# $p$-adic Methods in Number Theory

**Tata Institute of Fundamental Research**

November 25 – 29, 2024

## 1 Kenichi Bannai (Keio University, Japan)

TITLE: **On the Shintani Generating Class and special values of Hecke $L$-functions for totally real fields**

ABSTRACT: The classical Dirichlet $L$-function associated to a Dirichlet character may be expressed as a linear sum of Lerch Zeta Functions, and there exists a rational function on the multiplicative group which is a universal generating function for special values of the Lerch Zeta Functions at various integer points and roots of unity. Generalizing this result to the case of totally real fields, we re-interpreted the classical result of Takuro Shintani to construct a certain universal equivariant class which we call the Shintani Generating Class on an algebraic torus associated to a totally real field which universally generates the special values of the Lerch Zeta Function at various integer points and torsion points of the algebraic torus. We give a construction of the polylogarithm in this case, and give some observations about the relation of this class to the Beilinson conjecture for Hecke characters of totally real fields (joint work with Hohto Bekki, Kei Hagihara, Tatsuya Ohshita, Kazuki Yamada and Shuji Yamamoto).

## 2 Ashay Burungale (University of Texas, USA)

TITLE: **A local sign decomposition for symplectic self-dual Galois representations,** II

ABSTRACT: The eponymous Galois representations naturally arise in arithmetic geometry and the backdrop of automorphic forms. The Bloch-Kato conjecture suggests the associated epsilon factors to be pivotal to their arithmetic. We present a functorial decomposition of Galois cohomology of such $p$-adic representations of $G_{\mathbb{Q}_p}$ of rank two, mirroring an underlying symplectic structure.

The decomposition has local as well as global applications. This includes: compatibility of the Mazur-Rubin arithmetic local constant with the epsilon factor, parity conjecture at primes of non-semistable reduction, and framework for an integral Iwasawa theory at non-semistable primes in terms of a $p$-adic $L$-function. The decomposition encodes $p$-adic variation of Bloch-Kato subgroups of $p$-adic families of de Rham representations in terms of epsilon factors.

In the case of induced representations the decomposition recovers a 1987 conjecture of Rubin on Galois cohomology of anticyclotomic local units, and its ramified counterpart. The former was established by us in 2021 (with S. Kobayashi, K. Ota), and the latter yields the framework for Iwasawa theory alluded to above. This is a joint work with S. Kobayashi, K. Nakamura and K. Ota.

## 3 Kazim Büyükboduk (University College Dublin, Ireland)

TITLE: **On the behaviour of $p$-adic $L$-functions at critical points**

ABSTRACT: In the first half of this talk, I will review basic constructions (which may be considered as an incarnation of Hida's $p$-adic Rankin–Selberg method) from joint work with D. Benois on $p$-adic $L$-functions about a theta-critical point $x$ on the $GL_2$-eigencurve. In the second part, I will propose a definition (à la Perrin-Riou) of the adjoint $p$-adic $L$-function about $x$, focusing on the difficulties that are germane to this scenario.

## 4 Anand Chitrao (Harish-Chandra Research Institute, Prayagraj)

TITLE: **Reductions mod $p$ of semi-stable representations**

ABSTRACT: We describe a new method to compute the reductions mod $p$ of irreducible two-dimensional semi-stable representations of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. This method uses the compatibility with respect to reduction mod $p$ between the $p$-adic Local Langlands Correspondence and the Iwahori theoretic version of the mod $p$ Local Langlands Correspondence. By estimating certain logarithmic functions on $\mathbb{Q}_p$ by polynomials on open subsets of $\mathbb{Z}_p$, we compute the reductions mod $p$ completely for weights at most $p+1$. We also state how this method can be used, in theory, to compute the reductions mod $p$ of semi-stable representations of arbitrarily large weights. This talk is based on a joint work with Eknath Ghate. For details, please refer to `https://arxiv.org/abs/2311.03740`.

## 5 Michael Harris (Columbia University, USA)

TITLE: **Square root $p$-adic $L$-functions**

ABSTRACT: I explain the construction of a $p$-adic analytic function interpolating the normalized central critical values of the function $L(s, \pi \times \pi')$, where $\pi$ is a holomorphic automorphic representation of $U(V)$, $V$ is a hermitian space of dimension $n$, and $\pi'$ is a holomorphic automorphic representation of $U(V')$ where $V'$ is of codimension 1 in $V$. The construction is based on the Ichino-Ikeda formula and was completed a few years ago, using a method that goes back to Hida's construction of $p$-adic Rankin-Selberg $L$-functions. More recently, there have been more complete results when $n = 3$, due to Hsieh-Yamana when $V$ is positive definite and to myself with Hsieh and Yamana when both $V$ and $V'$ are indefinite. At the end of the talk I will explain how the $p$-adic Fourier theory of Graham, Howe, and van Hoften clarifies the computation of the local Euler factor at $p$.

