On the low-density limit of the Lorentz gas for general scatterer configurations

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based on joint work with Jens Marklof

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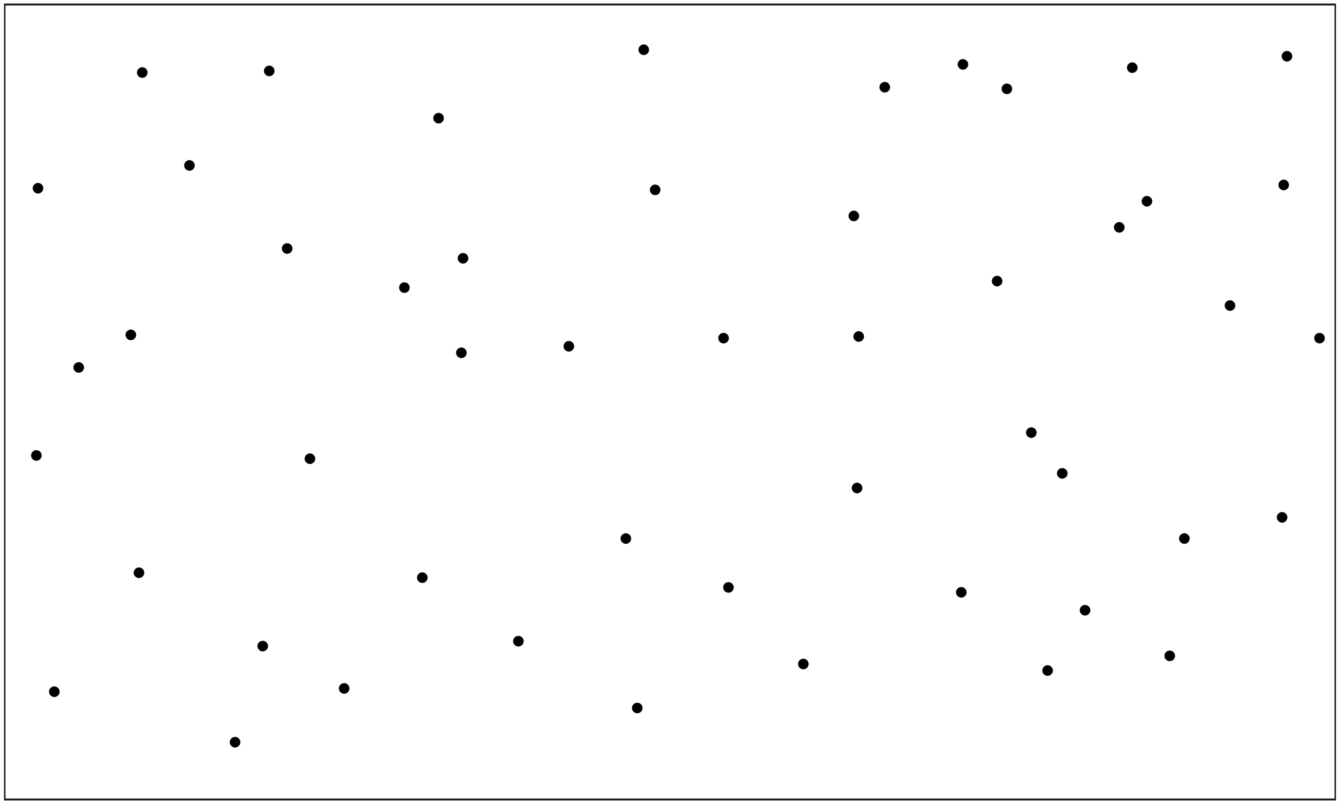
The Lorentz gas

(Hendrik Lorentz, 1905, “The motion of electrons in metallic bodies”)



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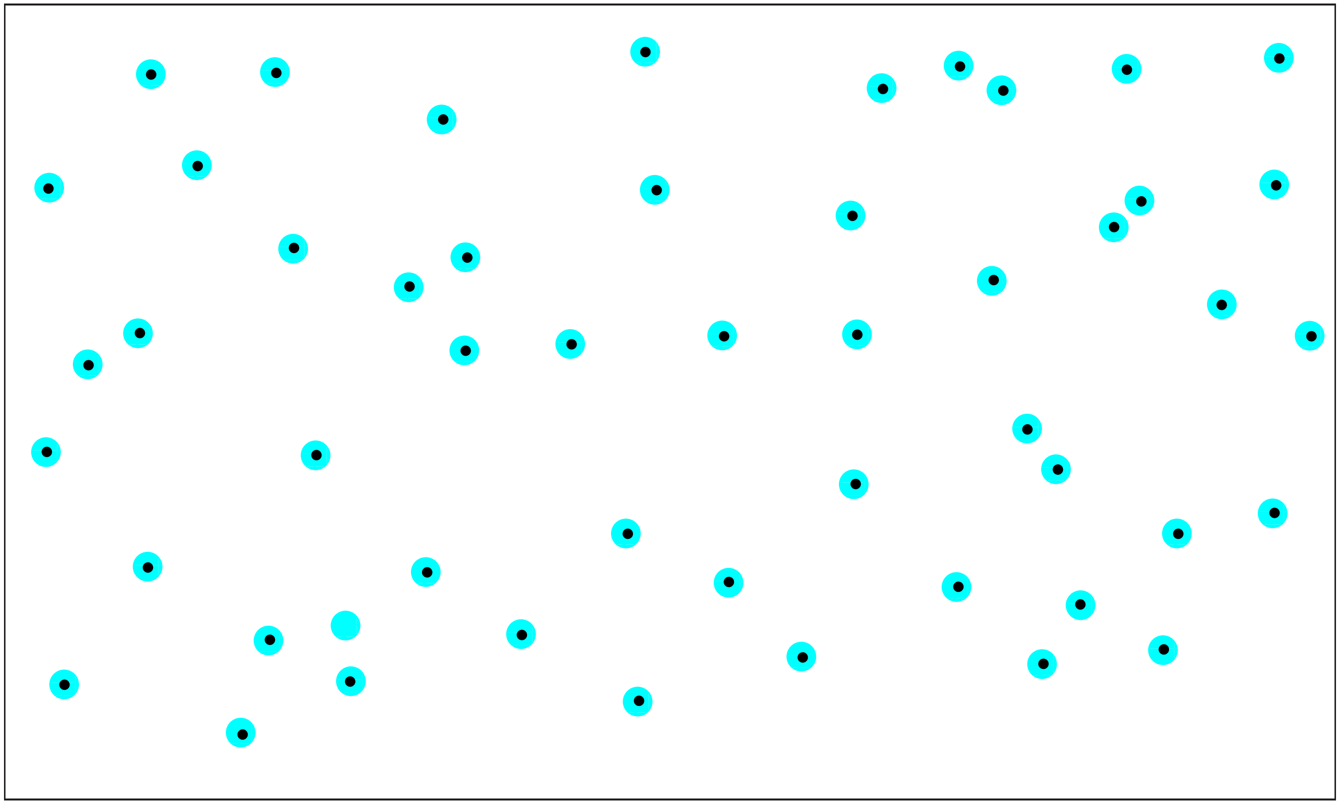

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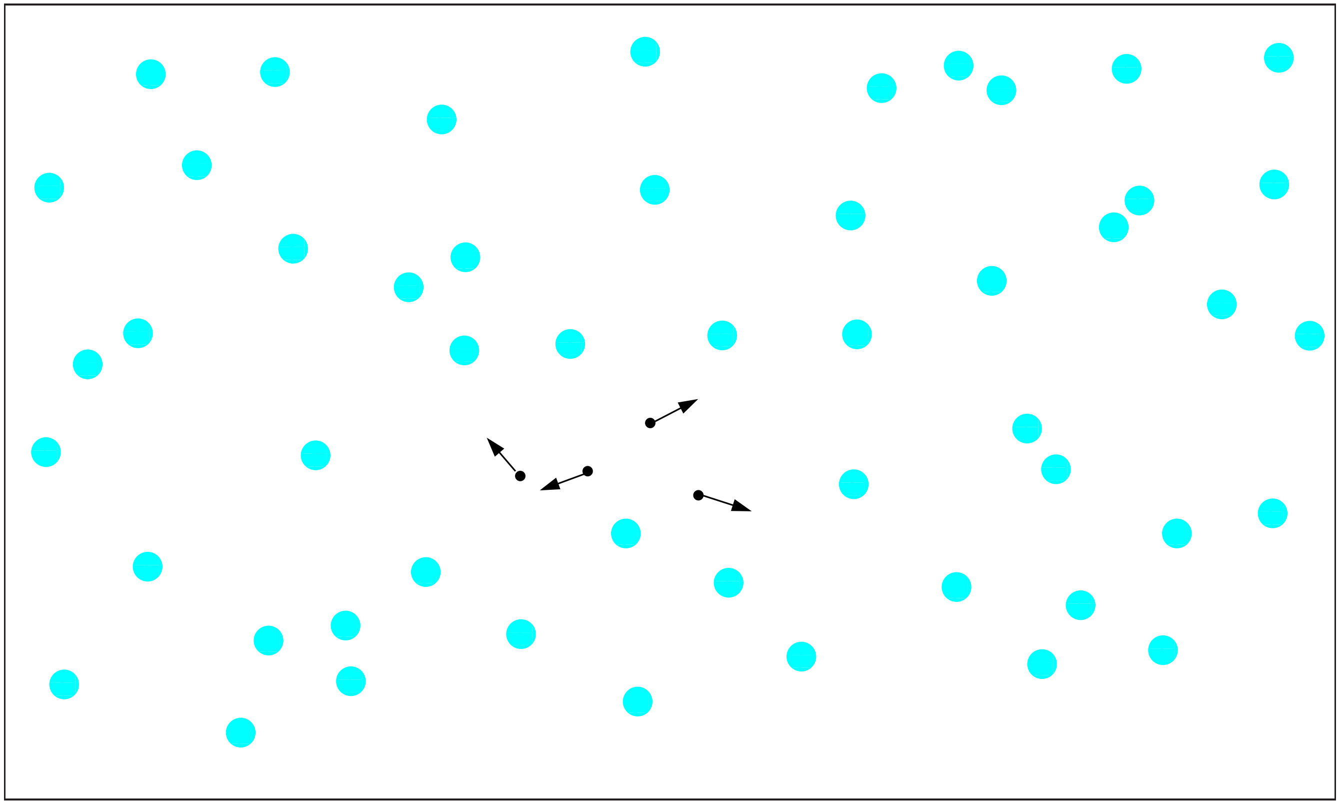

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Non-interacting point particles moving in $K_r := \mathbb{R}^d - (\mathcal{P} + \mathcal{B}_r^d)$.

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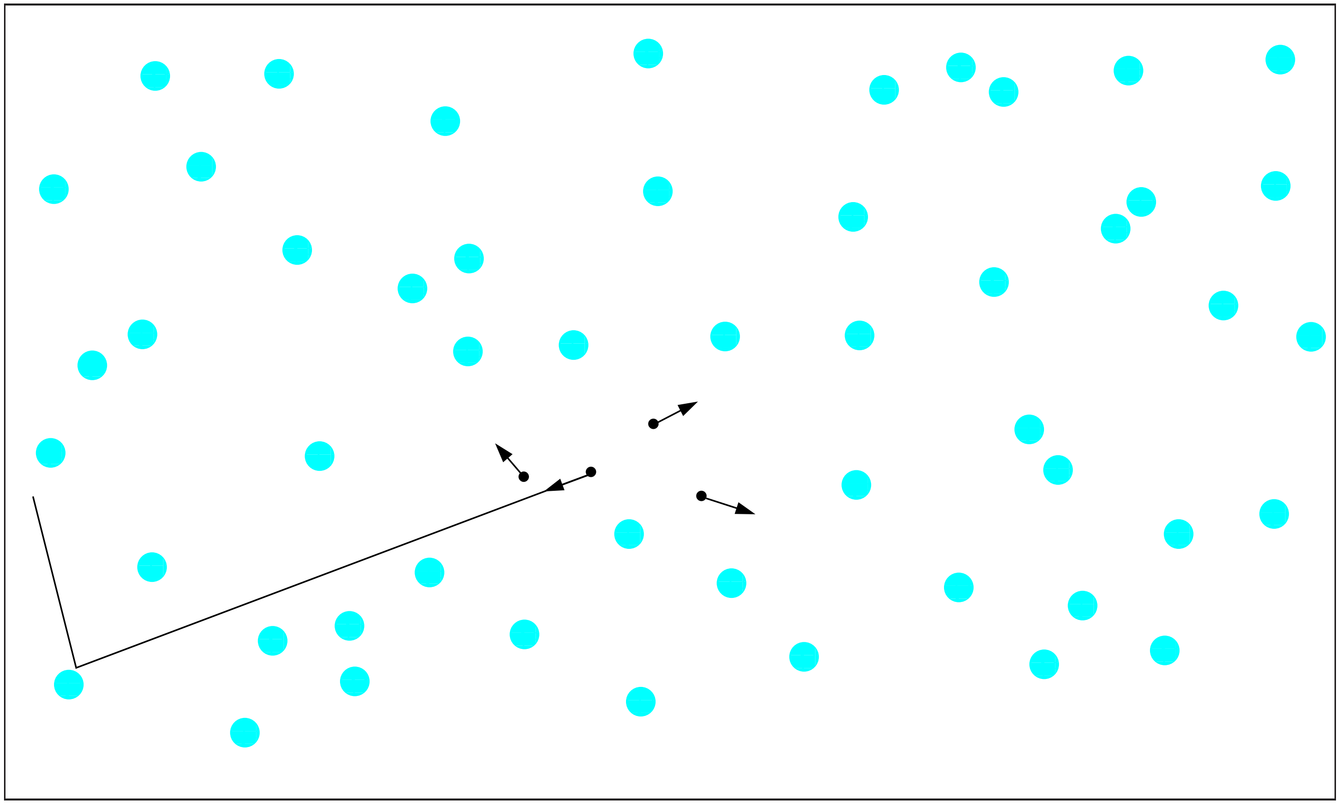

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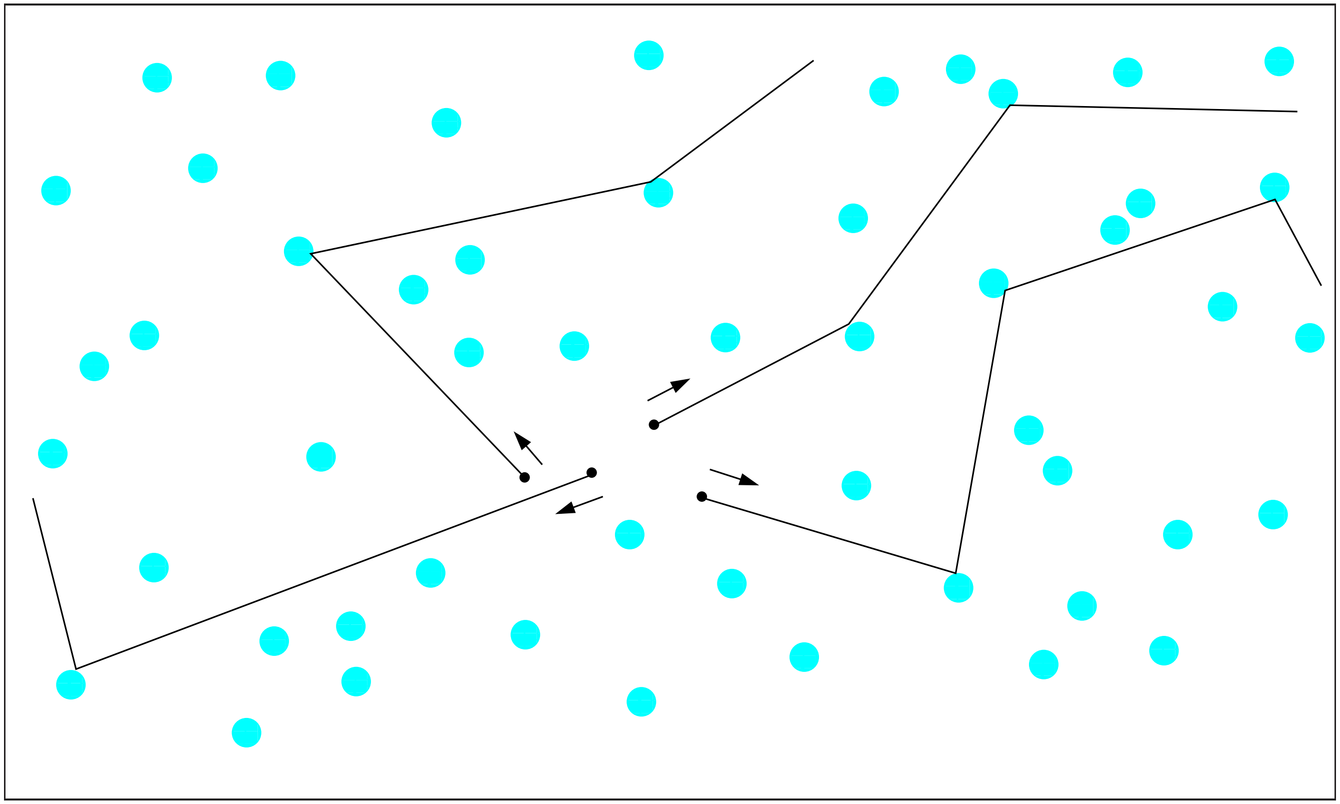

