A Glimpse into connections between PDEs and Fourier

Vigyan Vidhushi Program, 2023
TIFR, Mumbai

Mythily Ramaswamy
ICTS - TIFR, Bangalore

Contents

Joseph Fourier . (1768 - 1830)

Fourier - a brief biography

- A French Mathematician and Physicist ;
- Worked in prestigious institutes like Ecole Normale and Ecole Polytechnique.
- He was close to Napoleon Bonaparte and was appointed as Governor of Grenoble; His experiments on heat conduction started.
- In 1807, he submitted his work " Theory of propagation of heat in solid bodies" to Institut de France.
- He had proposed a PDE to describe heat conduction and solved it by using an infinite sum of trigonometric series.
- The committee (Laplace, Lagrange, Lacroix, Monge) that reviewed it, dismissed it.

Fourier's work

- One of the objection was that the function to be expanded was not periodic.
- Further, Fourier used the trigonometric sum for even discontinuous functions. This was another objection.
- The treatment lacked rigour according to the committee.
- Even before Fourier, the idea of using an infinite trigonometric sum was used by Daniel Bernoulli, in his work on string equation and probably some others too.
- It was Fourier who boldly claimed that any function, continuous or even discontinous could be expanded this way.
- He also showed how to calculate the coefficients; he further computed a few term and showed the convergence of the series to the function.
- He published his work in 1822 " The Analytic Theory of heat ".

Fourier's work

- He modeled heat conduction in a rod by the PDE

$$\frac{\partial u}{\partial t}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad x \in (0,\pi),\ t > 0,$$

with boundary and initial conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \qquad u(x,0) = f(x).$$

- His solution : $u(x,t) = e^{-c^2k^2t}\text{sin}\ (kx)$, if $f(x) = \text{sin}(kx)$.
- If $f(x) = \sum_1^\infty a_k sin(kx)$, then solution is

$$u(x,t) = \sum_1^\infty a_k e^{-c^2k^2t}\text{sin}\ (kx)$$

- Further, $a_k = \frac{2}{\pi}\int_0^\pi f(x)\text{sin}(kx)dx$.

After Fourier's work

- At the time of Fourier's work, even basic notions like, continuity, convergence, integration lacked rigorous treatment.
- Cauchy (1789 - 1857) contributed a lot to bring rigour to basic mathematical definitions.
- Dirichlet in 1829 proved the convergence of Fourier series to a piecewise monotonic function in a closed bounded interval.
- Riemann (1826 - 1866) investigated the existence the integral defining Fourier coefficients and gave a nearly satisfactory answer.
- One century later, Lebesgue (Thesis in 1902) gave a completely satisfactory answer by developing Lebesgue measure and integration: The integral exists in a bounded interval if the set of discontinuities of the function is of measure zero.

Different Perspectives

Spectral Theory

- For $n \times n$ symmetric matrix A, unit eigenvectors satisfying $Ax = \lambda x$ form an orthonormal basis.
- For the linear operator $\frac{d^2}{dx^2}$, on C^2 functions vanishing at both end points of the interval $(0, \pi)$, eigenfunctions are sine functions :

$$\frac{d^2}{dx^2}(\sin(kx)) \; = \; (-k^2)\sin(kx)$$

- If the inner product is taken to be $< f, g >= \int f(x)g(x)dx$, then in this function space

$$< \frac{d^2 f}{dx^2}, g > = < f, \frac{d^2 g}{dx^2} > .$$

- Fourier's conjecture is a generalization of the fact for symmetric matrices, in a suitable function space.

Different Perspectives

Transforms

- Fourier Series : Transforms a 2π periodic function f defined on $(-\pi, \pi)$ into a function defined on integers : $\hat{f}(k)$.

$$f(x) = \sum_{-\infty}^{\infty} a_k e^{ikx}; \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

- Fourier Transform : Considering function f defined on the whole real line, associate it to

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx; \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi.$$

- FT converts differentiation into a simple algebraic operation ; PDEs converted to ODEs.
- Discrete Fourier Transform: For $a = (a_0, a_1, \cdots, a_{N-1})$

$$F_N(a) \ = \ \sum_{0}^{N-1} e^{-\frac{2\pi imn}{N}} a_n; \quad F_N^{-1}(b) \ = \ \sum_{0}^{N-1} e^{\frac{2\pi imn}{N}} b_n.$$

Different Perspectives

- Fourier Series maps functions on $G = (\mathbb{R}/2\pi\mathbb{Z}, +)$ to functions on $\hat{G} = (\mathbb{Z}, +)$
- For Fourier Transform, G and $\hat{G}$ both are $(\mathbb{R}, +)$.
- In our examples, $E(x, n) = e^{-inx}$ and $E(x, y) = e^{-ixy}$.
- Harmonic Analysis studies the relation between the function and its Fourier transform.
- More generally, if G is a locally compact Abelian group, then there is another group $\hat{G}$, the dual group and a map $E(u, v)$ from $G \times \hat{G}$ into multiplicative group of complex numbers of modulus one, which is a group homomorphism and forward and inverse formulas hold.

