Sums of reciprocals of fractional parts and applications to Diophantine approximation: Part 1

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Goa: 1 February 2016

Others

Joint work with Victor Beresnevich and Alan Haynes (York)

The sums of interest

We investigate the sums

$$S_N(\alpha,\gamma) := \sum_{n=1}^{N} \frac{1}{n\|n\alpha-\gamma\|} \quad \text{and} \quad R_N(\alpha,\gamma) := \sum_{n=1}^{N} \frac{1}{\|n\alpha-\gamma\|},$$

where α and γ are real parameters and $\|\cdot\|$ is the distance to the nearest integer. The sums are related (via partial summation):

$$S_N(\alpha,\gamma) = \sum_{n=1}^{N} \frac{R_n(\alpha,\gamma)}{n(n+1)} + \frac{R_N(\alpha,\gamma)}{N+1}.$$

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Schmidt (1964): for any $\gamma \in \mathbb{R}$ and for any $\varepsilon > 0$

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for almost all $\alpha \in \mathbb{R}$.

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Schmidt (1964): for any $\gamma \in \mathbb{R}$ and for any $\varepsilon > 0$

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for almost all $\alpha \in \mathbb{R}$. In the homogeneous case ($\gamma = 0$), easy to see that the ε term in (1) can be removed if α is badly approximable.

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We show that when $\gamma = 0$, the l.h.s. (1) is true for all irrationals while the r.h.s. (1) is true with $\varepsilon = 0$ for a.a. irrationals.

More precisely:

Homogeneous results: $S_N(\alpha, 0)$

Theorem. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then for $N \geqslant N_0$

$$\frac{1}{2}(\log N)^2 \overset{\forall}{\leqslant} S_N(\alpha,0) := \sum_{n=1}^{N} \frac{1}{n\|n\alpha\|} \overset{a.a}{\leqslant} 34\,(\log N)^2.$$

In fact, the upper bound is valid for any $\alpha := [a_1, a_2, \ldots]$ such that

$$A_k(\alpha) := \sum_{i=1}^{k} a_i = o(k^2)\,.$$

(Diamond + Vaaler: For a.a. α, for k sufficiently large $A_k \leqslant k^{1+\varepsilon}$.)

Homogeneous results: $R_N(\alpha, 0)$

Theorem. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then for $N \geqslant N_0$

$$N \log N \overset{\forall}{\ll} R_N(\alpha, 0) := \sum_{n=1}^{N} \frac{1}{\|n\alpha\|}.$$

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Hardy + Wright: $R_N(\alpha, 0) \ll N \log N$ for badly approximable α. In general, not even true a.a. Indeed:

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Now to some inhomogeneous statements.

Inhomogeneous results: a taster

Theorem. *For each* $\gamma \in \mathbb{R}$ *there exists a set* $\mathcal{A}_\gamma \subset \mathbb{R}$ *of full measure such that for all* $\alpha \in \mathcal{A}_\gamma$ *and all sufficiently large* N

$$S_N(\alpha,\gamma) := \sum_{1\leqslant n\leqslant N} \frac{1}{n\|n\alpha-\gamma\|} \ll (\log N)^2 .$$

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Theorem. *Let* $\alpha \in \mathbb{R} \setminus (\mathfrak{L} \cup \mathbb{Q})$*. Then, for all sufficiently large* N *and any* $\gamma \in \mathbb{R}$

$$S_N(\alpha,\gamma) \gg (\log N)^2 .$$

Schmidt's lower bound is a.a. and depends on γ. The above is for all irrationals except possibly for Liouville numbers $\mathfrak{L}$.

Inhomogeneous results: a taster continued

In the lower bound result for $S_N(\alpha,\gamma)$ we are not sure if we need to exclude Liouville numbers. However, it is necessary when dealing with $R_N(\alpha,\gamma)$.

