On the volumes of modular geodesics

Tali Pinsky

(Joint work with Maxime Bergeron and Lior Silberman)

- Geodesic flows and template theory
- The template for the modular surface
- Bounding the volumes of modular geodeiscs
- Questions

Consider a surface S.
Its unit tangent bundle T^1S is the bundle of tangent vectors of norm 1 at each point.
e.g., the unit tangent bundle of $\mathbb{H}^2$ is $PSL_2(\mathbb{R})$.

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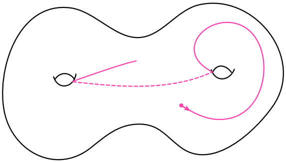
A geodesic on S can be lifted to the unit tangent bundle, by attaching a tangent unit vector at each point.

A point and tangent direction on S can be slid along the geodesic they define. This is the *geodesic flow* on T^1S.



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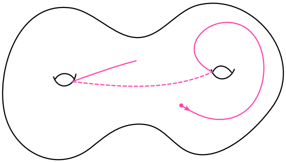

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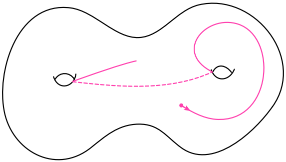

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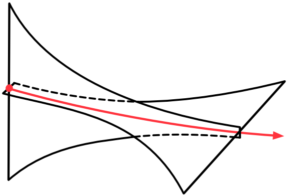
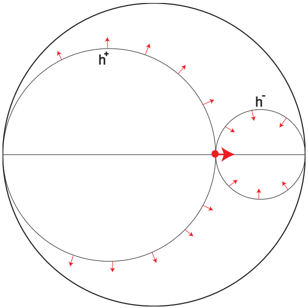
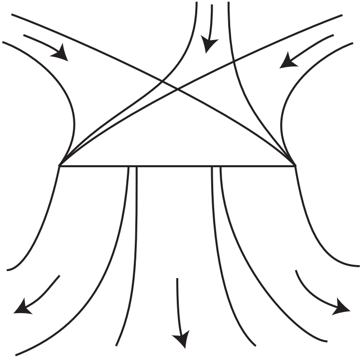
For a hyperbolic surface $S = \mathbb{H}^2/\Gamma$, $T^1S = PSL_2(\mathbb{R})/\Gamma$ and the geodesic flow is given by the action of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, and the flow is always *Anosov*, i.e. it has an expanding direction and a contracting direction.



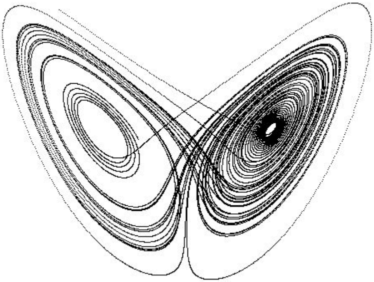


Templates

A *template* is a compact branched two-manifold with boundary and a smooth expanding semi flow built from a finite number of branch line charts.



The Lorenz attractor



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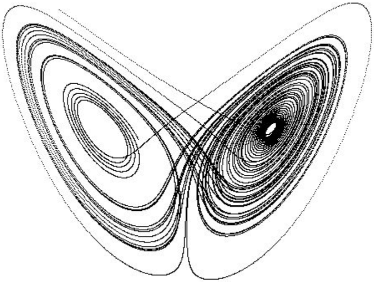
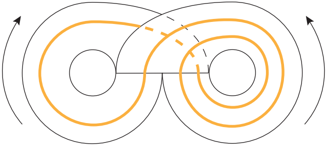
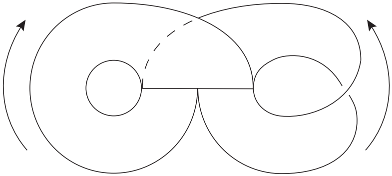


And the Lorenz template



By Birman and Williams, and Dehornoy:

- Every Lorenz link is a fibered link.
- Every Lorenz knot is prime.
- Every Lorenz link is a closed positive braid.
- Non-trivial Lorenz links have positive signature.
- Out of the first 1,701,936 knots, only 20 are Lorenz.