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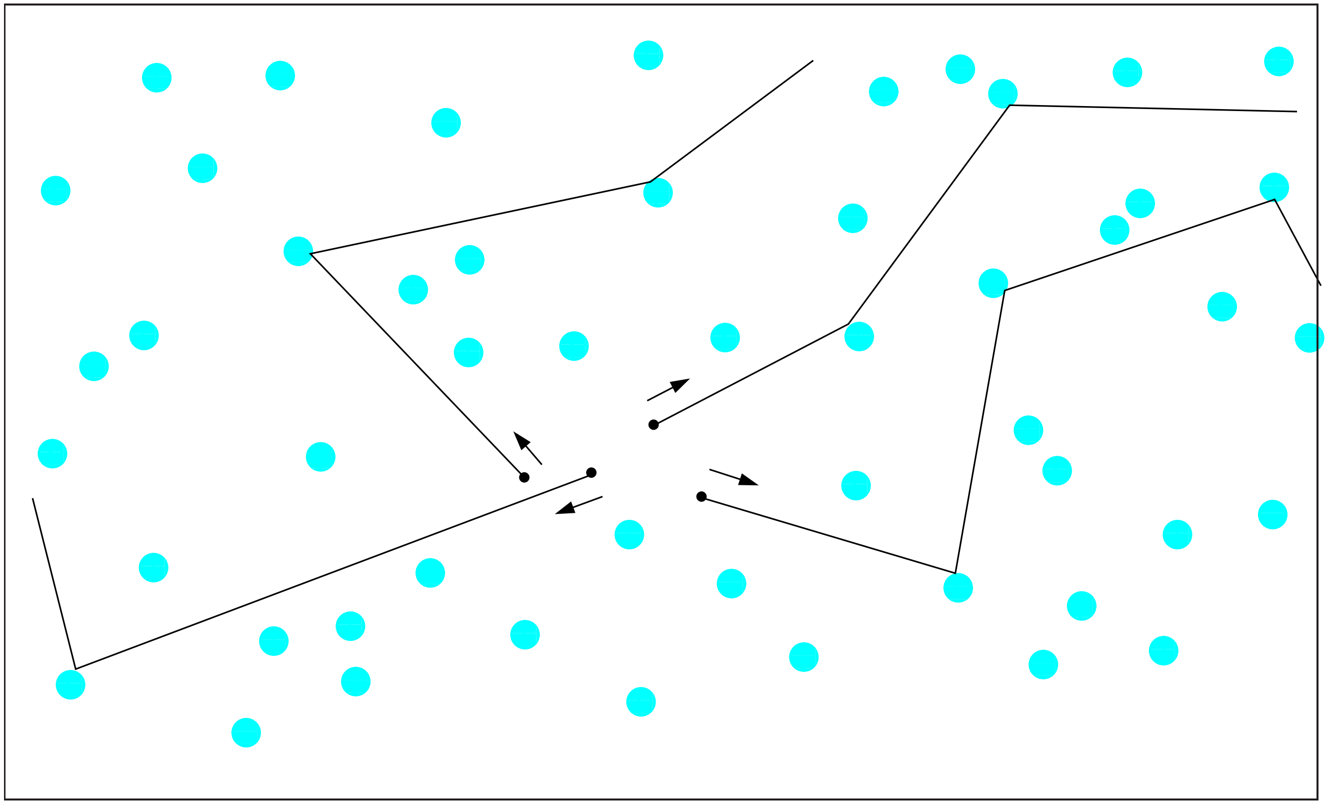

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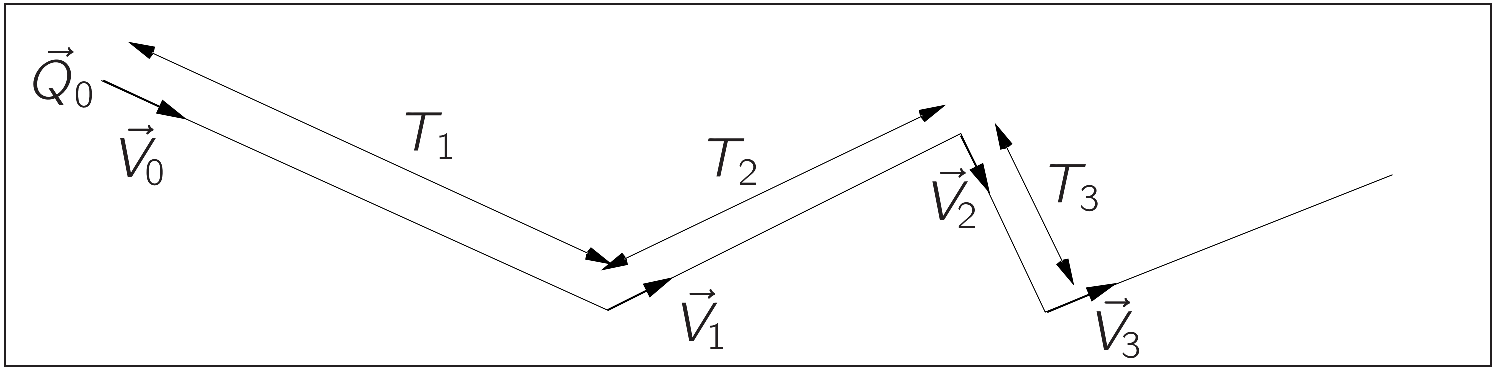
Boltzmann-Grad limit:

Let $r \to 0$ in macroscopic coordinates $(\vec{Q}, \vec{V}) = (r^{d-1}\vec{q}, \vec{v})$

Boltzmann-Grad limit for $\mathcal{P}$ "Poisson"

Let Λ be a.c. prob. measure on $\mathrm{T}^1(\mathbb{R}^d) = \mathbb{R}^d \times \mathrm{S}_1^{d-1}$.

For $(\vec{Q}_0, \vec{V}_0)$ random in $(\mathrm{T}^1(\mathbb{R}^d), \Lambda)$, define:



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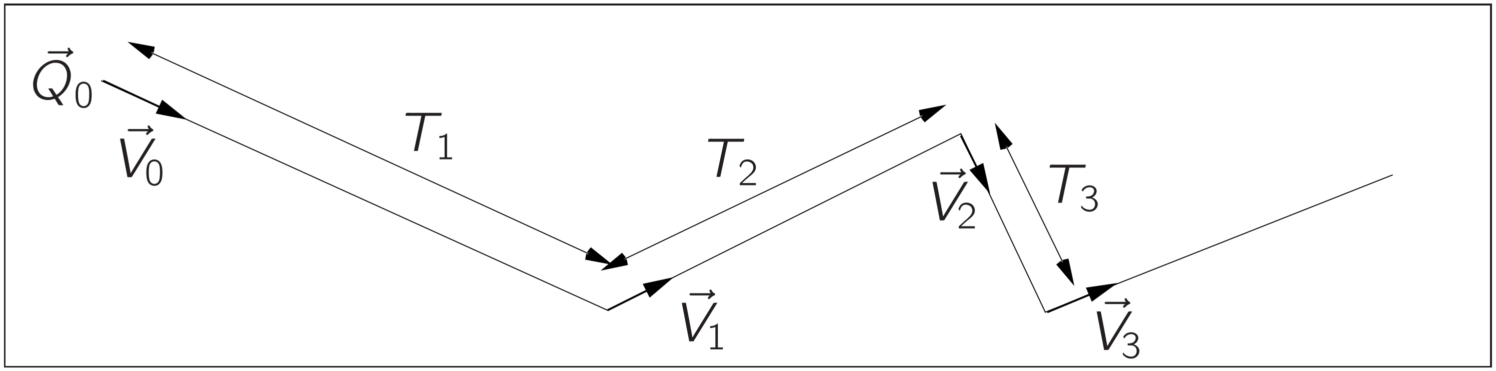
Theorem (Boldrighini-Bunimovich-Sinai, 1983) For $\mathcal{P}$ a *fixed realization of a Poisson process in* $\mathbb{R}^d$ *of intensity* 1, one has almost surely:

$$\langle (\vec{Q}_0, \vec{V}_0), (T_1, \vec{V}_1), \ldots, (T_n, \vec{V}_n) \rangle$$

$$\xrightarrow[(r \to 0)]{\mathrm{d}} \quad \text{r.v. with density } \Lambda'(\vec{Q}_0, \vec{V}_0) \prod_{j=1}^{n} \Big(\sigma(\vec{V}_{j-1}, \vec{V}_j) e^{-\overline{\sigma} T_j} \Big).$$

($\sigma(\vec{V}', \vec{V})$ = differential cross section of a scatterer; $\overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1})$)



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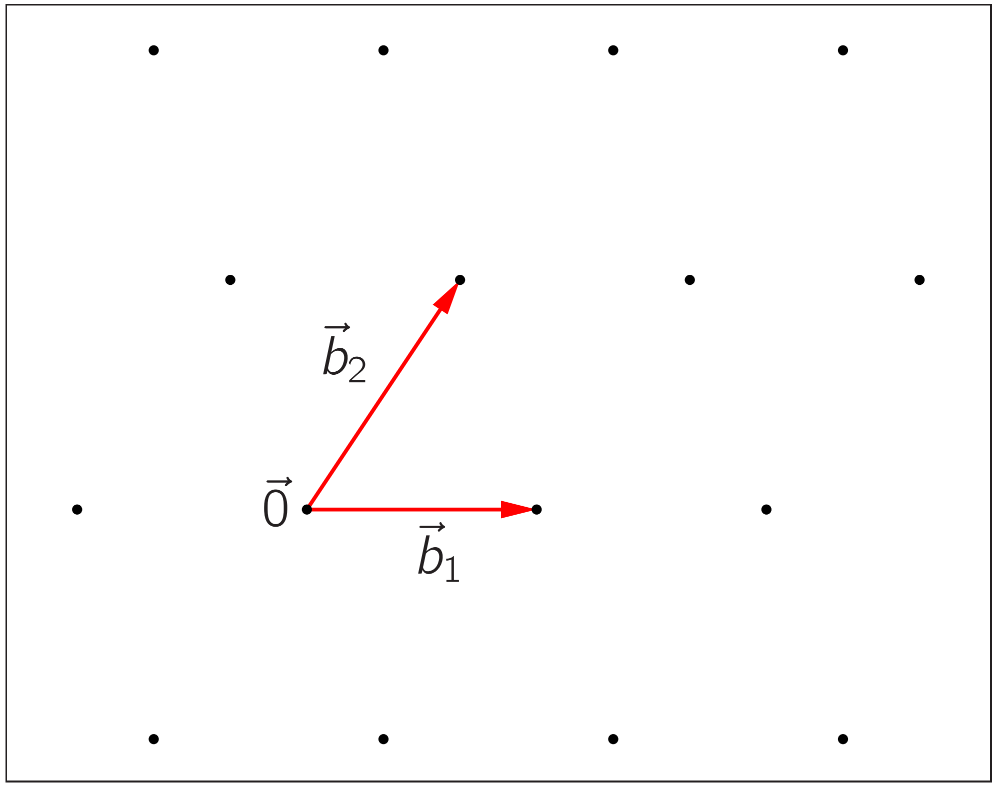
Hence in the Boltzmann-grad limit, the evolution of an initial particle density $f_0 \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ $(f_0 \geqslant 0)$ is governed by the *linear Boltzmann equation*,

$$(\partial_t + \vec{V}\nabla_{\vec{Q}}) f_t(\vec{Q}, \vec{V}) = \int_{\mathrm{S}_1^{d-1}} (f_t(\vec{Q}, \vec{V}') - f_t(\vec{Q}, \vec{V}))\sigma(\vec{V}', \vec{V})\, d\vec{V}'.$$

Boltzmann-Grad limit for $\mathcal{P}$ a lattice

Now let $\mathcal{P}$ be a d-dimensional lattice, i.e. $\mathcal{P} = \mathbb{Z}\vec{b}_1 + \ldots + \mathbb{Z}\vec{b}_d$ for some ($\mathbb{R}$-linear) basis $\vec{b}_1, \ldots, \vec{b}_d$ of $\mathbb{R}^d$.



In this case, F. Golse proved (2006): *The linear Boltzmann equation cannot hold in the limit.* (Using previous work by Bourgain - Golse - Wennberg, 1998.)

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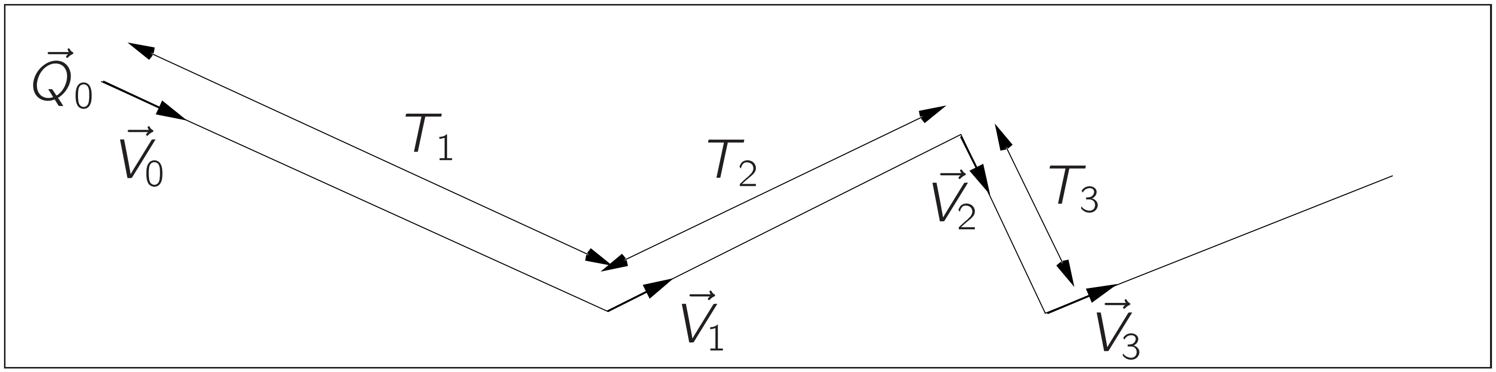

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Theorem (Marklof - S, 2011) For $\mathcal{P}$ a lattice,

$$\langle (\vec{Q}_0, \vec{V}_0), (T_1, \vec{V}_1), \ldots, (T_n, \vec{V}_n) \rangle$$

$$\xrightarrow[(r\to 0)]{\mathrm{d}} \quad \text{r.v. with density } \Lambda'(\vec{Q}_0, \vec{V}_0) p(\vec{V}_0; T_1, \vec{V}_1) \prod_{j=2}^{n} p_0(\vec{V}_{j-2}, \vec{V}_{j-1}; T_j, \vec{V}_j).$$

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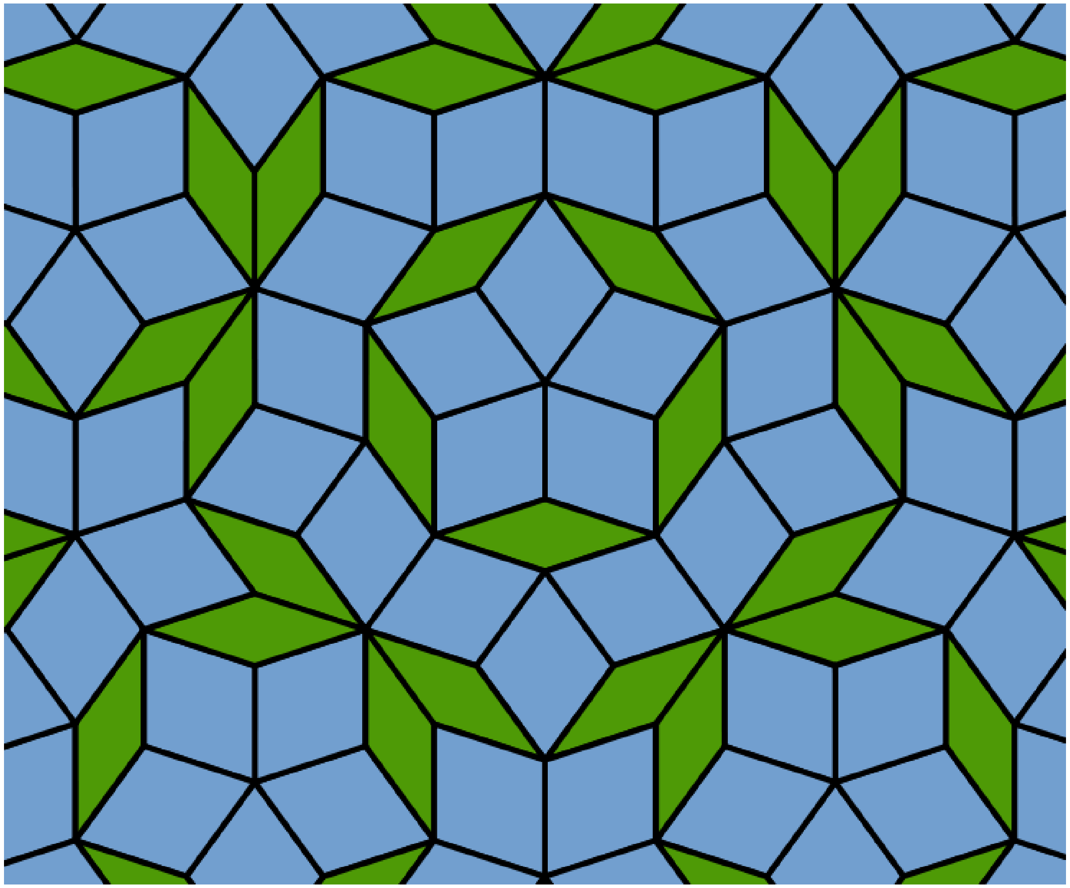
E.g. take $\mathcal{P}$ to the vertex set of a Penrose tiling.



