Is it easy to be fair?

Kavitha Telikepalli

School of Technology and Computer Science
TIFR, Mumbai

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Fair division

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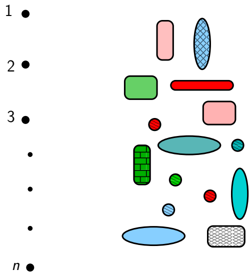

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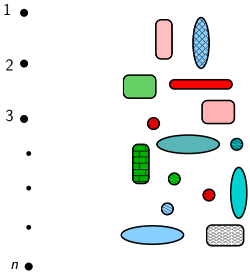

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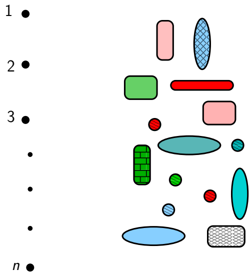


A fair division problem:

- there are n agents
- and there are m goods.

We want to distribute the m goods <u>*fairly*</u> among the n agents.

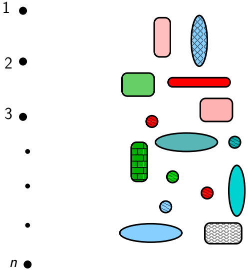
Fair division

Many applications:

- Partnership dissolutions;
- Dividing inheritance and so on.

Check www.spliddit.org for more details.

Fair division



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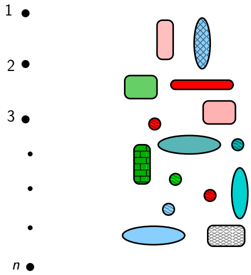
Fair division



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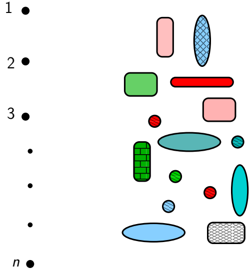

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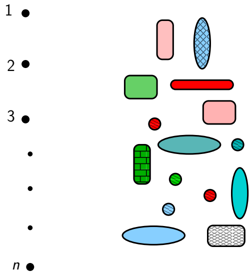
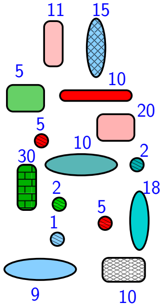


How do we measure fairness?

- Let M be the "grand bundle", i.e., the entire set of m goods.
- Every agent has a value associated with each subset of M.
  - So for every agent i, there is a valuation function $v_i : 2^M \to \mathbb{R}_{\geq 0}$.

An example of a valuation function



An additive valuation v_i: for any subset $S = \{g_1, \ldots, g_k\}$ of M, we have

$$v_i(S) = v_i(g_1) + \cdots + v_i(g_k).$$

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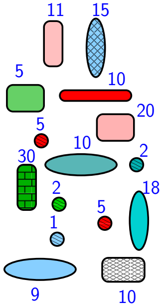

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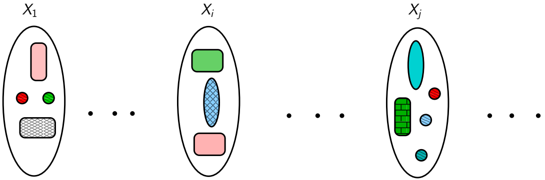
Valuations can be more general – the only rule v_i has to obey is:

- for any $S \subseteq T \subseteq M$, we have $v_i(S) \leq v_i(T)$.

An allocation



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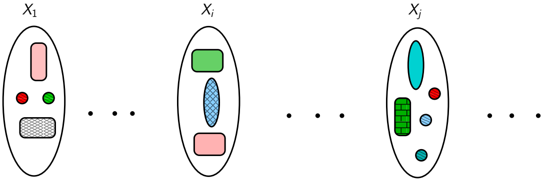

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We say agent i envies agent j if $v_i(X_i) < v_i(X_j)$, i.e., i values X_j more than X_i.

