Lectures on
Siegel's Modular Functions

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Siegel's Modular Functions

The theory of modular functions of degree n is not quite new. The conception of this theory is due to Siegel who gave a first introduction of this in 1939 (Einfuhrung in die Theorie der Modulfunktionen n-ten Grades, Math. Ann., Vol 116(1939)). Since then, numerous contributions have been made by various authors. My desire to give a course on this special branch of mathematics arose out of more than one consideration. Siegel's modular functions constitute one of the most important classes of analytic functions of several variables, and we are able to look forward to far reaching results out of this class. The modular functions of degree n are connected with the manifolds of the closed Riemann surfaces of genus n in just the same way as the elliptic modular functions are, with the manifolds of Riemann surfaces of genus 1. Moreover we realise the excellent use of modular forms of degree n in the analytic theory of quadratic forms, first positive quadratic forms and then, indefinite forms too. But in trying to extend this theory to the case of indefinite quadratic forms, one meets with new types of functions defined by partial differential equations and one is led to a variety of unsolved problems in this direction. Finally, the researches of Siegel have becomes representative of those on a general class of automorphic functions of several variables. 1

Chapter 1

The Modular Group of Degree *n*

Let $\mathfrak{f}$ be a closed Riemann surface of genus n; $d\omega_i, i = 1, 2, \ldots n$, a basis of the Abelian differentials of the first kind on $\mathfrak{f}$, and ψ_ν, ψ'_ν, $\nu = 1, 2 \ldots n$, a canonical system of curves which dissect $\mathfrak{f}$ into a simply connected surface bounded by one closed curve. We ask for the transformation properties of the periods

$$q_{\mu\nu} = \int_{\psi'_\mu} d\omega_\nu, \, p_{\mu\nu} = -\int_{\psi_\mu} d\omega_\nu (\mu, \nu = 1, 2 \ldots n)$$

concerning a replacement of $d\omega_\nu (\nu = 1, 2 \ldots n)$ by another basis $d\omega^*_\nu$ 2
$(\nu = 1, 2 \ldots n)$, and ψ_ν, ψ'_ν respectively by another canonical system of curves ψ^*_ν, $\psi^*_\nu (\nu = 1, 2 \ldots n)$. We introduce the period matrices

$$P = (p_{\mu\nu}), \; Q = (q_{\mu\nu}), \; (\mu, \nu = 1, 2 \ldots n),$$

and let P^*, Q^* denote the corresponding matrices in the changed system. Let $\mathscr{L}_\mu$, $\mathscr{L}'_\nu$ be arbitrary oriented closed curves on $\mathfrak{f}$ and C_μ, C'_ν be arbitrary complex numbers. *A* homology

$$\sum_{\mu=1}^{r} C_\mu \mathscr{L}_\mu \sim \sum_{\gamma=1}^{s} C'_\nu \mathscr{L}'_n u$$

means that for every integrable function F, the equation

$$\sum_{\mu=1}^{r} C_\mu F(\mathscr{L}_\mu) = \sum_{\nu=1}^{r} C'_\nu F(\mathscr{L}'_\nu) \qquad \text{holds.}$$

For every closed curve $\mathscr{L}$, there exists, as is well known, a representation

$$\mathscr{L} \sim \sum_{\mu=1}^{n} C_\mu \psi_\mu + \sum_{\nu=1}^{n} C'_\nu \psi'_\nu$$

with uniquely determined integers $C_\mu, C'_\nu(\mu, \nu = 1, 2 \ldots, n)$. That is to say that the curves of a canonical system represent a basis of the homology classes of all closed curves. Thus we have in particular

$$\left.\begin{aligned} \psi_i^* &\sim \sum_\mu (a_{i\mu}\psi_\mu - b_{i\mu}\psi'_\mu) \\ \psi_i^{*'} &\sim \sum_\mu (-C_{i\mu}\psi_\mu + d_{i\mu}\psi'_\mu) \end{aligned}\right\} \tag{1}$$

with integers $a_{\mu\nu}, b_{\mu\nu}, C_{\mu\nu}, d_{\mu\nu}(\mu, \nu = 1, 2 \ldots n)$. Since the change from the homology basis ψ_ν, ψ'_ν, to the basis ψ^*_ν, $\psi^*{}_{\nu'}$ will always be effected by a matrix which is unimodular, the determinant of the matrix $M =$
3 $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A = (a_{\mu\nu})$, $B = (b_{\mu\nu})$, $C = (C_{\mu\nu})$, $D = (d_{\mu\nu})$, denoted by $|M|$, is equal to ± 1, i.e. $|M| = \pm 1$.

Furthermore,

$$d\omega^*_\mu = \sum_{\nu=1}^{n} d\omega_\nu t_{\nu\mu} (\mu = 1, 2 \ldots n)$$

with a non-singular matrix $T = (t_{\mu\nu})$

Now we express the periods

$$q^*_{i\mathscr{R}} = \int_{\psi_i^{*'}}, d\omega^*_{\mathscr{R}}, P^*_{i\mathscr{R}} = -\int_{\psi_i^*} d\omega^*_{\mathscr{R}}$$

in terms of $P_{\mu\nu}$, $q_{\mu\nu}$ as follows.

$$
\begin{aligned}
p^*_{i\mathscr{R}} &= -\sum_{\mu}(a_{i\mu}\int\limits_{\psi_\mu} d\omega^*_{\mathscr{R}} - b_{i\mu}\int\limits_{\psi'_\mu} d\omega^*_{\mathscr{R}})\\
&= -\sum_{\mu\nu}(a_{i\mu}t_{\mu\mathscr{R}}\int\limits_{\psi_\mu} d\omega_\nu - b_{i\mu}t_{\nu\mathscr{R}}\int\limits_{\psi'_\mu} d\omega_\nu)\\
&= \sum_{\mu\nu}(a_{i\mu}\psi_{\mu\nu}t_{\nu\mathscr{R}} + b_{i\mu}q_{\mu\nu}t_{\nu\mathscr{R}},\\
q^*_{i\mathscr{R}} &= \sum_{\mu}(-C_{i\mu}\int\limits_{\psi_\mu} d\omega^*_{\mu} + d_{i\mu}\int\limits_{\psi'_\mu} d\omega^*_{\mathscr{R}})\\
&= \sum_{\mu\nu}(-C_{i\mu}t_{\nu\mathscr{R}}\int\limits_{\psi_\mu} d\omega_\nu + d_{i\mu}t_{\nu\mathscr{R}}\int\limits_{\psi'_\mu} d\omega_\nu)\\
&= \sum(C_{i\mu}p_{\mu\nu}t_{\nu\mathscr{R}} + d_{i\mu}q_{\mu\nu}t_{\nu\mathscr{R}})
\end{aligned}
$$

Rewritten in terms of the matrices, these relations give

$$P^* = (p^*_{\mu\nu}) = (AP + BQ)T, \;\; Q^* = (q^*_{\mu\nu}) = (CP + DQ)T.$$

As is well known, a differential of the first kind is uniquely determined by its periods $\int\limits_{\psi'_\mu} d\omega(\mu = 1, 2 \ldots n)$ so that $|Q| \neq 0$. The choice $T = Q^{-1}$ is therefore permissible and this leads to the relations $P^* = APQ^{-1} + B = AZ + B$, where $Z = PQ^{-1}$, and $Q^* = CPQ^{-1} + D = CZ + n$.

If we change only the basis of the differentials and keep the canonical basis unchanged, the above relations lead to the normalized period 4
matrices $P^* = PQ^{-1} = Z$; $Q^* = E^{(n,n)}$, the unit square matrix of order n, as in this case we have

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E^{(n,n)} & o \\ o & E^{(n,n)} \end{pmatrix} = E^{(2n,2n)}.$$

We denote by X, Y, the real and imaginary parts of the matrix Z and by Z' the transpose of Z. From the theory of algebraic functions it is known that Z is a symmetric matrix with a positive imaginary part (i.e. the quadratic form $\mathscr{Y}'\ Y\mathscr{Y}$, for any real vector $\mathscr{Y}$ is always positive). In symbols, these mean

$$\boxed{\text{Z = Z' ; Y, 0.}} \tag{2}$$

It is obvious that in the general case (M arbitrary, unimodular; T arbitrary, non-singular) we have

$$Z^* = P^* Q^{*^{-1}} = (AP + BQ)(CP + DQ)^{-1} = (AZ + B)(CZ + D)^{-1}$$

and Z^* also satisfies the same properties as Z, viz. it is symmetric with a positive imaginary part.

In the sequel we shall denote by $\mathscr{Y}$ the set of all symmetric matrices with a positive imaginary part.

In order to obtain the typical relations for the coefficients of the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we consider the intersection properties of the canonical system $\psi_\nu, \psi'_\nu (\nu = 1, 2 \ldots n)$, described by means of the notion of the *characteristic* (Kronecker). For every pair of closed curves $\mathscr{L}_1$, $\mathscr{L}_2$ on $\mathfrak{f}$, there exists a number $\mathscr{S}(\mathscr{L}_1,\ \mathscr{L}_2)$ called the characteristic of $\mathscr{L}_1$ with regard to $\mathscr{L}_2$, which satisfies the following conditions.

1) $\mathscr{S}(\mathscr{L}_1, \mathscr{L}_2)$ is an integer depending only on the homology classes of $\mathscr{L}_1$ and $\mathscr{L}_2$.

5 We may now define $\mathscr{S}(\mathscr{L}_1 + \mathscr{L}_2, \mathscr{L}_3)$ as $\mathscr{S}(\mathscr{L}, \mathscr{L}_3)$ where $\mathscr{L} \sim \mathscr{L}_1 + \mathscr{L}_2$.

2) $\mathscr{S}(\mathscr{L}_1, \mathscr{L}_2) = -\mathscr{S}(\mathscr{L}_2, \mathscr{L}_1)$. This implies the additivity of $\mathscr{S}$, viz.

$$\mathscr{S}(\mathscr{L}_1 + \mathscr{L}_2, \mathscr{L}) = \mathscr{S}(\mathscr{L}_1, \mathscr{L}) + \mathscr{S}(\mathscr{L}_2, \mathscr{L}) \tag{3}$$

3) For every canonical basis $\psi_\nu, \psi_\nu (\nu = 1, 2 \ldots n)$ of the homology classes we have

$$\begin{aligned} \mathscr{S}(\psi_\mu, \psi_\nu) &= \mathscr{S}(\psi'_\mu, \psi'_\nu) = 0 \\ \mathscr{S}(\psi_\mu, \psi'_\nu) &= \delta_{\mu\nu} \quad \text{where} \end{aligned} \tag{4}$$

the $\delta'_{\mu\nu}\mathscr{S}$ are the Kronceker $\delta'\mathscr{S}$.

4) If $\mathscr{L}_1$ crosses $\mathscr{L}_2$, r times from the right side to the left, and l times from the left to the right, then

$$\mathscr{S}(\mathscr{L}_1, \mathscr{L}_2) = r - \ell \tag{5}$$

as far as r and ℓ are computable.

As we only intend to sketch the way which leads to the typical relations for M, we do not prove here the existence of a characteristic (cf:H.Weyl, Die Idee der Riemanns chen Flanche).

From the above properties it is clear that the characteristic defines a bilinear form on the class of all closed curves and that the matrix associated with this bilinear form with respect to a canonical basis is $I = \begin{pmatrix} o & E^{(n,n)} \\ -E^{(n,n)} & o \end{pmatrix}$.

If we transform this basis by means of the matrix M, the matrix associated with the above bilinear form with respect to the transformed basis is clearly MIM'. But M is so chosen that the transformed basis is again canonical so that the matrix associated with the bilinear form with respect to the transformed basis should again be I. Hence we conclude that

$$MIM' = I \tag{6}$$

This clearly characterises M, since the matrix associated with the 6
above bilinear form with respect to a certain basis can be equal to I if and only if that basis is canonical. As a consequence of (6) we obtain the desire characteristic relations as

$$AD' - BC' = E; AB' - BA' = 0; CD' - DC' = 0 \tag{7}$$

A matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is said to be *symplectic* if it satisfies the relation (6) or equivalently the relation (7). We denote by $S = S_n$ the class of all symplectic matrices. In view of the relations

$$\begin{aligned} I^{-1} &= -I \\ M^{-1} &= -IM'I \end{aligned} \tag{8}$$

it is easily seen that S is a group, called the *symplectic group of degree n*. The integral matrices $M \in S$ form a sub-group, called the *Modular Group of degree n*. We denote the modular group by $M = M_n$. We may note that if one replaces M by M' in (7) one obtains

$$A'D - C'B = E; A'C - C'A = 0; B'D - D'B = 0 \tag{9}$$

These relations are equivalent to (7). For, from (8) we have

$$\begin{aligned} M' &= I^{-1}M^{-1}I^{-1} \\ &= I^{-1}M^{-1}I \end{aligned} \tag{8$'$}$$

Clearly $I \in S$. Since S is a group, the relation (8)$'$ signifies that $M \in S$ if and only if $M' \in S$ which is precisely the content of our claim that the relations (7) and (9) are equivalent. In view of (8) we also obtain

$$M^{-1} = \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix} \tag{10}$$

With a view to fixing our ideas, we notice the following aspect (cf. Siegel, Über die analytische Theorie der quadratischen Formen,
7 Ann.Math., Vol. 36(1935)). We define two points $Z, Z^* \in \mathscr{Y}$ to be equivalent (with regard to M) if

$$Z^* = M\langle Z\rangle = (AZ + B)(CZ + D)^{-1} \tag{11}$$

for a suitable matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M$

Then our earlier discussion shows that to every closed Riemann surface, there corresponds a uniquely determined set of equivalent points in $\mathscr{Y}$. The converse is not always true. Let $\mathscr{Y}_0$ be the submanifold of $\mathscr{Y}$ consisting of all sets of equivalent points Z, which appear from period matrices in the manner described above. As Riemann has shown, $\mathscr{Y}_0$ is a complex analytic manifold depending on $3n - 3$ independent complex variables when $n > 1$, while $\mathscr{Y}$ depends on $n(n + 1)/2$ independent complex variables. By a *module* of $\mathfrak{f}$ we shall understand a complex valued function $\mathfrak{f}(\mathfrak{f})$ defined on all closed Riemann surfaces of genus n

and possessing certain analytic properties. For instance, every function $\mathfrak{f}(Z)$ meromorphic in $\mathscr{Y}$ and invariant under M is such a modul. We call such an $\mathfrak{f}$ a modular function of degree n, but later we shall give a precise definition of this concept. The modular functions of degree n constitute a function field. More over we shall prove that this function field is generated by a special set of modular function

$$\mathfrak{f}_o(Z), \mathfrak{f}_1(Z), \ldots \mathfrak{f}_{\mathscr{R}}(Z), \mathscr{R} = n(n+1)/2 \tag{12}$$

The degree of transcendence of this function field is k, that is to say, the functions (12) satisfy one irreducible algebraic equation

$$A_o(\mathfrak{f}_o, \mathfrak{f}_1 \ldots \mathfrak{f}_k) = 0 \tag{13}$$

It is possible to find a system of algebraic equations

$$A_\nu(\mathfrak{f}_o, \mathfrak{f}_1 \ldots \mathfrak{f}_{\mathscr{R}}) = 0, \nu = 1, 2 \quad (n-2)(n-3)/2 \tag{14}$$

which give a necessary and sufficient condition for Z to belong to $\mathscr{Y}_o$. The field of modular functions will be changed into the field of the moduls by these relations. The field of moduls has the degree of transcendence $(n(n+1)/2) - ((n-2)(n-3)/2) = 3n-3$. Since every closed **8**
Riemann surface of genus n is biuniquely determined by the values of $\mathfrak{f}_o, \mathfrak{f}_1, \ldots \mathfrak{f}_n$, the manifolds of all closed Riemann surfaces of genus n represent an algebraic manifold by (13) and (14). The explicit computation of these equations is still on unsolved problem.

We proceed to investigate the arithmetic properties of the matrices $M \in M$. First we consider the more general group S. Let $M \in S$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We call (AB) the first matrix row of M and (C, D), the second. Two complex matrices P, Q are said to from a *symmetric pair* if we have $PQ' = QP'$. We note that if (P, Q) is a symmetric pair and $|Q| \neq 0$, then $Q^{-1}P$ is a symmetric matrix.

A matrix A will be called *integral* if all the elements of A are integers. A pair of n-rowed matrices A, B will be called *coprime* if the matrix products GA, GB are integral when and only when the matrix G is integral. This in particular implies that both A, B are integral if (A, B) is coprime.

Let (C, D) be a coprime pair of matrices and let $\mathscr{U}_1$, a n-rowed and $\mathscr{U}_2$, a $2n$-rowed unimodular matrix. Then the matrices C_1, D_1 defined by $(C_1 D_1) = \mathscr{U}_1(C, D)\mathscr{U}_2$ are coprime too.

Indeed, GC_1, GD_1 are integral if and only if $(G\mathscr{U}_1 C, G\mathscr{U}_1 D)\mathscr{U}_2$ are integral which in turn is true if and only if $G\mathscr{U}_1 C$, $G\mathscr{U}_1 D$ are integral as $\mathscr{U}_2$ is unimodular. Since by assumption (C, D) is a coprime pair, the above holds if and only if $G\mathscr{U}_1$ and consequently G are integral, and this proves our contention.

We will now choose $\mathscr{U}_1$ and $\mathscr{U}_2$ so that $D_1 = 0$ and C_1 becomes a diagonal matrix with non-negative integers as diagonal elements. The choice is possible by the elementary divisor theorem. Then, of necessity $C_1 = E$, as otherwise we can always find a non integral matrix G such
9 that GC_1 is integral, contradicting the fact the pair $(C_1, 0)$ is coprime. Thus we obtain

$$(CD)\mathscr{U}_2 = \mathscr{U}_1^{-1}(C_1, D_1) = \mathscr{U}_1^{-1}(E, 0) = (\mathscr{U}_1^{-1}, 0)$$

which shows that the matrix (C, D) is of rank n. This says even more, viz. that there exist integral matrices X, Y satisfying the relation

$$CX + DY = E \tag{15}$$

Indeed, we have only to define X, Y by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathscr{U}_2 \begin{pmatrix} \mathscr{U}_1 \\ 0 \end{pmatrix}$$

and then

$$CX + DY = (CD)\begin{pmatrix} X \\ Y \end{pmatrix} = (CD)\mathscr{U}_2 \begin{pmatrix} \mathscr{U}_1 \\ 0 \end{pmatrix} = E.$$

The converse is true too, viz. if (C, D) is a pair of integral matrices for which the equation (15) is solvable for integral X, Y, then (C, D) is a coprime pair. For, in this case if G is integral then GC, GD are trivially integral, while if GC, GD are integral, then GCX, GDY are integral which in turn implies that $GCX + GDY = GE = G$ is integral.

We now note that matrix rows of a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}$ consist of coprime pairs in view of our above inference and relations (7). They are trivially symmetric pairs, again a consequence of (7). The partial converse is also true, viz. every symmetric coprime pair (C, D) is a second matrix row of some matrix $\mathcal{M} \in \mathcal{M}$. For, we can determine integral X, Y such that $CX + DY = E$ and then we have only to set

$$A = Y' + X'YC, B = -X' + X'YD.$$

Then

$$\begin{aligned} AB' - BA' &= (Y' + X'YC)(-X + D'Y'X) - (-X' + X'YD)(Y + C'Y'X) \\ &= (X'Y - Y'X) + (X'C' + Y'D')Y'X - X'Y(CX + DY) \\ &= (X'Y - Y'X) + Y'X - X'Y = 0 \end{aligned}$$

and **10**

$$\begin{aligned} AD' - BC' &= (Y' + X'YC)D' - (-X' + X'YD)C' \\ &= Y'D' + X'C' = (CX + DY)' = E \end{aligned}$$

which show that A, B, C, D satisfy (7) and consequently

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}.$$

Let us inverstigate how far (C, D) determines $\mathcal{M}$ uniquely. Let $\mathcal{M}$, $\mathcal{M}_1$ be two modular matrices with the same second matrix row, say $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\mathcal{M}_i = \begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}$. Then

$$\mathcal{M}_1\mathcal{M}^{-1} = \begin{pmatrix} A_1 & B_1 \\ C & D \end{pmatrix}\begin{pmatrix} D' & -B' \\ -C & A' \end{pmatrix} = \begin{pmatrix} E & S \\ 0 & R \end{pmatrix}$$

where $R = -CB' + DA'$, $S = -A_1B' + B_1A'$.

Since $\mathcal{M}_1\mathcal{M}^{-i} \in \mathcal{M}$ ($\mathcal{M}$ being a group), it is necessary that $R = E$ and $S = S'$. Thus $\mathcal{M}_1 = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}\mathcal{M}$ where S is a symmetric matrix conversely also, if $\mathcal{M}, \mathcal{M}_1 \in \mathcal{M}$ and $\mathcal{M}_1 = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}\mathcal{M}$ for a symmetric matrix S, then $\mathcal{M}$ and $\mathcal{M}_1$ have the same second matrix row as is seen by direct multiplication. Let T be the Abelian sub-group of $\mathcal{M}$ consisting of all matrices of the form $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$ where $S = S'$. Then the class of all modular matrices with different second rows provides a complete representative system of the set of all right cosets of T in $\mathcal{M}$.

We proceed to consider another sub-group A of $\mathcal{M}$ of which T will be a normal sub-group. This group A consists precisely of all elements $\mathcal{M} \in \mathcal{M}$ of the form $\mathcal{M} - \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. The conditions (7) will then imply that $AD' = E$ and $AB' = BA'$. The first condition means that $D' = A^{-1}$. Since A, D' are integral matrices it is now immediate that they are unimodular, say $A' = u$ and $D = u^{-1}$. Let $S = Bu = BA' = AB'$. The second condition then implies that S, defined as above, is symmetric. Thus A is the sub-group of $\mathcal{M}$ consisting precisely of the matrices of the form $\mathcal{M}_0 = \begin{pmatrix} u' & su^{-1} \\ 0 & u^{-1} \end{pmatrix}$ where S is symmetric and u is unimodular. It is easily seen that T is a normal sub-group of A. Then since the mapping
11 which taken the matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathcal{M}$ to $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is a homomorphism of which T is the kernel, it follows that A/T is isomorphic to the group of matrices

$$\begin{pmatrix} u' & 0 \\ 0 & u^{-1} \end{pmatrix}, \qquad u - \text{ unimodular}. \tag{16}$$

Let us decompose $\mathcal{M}$ into right cosets modulo A. The matrices $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\mathcal{M}_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ belong to the same right coset of A if and only if $\mathcal{M}_o\mathcal{M} = \mathcal{M}_1$ for some $\mathcal{M}_o \in A$.

This implies that

$$u(C_1, D_1) = (C, D), \qquad u - \text{ unimodular }, \tag{17}$$

and hence $CD_1' = uC_1D_1' = uD_1C_1' = DC_1'$

Thus we have

$$CD_1' = DC_1'. \tag{18}$$

On the other hand, if $\mathcal{M}$, $\mathcal{M}_1 \in \mathcal{M}$ be any two matrices for which (18) holds, then $\mathcal{M}\mathcal{M}_1^{-1} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ so that $\mathcal{M}\mathcal{M}_1^{-1} \in A$; in other words $\mathcal{M}$ and $\mathcal{M}_1$ determine the same right coset of A. Thus (17) and (18) are equivalent and either of them gives a necessary and sufficient condition that $\mathcal{M}$ and $\mathcal{M}_1$ belong to the same right coset modulo A. Let us now define two pairs (C, D), (C_1, D_1) to be *associated* if

1. Both of them are symmetric co-prime pairs and

2. $CD_1' = DC_1'$

This relation of being associated pairs is reflexive, symmetric and transitive - in other words, and equivalence relation. We can now say that in the coset AM lie exactly those matrices whose second matrix rows are associated with the second matrix row of $\mathcal{M}$.

Let $\{C, D\}$ denote the class of all coprime symmetric pairs of matrices associated with (C, D). Let C be of rank r, $O < r \leq n$. By the elementary divisor theorem, we can can determine unimodular matrices u_1, u_2 such that

$$u_1 C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} u_2', |C_1| \neq 0 \tag{19}$$

Here C_1 is a $r \times r$ square matrix. 12

In analogy with (19) we write

$$u_1 D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} u_2^{-1} \tag{19$'$}$$

and determine the nature of D_i, $i = 1, 2, 3, 4$.

Since $CD' = DC'$, we have

$$\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D_1 & D_3 \\ D_2 & D_4 \end{pmatrix} = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \begin{pmatrix} C_2' & 0 \\ 0 & 0 \end{pmatrix}$$

which gives $C_1 D_1' = D_1 C_1'$ and $D_3 = 0$

We now contend that D_4 is unimodular. For, in view of (15) we need only verify that the pair $(D_4, 0)$ is coprime. Let then GD_4 be integral. Of necessity, G is a matrix with $(n - r)$ columns. We complete G with zeros to a matrix $(O \ \ G)$ with n columns. From (19) and ((19)′) we obtain that $(O \ G)u_1C$ and $(OG)u_1D$ are integral. But (C,D) is a coprime pair and consequently (u_1C, u_1D) is also coprime, so that G should be integral. It is now immediate that $(D_4, 0)$ is coprime and consequently D_4 is unimodular. Let now $u_1 = \begin{pmatrix} E & D_2 \\ 0 & D_4 \end{pmatrix} u_1^*$. Then u_1^* is unimodular. Also

$$u_1^*C = \begin{pmatrix} E & D_2 \\ 0 & D_4 \end{pmatrix}^{-1} u_1C = \begin{pmatrix} E & -D_2D_4^{-1} \\ 0 & D_4^{-1} \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} u_2' = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} u_2'$$

and

$$u_1^*D = \begin{pmatrix} E & -D_2D_4^{-1} \\ 0 & D_4^{-1} \end{pmatrix} u_1D = \begin{pmatrix} E & -D_2D_4^{-1} \\ 0 & D_4^{-1} \end{pmatrix} \begin{pmatrix} D_1 & D_2 \\ 0 & D_4 \end{pmatrix} u_2^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix} u_2^{-1}$$

We will now show that the pair $C_1, D_1)$ is coprime. Let GC_1, GD_1 be integral. Then $(G \ \ O)u_1^*C$ and $(G \ \ O)u_1^*D$ are integral. But (C,D) is a coprime pair and hence also $(u_1^*C, \ u_1^*D)$. Hence we conclude that G is integral. It is now immediate that (C_1, D_1) is coprime too. We may summarise our results into the following statement:

13 If (C,D) is a coprime symmetric pair with rank $C = r$, $O < r \leq n$, there exist square matrices C_1, D_1 of order r which again form a coprime symmetric pair, and unimodular matrices u_1, u_2 such that

$$u_1C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} u_2', u_1D = \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix} u_2^{-1} \tag{20}$$

We will now fix u_2 more precisely. If we replace u_2 in (20) by a matrix $u_2 \begin{pmatrix} u_3 & 0 \\ 0 & E \end{pmatrix}$ where u_3 is an $r \times r$ unimodular matrix, we still obtain the same form. This replacement amounts to changing Q to Qu_3 where Q consists of the first r columns of u_2. We shall call two matrices Q,

Q_1 *right associated* if $Q_1 = Qu_3$ for some unimodular matrix u_3. Let $\{Q\}$ denote the class of all matrices right associated with Q. A change of u_1 to $u_1^* = \begin{pmatrix} u_4 & 0 \\ 0 & E \end{pmatrix} u_1$ where u_4 is unimodular carries the pair (C_1, D_1) into $(u_4C_1, u_4D_1) \in \{C_1, D_1\}$. A proper choice of u_3 and will transform Q into a fixed matrix in the class Q and u_4 the pair (C_1, D_1) into a fixed pair in $\{C_1, D_1\}$. This settles how for C_1, D_1 and u_1 characterise C, D in (20). We may note that the matrices Q which enter into our discussion are precisely those that can be completed to a unimodular matrix $(Q\ R)$. A matrix with this property will be called *primitive*. The elementary divisors of a primitive matrix are all equal to 1 as $u_2^{-1}Q = \begin{pmatrix} E \\ 0 \end{pmatrix}$ shows. Conversely, if all the elementary divisors of Q are equal to 1, then Q is primitive. For, then we can determine unimodular matrices u_2, u_3 so that $Q = u_2 \begin{pmatrix} E \\ 0 \end{pmatrix} u_3$. We write $u_2 = (Q_1, R_1)$ where Q_1 is r-columned and obtain

$$(Q\ R_1) = (Q_1u_3R_1) = u_2 \begin{pmatrix} u_3 & 0 \\ 0 & E \end{pmatrix}$$

We use the notation $Q = Q^{(n,n)}$ to signify that Q is a matrix with n rows and r columns. We will denote $Q^{(n,n)}$ simply by $Q^{(n)}$. A parametric representation of all classes $\{C, D\}$ of coprime symmetric pairs C, D, with rank $C = r$, $0 < r \leq n$ is give by

Lemma 1. *Let $Q = Q^{(n,r)}$ run through a complete representative system of all primitive classes $\{Q\}$ of $n \times r$ matrices and let (C_1, D_1) run independently through a complete representative system of all classes $\{C_1, D_1\}$ of coprime symmetric pairs of matrices $C_1 = C_1^{(r)}$ and $D_1 = D_1^{(r)}$ with rank $C_1 = r$. To every Q we make correspond only one of the matrices, say u_2, obtained by completing Q arbitrarily to a unimodular matrix. Then we obtain a complete representative system of all classes $\{C, D\}$ of coprime symmetric pairs (C, D) where $C = C^{(n)}$, $D = D^{(n)}$ and rank $C = r$, in the form* 14

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} u_2', D = \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix} u_2^{-1}, \tag{21}$$

different choices of C_1, D_1, Q Leading to different classes.

Proof. Only the last part of the Lemma remains to be proved.

$$\text{Let} \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & O \end{pmatrix} u_2', D = \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix} u_2^{-1}$$

and

$$C^* = \begin{pmatrix} C_1^* & 0 \\ 0 & 0 \end{pmatrix} u_2^{*\prime}, D^* = \begin{pmatrix} D_1^* & 0 \\ 0 & E \end{pmatrix} u_2^{*^{-1}}$$

and let $\{C, D\} = \{C^*, D^*\}$ i.e. to say

$$C^* D' - D^* C' = 0 \tag{22}$$

□

Then we have to show that $\{C_1, D_1\} = \{C_1^*, D_1^*\}$ and $\{Q\} = \{Q^*\}$ in an obvious notation.

From (22) we get

$$\begin{pmatrix} C_1^* & 0 \\ 0 & 0 \end{pmatrix} u_2^{*\prime} u_2^{-1} \begin{pmatrix} D_1' & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} D_1^* & 0 \\ 0 & E \end{pmatrix} u_2^{*^{-1}} u_2 \begin{pmatrix} C_1' & 0 \\ 0 & 0 \end{pmatrix}$$

or with the notation

$$u_2^{*\prime} u_2'^{-1} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}, u_2^{*^{-1}} u_2 = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$$

15 We have $\begin{pmatrix} C_1^* V_1 D_1' & C_1^* V_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D^* W_1 C_1' & 0 \\ W_3 C_1 & 0 \end{pmatrix}$ from which it follows that $C_1^* V_2$ and $W_3 C_1'$ and consequently V_2 and W_3 are zero (as C_1, C_1^* are nonsingular).

Since $u_2^{*^{-1}} u_2$ is unimodular, the above implies that W_1 is unimodular. Let $u_2 = (Q\ R)$ and $u_2^* = (Q^* R^*)$ where $Q = Q^{(n,r)}$ and $Q^* = Q^{*^{(n,r)}}$. Then

$$(Q\ R) = u_2 = u_2^* \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} = (Q^* R^*) \begin{pmatrix} W_1 & W_2 \\ 0 & W_4 \end{pmatrix} = (Q^* W_1 *)$$

So that $Q = Q^* W_1$. This in turn implies that $\{Q\} = \{Q^*\}$ which is one of the two results we were after. Since Q has been assumed to run through a representative system of the classes $\{Q\}$, it follows that $Q = Q^*$ and therefore $u_2 = u_2^*$. This in its implies that $W_1 = E$ and $V_1 = E$. Hence $C_1^* D_1' = D_1^* C_1'$ or $\{C_1, D_1\} = \{C_1^*, D_1^*\}$. If we assume that just one element is taken from each class $\{C, D\}$, then of course we should have $C_1 = C_1^*$ and $D_1 = D_1^*$.

It remains to prove that the class $\{C, D\}$ does not depend on the manner in which Q is completed to a unimodular matrix. For this, we suppose in the above that $C_1 = C_1^*$ and $D_1 = D_1^*$ and $Q = Q^*$. Then if follows that $W_1 = E$, $W_3 = 0$, $V_1 = E$ and $V_2 = 0$. We have to deduce that $\{C, D\} = \{C^*, D^*\}$, whatever R, R^* be. Now

$$\begin{aligned} C^*D' - D^*C' &= \begin{pmatrix} C_1^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} D_1' & 0 \\ 0 & E \end{pmatrix} \\ &\quad - \begin{pmatrix} D^1 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} \begin{pmatrix} C' & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} C_1^* D_1' & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} D_1^* C_1' & 0 \\ 0 & 0 \end{pmatrix} = 0 \end{aligned}$$

This completes the proof.

Chapter 2

The Symplectic group of degree n considered as a Group of mappings

Let Z be a point of the manifold $\mathscr{Y}$ defined by $Z = Z'$, $Y > 0 (Z = X + iY)$ **16**
and $\mathcal{M}$, a symplectic matrix: $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S$. We prove that the transformation

$$Z \to \mathcal{M}\langle Z \rangle = (AZ + B)(CZ + D)^{-1} \tag{23}$$

is a $1 - 1$ mapping of $\mathscr{Y}$ onto itself. For brevity, we put

$$P = AZ + B, Q = CZ + D$$

In view of (9) we obtain

$$\begin{aligned} \frac{1}{2i}(P'\bar{Q} - Q'\bar{P}) &= \frac{1}{2i}((Z'A' + B')(C\bar{Z} + D) - (Z'C' + D')(A\bar{Z} + B)) \\ &= \frac{1}{2i}(Z' - \bar{Z}) = \frac{1}{2i}(Z - \bar{Z}) = Y > O \end{aligned}$$

Also the matrix $CZ + D$ is non-singular so that the above mapping is well defined. For, if $\mathfrak{z}$ be any n-rowed complex column which satisfies the equation $Q\mathfrak{z} = 0$, then $\overline{Q}\overline{\mathfrak{z}} = 0$ and $\mathfrak{z}'Q' = 0$ so that

$$\mathfrak{z}'Y\overline{\mathfrak{z}} = \frac{1}{2i}(\mathfrak{z}'P'\overline{Q}\overline{\mathfrak{z}} - \mathfrak{z}'Q'\overline{P}\overline{\mathfrak{z}}) = 0. \qquad \text{But } Y > 0.$$

Hence it follows that $\mathfrak{z} = 0$ and this in turn implies that Q is non-singular.

Let then $Z_1 = PQ^{-1}$. In view of (9) It follows that

$$P'Q = (Z'A' + B')(CZ + D) = (Z'C' + D')(AZ + B) = Q'P$$

in other words, Z_1 is a symmetric matrix.

Further

$$\begin{aligned} Y_1 &= \frac{1}{2i}(Z_1 - \overline{Z}_1) = \frac{1}{2i}(Z_1' - \overline{Z}_1) = \frac{1}{2i}(Q'^{-1}P' - \overline{P}\overline{Q}^{-1}) \\ &= \frac{1}{2i}Q'^{-1}(P'\overline{Q} - Q'\overline{P})\overline{Q}'^{-1} = Q'^{-1}Y\overline{Q}^{-1} > 0 \end{aligned} \tag{24}$$

Thus it follows that $Z_1 \in \mathscr{Y}$; in other words, the transformation (23) takes $\mathscr{Y}$ into $\mathscr{Y}$.

A simple computation shows that $\mathcal{M}_1\mathcal{M}_2\langle Z\rangle = \mathcal{M}_1\langle\mathcal{M}_2\langle Z\rangle\rangle$, $\mathcal{M}_i \in$
17 S. Since $\mathcal{S}$ is a group implies that the mapping $Z \to \mathcal{M} < Z >$ is actually onto and that this mapping has an inverse.

We showed that $|CZ + D| \neq o$ for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \varepsilon \mathcal{S}$. Specialising A, D to be 0, and $B, -C$ to be E, the above implies that, for $Z \varepsilon \mathscr{Y}$

$$|Z| \neq 0 \tag{25}$$

Now for every $Z \in \mathscr{Y}$, $Z + iE$ also lies in $\mathscr{Y}$ so that we conclude that

$$|Z + iE| \neq o, Z \in \mathscr{Y} \tag{25$'$}$$

We can then introduce the mapping

$$W = T < Z > \qquad \text{with } T = \begin{pmatrix} E & -iE \\ E & iE \end{pmatrix} i.e \;\; W = (Z - iE)(Z + iE)^{-1}$$

Let $\mathfrak{R}$ be the domain onto which $\mathscr{Y}$ in mapped by T. Then

$$W - iE = (Z - iE)(Z + iE)^{-1} - E$$

$$= \{(Z + iE) - (Z - iE)\}(Z + iE)^{-1} = -2i(Z + iE)^{-1}$$

so that $|W - iE| \neq 0$ Also $W(Z + iE) = Z - iE$ i.e $i(E + W) = (E - W)Z$ so that $Z = i(E - W)^{-1}(E + W)$. This in particular implies that the correspondence between $\mathscr{Y}$ and $\mathfrak{K}$ is $1-1$. Also is it easily seen that the relations $Z = Z'$ and $W = W'$ are equivalent. Now

$$\begin{aligned}
&E - W\overline{W} = E - (Z + iE)^{-1}(Z - iE)(\overline{Z} + iE)(\overline{Z} - iE)^{-1}\\
&= (Z + iE)^{-1}\{(Z + iE)(\overline{Z} - iE) - (Z + iE)(\overline{Z} + iE)\}(\overline{Z} - iE)^{-1}\\
&= Zi(Z + iE)^{-1}(\overline{Z} - Z)(\overline{Z} - iE)^{-1}\\
&= 4(Z + iE)^{-1}Y(\overline{Z} - iE)^{-1} > 0
\end{aligned}$$

Hence $\mathfrak{z}'(E - W'\overline{W})\overline{\mathfrak{z}}$ is a hermitian form for any complex column $\mathfrak{z}$. Thus $W = T < Z >$ satisfies the relations.

$$W = W', E - W\overline{W} > o \tag{26}$$

We claim that these relations characterise the elements of $\mathfrak{K}$. For, if W is any matrix satisfying relations (26), then the matrix $E - W$ is non-singular, since the relation $(E - W)\mathfrak{z} = 0$ for any complex column $\mathfrak{z}$ implies along with (26) that $\mathfrak{z}'W = \mathfrak{z}'$ and $\overline{W\mathfrak{z}} = \overline{\mathfrak{z}}$ so that $\mathfrak{z}'(E - W\overline{W})\overline{\mathfrak{z}} = 0$ and consequently $\mathfrak{z} = 0$. Hence the matrix $Z = i(E - W)^{-1}(E + W)$ is well defined, and precisely as in the earlier case it can be shown that $Z \in \mathfrak{H}$ and that $W = T < z >$. This allows us to conclude that the domain $\mathfrak{K}$ is characterised by the relations (26), viz. $W = W'E - W\overline{W} > 0$ for any $W \in \mathfrak{K}$. This domain is one of the four main types of E. Cartan's irreducible bounded symmetric domains. We may remark in this connection that the domain $\mathscr{Y}$ is called the *generalized upper half plane* and the domain $\mathfrak{K}$, the *generalised unit circle.* 18

We have seen that the symplectic substitutions (24) form a group of $1 - 1$ mappings of $\mathscr{Y}$ onto itself. By means of the transformation $W = T < Z >$ which is a $1 - 1$ map of $\mathscr{Y}$ onto $\mathfrak{K}$, the above group can be transformed into a group of $1 - 1$ mappings of $\mathfrak{K}$ onto itself. In fact, which corresponds to the mapping $Z \to \mathcal{M} < Z >$ of $\mathscr{Y}$ will be given by $W \to \mathcal{M}_1 < W >$ where $\mathcal{M} < T^{-1} < W >>= T^{-1} < \mathcal{M}_1 < W >>$. In other words $\mathcal{M}_1 < W >= T\mathcal{M}T^{-1} < W >$ (each of the above groups will be shown to be full group of analytic homeomorphisms of the respective domain). Thus to the symplectic matrix $\mathcal{M}$ corresponds the matrix $\mathcal{M}_1 = T\mathcal{M}T^{-1}$ and to the group $\mathcal{S}$ corresponds the group $S_1 = TST^{-1}$. Using (6) which characterises the elements of $\mathcal{S}$ we shall obtain a characterisation of the elements of $\mathcal{S}_l$. $\mathcal{M} \in S$ is characterised by $\mathcal{M} = \bar{\mathcal{M}}$, $\mathcal{M}' I \mathcal{M} = I, I \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ With $\mathcal{M} = T^{-1}\mathcal{M}_1 T$ the above give

$$\left.\begin{aligned} \mathcal{M}_1' T'^{-1} I T^{-1} \mathcal{M}_1 &= T'^{-1} I T^{-1} \\ \mathcal{M}_1' T'^{-1} I \bar{T}^{-1} \bar{\mathcal{M}}_1 &= T'^{-1} I \bar{T}^{-1} \end{aligned}\right\}.$$

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By a simple computation we find that

$$T^{-1} = \frac{1}{2i}\begin{pmatrix} iE & iE \\ -E & E \end{pmatrix}, -T'^{-1} I T^{-1} = \frac{i}{2} I, T'^{-1} I \bar{T}^{-1} = \frac{i}{2} H \tag{27}$$

where

$$H = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

Thus $\mathcal{M}_1$ can be characterised by

$$\mathcal{M}_1' I \mathcal{M}_1 = I, \mathcal{M}_1' H \bar{\mathcal{M}}, = H$$

We can replace the second relation, in view of the first, by

$$I \mathcal{M}_1^{-1} I^{-1} H \bar{\mathcal{M}}_1 = H \quad \text{or} \quad I^{-1} H \bar{\mathcal{M}}_1 = \mathcal{M}_1 I^{-1} H.$$

With $K = I^{-1} H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ we obtain finally the relations

$$\mathcal{M}_1' I \mathcal{M}_1 = I, K \bar{\mathcal{M}}_1 = \mathcal{M}_1 k \tag{28}$$

The decomposition $\mathcal{M}_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ leads to the relations

$$B_1 = \bar{C}_1, D_1 = \bar{A}_1, A_1' \bar{A}_I - C_1' \bar{C}_1 = E, A_1' C_1 = C_1' A_1 \tag{29}$$

The relations (28) and (29) are clearly equivalent.

We now show that the domains $\mathscr{Y}$ and $\mathfrak{R}$ are homogeneous, in other words, that the groups $\mathcal{S}$ and $\mathcal{S}_1$ are transitive in their respective domains. It clearly suffices to prove for one, say $\mathfrak{R}$, and this in turn only requires the determination of a substitution $\mathcal{M}_1$ which takes the point 0 to any assigned point $W \in \mathfrak{R}$. If $\mathcal{M}_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$, the relation $W = \mathcal{M}_1 < 0 >$ implies that $W = B_1 D_1^{-1}$ i.e $B_1 = WD_1$. Then in view of (29) we should have $C_1 = W\bar{D}_1$, $A_1 = \bar{D}_1$ and $\bar{D}_1 D_1 - \bar{D}_1 \bar{W} W D_1 = E$. Consequently $\bar{D}'_1(E - \bar{W}'W)D_1 = E$ or $D_1\bar{D}_1 = (E - \bar{W}'W)^{-1}$. The last equation is certainly solvable for D_1 as $E - \bar{W}'W$ is positive Hermitian. If D_1 is obtained as any solution of the above equation, then the earlier **20**
equations define A_1, B_1 and C_1 and the matrix $\mathcal{M}_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ will have the desired properties.

We proceed to establish a result we promised earlier, viz that the group of symplectic substitutions is the full group of analytic mapping of the domain $\mathscr{Y}$ onto itself.

By an *analytic mapping* $Z \to Z^{\times}$ of $\mathscr{Y}$ onto itself, we mean a mapping with the following properties

i) it is topological,

ii) $Z^*_{\mu\nu} = Z^*_{\mu\nu}(Z)$ is a regular function of the independent elements $Z_{\mu\nu}(\mu \leq \nu)$ for $\mu\nu = 1, 2 \ldots \eta$

iii) $Z_{\mu\nu} = Z_{\mu\nu}(Z^*)$ is a regular function of the independent variables $Z^*_{\mu\nu}(\mu \leq \nu)$ of Z^* for $\mu\nu = 1, 2 \ldots \eta$.

Then every symplectic substitution $Z \to Z^* = \mathcal{M} < Z >$ is clearly and analytic mapping of $\mathscr{Y}$. We prove the converse in the following

Theorem 1. *Every analytic mapping of $\mathscr{Y}$ onto itself is a symplectic substitution*

In view of the $1 - 1$ correspondence between $\mathscr{Y}$ and $\mathfrak{R}$ by means of the transformation $Z \to T < Z >$, the above will imply a similar result of the domain $\mathfrak{R}$- in fact, it suffices to prove the corresponding result for

$\mathfrak{R}$ to infer Theorem 1. Since the group $\mathcal{S}_1$ is transitive with regard to $\mathfrak{R}$ we may even assume that the given analytic mapping

$$W_o \to W_o^* \tag{30}$$

where $W_o = (\omega_{\mu\nu})$ of $\mathfrak{R}$ has 0 as a fixed point. Let $W = tW_o$ where t is a complex variable and let the characteristic roots of the hermitian matrix $W_0'\overline{W}$ be $r_1, r_2 \ldots r_n$ where $o < r_1 \leq r_2 \leq \cdots \leq r_n$.

It is well known that there exists a unitary matrix u such that

$$\mathfrak{A}' W_o' \overline{W}_o \overline{\mathfrak{A}} = \begin{pmatrix} r_1 & & 0 \\ & r_{2_{..}} & \\ 0 & & r_n \end{pmatrix}$$

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We then have

$$\mathfrak{A}'(E - W\overline{W})\overline{\mathfrak{A}} = E - t\overline{t} \begin{pmatrix} r_1 & & 0 & \\ & r_2 & & \\ & & r_n & \\ 0 & & & \end{pmatrix} > 0$$

if and only if $t\overline{t}r_n < 1$. But for $t = 1$ $W = W_o$ and we know that $E - W_o'\overline{W}_o$. Hence $r_n < 1$.It is now immediate that if $|t| \leq 1$ then $E - W'\overline{W} > o$ and consequently $W \in \mathfrak{R}$. Then to each point $W = t\ W_o$ with $|t| \leq 1$ there corresponds an image point $W^* \in \mathfrak{R}$. The mapping : $t \to (tw_o)^* = W^*$ is regular function of the single variable t in $t\overline{t}r_n < 1$, meaning each element of W^* is a regular function of t, so that there exists a power series expansion for the elements of W^* in the form

$$W^* = \sum_{\mathfrak{R}=1}^{\infty} t^{\mathfrak{R}} W_{\mathfrak{R}}^* \tag{31}$$

which converges for $t\overline{t}r_n < 1$ and, a fortiori for, $|t| \leq 11$ where $W_{\mathfrak{R}}^*$ for each $\mathfrak{R}$ is a matrix whose elements are functions (polynomials) of the elements of W_o alone. One the other hand, the elements of W^* are regular functions of the variables $t\omega_{\mu\nu}$ so that they may be developed into a power series in $t\omega_{\mu\nu}$ (about the origin) converging for sufficiently

small values of the variables. But the $\omega_{\mu\nu}$ occurring in $W_o \in \mathfrak{K}$ can all be shown to be uniformly bounded so that we need only restrict $|t|$ to be small while the $\omega'_{\mu\nu}s$ may be arbitrary in their domain. Since a power series expansion (for the elements 0 W^*) is unique, it follows that (31) again is the desired power series and that $t^{\mathfrak{K}} W^*_{\mathfrak{K}}$ is precisely the aggregate 22
of all terms $\mathfrak{K}$ in it. Consequently, the series

$$W_o^* = \sum_{\mathfrak{K}=1}^{\infty} W^*_{\mathfrak{K}} \tag{32}$$

obtained from (31) by specialising t, converges everywhere in $\mathfrak{K}$ if we do not spilt up the polynomials which are elements of $W^*_{\mathfrak{K}}$ into their single terms. Since $E - W^* \overline{W}^* > 0$ for $t\overline{t} = 1$,we obtain by integration over the circle $t\overline{t} = 1$ that

$$\frac{1}{2\pi i} \int\limits_{t\overline{t}=1} (E - W^* \overline{W}^*) \frac{dt}{t} > 0 \tag{33}$$

where by $\int A dt$ with any matrix $A = (a_{\mu\nu}(t))$ we mean the matrix $(\int a_{\mu\nu} dt)$. Substituting for W^* the series (31) which converges uniformly for $|t| \leq 1$, termwise integration yields

$$E - \sum_{\mathfrak{K}=1}^{\infty} W^*_{\mathfrak{K}} \overline{W}^*_{\mathfrak{K}} > 0 \tag{34}$$

the rest of the terms all vanishing. In particular we have

$$E - W_1^* \overline{W}_1^* > 0 \tag{35}$$

The elements of $W^*_{\mathfrak{K}}$ are homogeneous polynomials of degree $\mathfrak{K}$ in $\omega_{\mu\nu}(\mu \leq \nu)$ so that the $n(n+1)/2$ elements of $W_1^* = (W^{(1)*}{}_{\mu\nu})$ are liner functions of the independent elements $\omega_{\mu\nu}(\mu \leq \nu)$. Let D be the determinant of this linear transformation. Since W_1^* is the linear part of the power series (32), the functional determinant of the $n(n+1)/2$ independent elements of W_o^* with respect to the variables $\omega_{\mu\nu}(\mu \leq \nu)$ at the point $W_o = 0$ is also D. As the mapping $W_o \to W_o^*$ is invertible it follows that $D \neq 0$. Also if we replace the mapping $W_o \to W_o^*$ by its inverse, the determinant change into D^{-1} so that we can, without loss of generality,assume that $D\bar{D} \geq 1$.

Consider now the liner mapping $\varphi : W_o \to W_1^*$. Let $\mathfrak{R}_1$ the image of $\mathfrak{R}$ by φ. We denote by $\vartheta(\mathfrak{R}_1)$ the Euclidean volumes of $\mathfrak{R}$ and $\mathfrak{R}_1$, the real and imaginary parts of $\omega_{\mu\nu}$ being the rectangular cartesian coordinates. Then we have $\vartheta(\mathfrak{R}_1) = D\bar{D}\vartheta(\mathfrak{R}) \geq \vartheta(\mathfrak{R})$. But $\mathfrak{R}_1 \subset \mathfrak{R}$ so that $\vartheta(\mathfrak{R}_1) \leq$
23 $\vartheta(\mathfrak{R})$. Hence it follows that $\vartheta(\mathfrak{R}) = \vartheta(\mathfrak{R}_1)$ and $D\bar{D} = 1$. We are going to conclude from this that $\mathfrak{R} = \mathfrak{R}_1$. Our argument is as follows. Let $\mathfrak{R}^* =$ Exterior $\mathfrak{R} = (\bar{\mathfrak{R}})^c$ and $\mathfrak{R}_1^* =$ Ext $\mathfrak{R}_1 = (\bar{\mathfrak{R}})^c$ the superscript 'c' denoting complements, and the bars denoting the closures. Since $\mathfrak{R}_1 \subset \mathfrak{R}$ and $\vartheta(\mathfrak{R}_1) = \vartheta(\mathfrak{R})$ it is immediate that ϑ(Interior $(\mathfrak{R} - \mathfrak{R}_1) = 0$) and consequently.

Int $(\mathfrak{R} - \mathfrak{R}_i) = 0$ (void). A point of $\mathfrak{R}$ is then either a point of $\mathfrak{R}_1$ or a limit point of $\mathfrak{R}_1$ and so $\bar{\mathfrak{R}} = \bar{\mathfrak{R}}_1$. We need only show then that the boundary $Bd\mathfrak{R} = Bd\mathfrak{R}_1$. Being a linear map, φ is a topological map in the large so that if it maps $\mathfrak{R}$ onto $\mathfrak{R}_1$ then it maps $Bd\mathfrak{R}$ onto $Bd\mathfrak{R}_1$, Ext. $\mathfrak{R}$ onto Ext. $\mathfrak{R}_1$, and Bd. (Ext. $\mathfrak{R}_1$) onto Bd. (Ext $\mathfrak{R}_1$). Thus

$$\begin{aligned} Bd.\mathfrak{R}_1 &= \varphi(Bd.\mathfrak{R}) = \varphi(Bd.\mathfrak{R}^*) = Bd.\mathfrak{R}_1^* = Bd.(\mathfrak{R}_1)^c \\ &= Bd.(\bar{\mathfrak{R}})^c = Bd.\mathfrak{R}^* = Bd.\mathfrak{R}. \end{aligned}$$

We have incidentally shown that φ maps $Bd.\mathfrak{R}$ onto itself.

We prove in Lemma 2 that any complex symmetric matrix can be represented in the form $W_o = u'\mathcal{P}u$, u-unitary and $\mathcal{P}$, a diagonal matrix with diagonal elements $p_1, p_2 \ldots p_n$ where $p_2^2, \nu - 1, 2 \ldots n$ are the characteristic roots of $W_o\bar{W}_o$. Then $E - W_o\bar{W}_o = u'\left(E - \begin{pmatrix} 2 & 0 & \\ p_\nu & & \\ 0 & & \end{pmatrix}\right)\bar{u}$ so that $W_o \in \mathfrak{R}$ if and only if $p_\nu^2 < 1$, $\nu = 1, 2 \ldots n$. The boundary points of $\mathfrak{R}$ are therefore precisely those for which $p_\nu^2 \leq 1$, $\nu = 1, 2 \ldots n$, the equality holding for at least one ν. Since $\varphi(Bd.\mathfrak{R}) = Bd.\mathfrak{R}$ it is immediate that $|E - W_1^*\bar{W}_1^*| = 0$ for $W_0 \in Bd.\mathfrak{R}$. In other words, $|E - W_1^*\bar{W}_1^*|$ considered a polynomial in p_ν, $\nu = 1, 2, \ldots n$ vanishes if $p_\nu^2 = 1$ for some ν. Hence it is divisible by $\prod_{\nu=1}^{n}(1 - p_\nu^2)$. Considering the degree of $|E - W_1^*\bar{W}_1^*|$ in
24 the $p_i's$ we have $|E - W_1^*\bar{W}_1^*| = c\prod_{\nu=1}^{n}(1 - p_\nu^2)$, $c - a$ constant. The choice

of P as 0 leads to the determination of c to be 1. Hence

$$|E - W_1^* \overline{W}_1^*| = \prod_{\nu=1}^{n} (1 - p_\nu^2) = |EW_o \overline{W}_o| \tag{36}$$

Replacing W_o in (36) by $\dfrac{1}{\sqrt{\lambda}} W_o$, in view of the linearity of $\varphi : W_o \to W_1^*$ we obtain

$$|\lambda E W_o \overline{W}_o| = |\lambda E - W_1^* \overline{W}_1^*|$$

identically in λ and therefore that $W_o \overline{W}_0$ and $W_1^* \overline{W}_1^*$ have the same characteristic roots. Lemma 2 (proved below) will then imply that

$$W_1^* = u' W_o u \tag{37}$$

for some unitary matrix u. We stop here to prove

Lemma 2. *Every complex symmetric matrix W_o admits a representation of the kind $W_o = u'Pu$ with $u'u = E$, $P = \begin{pmatrix} p_1 & & 0 \\ & p_2 & \\ & & \ddots & \\ 0 & & & p_n \end{pmatrix}$ p_ν^2 being the characteristic roots of $W_o \overline{W}_o$.*

Proof. $W_o \overline{W}_o$ being hermitian, there exists a unitary matrix u_1 such that $W_o \overline{W}_o = \underline{u_1'} \mathcal{P}^2 \overline{u}$. Then $F = u_1'^{-1} W_o u_1^{-1}$ is symmetric and satisfies the relation $F\overline{F} = P^2$. Writing $F = F_1 + iF_2$ (F_ν being real), it follows that $F_1 F_2 - F_2 F_1 = 0$. Hence by a well know result, an orthogonal matrix Q can be found which transform F_1 and F_2 simultaneously into diagonal matrices $Q' F_1 Q$ and $Q' F_2 Q$. Then $R = Q' F Q$ is also diagonal. If r_ν, $\nu = 1, 2, \ldots n$ be the diagonal elements of R the relation $R\overline{R} = Q' P^2 Q$ implies that $r_\nu \overline{r}_\nu (\nu = 1, 2, \ldots n)$ are identical with $P_\nu^2 (\nu = 1, 2, \ldots \eta)$. Then we can find a unitary matrix u_2 such that $R = u_2' p u_2$. Now $W_o = u_1' F u_1 = u_1' Q'^{-1} R Q^{-1} u_1 = u_1' Q'^{-1} u_2' p u_2 Q^{-1} u_1$. Taking $u = u_1' Q^{-1} u_2'$ **25**
we obtain $W_o = u'Pu$. Clearly u is a unitary matrix. □

We revert now to the proof of the main theorem. From (34), we have

$$E - W_1^* \overline{W}_1^* - W_{\mathfrak{K}}^* \overline{W}_{\mathfrak{K}}^* > 0 \tag{38}$$

for $\mathfrak{K} = 2, 3, \ldots$, and any W_o. Choosing in particular $W_o = uc^{is}$ with $o < u < 1$ and $s = s' = \overline{s}$ we get from (37) that

$$W_1^* \overline{W}_1^* = uu' e^{is} uu\overline{u}' e^{-is} \overline{u} = u^2 e^{is} e^{-is} = u^2 E$$

But $W_o \overline{W}_o = u^2 E$ and form (38)

$$(1 - u^2) E W_{\mathfrak{K}}^* - \overline{W}_{\mathfrak{K}}^* > O \text{ for } \mathfrak{K} > 1, o < u < 1.$$

Letting $u \to 1$ this gives $W_{\mathfrak{K}}^* = 0$ for $\mathfrak{K} > 1$, $W_o e^{is}$, $S = S'\overline{S}$

Since the elements of $W_{\mathfrak{K}}^*$ are analytic functions of the elements of S and they vanish for real values of these elements, it follows that they vanish for complex S too. Since the mapping $S \to W_o = e^{is}$ maps a neighborhood of 0 onto a neighborhood of E so that $W_{\mathfrak{K}}^* = 0$ in a neighborhood of E, so that $W_{\mathfrak{K}}^* = 0$ in a neighborhood of for $\mathfrak{K} > 1$, we conclude that $W_{\mathfrak{K}}^* = 0$ identically for $\mathfrak{K} > 1$, and $W_o^* = \sum W_{\mathfrak{K}}^* = W_1^*$. Thus the mapping $W_o \to W_0^* = W_1^*$ is linear. If now we show that in (37) the matrix u does not depend on W_o, then $W_1^* = u' W_o u = \mathcal{M}_1 < W_o >$ where $\mathcal{M}_1 = \begin{pmatrix} u' & 0 \\ 0 & u^{-1} \end{pmatrix} \in S_1$ and our theorem would have been proved. We proceed to establish this.

