Dirac Operator

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ATIYAH-SINGER INDEX THEOREM

There are several proofs of the Atiyah-Singer index theorem :

- The first proof by M. F. Atiyah and I. M. Singer.
  Published in the R. Palais book *Seminar on the Atiyah-Singer Index Theorem*, Princeton University Press (1965).
  The proof uses cobordism theory i.e. Pontrjagin-Thom construction and calculation of cobordism groups.
  A number of changes-simplifications in the proof have been made by M. S. Raghunathan.
  M. S. Raghunathan *The Atiyah-Singer Index Theorem*, published in "Contemporary Mathematics", Vol 522, (Vector Bundles and Complex Geometry) (2008).

- The K-theory proof by Atiyah-Segal-Singer.
  Five papers in Ann. of Math. (1968) (1971).
  Proves Atiyah-Singer for the equivariant case
  and the families case.
- The heat equation proof by Atiyah-Bott-Patodi.
  On the heat equation and the index theorem,
  Invent. Math. 19 (4): 279 - 330, (1973).
  Based on an idea originally proposed by McKean and Singer.
  Uses Chern-Weil theory.
  Does not in and of itself prove the general case of
  Atiyah-Singer — but proves a very strong result for Dirac
  operators. Sets the stage for the eta-invariant and index
  theory on compact manifolds with boundary.

- Bott periodicity proof by Baum - van Erp.
  Reveals exactly how and why the Atiyah-Singer index theorem can be proved as a corollary of Bott periodicity.
  Does not use Pontrjagin-Thom construction and calculation of cobordism groups.
- Tangent groupoid proof by Alain Connes.
  Published in the book *Noncommutative Geometry* by Alain Connes, Academic Press, (1994).
  See also the cyclic cohomology theory of Alain Connes.
- Bivariant K-theory proof by Joachim Cuntz.
  Uses Kasparov bivariant K-theory.

Five lectures:

1. Dirac operator
2. Atiyah-Singer revisited
3. What is K-homology?
4. Beyond ellipticity
5. The Riemann-Roch theorem

DIRAC OPERATOR

The Dirac operator of $\mathbb{R}^n$ will be defined. This is a first order elliptic differential operator with constant coefficients. Next, the class of differentiable manifolds which come equipped with an order one differential operator which at the symbol level is locally isomorphic to the Dirac operator of $\mathbb{R}^n$ will be considered. These are the Spin^c manifolds. Spin^c is slightly stronger than oriented, so Spin^c can be viewed as "oriented plus epsilon". Most of the oriented manifolds that occur in practice are Spin^c. The Dirac operator of a closed Spin^c manifold is the basic example for the Hirzebruch-Riemann-Roch theorem and the Atiyah-Singer index theorem.

What is the Dirac operator of $\mathbb{R}^n$?

To answer this, shall construct matrices $E_1, E_2, \ldots, E_n$ with the following properties :

Properties of $E_1, E_2, \ldots, E_n$

- Each E_j is a $2^r \times 2^r$ matrix of complex numbers, where r is the largest integer $\leq n/2$.
- Each E_j is skew adjoint, i.e. $E_j^* = -E_j$ (* $=$ conjugate transpose)
- $E_j^2 = -I \qquad j = 1, 2, \ldots, n$ (I is the $2^r \times 2^r$ identity matrix.)
- $E_j E_k + E_k E_j = 0$ whenever $j \neq k$.
- For n odd, $(n = 2r + 1)$ $i^{r+1} E_1 E_2 \cdots E_n = I \qquad i = \sqrt{-1}$
- For n even, $(n = 2r)$ each E_j is of the form

$$E_j = \begin{bmatrix} \mathbf{0} & * \\ * & \mathbf{0} \end{bmatrix} \quad \text{and} \quad i^r E_1 E_2 \cdots E_n = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{bmatrix}$$

These matrices are constructed by a simple inductive procedure.

$n = 1$, $E_1 = [-i]$

$n \rightsquigarrow n+1$ with n odd $\qquad (r \rightsquigarrow r+1)$

The new matrices $\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{n+1}$ are

$$\widetilde{E}_j = \begin{bmatrix} \mathbf{0} & E_j \\ E_j & \mathbf{0} \end{bmatrix} \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \widetilde{E}_{n+1} = \begin{bmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{bmatrix}$$

where $E_1, E_2, \ldots, E_n$ are the old matrices.

$n \rightsquigarrow n+1$ with n even $\qquad$ (r does not change)

The new matrices $\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{n+1}$ are

$$\widetilde{E}_j = E_j \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \widetilde{E}_{n+1} = \begin{bmatrix} -iI & \mathbf{0} \\ \mathbf{0} & iI \end{bmatrix}$$

where $E_1, E_2, \ldots, E_n$ are the old matrices.

Example

$n = 1$: $E_1 = [-i]$

$n = 2$: $E_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$n = 3$: $E_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

Example

$$n = 4:\ E_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$D =$ Dirac operator of $\mathbb{R}^n$

$$\begin{cases} n = 2r & n \text{ even} \\ n = 2r+1 & n \text{ odd} \end{cases}$$

$$D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}$$

D is an unbounded symmetric operator on the Hilbert space
$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)$ (2^r times)

To begin, the domain of D is
$C_c^\infty(\mathbb{R}^n) \oplus C_c^\infty(\mathbb{R}^n) \oplus \ldots \oplus C_c^\infty(\mathbb{R}^n)$ (2^r times)

D is essentially self-adjoint
(i.e. D has a unique self-adjoint extension)
so it is natural to view D as an unbounded self-adjoint operator on the Hilbert space
$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)$ (2^r times)

QUESTION : Let M be a C^∞ manifold of dimension n.
Does M admit a differential operator which (at the symbol level) is locally isomorphic to the Dirac operator of $\mathbb{R}^n$?