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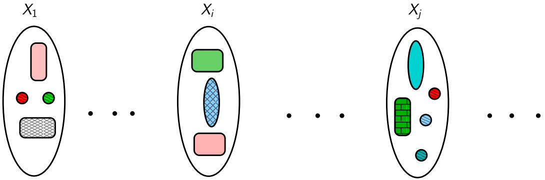

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- For v_i in the previous slide: $v_i(X_i) = 40$ and $v_i(X_j) = 56$; so i envies j.

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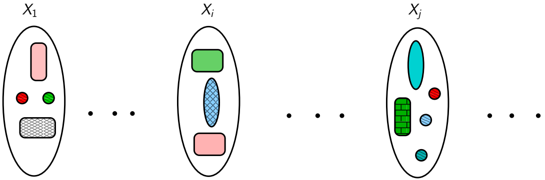

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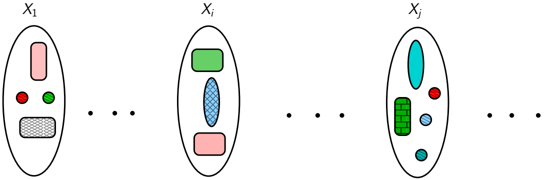
- $\underbrace{v_i(X_i) \geq v_i(X_j)}_{i \text{ likes } X_i \text{ as much as } X_j}$ for all $i, j \Rightarrow \langle X_1, \ldots, X_n \rangle$ is an envy-free allocation.

An envy-free allocation

We want a partition $\langle X_1, \ldots, X_n \rangle$ of M that is envy-free.



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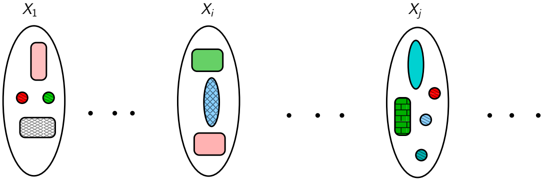

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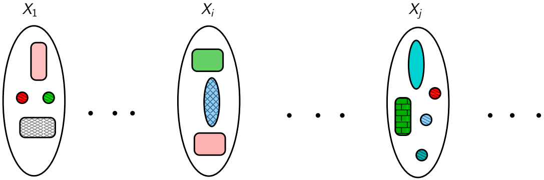

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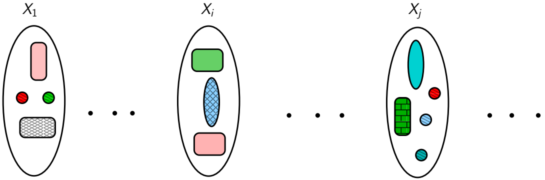

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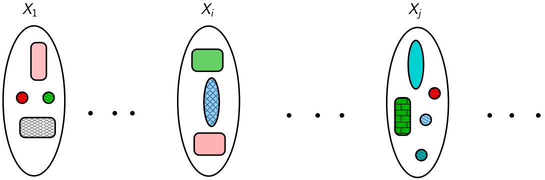

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  - only one of them gets the good and the other agent envies her.

History

Envy-free allocations always exist for two agents and *divisible* goods such as cake.

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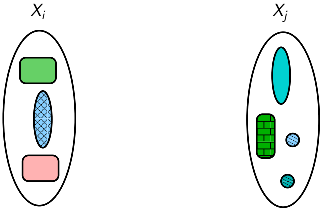
The cut-and-choose protocol: (this dates back to the Bible)

- Abraham partitions the land into two parts;
- Lot chooses which part he would like to keep.