When you add twists to the two bands, the template is called *Lorenz-like*. Some are prime, some are universal, and many are neither.

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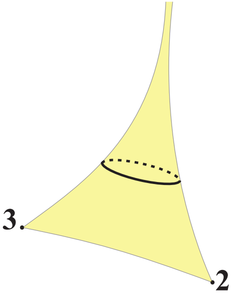
$\Rightarrow$ Any geodesic flow on a hyperbolic surface has a template.

The modular surface

$M = \mathbb{H}^2 / PSL_2(\mathbb{Z})$



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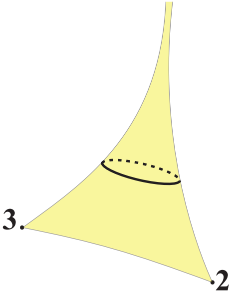

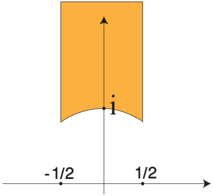

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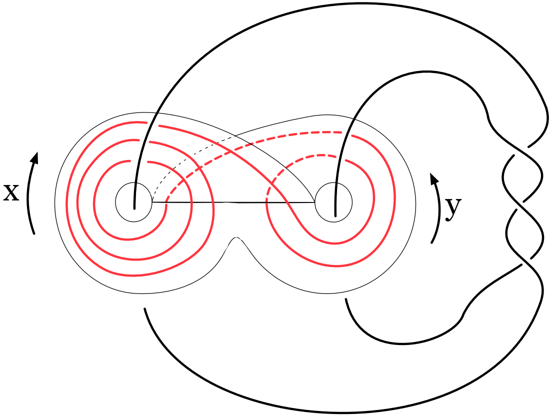
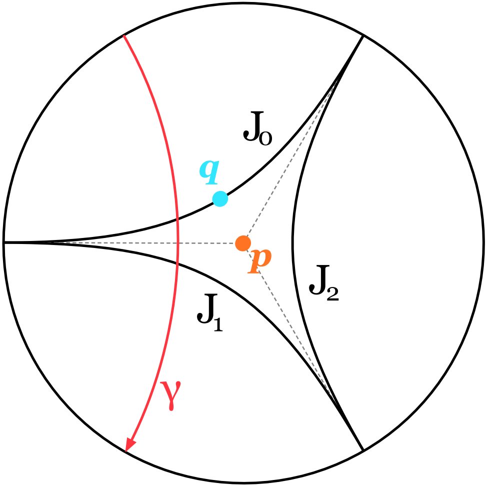
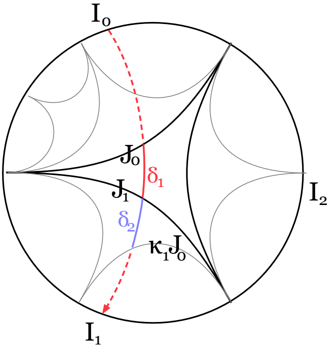
Theorem (Ghys, 2006)

The modular flow has a template identical to the Lorenz template. Thus, modular knots and Lorenz knots coincide.

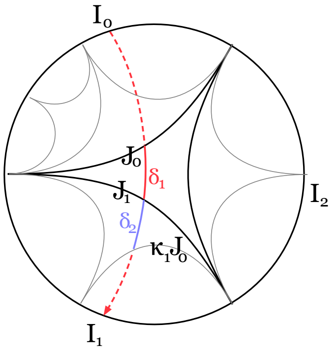


The construction of the template:





One gets a code $\ldots x^{n_{-2}} y^{n_{-1}} x^{n_0} y^{n_1} x^{n_2} y^{n_3} \ldots$



One gets a code $\ldots x^{n_{-2}}y^{n_{-1}}x^{n_0}y^{n_1}x^{n_2}y^{n_3}\ldots$ equivalent to the continued fraction expansion $(\ldots n_{-2}, n_{-1}, n_0, n_1, n_2, n_3\ldots)$.