Let

$$W_1^* = W_1^*(W_o) = \sum_{\mu \leq \gamma} \omega_{\mu\nu} A_{\mu\nu} \tag{39}$$

and introduce

$$W_1 = W_1(W_o) = \sum_{\mu \leq \gamma} \omega_{\mu\nu} \overline{A}_{\mu\nu} \tag{40}$$

26 where $A_{\mu\nu}$ are constant matrices. Then $\overline{W}_1^* = W_1(\overline{W}_o)$ and $W_1^* \overline{W}_1^* = u' W_o \overline{W}_o \overline{u} = E W_o \overline{W}_o = E$ Hence $W_1^*(W_o) W_1(W_o^{-1}) = E$ for W_o such that $W_o \overline{W}_o = E$. In particular then, this is true for $W_o = e^{is}$ where $S = S' = \overline{S}$ Since W_1 and W_1^* depend analytically upon their arguments, this relation holds for complex symmetric S too, that is to say identically in W_o, by an earlier argument.

For convenience, we shall usually write W_2 instead of $W_{\nu\nu}$ and A_ν instead of $A_{\nu\nu}$.

Let

$$W_{oo}\begin{pmatrix}\omega_1 & & o\\ & \omega_2 & \\ o & & \omega_n\end{pmatrix} W_{o1} = W_o - W_{oo}.$$

Employing the Taylor Series Expansion in the neighborhood of $W_{o1} = 0$ we find

$$\begin{aligned} W_o^{-1} &= (W_{oo} + W_{o1})^{-1} = W_{oo}^{-1}(E + W_{o1}W_{oo}^{-1})^{-1}\\ &= W_{oo}^{-1} - W_{oo}^{-1}W_{o1}W_{oo}^{-1} + \cdots \end{aligned}$$

Using the linearity of W_1 and W_1^* we therefore get

$$\begin{aligned} E &= W_1^*(W_o)W_1(W_o^{-1})\\ &= (W_1^*(W_{oo}) + W_1^*(W_{01}))(W_1(W_{oo}^{-1}) - W_1(W_{oo}^{-1}W_{o1}W_{oo}^{-1}) + \cdots) \end{aligned}$$

Comparing the terms of degree 0 and 1 we obtain

$$W_1^*(W_{oo})W_1(W_{oo}^{-1}) = E \tag{41}$$

$$W_1^*(W_{oo})W_1(W_{oo}^{-1}W_{o1}W_{oo}^{-1}) = W_1^*(W_{o1})W_{11}(W_{oo}^{-1}) \tag{42}$$

The first equation means $\sum_{\mu,\nu=1}^{n}\omega_\mu\omega_\nu^{-1}A_\mu\overline{A}_\nu = E$ identically in ω_ν, $\nu = 1, 2 \ldots n$

This leads to

$$A_\mu\overline{A}_\nu = 0, \mu \neq \nu \tag{43}$$

Let the 'u' which enters in (37) corresponding to $W_{o1} = 0$ be denoted by u_o **27**

From (37) and (39) we get

$$\sum_{\nu=1}^{n}\omega_\nu A_\nu = u_o' W_{oo} u_o$$

From this, in view of (43) we infer that

$$\sum_{\nu=1}^{n}\omega_\nu\overline{\omega}_\nu A_\nu\overline{A}_\nu = u_o' W_{oo}\overline{W}_{oo}\overline{u}_o \text{ and consequently}$$

$$|\lambda E - \sum_{\nu=1}^{n} \omega_\nu \bar{\omega}_\nu A_\nu \bar{A}_\nu| = |\lambda E - W_o \bar{W}_{oo}|$$

identically in λ. Choosing $\omega_\nu = 1$ and $\omega_\nu = 0$ for $\mu \neq \nu$ the above gives

$$|\lambda E - A_\nu \bar{A}_\nu| = \left|\lambda E - \begin{pmatrix} o & & o \\ & 1 & \\ o & & o \end{pmatrix}\right|$$

so that the characteristic roots of $A_\nu \bar{A}_\nu$ are $1, 0, 0 \ldots 0$. In virtue of Lemma 2, we can assert the existence of a unitary matrix ν such that

$$A_1 = \nu' \begin{pmatrix} 1 & & o & \\ & o & & \\ & & o & \\ o & & & o \end{pmatrix} \nu$$

Without loss of generality we can assume $\nu = E$, as in the alternative case, we need only consider the mapping $\bar{\nu} W_o^* \bar{\nu}$ instead W_o^*. Then $A_\nu \bar{A}_\nu = A_\nu A_1 = 0$ for $\nu > 1$ by (43) which means that for $\nu > 1$

$$A_\nu = \begin{pmatrix} o & o \\ o & B_\nu^{(n.1)} \end{pmatrix}$$

Successive application of this argument shows that we may assume

$$A_\nu \begin{pmatrix} o & & & & \\ & o & & & \\ & & 1 & & \\ & & & o & \\ & & & & o \end{pmatrix} (\nu = 1, 2 \ldots n)$$

28 Then we shall have $\sum_{\nu=1}^{n} A_\nu = E$ and

$$W_1^*(W_{oo}) = \sum_{\nu=1}^{n} \omega_\nu A_\nu = W_{oo} \qquad \textit{and}$$

$$W_1^*(W_{oo}) = \sum_{\nu=1}^{n} \omega_\nu A_\nu = W_{oo}$$

From (44) we then conclude that

$$W_1^*(W_{o1}) = W_{oo}W_1(W_{oo}^{-1}W_{o1}W_{oo}^{-1})W_{oo}$$

Now $$W_1^*(W_{o1}) = \sum_{\mu<\nu}\omega_{\mu\nu}A_{\mu\nu} \tag{44}$$

and $$W_{oo}W_1(W_{oo}^{-1}W_oW_{oo}^{-1})W_o = \sum_{\mu\nu}\frac{\omega_{\mu\nu}}{\omega_\mu\omega_\nu}W_{oo}\bar{A}_{\mu\nu}W_{oo} \tag{45}$$

A comparison of (44) and (45) yields

$$A_{\mu\nu} = \frac{1}{\omega_\mu\omega_\nu}W_{oo}\bar{A}_{\mu\nu}W_{oo}$$

or $$\omega_\mu\omega_\nu A_{\mu\nu} = W_{oo}\bar{A}_{\mu\nu}W_{oo} \qquad (\mu<\nu).$$

Since this holds identically in W_{oo}, setting $W_{oo} = E$, this in particular implies that $A_{\mu\nu} = \bar{A}_{\mu\nu}$ i.e $A_{\mu\nu}$ is real, and further $A_{\mu\nu} = a_{\mu\nu}(e_{\mu\nu} + e_{\nu\mu})$ with real $a_{\mu\nu}$,where $e_{\mu\nu}$ denotes the matrix with 1 at the $(\mu, \nu th)$ place and 0 else where. Since $A_\mu \begin{pmatrix} o & & \\ & 1 & \\ & & 0 \end{pmatrix}$ we have in particular $a_{\nu\nu} = \frac{1}{2}$.

Now $W_1^* = \sum_{\mu\leq\nu}\omega_{\mu\nu}A_{\mu\nu} = (a^*_{\mu\nu}\omega_{\mu\nu})$ with real $a^*_{\mu\nu}$ and $a^*_{\mu\nu} = 1$. In view of (37), the expression $|W_1^*||W_o|^{-1}$ has a constant absolute value. On the other hand it is a rational function of $\omega_{\mu\nu}$. Consequently it is a constant. Since the product $\omega_1\omega_2\ldots\omega_n$ appears in both the determinants with the factor 1, it is immediate that $|W_1^*| = |W_o|$. Since the matrix $(\pm\delta_{\mu\nu})$ is unitary, we can assume that $a^*_{1\nu} \geq 0$ for $\nu > 1$. The **28**
term $(\omega_1\omega_\mu\omega_\nu)^{-1}\omega\omega_{1\mu}\omega_{\mu\nu}$ has in $|W_1^*|$ the coefficient $2a^*_{\mu\nu}(1<\mu<\nu)$ and in $|W_0|$, the coefficient 2. Hence $a^*_{\mu\nu} = 1(1<\mu<\nu)$. Also the term $(\omega_1\omega_\nu)^{-1}\omega\omega^2_{1\nu}$ has in $|W_1^*|$ the coefficient, $-a^{*2}{}_{1\nu}$ and in $|W_o|$, the coefficient -1. Hence $a^*_{1\nu} = 1$ for $\nu =\geq 1$. We already know that $a^*_{\nu\nu} = 1$. Thus we conclude that $W_1^* = (\omega_{\mu\nu}) = W_o$. In other words we have shown that with the aid of appropriate symplectic transformations, any analytic map $W_o \to W_o^*$ of $\mathscr{Y}$ onto $\mathscr{Y}$ can be reduce to the identity map so that the analytic map we started with must itself be symplectic. This completes the proof of theorem 1.

Chapter 3

Reduction Theory of Positive Definite Quadratic Forms

By the reduction of a positive definite quadratic from we shall understand the reduction of the corresponding matrix $Y = Y^{(n)} > 0$. Investigations in this direction are needed if we wish to construct a suitable fundamental domain for the modular group of degree n in $\mathscr{Y}$. Let $\mathscr{Y}$ be the domain of all real symmetric matrices Y and $\mathfrak{P}$ the domain of all $Y > 0$ in $\mathscr{Y}$. Two matrices $Y, Y_1 \in \mathscr{Y}$ are said to be equivalent if $Y_1 = Y[u] = u'Yu$ for some unimodular matrix u. We consider only the classes of equivalent positive definite matrices, and the theory of reduction consists in fixing in each class a typical element, called a *reduced matrix*, satisfying certain extremal properties. **29**

Let $Y > 0$ be a given matrix in $\mathscr{Y}$. Choose a primitive column $\tilde{u}_1$ such that $Y[\tilde{u}_1] = \tilde{u}'_1 Y\tilde{u}_1$ is a minimum for all primitive (integral) columns. Such a $\tilde{u}_1$ clearly exists. Consider now all integral column vectors $\tilde{u}$ such that $(\tilde{u}_1\tilde{u})$ is primitive and choose $\tilde{u} = \tilde{u}_2$ so that $Y[\tilde{u}_2]$ is a minimum. We can further assume that $\tilde{u}_1 Y\tilde{u}_2 \geq 0$ as otherwise $-\tilde{u}_2$ will serve the role of $\tilde{u}_2$ and satisfy this. Continuing in this way, let $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_r$ be already determined so that in particular $(\tilde{u}_1, \tilde{u}_2 \ldots \tilde{u}_r)$ is primitive. Then we choose $\tilde{u}_{r+1}$ such that

i) $\tilde{u}'_r y\tilde{u}_{r+1} \geq 0$

ii) $(\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_{r+1})$ is primitive and

iii) $y[u_{r+1}]$ is a minimum among those $\tilde{u}_{r+1}$ satisfying (i) and (ii).

In this way we obtain a unimodular matrix $u = (\tilde{u}_1, \tilde{u}_2 \ldots \tilde{u}_n)$ with some extremal properties and we call $R = y[u]$ *a reduced matrix*

30 We shall now obtain explicitly the reduction conditions on the elements $r_{\mu\nu}$ of a reduced matrix R

Let $u_{\Re}$ be a unimodular matrix which has the same first $(\Re - 1)$ columns as u. Such an $u_{\Re}$ can be represented by

$$u_{\Re} = u \begin{pmatrix} E & A \\ 0 & B \end{pmatrix}$$

where $E = E^{(\Re-1)}$, A integral and B, unimodular.Let $\mathscr{Y}_{\Re}$ be the $\Re^{\text{th}}$ column of $u^{-1}u_{\Re}$. Then the first column of B is formed just by the $(n-\Re+1)$ last elements $g_{\Re}, g_{\Re+1} \ldots g_n$ of $\mathscr{Y}_{\Re}$ so that these elements are coprime. Conversely if $g_{\Re}, g_{\Re+1} \ldots g_n$ by any $n - \Re + 1$ elements which are coprime, there exists a unimodular matrix B with these elements precisely constituting the first column and consequently, *a* $u_{\Re}$ too. Since $u\mathscr{Y}_{\Re}$ is the $\Re^{th}$ column of $u_{\Re}$ we obtain

$$Y[u\mathscr{Y}_{\Re}] = R[\mathscr{Y}_{\Re}] \geq r_{\Re\Re} \equiv r_{\Re}, \Re = 1, 2 \ldots, n;$$

$$\tilde{u}'_{\Re} Y \tilde{u}_{\Re+1} = r_{\Re\Re+1} \geq 0, \ \Re = 1, 2 \ldots \eta.$$

This proves

Lemma 3. *In order that $R = (r_{\mu\nu})$ should be a reduced matrix, it is necessary and sufficient that*

$$R[\mathscr{Y}_{\Re}] \geq r_{\Re}, r_{\Re\Re+1} \geq 0, \ \Re = 1, 2 \ldots n \tag{46}$$

where $\mathscr{Y}_{\Re}$ denotes an arbitrary integral column whose last $n - \Re + 1$ elements are coprime.

While the necessity part has been shown above, the sufficiency is immediate since $R = R[E]$ and this is clearly a reduced matrix by virtue of the conditions (46).

In case $\mathscr{Y}_{\Re} = \pm n_{\Re}$ the $\Re^{\text{th}}$ unit vector, the equation $\Re[\mathscr{Y}_{\Re}] = \Re_{\Re}$ holds identically in $\Re$ so that one of the condition (46) will now be in-
31 nocuous. We may therefore assume that the $\mathscr{Y}_{\Re}$ in Lemma 3 is different from $\pm n_{\mathcal{R}}$.

Lemma 4. *Every reduced matrix $R = (r_{\mu\nu})$ satisfies the following in equalities*

$$r_{\mathcal{R}} \leq r_{\ell}, \Re \leq \ell \tag{47}$$

$$r_{\ell} \leq 2r_{\Re\ell} \leq r_{\ell}, \Re > \ell \tag{48}$$

$$r_1 r_2 \ldots r_n < c_1 \mid R \mid \text{ where } C_1 = C_1(n), \tag{49}$$

viz. a positive number depending only on n.

While (47) is immediate from Lemma 3 by choosing $\mathscr{Y}_{\mathcal{R}} = n_{\ell}(\ell \geq \Re$ to prove (48) we have only to set $\mathfrak{Y}_{\Re} = n_{\Re} \pm k_{\ell}(\ell < \Re)$ The proof of (49) will be by induction on n. Clearly (49) is true for the case $n = 1$. Let R_{ℓ} denote the matrix which arises from R by deleting the last $n - \ell$ rows and columns. It is clear that $R_{\ell} > 0$. Then by Lemma 3, it will follow that R_{ℓ} is reduced too. The induction hypothesis will now imply that

$$r_1 r_2 \ldots r_{n-1} < C_2 \mid R_{n-1} \mid, C_2 = C_2(n) \tag{50}$$

We denote by $D_{\Re\ell}$ the $(n-2)$ rowed sub-determinant of R_{n-1} obtained by deleting the row and column containing $e_{\Re\ell}$. Then $D_{\Re\ell}$ is the sum of $(n-2)!$ terms of the type $r_{\nu_1 1} r_{\nu_2 2} \ldots r_{\nu_{\ell-1}\ell-1} r_{\nu_{\ell+1}\ell+1} \ldots r_{\nu_{n-1}n-1}$, Majorising each term with the help of (48), we get

$$\pm D_{\Re\ell} r_{\ell} < C_3 r_1 r_2 \ldots r_{n-1}, C_3 = C_3(n).$$

Consequently, in view of (50) we have

$$\pm D_{\Re\ell} \mid R_{n-1} \mid^{-1} < C_2 C_3 r_{\ell}^{-1}$$

We now define $\mathscr{H}$ by $\begin{pmatrix} R_{n-1} & \mathscr{H} \\ \mathscr{H}' & r_n \end{pmatrix}$

and set $r = r_n - R_{n-1}^{-1}[\mathscr{H}]$

Then $R = \begin{pmatrix} R_{n-1} & 0 \\ 0 & r \end{pmatrix} \left[\begin{pmatrix} E & R_{n-1}^{-1}\mathscr{H} \\ 0 & 1 \end{pmatrix} \right] \mid R \mid = r \mid R_{n-1} \mid$.
Let now $\xi' = (\mathfrak{z}' x_n)$ be a row with n variable elements. 32
We have

$$\begin{aligned} R[\xi] &= \begin{pmatrix} R_{n-1} & 0 \\ 0 & r \end{pmatrix} \left[\begin{pmatrix} E & R_{n-1}^{-1}\mathscr{H} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{z} & \\ x_n & . \end{pmatrix} \right] \\ &= \begin{pmatrix} R_{n-1} & 0 \\ 0 & r \end{pmatrix} \left[\begin{pmatrix} \mathfrak{z} + R_{n-1}^{-1}\mathscr{H} xn \\ x_n \end{pmatrix} \right] \\ &= R_{n-1}[\mathfrak{z} + R_{n-1}^{-1}\mathscr{H} x_n] + r x_n^2. \end{aligned}$$

The elements of R_{n-1}^{-1} are $\pm D_{\mathfrak{K}\ell} \mid R_{n-1} \mid^{-1}$ so that by virtue of (47) and (48) and (50) we have

$$\begin{aligned} R_{n-1}^{-1}[\mathscr{H}] &< C_2 C_3 \sum \frac{1}{r_\ell} r_{n\mathfrak{K}} r_{n\ell} \\ &< C_4 r_{n-1}, C_4 = C_4(n) \end{aligned} \tag{51}$$

Also $r_n = r + R_{n-1}^{-1}[\mathscr{H}] < r + C_4 r_{n-1}$ so that $r_1 r_2 \ldots r_n < C_2 \mid R_{n-1} \mid r_n < C_2\left(1 + C_4 \frac{r_{n-1}}{r}\right) \mid R \mid$. We will now show that $r_{n-1} < C_5 r, C_5 = C_5(n)$ and then we would have proved (49).

Let

$$C_6 = 4(n-1)^2, C_7 = (2n-2)^{n-1} = C_6^{\frac{n-1}{2}} \tag{52}$$

Let $\mathfrak{K}$ be determined such that

$$r_{\ell+1} < C_6 r_\ell \tag{53}$$

for $\ell = n-2, n-3, \ldots \mathfrak{K}+1, \mathfrak{K}$ but not for $\ell = \mathfrak{K}-1$ (The statement will have the obvious interpretation in the border cases corresponding to $\mathfrak{K} = 0$ and $\mathfrak{K} = n-1$). Let $x_\nu + a_\nu x_n$ be the ν^{th} element of $\mathfrak{z} + R_{n-1}^{-1}\mathscr{H} x_n$. For each integer x_n' in the interval $0 \leq x_n' \leq C_7^{n-\mathfrak{K}}$ we determine a set of $n - \mathfrak{K}$ integers $x_\nu', \nu = \mathfrak{K}, \mathfrak{K}+1 \ldots n-1$ such that $0 \leq x_\nu' + a_\nu x_n' < 1$ for each ν. Thus corresponding to the $C_7^{n-\mathfrak{K}} + 1$ possible choices for $x'n$ we obtain $C_7^{n-\mathfrak{K}} + 1$ points in the $(n-\mathfrak{K})$ dimensional Euclidean space,
33 all lying in the half open unit cube $0 \leq \mathfrak{Y}_\nu < 1, \nu = \mathfrak{K}, \mathfrak{K}+1, \cdots n-1$. If we divide this cube into equal cubes each of whose sides is of length

C_7^{-1}, there will be $C_7^{n-\Re}$ such cubes and consequently, by a well known principle, one of these cubes contains at least two of the above $C_7^{n\ \Re}+1$ points. Their difference is clearly a point with coordinates of the form $x_\nu + a_\nu x_n, \nu = \Re, \Re+1, \cdots, n-1$ where $x'_\nu s$ are integers,

$$| x_\nu + a_\nu x_n |< C_7^{-1}, 0 < x_n \leq C_7^{n-\Re} \tag{54}$$

In other words we have solved (54) with integral $x_\nu, x_n, \nu = \Re, \Re + 1, \cdots n-1$. We can assume that $x_\Re, \ldots x_n$ are coprime. Now we choose integers $x_1, x_2 \ldots x_{\Re-1}$ such that

$$1 x_\nu + a_\nu x_n |< 1, \nu = 1, 2, \ldots \Re - 1 \tag{55}$$

From Lemma 3, we have $R\,[\xi] \geq r_\Re$. On the other hand, the relation $r_\Re > C_6 r_{\Re-1}$ and the relations (52) - (55) entitle us to conclude that

$$\begin{aligned} R[\xi] &= R_{n-1}[\mathfrak{z} + R_{n-1}^{-1}\mathscr{H} x_n] + r x_n^2 \\ &< (\Re-1)^2 r_{\Re-1} + (\Re-1)(n-\Re) r_{\Re-1} C_7^{-1} + \\ &+ (n-\Re)^2 C_7^{-2} C_6^{b-\Re-1} r_\Re + r C_7^{2(n-\Re)} \\ &\leq (\Re-1)(n-1) C_6^{-1} r_\Re + (n-\Re)^2 C_6^{-\Re} r_\Re + C_7^{2n-2} r \end{aligned}$$

Thus $r_\Re < C_8 r$, $C_8 = C_8(n)$ and by (53), $r_{n-1} < C_5 r$. This completes the proof.

Having thus settled the arithmetical properties of reduced matrices, we proceed with their existential nature. We have already seen in the beginning of this section that given any matrix $Y = Y^{(n)} > 0$ there exists a unimodular matrix u such that $Y[u]$ is reduced. In other words, for any matrix $Y > 0$ there always exists an equivalent reduced matrix R. Such a matrix R is by no means unique. However, the number of matrices R equivalent to a given matrix $Y - Y^{(n)} > 0$ is finite and this number is **34**
bounded by an integer which depends only on n. It is our aim now to establish this result.

Given any quadratic form $Y[\xi]$, by the method of completion of squares, it can always be rewritten uniquely as

$$Y[\xi] = d_1(x_1 + b_{12}x_2 + \ldots b_{1n} \times_n)^2 +$$

$$+d_2(x_2 + b_{23}x_3 + \dots b_{2n}x_n)^2$$
$$+\dots\dots\dots$$
$$+d_n x_n^2$$

and, $Y > 0$ if and only if the $d_i's$ are positive. This shows that any matrix $Y > 0$ has a unique representation in the form

$$\gamma = D[B] \tag{56}$$

where D is a diagonal matrix $(\delta_{\mu\nu}d_{\mu\nu})$ with the diagonal elements $d_{\nu\nu} \equiv d_\nu$, all positive; and B is a matrix $(b_{\mu\nu})$ with $b_{\mu\mu} = 1$ and $b_{\mu\nu} = 0$ for $\mu > \nu$. A matrix $B = (b_{\mu\nu})$ whose elements satisfy the above conditions will be referred to as a *triangular matrix*. Assume now that Y is a reduced matrix $R = (r_{\mu\nu})$. From (56) we will have

$$r_\ell = d_\ell + \sum_{\nu=1}^{\ell-1} d_\nu b_{\nu\ell}^2, \ell = 1, 2, \dots, n$$
$$\text{and } \mid R \mid = d_1 d_2 \dots d_n.$$

In view of (49), this implies that

$$1 \leq \frac{r_\ell}{d_\ell} \leq \prod_{\nu=1}^{n} \frac{r_\nu}{d_\nu} < C_1 \tag{57}$$

and consequently, with the help of (47) we have for $\Re \leq \ell$

$$0 < \frac{d_\Re}{d_\ell} < C_1 \frac{r_\Re}{r_\ell} \leq C_1 \tag{58}$$

We use these to prove that in the case of reduced matrices Y represented in the form (56), the $b'_{\mu\nu}s$ have an upper bound depending only on n. The proof is by induction on u. Assume then that $\pm b_{\mathcal{P}\ell} < C_9$.
35 $C_9 = C_9(n)$ for $p = 1, 2, \dots \Re - 1$ and $\ell > p$. By means of the relation

$$r_{\Re\ell} = d_\Re b_{\Re\ell} + \sum_{p=1}^{\Re-1} d_p b_{p\Re} b_{p\ell},$$

Our assumption will imply in view of (57), (58) that for $\ell > \Re$,

$$\pm b_{\Re\ell} \leq \frac{r_\Re}{2d_\Re} + \sum_{p=1}^{\Re-1} \frac{d_p}{d_\Re} C_9^2 < \frac{1}{2} C_1 + (n-1) C_1 C_9^2 = C_{10}(n).$$

Thus assuming the result for $p = 1, 2, \ldots \Re - 1, \ell > p$, we have established it for $p = \Re$, $\ell > p$ and by the principle of induction, this completes the proof. We have now proved

Lemma 5. *Let $D = (\delta_{\mu\nu} d_{\mu\nu})$ be a diagonal matrix and $B = (b_{\mu\nu})$ a triangular matrix such that $\gamma = D[B] > 0$ is reduced. Then*

$$d_\nu < C_{11} d_{\nu+1}, \nu = 1, 2, \ldots n - 1, \quad (59)$$
$$\pm b_{\mu\nu < C_{11}}, \mu < \nu, C_{11} = C_{11}(n),$$

The possible converse to this is false, viz. if D^*, B^* be two other matrices whose elements satisfy (59) (with the same constant C_{11}) and D^* is diagonal while B^* is triangular, we cannot conclude that $R^* = D^*[B^*]$ is a reduced matrix. In this direction, however, we have

Lemma 6.

$$I f D* = (\delta_{\mu\nu} d^*_{\mu\nu}), d^*_{\nu\mu} \equiv d^*_\mu > 0; B^* = (b^*_{\mu\nu})$$

a triangular matrix and G integral matrix with $| \ G \ | \neq 0$ such that $D^*[B^*G]$ is reduced, then the elements of G all lie between two bounds which depend only on μ and $n - \mu$ being a common upper bound for the absolute values of $\dfrac{d^*\nu}{d^*\nu + 1}, \nu = 1, 2, \ldots, n-1, b_{\mu\nu}(\nu > \mu)$ and $| \ G \ |$ and n being the order of D^* or b^*.

Proof. We again resort to induction, this time on it. For $n = 1$ the lemma is clearly true. Since $D^*[B^*G]$ is reduced have by (56), $D^*[B^*G] = D[B]$ **36**
for some diagonal matrix D and triangular matrix B. Let $G = (g_{\mu\nu})$, $B^*GB^{-1} = Q = (q_{\mu\nu})$, $B^{*^{-1}} = (\beta_{\mu\nu})$ Then we have $D^*[Q] = D$ and $D[Q^{-1}] = D^*$ and therefore $d_\ell = \sum_{\Re=1}^n d^*_\Re q^z_{\Re\ell} (\ell = 1, 2, \ldots, n)$ Consequently, □

$$d^*_{\Re} q^2_{\Re\ell} \leq d_\ell, \Re, \ell = 1, 2, \ldots n. \tag{60}$$

Since $G = B^{*^{-1}} QB$ and B, B^* are triangular matrices, we have

$$g_{\Re\ell} = \sum_{\lambda=\Re}^{n} \sum_{\lambda=1}^{\ell} \beta_{\Re\lambda} q_{\lambda\lambda} \ell_{\lambda\ell}; \Re, \ell = 1, 2, \ldots n$$

Hence

$$d^*_{\Re} g^z_{\Re\ell} = d^*_{\Re} (\sum_{\lambda=\Re}^{n} \sum_{\lambda=1}^{n} \beta_{\Re\lambda} q_{\lambda\lambda} \ell_{\lambda\ell})^2$$

By assumption the $\ell'^* s$ are bounded, by μ so that the $\beta' s$ which are rational functions of the $\ell'^* s$ are also bounded, and by Lemma 5, the $\ell' s$ are bounded, in both the cases the bound depending only on μ and n.

We can therefore write

$$d^*_{\Re} g^2_{\Re\ell} < \mu^* (\sum_{\chi=\Re}^{n} \sum_{\lambda=\ell}^{\ell} q_{\chi\lambda})^2 d^*_{\Re}, \mu^* = \mu^*(n)$$
$$< \mu^* (\sum q_{\chi\lambda} q_{\chi'\lambda'} d^*_{\Re}), \chi\chi' \geq \Re$$

Majorising $q_{\chi\lambda} q_{\chi'\lambda'}$ by $(q^2_{\chi\lambda} + q^2_{\chi'\lambda'})$ and $q^2_{\chi\lambda} d^*_{\Re} (\chi \geq \Re)$ by $q^2_{\chi\lambda} d^*_{\chi}$, with the aid of (59), we get $d^2_{\Re} g^2_{\Re\ell} < \mu^*_1 \sum (q^2_{\chi\lambda} d^*_{\chi} + q^2_{\chi'\lambda'} d^*_{\chi'})$ and using (60), we have finally,

$$d^*_{\Re} g^2_{\Re\ell} < \mu_1 d_\ell \Re\ell = 1, 2, \ldots N \tag{61}$$

(μ_1 and the $\mu'_\nu s$ that occur subsequently in the course of the proof are positive constants depending only μ and R).

37 Replacing the equation xxxxx $D[Q^{-1}] = D^*$ and repeating the earlier arguments we have

$$d_{\Re} f^2_{\Re\ell} < \mu_2 d^*_\ell \tag{62}$$

where $(f_{\mu\nu}) = G^{-1}$.

Since $|\ (f_{\mu\nu})\ | = |\ G^{-1}\ | \neq 0$ there exists a permutation of the indices $1, 2, \ldots$ into $\ell_1, \ell_2 \ldots, \ell_n$ with $\prod\limits_{\Re=1}^{n} f_{\Re\ell_{\Re}} \neq 0$ Since $|\ G\ |\ f_{\Re\ell_{\Re}}$ is an integer, its absolute value is at least 1. Hence $1/f_{\Re\ell\Re}$ is bounded by the absolute value of $\mid G \mid$ and a fortiori by μ. Consequently, from (62) we have $d_{\Re} < \mu_3 d^*_{\ell_{\Re}}, \Re = 1 \cdots n$. Among the $(n - \Re + 1)$ indices $\ell_{\Re}, \ell_{\Re+1} \ldots \ell_n$, there should be at least one not exceeding $\Re$ so that

$$\min(d_{\Re}, d_{\Re+1}, \ldots d_n) < \mu \max(d^*_1, d^*_2, \ldots d^*_{\Re})$$

and hence, by means of (59) and the analogous assumption on the $d^{*'} s$ We have $d_{\Re} < \mu_4 d^*_{\Re}, \Re = 1, 2 \ldots n$. The relation (61) now allows us to conclude that

$$d_{\Re} g^2_{\Re\ell} < \mu_5 d_{\ell}, \Re, \ell = 1, 2, \ldots n \tag{63}$$

Let p denote the largest number among $1, 2, \ldots n$ such that the relation $d_{\Re} \geq \mu_5 d_{\ell}$ holds for $\Re = p, p+1, \ldots n$ and $\ell = 1, 2, \ldots p - 1$. For each g among $p + 1, p + 2, \ldots n$ there exists then a $\Re = \Re(g) \geq q$ and an $\ell = \ell_g < g$ with $d_{\Re} < \mu_5 d_{\ell}$ (with the appropriate interpretation for the border cases $p = 1, n$). In view of (59) we have then

$$d_g < \mu_6 d_{g-1}, g = p + 1, p + 1, \ldots n \tag{64}$$

(59) and (64) together imply that $d_{\ell} \mid d_{\Re}$ is a bounded quotient for

$$\ell, \Re = p, p + 1, \ldots n. \quad \text{i.e.} \quad d_{\ell} < \mu_7 d_{\lambda}$$

and consequently, from (63).

$$g^2_{\Re\ell} < \mu_5 \mu_4, \ell = p, p + 1, \ldots, n$$

By choice of $p, d_{\Re} \geq \mu_5 d_{\ell}, \ell = 1, 2, \ldots, p-1$ and $\Re = p, p+1, \ldots n$. **38**
Also $g_{\Re\ell}$ is an integer for each $\Re, \ell$. Hence (63) can hold only if $g_{\Re\ell} = 0$ for these indices, viz. $\Re = p, p + 1, \ldots n$; $\ell = 1, 2, \ldots p - 1$.

Thus G is a a matrix of the type

$$G = \begin{pmatrix} G_1 & G_{12} \\ 0 & G_2 \end{pmatrix}$$

where the elements of $G_2 = G_2^{(n-p+1)}$ are bounded by a μ_8. In case $p = 1$, the proof of Lemma 6 is already complete. In the alternative case, we write, analogous to G,

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad D^* = \begin{pmatrix} D^*_1 & 0 \\ 0 & D^*_2 \end{pmatrix}$$

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_{12} \end{pmatrix}, \qquad B^* = \begin{pmatrix} B_1^* & B_{12}^* \\ 0 & B_2^* \end{pmatrix}$$

and obtain

$$D_1^*[B_1^* G_1] = D_1[B_1] \tag{65}$$

by equating

$$D^*[B^* G] = \begin{pmatrix} D_1^*[B_1^* G_1] & * \\ * & * \end{pmatrix} \text{ and } D[B] = \begin{pmatrix} D_1[B_1] & * \\ * & * \end{pmatrix}$$

By assumption $D^*[B^*G]$ is reduced so that in particular $D_1^*[B_1^* G_1]$ is reduced. It is now immediate that our assumptions on D^*, B^* and G are also true of D_1^*, B_1^* and G_1, so that by induction assumption the elements of G_1 are all bound. What remains then to complete the proof is only to show that the elements of G_{12} are bounded.

From the matrix relation

$$G_1' D_1^*[B_1^*] G_{12} + G_1' B_1^{*'} D_1^* B_{12}^* G_2$$

with the help of (65), we get $G_{12} = G_1 B_1^{*'} B_{12} - B_1^{*'^{-1}} B_{12}^* G_1$,

39 As the elements of all the matrices occurring on the right side are bounded, the same is true of the elements of G_{12}, and in all these cases the bound depends only on μ and n. The proof is now complete. Lemmas 5 and 6 now yield

Lemma 7. *If $Y = Y^{(n)} > 0$ is a reduced matrix and $\mathcal{U}$, a unimodular matrix such that $Y[\mathcal{U}]$ is also reduced, then the elements of $\mathcal{U}$ all lie between bounds which depend only on n.*

As an immediate consequence, we infer that, equivalent to a given matrix $Y > 0$ there exists only a finite number of reduced matrices, and further, this number is bounded by a constant which depends only upon n.

We now proceed to determine the structure of the space $\mathscr{R}$ of the reduced matrices and its relationship to the space p of all real positive symmetric matrices. We first observe that p is an open domain in the space $\mathscr{Y}$ of all real symmetric matrices. For if $G_{\mathfrak{R}} \in \mathscr{Y}, G_{\mathfrak{R}} \notin \mathfrak{P}\mathfrak{R} =$

$1, 2, \ldots$, and if $G_{\Re} \to G \in \mathscr{Y}$, then for each $\Re$, there exists a vector $\mathscr{E}$ which may be supposed to be of length 1 with $G_{\Re}[\mathscr{E}_{\Re}] \leq 0$. Then a subsequence $\mathscr{E}_{\nu_{\Re}}$ clearly converges to a vector $\mathscr{E}$, again of length 1, and

$$G[\mathscr{E}] = \lim_{\Re} G_{\nu_{\Re}}[\mathscr{E}\nu_{\Re}] \leq 0$$

whence $G \notin \mathfrak{P}$. This certainly implies that $\mathfrak{P}$ is an open do main. Let now G be a boundary point of $\mathfrak{P}$ so that $G \notin \mathfrak{P}$. Then there exists a sequence $y_{\Re} \in \mathfrak{P}$ with $y_{\Re} \to G$. Since $\gamma_{\Re}[\mathscr{E}] > 0$ for every $\mathscr{E} \neq 0$, we get $G[\mathscr{E}] \geq 0$ for $\mathscr{E} \neq 0$. Also $G[\mathscr{E}] = 0$ for at least one $\mathscr{E}$, as otherwise G will belong to $\mathcal{P}$. Every symmetric matrix G with this property, viz., $G[\mathscr{E}] \geq 0$ for every $\mathscr{E} \neq 0$, the equality holding for at least one such $\mathscr{E}$, shall be called *semi positive*. We can now state that any boundary point of $\mathfrak{P}$ is semi positive. Conversely too, every semi positive matrix G is a **40**
boundary point of $\mathfrak{P}$, since $G+ \in E$ lies in $\mathfrak{P}$ for every $\in> 0$ but $G \notin \mathfrak{P}$. In $\mathfrak{P}$, we now consider the domain $\mathscr{R}$ of all reduce $\gamma > 0$. Let $R = (\mathscr{R}_{\mu\nu})$ be a point of $\mathscr{R}$. This means by Lemma 3 that

$$R[\mathscr{Y}_{\Re}] \geq r_{\Re\Re} \equiv, n_{\Re\Re+2} \geq 0 \tag{46$'$}$$

for any integral $\mathscr{Y}_{\Re}$ whose last $n - \Re + 1$ elements are coprime, $\Re = 1, 2, \ldots n - 1$.

Interpreting the $\dfrac{1}{2}n(n+1)$ independent elements of R as the cartesian coordinates in the Euclidean space of the same dimension, the above inequalities define a cone with the apex at the origin. We shall show that much more is true.

Let R_o be a boundary point of $\mathscr{R}$. Then either $R_o \in \mathfrak{P}$ in which case it is positive or $R_0 \in Bd\mathfrak{P}$ in which case it is semi positive. Consider first the case when R_0 is positive. Then it satisfies (46) and also there exists a sequence $y_{\Re} > 0$ in $\mathscr{Y}$ such that $y_{\Re} \notin \Re$ and $y_{\Re} \to R_o$. We represent R_o according to (56) in the form $R_o = D[B]$ with a diagonal matrix $D = (d_{\mu\nu}\ d_{\mu\nu})$ and a triangular matrix $E = (b_{\mu\nu})$. The transformation $(r_{\mu\nu}) \to (d_\nu, b_{\mu\nu})$ defines a topological mapping of a neighbourhood of R_o on to a neighbourhood of $(d_\nu, b_{\mu\nu})$ (where, in the indices for b, we assume $\mu < \nu$. We represent $y_{\Re}$ in the same way, viz. $y_{\Re} = D_{\Re}[B_{\Re}]$, and in view of the topological character of the above mapping, we conclude that $D_{\Re} \to D$

and $E_{\Re} \to B$. Since $D[P]$ is reduced, the elements of D, B satisfy (59) and the same is therefore true of the elements of $D_{\Re}, B_{\Re}$ for sufficiently large $\Re$. Let $u_{\Re}$ be unimodular such that $y_{\Re}[u_{\Re}]$ is reduced. $y_{\Re} \notin \mathscr{R}$ by assumption so that $u \neq \Re$. Lemma 6 then shows that for sufficiently
41 large $\Re$, $d_{\Re}$ belongs to a finite set of matrices and in particular, there exist an infinity of $\Re's$ for which the $\mathcal{U}'_{\Re}s$ are the same, say, $\mathcal{U}_{\Re} = \mathcal{U} \neq \pm E$. As $\Re \to \infty$ through this sequence of values, we have

$$R_o[\mathcal{U}] = \lim_{\Re} \gamma_{\Re}[\mathcal{U}] \in \mathscr{R}$$

since each $y_{\Re}[\mathcal{U}] \in \mathscr{R}$ and $\mathscr{R}$ is easily seen to be a closed set. Thus what we have shown is that if R_o is a positive boundary point of $\mathscr{R}$, there exists a unimodular matrix $u \neq \pm E$ such that

$$R_o[\mathcal{U}] \in \Re. \tag{66}$$

Indeed $R_o[\mathcal{U}]$ is a boundary point of $\mathscr{R}$ and this, we proceed to : establish. More generally we show that *if $R_o, R_1 \in \mathscr{R}$ and $\Re_1 = \Re_o[\mathcal{U}]$ for some unimodular matrix $\mathcal{U} \neq \pm E$, then both R_o and R_1 are boundary points of $\mathscr{R}$.*

Assume first that $\mathcal{U}$ is not a diagonal matrix. Let $\mathscr{Y}_1, \mathscr{Y}_2, \ldots \mathscr{Y}_n$ be the columns of $\mathcal{U}$ and let $\mathscr{Y}_{\Re}$ be the first column which is different form $\pm n_{\Re}$ (If no such $\mathscr{Y}_{\Re}$ exists, then $\mathcal{U}$ would be diagonal, contrary to assumption). Then the $\Re^{th}$ column $\mathscr{Y}_{\Re}$ of $\mathcal{U}^{-1}$ has also the same property. Let us write $R_o = (r_{\mu\nu})$, $R_1 = (\mathscr{S}_{\mu\nu})$. Then we have

$$\mathscr{S}_{\Re} = R_o[\mathscr{Y}] \geq = R_1[\mathscr{Y}_{\Re}] \geq \mathscr{S}_{\Re}$$

Hence the equality holds throughout and

$$R_o[\mathscr{Y}_{\Re}] = r_{\Re} = \mathscr{S}_{\Re} = R_1[\mathscr{S}_{\Re}]$$

If R_o is an interior point of $\mathscr{R}$, then the strict inequality must hold in (46) for all $\mathscr{Y}_{\Re}$. As we have shown the equality to hold for one $\mathscr{Y}_{\Re}$ it follows that $R_o \in Bd\mathscr{R}$. The same is of course true of R_1 too. We stop here to make the following remark.

From Lemma 7, it is clear that $\mathscr{Y}_{\Re}$ and $\mathcal{S}_{\Re}$ belong to a finite set of primitive vectors. So (66) allows us to conclude that from the infinite

42 set of inequalities ((46)′) defining $\mathscr{R}$, it is possible to determine a finite subset such that at least one of this finite set of inequalities reduces to an equality in the case of any positive boundary point of $\mathscr{R}$. In other words, the positive boundary points of $\mathscr{R}$ all lie on finite number of planes and these planes bound a convex pyramid $\mathscr{E}$ containing $\mathscr{R}$. Of course we have still the case when $\mathcal{U}$ is a diagonal matrix to settle, to complete the proof. In this case all diagonal elements are ± 1, the sign changing at least once. Let then the sign change for the first time from the q^{th} to $(q+1)^{\text{th}}$ element. By changing $\mathcal{U}$ to $-\mathcal{U}$ if necessary, we shall have

$$\mathscr{S}_{qq+1} = \mathscr{Y}'_{q+1} R_0 \mathscr{Y}_q = x'_{q+1} R_0 x_q = -r_{qq+1}.$$

But due to one of the reduction in equalities $r_{qq+1} \geq 0, \mathscr{S}_{qq+1} \geq 0$. Hence it follows that

$$r_{qq+1} = 0 = \mathscr{S}_{qq+1} \tag{67}$$

It is immediate that R_o and R_1 are again boundary points of $\mathscr{R}$ and one of the inequalities in (46) reduces to an equality. Now our earlier remark is unreservedly valid.

We now show the interior of $\mathscr{R}$ is non void. Consider a compact set $\mathscr{L} \subset \mathfrak{P}$ with non null interior. Then its interior is of the highest dimension, viz. $n(n+1)/2$. Let $y \in \mathscr{L}$. Then by (56) we can represent y in the form $y = D[B]$ and then, since $\mathscr{L}$ is compact the ratios $d_\nu \mid d_{\nu+1}$ and $\pm b_{\mu\nu}$ are bounded by μ (say), μ depending only on $\mathscr{L}$. So now, if we determine a unimodular $\mathcal{U}$ such that $y[\mathcal{U}]$ is reduced, then by Lemma 6, the elements of $\mathcal{U}$ are all bounded, the bound depending only on $\mathscr{L}$ and n : in other words, even though the $y's$ belonging to $\mathscr{L}$ may be infinite, there are only a finite number of $\mathcal{U}'s$ such that $y[\mathcal{U}]$ is reduced. If $\mathscr{R}$ has no interior points, then the $y[\mathcal{U}]'s$ are all boundary points of $\mathscr{R}$ and hence belong to a finite set of planes. This would then 43
mean that a finite number of planes is mapped by a finite set of $\mathcal{U}'s$ on to a set of dimension $n(n+1)/2$ which is clearly impossible. Hence we conclude that $\mathscr{R}$ has interior points.

Let now $\mathscr{R}$ be an interior point of $\mathscr{R}$. We consider the segment $(1-\lambda)T + \lambda R, 0 \leq \lambda \leq 1$ where $T \notin \mathscr{E}$ but $T \in \mathscr{R}$. Our aim is to show that such a T is semi positive. Since $\mathscr{E}$ is a convex set, all the points on the above segment are points of $\mathscr{E}$ and all, except possibly T, are interior points of $\mathscr{E}$. On this line there exists a boundary point R_o of $\mathscr{R}$. R_o cannot be positive as otherwise R_o would lie on one of the planes which bound $\mathscr{E}$ so that it is a boundary point of $\mathscr{E}$ too, and consequently $R_o = T \in \mathscr{R}$ - a contradiction to the choice of T. Hence R_0 is semi positive. The characteristic roots of a semi positive matrix are all ≥ 0 and one at least among them is zero. Hence $| R_o |= o$ so that writing $R_o = (r^o_{\mu\nu})$ and approaching R_o through a sequence of matrices $R = (r_{\mu\nu}) \in \mathscr{R}$ we obtain, in view of the reduction conditions, viz.

$$\pm 2r_{1\nu} \leq r_1 \leq r_2 \leq \ldots \leq r_n,$$
$$r_1 r_2 \ldots r_n < C_1 \mid R \mid \to C_1 \mid R_0 \mid = 0$$

that $r^o_{1\nu} = 0, \nu = 1, 2, \ldots n$

But $R_o = (1 - \lambda_o T) + \lambda_o R$ for some $\lambda_0, 0 \leq \lambda_0 \leq 1$; we have therefore in particular

$$0 = (1.\lambda_o)t_{1\nu} + \lambda_0 r_{1\nu}, \nu = 1, 2, \ldots n$$

We may consider the above as linear equations in the variable λ so that the determinant $\begin{vmatrix} t_{1\nu} & r_{1\nu} \\ t_{11} & r_{11} \end{vmatrix} = 0$ or $b_{1\nu} = t_\mu \dfrac{r_{1\nu}}{r_{11}} \nu = 2, \ldots n$.

But Γ is a given point and R is an arbitrary point in the segment. We therefore conclude that $t_{11} = 0 = t_{1\nu}$ and then from an earlier equation
43 $\lambda_0 = 0$. Thus $T = R$ and hence is semi positive. We have therefore shown that $\varepsilon - \mathscr{R}$ consists of semi positive boundary points of $\mathscr{R}$. Conversely, any such point lies in $\varepsilon - \mathscr{R}$ since an arbitrary neighborhood of this point includes exterior points of $\mathscr{R}$. Now we obtain the main result of Minkowski's reduction theory, viz.

Theorem 2. *The space $\mathscr{R}$ of the reduced matrices $y > 0$ is a convex pyramid with the vertex at the origin. If $\mathcal{U}$ runs over all unimodular matrices, wherein $-\mathcal{U}$ will not be considered as different from $\mathcal{U}$, then the images of $\mathscr{R}$ under the group of mappings $y \to y[\mathcal{U}]$ cover the space $\mathfrak{P}$ of all symmetric matrices $y < 0$ without over lapping (except for boundary points). In other words, $\mathscr{R}$ is a fundamental domain in $\mathfrak{P}$ for the group of mappings $Y \to Y[\mathcal{U}]$ acting on it.*

Further, $\mathscr{R}$ has a non void interior and its boundary lies on a finite set of planes.

Chapter 4

The Fundamental Domain of the Modular Group of Degree n

The tools now at our disposal enable us to construct a fundamental domain in the generalised upper half plane for the modular group acting on it. We need a few preliminaries. **44**

We first prove that given any point $Z \in \mathscr{Y}$ there exists only a finite number of classes $\{C, D\}$ of coprime symmetric pairs C, D for which the absolute value of $| CZ + D |$, written as $\| CZ + D \|$, has a given upper bound $\mathfrak{K}$, i.e. such that $\| CZ + D \| \leq \mathfrak{K}$. Let r denote the rank of C. We employ the parametric representation of (C, D) given by Lemma 1. Then

$$C = \mathcal{U}_1 \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{U}_2', D = \mathcal{U}_1 \begin{pmatrix} 0_1 & 0 \\ 0 & E \end{pmatrix} \mathcal{U}_2^{-1} \tag{68}$$

where $\mathcal{U}_2 = (QR)$, $Q = Q^{(n,r)}$

We have then

$$CZ + D = \mathcal{U}_1 \left\{ \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{U}_2' Z \mathcal{U}_2 + \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix} \right\} \mathcal{U}_2^{-1} \qquad \text{and}$$

$$\mathcal{U}_2' Z \mathcal{U}_2 = \begin{pmatrix} Q' \\ R' \end{pmatrix} Z(QR) = \begin{pmatrix} Z[Q] & Q'ZR \\ R'ZQ & Z[R] \end{pmatrix}$$

Thus $CZ+D=\mathcal{U}_1\begin{pmatrix} C_1Z[Q]+D_1 & * \\ 0 & E \end{pmatrix} u_2^{-1}$ and

$$| CZ+D | = | \mathcal{U}_1 || \mathcal{U}_2 |^{-1} | C_1Z[Q]+D_1 | = \pm | C_1Z[Q]+D_1 |$$
$$= \pm | C_1 || Z[Q]+P | \text{ where } P = C_1^{-1}D_1$$

We shall subsequently need this relation

$$| CZ+D | = \pm | C_1 | \left| Z[Q]+P \right| \tag{69}$$

We may observe that P is a rational symmetric matrix which uniquely determines and is uniquely determined by the class $\{C_1, D_1\}$. For if $C_0^{-1}D_0 = C_1^{-1}D_1 = P$, then $D_0C_1' = C_0D_1'$ which means that $\{C_0, D_0\}$
45 $= \{C_1, D_1\}$, while if A, B is any pair in the class $\{C_1, D_1\}$, then $A = \mathcal{U}C_1$, $B = \mathcal{U}D_1$, for some unimodular $\mathcal{U}$, and $A^{-1}B = C_1^{-1}D_1 = P$. What is more, while $P = C_1^{-1}D_1$ is rational symmetric for any coprime symmetric pair (C_1, D_1), conversely too, any rational symmetric matrix P is of the above form. We need to only observe that in this case, we can choose unimodular matrices $\mathcal{U}_3$, $\mathcal{U}_4$ such that

$\mathcal{U}_3P\mathcal{U}_4 = \left(\delta_{\mu\nu}\dfrac{a_\nu}{b_\nu}\right)$, viz. a diagonal matrix with the diagonal elements a_{ν/b_ν}, $\nu = 1, 2, \ldots n$ where a_ν, b_ν are coprime integers with $\ell_\nu > O$ and then, defining $C_1 = (\delta_{\mu\nu}b_\nu)\mathcal{U}_3^{-1}$, $D_1 = (\delta_{\mu\nu}a_\nu)\mathcal{U}_4$, we have $P = C^{-1}D_1$.

We now return to the representation (68) of C, D. Let X, Y denote the real and imaginary parts of Z. Since Q in the above representation can always be replaced by $Q\mathcal{U}_3$, $\mathcal{U}_3$-unimodular, while preserving the form of (68), we can assume that $Y[Q]$ is reduced. Let then

$$\left.\begin{aligned} S &= X[Q]+P \\ T &= Y[Q] \end{aligned}\right\} \tag{70}$$

and lat $F^{(r)}$ a real matrix with $T[F] = E, S[F] = H = (\delta_{\mu\nu}b_\nu)$. Then $(S+iT)[F] = H+iE$ and $(S+iT) = (H+iE)[F^{-1}]$

Now $|Z[Q]+P| = |S+iT| = |T| \prod_{\nu=1}^{r}(i+h_\nu)$ and

$$\|Z[Q]+P\|^2 = |T|^2 \prod_{\nu=1}^{r}(l+h_\nu^2).$$ Hence it follows that

$$\|CZ+D\|^2 = |C_1|^2|T|^2 \prod_{\nu=1}^{r}(1+h_\nu^2) \tag{71}$$

By assumption $\|CZ+D\| \leq \mathfrak{K}$; also, C_1 being integral, $|C_1| > 1$, and trivially $1+h_\nu^2 > 1$. (71) now implies that $|T|$ is bounded. But T is reduced so that one of the inequalities (47-49) will imply that

$\prod_{\nu=1}^{r} Y[\mathscr{Y}] < Q_1|T|$ where we denote $Q = \mathscr{Y}_1\mathscr{Y}_2\mathscr{Y}_n)$ Since each of the factor $Y[\mathscr{Y}_\nu]$ has a positive lower bound, vis $Y[\mathscr{Y}\nu] \geq \lambda\mathscr{Y}_\nu'\mathscr{Y}_\nu \geq \lambda > o$ **46**
where λ denotes the smallest characteristic root of Y, we conclude from the above that each factor $Y[\mathscr{Y}_\nu]$ is bounded. As a consequence, the $\mathscr{Y}_\nu'\mathscr{S}$ belong to a finite set of primitive columns and there are only a finite number of possible choices for Q. In particular, therefore, the elements of $T = Y[Q]$ are bounded and since $T = F'^{-1}F^{-1}$ the same is true of the elements of F too. Also, the number of distinct $T's$ being finite, $|T|$ has a positive lower bound and being an integer, $|C_1|^2 \geq 1$ so that we infer from (71) that the $h_\nu's$ and what is the same, the matrix H are bounded. But

$S+iT = (H+iE)[\Gamma^{-1}]$ and we already know that F^{-1} is bounded. Hence $S+iT = Z[Q]+P$ is bounded and so also $P = S - \times[Q]$ Since $|C_1|P$ is integral and $|C_1|$ is bounded as is seen from the relation

$$|C_1|^2|T|^2 \prod_{\nu=1}^{r}(1+h_\nu^2) < \mathfrak{K}$$

it follows that the number of distinct $P's$ occurring here is finite. We know however that distinct classes $\{C_1, D_1\}$ correspond to distinct $P's$ so that the number of distinct classes $\{C_1, D_1\}$ occurring are finite. As we have shown already that the number of classes $\{Q\}$ is also finite we conclude that the number of classes $\{C, D\}$ in our discussion, viz. those which satisfy $\|CZ+D\| < \mathfrak{K}$ for a given $\mathfrak{K} > O$ is finite and this was what we were after.

Let Y, Y^* be the imaginary parts of two points Z, $Z^* \in \mathscr{Y}$ which are equivalent with regard to $\mathcal{M}$. We call Z^* *higher than* Z if we have

$|Y^*| > |Y|$. If $Z^* = (AZ + B)(CZ + D)^{-1}$, then we have from (24) that $Y^* = (ZC' + D')^{-1} Y(C\overline{Z} + D)^{-1}$ so that

$$|Y^*| = |Y| \left| CZ + D \right|^{-2} \tag{72}$$

47 Thus Z^* is higher than Z if and only if $\|CZ + D\| \geq 1$ where we assume $Z^* = (AZ + B)(CZ + D)^{-1}$ As a consequence we have that in each class of equivalent points there exist at least one highest point. For, in the alternative case, there would exist a sequence of equivalent points $Z_{\Re}$,

$$Z_{\Re} = X_{\Re} + iY_{\Re} = (A_{\Re} Z_1 + B_{\Re})(C_{\Re} Z_1 + D_{\Re})^{-1} \begin{pmatrix} A_{\Re} B_{\Re} \\ C_{\Re} D_{\Re} \end{pmatrix} \in \mathcal{M}$$

such that $|Y_1| < |Y_2| < \ldots ad \inf.$ and this will imply in its turn by (72) that

$1 > \|C_Z Z_1 + D_2\| > \|C_3 Z_1 + D_3\| > \ldots ad \inf$. In other words we end up with the conclusion that an infinite number of classes $\{C_{\Re}, D_{\Re}\}$ have the property that $\|C_{\Re} Z_1 + D_{\Re}\| < 1$ a contradiction to an earlier result. Now we state

Theorem 3. *The domain $\mathfrak{f}$ defined by the following inequalities represents a fundamental domain in $\mathcal{Y}$ with regard to $\mathcal{M}$. We denote $Z = x + iy$, $X = (x_{\mu\nu})$, $Y = (y_{\mu\nu})$ and then $Z \in \mathcal{F}$ is defined by*

(i) $\|CZ + D\| \geq 1$ *for all coprime symmetric pairs C, D,*

(ii) $\left.\begin{array}{l} Y[y_{\Re}] \geq y_{\Re\Re}, \\ y_{\Re\Re+1} \geq 0, \\ \quad \Re = 1, 2, \ldots \end{array}\right)$ *(Minkowski's reduction conditions) where $y_{\Re}$ is an arbitrary integral column with its last $n - k + 1$ elements coprime.*

(iii) $-\dfrac{1}{2} \leq x_{k\ell} \leq \dfrac{1}{2}, \Re, \ell = 1, 2, \ldots n$

Also $\mathcal{F}$ is a connected and closed set which is bounded by a finite number of algebraic surfaces.

