Seshadri constants on algebraic surfaces

Praveen Kumar Roy

TIFR

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- A line bundle $\mathcal{L}$ on X is said to be ample if $\mathcal{L}^{\otimes m}$ is very ample for some $m \in \mathbb{Z}$.
- A line bundle $\mathcal{L}$ on X is said to be *nef if* $\mathcal{L} \cdot C \geq 0$ *for every curve* $C \subset X$.

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- $Num(X) = \frac{Div(X)}{\sim}$ denotes the group of divisors modulo numerical equivalence.
- Seshadri criterion for ampleness of a line bundle.

Theorem (Seshadri criterion)

Let X be a projective variety and $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is ample if and only if $\exists\ \varepsilon > 0$ such that

$$\mathcal{L} \cdot C \geq \varepsilon \cdot mult_x C,$$

for every $x \in X$ and every irreducible curve $C \subset X$ passing through x.

- Motivated by the above criteria, Demailly defined the Seshadri constants of an ample line bundle to quantify the positivity of ample line bundle.

Definition (Seshadri constant at a point, Demailly, 1987)

Let X be a projective variety and $\mathcal{L}$ be an ample (nef) line bundle on X, then the Seshadri constant of $\mathcal{L}$ at a point $x \in X$ is defined as

$$\varepsilon(X; \mathcal{L}, x) := \inf_{x \in C \subset X} \left\{ \frac{\mathcal{L} \cdot C}{mult_x C} \right\}.$$

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$$\pi : Bl_x X \to X$$

such that E is the exceptional divisor. Then,

$$\varepsilon(X; L, x) = \max\{\lambda : \pi^* L - \lambda \cdot E \textit{ is nef}\}.$$

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- Note that $\pi^* L - \lambda \cdot E$ is nef gives

$$\begin{aligned} (\pi^* L - \lambda)^n &\geq 0, \\ \Rightarrow \quad L^n &\geq \lambda^n, \\ \Rightarrow \quad \sqrt[n]{L^n} &\geq \lambda. \end{aligned}$$

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- $\varepsilon(X; L, x_1, x_2, \ldots, x_r) := \inf_{C \cap \{x_1, x_2, \ldots, x_r\} \neq \phi} \frac{L \cdot C}{\sum_{i=1}^{i=r} mult_{x_i} C}$

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- It is easy to see that the following set of inequalities holds

$$0 < \varepsilon(X; L) \leq \varepsilon(X; L, x) \leq \varepsilon(X; L, 1) \leq \sqrt[n]{L^n}.$$

Here n is the dimension of X.

Example

Let $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$ be a an ample line bundle on X. Then

$$\varepsilon(X; L, x) = 1,$$

for every $x \in X$.

Proof.

Fix $x \in X$. Let $\ell \subset X$ denote the line passing through x. Therefore we get $\varepsilon(X, L, x) \leq (L \cdot \ell)/1 = 1$. Now, let $C \subset X$ be a curve of degree d passing through x with multiplicity m, then Bézout's theorem says that $d \geq m$. Therefore $d/m \geq 1$. Hence $\varepsilon(X, L, x) = 1$. $\square$

Example

Let $p_1, p_2, p_3 \in \mathbb{P}^2$ are three points and $\ell \subset \mathbb{P}^2$ denotes a line. Then the multi-point Seshadri constant

$$\varepsilon(\mathbb{P}^2; \ell, p_1, p_2, p_3) = \begin{cases} \frac{1}{3}, & \text{if the points are collinear} \\ \frac{1}{2}, & \text{else.} \end{cases}$$

Sketch of the proof:

Collinear: Let $l \subset \mathbb{P}^2$ be the line containing those points. Then $\varepsilon(\mathbb{P}^2; \ell, p_1, p_2, p_3) \leq (\ell \cdot l)/3 = 1/3$. If $C \subset \mathbb{P}^2$ be any curve containing these points with multiplicities m_1, m_2, m_3 then Bézout's theorem gives

$$\begin{aligned} deg(C) = C \cdot l &\geq m_1 + m_2 + m_3, \\ \Rightarrow \frac{deg(C)}{m_1 + m_2 + m_3} &\geq 1 > \frac{1}{3}. \end{aligned}$$

Non-collinear: Let $l \subset \mathbb{P}^2$ denote the line passing through any two of the points. Then we get $\varepsilon(\mathbb{P}^2; \ell, p_1, p_2, p_3) \leq (\ell \cdot l)/2 = 1/2$. As in the previous case take C be any curve of degree d passing with p_i with multiplicity m_i. Then using bezouts theorem

$$\begin{aligned} d = C \cdot l_3 &\geq m_1 + m_2, \\ d = C \cdot l_2 &\geq m_1 + m_3, \\ d = C \cdot l_1 &\geq m_2 + m_3, \\ \Rightarrow 3d &\geq 2(m_1 + m_2 + m_3), \\ \Rightarrow d/(m_1 + m_2 + m_3) &\geq 2/3 > 1/2. \end{aligned}$$

Seshadri constant was introduced by Demailly to understand the following conjecture.