Estimating volumes

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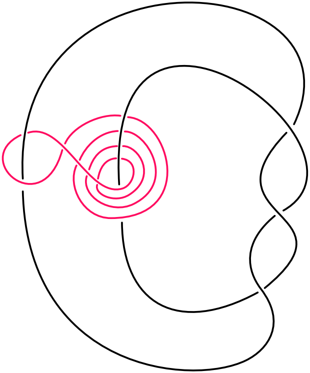
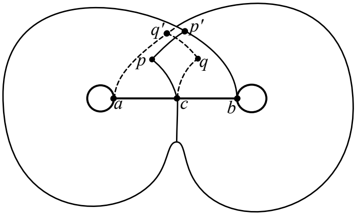
Thus its complement has a well defined volume.
The volume of a modular geodesic generally grows when the geodesic gets longer.

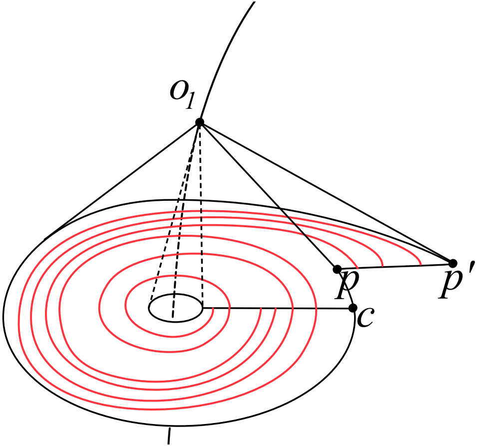
But not always. For example, the geodesics with codes xy^n on the modular surface.



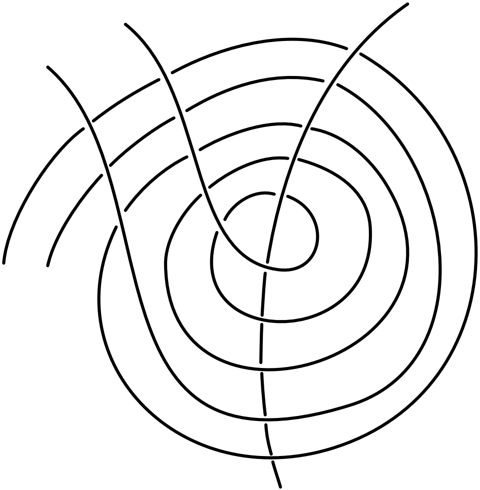


We will estimate separately the volume contribution for the region above the left band and for the region above the right band of the template

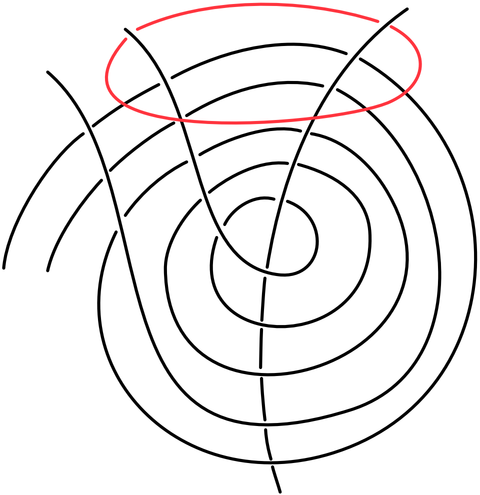
The left band:



The right band:



The right band:



