Geometry of Complex Algebraic Varieties

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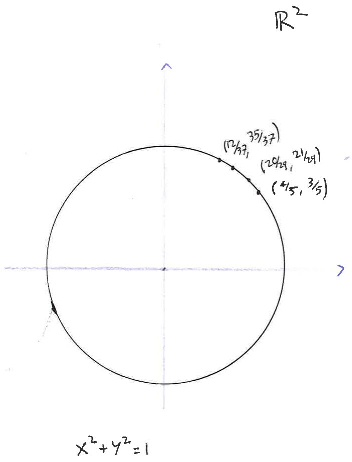
Introduction

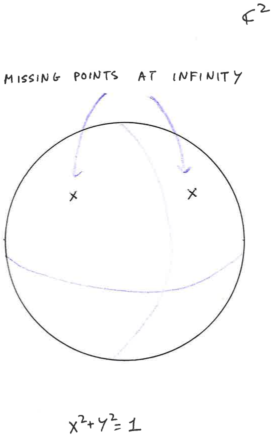
- An **algebraic variety** is a geometric object defined by polynomial equations.
- Typical examples are a parabola, circle, Fermat's equation

$$y = x^2, \qquad x^2 + y^2 = 1, \qquad x^n + y^n = z^n.$$

- One can look for different kinds of solutions.
- For example, the rational/$\mathbb{Q}$ solutions of $x^2 + y^2 = 1$ corresponds to Pythagorean triples $a^2 + b^2 = c^2$ so that $x = a/c$, $y = b/c$.
- The real/$\mathbb{R}$ solutions give a circle.
- The complex/$\mathbb{C}$ solutions give a sphere (minus 2 points at infinity).



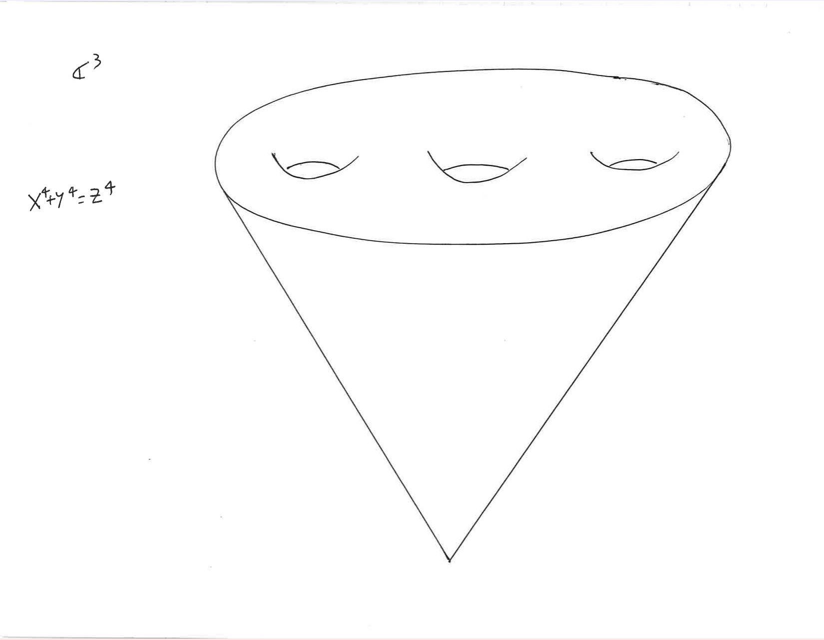
ℝ²
(12/37, 35/37)
(20/29, 21/29)
(4/5, 3/5)
x²+y²=1