Conjecture (Fujita, 1987)

Let X be a smooth projective algebraic variety of dimension n over an algebraically closed field of characteristic zero. Let K_X be the canonical line bundle on X and A be an ample Cartier divisor on X. Then the adjoint line bundle

1. $\mathcal{O}_X(K_X + mA)$ *is globally generated for all* $m \geq n+1$,
2. $\mathcal{O}_X(K_X + mA)$ *is very ample for all* $m \geq n+2$.

The following theorem relates the Fujita's conjecture with the Seshadri constants.

Theorem (Ein, Lazarsfeld)

Let X be a smooth projective variety of dimension n. Let K_X be the canonical line bundle on X and L be a big and nef line bundle on X. Then,

1. $\varepsilon(L,x) > n$ *for all* $x \in X \implies K_X + L$ *is globally generated.*
2. $\varepsilon(L,x) > n+1$ *for all* $x \in X \implies K_X + L$ *is very ample.*

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- Note that

$$\varepsilon(L,x) > 1, \ \forall\, x \in X \ \Rightarrow \ \varepsilon(mL,x) > n, \ \ \forall\, m \geq n+1.$$

Since,

$$\varepsilon(L,x) > 1 > n/m \Rightarrow \varepsilon(mL,x) > n.$$

- By the previous theorem of Ein and Lazarsfeld we get
  1. $\varepsilon(mL, x) > n$ for all $x \in X \Rightarrow K_X + mL$ is globally generated for $m \geq n+1$,
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- But it is too optimistic to prove that $\varepsilon(L, x) > 1$ for all $x \in X$, as the following example of Miranda suggests.

Example (Miranda)

For any given $\delta > 0$ there exists a smooth projective surface X, a point $x \in X$ and an ample line bundle L on X such that

$$\varepsilon(X; L, x) < \delta.$$

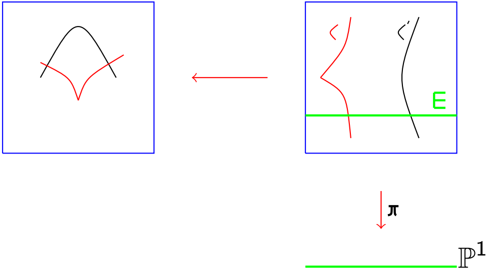
Sketch of the proof : Consider a curve $\Gamma \subset \mathbb{P}^2$ of degree d having an m-fold point $p \in \Gamma$, with $\delta > 1/m$. Choose a second curve $\Gamma' \subset \mathbb{P}^2$ of degree d that meets Γ transversely. We can choose $d >> 0$ and Γ' sufficiently general so that the linear system $\mathcal{L}$ generated by Γ and Γ' are irreducible. We then consider the surface obtained after the blow up of base points of this system.

$$\mu : X := Bl_{\Gamma \cap \Gamma'}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$$

The linear system $\mathcal{L}$ defines a mapping

$$\pi : X \longrightarrow \mathbb{P}^1$$

Let C and C' are curves in X isomorphic to Γ and Γ' appearing as fibres of π.



Let $L = aC + E$ for $a \geq 2$, then it can be verified that L is ample and

$$\varepsilon(X; L, x) \leq \frac{L \cdot C}{mult_x C} = \frac{1}{m} < \delta.$$

So the natural questions arises are as follows

- Can we compute $\varepsilon(X; L, x)$, $\varepsilon(X; L)$, $\varepsilon(X; L, r)$, $\varepsilon(X; L, x_1, x_2, ..., x_r)$?

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- Or at least can we bound them ?
- Is $\varepsilon(X; L) \in \mathbb{Q}$?
- Open questions
  1. Is there a pair (X, L) such that $\varepsilon(X; L) = 0$?
  2. Is there a triple (X, L, x) such that $\varepsilon(X; L, x) \notin \mathbb{Q}$?