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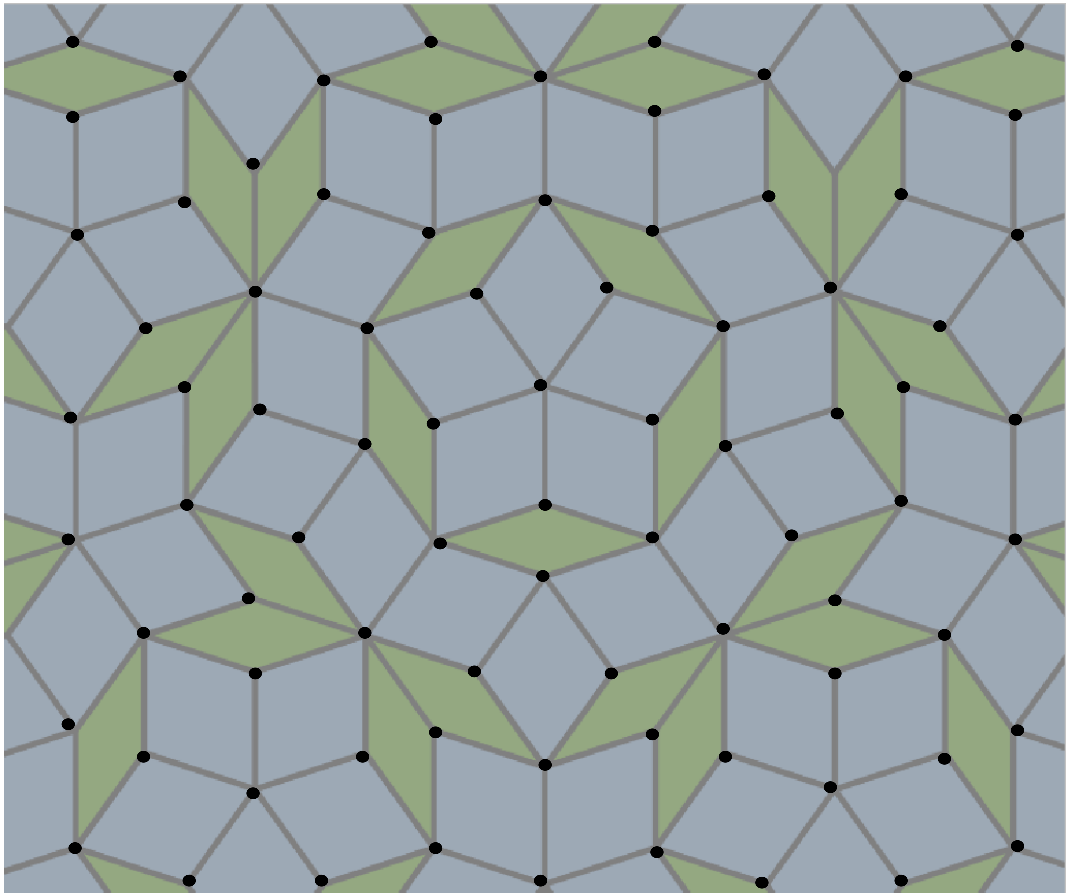

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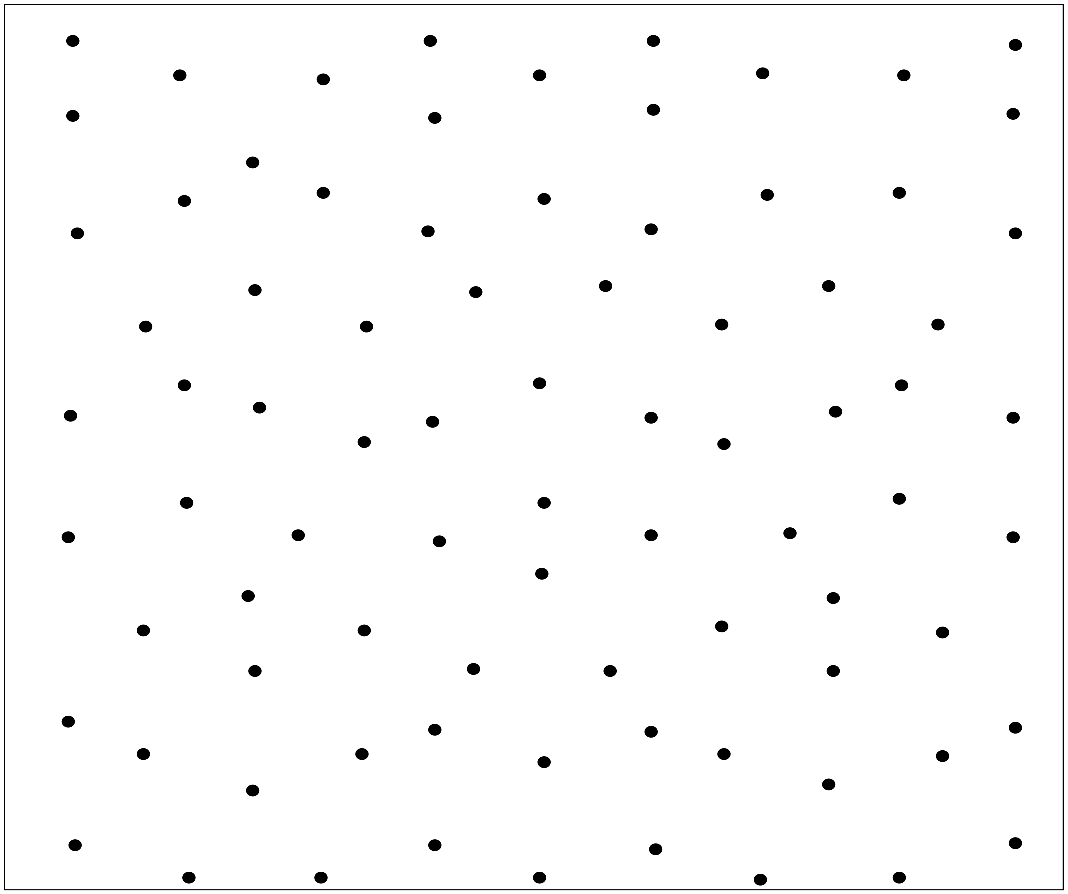


Other choices of $\mathcal{P}$?

- $\mathcal{P}$ = a *quasicrystal*, for example a "cut-and-project" set.

- $\mathcal{P}$ = a lattice with each point removed with probability p.

- $\mathcal{P}$ = a union of lattices.

- $\mathcal{P}$ = a lattice with each point perturbed slightly.

KEY ASSUMPTION ON $\mathcal{P}$ (simplified form)

For any $\vec{q} \in \mathcal{P}$ and $\lambda \in P(S_1^{d-1})$, $\lambda \ll \text{vol}_{S_1^{d-1}}$,

for $\vec{v}$ *random in* (S_1^{d-1}, λ), we assume $\boxed{\Pi_r(\vec{q}, \vec{v}) \xrightarrow{\text{d}} \Pi \text{ as } r \to 0}$ (*),

where the random point set Π is *independent of* $\vec{q}$, λ.

(*) Convergence in distribution of random elements in N_s.

— In fact we need to assume that (*) holds *uniformly* over $\vec{q} \in \mathcal{P}$.

Total list of assumptions on $\mathcal{P}$ (simplest & strongest form)

• $\mathcal{P} \subset \mathbb{R}^d$ is *locally finite* and has *asymptotic density* $c_{\mathcal{P}} > 0$ (viz., for every Jordan set $B \subset \mathbb{R}^d$, $\lim_{T\to\infty} T^{-d}\#(\mathcal{P} \cap TB) = c_{\mathcal{P}}\mathrm{vol}(B)$).

• There exists a random element Π in N_s such that for any fixed $\lambda \in P(\mathrm{S}_1^{d-1})$ with $\lambda \ll \mathrm{vol}_{\mathrm{S}_1^{d-1}}$, if $\vec{v}$ is random in $(\mathrm{S}_1^{d-1}, \lambda)$ then

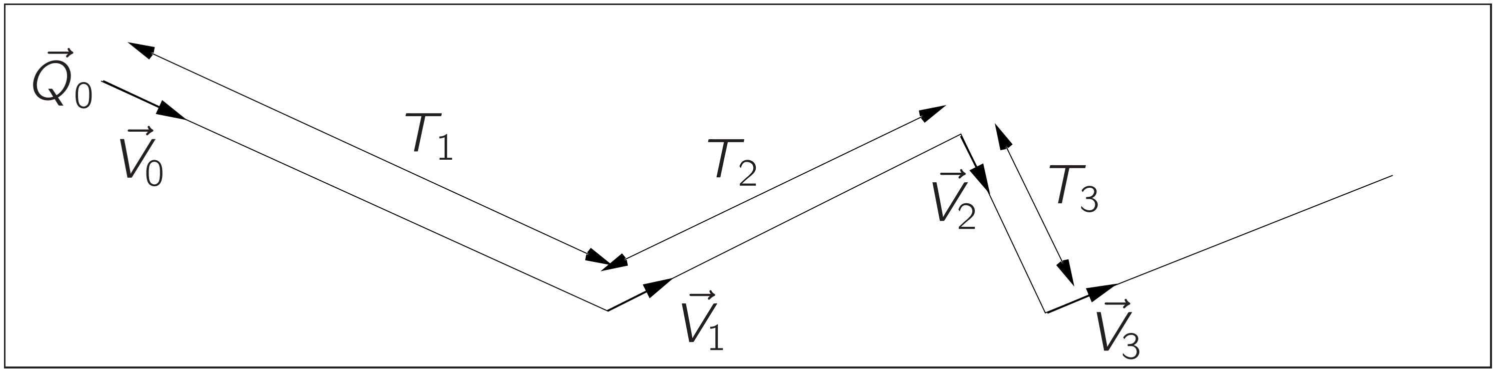
$$\boxed{\Pi_r(\vec{q}, \vec{v}) \xrightarrow[r\to 0]{\mathrm{d}} \Pi, \text{ uniformly over all } \vec{q} \in \mathcal{P}.}$$

$$\begin{cases} \bullet \text{ The law of } \Pi \text{ is invariant under } \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(d-1) \end{pmatrix}. \\ \bullet\ \forall \varepsilon > 0\text{: } \exists R > 0\text{: } \forall \vec{x} \in \mathbb{R}^d\text{: } \mathrm{Prob}(\Pi \cap (\mathcal{B}_R^d + \vec{x}) = \emptyset) < \varepsilon. \\ \bullet \text{ The image of } \Pi \text{ under } (x_1, \ldots, x_d) \mapsto x_1 \text{ is simple.} \end{cases}$$

• "$T_1 < \infty$ a.s. for macroscopic initial conditions":
Set $C_\xi = (0, \xi) \times \mathcal{B}_1^{d-1}$. For any bounded Borel set $B \subset \mathbb{R}^d$,

$$\lim_{\xi\to\infty} \limsup_{r\to 0} [\mathrm{vol} \times \mathrm{vol}_{\mathrm{S}_1^{d-1}}]\{(\vec{Q}, \vec{V}) \in B \times \mathrm{S}_1^{d-1} \,:\, \Pi_r(r^{1-d}\vec{Q}, \vec{V}) \cap C_\xi = \emptyset\} = 0.$$

Given $(\vec{Q}_0, \vec{V}_0)$, define $T_j \in \mathbb{R}_{>0}$, $\vec{V}_j \in S_1^{d-1}$:



(The picture is in macroscopic coordinates; $(\vec{Q}, \vec{V}) = (r^{d-1}\vec{q}, \vec{v})$)

Theorem 1 (Marklof & S, '16). Assume that the point set $\mathcal{P} \subset \mathbb{R}^d$ satisfies all the previous assumptions. Let Λ be a.c. probability measure on $\mathrm{T}^1(\mathbb{R}^d) = \mathbb{R}^d \times S_1^{d-1}$. Let $n \geqslant 1$. Then for $(\vec{Q}_0, \vec{V}_0)$ random in $(\mathrm{T}^1(\mathbb{R}^d), \Lambda)$,

$$\left\langle (\vec{Q}_0, \vec{V}_0), (T_1, \vec{V}_1), \ldots, (T_n, \vec{V}_n) \right\rangle$$

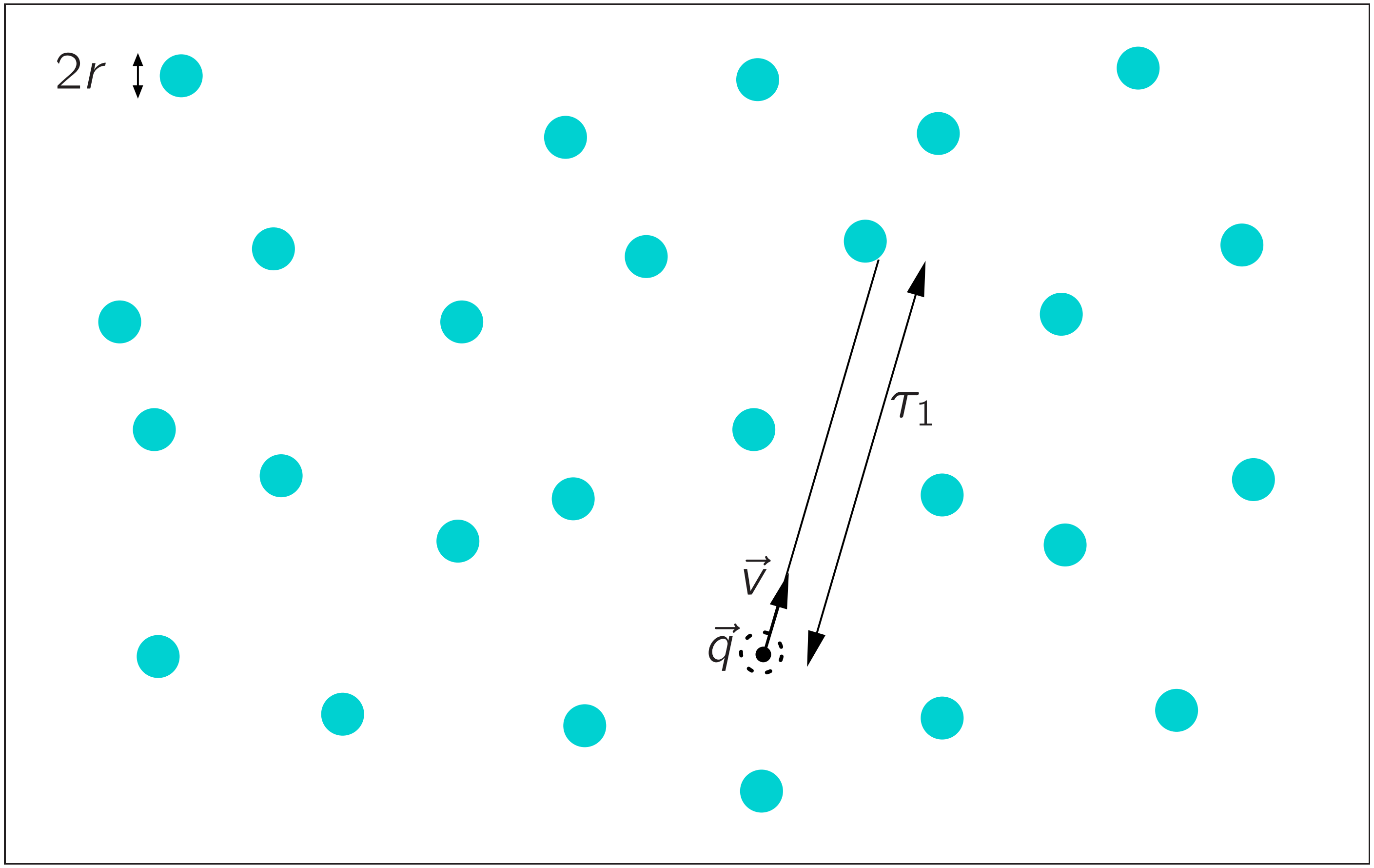
$$\xrightarrow[r \to 0]{\mathrm{d}} \text{r.v. with density } \Lambda'(\vec{Q}_0, \vec{V}_0)\, p(\vec{V}_0; T_1, \vec{V}_1) \prod_{j=2}^{n} p_0(\vec{V}_{j-2}, \vec{V}_{j-1}; T_j, \vec{V}_j),$$

where the collision kernels p and p_0 depend only on Π.

