Sums of reciprocals: applications to Diophantine approximation

Victor Beresnevich

(University of York)
victor.beresnevich@york.ac.uk

GOA - Feb 2016

The theorem to be discussed

B., Haynes, Velani (2015, preprint): *Given any* $\alpha \in \mathbb{R}$ *with the Diophantine exponent* $w(\alpha) < 3$, *i.e.*

$$\limsup_{n\to\infty} \frac{-\log\|n\alpha\|}{\log n} < 3\,,$$

where $\|\cdot\|$ *is the distance to the nearest integer, and given any monotonic* $\psi : \mathbb{N} \to (0, +\infty)$ *such that*

$$\sum_{n=1}^{\infty} \psi(n)\log n = \infty\,,$$

we have that for almost every $\beta \in \mathbb{R}$

$$\|n\alpha\|\cdot\|n\beta\| < \psi(n) \qquad \text{for infinitely many } n \in \mathbb{N}\,.$$

Note: With a slight restriction on ψ, i.e. requiring that $n \mapsto n\psi(n)$ is decreasing, the above becomes true for ALL α.

Convergence

Fact: *Let* $n \mapsto n\psi(n)$ *be decreasing and*

$$\sum_{n=1}^{\infty} \psi(n) \log n < \infty \, .$$

Let $\alpha \in \mathbb{R}$ *be any irrational real number such that*

$$\sum_{n=1}^{N} \frac{1}{n\|n\alpha\|} \ll (\log N)^2 \qquad \textit{for all } N \geq 2 \, . \tag{1}$$

Then for almost all $\beta \in \mathbb{R}$*, there exist only finitely many* $n \in \mathbb{N}$ *such that*

$$\|n\alpha\| \, \|n\beta\| < \psi(n) \, .$$

Note, (1) is true for Lebesgue almost all α, and in particular, for all badly approximable α.

Convergence

Proof. By the Borel-Cantelli lemma, it is enough to verify that

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{\|n\alpha\|} < \infty.$$

By partial summation

$$\sum_{n\le N} \frac{n\psi(n)}{n\|n\alpha\|} = \sum_{n\le N} \big(n\psi(n) - (n+1)\psi(n+1)\big) \sum_{m\le n} \frac{1}{m\|m\alpha\|}$$

$$+ (N+1)\psi(N+1) \sum_{m\le N} \frac{1}{m\|m\alpha\|}$$

$$\asymp \sum_{n=1}^{N} (n\psi(n) - (n+1)\psi(n+1))(\log n)^2 + (N+1)\psi(N+1)(\log N)^2$$

$$\asymp \sum_{n=1}^{N} (n\psi(n) - (n+1)\psi(n+1)) \sum_{m=1}^{n} \frac{\log m}{m} + (N+1)\psi(N+1) \sum_{m=1}^{N} \frac{\log m}{m}$$

$$\asymp \sum_{n\le N} \psi(n) \log n \ll 1 \text{ for all } N \text{ as required.}$$

Khintchine, Gallagher or Duffin and Schaeffer?

What we deal with: α is fixed, β is random and $\|n\alpha\|\cdot\|n\beta\| < \psi(n)$ for infinitely many $n \in \mathbb{N}$. Consider the following three collections of sets:

$$K(n,\Psi) := \Big\{\beta \in [0,1) : |n\alpha - s| < \Psi(n) \text{ for some } s \in \mathbb{Z}\Big\}$$

$$D(n,\Psi) := \Big\{\beta \in [0,1) : |n\alpha - s| < \Psi(n) \text{ for some } (s,n) = 1\Big\},$$

$$G(n,\psi) := \Big\{(\alpha,\beta) \in [0,1)^2 : \|n\alpha\| \cdot \|n\beta\| < \psi(n)\Big\},$$

where $n \in \mathbb{N}$, and let

$$\mathcal{K}(\Psi) := \limsup_{n\to\infty} K(n,\Psi),$$

$$\mathcal{D}(\Psi) := \limsup_{n\to\infty} D(n,\Psi),$$

$$\mathcal{G}(\psi) := \limsup_{n\to\infty} G(n,\psi).$$

Khintchine, Gallagher or Duffin and Schaeffer?