Relaxing envy-freeness for indivisible goods

Agent i may envy agent j, i.e., $v_i(X_i) < v_i(X_j)$



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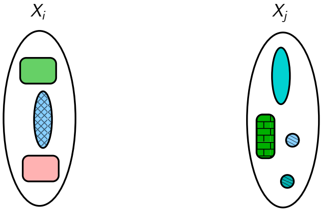

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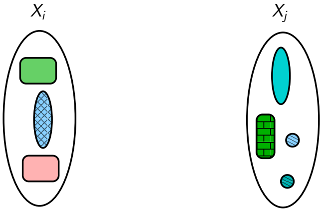

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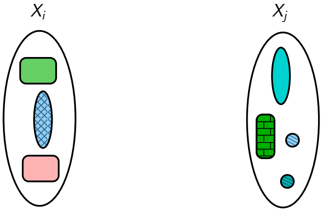

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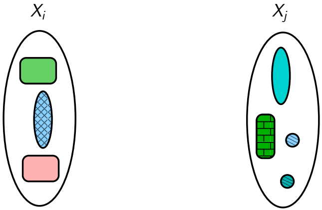

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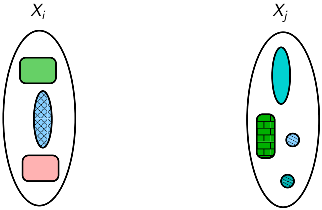

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- Such an allocation is called EF1: envy-free up to *one* good.

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Consider the following instance with additive valuations:

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Agent 1	100	70	20	5
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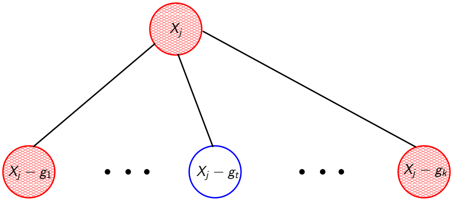

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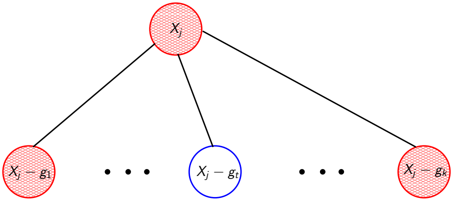

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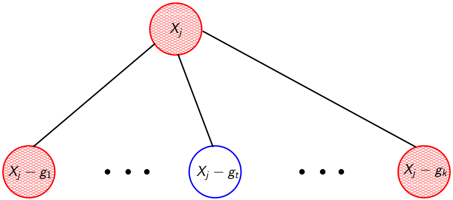

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Good news: An EF1 allocation $X = \langle X_1, \ldots, X_n \rangle$ always exists.

Constructing an EF1 allocation

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Claim: This is an EF1 allocation.

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So we get $X_1 = \{a, c\}$ and $X_2 = \{b, d\}$.

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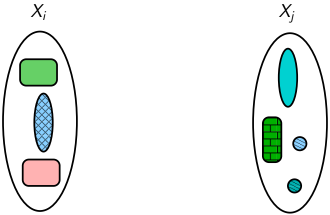
- This is indeed an EF1 allocation.

The correctness of round-robin

Consider any pair of agents i and j:



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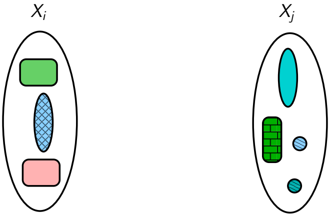

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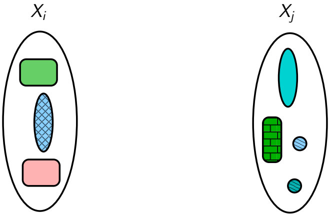

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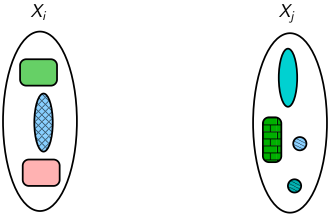

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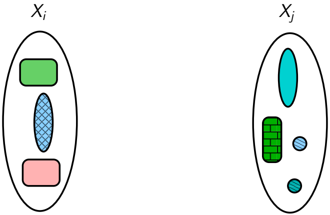

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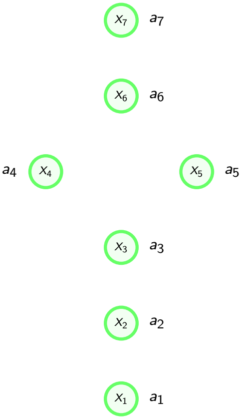
In case 1, i does not envy j.

