# Bounded Orbits in the Space of Unimodular Lattices

(Joint work with Lifan Guan and Dmitry Kleinbock)

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Let $X_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.

- $X_n \cong$ space of unimodular lattices in $\mathbb{R}^n$.
- $X_n$ is noncompact.
- $X_n$ admits an $SL_n(\mathbb{R})$-invariant probability measure (Haar measure).

Let

- $F = \{g_t : t \in \mathbb{R}\}$ be a 1-parameter subgroup of $SL_n(\mathbb{R})$ with noncompact closure,
- $F^+ := \{g_t : t \geq 0\}$ be the corresponding 1-parameter subsemigroup.

We are interested in the sets

$$E(F, X_n) := \{x \in X_n : Fx \text{ is bounded}\}$$

and

$$E(F^+, X_n) := \{x \in X_n : F^+x \text{ is bounded}\}.$$

- Note that $E(F, X_n) \subset E(F^+, X_n)$.

- Moore's Ergodicity Theorem
  $\implies$ the $F$-action on $X_n$ is strongly mixing
  $\implies$ $E(F^+, X_n)$ has zero Haar measure.

- If $F$ is quasiunipotent (i.e., all eigenvalues of every $g_t$ are of modulus 1), then
  Ratner's Theorem
  $\implies$ $E(F^+, X_n) \subset$ a countable union of proper submanifolds
  $\implies$ $\dim_H E(F^+, X_n) < \dim X_n$.

We say that

- a subset $S \subset X_n$ is thick if $\dim_H S \cap U = \dim X_n$ for any open subset $U \subset X_n$.

The following was conjectured by Margulis in 1990.

**Theorem (Kleinbock-Margulis, 1996)**

*If $F$ is non-quasiunipotent, then $E(F, X_n)$ is thick in $X_n$.*

We are interested in the following

## Question

Given countably many 1-parameter non-quasiunipotent $F_k$ ($k = 1, 2, \ldots$). Are the sets

$$\bigcap_{k=1}^{\infty} E(F_k, X_n) = \{x \in X_n : F_k x \text{ is bounded for every } k\}$$

and

$$\bigcap_{k=1}^{\infty} E(F_k^+, X_n) = \{x \in X_n : F_k^+ x \text{ is bounded for every } k\}$$

thick in $X_n$?

# Relation to Diophantine approximation

Given $d \geq 1$.

- $\mathcal{R}_d := \{(r_1, \ldots, r_d) \in \mathbb{R}^d : r_i \geq 0, \sum_i r_i = 1\}$.
- For $\mathbf{r} \in \mathcal{R}_d$, consider the 1-parameter subsemigroup
$$F_{\mathbf{r}}^+ := \{\mathrm{diag}(e^{r_1 t}, \ldots, e^{r_d t}, e^{-t}) : t \geq 0\}$$
of $SL_{d+1}(\mathbb{R})$.
- For $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$, denote
$$x_{\mathbf{a}} = \begin{pmatrix} \mathrm{I}_d & \mathbf{a}^{\mathrm{T}} \\ 0 & 1 \end{pmatrix} SL_{d+1}(\mathbb{Z}) \in X_{d+1}.$$

**Theorem (Dani 1985; Kleinbock 1998)**

$x_{\mathbf{a}} \in E(F_{\mathbf{r}}^+, X_{d+1}) \iff \mathbf{a}$ *is* $\mathbf{r}$*-badly approximable, i.e.,*

$$\mathbf{a} \in \mathbf{Bad}(\mathbf{r}) := \{(a_1, \ldots, a_d) \in \mathbb{R}^d : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq d} q^{r_i} \|q a_i\| > 0\}.$$

Consider the $d$-dimensional torus

$$T_d := \{x_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^d\} \subset X_{d+1}.$$

**Theorem (Badziahin-Pollington-Velani (2010) for $d = 2$; Beresnevich (2013) for arbitrary $d$)**

*Let $\mathcal{S} \subset \mathcal{R}_d$ be a countable subset. Suppose*

$$dist(S \smallsetminus \partial\mathcal{R}_d, \partial\mathcal{R}_d) > 0. \tag{1}$$

*Then $\bigcap_{\mathbf{r}\in\mathcal{S}} \mathbf{Bad}(\mathbf{r})$ is thick in $\mathbb{R}^d$. Equivalently, the set*

$$T_d \cap \bigcap_{\mathbf{r}\in\mathcal{S}} E(F_{\mathbf{r}}^{+}, X_{d+1})$$

*is thick in $T_d$.*

- $d = 2 \implies$ Schmidt's Conjecture: $\mathbf{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \mathbf{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \varnothing$.
- When $d = 2$, condition (1) can be removed (A., 2012).
- When $d > 2$, removing (1) is a challenging open problem.

In the whole space $X_n$ (instead of in $T_d$), we have

**Theorem (Kleinbock-Weiss, 2013)**

*For any countably many 1-parameter diagonalizable subgroups $F_k$ of $SL_2(\mathbb{R})$, the set*

$$\bigcap_{k=1}^{\infty} E(F_k, X_2)$$

*is thick in $X_2$.*

**Theorem ($*$) (A.-Guan-Kleinbock, 2015)**

*For any countably many 1-parameter diagonalizable subgroups $F_k$ of $SL_3(\mathbb{R})$, the set*

$$\bigcap_{k=1}^{\infty} E(F_k, X_3)$$

*is thick in $X_3$.*

# HAW subsets in a manifold (Kleinbock-Weiss, 2013)

Essential tool in proving both theorems: hyperplane absolute winning (HAW) subsets in a $C^1$-manifold $M$.