ℂ²
MISSING POINTS AT INFINITY
x²+y²=1

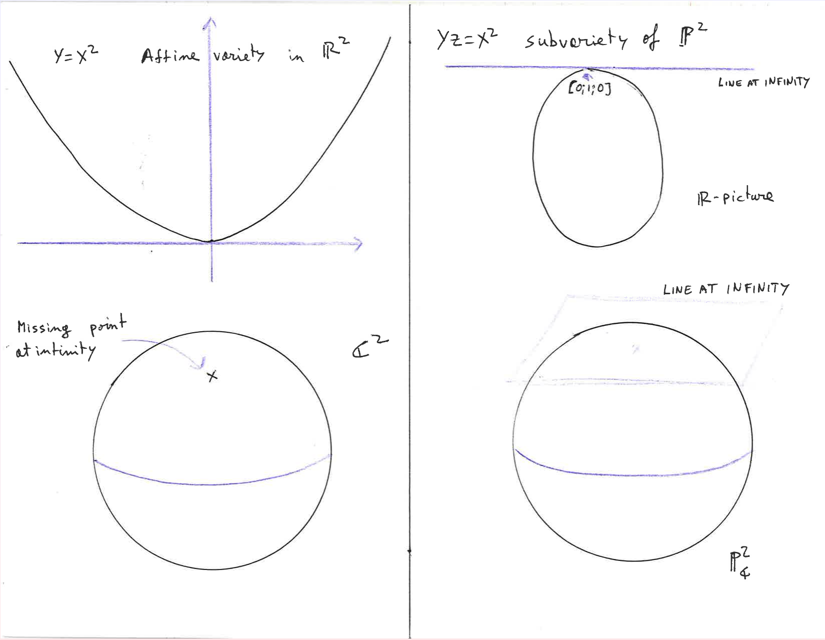
Introduction

- By a famous result of Wiles, for $n \geq 3$ there are no non-trivial integer (rational) solutions to $x^n + y^n = z^n$,
- However, there are plenty of complex solutions corresponding to a (cone over a) Riemann surface of genus $g = (n-1)(n-2)/2$.
- In this talk we will focus on complex/$\mathbb{C}$ solutions.

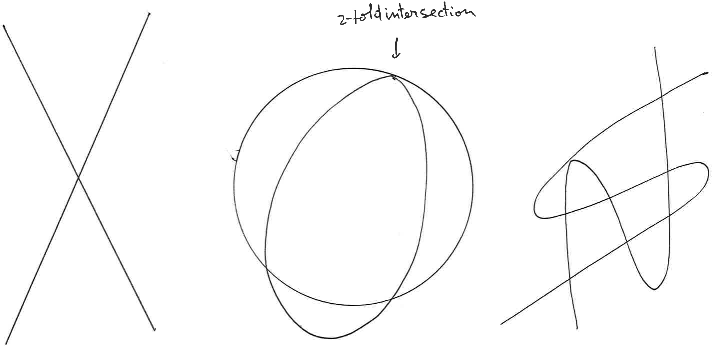

$\mathbb{C}^3$
$x^4+y^4=z^4$

Projective varieties

- **Affine varieties** $X \subset \mathbb{C}^N$ are defined by polynomials $Q_1, \ldots, Q_r \in \mathbb{C}[x_1, \ldots, x_N]$.
- A **projective variety** $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, \ldots, P_r \in \mathbb{C}[x_0, \ldots, x_N]$.
- $\mathbb{P}^N_{\mathbb{C}} = (\mathbb{C}^{N+1} - \bar{0})/\mathbb{C}^*$ is a natural compactification of $\mathbb{C}^N$ corresponding to lines in $\mathbb{C}^{N+1}$.
- $\mathbb{P}^N_{\mathbb{C}} = \mathbb{C}^N \cup \mathbb{P}^{N-1}_{\mathbb{C}}$.
- We think of $\mathbb{P}^{N-1}_{\mathbb{C}}$ as the hyperplane at infinity whose points correspond to all directions in **projective space** $\mathbb{P}^N_{\mathbb{C}}$.
- Projective varieties are natural compactifications of affine varieties obtained by homogenization $P_i(x_0, \ldots, x_N) = x_0^{d_i} Q_i(x_1/x_0, \ldots, x_1/x_n)$.
- This has many advantages

