A FLOER THEORY FOR TORIC ORBIFOLDS

- MAINAK PODDAR

BASED ON JOINT WORK WITH C.-H. CHO.

Symplectic Manifold

(M^{2n}, ω)

- $\omega \in \Omega^2(M)$, $d\omega = 0$
- $\omega|_p : T_pM \longrightarrow T_p^*M$ is an isomorphism $\forall\, p \in M$.

Lagrangian Submanifold

$L^n \subset M^{2n}$.

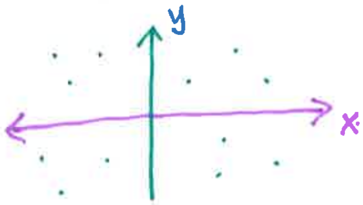
- $\omega(X, Y) = 0 \quad \forall\, X, Y \in T_pL$. $\forall\, p \in L$.

Examples.

$M = \mathbb{R}^{2n}$

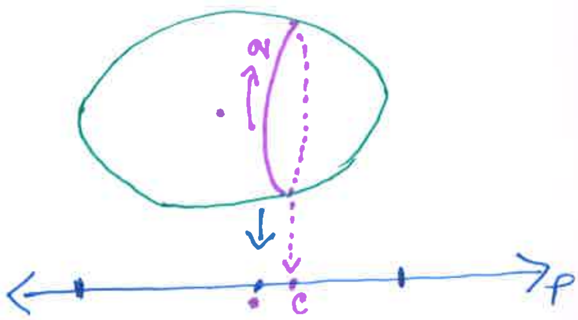
$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$

$L = \{ y_i = 0 \mid 1 \le i \le n \}$



$M = S^2 = \mathbb{C}P^1$.

$\omega = dp \wedge dq$



$L = \{ p = c \}$

$\begin{cases} x = \sqrt{p}\cos q \\ y = \sqrt{p}\sin q \end{cases}$

Hamiltonian vector field.

A vector field X on (M,ω) is called Hamiltonian if $\exists$ a smooth fn $H : M \to \mathbb{R}$ s.t.

$$dH = \omega(X, \cdot) =: i_X \omega$$

- Note if X is Hamiltonian then

$$\mathcal{L}_X \omega = d \circ i_X \omega + i_X \circ d\omega$$
$$= d\, dH + 0$$
$$= 0.$$

$\Rightarrow$ Flow of X preserves ω.

Let ϕ be the time $t=1$ diffeo generated by the flow of X.

Arnold's Conj: If M is closed, then

$$\# \operatorname{Fix} \phi \geq \sum_k \operatorname{rank} H_k(M) \quad (1)$$

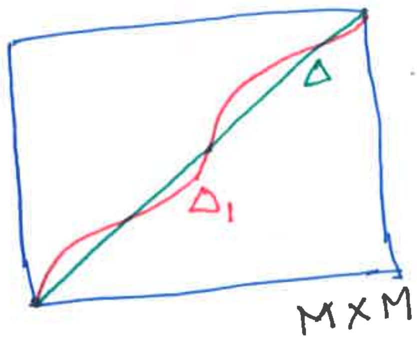
(Rem: Hamiltonian H may be time-dep, periodic.) with period $= 1$.

* Contrast (1) with Poincaré Hopf Theorem.

Lagrangian Intersections.

Consider M ~~as~~ embedded as the diagonal

$\Delta \subset (M \times M, \ \omega \oplus -\omega)$.



Δ_1 = Graph of ϕ.

$$\boxed{\mathrm{Fix}\,\phi = \Delta \cap \Delta_1}$$

Δ, Δ_1 are both Lagrangian in $(M \times M, \ \omega \oplus -\omega)$

This was Floer's original approach to Arnold's conj.

$\Rightarrow$ Lagrangian intersections have important information about Hamiltonian dynamics.

Morse Complex (Witten / Floer)

A C^∞ fn $f : M \to \mathbb{R}$ is Morse if

(i) Crit(f) is discrete

(ii) For any $p \in$ Crit (f), the Hessian $D^2 f(p)$ is nonsingular.