Different Perspectives

Other Applications

- Pseudo-differential operators are developed as extensions of partial differential operators :

$$P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(x, \xi)\hat{u}(\xi)\ d\xi$$

- Fourier integral operators, generalizations of differential and integral operators ;
- Applications in Signal Processing, Image compression and many other engineering applications.
- Fast Fourier Transform connects Fourier and Numerical analysis and is very important in applications.
- The completeness of more general exponential families of functions has taken root as Non-Harmonic Analysis having connections to Number Theory.

Orthonormal Basis

Suppose H is a Hilbert space.

$\{e_n\}_1^\infty$ is an orthonormal basis in H:

$$\langle e_i, e_j \rangle = \delta_{i,j}$$

Parseval's identity

For any $f \in H$

$$\|f\|^2 = \sum_1^\infty |\langle f, e_n \rangle|^2$$

Wave Equation

- Model the free vibrations of a string with both end points free.
- Initial Boundary Value Problem for $u(x,t)$, for $(x,t) \in (0,\pi) \times \mathbb{R}$

$$u_{tt} - u_{xx} = 0,$$

with boundary conditions

$$u_x(0,t) = 0, \qquad u_x(\pi,t) = 0$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x).$$

- For PDE's, Fourier Series help in solving both direct and inverse problems.

Wave Equation - Direct Problem

- Separation of variables leads to a solution :

$$u(x,t) = \sum_{k=1}^{\infty} (a_k cos(kt) + b_k sin(kt))\ cos(kx).$$

- Using the initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$,

$$u_0(x) \ = \ \sum a_k \cos(kx), \quad u_1(x) \ = \ \sum k\ b_k \cos(kx)$$

- Then a_k, b_k can be determined in terms of the Fourier coefficients of u_0, u_1.
- Using Parseval's identity, : $\|u\|^2 \leq C(\|u_0\|^2 + \|u_1\|^2)$
- Thus we can determine the solution using given data.

Wave Equation - Inverse Problem

Inverse Problem:

- Suppose that oscillation $u(0,t)$, for $0 \leq t \leq T$ is known.
- Can we determine the unknown initial data u_0, u_1 ?
- Suppose that (u_1, u_0) lies in $H \times V$,

$$H = \{f \in L^2(0,\pi) \mid \int f = 0\}$$

$$V = H^1(0,\pi) = \{f \in H \mid f' \in L^2\}.$$

- We will need to consider $u(0,t)$ in $H^1(0,T)$.

Wave Equation

- If $T \geq 2\pi$, then this map can be shown to be one-one and continuous.
- Again Fourier series helps : Parseval's identity gives

$$\|u_1\|_H^2 = \int_0^\pi |u_t(x,0)|^2 dx = \frac{\pi}{2} \sum_1^\infty k^2 b_k^2$$

$$\|u_0\|_V^2 = \int_0^\pi |u_x(x,0)|^2 dx = \frac{\pi}{2} \sum_1^\infty k^2 a_k^2$$

- Then compute the norm of

$$u(0,t) = \sum_{k=1}^\infty a_k cos(kt) + b_k sin(kt)$$

Wave Equation - Inverse Problem

- The family $\{cos(kt), sin(kt)\}$ forms an orthogonal system in $L^2(0, 2m\pi)$ for any integer m.
- Using Parseval's identity for $u(0,t)$ in $H^1(0,T)$

$$\int_0^T |u_t(0,t)|^2 dt \geq m\pi \sum_1^\infty k^2(a_k^2 + b_k^2)$$

- Here m is the integral part of $\frac{T}{2\pi}$
- Now bound $\int_0^T |u_t(0,t)|^2 dt$ from below by norms of u_0, u_1.
- $\|u_0\|^2 + \|u_1\|^2 \leq C\|u(0,t)\|^2$

Wave Equation with a lower order term

- A slightly different model for $u(x,t)$, for $(x,t) \in (0,\pi) \times \mathbb{R}$

$$u_{tt} - u_{xx} + u = 0, \quad u(x,0) = u_0(x),\; u_t(x,0) = u_1(x),$$

$$u_x(0,t) = 0, \quad u_x(\pi,t) = 0.$$

- Separation of variables leads to a Fourier series

$$u(x,t) = \sum_{1}^{\infty} (a_k cos(\omega_k t) + b_k sin(\omega_k t)) cos(kx)$$

  with $\omega_k = \sqrt{k^2+1}$
- The family $\{cos(\omega_k t), sin(\omega_k t)\}$ is no longer orthogonal in any interval $(0,T)$.
- Can Parseval's identity be relaxed to an inequality for such $\{\omega_k\}$? Under what conditions on the sequence?