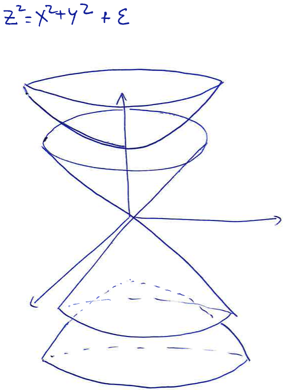

$y=x^2$
Affine variety in $\mathbb{R}^2$
$yz=x^2$ subvariety of $\mathbb{P}^2$
LINE AT INFINITY
[0;1;0]
$\mathbb{R}$-picture
LINE AT INFINITY
Missing point at infinity
$\mathbb{C}^2$
$\mathbb{P}^2_{\mathbb{C}}$

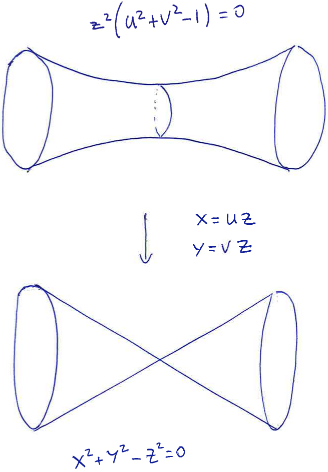
BEZOUT'S THEOREM: Two distinct curves $C, D \subset \mathbb{P}^2_{\mathbb{C}}$ of degrees d, c intersect in exactly $d \cdot c$ pts (when counted with mult.)



$c=1 \quad d=1$ $\qquad$ $c=2 \quad d=2$ $\qquad$ $c=3 \quad d=3$

- A variety X is **irreducible** if it is not the union of two distinct varieties.
- We will focus on smooth irreducible varieties, i.e. complex manifolds.
- This is reasonable because:
  - Most varieties are smooth (if you perturb the coefficients of your equations you get a smooth variety).
  - Every variety is smooth outside a closed (measure 0) subset.
  - By Hironaka's theorem every variety is birational to a smooth one (i.e. can be smoothed by an appropriate "non-invertible change of variables").


$z^2 = x^2 + y^2 + \varepsilon$

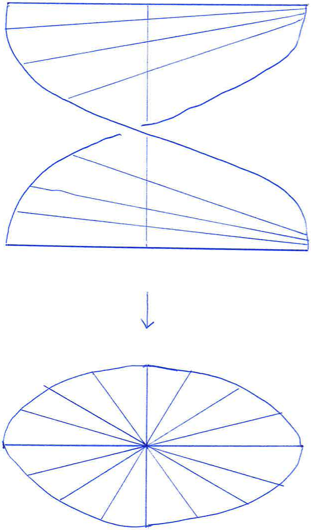

$z^2(u^2+v^2-1) = 0$
$x = uz$
$y = vz$
$x^2+y^2-z^2=0$

Birational varieties

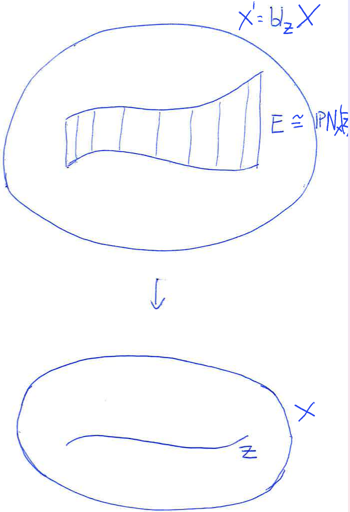
- To study X we consider **meromorphic functions** $\mathbb{C}(X)$ on X (since X is a compact, global holomorphic functions are constant $H^0(\mathcal{O}_X) \cong \mathbb{C}$).
- Two varieties X, X' are **birational** if
  1. they have isomorphic fields of rational (meromorphic) functions $\mathbb{C}(X) \cong \mathbb{C}(X')$ or equivalently if
  2. they have isomorphic open subsets $U \cong U'$.
- We work in the **Zariski topology**, so that a closed set $Z = X \setminus U$ is defined by the vanishing of polynomial equations.
- If $\dim X = \dim X' = 1$, then X and X' are birational iff they are isomorphic (outside finitely many points).
- In higher dimensions this is no longer true (so we need to understand birational equivalence in more detail).