To answer this question, will define Spin^c vector bundle.

What is a Spin^c vector bundle?

Let X be a paracompact Hausdorff topological space.
On X let E be an $\mathbb{R}$ vector bundle which has been oriented.
i.e. the structure group of E has been reduced from
$GL(n, \mathbb{R})$ to $GL^+(n, \mathbb{R})$

$$GL^+(n, \mathbb{R}) = \{[a_{ij}] \in GL(n, \mathbb{R}) \mid \det[a_{ij}] > 0\}$$

$n=$ fiber dimension (E)

Assume $n \geq 3$ and recall that for $n \geq 3$

$$H^2(GL^+(n, \mathbb{R}); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

Denote by $\mathcal{F}^+(E)$ the principal $GL^+(n, \mathbb{R})$ bundle on X consisting of all positively oriented frames.

A point of $\mathcal{F}^+(E)$ is a pair $\big(x, (v_1, v_2, \ldots, v_n)\big)$ where $x \in X$ and $(v_1, v_2, \ldots, v_n)$ is a positively oriented basis of E_x. The projection $\mathcal{F}^+(E) \to X$ is

$$\big(x, (v_1, v_2, \ldots, v_n)\big) \mapsto x$$

For $x \in X$, denote by

$$\iota_x \colon \mathcal{F}^+_x(E) \hookrightarrow \mathcal{F}^+(E)$$

the inclusion of the fiber at x into $\mathcal{F}^+(E)$.

Note that (with $n \geq 3$)

$$H^2(\mathcal{F}^+_x(E); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

A Spin^c vector bundle on X is an $\mathbb{R}$ vector bundle E on X (fiber dimension $E \geq 3$) with

1. E has been oriented.
2. $\alpha \in H^2(\mathcal{F}^+(E); \mathbb{Z})$ has been chosen such that $\forall x \in X$

$$\iota_x^*(\alpha) \in H^2(\mathcal{F}_x^+(E); \mathbb{Z}) \text{ is non-zero.}$$

Remarks

1.For $n = 1, 2$ "E is a Spin^c vector bundle" $=$"E has been oriented and an element $\alpha \in H^2(X; \mathbb{Z})$" has been chosen. ($\alpha$ can be zero.)

2. For all values of $n =$ fiber dimension(E), E is a Spin^c vector bundle iff the structure group of E has been changed from $GL(n, \mathbb{R})$ to Spin$^c(n)$.
i.e. Such a change of structure group is equivalent to the above definition of Spin^c vector bundle.

By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin^c vector bundle

$$\begin{array}{ccc} & & \text{complex} \\ & & \Downarrow \\ \text{Spin} & \Rightarrow & \text{Spin}^c \\ & & \Downarrow \\ & & \text{oriented} \end{array}$$

A Spin^c structure for an $\mathbb{R}$ vector bundle E can be thought of as an orientation for E plus a slight extra bit of structure. Spin^c structures behave very much like orientations. For example, an orientation on two out of three $\mathbb{R}$ vector bundles in a short exact sequence determines an orientation on the third vector bundle. An analogous assertion is true for Spin^c structures.

Two Out Of Three Lemma

Lemma

Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{R}$*-vector bundles on* X. *If two out of three are Spin*^{c} *vector bundles, then so is the third.*

Definition

Let M be a C^∞ manifold (with or without boundary). M is a Spin^c manifold iff the tangent bundle TM of M is a Spin^c vector bundle on M.

The Two Out Of Three Lemma implies that the boundary ∂M of a Spin^c manifold M with boundary is again a Spin^c manifold.

Various well-known structures on a manifold M make M into a Spin^c manifold.

$$
\begin{array}{ccc}
 & & \text{(complex-analytic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Rightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Rightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}
$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

A Spin^c manifold comes equipped with a first-order elliptic differential operator known as its Dirac operator. This operator is locally isomorphic (at the symbol level) to the Dirac operator of $\mathbb{R}^n$.

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q} T^*_{\mathbb{C}} M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p, q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^* \colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

TWO POINTS OF VIEW ON SPIN^c MANIFOLDS

1. Spin^c is a slight strengthening of oriented. Most of the oriented manifolds that occur in practice are Spin^c.

2. Spin^c is much weaker than complex-analytic. BUT the assempled Dolbeault complex survives (as the Dirac operator). AND the Todd class survives.

$$M \quad \mathrm{Spin}^c \Longrightarrow \quad \exists \quad Td(M) \in H^*(M;\mathbb{Q})$$

If M is a Spin^c manifold, then $Td(M)$ is

$$Td(M) := \exp^{c_1(M)/2} \widehat{A}(M) \qquad\qquad Td(M) \in H^*(M; \mathbb{Q})$$

If M is a complex-analyic manifold, then M has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2} \widehat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$

WARNING!!!

The Todd class of a Spin^c manifold is not obtained by complexifying the tangent bundle TM of M and then applying the Todd polynomial to the Chern classes of $T_{\mathbb{C}}M$.

$$Td(T_{\mathbb{C}}M) = \widehat{A}(M)^2 = \widehat{A}(M) \cup \widehat{A}(M)$$

Correct formula for the Todd class of a Spin^c manifold M is:

$$Td(M) := \exp^{c_1(M)/2} \widehat{A}(M) \qquad\qquad Td(M) \in H^*(M; \mathbb{Q})$$

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.

D_E denotes the Dirac operator of M tensored with E.

$$D_E \colon C^\infty(M, \mathcal{S}^+ \otimes E) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes E)$$

$\mathcal{S}^+, (\mathcal{S}^-)$ are the positive (negative) spinor bundles on M.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M. D_E denotes the Dirac operator of M tensored with E.