Riesz Basis

Suppose now that $\{e_n\}_1^\infty$ is not an orthonormal basis,

but just a Riesz basis :

Image of an orthonormal basis under a linear invertible map.

Then Ingham inequality gives the relation between the norm of the vector and the norm of the coefficients.

Either Direct problems, Inverse Problems,

Or to establish Controllability (steer the trajectory from a given initial state to a desired final state in a given time.)

Statement of Ingham Inequality

Theorem

Let $(\omega_k)_{k\in\mathbb{Z}}$ *be a family of real numbers, satisfying the uniform gap condition*

$$\gamma := inf_{k\neq n}|\omega_k - \omega_n| > 0.$$

If I *is a bounded interval of length* $|I| > \frac{2\pi}{\gamma}$*, there exist positive constants* C_1*,* C_2 *depending on* T, γ *such that for all functions given by*

$$x(t) = \sum_{k\in\mathbb{Z}} x_k e^{i\omega_k t},$$

with $(x_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$ *satisfy*

$$C_1 \sum_{k\in\mathbb{Z}} |x_k|^2 \leq \int_I |x(t)|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} |x_k|^2.$$

Wave Equation

The wave equation $u_{tt} = u_{xx}$ with Dirichlet boundary conditions can be written as an operator equation in $\mathbf{Z} = H_0^1(0,\pi) \times L^2(0,\pi)$ with $\mathbf{U}(t) = (u(t), u_t(t))$

$$\begin{aligned} \frac{d\mathbf{U}(t)}{dt} &= A\mathbf{U}(t), \quad t > 0 \\ \mathbf{U}(0) &= \mathbf{U}_0 \ \in \ \mathbf{Z}. \end{aligned}$$

and domain of A as

$$\mathcal{D}(A) \; = \; (H^2 \cap H_0^1) \times H_0^1(0,\pi)$$

Define $A : \mathcal{D}(A) \to \mathbf{Z}$:

$$A \; = \begin{bmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{bmatrix}.$$

The spectrum of the operator A is $\{\pm in\}_1^\infty$

Compressible Navier-Stokes system

A model for flow of compressible fluid in $\Omega \subset \mathbb{R}$:

Density $\rho(x,t)$, velocity $u(x,t)$ of the fluid in $\Omega \times \ (0,T)$:

$$\rho_t + (\rho u)_x \ = \ 0$$

$$(\rho u)_t \ + \ (\rho u^2)_x \ + \ (p(\rho))_x \ - \nu u_{xx} \ = \ 0.$$

Pressure p is assumed to be

$$p(\rho) \ = \ (a \ \rho^{\gamma}) \ \text{ for } \gamma > 0, a > 0.$$

Initial boundary value problem for the linearized system

- Domain $\Omega = (0, 2\pi)$
- $(\rho_s,\ u_s)$: a constant steady state solution with $\rho_s > 0, u_s > 0$
- Linearized system around this solution :

$$\begin{aligned} \partial_t \rho \ &+\ u_s\ \rho_x \ +\ \rho_s\ u_x \ = 0 \\ \partial_t u \ &-\ \frac{\nu}{\rho_s} u_{xx} + u_s\ u_x + a\gamma\ \rho_s^{\gamma-2} \rho_x \ = \ f \chi_O \end{aligned}$$

with $O \subset \Omega$

- Initial conditions :

$$\rho(x,0) = \rho_0(x)\ ; \quad u(x,0) \ = \ u_0(x)$$

- Periodic boundary conditions for ρ, u and u_x
- Distributed (internal) control : f, localised in a given open set O
- Is the system Controllable/Observable ?

Linearized system at (ρ_s, u_s)

For the linearized system around (ρ_s, u_s) with periodic boundary conditions for ρ, u and u_x in $(0, 2\pi)$

The Point Spectrum of A

- Consists of eigenvalues $\{-\lambda_n\}$, in the left side of the complex plane
- One sequence is

