Boundaries of mapping class groups

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Γ a finitely generated infinite group

- Is there a way to compactify Γ?
- Can one obtain information on Γ from the boundary of such a compactification?
- Can one obtain such a compactification from geometric information on a space on which the group acts properly by isometries?

Example

1) A *surface group* Γ is a cocompact torsion free lattice in the group $PSL(2,\mathbb{R})$ acting freely, properly and cocompactly on the hyperbolic plane $\mathbb{H}^2$.
2) Γ acts on a Cayley graph for some finite symmetric generating set.

The hyperbolic plane $\mathbb{H}^2$:

- $\mathbb{H}^2$ is contractible.
- The *hyperbolic metric* g on $\mathbb{H}^2$ is $PSL(2, \mathbb{R})$-invariant.
- g is a complete Riemannian metric of constant curvature -1.
- A geodesic ray for g starting from a fixed point $x \in \mathbb{H}^2$ is globally minimizing, and two distinct such geodesic rays γ, η diverge linearly: $d(\gamma(t), \eta(t)) \geq 2t - C$.
- $\mathbb{H}^2$ can be compactified by adding a *circle* S^1 *at infinity*, the space of geodesic rays starting from x. The closure $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup S^1$ is homeomorphic to a closed disk in $\mathbb{C}$.
- The isometric action of $PSL(2, \mathbb{R})$ extends to an action on $\overline{\mathbb{H}^2}$ by homeomorphisms.

A surface group Γ acting on $\mathbb{H}^2$:

The action of Γ on

- $\mathbb{H}^2$ is free and cocompact
- on $S^1 = \partial\overline{\mathbb{H}^2}$ is *minimal*
- on S^1 is *strongly proximal*
- on S^1 is *topologically free*
- $\mathcal{U}$-small for every open covering $\mathcal{U}$ of $\overline{\mathbb{H}^2}$.
- $H^1(S^1, \mathbb{Z}) = H^2(\Gamma, \mathbb{Z}) = \mathbb{Z}$.

Definition

An action of a group Γ by homeomorphisms on a space X is

- *minimal* if every Γ-orbit is dense
- *strongly proximal* if the closure of every Γ-orbit on the space $\mathcal{P}(X)$ of probability measures on X contains a Dirac measure
- topologically free if the fixed point set of any $\gamma \neq \mathrm{Id}$ has empty interior.

Example

A *non-elementary hyperbolic group* Γ is group which is not virtually cyclic and whose Cayley graph for some finite generating set satisfies the thin triangle condition.

The Gromov boundary ∂X of a hyperbolic space X is a set of equivalence classes of sequences (x_i) with $d(x_0, x_i) \to \infty$ $(i \to \infty)$ and

$$(x_i \mid x_j)_{x_0} = \frac{1}{2}(d(x_i, x_0) + d(x_0, x_j) - d(x_i, x_j)) \to \infty \, (i, j \to \infty)$$

If X is proper then ∂X is compact. If Γ acts on X properly as a group of isometries, then this action extends to an action on ∂X.

If $\gamma \in \Gamma$ is *loxodromic*, then γ acts on ∂X with north-south dynamics

Definition

The dimension of a compact metrizable space X is the smallest number n such that any open cover $\mathcal{U}$ of X admits a refinement $\mathcal{V}$ such that any point in X is contained in at most $n+1$ sets from $\mathcal{V}$.

Definition

A compact, metrizable, finite dimensional, contractible and locally contractible space is an Euclidean retract.

Definition

A $\mathcal{Z}$-structure for a torsion free group Γ is a pair $(\overline{X}, Z)$ of spaces s.th. $\overline{X}$ is an Euclidean retract, $X = \overline{X} - Z$ admits a covering space action of Γ with compact quotient, every point in Z has a neighborhood basis in $\overline{X}$ consisting of sets whose intersections with X are contractible, and the Γ-action is $\mathcal{U}$-small for every open cover $\mathcal{U}$ of $\overline{X}$.