THEOREM Index$(D_E) = (ch(E) \cup Td(M))[M]$.

This theorem will be proved in the next lecture as a corollary of Bott periodicity.

In particular, this will prove the Hirzebruch-Riemann-Roch theorem.

Also, this will prove (for closed even-dimensional Spin^c manifolds) the Hirzebruch signature theorem.

From $E_1, E_2, \ldots, E_n$ obtain :

1) The Dirac operator of $\mathbb{R}^n$

2) The Bott generator vector bundle on S^n (n even)

3) The spin representation of $\mathrm{Spin}^c(n)$

W finite dimensional $\mathbb{C}$ vector space $\qquad \dim_{\mathbb{C}}(W) < \infty$

$T : W \to W \qquad T \in \mathrm{Hom}_{\mathbb{C}}(W, W) \qquad T^2 = I$

$\Longrightarrow$ The eigen-values of T are ± 1

$W = W_1 \oplus W_{-1}$

$W_1 = \{v \in W \mid Tv = v\} \qquad W_{-1} = \{v \in W \mid Tv = -v\}$

Bott generator vector bundle

n even $\qquad n = 2r \qquad S^n \subset \mathbb{R}^{n+1} \qquad S^n \to M(2^r, \mathbb{C})$

$$S^n = \{(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_1^2 + a_2^2 + \cdots + a_{n+1}^2 = 1\}$$

$$(a_1, a_2, \ldots, a_{n+1}) \mapsto i(a_1 E_1 + a_2 E_2 + \cdots + a_{n+1} E_{n+1})$$

$$i = \sqrt{-1}$$

$$\begin{aligned} & (i)^2 (a_1 E_1 + a_2 E_2 + \cdots + a_{n+1} E_{n+1})^2 \\ = \;& (-1)(-a_1^2 - a_2^2 - \ldots - a_{n+1}^2)\, I \\ = \;& I \end{aligned}$$

$\Longrightarrow$ The eigenvalues of $i(a_1 E_1 + a_2 E_2 + \cdots + a_{n+1} E_{n+1})$ are ± 1.

Bott generator vector bundle β on S^n $\qquad$ n even $\qquad$ $n = 2r$

$$
\begin{aligned}
& \beta_{(a_1,a_2,\ldots,a_{n+1})} \\
= \; & (+1 \text{ eigenspace of } i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1}))^* \\
= \; & \mathrm{Hom}_{\mathbb{C}}(\{v \in \mathbb{C}^{2^r} \mid i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})\, v = v\}\,,\, \mathbb{C})
\end{aligned}
$$

$$
\begin{array}{ccccc}
K^0(S^n) & = & \mathbb{Z} & \oplus & \mathbb{Z} \\
 & & \mathbf{1} & & \beta
\end{array}
$$

$\mathbf{1} = S^n \times \mathbb{C}$

Bott generator vector bundle β on S^n n even $n = 2r$
β is determined by:

1. $\forall p \in S^n\ ,\ \dim_{\mathbb{C}}(\beta_p) = 2^{r-1}$
2. $\mathrm{ch}(\beta)[S^n] = 1$

n even $\qquad n = 2r \qquad S^n \subset \mathbb{R}^{n+1}$

With the Spin (or Spin^c) structure S^n has as the boundary of the unit ball B^{n+1} of $\mathbb{R}^{n+1}$, the Spinor bundle $\mathcal{S}$ of S^n is:

$$\mathcal{S} = S^n \times \mathbb{C}^{2^r}$$

The positive (negative) Spinor bundles $\mathcal{S}^+(\mathcal{S}^-)$ are defined by :

$$\mathcal{S}^+_{(a_1,a_2,\ldots,a_{n+1})} = +1 \quad \text{eigenspace of} \;\; i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})$$

$$\mathcal{S}^-_{(a_1,a_2,\ldots,a_{n+1})} = -1 \quad \text{eigenspace of} \;\; i(a_1E_1 + a_2E_2 + \cdots + a_{n+1}E_{n+1})$$

$$\mathcal{S} = S^n \times \mathbb{C}^{2^r} = \mathcal{S}^+ \oplus \mathcal{S}^-$$

$$\beta = (\mathcal{S}^+)^*$$

M Spin^c manifold

∂M might be non-empty

$TM =$ the tangent bundle of M

Dirac operator

$$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$$

$\boxed{\mathcal{S} \text{ is the Spinor bundle}}$

$C_c^\infty(M, \mathcal{S}) = \{C^\infty \text{ sections with compact support of } \mathcal{S}\}$

$$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$$

such that

(1) D is $\mathbb{C}$-linear
$D(s_1 + s_2) = Ds_1 + Ds_2 \qquad s_j \in C_c^\infty(M, \mathcal{S})$
$D(\lambda x) = \lambda Ds \qquad \lambda \in \mathbb{C}$

(2) If $f : M \to \mathbb{C}$ is a C^∞ function, then

$$D(fs) = (df)s + f(Ds)$$

(3) If $s_j \in C_c^\infty(M, \mathcal{S})$ then

$$\int_M (Ds_1x, s_2x) = \int_M (s_1x, Ds_2x)dx$$

(4) If dim M is even, then D is off-diagonal $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$ is an elliptic first-order differential operator.