Outline of proof of Theorem 1

First consider a particle starting at $\vec{q} \in \mathcal{P}$ (ignoring the scatterer at $\vec{q}$), with $\vec{v}$ random in $(\mathrm{S}_1^{d-1}, \lambda)$.

$\mathrm{Prob}(\tau_1 \geqslant \xi r^{1-d}) = ?$



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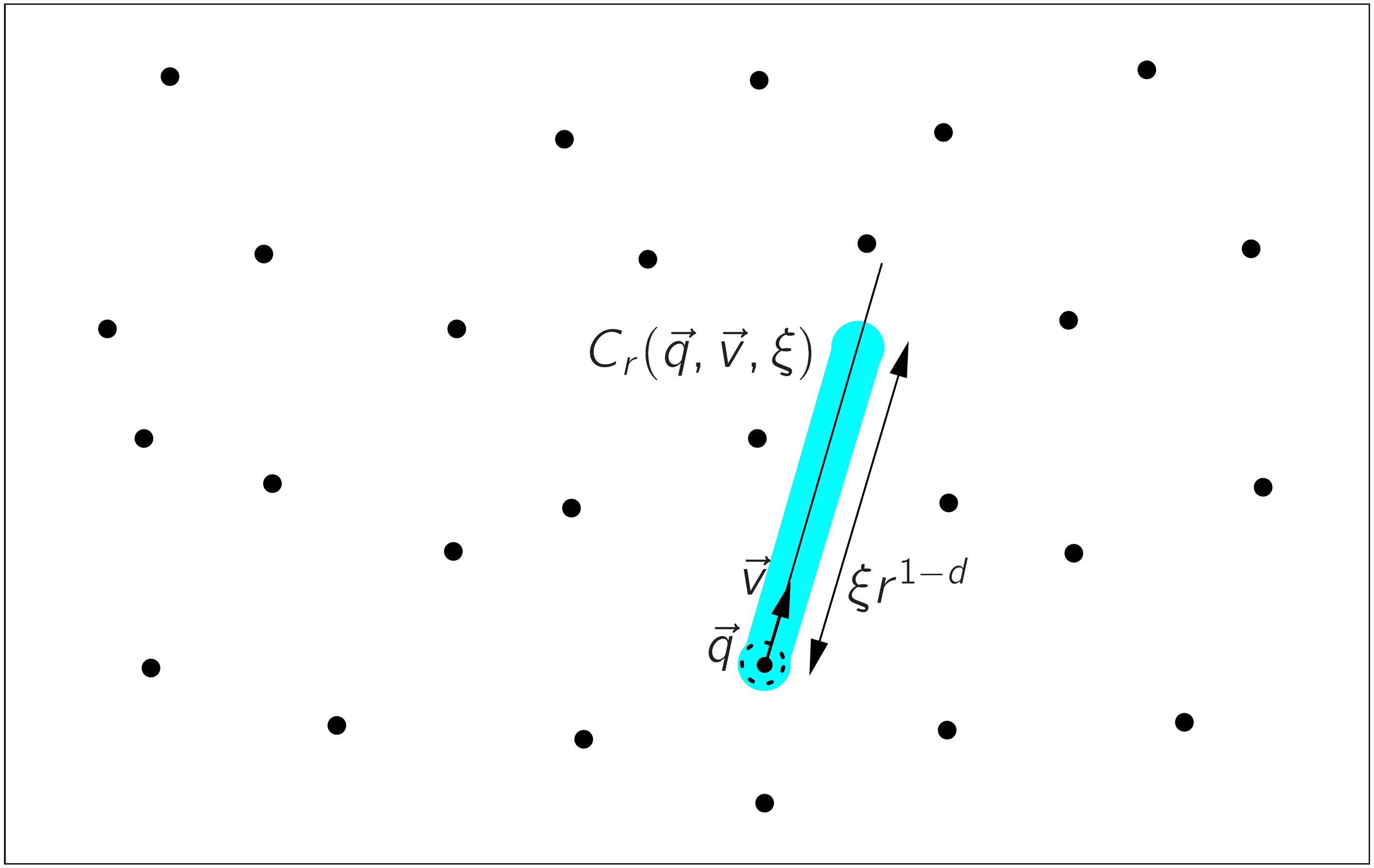
$$\mathrm{Prob}(\tau_1 \geqslant \xi r^{1-d}) = \mathrm{Prob}(C_r(\vec{q}, \vec{v}, \xi) \cap \mathcal{P} = \emptyset)$$



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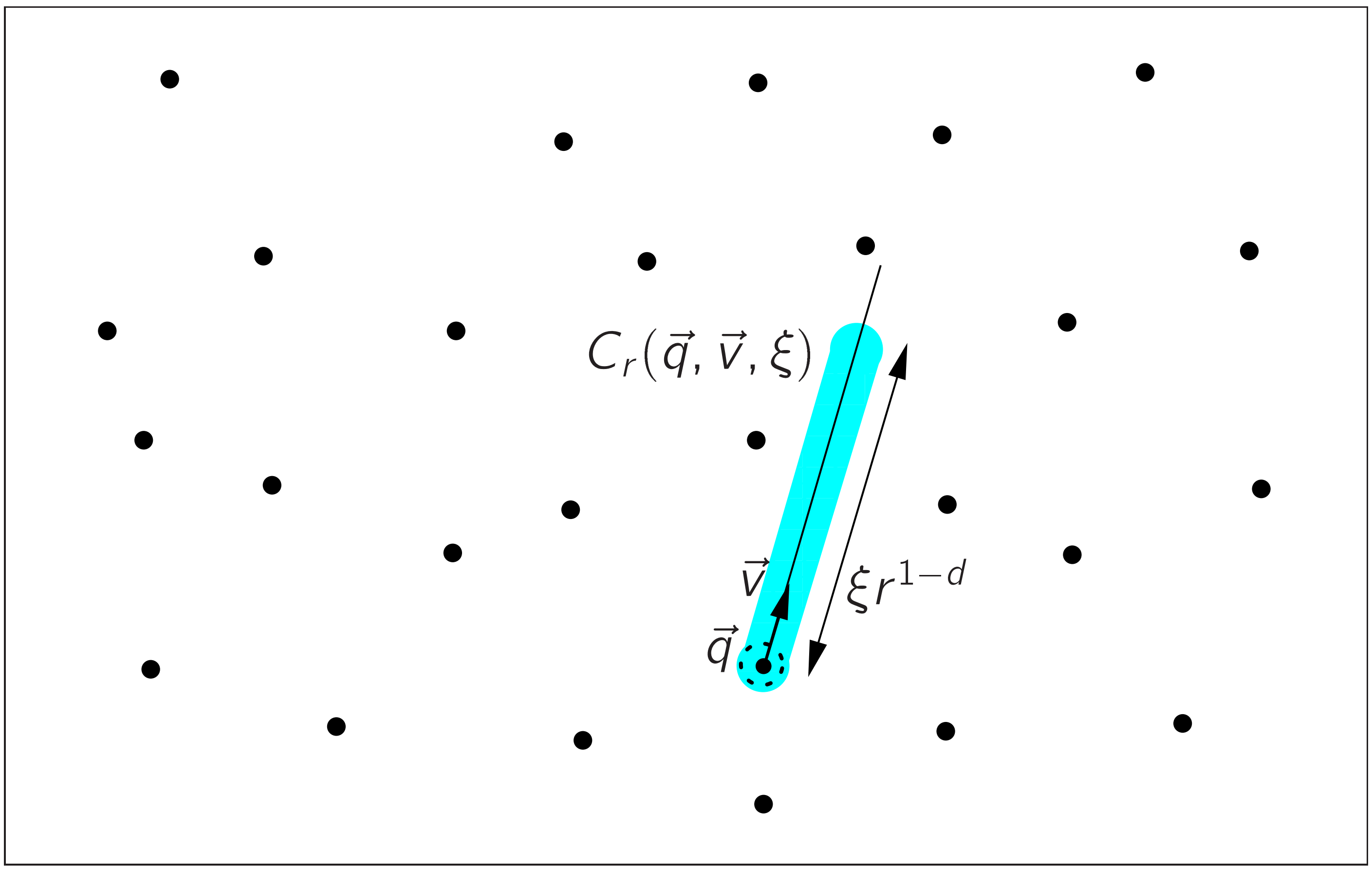
$$\begin{aligned}\mathrm{Prob}(\tau_1 \geqslant \xi r^{1-d}) &= \mathrm{Prob}(C_r(\vec{q}, \vec{v}, \xi) \cap \mathcal{P} = \emptyset) \\ &= \mathrm{Prob}((C_r(\vec{q}, \vec{v}, \xi) - \vec{q}) R_{\vec{v}} D_r \cap \Pi_r(\vec{q}, \vec{v}) = \emptyset)\end{aligned}$$



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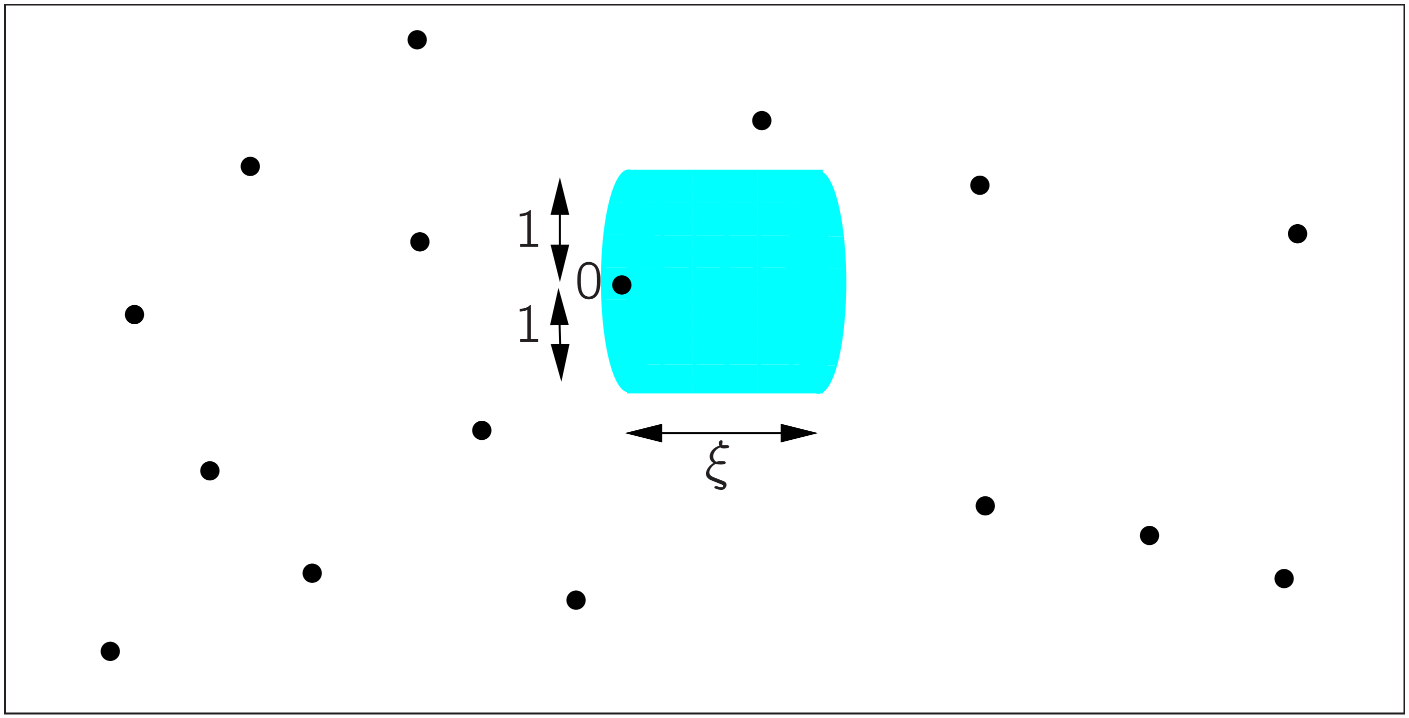