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- $\varepsilon(X; L) \in \mathbb{Q}$ for
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  - K3 surfaces of degree 6 and 8
  - Most hyperelliptic surfaces
  - Some surfaces of general type

In the direction of irrational Seshadri constant

Following is known in the direction of finding an irrational Seshadri constant.

Theorem (Dumnicki, Küronya, Maclean, Szemberg)

Let $\pi : X \to \mathbb{P}^2$ denotes the blow up of $\mathbb{P}^2$ at s (≥ 9) very general points for which the SHGH Conjecture holds true. Then

1. *either there exists an ample line bundle L on X and a point $p \in X$ such that*

$$\varepsilon(X; L, p) \text{ is irrational,}$$

2. *or the SHGH conjecture fails for $s + 1$ points.*

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$\star$ Any prime divisor C on Y with $C^2 < 0$ is an exceptional curve.

Theorem (Hanumanthu, Harbourne)

Assumption of $\star$ *on* Y *implies the existence of an ample line bundle* L *on* X *with* $\varepsilon(X; L, x)$ *being irrational if and only if* $s \geq 9$.

Let X be smooth complex projective variety and L be a line bundle on X. Consider the linear system $|mL|$ for $m \in \mathbb{N}$. The global sections of mL defines a rational map

$$\phi_{mL} : X \dashrightarrow \mathbb{P}(H^0(X, mL)).$$

Clearly, the $dim(\phi_{mL}(X)) \leq dim(X)$.

Definition

$\kappa(X, L) := \max\{dim(\phi_{mL}(X)) : m \in \mathbb{N}\}$.

Definition

Given a smooth complex projective variety X with canonical divisor K_X, the *Kodaira dimension* of X is defined as $\kappa(X, K_X)$.

Introduction of Hyperelliptic surfacces

Definition

A hyperelliptic surface X is a minimal smooth surface with Kodaira dimension $\kappa(X) = 0$ satisfying $h^1(X, \mathcal{O}_X) = 1$ and $h^2(X, \mathcal{O}_X) = 0$.

Alternate characterization

A smooth surface X is hyperelliptic if and only if $X \cong (A \times B)/G$, where A and B are elliptic curves and G is a finite group of translation of A acting on B in such a way that $B/G \cong \mathbb{P}^1$. We have the following diagram:

$$\begin{array}{ccc} X \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \downarrow{\scriptstyle \Psi} & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

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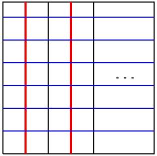

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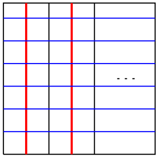
- 
- Let $m_1, m_2, ...m_s$ denote the multiplicities of the singular fibres and s denotes the number of them.

- Let $\mu = lcm(m_1, m_2, ...m_s)$ and $\gamma = |G|$, then $Num(X)$ is generated by A/μ and $(\mu/\gamma)B$.

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- L is ample if and only if $a, b > 0$.
- There are seven types of hyperelliptic surfaces depending on G.

Theorem

Let $X \cong (A \times B)/G$ be a hyperelliptic surface. A basis for the group $Num(X)$ of divisors modulo numerical equivalence and the multiplicities of the singular fibres of $\Psi : X \to B/G$ in each type are given in the following table.

Type of X	G	$m_1, m_2, \ldots, m_s$	*Basis of* $Num(X)$
1	$\mathbb{Z}_2$	$2, 2, 2, 2$	$A/2,\ B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2, 2, 2, 2$	$A/2,\ B/2$
3	$\mathbb{Z}_4$	$2, 4, 4$	$A/4,\ B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2, 4, 4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3, 3, 3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3, 3, 3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2, 3, 6$	$A/6, B$

Theorem (Hanumanthu, Roy)

Let X be a hyperelliptic surface of type different from 6 and let L be an ample line bundle on X. Then $\varepsilon(L) \in \mathbb{Q}$.

