On the second moment of twisted L-functions

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Dirichlet L-functions

To $\chi \pmod q$ a primitive Dirichlet character of modulus $q \geq 1$ is associated a Dirichlet L-function

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$$

The function $L(\chi, s)$ has analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(\chi, s) = \varepsilon(\chi)\Lambda(\overline{\chi}, s)$$

$$\Lambda(f \otimes \chi, s) = q^{s/2} L_\infty(\chi, s) L(\chi, s)$$

where $|\varepsilon(\chi)| = 1$ and $L_\infty(\chi, s)$ is a Gamma-function.

Dirichlet L-functions

The function $L(\chi, s)$ is very well understood for $\Re s > 1$ (and therefore for $\Re s < 0$ by the fct. eqn.); indeed most fundamental for applications are the analytic properties and values of $L(\chi, s)$ for $\Re s \in [0, 1]$ (the critical strip):
For instance Dirichlet's Thm. on primes in arithmetic progressions follows from

Theorem (Dirichlet)

$$\forall q,\ \forall \chi \pmod q,\ L(\chi, 1) \neq 0.$$

The closer $\Re s$ gets to $1/2$ the harder investigation become.

Moments of Dirichlet L-functions

Besides individual analytic properties, it is also important to understand the variations of $L(\chi, s)$ as χ varies over the characters of given modulus.
For instance a very challenging problem is to understand the moments:

$$\frac{1}{\varphi^*(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \textit{primitive}}} |L(\chi, 1/2)|^k \stackrel{?}{=} MT_k(q) + O(q^{-\delta_k})$$

for some $\delta_k > 0$ and $MT_k(q) = (\log q)^{O_k(1)}$.
The moments of order $k = 0, 1, 2, 3$ (without absolute values in the odd case) are not too hard. The case $k = 4$ is much harder.

The fourth moment of Dirichlet L-functions

Theorem (M. Young, 2011)

For q prime, one has

$$\frac{1}{q-2} \sum_{\substack{\chi \pmod q \\ \chi \neq 1}} |L(\chi, 1/2)|^4 = P_4(\log q) + O(q^{-\delta})$$

with $\delta = 5/513$ and P_4 has degree 4.

- The case q prime is the hardest, although the main term is more complicated for q composite.
- The exponent $5/513$ has recently been improved to $1/33$ (2015) and to $1/23$ (2016) by BFKMM.

Twisted L-functions

M. Young's proof builds crucially on the following facts

1. $L(\chi, s)^2 = \sum_{n\geq 1} \frac{d(n)\chi(n)}{n^s}$ where $d(n)$ is the divisor function
$$d(n) = 1 \star 1(n) = \sum_{ab=n} 1$$
2. The divisor function $d(n)$ is the n-th Hecke eigenvalue of the Eisenstein series at the central point
$$\frac{\partial}{\partial s} E(z, \frac{1}{2} + s)_{|s=0}$$
so that
$$L(\chi, s)^2 = L(E'_{s=0} \otimes \chi, s)$$
is the twisted L-function of $L(E'_{s=0}, s) = \zeta(s)^2$ by the character χ.

Twisted L-function

Given

$$f(z) = \sum_{n \in \mathbb{Z}} \lambda_f(n) W_f(n, z)$$

be a modular form for the group $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$: ie.

- an holomorphic cusp form of weight $k \in 2\mathbb{N}_{\geq 1}$,
- a non-holomorphic Maass cusp form of weight 0,
- a non-holomorphic Eisenstein series.

which is also eigenvalue of the (suitably normalized) Hecke operators $(T_n)_{n\geq 1}$ with eigenvalue $(\lambda_f(n))_{n\geq 1}$.

Twisted L-function

The twisted L-function is

$$L(f \otimes \chi, s) = \sum_{n \geq 1} \frac{\lambda_f(n)\chi(n)}{n^s} = \prod_p (1 - \frac{\lambda_f(p)\chi(p)}{p^s} + \frac{\chi(p)}{p^2 s})^{-1}.$$

The function $L(f \otimes \chi, s)$ has analytic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(f \otimes \chi, s) = \varepsilon(f \otimes \chi)\Lambda(f \otimes \chi, s)$$

$$\Lambda(f \otimes \chi, s) = q^s L_\infty(f \otimes \chi, s) L(f \otimes \chi, s)$$

where $|\varepsilon(f \otimes \chi)| = 1$ and $L_\infty(f \otimes \overline{\chi}, s)$ is a product of Gamma-functions.