D can be viewed as an unbounded operator on the Hilbert space $L^2(M, \mathcal{S})$

$$(s_1, s_2) = \int_M (s_1 x, s_2 x) dx$$

$$D : C_c^\infty(M, \mathcal{S}) \to C_c^\infty(M, \mathcal{S})$$

is a symmetric operator

EXAMPLE. Let M be a compact complex-analytic manifold. The positive (negative) Spinor bundles of the underlying Spin^c manifold are :

$$\mathcal{S}^+ = \bigoplus_j \Lambda^{0,\,2j}\, T^*_{\mathbb{C}} M$$

$$\mathcal{S}^- = \bigoplus_j \Lambda^{0,\,2j+1}\, T^*_{\mathbb{C}} M$$

$$D^+ : C^\infty(M, \mathcal{S}^+) \to C^\infty(M, \mathcal{S}^-) \ \text{ is}$$

$$\bar\partial + \bar\partial^* \colon C^\infty(M,\, \bigoplus_j \Lambda^{0,\,2j}\, T^*_{\mathbb{C}} M) \longrightarrow C^\infty(M,\, \bigoplus_j \Lambda^{0,\,2j+1}\, T^*_{\mathbb{C}} M)$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

EXAMPLE. Let M be a compact even-dimensional Spin^c manifold without boundary.

$$D^+_{\mathcal{S}^*} \colon C^\infty(M, \mathcal{S}^+ \otimes \mathcal{S}^*) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes \mathcal{S}^*)$$

is the Hirzebruch signature operator of M.
If the dimension of M is divisible by 4, the index of this operator is the signature of the quadratic form

$$H^r(M;\mathbb{R}) \otimes_{\mathbb{R}} H^r(M,\mathbb{R}) \longrightarrow \mathbb{R} \qquad n = 2r \quad r \text{ even}$$

$$a \otimes b \mapsto (a \cup b)[M]$$

Example.

$$n \text{ even} \qquad n = 2r \qquad S^n \subset \mathbb{R}^{n+1}$$

$D =$ Dirac operator of S^n

$\mathcal{S} =$ Spinor bundle of $S^n = S^n \times \mathbb{C}^{2^r}$

$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$

$D : C^\infty(S^n, \mathcal{S}) \to C^\infty(S^n, \mathcal{S})$

$$D = \begin{bmatrix} 0 & D^- \\ & \\ D^+ & 0 \end{bmatrix}$$

$D^+ : C^\infty(S^n, \mathcal{S}^+) \to C^\infty(S^n, \mathcal{S}^-)$

$\mathrm{Index}\,(D^+) := \dim_{\mathbb{C}}(\mathrm{Kernel}\, D^+) - \dim_{\mathbb{C}}(\mathrm{Cokernel}\, D^+) = 0$

Theorem. $\mathrm{Index}\,(D^+) = 0$

Tensor D^+ with the Bott generator vector bundle β

$$D^+_\beta : C^\infty(S^n, \mathcal{S}^+ \otimes \beta) \to C^\infty(S^n, \mathcal{S}^- \otimes \beta)$$

Theorem. On S^n, with n even,
Index$(D^+) = 0$ and $\mathrm{Index}\,(D^+_\beta) = 1.$

BOTT PERIODICITY

$$\pi_j GL(n,\mathbb{C}) = \begin{cases} \mathbb{Z} & \text{j odd} \\ \\ 0 & \text{j even} \end{cases}$$

$$j = 0, 1, 2, \ldots, 2n-1$$

Why ???? does Bott periodicity imply

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M. D_E denotes the Dirac operator of M tensored with E.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

This will be explained in the next lecture — tomorrow (i.e. Tuesday, 4 August).

Two out of three lemma.

Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of $\mathbb{R}$ vector bundles on X. If Spin^c structures are given for any two of E', E, E'' then a Spin^c structure is determined for the third.

Corollary. If M is a Spin^c manifold with boundary ∂M, then ∂M is (in a canonical way) a Spin^c manifold.

Proof.

set $\mathbf{1} = \partial M \times \mathbb{R}$

exact sequence

$0 \to T(\partial M) \to TM \mid \partial M \to \mathbf{1} \to 0$

By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin^c vector bundle

$$
\begin{array}{ccc}
 & & \text{complex} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{oriented}
\end{array}
$$

A Spin^c structure for the $\mathbb{R}$ vector bundle E can be thought of as an orientation for E plus a slight extra bit of structure. Spin^c structures behave very much like orientations. For example, an orientation on two out of three $\mathbb{R}$ vector bundles in a short exact sequence determines an orientation on the third vector bundle. Analogous assertions are true for Spin^c structures.

Two out of three lemma.

Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of $\mathbb{R}$ vector bundles on X. If Spin^c structures are given for any two of E', E, E'' then a Spin^c structure is determined for the third.

Various well-known structures on a manifold M make M into a Spin^c manifold

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\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

$$\mathrm{Cliff}_{\mathbb{C}}(V) = \mathbb{C} \underset{\mathbb{R}}{\otimes} \mathrm{Cliff}(V)$$

$$V \subset \mathrm{Cliff}(V) \subset \mathrm{Cliff}_{\mathbb{C}}(V)$$

$\mathrm{Cliff}_{\mathbb{C}}(V)$ is a C^* algebra

$v \in V \qquad\qquad v^* = -v$

Choose an orthogonal basis $e_1, e_2, \ldots, e_n$ for V

$n = \dim_{\mathbb{R}}(V)$

n even $\qquad n = 2r$

$E_1, E_2, \ldots, E_n$ $\qquad$ $2^r \times 2^r$ matrices

$e_j \mapsto E_j$ $\qquad$ $\mathrm{Cliff}_{\mathbb{C}}(V) \xrightarrow[\cong]{} M(2^r, \mathbb{C})$

Isomorphism of C^* algebras

$\underline{n \text{ odd} \qquad n = 2r+1}$ $\qquad n = \dim_{\mathbb{R}}(V)$

$$\varphi_+ : \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$E_1, E_2, \ldots, E_n \qquad 2^r \times 2^r$ matrices

$$\varphi_+(e_j) = E_j \qquad j = 1, 2, \ldots, n$$

$$\varphi_- : \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$$\varphi_-(e_j) = -E_j \qquad j = 1, 2, \ldots, n$$

$$\varphi_+ \oplus \varphi_- : \mathrm{Cliff}_{\mathbb{C}}(V) \underset{\cong}{\longrightarrow} M(2^r, \mathbb{C}) \oplus M(2^r, \mathbb{C})$$

Isomorphism of C^* algebras

Remark. These isomorphisms are non-canonical since they depend on the choice of an orthornormal basis for V.