## 6 Haruzo Hida (University of California, Los Angeles, USA)

TITLE: **Iwasawa's dream** (Colloquium talk)

ABSTRACT: We first discuss Conjectures/Problems (in cyclotomic theory) Iwasawa described in one of his final unpublished manuscripts. Then if time allows, we indicate which of his problems has generalizations in more general settings of adjoint Selmer groups via the theory of modular forms. In the general case, the Iwasawa algebra is replaced by a universal deformation ring (which is a $p$-adic Hecke algebra by a theorem of Taylor–Wiles). We encounter new interesting features related to Iwasawa's question in the general case.

## 7 Ming-Lun Hsieh (National Taiwan University, Taiwan)

TITLE: **Hida families of Yoshida lifts and the congruence between modular forms**

ABSTRACT: In this talk, I will report on a work in progress joint with Zheng Liu. I will first give a construction of a particular Hida family of Yoshida lifts and explain its non-vanishing mod $p$. Then I will discuss how to establish a non-trivial congruence between Yoshida lifts and stable forms by applying Furusawa's pull-back formula.

## 8 Wansu Kim (Korean Advanced Institute of Science and Technology, South Korea)

TITLE: **On degeneration of $D$-shtukas over ramified legs**

ABSTRACT: Shimura varieties associated to a reductive group totally anisotropic modulo centre are always compact, and the "canonical" integral models are also expected to be proper. However, the analogous property does not hold for the moduli of $G$-shtukas in the function field setting. Given a curve $X$ over a finite field and a parahoric group scheme $G$ over $X$ associated to a maximal order of a central division algebra $D$ over the function field of $X$, Eike Lau obtained a numerical criterion for properness of the moduli of "bounded $G$-shtukas" (with legs in the split locus for $D$), and showed that the leg morphism may not be proper.

By the work of Arasteh Rad-Hartl and Bieker, one can extend the notion of bounded $G$-shtukas to allow legs in the ramification locus for $D$. We extend the result of Lau to the case when the legs are allowed to be in the ramification locus, and show in particular that the moduli of ($r$-legged) $G$-shtukas could be proper when the legs are restricted to the split locus of $D$, but not proper over $X^r$. This is joint work with Yong-Gyu Choi and Junyeong Park.

## 9 Shinichi Kobayashi (Kyushu University, Japan)

TITLE: **A local sign decomposition for symplectic self-dual Galois representations, I**

ABSTRACT: The eponymous Galois representations naturally arise in arithmetic geometry and the backdrop of automorphic forms. The Bloch-Kato conjecture suggests the associated epsilon factors to be pivotal to their arithmetic. We present a functorial decomposition of Galois cohomology of such $p$-adic representations of $G_{\mathbb{Q}_p}$ of rank two, mirroring an underlying symplectic structure.

The decomposition has local as well as global applications. This includes: compatibility of Mazur-Rubin arithmetic local constant with the epsilon factor, parity conjecture at primes of non-semistable reduction, and framework for an integral Iwasawa theory at non-semistable primes in terms of a $p$-adic $L$-function. The decomposition encodes $p$-adic variation of Bloch-Kato subgroups of $p$-adic families of de Rham representations in terms of epsilon factors.

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## 10 Arthur-César Le Bras (Université de Strasbourg, France)

TITLE: **Whittaker patterns**

ABSTRACT: Automorphic forms can be analyzed through their Fourier expansions. This leads, in representation-theoretic terms, to the theory of Whittaker and Kirillov models. I'd like in this talk to explain how such ideas still play a fruitful role in the geometrization of the Langlands program, both in the original global function field setting and in its much more recent local avatars.

## 11 Aprameyo Pal (Harish-Chandra Research Institute, Prayagraj)

TITLE: **On the Artin formalism for triple product $p$-adic $L$-functions**

ABSTRACT: Dasgupta has proved the factorization of the tensor square $p$-adic $L$-function of a $p$-ordinary modular form using an interplay between the Euler system of $GL_2 \times GL_2$ (Beilinson-Flach class) and the Euler system of circular units. Along with Kazim Büyükboduk, Daniele Casazza, and Carlos de Vera-Piquero, we show that the factorization formula ($p$-adic Artin formalism) for triple product $p$-adic $L$-functions will hold assuming some conjectures on the Gross-Zagier formula for the triple product of modular forms. This is done via a comparison of Chow-Heegner points (coming from diagonal cycles) and twisted Heegner points in Hida families, via formulae of Gross-Zagier type.

## 12 Chol Park (Ulsan National Institute of Science and Technology, South Korea)

TITLE: **Families of strongly divisible modules of rank** 2

ABSTRACT: Strongly divisible modules correspond to Galois-stable lattices within semi-stable representations with Hodge–Tate weights in the Fontaine–Laffaille range. In this talk, we introduce pseudo-strongly divisible modules of rank 2 and provide explicit conditions under which these modules become genuine strongly divisible modules. These conditions are formulated in terms of basic equations and inequalities. This is a joint work with Seongjae Han.