Birational maps

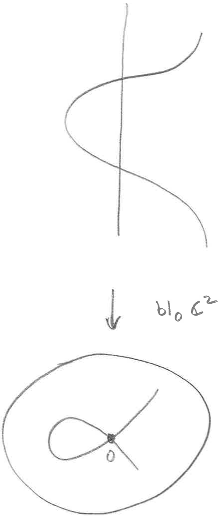
- Typical examples of birational maps are given by **blowing up**.
- If $Z \subset X$ is a smooth subvariety of a smooth variety, then the **blow up of** X **along** Z is a "surgery" that replaces Z by $\mathbb{P}(N_{Z/X})$.
- So every point of Z is replaced by its normal directions in X.
- Two varieties are birational if and only if they are related by a sequence of blow ups and blow downs (inverse of blowing up).

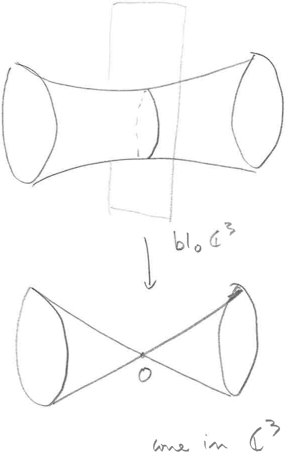


Blow up a point in $\mathbb{C}^2$



Blow up a curve in a threefold


$bl_0 \mathbb{C}^2$
0


$bl_0 \mathbb{C}^3$
0
cone in $\mathbb{C}^3$

Line bundles

- Since global holomorphic functions are constant ($H^0(\mathcal{O}_X) = \mathbb{C}$), to study X we consider **global sections** $s \in H^0(X, \mathcal{L})$ of a **line bundle** $\mathcal{L}$.
- Locally, $X = \cup U_\alpha$, $\mathcal{L}|_{U_\alpha} \cong \mathbb{C} \times U_\alpha$, $\mathcal{L}$ is defined by $g_{\alpha,\beta} \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$ and $s|_{U_\alpha} = s_\alpha \in \mathcal{O}_{U_\alpha}$ such that $s_\alpha = g_{\alpha,\beta} s_\beta$.
- $H^0(X, \mathcal{L})$ is a finite dimensional $\mathbb{C}$ vector space (which can be identified to a subspace of $\mathbb{C}(X)$).
- Eg. $\mathcal{O}_{\mathbb{P}^n}(k)$ is the line bundle whose sections $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ are homogeneous polynomials of degree k in $\mathbb{C}[x_0, \ldots, x_n]$.
- The difficulty is to choose an appropriate line bundle. There is essentially only one possible choice: the **canonical line bundle**.

The canonical line bundle

- If X is smooth and $\dim X = d$, then let $\omega_X = \wedge^d T_X^\vee$ be the **canonical line bundle**.
- For any $m > 0$ consider **pluricanonical forms** $s \in H^0(\omega_X^{\otimes m})$.
- Locally $s|_U = f \cdot (dz_1 \wedge \ldots \wedge dz_d)^{\otimes m}$.
- If $s_0, \ldots, s_N$ is a basis of $H^0(\omega_X^{\otimes m})$, then define the m-th pluricanonical map

$$\phi_m : X \dashrightarrow \mathbb{P}^N, \qquad x \to [s_0(x) : \ldots : s_N(x)].$$

- ϕ_m is not defined at common zeroes of $s_0, \ldots, s_N$.
- If X, X' are smooth birational varieties, then