Assume $\begin{cases} X \times G \to X \\ G \text{ acts on } X \text{ by a right action} \\ \\ G \times Y \to Y \\ G \text{ acts on } Y \text{ by a left action} \end{cases}$

Notation. $X \underset{G}{\times} Y = X \times Y / \sim \quad (xg, y) \sim (x, gy)$

Example. $E \quad \mathbb{R}$ vector bundle on X

$$\Delta(E) \underset{\mathrm{GL}(n,\mathbb{R})}{\times} \mathbb{R}^n \cong E$$

$$((p, v_1, v_2, \ldots, v_n), (a_1, a_2, \ldots, a_n)) \mapsto a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$E \quad \mathbb{R}$ vector bundle on X

A Spin^c datum $\eta : P \to \Delta(E)$ determines
a Spinor system $(\epsilon, \langle\ ,\ \rangle, F)$ for E.

$\underline{\epsilon \text{ and } \langle\ ,\ \rangle}$ $\quad p \in X \quad$ An $\mathbb{R}$ basis $v_1, v_2, \ldots, v_n$ of E_p is positively oriented and orthonormal iff

$$(v_1, v_2, \ldots, v_n) \in \mathrm{Image}\,(\eta)$$

$\underline{\text{Spinor bundle } F}$

$n = 2r$ or $n = 2r + 1$

$$F = P \underset{\mathrm{Spin}^c(n)}{\times} \mathbb{C}^{2^r}$$

How does $\mathrm{Spin}^c(n)$ act on $\mathbb{C}^{2^r}$?

$\underline{n \text{ odd}}$

$\mathsf{Spin}^c(n)$ has an irreducible representation known as its spin representation

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^r, \mathbb{C})$$

$$n = 2r + 1$$

$\underline{n \text{ even}}$

$\mathsf{Spin}^c(n)$ has two irreducible representations known as its $1/2$-spin representations

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C})$$

$$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C})$$

The direct sum $\mathrm{Spin}^c(n) \to \mathrm{GL}(2^r, \mathbb{C})$ of these two representations is the spin representation of $\mathrm{Spin}^c(n)$

Consider $\mathbb{R}^n$ with its usual inner product and usual orthonormal basis $e_1, e_2, \ldots, e_n$

$$\varphi : \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \to M(2^r, \mathbb{C})$$

$$\varphi(e_j) = E_j \qquad\qquad j = 1, 2, \ldots, n$$

There is a canonical inclusion

$$\mathrm{Spin}^c(n) \subset \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$$

$\varphi : \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \to M(2^r, \mathbb{C})$ restricted to $\mathrm{Spin}^c(n)$ maps $\mathrm{Spin}^c(n)$ to $2^r \times 2^r$ unitary matrices

$$\mathrm{Spin}^c(n) \to U(2^r) \subset \mathrm{GL}(2^r, \mathbb{C})$$

This is the *Spin representation* of $\mathrm{Spin}^c(n)$

$\mathrm{Spin}^c(n)$ acts on $\mathbb{C}^{2^r}$ via this representation

M C^∞ manifold

∂M might be non-empty

$TM =$ the tangent bundle of M

$$
\begin{pmatrix} \mathrm{Spin}^c \text{ datum for } TM \\ \eta : P \to \Delta(TM) \end{pmatrix}
$$

$$
\Big\downarrow
$$

$$
\begin{pmatrix} \text{Spinor system for } TM \\ (\epsilon, \langle\ ,\ \rangle, F) \end{pmatrix}
$$

$$
\Big\downarrow
$$

$$
\begin{pmatrix} \text{Dirac operator} \\ D : C_c^\infty(M, F) \to C_c^\infty(M, F) \end{pmatrix}
$$

$\boxed{F \text{ is the Spinor bundle}}$

$C_c^\infty(M, F) = \{C^\infty \text{ sections with compact support of } F\}$

$$D : C_c^\infty(M, F) \to C_c^\infty(M, F)$$

such that

(1) D is $\mathbb{C}$-linear
$D(s_1 + s_2) = Ds_1 + Ds_2 \qquad s_j \in C_c^\infty(M, F)$
$D(\lambda x) = \lambda Ds \qquad \lambda \in \mathbb{C}$

(2) If $f : M \to \mathbb{C}$ is a C^∞ function, then

$$D(fs) = (df)s + f(Ds)$$

(3) If $s_j \in C_c^\infty(M, F)$ then

$$\int_M (Ds_1 x, s_2 x) = \int_M (s_1 x, Ds_2 x) dx$$

(4) If dim M is even, then D is off-diagonal $F = F^+ \oplus F^-$

$$D = \begin{bmatrix} 0 & D^- \\ \\ D^+ & 0 \end{bmatrix}$$

Existence of D ?

YES – Construct D locally and patch together with a C^∞ partition of unity.

Uniqueness of D ?

YES – If D_0 and D_1 both satisfy (1)-(4) then $D_0 - D_1$ is a vector bundle map

$$D_0 - D_1 : F \to F$$

Hence D_0 and D_1 differ by lower order terms

$D : C_c^\infty(M, F) \to C_c^\infty(M, F)$ is an elliptic first-order differential operator.