For $p \in$ Crit (f), index (p) = # negative e.values of $D^2 f(p)$

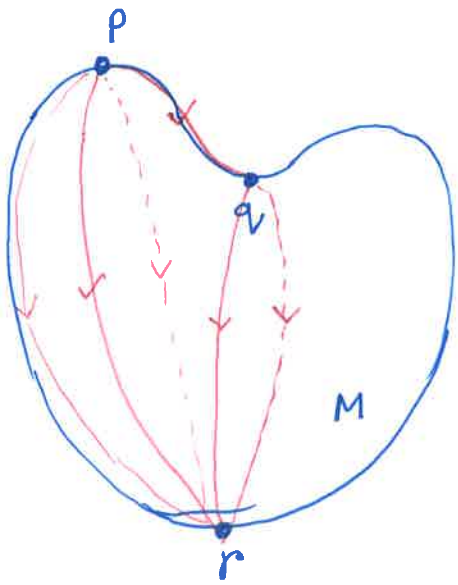
Morse chain complex (over $\mathbb{Z}$)

Generators = Crit(f)

$$\partial \langle p \rangle = \sum_q n_{pq} \langle q \rangle$$

s.t. index(q) = index$(p) - 1$.

n_{pq} = # −ve gradient flow lines from p to q.



f = height.

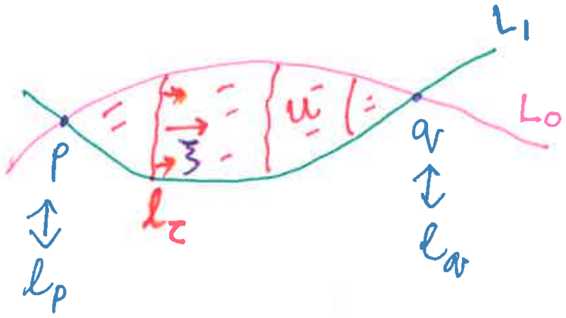
$$\boxed{\partial \circ \partial = 0}$$

$\Updownarrow$

For r s.t. index(r) = index$(p) - 2$

moduli of flow lines $p \to r$ is 1 dim manifold with $\{$broken trajectories $p \to q \to r\}$ as boundary.

Floer Chain Complex: $CF^\cdot(L_0, L_1)$



$\ell_\tau(t) = u(\tau, t)$.

u : J-holo.

J: Almost complex structure on M
compatible with ω $\Leftrightarrow$ $\omega(JX, JY) = \omega(X, Y)$.

(ω, J) Defines a metric $g = \omega(\cdot, J\cdot)$ on M.

L_0, L_1 : a pair of lagrangians that intersect transversally.

Then $CF^\cdot(L_0, L_1)$ generated (over a suitable ring) by elements of $L_0 \cap L_1$ or equivalently by constant paths from L_0 to L_1.

$$\mathcal{P} = \{ \ell : [0,1] \to M \mid \ell(0) \in L_0,\ \ell(1) \}.$$

Action functional

$$\mathcal{A}(u) = \int u^* \omega.$$

Energy

The surface u is given by a path in $\mathcal{P}$.

$$u : \underset{\tau}{\mathbb{R}} \times \underset{t}{[0,1]} \longrightarrow M.$$

(or $[0,1] \times [0,1] \to M$)
(or $D^2 \longrightarrow M$.)

Metric on $\mathcal{P}$: $\langle \xi_1, \xi_2 \rangle = \int_0^1 g(\xi_1(t), \xi_2(t))\, dt$

$$= \int_0^1 \omega(\xi_1(t), J\xi_2(t))\, dt$$

Gradient flow of $\mathcal{A}$ corresponds to J-holo u:

$$\frac{du}{d\tau} + J\frac{du}{dt} = 0.$$

Differential:

Maslov index: $\mu(u)$ — Consider a symplectic trivialization of u^*TM.

Any lagrangian in $\mathbb{C}^n = (\mathbb{R}^{2n}, \Sigma dx_i \wedge dy_i)$ can be expressed as $A\mathbb{R}^n$ where $A \in U(n)$.

So lagrangians are parametrized by $U(n)/O(n)$.

$\pi_1(U(n)/O(n)) \cong \mathbb{Z}$.