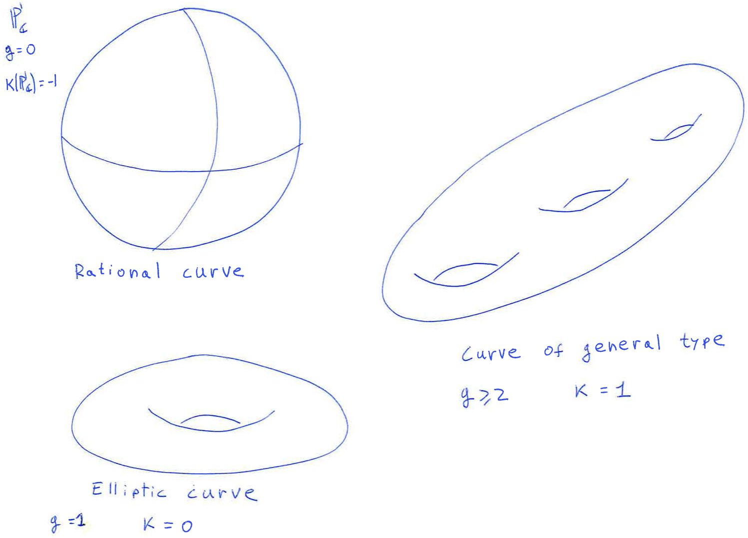
$$H^0(\omega_X^{\otimes m}) \cong H^0(\omega_{X'}^{\otimes m}).$$

The canonical ring

- $R(\omega_X) = \oplus_{m\geq 0} H^0(\omega_X^{\otimes m})$ is the **canonical ring**.
- $\kappa(X) = tr.deg._{\mathbb{C}} R(\omega_X) - 1 \in \{-1, 0, 1, \ldots, \dim X\}$ is the **Kodaira dimension**.
- We have $\kappa(X) = \max\{\dim \phi_m(X)\}$.
- X is of **general type** if
  1. $\kappa(X) = \dim X$, or equivalently
  2. ϕ_m is birational for all $m \gg 0$ (i.e. $\phi_m|_U : U \to \phi_m(U)$ is an isomorphism for some non-empty open subset $U \subset X$), or equivalently
  3. $\dim H^0(\omega_X^{\otimes m}) = \frac{v \cdot m^d}{d!} + L.O.T.$ where $v := \mathrm{vol}(\omega_X)$.

The canonical ring of a curve

- When $d = \dim X = 1$, we say that X is a **curve**.
- A curve X is topologically a Riemann surface of genus g and there are 3 main cases:
- $\kappa(X) = -1$: Then $X \cong \mathbb{P}^1_{\mathbb{C}}$ is a **rational curve**. Note that $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and so $H^0(\omega_{\mathbb{P}^1}^{\otimes m}) = 0$ for all $m > 0$ i.e. $R(\omega_X) \cong \mathbb{C}$.
- $\kappa(X) = 0$: Then $\omega_X \cong \mathcal{O}_X$ and so $H^0(\omega_X^{\otimes m}) \cong \mathbb{C}$ for all $m > 0$ i.e. $R(\omega_X) \cong \mathbb{C}[t]$.
- In this case X is an **elliptic curve**. There is a one parameter family of these $x^2 = y(y-1)(y-\lambda)$.


$\mathbb{P}^1_{\mathbb{C}}$
$g=0$
$K(\mathbb{P}^1_{\mathbb{C}})=-1$
Rational curve
Curve of general type
$g \geq 2$
$K=1$
Elliptic curve
$g=1$
$K=0$

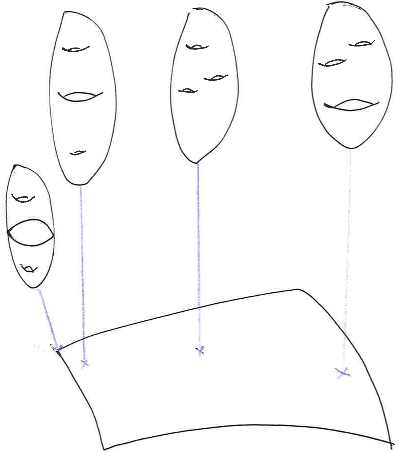
- If $\kappa(X) = 1$, then we say that X is a curve of **general type**. These are Riemann surfaces of genus $g \geq 2$.
- For any $g \geq 2$ there is a $3g - 3$ irreducible algebraic family of these curves.
- We have $\deg(\omega_X) = 2g - 2 > 0$.
- By Riemann Roch, it is easy to see that $\omega_X^{\otimes m}$ is **very ample** for $m \geq 3$. This means that if $s_0, \ldots, s_N$ are a basis of $H^0(\omega_X^{\otimes m})$, then

$$\phi_m : X \to \mathbb{P}^N, \qquad x \to [s_0(x) : s_1(x) : \ldots : s_N(x)]$$

  is an embedding.
- Thus $\omega_X^{\otimes m} \cong \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$, hence $\mathbb{C}[x_0, \ldots, x_N] \to R(\omega_X^{\otimes m})$ is surjective (in high degree) and so $R(\omega_X)$ is finitely generated.