D can be viewed as an unbounded operator on the Hilbert space $L^2(M, F)$

$$(s_1, s_2) = \int_M (s_1 x, s_2 x) dx$$

$$D : C_c^\infty(M, F) \to C_c^\infty(M, F)$$

is a symmetric operator

Example.

n even

$$S^n \subset \mathbb{R}^{n+1}$$

$D =$ Dirac operator of S^n

$F =$ Spinor bundle of S^n

$F = F^+ \oplus F^-$

$D : C^\infty(S^n, F) \to C^\infty(S^n, F)$

$$D = \begin{bmatrix} 0 & D^- \\ & \\ D^+ & 0 \end{bmatrix}$$

$$D^+ : C^\infty(S^n, F^+) \to C^\infty(S^n, F^-)$$

$$\mathrm{Index}\,(D^+) := \dim_{\mathbb{C}}(\mathrm{Kernel}\, D^+) - \dim_{\mathbb{C}}(\mathrm{Cokernel}\, D^+)$$

Theorem. $\mathrm{Index}\,(D^+) = 0$

Tensor D^+ with the Bott generator vector bundle β

$$D^+_\beta : C^\infty(S^n, F^+ \otimes \beta) \to C^\infty(S^n, F^- \otimes \beta)$$

Theorem. $\mathrm{Index}\,(D^+_\beta) = 1$

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or pseudo-differential) operator on M

$E^0, E^1, \quad C^\infty \quad \mathbb{C}$ vector bundles on M

$C^\infty(M, E^j)$ denotes the $\mathbb{C}$ vector space of all C^∞ sections of E^j.

$D\colon C^\infty(M, E^0) \longrightarrow C^\infty(M, E^1)$

D is a linear transformation of $\mathbb{C}$ vector spaces.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or pseudo-differential) operator on M

Index$(D) := \dim_{\mathbb{C}}$ (Kernel $D) - \dim_{\mathbb{C}}$ (Cokernel D)

Theorem (M.Atiyah and I.Singer)

Index (D) = (a topological formula)

Example

$M = S^1 = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 = 1\}$

$D_f \colon L^2(S^1) \longrightarrow L^2(S^1)$ is

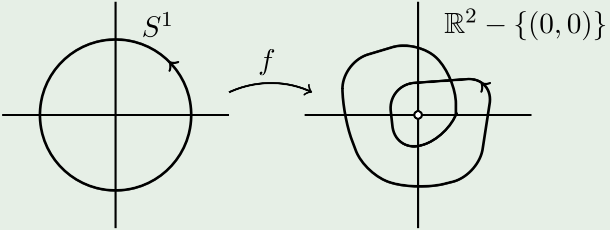
T_f	0
0	I

where $L^2(S^1) = L^2_+(S^1) \oplus L^2_-(S^1)$.

$L^2_+(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = 0, 1, 2, \ldots$
$L^2_-(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = -1, -2, -3, \ldots$.

Example

$f\colon S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ is a C^∞ map.



$T_f\colon L^2_+(S^1) \longrightarrow L^2_+(S^1)$ is the composition
$L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \longrightarrow L^2_+(S^1)$

$T_f\colon L^2_+(S^1) \longrightarrow L^2_+(S^1)$ is the Toeplitz operator associated to f

Example

Thus T_f is composition
$T_f \colon L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \xrightarrow{P} L^2_+(S^1)$

where $L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1)$ is $v \mapsto fv$
$fv(t_1, t_2) := f(t_1, t_2)v(t_1, t_2) \qquad \forall (t_1, t_2) \in S^1 \qquad \mathbb{R}^2 = \mathbb{C}$

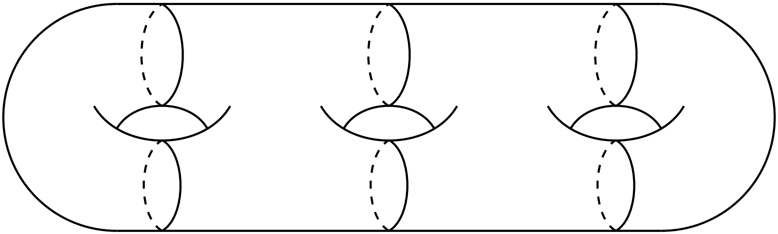
and $L^2(S^1) \xrightarrow{P} L^2_+(S^1)$ is the Hilbert space projection.

$D_f(v + w) := T_f(v) + w \qquad v \in L^2_+(S^1), \quad w \in L^2_-(S^1)$

Index(D_f) = -winding number (f).

RIEMANN - ROCH

M compact connected Riemann surface



$$\text{genus of } M = \# \text{ of holes}$$
$$= \frac{1}{2}\left[\mathrm{rank} H_1(M;\mathbb{Z})\right]$$

D a divisor of M

D consists of a finite set of points of M $p_1, p_2, \ldots, p_l$ and an integer assigned to each point $n_1, n_2, \ldots, n_l$

Equivalently

D is a function $D \colon M \to \mathbb{Z}$ with finite support

Support$(D) = \{p \in M \mid D(p) \neq 0\}$

Support(D) is a finite subset of M

D a divisor on M

$\deg(D) := \sum_{p \in M} D(p)$

Remark

D_1, D_2 two divisors

$$D_1 \geqq D_2 \text{ iff } \forall p \in M, D_1(p) \geqq D_2(p)$$

Remark

D a divisor, $-D$ is

$$(-D)(p) = -D(p)$$

Example

Let $f\colon M \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function.