Proof

Let X be of type 1 hyperelliptic surface. Let $L \equiv (a, b)$ be an ample line bundle. We can also write L as $a \cdot \frac{A}{2} + b \cdot B$ where $A/2$ and B are the generators of $Num(X)$. Notice that, through any point $x \in X$ there are copies of A and B passing through it. Therefore,

$$\varepsilon(X, L) \leq \frac{L \cdot B}{mult_x B} = a,$$

$$\varepsilon(X, L) \leq \frac{L \cdot A}{mult_x A} = \frac{2b}{2} = b.$$

Therefore we see that $\varepsilon(X, L) \leq \min(a, b)$. Now, we will show that $\varepsilon(X, L) \geq \min(a, b)$. To this end, let $C \equiv s \cdot A/2 + t \cdot B$ ($\neq A$, B) be a reduced and irreducible curve in X passing through x with multiplicity m. Then by Bézout's theorem we get

$$\begin{aligned} s = C \cdot B \geq m \cdot 1 \Rightarrow bs \geq bm, \\ 2t = C \cdot A \geq m \cdot 2 \Rightarrow at \geq am. \end{aligned}$$

Thus we have

$$\frac{L \cdot C}{m} = \frac{at + bs}{m} \geq a + b \geq \min(a, b),$$

and therefore $\varepsilon(X, L) = \min(a, b)$.

In fact it is true that $\varepsilon(X; L) = \min(a, b)$ for all ample line bundle on every hyperelliptic surfaces of odd type. For even type, except of type 6, we show that $\varepsilon(X; L, x) < \sqrt{L^2} = \sqrt{2ab}$ for some $x \in X$. Then the proof follows from the following theorem

Theorem (Bauer, Szemberg)

Let X be a smooth projective surface and L be an ample line bundle on X. If there exists a point $x \in X$ such that $\varepsilon(X, L, x) < \sqrt{L^2}$, then $\varepsilon(X, L) \in \mathbb{Q}$.

Theorem (Hanumanthu, Roy)

Let X be a hyperelliptic surface and let L be an ample line bundle on X. If $\varepsilon(L,1) < (0.93)\sqrt{L^2}$, then $\varepsilon(L,1) = \min(L \cdot A, L \cdot B)$.

Definition (Surface of General type)

A smooth complex algebraic surface X is said to be of *general type* if the Kodaira dimension $\kappa(X) = 2$.

Example

Let $S = C \times D$ where C and D are two curves of genus ≥ 2 then S is a surface of general type. Now let p and q denotes the two projection then $K_S = p^*(K_C) \otimes q^*(K_D)$ and the rational map $\phi_{nK_S} : S \dashrightarrow \mathbb{P}^N$ factorises as

$$\phi_{nK_S} : C \times D \overset{(\phi_{nK_C}, \phi_{nK_D})}{\dashrightarrow} \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \overset{s}{\hookrightarrow} \mathbb{P}^N$$

where s is the Segre embedding (defined by $(x_i), (y_j) \mapsto (x_i y_j)$). Since $\kappa(C) = \kappa(D) = 1$ we see that $\kappa(S) = 2$ and hence S is a surface of general type. In fact if S is any surface fibred over a curve of genus at least 2, whose generic fibre is of genus at least two, is a surface of general type.

Multi-point Seshadri constant

Theorem

Let X be a surface of general type and K_X be the canonical line bundle on X. If K_X is big and nef and $x_1, x_2, ..., x_r \in X$ are $r \geq 2$ points, then we have the following.

1. $\varepsilon(X; K_X, x_1, x_2, ..., x_r) = 0 \Leftrightarrow$ *at least one of x_i lies on one of the finitely many (-2)-curves on X.*
2. *If $0 < \varepsilon(X; K_X, x_1, x_2, ..., x_r) < \frac{1}{r}$, then the Seshadri curve C satisfies $K_X \cdot C \leq 2$ and*

$$\varepsilon(X; K_X, x_1, x_2, ..., x_r) = \begin{cases} \frac{1}{3} \text{ or } \frac{1}{4} \text{ or } \frac{2}{5} & \text{if} \quad r = 2, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} & \text{if} \quad 3 \leq r < 9, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} \text{ or } \frac{1}{r+3} & \text{if} \quad r \geq 9. \end{cases}$$

Theorem

Let $X = C \times C$, where C is a general member of moduli of smooth curves of genus $g \geq 2$. Let $L \equiv_{num} a_1 F_1 + a_2 F_2 + a_3 \delta$ be an ample line bundle satisfying any of the following conditions on a_1, a_2 and a_3.

1. $a_3 = 0$,
2. $a_3 > 0$, $a_1 \leq a_2$ and $a_1^2 + a_3^2 < 2a_1 a_2$,
3. $a_3 < 0$ and $a_2 \geq \left(\frac{2gk^2+2k+1}{2(k+1)}\right) \cdot a_1$, where $k = \lceil \frac{|a_3|/a_1}{1-|a_3|/a_1} \rceil$.

Then $\varepsilon(X; L) \in \mathbb{Q}$.

Thank You!