Curves of general type

- It follows from Riemann Roch that $X \subset \mathbb{P}^{5g-6} = \mathbb{P}H^0(\omega_X^{\otimes 3})$ has degree $6g - 6 = 3\mathrm{vol}(\omega_X)$.
- Thus X depends on finitely many parameters (coefficients of the corresponding polynomials).
- We would like to generalize this picture to any dimension.



M_g $\dim = 3g-3$

CURVES OF GENERAL TYPE

$g \geq 2$

$\kappa(X) = 1$

$\deg(\omega_X) = 2g-2 > 0$

Finite generation

Theorem (Birkar-Cascini-Hacon-M^cKernan, Siu 2010)

Let X be a smooth complex projective variety, then the canonical ring $R(K_X)$ is finitely generated.

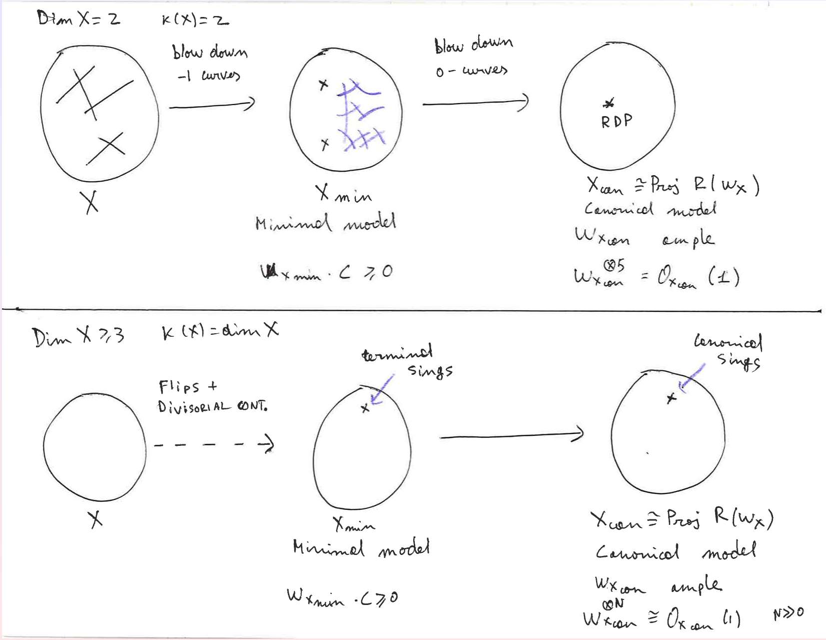
- If X is of general type ($\kappa(X) = \dim X$) then the **canonical model** $X_{\rm can} := \mathrm{Proj}(R(\omega_X))$ is a distinguished "canonical" (unique) representative of the birational equivalence class of X which is defined by the generators and relations in the finitely generated ring $R(\omega_X)$.
- $X_{\rm can}$ may be singular, but its singularities are mild (canonical). In particular they are cohomologically insignificant (rational sings) so that e.g. $H^i(\mathcal{O}_X) \cong H^i(\mathcal{O}_{X_{\rm can}})$.
- The "canonical line bundle" is now a $\mathbb{Q}$**-line bundle** which means that $\omega_{X_{\rm can}}^{\otimes n}$ is a line bundle for some $n > 0$.
- $\omega_{X_{\rm can}}$ is ample so that $\omega_{X_{\rm can}}^{\otimes m} = \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$ for some $m > 0$.