48 *Proof.* We first show that given any point $Z \in \mathscr{Y}$, $\mathcal{F}$ contains an equivalent point $Z_1 = M < Z >$. In fact, since we know that any class of equivalent points contains a highest point, we can assume that Z is a highest point and then $\|CZ + D\| \geq 1$ for all coprime symmetric pairs C, D. If now $M = \begin{pmatrix} \mathcal{U}' & S\mathcal{U}^{-1} \\ 0 & \mathcal{U}^{-1} \end{pmatrix}$ where $\mathcal{U}$ is unimodular and S an integral symmetric matrix to be specified presently, we shall have

$Z_1 = Z[\mathcal{U}] + S, Y_1 = Y[\mathcal{U}] = \mathcal{U}'Y\mathcal{U}$ and $X_1 = X[\mathcal{U}] + S$ where we assume $Z_1 = X_1 + iY_1$. We can determine $\mathcal{U}$ such that $Y_1 = Y[\mathcal{U}]$ is reduced and then S can be chosen such that X_1 satisfies the last of the three conditions in the theorem. The first condition is then automatically ensured as $|Y_1| = |Y|$ so that Z_1 is a highest point as Z is. The Z_1 thus determined clearly serves. □

Further, no two distinct interior points of $\mathcal{F}$ can be equivalent. For let $Z, Z_1 \in Int.\mathcal{F}$ and $Z_1 = M < Z >, M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then, in the conditions stipulated in theorem 3 for Z, we have strict inequality throughout except in the cases where these reduce to identities in Z, viz. when $\mathscr{Y}_{\mathfrak{R}} = \pm n_{\mathfrak{R}}$ and $(C, D) = (0, \mathcal{U}^{-1})$, $\mathcal{U}$ being unimodular. Now $Z_1 = (AZ + B)(CZ + D)^{-1}$ so that

$$(-C'Z_1 + A')(CZ + D) = E \tag{73}$$

Since $Z \in \mathcal{F}$ and $(C, D), (-C', A')$ are coprime symmetric, we have $\|CZ + D\| \geq 1$ and $\| - C'Z_1 + A'\| \geq 1$. Since their product is equal to 1 by (73), we conclude that the equality holds in both cases. It therefore follows that (C, D) is one among the exceptional pairs singled out earlier, 49
viz. $C = o$, $D = \mathcal{U}^{-1}$ for some unimodular matrix $\mathcal{U}$. Then from (73) we have

$$\left.\begin{aligned} Z_1 &= Z[\mathcal{U}] + S \\ A &= D'^{-1} = \mathcal{U}' \end{aligned}\right\} \tag{74}$$

where we write $S = B\mathcal{U}$. In particular, writing $Z_1 = X_1 + iY_1$ we have $X_1 = X[\mathcal{U}] + S$, $Y_1 = Y[\mathcal{U}]$. Since Y and Y_1 are both reduced matrices and Y is an interior point of $\mathscr{R}$, theorem 2 states that $\mathcal{U} = \pm E$. Then $X_1 = X + S$ and condition (iii) of theorem 3 will require that $S = O$.

Thus $Y_1 = Y$ and $X_1 = X$ and consequently $Z_1 = Z$. We have therefore shown that two points of $\mathcal{F}$ can be equivalent only if both are boundary points of $\mathcal{F}$, a property typical of a fundamental domain and this settles the first part of theorem 3.

We now show that $\mathcal{F}$ is a closed set. Let $Z_{\Re} \in \mathcal{F}, \Re = 1, 2, \ldots,$ and $Z_{\Re} \to Z$. Clearly the conditions (i) and (iii) are satisfied by Z as they are true of each $Z_{\Re}$. Condition (ii) also would be fulfilled by Z except that we do not know from the fact $Y_{\Re} > o$ for each $\mathscr{R}$, that $Y > o$. We shall show that this is so. More generally we show that $|Y|$ has a positive lower bound for all Y such that $Z = X + iY \in \mathcal{F}$. If $(C_1^{(r)}, D_1^{(r)})$ is any coprime pair with $|C_1| \neq o$ and $Q^{(n,r)}$, a primitive matrix, then the pair (C, D) given by (68) is always symmetric and coprime so that for (69) we have $\|CZ + D\| = \|C_1 Z[Q] + D_1\| \geq 1$. Choosing in particular $C_1 = E^{(r)}, D_1 = o$ and $Q = \begin{pmatrix} E^{(r)} \\ 0 \end{pmatrix}$, we have $\|CZ + D\| = \|Z_r\|, Z_r$ being the matrix which results when Z is deprived of its last $(n - r)$ rows and
50 columns and this is true for each r. If now $Z \epsilon \mathcal{F}$, then $\|CZ + D\| \geq 1$ by one of the condition of $\mathcal{F}$ so that we can conclude that $\|Z_r\| \geq 1$ for each r and every $Z \epsilon \mathcal{F}$. In particular, setting $r = 1$, this means that $\|Z_1\| = |Z_{11}| \geq 1$ which in its turn implies by reason of $|X_{11}|$. being less than or equal to $\frac{1}{2}$ that

$$\boxed{y_{11} \geq \frac{1}{2}\sqrt{3}} \tag{75}$$

where $x_{11} + iy_{11} = Z_{11}$

In view of the reduction conditions (47 - 49) we have

$y_{11}^n \leq \prod\limits_{r=1}^{n} y_{rr} < C_1 |Y|$ and the above inequality now implies the desired result. Specifically we have

$$|Y| > C_1^{-1} (\frac{1}{2}\sqrt{3})^n > o \tag{76}$$

for every $Z = X + iy \in \mathcal{F}$

The proof of the connectedness of $\mathcal{F}$ is a bit more involved.

let (C, D) be a given coprime symmetric pair and use the representation (69) for C, D. We recall the following notation.

$$P = C_1^{-1} D_1, S = X[Q] + P, T = Y[Q], T[F] = E, S[F] = R = (\delta_{\mu\nu} h_\nu)$$

We further introduce $Z_1 = X + i\lambda Y$ with $\lambda \geq 1$. Then we have, as in deriving (71),

$Z[Q] + C_1^{-1} D_1 = (H + iE)[F^{-1}], Z_1[Q] + C_1^{-1} D_1 = (H + i\lambda E)[F^{-1}]$, $\|CZ + D\|^2 = |C_1|^2 |T|^2 \prod_{\nu=1}^{r} (1 + h_\nu)^2$ and

$$\|CZ_1 + D\|^2 = |C_1|^2 |T|^2 \prod_{\nu=1}^{r} (\lambda^2 + h_\nu^2) \tag{71$'$}$$

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It is immediate that $\|CZ_1 + D\| \geq \|CZ + D\|$ and that $Z \in \mathcal{F}$ implies that $Z_1 \in \mathcal{F}$ for $\lambda \geq 1$, $Z_1 = X + i\lambda Y$.

We now ask how to choose λ such that $Z_1 = X + i\lambda Y \in \mathcal{F}$ for two matrices X, Y which satisfy the condition (iii) and condition (ii) respectively of theorem 3, and we show that this will be fulfilled if λ is chosen sufficiently large-specifically if

$$\lambda \geq \frac{C_1}{y_{11}}, C_1 = C_1(n) \tag{77}$$

Let λ satisfy the above condition (77). Then from ((71)$'$) we have $\|CZ_1 + D\|^2 \geq \lambda^{2r} |T|^2$ so that

$$\|CZ_1 + D\| > \lambda^r |T| = \lambda^r |Y[Q]|$$

As in earlier contexts we can assume that $T = Y[Q]$ is reduced by an appropriate choice of Q. Then, in view of (47 - 49), $|T| \geq C_1^{-1} \prod_{\nu=1}^{r} Y[y_\nu] \geq C_1^{-1} y_{11}^r \geq \lambda^{-r}$ where we denote $Q = (y_1 y_2 \ldots y_r)$ and assume without loss of generality that $C_1 > 1$. Hence $\|CZ_1 + D\| \geq \lambda^r |T| \geq 1$ for all symmetric coprime pairs (C, D) and this is precisely what ensures us that $Z_1 \in \mathcal{F}$

Let now $Z_\nu \in \mathscr{F}$, $Z_\nu = X_\nu + iY_\nu$, $\nu = 1, 2$, and let $\lambda_o \geq 2C_{1/\sqrt{3}}$ join Z_1 and Z_2 by the polygon consisting of

$$Z = X_1 + i\lambda Y_1, 1 \leq \lambda \leq \lambda_o, \tag{78}$$

$$Z = (1-\lambda)(X_1 + i\lambda_0 Y_1) + \lambda(X_2 + i\lambda_o Y_2), o \leq \lambda \leq 1, \tag{79}$$
$$Z = X_2 + i\lambda_{Y_2}, i \leq \lambda \leq \lambda_o \tag{80}$$

52

We prove that this polygon lies completely in $\mathscr{F}$. The result we proved above and (75) together imply that the lines defined by (78) and (80) both lie in $\mathscr{F}$. It remains then to consider only the line determined by (79). Since y_1, y_2 are reduced, and $\mathscr{R}$ is convex, $(1-\lambda)Y_1 + \lambda Y_2 \in \mathscr{R}$ for $0 \leq \lambda \leq 1$. Also since x_1, x_2 satisfy the condition (iii) of theorem 3, so does $(1-\lambda)x_1 + \lambda x_2, o \leq \lambda \leq 1$. Hence, in view of our earlier result, $Z = \{(1-\lambda)x_1 + \lambda x_2\} + i\lambda_o\{(1-\lambda)y_1 + \lambda y_2\}$ belongs to $\mathscr{F}$ provided λ_o satisfies (77). We shall show that it does. This is in fact immediate since

$$\lambda_o \geq \frac{2C_1}{\sqrt{3}} = \frac{C_1}{(1-\lambda)\frac{\sqrt{3}}{2} + \lambda\frac{\sqrt{3}}{2}}$$
$$\geq C_I/(1-\lambda)y'_{11} + \lambda y''_{11}$$

It follows therefore that the points Z in (79) all belong to $\mathscr{F}$. We can now conclude that $\mathscr{F}$ is a connected set.

Our assertion concerning the boundary of $\mathscr{F}$ still remains to be settled. Specifically, we have got to show that the boundary of $\mathscr{F}$ consists of a finite number of algebraic surfaces. First we note that every positive matrix $\mathscr{Y}$ satisfies the inequality

$$|y| \leq y_{11}y_{22}..y_{nn} \tag{81}$$

53 where we assume $y = (y_{\mu\nu})$

For we can write $y = \mathcal{R}'\mathfrak{K}$ with a non-singular real matrix $\mathfrak{K} = (\mathscr{W}_1\mathscr{W}_2 \cdots \mathscr{W}_n)$ Then it is known that

$$|\mathfrak{K}|^2 \leq \prod_{\nu=1}^{n} \mathscr{W}_\nu{}'\mathscr{W}_\nu = \prod_{\nu=1}^{n} y_{\nu\nu}$$

But $|\mathfrak{K}|^2 = |\mathcal{Y}|$ and this proves what is desired. We proceed to determine a lower bound for the smallest characteristic root λ of a positive reduced matrix Y. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the characteristic roots of

$\mathcal{Y}$ and let $\mathcal{Y}_\nu$ denote the matrix which arises from $\mathcal{Y}$ by deleting its ν^{th} row and column. It is known then that the characteristic roots of $\mathcal{Y}^{-1}$ are $\lambda_\nu^{-1}, \nu = 1, 2, \ldots, n$. Denoting by $\sigma(A)$, the trace of a square matrix A, we have from (49), (81) and (47)

$$\begin{aligned} \frac{1}{\lambda} &= \max_\nu \frac{1}{\lambda_\nu} \leq \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots \frac{1}{\lambda_\nu} = \sigma(Y^{-1}) \\ &= \sum_{\nu=1}^{n} \frac{|y_\nu|}{|y|} \leq C_I \sum_{\nu=I}^{n} \frac{y_{11} y_{22} \ldots y_{\nu-1} y_{\nu-1} \cdot y_{\nu+1} y_{\nu+1} \ldots y_{nn}}{y_{11} \cdot y_{22} \ldots y_{nn}} \\ &= C_1 \sum_{\nu=1}^{n} \frac{1}{y_{\nu\nu}} \leq \frac{nC_1}{y_{11}} \end{aligned} \tag{82}$$

Thus if $y > o$ is reduced and $y = (y_{\mu\nu})$ and if λ be the minimum of the characteristic roots of y, then

$$\lambda \geq y_{11}/nC_1 \tag{82}$$

If now $Z = X + iY \in \mathcal{F}$ then $y_{11} \geq \dfrac{1}{2}\sqrt{3}$ from (75) so that in this case

$$\lambda \geq \sqrt{3}/2nC_1, C_1 = C_1(n) \tag{83}$$

Let $Z \in Bd \cdot \mathcal{F}$ and let $Z_\mathfrak{R} \to Z, Z_\mathfrak{R} \notin \mathcal{F}$. To each $Z_\mathfrak{R}$ we can 54
determine an equivalent point $\mathcal{W}_\mathfrak{R} \epsilon \mathcal{F}$ say, $\mathcal{W}_\mathfrak{R} = (A_\mathfrak{R} Z_\mathfrak{R} + B_\mathfrak{R})(C_\mathfrak{R} Z_\mathfrak{R} + D_\mathfrak{R}^{-1})$. Then

$$\begin{pmatrix} A_\mathfrak{R} & B_\mathfrak{R} \\ C_\mathfrak{R} & D_\mathfrak{R} \end{pmatrix} \neq \pm \begin{pmatrix} E & o \\ o & E \end{pmatrix}$$

Case. i: Let $C_\mathfrak{R} \neq O$ for an infinity of indices. By passing to a subsequence, we may assume this to be true for all $C'_\mathfrak{R} s$. Also, since their ranks belong to the finite set $1, 2, \ldots n$, we can assume all $C_\mathfrak{R}, s$ to be of the same rank $r > o$. To each class $\{C_\mathfrak{R}, D_\mathfrak{R}\}$, we determine the corresponding classes $\{C_o^{(r)}, D_o^{(r)}\}$ and $\{Q^{(n,r)}\}$ with $|C_o| \neq 0$, given by Lemma 1. Both classes $\{C_o, D_o\}$ and $\{Q\}$ depend on $\mathfrak{R}$ though the notation is not suggestive. Then from (70) and (71) we have

$$\|C_\mathfrak{R} Z_\mathfrak{R} + D_\mathfrak{R}\|^2 = \|C_o Z_\mathfrak{R}[Q] + D_o\|^2$$

$$= |C_o|^2|T|^2 \prod_{\nu=1}^{r}(1+h_\nu^2) \tag{84}$$

where $T = y_{\mathfrak{R}}[Q]$, $S = x_{\mathfrak{R}}[Q] + C_o^{-1}D_o$,

$$x_{\mathfrak{R}} + iy_{\mathfrak{R}} = Z_{\mathfrak{R}}, T[F] = E$$

and $S[F] = H = (\delta_{\mu\nu}h_\nu)$

As usual we assume that T is reduced. Let $y_1, y_2, \ldots y_r$ be the columns of C. Then $T = (\mathscr{U}_\nu' \mathcal{Y}_{\mathfrak{R}} \mathscr{U}_\nu)$ so that by (49),

$$\prod_{\nu=1}^{r} \mathcal{Y}_{\mathfrak{R}}[\mathscr{U}_\nu] < C_1|\top| \tag{85}$$

55 As is well known, the smallest characteristic root $\lambda^{(\mathfrak{R})}$ of $y_{\mathfrak{R}}$ is given by

$$\lambda^{(\mathfrak{R})} = \min_{\mathcal{E}'\mathcal{E}=1} \mathcal{Y}_{\mathfrak{R}}[\mathcal{E}] \qquad (\mathcal{E} - \text{ real column }).$$

Since $\lambda^{(\mathfrak{R})}$ is a continuous function of $\mathcal{Y}_{\mathfrak{R}}$ and $\mathcal{Y}_{\mathfrak{R}} \to \mathcal{Y}$ we have $\lim\limits_{\mathfrak{R}} \lambda^{(\mathfrak{R})} = \lambda$ the smallest characteristic root of Y, and further $\lambda \geq \sqrt{3}/2nC_1$, from (83), so that

$$\lambda^{(\mathfrak{R})} \geq \sqrt{3}/4nc_1 \tag{86}$$

for sufficiently large k. We shall be concerned only with these $k's$ in the rest of the proof. Then we have

$$\mathcal{Y}_{\mathfrak{R}}[\mathscr{U}_\nu] \geq \lambda^{(\mathfrak{R})} \mathscr{U}'_\nu \mathscr{U}_\nu \geq \lambda^{(\mathfrak{R})} \geq \sqrt{3}/4nC_1$$

and therefore from (85),

$$|\top| > \left(\frac{\sqrt{3}}{4^n}\right)^r C_1^{-r-1} > o \tag{87}$$

Clearly $\mathcal{W}_{\mathfrak{R}}$ satisfies the relation

$$(-C'_{\mathfrak{R}} \mathcal{W}_{\mathfrak{R}} + A'_{\mathfrak{R}})(C_{\mathfrak{R}} Z_{\mathfrak{R}} + D_{\mathfrak{R}}) = E$$

and since $\mathcal{W}_{\mathfrak{K}} \in \mathcal{F}$ we have $\| - C'_{\mathfrak{K}}\mathcal{W}_{\mathfrak{K}} + A'_{\mathfrak{K}}\| \geq 1$

We therefore conclude that $\|C_{\mathfrak{K}}Z_{\mathfrak{K}} + D_{\mathfrak{K}}\| \leq 1$ Hence

$$1 \geq \|C_{\mathfrak{K}}Z_{\mathfrak{K}} + D_{\mathfrak{K}}\|^2 = |C_o|^2|\top|^2\pi(1 - h_\nu^2)$$

which implies that $|C_o|$ is bounded as $|\top|$ and $(1 + h_\nu^2)$ have positive lower bounds as a result of (87). By appealing to (84) and (87) we then conclude that $|C_o|, |\top|, h_1, h_2, \ldots h_r$ all lie between bounds which do not depend upon $\mathfrak{K}$ and Z. Since

$$\frac{\sqrt{3}}{4nc_1}\mathscr{U}'_\nu\mathscr{U}_\nu \leq \lambda^{(\mathfrak{K})}\mathscr{U}'_\nu\mathscr{U}_\nu \leq \mathcal{Y}[\mathscr{U}_\nu]$$
$$\leq \frac{C_1|\top|}{\prod_{\mu\neq\nu}\mathcal{Y}_{\mathfrak{K}}[\mathscr{U}_\mu]} \leq (\frac{4n}{\sqrt{3}})^{r-1}C_1^r|\top|$$

the vector $\mathscr{U}_\nu$ is bounded. This proves that the matrix Q belongs to a finite set of matrices which does not depend on Z. Since the diagonal elements $\mathscr{U}_\nu Y_{\mathscr{R}}\mathscr{U}_\nu$ of T are bounded as a consequence of (84) and (87) and $T > 0$, it follows that all the elements of T are bounded and this in its turn, in view of the relations $T = E[F^{-1}]; S = H[F^{-1}]$ implies that the elements of F^{-1} and S are bounded. Now we observe that $|x_{\mathscr{R}}|x_{\mu\nu}^{(\mathscr{R})}| \leq \frac{1}{2}$ and $x_{\mathscr{R}} \to x = (x_{\mu\nu})$ so that $|x_{\mu\nu}| \leq \dfrac{1}{2}$ and consequently X is bounded. But $S = x_{\mathscr{R}}[Q] + C_o^{-1}D_o i.e. C_o^{-1} = \delta - x_{\mathscr{R}}[Q]$. The above then shows that $P = C_o^{-1}D_o$ is bounded; in their words only a finite number of choices exist for P. But we know that P determines the class $\{C_o, D_o\}$ uniquely. Hence the number of choices for $\{C_o, D_o\}$ is also finite. As we have already shown that the number of classes $\{Q\}$ is finite, it follows that the number of distinct classes $\{C_{\mathscr{R}}, D_{\mathscr{R}}\}$ is finite; in other words the classes $\{C_{\mathscr{R}}, D_{\mathscr{R}}\}$ are equal for an infinity of $k's$.We shall denote this common class by $\{C, D\}$. Then, taking the appropriate sequence of $k's$, 56

$$\|CZ + D\| = \lim_{\mathscr{R}} \|C_{\mathscr{R}}Z_{\mathscr{R}} + D_{\mathscr{R}}\| \leq 1, \text{ as } \|C_{\mathscr{R}}Z_{\mathscr{R}} + D_{\mathscr{R}}\| \leq 1$$

for each k.

Since $Z \in \mathfrak{f}$, we have the reverse inequality and the equality results. Hence in this case $Z \in Bd.\mathfrak{f}$ satisfies the equation

$$\|CZ + D\| = 1 \tag{88}$$

56 for some out of a finite number of paris (C, D) which are not mutually equivalent.

Case ii: Suppose $C_{\mathfrak{K}} = O$ for all sufficiently large $k's$. Considering only these $k's$, we have

$$W_{\mathfrak{K}} = Z_{\mathfrak{K}}[\mathcal{U}_{\mathfrak{K}}] + S_{\mathfrak{K}}$$

with a unimodular matrix $\mathcal{U}_{\mathfrak{K}}$ and a symmetric integral matrix $S_{\mathfrak{K}}$ as in (74). If $V_{\mathfrak{K}}$ be the imaginary part of $W_{\mathfrak{K}}$, then $Y_{\mathfrak{K}}[\mathcal{U}_{\mathfrak{K}}] = V_{\mathfrak{K}} \in \mathscr{R}$. We wish to show that the number of $[u_{\mathfrak{K}}]'s$ is finite. Since the imaginary part Y of Z belongs to $\mathscr{R}$, representing Y in the form $Y = D[B]$, $D = (\delta_{\mu\nu} d_\nu)$, and $B = (b_{\mu\nu})$, we have from (59) that $\dfrac{d_\nu}{d_{\nu+1}}$, $b_{\mu\nu}$ are all bounded by $C_{11} = C_{11}(n)$. But $\mathcal{Y}_{\mathfrak{K}} \to \mathcal{Y}$ so that representing $\mathcal{Y}_{\mathfrak{K}}$ analogous to Y as $y_{\mathfrak{K}} = D_{\mathfrak{K}}[B_{\mathfrak{K}}]$, $D_{\mathfrak{K}} = (\delta_{\mu\nu} d_\nu^{(\mathfrak{K})}$, and, $B_{\mathfrak{K}} = (\ell_{\mu\nu}^{(\mathfrak{K})})$. we have $D_{\mathfrak{K}} \to D$ and $B_{\mathfrak{K}} \to B$. We therefore deduce that the ratios $d_\nu^{(\mathfrak{K})}/d_{\nu+1}^{(\mathfrak{K})}$ and$(\ell_{\mu\nu}^{(\mathscr{R})})$. are bounded, say, by $2C_{11}$. Lemma 6 now implies that there can be only a finite number of $\mathscr{U}'_{\mathscr{R}}s$ with $y_{\mathscr{R}}[\mathscr{U}_{\mathscr{R}}] = D_{\mathscr{R}}[B_{\mathscr{R}}\mathscr{U}_{\mathscr{R}}]$ reduced, in other words, for an infinity of $k's$, the $\mathscr{U}'_{\mathscr{R}}$ are the same, say $\mathscr{U}_{\mathscr{R}} = \mathscr{U}$. As $\mathscr{R} \to \infty$ through the sequence of these values, we have $y[\mathscr{U}] = \lim_{\mathscr{R}} y_{\mathscr{R}}[\mathscr{U}]$ and since each $y_{\mathscr{R}}[\mathscr{U}] \in \mathscr{R}$ and $Y > 0$, it follows that $y[\mathscr{U}] \in \mathscr{R}$. But we already know that $y \in \mathscr{R}$ and this, by theorem 2, is possible only if both Y and $Y[\mathscr{U}]$ belong to the boundary of $\mathscr{R}$. Hence

$$y[\mathcal{U}_{\mathscr{R}}] = \mathscr{Y}_{\mathscr{R}} \tag{89}$$

57 where $\mathscr{U}_{\mathscr{R}}$ is the first column of $\mathscr{U}$ such that $\mathcal{U}_{\mathscr{R}} \neq \pm \mathscr{H}\mathscr{R}$ provided such a $\mathscr{U}_{\mathscr{R}}$ exists. This is therefore true if $\mathscr{U}$ is not a diagonal matrix.

Let now $\mathcal{U}$ be a diagonal matrix. Two cases arise, viz. either $\mathcal{U} \neq \pm E$ or $\mathcal{U} = \pm E$. In the former case it follows, as in (68) that

$$\mathscr{Y}_{\mathscr{R}\mathscr{R}+1} = 0 \tag{90}$$

for some k. In the latter case, taking the k's for which $\mathscr{U}_{\mathscr{R}} = \mathscr{U} = \pm E$ we have $W_{\mathscr{R}} = Z_{\mathscr{R}} + S_{\mathscr{R}}$ with $S_{\mathscr{R}} \neq O$ as $W_{\mathscr{R}} \in \mathfrak{f}$ while $Z_{\mathscr{R}} \notin \mathfrak{f}$. Then the condition (i) of theorem 3 for the elements of $\mathfrak{f}$ implies that

$$x_{\mu\nu} = \pm\frac{1}{2} \tag{91}$$

for at least one pair (μ, ν).

Thus any point $Z \in Bd, \mathfrak{f}$ satisfies one or the other of the system of equations (88) - (91), in other words, Z satisfies one of a finite set of equalities, and each one of them defines an algebraic surface. It therefore follows that a finite number of algebraic surfaces bound $\mathfrak{f}$ and this completes the proof of theorem 3.

We insert here the following remarks for future reference. A group of symplectic matrices shall be called *discrete* if every infinite sequence of different elements of the group diverges, or equivalently any compact subset of it contains only finitely many distinct elements. The modular group of degree n is then clearly discrete.

We shall call a group of symplectic substitutions *discontinuous* in **58**
$\mathscr{Y}$, if for every $z \in \mathscr{Y}$, the set of images of z relative to the given group has no limit point in $\mathscr{Y}$. Concerning the modular group then, we have

Lemma 8. *The modular group of degree n is discontinuous in $\mathscr{Y}_n$*

The proof is by contradiction. Let $z_1, z_2 \cdots$ be a sequence of equivalent points, all different, converging to a point $z \in \mathscr{Y}$. Then $Z_{\mathscr{R}} = M_{\mathscr{R}} < Z_1 >$ for some $M_{\mathscr{R}} = \begin{pmatrix} A_{\mathscr{R}} & B_{\mathscr{R}} \\ C_{\mathscr{R}} & D_{\mathscr{R}} \end{pmatrix} \in \mathcal{M}_n$

Let $z_{\mathscr{R}} = x_{\mathscr{R}} + Y_{\mathscr{R}}$ and $Z = X + iY$.

Then $Y_{\mathscr{R}} = (\overline{Z}'_1 C'_{\mathscr{R}} D^{-1}_{\mathscr{R}} y_1 (C_{\mathscr{R}} z_1 + D_{\mathscr{R}})^2$ by (24). If $Y_1 = R'R$ this gives $Y_{\mathscr{R}} = \overline{\Omega}' \Omega$ with $\Omega = R(C_{\mathscr{R}} Z_1 + D_{\mathscr{R}})^{-1}$. Also $y'_{\mathscr{R}} s$ are bounded as $Y_{\mathscr{R}} \to Y$, and it follows that Ω is bounded. This in its turn implies that $(C_{\mathscr{R}} Z_1 + D_{\mathscr{R}})^{-1}$ is bounded. Also $\|C_{\mathscr{R}} z_1 + D_{\mathscr{R}}\|$ is bounded as $\|C_{\mathscr{R}} Z_1 + D_{\mathscr{R}}\|^2 = |Y_1|/|Y_{\mathscr{R}}|$ from (72). We therefore infer that $C_{\mathscr{R}} Z_1 + D_{\mathscr{R}}$ is bounded. Considering the real and imaginary part separately this implies that $C_{\mathscr{R}}$and hence also $D_{\mathscr{R}}$ are bounded and consequently also $A_{\mathscr{R}}$ and $B_{\mathscr{R}}$ as the relation $Z_{\mathscr{R}}(C_{\mathscr{R}} Z_1 + D_{\mathscr{R}}) = A_{\mathscr{R}} Z_1 + B_{\mathscr{R}}$ shows. Thus the $M'_{\mathscr{R}} s$ are all bounded for $\mathfrak{K} = 1, 2, \ldots$ and therefore for an infinity of $k's$ the $M'_{\mathscr{R}} s$ are identical, and for these $k's$ the $Z'_{\mathscr{R}} s$ are identical, contradicting our assumption. This proves the Lemma.

Chapter 5

Modular Forms of Degree n

The possible of developing a modular form into a Fourier series rests upon a general theorem in complex function theory. First of all we have to deal with the following facts. 59

Let $\mathscr{E}$ be the Gaussian plane. A subset $\mathfrak{R}$ of direct product $\mathscr{E}^n = \mathscr{E} \times \mathscr{E} \times \cdots \times \mathscr{E}$ (n times) shall be called a *Reinhardt domain* with zero centre, when the following conditions are satisfied.

1) $\mathcal{R}$ is a domain in the sense of function theory, viz. an open connected non empty set.

2) $\mathcal{R}$ is invariant under the group of transformations $(Z_1, Z_2 \cdots Z_n) \to (Z_1 e^{i\varphi_1}, Z_2 e^{i\varphi_2}, \ldots Z_1 e^{i\varphi_n})\varphi_\nu$- real. We now stare

Lemma 9. *Let R be a Reinhardt domain with $(o, o \cdots o)$ and $f(Z_1, Z_2, \ldots, Z_n)^a$ functions regular in $\mathcal{R}$. Then we can develop f into a power series*

$f(Z_1, Z_2, \ldots Z_n) = \sum_{\nu_1, \nu_2, \cdot, \nu_n = -\infty}^{\infty} o_{\nu_1 \nu_2 \cdots \nu_n} .0 Z_n^{\nu_1} Z_2^{\nu_2} \cdots Z_n^{\nu_n}$

This representation is valid in the whole domain $\mathcal{R}$ and the power series converges uniformly on the manifold $|Z_\nu| = r_\nu, \nu = 1, 2, \ldots n$ when $(r_1, r_2, \ldots, r_n) \in \mathcal{R}$.

Proof. We first prove the Lemma in the special case of the domain

$$\mathcal{R} :^{|Z_\nu| < \rho_\nu, 1 \leq \nu \leq h}$$

$$o < \sigma_\nu < |Z_\nu| < \rho_\nu, h < \nu \leq n.$$

60 Consider the contracted domain

$$|Z_\nu| < \rho'_\nu < \rho_\nu \ (1 \leq \nu \leq h)$$
$$o < \sigma_\nu < \sigma'_\nu < |Z_\nu| < \rho'_\nu < \rho_\nu (h < \nu \leq n).$$

Let $\mathcal{L}^1_\nu$ be the circle $|Z_\nu| < \sigma'_\nu, h < \nu \leq n$ assigned with the negative sense of rotation and let $\mathcal{L}^1_\nu$ be the circle $|Z_\nu| = \rho'_\nu (1 \leq \nu \leq n)$ assigned with the positive sense of rotation. Then we have by mean of the Cauchy integral formula, $f(Z_1, Z_2, \ldots Z_n)$=

$$= (\frac{1}{2\pi i})^n \int\limits_{\mathcal{L}^2_1} \cdots \int\limits_{\mathcal{L}^2_h} \int\limits_{\mathcal{L}^2_{h+1}} \cdots \int\limits_{\mathcal{L}^2_{n+\mathcal{L}^2_n}} \frac{f(\zeta_1, \ldots \zeta_1)}{(\zeta_1 - Z_1) \cdot (\zeta_1 - Z_n)} ds_1 \cdot d$$

$$= \sum_{\nu_{h+1}, \ldots \nu_n = 1}^{2} ((\frac{1}{2\pi i})^n) \int\limits_{\mathcal{L}^2_1} \int\limits_{\mathcal{L}^2_h} \int\limits_{h+1}^{\nu_{h+1}} \cdot \int\limits_{n}^{\nu,h} \frac{f(\zeta_1, \ldots \zeta_1)}{(\zeta_1 - Z_1) \cdot (\zeta_1 - Z_n)} ds_1 \cdot d\zeta$$

By expanding the quotient $1/\rho_\nu - Z_\nu$ as a convergent power series as in the one variable case of each ν, a simple argument shows that every integral in the above sum is expressible as a power series, a typical term of which contains non negative powers of

$Z_1, Z_2 \cdots Z_h, Z^{\pm 1}_{h+1}, Z^{\pm 1}_{h+2}, \ldots Z^{\pm 1}_n$ where in $Z^{\pm 1}_r (r \geq h + 1)$ the upper sign holds in cases where $\nu_r = 2$ and the lower sign holds in cases where $\nu_n = 1$. Clearly, the coefficients of these power series do not depend upon the choice of ρ^1_ν, σ^1_ν. The uniform convergence as stated in the Lemma is immediate from a consideration of the derived series as in the one variable case.

61 In the general case of any Reinhardt domain with center $(0, 0, \ldots 0)$ by the result shown above, in every subset $R' \subset R$ of the type $|Z_{\nu_i}| = \rho_i$, $1 \leq i \leq h$,

$$0 < \sigma_\nu < |Z_\nu| < \rho_\nu, \nu \neq \nu_1, \nu_2 \cdots \nu_h,$$

there exists a representation of $f(Z_1, Z_2, \ldots Z_n)$ as a power series with the desired properties. We need then only prove that two representations in two subsets $\mathcal{R}'_1$, $\mathcal{R}'_2$ are identical in their intersection and then the representation in any two subsets $\mathcal{R}'_1, \mathcal{R}''_1$ are identical as we can always find a sequence of such subsets, viz.

$$\mathcal{R}'_1 = R_1, R_2, \ldots R_m = \mathcal{R}''$$

with $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ non empty for any two indices i, $i+1$. But for any two such subsets $\mathcal{R}'_i, \mathcal{R}'_2$ their intersection $\mathcal{R}'_2 \cap \mathcal{R}'_2$ is again a subset of the same type and it is then immediate that the power series representing $f(Z_1, Z_2, \ldots Z_n)$ in the two sets are identical. Lemma 9 now follows.

Let $\mathfrak{K}$ be given integer and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, a symplectic matrix. We introduce the notation

$$f(Z)|M = f(M < Z >); CZ + D_1^{\mathcal{K}} \tag{92}$$

where $f(Z)$ is an arbitrary function defined on $\mathscr{Y}$. It is easy to see that $(f(Z)|M_1)|M_2 = f(Z)|(M_1 M_2)$. We now fix the conception of a modular form by the following. □

Definition. *A modular form of degree $n \geq 1$ and weight $\mathfrak{K}$, is a function $f(Z)$ satisfying the following conditions.*

1) $f(Z)$ is defined in $\mathscr{Y}$ and is a regular function of the $\dfrac{n(n+1)}{2}$ independent elements $Z_{\mu\nu}$ $(\mu \leq \nu)$ of Z. 62

2) $f(Z)|M = f(Z)$ for $M \in M$

3) In this case $n = 1$, $f(Z)$ is bounded in the fundamental domain $\mathcal{F}$ on $\mathscr{Y}$ relative to M.

We shall show later that the last condition is a consequence of the earlier conditions in the case $n > 1$.

Considering a matrix M of the form $M = \begin{pmatrix} E & S \\ O & E \end{pmatrix} \in M$ where $\mathcal{S}$ is an integral symmetric matrix, we infer from condition (2) above that $f(Z+S) = f(Z)$ for an integral symmetric matrix S; in other words a modular form is a periodic function of period 1 in each of the variables. Hence $g(\zeta_{\mu\nu}) = f(e^{2piiZ}\mu\nu)$ is a single valued regular function in the domain into which $\mathscr{Y}$ is mapped by the mapping $\zeta_{\mu\nu} = f(e^{2piiZ}\mu\nu$ viz. the domain defined by the inequalities $\zeta_{\mu\nu} = \zeta_{\nu\mu} \neq O, (-\ell oc|\zeta_{\mu\nu}|) > O$. It is easy to see that this is a Reinhard domain and then Lemma 9 assures us of a power series representation for $g(\zeta_{\mu\nu})$ valid throughout this domain. This power series can be looked upon as a Fourier series of $f(Z)$ and can be written in the form

$$f(Z) = \sum_T a(T)e^{2\pi i\sigma(TZ)} \tag{93}$$

where $T = (t_{\mu\nu})$ runs over all rational symmetric matrices such that $t_{\mu\mu}$ and $2b_{\mu\nu}(\mu \neq \nu)$ are integral. Such matrices are called *semi integral*. To verify the above fact, we need only observe that

$$\sigma(TZ) = \sum_{\mu,\rho=1}^{n} t_{\mu\rho}Z_{\rho\mu} = \sum_{\mu=1}^{n} t_{\mu\mu}Z_{\mu\mu} + 2\sum_{\mu<\nu} t_{\mu\nu}Z_{\mu\nu}$$

63 so that

$$\begin{aligned} e^{2\pi i\sigma(TZ)} &= e^{2\pi i(\sum_{\nu=1}^{n} t_{\mu\mu}+2\sum_{\mu<\nu} t_{\mu\nu}Z_{\mu\nu})} \\ &= \prod_{\mu=1}^{n} \zeta_{\mu\nu}^{t_{\mu\nu}} \prod_{\mu<\nu} \zeta_{\mu\nu}^{zt_{\mu\nu}} \end{aligned}$$

The condition (2) in the definition of a modular form yields in the case of the modular substitution $Z_1 = Z[\mathcal{U}], \mathcal{U}$-unimodular the transformation formula

$$f(Z_1) = |\mathcal{U}|^{-\mathfrak{K}} f(Z). \tag{94}$$

Observing that the trace of a matrix product is invariant under a cyclic change in the succession of the factors and in particular $\sigma(AB) = \sigma(BA)$ for any two matrices A, B, we obtain

$$\begin{aligned} f(Z_1) &= \sum_T a(T)e^{2\pi i\sigma(TZ[\mathcal{U}])} \\ &= \sum_T a(T)e^{2\pi i\sigma(T\mathcal{U}'Z\mathcal{U})} \\ &= \sum_T a(T)e^{2\pi i\sigma(\mathcal{U}T\mathcal{U}'Z)} \end{aligned}$$

$$= \sum_T a(T[\mathcal{U}'^{-1}])e^{2\pi i\sigma(TZ)}$$

On the other hand,

$$|\mathcal{U}'|^{-\mathfrak{K}} f(Z) = \sum_T |\mathcal{U}'|^{-\mathfrak{K}} \alpha(T) e^{2\pi i\sigma(TZ)}$$

A comparison of coefficients by mean of (94) now yields

$$|\mathcal{U}|^{\mathfrak{K}} a(T[\mathcal{U}'^{-1}]) = a(T)$$

where $\mathcal{U}$ is an arbitrary unimodular matrix. Replacing $\mathcal{U}$ by $\mathcal{U}'^{-1}$ we get **64**

$$a(T[\mathcal{U}]) = |\mathcal{U}|^{\mathfrak{K}} a(T) \tag{95}$$

The special choice $\mathcal{U} = -E$ leads to

$$a(T) = (-1)^{n\mathfrak{K}} a(T) \tag{96}$$

This proves that modular forms not vanishing identically can exist only in the case $n\mathfrak{K} \equiv 0(2)$. Also we deduce from (95) that for *properly unimodular matrices* i.e. $\mathcal{U}$ such that $|\mathcal{U}| = 1$,

$$a(T[\mathcal{U}]) = a(T) \tag{97}$$

We apply these results to prove.

Lemma 10. *A modular form $f(Z)$ which is bounded in the fundamental domain $\mathcal{F}$ of $\mathscr{Y}$ relative to M has a representation of the form*

$$f(Z) = \sum_{T \geq O} a(T) e^{2\pi i\sigma(TZ)} \tag{98}$$

and conversely any modular form representable by (98) and in fact any series (98) which converges everywhere in $\mathscr{Y}$ is bounded in $\mathcal{F}$.

We prove the direct part first. We shall denote by $\mathcal{H}$ the cube $-\frac{1}{2} \leq x_{\mu\nu} \leq \frac{1}{2} (\mu \leq \nu)$ and put $[dx] = \prod_{\mu \leq \nu} dx_{\mu\nu}$

Since the Fourier series of $f(Z)$ converge uniformly in $\mathcal{H}$, we obtain

$$a(T) = \int\limits_{\mathcal{H}} \cdots \int f(Z) e^{-2\pi i \sigma(TZ)} [dx]$$

65 and consequently

$$a(T) e^{-2\pi\sigma(Ty)} = \int\limits_{\mathcal{H}} \cdots \int f(Z) e^{-2\pi i \sigma(Tx)} [dx]$$

where $Z = X + iY \in \mathscr{Y}$

Let y_o be a reduced matrix such that $\sigma(Ty_o) < O$ for some fixed T. By (77), if λ is chosen sufficiently large, the largeness depending only on y_o then $Z = X + i\lambda Y_o \in \mathcal{F}$ for $x \in \mathcal{H}$ and y-reduced. With such a choice of λ, the assumption $|f(Z)| < C$ for $Z \in \mathcal{F}$ implies by means of the relation

$$a(T) e^{-2\pi\gamma\sigma(Ty_o)} = \int\limits_{\mathscr{X}} \cdots \int f(Z) e^{-2\pi i \sigma(Ty_o)} [dx]$$

that

$$|a(T)| e^{-2\pi\lambda\sigma(Ty_o)} \leq C.\vartheta o\ell\mathcal{H} = C.$$

As $\lambda \to \infty$ our assumption $\sigma(Ty_o) < 0$ implies that

$$a(T) = 0 \tag{99}$$

Let now Y be an arbitrary positive matrix with $\sigma(Ty) < 0$ and choose a unimodular matrix $\mathcal{U}$ such that $y = y_o[\mathcal{U}']$, $y_o \in \mathscr{R}$, the space of reduced matrices. Then we have

$$0 > \sigma(Ty) = \sigma(Ty_o[\mathcal{U}']) = \sigma(T[\mathcal{U}]y_o)$$

and (99) and (95) now imply that

$$a(T[\mathcal{U}]) = \pm a(T) = 0$$

for those T for which there exist positive matrices Y with $\sigma(Ty) < 0$. Hence it is sufficient that the matrix T in the Fourier series (93) runs only over those matrices with the property $\sigma(Ty) \geq 0$ for all $y > 0$.

66 Let $R = R^{(n)}$ be any nonsingular real matrix, $R = (\mu_1, \mathscr{W}_2, \ldots \mathscr{W}_n)$ and let $y = RR'$. Then $Y > 0$ and the above implies for the matrices T occuring in (93) that $\sigma(Ty) = \sigma(T[R]) = \sum_{\nu=1}^{n} T[W_\nu] \geq 0$. This is true of all non-singular real matrices R and $\sigma(T[R])$ is a continuous function of R so that it is true of singular matrices too with real elements. In other words $T[\varepsilon] \geq 0$ for all real column $\mathscr{E}$ and what is the same, $T \geq 0$. This settles the first part of the Lemma.

Conversely we have to prove that the sum of a Fourier series (98) which converges everywhere in $\mathscr{Y}$ is bounded in $\mathcal{F}$. The matrices T occurring in (98) are all semi integral matrices which are further positive. Let $T = RR', R = (\mathscr{W}_1, \mathscr{W}_2 \cdots \mathscr{W}_n)$-real and let $Z = X + iY \in \mathcal{F}$. The characteristic roots of y have by (83) a positive lower bound $C = \dfrac{\sqrt{3}}{2nc_1}, C_1 = C_1(n)$. Hence

$$\begin{aligned} \sigma(Ty) = \sigma(y[R]) = \sum_{\nu=1}^{n} y[\mathscr{W}_\nu] &\geq C \sum_{\nu=1}^{n} \mathscr{W}_\nu' \mathscr{W}_\nu \\ &= C\sigma(R'R) = C\sigma(T). \end{aligned}$$

The convergence of the series (93) at the particular point $Z = \dfrac{iC}{2}E$ implies that $|a(T)|e^{-\pi C\sigma(T)} < \mathscr{C}$ for all T, where $\mathscr{C}$ is a suitable constant. If now $Z \in \mathcal{F}$ then

$$\begin{aligned} |a(T)e^{-\pi C\sigma(TZ)}| &= |a(T)|e^{-\pi C\sigma(Ty)} \\ &\leq |a(T)|e^{-\pi C\sigma(T)} \\ &\leq \mathscr{C} e^{-\pi C\sigma(T)} \end{aligned}$$

and consequently, 67

$$\begin{aligned} |f(Z)| &= |\sum_{T\geq 0} a(T)e^{2\pi i\sigma(TZ)}| \\ &\leq |\sum_{T\geq 0} a(T)e^{2\pi\sigma(Ty)}| \\ &\leq \sum_{T\geq 0} \mathscr{C} e^{-\pi c\sigma(T)} \end{aligned}$$

The last sum is independent of Z and we will be through if only we show that this series is convergent. The convergence of this series is a consequence of the fact that the number of semi integral matrices $T \geq 0$ whose trace (which is always an integer) is equal to a given integral value t increases at most as a fixed power of t as $t \to \infty$. For it $\sigma(T) = t$, $T \geq 0-$ semi integral, then writing $T = (t_{\mu\nu})$ we have $t_{\nu\nu} \leq t$ for all indices ν so that the number of possible choices for all $t'_{\nu\nu}s$ together is atmost $(t+1)^n$. Since $T \geq 0$ we have $t_{\nu\nu} \leq t_{\mu\mu}t_{\nu\nu} \leq (t+1)^2$ so that $\pm 2t_{\mu\nu} \leq 2(t+1)$ and the number of $t'_{\mu\nu}s$ for a given (μ,ν) consistent with this inequality is at most $4t+1$. It therefore follows that the number of T' satisfying our requirements can be majorised by

$$(t+1)^n(4t+1)^{n(n-1)/2} \leq \mathscr{C}_1 t^{n+n(n-1)/2} = \mathscr{C}_1 t^{n(n+1)/2}$$

$\mathscr{C}_1$ being suitable positive constant. We can now estimate

$$\sum_{T\geq 0} \mathscr{C} e^{-\pi c\sigma(T)} = \mathscr{C} \sum_{t=0}^{\infty} \sum_{T \text{ semi integral,}} e^{-\pi ct}$$
$$\leq \mathscr{C}\mathscr{C}_1 \sum_{t=0}^{\infty} e^{-\pi ct} t^{n(n+1)/2}$$

68 and the last sum is clearly convergent. The proof of Lemma 10 is now complete.

We now apply Lemma 10 to show that every modular form is bounded in the fundamental domain $\mathscr{F}$ of the modular group. In case $n = 1$ this is true by the definition of a modular form. Assume $n > 1$. Since the Fourier series of a modular form may be considered as a power series, the convergence of such a series is absolute. Hence every partial series of (93), viz $\sum_{T\in K} a(T)e^{2\pi i\sigma(T>.)}$ where $\mathscr{K}$ denotes an arbitrary set of semi integral matrices, converges absolutely. Let T be a fixed matrix such that $a(T) \neq 0$ and k_T the set of matrices $T_1 = T[\mathcal{U}]$ where $\mathcal{U}$ denotes an arbitrary proper unimodular matrix. Then $a(T_1) = a(T)$ by (95) and the series

$$g(Z,T) = \sum_{T_1\in k_T} e^{2\pi i\sigma(T_1 Z)} \tag{100}$$

converges absolutely. We shall show that this is possible only if $T \geq 0$. Let $\nu(T, m)$ denote the number of matrices $T_1 \in k_T$ with $\sigma(T_1) = m$. We shall use the abbreviation t for $e^{2\pi}$.

From (101) we get

$$g(\text{i.e}, T) = \sum_{T_1 \in k_T} e^{-2pi\sigma(T_1)} = \sum_{m=-\infty}^{\infty} \nu(T, m) t^{-m}$$
$$\geq \sum_{m=1}^{\infty} \nu(T, -m) t^m \geq \sum_{m=1}^{\infty} \nu(T, -m),$$

If $T \not\geq 0$ we shall show that $\nu(T, -m) \geq 1$ for an infinity of $m's$ and this will provide a contradiction as we know that the series (100) is convergent. If $T \not\geq 0$ we can find an integral column $\mathscr{Y}$ such that $T[\mathcal{Y}] < 0$. Let $\mathcal{U} = E + h(h_1\mathcal{Y}, h_2\mathcal{Y} \cdots \beta_n\mathcal{Y})$ with integers $h, h_i, i = 12, \ldots n$. Since $\mathcal{U} - E$ has rank 1, we contend **69**

$$\text{that } |\mathcal{U}| = 1 + \text{ trace } \{h(h_1, \mathcal{Y}, h_2\mathcal{Y}, \ldots h_n\mathcal{Y})\}$$
$$= 1 + h\sigma(h_1\mathcal{Y}, h_2\mathcal{Y}, \ldots, h_n y).$$

To verify this we need only observe that while for any matrix A we have

$$|tE + A| = t^n + \sigma(A)t^{n-1} + \cdots,$$

if A is of rank 1, the coefficient of t^{n-2} and lower powers of t which depend on subdeterminants of A all vanish and consequently $|tE + A| = t^n + \sigma(A)t^{n-1}$,

In particular, choosing $t = 1$ we obtain $|E + A| = 1 + \sigma(A)$ which is precisely what we desired.

Now $\sigma(h_1\mathcal{Y}, h_2\mathcal{Y}, \ldots h_n\mathcal{Y})$ is a linear form in the $h's$ and for $n > 1$, there exists a non trivial integral solution of the equation

$$\sigma(h_1\mathcal{Y}, h_2\mathcal{Y}, \ldots h_n\mathcal{Y}) = 0$$

and then $\sum_{\nu=1}^{n} h_\nu^2 > 0$.

With these $h_\nu' s$ and the free variable h we compute

$$\sigma(T_1) = \sigma(T[\mathcal{U}])$$

$$
\begin{aligned}
&= \sigma\{T[E + h(h_1\mathcal{Y}, h_2\mathcal{Y}, \dots h_n\mathcal{Y})]\}\\
&= \{T + hT(h_1\mathcal{Y}, h_2\mathcal{Y}, \dots h_\mu\mathcal{Y})\\
&\qquad + h'k_i\mathcal{Y}\beta_\nu\mathcal{Y}, \dots h_n\mathcal{Y}'\pi + h^2T[h_1\mathcal{Y}, \cdot h_n\mathcal{Y}]\}\\
&= \sigma(T) + 2h\sigma(\lambda(h_1\mathcal{Y}, h_2\mathcal{Y}, \dots h_n\mathcal{Y})) + h^2\sigma T[(h_1\mathcal{Y}, \dots, h_n\mathcal{Y})]\\
&= \sigma(T) + 2h\sigma(T(h_y, \dots h_n\mathcal{Y})) + \rho^2T[\mathcal{Y}]\sum_{\nu=1}^{n} h_\nu^2
\end{aligned}
$$

70 Choosing h suitable, this shows clearly that $\sigma(T[\mathcal{U}]) = -m, |\mathcal{U}| = 1$ is solvable for an infinite number of positive integers m and this is equivalent to saying $\nu(T, -m) \geq 1$ for an infinity of $m's$. Thus we have shown that, for the series (101) to converge absolutely, we must have $T \geq 0$ and this in its turn implies that only such $T's$ occur in the series (93) representing $f(Z)$. Lemma (10) implies that $f(Z)$ is bounded in $\mathfrak{f}$. We have now proved

Theorem 4. *Every modular form is bounded in the fundamental domain $\mathcal{F}$ of the modular group acting on $\mathscr{Y}$.*

We proceed to show that the modular forms of negative weight ($\mathfrak{K} < 0$) must necessarily vanish identically.

Let $\mathfrak{f}(Z)$ be a modular form of degree n and weight $\mathfrak{K}$. Then as a result of (72) it is easy fo see that $h(Z) = |Y|^{\mathfrak{K}/2}|f(Z)|$ is invariant under all modular substitutions, viz.

$$h(M < Z >) = h(Z) \tag{101}$$

for $M \in M$. If we assume $\mathfrak{K} < 0$, then by theorem (4) $\mathfrak{f}(Z)$ is bounded in $\mathcal{F}$ and by (76) $|y|$ has a positive bound in $\mathcal{F}$. It follows therefore that $h(Z)$ is bounded in $\mathcal{F}$ and hence also throughout $\mathscr{Y}$. Let then $h(Z) \leq \mathscr{C}$ for $Z \in \mathscr{Y}$.

By means of the representation

$$a(T)e^{-2\pi\sigma(Ty)} = \underset{\mathscr{H}}{\int \cdots \int} f(Z)e^{-2\pi i\sigma(Tx)}[dx]$$

we conclude that

$$|a(T)|e^{-2\pi\sigma(Ty)} \leq \sup_{\substack{x\in\mathscr{H} \\ Z=x+ty}} |f(Z)| = \mathscr{C}|\lambda|^{-\mathfrak{K}/2}.$$

and letting $\in\to 0$ the limit process yield that $a(T) = 0$. We then have **71**

Theorem 5. *A modular form of negative weight vanishes identically. We can therefore assume in the sequel that the weight $\mathfrak{K}$ of a modular form $\mathscr{F}$ is non negative. Later on we shall show that if $\mathfrak{K} = 0$ then $\mathscr{F}$ is necessarily a constant.*

We now introduce an operator which maps the modular forms of degree $n > 1$ into those of degree $n-1$ with the same weight. This operators will be denoted by ϕ and the image of $\mathscr{F}(Z)$ under will be denoted by $\mathscr{F}(Z)|\phi$. The use of this operator will be particularly felt in such cases where proofs are based on induction on n

We write, in place like these, where we are concerned with modular forms of different degrees,

$$\mathscr{Y} = \mathscr{Y}_n, \mathcal{F} = \mathcal{F}_n, M = M_n.$$

It is straight forward verification that if $Z \in \mathscr{Y}_n$ the matrix Z_1 arising from Z by cancelling its last row and column belongs to $\mathscr{Y}_{n-t}$ and if $Z_1 \in \mathscr{Y}_{n-1}$ then the matrix $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda \end{pmatrix} \in y_n$ provided $\lambda > 0$

We can then form the function $f\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}$ define for every $Z_i \in y_{n-1}$ and $\mathfrak{f}$, a modular form of degree $n-1$. We shall show that $\lim\limits_{\lambda\to\infty} \mathfrak{f}\begin{pmatrix} Z & C \\ 0 & c\lambda \end{pmatrix}$ exists, denoted by $f(Z_t)$ and this will be the modular form $f(z)|\phi$ of degree $n-1$ and weight $\mathfrak{K}$.

Let $\mathcal{L}$ be a compact subset of y_{n-1} and $Z_1 = X_1 + iY_1 \in \mathcal{L}$.

We show first that for any $T_1 \geq 0$, **72**

$$\sigma(T_1 y_1) \geq \lambda\sigma(T_1) \tag{102}$$

where $\gamma = \gamma(\mathcal{L}) > 0$. We need only consider the case $\sigma(T_1) > 0$ as otherwise $T_1 = (0)$ and the inequality reduces to a trivial equality. Then by reason of homogeneity of both sides in T_1, we can assume $\sigma(T_1) = 1$ The equations:

$$\sigma(T_1) = 1, T_1 \geq 0, Z_1 \in \mathcal{L} \subset \mathscr{Y}_{n-1}$$

define a compact Z_1, T_1-set and on this set, $\sigma(T_1 Y_1)$ is a function, continuous in T_1 and Y_1-If we show that this function is positive at every point, then it has a positive minimum γ in this set and we would have proved (102). Let $R = R^{n-1} \neq 0$ be determined with $T_1 = RR'$ and let $R = (\mathscr{W}_1, \mathscr{W}_2, \ldots \mathscr{W}_{n-1})$.

Then $\sigma(T_1 Y_1) = \sigma(Y_1[R]) = \sum^{n-1} \nu = 1 y_1[\mathscr{W}_\nu]$, and the last sum is positive as $Y_1 > 0$ and at least one of the columns is nonzero. This settles our contention. Since the Fourier series

$$f(Z) = \sum_{T \geq 0} a(T) e^{2\pi i \sigma(TZ)}$$

converges everywhere and in particular at the point $Z = \dfrac{i}{2}\gamma E$, we have $|a(T)| \leq \mathscr{C} e^{\pi\gamma\sigma(T)}$ for $T \geq 0$ and a certain positive constant $\mathscr{C}$.