Khintchine (1924): *Let* $\Psi:\mathbb{N}\to[0,+\infty)$ *be decreasing. Then*

$$\lambda(\mathcal{K}(\Psi))=\begin{cases}0, & \textit{if } \sum_{n=1}^{\infty}\Psi(n)<\infty\,,\\ 1, & \textit{if } \sum_{n=1}^{\infty}\Psi(n)=\infty\,.\end{cases}$$

Duffin-Schaeffer (1941): *Let* $\Psi:\mathbb{N}\to[0,+\infty)$ *satisfy the condition*

$$\sum_{n=1}^{N}\frac{\varphi(n)\Psi(n)}{n}\geq Const\times\sum_{n=1}^{N}\Psi(n) \tag{2}$$

for i.m. $N\in\mathbb{N}$. *Then*

$$\lambda(\mathcal{D}(\Psi))=\begin{cases}0, & \textit{if } \displaystyle\sum_{n=1}^{\infty}\frac{\varphi(n)\Psi(n)}{n}<\infty\,,\\ 1, & \textit{if } \displaystyle\sum_{n=1}^{\infty}\frac{\varphi(n)\Psi(n)}{n}=\infty\,.\end{cases} \tag{3}$$

Here λ is Lebesgue measure, φ is the Euler function.

Note: $\mathcal{D}(\Psi)\subset\mathcal{K}(\Psi)$, hence the D-S theorem is a stronger statement!

Duffin-Schaeffer conjecture (1941): (3) *holds without assuming* (2).

Khintchine, Gallagher or Duffin and Schaeffer?

Gallagher (1962): *Let* $\psi : \mathbb{N} \to [0, +\infty)$ *be decreasing and tending to zero at infinity. Then*

$$\lambda_2(\mathcal{G}(\psi)) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \psi(n) \log n < \infty\,, \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(n) \log n = \infty\,. \end{cases}$$

Q: $\mathcal{G}_\alpha(\psi) = \mathcal{G}(\psi) \cap \{(x, y) \in \mathbb{R}^2 : x = \alpha\}$ Given a fixed $\alpha \in [0, 1)$ what is the size of the set $\mathcal{G}_\alpha(\psi)$ of $\beta \in [0, 1)$ such that

$$\|n\alpha\| \cdot \|n\beta\| < \psi(n) \text{ for i.m. } n \in \mathbb{N}\,???$$

E.g., given any $\alpha \in [0, 1)$ is it true that for almost every $\beta \in [0, 1)$

$$\liminf_{n\to\infty} (\log n)^2 \|n\alpha\| \cdot \|n\beta\| = 0 \quad ??? \tag{4}$$

Badziahin (2013): *For any badly approximable* α *the set of* $\beta \in [0, 1)$ *such that*

$$\liminf_{n\to\infty} \log n \log\log n \, \|n\alpha\| \cdot \|n\beta\| > 0$$

has Hausdorff dimension 1.

Can the Duffin-Schaeffer conjecture be useful? Let $\Psi(n) := \dfrac{\psi(n)}{\|n\alpha\|}$. Then

$$\mathcal{G}_\alpha(\psi) = \mathcal{K}(\Psi) \supset \mathcal{D}(\Psi)\,.$$

One may try to use the DS theorem, but verifying (2) for specific α does not seem easy (if feasible at all). If we had a proof to the DS conjecture, then we could 'eliminate' φ by using the inequality

$$\varphi(n) \gg \frac{n}{\log\log n}$$

and then we could at least prove that $\lambda(\mathcal{G}_\alpha(\psi)) = \lambda(\mathcal{D}(\Psi)) = 1$ when

$$\psi(n) = \frac{1}{n(\log n)^2 \log\log\log n}$$

Hence we would be able to conclude (4). But... D-S is still a conjecture!