- Remark: It is difficult to prove Theorem $(*)$ using Schmidt's $(\alpha, \beta)$-game directly.

The definition of HAW subsets in $M$ consists of 2 steps.

Step 1. Define the hyperplane absolute game on an open subset $U \subset \mathbb{R}^d$ (generalizing the same game on $\mathbb{R}^d$ introduced by Broderick-Fishman-Kleinbock-Reich-Weiss, 2012):

- Two players: Alice and Bob.
- Given a target set $S \subset U$ and $\beta \in (0, \frac{1}{3})$.
- Bob starts by choosing a closed ball $B_0$ in $\mathbb{R}^d$ contained in $U$.
- After Bob chooses a closed ball $B_i$ of radius $\rho_i$, Alice chooses a $\rho_i'$-neighborhood $L_i^{(\rho_i')}$ of an affine hyperplane $L_i \subset \mathbb{R}^d$ with $\rho_i' \leq \beta\rho_i$.
- Then Bob chooses a closed ball $B_{i+1} \subset B_i \setminus L_i^{(\rho_i')}$ of radius $\rho_{i+1} \geq \beta\rho_i$.
- Alice wins if

$$\bigcap_{i=0}^{\infty} B_i \cap S \neq \emptyset.$$

- $S$ is HAW on $U$ if Alice has a winning strategy for every $\beta \in (0, \frac{1}{3})$.

Step 2. A subset $S \subset M$ is HAW on $M$ if for any $x \in M$, there is a $C^1$ diffeomorphism $\phi$ from an open neighborhood $V$ of $x$ onto an open subset $U \subset \mathbb{R}^{\dim M}$ such that $\phi(S \cap V)$ is HAW on $U$.

### Lemma

1. *An HAW subset is thick.*
2. *A countable intersection of HAW subsets is HAW.*
3. *The image of an HAW subset under a $C^1$ diffeomorphism of $C^1$ manifolds is HAW.*

Thus, Theorem $(*)$ $\Longleftarrow$

### Theorem $(**)$

*For any 1-parameter diagonalizable subgroup $F$ of $SL_3(\mathbb{R})$, the set $E(F^+, X_3)$ is HAW on $X_3$.*

# Proof of Theorem $(**)$

- Let $\mathbf{r} = (r_1, r_2) \in \mathcal{R}_2$ with $r_1 > r_2 > 0$.
- Given $b \in \mathbb{R}$, a vector $(a_1, a_2) \in \mathbb{R}^2$ is *$b$-twisted $\mathbf{r}$-badly approximable* if

$$\inf_{(p_1,p_2,q)\in\mathbb{Z}^2\times\mathbb{N}} \max\{q^{r_1}|qa_1-p_1-b(qa_2-p_2)|, q^{r_2}|qa_2-p_2|\} > 0.$$

- Let $\mathbf{Bad}^b(\mathbf{r})$ denote the set of such vectors.
- Note that $\mathbf{Bad}^0(\mathbf{r}) = \mathbf{Bad}(\mathbf{r})$.
- Recall

$$F_{\mathbf{r}}^+ := \{\mathrm{diag}(e^{r_1 t}, e^{r_2 t}, e^{-t}) : t \geq 0\}.$$

**Lemma (Kleinbock, 1998)**

$$F_{\mathbf{r}}^+ \begin{pmatrix} 1 & b & a_1 \\ & 1 & a_2 \\ & & 1 \end{pmatrix}^{-1} SL_3(\mathbb{Z}) \text{ is bounded} \iff (a_1, a_2) \in \mathbf{Bad}^b(\mathbf{r}).$$

$$H := \left\{ \begin{pmatrix} 1 & b & a_1 \\ & 1 & a_2 \\ & & 1 \end{pmatrix} : a_1, a_2, b \in \mathbb{R} \right\}.$$

Theorem $(**) \Longleftarrow$

**Theorem**

*The set $\{h \in H : F_{\mathbf{r}}^{+} h \, SL_3(\mathbb{Z})$ is bounded$\}$ is HAW on $H$. Equivalently, the set*

$$\{(a_1, a_2, b) \in \mathbb{R}^3 : (a_1, a_2) \in \mathbf{Bad}^b(\mathbf{r})\}$$

*is HAW on $\mathbb{R}^3$.*

We also have

**Theorem**

*For any $b \in \mathbb{R}$, $\mathbf{Bad}^b(\mathbf{r})$ is HAW on $\mathbb{R}^2$.*

This generalizes:

- Nesharim-Simmons (2013): $\mathbf{Bad}(\mathbf{r})$ is HAW on $\mathbb{R}^2$.

For arbitrary $d \geq 2$, we have

**Theorem**

*Suppose* $\mathbf{r} = (r_1, \ldots, r_d) \in \mathcal{R}_d$ *and*

$$\#\{i : r_i = \max_{1 \leq j \leq d} r_j\} \geq d - 1.$$

- *(Guan-Yu, 2015)* $\mathbf{Bad}(\mathbf{r})$ *is HAW on* $\mathbb{R}^d$*.*
- *(Guan-Wu, 2016)* $E(F_{\mathbf{r}}^+, X_{d+1})$ *is HAW on* $X_{d+1}$*.*

# Thank You!