Theorem. *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. *Then,* $\alpha \notin \mathfrak{L}$ *if and only if for any* $\gamma \in \mathbb{R}$,

$$R_N(\alpha,\gamma) := \sum_{1 \leqslant n \leqslant N} \frac{1}{\|n\alpha - \gamma\|} \gg N \log N \quad \text{for } N \geqslant 2.$$

The counting function: $\#\{1 \leqslant n \leqslant N : \|n\alpha - \gamma\| < \epsilon\}$

Related to the sums, given $\alpha, \gamma \in \mathbb{R}$, $N \in \mathbb{N}$ and $\varepsilon > 0$, we consider the cardinality of

$$N_\gamma(\alpha, \varepsilon) := \{n \in \mathbb{N} : \|n\alpha - \gamma\| < \varepsilon,\ n \leqslant N\}.$$

Observing that in the homogeneous case, when $\varepsilon N \geqslant 1$, Minkowski's Theorem for convex bodies, implies that

$$\#N(\alpha, \varepsilon) := \#N_0(\alpha, \varepsilon) \geqslant \lfloor \varepsilon N \rfloor.$$

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Under which conditions can this bound can be reversed?

The homogeneous counting results

Theorem. *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and let* $(q_k)_{k \geqslant 0}$ *be the sequence of denominators of the convergents of* α. *Let* $N \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that* $0 < 2\varepsilon < \|q_2\alpha\|$. *Suppose that*

$$\tfrac{1}{2\varepsilon} \leqslant q_k \leqslant N$$

for some integer k. *Then*

$$\lfloor \varepsilon N \rfloor \; \leqslant \; \# N(\alpha, \varepsilon) \; \leqslant \; 32\,\varepsilon N \, . \tag{2}$$

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In terms of the Diophantine exponent $\tau(\alpha)$: Let $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ and let $\nu \in \mathbb{R}$ satisfy

$$0 < \nu < \frac{1}{\tau(\alpha)}\,.$$

Then, $\exists\, \varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that for any sufficiently large N and any ε with $N^{-\nu} < \varepsilon < \varepsilon_0$, estimate (2) is satisfied.

The inhomogeneous counting results

Estimates for $\#N_\gamma(\alpha, \varepsilon)$ are obtained from the homogenous results via the following.

Theorem. *For any $\varepsilon > 0$ and $N \in \mathbb{N}$, we have that*

$$\#N_\gamma(\alpha, \varepsilon) \leqslant \#N(\alpha, 2\varepsilon) + 1.$$

If $N'_\gamma(\alpha, \varepsilon') \neq \emptyset$, where $N' := \frac{1}{2}N$ and $\varepsilon' := \frac{1}{2}\varepsilon$, then

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UPSHOT: If $\#\{1 \leqslant n \leqslant N/2 : \|n\alpha-\gamma\| < \epsilon/2\} > 0$, then under the conditions of the homogeneous results

$$\lfloor \tfrac{1}{4}\varepsilon N \rfloor \;\leqslant\; \#N_\gamma(\alpha,\varepsilon) \;\leqslant\; 64\,\varepsilon N+1\,.$$

Main Tools: Ostrowski

- Ostrowski expansion of real numbers: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then, for every $n \in \mathbb{N}$ there is a unique integer $K \geqslant 0$ such that

$$q_K \leqslant n < q_{K+1},$$

and a unique sequence $\{c_{k+1}\}_{k=0}^{\infty}$ of integers such that

$$n = \sum_{k=0}^{\infty} c_{k+1} q_k, \tag{3}$$

$$0 \leqslant c_1 < a_1 \quad \text{and} \quad 0 \leqslant c_{k+1} \leqslant a_{k+1} \quad \forall\, k \geqslant 1,$$

$$c_k = 0 \quad \text{whenever} \quad c_{k+1} = a_{k+1} \quad \text{with } k \geqslant 1,$$

$$c_{k+1} = 0 \quad \forall\, k > K.$$

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Let $\alpha \in [0,1) \setminus \mathbb{Q}$, $n \in \mathbb{N}$ and, let m be the smallest integer such that $c_{m+1} \neq 0$ in the Ostrowski expansion of n. If $m \geqslant 2$, then

$$\|n\alpha\| = \left| \sum_{k=m}^{\infty} c_{k+1} D_k \right| \qquad (D_k := q_k \alpha - p_k \, (k \geqslant 0))$$