$\gamma \in U(n)/O(n)$, $\boxed{\mu(\gamma) := \deg(\det^2 \circ \gamma)}$

example: For standard disc $\subset \mathbb{C}$, $\mu = 2$.

$\boxed{\mu(u) = \mu(\gamma) \text{ where } \gamma \text{ is the loop formed by } (\partial u)^* TL.}$

Differential (contd)

Given $\langle p \rangle = \langle \ell_p \rangle$, the Floer differential should count $\langle \ell_q \rangle$ with multiplicity given by holomorphic discs "u" s.t.

~~st~~ 

, & $\mu(u) = 1$.

However, there are infinite possibilities, as u may have arbitrary energy.

However, by <u>Gromov compactness theorem</u> given <u>any</u> energy bound, there are only finitely many possibilities up to homotopy type (and reparametrization/automor of the domain) with markings if necessary

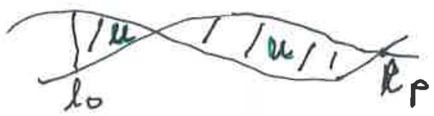
<u>Novikov Ring(s):</u>

$$\Lambda = \left\{ \sum_\beta a_\beta T^{\omega(\beta)} \;\middle|\; a_\beta \in \mathbb{R} \text{ (or } R\text{)},\ \beta \in \pi_2(M) \text{ sub. to (f.c.)} \right\}$$

(f.c.) $\#\{\beta \in \pi_2(M) : a_\beta \neq 0,\ \omega(\beta) \leq c\} < \infty$

for ~~#~~ all $c > 0$.

In practice, it is convenient to fix a base path $\langle l_0 \rangle$, and describe generators/elements of $CF^\bullet$ with respect to it



One also uses (subring or quotient) of

$$\Lambda_{0,nov} = \left\{ \sum a_i T^{\lambda_i} e^{n_i} \,\middle|\, a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R}_{\geq 0},\ n_i \in \mathbb{Z},\ \lim_{i\to\infty} \lambda_i = \infty \right\}$$

as coeff. ring of $CF^\bullet$.

The variable e is to capture maslov index : $n_i = \mu(\beta)/2$ $(\deg T = 0,\ \deg e = 2)$

Bubbling / Compactness of moduli.

$|du|$ is not bounded even if energy is bdd.

$\exists$ energy threshold (min.) for a bubble :

In a Darboux nbhd $\omega = d\lambda$.

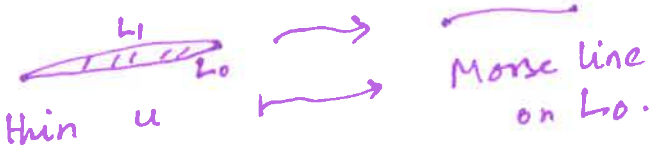
$$\int_{S^2} u_n^* \omega = \int_{S^2} d(u_n^* \lambda) = 0 .$$

Anomaly: • Bubbles break the symmetry observed in $\partial(\text{Moduli})$ in Morse complex.

In general $\partial \circ \partial \neq 0$.

• Bubbling is a codimension one phenomenon in Lag. Floer complex unlike in GW theory.
It has to be dealt with at chain level,
cannot be dealt at homology level.

• Floer had success in tackling Arnold's conj in some cases where bubbling is absent or balanced. Idea: $L_1 = \phi_1^H L_0 \simeq L_0$.
By thick-thin decomposition



• By early 2000's Fukaya-Oh-Ohta-Ono (FOOO) developed a very beautiful machinery of Filtered A^∞-algebra & its deformation to perturb $\partial \longmapsto \partial^b$ s.t. $\partial^b \circ \partial^b = 0$.

Filtered A^∞-algebra $(m_0 \neq 0)$

$C^k(L)$ = Free $\mathbb{Q}$-module gen. by "all" smooth singular k-simplices of L.

$$\boxed{C^\bullet = C^\bullet(L) \otimes \Lambda_{0,nov}}$$

degree in $C^\bullet$ = deg in $C^\bullet(L)$ + deg in $\Lambda_{0,nov}$.