Writing $Z = \begin{pmatrix} Z_i & 0 \\ 0 & t\lambda \end{pmatrix}, Z_1 \in \mathcal{L}$ and decomposing T analogously as $T = (t_{\mu\nu}) = \begin{pmatrix} T_1 & \mu \\ \nu & t_{nn} \end{pmatrix}$ we get from the above that

$$|a(T)| e^{2\pi i \sigma(TZ)} \leq \mathscr{C} e^{\pi\nu\sigma(T)} e^{-2\pi\sigma(YT)}$$

$$\begin{aligned}
&= \mathscr{C} e^{\pi\lambda\sigma(T)} e^{2\pi(\sigma(T_1 y_1) + \lambda tnn)} \\
&= \mathscr{C} e^{\pi\gamma\sigma(T_1) - 2\pi\sigma(T_1 y_1) - \pi(2\lambda - \gamma) t_) nn} \\
&\leq \mathscr{C} e^{-pi\gamma\sigma(T_1) - \pi\gamma t_{nn}} \\
&= \mathscr{C} e^{-pi\gamma\sigma(T)}
\end{aligned}$$

73 assuming $\lambda \geq \gamma$. Thus if $Z_1 \in \mathcal{L}, \lambda \geq \gamma(\mathcal{L}), Z = \begin{pmatrix} Z_1 & 0 \\ 0 & t\lambda \end{pmatrix}$ then $|a(T)| e^{2\pi i \sigma(TZ)}! \leq \mathscr{C}^{-pi\gamma\sigma(T)}$

It is now immediate that the series

$$f(Z) = \sum_{T\geq 0} \alpha(T) e^{2pii\sigma(Tz)}$$

which is majorised by $\mathscr{C} \sum_{T\geq 0} e^{-pi\gamma\sigma(T)}$ independent of Z, converges uniformly for Z_1 in every compact domain $\mathcal{L} \subset \mathscr{Y}_{n-1}$ and $\lambda \geq \gamma(\mathcal{L})$. Then

$$\begin{aligned}
\lim_{\lambda\to\infty} \mathfrak{f}(Z) &= \lim_{\lambda\to\infty} \mathfrak{f}\begin{pmatrix} Z_1 & 0 \\ 0 & \ell\lambda \end{pmatrix} \\
&= \sum_{T\geq 0} a(T) \lim_{\lambda\to\infty} e^{2\pi i\sigma(TZ)} \\
&= \sum_{T\geq 0} a(T) \lim_{\lambda\to\infty} e^{2\pi i\sigma(T_1 Z_2) - 2\pi\lambda tnn}
\end{aligned}$$

In the last series, the terms involving T for which $t_{nn} > 0$ vanish in the limit and only there terms for which $t_{nn} = 0$ survive. We than obtain

$$\begin{aligned}
\lim_{\lambda\to\infty} \mathfrak{f}(Z) &= \mathfrak{f}_1(Z_1) \\
&= \sum_{T\geq 0 t_{nn}=0} a(T) e^{2\pi i 5(T_1 z_1)} \\
&\sum_{T_\gamma 0\geq 0} a(T_1) e^{2\pi i\sigma(T_1, z_1)} \qquad (103)
\end{aligned}$$

where by definition $a(T_1) = a\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ **73**

It is clear that t $\mathfrak{f}_i(Z_i)$ is regular in $\mathscr{Y}_{n-1}$ as the corresponding Fourier series converges uniformly in every compact subset of $\mathscr{Y}_{n-1}$. Further $\mathfrak{f}_1(Z_2)$ is bounded in the fundamental domain $\mathfrak{f}_{n-1}$ as in the series (103) only those $T_1's$ occur which are semi positive. It remains to show that $\mathfrak{f}_1(Z_1)$ is actually a modular form of degree $n-1$ and weight $\mathscr{R}$

Let $M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in M_{n-1}$ We complete M_1 to a modular matrix M of degree n as follows.

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ where } A = \begin{pmatrix} A_1 & o \\ o & 1 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } D = \begin{pmatrix} D_1 & 0 \\ 0 & 1 \end{pmatrix}$$

If the $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & \ell\lambda \end{pmatrix}$, $L_1 \in \mathscr{Y}_{n-1}$ we have $M < Z >= (AZ + B)(CZ + D)^{-1} =$

$$\begin{aligned}
&= \left\{ \begin{pmatrix} A_I & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & i\lambda \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \left\{ \star \right\}^{-1} \\
&= \begin{pmatrix} A_1 Z_1 + B_1 & 0 \\ 0 & i\lambda \end{pmatrix} \begin{pmatrix} C_i Z_i + 0_1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} A_1 Z_1 + B_1 & 0 \\ 0 & i\lambda \end{pmatrix} \begin{pmatrix} (C_i Z_i + 0_1)^1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} M_1 < Z_1 > & 0 \\ 0 & i\lambda \end{pmatrix}
\end{aligned}$$

74

Also $|CZ + D1 = |C_1 Z_1 + D_1|$. Hence we obtain from the relation $f(M < Z >)|CZ + D1^{-\mathscr{R}} = f, z)$ that $\begin{pmatrix} M_1 < Z_1 > & 0 \\ 0 & \lambda \end{pmatrix} |C_1 Z_1 + D_1|^{\mathscr{R}} = f \begin{pmatrix} Z_1 & 0 \\ 0 & \lambda \end{pmatrix}$ which as $\lambda \to \infty$ yields

$$f_1(M_1 \langle Z_1 \rangle) |C_1 Z_1 + D_1|^{-\mathscr{R}} = f_1(Z_1),$$

That is to say $f_1(Z_1)|M_1 = f_1(Z_1)$ for $M_1 \in M_{n-1}$ and it is immediate that $f_1(Z_1)$ is a modular form of degree $n - 1$ and weight $\mathscr{R}$.

Chapter 6

Algebraic dependence of modular forms

We are interested here in the question: when a modular form of degree n **75**
vanishes identically, in other words, when two when two given modular forms of the same degree are identical. The following theorem provides a useful criterion in this direction.

Theorem 6. *Let s_n denote the least upper bound of $\sigma(Y^{-1})$ for Y^{-1} such that $X + iY \in f_n$ and let*

$$\mathfrak{f}(Z) = \sum_{T \geq 0} a(T) e^{2\pi i \sigma(TZ)}$$

be a modular form of degree n and weight $k \geq 0$. If $a(T) = 0$ for T such that $\sigma(T) \leq \dfrac{\mathscr{R}}{4\pi} s_n$ (i.e. if a certain finite set of Fourier coefficients vanish), then $f(z)$ vanishes identically.

Proof. First we note that $\{s_n\}, n = 1, 2, \ldots,$ is an increasing sequence. For, we know from (82) that

$$\sigma(y^{-1}) \leq \frac{na}{y_M} \leq \frac{2nC_1}{\sqrt{3}} \text{ for } Z = x + oy \in \mathcal{F}_n.$$

□

In particular then, $S_n \leq \dfrac{2nc_1}{\sqrt{3}} < \infty$ Let now $Z_1 \in \mathcal{F}_{n-1}$. We claim that $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & M \end{pmatrix} \in \mathcal{F}_n$ provided λ s sufficiently large. We prove this as follows. Let $Z = x + iyu, Z_1 = x_1 + y_1$ Then $y = \begin{pmatrix} y_1 & 0 \\ 0 & \lambda \end{pmatrix} = (y_m)$ say.

We have to verify the reduction condition of Minkowski for Y. Let $\mathscr{Y}_{\mathscr{R}} = \begin{pmatrix} \mathscr{Y}^*_{\mathscr{R}} \\ g_n \end{pmatrix}$ be a primitive column with the integers $g_1, g_2, \cdots g_n$ as elements. Then we have

$$y[Y_{\Re}] = y_1[\mathscr{Y}_{\Re}] + \lambda g_n^2$$

76

If $g_n = 0$, then $g_{\Re}, g_{\mathfrak{k}+1} \cdots g_{n-1}$ are themselves coprime and then $Y_1[y_{\Re}] \geq y_{\mathscr{K}\mathscr{B}}$ as y_1 is reduced and k is necessarily less than n. It then follows that $Y[\mathscr{Y}_{\Re}] = y_1[\mathscr{Y}_{\Re}] \geq y_{\Re\Re}$ If $y_n \neq 0$, then $y[\mathscr{Y}_{\Re}] = y[\mathscr{Y}_{\Re}] + \lambda g_{\Re} + \lambda$, so that $y[\mathscr{Y}_{\Re}] \geq y_{\Re\Re}$ provided we choose $\lambda \geq g_{n-1}, n-1$ Trivially $y_{\Re\Re+1} \geq o$ for all k as this is true of y_1.

It is now immediate that if

$$\lambda \geq y_{n-1,n-1} \tag{104}$$

then $y = \begin{pmatrix} y_1 & 0 \\ 0 & \lambda \end{pmatrix}$ satisfies Minkowski's reduction cond

itions. Clearly, since y_1 is reduced modulo 1, χ is also reduced modulo l. It only remains therefore to verify that Z is a highest point in the set of all points which are equivalent to Z relative to M_n provided λ is sufficiently large; in other words

$$||CZ + D|| \geq 1$$

As in (71) we have , in the usual notations in (70),

$$||CZ + D||^2 = |C_1|^2 |T|^2 \prod_{\nu=1}^{r} (1 + h_\nu^2)$$

and to infer that $||CZ + D||^2 \geq 1$ it suffices to ensure ourselves that $||T|| \geq 1$.

77 As usual, we assume that T is reduced. Then by (49), $C_1|T| > \prod_{\nu=1}^{r} y[y_\nu]$, $Q = (y_1, y_2, \ldots y_r) = (Q^* y')$ (say) where $Y' = (q_1, q_2, \ldots q_r)$ is the last row of Q and let $Q^* = (y_1^*, y_2^*, \ldots y_r^*)$ where $y_\nu = (y_\nu^*, q_\nu)$. Then $Y[y_\nu] = Y_1[y_\nu^*] + \lambda q_r^*$ and by (83), since the smallest characteristic root λ of λ_1 is at most equal to $\dfrac{\sqrt{3}}{2(n-1)C_1} = C$, we have $y_1 = y_\nu^*| \geq C y_\nu^* y_\nu^*$ with a positive constant $C = 1$ which depends only on u. Hence choosing $\lambda \geq C$, we have

$$y[\mathscr{U}] = y_1[\mathscr{U} *_\nu] + \lambda q_n^2 \geq C(y_\nu^*, y_\nu^* + y_\nu^2) = c y_\nu' y_\nu \geq C$$

since $\mathscr{U}_\nu' \mathscr{U}_\nu \geq$ being a primitive vector.

Thus

$$y[\eta_\nu] \geq \begin{cases} c, & \text{in any case} \\ \lambda, & \text{if } q_\nu = C \end{cases}$$

Hence if $xxxx \neq o$ at least one $q_\nu \neq *****$ so that one of the factors of the product ***** is greater than or equal to λ while the others are at least equal to C. Consequently ***** and a fortiri ***** if $\mathscr{U} \neq 0$. This means that $|T| \geq xxxxxxx$ choosing λ with $\lambda \geq \max *****$ and this in its turn implies that $\|CZ + 0\| \geq 1$ as desired. If however ***** then ***** is itself primitive and *****. The pair $\{C_1, C_2\}$ and ***** then determine a class ***** of coprime symmetric pairs of order $n-1$. A simple consideration shows that

$$\|C_o Z_1 + D_o\|^2 = |C_o|^2 |T|^2 \prod_{\nu=1}^{r} (1 + h_\nu^2) = \|CZ + D\|^2.$$

Since $Z_i \in f_{n-1}, \|C_o Z_1 + D_2\| \leq 1$- the same is therefore true of $\|CZ + D\|$ 78
too. Thus if $\lambda \geq \max(y_{n-1,n_1}, C, C_1 C^{1-n}$ and $Z_i \in$, then

$$Z = \begin{pmatrix} Z_i & 0 \\ 0 & \lambda \end{pmatrix} \in f_n. \tag{105}$$

Let $Z_1 \in \mathcal{F}_{n-1}$ be chosen such that $\sigma(\gamma_1^{-1}) \geq s_{n-1}^{-\varepsilon}$ where $Z_1 = X_1 + iy_1$ and ε is a given positive number.

Then, if $Z = \times + iy$ is determined by (105) we have $y = \begin{pmatrix} Z_i & 0 \\ 0 & \lambda \end{pmatrix}$ and $\sigma(y^{-1}) = \sigma_1^{-1}) = 1_{/\lambda}$.

Now $\mathcal{S}_n \geq \sigma(y^{-1}) = \sigma(y^{-1}) + 1_{/\lambda} \geq \mathcal{S}_{n-1} + 1_{/\lambda} - \varepsilon$. Since this is true for all $\lambda \geq \max *****$ letting $\lambda \to \infty$ we obtain that $s_n \geq s_{n-1} - \varepsilon$

The arbitrariness of $\in$ now implies that $z_n \geq s_{n-1}$, in other words, the sequence ***** is monotone increasing.

We shall need one more fact for proving our theorem, viz. that if $T = T^{(n)}$ is a semi positive matrix, and $0 <$ rank $T = n < \eta$ then there exists a positive matrix $T_i = T^{(n)} = T_1$ and a unimodular matrix V such that

$$T[V] = \begin{pmatrix} T_1^{(n)} & 0 \\ 0 & 0 \end{pmatrix} T_1^{(r)} > 0. \tag{106}$$

Indeed, due to our assumption we have $|T| = 0$ so that there exists a rational column $\mathcal{W}$ with $T[\mathcal{W}]$. Clearly we can assume $\mathcal{W}$ to be
79 primitive. Then $\mathcal{W}$ can be completed to a unimodular matrix V_o with $\mathcal{W}$ as the last column and then $T[V_o] = (t^{(o)}_{\mu\nu})$ is a matrix all the elements of whose last row and column vanish; in fact, $t^{(o)}_{nn} = 0$ since $T[\mathcal{W}] = 0$ and then $t^{(o)}_{n\nu} = 0 = (t^{(o)}_{\nu n} = 1, 2, \ldots n$ as $T \geq 0$. Let then

$$T[V] = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} T_o = T'_o = t_o^{(n-1)} \geq 0$$

If $r = n-1$, then clearly $T_o > 0$ and we are through. In the alternative case we can repeat the above with T_0 in the place of T and this can be continued until we arrive at $T_1 = T_1^{(r)}$ satisfying (106).

We now take up the proof of the main theorem. The proof is by induction on n. We assume that either $n = 1$ or if $n > 1$ then the theorem is true for all modular forms of degree $n - 1$. With this assumption on n we shall prove that the theorem is true for modular forms of degree n.

Let

$$f(Z) = \sum_{T \geq 0} a(T) e^{2\pi i \sigma(TZ)} \tag{107}$$

In the second case, viz. when $n > 1$ we have

$$f(Z)|\phi = \sum_{T_1 \geq 0} a(T_1) e^{2\pi i \sigma(TZ)}$$

where

$$a(T_1) = a\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

If $\sigma(T_i) < \dfrac{\Re}{4\pi} s_{n-i}$ then $\sigma\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = \sigma(T_1) < k_\nu S_{n-1}$ and our hy- **80**
pothesis now implies that $a(T_1) = a\begin{pmatrix} T_1 & 0 \\ 0 & C \end{pmatrix} = 0$. In other words, $f(z)|\phi$ satisfies the conditions of Theorem 6 so that, due to our assumption $f(z)|\phi \equiv 0$ as it is a modular form of degree $n-1$. This means that $a(T) = 0$ for T such that $|T| = 0$ For, by (106), $T = \begin{pmatrix} T_1 & 0 \\ 0 & C \end{pmatrix}[u]$, $\mathcal{U}$-unimodular, and by (95), since $T_1 \geq 0$,

$$\pm a(T) = a\begin{pmatrix} T_1 & 0 \\ 0 & C \end{pmatrix} = a(T_1) = 0 \tag{108}$$

For $n = 1$, the above is true by one of the assumptions in Theorem 6. Thus in any case $|T| = 0$ implies $a(T) = 0$ so that in (107) only those $T's$ with $T > 0$ survive; in other words

$$f(Z) = \sum_{T>0} a(T) e^{2\pi i \sigma(TZ)}$$

We now wish to prove a preliminary result, viz. if $Z = x, iy \in \mathscr{F}_n$ then

$$\lim_{|y|\to\infty} |y|^{\Re/2} \int (Z) = 0 \tag{109}$$

Let $y = (y_{\mu\nu}), T = (t_{\mu\nu}) > 0$ and introduce y_1, T_1 by requiring that $y = y_1[K]$, $T_1 = T[K] = [K]$ where $K = (\delta_{\mu\nu}\sqrt{y_{\mu\nu}})$. Then $\mathfrak{f} = y[K^{-1}]$ is a matrix of the type $\begin{pmatrix} 1_1 * & \\ * & 1 \end{pmatrix}$ and is positive. Hence y_1 is a bounded matrix with $|y_2| = |y|(y_{11}, y_{22}, y_{nn})^1 \geq C_1^{-1} > 0$ as a result of (49). These show that y_1 belongs to a compact subset y of the space of all positive symmetric matrices, which set depends only upon n.

According to (102) we then have

$$\sigma(Ty) = \sigma(T_1[k^{-1}]y_1[K]) = \sigma(T_1 y_1)$$

$$\geq \gamma\sigma(T_1) = \gamma \sum_{\nu=1}^{\gamma} y_{\nu\nu} t_{\nu\nu} \tag{110}$$

81 with a positive constant γ depending only on n. Since the series (107) converges everywhere and in particular at the point $z = \frac{\sqrt{3}}{2}\frac{\gamma}{2} iE$ we get that $a(T)e^{-\frac{\sqrt{3}}{2}\pi\gamma\sigma(T)}$ is bounded, and by multiplying $\mathfrak{f}(Z)$ by a constant factor if necessary, we can assume the bound to be 1. We may now estimate the general term of (107) as follows :

$$\begin{aligned} |a(T)e^{2\pi i\sigma(TZ)}| &\leq e^{\frac{\sqrt{3}}{2}\pi\gamma\sigma(T) - 2\pi\sigma(TY)} \\ &= e^{-\frac{\sqrt{3}}{2}}(t_{11} + t_{22} + \cdots t_{nn} - 2\pi\sigma(TY) \\ &\leq e^{\pi\gamma(\sum_{\nu=1}^{n} t_{\nu\nu}) - 2\pi\gamma(\sum_{\nu=1}^{n} t_{\nu\mu} y_{\nu\mu}})\\ &e^{-\pi\gamma(\sum_{\nu=1}^{n} t_{\nu\nu} y_{\nu\nu})}. \end{aligned}$$

In the above we have made use of (75) and (110).

Since $\sum\limits_{\nu=1}^{n} t_{\nu\nu} y_{\nu\nu} \geq n \sqrt[n]{\prod_{\nu=1}^{n} t_{\nu\mu} y_{\nu\nu}}$ and $|y| \leq \prod\limits_{\nu=1}^{n} y_{\nu\nu}$ by (81), Y being positive, we have from the above

$$\begin{aligned} |a(T)e^{2\pi i\sigma(TZ)}| &\leq e^{-\pi\gamma n \sqrt[n]{n\pi_{\nu=1}^{n} t_{\nu\nu} y_{\nu\nu}}} \\ &\leq e^{-\pi\gamma n \sqrt[n]{|Y|} \sqrt[n]{t_{11} t_{22}, \ldots t_{nn}}} \end{aligned}$$

Since $\mathfrak{K} \geq 0$ for a suitable constant $\mathscr{C}$ we have

$$|Y|^{\frac{\mathfrak{K}}{2}} e^{\pi n \frac{\gamma}{2} \sqrt[n]{|Y|}} < \mathscr{C}$$

and then $|Y|^{\frac{\mathfrak{K}}{2}} \mathfrak{f}(Z) = \sum\limits_{T>0} |Y|^{\frac{\mathfrak{K}}{2}} a(T) e^{2\pi i\sigma(TZ)}$ is majorised by the series $\mathscr{C} \sum\limits_{T>0} e^{\dfrac{-\pi\gamma n}{2} \sqrt[n]{|\gamma|} \sqrt[n]{t_{11} t_{22} \ldots t_{nn}}}$

82 The last series converges by arguments as in pages (66 - 67) since the number of semi integral positive matrices $T = (t_{\mu\nu})$ with $t_{11} t_{22} \ldots t_{nn} = t$ for a given t, can be estimated by $\mathscr{C} t^{\frac{n(n+1)}{2}}$, viz. a fixed power of t and then, the last series is further majorised by the convergent series

$$\mathscr{C} \sum_{t=1}^{\infty} \mathscr{C}_1 t^{n(n+1)/2} e^{-T\gamma/2*****}$$

It now turns out that for $Z \in \mathcal{F}_n$

$$|Y|^{\Re/2}\mathcal{F}(x) = 0(e^{-e*****} \qquad (111)$$

with some $\in > 0$ as $|y| \to \infty$ and this, in particular, implies (109). If $h(Z) = |y|^{\Re/2}|\mathcal{F}(Z)|$, Theorem (4) and (109) imply that $h(Z)$ is bounded in the fundamental domain. But we know from (101) that $h(Z)$ is invariant under modular substitutions and the $h(Z)$ is bounded throughout its domain. Since further $h(Z)$ is continuous in $\mathscr{Y}$ it attains its maximum; in other words there exists a point $Z_\nu \in$ such that $h(Z) \leq h(Z_o)$ for $z \in \mathscr{Y}$. Consider $h(Z)$ in a neighbourhood of Z_o. Let $z = x - l_y$ be a complex variable $Z = Z_o - zE$ and $t = e^{z\pi iz}$.

Let $g(t) = \mathcal{F}(z)e^{-\lambda\sigma(Z)}$ with λ determined by

$$\frac{\eta\lambda}{2\pi} = 1 + [k/_{4\pi}s_n] \text{ where } [x]$$

denotes the integral part of x, viz. the largest integer not exceeding x. Using our assumption that $a(T) = 0$ if $\sigma(T) < \frac{\Re}{4\pi}s_n$ we have

$$\begin{aligned} g(t) &= \sum_{\sigma(T) > \frac{\Re}{4\pi}s_n} a(T)e^{2\pi i\sigma(TZ) - i\lambda\sigma(z)} \\ &= \sum_{\sigma(T)\frac{\Re}{4\pi}\mathcal{S}_n} a(T)e^{2\pi i\sigma(Tz_o)}t^{\sigma(T)}e^{-i\lambda\sigma(z)} \\ &= \sum_{\sigma(T)\frac{\Re}{4\pi}\mathcal{S}_n} a(T)e^{2\pi i\sigma(Tz_o) - i\lambda\sigma(z_o)}t^{\sigma(T) - \frac{\lambda n}{2\pi}} \end{aligned}$$

Here the exponent $\sigma(T) - \dfrac{\lambda n}{2\pi} > \dfrac{\Re}{4\pi}\mathcal{S}_n - [\dfrac{\Re}{4\pi}\mathcal{S}_n] - 1 \geq -1$ which **83**
means, as $\sigma(T) - \dfrac{\lambda n}{2\pi}$ is an integer, that $\sigma(T) - \dfrac{\lambda n}{2\pi} \geq 0$.

This shows that the function $g(t)$ is regular in a circle $|t| \leq \rho$ and we can assume $\rho > 1$ by choosing $y < 0$ with its absolute value sufficiently small.

By the maximum principle, there exists then a point t_1 with $|t_1| = \rho > 1$ such that $|g(t_1)| \geq |g(1)|$. If the z-point corresponding to t_1 is denoted by z_1, we have

$$|g(t)| = |\mathfrak{f}(Z)|e^{\lambda\sigma(y)} = h(z)|y|^{\frac{\Re}{2}}e^{\lambda\sigma(y)}$$

and putting $t = 1$ the above now implies that

$$h(z_o)|Y_o|^{\frac{\Re}{2}} e^{\lambda\sigma(y_o)} \leq h(z_1)|Y_o|^{\frac{\Re}{2}} e^{\lambda\sigma(y_1)}$$

where $Z_1 = Z_o + z_1 E, y_1 = y_o + yE, \int^{-2\pi y} = |t_1| = \rho$.

$$y = -\frac{1}{2\pi} \log \rho$$

Let $h(z_o) = \sup_{z \in y} h(z) = M$ (say). Then the above yields

$$M \leq M|y|^{\frac{\Re}{2}} |y_0|^{\frac{\Re}{2}} e^{\sigma(y_1 - y_o)} = Me^{\psi(y)} \tag{112}$$

84 where

$$\psi(y) = \lambda ny - \frac{\Re}{2} \log(|y_1||y_o|^{-1})$$

$$= \psi(y) \qquad\qquad \lambda ny - \frac{\Re}{2} \log |E|y\gamma_o^{-1}$$

Now $\psi(0) = 0$ and $\psi'(o) = \lambda n \frac{\Re}{2} \sigma(\gamma_o^{-1})$

$$\geq \lambda n - \frac{\Re}{2} s_n$$

$$= 2\pi(\frac{\lambda n}{2\pi} - \frac{\Re}{4\pi} s_n) > 0.$$

Hence it follows that $\psi(y)$ is monotone increasing in a neighbourhood of $y = 0$ so that it is negative for sufficiently small $y < 0$. But $\psi(y) < 0$ implies from (112) that $M = 0$ which in its turn means that $h(z)$ and consequently $f(z)$ vanishes identically. The proof of theorem 6 is now complete.

The above theorem has several interesting consequences. In the first instance it implies that *all modular forms of weight* $0\Re = 0$) *are necessarily constants*. For if $\mathcal{F}(z) = a(o) + \sum_{T \neq 0} a(T) e^{2\pi i \sigma(TZ)}$ be such a form, then $\mathcal{F}(Z) - a(0)$ is a form satisfying the conditions of theorem 6. Hence it follows from theorem 6 that $\mathcal{F}(Z) - a(o) \equiv 0$, in there words $\mathcal{F}(Z) \equiv a(o)$. We therefore assume in the sequel that $k > 0$.

Now consider modular forms $\mathcal{F}(Z)$ of degree $1(n = 1)$ and weight $\mathfrak{K} \leq \delta$. Then from (75) we deduce that $\mathscr{S}_1 \frac{2}{\sqrt{3}}$. Then $\sigma(T) \leq \frac{\mathfrak{K}}{4\pi}\mathscr{S}_1$ implies that $\sigma(T) \leq \frac{2}{\pi}\frac{2}{\sqrt{3}} = \frac{4}{\pi\sqrt{3}} < 1$ which in its turn means that $\sigma(T) = 0$ as $sigma(T)$ is an integer, and consequently $T = 0$. Hence if

$$\mathfrak{f}(z) = a(0) + \sum_{T \neq 0} a(T) e^{2\pi i \delta(TZ)}$$

and if $a(0) = 0$, then the conditions of theorem 6 are satisfied so that by **85**
its conclusion, $\mathfrak{f}(z) \equiv 0$. In other words, modular forms of degree 1 and weight $\mathfrak{K} \leq 8$ are uniquely determined by their 'first' Fourier coefficient $a(0)$. It is now immediate that any two modular forms of degree 1 and weight $\mathfrak{K} \leq 8$ are proportional and what is the same, a modular form of degree 1 and weight $\mathfrak{K} \leq 8$ is unique, except for multiplication by a constant. Indeed, the same result is true of modular forms of degree 2 too. For if $Z = x + iy \in \mathfrak{f}_2$ and $f(Z)$ is a modular form of degree 2 and weight $\mathfrak{K} \leq 8$, then writing $Y = (y_{\mu\nu})$ we have, by Minkowski's reduction conditions,

$$0 \leq 2y_{12} \leq y_{y11} \leq y_{22}$$

and then $|Y| = y_{11}y_{22} - y_{12}^2 \geq \frac{3}{4} y_{11} y_{22}$.

Since $y_{11} \geq \frac{\sqrt{3}}{2}$ by (75), this means that

$$\sigma(Y^{-1}) = \frac{y_{11} + y_{22}}{|Y|} \leq \frac{4}{3}(\frac{1}{y_{11}} + \frac{1}{y_{22}}) \leq \frac{8}{3}\frac{1}{y_{11}} \leq \frac{16}{3\sqrt{3}}$$

so that $s_2 \leq \frac{16}{3\sqrt{3}}$. Also

$$\frac{\mathfrak{K}}{4\pi} s_2 \leq \frac{2}{\pi} \leq \frac{32}{3\sqrt{3}} < 2 \text{ and then }, \sigma(T) < \frac{\mathfrak{K}}{4\pi} s_2$$

implies that $\sigma(T) < 2$ This means that at least one of the two diagonal elements and consequently one of the elements t_{12}, t_{21} where $T = (t_{\mu\nu})$

also vanish. Then it is clear that $|T| = o$. Then assumption $a(o) = o$ will imply by our earlier result on modular forms of degree 1 that $\mathfrak{f}(Z)|\phi \equiv 0$ and then as in (106) and (108), we have $a(T) = 0$ for $|T| = 0$. In
86 other words if we assume $a(0) = 0$ then the conditions of theorem 6 are satisfied and we are able to conclude that $\mathfrak{f}(Z) \equiv 0$. It now follows as in the earlier case that *every modular form of degree 2 and weight $R \leq 8$ is uniquely determined by the first Fourier coefficient* $a(0)$.

The above results are for the moment hypothetical. In other words, their significance is based on the assumption that modular forms of the desired degree, not vanishing identically actually exist. Their existence we prove later, by constructing the so called *Eisenstein Series*.

An interesting application of theorem 6 is to prove the algebraic dependence of any set of sufficiently large number of modular forms. Specifically we state

Theorem 7. *Let $h = \frac{n(n+1)}{2} + 2$ and let $\mathfrak{f}_\nu(Z)$ be a modular form degree n and weight $\mathfrak{K}_\nu > 0, \nu = 1, 2, \ldots h$. Then there exists an isobaric algebraic relation*

$$\sum C_{\nu_1 \nu_2}, \ldots \nu_2 \mathfrak{f}^1 \mathfrak{f}_Z^{\nu 2} . \mathfrak{f}_h^{\nu h} = 0 \tag{113}$$

not all of whose coefficients vanish, the summation extending over all integers $\nu_i \geq o$ with the property

$$\sum_{l=1}^{h} \nu_\varepsilon \mathfrak{K}_i = m \mathfrak{K}_i \mathfrak{K}_2, \mathfrak{K}_h \tag{114}$$

where m is an integer which depends only upon n

We may remark that the product of two modular forms of weight k and ℓ is a modular form of weight $\mathfrak{K}K$, their degrees being the same . Consequently all the power products $\mathfrak{f}_1^{\nu_1} \mathfrak{f}_2^{\mu_2} \ldots, \mathfrak{f}_h^{\mu_h}$ that occur in (113) are modular forms of the same weight in view of (114) and hence the name *isobaric* for the relation (113).

86 *Proof.* All modular forms of degree n and weight k form a linear space $m_{\mathfrak{K}}^n$. By means of theorem 6 we infer that this space is of finite dimension

$\alpha_n(\Re)$. In fact $\alpha_n(\Re)$. is at most equal to the number of solutions of the relation $\sigma(T) \leq \frac{(\Re)}{4\pi}\mathscr{S}_n$ with semi positive integral T's, a rough upper estimation of which is provided by $\frac{\Re)}{4\pi}\mathscr{S}_n^{*****}$ by arguments as at the end of page (66); in other words we have

$$d_\eta(\Re)(\frac{\Re)}{4\pi}\mathscr{S}_n + i)^{n(n+1)/2} \tag{115}$$

We now find a lower estimation of the number of possible power products $F = \mathfrak{k}_1^{\nu_1}\mathfrak{k}_2^{\nu_2}, \ldots \mathfrak{k}_h^{\nu_h}$ subject to the conditions □

$$\left.\begin{aligned} &\mu_i \geq C, i = 1.2\ldots 2 \text{ and integral,}\\ &\sum_{i=1}^{h} \nu_i\Re_i = \eta\Re_i\Re_2, \ldots \Re_h m = m(n) \end{aligned}\right\} \tag{116}$$

Let us assume for the moment that m is divisible by $2h - 2$.

Let $(x_1, x_2, *****)$ be a system of integers with

$$o \leq x_\nu \leq ***** \tag{117}$$

where $K = \Re_1\Re_2 \cdots \Re_h$. The number of such systems is clearly $H*****$

There exists then one coset modulo $\Re_1$ which contains the sums $\Re_1 x_1 \Re_2 x_2 + \ldots \Re_{r-1} x_{h-1}(\text{mod})\Re_n$ for at least $\frac{H-1}{\Re_h} + 1$ different systems *** as otherwise, the total number of different systems $(x_1, x_2, \ldots x_{h-1})$ could be at most $H - 1_{/\Re_h}, \Re_h = H - 1$ while actually there are H such systems. Consider then the systems $(x_1, x_2, \ldots x_{h-1})$ corresponding to this coset. Let $(\xi_1, \xi_2, \ldots \xi_{h-1})$ denote a fixed system among these and let $(\eta_1, \eta_2, \ldots \eta_{h-1})$ a variable system. Then clearly we have **87**

$$\sum_{i=1}^{h-1} k_i\eta_i - \sum_{i=1}^{h-1} \Re_i\xi_i \equiv o \bmod \Re_h \tag{118}$$

We shall further assume $o \leq \xi_i < \Re_i, l = 1, 3, \ldots h - 1$

We introduce now $\nu_i = \eta_i\Re_h - \xi_i, = 1, 2, \ldots h - 1$ and $\nu_h = \frac{m-k}{\eta_n} - \frac{1}{h_k}(\sum_{i=1}^{h-1} \Re_i\nu_i$

We shall verify that the system $h\{\nu\}$ satisfies the conditions (116). Clearly $\nu_1, \nu_2, \ldots \nu_{h-1}$ are non negative and integral. Also using (117) we have

$$\begin{aligned} \nu_h &= \frac{mK}{\Re_h} - \frac{1}{\Re_h}\sum_{i=1}^{h-1}(\Re_i\eta_i + \Re_i(\Re_h - \xi_i)) \\ &\leq \frac{mK}{\Re_h} - \frac{1}{\Re_h}\sum_{i=1}^{h-1} - \frac{mK}{2h-2} - \sum_{i=1}^{h-1} - \Re_i \\ &= mK_{/2\Re_h} - \sum_{i=1}^{h-1} - \Re_i \\ &= \frac{(h-1)K}{\Re_h} - \sum_{i=1}^{h-1} - \Re_i \geq 0 \end{aligned} \tag{119}$$

The last but one of these relations is a consequence of the fact that m is divisible by $2h-2$ and a fortiori $\frac{m}{2} \geq h-1$ while the last step
87 is immediate by observing that $\frac{K}{k_h} \geq \Re_i 0 = 1, 2, \ldots h-1. \nu_h \nu_h$ is certainly integral as seen from (119) by means of (118). We have therefore shown that each system $(\eta_1, \eta_2, \ldots \eta_{h-1})$ leads to a permissible system of exponents $\nu_1, \nu_2, \ldots \nu_h$. Consequently,the number of possible power products $\mathfrak{f}_1^{\nu 1}\mathfrak{f}_2^{\nu 2} \cdots \mathfrak{f}_h^{\nu h}$ is at least as great as the number of the systems $(\eta_1, \eta_2, \ldots \eta_{h-1})$ which is at least $\frac{H-1}{\Re_h} + 1$. Denoting $\frac{H-1}{\Re_h}$ by q we have proved that there exist at least $q+1$ modular forms of the kind $\mathfrak{f}_1^{\nu 1}\mathfrak{f}_2^{\nu 2} \cdots \mathfrak{f}_h^{\nu h}$

Therefore, in case

$$q \geq d_n(mK) \tag{120}$$

there exists a non trivial relation

$$\sum C_{nu1\nu_2} \cdots \nu_h \mathfrak{f}_1^{\nu 1} \cdots \mathfrak{f}_h^{\nu h} = 0$$

with constant coefficients C_ν.

As a result of (115), (120) will be satisfied provided we have $q \geq (\frac{mK}{\pi} s_n + 1)^{2-2}$ which in turn will be true if

$$q \geq m^{h-8}(\frac{K}{\pi}\mathscr{S}_n + 1)^{h-2} \tag{121}$$

Since $H = \prod_{\nu=1}^{h}(1 + \frac{mK}{(2,p,-2)\Re_\nu})$ we have

$$q = \frac{H-1}{\Re_h} \geq \prod_{\nu=1}^{h}(1 + \frac{mK}{(2, p, -2)\Re_\nu})$$
$$= \frac{m^{h-1}K^{h-2}}{(2h-2)^{h-1}}$$

Hence (121) is certainly satisfied if

$$\frac{m^{h-1}K^{h-z}}{(2h-2)^{h-1}} \geq m^{h-z}(\frac{K}{\pi}\mathscr{S}_n + 1)^{h-z}$$
$$i.e. \qquad \text{if } m \geq (2h-2)^{h-1}(\frac{\mathscr{S}n}{\pi} + 1)^{h-2},$$

Obviously, consistent with this requirement, the assumption that m is **88**
divisible by $2h-2$ is permissible and theorem 7 is established.

Chapter 7

The symplectic metric

The paragraph deals with the symplectic metric in $\mathscr{Y}$ defined by a positive quadratic differential form which is invariant under all symplectic substitutions. Let Z_1, $Z_2 \in \mathscr{Y}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_n$. Let $Z^*_\nu = M < Z_\nu >$, $\nu = 1, 2$. A simple computation yields by means of the relation **89**

$$Z^*_\nu = (AZ_\nu + B)(CZ_\nu + D)^{-1} = (Z_\nu C' + D')^{-1}(Z_\nu A' + B')$$

and the typical relations for a symplectic matrix established in §1, the formulae

$$\begin{aligned}
Z^*_2 - Z^*_1 &= (Z_1 C' + D')^{-1}(Z_2 - Z_1)(CZ_z + D)^{-1}\\
Z^*_2 - \bar{Z}^*_1 &= (\bar{Z}_1 C' + D')^{-1}(Z_2 - \bar{Z}_1)(CZ_2 + D)^{-1}\\
\bar{Z}^*_2 - \bar{Z}^*_1 &= (\bar{Z}_1 C' + D')^{-1}(\bar{Z}_2 - \bar{Z}_1)(C\bar{Z}_2 + D)^{-1}\\
\bar{Z}^*_2 - \bar{Z}^*_1 &= (Z_1 C' + D')^{-1}(\bar{Z}_2 - Z_1)(C\bar{Z}_2 + D)^{-1}
\end{aligned} \tag{122}$$

Now for any points $Z_1, Z_2 \in \mathscr{Y}$ it is easily seen that $Z_2 - \bar{Z}_1$ and $\bar{Z}_2 - Z_1$ also belong to $\mathscr{Y}$ and by (25), every $Z \in \mathscr{Y}$ is nonsingular. Thus the inverses $(Z^*_2 - \bar{Z}^*_1)^{-1}(\bar{Z}^*_2 - Z^*_1)^{-1}$ exist. From (122) we now obtain that

$$\begin{aligned}
&(Z^*_2 - Z^*_1)(Z^*_2 - \bar{Z}^*_1)^{-1}(\bar{Z}^*_2 - \bar{Z}^*_1)(\bar{Z}^*_2 - \bar{Z}^*_1)^{-1}\\
&= (Z_1 C' + D')^{-1}\varrho(Z_1, Z_2)(Z_1 C' + D')
\end{aligned} \tag{123}$$

where

$$\varrho(Z_1, Z_2) = (Z_2 - Z_1)(Z_2 - \bar{Z}_1)^{-1}(\bar{Z}_2 - \bar{Z}_1)(\bar{Z}_2 - Z_1)^{-1} \tag{124}$$

90 It is immediate from (123) and (124) that the characteristic roots of $\varrho(Z_1, Z_2)$ for any two Z_1, $Z_2 \in \mathscr{Y}$ are invariant under the symplectic mapping $Z_\nu \to M < Z_\nu >, M \in S_n$. In particular it follows that $\sigma(\varrho(Z_1, Z_2))$ is an invariant function of Z_1, Z_2 meaning

$$\sigma(\varrho(Z_1, Z_2)) = \sigma(\varrho(M < Z_1 >, M < Z_2 >)), M \in S_n \tag{125}$$

For any matrix Z, we define dZ the matrix of the differentials as $dZ = (dZ_{\mu\nu})$ where $Z = (Z_{\mu\nu})$. For two matrices Z_1, Z_2 the following relations are easily verified.

$$\begin{aligned} &1) \quad d(Z_1 + Z_2) = dZ_1 + dZ_2 \\ &2) \quad d(Z_1 + Z_2) = dZ_1.Z_2 + Z_1 dZ_2 \end{aligned} \tag{126}$$

As a consequence of (2) above we have, if $|Z| \neq 0$,

$$\begin{aligned} 0 = d(E) = d(ZZ^{-1}) = dZZ^{-1} + Z \cdot dZ^{-1} \text{ so that} \\ dZ^{-1} = Z^{-1} dZ \cdot Z^{-1} \end{aligned} \tag{126$'$}$$

With these preliminaries about differentials, we once again take up the main thread. In (124), we specialize Z_1, Z_2 as $Z_1 = Z$, $Z_2 = Z + dZ$ and obtain

$$\rho(Z, Z + dZ) = \frac{1}{4} dZ Y^{-1} d\bar{Z} y^{-1}, Z = X + iY. \tag{127}$$

From (125) it now follows that

$$d\mathscr{S}^2 = \sigma(dZ.Y^{-1} d\bar{Z} Y^{-1}) \tag{128}$$

is an invariant quadratic defferential form in the elements $dX_{\mu\nu}$, $dY_{\mu\nu}$ of
91 dX, dY respectively. We may also consider $d' s^2$ as a hermitian form in the elements $dZ_{\mu\nu}$ of dZ. Further, if R be a real matrix such that $Y^{-1} = RR'$ and if $\Omega = (\omega_{\mu\nu}) = R' dZR$ then $\Omega = \Omega'$ and $d\mathscr{S}^2 = \sigma(R' dZRR' d\bar{Z}R) = \sigma(\Omega' \bar{\Omega}) = \sum_{\mu,\nu} \omega_{\mu\nu} \bar{\Omega}_{\mu\nu} \geq 0$. If now $\sum_{\mu,\nu} \omega_{\mu\nu} \bar{\omega}_{\mu\nu} =$

0 then $\Omega = 0$ and this will require that $dZ = 0$ as $R \neq 0$. Thus $ds^2 > 0$ if $dZ \neq 0$ and what is the same, $ds^2 > 0$ represents a positive quadratic form. Now we can consider $\mathscr{Y}$ as a Riemannian space with the fundamental metric given by ds^2, called the *symplectic metric*. The decomposition $dZ = dX - dY$ yields

$$\begin{aligned} ds^2 &= \sigma\{(dX + dY)Y^{-1}(dX - dY)Y^{-1}\} \\ &= \sigma\{(dXY^{-1})dXY^{-1} + (dY - Y^{-1}dYY^{-1}\} \end{aligned}$$

taking only the real part, since the imaginary part has got to vanish as $ds^2 > 0$. Thus we get

$$ds^2 = \sigma(Y^{-1}dX)^2 + \sigma(Y^{-1}dY)^2 \tag{129}$$

as expression which is well known in the case $n = 1$.

We now carry over these results to the generalized unit circle, which we may recall is the set $\mathfrak{K}$ of $W's$ satisfying the condition

$$W = W', E - W'\overline{W} > 0 \tag{$2b$}'$$

we know that we can map $\mathscr{Y}$ into $\mathfrak{K}$ by means of the mapping

$$W = (Z - iE)(Z + iE)^{-1} = (Z + iE)^{-1}(Z - iE),$$

If W_1, W_2 correspond to Z_1, Z_2 we have

$$\begin{aligned} W_2 - W_1 &= 2i(Z_1 + iE)^{-1}(Z_2 - Z_1)(Z_2 + iE)^{-1} \\ E - \overline{W}_1 W_2 &= -2i(\overline{Z}_1 - iE)^{-1}(Z_2 - \overline{Z}_1)(Z_2 - iE)^{-1} \\ \overline{W}_2 - \overline{W}_1 &= -2i(\overline{Z}_1 - iE)^{-1}(\overline{Z}_2 - \overline{Z}_1)(\overline{Z}_2 - iE)^{-1} \\ E - W_1\overline{W}_2 &= 2i(Z_1 + iE)^{-1}(\overline{Z}_2 - Z_1)(\overline{Z}_2 - iE)^{-1} \end{aligned} \tag{130}$$

and then **92**

$$(W_2 - W_1)(E - \overline{W}_1 W_2)^{-1}(\overline{W}_2 - \overline{W}_1)(E - W_1\overline{W}_2)^{-1} = (Z_1 + iE)^{-1}\varrho(Z_1, Z_2)(Z_1 + iE)$$

In particular therefore, the matrix represented by the left side has the same characteristic roots as $\varrho(Z_1, Z_2)$ and hence also the same trace.

Setting $Z_1 = dZ$, $Z_2 = Z + Z$ we have $W_1 = W$, $W_2 = W + dW$ and then from the above, and from (127), (128) we have

$$\begin{aligned} ds^2 &= 4\sigma(\varrho(Z, Z + dZ)) \\ &= 4\sigma(dW(E - \overline{W}W)^{-1} d\overline{W}(E - W\overline{W})^{-1}) \end{aligned} \tag{131}$$

This provides another useful expression for ds^2

We observe at this stage that given two points Z_1, $Z_2 \in \mathscr{Y}$ we can always find a symplectic substitution which brings Z_1, Z_2 into the special position

$$Z_1 = iE, Z_2 = iD = i(\delta_{\mu\nu\delta_\nu}) \quad 1 \leq d_1 \leq d_2 \leq \cdots \leq d_n$$

For, in view of the $1-1$ correspondence between $\mathscr{Y}$ and $\mathfrak{K}$ it suffices to determine a permissible substitution which takes two assigned points
93 W_1, W_2 into 0 and a special points $D_1 = (\delta_{\mu\nu}, \dfrac{d_\nu - 1}{d_\nu + 1})$ and in view of the homogenity of the space $\mathfrak{K}$ it suffices to determine a mapping which leaves the origin fixed and takes a given point W into a point of the type D_1 viz, a diagonal matrix with positive diagonal elements in the increasing order. This we know is possible by Lemma 2 by a mapping of the kind

$$W \to u'Wu, u - \text{unitary},$$

and this settles our claim,

We now introduce a parametric representation for positive matrices $y > 0$. We choose matrices $F = F^{(r)}$, $G = G^{(r)}$ and $H = H^{(r,n-r)}$ where r is a fixed integer in $1 \leq r < n$ to satisfy

$$y = \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix} \left[\begin{pmatrix} E & H_{nr} \\ 0 & E \end{pmatrix} \right] = \begin{pmatrix} F & FH \\ H'E & G + F[H] \end{pmatrix} \tag{132}$$

Such a choice is always possible as a comparison of the two extremes shows. Also from (132) it is clear that $Y > 0$ is equivalent with $F > 0$, $G > 0$. By means of (132) we also have

$$y^{-1} = \begin{pmatrix} F^{-1} + G^{-1}[H'] & HG^{-1} \\ -G^{-1}H' & G^{-2} \end{pmatrix}$$

and obtain by a simple computation, using properties (126), ((126)′), that

$$\sigma(y^{-1}dy)^2 = \sigma(F^{-1}dF)^2 + \sigma(G^{-1}dG)^2 + 2\sigma(G^{-1}dH'FdH) \qquad (133)$$

We wish to remark at this stage that all terms in (133) are positive quadratic forms in the appropriate elements; a possible doubt can only be about the last term but by specialising dF, dG it is easily seen that the last term represents a positive quadratic form in the elements of dH. 94

We proved to consider the existence and uniqueness of geodesic lines in $\mathscr{Y}$ and we prove the following

Theorem 8. *Given any two points Z_1, $Z_2 \in \mathscr{Y}$ there exists a uniquely determined geodesic line joining them. The length $s(Z_1, Z_2)$ of this geodesic line is given by*

$$s(Z_1, Z_2) = \sqrt{\sum_{\nu=1}^{n}(\log \frac{1+\lambda_\nu}{1-\lambda_\nu})^2} \qquad (134)$$

where $\lambda_\nu^2, \nu = 1, 2, \ldots, n$ are the characteristic roots of

$$\rho(Z_1, Z_2) = (Z_2 - Z_1)(Z_2 - \overline{Z}_1)^{-1}(\overline{Z}_2 - \overline{Z}_1)(\overline{Z}_2 - Z_1)^{-1}.$$

In the special case $Z_1 = iE$, $Z_2 = iD = l(\delta_{\mu\nu})$ a parametric representation of the geodesic line given by

$$Z = Z(t) = (\delta_{\mu\nu} d_\nu^t; 0 \leq \varepsilon \leq 1$$

Proof. Since we know that there always exists a symplectic substitution which takes Z_1, Z_2 into iE, iD and that the characteristic roots λ_ν^2 of $\rho(Z_1, Z_2)$ are invariant under such a substitution, it clearly suffices to prove the theorem for these special values of Z_1, Z_2 Assume for a moment that a geodesic line joining $Z_1 = iE$ to $Z_2 = iD$ exists with a parametric representation $Z = Z(t), 0 \leq t \leq 1$ such that the elements $Z_{\mu\nu} = Z_{\mu\nu}(t)$ have continuous derivatives with respect to t. We shall show 95
that such a line is unique. Also, in the course of the proof we explicitly determine what $Z(t)$ is. Then the existence of a geodesic line with the desired properties is immediate - in fact, the curve represented by $Z(t)$ which we have explicitly determined is the desired geodesic line. □

We shall denote the differential coefficient with respect to t as $\dfrac{d}{dt}(*) = (\dot{*})$ By means of (128) and (133) we have

$$
\begin{aligned}
s(Z_1, Z_2) = \int_{Z_1}^{Z_2} (ds^2)^{1/2} &= \int_0^1 \{\sigma(Z)^{-1}\bar{Z}Y^{-1}\}^{1/2} dt \\
&= \int_0^1 \{\sigma(Y)^{-1}ZY^{-1}\bar{Z}\}^{1/2} dt \\
&= \int_0^1 \{\sigma(Y^{-1}x)^2 + \sigma(Y^{-1}y)^2\}^{1/2} dt \\
&= \int_0^1 \{\sigma(Y^{-1}x)^2 + \sigma(F^{-1}\dot{F})^2 + \sigma(G^{-1}\dot{G})^2 \\
&\qquad + 2\sigma(G^{-1})\dot{H}^1 F\dot{H})\}^{1/2} dt \\
&\geq \int_0^1 \{\sigma(F^{-1}\dot{F})^2 + \sigma(G^{-1}\dot{G})^2\} dt
\end{aligned}
$$

We claim that the last inequality is actually a equality. For otherwise, in the equation $Z_{(t)} = \begin{pmatrix} F(t) & C \\ 0 & G(t) \end{pmatrix}$ we will have a curve joining Z_1 and Z_2 whose length will be actually

$$\int_0^1 \{\sigma(F^{-1}\dot{F})^2 + \sigma(G^{-1}\dot{G})^2\} dt < s(Z_1, Z_2)$$

contradicting our assumption. But then, the equality can hold in the above when and only when

$$\dot{X} = 0 = \dot{H} \tag{135}$$

We therefore conclude that (135) holds identically in t, which means
96 that $X \equiv 0 \equiv H$ as $X(0) = 0 = H(0)$ then the parametric representation of our geodesic line is given by

$$Z(t) = i\begin{pmatrix} F(t) & 0 \\ 0 & G(t) \end{pmatrix} \tag{136}$$

where $F = F^{(r)}$, $G^{(n-r)}$ and r is arbitrary with $1 \leq r < n$. According to the permissible cases $r = 1, 2, \ldots, n-1$ the same geodesic line will admit of $(n-1)$ formally different representations so that it follows from (136) that $Z(t)$ is necessarily a diagonal matrix $Z(t) = i(\delta_{\mu\nu} y_{\mu\nu}(t))$ Consequently we obtain

$$s(Z_1, Z_2) = \int_0^1 \sqrt{\sum_{\nu=1}^{k} \mathscr{Y}_{\nu\nu}^{2} \mathscr{Y}_{\nu\nu}^{-2} dt} = \int_0^1 \sqrt{\sum_{\nu=1}^{n} \mathscr{Y}_{\nu\nu}^{2} dt}$$

putting $n_{\mu\nu} = \log \mathscr{Y}_{\nu\nu}$. In the n dimension Euclidean space with $n_{\nu\nu}, \nu = 1, 2, \ldots n$ as rectangular cartesian coordinates, $s(Z_1, Z_2)$ then represent the Euclidean length of our curve. But in this case we know that the curve of shortest length is the straight line segment joining the points corresponding to Z_1, Z_2. Hence we conclude that

$$\log \mathscr{Y}_{\nu\nu}(t) = \eta_{\nu\nu}(t) = t \log d_2 \quad (\nu = 1, 2, n) 0 \leq t \leq 1.$$

In other words we have the parametric representation of the geodesic line joining $Z_1 = iE$, $Z_2 = iD = i(\delta_{\mu\nu} d_\nu)$ as

$$Z(t) = i(\delta_{\mu\nu}\ d_\nu^t), 0 \leq t \leq)$$

Now we compute the characteristic roots λ_ν^2 of (iE, iD)

$$\begin{aligned} \varrho(iE, iD) &= (D-E)(D+E)^{-1}(D-E)(D+E)^{-1} \\ &= (\delta_{\mu\nu}(\frac{d_\nu - 1}{d_\nu + 1})^2) \end{aligned}$$

so that $\lambda_\nu^2 = (\frac{d_\nu - 1}{d_\nu + 1})^2$. Consequently $\pm\lambda_\nu = \dfrac{d_\nu - 1}{d_\nu + 1}$ which means that **97**
$(\log \dfrac{1+\lambda\nu}{1-\lambda\nu})^2) = (\log d_\nu)^2$.

We know that $s(Z_1, Z_2)$ is the Euclidean length of the segment joining $(0, 0, \ldots 0)$ and $(\log d_1, \ldots \log d_n)$.

$$\begin{aligned} \text{Hence } s(Z_1, Z_2) &= \left(\sum_{\nu=1}^{n} (\log d_\nu)^2\right)^{1/2} \\ &= \left\{\sum_{\nu=1}^{n} (\log \frac{1+\lambda_\nu}{1-\lambda_\nu})^2\right\}^{1/2} \end{aligned}$$

and the proof is complete.

Having thus obtained the 'shortest distance' $s(Z_1, Z_2)$ between any two points Z_1, Z_2 we wish to remark that the *symplectic sphere* defined as the set of points Z with $Z \in \mathscr{Y}$, $s(Z, Z_0) \leq r$ – Z_0 being a fixed point in $\mathscr{Y}$ – is a compact set. Certainly we can assume that $Z_0 = iE$.

Then

$$s(Z, iE) = \left\{ \sum_{\nu=1}^{n} (\log \frac{1+\lambda_\nu}{1-\lambda_\nu})^2 \right\}^{1/2} \leq r \tag{137}$$

where λ_ν^2 are the characteristic roots of $\rho(Z, iE)$. But

$$\varrho(Z, iE) = (Z - iE)(Z + iE)^{-1}(\bar{Z} + iE)(\bar{Z} - iE)^{-1} = W\bar{W}$$

where W is the point the in the Generalised unit circle which corresponds to Z under the usual mapping and we know from (26) that $W\bar{W} <$
98 E. We therefore conclude that $\lambda_\nu^2 < 1, \nu = 1, \ldots r$. From (137) it is clear that λ_ν cannot be arbitrarily close to $\underline{1}$ so that we should have $\lambda_\nu^2 \leq 1 - \delta < 1$ for some $\delta > 0$. Then $W\bar{W} \leq (1 - \delta)E$ and the set of $W's$ consistent with this inequality is clearly compact. Therefore the same is true of the corresponding Z set too and hence also of our symplectic sphere.

We add here one more result for future reference, viz.

