

Diophantine approximation and products of linear forms

Work in progress with Emmanuel Breuillard

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