In case 2, i does not envy j after removing h_1 from X_j, i.e., $v_i(X_i) \geq v_i(X_j - h_1)$.

EF1 for general valuations

The envy graph G: (Lipton, Markakis, Mossel, and Saberi, 2004)

- agents are vertices in G.



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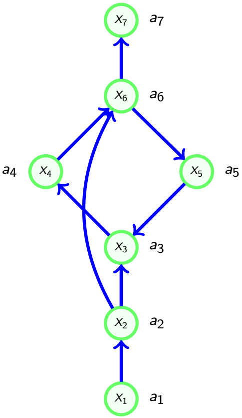
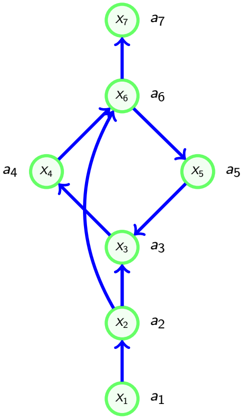


G has an edge from a_i to a_j $\iff$ agent i envies agent j.

The envy graph G

If G has *directed cycles* then we can eliminate them.



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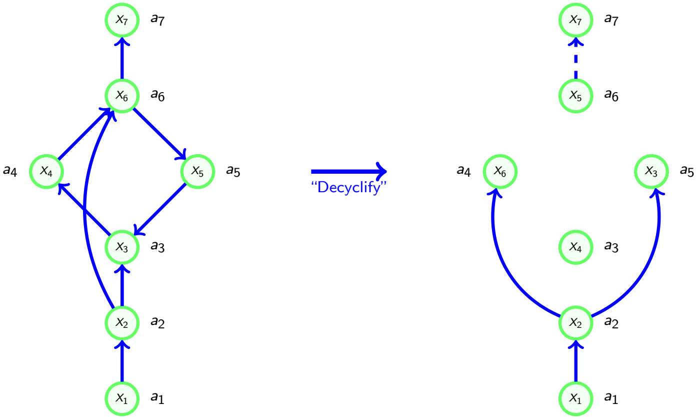

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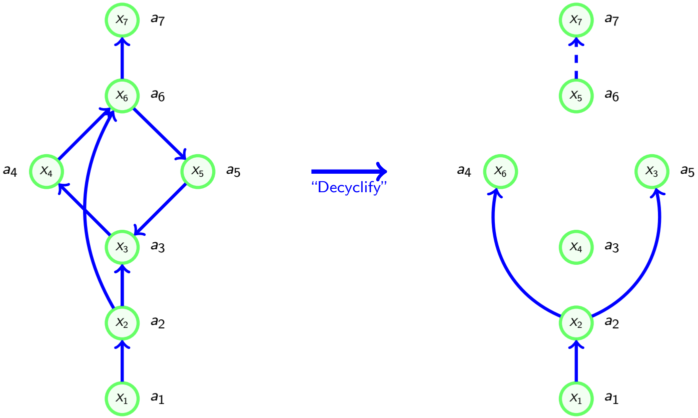

The number of edges decreases.

The EF1 algorithm

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Thus an EF1 allocation can be easily computed.

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- however X is quite unfair towards agent 1;
- the allocation $Y = \langle\{a, b\}, \{c\}\rangle$ seems fairer.

Our old example

Recall the following instance:

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Can we come up with a stronger relaxation of "envy-freeness" that always exists?