In particular

$$(c_{m+1} - 1)|D_m| \ \leqslant \ \|n\alpha\| \ \leqslant \ (c_{m+1} + 1)|D_m|. \tag{4}$$

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Since $\frac{1}{2} \leqslant q_{k+1}|D_k| \leqslant 1$, it follows that

$$\tfrac{1}{2}(c_{m+1} - 1) \;\leqslant\; q_{k+1}\|n\alpha\| \;\leqslant\; (c_{m+1} + 1).$$

Why should we care?

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- They have elegant applications to metrical Diophantine approximation; in particular the multiplicative theory.

Khintchine's Theorem: 1-dimensional

Let $\mathrm{I} := [0,1]$, $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real positive function and

$$W(\psi) := \{x \in \mathrm{I} : \|qx\| \leq \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$$

– the set of ψ-*well approximable numbers.*

Khintchine's Theorem (1924) *If ψ is monotonic, then*

$$m(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty \,, \\ \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \,. \end{cases}$$

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- Divergence part requires monotonicity.

• Put $\psi(q) = \frac{1}{q \log q}$. Divergent part implies for almost all x $\exists$ infinitely many $q > 0$ such that $q \,\|qx\| \leq 1/\log q$.

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Overcome this by insisting that p, q are co-prime: let $W'(\psi)$ be the set of $x \in \mathrm{I}$ such that $|qx - p| \leq \psi(q)$ for infinitely many p/q with $(p, q) = 1$.

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The Duffin-Schaeffer Conjecture (1941) *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real positive function. Then*

$$m(W'(\psi)) = 1 \quad \textit{if} \quad \sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} = \infty \,.$$

- Gallagher (1965): $m(W'(\psi)) = 0$ or 1
- Various partial results are know:

The Duffin-Schaeffer Theorem

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and

$$\limsup_{N\to\infty} \left(\sum_{q=1}^{N} \frac{\varphi(q)}{q} \psi(q) \right) \left(\sum_{q=1}^{N} \psi(q) \right)^{-1} > 0 \,. \qquad (5)$$

Note that (5) implies that the convergence/divergence behavior of $\sum_{q=1}^{\infty} (\varphi(q)\psi(q))/q$ and $\sum_{q=1}^{\infty} \psi(q)$ are equivalent.

The Inhomogeneous Duffin-Schaeffer Conjecture

Let $\gamma \in \mathbb{R}$ and let $W'(\psi, \gamma)$ be the set of $x \in \mathrm{I}$ such that $|qx - p - \gamma| \leq \psi(q)$ for infinitely many p/q with $(p, q) = 1$.

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- No inhomogeneous analogue of Gallagher's $0 - 1$ law. Ramírez (2015): $\exists$ integer $t \geqslant 1$ so that $m(W'(\psi, t\gamma)) = 0$ or 1.
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- No inhomogeneous analogue of the Duffin-Schaeffer Theorem. (We show that such a theorem would imply inhomogeneous Gallagher for multiplicative approximation)

Khintchine in $\mathbb{R}^2$: the statement

Let $\mathrm{I}^2 = [0,1)^2$ and given $\psi : \mathbb{N} \to \mathbb{R}^+$ let

$$W(2,\psi) := \{(\alpha,\beta) \in \mathrm{I}^2 : \ \max\{\|q\alpha\|, \|q\beta\|\} < \psi(q) \text{ for i. m. } q \in \mathbb{N}\} \ .$$

Throughout, m_2 will denote 2-dimensional Lebesgue measure.