The theorem: ideas

B., Haynes, Velani (2015, preprint): *Given any $\alpha \in \mathbb{R}$ with the Diophantine exponent $w(\alpha) < 3$ and any monotonic $\psi : \mathbb{N} \to (0, +\infty)$ such that*

$$\sum_{n=1}^{\infty} \psi(n) \log n = \infty \,,$$

we have that $\lambda(\mathcal{G}_\alpha(\psi)) = 1$, e.i. for almost every β

$$\|n\alpha\| \cdot \|n\beta\| < \psi(n) \qquad \textit{for i.m. } n \in \mathbb{N} \,.$$

Some ideas:

1. Zero-one law (Cassles, Gallagher):
$\lambda(\mathcal{K}(\psi)) \in \{0, 1\}$, $\lambda(\mathcal{D}(\Psi)) \in \{0, 1\}$ for any ψ and Ψ.
All we need:

$$\lambda(\mathcal{G}_\alpha(\psi)) > 0$$

The theorem: ideas

2. Borell-Cantelli Lemma (a version of): *For any sequence of intervals $E_i \subset [0,1]$ such that*

$$\sum_{i,j=1}^{N} \lambda(E_i \cap E_j) \le \textsc{Const} \times \left(\sum_{i=1}^{N} \lambda(E_i)\right)^2 \qquad (5)$$

for infinitely many $N \in \mathbb{N}$ we have that

$$\lambda(\limsup_{i\to\infty} E_i) \ge \frac{1}{\textsc{Const}} > 0\,.$$

Condition (5) — Quasi Independence on Average (QIA).

Is there a better way to prove positive measure? Almost No:

B.-Velani (yet unpublished): *For any sequence of intervals $E_i \subset [0,1]$ we have that*

$$\lambda(\limsup_{i\to\infty} E_i) > 0$$

if and only if $(E_i)_{i\in\mathbb{N}}$ contains a subsequence satisfying QIA.

A proof of Khintchine's theorem

Proposition 1: *Let $R = 6$. Then for all sufficiently large $t \in \mathbb{N}$*

$$\lambda\left(\bigcup_{R^{t-1}<q\le R^t}\ \bigcup_{p\in\mathbb{Z}} B\left(\tfrac{p}{q},\tfrac{R}{R^{2t}}\right)\right)\ \ge\ \frac{1}{2}\,.$$

For sufficiently large t choose a maximal subcollection of rationals

$$\frac{p_1}{q_1},\dots,\frac{p_m}{q_m}\in[0,1],\qquad m=m_t$$

such that $R^{t-1} < q_i \le R^t$ and

$$\left|\frac{p_i}{q_i}-\frac{p_j}{q_j}\right|\ge\frac{1}{R^{2t}}\qquad(i\ne j)\,.$$

By its maximality and Proposition 1 we have that

$$\lambda\left(\bigcup_{i=1}^{m} B\left(\frac{p_i}{q_i},\frac{R+1}{R^{2t}}\right)\right)\ \ge\ \frac{1}{2}\,.$$

Hence $m = m_t \ge \dfrac{1}{4(R+1)}\,R^{2t}$.

A proof of Khintchine

Define

$$E_t\left(\frac{p_i}{q_i}\right) = B\left(\frac{p_i}{q_i}, \frac{\Psi(R^t)}{R^t}\right) \qquad \text{and} \qquad E_t = \bigcup_{i=1}^{m_t} E_t\left(\frac{p_i}{q_i}\right)$$

Then

$$\limsup_{t\to\infty} E_t \subset \mathcal{K}(\Psi) \cup \mathbb{Q}$$

and enough to show that $\lambda(\limsup_{t\to\infty} E_t) > 0$. Next

$$\lambda(E_t) \asymp R^t\Psi(R^t)\,. \tag{6}$$

And so

$$\sum_{t=t_0}^{\infty} \lambda(E_t) \asymp \sum_{t=t_0}^{\infty} R^t\Psi(R^t) \asymp \sum_{n=1}^{\infty} \Psi(n) = \infty.$$

Next, let t, ℓ are sufficiently large, assume and w.l.o.g. $t > \ell$. Then for any $i \le m_\ell$ the interval $E_\ell(p_i/q_i)$ intersects at most

$$\frac{|E_\ell(p_i/q_i)|}{R^{-2t}} + 3 \le R^{-\ell+2t+1}\Psi(R^\ell) + 3 \quad \text{intervals } E_t(p_j'/q_j').$$

A proof of Khintchine

Hence,
$$\lambda(E_\ell(p_i/q_i)\cap E_t) \ll \Big(R^{-\ell+2t}\Psi(R^\ell)+1\Big)\times R^{-t}\Psi(R^t)$$

and
$$\lambda(E_\ell\cap E_t) \ll \Big(R^{-\ell+2t}\Psi(R^\ell)+1\Big)\times R^{-t}\Psi(R^t)R^{2\ell} \le$$
$$\ll R^t\Psi(R^t)R^\ell\Psi(R^\ell)+R^t\Psi(R^t)R^{-2(t-\ell)} \le$$
$$\le \lambda(E_t)\lambda(E_\ell)+\lambda(E_t)R^{-2(t-\ell)}\,.$$