EFX

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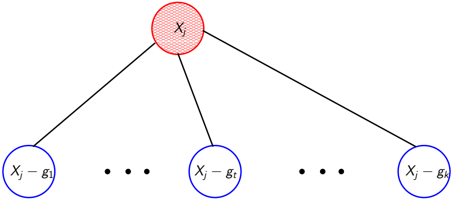

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$$v_i(X_i) \geq v_i(X_j - g) \text{ for all } g \in X_j.$$

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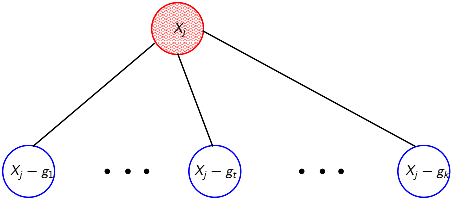

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- EFX is a stronger notion than EF1.

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"*Fair division's biggest problem*." (Ariel Procaccia, 2020)

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- However no such instance is currently known.

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Claim: $X = \langle X_1, \ldots, X_n \rangle$ is EFX.

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  - This allocation is EFX.

Proof of the claim

Let signature$(X) = (v(X_1), |X_1|, v(X_2), |X_2|, \ldots)$.

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The new allocation (with possibly some swapping of bundles) has a larger signature, a contradiction.

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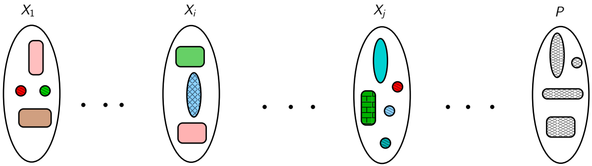
- However finding such an allocation can be hard.

A relaxation of EFX



EFX-with-charity (Caragiannis, Gravin, Huang, 2019)

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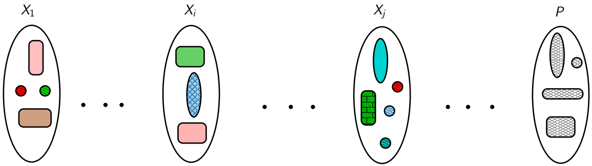

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- partition M into $X_1, \ldots, X_n$ and left-over goods P (the pool) such that:

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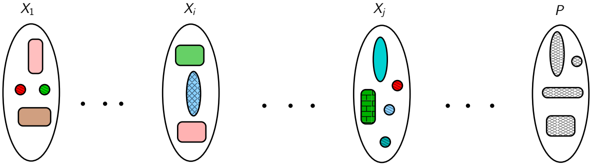

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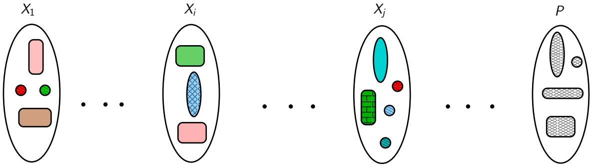
Can we show such an allocation where nobody envies P and the size of P is small?

EFX with charity



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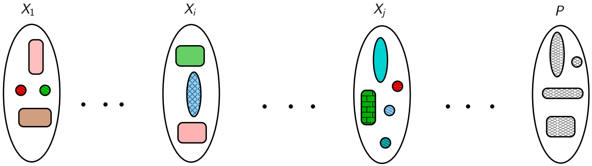

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So if there exists one agent (say, i) who is beyond envy, i.e., $v_i(S) = 0$ for all $S \subseteq M$:

- then EFX allocations exist!

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We will always maintain a partial allocation $X = \langle X_1, \ldots, X_n \rangle$ that is EFX.