Surfaces of general type

- In dimension 2, the canonical model of a **surface** is obtained by first contracting all -1 curves ($E \cong \mathbb{P}^1$, $c_1(\omega_X) \cdot E = -1$) to get $X \to X_{\min}$,
- Then, we contract all 0-curves ($E \cong \mathbb{P}^1$, $c_1(\omega_X) \cdot E = 0$) to get $X_{\min} \to X_{\mathrm{can}}$.
- Bombieri's Theorem says that ϕ_5 embeds X_{can} in $\mathbb{P}^N = \mathbb{P}H^0(\omega_X^{\otimes 5})$ as a variety of degree $25c_1(\omega_{X_{\mathrm{can}}})^2$.
- So for any fixed integer $v = c_1(\omega_{X_{\mathrm{can}}})^2$, canonical surfaces depend on finitely many algebraic parameters.
- The number

$$v = c_1(\omega_{X_{\mathrm{can}}})^2 = \lim \frac{\dim H^0(\omega_X^{\otimes m})}{m^2/2},$$

is the **canonical volume**.


Dim X = 2 κ(X) = 2
blow down
-1 curves
X
X min
Minimal model
$\omega_{X_{min}} \cdot C \geq 0$
blow down
0 - curves
RDP
$X_{can} \cong \mathrm{Proj}\, R(\omega_X)$
Canonical model
$\omega_{X_{can}}$ ample
$\omega_{X_{can}}^{\otimes 5} = \mathcal{O}_{X_{can}}(1)$
Dim X ≥ 3 κ(X) = dim X
Flips +
DIVISORIAL CONT.
X
terminal
sings
X_{min}
Minimal model
$\omega_{X_{min}} \cdot C \geq 0$
canonical
sings
$X_{can} \cong \mathrm{Proj}\, R(\omega_X)$
Canonical model
$\omega_{X_{can}}$ ample
$\omega_{X_{can}}^{\otimes N} \cong \mathcal{O}_{X_{can}}(1)$ N ≫ 0

Surfaces of Kodaira dimension 1

- If $\kappa(X) = 1$, then there is a morphism to a curve, $X_{\min} \to C = \mathrm{Proj} R(\omega_X)$ whose fibers are elliptic curves (genus 1).
- The typical example is $E \times C$ where $g(E) = 1$ and $g(C) \geq 2$.
- These can be classified by studying families of elliptic curves.

Surfaces of Kodaira dimension 0

If $\kappa(X) = 0$, then $X_{\min}$ belongs to one of 4 well understood cases:

- **Abelian surfaces** $\mathbb{C}^2/\Lambda$,
- **K3 surfaces** (eg. degree 4 surface in $\mathbb{P}^3$),
- **bi-elliptic surfaces** (the quotien of an abelian surface by a finite group) and
- **Enriques surface**, (the quotien of a $K3$ by $\mathbb{Z}/2\mathbb{Z}$).
- In any case $\omega_X^{\otimes 12} \cong \mathcal{O}_X$.

- If $\kappa(X) = -1$, then there is a morphism to a curve, $X_{\min} \to C$ whose fibers are rational curves (genus 0).
- The typical example is $\mathbb{P}^2$ and $\mathbb{P}^1 \times C$ where $g(C) = \dim H^0(\Omega^1_X)$.
- In any case we have $\omega_{X_{\min}}|_F \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ so that $H^0(\omega_X^{\otimes m}) = 0$ for any $m > 0$, i.e. $R(\omega_X) \cong \mathbb{C}$.

Varieties of general type

- In higher dimensions, if X is of general type $d = \dim X = \kappa(X)$, then we define the canonical volume

$$\mathrm{vol}(X) = c_1(\omega_{X_{\mathrm{can}}})^d = \lim \frac{\dim H^0(\omega_X^{\otimes m})}{m^d/d!}.$$

- When $d = 1$, we have $\mathrm{vol}(X) = 2g - 2$.

Theorem (Hacon-McKernan, Takayama, Tsuji)

Let V_d be the set of canonical volumes of smooth projective d-dimensional varieties. Then V_d is discrete. In particular $v_d := \min V_d > 0$.

Thus $\mathrm{vol}(X)$ is the natural higher dimensional analog of the genus of a Rieman surface.