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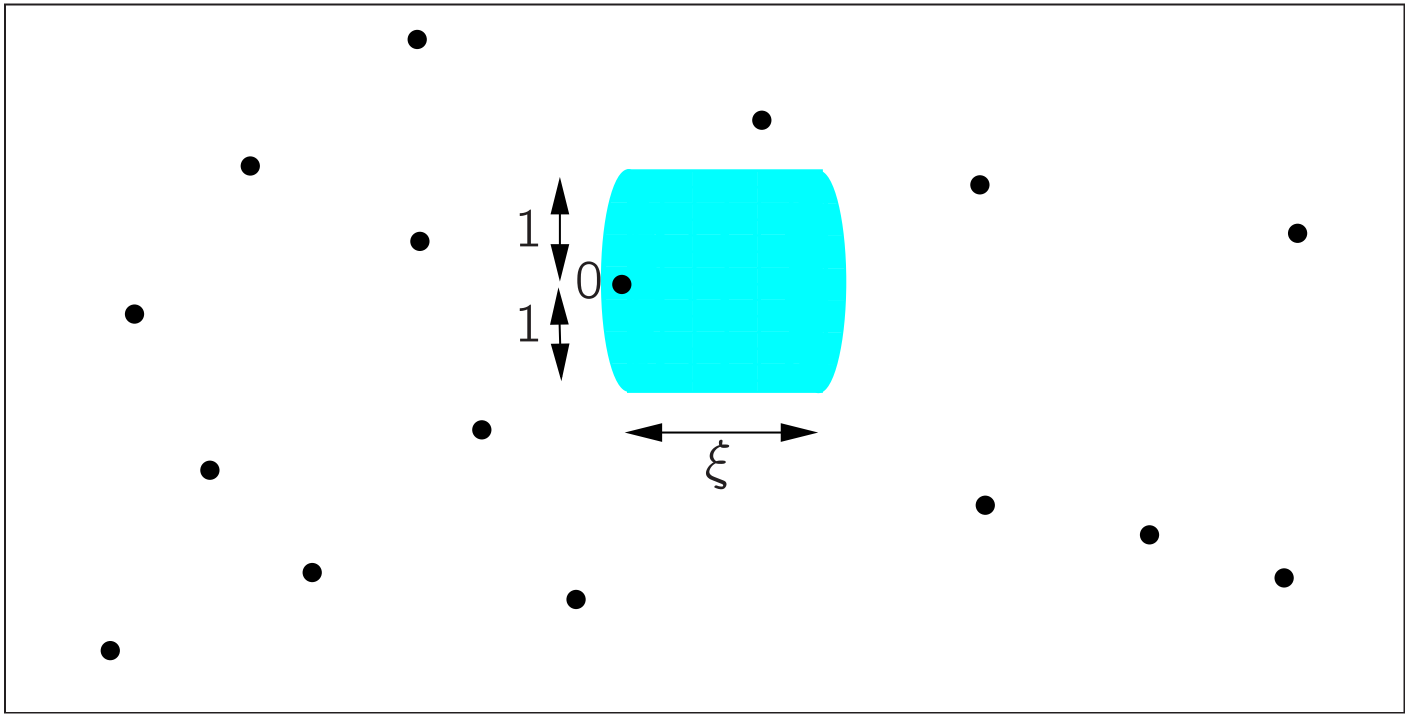
\begin{aligned}
\operatorname{Prob}(\tau_1 \geqslant \xi r^{1-d}) &= \operatorname{Prob}(C_r(\vec{q}, \vec{v}, \xi) \cap \mathcal{P} = \emptyset) \\
&= \operatorname{Prob}((C_r(\vec{q}, \vec{v}, \xi) - \vec{q}) R_{\vec{v}} D_r \cap \Pi_r(\vec{q}, \vec{v}) = \emptyset) \\
&\approx \operatorname{Prob}(C_\xi \cap \Pi_r(\vec{q}, \vec{v}) = \emptyset)
\end{aligned}
$$

$$C_\xi := (0, \xi) \times \mathcal{B}_1^{d-1}$$



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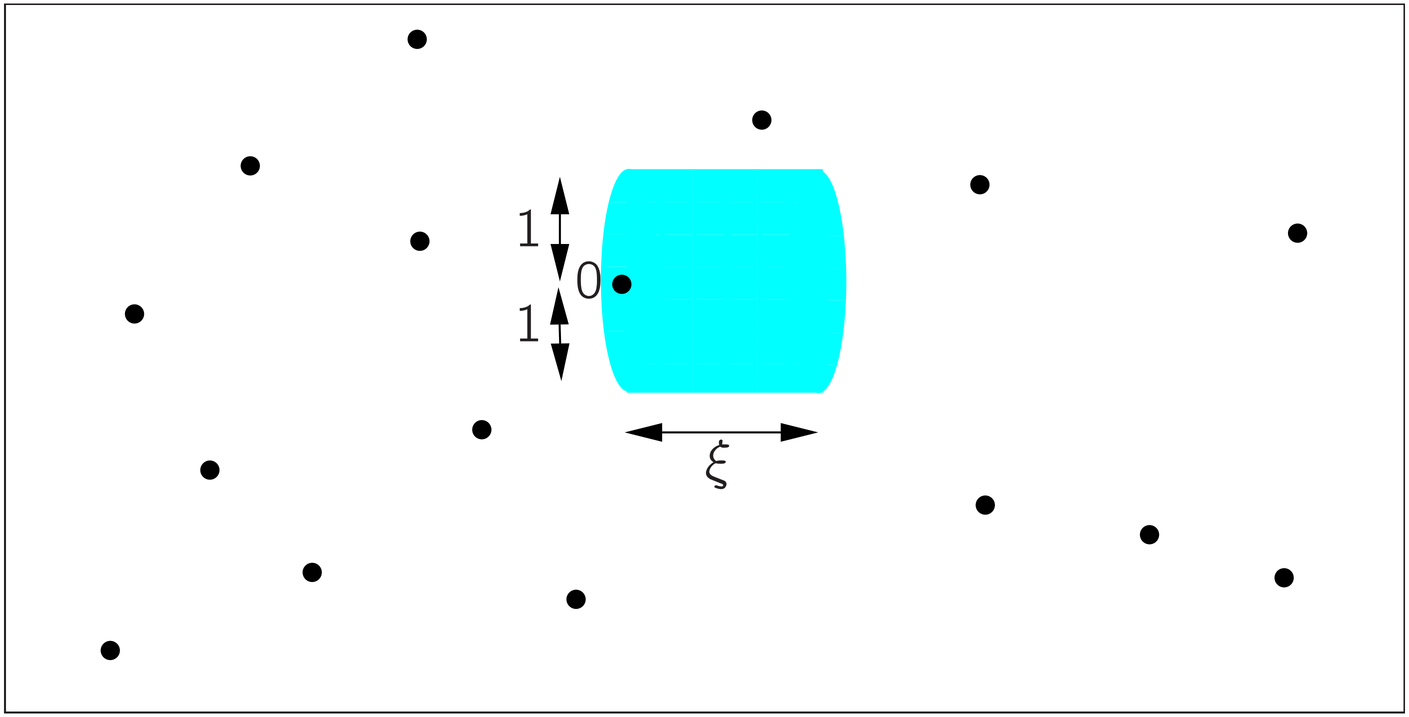

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&\boxed{\to \mathrm{Prob}(C_\xi \cap \Pi) = \emptyset} \qquad \text{as } r \to 0.
\end{aligned}$$

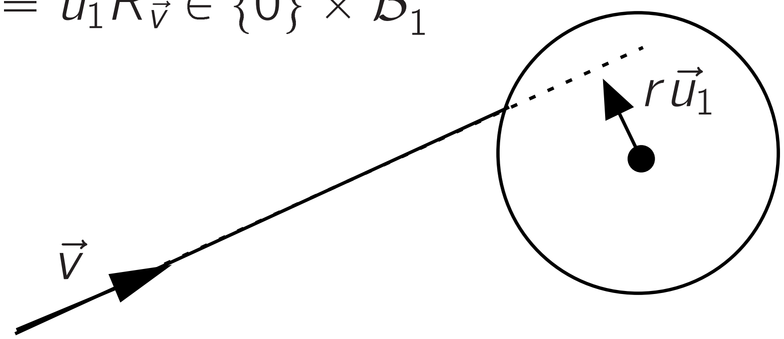
The key input for this is $\boxed{\Pi_r(\vec{q}, \vec{v}) \xrightarrow[r \to 0]{\mathrm{d}} \Pi}$

One also uses the fact that *the intensity measure of* Π *is* $\ll$ *Lebesgue*, i.e. $\mathbb{E}(\#\Pi \cap B) \leqslant c_{\mathcal{P}} \mathrm{vol}(B)$ for any Borel $B \subset \mathbb{R}^d$.

Outline of proof of Theorem 1

We can even get convergence of the *joint* distribution of free path length $T_1 = r^{d-1}\tau_1$ and the (normalized) *impact parameter* $\vec{w}_1$*:*

Def: $\vec{w}_1 = \vec{u}_1 R_{\vec{v}} \in \{0\} \times \mathcal{B}_1^{d-1}$



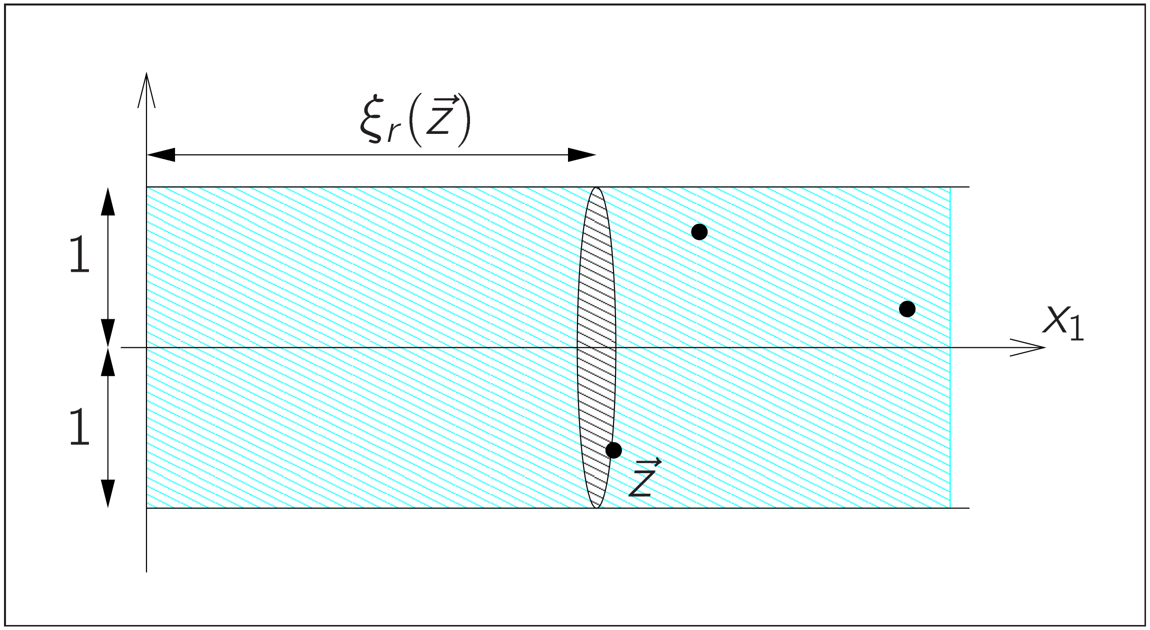
Outline of proof of Theorem 1

Indeed, for $\vec{z}$ in $C_\infty = (0, \infty) \times \mathcal{B}_1^{d-1}$, set

$$\xi_r(\vec{z}) = \inf\{\xi > 0 \ : \ z \in \xi \vec{e}_1 + \mathcal{B}_r^d D_r\}$$

and define $F_r : N_s \to C_\infty \sqcup \{\text{undef}\}$ by:

$F_r(Y) = (\xi_r(\vec{z}), -(0, z_2, \ldots, z_d))$ for the unique $\vec{z} \in C_\infty \cap Y$ which minimizes $\xi_r(\vec{z})$.



Then $\langle T_1, \vec{w}_1 \rangle = F_r(\Pi_r(\vec{q}, \vec{v}))$ whenever $F_r(\Pi_r(\vec{q}, \vec{v})) \neq \text{undef}!$

Outline of proof of Theorem 1

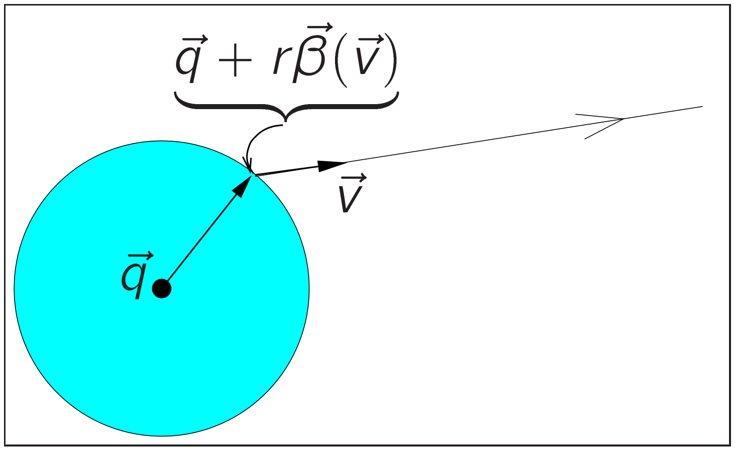
Now: If $Y_n \to Y$ in N_s and $F_0(Y) \neq \text{undef}$ and $Y \cap \partial C_\infty = \emptyset$, and if $r_n \to 0$, then $F_{R_n}(Y_n) \to F_0(Y)$ in $\mathbb{R}^d$.

Also by our *assumptions*, $\text{Prob}(F_0(\Pi) \neq \text{undef}) = 1$.

Hence $\boxed{\langle T_1, \vec{w}_1 \rangle = F_r(\Pi_r(\vec{q}, \vec{v})) \xrightarrow[r \to 0]{\text{d}} F_0(\Pi)}$ as desired!

More general initial condition

Consider a point particle starting from $(\vec{q} + r\vec{\beta}(\vec{v}), \vec{v})$, where $\vec{\beta} : S_1^{d-1} \to S_1^{d-1}$ is a continuous function (subject to $(\vec{\beta}(\vec{v}) + \mathbb{R}^+\vec{v}) \cap \mathcal{B}_1^d = \varnothing$).



$\vec{x}_\perp := (0, x_2, \ldots, x_d)$

Note $\Pi_r(\vec{q}, \vec{v}) - r\vec{\beta}(\vec{v}) R_{\vec{v}} D_r \xrightarrow[r\to 0]{\mathrm{d}} \Pi^{(\vec{\beta},\lambda)} := \Pi - (\vec{\beta}(\vec{v}) R_{\vec{v}})_\perp,$

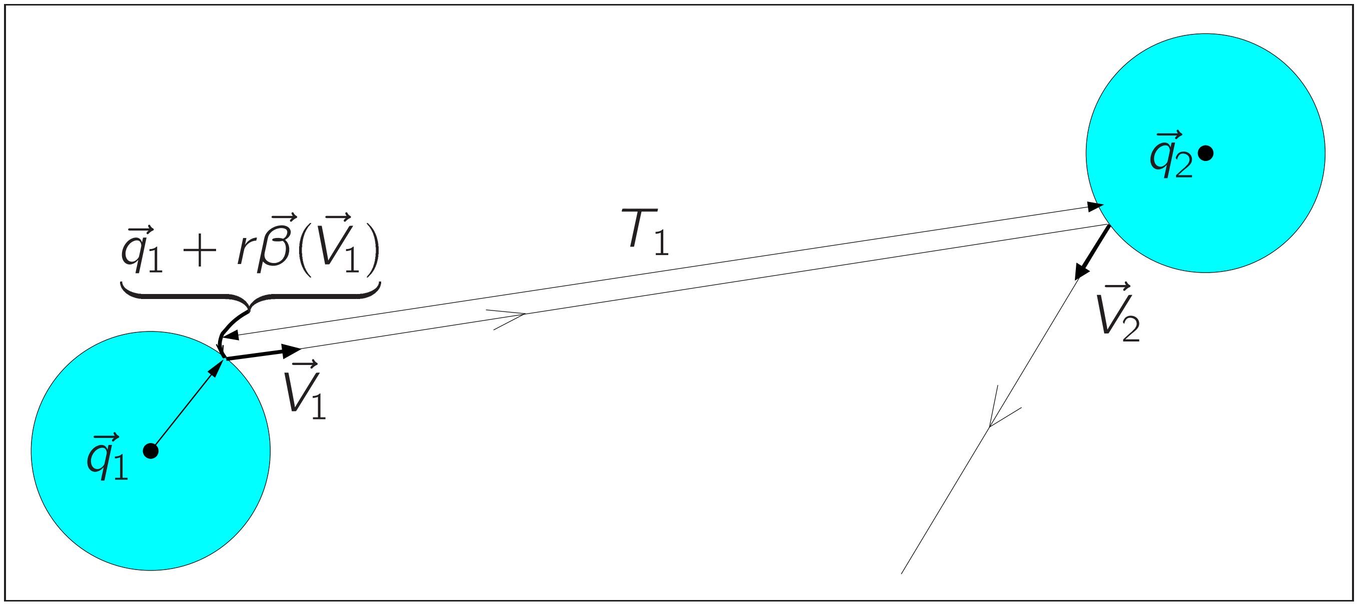
with $\vec{v}$ independent from Π.