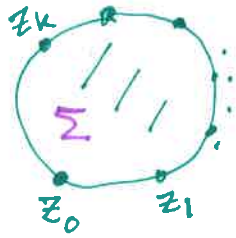
recall $\deg T^\lambda e^\mu = 2\mu$.

$F^\lambda \Lambda_{0,nov} = T^\lambda \Lambda_{0,nov}$

naturally induces a filtration F^λ on $C^\bullet$.

Bordered, marked discs:

$$\mathcal{M}^{main}_{k+1} = \Big\{$$



$$\Big\} \Big/ \text{auto}$$

$k+1$ marked points on $\partial\Sigma$ ordered in anticlockwise manner.

$$\mathcal{M}^{main}_{k+1}(\beta) = \{ u : (\Sigma, \partial\Sigma) \longrightarrow (M, L) \mid u \text{ is } J\text{-holo \& has homotopy class } \beta \in \pi_2(M, L) \}.$$

$$ev_i : \mathcal{M}^{main}_{k+1}(\beta) \longrightarrow L, \quad u \longmapsto u(z_i).$$

The operators m_k :

$$M^{main}_{k+1}(\beta;\vec{P}) \text{ "=" } \{u \in M^{main}_{k+1}(\beta) \mid ev_i(u) \in P_i\}$$

(ignoring transversality issues & virtual classes)

Here $\vec{P} = (P_1, P_2, \ldots, P_k)$

Def: $m_k : (C^\bullet)^{\otimes k} \longrightarrow C^\bullet$

$$\boxed{m_k = \sum_\beta m_{k,\beta} \otimes T^{\omega(\beta)} \otimes e^{\mu_L(\beta)/2}}$$

where

$k=0$ $$\boxed{m_{0,\beta}(1) = \begin{cases} ev_{0*}[M_1(\beta)] & \text{if } \beta \neq 0 \\ 0 & \text{if } \beta = 0 \end{cases}}$$

$k=1$ $$\boxed{m_{1,\beta}(P) = \begin{cases} ev_{0*}\left[M_2^{main}(\beta;P)\right] & \text{if } \beta \neq 0 \\ (-1)^n \partial P & \text{if } \beta = 0 \end{cases}}$$

for $k \geqslant 2$

$$\boxed{m_{k,\beta}(P_1, \ldots, P_k) = ev_{0*}\left[M^{main}_{k+1}(\beta;\vec{P})\right]}$$

A^∞ - relations.

Define $\hat{d} : C^{\bullet \otimes} \longrightarrow C^\bullet$ by $\hat{d} = \sum_0^\infty m_k$

A^∞-relations: $\hat{d} \circ \hat{d} = 0$

$$\Leftrightarrow \sum_{\beta_1+\beta_2=\beta} \sum_{k_1+k_2=k+1} \sum_i (-1)^{\Sigma \deg P_j + i - 1}$$

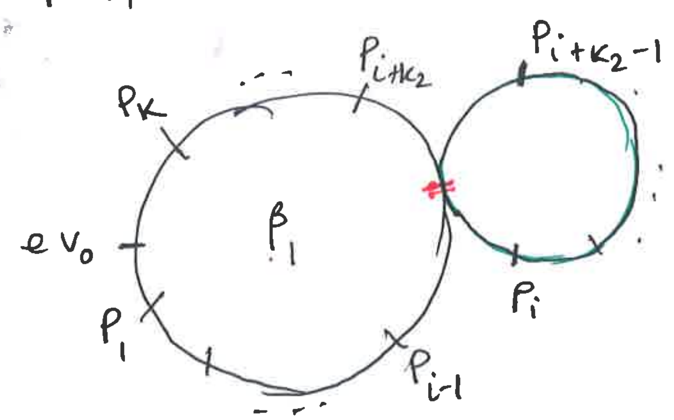
$$m_{k_1,\beta_1}(P_1 \ldots, m_{k_2,\beta_2}(P_i, \ldots, P_{i+k_2-1}) \ldots P_k) = 0$$

$$\Leftrightarrow m_{1,0}\, m_{k,\beta}(P_1, \ldots P_k) + \sum (-1)^{\cdots} m_{k\beta}(P_1 \ldots, m_{10}(P_i), \ldots P_k)$$