Lemma 11. *Given a point $Z_0 \in \mathscr{Y}$ there are at most a finite number of modular substitution which have Z_o as a fixed point.*

Proof. We first observe that the modular group is discrete and hence also countable. Let $M \neq \pm E$ be any modular substitution. Then the equation $M < Z >= Z$ define for a given M a complex analytic manifold of dimension less than $\dfrac{n(n+1)}{2}$. Consider now all analytic manifolds of this kind, viz those formed by the fixed points of a given modular substitution. In view of the above fact, given any point $Z_o \in \mathscr{Y}$ there always exist points Z in any neighbourhood of Z_o with $M < Z > \neq Z$ for any $M \in M_n$, $M \neq \pm E$. If now $M_k < Z_0 >= Z_0$ for an infinity of $\Re' s$, say $\Re = 1, 2 \ldots, and M_\Re \neq \pm M_l \Re \neq \ell$ then choosing Z with $M < Z > \neq Z$ for any M different from $\pm E$ we have $M_\Re^{-1} < Z > \neq Z$ for $\ell \neq \Re$ i.e. $M_\Re < Z > \neq M_\ell < Z >$ for $\neq \Re$. On the other hand, $\mathscr{S}(M_\Re < Z >$

$, Z_0) = \mathscr{S}(M_{\Re} < Z >, M_{\Re} < Z_\nu >) = \mathscr{S}(Z, Z_0)$. Consequently the points $M_{\Re} < Z >$, $\Re = 1, 2$, are all distinct and lie on a compact subset of $\mathscr{Y}$ so that they have at least one limit point contradicting Lemma 8. We therefore conclude that $M_{\Re}$ are identical after a certain stage and the Lemma is proved. □

We are now on the look out for the invariant volume element of the symplectic geometry. We make a few preliminary remarks. Let $R = (r_{\mu\nu})$ denote a variable $n-$ rowed symmetric matrix and $\Omega = (\omega_{\mu\nu})$ a $n \times n$ square matrix. Let $S = (s_{\mu\nu}) = R[\Omega]$ and let $\dfrac{\partial(S)}{\partial(R)}$ denote the functional determinant of the $n(n+1)/2$ independent linear functions $\mathscr{S}_{\mu\nu}(p \leq \nu)$ with respect to the independent variables $r_{\mu\nu(\mu\leq\nu)}$. Then $\dfrac{\partial(S)}{\partial(R)} \neq C$ if and only if the mapping $R \to S$ is $1-1$. But this mapping is $1-1$ when and only when $|\Omega| \neq 0$. For, while the one way implication is trivial, viz. when $|\Omega| \neq 0$ the mapping $R \to S$ is actually invertible as $R = S[\Omega^{-1}]$, to realize the converse, we argue that if $|\Omega| = 0$ there exists a row $XXXXX$ with $W\Omega = 0$. The choice of has $R = \nu\nu$ leads to $R[Q] = \Omega' R \Omega = 0$ without R being zero, verifying that the mapping $R \to S = R[\Omega]$ is not $1-1$. It therefore follows that $\dfrac{\partial(S)}{\partial(R)} = 0$ when and only when $|\Omega| = 0$. In other words, considered as polynomials in the n^2 variables $\Omega_{\mu\nu}$, $\dfrac{\partial(S)}{\partial(R)}$ and $|\Omega|$ have the same zeros. Since $|\Omega|$ is an irreducible polynomial in these n^2 variable, we conclude by means of a well known algebraic result that $\dfrac{\partial(S)}{\partial(R)} = C|\Omega|^{n+1}$ with a constant $C \neq 0$. The index $(n+1)$ in the right side is suggested by a comparison of the degrees of both sides in $\omega_{\mu\nu}$. The special choice $\Omega = E$ yields to the determination $C = 1$. Thus we have 99

$$\frac{\partial(S)}{\partial(R)} = |\Omega|^{n+1} \text{ for } S = R[\Omega]. \tag{138}$$

Let now $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Z\mathscr{Y}$ and consider the symplectic substitution $Z \to Z^* = M < Z >$. One of the relations 122 gives

$$Z_2^* - Z_1^* = (Z_1 C' + D')^{-1}(Z_2 - Z_1)(CZ_2 + D)^{-1}$$

100 and it then follows that

$$dZ^* = (ZC' + D)^{-1} dZ(CZ + D)^{-1})$$

The computation of the functional determinant $\dfrac{\partial(Z^*)}{\partial(Z)}$ requires only the knowledge of the linear relation between dZ and dZ^* given by the last formula. By means of (138) we then get

$$\frac{\partial(Z^*)}{\partial(Z)} = (C + D)^{-n-1} \tag{139}$$

Decomposing $Z, Z^*, \overline{Z}, \overline{Z}^*$ into their real and imaginary parts we get

$$Z = X + iY, Z^* = X^* + iY^*$$
$$\overline{Z} = X + iY, \overline{Z}^* = X^* + iY^*$$

and it is immediate that

$$\frac{\partial(Z, \overline{Z})}{\partial(X, Y)} = \frac{\partial(Z^*, \overline{Z}^*)}{\partial(X^*, Y^*)}$$

From $Z^* = M < Z >$ and $\overline{Z}^* = M < \overline{Z} >$ we then obtain using (139) and (72) that

$$\begin{aligned}\frac{\partial(X^*, Y^*)}{\partial(X, Y)} &= \frac{\partial(X^*, Y^*)}{\partial(Z^*, Z^*)} \quad \frac{\partial(Z^*, \overline{Z}^*)}{\partial(Z, \overline{Z})} \quad \frac{\partial(Z, \overline{Z})}{\partial(X, Y)}\\ &= \frac{\partial(Z^*, \overline{Z})}{\partial(Z, \overline{Z})} = \frac{\partial(Z, \overline{Z}^*)}{\partial(Z)}^*\\ &= \|CZ + v\|^{2r2} = (\frac{Z^*}{Z})\end{aligned}$$

101 It now follows that the volume element

$$dv = |y|^{n-1}[dx][dy] \tag{140}$$

$$\text{with}\begin{cases}[dx] = \pi_{\mu\leq\nu} dr_{\mu\nu}\\ [dy] = \pi_{\mu\leq\nu} dy_{\mu\nu}\end{cases}$$

is invariant relative to symplectic substitutions.

We supplement our above results with the following two lemmas.

Lemma 12. *If Z, Z^* $\in \mathscr{Y}_n$ and $Z_1, Z_1^* \in \mathscr{Y}_r$ denote the matrices which arise from Z, Z^* by deleting their last* $(n-2)$ *rows and columns, where $r \leq n$ then*

$$s(Z_1, Z_1^*) \leq s(Z, Z^*) \tag{141}$$

Proof. Since we know that the geodesic lines are uniquely determined, it clearly suffices to prove the corresponding inequality for ds, viz. $ds_1 \leq ds$ in an obvious notation. In other words we need only show that

$$\sigma(y_1^{-1} dy_1)^2 + \sigma(x_1^2 dx_1)^2 \leq \sigma(y^{-1} dy)^2 + \sigma(y^{-1} dx)^2$$

where $Z = Z^{(n)} = x + iy$ and $Z_1 = Z_1^{(r)} = x_1 + iy_1$. We use the parametric representation we had for positive matrices to write $y = \begin{pmatrix} y_1^* \\ x^* \end{pmatrix}\left[\begin{pmatrix} E & H \\ 0 & E\end{pmatrix}\right]$ and appeal to (133) to infer that $\sigma(y^{-1} dy)^2 \geq \sigma(y^{-1_1} dx_1)^2$ ◻

Since $X = \begin{pmatrix} x_1^* \\ x^* \end{pmatrix}$ we also have, as in the above case, $\sigma(y^{-1} dx)^2 \geq \sigma(y_1^{-1} dx_1)^2$

The desired result is now immediate. **101**

Lemma 13. *If $Z, Z^* \in y_n$ and $Z_1 = X_1 + y_1$, $Z_1^* = X_1^* + LY_1^*$ be as in the previous Lemma, and if $\mathscr{S}(Z, Z^*) \leq \varrho$ then there exist positive constants $M_\nu = M_\nu(\varrho\nu)\nu = 1, 2$ with $\dfrac{1}{M_1} \leq \dfrac{\sigma(Y_1^*)}{\sigma(Y_1)} \leq M_1$, $\dfrac{1}{M_2} \leq \dfrac{|Y_1^*|}{|Y_1|} \leq M_2$.*

Proof. Since the condition $s(Z, Z^*) \leq \varrho$ implies by Lemma 12 that $s(Z_1, Z_1^*) \leq \varrho$ we need only consider the case $r = n$. In view of the interchangeability of Y and Y^* it clearly suffices to prove one part of each of the two sets of inequalities. Let then W be an orthogonal matrix $(WW' = E)$ so determined that $y^*[W] = D = (\delta_{\mu\nu}, d_\nu), d_\nu > c\nu = 1, 2, \ldots n$. Let $\tilde{Z} = (Z - X^*)[WD^{-1}] = \tilde{X} +$ and $\tilde{Z}^* = (Z^* - X^*)[WD^{-1}] =$

$iy^*[WD^{-1}] = iE$. Since s is invariant under symplectic substitutions, we have $s(\tilde{Z}, iE) = s(\tilde{Z}, \tilde{Z}^*) = s(Z, Z^*) \leq \varrho$

In other words $\tilde{Z}$ belongs to a symplectic sphere which we know is a compact subset of $\mathscr{Y}_n$ and consequently

$$\sigma(\tilde{Y}) \leq m_1(n, \varrho), \frac{1}{m_2} \leq (\tilde{y}) \leq m_2(n, \varrho) \tag{142}$$

with suitable constants M_ν $\nu = 1, 2$. Also $|\tilde{Y}| = |y|Wd^{-1}| = |y|D^{-1,2} = |y|Y^*$. Then (142) implies that

$$\frac{1}{M_2} \leq |y||y^*|^{-1} \leq M_2$$

Further, if $y[W] = H = (h_{\mu\nu})$ then

$$\begin{aligned} \sigma(\tilde{Y}) = \sigma(H[D^{-1}]) &= \sum_{\nu=1}^{n} \frac{h_{\nu\nu}}{d_\nu^2} \\ &\geq \frac{\sum_{\nu=1}^{n} h_{\nu\nu}}{\sum_{\nu=1}^{n} d_\nu^2} = \frac{\sigma(H)}{\sigma(Y^*)} = \frac{\sigma(y)}{\sigma(y^*)}. \end{aligned}$$

102 (142) again implies now that $\sigma(\gamma)\sigma(y^*)^{-1} \leq m_1(n, \varrho)$ and as remarked earlier, the interchange of y and y^* leads to the other half of our inequality. The lemma is thus established. □

Chapter 8

Lemmas concerning special integrals

We need to settle some special integrals to be applied subsequently in our consideration of the Poincare′-theta series. First we generalize Euler's well known Γ-integral **103**

$$\int_o^\infty e^{-ty}y^{-s-1} = \Gamma(s)t^{-s}, t > o, Re.s > o$$

to the case of a matrix variable.

Lemma 14. *If $T = T^{(n)} > 0$ and $Re.s > \dfrac{n-1}{2}$, we have*

$$\int\limits_{\gamma=\gamma^{(n)}>o} e^{-\sigma(T\gamma)}|y|^{s-\frac{n+1}{2}}[dy] = \pi^{\frac{n(n-1)}{4}}\prod_{\nu=o}^{n-1}\Gamma(s-\nu_{/2})|T|^{-s} \tag{143}$$

The proof is by induction on n. We first observe that it suffices to consider only the case $T = E$, as in the general case, we can write $T = RR'$ and changing the variable y in the above integral to $y^* = y[R]$

we shall have $\dfrac{\partial(y^*)}{\partial(y)} = |R|^{n+\ell}$ by (138) so that

$$|y^*|^{\frac{n+1}{2}}[dy^*] = |[R]|^{-n+\ell}2[dy] = |y|^{-n+\ell}2|R|^{-n-1}\frac{\partial(y^*)}{\partial(y)}[dy]$$
$$= |y|^{-n+\ell}2[dy]$$

and $|y|^s = |y^*|^s|T|^{-s}$ as $y = y^*[R^{-1}]$.

Then the integral we had, transforms into

$$|T|^{-s}\int\limits_{y^*>o} e^{-\sigma(y^*)}|y^*|^{s-\frac{n+\ell}{2}}[dy^*]$$

and what we claimed is now immediate. We shall assume then that $T = E$. Let

$$Y = \begin{pmatrix} y_1 & \mathscr{Y} \\ \mathscr{Y}' & y_{nn}\end{pmatrix} - \begin{pmatrix} y_1 & o \\ o & y\end{pmatrix}\left[\begin{pmatrix} E & w \\ o & 1\end{pmatrix}\right].$$

104 The above is a special case of our representation (132) where we have set $r = n - 1$. The following relations are immediate.

$$\left.\begin{aligned} y_{nn} &= y + y_1[w] \\ &= y_1[w]\end{aligned}\right\} \tag{144}$$

Clearly then, the functional determinant

$\dfrac{\partial(y)}{\partial(y_1, y, w)} = \dfrac{\partial(y_1, y_{nn})}{\partial(y_1, y, w)}$ is of the type $\begin{vmatrix} E & 0 & 0 \\ * & 1 & * \\ * & 0 & y_1\end{vmatrix}$ so that

$$\frac{\partial(y)}{\partial(y_1, \mathcal{Y}, \mathscr{W})} = |y_1| \tag{145}$$

Also $|y| = |y_1|\mathcal{Y}, \sigma(y) = \sigma(y_1) + \mathcal{Y} + y_1[\mathscr{W}]$, and $y > o$ when and only when $y_1 > o$ and $\mathcal{Y} > o$ all these being immediate consequences of (144). Hence we have, writing $d[\mathscr{W}] = \pi_{\nu=1}^{n-1} d\vartheta_\nu$ there $\mathscr{W}' = (\vartheta_1.\vartheta_2, \ldots \vartheta_{n-1})$,

$$\int\limits_{y>0} e^{-\sigma(y)}|y|^{s-\frac{n+1}{2}}[dy] = \int\limits_{y_1>o}\int\limits_{y>o}\int\limits_{\mathscr{W}} e^{-\sigma(y_1)-y-y_1[\mathscr{W}]}$$

$$|y_1|^{s-\frac{n-1}{2}} \mathscr{Y}^{s-\frac{n+1}{2}} [d\mathscr{W}] d\mathscr{Y} [ny_1]$$

$$= y(s - \frac{n+1}{2}) \int\limits_{y_1 > o} e^{-\sigma(y_1)} |y_1|^{s-\frac{n-1}{2}} \Big(\int\limits_{\mathscr{W}} e^{-y_1[\mathscr{W}]} [d\mathscr{W}] \Big) [dy_1]$$

$$= \pi^{\frac{n-1}{2}} y(s - \frac{n-1}{2}) \int\limits_{y_1 > o} e^{-\sigma(y_1)} |y_1|^{s-\frac{(n-2)+1}{2}} [dy_1]$$

observing that, if we write $y = R'R$ on $\mathscr{Y} = (w_1, w_2, \ldots w_{n-1}) = (R\mathscr{W})$ then the integral within the parenthesis, upto a factor, reduces to 105

$$\int e^{w_1^2 - w_{n-1}^2} dw_1 \ldots dw_{n-1} = \Big(\int\limits_{\sigma}^{\alpha} e^{-w^2} dw \Big)^{n-1} = \Big(\pi^{1/2} \Big)^{n-l}.$$

By the induction assumption, the last integral, viz.

$$\int\limits_{y_1 > o} e^{-\sigma(y_1)} |dy_1| = \pi^{\frac{n-1}{2}, \frac{n-2}{2}} \prod_{\nu=0}^{n-2} y(\mathcal{S} 0 \nu/2).$$

Substituting this in the above we get

$$\int\limits_{y > 0} e^{-\sigma(y)} |y|^{\mathcal{S} - \frac{n+1}{2}} [dy] = \Pi^{\frac{n}{2} \cdot \frac{n-1}{2}} \prod_{\nu=0}^{n-1} y(\mathcal{S} - \nu/2)$$

and the proof is complete.

With $\mathscr{R}_n$ denoting space of all reduced matrices $y = y^n > 0$ we state

Lemma 15. *Let $f(y)$ be a real and continuous functions on the ray $y \geq o$ and let $s + \dfrac{n-2}{2} > 0$. Then*

$$\int\limits_{\substack{y \in \mathfrak{R}_n \\ |y| \geq t}} f(|y|) |y|^s [dy] = \frac{n+1}{2} \vartheta_n, \int\limits_{0}^{1} f(y) y^{s + \frac{n-1}{z}} dy \tag{146}$$

where ϑ_n is a certain positive constant, depending only on n.

Proof. We first prove the lemma in the special case when $f(y) \equiv 1$ and $s = o$. Then (146) reduces to

$$\int\limits_{\substack{y\in\Re_n\\ ||y|\leq t|}} [dy] = \vartheta_n t \frac{n+1}{2} \tag{146$'$}$$

If only we know that the integral on the left of ((146)′), exists then in view of its homogenity in the variables y, ((146)′) certainly holds with $\vartheta_n = \int\limits_{\substack{y\in\Re_n\\ |y|\leq L}} [dy]$. □

106 We shall now show that the integral

$$y_n(t) = \int\limits_{\substack{y\in\Re\\ |y|\leq\in}} [dy] \text{ exists.}$$

Towards this end, we approximate $y_n(t)$ by the following integral $y_n(\delta, t) = \int\limits_{\substack{y\in\Re_n\\ y_l l\geq\delta}} |y \leq\in [dy]$ where δ denotes a small positive number. The above requirements in y will imply, in view of the reduction conditions that all $y'_{\nu\nu}s$ are bounded as

$$\delta \leq y_{i1} \leq \ldots \leq y_{nn}.\Pi_{\nu=1}^{n} y_{\nu\nu} \leq c, |\lambda| \leq C_i t$$

and as a consequence, all the $y'_{Nn}s$ are bounded. In other words the domain of integration for $J_n(s, \in)$ is finite (i.e. $J_n(\delta, \in)$) is a proper integral) so that $J_n(\delta, \in)$ certainly exists. We need then only show that $J_n(\delta, \in)$ has a limit as $\int \to o$ and then this limit is obviously $J_n(\delta, \in)$. Specifically we show that if $0 < \delta < \delta'$, then the difference $J_n(\delta, \in), J_n(\delta, \in)$ (which is positive) can be made arbitrarily small by taking δ sufficiently small. This difference is representable as the integral.

$$\int\limits_{y\in\Re_n, |y|\leq\in, \delta\leq y_{11}\leq\delta} [dy] \tag{147}$$

If we write $\lambda = \begin{pmatrix} y_{11} & x \\ x & y_i \end{pmatrix}, y_i = y_i^{(n-1)}$, then the domain of integration on (147) is contained in the following domain

$$'\delta \leq y_{ii} \leq \delta$$
$$|y_{2\nu}| \leq \frac{1}{2} y_{i1} \nu = 2, 3, \ldots, , .$$
$$|y_i > 0|$$

as the inequalities $y_{i1}|y_i| \leq y_{i1}y_{22} \cdots \leq c_i|\lambda| \leq t$ which are consequences of (81) and (49), show. Denoting the last domain as $\mathscr{L}$, we have, in view of the induction hypothesis , **107**

$$\begin{aligned}
\int_{\mathscr{L}} [dy] &= \int_{\delta \leq y_{11} \leq \delta} \int_{\substack{|y_i \nu \leq ^1 \backslash 2 |y_{11} \\ \nu = 2.3, n}} \int_{\substack{y_1 \in \Re_{n-1} \\ |r_i| \leq C, t \backslash y11}} [dy,] dy, z..dy_i ndy_i i \\
&= \vartheta_{n-1} \int_{y < y_{11} \delta} \int_{|y_1 1| \leq ^1 /_2 y_{11}|} \left(\frac{C_1 t}{y_1 1}^{n/2} \right) dy_{12} \ldots dy_{1n} dy_{ii} \\
&= \vartheta_{n-1} \int_{\delta \leq y_{11} \leq \delta} \left(\frac{C_1 t}{y_1 1} \right)^{n/2} dy_{12} \ldots, dy_{1n} dy_{11} \\
&= (c_i)^{n \backslash 2} \vartheta_{n-1} \int_{\delta' \leq y_{11} \leq \delta} y_{11}^{n/2-1} dy_{11} <\in'
\end{aligned}$$

if δ is sufficiently small, ((146)′) now follows.

It is also clear that the existence of $\int\limits_{\substack{y \in \Re n \\ |y| \leq t}} [dy]$ which we proved above already implies the existence of the integral on the left side of (146), taking into consideration the continuity on $f(y)$ in $y \geq 0$, so that it remains only to establish the equality of the two side of (146). Let us introduce the following truncated integral

$$I(a, t) = \int_{\substack{y \in \Re_n \\ a \leq |y| \leq t}} f(|y|)|y|^s [dy]$$

and observe that

$$I(a,t+h)-I(a,t)=I(t,t+h)=\int\limits_{\substack{y\in\Re_n\\ t\leq|y|\leq t+\ell}} f(|y|)|y|^A[dy]$$

108 By a mean theorem, in view of the continuity of the function $f(y).y^2$ in the closed interval $[t,t,\Re]$, the last integral is equal to

$$f(t+\vartheta h)(t+\vartheta+h)^s \int\limits_{y\in\Re n, t\leq|y|\leq t,h} [dy]$$

with a certain ϑ in the interval [0,1]

Thus we have from ((146)′),

$$\begin{aligned} I(a,t+h)-I(a,t) &= f(t+\vartheta|)(t+\vartheta)^s\Big\{J(t+h)-7(t)\Big\}\vartheta \\ &= \mathfrak{f}(t+\vartheta h)(t-\vartheta h)^s\{(t-h)^{\frac{n+1}{2}}-t\frac{n+1}{2}\}\vartheta_n \end{aligned}$$

and therefore

$$\begin{aligned} \lim_{h\to 0}\frac{I(a,t+R)-I(a,t)}{h} &= f(t)t^s\vartheta_n \lim_h \to o\frac{(t+h\frac{n+1}{2})t\frac{n+1}{2}}{h} \\ &= \frac{n+1}{2}\vartheta_n f(t)t^s+\frac{n-1}{2} \end{aligned}$$

The limit in the left side is clearly equal to $\dfrac{\partial(t(0.t))}{\partial t}$. It now follows that

$$I(a,t)=\int\limits_a^t \frac{\partial I(a,t)}{\partial t}dt=\frac{n+1}{2}\vartheta_n\int_o^t f(t)t^s+\frac{n+1}{2}dt$$

and since $s+\dfrac{n+1}{2}>0$ by assumption, letting $a\to 0$, and since $s+\dfrac{n+1}{2}>0$ by assumption, letting $a\to 0$, what results is precisely what we wanted. We single out important special cases of (146), viz.

1) Choosing $f(t) \equiv 1$ in the above, (146) yields that

$$\int\limits_{\substack{y\in\Re_n \\ |y|\leq t}} |y|^s[dy] = \frac{n+1}{2}\vartheta_n \int_o^t y^{s+\frac{n-1}{2}} xy = \frac{n+1}{2s+n+1}$$

$$\vartheta_n \tau_n t^{s+\frac{n+1}{s}} \text{ provided } s + \frac{n+1}{2} > 0. \qquad (148)$$

2) Setting $f(y) = e^{-ay^d}(a, 0, \partial > 0)$ in the above, and letting $t \to \infty$ in (146), that the limit exists is seen by considering the right side and **109**
we get

$$\int\limits_{y\in\Re n} e^{-a|y|^d}|y|^s[dy] = \frac{2+1}{2} 2_n \int\limits_0^y e^{-ay^\partial} y^{s+\frac{n-1}{2}} dy$$

$$= \frac{n+1}{2d}\vartheta_n|(\frac{s}{2} + \frac{n+1}{2d})d^{-s\partial}-\frac{n+1}{2d} \qquad (149)$$

provided $a > 0, \partial > 0$ and $s + \dfrac{n+1}{2} > 0C$.

It is remarkable that all integrals which we use and need for a consideration of the Poincare' series are easily computed as seen from the above.

Chapter 9

The Poincare' Series

This section is devoted to the construction of modular forms in the shap of the Poincare' series. Let $T \geq 0$ be a semigroup of modular matrices of the form 110

$$\begin{pmatrix} u & S_o u'^{-1} \\ o & u'^{-1} \end{pmatrix} \tag{150}$$

where u is a *unit* of T meaning $T[u] = T$, and S_o is symmetric and integral. Let $V(T)$ denote a set of modular matrices which constitutes a complete system of representatives of the left cosets of M modulo $\mathcal{A}(T)$. Then

$$M = \sum_{S \in V(T)} A(T)S \tag{151}$$

With k standing for an *even integer*, we introduce the Poincare' series as

$$g(Z, T) = \sum_{S \in V(T)} e^{2\pi\sigma(TS<z>)} |cz + D|^{-\mathfrak{K}} \tag{152}$$

where we assume $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

First we note that the Poincare' series does not depend on the choice of $V(T)$. For, this only requires that (152) is left invariant if we replace the matrices S occurring in (152) by MS where

$M = \begin{pmatrix} u & S_o u'^{-1} \\ 0 & u'^{-1} \end{pmatrix}$ is a matrix of the type occurring in (150). Due to 111 this replacement $\sigma(TS<Z>)$ will go over into $\sigma(TMS<Z>)$ and

$$\begin{aligned}\sigma(TMS<Z>) &= \sigma\{T(S<Z>[u']+S_o)\}\\ &= T(S<Z>[u']+S_o)\\ &= \sigma(T[u]S<Z>)+\sigma(TS_o)\\ &= \sigma(TS<Z>)+\sigma(TS_o)\end{aligned}$$

since u is a unit of T. Also

$$\sigma(TS_o) = \sum_{\mu,\nu} t_{\mu\nu}s^o_{\nu\mu} = \sum t_{\mu\mu}s^o_{\mu\mu} - \sum_{\mu<\nu} t_{\mu\nu}2s^o_{\mu\nu}$$

so that $\sigma(TS_o)$ is an integer as $S_o = (s^o_{\mu\nu})$ is semintegral. It therefore follows that the exponential factors occurring in (152) are left unaltered by such a replacement. The same is trivially true of the other factor $|cZ+D|^{\mathfrak{K}}$ as it goes over into $|u^{-1}cZ+u^{-1}D| = |u'|\mathfrak{K}|cZ+D|^{\mathfrak{K}} = |cZ+D|^{\mathfrak{K}}$, u being unimodular and k being an even integer. It is immediate that $g(Z,T)$ is independent of the choice of $V(T)$. We now contend that

$$g(Z,T)|M = g(Z,T)| \tag{153}$$

for any $M = \begin{pmatrix} A_o & B_o \\ C_o & D_o \end{pmatrix} \in M$ where

$$g(Z,T)|M = g(M<Z>,T)|C_0Z+D_0|^{-\mathfrak{K}}$$

For, $g(Z,T)|M = \sum_{S\in V(T)} e^{2\pi i(\sigma(TS)<Z>)}|C_0Z+D_0|^{-\mathfrak{K}}$

$$\begin{aligned}&= \sum_{S\in V(T)M} e^{2\pi(\sigma(TS)<Z>)}|cZ+D|^{-\mathfrak{K}}\\ &= g(Z,T)\end{aligned}$$

112 since $V(T)M$ represents a set of representatives of the left cosets of $A(T)$ in M in view of $V(T)$ doing so. What is more, $g(Z,T)$ depends only on the equivalence class of T. In other words a replacement of T by $T[y]$ where V is unimodular but arbitrary, does not affect $g(Z,T)$. We may remark are this stage that all our considerations are formal for the

present. V being any unimodular matrix, the group $A(T[V])$ consists of all modular matrices

$$\begin{pmatrix} u_1 & S_1 u_1'^{-1} \\ 0 & u_1^{-1} \end{pmatrix} \text{ with } (T[v])[u_1] = T[V]$$

and $S_1 = S_1'$ (integral). This means that if we introduce

$$u = vu_1v^{-1}, S = vS_1v' \text{ and } R = \begin{pmatrix} v & C \\ 0 & v^{-1} \end{pmatrix}$$

then $R \in M$ and, with $T[u] = T$,

$$R\begin{pmatrix} u_1 & S-1u_1^{-1\prime} \\ 0 & u_1^{-1} \end{pmatrix}R^{-1} = \begin{pmatrix} u & Su'^{-1} \\ 0 & u^{-1\prime} \end{pmatrix}.$$

An argument in the reverse direction is also valid and thus we obtain

$$RA(T[v])R^{-1} = A(T) \tag{154}$$

From (151) we have

$$M = \sum_{S \in V(T)} A(T)S = \sum_{S \in V(T[v])} A(T[v]S)$$

so that using (154) **113**

$$\begin{aligned} M = RMR^{-1} &= \sum_{S \in V(T[v])} RA(T[v])R^{-1}RSR^{-1} \\ &= \sum_{S \in RV(T[v])R^{-1}A(T)S} . \end{aligned}$$

Thus $RV(T[v])R^{-1}$ is a set of the kind $V(T)$ and multiplication on the right by a modular substitution will again lead to a set of the same kind so that $RV(R[v])$ is itself a set of the type $V(T)$. Now

$$g(Z, T[V]) = \sum_{S \in V(T[v])} e^{2\pi\sigma(T[v]S<Z>)}|cZ + D|^{\mathfrak{K}}.$$

Since

$$\sigma(T[v]S<Z>) = \sigma(TS<Z>[v]) = \sigma(TRS<Z>)$$

we conclude by arguments as in earlier contexts that

$$g(Z, T[v]) = g(Z, T) \tag{155}$$

where V is an arbitrary unimodular matrix.

We are heading towards the proof of the absolute and uniform convergence of the Poincare' series (152) in every compact subset of $\mathscr{Y}$ provided

$$\Re > \min(2n, n+1+\text{ rank } T) \tag{156}$$

and then for such $R's, g(Z,T)$ is a regular function in $\mathscr{Y}$. The absolute convergence will in particular justify all our earlier considerations which were hitherto formal and then by means of (153), we would have proved that $g(Z,T)$ represents a modular form of degree $n > 1$ and weight k.

114 The case $n=1$ requires some further consideration.

In view of (155) and (106), it suffices to consider the convergence of the Poincare' series in the special case when

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, T_1 = T_1^{(r)} > 0. \tag{157}$$

If u is a unit of T, we decompose u analogous to T as $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and use the relation $T[u] = T$ to infer that $u_2 = 0$ and $T_1[u_1] = T_1$. Let us denote by $A_r(r \geq 0)$ the group of modular matrices

$$M = \begin{pmatrix} u & S_0 u'^{-1} \\ 0 & u'^{-1} \end{pmatrix} \tag{158}$$

with $u = \begin{pmatrix} E^{(r)} & o \\ u_3 & u_4 \end{pmatrix}$ u- unimodular and S_o symmetric integral. If T be as in (157) which it is throughout our subsequent discussion, then A_r is actually a sub-group of $A(T)$ of finite index $(A(T) : A_r)$. In fact, if $\varepsilon(T_1)$

denotes the number of units of $T_1 = T_1^{(r)}$ for $r > 0$ and denotes unity if $r = 0$, then

$$(A(T) : A_r) = \varepsilon(T_1).$$

Let V_r be a set of representatives of the left cosets of A_r in M so that

$$M = \sum_{S \in V_r} A_r S, r \geq 0. \tag{159}$$

It is clear that to each $S \in V(T)$ there corresponds $\varepsilon(T_1)$ elements of V_r say $S_1, S_2, \ldots S_{\varepsilon(T_1)}$ all of which belong to the same left coset of **115**
$A(T)$ in M and consequently,

$$g(Z,T) = \frac{1}{\varepsilon(T_1)} \sum_{S \varepsilon V_r} e^{2\pi\sigma(TS<Z>)} |cZ + D|^{-\mathfrak{K}}, S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

By (159), $M = \sum_{S \in V_o} A_0 S$ so that if we assume

$A_o = \sum_R A_r R$, then $M = \sum_{RS} A_r RS$. Comparing this with (159) we conclude that V_r is obtained as all possible products RS where R runs through a set of representatives of left cosets of A_r in A_o and S run through all the elements of V_o. In view of an earlier result (pp. 11) we can take for V_o the set of all matrices whose second matrix rows constitute a class of coprime symmetric mutually non-associated pairs. We now determine a representative system for R. If $R_\nu = \begin{pmatrix} u_\nu & * \\ 0 & * \end{pmatrix}$, $\nu = 1, 2$ are two matrices which belong to the same coset of A in A_o then there exists a matrix $M = \begin{pmatrix} u & S_o u^{-1} \\ ou^{-1} & \end{pmatrix}$ of the type occurring in (158) with $MR_1 = R_2$. This is easily seen to mean that $uu_1 = u_2$ and consequently $u_1' u' = u_2'$. Since $u = \begin{pmatrix} E^{(r)} & u_3 \\ o & u_4 \end{pmatrix}$ the above gives $u_1{}' \begin{pmatrix} E^{(r)} & u_3 \\ o & u_4 \end{pmatrix} = u_2{}'$. If $u_\nu' = (p^{(r,r)} w)$, $\nu = 1.2$, such an equation can hold only when $p_1 = p_2$. It is now easy to infer that if $P = P^{(r,r)}$ runs through all primitive matrices and to each P we make correspond a unique matrix u^* obtained by completing P to a unimodular matrix in an arbitrary way, then the class **116**
of all matrices

$$R = \begin{pmatrix} u^* & o \\ o & u^{*\prime -1} \end{pmatrix} \tag{161}$$

provides a compete representative system of the left cosets of A_r in A_o. We now have

$$g(Z,T) = \frac{1}{\varepsilon(T_1)} \sum_R \sum_{**V_o} e^{2\pi i \sigma(TRS<Z>)} |CZ+D|^{-\Re}$$

where (C, D) which should strictly denote the second matrix row of the product RS can be assumed to be the second matrix row of S in view of the special form (161) of R. Since to each primitive $P = P^{(n,r)}$ we have corresponded a unique R, the summation in the last series can be regarded as one over the set of $P's$ instead of the $R's$. In fact the general term of the last series depends only on P and not on the rest of the columns of R as the equations

$$\begin{aligned}\sigma(TRS<Z>) &= \sigma\{T(S<Z>)[u'^*]\}\\ &= \sigma\{T[u^*]S<Z>\}\\ &= \sigma(T_1[P']S<Z>)\end{aligned}$$

show. We have thus shown that

$$g(Z,T) = \frac{1}{\in(T_1)} \sum_P \sum_{S\in V_o} e^{2\pi i \sigma(T_1[P']S<Z>)} |CZ+S|^{-\Re} \tag{162}$$

where $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

We proceed to construct a suitable fundamental domain for A_r in $\mathscr{Y}$. We use the parametric representation (132) for $y > o$ as

$$y = \begin{pmatrix} y_1 & o \\ o & y_2 \end{pmatrix}\left[\begin{pmatrix} E & y \\ o & E \end{pmatrix}\right] \tag{132$'$}$$

117 where $y_1 = y_1^{(r)}, y_2 = y_2^{(n-r)}$ and $v = V^{(r,n-n)}$.

The substitution (158) takes $Z = X - iy \in \mathscr{Y}$ into

$$\begin{aligned}Z[u'] + S_o &= X[u'] + S_o + i\begin{pmatrix} y_1 & o \\ o & y_2 \end{pmatrix}\left[\begin{pmatrix} E & o \\ o & y_2 \end{pmatrix}\begin{pmatrix} E & u_3' \\ o & u_4' \end{pmatrix}\right]\\ &= X[u'] + S_o + i\begin{pmatrix} y_1 & o \\ o & y_2 \end{pmatrix}\left[\begin{pmatrix} E & u_3' + vu_4' \\ o & u_4' \end{pmatrix}\right]\end{aligned}$$

$$= X[u'] + S_o + i\begin{pmatrix} y_1 & o \\ o & y_2[u_1'] \end{pmatrix}\left[\begin{pmatrix} E & u_3' + vu_4 \\ o & u_4' \end{pmatrix}\right]$$

Comparing this with

$$Z = X + iy = \times + i\begin{pmatrix} y_1 & o \\ o & y_2 \end{pmatrix}\left[\begin{pmatrix} E & V \\ o & E \end{pmatrix}\right]$$

we deduce that a transformation of the type (158) effects the following changes.

$$\left.\begin{array}{ll} \times \to \times[u'] + S_o & y_1 \to y_1 \\ Y_2 \to y_2[u_4']. & V \to u_3 + Vu_4' \end{array}\right\}. \tag{163}$$

We need a few notations. We shall denote by

1) $\Re_{\Re, n-n}$ the set of all matrices $V = V^{(\Re, n-n)} = (\vartheta_m v)$ satisfying the conditions: $-\dfrac{1}{2} \leq \vartheta_{\mu\nu} \leq +\dfrac{1}{2}$ for all μ, ν,

2) $\Re_n$ the set of all matrices $\times = x^{(n)}$ with

$$\times = \times', -\frac{1}{2} \leq \times_{\mu\nu} \leq +\frac{1}{2}, \mu, \nu = 1, 2, \ldots r$$

3) $\Re_{n'}$ the set of all reduced positive matrices $y - y^{(n)}$ (in the sense of Minkowski)

By an appropriate choice of u_4 in (158), we can obtain $y_2[u_4']$ in (163) as a reduced matrix, in other words $Y_2[u_4'] \in \Re_{n-n}$ and in general, u_Λ is uniquely determined. We can then choose an integral matrix u_3 such that in (163), $u_3 vu_4' \in \Re_{\Re, n-n}$ and u_3 is also uniquely determined in general. The choice of u_3 and u_4 fixes u and then we can choose S_o such that $[u'] + S_o \in \Re_n$. In general S_o is also uniquely determined. These considerations prompt us to define a fundamental domain y_u for A_2 on $\Re$ as the set of Z. **118**

$Z = \times + iy = \times + i\begin{pmatrix} y_1 & o \\ o & y_2 \end{pmatrix}\left[\begin{pmatrix} E & V \\ o & E \end{pmatrix}\right]$ where

$$\times \in yW_n, y_2 \in \Re_{n-n}, V \in y_{n-n}. \tag{164}$$

For purposes of proving the convergence of the Poincare' series, we in fact consider a majorant of it, viZ. the series

$$h(Z,T) = \frac{1}{\varepsilon(T_Z)} \sum_{S\in V_n} e^{-2\pi\sigma(Ty_S)} |y_2|^{h/2} \tag{165}$$

which arises from (**??**) by replacing each term of (**??**) by its absolute value multiplied by a factor $|y|^{\Re} = |y|^{\Re/2} ||CZ+D||^{\Re}$ where we denote $S < Z >= \times_S + y_2$. The absolute or uniform convergence of the above series implies the corresponding fact for the Poincare' series. In the sequel therefore, we shall concern ourselves with this new series.

We first prove the uniform convergence of $h(Z,T)$ on special compact subsets of $\mathscr{Y}$, viz. the symplectic spheres and the general case will
119 immediately follow. Let $Z_0 \in \mathscr{Y}$ be fixed and let $\Re_o$ denote the symplectic sphere $s(Z,Z) \leq \frac{1}{2}\rho$ Assume ρ to be so chosen that $\Re_o$ does not have a non empty intersection with any of the $S(\Re_o)\mathscr{Y}$ without being identical with it, S denoting an arbitrary modular substitution. The validity of this assumption is an immediate consequence of the modular group being discontinuous on $\mathscr{Y}$ (cf. Lemmas 8). Let Z, Z^* be arbitrary points of $\Re_\nu$. Then we have $s(Z,Z^*) \leq \rho$ and $s(s < Z >), S < Z^* > \leq S, S \in M$. With a view to obtaining a convergent majorant for $h(Z^*T)$ which does not depend on $Z^* \in \Re_o$ we work out the following estimations.

If $Z = \times + iy$ and $y = \begin{pmatrix} r_n r \\ *y \end{pmatrix}$ with $y_1 = y_1^{(r)}$ then in view of the special form (157) for T we have

$$\sigma(Ty) = \sigma(T_1 y_1).$$

Let $R = R^{(n)}$ be a real matrix such that $y = RR'$ and let $w_\nu, u = 1, 2, \ldots r$ denote the columns of R. $\mathscr{E}$ denoting any real column, let $\mu_1 = \min_{\nu=1} r_1[\in]$ and $\mu_2 = \mu_{\varepsilon=1} T_1[\varepsilon]$, Then we have $\sigma(T_1 Y_2) = \sigma(T_1(\varepsilon)) = \sum_{\nu=1}^{\nu} \sigma(\Gamma_1[W_\nu])$. Also

$$\mu_1 \sigma(y_1) = \mu_1 \sum_{\nu=1}^{r} 4'_\nu W_\nu \leq \sum_{\nu=1}^{r} T_1[W_\nu]$$

$$\leq \Gamma_2 \sum_{\nu=1}^{r} W'_\nu W_\nu = \mu_2 \sigma y_1$$

so that

$$\Gamma_1 \sigma(y_1) \leq (Ty) \leq \mu_\nu \sigma(y_\lambda)$$

Replacing Z by $Z^\nu = x^r + 0y^\nu, y^\nu = \left(z_\nu^{\nu(\mu)} y_r\right)$.
we have from the above

$$\mu_1 \sigma(y_1^\nu) \leq \sigma(Tj^2) \leq \mu_2 \sigma : y_1^\nu$$

Since $\mathscr{S}(Z, Z^* = \varrho)$, by means of Lemma 13, we have **120**

$$\frac{1}{m_1} \leq \frac{\sigma(y_1^*)}{\sigma(y_1)} \leq m_1$$

Combining all these it results that

$$\sigma(Ty^*) \geq \mu_1 \sigma(y_1^*) \geq \frac{\mu_1}{m_1} \sigma(y_1) \geq m\sigma(Ty', m\rangle 0$$

The interchange of y and y^* leads to an inequality in the opposite direction and we have

$$m \leq \frac{\sigma(Ty^*)}{\sigma(Ty)} \leq \frac{1}{m}$$

The assumption on Z, Z^* used to prove the above inequalities are also true of $S\langle Z\rangle$, $S\langle Z^*\rangle$, $S \in \mathscr{M}$ so that, writing

$$S\langle Z\rangle = x_s + tY_s, S\langle Z^*\rangle = \lambda_s^* + iy_s^*$$

we also have

$$m \leq \frac{\sigma(Ty_s^*)}{\sigma(Ty_s)} \leq \frac{1}{m}$$

From Lemma 13 we have an estimate for $|y_s^*|/|y_s|$ viz.

$$\frac{1}{m_Z} \leq \frac{|y_s^*|}{|y_s|} \leq m_s$$

and consequently $|y_s^*|^{k/Z} \leq \mathscr{M} |x_s|^{k/Z}$ with $\mathscr{M} = m_s^{k/Z}$.

The general term of (165) allows then the following estimation for $Z \in \mathsf{E}_o$, viz

$$e^{-2\pi\sigma(Ty_s^*)}|y_s^*|^{k/2} \leq \mathscr{M} e^{-2\pi m\sigma(Ty_s,)}|y_s|^{k/2}$$

On integration, the above yields

$$e^{-2\pi\sigma(Ty_s^*)}|y_s^*|^{k/2} \leq \frac{\mathscr{M}}{\urcorner_o} \int\limits_{y\in R_o} e^{-2\pi m\sigma(Ty_s,)}|y_s|^{k/2}|y_s|^{n-1} dxTo$$

where $\urcorner_o$ is the volume of E_o and is given by

$$\urcorner_o = \int\limits_{\urcorner_o} |y|^{-n-1}[dx][dy] - \int\limits_{Z\in\mathsf{E}_o} |y_Z|^n [dx][dy]$$

121 Hence it follows that

$$h(Z^*, T) \leq \frac{\mathscr{M}}{\urcorner_o}\frac{1}{\varepsilon(T_1)} \sum_{\varepsilon\in V_r} \int\limits_{Z\in E_2} \in^{-2\pi m\sigma(Ty_s)} |y_s|^{k/Z-n-1}[dx_s][dy_s]$$
$$= \frac{\mathscr{M}}{\urcorner_o}\frac{1}{\varepsilon(T_1)} \sum_{S\in V_r} \int\limits_{Z\in\in\langle E_2\rangle} \in^{-2\pi m\sigma(Ty)} |y|^{k/Z-n-1}[dx][dy]_{01\in s} \tag{166}$$

We now break the integral

$$\int\limits_{s\langle E,o\rangle} e^{-2\pi m\sigma(Ty)}|y|^{k/Z-n-1}[dx][dy]$$

into a sum of integrals the domain for each of which can be brought into a subset of the fundamental domain $'o_n'$ of $\mathscr{A}_n$ by means of an appropriate substitution in $\mathscr{A}_n$. Noting that the integrated above is invariant under the substitutions of $\mathscr{A}_n$ we would have then expressed the above integral as a sum of integrals, all taken over appropriate subsets of y_n^o. This is done as follows. Since y_n^o is a fundamental domain for $\mathscr{A}_n$ in ξ we have $\xi = \bigcup\limits_{\mathscr{M}\in\mathscr{A}_n} \mathscr{M}\langle y_n^O\rangle$. Hence

$$S\langle\Re_o\rangle = \varrho\langle\Re_o\rangle \cap \mathscr{Y} \bigcup_{\mathscr{M}\in\mathscr{A}_n} \varrho\langle\Re_o\rangle \cap M\langle y_n^o\rangle$$

Let $s\langle\Re_o\rangle \cap M_s^{\sim}\langle y_n^o\rangle$ be non empty for $\nu = 1, 2, \ldots, a_s$, the suffix S in M_s^{∞}, a_s denoting that these depend on S. (It is not difficult to show that μ_s is finite in each case but we do not need this). Let further

$$\mathscr{L}_s^{(\nu)} = S\langle\tilde{\Re}\rangle \cap M_{\in}^{(\nu)}\langle y_n^o\rangle, \nu = 1, R, \ldots, R_s.$$

Then $S\langle\mathscr{R}\rangle = \cup_{\nu=1}^{a}\mathscr{L}_S^{(\nu)}$. If $\mathscr{L}_s^{(\nu)} = M^{(\nu)^{-I}}\langle\mathscr{L}_y^{(\nu)}\rangle$, then $\mathscr{L}_s^{(\nu)} \subset \mathscr{L}$ for each λ and we have clearly

$$\int\limits_{S\langle\Re_o\rangle} e^{-2\pi m\sigma(Ty)}|y|^{\frac{\Re}{2}-n-1}[dx][dy]$$

$$= \sum_{\nu=I}^{a_s}\int\limits_{\mathscr{L}_{\mathscr{S}^{(\nu)}}} e^{-2\pi m\sigma(Ty)}|y|^{\frac{\Re}{2}-n-1}[dx][dy]$$

and consequently, from (166) **122**

$$h(Z^*, T) \leq \frac{\mu}{\mathcal{J}_o\varepsilon(T_1)}\sum_{S\varepsilon V_r}\sum_{\nu=1}^{a_s}\int\limits_{\mathscr{L}_{\mathscr{S}}^{(\nu)}} e^{-2\pi m\sigma(Ty)}|y|^{\frac{\Re}{2}-n-1}[dx][dy] \quad (167)$$

We certainly know that the union $\bigcup\limits_{\substack{S\in V_r\\ \nu=1,2\ldots a_s}} \mathscr{L}_{\mathscr{S}}^{(\nu)}$ is contained in y_r^o. But something more is true, viz. this union is actually contained in the set of $Z = \times + iy$ defined by:

$$Z \in y_r, |y| \leq t$$

if t is chosen sufficiently large. For $|y|$ is invariant under the substitutions in $\mathscr{A}_r$ and then we need only prove that $|y| \leq t$ for

$$Z = \times + \tau y \in \cup_{\nu=1}^{a_s}\mathscr{L}_{\mathscr{S}}^{(\nu)} = S.$$

Consider now the set of points $\{S < Z_0 > /S \in V_r\}$. Being a set of equivalent points it has a highest point in it, say $\mathscr{S}_i < Z_0 >$. Then if $\mathscr{S}_i < Z_0 >= \times_{os} + iy_o s$, we have $||y_{os_i}|| \geq ||y_{os}||$ for every$S \in v_r$ Let Z

be an arbitrary point is $S < \Re_0 >$. Then $(Z, s < Z_0 >) \leq \frac{1}{2}\mathcal{P}$ so that by Lemma 13

$$\frac{1}{m_2} \leq \frac{|y|}{|y_{os}|} \leq m_2, m_2 = n2(\rho, n).$$

Thus $|y| \leq m_2|y_{os}| \leq m_2|y_{os_i}| = t$ (say). Then with this t (which
123 is a fixed number) we have that $\bigcup\limits_{\substack{s\in V_r\\ u=1,2\ldots 0_S}}$ is contained in the set of $Z's$ defined by: $Z \in r_r$, $|y| \leq t$ If now $\in_0$ denotes the number of modular matrices s having Z_0 as a fixed point, viz. $s < Z_0 >= Z_0$, we know from Lemma 11 that $\in_0$ is finite and we want to show that every point of y_r appears at most e_0^2 times in the union $\bigcup\limits_{\substack{s\in V_r\\ u=1,2,}} \mathcal{L}^{\nu}_{s_{a_s}}$ In other words, we have got to show that if $\exists$ denotes the set of all pairs (ν, s) with $Z \in \mathcal{L}^{(\nu)}_s$ where $>$, is a fied point in $\mathfrak{f}_r$, then this set has at most e_0^2 elements. If $(\nu, s), (\nu_0, s_0)$ be two pairs in $\mathfrak{f}$, then $Z \in M_s^{\nu^{-1}}\langle\mathcal{L}^{(\nu)}_s\rangle \subset m_s^{(\nu)^{-1}} s\langle\Re_0\rangle$ and similarly $Z \in M_{s_0}^{(\nu)_0^{-1}}\langle\mathcal{L}^{(\nu_0)}_{s_0}\rangle \subset M_{s_0}^{(\nu)^{-1}} s_0\langle\Re_0\rangle$. This means that the two spheres $M_s^{(\nu)^{-1}} S\langle\Re_0\rangle, M_{s_0}^{(\nu_0)^{-1}}\langle\Re_0\rangle$ have one point in common and hence the spheres themselves are identical. In particular, their centres are equal so that

$$M_{s_0}^{(\nu_0)^{-1}} S_0\langle Z_0\rangle = M_S^{(\nu)^{-1}} S\langle Z_0\rangle.\text{i.e.} Z_0 = S_0^{-1} M_{s_0}^{(\nu_0)} M_s^{(\nu)^{-1}} S\langle Z_0\rangle.$$

Keeping (ν_0, S_0) fixed, we have now produced for each pair $(\nu, S) \in \mathfrak{f}$ a substitution $S_0^{-1} M_{s_0}^{(\nu_0)} M_s^{(}\nu)_s^{-1} S$ which has Z_0 as a fixed point and
124 which consequently belongs to a finite set of e_0 elements. It easily follows then that the possible choices for (ν, S) are at most e_0^2 in number. Turning to our series $h(Z^*, T)$ and its majorant in (167), we can now state that

$$h(Z^m, T) \leq \frac{M}{\mathcal{J}_0 \in (T_1)} \sum_{s\in v_r} \sum_{\nu=1}^{a_s} \int\limits_{\substack{Z\in\mathcal{L}^{(\nu)}_s\\ |y|\leq t}} e^{-2\pi m\sigma(Ty)}|y|^{\frac{R}{2}} - n - 1[dx][dy]$$

$$\leq \frac{Me_0^2}{\mathcal{J}_0 \in (T_1)} \int\limits_{\substack{Z\in y_r\\ |y|\leq t}} e^{-2\pi m\sigma(Ty)}|y|^{\frac{R}{2}} - n - 1[dx][dy]$$

$$= \frac{Me_0^2}{\mathcal{J}_0 \in (T_1)} F(mT_1, t)(\text{say})$$

It only remains therefore to investigate under what conditions the last integral $F(mT_1, T)$ exists. First we treat the case $0 < r < n$. In this case we have the parametric representation for $y > 0$ from (132) as

$$y = \begin{pmatrix} y_1 & o \\ o & y_2 \end{pmatrix} \left[\begin{pmatrix} E & v \\ o & E \end{pmatrix} \right]$$

where $y_1 = y_1(r)$, $y_2 = y_2^{(n-r)}$, $v = v^{(r,n-r)}$ and if $Z = \times + iy \in y_r$ then $\times \in XXXXX\rho_n$, $v \in \Re\rho_{r,n-r}$ and $y_2 \in \Re_{n-r}$ As with (145), we can show that $\dfrac{\alpha(y)}{\alpha(y_1, y_2, v)} = |y_1|^{n-r}$ and consequently

$$[dy] = |y_1|^{n-r}[dy_1][dy_2][dv]$$

(We may remark that in $[dv]$ we have the product of the differentials of all the $r(n-r)$ elements of V as v is not in general symmetric). Also $|y| = |y_1||y_2|$and$\sigma(Ty) = \sigma(T_1 y_1)$. Hence

$$F(mT_1, t) \equiv \int\limits_{\substack{Z \in \mathcal{L}_r \\ |y| \leq t}} \varphi^{-Zam\sigma(Ty)} |y|^{k/Z-n-1} [d\times][dy]$$

$$= \int\limits_{\substack{\times \in \Re\varphi_n . v \in \Re\varphi_n, n.r, |y_1||y_2| \leq t \\ y_1 > 0, y_2 \in \Re_{n-r} |y_1||y_2| \leq t}} e^{-2\pi m\sigma(T_i y_i)} |y_i|^{\Re/2, n-1} [dx][dy_i][dy_i][dy]$$

$$= \int\limits_{\substack{y_i > 0, y_2 \in \Re_{n-r} \\ |y_i||y_2| \leq t}} e^{-2\pi m\sigma(T_i y_i)} |y_i|^{\Re/2-r-1} |y_2|^{\Re/2-n-1} [dy_i][dy_2]$$

since the above integrand is independent of $\times$, v and the volumes of $\Re\varphi_n$, $\Re\varphi_r$, $n-r$ which are the domains for $\times$, v are both unity. Thus, using (146) and (143) we can now state that

$$F(mT_i, t) = \int\limits_{y_i > 0} e^{-2\pi me(T, y_i)} |y_i|^{\Re/2-m-1} \Bigg(\int\limits_{\substack{y_2 \in \Re_{n-r} \\ |y_2| \leq t/|y_i|}} |y_2|^{\Re/2-r-e} [iy_2] \Bigg) [dy_i]$$

$$= \frac{n-r+1}{\Re - n - r - 1}\vartheta_{n-r} t^{\frac{\Re-n-r-1}{2}} \int y_i > 0 e^{-2\pi m\sigma(T_i y_i)} |y_i| \frac{n-r-1}{2} [dy_i]$$

$$= \frac{n-r+1}{\Re - n - r - 1}\vartheta_{r-r} t^{\frac{\Re-n-r-1}{2}} \pi^{\frac{n(r-1)}{4}} (2\pi m)^{\frac{-rt}{2}} |r|^{-n/2} \prod_{\nu=a}^{r-1} r(\frac{n-\nu}{2})$$

provided $k > n + r + 1$.

In particular we have shown the existence of $F(mT_i t)$ under the assumption $k > n + r + 1$ $0 < r < n$. We now discuss the border cases $r = 0, n$.

125 Let $r = 0$. Then using (146)

$$F(mT_i, t) = \int_{\times\in n\varphi_n, y\in\Re n, |y|\leq t} |y|^{\Re/2-n-1} [d\times][dy]$$
$$= \int_{y\in\Re_n, |y|\leq t} |y|^{H/2-n-1} [dy]$$
$$= \frac{n+1}{\Re - n - 1}\vartheta_n t^{\frac{\Re-n-r-1}{2}}$$

provided $k > n + 1$ Let now $r = n$. Then

$$F(mT_i, t) = \int\limits_{\times\in\Re\varphi n, y>0, |y|\leq t} e^{-2\pi m\sigma(Ty)} |y|^{\Re/2-n-1} [d\times][dy]$$
$$\leq \int_{y>0} e^{-2\pi m\sigma(Ty)} |y|^{\Re/2-n-1} [dy]$$
$$= \pi^{\frac{n(n-1)}{4}} (2\pi m)^{\frac{n(n+1-\Re}{2}} |T|^{\frac{n+1-\Re}{2}} \pi^n_{\nu=i} \Gamma(\frac{\Re - n - \nu}{2})$$

provided $k > 2n$.

It follows that in any case provided $k > \min(2n, n + r + 1)$, $r =$ rank T the Poincare' series $g(Z, T)$ converges absolutely and uniformly in a neighbourhood of every point Zey and hence also in every compact subset of y.

Since by our assumption k is an even integer, the above condition on k actually simplifies into

$$k > n + 1 + \text{rank} T.$$

In the case $n = 1$, we can say something more. In fact, the Poincare' series in this case has a convergent majorant which does not depend on $Z \in\in_n$ so that $g(Z, T)$ is actually bounded in $\lceil_n$. This result is a consequence of the following

Lemma 16. *Let $\mu, \vartheta, \times, y$ be real numbers, and let $|x| \leq b$ and $y \geq \delta > 0$ for suitable constants $\mathscr{C}, \xi$. If $Z = x + iy$, then* 126

$$|\mu Z + \vartheta| \geq\in |\nu i + \vartheta| \tag{168}$$

with a certain positive constant $\in=\in (\delta, \mathscr{C}$.

If $\mu^2+\nu^2 = 0$, (168) is trivially true so that we can assume $\mu^2+\nu^2 > 0$. Then, due to the homogenity of (168) in μ, ν we can in fact assume that $\mu^2 + \nu^2 = i$. The proof is by contradiction. If the Lemma were not true, there exists sequences $Z_n = \mu_n + iy_n, u_n, \vartheta_n, n = 1, 2 \ldots$, with $\mu_n^2 + \nu_n^2 = |x_n| \leq \mathscr{E}, y_n \geq \delta$ and finally $|u_n 2_n + \vartheta_n|^2 = (\nu_n y_n)^2 + (\nu_n x_n + \vartheta_n)^2 < 1/n$. So, in particular $u_n^2 < 1/ny_n^2 \leq 1/n\delta^2$ so that $u_n \to 0XXXXXn \to \infty$ This implies, since $|x_n| \leq \zeta$ that $u_n x_n \to o$ and consequently $\vartheta_n \to o$. But this is impossible as $u_n^2 + \vartheta_n^2 = 1\ 1$ for every n. This proves our Lemma.

Reverting to the Poincare' series in the case $n = 1$, we have

$$g(Z, T) = \frac{1}{\varepsilon(T_1)} \sum_{o_{1V_n}} e^{Z,*****} |eZ + D1$$

If $r = 1$, the above series is majorised by the series

$$\frac{1}{\varepsilon(t)} \sum_{SoV_1} e^{2\pi toy} |c2 - D1^{-R}$$

obtained by taking absolute values termwise and in the alternative case, vie, $r = 0$, it is injorised by $\sum\limits_{SgV_o} |cL + D1$

For $Z \in \delta_1$, whence $|x| \leq \zeta = 1/2$ and $|y| \geq \delta = \sqrt{3}/2, Z = x + y$, 127
either of these has the convergent majorant $\zeta \sum |ei + d1^{-\mathfrak{R}}$ independent of Z by means of Lemma 15.

We one summarize our results into the following

Theorem 9. *The Poincare, series (152), Viz.*

$$g(Z,T) = \sum_{S \in V(T)} e^{2\pi\sigma(TS<Z>)} |CZ|D1^{\mathfrak{K}}, S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

converges absolutely and uniformly in every compact subset of $\mathscr{Y}$ and represent a modular form of weight k for $\mathfrak{K} \equiv o(2)$, $k > n + 1 + rankT$, where T stands for an arbitrary semi integral semi positive matrix.

Later, we shall prove that Poincare' series actually generates all modular forms of degree n and weight k provided $k \equiv 0(2)$ and $k > 2n$. This proof requires a new tool, viz, the metrization of modular forms and the next section is devoted this topic.