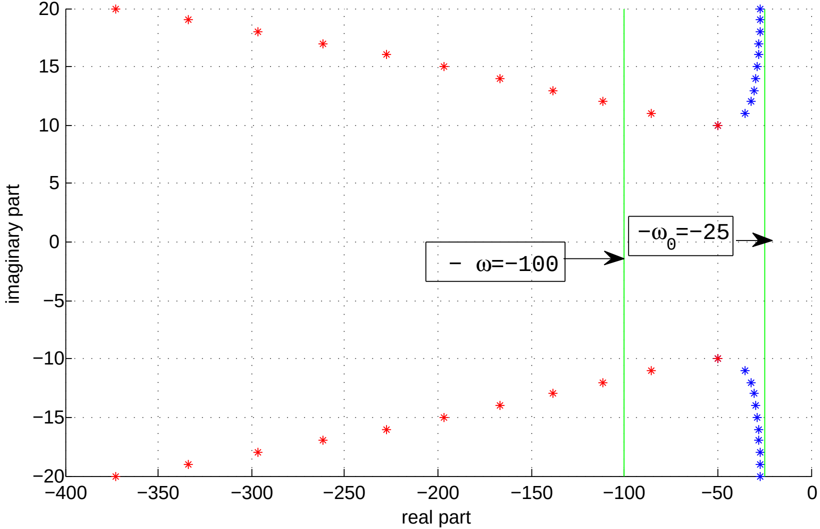
$$\lambda_n^h = \omega_0 - \varepsilon_n^h - i\ n\ u_s$$

  with $\varepsilon_n^h \to 0$, as $|n| \to \infty$, for $n \in \mathbb{Z}$;
- The other sequence is

$$\lambda_n^p = \nu_0 n^2 - \omega_0 + \varepsilon_n^p - i\ n\ u_s$$

  with $\varepsilon_n^p \to 0$, as $|n| \to \infty$, for $n \in \mathbb{Z}$;
- No accumulation point in the spectrum
- Absolute value of the eigenvalues goes to infinity.

$u_s = 1 = \rho_s = a = \nu$, $\gamma = 2.5$, $\omega_0 = 25$ and $\omega = 100$



Ingham type inequalities for "hyperbolic eigenvalues"

Ingham Inequality deals with $\{e^{i\omega_n t}\}$ for ω_n real. Here

$$\omega_n = i\overline{\lambda_n^h} \,,$$

a small perturbation with the addition of a bounded imaginary part!

Theorem

Let $T > \frac{2\pi}{V_0}$. There exist N and positive constants C and C_1 depending on T such that for

$$\lambda_n = \overline{\lambda_n^h} = \omega_0 - \varepsilon_n - inV_0$$

with $\varepsilon_n \to 0$, as $|n| \to \infty$, we have

$$C \sum_{|n|>N} |\alpha_n|^2 \leq \int_0^T |\sum_{|n|>N} \alpha_n e^{-\lambda_n t}|^2 \, dt \leq C_1 \sum_{|n|>N} |\alpha_n|^2.$$

Ingham type inequality for "parabolic" eigenvalues

Unable to use Ingham inequality for the parabolic eigenvalues !

Theorem

Let $\{\lambda_k\}_{k=1}^{\infty}$, $0 = \lambda_1 < \lambda_2 < \cdots$ *be a strictly increasing sequence of non-negative numbers. Then span* $\{x^{\lambda_k}\}$ *is dense in* $C[0,1]$ *if and only if*

$$\sum \frac{1}{\lambda_k} = \infty.$$

Munz Theorem connects topological concept of denseness and

arithmetic concept of series convergence.

The sequence $\{1/n\}$ gives Weirstrass theorem.

Muntz-Szasz Theorem treats complex sequences.

Similar results hold for $\{e^{-\lambda_k}\}$ in $[0,\infty)$ and also in $(0,T)$.

Ingham type inequality for "parabolic" eigenvalues

The closure of the linear span of $\{e^{-\lambda_k t}\}$ is a proper subspace of $L^2(0,T)$ if and only if

$$\sum \frac{1}{\lambda_k} < \infty$$

Then the complement can be shown to be the span of a bi-orthogonal sequence of functions.

Theorem

Let $\sum \frac{1}{\lambda_k} < \infty$. Then for $T > 0$, there exists a positive constant C depending on T, such that uniformly for all $(a_k) \in l^2(C)$ and $T > 0$, we have

$$\sum |a_k|^2 e^{-2\lambda_k T} \leq C \int_0^T |\sum a_k e^{-\lambda_k t}|^2 \, dt$$

Summary

- Fourier's conjecture has led to important advances in mathematics.
- New directions have developed as generalizations of these ideas.
- Parseval's identity useful in the direct and inverse problems for PDE.
- Ingham inequality useful in some inverse problems for PDE.
- It also helps to prove controllability via observability inequality.
- For linearized compressible Navier -Stokes system, a coupled system of parabolic and hyperbolic types, spectrum has 2 sequences of eigenvalues with different behaviour.
- For eigenvalue sequence like $\{in\}$, Ingham inequality helps.
- For eigenvalue sequence like $\{-n^2\}$, Muntz-Szasz type Theorem helps.

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