$$+ \sum_{\substack{\beta_1+\beta_2=\beta \\ \text{other terms}}} \sum (-1)^{\cdots} m_{k_1,\beta_1}(P_1 \ldots, m_{k_2\beta_2}(P_i, \ldots), \ldots P_k) = 0$$

$$\Leftrightarrow (-1)^{\cdots} \partial \left(ev_{0*} \left(\mathcal{M}^{main}_{k+1}(\beta, \vec{P}) \right) \right) + \sum (-1)^{\cdots} ev_{0*} \mathcal{M}^{main}_{k+1}(\beta, P_1, \ldots, \partial P_i, \ldots P_k)$$

$$+ \sum_{\substack{\text{other terms} \\ \beta=\beta_1+\beta_2}} (-1)^{\cdots} ev_{0*} \mathcal{M}^{main}_{k_1+1}(\beta_1; P_1, \ldots P_{i-1}, ev_{0*}(\mathcal{M}^{main}_{k_2+1}(\beta_2; P_i, \ldots)) \ldots P_k) = 0$$



Observe:

$$m_1 \leftrightarrow \partial.$$

But $m_2(m_0(1), x) \pm m_2(x, m_0(1)) + m_1(m_1(x)) = 0$

Deformation of ∂ or m_1:

• Def: $e_0 \in C^0$ is a unit if

- $m_2(x, e_0) = (-1)^{\deg x} m_2(e_0, x) = x$.
- $m_{k+1}(x_1, \dots, e_0, \dots x_k) = 0$ for $k=0$ or $k \geq 2$.

Thm: (FOOO) If $\exists\, b \in C^0$ with positive filtration level s.t. $m(e^b) := \sum_0^\infty m_k(b, \dots, b) = p\, e_0$

for some $p \in \Lambda_{0,nov}$

then the deformed operator

$$m_1^b : C^\bullet \to C^\bullet,$$

$$p \mapsto \sum m_{k_1+k_2+1}(\underbrace{b, \dots, b}_{k_1}, p, \underbrace{b, \dots, b}_{k_2})$$

has square zero.

Rem: A homotopy unit suffices.

<u>Thm (FOOO)</u>: Obstructions to existence of b lies in

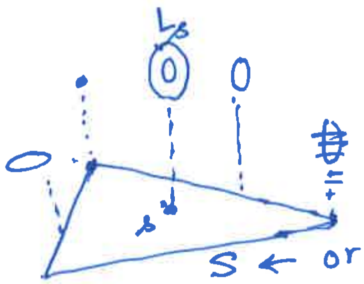
$$\frac{H^{ev}(L;\mathbb{Q})}{\mathrm{Im}\left(H^{ev}(M;\mathbb{Q}) \to H^{ev}(L;\mathbb{Q})\right)}.$$

<u>Bulk-deformation:</u> (FOOO)

One may also use cycles in the ambient manifold M to deform m_1.

The idea is to let a marked point migrate from $\partial\Sigma$ to $\mathrm{Int}(\Sigma)$.

<u>Calculations:</u> Cho-Oh, FOOO for toric manifolds.



$\exists\, s \in S$ s.t. L_s is non displaceable and

$\#(L_s \cap \phi L_s) \geq 2^n$.

$\underbrace{\text{Parameter space of such } s}_{\text{Rigidity locus}}$ has codim ≥ 1 in S.

Orbifold (toric) version (Cho, −)

- Use discs with orbifold singularities at interior points
- Use J-holo good maps (Chen-Ruan).
- Use "desingularized Maslov index" instead of Maslov index. (⇒ Use "desingularization" of the orbifold bundle "$\Omega^{0,1}(\Sigma) \otimes u^*TM$") (Chen-Ruan)

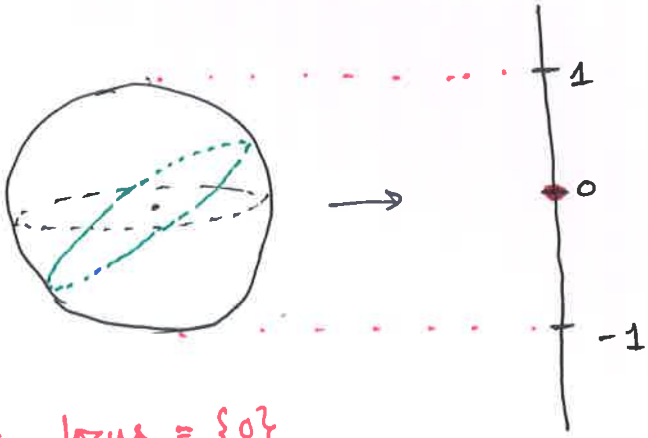
- Use Chen-Ruan cohomology classes of M for bulk deformation.