Second Moment of twisted L-function

The fourth moment of Dirichlet L-functions is a special case of the longstanding problem of evaluating asymptotically the second moment

$$\frac{1}{\varphi^*(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \textit{primitive}}} |L(f \otimes \chi, 1/2)|^2 \stackrel{?}{=} MT(f,q) + O(q^{-\delta})$$

for some $\delta > 0$ and $MT(f,q) = (\log q)^{O(1)}$. When q is a prime (hardest case)

$$MT(f,q) = P_f(\log q).$$

Second Moment of twisted L-function

Young's proof exploit crucially the fact that the divisor function is a Dirichlet convolution; nothing like this when f is cuspidal. Khan-Ricotta-Zhao and Blomer-Milicevic have obtained asymptotic formula (sometimes with weak error terms) when q is composite:

Theorem (BFKMM + KMS)

Let f, g be cusp forms and q a prime; one has

$$\frac{1}{q-2} \sum_{\substack{\chi \pmod q \\ \chi \neq 1}} L(f \otimes \chi, 1/2)\overline{L(g \otimes \chi, 1/2)} = P_{f,g}(\log q) + O(q^{-\delta})$$

for $\delta = 1/145$ and $P_{f,g}(X)$ of degree $1, 0$ or $-\infty$.

Second Moment of twisted L-function: idea of the proof

The problem has a dynamical flavor which will be discussed below; for now we would like to give an idea of the current proof. Standard methods reduce to the evaluation of $O(\log^2 q)$ sums of the shape

$$\frac{1}{q-1}\sum_{\chi \pmod q}\sum\sum_{m,n}\frac{\lambda_f(m)\lambda_g(n)}{m^{1/2}n^{1/2}}W_1(\frac{m}{M})W_2(\frac{n}{N})\chi(m)\overline{\chi}(n)$$

$$=\sum_{m\equiv n \pmod q}\frac{\lambda_f(m)\lambda_g(n)}{m^{1/2}n^{1/2}}W_1(\frac{m}{M})W_2(\frac{n}{N})\stackrel{?}{=}MT(M,N)+O(q^{-\delta}).$$

where $W_1, W_2\in\mathcal{C}^\infty([1,2])$ and $MN\leq q^2$. The main case is

$$MN\approx q^2.$$

The shifted convolution problem

The method of evaluation differs depending on the relative values of M, N:
If M, N are not too far apart (eg. $M \approx N \approx q$) we rewrite the congruence $m \equiv n \pmod q$ as $m - n = qh$ with $h \ll (M+N)/q$.

- For $h = 0$ the evaluation is easy, using the Rankin-Selberg method and yields the main term $MT(M, N)$ plus an admissible error term.
- For $h \neq 0$ the equality

$$\sum_{m-n=qh} \frac{\lambda_f(m)\lambda_g(n)}{m^{1/2}n^{1/2}} W_1(\frac{m}{M}) W_2(\frac{n}{N}) \stackrel{?}{=} \delta_{f,g=Eis} MT + O(q^{-\delta})$$

is an instance of the Shifted convolution problem.

The shifted convolution problem

The shifted convolution problem may be approached either by harmonic analysis on $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$ or by a variant of the circle method:

$$\sum_{m-n=qh} \frac{\lambda_f(m)\lambda_g(n)}{m^{1/2}n^{1/2}} W_1\left(\frac{m}{M}\right) W_2\left(\frac{n}{N}\right)$$

$$= \int_0^1 W_f\left(n(x)a\left(\frac{1}{M}\right)\right)\overline{W_g\left(n(x)a\left(\frac{1}{N}\right)\right)}e(-hx)dx$$

where W_f and W_g are vectors in (the Whittaker models of) the automorphic representations generated by f and g and

$$n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \ a(y) = \begin{pmatrix} y & 0 \\ & 1 \end{pmatrix}.$$

The shifted convolution problem

Decomposing spectrally the inside product before computing the horocycle integral and taking further advantage of the h averaging, one has

Theorem (Blomer-Milicevic)

Assume $M \leq N$

$$\sum_{m-n\equiv 0 \pmod q} \frac{\lambda_f(m)\lambda_g(n)}{m^{1/2}n^{1/2}} W_1(\frac{m}{M})W_2(\frac{n}{N})$$

$$= \delta_{f,g=Eis}MT + (MNq)^{o(1)}[(\frac{N/M}{q})^{1/4} + (\frac{N/M}{q})^{1/2}].$$

which is good as long as $M \geq q^{1/2+\eta}$, $\eta > 0$.