Hence $\boxed{\langle T_1, \vec{w}_1 \rangle \xrightarrow[r\to 0]{\mathrm{d}} F_0(\Pi^{(\vec{\beta},\lambda)})}$

Getting the joint limit distribution of $\vec{V}_0, (T_1, \vec{V}_1), (T_2, \vec{V}_2)$ – OUTLINE

From above, get the limit distribution of $\boxed{\langle \vec{V}_1, (T_2, \vec{V}_2)\rangle}$ as $r \to 0$,

if $\vec{V}_1$ is picked at random according to a *given* $\lambda \in P(S_1^{d-1})$, and the particle starts at $\vec{q}_1 + r\vec{\beta}(\vec{V}_1)$, for *given* $\vec{q}_1 \in \mathcal{P}$ and $\vec{\beta} : S_1^{d-1} \to S_1^{d-1}$:

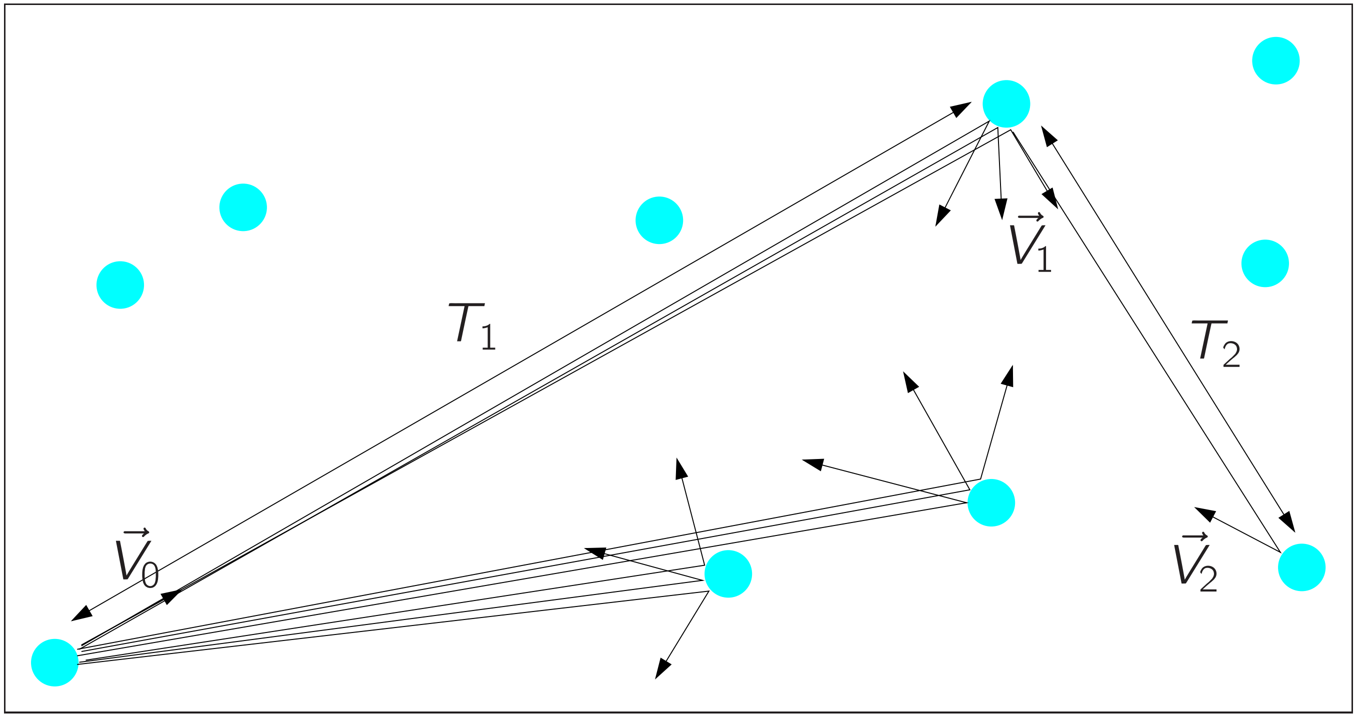


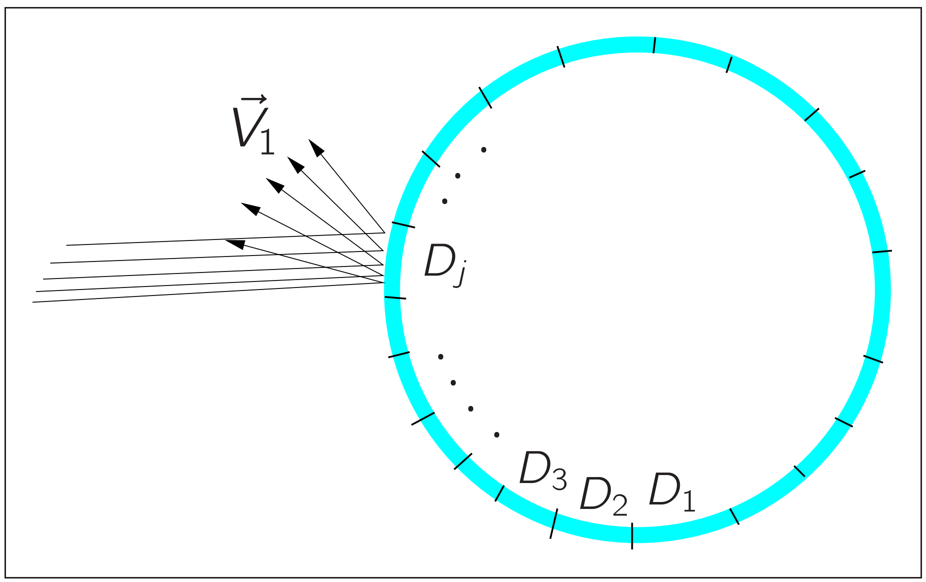
Decompose $S_1^{d-1} = \sqcup_{j=1}^{N} D_j$, with each D_j "nice" and of small diameter.

Make the limit result *uniform* over:

- $\vec{q}_1 \in \mathcal{P}$;
- $\vec{\beta}$ in compact families, and
- $\lambda = \lambda' \cdot \mathrm{vol}_{S_1^{d-1}|D_j}$ for $j \in \{1, \ldots, N\}$, $\lambda' \in$ compact $\subset C(D_j)$

Getting the joint limit distribution of $\vec{V}_0, (T_1, \vec{V}_1), (T_2, \vec{V}_2)$ – OUTLINE





After removing a set of $\vec{V}_0$'s of small λ-measure, the remaining $\vec{V}_1$'s emanate from "fully lighted D_j-sets", and the previous result can be applied!

Macroscopic initial condition $((\vec{Q},\vec{V}) = (r^{d-1}\vec{q},\vec{v}))$

Fix an a.c. prob. measure Λ on $\mathrm{T}^1(\mathbb{R}^d) = \mathbb{R}^d \times \mathrm{S}_1^{d-1}$.

Consider $\boxed{\Pi_r(r^{1-d}\vec{Q},\vec{V}) \text{ for } (\vec{Q},\vec{V}) \text{ random in } \langle \mathrm{T}^1(\mathbb{R}^d),\Lambda\rangle}$

Limit as $r \to 0$?

— In other words, for given $f \in \mathrm{C}_b(N_s)$:

$$\lim_{r\to 0} \mathbb{E}\, f(\Pi_r(r^{1-d}\vec{Q},\vec{V}))$$

$$= \lim_{r\to 0} r^{d(d-1)} \int_{\mathrm{S}_1^{d-1}} \int_{\mathbb{R}^d} f\big(\Pi_r(\vec{q},\vec{v})\big)\, \Lambda'(r^{d-1}\vec{q},\vec{v})\, d\vec{q}\, d\vec{v} = ???$$

(May assume $\Lambda' \in \mathrm{C}_c(\mathrm{T}^1(\mathbb{R}^d))$.)

Macroscopic initial condition. Given $f \in C_b(N_s)$, $\Lambda' \in C_c(\mathrm{T}^1(\mathbb{R}^d))$:

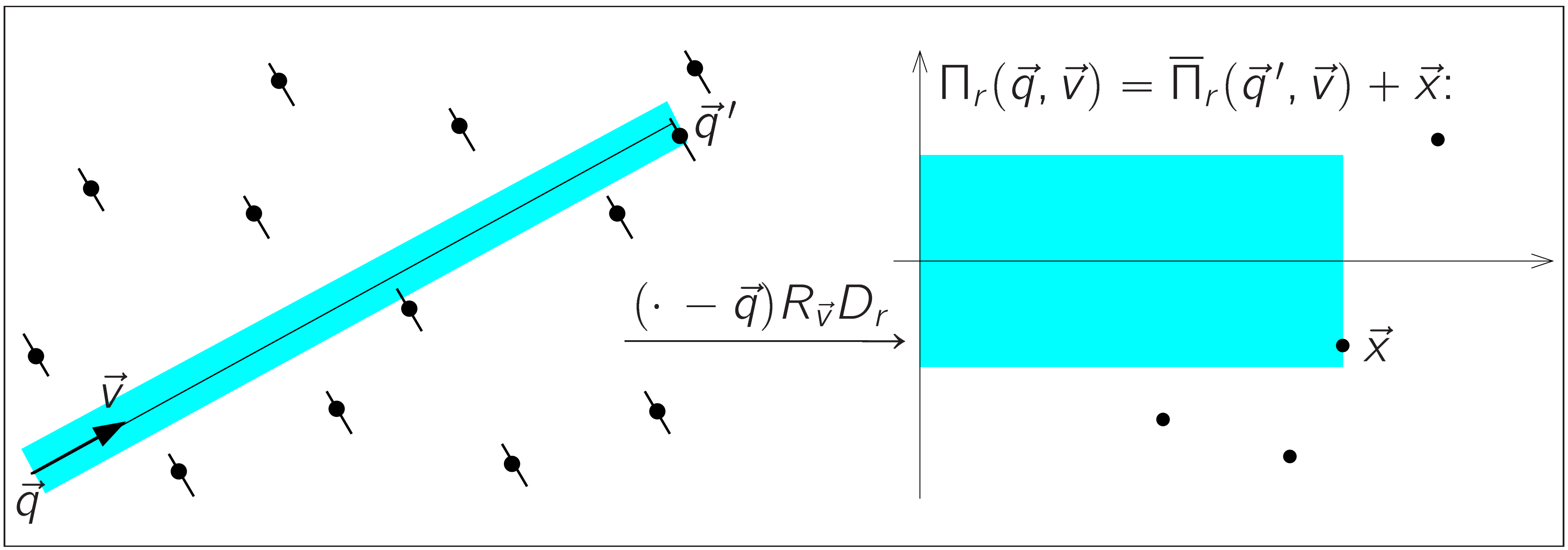
$$\lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \int_{\mathbb{R}^d} f\big(\Pi_r(\vec{q},\vec{v})\big)\, \Lambda'(r^{d-1}\vec{q},\vec{v})\, d\vec{q}\, d\vec{v} = ?$$

For fixed $\vec{v}$: Draw a "flat scatterer" $\vec{q}\,' + (\{0\} \times \mathcal{B}_r^{d-1})R_{\vec{v}}$ at each $\vec{q}\,' \in \mathcal{P}$. To each $\vec{q} \in \mathbb{R}^d$, associate that $\vec{q}\,' \in \mathcal{P}$ whose scatterer is first hit by $\vec{q} + \mathbb{R}_{>0}\vec{v}$.

Then $\quad (\vec{q}\,' - \vec{q})R_{\vec{v}}D_r = \vec{x} \quad$ for some $\vec{x} \in C_\infty$; thus

$$\Pi_r(\vec{q},\vec{v}) = \overline{\Pi}_r(\vec{q}\,',\vec{v}) + \vec{x},$$

where $\overline{\Pi}_r(\vec{q}\,',\vec{v}) := (\mathcal{P} - \vec{q}\,')R_{\vec{v}}D_r = \Pi_r(\vec{q}\,',\vec{v}) \cup \{\vec{0}\}$.



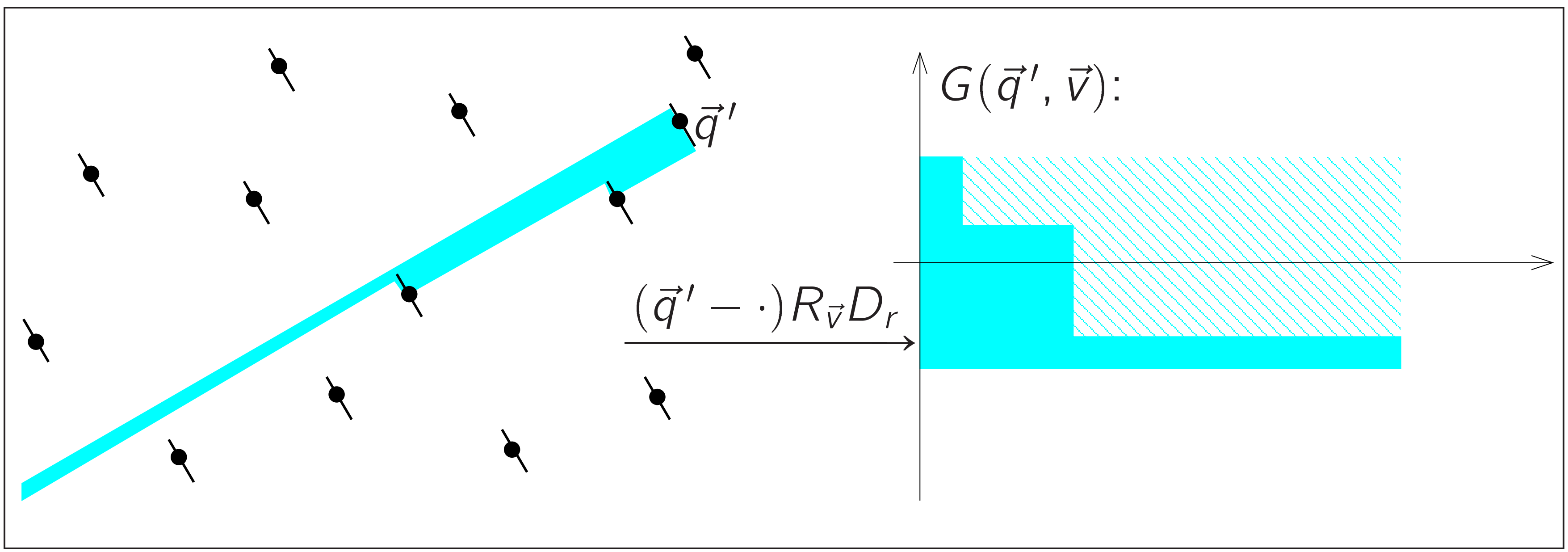
Macroscopic initial condition. Given $f \in C_b(N_s)$, $\Lambda' \in C_c(\mathrm{T}^1(\mathbb{R}^d))$:

$$\lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \underbrace{\int_{\mathbb{R}^d} f\big(\Pi_r(\vec{q},\vec{v})\big)\, \Lambda'(r^{d-1}\vec{q},\vec{v})\, d\vec{q}}_{(*)}\, d\vec{v} = ?$$

For fixed $\vec{v}$: Contribution from one $\vec{q}\,' \in \mathcal{P}$ to $(*)$:

$$\approx \int_{G(\vec{q}\,',\vec{v})} f\big(\overline{\Pi}_r(\vec{q}\,',\vec{v}) + \vec{x}\big)\, \Lambda'(r^{d-1}\vec{q}\,' - x_1\vec{v},\vec{v})\, d\vec{x},$$

where $G(\vec{q}\,',\vec{v}) = \big\{\vec{x} \in C_\infty \;:\; (\Pi_r(\vec{q}\,',\vec{v}) + \vec{x}) \cap C_{x_1} = \varnothing\big\}$.



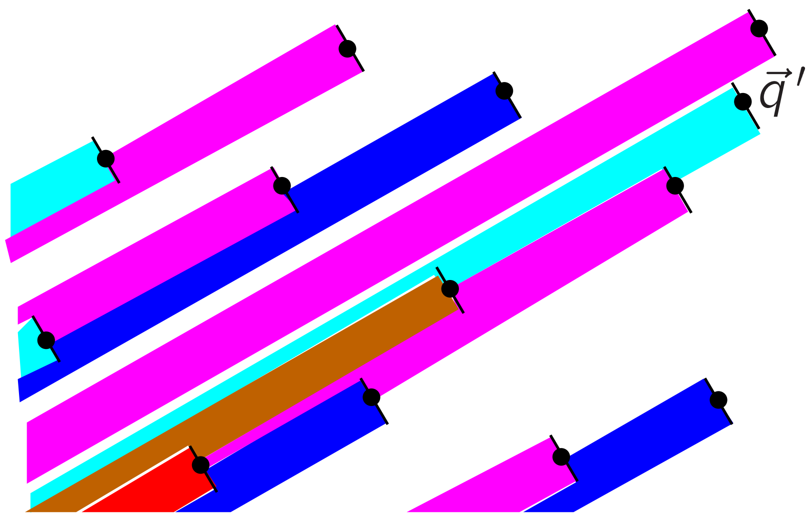
Macroscopic initial condition. Given $f \in C_b(N_s)$, $\Lambda' \in C_c(\mathrm{T}^1(\mathbb{R}^d))$:

$$\lim_{r \to 0} r^{d(d-1)} \int_{S_1^{d-1}} \underbrace{\int_{\mathbb{R}^d} f\left(\Pi_r(\vec{q}, \vec{v})\right) \Lambda'(r^{d-1}\vec{q}, \vec{v})\, d\vec{q}}_{(*)}\, d\vec{v} = ?$$

For fixed $\vec{v}$: Contribution from one $\vec{q}\,' \in \mathcal{P}$ to $(*)$:

$$\approx \int_{G(\vec{q}\,', \vec{v})} f\left(\overline{\Pi}_r(\vec{q}\,', \vec{v}) + \vec{x}\right) \Lambda'(r^{d-1}\vec{q}\,' - x_1\vec{v}, \vec{v})\, d\vec{x},$$

where $G(\vec{q}\,', \vec{v}) = \left\{\vec{x} \in C_\infty \ : \ (\Pi_r(\vec{q}\,', \vec{v}) + \vec{x}) \cap C_{x_1} = \varnothing\right\}$.



