GOA TALK

LEI YANG

This is a joint work with Ralf Spatzier.

Exponential mixing of higher order actions. Let N be a nilpotent algebraic group defined over $\mathbb{Q}$. $M = N(\mathbb{Z}) \setminus N(\mathbb{R})$ is a compact nilmanifold. Let $\pi : N(\mathbb{R}) \to M$ denote the canonical projection. Consider a $\mathbb{Z}^k$ $(k \geq 2)$ action on M by affine automorphisms. Gorodnik and Spatzier proved that if the action does not have rank one factors, then the following exponential mixing result holds: there exist constants $C > 0$ and $\eta > 0$ such that for any $f, g \in C^\theta(M)$, and any $a \in \mathbb{Z}^k$, we have that

$$|\langle f \circ a, g\rangle - \langle f, 1\rangle \cdot \langle g, 1\rangle| \leq Ce^{-\eta\|a\|}\|f\|_\theta\|g\|_\theta.$$

We want to study the mixing property of $\mathbb{Z}^k$ actions on S-adic nilmanifolds. Consider a $\mathbb{Z}^k$ action ρ_l on a S-adic nilmanifold $\mathcal{S}(M) = N(\mathbb{Z}) \setminus N(\mathbb{R}) \times \prod_{p\in S} N(\mathbb{Z}_p)$ by affine automorphisms. Do there exist constants $C > 0$ and $\eta > 0$ such that for all $f, g \in C^\theta(\mathcal{S}(M))$, the following holds:

$$|\langle f \circ a, g\rangle - \langle f, 1\rangle \cdot \langle g, 1\rangle| \leq Ce^{-\eta\|a\|}\|f\|_\theta\|g\|_\theta.$$

We could prove the following weaker statement: for any $f, g \in C^\theta(M)$, if we consider them as functions on $\mathcal{S}(M)$, we have the above estimate.

Smooth classification of higher rank expanding actions. Consider a smooth $\mathbb{Z}^k$ $(k \geq 2)$ action on M that does not have any rank one factors and contains an Anosov element. The work of Fisher-Kalinin-Spatzier and Rodriguez.Hertz-Wang says that the action is smoothly conjugate to an action by affine automorphisms. We wonder what happens if we replace $\mathbb{Z}^k$ actions by $\mathbb{Z}^k_+$ actions, and replace M by any compact manifold $\overline{M}$. Let ρ be a smooth $\mathbb{Z}^k_+$ $(k \geq 2)$ semigroup action on $\overline{M}$ that does not have any rank one factors. Suppose there exists $a \in \mathbb{Z}^k_+$ such that $\rho(a)$ is expanding, then by a result of Gromov (as well as Shub), there exists an infra-nilmanifold M and a $\mathbb{Z}^k_+$ action ρ_l on M by affine infra-nilendomorphisms, such that by a homeomorphism $\phi : \overline{M} \to M$, ρ is conjugated to ρ_l.

In general, ϕ is Hölder continuous, without the higher rank assumption. Can we make ϕ differentiable with the higher rank assumption? As an application of the exponential mixing, we prove the following:

Theorem. When $\dim M \geq 5$, ϕ is C^∞.

To prove the above theorem, we need to apply the technique developed by Fisher-Kalinin-Spatzier. We first extend the action ρ and ρ_l to $\mathbb{Z}^k$ actions on the solenoids of M and $\overline{M}$, denoted by $\mathcal{S}(M)$ and $\mathcal{S}(\overline{M})$ respectively. The conjugacy ϕ does not do anything to the non-archemedean fibers. We write $\phi(z)$ as $zh(z)$ and write down the equation $h(z)$ satisfies:

$$h(z) = Q_a(z)\rho_l(a)^{-1}h(\rho(a)z).$$

This equation can be iterated to get a formal series of $h(z)$:

$$h = \sum_{i=0}^{\infty} \rho_l(a)^{-i} Q_a \circ \rho(ia).$$

Let V be a generalized eigenspace of ρ_l, let h_V denote the projection of h on V. Then h_V can be written as the following series:

$$h_V = \sum_{i=0}^{\infty} \rho_l(a)^{-i} (Q_a)_V \circ \rho(ia).$$

To show h is smooth, it suffices to show that h_V is smooth for all possible V's. In order to show h_V is smooth, we need to consider the Lyapunov foliations of the action ρ, and prove that for any Lyapunov foliation $\mathcal{W}$, all partial derivatives of h_V along $\mathcal{W}$ exists and is Hölder continuous. For each $k \geq 1$, V and $\mathcal{W}$, we need to choose a such that $\rho_l(a)^{-1}$ does not expand V too much, and $D\rho(a)$ does not expand $T(\mathcal{W})$ too much, then for any $g \in C^\theta(M)$, we have that

$$\langle \partial_{\mathcal{W}} h_V, g \rangle = \sum_{i=0}^{\infty} \langle \rho_l(a)^{-1} \partial_{\mathcal{W}} (Q_a)_V \circ \rho(ia) D\rho(ia)|_{\mathcal{W}}, g \rangle.$$

The exponential mixing for linear action implies the same result for general actions, thanks to the existence of the Hölder conjugacy ϕ. This shows that $\partial_{\mathcal{W}}^k$ is in the dual space of $C^\theta(M)$. Combined with a result of Taylor and Rauch, this shows that h_V is smooth.

Sketch of the proof. We first reduce the problem to proving the exponential mixing when every $\rho_l(a)$ is semisimple and irreducible. Then we divide the whole space into small boxes. In each box, the value of g can be estimated by evaluating at a typical point. Then we need to show that the integral of $f \circ \rho_l(a)$ over the box is close to the integral f over the whole space. There are two cases. The first case is when some real eigenspace is expanded exponentially by $\rho_l(a)$, then the proof is the same as that in the work of Gorodnik and Spatzier. The second case is when no real eigenspace is expanded exponentially. Then some p-adic eigenspace is expanded exponentially. Let us denote it by V. Then we consider the image of the small box under the action of $\rho_l(a)$, and its projection on M. We will get a polynomial sequence on M. By Green-Tao theorem, the obstruction of the sequence being effectively equidistributed is that V "almost" satisfies a integer linear equation with bounded coefficients. Since $\rho_l(a)$ is irreducible, V is not defined over $\mathbb{Z}$. We could find a vector $v \in V$ whose coordinates are $\mathbb{Q}$ independent algebraic numbers (in $\mathbb{Q}_p$). For any integer vector z, the linear form $|\langle v, z \rangle|_p$ is bounded from below by $C\|z\|^{-\gamma}$. This means that the obstruction cannot hold for appropriate exponential index.