ if and only if

$$Z_{\alpha H, B_0+\beta H} = -\left(ch_3^{B} - s\alpha^2 H^2 ch_1^{B}\right) + i\left(H ch_2^{B} - \frac{\alpha^3}{2} H^3 ch_0^{B}\right)$$

is the charge of a stability condition on $\mathcal{A}_{\alpha H, B_0+\beta H}$, i.e., if and only if the GBG inequality holds on X for the classes $B_0 + \beta H$ and αH.

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- If we know that the relative FM transform send a tilt of $Coh(X)$ to a double tilt of $Coh(X)$, then we can study the image of the Bogomolov form by the FM transform. In the case that X is elliptically fibered over a del Pezzo surface and $\omega = \Theta - 2p^*(K_B)$ then the image of the discriminant by the FM transform is

$$\left(3ch_0(\Phi E)K_B^2 - ch_2(\Phi E)\cdot p^*K_B\right)^2 + 7K_B^2 ch_1(\Phi E)\cdot f\left(\left(1+\frac{1}{24}\right)K_B^2 ch_1(\Phi E)\cdot f - ch_1(\Phi E)\cdot\Theta^2 - ch_3(\Phi E)\right) \geq 0.$$

Thank you!