## 13 Julian Quast (Universität Duisburg-Essen, Germany)

TITLE: **On local Galois deformation rings**

ABSTRACT: In joint work with Vytautas Paškūnas, we show that the universal framed deformation ring of an arbitrary mod $p$ representation of the absolute Galois group of a $p$-adic local field valued in a possibly disconnected reductive group $G$ is flat, local complete intersection and of the expected dimension. In particular, any such mod $p$ representation has a lift to characteristic 0. The work extends results of Böckle, Iyengar and Paškūnas in the case $G = GL_n$. We give an overview of the proof of this main result.

## 14 V. Ravitheja (Indian Institute of Science, Bengaluru)

TITLE: **$p$-adic Rankin Selberg $L$-function**

ABSTRACT: The $p$-adic Rankin Selberg $L$-function associated with two cusp forms was independently constructed by H. Hida and A. Panchishkin. We noticed that there is a sign discrepancy between the interpolation formulae of Hida and Panchishkin. In this talk, we address this inconsistency by fixing a sign error in Panchishkin's original argument, without which it does not seem possible to verify the Kummer congruences and obtain the $p$-adic $L$-function. This is a joint work with E. Ghate.

## 15 Tristan Ricoul (Université Sorbonne Paris Nord, France)

TITLE: **Integral period relations for real quadratic base changes to $GL_3$**

ABSTRACT: A few years ago, Tilouine and Urban proved a conjecture of Hida regarding an integral period relation for the base change of a classical cusp form to some real quadratic field $E$. In this talk, we will be interested in the generalization of these results to $GL_3$. In this case, along with Arthur-Clozel's base change, there is Rogawski's stable base change from the quasi-split unitary group associated with $E$. We will formulate some similar conjectural integral period relations, for these two base changes. These relations involve a new kind of automorphic period, defined using the middle degree cuspidal cohomology of $GL_3$ over $E$, instead of the bottom or top degrees. We will then explain how to establish one divisibility toward these conjectures.

## 16 Hae-Sang Sun (Ulsan National Institute of Science and Technology, South Korea)

TITLE: **Non-vanishing mod $p$ of Dirichlet $L$-values for characters with large kernels**

ABSTRACT: In this talk, I will discuss how to study the residual non-vanishing problem for Dirichlet $L$-values with characters whose kernels are unbounded as the conductors grow. Main ingredients are a dynamical reformulation of the problem and a study on the spectral properties of relevant transfer operators. This is ongoing research joint with A. Burungale.

## 17 Jacques Tilouine (Université Sorbonne Paris Nord, France)

TITLE: **Integral period relations for imaginary quadratic base changes**

ABSTRACT: In a previous work with E. Urban, we proved integral period relations for quadratic base changes for $GL_2$. In the present joint work with B. Balasubramanyam, we study the case of imaginary quadratic base changes for $GL_n$. We establish two generalizations of our results with E. Urban. One result is valid for any odd $n$, but more precise for $n = 3$, while the other requires rather essentially (for the moment) the assumption $n = 3$.

## 18 Eric Urban (Columbia University, USA)

TITLE: **Zeta elements for adjoint modular Galois representations in the supersingular case**

ABSTRACT: In a recent joint work with S. Iyengar, C. Khare and J. Manning, we use their notion of congruence modules in higher codimension to give a new construction of the bottom class of the rank $d = [F : \mathbb{Q}]$ Euler system attached to nearly ordinary Hilbert modular forms for a totally real field $F$ that I constructed a few years ago. In this talk, I will discuss the non ordinary case for elliptic cusp forms and the construction of Zeta elements in the exterior square of the Galois cohomology of adjoint modular Galois representations via an integral control theorem for the $p$-adic Hecke algebra in the supersingular case which replaces the classical control theorem in Hida theory used in that previous work.

## 19 Alex Youcis (National University of Singapore, Singapore)

TITLE: **Serre-Tate theory for Shimura varieties of abelian type**

ABSTRACT: The celebrated Serre-Tate theorem says that deformations of an abelian variety are naturally parameterized in terms of deformations of the abelian variety's Barsotti-Tate group. In particular, this says that the natural functor from Mumford's moduli spaces of principally polarized abelian varieties to the moduli stack of Barsotti-Tate groups is formally étale. In this talk I will discuss joint work with Naoki Imai and Hiroki Kato which shows a similar result holds true for integral canonical models of arbitrary Shimura varieties of abelian type (at hyperspecial level), and how this uniquely characterizes such models (at individual level). This involves the construction of a 'syntomic realization functor' on such integral canonical models.