Khintchine in $\mathbb{R}^2$. *If ψ is monotonic, then*

$$m_2(W(2,\psi)) = \begin{cases} 0 & \text{if } \ \sum_{q=1}^{\infty} \psi^2(q) \ < \infty \ , \\ \\ 1 & \text{if } \ \sum_{q=1}^{\infty} \psi^2(q) \ = \infty \ . \end{cases}$$

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- Convergence not true in general for fixed α.

Convergence not true for rational α

Let L_α be the line parallel to the *y*-axis passing through the point $(\alpha, 0)$. Suppose $\alpha = \frac{a}{b}$.

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$$\|bq\beta\| < \frac{b}{q} = \frac{b^2}{bq} \quad \text{and} \quad \|bq\alpha\| = \|aq\| = 0 < \frac{b^2}{bq} .$$

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The upshot of this is that every point on the rational vertical line L_α is $\psi(q) = b^2 \; q^{-1}$ - approximable and so

$$m_1(W(2,\psi) \cap \mathrm{L}_\alpha) = 1 \quad \text{but} \quad \sum_{q=1}^{\infty} \psi(q)^2 = b^4 \sum_{q=1}^{\infty} q^{-2} < \infty .$$

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- Convergence not true in general for fixed α.
- Is divergence true for fixed α?

Khintchine on fibers

Fix $\alpha \in \mathrm{I}$ and let L_α be the line parallel to the y-axis passing through the point $(\alpha, 0)$ and let ψ is monotonic. The claim is that

$$m_1(W(2,\psi) \cap \mathrm{L}_\alpha) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi^2(q) = \infty\,. \tag{6}$$

Theorem (Ramírez, Simmons, Süess)

A. If $\tau(\alpha) < 2$, then (6) is true.

B. If $\tau(\alpha) > 2$ and for $\epsilon > 0$, $\psi(q) > q^{-\frac{1}{2}-\epsilon}$ for q large, then $W(2,\psi) \cap \mathrm{L}_\alpha = \mathrm{I}^2 \cap \mathrm{L}_\alpha$. In particular, $m_1(W(2,\psi) \cap \mathrm{L}_\alpha) = 1$.

- (A) requires estimates for $\#\{q \leqslant N : \|q\alpha\| \leqslant \psi(q)\}$

Multiplicative approximation: Littlewood's Conjecture

Littlewood Conjecture (c. 1930): For every $(\alpha, \beta) \in \mathrm{I}^2$

$$\liminf_{q \to \infty} q \|q\alpha\| \|q\beta\| = 0.$$

- Khintchine's theorem implies that

$$\liminf_{q \to \infty} q \log q \, \|q\alpha\| \|q\beta\| = 0 \quad \forall \, \alpha \in \mathbb{R} \text{ and for almost all } \beta \in \mathbb{R} \, .$$

Multiplicative approximation: Gallagher's Theorem

Given $\psi : \mathbb{N} \to \mathbb{R}^+$ let

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The following result is the analogue of Khintchine's simultaneous approximation theorem within the multiplicative setup.

Theorem (Gallagher, 1962)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function. Then

$$m_2(W^\times(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) \log q < \infty \,, \\ \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} \psi(q) \log q = \infty \,. \end{cases}$$

Multiplicative approximation: a moments reflection

Gallagher's theorem implies that

$$\liminf_{q\to\infty} q \log^2 q \, \|q\alpha\| \|q\beta\| = 0 \quad \text{for a.a. } \alpha \in \mathbb{R} \text{ and for a.a. } \beta \in \mathbb{R}\,. \tag{7}$$

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The extra log factor from Gallagher comes at a cost of having to sacrifice a set of measure zero on the α side. Thus, unlike with (8) which is valid for any α, we are unable to claim that the stronger 'log squared' statement (7) is true for say when $\alpha = \sqrt{2}$.