Chapter 10

The metrization of Modular forms of degree

In §.5 we defined modular forms of degree $n \geq 1$ and weight k and we set up an operator ϕ which maps the linear space $\mathscr{M}_{\mathfrak{K}}^{(n)}$ of such forms into $\mathfrak{M}_{\mathfrak{K}}^{(n-1)}$ when $n > 1$. With a view to facilitating the statement of our subsequent results we agree to call constants modular forms of degree 0 and a given weight k and extend the domain of ϕ to include $\mathscr{M}_{\mathfrak{K}}^{(1)}$ by specifying that for modular forms $f(Z) \in \mathscr{M}_{\mathfrak{K}}^{1}$, $f(Z)/\phi = a, (o)$, viz, its 'first' Fourier coefficient . We can now state that 128

$$\mathscr{M}_{\mathfrak{K}}^{(n)}\Big|\phi \subset \mathscr{M}_{\mathfrak{K}}^{(n-1)}, n \geq 1.$$

It is of interest to investigate when the reverse inclusion is also true, viz, when the inclusion relation above is actually a relation of equality. This turns out to be true in all the cases where the Poincare' series $g(Z, T)$ which depend on k and n have been proved to exist (as modular forms of degree n and weight k), as we shall see subsequently . A modular form of degree $n > 0$ shall be called a *cusp form* if

$$f(Z)|\phi \equiv o$$

Any modular form of degree 0 shall by definition be a cusp form too. If $f(Z), g(Z)$ be two modular forms of degree $n \geq 0$ and weight k, of which at least one is a cusp form, we define a scalar product (f, g) by

$$(f,g)=\begin{cases}\int\limits_{f_n} f(Z)\overline{g(Z)} & |y|^{\mathfrak{R}-n-1}[d\times][dy],\ fnn>o, Z-x+y \\ f\overline{g}, & \text{if } n=0\end{cases} \quad x \tag{169}$$

129 We prove the existence of the right side of (169) to make our definition meaningful. Let, of the two, $g(Z)$ be a cusp form.

It follows from (111) that

$$|g(Z)| \leq \zeta e^{-2\in\mathscr{Y}\overline{|y|}} \text{ for } Z=\chi+\in y\in F_n$$

with certain constants $\in, \zeta 0$. Clearly $f(Z)$ and $\in^{-\in\sqrt{|y|}}|y|^{\mathfrak{R}-n-1}$ are bounded in f_n. We have therefore only to prove the existence of the integral

$$\int\limits_{f_n} e^{\in u\sqrt{|y|}}[dy][dx].$$

Since f_n is contained in the set of $zXXXXX$ defined by:

$$Z=x+iy, \chi\in\mathscr{M}_n, y\in\mathfrak{R}_n$$

the above integral is majorised by $\int\limits_{y\in\mathfrak{R}_n}\varepsilon^{-\in u\sqrt{|y|}}[dy]$ and this integral, we know, exists by (149).

We also observe that the integrand $f(Z, g, \overline{Z})|y|^{\mathfrak{R}-n-1}[dx][dy]$ is invariant with respect to the modular substitutions, as the product $\overline{g(Z)}$ $f(Z)|y|^{\mathfrak{R}}$ and $dy=|y|^{-n-1}[d\chi][dy]$ are so, and consequently it is permissible to replace the domain of integration f_n occurring in the right side of (169) by any other fundamental domain for M_n

The cusp forms of degree n and weight k constitute a linear space $y_{\mathfrak{R}}^{(n)}$, a sub space of $\mathscr{M}_{\mathfrak{R}}^{(n)}$. We define the orthogonal space $y_{\mathfrak{R}}^{(n)}$ of $y_{\mathfrak{R}}^{(\mathfrak{R})}$ in $\mathscr{M}_{\mathfrak{R}}^{(n)}$ in the usual way, is the set of forms in $\mathscr{M}_{\mathfrak{R}}^{()}$, which are orthogonal
130 to all the forms in $y_{\mathfrak{R}}^{(n)}$ in the sense of our inner product. It is well known that $\mathscr{M}_{\mathfrak{R}}^{(n)}$ is the direct sum of $y_{\mathfrak{R}}^{(n)}$ and ${}_{\mathfrak{R}}^{(n)}$ i.e.

$$\mathscr{M}_{\mathfrak{R}}^{(n)}=y_{\mathfrak{R}}^{(n)}+\mathscr{N}_{\mathfrak{R}}^{(n)}.$$

A much finer decomposition of $\mathscr{M}_{\mathfrak{R}}^{(n)}$ is given in

Theorem 10. *There exists a uniquely determined representation of* $\mathscr{M}_{\Re}^{(n)}$ *as a direct sum*

$$\mathscr{M}_{\Re}^{(n)} = y_{\mathscr{R}0}^{(n)} + y_{\Re 1}^{(n)} + \cdots + y_{\Re n}^{(n)} \tag{170}$$

with

$$y_{\Re n}^{(n)} \subset y_{\Re}^{(n)}$$

$$\text{for } \nu < n \text{ and } y_{\Re\nu}^{(n)} \subset \mathscr{N}_{\Re}^{(n)}, y_{\Re o}^{(n)} \& \subset y_{\mathscr{R}''}^{(n-1)} \tag{171}$$

Proof. We first remark that the mapping ϕ is $1-1$ on $\mathscr{N}_{\Re}^{(n)}$. □

Indeed, due to linearity of ϕ, this is ensured if $f(Z) \in \mathscr{N}_{\Re}^{(n)}$, $f(Z)\phi \equiv 0$ together imply that $f(Z) \equiv 0$, but this is immediate as, in this case, $f(Z) \in \mathscr{N}_{\Re}^{(n)} o y_{\Re}^{(n)} = (0)$.

Also the theorem is innocuous in the case $n = 0$. We now base our proof on induction on n. We therefore assume given $y_{\Re\nu}^{(n-1)}$, $\nu = 1, 2\cdots(n-1)$ satisfying (170)- (171) with n replaced by $n-1$. Let $y_{\Re\nu}^{(n)}(\nu < n)$ denote the linear space all $f(Z) \in \mathscr{N}_{\Re}^{(n)}$ such that $f(Z)\ \phi y_{\Re\nu}^{(n-1)}$.

Since ϕ is invertible on $\mathscr{N}_{\Re}^{(n)}$, $\overline{y}_{\Re}^{(n)}$ can be characterized by the relations:

$$\overline{y}_{\Re\nu}^{(n)} \subset \mathscr{N}_{\Re}^{(n)}, \overline{y}_{\Re\nu}^{(n)}|\phi = \mathscr{N}_{\Re}^{(n)}|\phi \cap y_{\Re\nu}^{(n-1)}$$

By assumption 131

$$\mathscr{M}_{\Re}^{(n-1)} = \sum_{\nu=o}^{n-1} y_{\Re\nu}^{n-1}$$

so that

$$\mathscr{N}_{\Re}^{(n)}|\& = \mathscr{N}_{\Re}^{(n)}|\& \cap \mathscr{M}_{\Re}^{(n-1)} = \sum_{\nu=o}^{(n-1)} \mathscr{N}_{\Re}^{(n)}|\& \cap y_{\Re\nu}^{(n-1)}$$

$$= \sum_{\nu=o}^{n-1} y_{\Re\nu}^{(n)}|\phi$$

where all the sums occurring are direct.

Since ϕ is $1-1$ on $\mathscr{N}_{\Re}^{(n)}$ this means that

$$\mathscr{N}_{\Re}^{(n)} = \sum_{\nu=o}^{n-1} \overline{y^{(n)}}_{\Re\nu}$$

and then

$$\mathscr{M}_{\Re}^{(n)} = \mathscr{N}_{\Re}^{(n)} + y_{\Re}^{(n)} = \sum_{\nu=o}^{n-1} \bar{y}_{\Re\nu}^{(n)} + y_{\Re}^{(n)}$$

again all the sums direct. The last decomposition has clearly all the desired properties.

It remains to show that decomposition is unique .

Let $\mathscr{M}_{\Re}^{(n)} = \sum_{\nu=o}^{n} y_{\Re\nu}^{(n)}$ be another decomposition satisfying (171). Then we have for $\nu < n$,

$$y_{\Re\nu}^{(n)}|\phi \subset \mathscr{N}_{\Re}^{(n)}|\phi \cap \mathscr{N}_{\Re\nu}^{(n-1)} = \bar{y}_{\Re}^{(n)}|\varphi$$

and both $y_{\Re\nu}^{(n)}$, $\bar{y}_{\Re\nu}^{(n)}$ are contained in $\mathscr{N}_{\Re}^{(n)}$ on which ϕ is $1-1$. Hence $y_{\Re\nu}^{(n)} \subset \bar{y}_{\Re\nu}^{(n)}$ for $\nu < n$, and by assumption, $y_{\Re\nu}^{(n)} \subset y_{\Re}^{(n)}$. Since $\sum_{\nu=o}^{n} y_{\Re\nu}^{(n)} = \sum_{\nu=o}^{n} \bar{y}_{\Re\nu}^{(n)} + y_{\Re}^{n}$ and both sums are direct, it easily follows that $y_{\Re\nu}^{(n)} = \bar{y}_{\Re\nu}^{(n)}$ for $\nu < n$ and $y_{\Re\nu}^{(n)} = y_{\Re\nu}^{(n)}$.

The proof of theorem 10 is now complete.

132 As a consequence of (171) we deduce that

$$y_{\Re\nu}^{(n)}|\varphi^{n-\nu} \subset y_{\Re\nu}^{(n-1)}|\varphi^{n-\nu-1} \subset y_{\Re\nu}^{(n-1)}|\varphi^{n-\nu-1} \cdots \subset y_{\Re\nu}^{(n)} = y_{\Re}^{(n)}.$$

This means in words that $y_{\Re\nu}^{(n)}|\varphi^{n-\nu}$ consists of cusp forms of degree ν. Further $\phi^{n-\nu}$ is $1-1$ on $y_{\Re\mu}^{(n)}$ for $\mu \leq \nu \leq n$ and in particular on $y_{\Re\nu}^{(n)}$. For, we need only show that the conditions $f(Z) \subset y_{\Re\mu}^{(n)}$, $f(Z)|\phi^{n-\nu} = o$, $\mu \leq \nu \leq n$, together imply that $f(Z) \equiv o$. Since in the case $\nu = n$, the assertion is trivial, we assume $\nu < n$. We resort to induction on n.

Let $g(Z) = f(Z)|\phi$. Then our assumption implies in view of (171) that

$$g(Z) \in y_{\Re\mu}^{(n-1)}, g(Z)|\phi^{n-1-\nu} = o$$

The induction assumption then implies that $g(Z) = 0$ which in its turn means that $f(Z) = 0$ as we know that ϕ is $1-1$ on $\mathscr{N}_{\Re}^{(n)}$ and in particular on $y_{\Re\mu}^{(n)} \subset \mathscr{N}_{\Re}^{(n)}(\mu < n)$.

We shall now introduce a scalar product for arbitrary pairs of modular forms. In view of theorem 10, we have a decomposition for any modular form $f(Z) \in \mathscr{M}^{(n)}_{\Re}$ as

$$f(Z) = \sum_{\nu=o}^{n} f_\nu(Z),\ f(Z),\ f_\nu \in y^{(n)}_{\Re\nu},\ \nu = 0.1.2\cdots n,$$

and this decomposition in unique. Let $g(Z)$ be another form in $\mathscr{M}^{(n)}_{\Re\nu}$ and let $g(Z) = \sum_{\nu=o}^{n} g_\nu(2)$, $g_\nu(2) \in y^{(n)}_{\Re\nu}$.

Then we define the scalar product $(f(Z), g(Z))$ as

$$(f(Z), g(Z)) = \sum_{\nu=o}^{n} (f_\nu(Z)|\varphi^{n-\nu} g_\nu(2)|\varphi^{n-\nu}) \tag{172}$$

where the scalar product occurring in the right side have the meaning 133
given earlier. If $g(Z)$ is a cusp form, then $g(Z) = g_n(Z)$ and all but the last term vanish in the summation on the right side of (172). Thus (172) reduces in this case to: $(f(Z), g(Z)) = (f_n(Z), g_n(Z))$, the right side being interpreted in the sense of (169). This proves the consistency of our present definition of the scalar product $(f(Z), g(Z))$ with the earlier ones given by (169) whenever applicable. Also it is clear from (172) that $(p, g) = (\overline{g, p})$ and that the assumption $(f, f) = 0$ implies $p_\nu(Z)|\phi^{n-\nu} = 0$ for $o \leq \nu \leq n$ and consequently $f_\nu, \nu \leq n$ and $f = \sum_{\nu=o}^{n} f_\nu$ vanish identically.

We thus conclude that the metric which (172) gives rise, to, is a *positive hermitian* metric.

Let us compute the scalar product $((f, Z), g(Z))$ where $g(Z)$ is represented by the Poincare' series $g(Z, T)$ and $f(Z)$ is a cusp form, i.e. $f(Z) \in y^{(n)}_{\Re}$. We assume of course that the Poincare' series converges. Since $f(Z)$ is a cusp form, we have from (111) that

$$f(Z) = 0(r^{-2m\sqrt[n]{|y|}}) \text{ for } Z = x + y \in f_n$$

where m is a suitable positive constant. Hence

$$|y|^{R/2}|f(Z)| \leq \mathcal{M} e^{-m\sqrt[n]{|y|}} \text{ for } Z \in f_n$$

with a sufficiently large $\mathcal{M}$. Since $|y|^{R/2}|f(Z)| = h(Z)$ is invariant by (101) and $|y|$ does not increase when we replace $Z \in f_n$ by an equivalent point with respect to M_n, it follows that 134

$$|y|^{k/2}|f(Z)| \leq \mathcal{M} \in^{n\sqrt[n]{iy'}} \text{ for } > \in \mathscr{Y}_r.$$

We require these facts to prove that the function $\rho(Z) \in^{2\pi\sigma(T\varepsilon)} |y|^{k-r-1}$ with $T = \begin{pmatrix} T_1^{(n)} & o \\ o & o \end{pmatrix}, T_1^{(n)} > o$ is absolutely integrable over the fundamental domain y_r of the groups $\mathscr{A}_n$. This is certainly true as a result of our above assertions if the integral.

$$G(T_\ell, m) = \int\limits_{y_n} e^{-Z\pi\sigma(\chi y) - m\sqrt[n]{|y|}} |y|^{k/2} - n - 1, k\times][dy] \tag{173}$$

exists. We shall show that this integral does exist.

First we consider the case $0 < r < n$. In this case we can use the usual parametric representation for $y > o$ as

$$Y = \begin{pmatrix} y_1 & o \\ o & y_w \end{pmatrix} \left[\begin{pmatrix} E & v \\ o & E \end{pmatrix} \right]$$

Transforming (173) in terms of the new variables and carrying out the integrations with suspect to x and v as in earlier contexts, we have

$$\begin{aligned}
G(T, m) = & \int\limits_{\substack{y_1 > o \\ y_2 \in \Re_{n-r}}} \in^{-Z\pi\sigma(T_1 y_1 - m\sqrt[n]{|y_1|}\sqrt[n]{|y_2|}|y_1|^{k/2-r-1}} |y_2|^{k/2-r-1} [dy_1][dy_2] \\
&= \int\limits_{y_1 > o} e^{2\pi\sigma < T_1 y_1 >} |y_1|^{k/2-r-1} \left(\int\limits_{y_2 o \Re_{n-r}} e^{m\sqrt[n]{|y_1|}\sqrt[n]{|y_2|}} |y_2|^{k/2-n-1} [dy_2] \right) [dy] \\
&= \frac{n(n-r+1)}{2} \vartheta_{n-r} m^{\frac{-n(k-n-n-1}{2}} \chi \left(\frac{n(k-r-r-1}{2} \right) \\
&\qquad \times \int\limits_{y_1 > o} \in^{2\pi\sigma < T_1 y_1 >} |y_2|^{k-n-r} 2 [dy_1] \\
&= \frac{n(n-r+1}{2} \vartheta_{n-r} m^{\frac{-n(k-n-r-1}{2}} y \left(\frac{n(k-n-r-1}{2} \right) \Pi^{r(r-o/4} \\
&\qquad (2\Pi)^{r0/2} |T|^{\frac{1}{2}} \mathscr{V}_{v-o}^{k-2} y \left(\frac{n-u}{2} \right)
\end{aligned}$$

135 The last two relations which are consequences of (149) and (143) are true provided $k > n + r + 1$.

We shall now turn to the border cases $r = o, n$. In the first case, viZ. $r = o$

$$G(T_1, m) = \int\limits_{y\in\Re_n} e^{-m\sqrt[n]{|y|}}|y|^{k/2-r-1}[dy]$$
$$= \frac{n(n+1)}{2}\vartheta_n y(\frac{n(k-n-o}{2} m^{-\frac{n(k-n-1}{2}}$$

provided $k > n + 1$ and in the alternative case when $r = n$,

$$G(T_1 m) = \int\limits_{y>o} e^{2\pi\sigma(Ty)-m\sqrt[n]{|y|}}|y|^{k/2-n-1}[dy]$$
$$\leq \int\limits_{y>o} e^{-2\pi\sigma(Ty)}|y|^{k/2-n-1}[dy]$$
$$= \Pi\frac{n(n-1}{4}(2\pi)\frac{n(n+1-k}{2}|T|\frac{n+1-k}{2} ***^n_{\nu=1}\chi(\frac{k-n-\nu}{2}$$

provided $k > 2n$.

The conditions in all these cases under which we have proved (173) to exist are precisely those under which we proved the Poincare' series to exist and in particular we have shown that the integral

$$H(T_1,\rho) = \int\limits_{Z\in y_r} \rho(Z)e^{2\pi i\sigma(T\bar{Z})}|y|^{k/2-n-1}[dx][dy] \tag{174}$$

exists provided $k \equiv 0(2)$ and $k > n + 1 + r$, where $r =$ rank T. We observe that the integrand in $H(T_1, \dagger$ is invariant under the substitutions in $\mathcal{A}_r$ so that we can choose for the domain of integration any convenient fundamental domain of $\mathcal{A}_r$ in the place of y_r. In fact we compute 136
$H(T_2, \Re)$ by choosing "y_r" in two different ways and this will lead us to an interesting result.

From the definition of $\mathcal{A}_r (r \geq 0)$ it is clear that, while $\mathcal{A}_0$ contains with every $\mathcal{M}$, - $\mathcal{M}$ too, $\mathcal{A}_r$ for $r > o$ contains at most one of the matrices

$\mathcal{M}$, - $\mathcal{M}$. So we may assume that V_r in the case $n > o$ contains $-S$ with S. Then $U_{S\in V_r} S < S_n >$ is a fundamental domain for $\mathcal{A}_r$ which is covered but once if $r = o$, while it is covered twice if $r > 0$.

Let us introduce

$$\delta_r = \begin{cases} 1. \text{ if } r = 0 \\ 2. \text{ if } r > 0 \end{cases}$$

and obtain from (174), in view of the invariance of the integrand appearing in it under the substitutions in $\mathcal{A}_r$,

$$\begin{aligned} H(T_1, \mathfrak{f}) &= \frac{1}{\delta_r} \sum_{S\in V_r} \int_{S\in S<\mathfrak{f}_n>} \mathfrak{f}(Z) \in^{-2\pi i\sigma(T\bar{Z})} |y|^{k-n-1}[dx][dy] \\ &= \frac{1}{S_r} \sum_{S\in V_r} \int_{Z\in\mathfrak{f}_n} \mathfrak{f}(S < Z >) e^{-2\pi\sigma(TS<Z>)} |y_s|^{k-n-1}[dx_s][dy_s] \end{aligned}$$

where we assume $S < Z >= X_s + iy_s$

Since $dv = |y|^{-n-1}[dx][dy]$ is the invariant volume element, using (92) and (**??**) we get

$$\begin{aligned} H(T_1, \mathfrak{f}) = \frac{1}{\delta_r} \sum_{S\in V_r} \int_{Z\in\delta_n} \mathfrak{f}(S < Z > e^{-2\pi i\sigma(T\overline{S<Z>})}|cZ \\ + D|^{-2k}|y|^{k-n-1}[dx][dy] \\ = \frac{1}{\delta_r} \sum_{S\in V_r} \int_{Z\in\delta_r} \mathfrak{f}(S < Z >)|cZ + d|^{-k} e^{-2\pi i\sigma(T\overline{S<Z>})}|c\overline{Z} \\ + D||y|^{\mathfrak{K}-n-1}[dx][dy] \\ = \frac{1}{\delta_r} \sum_{S\in V_r} \int_{Z\in S_n} \rho(Z) e^{-2\pi i\sigma(T\overline{S<Z>})}|c\overline{Z} + D|^{-\mathfrak{K}}|y|^{\mathfrak{K}-r-1}[dx][dy] \\ = \frac{1}{\delta_r} \int_{Z\in\mathfrak{f}_n} \mathfrak{f}(Z)|y|^{\mathfrak{K}-n-1}(\sum_{S\in V_r} e^{-2\pi i\sigma(T\overline{S<Z>})}|c\overline{z} \\ + D|^{-\mathfrak{K}}[dx][dy] \\ = \frac{\varepsilon(T_1)}{\delta_r} \int_{Z\in\mathfrak{f}_n} \mathfrak{f}(Z)\overline{g(Z, T}|y|^{\mathfrak{K}-n-1}[dx][dy] \end{aligned}$$

$$= \frac{\varepsilon(T_1)}{\delta_r}(\mathfrak{f}(Z), g(z,T)) \tag{175}$$

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On the other hand we can computer $H(T_1, \mathfrak{f})$ directly as follows :
The part of the integral involving $\times$ in (174) is clearly

$$\int\limits_{\times\in n0_n} \mathfrak{f}(Z)e^{-2\pi i\sigma(T\bar{Z})}[ax] = e^{-4\pi e(Ty)} \int\limits_{\times\in\delta S_r} \mathfrak{f}(z)e^{-2\pi i\sigma(TZ)}$$
$$= e^{-\triangle\pi\sigma(Ty)}a(T)$$

where $a(T)$ is the Fourier coefficient of $f(z)$ with respect to the exponent - matrix T.

But $a(T) = 0$ by (108) if $|T| = 0$ since by assumption $f(z)$ is a cusp form so that, in particular if $\mathfrak{R} < \mathfrak{R}$, $H(T_1, \mathfrak{f}$ and consequently the scalar product $(\mathfrak{f}(Z)g(z,T))$ vanish. Assume then that $r = n$. Then

$$H(T_1, \mathfrak{f}) = \int\limits_{\substack{\times\in xyw\\ y>o}} \mathfrak{f}(Z)e^{-2\pi\sigma(Tz)}[dx]e^{-\triangle\pi\sigma(Ty)}|y|^{\mathfrak{R}-n-1}[dy]$$
$$= a(T)\int\limits_{y>o} e^{-4\pi\sigma(Ty)}|y|^{\mathfrak{R}-n-1}[dy]$$
$$= a(T)\pi^{\frac{n(n-1)}{4}}(4\pi)^{\frac{n(n+1}{2}-n\mathfrak{R}}\prod_{\nu=s}^{n} y(\mathfrak{R}-\frac{n+\nu}{2}|T|^{n+1}2 \tag{176}$$

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Comparing with (175) we get

Theorem 11. *If $\mathfrak{f}(Z) \in y_{\mathfrak{R}}^{(n)}$ and $\mathfrak{R} \equiv 0(2), \mathfrak{R} > n+1+$ rank T, then $(f(z)g(z,T))$ is equal to*

$$\begin{cases} \frac{2}{\varepsilon(T)}a(T)\pi^{\frac{n(n-1}{4}}(4\pi)^{\frac{n(n+1}{2}-n\mathfrak{R}}\prod_{\nu=1}^{n} y(\mathfrak{R}-\frac{n+\nu}{2})|T|^{\frac{n+1}{2}-\mathfrak{R}} \text{for } T > o \\ o \qquad \textit{for } T = o \end{cases}$$

where $a(T)$ represents the Fourier coefficient of $f(z)$ with respect to the exponent matrix T.

We generalize the above result to the case of $f(z) \in y_{\mathfrak{R}}(\nu \leq n)$ in the next section.

Chapter 11

The representation theorem

Our aim in this section is to establish a result we promised earlier at the end of §9, viz. that the Poincare series actually generate all modular form of degree n under suitable assumptions. As a first step we compute $\mathscr{Y}(Z, T)|\mathscr{Y}$. Let $z^* \in \mathscr{Y}_{n-1}$ and $x \in \mathscr{Y}$ so that $Z = \begin{pmatrix} Z & 0 \\ 0 & r \end{pmatrix} c\mathscr{Y}_n$. For fixed $Z*$ let $\varphi(z) = g(Z, T)$. Then $\varphi(z)$ is a modular form of degree 1 and weight k. For, if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a modular substitution of degree 1 and we set 139

$$A = \begin{pmatrix} E & 0 \\ 0 & a \end{pmatrix} = A^{(n)}, B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = B^{(n)}$$
$$C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} = C^{(n)}, D = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = D^{(n)}$$

then $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}_n$ Also

$$M(Z) = (AZ + B)(CZ + o) = \begin{pmatrix} Z & 0 \\ 0 & N < z > \end{pmatrix} \text{ where } \tag{177}$$
$$N(z) = (az + b)(az + d)^{-1}$$

so that the mapping $z \to \mathcal{M} < z >$ is equivalent with the pair of mappings:

$$Z^* \to Z^*, z \to (az+b)(cz+d)^{-1}$$

Besides, we have also $|CZ+D| = cz+d$ so that from (177)

$$\begin{aligned} \varphi(z) = g(Z,T) = g(z,T)|M &= g(M < z > T)|CZ+D|^{-\mathfrak{K}} \\ &= \varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^{-\mathfrak{K}} \\ &= \varphi(z)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned} \tag{178}$$

140 Clearly $\varphi(Z)$ is regular in the fundamental domain $\mathfrak{f}_1$ of M_1 It only remains then to verify that $\varphi(Z)$ is bounded in $\mathfrak{f}_1$. This is obvious at least in the case of $Z^* \in \mathfrak{f}_{n-1}$ since in this case $Z \in \mathfrak{f}_n$ when the imaginary part $\mathscr{Y}$ of z is sufficiently large, and $g(z,T)$, being a modular form of degree n, is bounded in $\mathfrak{f}_n$.

The general case does not present any difficulty either, as in this case there exists a substitution

$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in M_{n-1}$ so that $M_1 < Z* > \in \mathfrak{f}_{n-1}$ and if we set $A = \begin{pmatrix} A_1 & o \\ o & 1 \end{pmatrix}, B = \begin{pmatrix} B_1 & o \\ o & o \end{pmatrix}, C = \begin{pmatrix} C_1 & o \\ o & o \end{pmatrix}$ and $D = \begin{pmatrix} D_1 & o \\ o & 1 \end{pmatrix}$ then $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_1, M < z > = \begin{pmatrix} M_1 < z > & o \\ o & z \end{pmatrix}$ and $|CZ+D| = |c, z^* + D_1|$.

$$\begin{aligned} \text{Finally } \varphi(Z) = g(z,T) &= g(z,T)|M \\ &= g(M < z > T)|cz+D|^{-\mathfrak{K}} \\ &= g\left(\begin{pmatrix} M_I < Z* > & o \\ o & z \end{pmatrix}, T\right)|C_1 z * + D_1|^{-\mathfrak{K}}. \end{aligned}$$

The term to the extreme right is bounded as $M_1 < Z^* > \in \mathfrak{f}_{n-1}$ and then $\varphi(z)$ is clearly bounded. It is now immediate that $\varphi(z)$ is a modular form of degree 1 and weight k.

141 Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an arbitrary modular substitution of degree n

(we shall stick to this notation throughout this section). It is easy to see that the matrix $A^Z + B$ depends linearly in z and further z appears only in its last column. The same is true of $(CZ + D)$ too so that $|CZ + D|$ is a linear function of z, say $|cz + D| = cz + D$. It follows then that $|CZ + D|(CZ + D)^{-1}$ too is a matrix whose elements are linear in z and which has its last row independent of z; consequently

$$|CZ + D|S < Z >= |CZ + D|(AZ + B)(cz + D)^{-1} \tag{179}$$

is also linear in z. Let then $\sigma(TS < Z >= \dfrac{az + b}{cz + d}$ with $|CZ + D| = cz + d$ where a, b, c, d are all complex numbers not depending on z. We shall denote by $L_{\mathscr{S}}$ the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defined as above. Since $Z \in \mathscr{Y}_n$ we have $|CZ + D| \neq o$ and hence $cz + d \neq 0$. Since $ImL_S < Z >= Im\dfrac{ax + b}{cz + d} \geq 0$ for $Z \in \mathscr{Y}_1$ it follows that either $o = o$ in which case $\dfrac{a}{d}$ is real and $\dfrac{a}{d} \geq 0$ or if $c \neq 0$ then $Im - \dfrac{d}{c} \geq 0$. The Poincare' series (152) now takes the form

$$\varphi(z) = g(Z, T) = \sum_{S \in V(T)} e^{2\pi i \frac{az+b}{cz+d}} (cz + d)^{-\mathfrak{K}}. \tag{180}$$

We construct a suitable $V(T)$ as follows. Let $\triangle$ be the cyclic group generated by the modular substitution

$$G = \begin{pmatrix} E^{(n)} & \begin{pmatrix} o & o \\ o & 1 \end{pmatrix} \\ o & E^{(n)} \end{pmatrix}$$

The mapping $Z \to G < Z >$ leaves Z^* fixed and takes z into $z + 1$. **142**
With the notation

$$\begin{aligned} \omega_S(Z) &= e^{2\pi i \sigma(TS<Z>)} |CZ + D|^{-\mathfrak{K}} \\ &= e^{2\pi i \sigma(TZ)} |S, S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

we have $\omega_{SG} \in (Z) = e^{2\pi i\sigma(TZ)}|SG$

$$= e^{2\pi i\dfrac{az+b}{cz+d}}(cz+d)^{-\Re}|\begin{pmatrix}1 & o\\ o & 1\end{pmatrix}$$

$$= e^{2\pi i\dfrac{a(z+d)+d}{c(z+c)+d}}(c(z+b)+d)^{-\Re}$$

where we recall $\begin{pmatrix}a & b\\ c & d\end{pmatrix} = L_S$.

Let

$$M = \sum_{S*\in V*(T)} A(T)S^*.$$

If n_S denotes the set of the exponents t with the property that $(S^*GS^{*-1})^t = S*G^tS^{*-1} \in \mathcal{A}(T)$, then it is easily seen that the products S^*G^t with $S^* \in V^*(T)t = t \mod n_S$ constitute a set of the type $V(T)$. So we obtain from (180) that

$$\varphi(Z) = \sum_{S\in V*(T)} \sum_{t \mod n_S} \omega_{SG}t(z) = \sum_{S\in V*(T)} \varphi(z) \tag{181}$$

where

$$\begin{aligned}\varphi_{\Re}(z) &= \sum_{t \mod n_S} \omega_{SG}t(Z)\\ &= \sum_{t \mod n_S} e^{2\pi i\dfrac{a(z+t)+b}{a(z+t)+d}}(c(z+b)+d)^{-\Re}\\ &= \sum_{t \mod n_S} e^{2\pi cL_S<z+b>}(c(z+t)+d)^{-\Re}\end{aligned} \tag{182}$$

142 It is obvious that $\varphi_S(z+1) = \varphi_S(z)$ in other words, the functions $\varphi_S(z)(S \in V*(T)$ and consequently $\varphi(z)$ are periodic with period 1. Hence we can introduce the new variable $S = \in^{2\pi iz}$ and then $\varphi_S(Z) = \lambda_{(S)}$ and $\varphi(z) = \lambda(S)$ are single valued functions of S. We shall see that they are also regular in $|S| < 1$.

We first prove that $c \neq 0$ when and only when $n_S = 0$. Indeed if $c = 0$ then $\dfrac{a}{d}$ is real and > 0 so that all the terms occurring in the right side of (182) have the same absolute value, independent of t, and this in its turn means that the series in the right of (182) is actually a finite sum as we know the series to converge and consequently $n_S \neq 0$. Conversely, if $n_S \neq 0$ then $\omega_S(z) = \omega_{S_S^n}(z)$ or

$$e^{2\pi i L_S<z>}(cz+d)^{-\mathfrak{K}} = e^{2\pi L_S<+n_S}(c(z+n_S)+d)^{-\mathfrak{K}}$$

so that

$$e^{2\pi i L_S<Z+n_S-\pi**L_S<z>} = \left(\frac{c(z+b_S+d}{cz+a}\right)^{\mathfrak{K}}.$$

The right side of the last equality is meromorphic in z and this can be said of the left side only when $0 = 0$. This proves our assertion.

We shall now appeal to Lemma 16 to show that

$$\lim_{y\to r}\varphi_S(z) = \sum_{t \mod n_S} \lim_{y\to\infty} e^{2\pi i L_S<Z+b>}(c(z-t)+d)^{-\mathfrak{K}} \tag{183}$$

provided the limits in the right side all exist. This is trivially true if $n_S > o$ as in this case we face only a finite sum in (183). In the case $n_S = o$ the right side of (183) is actually an infinite series, t ranging from $-\infty$ to $+\infty$, but in this case we show that the right side of (182) has a convergent majorant not depending on z in a domain of the type $|x| \leq \varphi, \mathscr{Y} \geq \delta > o$ where $x + iy = z$ and this will validate our taking the limit in (183) under the summation sign. 143

Since $ImL_S < z + t > \geq o$ we have

$$|e^{2\pi i L_S<z+t>}| \leq I.$$

Also $|c(z+t)+d| \geq \in ic(c+t)+d|$ for $|x| \leq \mathscr{E}, y \geq \delta > o$ with a certain positive $\in = \in (\mathscr{E}, \delta)$. For, since by assumption $n_S = o$, we have $c \neq 0$ and then the above inequality is equivalent with $|z+t+\dfrac{d}{c}| \geq \in |c+c+\dfrac{d}{c}|$ or equivalent with $|x+t+p+c(y+q)| \geq \in |c+p+c(i+q)$ where we write $d/c = p + \nu$, i.e. with

$$\left|\frac{x+i(y+q)}{1+q} + \frac{p+t}{1+q}\right| \geq \varepsilon\left|c + \frac{p+t}{1+q}\right|$$

and the last inequality is clearly true by Lemma 15. Thus we obtain 144

$$\left|e^{2\pi i L_S <z+t>}(c(z+c)+d)^{-\Re}\right| \leq \in^{-\Re} |c(+()+d|^{-\Re}$$

and consequently $\sum_{t=-\infty}^{\infty} \in^{-\Re} |c(i+t)+d|$ provides a convergent majorant we were after.

Reverting to our functions $\varphi_S(\zeta) = \varphi_S(z)$ and $x(S) = \varphi(z)$ we observe that they are regular in $|\zeta| < 1$. The only point of doubt is the origin but all these functions are bounded in a neighbourhood of the origin $-\varphi(z)$ is so because it is a modular form and the $\varphi_S(z)r$ are so in view of their having a convergent majorant not depending on z as shown earlier. Hence the origin is also free from being a singularity for any of these functions. We can therefore conclude that

$$\chi(o) = \lim_{y\to\infty} \varphi(z), \chi_S(o) - \lim_{y\to\infty} \varphi_S(z) \tag{184}$$

Since the Poincare' series converges uniformly in every compact subset of $\mathscr{Y}$, in particular

$$\chi(\zeta) = \sum_{S V*(T)} \chi_\infty(\zeta)$$

converges uniformly on $|\zeta| = \rho(o < \rho < 1)$.

145 Cauchy's integration formula then yields that

$$\chi(o) = \frac{1}{2}\pi i \int_{|\zeta|=S} \frac{\psi(\zeta)}{\zeta} o\zeta = \sum_{S\in V*(T)} \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{\psi_S(\zeta)}{\zeta} \alpha\zeta = \sum_{S\in V*(T)} \chi_S(o) \tag{185}$$

The results (184) and (185) together imply that

$$\lim_{y\to\infty} \varphi(z) = \sum_{S\in V*(T)} \lim_{y\to\infty} \varphi_S(z) \tag{186}$$

and coupled with (183) this gives

$$g(Z,T)|\phi = \lim_{y\to\infty} g(Z,T)$$

$$= \sum_{S \in V(T)} \lim_{y \to \infty} e^{2\pi i\sigma(TS<Z>)} |cz + D|^{-\Re} \tag{187}$$

with $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ provided the limits occurring in the right all exist and the series converges absolutely. To compute $g(Z, T)|\phi$ therefore, and this was the purpose with which we set out in the beginning of this section, we need only compute

$$\lim_{y \to \infty} e^{2\pi i\sigma(TS<2>)} |CZ + D|^{-\Re}, S \in V(T).$$

For this purpose we use the representation (162) which can be rewritten as

$$g(z, T) = \frac{1}{\sum(T,)} \sum_{p} e^{2\pi i\sigma(T_1 S<z>[p])} |cz + D|^{-\Re} \tag{162$'$}$$

where $T = \begin{pmatrix} T_1 & o \\ o & o \end{pmatrix}, T_1 = T_1^{\Re} > o$, r being the rank of T and $S =$ **146**
$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and the summation for P runs over all primitive matrices $\rho = \rho^{(n,r)}$ while that for S runs over all the elements of V_o, viz. a complete set of modular matrices whose second rows are non associated.

A representation for V_o is given by Lemma 1 as follows.

Let $\{c_o, D_o\}$ run through a complete set of non associated coprime symmetric pairs of square matrices of order $s(1 \leq s \leq n)$ with $|c_o| \neq o$ and let $\{Q\}$ run through a complete set of non right associated primitive matrices $Q = Q^{(n,r)}$. Let $(C_o D_o)$ be completed to a modular matrix $\begin{pmatrix} A_o & B_o \\ C_o & D_o \end{pmatrix} \in M_r$ and Q to an unimodular matrix $u = (QR)$ in any one arbitrary way and set

$$A = \begin{pmatrix} A_o & o \\ o & E \end{pmatrix} u', \qquad B = \begin{pmatrix} B_o & o \\ o & o \end{pmatrix} u^{-1},$$

$$C = \begin{pmatrix} C_o & o \\ o & o \end{pmatrix} u', \qquad D = \begin{pmatrix} D_o & o \\ o & E \end{pmatrix} u^{-1}$$

with $E = E^{n-r}$.

The resulting matrices $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as s runs through all integers $| \leq r \leq n$ together with the unit matrix $E = E^{(2n)}$ corresponding to the case $s = \text{rank } C = o$ form a class of the desired type. We can now write ((162′)) using (69) as $g(z, T) = \frac{1}{\varepsilon(T,)} \sum_{\rho} e^{2\pi i\sigma(T, Z[\rho])}$

$$\frac{1}{\sum(T,)} \sum_{r=1}^{n} \sum_{\rho} \sum_{\substack{\rho C_o, D_o \\ \{Q\}}} e^{2\pi i\sigma(T, S<z>[\rho])} |c, z[Q] + D_o|^{-\mathfrak{K}} \tag{188}$$

147 We introduce again $Z = \begin{pmatrix} Z* & o \\ o & z \end{pmatrix}$ with $Z = X + iy, z = x + iy, Z^* = X^* + iy^*$ and denote Q as $Q = (\mathscr{Y}_{m\nu} = (\mathscr{Y}_1 \mathscr{Y}_2 \cdots \mathscr{Y}_r$. We assume without loss of generality that $y[Q]$ is reduced. Let Q^* denote the matrix arising from Q by deleting its last row and let $Q^* = (\mathscr{Y}_1^* \cdots \mathscr{Y}_r^*$.

Then using (47 - 49) we have

$$||C_o Z[Q] + D_o|| \geq |Y[Q]| \geq \frac{1}{c_1} \pi_{\nu=1}^{r} Y[\mathscr{Y}_\nu] = \frac{1}{c_1} \mathscr{Y}_{\nu=1}^{r} (Y^* L \mathscr{Y}_o^*) + y q_{n\nu}^{z}) \tag{189}$$

with a certain constant $c_1 > o$.

It is clear from (189) that if $q_{n1} q_{n2} \cdots q_{nr}) \neq (oo \cdots o)$ then $||C_o Z [Q] + D_o|| \to \infty$ as $y \to x$ and consequently

$$\lim_{y \to \infty} e^{2\pi i\sigma(T_1 S<z>[P]} |C_o Z[Q] + D_o|^{-\mathfrak{K}} = o$$

The alternative case, viz. $q_{n1} q_{n2} \cdots q_{nr}) = (oo \cdots o)$ can occur only if $s < n$. For, if $s = n$, then Q is a primitive square matrix and hence is itself unimodular and its last row cannot therefore consist of all zeros. Let then $s < n$ and $q_{n1} q_{n2} \cdots q_{nr}) = (oo \cdots o)$ so that $Q = \begin{pmatrix} Q^* \\ o \end{pmatrix}$. Then Q^* is itself primitive and can be completed to a unimodular matrix $u^* = (Q^* R^*)$. The choice $R = \begin{pmatrix} R^* o & \\ o & 1 \end{pmatrix}$ is permissible and then $u = \begin{pmatrix} u^* & o \\ o & 1 \end{pmatrix}$

148 A simple computation yields that

$$s\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$$

$$= \left\{ \begin{pmatrix} A_0 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} Q' \\ R' \end{pmatrix} Z + \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} (QR)^{-1} \right\}$$

$$\times \left\{ \begin{pmatrix} 0_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q' \\ R' \end{pmatrix} Z + \begin{pmatrix} D_0 & 0 \\ 0 & E \end{pmatrix} (QR)^{-1} \right\}^{-1}$$

$$= \left\{ \begin{pmatrix} A_0 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} Q' \\ R' \end{pmatrix} Z(QR) + \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\times \left\{ \begin{pmatrix} C_0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q' \\ R' \end{pmatrix} Z(QR) + \begin{pmatrix} D_0 & 0 \\ 0 & E \end{pmatrix} \right\}^{-1}$$

$$= \begin{pmatrix} A_0 Z[Q] + B_0 & A_0 Q' Z R \\ R' Z Q & Z[A] \end{pmatrix} \begin{pmatrix} C_0 Z[Q] + D_0 & C_0 Q' Z A \\ R' Z Q & E \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} A_0 Z^*[Q^*] + B_0 & A_0 Q^{*^1} Z^* R^* & 0 \\ R^{*^1} Z^* Q^* & Z^*[R^*] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} C_0 Z^*[Q^*] + D_0 & C_0 Q^{*^1} Z^* R^* & 0 \\ 0 & E & 0 \\ 0 & 0 & 1 \end{pmatrix}^2$$

$$= \begin{pmatrix} S^*\langle Z^* \rangle & 0 \\ 0 & Z \end{pmatrix}$$

where S^* represents a modular substitution of degree $n-1$ which has the same relationship to the classes $\{C_0, D_0\}$, $\{Q^*\}$ as S has, to $\{C_0, D_0\}$ and $\{Q\}$. This means that

$S^* = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix}$ with

$$A^* = \begin{pmatrix} A_0 & 0 \\ 0 & E \end{pmatrix} u^{*'}, \qquad B^* = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} u^{*^{-1}}$$

$$C^* = \begin{pmatrix} C_0 & 0 \\ 0 & 0 \end{pmatrix} u^{*'}, \qquad D^* = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix} u^{*^{-1}},$$

where $u^* = (Q^* R^*)$ and $E = E^{(n-1-s)}$. **149**

Writing $S\langle Z \rangle = X_3 + S Y_S$ and $S^*\langle Z^* \rangle = X^*_{S*} + S Y^*_{S^*}$ we have $Y_S = \begin{pmatrix} Y^*_{S^*} & 0 \\ 0 & y \end{pmatrix}$ and $|C_0 Z[Q] * D_0| = |C_0 Z^*[Q^*] + D_0|$ Let $P = (P_{\mu\nu}) = (y_n, y_2, \ldots, y_n)$ and $P^* = (y^*_n, y^*_2, \ldots, y^*_n)$ be the matrix arising from P by deleting its last row. From (102) we have

$$\begin{aligned}\sigma(\tau, Y_s[P]) \geq y\sigma(Y_s[P]) &= y\sum_{\nu=1}^{r} Y_S[Y_\nu]\\ &= y\sum_{\nu=1}^{r}(Y^*_{S^*}[Y^*_\nu] + YP^2_{\mu\nu})\end{aligned}$$

with a positive constant $y = y(\tau_1)$. Hence $\sigma(\tau, Y_s[P]) \to \infty$ as $y \to \infty$ and consequently $\lim_{y\to\infty} \in^{2\pi s\in(T_1 S\langle Z\rangle[P])} = 0$ provided $(P_{n1}, P_{n2}, .P_{nr}) \neq (00, .0)$.

Thus in this case too, the general term $e^{2\pi s\sigma(\tau_1 S\langle Z\rangle[P])}|C_0 Z[Q] + D_0|^{-k}$ of ((162′))

tends to zero as $y \to \infty$. The only case which remains to be settled is when both $(q_{n1}, q_{n2}., q_{ns})$ and $(p_{n1}, p_{n2}., p_{nr})$ are the respective zero vectors simultaneously. In this case we have seen that $s < n$ and by an analogous reasoning $r < n$ and $P = \begin{pmatrix} P^* \\ 0 \end{pmatrix}, Q = \begin{pmatrix} Q^* \\ 0 \end{pmatrix}$ so that P^* and Q^* are themselves primitive. Further the general term of the Poincare series ((162′)) does not involve Z in this case as the relations $S\langle Z\rangle[P] = \begin{pmatrix} s^*\langle Z^*\rangle & 0 \\ 0 & Z \end{pmatrix}\begin{pmatrix} P^* \\ 0 \end{pmatrix} = S^*\langle Z^*, [P^*]$ and $|C_0 Z[Q] + D_0| = |C_0 Z^*[Z^*] + D_0|$
150 show. Trivially then does the limit $\lim\limits_{y\to\infty} e^{2\pi i\sigma(\tau_1 S\langle Z\rangle[P])}|C_0 Z[Q] + D_0|^{-k}$ exist in this case too. We have therefore shown that when $Q \neq \begin{pmatrix} Q^* \\ 0 \end{pmatrix}$ or $P \neq \begin{pmatrix} P^* \\ 0 \end{pmatrix}$ and in particular when either $r = n$ or $s = n$, the general term of the Poincare' series ((162′)) tends to zero as $y \to \infty$ while in the alternative case viz. when $Q = \begin{pmatrix} Q^* \\ 0 \end{pmatrix}, P = \begin{pmatrix} P^* \\ 0 \end{pmatrix}$ simultaneously, whence $r < n$ and $s < n$, it assumes as the limit $e^{2\pi i\sigma(\tau_1 S^*\langle Z^*\rangle[P^*])}|C_\circ Z^*[Q^*] + D_\circ|^{-k}$, being in fact independent of z in this case. Since $g(Z, T)|\phi = \lim_{y\to\infty} g(Z, T)$

we can now state that $g(Z,T)|\phi = 0$ in case $r = n > 1$ and

$$g(Z,T)|\phi = \frac{1}{\varepsilon(T_1)} \sum e^{2\pi i\sigma(T_1 Z^*[P^*])} + \frac{1}{\varepsilon(T_1)} \sum_{\mathscr{S}=1}^{p^* n-1} \sum_{p^*} \sum_{\substack{\{C_\omega, 0_0\}\\ \{Q^*\}}} e^{2\pi i\sigma(T_1 S^*\langle Z^*\rangle[P^*])} |C_0 Z^*[Q^*] + D_0|^{-k} = g(Z^*, T^*)$$

in case $r < n$ where $T^* = T^{*(n-1)} = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, viz. the matrix which arises from T on depriving it of its last row and column. $g(Z^*, T^*)$ is then just the Poincare' series obtained from ((162′)) on replacing n by $n-1$. We may define formally $g(Z,T) = 1$ for $n = 0$ so as to validate the above results for all $n \geq 1$ and state

Theorem 12. *Let* $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $T_1 = T_2^{(r)} > 0$, $T = T^{(r)} n > 0$ *and* $k \equiv 0(2)$, $k > n + r + 1$. *Let* $g(Z,T)$ *be a Poincare series of degree n and weight k and let* $Z^* \cdot T^*$ *the matrix which arises from Z, T by depriving them of their last row and column.* **151**

Then

$$g(Z,T)|\phi = \begin{cases} g(Z^*, T^*) & \text{, for} \quad r < n \\ 0 & \text{, for} \quad r = n \end{cases} \tag{190}$$

The theorems 11 and 12 lead to interesting consequences. An immediate consequence is that the Poincare' series $g(Z,T)$ with $T > 0$ generate the space $\mathscr{P}^{(n)}_{\mathfrak{K}}$ of all cusp forms. For, from (190) we know that such a $g(Z,T)$ belongs to $\mathscr{P}^{(n)}_{\mathfrak{K}}$ If $\mathfrak{k}^{(n)}_k$, denotes the space generated by $g(Z,T), T > 0$ and $\mathfrak{K}^{(n)}_k$, the orthogonal space of $\mathfrak{k}^{(n)}_{\mathfrak{K}}$ in $\mathscr{P}^{(n)}_{\mathfrak{K}}$, then $\mathscr{P}^{(n)}_{\mathfrak{K}} = \mathfrak{k}^{(n)}_{\mathfrak{K}} + \mathfrak{R}^{(n)}_{\mathfrak{K}}$, the sum being direct. If $\mathfrak{f}(Z)\varepsilon\mathfrak{R}^{(n)}_{\mathfrak{K}}$ then $\mathfrak{f}(Z)\varepsilon\mathscr{P}^{(n)}_{\mathfrak{K}}$ and is therefore a cusp form. This means that $a(T) = 0$ for $|T| = 0$ in the standard notation by (108), while if $T > 0$, $(\mathfrak{f}(Z), g(Z,T)) = 0$, as then $\mathfrak{f}(Z) \in \mathfrak{R}^{(n)}_{\mathfrak{K}}$ and $g(Z,T) \in \mathfrak{k}^{(n)}_{\mathfrak{K}}$ so that from Theorem 11 we have again

$$a(T) = \frac{1}{\mathscr{C}} (\mathfrak{f}(Z), g(Z,T)) = 0.$$

Thus all the Fourier coefficients $a(T)$ of $\mathfrak{f}(Z)$ vanish and $\mathfrak{f}(Z) \equiv 0$. This means that $\mathfrak{N}^{(n)}_{\mathfrak{K}} = 0$ and consequently $\mathscr{P}^{(n)}_{\mathfrak{K}} = \mathfrak{f}^{(n)}_{\mathfrak{K}}$. We now go a step further and prove

Theorem 13. *The Poincare' series $g(Z, T)$ with rank $T = r$ (fixed) generate $\mathscr{P}^{(n)}_{\mathfrak{K}r} (r \leq n)$ provided $k \equiv 0(2)$ and $k > n + r + 1$.*

152 *Proof.* The case $r = n$ was just now settled. Assume than that $r < n$. Let $\mathfrak{f}^{(n)}_{km}$ denote the space generated by $g(Z, T)$ with rank $T = r < n$ and apply induction on n. From theorem 11 it easily follows that $\mathfrak{f}^{(n)}_{\mathfrak{K}r} \subset \mathscr{M}^{(n)}_{\mathfrak{K}}$ and due to induction assumption $\mathfrak{f}^{(n)}_{\mathfrak{K}r}|\phi = \mathscr{P}^{(n-1)}_{\mathfrak{K}r}$. Since by construction, $\mathscr{P}^{(n)}_{\mathscr{K}r}|\phi \subset \mathscr{P}^{(n-1)}_{\mathfrak{K}r}$ and ϕ is $1 - t$ on $\mathscr{M}^{(n)}_{\mathfrak{K}}$ it follows that $\mathscr{P}^{(n)}_{kn}|\phi \subset \mathscr{P}^{(n-1)}_{\mathfrak{K}r} = \mathfrak{f}^{(n)}_{\mathfrak{K}n}|\phi$ and consequently $\mathscr{P}^{(n)}_{\mathfrak{K}r} \subset \mathfrak{f}^{(n)}_{\mathfrak{K}n}$. The reverse inclusion is also immediate as is seen from the relation

$$\mathfrak{f}^{(n)}_{\mathfrak{K}r}|\phi \subset \mathscr{M}^{(n)}_{\mathfrak{K}}|\phi \cap \mathscr{P}^{(n-1)}_{\mathfrak{K}r} = \mathscr{P}^{(n)}_{\mathfrak{K}r}|\phi$$

establishing thereby theorem 13. In conjunction with Theorem 11 this yields ◻

Theorem 14. *The Poincare' series $g(Z, T)$ generate $\mathscr{M}^{(n)}_{k}$, the space of all modular form of degree n and weight k provided $k \equiv 0(Z)$, $k > 2n$ (the usual condition for the convergence of $g(Z, T)$).*

This is precisely the representation theorem we were looking for. Further since $\mathfrak{f}^{(n)}_{\mathfrak{K}}|\phi = \mathscr{P}^{(n)}_{\mathfrak{K}}$ and we have shown that $\mathfrak{f}^{(n)}_{\mathfrak{K}n} = \mathfrak{f}^{(n)}_{\mathfrak{K}r} = \mathscr{P}^{(n)}_{\mathfrak{K}r}$ we have

$$\mathscr{P}^{(n)}_{\mathfrak{K}r}|P = \mathscr{P}^{|n-1|}_{\mathfrak{K}r}(\mathfrak{K} \equiv 0(Z), \mathfrak{K} > n + r + 1) \tag{191}$$

Repeated application of (191) yields that

$$\mathscr{P}^{(n)}_{\mathfrak{K}r}|\phi^{n-r} = \mathscr{P}^{(n)}_{\mathfrak{K}r} + \mathscr{P}^{(n)}_{\mathfrak{K}}(\mathfrak{K} \equiv (2)\mathfrak{K} = n + r + 1)$$

153 We know that $\phi^{n \cdot r}$ is $1 - 1$ on $\mathscr{P}^{(n)}_{\mathfrak{K},r}$ and we therefore infer that

$$\text{rank } \mathscr{P}^{(n)}_{\mathfrak{K}r} = \text{ rank } \mathscr{P}^{(r)}_{\mathfrak{K}}$$

In particular taking $r = 0$, $\mathscr{P}^{(0)}_{\mathfrak{K}}$ is the space of all constants so that rank $\mathscr{P}^{(n)}_{\mathfrak{K}u} =$ rank $\mathscr{P}^{(0)}_{\mathfrak{K}} = 1(k \equiv 0(2), k > n + 1)$. $\mathscr{P}^{(n)}_{\mathfrak{K},u}$ is thus generated by a single element $g(Z,0)$, the so called *Einstein Series* which converges for $\Re > n+1 \equiv 0(2)$, and represents a modular form not vanishing identically under these conditions. Setting now $r=1$, $\mathscr{P}^{(1)}_{\Re}$ is the space of all cusp forms of the (classical) elliptic modular forms and from a well known result

$$\text{rank } \mathfrak{P}^{(1)}_{\Re} = \begin{cases} \left[\frac{\Re}{12}\right] - 1, \text{ if } \Re \equiv Z(12), \Re > n+2, \\ \left[\frac{\Re}{12}\right], \text{ if } \Re \neq 2(12), \Re \equiv 0(Z), \Re > n+2 \end{cases}$$

and the same is therefore true of rank $\mathscr{P}^{(n)}_{\Re l}$.

We proceed to generalize the fundamental metric formula given in Theorem 11.

Let $\mathfrak{f}(Z) \in \mathscr{P}^{(n)}_{\Re r}$, $0 \leq r \leq n$. If rank $T = s$, then $g(\tau, T) \in \mathscr{P}^{(n)}_{\Re, s}$ and we state that $(P(Z), g(Z,T)) = 0\ yr \neq s$
while if $r = s$ we have in conformity with (172),

$$(f(Z), g(Z,T)) = (f(Z)|\phi^{r-r}.g(Z)|\phi^{n-r})$$

We can apply theorem 11 to the scalar product in the right side above, as follows. Assume $r > 0$ and assume for a moment that $T = \begin{pmatrix} f, & 0 \\ 0 & 0 \end{pmatrix}, y_1 = y_1^{(r)} > 0$.

Then, as a result of theorem 12, $g(Z,T)|\varphi^{n-r} = g(Z_1, T_1)$ where Z_i **154**
denotes the matrix arising from Z by deleting its last $(n-r)$ rows and columns and T_1 is as defined above. Hence, by means of theorem 11,

$$\begin{aligned} (\mathfrak{f}(Z), g(Z,T)) &= (\mathfrak{f}(Z)|\varphi^{n-r}, g(Z,T)|\varphi^{n-r}) \\ &= (\mathfrak{f}(Z)|\varphi^{n-r}, g(Z_1, T_1)) \\ &= \frac{2}{\in (T_1)} Q(T_1) \pi \frac{n(n-1)}{4(4\pi)} \frac{r(r+1)}{\Re} - r\Re \\ &\qquad \times \pi^{\Re}_{n=1} y(\Re - \frac{r+\nu}{2})|T_1| \frac{n+1}{2} - \Re \end{aligned}$$

where the Fourier coefficient $a(T)$ of $\mathfrak{f}(Z)|\varphi^{n-r}$ is identical with $a\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = a(T)$, the Fourier coefficient of $\mathfrak{f}(Z)$ In the case of a general T we can always assume that $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} [\mathcal{U}]$ with an appropriate unimodular matrix $\mathcal{U}$. Then $\triangle(T) = |T_2|$ and $\in(T_1)$ are uniquely determined by T and we have as above

$(\mathfrak{f}(Z), g(Z,T)) = \dfrac{\Re}{\in(T)} a(T)\pi \dfrac{r(n-1)}{4}_{(4\pi)} n(r+1)/\Re - r\Re_x \times \pi^r_{\nu=1} y(\Re - \dfrac{r+\nu}{\Re})(\triangle(T))\dfrac{n+1}{\Re-\Re}$,

our specific assumptions being $\Re \equiv 0(2)$, $\Re > n+r+1$, rank $T = r \geq 1$, $\mathfrak{f}(Z) \in \mathscr{P}^{(n)}_{\Re r}$, $g(Z,T) \in \mathscr{P}^{(n)}_{\Re n}$. The case when $r = s = 0$ still remains. In this case $T = (0)$ and $(\mathfrak{f}(Z), g(Z,0)) = (\mathfrak{f}(Z)|\varphi^{(n)}, g(Z,0)|\varphi^n)$ as an immediate consequence of our definition. Now $\mathfrak{f}(Z)|\varphi^n$ is a modular form of degree 0 and hence is equal to $C : (0)$ while with regard to $g(Z,0)|\varphi^n$ we can be a bit more specific and state that $g(Z,0)|\varphi^n = 1$ as
155 is easily verified by a consideration of the Eisenstein series of degree 1 Hence $(\mathfrak{f}(Z), g(Z,0)) = a(0)$. We thus have

Theorem 15. *Assume $\Re \equiv 0(2)$, $\Re > n+r+1$ rank $T = r$ and $\mathfrak{f}(Z) \in \mathscr{P}^{(r)}_{\Re r}$. Then*

$$(\mathfrak{f}(Z), g(Z,T)) = \begin{cases} \frac{2}{\in(T)} a(T)\pi^{\frac{r(n-1)}{4}} {}_{(4\pi)}{}^{\frac{r(r+1)}{2}} n\Re \prod\limits_{\substack{\nu=1\\ \text{for } r>0}} y(\Re - \frac{r+\nu}{\Re})(\triangle(T))^{\frac{r+1}{2}} \\ a(0) \qquad \text{for} \qquad r = 0 \end{cases}$$

where $a(T)$ denotes the Fourier coefficient of $\mathfrak{f}(Z)$ corresponding to the matrix T.