Bilinear sums of Kloosterman sums

To handle the case where $M < N$ are far apart we apply the (Voronoi) summation formula to the longest variable (say $N \geq q^{3/2-\eta}$), the above sum becomes essentially

$$\frac{N^{1/2}}{q^{3/2}M^{1/2}} \sum\sum_{m\sim M, n\ll N^*} \lambda_f(m)\lambda_g(n)\mathrm{Kl}_2(mn;q)$$

with $N^* = q^2/N \leq q^{1/2+\eta}$ and

$$n \mapsto \mathrm{Kl}_2(n;q) = \frac{1}{q^{1/2}} \sum_{x\in\mathbb{F}_q^\times} e_q(x^{-1}+nx)$$

is a Kloosterman sum.

Bilinear sums of Kloosterman sums

Kloosterman sums satisfy the bound (Weil)

$$|\mathrm{Kl}_2(mn; q)| \leq 2$$

which together with the fact that the $\lambda_f(m)$ and $\lambda_g(n)$ are essentially bounded on average yields

$$\frac{N^{1/2}}{q^{3/2}M^{1/2}} \sum\sum_{m\sim M, n\ll N^*} \lambda_f(m)\lambda_g(n)\mathrm{Kl}_2(mn; q) \ll q^{o(1)}(\frac{qM}{N})^{1/2}.$$

This bound is "good" as long as $N \geq q^{3/2+\eta}$ or equivalently $M \leq q^{1/2-\eta}$. It remain to cover the range

$$q^{1/2-\eta} \leq M, N^* \leq q^{1/2+\eta}.$$

Bilinear sums of Kloosterman sums

We need to bound the bilinear sum

$$\sum_{m\sim M,}\sum_{n\sim N^*} \lambda_f(m)\lambda_g(n)\mathrm{Kl}_2(mn;q)$$

in the range

$$q^{1/2-\eta} \leq M, N^* \leq q^{1/2+\eta}.$$

The trivial bound $\ll (MN^*)^{1+o(1)}$ is "just" not sufficient and it would be enough to get the bound $\ll (MN^*)^{1-\delta}$ for some $\delta > 0$: we have to exploit oscillations of Kloosterman sums and show the absence of correlations.

Remark

At this point we don't know how to exploit the automorphic origin of the coefficients $\lambda_f(m), \lambda_g(n)$ and we treat these as arbitrary, essentially bounded, complex numbers.

Bilinear sums of Kloosterman sums

– Oscillations of Kloosterman sums have been studied in depth by Katz using ℓ-adic cohomology; indeed the Kloosterman sum is the trace function of ℓ-adic sheaf (the Kloosterman sheaf $\mathcal{KL}_2$).
– These oscillations are well understood when the Kloosterman function $n \mapsto \mathrm{Kl}_2(n;q)$ is considered over the full interval $[0, q-1] \simeq \mathbb{A}^1(\mathbb{F}_q)$; here we have to deal with short intervals and against arbitrary coefficients.
– There exist completion techniques (Polya-Vinogradov method) allowing to handle non-trivially bilinear sums like

$$\sum_{m \sim M, n \sim N^*} \alpha_m \beta_n \mathrm{Kl}_2(mn;q)$$

but these produce non-trivial bounds only when

$$\max(M, N^*) \geq q^{1/2+\eta}.$$

Bilinear sums of Kloosterman sums

To get beyond we use a more advanced completion method (the $+ab$ trick of Vinogradov-Karatsuba) which introduce "additional" artificial summations points using that an additive translate of an interval is an interval and that the image of the product of two intervals is almost an interval.

At the end of the process, we need to bound sums of product of Kloosterman sums in three variable and there is a high price to pay in terms of ℓ-adic cohomology:

Bilinear sums of Kloosterman sums

For $\mathbf{b} = (b_1, b_2, b_2, b_4) \in {\mathbb{F}_q}^4$ and $r, h \in \mathbb{F}_q$, set

$$R_{\mathbf{b},h}(r; q) = \frac{1}{q^{1/2}} \sum_{s \in \mathbb{F}_q} \prod_{i=1}^{4} \mathrm{Kl}(s(r + b_i); q) e_q(hs)$$

Theorem (KMS)

There exist a codimension 1 subvariety $V^{bad} \subset \mathbb{A}^4_{\mathbb{F}_q}$ such that for any $\mathbf{b} \notin V^{bad}(\mathbb{F}_q)$, and any $h, h' \in \mathbb{F}_q$

$$\frac{1}{q} \sum_{r \in \mathbb{F}_q} R_{\mathbf{b},h}(r; q) \overline{R_{\mathbf{b},h'}(r; q)} = \delta_{h=h'} + O(q^{-1/2}).$$