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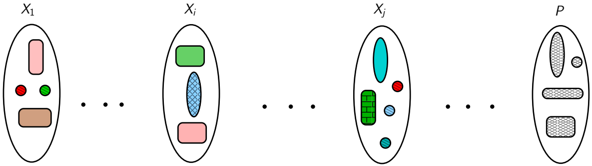

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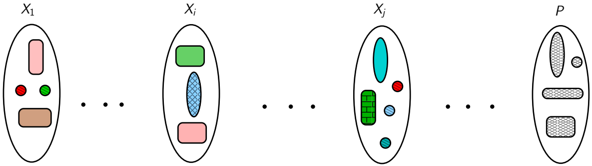

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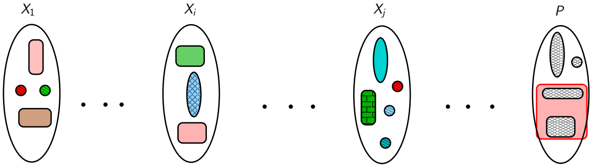

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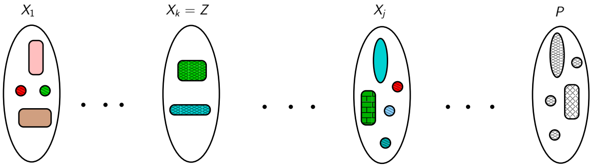

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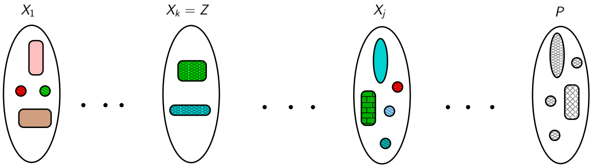

While there is envy towards P, we run this step.

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- It is known that $|P| \leq n - 2$ (Mahara, 2021).

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When valuations are additive:

- an epistemic EFX allocation always exists;
- we can efficiently find one.

A probability distribution over EF1 allocations

Consider the following instance with additive valuations:

	a	b	c
Agent 1	10	2	5
Agent 2	11	4	1
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	a	*b*	*c*
Agent 1	10	2	5
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The following fractional allocation is envy-free:

- Agent 1 gets $1/2$ of a and $1/2$ of c.
- Agent 2 gets $1/2$ of a, $1/4$ of b, and $1/4$ of c.
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- Take X with probability $1/4$, Y with probability $1/4$, and Z with probability $1/2$.

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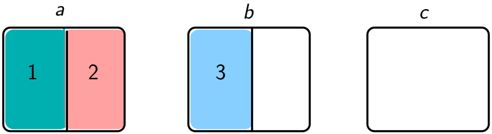

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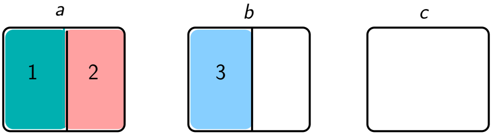

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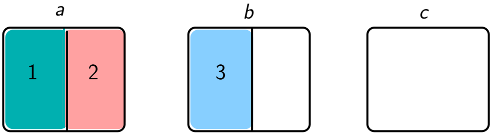
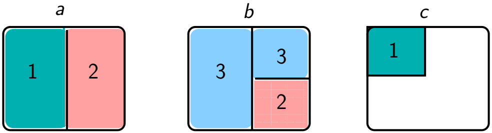


Once a good is completely consumed by a subset of agents:

- each of those agents then eats her favourite available good at the same speed.



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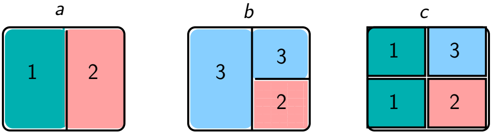

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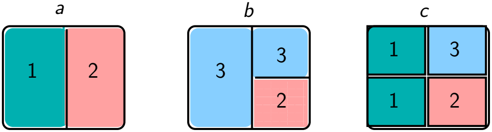

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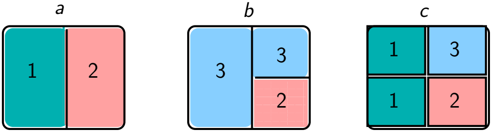

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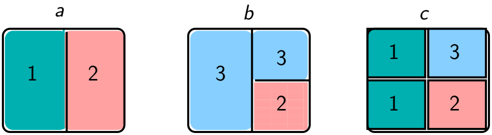

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