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Divergent Gallagher on fibres

Theorem (Beresnevich-Haynes-V, 2015)

Let $\alpha \in \mathrm{I}$ *and* $\psi : \mathbb{N} \to \mathbb{R}^+$ *be a monotonic function such that*

$$\sum_{q=1}^{\infty} \psi(q) \log q \; = \; \infty \tag{9}$$

and

$$\exists\ \delta > 0 \qquad \liminf_{n\to\infty} q_n^{3-\delta}\psi(q_n) \geqslant 1\,, \tag{10}$$

where q_n *denotes the denominators of the convergents of* α*. Then for almost every* $\beta \in \mathrm{I}$*, there exists infinitely many* $q \in \mathbb{N}$ *such that*

$$\|q\alpha\|\,\|q\beta\| < \psi(q)\,. \tag{11}$$

Condition (10) holds for all α with Diophantine exponent $\tau(\alpha) < 3$. Note that $\dim\{\alpha \in \mathbb{R} : \tau(\alpha) \geqslant 3\} = 1/2$.

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It follows that for every $\alpha \in \mathbb{R}$

$$\liminf_{q\to\infty} q \log^2 q\,\|q\alpha\|\,\|q\beta\| = 0 \quad \text{for almost all } \beta \in \mathbb{R}\,.$$

Pseudo sketch proof of divergent Gallagher on fibres

Given α and monotonic ψ, consider

$$\|q\beta\| < \Psi_\alpha(q) \quad \textit{where} \quad \Psi_\alpha(q) := \frac{\psi(q)}{\|q\alpha\|} .$$

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$$m_1(W(\Psi_\alpha)) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \Psi_\alpha(q) \ = \ \infty \ .$$

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We need to show that

$$\sum_{q=1}^{\infty} \psi(q) \, \log q \; = \; \infty \quad \Longrightarrow \quad \sum_{q=1}^{\infty} \Psi_\alpha(q) \; = \; \infty \,.$$

This follows by partial summation and the fact that for any irrational α and $Q \geqslant 2$

$$R_Q(\alpha; 0) := \sum_{q=1}^{Q} \frac{1}{\|q\alpha\|} \; \gg \; Q \log Q \,.$$

Covergent Gallagher on fibres

Theorem (Beresnevich-Haynes-V, 2015)

Let $\gamma, \delta \in \mathbb{R}$ *and* $\alpha \in \mathrm{I}$ *be irrational and let* $\psi : \mathbb{N} \to \mathbb{R}^+$ *be such that* $\sum \psi(q) \log q$ *converges. Furthermore, assume either:*

(i) $n \mapsto n\psi(n)$ *is decreasing and*

$$S_N(\alpha; \gamma) \ll (\log N)^2 \qquad \textit{for all } N \geqslant 2\,;$$

(ii) $n \mapsto \psi(n)$ *is decreasing and*

$$R_N(\alpha; \gamma) \ll N \log N \qquad \textit{for all } N \geqslant 2\,.$$

Then for almost all $\beta \in \mathrm{I}$*, there exist only finitely many* $q \in \mathbb{N}$ *such that*

$$\|q\alpha - \gamma\| \, \|q\beta - \delta\| < \psi(q)$$

$$(i.e. \;\; m_1(W^{\times}(\psi, \gamma, \delta) \cap \mathrm{L}_\alpha) = 0\,).$$

Taking $\alpha \in \mathbf{Bad}$ and $\gamma = 0$ works.

Inhomogeneous Divergent Gallagher

Conjecture. Let $\gamma, \delta \in \mathbb{R}$ and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be monotonic Then

$$m_2(W^\times(\psi,\gamma,\delta)) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi(q) \log q \; = \; \infty$$

i.e. for almost all $(\alpha, \beta) \in \mathrm{I}^2$, there exist infinitely many $q \in \mathbb{N}$ such that

$$\|q\alpha - \gamma\| \, \|q\beta - \delta\| < \psi(q)\,.$$

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$$\|q\alpha - \gamma\| \, \|q\beta - \delta\| < \psi(q) \, .$$

- Duffin-Schaeffer Theorem implies Conjecture true with $\delta = 0$.
- Inhomogeneous Duffin-Schaeffer Theorem (20??) implies Conjecture true in general.