Bilinear sums of Kloosterman sums

Theorem (KMS)

Given $M, N \leq q-1$ (+ mild additional conditions) and sequences $(\alpha_m)_{m \leq M}$, $(\beta_N)_{n \leq N}$ of complex number bounded by 1

$$\sum\sum_{m \leq M, n \leq N} \alpha_m \beta_n \mathrm{Kl}_2(mn; q) \ll q^{\varepsilon} MN\left(\frac{1}{M} + \frac{q^{11/32}}{(MN)^{12/32}}\right)^{1/2}.$$

Remark

In particular when $M = N = q^{1/2}$, one gains a factor $q^{1/64}$ over the trivial bound $MN = q$.

An ergodic proof ?

The above proof uses harmonic analysis, analytic number theory and algebraic geometry. On the other hand the question is very much dynamic flavored: a twisted central L-value can be represented as an adelic torus period: for $t = O(1)$

$$\frac{L(f \otimes \chi, 1/2 + it)}{q^{1/2}} \approx \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi_f(a(y) n_q(\frac{1}{q})) \chi(y) |y|_{\mathbb{A}}^{it} d^\times y$$

where $\varphi_f \in \pi_f \subset L^2(\mathrm{PGL}_2(\mathbb{Q}) \backslash \mathrm{PGL}_2(\mathbb{A}))$ is the " new vector" in the representation generated by f and

$$a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \; n_q(1/q) = \begin{pmatrix} 1 & 1/q \\ 0 & 1 \end{pmatrix}_q .$$

An ergodic proof ?

Therefore a naive attemp would be to write the second moment as

$$\frac{1}{q}\sum_{\chi \pmod{q}} |L(f\otimes\chi,1/2)|^2 \approx \sum_{\chi \pmod{q}} \Big|\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \varphi_f(a(y)n_q(\frac{1}{q}))\chi(y)d^\times y\Big|^2$$

and to apply Plancherel formula... but the χ-sum does not include the characters with non-trivial archimedean component $\chi.|\cdot|_{\mathbb{A}}^{it}!$. This is possible with an extra averaging:

$$\frac{1}{q}\int_{\mathbb{R}}\sum_{\chi \pmod{q}} |L(f\otimes\chi,1/2+it)|^2 h^2(t)dt$$

for $h\in\mathcal{C}^\infty([-\varepsilon,\varepsilon])$ with $\int h^2(t)dt=1$.

Problem

Can one let $\varepsilon\to 0$? give a ergodic proof of the second moment asymptotics ?

A similar problem for closed geodesics

Let $\mathbb{Q}(\sqrt{d})$ be a real quadratic field with ideal class group $H(d)$. For $\chi \in \widehat{H(D)}$, let g_χ be the theta function associated to χ and for $f(z)$ a Maass cusp-form let $L(f \otimes g_\chi, s)$ be the associated Rankin-Selberg convolution:

Problem

Assume that $h(d)$ is very large (eg. $d = 4n^2 + 1$, and squarefree); evaluate asymptotically the first moment:

$$\frac{1}{h(d)} \sum_{\chi \in \widehat{H(d)}} L(f \otimes g_\chi, 1/2) \text{ as } d \to \infty.$$

A similar problem for closed geodesics

I don't know how to solve this problem using analytic number theory yet; a possible approach could be geometric:
Let $\Gamma(d)$ the set of closed geodesics (all of length $\ell(d)$) indexed by $H(d)$ attached to (the maximal order of) $\mathbb{Q}(\sqrt{d})$.

Theorem (Waldspurger)

For any Hecke character χ of conductor d, the Rankin-Selberg L-value $L(f \otimes g_\chi, 1/2)$ is proportional to

$$|\frac{1}{h(d)} \sum_{\gamma} \int_{\gamma} f(t)\chi(t)dt|^2.$$

A similar problem for closed geodesics

By Plancherel formula, one can show that a suitably weighted average of the $L(f \otimes g_\chi, 1/2)$ over all Hecke characters of $\mathbb{Q}(\sqrt{d})$ is proportional to

$$\frac{1}{h(d)} \sum_{\gamma} \int_{\gamma} |f(t)|^2 dt \to \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |f(z)|^2 dz, \ d \to \infty$$

by Duke's equidistribution theorem; therefore there exists some Hecke character χ such that $L(f \otimes g_\chi, 1/2) \neq 0$ and so there exists $\gamma \in \Gamma(d)$ and a character χ_k of γ (identified with S^1) such that $\int_\gamma f(t)\chi_k(t)dt \neq 0$.