Add over all $\vec{q}\,' \in \mathcal{P}$!

$\left(\begin{array}{l}\text{Full coverage} \\ \text{– by our assumption} \\ \text{“}T_1 < \infty \text{ a.s.”!}\end{array}\right)$

Macroscopic initial condition

$$\cdots \implies \qquad \lim_{r\to 0} \mathbb{E}\, f(\Pi_r(r^{1-d}\vec{Q},\vec{V})) = c_{\mathcal{P}} \int_{(\vec{x},Y)\in\mathcal{G}} f(Y+\vec{x})\, d\mu_0(Y)\, d\vec{x},$$

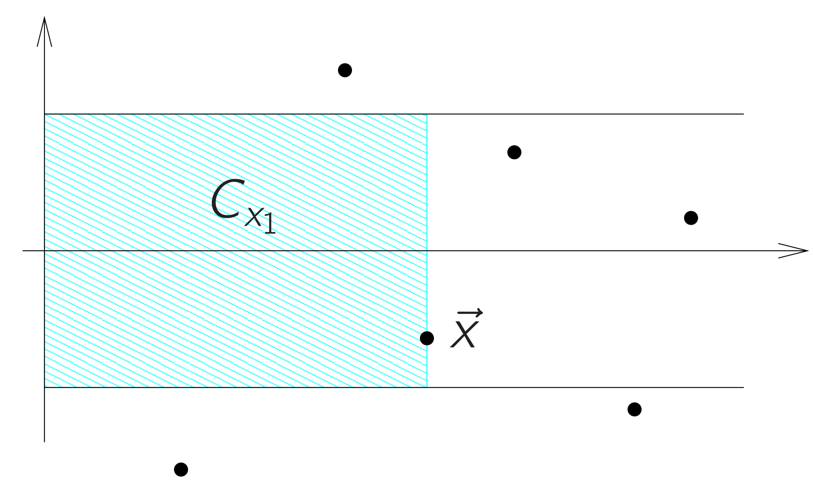
where $\quad \mathcal{G} := \{(\vec{x},Y) \,:\, (Y+\vec{x}) \cap C_{x_1} = \varnothing\} \subset C_\infty \times N_s,$

and $\quad \mu_0 \in P(N_s)$ is the distribution of $\overline{\Pi}$.

Note in particular: $\nu(\mathcal{G}) = 1$, where $\quad \nu = c_{\mathcal{P}} m \times \mu_0 \quad (m = \text{Lebesgue})$.

ANSWER: Let $\tilde{\mu} \in P(N_s)$ be the pushforward of $\nu_{|\mathcal{G}}$ by $(\vec{x},Y) \mapsto Y+\vec{x}$, and let $\tilde{\Pi}$ be a random element in N_s with law $\tilde{\mu}$. Then $\boxed{\Pi_r(r^{1-d}\vec{Q},\vec{V}) \xrightarrow[r\to 0]{\text{d}} \tilde{\Pi}}$

“ $\tilde{\Pi} = \overline{\Pi} + \vec{x}$ ”:



Macroscopic initial condition

ANSWER: Let $\widetilde{\mu} \in P(N_s)$ be the pushforward of $\nu_{|\mathcal{G}}$ by $(\vec{x}, Y) \mapsto Y + \vec{x}$, and let $\widetilde{\Pi}$ be a random element in N_s with law $\widetilde{\mu}$. Then $\boxed{\Pi_r(r^{1-d}\vec{Q}, \vec{V}) \xrightarrow[r\to 0]{\text{d}} \widetilde{\Pi}}$

Properties of $\widetilde{\Pi}$:

- The law of $\widetilde{\Pi}$ (viz., $\widetilde{\mu}$) is invariant under *translations*, and also under $\{D_r\}_{r>0}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(d-1) \end{pmatrix}$.

- $\widetilde{\Pi}$ has intensity $c_{\mathcal{P}}$.

- $\mathrm{Prob}(\widetilde{\Pi} \cap \mathcal{B}_R^d = \emptyset) \xrightarrow[R\to\infty]{} 0$.

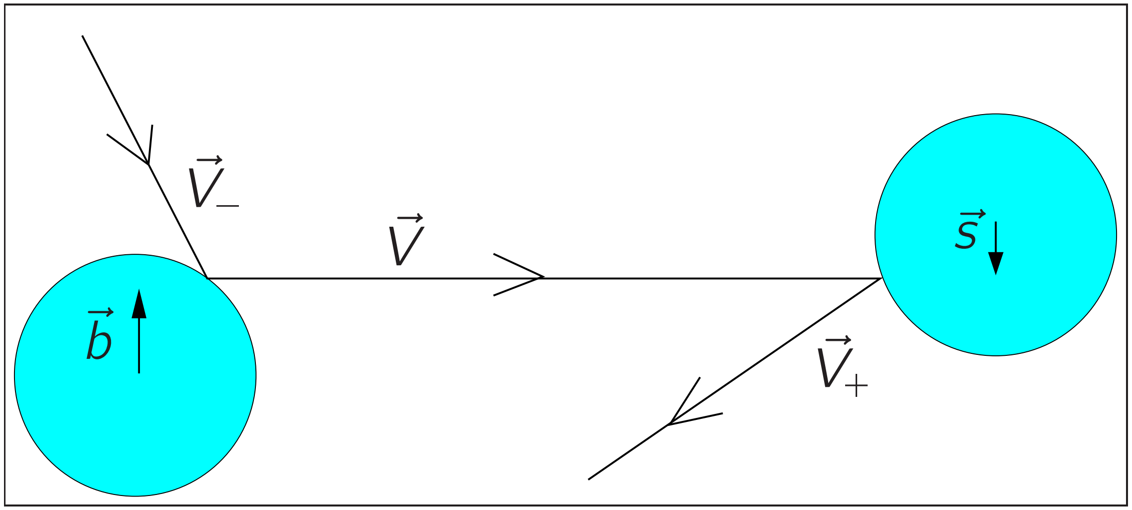
Transition kernel for the limiting process $(T_1, \vec{V}_1), (T_2, \vec{V}_2), \ldots$

For $\vec{x} \in \{0\} \times \mathcal{B}_1^{d-1}$, let the distr of $F_0(\Pi - \vec{x})$ be $k_0(\vec{x}, \xi, -\vec{y})\, d\xi\, d\vec{y} \in P(C_\infty)$.

Thus: $\text{Prob}(F_0(\Pi - \vec{x}) \in B) = \int_B k_0(\vec{x}; \xi, -\vec{y})\, d\xi\, d\vec{y}, \qquad \forall \vec{x} \in \mathcal{B}_1^{d-1}, B \subset C_\infty.$

Then the transition kernel for the limiting process $(T_1, \vec{V}_1), (T_2, \vec{V}_2), \ldots$ is:

$$p_0(\vec{V}_-, \vec{V}; \xi, \vec{V}_+) = \sigma(\vec{V}, \vec{V}_+) k_0\Big(\vec{b}[\vec{V}_-, \vec{V}] R_{\vec{V}}\; ;\; \xi\; ,\; \vec{s}[\vec{V}, \vec{V}_+] R_{\vec{V}}\Big)$$



More about $k_0(\vec{x};\xi,\vec{y})$

Assume Π has constant intensity $c_{\mathcal{P}}$. Then $k_0(\vec{x};\xi,\vec{y})$ can be expressed in terms of the **Palm distributions** of Π; $\nu : \mathbb{R}^d \times \mathcal{N} \to [0,1]$.

($\mathcal{N}$ = the Borel σ-field of N_s. Intuitively: $\nu(\vec{x}, A) =$ "Prob($\Pi \in A \mid \vec{x} \in \Pi$)")

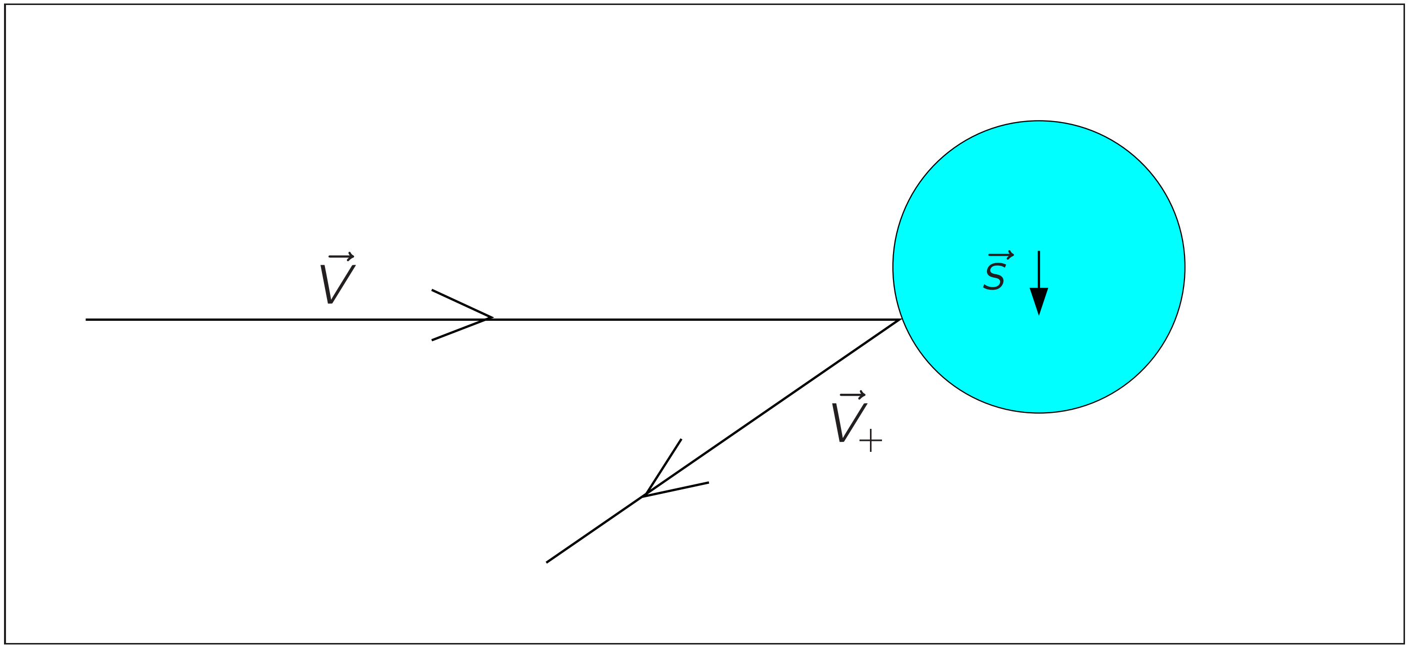
$$k_0(\vec{x};\xi,\vec{y}) = c_{\mathcal{P}} \cdot \nu\Big((\xi, \vec{x}-\vec{y}), \{Y \in N_s \;:\; (Y-\vec{x}) \cap C_\xi = \varnothing\}\Big).$$

Similarly: Transition kernel for macroscopic initial condition

Let the distribution of $F_0(\widetilde{\Pi})$ be $k(\xi, -\vec{y})\, d\xi\, d\vec{y} \in P(C_\infty)$.

Then the transition kernel for macroscopic initial conditions is:

$$p(\vec{V}; \xi, \vec{V}_+) = \sigma(\vec{V}, \vec{V}_+)\; k\Big(\, \xi\;,\; \vec{s}[\vec{V}, \vec{V}_+] R_{\vec{V}} \Big)$$



In fact, $\quad k(\xi, \vec{y}) = c_{\mathcal{P}} \displaystyle\int_{\xi}^{\infty} \int_{\mathcal{B}_1^{d-1}} k_0(\vec{x}; \xi', \vec{y})\, d\vec{x}\, d\xi'.$

Some more relations for k_0, k:

- $k(\xi, \vec{y}) = c_{\mathcal{P}} \int_{\xi}^{\infty} \int_{\mathcal{B}_1^{d-1}} k_0(\vec{x}; \xi', \vec{y}) \, d\vec{x} \, d\xi'$.

- Mean free path length: $\frac{1}{\overline{\sigma}} \int_{\mathcal{B}_1^{d-1}} \int_0^{\infty} \int_{\mathcal{B}_1^{d-1}} \xi k_0(\vec{x}; \xi, \vec{y}) \, d\vec{x} \, d\xi \, d\vec{y} = \frac{1}{\overline{\sigma} c_{\mathcal{P}}}$.

- $k_0(\vec{x}R; \xi, \vec{y}R) = k_0(\vec{x}; \xi, \vec{y})$ for $R \in \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(d-1) \end{pmatrix}$.

- $k_0(\vec{y}; \xi, \vec{x}) = k_0(\vec{x}R_-; \xi, \vec{y}R_-)$ for $R_- \in \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{O}(d-1) \end{pmatrix}$ with $\det R_- = -1$.

Example: "$\mathcal{P}$ = Poisson"

Let $\mathbb{P}$ = the Poisson probability measure on N_s with parameter $c > 0$.

Then for $\mathbb{P}$-a.a. $\mathcal{P} \in N_s$, the list of assumptions holds!
(And the limiting random point set Π has law $\mathbb{P}$.)

$$\left\{\begin{array}{l}\textit{Weaker form of the key assumption:} \\ \text{There exists a subset } \mathcal{E} \subset \mathcal{P} \text{ of vanishing density such that for any fixed } T \geqslant 1 \\ \text{and } \lambda \in P(\mathrm{S}_1^{d-1}) \text{ with } \lambda \ll \mathrm{vol}_{\mathrm{S}_1^{d-1}}, \text{ if } \vec{v} \text{ is random in } (\mathrm{S}_1^{d-1}, \lambda), \text{ then} \\ \boxed{\Pi_r(\vec{q}, \vec{v}) \xrightarrow[r\to 0]{\mathrm{d}} \Pi, \text{ uniformly over all } \vec{q} \in \mathcal{P}_T(r) := \mathcal{P} \cap \mathcal{B}^d(T r^{1-d}) \setminus \mathcal{E}.}\end{array}\right.$$

We choose:

$$\mathcal{E} = \left\{\vec{q} \in \mathcal{P} \ : \ d_{\mathcal{P}}(\vec{q}) \leqslant \|\vec{q}\|^{-\alpha/(d-1)}\right\}$$

where $d_{\mathcal{P}}(\vec{q}) = \min\{\|\vec{p} - \vec{q}\| \ : \ \vec{p} \in \mathcal{P} \setminus \{\vec{q}\}\}$ and $0 < \alpha < 1$ (fixed).