Chapter 12

The field of modular Functions

We shall need the following generalization of Lemma 16, viz. **156**

Lemma 17. *Let* $Z = X - iy \in \mathscr{Y}_n, \sigma(xx') \leq m_1, \sigma(y^{-1}) \leq m_2$. *Then*

$$\|CZ + D\| \geq \in_0 \|Ci + D\| \tag{193}$$

with a certain positive constant $\in_n = \in_0 (n, m, m_2)$, *where* $(C\ D)$ *is the second matrix row of an arbitrary symplectic is matrix*

Proof. We prove the lemma in stages. □

First we show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a set of real numbers with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $R^{(n,m)} = (r_{\mu,\nu} = (\mathscr{W}_1 \mathscr{W}_2 \cdots \mathscr{W}_m))$, $S^{(n,m)} = (\mathscr{S}_{\mu\nu}) = (\sigma_1 \sigma_2 \cdots \sigma_m)$ with $m \leq n$ be real matrices such that $s_{\mu\nu} = \lambda_\mu r_{\mu\nu}$ then

$$\begin{aligned}(\lambda_{n-m+1}\lambda_{n-m+2}\cdots\lambda_n)^{-2}|\mathscr{P}'_\mu \mathscr{P}_\nu| &\leq |\mathscr{W}'_\mu \mathscr{W}_\nu| \\ &\leq (\lambda_1, \lambda_2, \ldots, \lambda_n)^{-2} |\mathscr{P}'_\mu \mathscr{P}_\nu|, \mu, \nu = 1, 2, \ldots, m \end{aligned} \tag{194}$$

By a well known development of a determinant, we have

$$|\lambda_\mu \mathscr{W}_\nu| = \sum_{1 \leq \rho_1 < \rho_2 < \cdots < \rho_m < n} \begin{vmatrix} r_{s,i} \cdots r_{s_2,m} \\ \cdots\cdots\cdots \\ r_{s,r} \cdots r_{s_m,n} \end{vmatrix}^2$$

$$= \sum_{1\le\rho_1<\rho_2<\cdots<\rho_m<r} (\lambda_{\rho_1}\lambda_{\rho_2},\ldots\lambda_{\rho_n}) \begin{vmatrix} \mathscr{S}_{s,i}\cdots\mathscr{S}_{s_2,m} \\ \cdots\cdots\cdots \\ \mathscr{S}_{s,r}\cdots\mathscr{S}_{s_m,n} \end{vmatrix}^2$$

Also

$$|\mathscr{P}'_\mu \mathscr{P}_\nu| = \sum_{1\le\rho_1<\rho_2<\cdots<\rho_m<n} \begin{vmatrix} \mathscr{S}_{s,i}\cdots\mathscr{S}_{s_2,m} \\ \cdots\cdots\cdots \\ \mathscr{S}_{s,r}\cdots\mathscr{S}_{s_m,n} \end{vmatrix}^2$$

(194) is now immediate.

157 We next prove that if $y = y^{(n)} > 0, T = T^{(n)} = T'$ real, and $\sigma(y^{-1}) \le m_2$, then

$$||T+iy|| \ge \varepsilon_1 ||T+LE||, \epsilon_1\ (n, m_\nu) > 0 \tag{195}$$

Determine an orthogonal matrix W such that $|y[W] = D^2$ with $D = (\delta_{\mu\nu}\lambda_\nu), 0 < \lambda_1 \le \lambda_2 \le \lambda_n$ and set $T[W] = S = (\mathscr{S}_{\mu,\nu}), Q = S[D^{-1}] = (q_{\mu,\nu})$ Then $|Y| = |D|^2 = (\lambda_1\lambda_2\cdots\lambda_n)^2$ Also $\mathscr{S}_{\mu,\nu} = \lambda_\mu\lambda_\nu q_{\mu\nu}$ and

$$\begin{aligned} ||T+iy||^2 &= ||T[W]+iy[W]||^2 &&= ||S+iD^2||^2 \\ &= ||Q+iE||^2|D|^4 &&= ||Q+iE||^2|Y|^2 \\ &= |Q+iE||Q-iE||Y|^2 &&= |Q'Q+E||Y|^2 \end{aligned} \tag{196}$$

Let $Q = (\mathscr{U}_1\mathscr{U}_2\cdots\mathscr{U}_n)S = (*************)$ and introduce $R = (\mathscr{W}_1\mathscr{W}_2\cdots\mathscr{W}_n) = (r_{\mu\nu})$ where $\mathscr{U}_2 = \dfrac{1}{\lambda_\nu}\mathscr{W}_\nu, \nu = 1,2,\ldots n$ Then we have $\mathscr{S}_{\mu,\nu} = \lambda_\mu\lambda_\nu q_{\mu\nu} = \lambda_\mu r_{\mu\nu}$ so that the result (194) is now applicable (with the obvious changes). Then

$$\begin{aligned} 1Q'Q+C1 &= 1 + \sum_{\nu=1}^{n}\sum_{\mu_1<\mu_2<\mu_2} |M'_{\mu_{\mathscr{S}}} n_{\mu_{\mathfrak{K}}}| \\ &= 1 + \sum_{\nu=1}^{n}\sum_{\mu_1<\cdots<\mu_2} (\lambda_{\mu_1}\lambda_{\mu_2}\lambda_{\mu_2})^{-2} 1 \mathscr{W}'_{\mu_2}\mathscr{W}'_{\mu_{\mathfrak{K}}} \end{aligned}$$

$$\geq 1 + \sum_{\nu=1}^{n} (\lambda_{n-\nu+1} \cdots \lambda_n)^{-2} \sum_{\mu < \cdots \mu_{[\nu]}} |1\mu_{\mu_g} \mathscr{W}_{\mu_\Re}|$$

$$\geq 1 + \sum_{\nu=1}^{n} (\lambda_{n-\nu+1} \cdots \lambda_n)^{-4} \sum_{\mu < \cdots \mu_2} |\mathscr{P}_{\mu_g} \mathscr{P}_{\mu_\Re}| \qquad (197)$$

Since by assumption $\sigma(y^{-1}) \leq m_2$ all characteristic roots $\lambda_n u^2$ of y have positive lower bound which depends only on m_2 so that for a suitable constant $\varepsilon_1 = \varepsilon_1(n_1 m_2)$ we have 158

$$(\lambda_1 \lambda_2 \ldots \lambda_m \mu)^4 > \varepsilon_1^2, \mu = 0 = 1, 2, , n \qquad (198)$$

The in view of (196) - (198) we have

$$\begin{aligned} \|T + iy\|^2 &= |Q'Q + iE\|Y|^2 \\ &= |Q'Q + E|(\lambda_1 \lambda_2 \ldots \lambda_n)^r \\ &\qquad \geq \varepsilon_1^2 (1 + \sum_{\mu=2}^{n} \sum_{m_1 >, < m_\lambda} |y_{m_y}, y_{m_\Re}|) \\ &= \varepsilon_1^2 |s's + E| = \varepsilon_1^2 |s + iE\|s - iE| \\ &= \varepsilon_1^2 |T +; E\| \ T - iE|\varepsilon_1^2\| = T + iE\|^2 \end{aligned}$$

Hence $\|T + iY\| > \varepsilon_4 iE\|$ as desired. We now contend that under the assumption $\sigma(xx') \leq m_1$, where $x = x^{(n)} = x'$ and $s = s^(n) = s'$ real, we have

$$\|x + s + iE\| \geq \sum_2 \|s + iE\| \qquad (199)$$

with a certain positive constant $\sum_2 = \sum_2 (n_1 m_2)$. This only means that the quotient $\|x + s + iE\|/\|s + iE\|$ has a positive lower bound and this is certainly ensured if we show that $L = \log \| \times + s + CE\|$ - $\log \|S + \ell E\|$ is bounded. By the mean value theorem of the differential calculus we have

$$L = \sum_{1 \leq \mu \leq \nu \leq n} \xi_{\mu\nu} \frac{\partial \log \|S^* + iE\|}{\partial \mathscr{S}^*_{\mu\nu}} \Big/_{S^* = S + \mu x}$$

with $\times = (x_{\mu\nu})$, $S^* = (\mathscr{S}^*_{\mu\nu})$ and $0 < \vartheta < 1$. Also the assumption 159

$\sigma(xx') \leq m_2$ entitles us to conclude that $\pm\chi_{\mu\nu} < \mathscr{C}, \mu, \nu = 1, 2 \cdots n$. Hence

$$= |L| \leq \mathscr{C} \sum_{1\leq\nu\leq\nu\leq n} \left|\frac{\partial \log ||S^* + iE||}{\partial s^*_{\mu\nu}}\right|_{S^*=S+\vartheta\chi}$$

$$= \mathscr{C} \sum_{1\leq\nu\leq\nu\leq n} \left|\frac{\partial \log |S^* + iE|}{\partial s^*_{\mu\nu}}\right|_{S^*=S+\vartheta\chi}$$

$$= \mathscr{C} \sum_{1\leq\nu\leq\nu\leq n} \left|Re\left\{|S^* + iE^{-1}\frac{\partial |S^* + iE|}{\partial s^*_{\mu\nu}}\right\}\right|_{S^*=S+\vartheta\chi}$$

$$=\leq \mathscr{C} \sum_{1\leq\nu\leq\nu\leq n} \left||S^* + iE1^{-1}\frac{\partial |S^* + iE|}{\partial s^*_{\mu\nu}}\right|_{S^*=S+\vartheta\chi} \tag{200}$$

Let $W = (\omega_{\mu\nu}$ be an orthogonal matrix such that

$$W'S^*w = D = (\delta_{\mu\nu} d\nu), d_\nu) \geq o.$$

Then $W'(S^* + iE)w = D + iE$, or equivalently

$$(S^* iE)^{-1} = W(D + iE)^{-1}W' \tag{201}$$

Let $(S^* + iE)^{-1} = (\omega_{\mu\nu})$ and observe that the diagonal elements of $D + iE$ have the lower bound 1 as they are of the form $1/d_{\nu+i}$, $d_\nu \geq o$. Then $\omega_{\mu\nu} = \sum_{\varrho} \omega_{\mu S} \overline{d_3 + i\omega_{\varrho\nu}}$ from (201) so that, considering absolute values, $(\omega_{\mu\nu}\bar{\omega}_{\mu\nu})^{1/2} \leq \sum_{\rho=1}^{n} |\omega_{\mu\rho}\omega_{\mu\rho}|$ and consequently $\pm\omega_{\mu\nu} \leq 1$. From (200) we have then

$$|L| \leq \mathscr{C} \sum_{\mu\nu=1}^{n} (\omega_{\mu\nu}\bar{\omega}_{\mu\nu})^{1/2} \leq \mathscr{C} n^3$$

In other words L is bounded and this is precisely what we wanted to show.

160 We are now in position to face the main Lemma. We first prove it in the case when $|c| \neq o$. What we want to show is then that $||CZ + D|| \geq \varepsilon_o ||Ci + D||$ or equivalently $||Z + C^{-1}D|| \geq \varepsilon_o ||iE + C^{-1}D||$. With $Z = X + iY$, this is the same as $||\chi + C^{-1}D + iY|| \geq \varepsilon_o ||C^{-1}D + iE||$. In other words

$||T + iY|| \geq \varepsilon_o ||S + iE||$ with $T = \chi + C^{-1}D$ and $S = C^{-1}D$. An appeal to (195) which is by way legitimate, yields $||T + iY|| \geq \varepsilon_i ||T + iE||$ and then in view of (199), $||T + iY|| \varepsilon_1 ||T + iE|| \geq \varepsilon_1 \varepsilon_2 ||S + iE||$. Setting $\varepsilon_1 \varepsilon_2 = \varepsilon_o$ we have the desired result.

The general case can be reduced to the above case as follows. If (CD) is the second matrix row λ the symplectic matrix M, then (D, C) is the second matrix row of $M_1 = M \begin{pmatrix} o & -E \\ E & o \end{pmatrix}$ and M_1 is again symplectic matrix. Hence in particular $|DZ - C1 \neq o$ an consequently $|DS + C|$ does not vanish identically in $S = S'$. We can therefore determine a sequence of real symmetric matrices $S_{\Re}$ such that $\lim\limits_{\Re \to \infty} S_{\Re} = (o)$ while $|DS_{\Re + C| \neq o}$. Then, in view of the truth of our Lemma for a special case we established above,

$$||(DS_{\Re} + C)Z + D|| \geq \varepsilon_o ||(DS_{\Re} + C)i + D||.$$

Proceeding to the limit as $\Re \to \infty$ we get the desired result. This completes the proof of Lemma 17.

With these preliminaries we proved to define the field of modular functions. It looks quite plausible and natural for one to define a modular function as a meromorphic function in $\mathscr{Y}_n$ which is invariant under the group of modular substitutions and which behave a specified manner as we approach the boundary of ρ_n.

It is also true that with this definition, the modular functions constitute a field. But unfortunately, it has not been found possible to determine the structure of this function field expect in the classical case $n = 1$. With a view to getting the appropriate theorem on their algebraic dependence we are forced to give a (possibly more restrictive) definition as follows:- A *modular function of degree n* is a quotient of two modular forms of degree n both of which have the same weight. It is obvious that such a function is meromorphic in $\mathscr{Y}_n$ and is invariant under modular substitutions. However, it is not known whether the converse is also true. We shall be subsequently concerned with determining the structure of the field of all modular function of degree n. Specifically we shall prove the existence of $\dfrac{n(n+1)}{2}$ modular functions of degree n which are algebraically independent. **161**

Let $f_{\Re}$ be a modular form degree n and weight $\Re$ not vanishing identically. Such a form exists for $\mathfrak{R} \equiv 0(\mod 2)$, $\mathfrak{R} > n+1$, as for instance $f_\mu(z) = g(z.o) = g_{\mathfrak{R}}(Z, o)$ is one such. We introduce the series

$$M(\lambda) = \sum_{\{C,D\}} (\lambda - f_{\mathfrak{R}} o)|CZ + D|^{\mathfrak{R}})^{-1} \tag{202}$$

where λ denotes a complex variable, and (C, D) runs over a complete set of non-associated coprime symmetric pairs. Since a Poincare' series converges uniformly in every compact subset of $\mathscr{Y}$, $M(\lambda)$ represents a meromorphic function of λ with all poles, simple. Assume $f_{\mathfrak{R}}\rho_{\mathfrak{R}} \neq 0$ in a neighbourhood of 0.

162 Then in such that neighbourhood we have

$$\begin{aligned}
-M(\lambda) &= \sum_{\{C,D\}} \mathfrak{f}(z)^{-1}|cz + D|^{-\mathfrak{R}}(1 - \frac{\lambda}{\mathfrak{f}_{\mathfrak{R}}(z)|2 + D|^{\mathfrak{R}}})^{-1}\\
&= \sum_{\{C,D\}} \sum_{m=1}^{\infty} \lambda^{m-1}(\mathfrak{f}_{\mathfrak{R}}(z)|cz + D|^{\mathfrak{R}})^{-n}\\
&= -\sum_{m=1}^{\infty}(\mathfrak{f}_{\mathfrak{R}}(2))^{-m}\{\sum_{\{C,D\}} \frac{1}{|cz + D|^{\mathfrak{R}}}\}\lambda^{m-1}\\
&= \sum_{m=1}^{\infty} \varphi_m(z)\lambda^{m-1}
\end{aligned}$$

with $\varphi_m(z) = g_{m\mathfrak{R}}(z, e)/\mathfrak{f}(2)^m$.

$\varphi_m(z)$ is clearly invariant under the modular substitutions so that two points Z and Z_1 which are equivalent with respect to M_n define the same function $M(\lambda)$. A partial converse is also true, viz :-

Lemma 18. *Provided Z does not lie on certain algebraic surfaces which have the property that any compact subset of $\mathscr{Y}$ is intersected by only a finite number of them, the equations $\varphi_m(z) = \varphi_m(z_1)$ for $m = 1, 2, \ldots$ imply the equivalence of Z and Z_1 with respect to M.*

Proof. Due to the invariance property of $\varphi_m(z)$, we may assume that $z, z, \varepsilon 5$, and in view of the proviso in our lemma, we can further assume that $z \in$ interior $\mathfrak{F}$ as the boundary of $\mathfrak{F}$ is composed of a finite number of algebraic surfaces. We then have

$$|CZ + D| > L, \qquad |CZ + D| \geq 1 \tag{203}$$

for $\{C, D\} \neq \{0, E\}$. The poles of $M(\lambda)$ are clearly at the points $\mathfrak{f}(\pi)|CZ+D|^{\mathfrak{K}}$ and then the first of the inequalities (203) implies that among these is a point nearest to the origin, viz. $\mathfrak{f}_{\mathfrak{K}}(Z)$ corresponding to the choice $\{C, D\} = \{0, E\}$ and the residue of $M(\lambda)$ at this point is 163

1. Due to one of our assumptions, $M(\lambda)$ is an invariant function of λ for change of Z to Z_1 so that the poles of the same $M(\lambda)$ (corresponding to Z) are also given by. $\mathfrak{f}(I_1)|CZ_1 + D|^{\mathfrak{K}}$. A consideration of the existence of a pole of $M(\lambda)$ nearest to the origin at $\mathfrak{f}_{\mathfrak{K}}(Z)$ with residue 1 implies now by means of (203) that

$$|CZ_1 + D| > 1 \text{ for } \{C, D\} \neq \{0, E\} \text{ and that } \mathfrak{f}_{\mathfrak{K}}(Z_1) = \mathfrak{f}_{\mathfrak{K}}(z).$$

As the poles permute among themselves by a change of Z to Z_1 it follows that

$$\mathfrak{f}_{\mathfrak{K}}(Z)|CZ + D|^{\mathfrak{K}} = \mathfrak{f}_{\mathfrak{K}}(Z_1)|C_1Z_1 + D|^{\mathfrak{K}}$$

and consequently

$$|CZ + D| = \in |C_1Z_1 + D_1|. \tag{204}$$

for a certain permutation $\{C_1, D_1\}$ of all classes $\{C, D\} \neq \{0, E\}$ where $\in$ denotes a $\mathfrak{K}^{\text{th}}$ root of unity which may depend on C, D, Z, Z_1. □

If $\mathscr{L}$ is any compact subset of $\mathscr{F}$ we can show that, Z being arbitrary in $\mathscr{L}$ and Z_1 arbitrary in $\mathscr{F}$, there exists only a finite number of classes $\{C_1, D_1\}$ consistent with (204) for any given class $\{C, D\}$.

The proof is as follows:

Let $\mathfrak{K}$ be so determined that $|CZ + D| \leq \mathfrak{K}$ for $Z \in XXXX$ fixed. With appropriate choices of the constants m_1, m_2, a domain of the type $\sigma(xx') \leq \pi m_1 \mathscr{F}(y^{-1}) \leq 1_2$ contains (in an obvious notation) so that Z_1 can be assumed to lie in one of these domains. Then by Lemma 17, 164

$$\mathfrak{K} \geq ||CZ + D|| = \varepsilon ||c, z_i + D_1|| \geq \varepsilon\varepsilon_o ||C_1 i + D_1||$$

where $\varepsilon = \varepsilon_o(n, m_1 m_2) > o$. In other words, the determinant $|C_1 i + D_1|$ is bounded (by $\Re/\varepsilon_o$) so that, by an earlier argument (p. 44), the number of possible choices for $\{C_1, D_1\}$ is finite. This settles our claim.

Assume now that rank $C = 1$. Then $|CZ + D| = CZ[g] + d$ where g is a primitive column and (c, d) is a pair of coprime integers. Also from Lemma 1, we shall have $|C, Z_1 + D_1| = |C_2 Z_2 + D_2|$ with $Z_z = Z_1[\theta_z]\theta_z = \theta_z^{n,m}$, primitive and $C_2 = C_z^n|C_2| \neq 0$, r denoting the rank of C_1. From (204) we then have

$$CZ[\eta] + d = \in |C_2 Z_z + D_2| \tag{205}$$

where to repeat our assumptions, η denotes an arbitrary primitive column, (c, d) denotes a pair of coprime integers, (C_2^r, D_2^r) denotes a coprime symmetric pair of matrices with $|C_2| \neq 0$ and finally Z_2 stands for Z_1 transformed with a certain primitive matrix $\theta_z^{(n,m)}$. Keeping η fixed, we choose $(c, d) = (1, 1)$ and $(0, 1)$ successively and obtain from (205), the relations

$$\begin{aligned} Z[\eta] + 1 &= \in_2 |c_2 z_2 + D_2|, z_2 = z_1[\theta_2^{(n,m)}] \\ Z[\eta] + 1 &= \in_3 |c_3 z_3 + D_3|, z_3 = z_1[\theta_3^{(n,m)}] \end{aligned} \tag{205$'$}$$

which by subtraction yields

$$\in_2 |C_2 Z_2 + D_2| - \in_3 C_3 Z_3 + D_3 = 1 \tag{206}$$

165 where $C_3 = C_3^{(s)}$, $D_3 = D_3^{(s)}$ form a pair of coprime symmetric matrices with $|C_3| \neq 0$, $Q_3^{(n,s)}$ denotes a primitive matrix and $\in_2$, $\in_3$ stand for k^{th} roots of unity.

Let $n_{\mu,\nu}$ be the column having 1 in the μ^{th} and ν^{th} places and 0 elsewhere. Then

$$Z[\eta_{\mu,\nu}] = \begin{cases} Z_{\mu\mu} & \text{if } \mu = \nu \\ Z_{\mu,\mu} + Z_{\nu\nu} + 2Z_{\mu\mu} & \text{if } \mu \neq \nu \end{cases} \tag{207}$$

Choosing $\eta_{\mu,\nu}$ in the place of η in (205)$'$, we find from the relation $Z[\eta] = \varepsilon_3 |C_3 Z_3 + D_3|$ that all the elements of Z can be represented as polynomials in the elements of Z_1 and the coefficients of these polynomials belong to a finite set of numbers. To realize the last part, we need only observe that the columns $\eta_{\mu\nu}$ are finite in number and so are the pairs (C_3, D_3) as the pairs (C_1, D_1) are so for a given (C, D). Consider now the equation (206), viz.

$$\in_2 |C_2Z_2 + D_2| - \in_3 |C_3Z_3 + D_3| = 1$$

This is an equation in Z_1 and all the coefficients belong to a finite set of numbers which means that the number of possible such that equations is finite. We can then assume that above holds identically in Z_1 as in the alternative case, the finite number of (non identical) relations in Z imply a finite number of relations among the $Z'_{\mu\nu}s$ which are representable as polynomials in the $\dfrac{n(n+1)}{2}$ independent elements of Z_1 which in its turn means that Z lies on a certain finite set of algebraic surfaces- a proviso already assumed in the statement of Lemma. Rewriting the above 166
equation then, in the form

$$|Z_2 + P_2| - \alpha|Z_3 + P_3| = \beta \tag{208}$$

with $\alpha\beta \neq 0$, $P_2 = C_2^{-1}D_2$ and $P_3 = C_3^{-1}D_3$, and comparing the terms of the highest degree in this identity in Z_1 we get $r = s$ and $|Z_2| = \alpha|Z_3|$. The last is again an identity in Z_1. Replacing Z_1 by $Z_1[\mu_1]$ where μ_1 is unimodular, and then transforming Z_2. Z_3 with a unimodular matrix μ_2 it results that

$$|Z_1[\mu_1 Q_2 \mu_2]| = \alpha|Z_1[\mu_1 Q_3 \mu_2]| \tag{209}$$

We can assume $\mathcal{U}_1, \mathcal{U}_2$ to be so chosen that $\mathcal{U}_1 Q_2 \mathcal{U}_2 = \begin{pmatrix} E^{(r)} \\ 0 \end{pmatrix}$

Writing $\mathcal{U}_1 Q_3 U_2$ analogously as $U_1 Q_3 U_2 = \begin{pmatrix} R \\ S \end{pmatrix}$ with a square matrix $R = R^{(r)}$, and choosing $Z_1 = \begin{pmatrix} E^{(r)} 0 \\ 00 \end{pmatrix}$ we obtain from (208) that $\alpha |R|^2 = 1$. In particular this means that $|R| \neq 0$ and then we can determine a non-singular ν such that

$$R'^{-1} S' S R^{-1} = \nu' H \nu, \nu = \nu^{(r)}, H = (\delta_{\mu,\nu} h_\nu).$$

The choice $Z_1 = \begin{pmatrix}\lambda v' v 0\\ 0E\end{pmatrix}$ with a variable λ leads by means of (209) to the relations

$$\begin{aligned}\lambda^r |v|^2 &= |Z_1|u_1 Q_2 u_2| = \lambda |Z_1[u_1 Q_3 u_2]|\\ &= \mathcal{L}|\lambda R' v' v R + SS| = |\lambda v' v + R^{-1} SSR^{-1}|\\ &= |\lambda v' v + v' H v| = |\lambda E + H||v|^2 = \prod_{v=1}^{r}(\lambda + h_v)|v|^2\end{aligned}$$

167 Comparing the extremes which provide an identity in λ, we deduce that $H = 0$ and therefore that $S = 0$.

With $Z^* = Z_1[\mathcal{U}_1 Q_2 \mathcal{U}_2]$, (208) yields

$$|Z' + P_2[u_2]| - |Z' + P_3[u_2 R^{-1}]| = \beta |u_2|^2 = \beta_1 \neq 0$$

identically in Z^*. Replacing Z^* by $Z * + P_2[u_2]$ we obtain

$$|Z*| - |Z* - P_1| = \beta_1 \tag{210}$$

identically in Z^* with a certain symmetric matrix P.

Setting $P = W' \begin{pmatrix}E^{(t)} & 0\\ 0 & 0\end{pmatrix} \mathfrak{R}$ and $Z^* = \lambda W' W$ with $|W| \neq 0$ and a variable λ, (210) yields

$$\lambda^r |w|^2 - (\lambda + 1)^t \lambda^{r-t} |w|^2 = \beta_1$$

identically λ. 0f necessity then, $r = t = 1$ and consequently $R^{(r)} = \pm 1$. Since we know that $\alpha |R^2| = L$, this means that $\alpha = 1$. Also $C_2 = C_2^{(r)}$ and $D_2 = D_2^{(r)}$ reduce to pure numbers and from (206).

$$\in_2 |C_2 Z_2 + D_2| \in_3 |C_3 Z_3 + D_3| = 1.$$

Considering the terms of the highest degree in the elements of Z_1 we get that

$$\in_2 C_2 Z_2 |\theta_2| - \in_3 C_3 Z_1 [\theta_3] = 0$$

Also

$$u_1 \theta_3 u_3 = \begin{pmatrix}R\\ S\end{pmatrix} = \begin{pmatrix}R\\ 0\end{pmatrix} = \begin{pmatrix}E\\ 0\end{pmatrix} R = \pm \begin{pmatrix}E\\ 0\end{pmatrix} = \pm u_1 Q_2 u_2$$

so that $Q_3 = \pm Q_2$. The last relation now gives

$$\varepsilon_2 C_2 - \varepsilon_3 C_3 \tag{211}$$

Choosing $Z_1 = 0$ in (206), (210) implies that **168**

$$\varepsilon_2 D_2 - \varepsilon_3 D_3 = 1 \tag{212}$$

Since C_ν, D_ν are integers ($\mathcal{U} = 2, 3$) it follows from (211) - (212) that $\in_2$, $\in_3$ are rational numbers and being roots of unity are therefore equal to ± 1. We can assume without loss of generality that $\in_2, \in_3 = 1$. It is now to be inferred from (206)- (207) that the elements $Z_{\mu\nu}$ of Z are linear functions of the elements $S_{\mu\nu}$ of Z_1 with coefficients all belonging to a finite set of numbers, say

$$Z_{\mu\nu} = a_{\mu\nu} + \sum_{\varrho,\sigma} a_{\mu\nu,\varrho\sigma}\zeta_{\varrho\sigma} \tag{213}$$

Since $Z = Z'$ and $Z_1 = Z_1'$ we may assume in the above that $a_{\mu\nu} = a_{\nu\mu}$ and $a_{\mu\nu\sigma\varrho} = a_{\mu\nu,\varrho\sigma}$. Choosing in (205), $c = 1$, $d = 0$ and $\eta = (q_1 q_2 \cdots q_n)'$ from among a given finite set of primitive columns, and setting $(C_2, D_2) = (C_1, d_1)$ a pair of coprime integers (we have already shown that C_2, D_2 are pure numbers) and $Z_2 = Z_3[\mu_2]$ with $Q_2 = (p_1 p_2, p_n)$, the $p_i' s$ being coprime we conclude from (213) that

$$\sum_{\mu\nu,\varrho\sigma} a_{\mu\nu,\varrho\sigma}\zeta_{\zeta\sigma} q_\mu q_\nu + \sum_{\mu,\nu} a_{\mu,\nu} q_\mu q_\nu = C_1 \sum_{\varrho\sigma} \zeta_{\varrho,\sigma} p_\varrho p_\sigma + d_1 \tag{214}$$

The possible choices for $c_1, c_2, p_\nu (\nu = 1, 2, \ldots n)$ are finite in number so that, by an earlier argument (p. 156) we can assume, in view of the proviso in our Lemma that (214) is an identity in the $\zeta' s$. Then (214) implies that

$$\sum_{\mu} \sum_{\mu,\nu} a_{\mu\nu,\varrho\sigma} q_\mu q_\nu = C_1 p_\varrho p_\sigma \tag{215}$$

and, $C_1 \neq 0$ as we know that rank $C_2 = 1$. This shows that the symmetric matrix $(Q_{\varrho\sigma}) = (\sum_{\mu,\nu} a_{\mu\nu,\varrho\sigma} q_\mu q_\nu)$ has the rank 1 for each primitive column g. Taking a sufficiently large number of g, s, the truth of our **169** above statement for all these will imply that $(Q_{\varrho\sigma})$ has rank 1 identically in g and consequently all sub-determinants of two rows and two columns vanish. In particular, for all ϱ, σ

$$Q_{\varrho\varrho} Q_{\sigma\sigma} = 0 \tag{216}$$

and all these are algebraic conditions.

We shall say polynomials F, G to be equivalent, written $F \sim G$, if they differ only by a constant factor. Let us assume that $Q_{11} \neq 0$ and $Q_{\Re\lambda} \neq 0$. Then from (216), $Q_{\Re k} a_{\lambda\lambda} \neq 0$, $Q_{1\Re} \neq 0$ and $Q_{1\lambda} \neq 0$. Thus all $Q_{\mathscr{H}\Re}$, $a_{\lambda\lambda}$, $a_{1\Re}$, $a_{1\lambda}$ differ from 0.

Two cases arise now.

Case. i: Let Q_{11} have no square divisor. Then from (216) $Q_{11} Q_{Hx} - \varrho^2_{1\Re} = 0$ and it immediate that

$$Q_{11} \sim Q_{1\Re} \sim Q_{HX} \sim Q_{\lambda\lambda} \sim Q_{H\lambda}.$$

This means that $Q_{\varrho\sigma} = C_\varrho C_\sigma Q$ with $Q = \sum_{\mu,\nu} q_\mu q_\nu$ and then, from (213), $Q_{\mu\nu,\varrho\sigma} = C_\varrho C_\sigma$ and $Z_{\mu\nu} = a_{\mu\nu} + C_{\mu,\nu}\xi$ with $\xi = \sum_{\varrho\sigma} C_\varrho C_\sigma \zeta_{\varrho\sigma}$ Since $a_{\mu\nu}, C_{\mu\nu}$ belong to a finite set of numbers, these equations represent a finite number of algebraic surfaces, and the assumptions of our theorem permit us to exclude this case

Case. ii: Let us assume then that Q_{11} assume then that Q_{11} is not square
170 free. Then $Q_{11} \sim L_1^2$, where L, denotes a linear from. Then using (216) $Q_{nn} \sim L^2_\Re$, $Q_{\lambda\lambda} \sim L^2_\lambda$ and $Q_{n\lambda} \sim L_n L_\lambda$ where the $L's$ are again linear forms. By suitably normalizing these forms we can assume that

$$a_{n\lambda} = L_n L_\lambda.$$

Let $L_\Re = \sum\limits_\mu \ell_{\mu n} q_\mu$ (The b' s may be complex constants).

Then

$$Q_{n\lambda} = \sum_{\mu\nu} \ell_{\mu n} \ell_{\nu\lambda} q_\mu q_\nu = \sum_{\mu\nu} a_{\mu\nu,n\lambda} q_\mu q_\nu$$

and being an identity in the q's, this gives

$$a_{\mu\nu,n\lambda} = \ell_{\mu\nu} \ell_{\nu\lambda} \tag{217}$$

In other words
$$Z = Z_1[B] + A \tag{218}$$
with certain matrices $A = (Q_{\mu\nu})$, $B = (\ell_{\mu\nu})$ which belong to a finite set. It remains only to show that B is unimodular and A is integral and then the Lemma would have been established.

Let us choose $y = y_{\mu\nu}$ in (215) and then we obtain
$$a_{\mu\nu,\varrho\sigma} + a_{\mu\nu,\varrho\sigma} + Za_{\mu\nu,\varrho\sigma} = C_{\mu\nu} p_{\mu\nu,\varrho} p_{\mu\nu,\sigma} (\mu \neq \nu)$$
and $a_{\mu\nu,\varrho\sigma} = C_\mu p_\mu p_{\mu\sigma}$ with integers $C_{\mu\nu}$, $p_{\mu\nu\varrho}$, C_μ, $p_{\mu\sigma}$ where $C_{\mu\nu} C_\mu \neq 0$. For $\sigma = \varrho$ these yield in particular, by means of (217),
$$(\ell_{\mu\sigma} + \ell_{\nu\sigma})^2 = C_{\mu\nu} p^2_{\mu\nu\sigma} (\mu \neq \nu)$$
and $\ell^2_{\mu\sigma} = C_\mu p^2_{\mu\sigma}$.

Hence it follows that 171
$$\sqrt{C_{\mu\nu}} p_{\mu\nu\sigma} = \sqrt{C_\mu} p_{\mu\sigma} + \sqrt{C_\nu} p_{\nu\sigma} \mu \neq \nu \tag{219}$$
and squaring,
$$C_{\mu\nu} p^2_{\mu\nu\sigma} = C_\mu p^2_{\mu\sigma} + C_\nu p^2_{\nu\sigma} + 2\sqrt{C_\mu C_\nu} p_{\nu\sigma}, \mu \neq \nu. \tag{220}$$

If at least one product $p_{\mu\sigma} p_{\nu\sigma} \neq 0 (\mu \neq \nu)$ then $C_\mu C_\nu$ is clearly the square of a rational number. This is also true if $p_{\mu\sigma} p_{\nu\sigma} = 0$ for all μ, $\nu (\mu \neq \nu)$ as then $C_{\mu\nu} C_{[} \mu]$ and $C_{\mu\nu} C_{[} \nu]$ are square of rational numbers as is seen from (220). In particular the numbers $c_1, c_2, \ldots c_n$ have all the same sign. Since $Z \in \mathscr{Y}$ and A is a real symmetric matrix, $|Z - A| \neq 0$ and consequently from (218), $|B| \neq 0$. It turns by means of (218) that Z_1 also belongs to a compact subset of $\mathfrak{f}$ as Z does by assumption. This means that all that we have proved above for Z hold also for Z_1 Since we can assume $Z_1 = Z[B'] + A^+$ where $B* = B^{-1}$ in view of the proviso in our lemma, this in particular means that a representation of the kind $B = (\pi_\mu p_{\mu\nu})$ can be found for B^{-1} too, say $B^{-1} = \sqrt{C_\mu} p'_{\mu\nu}$. Since $BB^{-1} = E$, from the above representation for B, B^{-1} we have
$$\sqrt{C_1 C_2 C_n C'_1 C'_2 C_n} |p_{\mu\nu}| |p'_{\mu\nu}| = 1$$

and as all the c's and p's are integers this gives

$$C_1 C_2 \cdots C_n C"_1 C'_2 \cdots C'_n = 1, |p_{\mu\nu}| = \pm 1.$$

172 We know that all c's are of the same sign and therefore $c_1 = c_2 = \cdots\cdots = c_n = 1$.

It is now immediate that B is unimodular.

We now specialize (C, D) in (204) to be $E, 0$ respectively and obtain from (218) that

$$\varepsilon |C_1 Z_1 + D_1| = |Z| = |Z_1[B] + A| = |Z_1 + A[B^{-1}]|$$

This again can be assumed to be an identity in Z_1 and we conclude that $\varepsilon |C_1| = 1$, so that $|C_1| = \pm 1$. Then

$$|Z_1 + A[B^{-1}]| = \varepsilon |C_1 Z_1 + D_1| = Z_1 + C_1^{-1} D_1|$$

Replacing $Z_1 + C_1^{-1} D_1$ in the above by Z_1 we get

$$|Z_1| = |Z_1 + C_1^{-1} D_1 - A[B^{-1}]|$$

identically in Z_1 and as a consequence, $A[B^{-1}] = C_1^{-1} D_1$ or $A = C_1^{-1} D_1 [B_1] C_1$ is a unimodular matrix as we have shown earlier and this means that $C_1^{-1} D_1$ and consequently A are integral. Thus, from (218), $Z = Z_1[u] + A$ with a unimodular u and integral A, in other words Z and Z_1, are equivalent with regard to M. By assumption Z is an interior point of $\mathfrak{f}$ and $Z_1 \in \xi$ and therefore Z and Z_1 coincide. This proves Lemma 18.

We now prove that any maximal set of algebraically independent functions in the sequence $\{\varphi_\mu(Z)\}, \nu = 1, 2, \ldots$, consists of $\dfrac{n(n+1)}{2}$ elements. Let such that a set comprise of the functions $\lambda_q(z) = \varphi_{\nu_a}(z) a, = 1, 2, \ldots$. We first show that the $\lambda'_a s$ are at least $n(n+1)/2$ in number. For if their number is a $q < n(n+1)/2$, then $\lambda\lambda \cdots \lambda_q \varphi_m$ are algebraically
173 dependent for each m. In other words, there exists a polynomial $F_m(t)$ - which we may assume to be irreducible of one variable over the integrityed main generated by $\lambda_a(z), a = 1, 2, \ldots q$, such that $F_m(\varphi_m) = 0$,

$m = 1, 2, \ldots$ identically in Z. These polynomials being separable, their discriminants $D_m(m = 1, 2, \ldots)$ are all polynomials in $\lambda_1, \lambda_2, \ldots \lambda_q$ not vanishing identically. By Lemma 18, it is possible to find a bounded domain $\mathscr{G} \subset \mathfrak{f}$ viz. an open connected non-empty subset of $\mathfrak{f}$ such that the condition $\varphi_m(z) = \varphi_m(z_1)$, $m \geq 1, Z \in \mathscr{G}, z, \in \mathfrak{f}$ will automatically imply that $Z = Z_1$. We now construct a sequence of sub-domains of $\mathscr{G}$ as follows. Let $\mathscr{G}_1 \subset \mathscr{G}$ be so determined that $D_1 \neq 0$ for any $Z \in \mathscr{G}_1$. Then let $\mathscr{G}_2 \subset \mathscr{G}_1$ be such that $D_2 \neq 0$ for any $z \in \mathscr{G}_2$. This may be continued and there exists at least one point Z_o which belongs to every $\mathscr{G}_m$ so that $D_{\mathfrak{R}}(z_o) \neq 0$ for any $\mathfrak{R}$.

Consider now the equations

$$\lambda_o(z) = \lambda_o(Z_o), a = 1, 2, \ldots q, \tag{221}$$

These define an analytic manifold of complex dimension at least $n(n+1)/2 - q > 0$. Thus in $\mathscr{G}$ there exists an analytic curve through Z_o on which (221) is satisfied at every point. It is then immediate that functions $\lambda_a(z)$ are all constants on this curve. This means as a consequence that the functions $\varphi_m(z)$ are also constants on this curve, For the polynomials $Z_m(t)$ have all their coefficients on this curve and being separable their zeros are all distinct. In a sufficiently small neighbourhood of Z_o on this curve we should then have $\varphi_m(z) = \varphi_m(z_o)$ for every m and by the choice of $\mathscr{G}$ which contains this curve we have $Z = Z_1$.In **174**
other words the curve reduces to a point- an obvious contradiction to the assumption that the dimension of the manifold defined by (221) is positive. Consequently the supposition $q < n(n+1)/2$ is not tenable. We now show that the number of $\lambda' s$ cannot exceed $n(n+1)/2$. For if it does, let $q > n(n+1)/2$ and consider the modular forms $\mathfrak{f}_k$, $g_{\nu_a \mathfrak{R}} = \mathfrak{f}_{\mathfrak{R}}^{\nu_a} \lambda_a$, $a = 1, 2, \ldots q$. These are at least $n(n+1)/2 + 2$ in number so that, by theorem 7, these satisfy a non trivial isobaric algebraic relation

$$\sum C_{\mu_1 \mu_2 \cdots \mu_{q+1}} g_{\nu_1 \mathfrak{R}}^{\mu_1} g_{\nu_2 \mathfrak{R}}^{\mu_2} \cdots g_{\nu_q \mathfrak{R}}^{\mu_q} \mathfrak{f}_{\mathfrak{R}}^{\mu_{q+1}} = 0$$

with $\mu_i \geq 0$ and $\sum \mu_1 \nu_1 + \mu_2 \nu_2 + \mu_{q+1} = \lambda$ (λ being a fixed number). From the above relation it is immediate that $\sum C_{\mu_1 \mu_2 \cdots \mu_{q+1}} \lambda_1^{\mu_1} \lambda_2^{\mu_2} \cdots \lambda_q^{\mu_q} =$

$$= \sum C_{\mu_1 \mu_2 \cdots \mu_{q+1}} \left(\frac{g_{\ni_1 \mathfrak{R}}}{\mathfrak{f}_{\mathfrak{R}}^{\nu_1}} \right)^{\mu_1} \left(\frac{g_{\ni_2 \mathfrak{R}}}{\mathfrak{f}_{\mathfrak{R}}^{\nu_2}} \right)^{\mu_2} \cdots \left(\frac{g_{\ni_q \mathfrak{R}}}{\mathfrak{f}_{\mathfrak{R}}^{\nu_q}} \right)^{\mu_q}$$

$$= \frac{1}{\mathfrak{f}_{\mathfrak{R}}^{\lambda}} \Big(\sum C_{\mu_1\mu_2\cdots\mu_{q+1}} g^{\mu_1}_{\nu_1\mathfrak{R}} g^{\mu_2}_{\nu_2\mathfrak{R}} g^{\mu_q}_{\nu_q} \mathfrak{f}^{\mu_{q+1}}_{\mathfrak{R}} \Big) = 0$$

In other words we have established a non trivial relation among the $\lambda' s$ so that cannot be independent - thus providing a contradiction. Thus $q \not< \frac{n(n+1)}{2}$ we conclude that

$$q = \frac{n(n+1)}{2}$$

We are now equipped with the relevant preliminaries needed to establish the main result of this section viz.

175 **Theorem 16.** *The field of modular functions of degree n is isomorphic to an algebraic function field of degree of transcendence $\frac{n(n+1)}{2}$. That is to say, every modular function of degree n is a rational function of $\frac{n(n+1)}{2}+1$ special modular functions. These functions are algebraically dependent but every $\frac{n(n+1)}{2}$ of them are independent.*

Proof. Let $q = \frac{n(n+2)}{2}$ and let

$$\mathfrak{R}_a(z) = \varphi_{\nu_a}(z) = g_{\nu_a\mathfrak{R}}(z)(\mathfrak{f}_{\mathfrak{R}}(z))^{-\nu_a}, a = 1, 2, \ldots, q$$

with $\mathfrak{f}_{\mathfrak{R}}(z) = g_{\mathfrak{R}}(z, o)$ be a set of algebraically independent modular functions. The existence of such a set has already been shown. We first prove that the $(q + 1)$ Eisenstein series $g_{\mathfrak{R}}(z, o), g_{\nu_a\mathfrak{R}}(z, o)$ are algebraically independent. For, let

$$\sum_{\mu=0}^{m} L_\mu(z) = 0 \tag{222}$$

with

$$L_\mu(z) = \sum_{\mu_o\cdots\mu_q} C^{(\mu)}_{\mu_o\mu_1\cdots\mu_q} g^{\mu_o}_{\mathfrak{R}} g^{\mu_1}_{\nu_q\mathfrak{R}} \cdots g^{\mu_q}_{\nu_q\mathfrak{R}}, \mu_c \geq 0,$$

$\mu + \mu_1\nu_1 + \cdots \mu_q\nu_q = \mu$, be any algebraic relation among the $g' s$. We wish to show that such a relation cannot be non trivial. First we shall show that $L_\mu(z) = 0$ for each μ. Applying the modular substitution $Z \to (AZ + B)(CZ + D)^{-1}$ to (222) we obtain

$$\sum_{\mu=0}^{m} |CZ + D|^{\mu|Z} L_\mu(z) = 0 \tag{223}$$

identically in Z for all symmetric coprime pairs $\{C, D\}$. We can choose $C_\nu, D_{\nu_1}\nu = 1, 2, \dots m$ such that $|C_\nu Z + D_n u|$ are all different. □

Then we shall have from (223) a system of linear equations in $L_\mu(Z)$ whose determinant is nonzero. This requires that $L_\mu(Z) = 0$ for $\mu = 0, 1, 2, \dots m$. Thus 176

$$\sum_{\mu_o,\mu_1,\dots\mu_q} C_{\mu_o,\mu_1,\dots\mu_q} g_{\Re}^{\mu_o} g_{\nu_1\Re}^{\mu_1} \cdots g_{\nu_q\Re}^{\mu_q} = 0, \mu = 1, 2, \dots m$$

where

$$\mu_o + \sum_{q=1}^{q} \mu_c \nu_c = \mu$$

Dividing $L_\mu(Z)$ by $g_{\Re}^{\mu} = g_{\Re}^{\mu_o} + \sum_{i=1}^{q} \mu_1\nu_1$ the above yields that $\sum C^{(\Re)}_{\mu_o,\mu_1,\dots\mu_q} \lambda_1^{\mu_1} \lambda_2^{\mu_2} \cdots \lambda_q^{\mu_q} = 0$ and consequently $C^{(\mu)}_{\mu_o,\mu_1,\dots\mu_q} = 0$ for all indices. Thus we have shown that $g_\Re, g_{\nu_a\Re}, a = 1 \cdots q$ are all independent.

Let now $\lambda(Z)$ be an arbitrary modular function and $\lambda(Z) = \dfrac{\mathfrak{f}(Z)}{g(Z)}$ a representation of λ as the quotient two modular forms f, g of the same weight ℓ. Since any $\dfrac{n(n+1)}{2} + 2$ modular forms are algebraically dependent by theorem 7, this is true in particular of the two sets $\mathfrak{f}, g_\Re, g_{\nu_1\Re} \cdots g_{\nu_q\Re}$ and $g, g_\Re, g_{\nu_1\Re}, \dots g_{\nu_q\Re}$ and there exist nontrivial isobaric relations

$$\varphi(\mathfrak{f}) = \sum C_{\mu\nu\varrho_1\cdots\varrho_q} \mathfrak{f}^\mu g_\Re^\mu g_{\nu_q\Re}^{\mu_1} = 0$$
$$\varphi(g) = \sum d_{\mu\nu\varrho_1\cdots\varrho_q} g^\mu g_\Re^\nu g_{\nu_q\Re}^{\mu_q} = 0$$

with all exponents non negative satisfying the condition

$$\mu\ell + (\nu + \rho_1\nu_1 + \cdots + \rho_\nu\nu_\nu)\Re = m\ell\nu_1\nu_2 \cdots \nu_\nu \Re^{q+l}$$

a certain integer $m = m(n)$. It is immediate that 177

$$\mu \leq m\nu_1\nu_2\cdots\nu_\nu\Re^{q+1}$$

so that the polynomials $\varphi(t)$ and $\psi(t)$ are of bounded degree. We can assume without loss of generality that $\psi(0) \neq 0$, $\psi(0) \neq 0$. In $\varphi(t)$ we replace t by tg and let $\psi_1(8) = \varphi(tg)$. The resultant of $\varphi(g)$ and $\psi_1(g)$ (as polynomials in 8) is a polynomial in t denoted by $\sigma(t)$ say. Since $\varphi(0) \neq 0$, $\psi(\varphi \neq 0)$ it follows that $\sigma(0) \neq 0$, in other words, σ does not vanish identically. Further $\varphi(g) = 0$ and $\{\psi_1(g)\}_{t=\lambda} = \varphi(f) = 0$ so that ψ and ψ_1 have a common zero at the point λ. Hence σ has a zero too at this point. The degree of σ is clearly bounded (for varying λ) as the degrees of φ and ψ are bounded. Also, the coefficients of σ are all isobaric polynomials of $g_\Re, g_{\nu_q rk}$, in other words, all of them are modular forms of the same weight. It is clear then as in earlier contexts that, for a suitable $\lambda, g_\Re^{-\lambda}\sigma(t) = \sigma_1$ has coefficients all of which can be written as polynomials in χ_1, λ_q and then σ_1 satisfies the following conditions:

i) the degree of σ_1 is bounded with respect to λ

ii) $\sigma_1 \not\equiv 0$, and specifically, $\sigma_1(0) \neq 0$

iii) $\sigma_1(\lambda) = 0$.

Let now $\lambda = \lambda_o$ be so chosen that it is a zero of an irreducible polynomial $F(t)$ which has all the above properties of $\sigma 1_1$ and whose degree in t is maximal (with regard to λ). The existence of λ_o is easily
178 proved. Then the $q + 1$ modular functions λ_o, λ_1, λ_q generate the field of all modular functions. For if $K = R[\lambda_1, \ldots, \lambda_q]$ is the field generated by $\lambda_1, \ldots, \lambda_q$ and $K[\lambda_o]$ that generated by $\lambda_o, \lambda_1, \ldots \lambda_q$ and if λ is an arbitrary modular function then

$$K \subset K[\lambda_o] \subset K[\lambda_o, \lambda].$$

All these are finite extensions over K and λ being algebraic over K, the last is clearly a simple extension over K, say, $K[\lambda_1]$. Then, unless $K[\lambda_o] = K[\lambda_1^*]$, the irreducible equation over K satisfied by λ_1 will have a degree higher than that of the irreducible equation that λ_o satisfies-a contradiction to the choice of λ_o. Hence $K[\lambda_o] = K[\lambda_o, \lambda^*]$ and consequently λ^* is a rational function of $\lambda_o, \lambda, \ldots, \lambda_q$. The proof of Theorem 16 is now complete.

Chapter 13

Definite quadratic forms and Eisenstein series

A first access to the theory of modular forms of degree n was provided 179
by Siegel's work on quadratic forms. The main result of this theory can be expressed in the shape of an analytic identity between Theta series and Eisenstein series. A brief account of their relationship is given in this section confining ourselves to positive quadratic forms. We may remark that we shall prefer to speak of symmetric matrices rather than of quadratic forms.

We first formulate without proof the main result of Siegel's theory. This requires a few preliminaries. Let $S = S^{(m)}$ and $T = T^{(n)}$ denote positive integral matrices, and let $m \geq n$. Denote by $\alpha(S, T)$ the number of integral $X^{m,n}$ such that $S[X] = T$ and let $\alpha(\varepsilon, S)$ be denoted by $\varepsilon(s)$. By $\alpha_q(S, T)$ we shall mean the number of integral matrices $X^{m,n}$ distinct modulo q such that $S[X] \equiv T(\mod q)$. We shall say that S and T belong to the *same class* if $S[u] = y$ for some unimodular matrix $\mathcal{U}$ and we shall denote by (S) a special matrix in the class of S. If $m = n$ and if to every integer $q > 0$, there exists a pair of integral matrices χ, y such that $S(\chi) \equiv T$ and $T(y) \equiv S(\mod q)$, then we say that S is *related* to T, in symbols: SnT. The set of all matrices related to S shall be called the *genus* of S. It is well known that the genus of every positive integral matrix decomposes into a finite number of distinct classes (If a given genus contains S, it contains the class of S also). We now introduce the measure $\mu(S)$ of the genus containing S as

$$\mu(S) = \sum_{(S_{\Re})n(S)} 1/\varepsilon(S_{\Re}) \tag{224}$$

180 Finally let

$$\alpha_\infty = \frac{\pi^{(2m-n+1)}}{y(\frac{m-n+1}{2})y(\frac{m-n+2}{2})\cdots y(\frac{m}{2})}|S|^{\frac{-1}{2}}|T|^{\frac{m-n-1}{2}} \tag{225}$$

and

$$\alpha_o(S,T) = \frac{1}{\mu(S)}\sum_{(S_{\Re})r(S)}\frac{\alpha(S_{\Re},T)}{\varepsilon(S_{\Re})} \tag{226}$$

The last expression is called the *mean representation number of T by the genus of* S.

We now quote Siegel's main result in

Theorem 17. *Let* $S = S^{(n)}, T = T^{(n)}$ *with* $m > n+1$ *be positive integral matrices. Then*

$$\alpha_o(S,T) = \alpha_\infty(S,T)\lim_{\Re\to\infty}\frac{\alpha_q(S,T)}{\frac{mnn(n+1)}{2}}, q = \Re! \tag{227}$$

We proceed to express thee q-adic representation number $\alpha_q(S,T)$ *by means of the Gauss sums.*

Let $W^{(n)}$ be a rational symmetric matrix and d the smallest positive integer such that the quadratic form $d, W[\varepsilon]$ has all integral coefficients. We shall call such $a''d''$, the *denominator* of $W[\varepsilon]$. We now introduce the *generalized Gauss sum* $g(S,W)$ for any integral symmetric matrix $S^{(m)}$ and rational symmetric matrix $W^{(n)}$ as follows.

$$g(S,W) = \sum_A e^{2\pi i\sigma(S[A]W)} \tag{228}$$

181 where in the summation, $A = A^{(m,n)}$ runs through a complete set of integral matrices which are all distinct mod d, d denoting the denominator

of $W[\varepsilon]$. We have get to show that the sum in (228) does not depend on the choice of the representatives of the cosets mod d. This will be fulfilled if we prove the invariance of this sum for a replacement of A by $A + dB$ where B is an arbitrary integral matrix. In this case we have

$$\begin{aligned}\sigma(S[A+dB]W) - \sigma(S[A]W) &= d^2\sigma(S[B]W) + d\sigma(B'SAW)\\ &\qquad + d\sigma(A'SBW)\\ &= d^2\sigma(S[B]W) + 2d\sigma(B'SAW).\end{aligned}$$

Since $S[B]$ is integral and symmetric and dW is semi integral, it follows that $d, \sigma(S[B]W) \equiv 0(\mod 1)$ and similarly, since $B'SA$ and $2dW$ are integral, $2d\sigma(B'SAW) \equiv 0(1)$. Hence

$$e^{2\pi i\sigma(S[A+dB]W)} = e^{2\pi i\sigma(S[A]W)}$$

and the desired result is immediate.