Define a divisor $\delta(f)$ by:

$$\delta(f)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } f \\ \text{order of the zero if } f(p) = 0 \\ -(\text{order of the pole}) \text{ if } p \text{ is a pole of } f \end{cases}$$

Example

Let w be a meromorphic 1-form on M. Locally w is $f(z)dz$ where f is a (locally defined) meromorphic function. Define a divisor $\delta(w)$ by:

$$\delta(w)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } w \\ \text{order of the zero if } w(p) = 0 \\ -(\text{order of the pole}) \text{ if } p \text{ is a pole of } w \end{cases}$$

D a divisor on M

$$H^0(M,D) := \left\{ \begin{array}{l} \text{meromorphic functions} \\ f\colon M \to \mathbb{C}\cup\{\infty\} \end{array} \middle| \; \delta(f) \geqq -D \right\}$$

$$H^1(M,D) := \left\{ \begin{array}{l} \quad \text{meromorphic 1-forms} \\ w \text{ on } M \end{array} \middle| \; \delta(w) \geqq D \right\}$$

Lemma

$H^0(M,D)$ and $H^1(M,D)$ are finite dimensional $\mathbb{C}$ vector spaces

$$\dim_{\mathbb{C}} H^0(M,D) < \infty$$
$$\dim_{\mathbb{C}} H^1(M,D) < \infty$$

Theorem (R. R.)

Let M be a compact connected Riemann surface and let D be a divisor on M. Then:

$$\dim_{\mathbb{C}} H^0(M, D) - \dim_{\mathbb{C}} H^1(M, D) = d - g + 1$$

$$d = \textit{degree } (D)$$
$$g = \textit{genus } (M)$$

HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E} =$ sheaf of germs of algebraic sections of E

$H^j(M, \underline{E}) :=$ j-th cohomology of M using $\underline{E}$,
$j = 0, 1, 2, 3, \ldots$

LEMMA

For all $j = 0, 1, 2, \ldots \dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$.

For all $j > \dim_{\mathbb{C}}(M), \quad H^j(M, \underline{E}) = 0$.

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

THEOREM[HRR] Let M be a non-singular projective algebraic variety / $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$

Hirzebruch-Riemann-Roch

Theorem (HRR)

Let M be a non-singular projective algebraic variety $/\ \mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M,E) = (ch(E) \cup Td(M))[M]$$

Various well-known structures on a C^∞ manifold M make M into a Spin^c manifold

$$\begin{array}{ccc}
 & & \text{(complex-analytic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Rightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Rightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

Two Out Of Three Lemma

Lemma

Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{R}$*-vector bundles on* X*. If two out of three are Spin*^{c} *vector bundles, then so is the third.*

Definition

Let M be a C^∞ manifold (with or without boundary). M is a **Spin^c manifold** iff the tangent bundle TM of M is a Spin^c vector bundle on M.

The Two Out Of Three Lemma implies that the boundary ∂M of a Spin^c manifold M with boundary is again a Spin^c manifold.

A Spin^c manifold comes equipped with a first-order elliptic differential operator known as its Dirac operator.

If M is a Spin^c manifold, then $Td(M)$ is

$$Td(M) := \exp^{c_1(M)/2} \widehat{A}(M) \qquad\qquad Td(M) \in H^*(M; \mathbb{Q})$$

If M is a complex-analyic manifold, then M has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2} \widehat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q} T^* M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p, q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^* \colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

TWO POINTS OF VIEW ON SPIN^c MANIFOLDS

1. Spin^c is a slight strengthening of oriented. The oriented manifolds that occur in practice are Spin^c.

2. Spin^c is much weaker than complex-analytic. BUT the assempled Dolbeault complex survives (as the Dirac operator). AND the Todd class survives.

$$M \quad \mathrm{Spin}^c \Longrightarrow \quad \exists \quad Td(M) \in H^*(M;\mathbb{Q})$$

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.

D_E denotes the Dirac operator of M tensored with E.

$$D_E \colon C^\infty(M, S^+ \otimes E) \longrightarrow C^\infty(M, S^- \otimes E)$$

S^+, S^- are the positive (negative) spinor bundles on M.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

$\mathrm{K}_0(\cdot)$

Definition

Define an abelian group denoted $\mathrm{K}_0(\cdot)$ by considering pairs (M, E) such that:

1. M is a compact even-dimensional Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.

Set $\mathrm{K}_0(\cdot) = \{(M, E)\}/\sim$ where the the equivalence relation $\sim$ is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Addition in $\mathrm{K}_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $\mathrm{K}_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

Isomorphism (M, E) is isomorphic to (M', E') iff $\exists$ a diffeomorphism

$$\psi\colon M \to M'$$

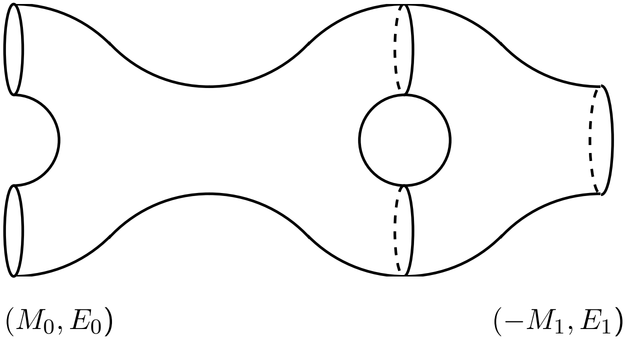
preserving the Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E.$$

Bordism (M_0, E_0) is **bordant** to (M_1, E_1) iff $\exists$ (Ω, E) such that:

1. Ω is a compact odd-dimensional Spin^c manifold with boundary.
2. E is a $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, E|_{\partial\Omega}) \cong (M_0, E_0) \sqcup (-M_1, E_1,)$

$-M_1$ is M_1 with the Spin^c structure reversed.