Example: "$\mathcal{P}$ = Poisson"

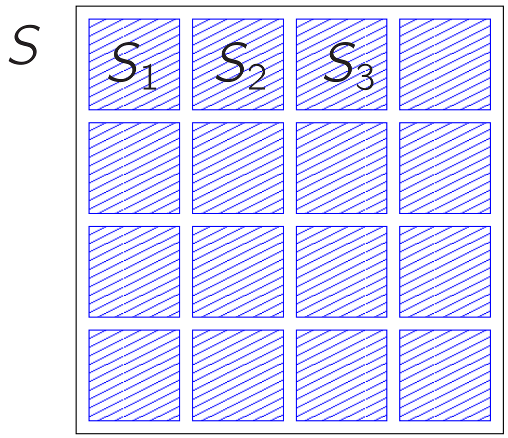
For $\lambda = \omega(S)^{-1}\omega_{|S}$ $\quad$ (S nice $\subset \mathrm{S}_1^{d-1}$; $\omega = \mathrm{vol}_{\mathrm{S}_1^{d-1}}$),

$A = \{Y \in N_s \,:\, \#(Y \cap B) \geqslant m\}$ $\quad$ ($m \in \mathbb{Z}^+$, $B \subset \mathbb{R}^d$, nice and bounded),

and $\vec{q} \in r^{1+\beta}\mathbb{Z}^d \cap \mathcal{B}^d_{r^{1-d-\beta}}$, $\quad$ we want to bound:

$$\boxed{\Delta = \lambda\Big(\big\{\vec{v} \in \mathrm{S}_1^{d-1} \,:\, \widehat{\Pi}_r(\vec{q},\vec{v}) \in A\big\}\Big) - \mathbb{P}(A)}$$

$$\widehat{\Pi}_r(\vec{q},\vec{v}) := ((\mathcal{P} \backslash (\vec{q} + \mathcal{B}^d_{r^{\alpha}/2})) - \vec{q})R_{\vec{v}}D_r$$



Decompose $S = \mathrm{supp}(\lambda) = \sqcup_{\ell=1}^{n} S_\ell \sqcup [\text{small}]$, where $\mathrm{diam}(S_\ell) \ll r^{\beta_1}$ and $\mathrm{dist}(S_\ell, S_{\ell'}) \gg r^{\beta_2}$ $\forall \ell \neq \ell'$.
$(0 < \beta_1 < \beta_2 < 1 - \alpha)$.

Write $\Delta = \sum_{\ell=1}^{n} [\text{contr. from } S_\ell] + [\text{small}]$.

Bernstein inequality $\Rightarrow$ $\boxed{\mathbb{P}(|\Delta| \leqslant r^{\delta}) \geqslant 1 - \exp(r^{-\delta})}$ (some fixed $\delta > 0$).

Use Borel-Cantelli to conclude! $\square$

Example: "$\mathcal{P}$ = Poisson"

Recall: The distribution of $F_0(\Pi - \vec{x})$ is $k_0(\vec{x}, \xi, -\vec{y})\, d\xi\, d\vec{y} \in P(C_\infty)$.

Thus we get $k_0(\vec{x}, \xi, \vec{y}) = c e^{-c\overline{\sigma}\xi}$.

Also $k(\xi, \vec{y}) = c e^{-c\overline{\sigma}\xi}$.

Hence we get back the linear Boltzmann equation:

$$(\partial_t + \vec{V}\nabla_{\vec{Q}}) f_t(\vec{Q}, \vec{V}) = c \int_{S_1^{d-1}} (f_t(\vec{Q}, \vec{V}') - f_t(\vec{Q}, \vec{V}))\sigma(\vec{V}', \vec{V})\, d\vec{V}'.$$

Example: $\mathcal{P}$ = a lattice

Assume $\mathcal{P}$ has covolume one for simplicity.

Then **the list of assumptions holds**, with "**Π = random lattice** $\setminus\{\vec{0}\}$":

$\boxed{\Pi = \mathbb{Z}^d g \setminus \{\vec{0}\} \text{ for } \Gamma g \text{ random in } (\Gamma\backslash G, \mu)}$ with $G = \mathrm{SL}_d(\mathbb{R})$, $\Gamma = \mathrm{SL}_d(\mathbb{Z})$,

μ = the invariant probability measure.

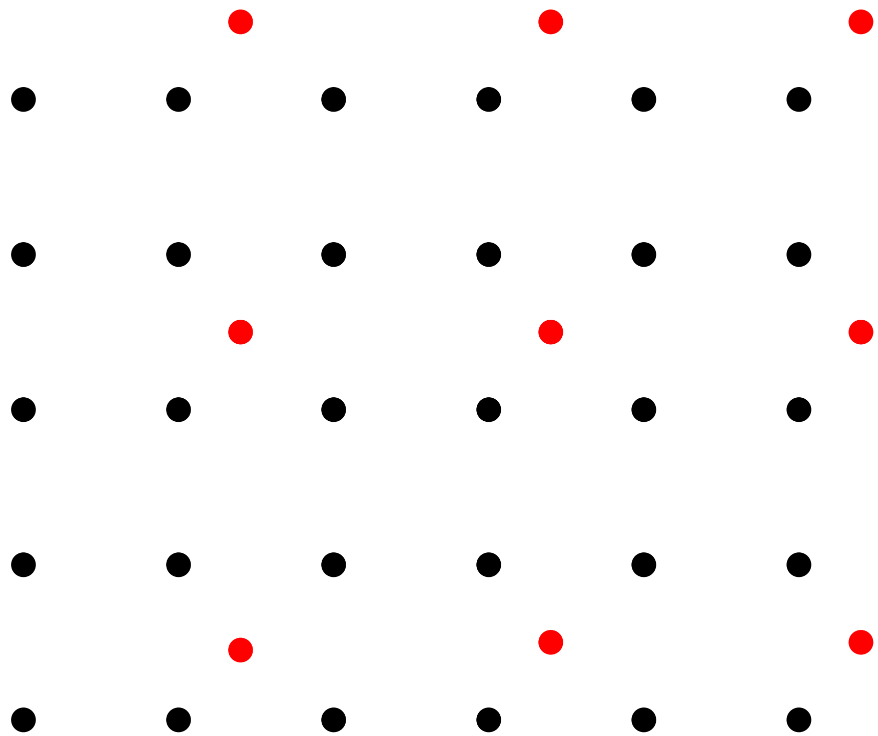
Palm distribution $\nu : \mathbb{R}^d \times \mathcal{B} \to [0,1]$:

For $\vec{z} \neq \vec{0}$; $\boxed{\nu(\vec{z}, B) = \sum_{m=1}^{\infty} \nu_{m,\vec{z}}(B)}$ where $\nu_{m,\vec{z}}$ is the invariant measure on

$X_{m,\vec{z}} = \{\Gamma g \in \Gamma\backslash G \;:\; \vec{z} \in m\vec{e}_1 \Gamma g\}$ with $\nu_{m,\vec{z}}(X_{m,\vec{z}}) = m^{-d}\zeta(d)^{-1}$.

Thus: $\qquad k(\vec{x}, \xi, \vec{y}) = \nu_{1,(\xi,\vec{x}-\vec{y})}(\{\Gamma g \in \Gamma\backslash G \;:\; (\mathbb{Z}^d g - \vec{x}) \cap C_\xi = \emptyset\})$.

Example requiring marking: $\mathcal{P} = \mathbb{Z}^d \cup (2\mathbb{Z}^d + (\sqrt{2}, \frac{1}{2}))$



The key assumption, allowing marking

Assume there is a compact metric space Σ, a map $\sigma : \mathcal{P} \to \Sigma$, and a continuous map $\Sigma \to P(N_s)$, $\sigma \mapsto \mu_\sigma$.

Let Π_σ be a random point set with law μ_σ.

We assume that for any fixed $T \geqslant 1$ and $\lambda \in P(\mathrm{S}_1^{d-1})$ with $\lambda \ll \mathrm{vol}_{\mathrm{S}_1^{d-1}}$, if $\vec{v}$ is random in $(\mathrm{S}_1^{d-1}, \lambda)$, then $\boxed{\Pi_r(\vec{q}, \vec{v}) \xrightarrow[r \to 0]{\mathrm{d}} \Pi_{\sigma(\vec{q})}}$, uniformly over all $\vec{q} \in \mathcal{P}_T(r)$.

Note here: $N_s = N_s(\mathbb{R}^d \times \Sigma)$, and

$$\Pi_r(\vec{q}, \vec{v}) := \left\{ \vec{p} \in \mathcal{P} \setminus \{\vec{q}\} \ : \ \left((\vec{p} - \vec{q}) R_{\vec{v}} D_r, \sigma(\vec{p}) \right) \right\} \subset \mathbb{R}^d \times \Sigma.$$

Example: $\mathcal{P}$ = a cut-and-project set

Fix a lattice L in $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$. $\quad n = d + m,\ m \geqslant 1.$

Fix a (nice) window $W \subset \mathbb{R}^m$.

$$\boxed{\mathcal{P} = \mathcal{P}(W, L) = \left\{\vec{q} \in \mathbb{R}^d \ : \ [\exists \vec{w} \in W \text{ s.t. } (\vec{q}, \vec{w}) \in L]\right\}}$$

For each $\vec{q} \in \mathcal{P}$, we assume there is a *unique* $\vec{w} = \vec{w}(\vec{q}) \in W$ giving $(\vec{q}, \vec{w}) \in L$.

Then, **the list of assumptions holds, with $\Sigma = \overline{W}$.**

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Write $L = \mathbb{Z}^n g$. Consider $\Gamma\backslash\Gamma g \begin{pmatrix} \mathrm{SL}_d(\mathbb{R}) & \\ & I_m \end{pmatrix}$.

Ratner's orbit closure theorem $\Rightarrow \exists H < G$ (connected, closed, $\Gamma \cap H$ lattice),

with $\overline{\Gamma\backslash\Gamma g \begin{pmatrix} \mathrm{SL}_d(\mathbb{R}) & \\ & I_m \end{pmatrix}} = \Gamma\backslash\Gamma H g$. Set μ = inv prob measure on $\Gamma\backslash\Gamma H$.

Now $\boxed{\Pi_{\vec{w}} = \mathcal{P}(W - \vec{w}, \mathbb{Z}^n h g)\setminus\{\vec{0}\}, \text{ for } \Gamma h \text{ random in } (\Gamma\backslash\Gamma H, \mu).}$

Some questions

- Which random point sets Π can occur?
- Invariance of Π? All $\mathrm{SL}_d(\mathbb{R})$?
- Characterize all (A)$\mathrm{SL}_d(\mathbb{R})$-invariant point processes?

END

Macroscopic initial condition. Hence get (here *formally*):

$$= \lim_{r\to 0} r^{d(d-1)} \int_{\mathrm{S}_1^{d-1}} \sum_{\vec{q}'\in\mathcal{P}} \int_{G(\vec{q}',\vec{v})} f(\overline{\Pi}_r(\vec{q}',\vec{v})+\vec{x})\, \Lambda'(r^{d-1}\vec{q}'-x_1\vec{v},\vec{v})\, d\vec{x}\, d\vec{v}$$

Macroscopic initial condition. Hence get (here *formally*):

$$= \lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \sum_{\vec{q}'\in\mathcal{P}} \int_{G(\vec{q}',\vec{v})} f\left(\overline{\Pi}_r(\vec{q}',\vec{v}) + \vec{x}\right) \Lambda'(r^{d-1}\vec{q}' - x_1\vec{v}, \vec{v})\, d\vec{x}\, d\vec{v}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \int_{S_1^{d-1}} I\Big(\big(\overline{\Pi}_r(\vec{q}',\vec{v}) + \vec{x}\big) \cap C_{x_1} = \emptyset\Big) f\left(\Pi_r(\vec{q}',\vec{v}) + \vec{x}\right) \Lambda'(\cdots)\, d\vec{v}\, d\vec{x}$$

Macroscopic initial condition. Hence get (here *formally*):

$$= \lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \sum_{\vec{q}'\in\mathcal{P}} \int_{G(\vec{q}',\vec{v})} f\big(\overline{\Pi}_r(\vec{q}',\vec{v})+\vec{x}\big)\,\Lambda'(r^{d-1}\vec{q}'-x_1\vec{v},\vec{v})\,d\vec{x}\,d\vec{v}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \int_{S_1^{d-1}} I\Big(\big(\overline{\Pi}_r(\vec{q}',\vec{v})+\vec{x}\big)\cap C_{x_1}=\emptyset\Big) f\big(\Pi_r(\vec{q}',\vec{v})+\vec{x}\big)\,\Lambda'(\cdots)\,d\vec{v}\,d\vec{x}$$

Here $\Pi_r(\vec{q}',\vec{v}) \xrightarrow[r\to 0]{\text{d}} \Pi$; hence $\overline{\Pi}_r(\vec{q}',\vec{v})+\vec{x} \xrightarrow[r\to 0]{\text{d}} \overline{\Pi}+\vec{x}$,

where $\boxed{\overline{\Pi} := \Pi \cup \{\vec{0}\}}$

Macroscopic initial condition. Hence get (here *formally*):

$$= \lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \sum_{\vec{q}'\in\mathcal{P}} \int_{G(\vec{q}',\vec{v})} f\left(\overline{\Pi}_r(\vec{q}',\vec{v}) + \vec{x}\right) \Lambda'(r^{d-1}\vec{q}' - x_1\vec{v},\vec{v})\, d\vec{x}\, d\vec{v}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \int_{S_1^{d-1}} I\Big(\big(\overline{\Pi}_r(\vec{q}',\vec{v}) + \vec{x}\big) \cap C_{x_1} = \emptyset\Big) f\left(\Pi_r(\vec{q}',\vec{v}) + \vec{x}\right) \Lambda'(\cdots)\, d\vec{v}\, d\vec{x}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \left(\int_{S_1^{d-1}} \Lambda'(r^{d-1}\vec{q}' - x_1\vec{v},\vec{v})\, d\vec{v} \right)$$

$$\times\, \mathbb{E}\Big(I\big((\overline{\Pi} + \vec{x}) \cap C_{x_1} = \emptyset\big) f\left(\overline{\Pi} + \vec{x}\right) \Big)\, d\vec{x}$$

Macroscopic initial condition. Hence get (here *formally*):

$$= \lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \sum_{\vec{q}'\in\mathcal{P}} \int_{G(\vec{q}',\vec{v})} f\left(\overline{\Pi}_r(\vec{q}',\vec{v})+\vec{x}\right) \Lambda'(r^{d-1}\vec{q}'-x_1\vec{v},\vec{v})\,d\vec{x}\,d\vec{v}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \int_{S_1^{d-1}} I\Big(\big(\overline{\Pi}_r(\vec{q}',\vec{v})+\vec{x}\big)\cap C_{x_1}=\emptyset\Big) f\left(\Pi_r(\vec{q}',\vec{v})+\vec{x}\right) \Lambda'(\cdots)\,d\vec{v}\,d\vec{x}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \left(\int_{S_1^{d-1}} \Lambda'(r^{d-1}\vec{q}'-x_1\vec{v},\vec{v})\,d\vec{v}\right)$$

$$\times\, \mathbb{E}\Big(I\big((\overline{\Pi}+\vec{x})\cap C_{x_1}=\emptyset\big) f(\overline{\Pi}+\vec{x})\Big)\,d\vec{x}$$

$\mathcal{P}$ has asymptotic density $c_{\mathcal{P}}$!

Macroscopic initial condition. Hence get (here *formally*):

$$= \lim_{r\to 0} r^{d(d-1)} \int_{S_1^{d-1}} \sum_{\vec{q}'\in\mathcal{P}} \int_{G(\vec{q}',\vec{v})} f\left(\overline{\Pi}_r(\vec{q}',\vec{v}) + \vec{x}\right) \Lambda'(r^{d-1}\vec{q}' - x_1\vec{v}, \vec{v})\, d\vec{x}\, d\vec{v}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \int_{S_1^{d-1}} I\Big(\big(\overline{\Pi}_r(\vec{q}',\vec{v}) + \vec{x}\big) \cap C_{x_1} = \emptyset\Big) f\Big(\Pi_r(\vec{q}',\vec{v}) + \vec{x}\Big) \Lambda'(\cdots)\, d\vec{v}\, d\vec{x}$$

$$= \lim_{r\to 0} r^{d(d-1)} \sum_{\vec{q}'\in\mathcal{P}} \int_{C_\infty} \left(\int_{S_1^{d-1}} \Lambda'(r^{d-1}\vec{q}' - x_1\vec{v}, \vec{v})\, d\vec{v} \right) \times \mathbb{E}\Big(I((\overline{\Pi} + \vec{x}) \cap C_{x_1} = \emptyset) f(\overline{\Pi} + \vec{x}) \Big)\, d\vec{x}$$

$$= \int_{C_\infty} \underbrace{\left(c_{\mathcal{P}} \int_{S_1^{d-1}} \int_{\mathbb{R}^d} \Lambda'(\vec{y} - x_1\vec{v}, \vec{v})\, d\vec{y}\, d\vec{v} \right)}_{\boxed{=1}} \times \mathbb{E}\Big(\cdots\cdots \Big)\, d\vec{x}$$

$$\boxed{= c_{\mathcal{P}} \int_{C_\infty} \mathbb{E}\Big(I((\overline{\Pi} + \vec{x}) \cap C_{x_1} = \emptyset) f(\overline{\Pi} + \vec{x}) \Big)\, d\vec{x}}$$