Example

Let M be a closed Riemannian n-manifold of non-positive curvature.
The universal covering $\tilde{M}$ of M is diffeomorphic to $\mathbb{R}^n$ and can be compactified to a $\pi_1(M)$-space B^n homeomorphic to a compact n-ball.
$(B^n, \partial B^n)$ is a $\mathcal{Z}$-structure for $\pi_1(M)$.

Example

(Bestvina-Mess) Let Γ be a torsion free hyperbolic group, with Gromov boundary $\partial\Gamma$. The pair $(\mathcal{R}(\Gamma), \partial\Gamma)$ is a $\mathcal{Z}$-structure for Γ where $\mathcal{R}(\Gamma)$ is the *Rips complex*.

Let Γ be a torsion free group which acts properly and cocompactly on a $\mathrm{CAT}(0)$ cube complex X. Then the visual boundary ∂X of X is finite dimensional (Swenson), and $(X, \partial X)$ is a $\mathcal{Z}$-structure for Γ.

The mapping class group $\mathrm{Mod}(S)$ of a closed surface S of genus $g \geq 2$ is the group of isotopy classes of diffeomorphism of S.

Fact: $\mathrm{Mod}(S)$ admits torsion free subgroups Γ of finite index.

Theorem

Γ admits an explicit $\mathcal{Z}$-structure $(\overline{X}, Z)$. The action of Γ on Z is minimal, strongly proximal and topologically free. If $S_0 \subset S$ is an essential subsurface, then the boundary of $\Gamma \cap \mathrm{Mod}(S_0)$ is embedded in Z.

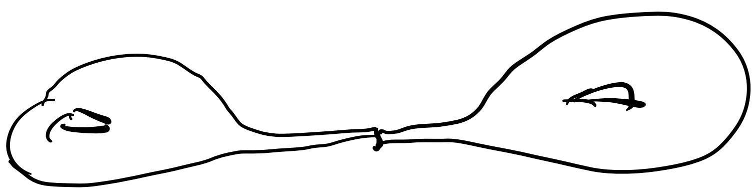
Idea: Use the strategy for $\mathrm{CAT}(0)$-groups

Definition

Teichmüller space $\mathcal{T}(S)$ for S is the space of all *marked hyperbolic metrics* on S. It is diffeomorphic to $\mathbb{R}^{6g-6}$, and $\mathrm{Mod}(S)$ acts properly on $\mathcal{T}(S)$ by change of markings.

Problem: The action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is not cocompact.

The *systole* of a hyperbolic surface X is a shortest closed geodesic on X. The systole of X may be arbitrarily small.



For small $\epsilon > 0$ define

$$\mathcal{T}_\epsilon(S) = \{X \in \mathcal{T}(S) \mid \ell_X(\mathrm{sys}(X)) \geq \epsilon\}$$

where ℓ_X denotes the length for the metric X.

Theorem (Ji-Wolpert)

$\mathcal{T}_\epsilon(S)$ is a manifold with corners which is a $\mathrm{Mod}(S)$-equivariant deformation retract of $\mathcal{T}(S)$. $\mathrm{Mod}(S)$ acts on $\mathcal{T}_\epsilon(S)$ properly and cocompactly.

Definition

A *geodesic lamination* on S is a *compact* subset of S which is foliated by *simple* geodesics.
A lamination is *minimal* if every leaf is dense.
It *fills* S is every complementary component is simply connected (or a boundary parallel annulus if S has boundary).

Theorem (Masur-Minsky, Klarreich, H-)

if they are disjoint

The curve graph $\mathcal{CG}(S)$ of S whose vertices are simple closed curves and where two curves are connected by an edge is hyperbolic. Its Gromov boundary is the space of minimal filling laminations, equipped with the coarse Hausdorff topology.