Summing over $t,\ell\le N$ gives
$$\sum_{t\le N}\sum_{\ell\le N}\lambda(E_t\cap E_\ell) \ll \Bigg(\sum_{t\le N}\lambda(E_t)\Bigg)^2 +$$
$$+\sum_{t\le N}\lambda(E_t)\underbrace{\sum_{\ell<t}R^{-2(t-\ell)}}_{\text{bounded}} \ll \Bigg(\sum_{t\le N}\lambda(E_t)\Bigg)^2$$

and so we are done!

Proof of BHV

WE SHALL TRY A SIMILAR APPROACH TO PROVE

BHV: *Given* α *..., and*

$$\sum_{n=1}^{\infty} \psi(n) \log n = \infty \,,$$

we have that for almost every β

$$\|n\alpha\| \cdot \|n\beta\| < \psi(n) \qquad \text{for infinitely many } n \in \mathbb{N} \,.$$

For simplicity I will assume α is badly approximable.

Proof of BHV: Setting up a limsup set

Proposition 2: *Let R be sufficiently large and fixed. Then for any sufficiently large integers $t > k$ there is $\Omega_{t,k} \subset [0,1]$ with $\lambda(\Omega_{t,k}) \geq \frac{1}{2}$ such that for any $\beta \in \Omega_{t,k}$ there exists a triple $(n,r,s) \in \mathbb{N} \times \mathbb{Z}^2$ of coprime integers such that*

$$\begin{cases} R^{-k-1} \leq |n\alpha - r| < R^{-k}, \\ \qquad\qquad |n\beta - s| < R^{-t+k}, \\ \qquad R^{t-1} < n \leq R^t. \end{cases} \tag{7}$$

Define the set $N(t,k)$ of $(n,r,s) \in \mathbb{N} \times \mathbb{Z}^2$ such that

$$\begin{cases} R^{t-1} < n \leq R^t, \\ R^{-k-1} < |n\alpha - r| < R^{-k}, \\ 0 \leq s \leq n, \\ \gcd(n,r,s) = 1 \end{cases} \tag{8}$$

Proof of BHV: Setting up a limsup set

Proposition 2 $\Rightarrow$

$$\lambda\left(\bigcup_{(n,r,s)\in N(t,k)}\left\{\beta\in\mathbb{R}:|\beta-s/n|<R^{-2t+k+1}\right\}\right)\geq\frac{1}{2}\,.$$

Recall from the proof of Khintchine: Proposition 1 $\Rightarrow$

$$\lambda\left(\bigcup_{R^{t-1}<q\leq R^t}\ \bigcup_{p\in\mathbb{Z}} B\left(\tfrac{p}{q},\tfrac{R}{R^{2t}}\right)\right)\ \geq\ \frac{1}{2}\,.$$

Let $Z(t,k)$ be a maximal subcollection of $N(t,k)$ such that

$$\left|\frac{s_1}{n_1}-\frac{s_2}{n_2}\right|>R^{-2t+k}$$

for $(n_1,r_1,s_1)\neq(n_2,r_2,s_2)$. By the maximality of $Z(t,k)$ and Proposition 2,

$$\#Z(t,k)\ \asymp\ R^{2t-k}\,.$$

Proof of BHV

Now, given $\xi \in \mathbb{R}$, let

$$E_{t,k}(\xi) = \{\beta \in \mathbb{R} : |\beta - \xi| < \psi(R^t)R^{-t+k}\}. \tag{9}$$

Furthermore, define

$$E_{t,k} := \bigcup_{(n,r,s)\in Z(t,k)} E_{t,k}(s/n) \tag{10}$$

and let $E_\infty = \limsup E_{t,k}$. Observe $E_\infty \subset \mathcal{G}_\alpha(\psi)$ and so we only need to prove $\boxed{\lambda(E_\infty) > 0}$. Important, we have

$$\sum_{t=1}^{\infty} tR^t\psi(R^t) = \infty.$$

We shall restrict k to lie in $\nu_1 t \le k \le \nu_2 t$ for some suitably chosen $\nu_1 < \nu_2$. Then

$$\sum_{t\le T}\sum_{k}\lambda(E_{t,k}) \asymp \sum_{t\le T} tR^t\psi(R^t) \to \infty \quad \text{as} \quad T\to\infty.$$