Results: We obtain examples of codim zero rigidity locus ~~of~~ of non displacable Lagrangian tori.

- Some of our results were obtained by Wilson − Woodward using a different approach.
  & Woodward

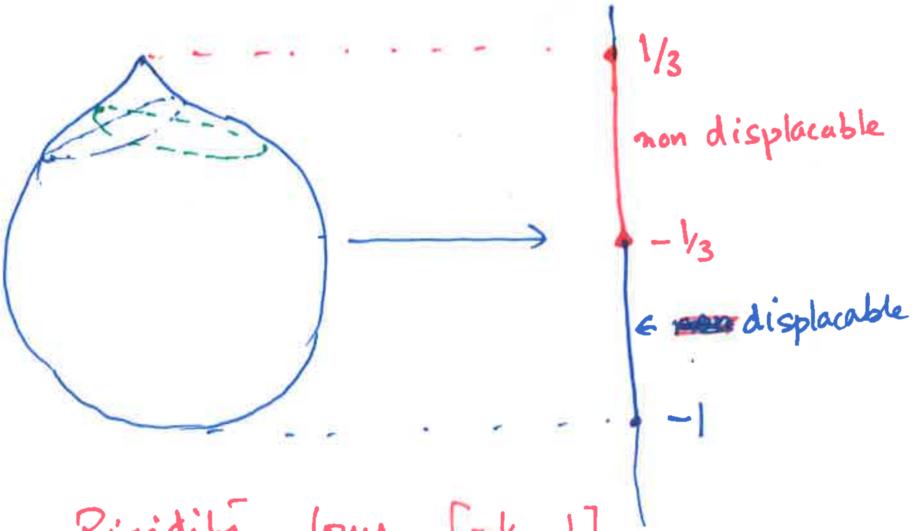
An example:

Ordinary S^2:



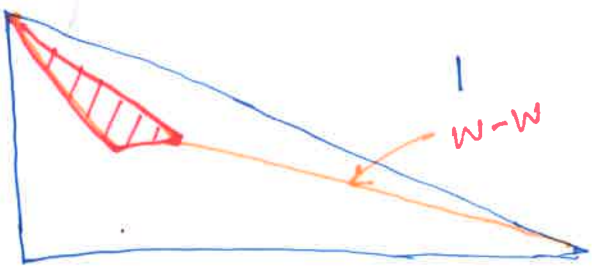
Rigidity locus = $\{0\}$

Orbifold S^2 with $\mathbb{Z}/3\mathbb{Z}$ singularity at N



Rigidity locus $[-\frac{1}{3}, \frac{1}{3}]$

— · —



$\mathbb{P}(1,3,5)$

References.

1) Fukaya, Oh, Ohta, Ono : Lagrangian intersection Floer theory – anomaly and obstruction, Part I, II. AMS/IP Studies in Adv. Math, 46.1. AMS, International Press, 2009.

2) Fukaya, Oh, Ohta, Ono: Lagrangian Floer theory on compact toric manifolds I, Duke Math J. vol. 151, no. 1, 2009.

3) Fukaya, Oh, Ohta, Ono: Lagrangian Floer theory on compact toric manifolds II : Bulk deformations. Selecta Math, New Series, Vol 17, no. 3 (2011), 609 – 711.

4) Cho, Poddar : Holomorphic orbidiscs and Lagrangian Floer cohomology of symplectic toric orbifolds, J. Differential Geometry, Vol 98, no 1 (2014), 21 – 116.

5) Wilson, Woodward : Quasimap Floer cohomology and varying symplectic quotients, Canad. J. Math. 65 (2013), 467 – 480.