Definition

A sequence (ξ_i) of geodesic laminations *converges in the coarse Hausdorff topology* to ξ if every limit of a subsequence which converges in the Hausdorff topology contains ξ as a sublamination.

$$Z = \{\sum_i a_i\xi_i \mid \xi_i \text{ fills } S_i, S_i \cap S_j = \emptyset (i \neq j), a_i \geq 0, \sum_i a_i = 1\}$$

Proposition

There exists a finite dimensional uniformly locally finite $\mathrm{CAT}(0)$ *cube complex* X *with visual boundary* Z*. The visual topology on* Z *is invariant under the action of* $\mathrm{Mod}(S)$*.*

Consequence: Z with this topology is finite dimensional, compact and metrizable.

Theorem

There exists a $\mathrm{Mod}(S)$*-invariant topology on* $\mathcal{T}_\epsilon(S) \cup Z$ *which restricts to the given topologies on* $\mathcal{T}_\epsilon(S)$ *and* Z*. The pair* $(\mathcal{T}_\epsilon(S), Z)$ *is a* $\mathcal{Z}$*-structure for* $\Gamma < \mathrm{Mod}(S)$*.*

The *virtual cohomological dimension* of $\mathrm{Mod}(S)$ equals

$$\mathrm{vcd}(\mathrm{Mod}(S)) = \sup\{n \mid H^n(\Gamma, M) \neq 0 \text{ for some } \Gamma - \text{module } M\}$$

where Γ is a torsion free f.i. subgroup

Fact: $\mathrm{vcd}(\Gamma)$ is the minimal dimension of a $K(\Gamma, 1)$-complex.

Theorem (Harer)

$\mathrm{vcd}(\mathrm{Mod}(S)) = 4g - 5.$

Fact: If $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure for Γ then

$$\tilde{H}^q(Z) = H_c^{q+1}(X) = H^{q+1}(\Gamma; \mathbb{Z}\Gamma) \quad \forall q$$

Fact: If Z is a compact space of finite covering dimension, then $\dim(Z) = \dim_{\mathbb{Z}}(Z)$.

Corollary (Gabai, Bestvina-Bromberg)

$\dim(\partial\mathcal{CG}(S)) \leq 4g - 6$.

Some ideas of proof:

Open questions:

1. The *asymptotic dimension* of a metric space X is defined to be $\leq n$ if for every $R > 0$ there exists a cover $\mathcal{U}$ of X by sets of uniformly bounded diameter and such that each closed ball of radius R intersects at most $n+1$ sets from $\mathcal{U}$.
   Bestvina-Bromberg-Fujiwara: $\mathrm{asym}(\mathrm{Mod}(S)) < \infty$.
   Conj: $\mathrm{asym}(\mathrm{Mod}(S)) = 4g - 5$.
2. Harer: Homological stability: In degree $k \leq 2g_0/3$, the homology $H_k(\mathrm{Mod}(S), \mathbb{Q})$ equals the homology of $\mathrm{Mod}(S_0)$ for a subsurface S_0 of genus $\geq g$.
   Church-Farb: $H_{4g-5}(\mathrm{Mod}(S), \mathbb{Q}) = 0$
   Is there homological stability with coefficients? Can one describe odd degree nontrivial homology classes?

3. Does $\mathrm{Mod}(S)$ have *property T*?
4. Is it true that for every finite index subgroup Γ of $\mathrm{Mod}(S)$ we have $H^1(\Gamma, \mathbb{Q}) = 0$?.
5. Can one construct $\mathcal{Z}$-structures for torsion free non-uniform lattices in higher rank Lie groups of non-compact type?
6. Is there a $\mathcal{Z}$-structure for $\mathrm{Out}(F_n)$? (Bestvina-Horbez)
7. If Γ is a torsion free *acylindrically hyperbolic group* with a $\mathcal{Z}$-structure Z, can one equivariantly embed the Gromov boundary of the hyperbolic space into Z? Mahan.

8. Is there a Markov chain s.th. the bd. of Γ for this chain equals the bd. of Γ?