Overlaps estimates for $E_{t,k}$
For different (t,k) and (t',k')

$$\lambda(E_{t,k}\cap E_{t',k'}) \ll \left(1+\frac{\psi(R^{t'})R^{-t'+k'}}{R^{-2t+k}}\right) R^{2t'-k'}\,\psi(R^t)R^{-t+k}$$

which is

$$\lambda(E_{t,k}\cap E_{t',k'}) \ll \lambda(E_{t,k})\,\lambda(E_{t',k'}) \qquad \text{(excellent!!)}$$

provided that

$$\frac{\psi(R^{t'})R^{-t'+k'}}{R^{-2t+k}} \geq 1$$

or equivalently when

$$\frac{\psi(R^{t})R^{-t+k}}{R^{-2t'+k'}} \geq 1\,.$$

In Khintchine, summation over the remaining terms still gives the required bound. This is not the case here.

Overlaps: another technique

$$E_{t,k}(s/n) \cap E_{t',k'}(s'/n') \neq \emptyset$$

implies that

$$\left|\frac{s}{n} - \frac{s'}{n'}\right| \leq 2\max\{\psi(R^t)R^{-t+k}, \psi(R^{t'})R^{-t'+k'}\}\,.$$

Then, for any fixed pair (n, n') the number of different (s, s') is $\ll \Delta$ provided that

$$n's - ns' \neq 0\,.$$

Then

$$\sum_{n's-ns'\neq 0} |E_{t,k}(s/n) \cap E_{t',k'}(s'/n')| \;\ll\; \psi(R^t)R^t\,\psi(R^{t'})R^{t'},$$

The remaining case

$$\frac{\psi(R^{t'})R^{-t'+k'}}{R^{-2t+k}} \le 1\,, \qquad \frac{\psi(R^{t})R^{-t+k}}{R^{-2t'+k'}} \le 1 \tag{11}$$

$$\lambda\left(\bigcup_{n's-ns'=0} E_{t,k}(s/n)\cap E_{t',k'}(s'/n')\right) \ll ???? \tag{12}$$

For

$$(A,B,C) := (n,r,s)\times(n',r',s') \ne \mathbf{0}\,,$$

we have that

$$B = -ns' + n's = 0.$$

and

$$|A|,|C| \ \ll\ \left(R^{-k+t'} + R^{-k'+t}\right)$$

and

$$|C\beta + A| = |(1,\alpha,\beta)\cdot(A,B,C)| \ll R^{-\nu_1 t}\,R^{-\nu_1 t'}\,.$$

Eventually we obtain

$$\Big|\bigcup_{n's-ns'=0} E_{t,k}(s/n)\cap E_{t',k'}(s'/n')\Big| \;\le\; R^{-\eta(t+t')} \qquad (13)$$

in the remaining case and thus altogether we get that

$$|E_{t,k}\cap E_{t',k'}| \;\ll\; \psi(R^t)R^t\,\psi(R^{t'})R^{t'} \;+\; R^{-\eta(t+t')}\,.$$

This is now enough to finish the proof.

Curves

BHV: *Given a fixed α..., we have that for almost every β*

$$\|n\alpha\| \cdot \|n\beta\| < \psi(n)$$

holds for infinitely/finitely many $n \in \mathbb{N}$ depending on convergence/divergence of

$$\sum_{n=1}^{\infty} \psi(n) \log n\,.$$

Problem: Find an analogue of Gallagher on curves and lines in $\mathbb{R}^2$ (or even more general manifolds in higher dimensions). That is given a 'reasonable' curve $\mathcal{C}$ in $\mathbb{R}^2$ of length 1 prove that

$$\lambda_{\mathcal{C}}(\mathcal{G}(\psi) \cap \mathcal{C}) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \psi(n) \log n < \infty\,, \\ 1, & \text{if } \sum_{n=1}^{\infty} \psi(n) \log n = \infty\,. \end{cases}$$

For non-degenerate curves the convergence case was proven in 2007 (B.-Velani), the divergence is open!