– The problem is to be able to restrict to class group characters.

Algebraic mixing

Theorem (Conrey-Iwaniec)

Let χ_q be the Legendre symbol and $k \geq 6$; one has

$$\frac{1}{q} \sum_{f \in S_k(q)} L(f \otimes \chi_q, 1/2)^3 \ll q^{o(1)}.$$

In particular (since $L(f \otimes \chi_q, 1/2)$ is non-negative) one has the Weyl type subconvex bound

$$L(f \otimes \chi_q, 1/2) \ll q^{o(1)+1/3}.$$

It is natural to look for an asymptotic formula with a power saving error term (this would allow to improve Weyl bounds above). This was carried out by Ian Petrow.

Algebraic mixing

A vague of his precise result is:

Theorem (Petrow)

A toy model for the main term in the Conrey-Iwaniec cubic moment is given by the twisted fourth moment

$$\frac{1}{q}\sum_{\chi \pmod q} |L(\chi,1/2)|^4 \tilde{K}(\chi)$$

where $\tilde{K}(\chi)$ is the Mellin transform (over $\widehat{\mathbb{F}_q^{\times}}$) of an explicit trace function K on $\mathbb{A}^1_{\mathbb{F}_q}$); in particular $\tilde{K}(\chi) \ll 1$ is absolutely bounded.

Petrow used the trivial bound and Young evaluation of the fourth moment to replace the $q^{o(1)}$ bound of Conrey-Iwaniec by $O((\log q)^4)$. However it would be interesting to look for an aymptotic formula with power saving error term: this would make it possible to improve the exponent $1/3$.

Algebraic mixing

Averaging over χ one obtain a sum of the shape ($f = E'_{|s=0}$)

$$\frac{1}{(MN)^{1/2}} \sum\sum_{m\sim M, n\sim N} \lambda_f(m)\lambda_f(n)\frac{K(m/n;q)}{q^{1/2}}$$

with $MN \sim q^2$. Observe that the trivial bound $\ll q^2/q^{3/2} = q^{1/2}$ is off by a factor $q^{1/2}$ from what we would like to improve on (in contrast with non-twisted moment where $q^{-1/2}K$ was replaced by $\delta_{m/n\equiv 1 \pmod{q}}$)!
In contrast to the above, the hardest case is when $M \sim N \sim q$.

Remark

Some recent general large sieve inequality for ℓ-adic sheaves of Xi Ping allows to save the annoying factor $q^{1/2}$ but we need more!

Algebraic mixing

Reversing the Fourier expansion process for modular form one find that the problem of bounding this sums is essnetially equivalent to the following horocycle algebraic mixing problem

$$\frac{1}{q^2}\sum_{a,b\in\mathbb{F}_q^\times} f\Big(\frac{a+z_0}{q}\Big)f\Big(\frac{b+z_0}{q}\Big)\widehat{\widehat{K}}(b/a) \overset{?}{\ll} q^{-1/2-\delta},\ \delta>0$$

or changing variables

$$\frac{1}{q^2}\sum_{a,\lambda\in\mathbb{F}_q^\times} f\Big(\frac{a+z_0}{q}\Big)f\Big(\frac{\lambda a+z_0}{q}\Big)\widehat{\widehat{K}}(\lambda) \overset{?}{\ll} q^{-1/2-\delta},\ \delta>0$$

Remark (Distribution of algebraically twisted horocycles, FKM)

For any trace function K on $\mathbb{A}^1(\mathbb{F}_q)$ which is not an additive character $\frac{1}{q}\sum_{a\in\mathbb{F}_q^\times} f(\frac{a+z_0}{q})K(a) \ll_f q^{-1/8+o(1)}$.

Algebraic mixing: a variant

One can also study the problem when $K(a,b)$ is the characteristic function of a curve in $\mathbb{A}^2(\mathbb{F}_q)$: in that case the expectation is that for $\mathcal{C}$ general enough:

$$\frac{1}{q}\sum_{(a,b)\in\mathcal{C}(\mathbb{F}_q)} f(\frac{a+z_0}{q})f(\frac{b+z_0}{q}) \overset{?}{\ll} q^{-\delta},\ \delta>0.$$

Remark

The work of Einsiedler-Lindenstrauss on mixing on products of modular curves gives non trivial results when the equation of $\mathcal{C}$ is of the shape $b=a^k,\ k\neq 0,1,-1$.