Theorem (Hacon-McKernan, Takayama, Tsuji)

Fix $d \in \mathbb{N}$ and $v \in V_d$, then the set $\mathcal{C}_{d,v}$ of d-dimensional canonical models $X_{\rm can}$ such that $vol(X_{\rm can}) = v$ is bounded (depends algebraically on finitely many parameters, and in particular has finitely many topological types).

Caution: For any fixed d and v, it is extremely hard/interesting to study the corresponding moduli space $\mathcal{C}_{d,v}$. Typically we don't even know if a given $\mathcal{C}_{d,v}$ is non-empty.

Conjecture

$\mathcal{C}_{3,v} = \emptyset$ *for all* $v < \frac{1}{420}$.

Caution: There is no integer $R(d) > 0$ such that $\omega_X^{\otimes R}$ is very ample for any d-dimensional canonical model. Such an integer must depend on both d and v.

Building on the above result one can show that in fact by work of Fujino, Hacon, Kollár, Kovács, McKernan, Patakfalvi, Xu and others, that

Theorem

The moduli space $\mathcal{C}_{d,v}$ can be compactified in a geometrically meaningful way by considering SLC-models, to a projective moduli space $\overline{\mathcal{C}}_{d,v}$.

One of the main steps is to show the boundedness of the moduli functor.

Theorem (Hacon-McKernan-Xu)

Fix $d \in \mathbb{N}$, then the set of volumes of SLC models

$$\mathcal{V}_d = \{\mathrm{vol}(X) | \dim X = d, \ X \text{ is a SLC model}\}$$

is well ordered, and for fixed d and $v \in V_d$, the set of d-dimensional SLC-models X such that $\mathrm{vol}(X) = v$ is bounded.

N.B. well ordered sets have no accumulation points from above but may have acumulation points from below e.g. $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$.

Varieties of negative Kodaira dimension

- At the opposite end of the spectrum we find varieties such that $\kappa(X) = -1$.

Theorem (Birkar-Cascini-Hacon-McKernan)

Assume that ω_X is not PSEF (conjecturally $\kappa(X) = -1$). Then there exists a birational map $X \dashrightarrow X'$ (given by a finite sequence of flips and divisorial contractions) and a Mori fiber space $X' \to Z$ such that the fibers F are positive dimensional, ω_F^{-1} is ample.

- When $d = 2$, then $F = \mathbb{P}^2$ or $\mathbb{P}^1$.
- Since $\omega_X|_F = \omega_F$ is negative, it follows easily that $H^0(\omega_X^{\otimes m}) = 0$ for all $m > 0$ and hence $R(\omega_X) = \mathbb{C}$ (which gives a geometric reason why $\kappa(X) = -1$).

Fano varieties

- The fibers of these Mori fiber spaces are **Fano varieties**, i.e. varieties F such that $\omega_F^\vee$ is ample.
- In order to understand Mori fiber spaces, we must hence study Fano varieties and their families.
- Note that we must allow F to have mild (terminal) singularities.
- The most important result in this direction is Birkar's celebrated solution of the BAB conjecture.

Theorem (Birkar)

The set of all terminal Fano varieties of dimension d is bounded.

Therefore, there exists a family $\mathcal{F} \to T$ such that for any d-dimensional Fano variety F with terminal singularities, then $F \cong \mathcal{F}_t$ for some $t \in T$.

Intermediate Kodaira dimension

- If $0 \leq \kappa(X) < \dim X$, then the **Iitaka fibration** $X \dashrightarrow Z = \mathrm{Proj} R(\omega_X)$ is a nontrivial fibration.
- The general fiber F has Kodaira dimension $\kappa(F) = 0$ and terminal singularities (if $\dim F = 1$, then $g(F) = 1$).
- It is thus important to study families of Calabi-Yau's.
- These varieties (Calabi-Yau's) are extremely interesting but very hard to study.
- It is known that they are not bounded (in dimension $d \geq 2$), never the less we have the following important conjecture:

Conjecture (Yau, $\sim$ late '70s)

The set of Calabi-Yau 3-folds ($\omega_X \sim \mathcal{O}_X$) has finitely many topological types.