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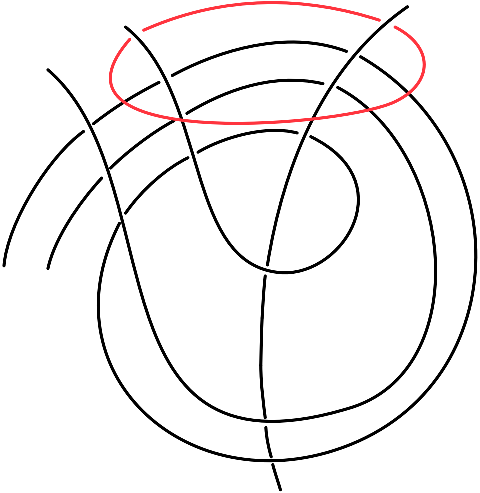

Theorem

There is a constant $C > 0$ such that, for any finite collection γ of periodic geodesics on M, we have

$$Vol(T^1M \setminus \gamma) \leq C\, n_\gamma \log(n_\gamma).$$

Where n_γ denotes the number of transitions from x to y in the set γ.

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Corollary

$$Vol(T^1 M \setminus \gamma) \leq C|\gamma| \log(|\gamma|)$$

Where $|\gamma|$ is the geometric length of γ.

For any hyperbolic surface, a closed geodesic is hyperbolic $\Leftrightarrow$ it is filling.

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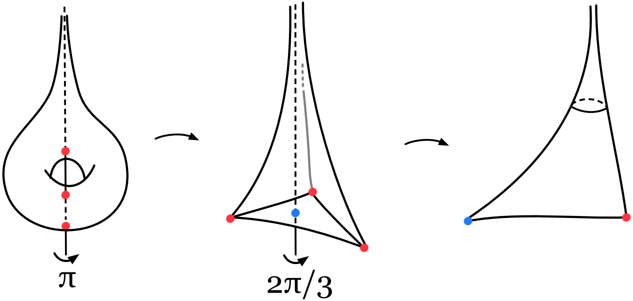
Any hyperbolic surface covers the modular surface, and so the bound for the modular surface lifts to a bound for any hyperbolic surface.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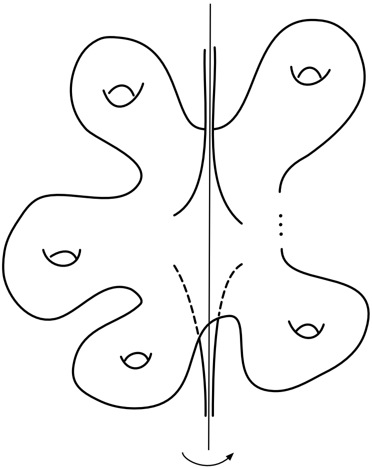
2π/3



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Note that the proof is topological, although $|\gamma|$ depends on the metric.

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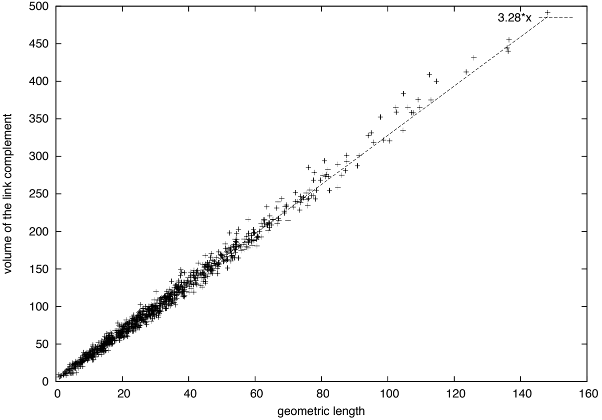
2. Can one obtain a linear upper bound?
3. Can one find a canonical/geometric triangulation?
4. There is a correspondence between closed geodesics on the modular surface and ideal classes of real quadratic fields. Do the volumes grow linearly in the length when taking the full set of geodesics corresponding to the same quadratic field?


3.28*x
volume of the link complement
geometric length
0
50
100
150
200
250
300
350
400
450
500
0
20
40
60
80
100
120
140
160

Thank you!