DIRICHLET'S THEOREM FOR INHOMOGENEOUS APPROXIMATION

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GOA 2016

Starting point: Dirichlet's Theorem

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Theorem (Dirichlet): for any $A \in M_{m \times n}(\mathbb{R})$ and any $T > 1$ $\exists$ $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$(1) \qquad \|A\mathbf{q} - \mathbf{p}\|^m \leq \frac{1}{T} \quad \text{and} \quad \|\mathbf{q}\|^n \leq T.$$

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Corollary (Dirichlet): for any $A \in M_{m\times n}(\mathbb{R})$
$\exists$ ∞ many $\mathbf{q} \in \mathbb{Z}^n$ such that

(2) $$\|A\mathbf{q} - \mathbf{p}\|^m \le \frac{1}{\|\mathbf{q}\|^n} \text{ for some } \mathbf{p} \in \mathbb{Z}^m.$$

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Questions: what happens if in (1) and (2) the RHS is replaced by a faster decreasing function of T and $\|\mathbf{q}\|$ respectively?
In particular, what happens for typical A?

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Well studied in the setting of (2), not so well for (1).

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Note: $W(\psi)$ is a **limsup set**. In fact it is easy to see that $A \in W(\psi)$ if and only if the system

$$(1\psi) \qquad \|A\mathbf{q} - \mathbf{p}\|^m \le \psi(T) \quad \text{and} \quad \|\mathbf{q}\|^n \le T$$

has a nontrivial integer solution for ∞ many $T \in \mathbb{N}$.

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(Yes, but E_k^c are way more complicated and harder to work with...)

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Mahlers's Criterion: $\delta(\Lambda)$ is very small $\leftrightarrow$ Λ is far far away in X.

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But the family of shrinking targets $\left\{\delta^{-1}\big((r,1]\big)\right\}$ is kind of complicated. Some partial results (for slowly decaying functions ψ) can be obtained, not a complete solution yet.

Example: put $\psi_c(T) = \frac{c}{T}$ where $c < 1$. This corresponds to

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has measure 0, therefore $DI(\psi_c)$ has measure zero (Davenport and Schmidt 1969).

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Inhomogeneous Dirichlet's Theorem

The set-up: we now have $A \in M_{m\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$, and look for the following statement:

for any $T > 1$ (or at least any large enough T)
$\exists\ \mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n$ such that

$$(\widehat{1}) \qquad \|A\mathbf{q} + \mathbf{b} - \mathbf{p}\|^m \le \ ???? \quad \text{and} \quad \|\mathbf{q}\|^n \le T$$

(there are no reasons to exclude $\mathbf{q} = 0$ anymore).

And what to put instead of ????
Turns out that no function which goes to 0 as $T \to \infty$ will work!
Just take $A = 0$ and $\mathbf{b} \in \frac{1}{2}\mathbb{Z}^m \smallsetminus \mathbb{Z}^m$.

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(which is not much).

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The work of Cassels and Schmidt gives precise conditions on ψ such that $\widehat{W}(\psi)$, or even $\{A : (A, \mathbf{b}) \in \widehat{W}(\psi)\}$ for fixed $\mathbf{b}$, has zero/full measure.

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However the dynamical approach works and produces a definitive result (so in some sense the inhomogeneous version is easier than its homogeneous counterpart!)

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The same principle works: good approximation to $(A, \mathbf{b})$

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small value of $\hat{\delta}(g_t\Lambda_{A,\mathbf{b}})$.

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Specifically, given $\psi(\cdot)$, there exists $r(\cdot)$ such that

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has an integer solution for some $T \iff \hat{\delta}(g_t \Lambda_{A,\mathbf{b}}) \leq r(t)$ with t explicitly depending on T.

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This (fixed) target $r(t) \equiv R$ corresponds to $\psi_C(T) = \frac{C}{T}$, $C > 0$.

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In particular, for any $R > 0$ the set $\hat{\delta}^{-1}\big((R, \infty)\big)$ has positive measure. This (fixed) target $r(t) \equiv R$ corresponds to $\psi_C(T) = \frac{C}{T}$, $C > 0$. By ergodicity of the g_t-action on $\hat{X}$, the set

$$\{\Lambda \in \hat{X} : \hat{\delta}(g_t \Lambda) \le R \text{ for all large enough } t\}$$

has measure 0, therefore $\widehat{DI}(\psi_C)$ has measure zero for any $C > 0$.

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- $\psi(k) = C\frac{(\log k)^{1+\varepsilon}}{k} \implies \widehat{DI}(\psi)$ has full measure.

Thank you for your attention!