(M_0, E_0)
$(-M_1, E_1)$

Direct sum - disjoint union

Let E, E' be two $\mathbb{C}$ vector bundles on M

$$(M, E) \sqcup (M, E') \sim (M, E \oplus E')$$

Vector bundle modification

(M, E)

Let F be a Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1}_{\mathbb{R}} = M \times \mathbb{R}$$

$$S(F \oplus \mathbf{1}_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus \mathbf{1}_{\mathbb{R}}$$

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \Big\downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers.

$F \oplus \mathbf{1}_{\mathbb{R}}$ is a Spin^c vector bundle on M with odd-dimensional fibers.

The Spin^c structure for F causes there to appear on $S(F \oplus 1_{\mathbb{R}})$ a $\mathbb{C}$-vector bundle β whose restriction to each fiber of π is the Bott generator vector bundle of that even-dimensional sphere.

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

Addition in $\mathrm{K}_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $\mathrm{K}_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

DEFINITION. (M, E) *bounds* $\iff \exists\, (\Omega, \widetilde{E})$ with :

1. Ω is a compact odd-dimensional Spin^c manifold with boundary.
2. $\widetilde{E}$ is a $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, \widetilde{E}|_{\partial\Omega}) \cong (M, E)$

REMARK. $(M, E) = 0$ in $K_0(\cdot) \iff (M, E) \sim (M', E')$ where (M', E') bounds.

Consider the homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

Notation

D_E is the Dirac operator of M tensored with E.

It is a corollary of Bott periodicity that this homomorphism of abelian groups is an isomorphism.

Equivalently, Index(D_E) is a complete invariant for the equivalence relation generated by the three elementary steps; i.e. $(M, E) \sim (M', E')$ if and only if $\mathrm{Index}(D_E) = \mathrm{Index}(D'_{E'})$.

BOTT PERIODICITY

$$\pi_j GL(n, \mathbb{C}) = \begin{cases} \mathbb{Z} & \text{j odd} \\ \\ 0 & \text{j even} \end{cases}$$

$$j = 0, 1, 2, \ldots, 2n-1$$

Why does Bott periodicity imply that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism?

To prove surjectivity must find an (M, E) with $\mathrm{Index}(D_E) = 1$.

e.g. Let $M = \mathbb{C}P^n$, and let E
be the trivial (complex) line bundle on $\mathbb{C}P^n$
$E{=}1_{\mathbb{C}} = \mathbb{C}P^n \times \mathbb{C}$
$\mathrm{Index}(\mathbb{C}P^n, 1_{\mathbb{C}}) = 1$

Thus Bott periodicity is not used in the proof of surjectivity.

Lemma used in the Proof of Injectivity

Given any (M, E) there exists an even-dimensional sphere S^{2n} and a $\mathbb{C}$-vector bundle F on S^{2n} with $(M, E) \sim (S^{2n}, F)$.

Bott periodicity is not used in the proof of this lemma.
The lemma is proved by a direct argument using the definition of the equivalence relation on the pairs (M, E).

Let r be a positive integer, and let $\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r)$
be the set of isomorphism classes of $\mathbb{C}$ vector bundles on S^{2n}
of rank r, i.e. of fiber dimension r.

$$\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1} GL(r, \mathbb{C})$$

PROOF OF INJECTIVITY

Let (M, E) have $\mathrm{Index}(M, E) = 0$.

By the above lemma, we may assume that $(M, E) = (S^{2n}, F)$.

Using Bott periodicity plus the bijection

$$\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1} GL(r, \mathbb{C})$$

we may assume that F is of the form

$$F = \theta^p \oplus q\beta$$

$\theta^p = S^{2n} \times \mathbb{C}^p$ and β is the Bott generator vector bundle on S^{2n}.

Convention. If $q < 0$, then $q\beta = |q|\beta^*$.

$\mathrm{Index}(S^{2n}, \beta) = 1 \quad \mathrm{Index}(S^{2n}, \theta^p) = 0$
Therefore

$$\mathrm{Index}(S^{2n}, F) = 0 \Longrightarrow q = 0$$

Hence $(S^{2n}, F) = (S^{2n}, \theta^p)$. This bounds

$$(S^{2n}, \theta^p) = \partial(B^{2n+1}, B^{2n+1} \times \mathbb{C}^p)$$

and so is zero in $\mathrm{K}_0(\cdot)$.
QED

Define a homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Q} \\ (M, E) &\longmapsto \big(ch(E) \cup \mathrm{Td}(M)\big)[M] \end{aligned}$$

where $ch(E)$ is the Chern character of E and $\mathsf{Td}(M)$ is the Todd class of M.

$ch(E) \in H^*(M, \mathbb{Q})$ and $\mathsf{Td}(M) \in H^*(M, \mathbb{Q})$.

$[M]$ is the orientation cycle of M. $[M] \in H_*(M, \mathbb{Z})$.

Granted that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism, to prove that these two homomorphisms are equal, it suffices to check one nonzero example.

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.

$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma) \in \mathrm{K}_0(\cdot)$, and

$$\mathrm{Index}(D_{E_\sigma}) = \mathrm{Index}(\delta).$$

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma)$$

$$\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup \mathrm{Td}((S(TX \oplus 1_{\mathbb{R}})))[(S(TX \oplus 1_{\mathbb{R}})]$$

and this is the general Atiyah-Singer formula.

$S(TX \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $TX \oplus 1_{\mathbb{R}}$.
$S(TX \oplus 1_{\mathbb{R}})$ is even dimensional and is — in a natural way — a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(TX \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.