A useful estimate for $g(S, W)$ when $|S| \neq 0$ is given by

Lemma 19. *Let $S = S^{(m)}$ be a nonsingular integral symmetric matrix, $W = W^{(n)}$ a rational symmetric matrix and d the denominator of the quadratic form $W[\varepsilon]$. Then*

$$|g(S, W| \leq 2^{m/2}||S||^{n/2} d^{mn-m/2} \tag{229}$$

Proof.

$$\begin{aligned}|g(S, W)|^2 &= g(S, W)\overline{g(S, W)}\\ &= \sum_{A_1, A_2 \mod d} e^{2\pi i\sigma(S[A_1]W) - 2\pi i\sigma(S[A_2]W)}\end{aligned}$$

□ 182

Keeping A_2 fixed we first sum over A_1 and this we can do after replacing A_1 by $A_1 + A_2$ We then get

$$\begin{aligned}|g(S, W)^2 &= \sum_{A_1, A_2 \mod d} e^{2\pi i\sigma(S[A_1+A_2]W) - 2\pi i\sigma(S[A_2]W)}\\ &= \sum_{A_1, A_2 \mod d} e^{2\pi i\sigma\{(A_1'S_2 + A_2 S A_1 + S[A_1]W)\}}\end{aligned}$$

$$= \sum_{A_1 \mod d} e^{2\pi i\sigma(S[A_1]W)} \sum_{A_2 \mod d} e^{2\pi i\sigma(2A_2^1 SA_1W)}$$
$$= \sum_{A_1 \mod d} e^{2\pi i\sigma(S[A_1]W)} \sum_{A_2 \mod d} e^{2\pi i\sigma(A_2^1 W_1)} \tag{230}$$

where $W_1 = 2SA_1W$

If $W_1 = (\omega_{\mu\nu})$ and $A_2 = (a_{\mu\nu})$ then

$$\sigma(A_2' W_1) = \sum_{\mu\nu} a_{\mu\nu}\omega_{\mu\nu}$$

and

$$\sum_{A_2 \mod d} e^{2\pi i\sigma(A_2^1 W_1)} = \sum_{a_{\mu\nu} \mod d} e^{2\pi i\Sigma_{\mu,\nu} a_{\mu\nu}\omega_{\mu\nu}}$$
$$= \prod_{\mu,\nu}\Big(\sum_{a_{\mu\nu} \mod d} e^{2\pi i\Sigma_{\mu,\nu} a_{\mu\nu}\omega_{\mu\nu}}\Big)$$

By a standard result, the sum within the parenthesis of d or 0 according as $\omega_{\mu\nu}$ is or is not an integer. Hence, the inner sum in the right side of (230) is 0 if at least one $\omega_{\mu\nu}$ is not integral and d^{mn} in the alternative case. Thus

$$|g(S, W)|^2 = d^{mn} \sum_{\substack{A \mod d \\ 2SA_1W-integral}} e^{2\pi i\sigma(S[A_1]W)} \leq d^{mn} L \tag{231}$$

183 where L denotes the number of cosets $\mod A_1$ such that $2SA_1W$ is integral.

We now wish to estimate L. For this we can assume that S and W are both diagonal matrices. For, in the alternative case, we can find unimodular matrices $\mathcal{U}_l, V_l, i = 12$, with $\mathcal{U}_1 S\mathcal{U}_2 = D = (d_{\mu\nu}d_\nu)$, $V_1WV_2 = H = (\delta_{\mu\nu}R_\nu)$. Then, if $A_3 = \mathcal{U}_2^{-1}A_1V_1^{-1}, A_3$ runs over a complete representative system of matrices $\mod d$ as A_1 does and $2DA_3H$ is integral when and only when $2SA_1W$ is integral. Consequently the replacement of S, W by D, H, viz. diagonal matrices, will not interfere with our estimation for L. We assume then that S and W are diagonal matrices, $S = (\delta_{\mu\nu s_\nu})$, $W = (\delta_{\mu\nu}\omega_\nu)$ and that $2SA_1W$ is integral. If

$A_1 = (a_{\mu\nu})$ this means that

$$a_{\mu\nu} 2d\omega_\nu s_\nu = 2s_\nu a_{\mu\nu} d_\nu \equiv 0 (\mod 1),$$

$$1 \leq \mu \leq m, 1 \leq \nu \leq n$$

Since $2d\omega_\nu$ is an integer, it is obvious that the number of distinct cosets $\mod d$ is just $(d, 2ds_\mu \omega_\nu)$ viz. the greatest common divisor of d and $2ds_\mu\omega_\nu$. Hence

$$L = \prod_{\mu\nu}(d, 2ds_\mu\omega_\nu) \leq \prod_{\mu,\nu}\{(d, 2d\omega_\nu)|s_\mu|\}$$

$$\begin{aligned}\leq \|S\|^n \prod_{\nu=1}^{n}(d, 2d\omega_\nu)^m \leq \|S\|^n\{(d, 2d\omega_1, 2d\omega_2, 2d\omega_2)d^{n-1}\}^n \\ \leq \|S\|^n(2d^{n-1})^m \qquad (232)\end{aligned}$$

In stating the above inequalities we have used the fact that $(g, g_1, \ldots, g_q)g^{q-1} \geq \prod_{\nu=1}^{q}(g, g_n)$ for any $q+1$ integers $g, g_i (l-1, q)$ and this is easily proved by induction on q. Combining (231) and (232) we get **184**

$|G(S, W)|^2 \leq \|S\|^n d^{mn}(2d^{n-1})^m$ and (229) is immediate. We obtain now a representation of the right side of (227) by an infinite series.

Lemma 20. *Let $W = W^{(n)}$ run through a complete set of rational matrices for which the quadratic forms $W[\in]$ are distinct modulo* 1 *and let d denote the denominator of $W[\in]$. Then*

$$\lim_{\Re\to\infty q} \frac{\alpha_q(S, T)}{mn - \frac{n(n+1)}{2}} = \sum_W d^{-mn} g(S, W) e^{-2\pi i\sigma(WT)}, q = \Re! \qquad (233)$$

where $S = S^{(m)} > 0, T = T^{(n)} > o, m > n^2 + n + 2$ and the series on the right side of (233) converges absolutely.

Proof. We first show that

$$q^{\frac{n(n+1)}{2}}\alpha_q(S,T) = \sum_W \sum_A e^{2\pi i\sigma\{(S[A]-T)W\}} \tag{234}$$

where A runs through all integral matrices distinct $\mod q$ and W runs over all rational matrices such that the quadratic forms $W[\in]$ are distinct mod 1 and $qW[\in]$ has all its coefficients integral. Let $S[A]-T = R = (r_{\mu\nu})$ and $W = (\frac{1}{2}(1+\delta_{\mu\nu})\omega_{\mu\nu})$ □

Then

$$\sum_W e^{2\pi i\sigma(RW)} = \sum_{\substack{\omega_{\mu\nu} \mod 1\\ q\omega_{\mu\nu}-\text{ integral}, \mu\leq\nu}} e^{2\pi\sum_{\mu\leq\nu} r_{\mu\nu}\omega_{\mu\nu}}$$

$$= \prod_{\mu\nu}\Big(\sum_{\substack{\omega_{\mu\nu} \mod 1\\ q\omega_{\mu\nu}\equiv 0(1)}} e^{2\pi i r_{\mu\nu}\omega_{\mu\nu}}\Big)$$

185 The sum inside the parenthesis is well known and is equal to q or 0 according as q is or is not a divisor of $q_{\mu\nu}$. It is now immediate that

$$\sum_W e^{2\pi i\sigma(RW)} = \begin{cases} q^{\frac{nm+1}{2}}, if S[A]\equiv T(\mod q)\\ 0 \qquad ,\text{otherwise}\end{cases}$$

The right side of (234), by a change of the order of summation clearly reduces then to $q\dfrac{n(n+1)}{2}\alpha_q(S,T)$ and (234) is established.

$$\text{Since} \sum_{A \mod q} e^{2\pi i\sigma(S[A]W)} = (\frac{q}{d})^n \sum_{A \mod d} e^{2\pi i\sigma(S[A]W)}$$

$$= (\frac{q}{d})^n g(S,W),$$

the right side of (234) can be rewritten as $q^{mn}\sum_W d^{-mn}g(S,W)e^{-2\pi i\sigma(TW)}$. Hence (234) now reads as

$$q^{\frac{n(n+1)}{2}}\alpha_q(S,T) = q^{-mn}\sum_{\substack{W[\varepsilon] \mod 1\\ qE[\varepsilon]\equiv 0(1)}} d^{-mn}g(S,W)e^{-2\pi i\sigma(TW)} \tag{235}$$

This is true for every integral $q = \Re!$ and as $\Re \to \infty$ the right side of (235) is just $\sum\limits_{W[\in] \mod 1} d^{-mn} g(S, W) e^{-\pi i\sigma(TW)}$ provided this infinite series converges. We shall show that this converges absolutely and then we would have proved Lemma 20. We proceed thus the number of quadratic forms $W[\in]$ which are distinct mod 1 and have a given denominator d can be roughly estimated from above by $d^{n(n+1)/2}$. Hence the series under question can be majorised by 186

$$\sum_{d=1}^{\infty} d^{\frac{n(n+1)}{2}} d^{-mn} g(S, W)$$

which again can be further majorised by means of (229) $K \sum_{d=1}^{\infty} d^{\frac{n(n+1)}{2} - \frac{m}{2}}$ with a suitable constant K. The last series is clearly convergent under our assumption on ^{m,n} viz. $m > n^2 + n + 2$ or $\frac{m}{2} - \frac{n(n+1)}{2} > 1$, and then the absolute convergence of the right side of (233) which we wanted to establish is immediate.

We wish to have a partial sum representation for the infinite sum $\sum\limits_{\substack{\tau > 0 \\ integral}} |T|^{\rho - \frac{n+1}{2}} e^{2\pi i\sigma(TZ)} (\rho > n+1)$ in Lemma 21 and as a preliminary, we quote *Poisson's Summation Formula*, viz. that if f is any function defined on the space $\mathscr{L}$ of all symmetric matrices Y, and T runs through all symmetric integral matrices while F runs through all symmetric semi integral matrices, then under suitable conditions,

$$\sum_{T} \mathfrak{f}(T) = \sum_{F} \int\limits_{\mathscr{L}} \mathfrak{f}(y) e^{-2\pi i\sigma(Fy)} [dy] \tag{236}$$

A formal proof of (236) may be furnished as follows.

If $f(y) = \sum_T \mathfrak{f}(T + y)$ then $g(y)$ is a periodic function of y and can be expanded in a Fourier series 187

$g(y) = \sum_F a(F) e^{2\pi i\sigma(Fy)}$ If $\mathcal{H}$ denotes the unit cube in $\mathscr{L}$, then

$\alpha(F) = \int\limits_{H} g(y) e^{-\pi i\sigma(Gy)} [dy]$. Replacing $g(y)$ by its infinite sum $\sum_T \mathfrak{f}$ $(T + Y)$ and changing the order of summation and integration (which can be justified under suitable conditions) we have

$$a(F) = \sum_T \int_T \mathfrak{f}(T+y)e^{-2\pi i\sigma(Fy)}[dy]$$
$$= \sum_t \int_H \mathfrak{f}(T+y)e^{-2\pi i\sigma(F(T+y))}[dy]$$
$$= \sum_T \int_{T+\mathcal{H}} \mathfrak{f}(y)e^{-2\pi i\sigma(Fy)}[dy]$$
$$= \int_{\mathscr{L}} \mathfrak{f}(y)e^{-2\pi i\sigma(Fy)}[dy]$$

Then

$$\sum_T \mathfrak{f}(T) = g(0) = \sum_F a(F) = \sum_F \int_{\mathscr{L}} \mathfrak{f}(y)e^{-2\pi i\sigma(Fy)}[dy]$$

and this establishes (246).

We apply this result in

Lemma 21. *Let $Z \in \mathfrak{f}g_n, \rho > n+1$ and let $T^{(n)}$ run through all positive integral matrices while $F^{(n)}$ runs through all semi integral matrices.*

Then

$$\sum_T |T|^{\rho-\frac{n+1}{2}} e^{2\pi i\sigma(TZ)} = \pi\frac{n(n-1)}{4} y(\rho)y(\rho-\frac{1}{2}).y(\rho-\frac{n-1}{2}) \sum_F |1\pi i(F-Z)|^{-\rho} \tag{237}$$

where we put $|2\pi L(F-Z)|^{-\rho} = e^{-\rho\log|2\pi i(F-Z)|}$ and understand by log $|2\pi L(F-Z)|$ that branch of the logarithm which is real for

188 *Proof.* We first note that since F is real and $Z \in \mathscr{Y}_n, |2\pi l(F-Z)| \neq 0$ so that the right side of (237) makes sense. Let us introduce the function $\mathfrak{f}(y)(y-$ symmetric) as

$$\mathfrak{f}(y) = \begin{cases} |y|p - \frac{n+1}{2} e^{2\pi i\sigma(yz),} & , if y > 0 \\ -0, \text{ otherwise} \end{cases}$$

and state that Poisson's summation formula is valid for this function. ◻

Then
$$\sum_{T} \mathfrak{f}(T) = \sum_{T>0} |T|^{p-\frac{n+1}{2}} e^{2\pi i\sigma(TZ)}$$
which is precisely the left side of (237) and by (236) this is equal to
$$\sum_{F} \int_{\mathscr{L}} \mathfrak{f}(y) e^{-2\pi i\sigma(Fy)}[dy] = \sum_{F} \int_{y>0} |y|^{p-\frac{n+1}{2}} e^{2\pi i\sigma(y(Z-F))}[dy]$$

By Lemma 14 we have
$$\int_{y>0} |y|^{p-\frac{n+1}{2}} e^{2\pi i\sigma(y(Z-F))} = \pi^{\frac{n(n-1)}{4}} y(\rho) y(\rho - \frac{1}{2}) y(\rho - \frac{n-1}{2}) |2\pi i(F-Z)|^{-\rho}$$
and then (237) is immediate. We may note that the convergence of the right side of (237) for $\rho > n+1$ is a consequence of the convergence of the Eisenstein series in this case. The proof of Lemma 21 is complete. In the following Lemma we are concerned with a parametric representation for integral matrices G with a given rank r.

Lemma 22. *Let $1 \leq r \leq n \leq m$ where r, n, m are all integral. Let $B = B^{n,m}$ run over all integral matrices of rank r and $a = a^{n,m}$ run over a complete set of right non associated primitive matrices.*

Then every integral matrix G rank r is obtained as the matrix prod- **189**
uct QB once and only once.

Proof. Let G be an integral matrix of rank n with suitable unimodular matrices $\mathcal{U}_1$, and $\mathcal{U}_2$ and a non-singular matrix $D = D^{|r|}$ we shall have $G = \mathcal{U}_1 \begin{pmatrix} 00 \\ 00 \end{pmatrix} \mathcal{U}_2$. Writing $\mathcal{U}_1 = (Q*)$ where $Q = Q^{(n,r)}$ and $\mathcal{U}_2 = \binom{R}{*}$ where $R = R^{(n,m)}$ we have
$$G = (a*) \begin{pmatrix} D0 \\ 00 \end{pmatrix} \binom{R}{*} = QDR = QB \qquad DR = B$$
□

Since R is primitive and D nonsingular we conclude that B is integral, and by choice Q is primitive. We may observe that we can subsequently replace Q by any given element of the equivalence class of

Q by absorbing the right factor (unimodular) introduced thereby, into B. To prove the uniqueness of the representation $G = QB$ we proceed thus. Let $Q_1 Q_B = Q_2 B_2$ where Q_ν, B_ν are matrices in the same sense as Q, B are. We determine R_1 such that $\mathcal{U} = (Q_1 R_1)$ is unimodular and put $\mathcal{U}^{-1} Q_2 = \binom{A}{H}$ with $A = A^{(r)}$. Then since $\mathcal{U}^{-1} Q_1 B_1 = \mathcal{U}^{-1} Q_2 B_2$ we have on substitution,

$$\begin{pmatrix} E^{(r)} \\ 0 \end{pmatrix} B_1 = \begin{pmatrix} A^{(r)} \\ H \end{pmatrix} B_2 \text{ or } B_1 = AB_2 \text{ and } 0 = HB_2 \text{ Since rank } B_2 = r$$

the last condition implies that $H = 0$ and consequently A is unimodular. Since $Q_2 = \mathcal{U}\binom{A}{0} = (Q_1 R_1)\binom{A}{0}$ it now follows that $Q_2 = Q_1$, $A = E$, and it is immediate that $B_1 = B_2$ by one of the earlier conditions.

Missing page 190

191 We now introduce that ϑ series $\vartheta(SZ)$ for any $S = S^{(m)} > 0$ integral, and $Z \in \mathscr{Y}_n$ with $m \geq n$. By definition

$$\vartheta(S, Z) = \sum_G e^{\pi i \sigma(S[G]Z)} \tag{238}$$

where the summation for G is extended over all integral $G = G^{(m,n)}$. It is obvious that the series in (238) is convergent, and it is also obvious that $\vartheta(S, Z)$ is a *class invariant* of S, in other words $\vartheta(S, Z) = \vartheta(S, Z)$ if S^* is any element of the class of S. Besides the class invariant $\vartheta(S, Z)$, we consider also the *genus invariant*

$$\mathfrak{f}(S, Z) = \frac{1}{\mu(S)} \sum_{(S_{\mathfrak{R}})r(S)} \frac{\vartheta(S_{\mathfrak{R}}, Z)}{\varepsilon(S_{\mathfrak{R}})} \tag{239}$$

This definition of $\mathfrak{f}$ is analogous to the definition (226) of the mean representation number of T by the genus of S.

Now $\vartheta(S, Z)$ can be rewritten as

$$\vartheta(S, Z) = 1 + \sum_{r=1}^{n} \sum_{\substack{G-integral \\ rank G = r}} e^{\pi i \sigma(S[G]Z)}$$

By Lemma 22, G' has a representation of the form $G' = QB'$ where $Q = Q^{(n,r)}$ is a primitive matrix and $B = B^{(m,n)}$ is an integral matrix of rank r. Then

$$\sigma(S[G]Z) = \sigma(S[BQ']Z) = \sigma(S[B]Z|Q)$$

and consequently

$$\vartheta(S, Z) = 1 + \sum_{r=1}^{n} \sum_{Q,B} e^{\pi i \sigma(S[B]Z)[Q]}$$

192 Introducing

$$\chi(S, Z) = \sum_G e^{\pi i \sigma(S[G]Z)} \tag{240}$$

where the sum extends over all integral G with rank $G = n$ (i.e. maximal rank) the above series for $\vartheta(S, Z)$ can be written as

$$\vartheta(S, Z) = 1 + \sum_{r=1}^{n} \sum_{Q} \chi(S, Z[Q])$$

Since

$$\chi(S, Z) = \sum_{G} e^{\pi i \sigma(S[G]Z)} = \sum_{\substack{T^{(n)}>0 \\ integral}} \alpha(S, T) e^{\pi i \sigma(TZ)}$$

it follows that

$$\vartheta(S, Z) = 1 + \sum_{r=1}^{n} \sum_{Q,T} \alpha(S, T) e^{\pi i \sigma(TZ[Q])} \tag{241}$$

$T = T^{(r)} > 0$, integral, $Q = Q^{(n,n)}$ primitive,

and consequently

$$\mathfrak{f}(S, Z) = 1 + \sum_{r=1}^{n} \sum_{Q,T} \alpha(S, T) e^{\pi i \sigma(TZ[Q])}$$

We now apply Siegel's main result (227) and then (233) in an obvious way to obtain that, for $m > n^2 + n + 2$,

$$\mathfrak{f}(S, Z) = 1 + \sum_{r=1}^{n} \sum_{Q,T} \alpha_\infty(S, T) \left(\lim_{\Re \to \nu} \frac{\alpha_q(S, T)}{q^{mn - r(r+\mu/xxxx)}} \right) \times e^{\pi i \sigma(TZ[Q])}$$

$$= 1 + \sum_{r=a}^{n} \sum_{T,Q} \frac{\pi^{\frac{r(2m-r+l)}{4}}}{y(\frac{m-r+1}{2}) y(\frac{m-r+1}{2)\cdots y(\frac{m}{2})}} |S|^{\frac{-r}{2}} |\tau| \frac{m-r}{2}$$

$$\times \sum_{w} d^{mn} g(s.w) e\pi\sigma(TZ[Q] - 2w)_{\substack{w=w^r rations \\ W[\varepsilon] \quad \text{mod } 1}}$$

$$= 1 + \sum_{r=1}^{n} \sum_{Q} \frac{\pi^{\pi(2m(-r+1/4))}}{y(\frac{m-r+1}{2}) y(\frac{m-r+1}{2}) \ldots y(\frac{-r}{2})} |S| \frac{-r}{2}$$

$$\times \sum_{w} \frac{g(S,w)}{\partial mr} \sum_{\tau} \frac{m-r+1}{2} e^{\pi\sigma(\tau i\sigma(Tz[Q]-2w))}$$

the last result being obtained by a change of the order of summation the justification for which will appear subsequently. By means of Lemma 21 we finally get 193

$$f(S,z) = 1+\sum_{r+1}^{n} |S|^{\frac{-r}{2}} \sum_{Q}\sum_{W} \frac{g(S,W)}{d^{mr}} \sum_{F} |\Pi(2f+2w-Z[(Q)])^{y\frac{m}{2}} F = F^{(r)}$$

semi integral

It is obvious that every quadratic form of r variables with rational coefficients has a representation of the form $(F+W)[\in]$above and conversely, so that, in view of $W[\in]$ and $(F+W)[\in]$ having the same denominator d and $g(S,W)$ being invariant for a replacement of W by $(F+W)$ we can finally write

$$f(s,z) = 1+\sum_{r+1}^{n} |S|\frac{-r}{2} \sum_{Q}\sum_{W} \frac{g(S,W)}{d^{mr}} |\pi i(dW - Z[Q]))|^{\frac{-m}{2}} \tag{242}$$

where $W = W^{(r)}$ now runs overall rational matrices. More explicitly we
194 obtain

Lemma 23. *Let $W^{(r)}$ run over all rational symmetric matrices and $Q^{n,r}$ run over a complete set of non right associated primitive matrices. Then*

$$f(s,z) = 1+\sum_{r=a}^{n} c^{\frac{-r}{w}} \sum_{W,Q} d^{\frac{-r}{2}} g(s,w)|z|Q| - 2w|^{\frac{-m}{2}} \tag{243}$$

provided $m > n^2+n+2$ where d denotes the denominator of $W[\in]$.

The reduction from (242) to (243) is straightforward and immediate. It remains only to show that the double series. $\sum\limits_{W,Q} \alpha^{-mr} g(S,W)|Z[Q] - 2W|^{-\frac{m}{2}}$ converges absolutely, to justify our earlier formal manipulations with this series. In the case $r=n$, we have $Q=E$ and this series reduces to $\sum\limits_{W(n)} d^{-mn} g(S,W)|Z-2W|^{-\frac{m}{2}}$. Since Z is fixed and $\|Z-2W\| \geq |Y|$ the absolute convergence of this series can be clearly made to depend on that of $\sum_{W^n} d^{-mn} g(S, W)$ and the latter series does converge absolutely under our assumption $m > n^2 + n + 2$, by means of Lemma 20. We have then only to consider the case $1 \leq r < n$. Given the point, Z, we can always assume Q so chosen that $y[Q]$ is reduced. We can determine a real non-singular matrix $F^{(r)}$ and a diagonal matrix, $H = (\delta_(\mu\nu))$ so that $Z[Q] - 2W = (H + iE)[F]$ and then

$$x[Q] - 2W = h[F] + y[Q] = F'F \text{ and } |Y[Q]| = |F|^2 \tag{244}$$

If $Q = (q\mu\nu) = (\mathscr{G}_1 \mathscr{G}_2 \ldots \mathscr{G})$, $q = \max_{\mu\nu} \pm q_{\mu\nu}$ and $h = \max_\nu \pm h_\nu$ we have from (47-49), **195**

$$|Z|[Q] - 2W = |Y[Q]| \prod_{\nu=q}^{r} (i + h_\nu) \text{ and}$$

$$C_i |Y[Q]| \leq \prod_{[} \nu = 1]^r Y[\mathscr{G}'_\nu, \mathscr{G}_\nu] \tag{245}$$

where C_1, denotes a positive constant depending only upon n and λ denotes the smallest characteristic root of Y. In stating the last of these inequalities we have only to observe that $\mathscr{G}'_\nu \mathscr{G}_\nu \geq 1$. If Q is a given primitive matrix t, a given integer, we ask for the number of quadratic forms $W[\in]$ with rational coefficients and with a given denominator d, consistent with the inequality $t - 1 \leq h \leq t$. From the relations (244) it is easy to see that this number has the upper estimate $(dtq^2)^{r(r+1)/2}$.

By lemma (19)

$$g(s, W) < \ell d^{mn - m/2} \text{ and}$$

$$||Z[Q] - 2w|| = |(y[Q])| \Pi_{\nu=1}^{r} (i + h\nu)| \leq b_1 t |y[Q]| \text{ with}$$

appropriate constants b, b_1, so that the absolute convergence of the series under consideration can be reduced to the convergence of the series .

$$\sum_{d=1}^{\infty} \sum_{t=1}^{\infty} \sum_{Q} (dtq^2)^{r(r+1)/2} d^{-mr} d^{mr - \frac{m}{2}} t^{-\frac{m}{2}} |y[Q]|$$

$$= \sum_{d=1}^{\infty}\sum_{t=1}^{\infty}(dt)^{r(r+1)/2-\frac{m}{2}}\sum_{Q} q^{r(r+1)}|y[Q]|^{-\frac{m}{2}}$$

$$= \left(\sum_{d=1}^{\infty} d^{\frac{r(r+1)}{2}-\frac{m}{2}}\right)^2 \sum_{Q} q^{r(r+1)}|y[Q]|^{-\frac{m}{2}}$$

196 The series within the parenthesis is clearly convergent under our assumption $m > n(n+1)+2 > r(r+1)+2$ so that we need only confine ourselves to the series $\Sigma_Q q^{r(r+s)}|y[Q]|^{\prime\frac{-m}{2}}$. From (245) it is clear that this last series, but for a constant multiplier, is majorised by $\Sigma_Q |y|[Q]|^{\frac{r(r+1)}{2}-\frac{m}{2}}$, and this in turn is further majorised by a constant multiple of

$$\Sigma_{\mathscr{G}_\nu \neq 0}(\prod_{\nu=1}^{r}))^{\frac{r(r+1)}{2}} = (\sum_{\mathscr{G}\neq 1}(\mathscr{G}'\mathscr{G}))^{\frac{r(r+1)}{2}\frac{-m}{2})^r}$$

The number of integral columns $\mathscr{G} \neq 0$ with$t^2 \leq \mathscr{G}'\mathscr{G} < (t+1)^2$ where t is a given integer, is of the order of t^{n-1} so that the last series can be compared with $\sum_{t=1}^{\infty} t^{n-1+r(r+1)-m}$

The series converges since $m > n^2 = (n-1)n + n \geq (r+1)+n$ and the proof of Lemma 23 is complete.

We now transform series (243) for the genus invariant in another form by means of the calculus matrices. Towards this effect, we put $-2W = C_1^{-1}D_1$ where $C_1^{(r)}, D_1^{(r)}$ denote symmetric coprime matrices with $|C_1| \neq 0$. The relation between W and the class $\{C_1, D_1\}$ is bi-unique as we have seen in pp. 44. According to Lemma 1, there is also is a $1-1$ correspondence between the classes $\{C_1^r D_1^r\}, \{Q^{(r,r)}\}$ with $|C_1| \neq 0$, and the classes $C_1^r D_1^r$ with rank $C = r$. Since from (69),

$$|Z[Q]-2W| = |Z[Q]+C_1^{-1} = |C_1^{-1}CZ|Cz+D| \qquad (246)$$

197 we conclude from (243) that

$$f(s,z) = \sum_{\{C^{(n)},D^{(n)}\}} h(c.d)|Czz+D|^{\frac{n}{2}}$$

with appropriate coefficients $h(C, D)$and it only remains to determine useful expressions for $h(C, D)$.

We stop her to establish

Lemma 24. *Let $C_l^{(r)}$ be an integral non-singular matrix and q a positive multiple of $|C_1|$. If $G^{m,r}$ runs through a complete set of integral matrices distinct mod. q the GC_1 runs through exactly $q^{mn}||C_1||^{-m}$ of them , each of these appearing the same number of times. i.e., $||C_1||$ times.*

Proof. We can assume that $C_1 i$ is a diagonal matrix , as otherwise there exists unimodular matrices u_1, u_2 such that the product u_1, Cu_2 is a diagonal matrix C_1^* and then the products $G_1 C_1 = G_1 u_1^{-1} C_1^* u_2^{-1}$ and $G_2 C_1 = G_2 u_1^{-1} C_1^* u_1^{-1}$ are distinct $\mod .q$ for two matrices G_1^*, G_2^* which are themselves distinct $\mod .q$. This in particular implies that for the purpose of determining the number of matrices GC distinct $\mod .q$ it is immaterial whether we argue with C_1 or C_1^*. We therefore assume that C_1 is a diagonal matrix, $C_1 = (\delta_{\mu\nu} C\{\nu\})$, $C_{\{}\nu\} > 0$. Let $G = (g_{\mu\nu})$ and $G^* = (g^*_{\mu\nu})$ and assume $GC_1 \equiv G * C_1 (modq)$. Then $g_{\mu\nu} C_\nu \equiv g_{\mu\nu} C_\nu (\mod .q), \mu = 1, 2, \ldots, m\nu = 1, \ldots r$. □

By assumption $|C_1|$ is a divisor of q and a fortiori , each C_ν is a divisor of q. It therefore follows from the above congruence that 198

$$g_{\mu\nu} \equiv g^*_{\mu\nu} (q|c_\nu) \tag{247}$$

The number of products $G^* C_1$ which are congruent to GC_1 $(\mod \cdot q)$ is then just the number of solutions of the congruence (247) and this number is clearly $\Pi_{\mu\ nu} c_\nu = ||C_1||^m$. Consequently , the number of products GC_1 distinct $\mod \cdot q$ is exactly $q^{mr} ||C_i||^{-m}, a^{mr}$ being the total number of $G = G^{(m,r)}$ distinct $\mod \cdot q$ and the Lemma is proved.

Before resuming the main thread we note the following : -

If A, G are integral matrices , then

$$\begin{aligned}
&\sigma(S[A + GC_1] C^{-1} D_1) p \sigma(S[A] C_1^{-1} D_1) \\
&= \sigma(C_i' G' S G D_i) + \sigma(C_1' G' S A C_1^{-1} D_i) + \sigma(A' S G D_i) \\
&= \sigma(C_i' G' S G D_i) + D_i) + 2\sigma(A' S G D_i) \\
&= \sigma(S[G] G_i C_i') + 2\sigma(A' S G D_i)
\end{aligned} \tag{248}$$

In stating the above relations we have only made use of the fact that $C_1^{-1}D_1$ is a symmetric matrix.

Consider now the sum

$$\sum_{A \mod (2C_i)|} e^{-\pi\sigma(S[A]C_1^{-1}D_1)} \tag{249}$$

199 where the sum is extended over a complete set of integral matrices A such that no two of them differ by a matrix of the form $G2C_1$ having $2C_1$ as a right divisor. It is immediate from (248) that the sum (249) is independent of the choice of representatives A mod.$(2C_1|)$. Two cases arise.

Case. i: Suppose that the congruence $\sigma(s[G]D_iC_i') \equiv 1(\mod .2)$ is solvable for an integral G. Then for that G, a replacement of A by $A + GC_1$ changes the sign of each term of the sum (249) as is seen from (248) while the sum itself is left unaltered due to such a replacement. It therefore follows that in this case

$$\sum A \mod (2C_1)| e^{\pi i\sigma(S[A]C_i^{-1}D_1)} = 0 \tag{250}$$

Case. ii: Suppose on the other hand $\sigma(S[G]D_1C_1') \equiv 0(\mod .2)$ for every integral G. Then the general term of (249) depends only on A mod $(C_\gamma$ and we can write the sum as

$$\sum_{A \mod (2C_i|)} e^{\pi i\sigma(S[A]C_1^{-1}D_1)} = 2^{mr} \sum_{A \mod (2C_i|)} e^{\pi i\sigma(S[A]C_1^{-1}D_1)} \tag{251}$$

Let us now consider the Gaussian sums

$$g(S, w) = g(s, \frac{1}{2}C^{-1}D_i)$$
$$= \sum_{\mod .d} e^{-\pi i\sigma S[A]C_1^{-1}D_i}$$

The denominator d of the quadratic form $W[\varepsilon] = \dfrac{1}{2}(C_i^{-1}D_1)[\varepsilon]$ is obviously a divisor of $2|C_1|$ and let us assume that $|2C_i|$ divides q. Then
200 $A \equiv A^*(\mod .q)$ implies that $A - A^* \equiv (\mod .12C_i|)$ in other words

$(A - A^*)|2C_i| \equiv 0(\mod .1)$ and $A \equiv A^* \mod .(2C_i)|$.

Thus we obtain that

$$g(s, -\frac{1}{2}C_1^{-1}D_i) = \sum_{A \mod .d} e^{\sigma(S[A]C_i^{-1}D_1)} e^{-\pi i\sigma(S[A]C_i^{-1}D_i)}$$
$$= (\frac{d}{q})^{mr} \sum_{A \mod q} e^{-\pi i\sigma(S[A]C_i^{-1}D_1)}$$
$$= (\frac{d}{q})^{mr} \sum_{A \mod (2C_i)} \sum_{\substack{B \mod q \\ B\equiv A \mod (2c_i)|}} e^{\pi i\sigma(S[B]C_1^{-1}D_i)}$$
$$= (\frac{d}{q})^{mr} e^{\pi i\sigma(S[B]C_1^{-1}D_i)} \sum_{A \mod (2C_i)} \sum_{\substack{B \mod q \\ B\equiv A \mod (2c_i)|}}$$

since the term $e^{\pi i\sigma(S[B]C_1^{-1}D_i)}$is invariant for a replacement of B by $B^* \equiv B \mod (2, c_i)|$ and in particular for a replacement of B by A. In order to compute the inner sum we put $B = A + 2GC_i$ with an integral G. If G runs over all cosets $\mod .q$ we obtain according to Lemma 24, $q^{mr}||2C_i^{-m}$ distinct cosets. $B \mod .q$, each one of them $||2C_1||^m$ times. The inner sum is thus equal to $q^{mr}||2C_i||^{-m}$ and we obtain that

$$d^{mr} g(S - \frac{1}{2}C_1^{-1}D_1) = ||2C||^{-m} \sum_{A \mod (2C_1)} e^{\pi i\sigma(S[A]C_1^{-1}D_i)}$$

In view of (250) - (251) it then turns out that **201**

$$d^{mr} g(S - \frac{1}{2}C_1^{-1}D_1) = \begin{cases} 0, or \\ ||C_i||^{-m} \sum_{A \mod (2C_1)} e^{\pi i\sigma(S[A]C_1^{-1}D_i)} \end{cases} \quad (252)$$

according as the congruence $\sigma(S[G]D_iC;_i) \equiv 1(12)$ is solvable for an integral G or not.

From lemma (23) and (246) we now obtain

Theorem 18. *Let $C^{(m)}$, $D^{(r)}$ runs over a complete set of non associated coprime symmetric pairs of matrices and let $m > n^2 + n + 2$. Then*

$$f(S, z) = \sum_{C,D} h(S, C, D)(CZ + D1)^{m/2} \quad (253)$$

where $h(s, oE) = 1$ *and for* $C \neq 0$,

$$h(S, C, D) = \begin{cases} i6mr/2\frac{-r}{2}\|C_i\|\frac{-m}{2} \sum_{A \mod (c_i)}, or e^{\pi i \sigma(S[A]C_1^{-1}D_i)} \\ 0 \end{cases} \tag{254}$$

according as the congruence $\sigma(S[G]D_iC_1') \equiv 1(2)$ *is not or is solvable for an integral matrix G where* $\{C_1^{(r)}D^{(r)_1}\}$ *denotes the unique class which corresponds to* $\{C, D\}$ *by lemma 1.*

We have now expressed Siegel's main theorem on the theory of modular forms in the shape of an analytic identity. It can be shown that $h(S, C, D)$ in (253) depends only on the cosets of C and $D \mod 4|5|$ and consequently

$$f(s, z) = \sum_{\{C,D\} \mod 4|5|} h(S.C.D) \sum_{\substack{\{C,D\} \\ C \equiv C| \mod 4|S|) \\ D \equiv D_c}} |CZ + D|^{-m/2}$$

202 In other words $f(S, Z)$ is expressed as a finite linear combination of the sums

$$\sum_{\substack{\{C.D\} \\ C \equiv C_o D \equiv D_o(\mod 4|5|)}}$$

of the Einstein series which converges for $m > 2n + 2$. The fourth power of these sums represent forms of degree n and weight $2m$ with respect to the *congruence group* $M(4|S|)$ consisting of all modular matrices,

$$M \equiv \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} (\mod .4|S|)$$

The same is also true of the fourth power of the Theta- series $\vartheta(S, z)$

A more detailed account of these results, in particular for $n = 1$, one finds in a paper of Siegel. "Über die analytische theorie der quadratischen formen." Ann. Math. 36(1935), 527 - 606.

Chapter 14

Indefinite Quadratic forms and modular forms

We proceed to investigate how far the results of the last section can be 203
carried over to the case of indefinite quadratic forms. The very first concept we introduced in the last section was that of the representation number $\alpha(S, T)$ which stood for the number of integral solutions of the matrix equation $S[x] = T$ where $S = S^{(m)}$ and $T = T^{(m)}$ are given positive integral matrices. The problem of determining the number of integral matrices $x^{(m,n)}$ satisfying the above equation in the case of an indefinite S is much more difficult. In fact an indefinite rational symmetric matrix S is known to have in general an infinity of units, viz. unimodular matrices u with $S[u] = s$. If now G is one integral solutions of the equation $S[X] = \tau$ so is uG for every unit u of S so that the number of integral solutions of this equation will in general then be no more meaningful, and we therefore replace it by the concept of *representation measures*. If we divide the set of all integral solutions G of $S[G] = T$ into equivalence classes, stipulating that two different $G's$, say G_1, G_2 belong to the same class if and only if $G_2 = uG_1$ for some unit u or S, the resulting number of distinct classes can be shown to be finite. To each to these classes we attach a positive weight in a particular way and then the sum of these weights for all the classes will yield the representations measures $\alpha_A(S, T)$. The following considerations are more general in

so far as they treat the notion of representation measure $\alpha_A(S,T)$ for the
204 set of solutions of the inhomogeneous equation $S[G + A] = T$.

Let $S = S^{(m)}$ be a real non-singular matrix and let (p,q) be the signature of S. That is to say, for a suitable non-singular matrix C we have

$$S = S_o[C] \text{ with } S_o = \begin{pmatrix} -\Sigma^{(p)} & 0 \\ 0 & \Sigma^{(p)} \end{pmatrix}$$

While $(p.q)$ is uniquely determined by S, C is not necessarily unique. Let us set $\mathscr{G} = C \in, \in = (x_u)$ and $\mathscr{G} = (y_m)$

Then we have $S[\in] = -y_1^2 - y_2^2 - -y_p^2 + y_{p+1}^2 + y_{p+1}^2, p + q, p + q = m$ Besides $S[\in]$, we also consider the positive form $P[\in] = y_1^2 + y_2^2 + quady_{p+q}^2$.

We call the symmetric matrix P defined by the above equation, a majorant of S. This majorant is not unique as it evidently depends on the choice of C. However, any majorant P of S is easily seen to be characterized by the relation

$$PS^{-1}P = S, P > 0 \tag{255}$$

The equation (255) is clearly invariant under the simultaneous transformation $S \to S[V]$, $P \to P[V]$ where V is an arbitrary non singular real matrix. It is then immediate that the majorants P_o of S_o defined by

$$P_o S_o^{-1} P_o = S_o, P_o > 0 \tag{255$'$}$$

yield all the majorants P of S in the form $P = P_o[C]$ where C, S_o have the same meaning as before. To determine a parametric representations
205 for the solutions of (255) it therefore suffices to consider the special case (255′). In the border cases $p = 0$ or $q = 0$ we should have clearly $S_o = E$ and $S_o = -E$ respectively so that the only possible solutions of (255) are $P = S$ and $P = -S$ in the respective cases. For $pq > 0$, a parametric representation for the solutions P_o of (255′) is given by

$$P_o = 2K - S_o, K = Y^{-1}[x' S_0], y = S_o[x] > 0 \tag{256}$$

where X is any real matrix of the type $x^{\mu\nu}$ with rank $x = q$ and the solutions of (255) are given by $P = P_o[C]$, P_o, being represented by (256). Writing $x' = (W_1 W_1)$ with and we observe that

$$y = S_o[X] = (W', W_2')S\binom{W_1}{W_2} = W_1' W_1 + W_2' W_2$$

so that the condition $Y > 0$ in (256) in particular implies that $W_2'W_2 > 0$ and consequently $|W_2| \neq 0$. It is easy to see that two matrices X_1, X_2 yield the same point in (256) if and only if $X_2 = X_1 H$ with a non-singular matrix H. We can therefore assume in (256) by replacing X by $XW_2^1 = (\begin{smallmatrix} W \\ E \end{smallmatrix})$ with $W = W_1 W_2^1$ that X is always of the normalized form $X = (\begin{smallmatrix} W \\ E \end{smallmatrix})$ Given, X, P_o is uniquely determined in (256) but not necessarily is the converse true in general. However P_o does determine W uniquely, in other words X is uniquely determined by P_o if we stipulate that X is always of the above normalized form. What is more, the transformation from W to $P = P_o[C]$ can be easily shown to be bi rational. Specially with $x = (\begin{smallmatrix} W \\ E \end{smallmatrix})$ we have

$$P = C \begin{pmatrix} (\frac{E+WW'}{E-WW'} & -2w(E-W-W)^1 \\ -2w(E-W-W)^1 & E+WW'E-WW' \end{pmatrix} C, E - WW > 0 \tag{257}$$

Let $u = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A = A^{(p)}$ be a real matrix which transforms S_o **206**
into itself, i.e., $S_o[u] = S_o$. Then the transformation $X \to uX$ has on P_0 and W the effect

$$P_o \to P_o[u^{-1}], W \to u < w >= (AW + B)(CW + D)^{-1} \tag{258}$$

The transformations $W \to W^* = u < W >$ constitute a group $1 - 1$ transformations of the domains $E - W'W > 0$ onto itself and in this group we have a sub-group formed by the units of S_o viz., those among the $u's$ which are further unimodular. This group of units can be shown to be a discontinuous group acting on the space $E - W'W > 0$ and an immediate question is about a fundamental domain for this group or one of its sub groups and the volume of the fundamental domain in an appropriate sense. Towards this effect we note that the space $E - W'W > 0$ can be considered as a Riemannian space with a metric, invariant relative to the transformation (258), given by

$$\begin{aligned} 8ds^2 &= \sigma(p^{-1}dp)^2 = \sigma(P_o^{-1}dP_o) \\ &= \sigma((E - WW')^{-1}dW(E - W'W)^{-1}dW') \end{aligned} \tag{259}$$

The invariant volume element in this metric can also be computed to be

$$d\vartheta = |E - WW'1^{\frac{m}{2}}[dW] \tag{260}$$

All these considerations are valid for every real matrix S with signature p, q. We now restrict the domain of S to suit our needs.

207 Consider the inhomogeneous equation

$$S[G+A]-T=0, G=G^{(m,n)} \text{ integral } (m \geq n) \tag{261}$$

and require that the left side of this equation should represent an integral valued matrix with the elements of G as variables, via., a matrix whose elements are polynomials in the elements of G with integral coefficients.

This means the following requirements on S, A and T

(i) S is semi integral

(ii) SA is integral for $n>1$ and $2SA$ is integral for $n=1$

(iii) $T \equiv S[A](\mod 1) i.e., S[A]-T$ is integral.

Confining ourselves to the above case, let us introduce the theta series

$$f_A(Z,P)=\sum_{B\equiv(\mod 1)} e^{2\pi i\sigma(S[B]x+iP[B]y)}, z=x+iy \tag{262}$$

where P denotes an arbitrary majorant of S. In the case $p=0$ we should have $P=S$ and then $\mathfrak{f}_A(Z,P)$ represents a theta series in the usual sense. The case $q=0$ presents no new difficulty either so that we confine our attention to the case $pq>0$. Since $\mathfrak{f}_A(Z,P)$ depends only upon the coset $A(\mod 3)$ and $A[\varepsilon]$ has a bounded denominator, it is clear that we only we have a finite number of different series $\mathfrak{f}_A(Z,P)$ corresponding to a given majorant P of S. Let $\mathfrak{f}(Z,P)$ denote the column with the $\mathfrak{f}(Z,P)'s$ as elements in a certain order. Due to an arbitrary

208 unimodular substitution $M=\begin{pmatrix} A & B \\ C & d\end{pmatrix} \in \mathcal{M}$ on $f(Z,P)$, one obtains

$$|CZ+D|^{-\frac{p}{2}}|C\overline{Z}+D|^{-\frac{q}{2}} f(M<Z>,\rho)=\mathcal{L}_M f(Z,P) \tag{263}$$

with a non-singular matrix $\mathscr{L}_M$. The matrices $\mathscr{L}_M$ belong to the modular group in case $p\equiv q\equiv o(\mod 2)$, and (263) is quite meaningful for positive definite S too. Finally, it can be shown that $\mathscr{L}_M$ is the unit matrix for $M \equiv \begin{pmatrix} E & o \\ o & E \end{pmatrix} \pmod{\lambda}$ for a suitable integer λ, so that the matrices $\mathscr{L}_M$ define a representative system for the quotient group $\mathcal{M}/\mathcal{M}(\lambda)$ where $M(\lambda)$ denotes the main congruence group to the level λ i.e. $M(\lambda)$ consists of the substitutions M satisfying the congruence $M \equiv \begin{pmatrix} E & o \\ o & E \end{pmatrix}$ $\pmod{\lambda}$

In the group of units $\mathcal{U}$ of S, viz. unimodular matrices $\mathcal{U}$ with $S[\mathcal{U}] = S$, let $y_A(S)$ be the sub group defined by $\mathcal{U}A \equiv A(\mod 1)$. We denote by $\mathcal{F}_A(S)$ a fundamental domain in $E - W'W > o$ relative to this sub-group and introduce the volume

$$V_A(S) = \int\limits_{\mathcal{F}_A(S)} d\vartheta \tag{264}$$

on $\mathcal{F}_A(S)$ computed with the volume element (260). This volume can be shown to be finite in all cases with one exception-the exceptional case being one for which m is 2 and $\sqrt{-|S|}$ is rational. From (262) it is easily seen that

$$\mathcal{F}_A(Z, P[\mathcal{U}]) = \mathcal{F}(Z, P) \tag{265}$$

for $\mathcal{U} \in y_A(S)$ Therefore it makes sense to form the integral mean value **209**

$$g_A(Z, S) = \frac{1}{V_{A(S)}} \int\limits_{\mathcal{F}_A(S)} f_{\mathfrak{R}}(Z, P) dV \tag{266}$$

This integral certainly exists provided $2n < m - 2$

It is remarkable that the column $\mathscr{G}(Z, s)$ with elements $g_A(Z, s)$ (in some order) also satisfies a transformation formula of the same kind as $\rho(Z, P)$ does in (263).

Specifically we have

$$|Cz + D|^{-\frac{p}{2}} |C\overline{Z} + D|^{-\frac{q}{2}} \mathscr{G}(M < Z >, S) = \mathscr{L}_M \mathscr{G}(Z, S) \tag{267}$$

Developing $g_A(Z, S)$ into a Fourier series there results an expansion which differs from the usual ϑ- series expansion (241) for the definite case only in so far as the representation numbers $\alpha(S, T)$ in (241) are to be replaced by the representation measures, $\alpha_A(S, T)$ and the exponential function $e^{-2\pi\sigma(tY[Q]})$ by certain generalized confluent hyper geometric functions.

If we replace the representation measures in the Fourier series of $g_A(z, s)$ by the products of the p-adic solution densities, consequent on a main theorem of Siegel concerning indefinite quadratic forms, we obtain a new representation of $g_A(z, S)$ which yields a partial fraction decomposition of the kind

$$g_A(Z, S) = \sum_{C,D} h_A(C, D, S)|CZ + D|^{\frac{-p}{2}} |C\overline{Z} + D|^{\frac{-q}{2}} \tag{268}$$

where (C, D) runs over a complete set of non associated symmetric coprime pairs and $h_A(C, D, S)$ denotes a certain sum of the kind (254). As against the definite case we have in (268) not only a means of expressing Siegel's main theorem as an analytic identity but actually it permits us to
210 prove Siegel's result by analytical means. Towards this effect, one has to show that the Eisenstein series on the right of (268) have the same transformation properties relative to the modular substitutions as $g_A(z, S)$ and that the Eisenstein series can be developed into a Fourier series of the same kind as that of $g_A(z, S)$. These facts coupled with some special properties of the generalized confluent hyper geometric function yield, as M.Koecher could prove (unpublished) that the two sides of (268) are identical upto a constant factor, which factor becomes 1 after a suitable normalization.

Now the question presents itself whether functions of the type (268) which no longer depend analytically on the elements of Z can be characterized in any way. As we shall see in the next section, such a characterization is possible by a system of partial differential equations with the invariance properties we have got to require of them. Another fact which appears at the outset in the case $n = 1$ can also be reasonably fused into this differential equation-theory. It concerns the following problem:-

In the case $n = 1$ if we apply the Mellin's transform to Siegel's zeta functions of indefinite quadratic forms with the signature (p, q) we obtain a function of the type

$$\sum_{(C,D)} h_A(C, D, S)|cz + D|^{-\alpha}|c\overline{z} + D|^{-\beta} \tag{269}$$

instead of (268), with certain exponents α, β which satisfy the relations

$$\alpha \equiv P/2, \beta \equiv q/2(\mod 1), d + \beta = 1/2(p + q)$$

We now ask for a process which yields a direct correspondence between the type (268) and (269) without the use of Dirichlit series. We **211**
shall in the following section that such a correspondence can be defined through certain differential operators.

For further details on some of the points raised in this section we refer to

1. H.Maass, Die Differentialgleichungen in der Theorie der elliptiochen Modulfunktionen, Math. Ann. 125(1953), 235-263.

2. C.L.Siegel, On the theory of indefinite quadratic forms, Ann. Math. 45(1944), 577-622.

3. ″ ″ , Indefinite quadratische Formen und Modulfunksionen, Studies Essays, Pres. to Courant, New York 1948, 395-406.

4. ″ ″ , Indefinite quaderatische Formen und Funtionen theoie I, Math. Ann. 124(1951), 17-54.

Chapter 15

Modular Forms of degree n and differential equations

Let $Z = (z_{\mu\nu})$ and $\overline{Z} = (\overline{z}_{\mu\nu})$ and consider the elements $z_{\mu\nu}$, $\overline{z}_{\mu\nu}$ as independent complex variables. We shall require however that $z_{\mu\nu} + \overline{z}_{\mu\nu}$ and $i(z_{\mu\nu}, \overline{Z}_{\mu\nu})$ are real. Let α, β be arbitrary complex numbers. By $|CZ + D|^{-\alpha}$ and $|C\overline{Z} + D|^{-\beta}$ where $Z \in \mathscr{Y}$, and (C, D) represents the second matrix row of a symplectic substitution we always understand the functions $e^{-\alpha log|CZ+D|}$ and $e^{-\alpha \log |C\overline{Z}+D|}$ with the principal value for the logarithm defined by **212**

$$\log z = \log |z| + i \arg z, \log |z| \text{ real }, -\pi < \arg z \leq \pi$$

for complex numbers $z \neq o$

We now ask for differential operators $\Omega_{\alpha\beta}$ which annihilate the Eisenstein series

$$g(Z, \overline{Z}, \alpha, \beta) = \sum_{C,D} h(C, D)|CD + D|^{-\alpha}|C\overline{Z} + D|^{-\beta} \tag{270}$$

for an arbitrary choice of the constants $h(C, D)$ where (C, D) denotes the second matrix row of a modular substitution or more generally of a symplectic substitution of degree, n. Thus we have to require that

$$\Omega_{\alpha\beta}|CZ + D|^{-\alpha}|C\overline{Z} + D|^{-\beta} = o \tag{271}$$

for all pairs (C, D) such that rank $(C, D) = n$, $CD' = DC'$

We shall also demand that the manifold of the functions satisfying

$$\Omega_{\alpha\beta} f(Z, \overline{Z}) = o \tag{272}$$

213 is invariant relative to the transformations

$$f(Z, \overline{Z}) \to f(Z, \overline{Z})|M = |CZ + D|^{-\alpha}|C\overline{Z} + D|^{-\beta} f(M < Z >, M < \overline{Z} >) \tag{273}$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S$ In other words if $f(z, \overline{z})$ is a solution of (271) so is $f(z, \overline{z})|M$. In view of this requirement, (272) now takes a particularly simple form. Prima facie, there is no loss of generality if we assume (271) to hold only for pairs (C, D) with $|C| \neq O$ as the submanifold defined by $|C| = 0$ in the manifold of all $(C, D)'s$ occurring in (271) is one of lower dimension. Then with $P = C_1^{-1} D_1$ we have $P = P'$ and (271) is equivalent with

$$\Omega_{\alpha\beta} |Z + P|^{-\alpha} |\overline{Z} + P|^{-\beta} = o \tag{271}$$

If we further assume that (272) is left invariant by the transformations (273) as we did, and in particular by the transformations $f(z, \overline{z}) \to f(z, \overline{z})|M$ where $M = \begin{pmatrix} E & -P \\ o & E \end{pmatrix}$ then (271) can be further simplified into

$$\Omega_{\alpha\beta} |z|^{-\alpha} |\overline{z}|^{-\beta} = o \tag{274}$$

and this is the equivalent form of (271) we sought for.

In order to construct the operators $\Omega_{\alpha\beta}$ with the desired properties we introduce the matrix operators

$$\frac{\partial}{\partial z} = (e_{\mu\nu} \frac{\partial}{\partial z_{\mu\nu}}), \frac{\partial}{\partial \overline{z}} = (e_{\mu\nu} \frac{\partial}{\partial \overline{z}_{\mu\nu}}) \tag{275}$$

with $e_{\mu\nu} = \frac{1}{2}(1 + \delta_{\mu\nu})$ and

$$K_\alpha = \alpha E + (z - \overline{z}) \frac{\partial}{\partial z} \Lambda_\beta = -\beta E + (z - \overline{z}) \partial / \partial \overline{z} \tag{276}$$

214 Then

$$\Omega_{\alpha\beta} = \Lambda_{\beta-\frac{1}{2}(n+1)} k_{\alpha+\alpha(\beta-\frac{1}{2}(n+1))E} \tag{277}$$

has the desired properties as we shall see presently.

We also show that $\Omega_{\alpha\beta}$ and

$$\tilde{\Omega}_{\alpha\beta} = K_{\alpha-\frac{1}{2}(n+m)} \Lambda_\beta + \beta(d - \frac{1}{2}(n+m))E \tag{278}$$

annihilate the same functions.

We first prove the following formulae.

$$\begin{cases} \Omega_{\alpha\beta} = (Z-\overline{Z})((Z,\overline{Z})\frac{\partial}{\partial Z})' \frac{\partial}{\partial Z} + \alpha(Z-\overline{Z})\frac{\partial}{\partial Z} - \beta(Z,\overline{Z})\frac{\partial}{\partial Z} \\ \tilde{\Omega}_{\alpha\beta} = (Z-\overline{Z})((Z,\overline{Z})\frac{\partial}{\partial Z})' \frac{\partial}{\partial Z} + \alpha(Z-\overline{Z})\frac{\partial}{\partial Z} - \beta(Z,\overline{Z})\frac{\partial}{\partial Z} \end{cases} \tag{279}$$

We may remark here concerning the use of $\frac{\partial}{\partial Z} f(Z,\overline{Z}) = (\frac{\partial}{\partial z_{\mu\nu}} f)$ that as an operator $\frac{\partial}{\partial z_{\mu\nu}} f = f\frac{\partial}{\partial z_{\mu\nu}} + \frac{\partial f}{\partial z_{\mu\nu}}$ while in other places $\frac{\partial}{\partial z_{\mu\nu}} f = \frac{\partial f}{\partial z_{\mu\nu}}$. The meaning of the symbol will be clear from the context.

We take up the proof of (279). We first need to establish the identities

$$\begin{aligned} \frac{\partial}{\partial z}(z-\overline{z}) &= \frac{n+1}{2}E + ((z-\overline{z})\frac{\partial}{\partial z})' \\ \frac{\partial}{\partial \overline{z}}(z-\overline{z}) &= -\frac{n+1}{2}E + ((z-\overline{z})\frac{\partial}{\partial \overline{z}})' \end{aligned} \tag{280}$$

Indeed by a simple transformation we have

$$\begin{aligned} \frac{\partial}{\partial z}(z-\overline{z}) &= (\sum_{\varrho=1}(e_{\mu\varrho}\frac{\partial}{\partial z_{\mu\varrho}}(z_{\varrho\nu}-\overline{z}_{\varrho\nu})) \\ &= (\sum_{\varrho=1}^{n}(z_{\varrho\nu}-\overline{z}_{\varrho\nu})e_{\mu\varrho}\frac{\partial}{\partial z_{\mu\varrho}} + (\sum_{\varrho=1}^{n} e_{\mu\varrho}\delta_{\mu\varrho}) \\ &= (\sum_{\varrho=1}^{n}(z_{\varrho\mu}-z_{\mu\varrho})e_{\varrho\nu}\frac{\partial}{\partial z_{\varrho\nu}}) + \frac{n+1}{2}\delta_{\mu\nu}) \end{aligned}$$

$$= (\sum_{\varrho=1}^{n}(z_{\mu\varrho} - \overline{z}_{\mu\varrho})e_{\varrho\nu}\frac{\partial}{\partial z_{\varrho\nu}}) + \frac{n+1}{2}E$$
$$= ((z-\overline{z})\frac{\partial}{\partial z})' + \frac{n+1}{2}E$$

215 The other half of (280) follows on similar lines.

Now consider $\Omega_{\alpha\beta}$. By means of (280) we have

$$\Omega_{\alpha\beta} = \left\{ -(\beta - \frac{n+1}{2})E + (z-\overline{z})\frac{\partial}{\partial \overline{z}}\right\}\left\{\alpha E + (z-\overline{z})\frac{\partial}{\partial \overline{z}}\right\} + \alpha(\beta - \frac{n+1}{2})E$$
$$= (z-\overline{z})\frac{\partial}{\partial \overline{z}}(z-\overline{z})\frac{\partial}{\partial z} + \alpha(z-\overline{z})\frac{\partial}{\partial \overline{z}} - (\beta - \frac{n+1}{2})(z-\overline{z})\frac{\partial}{\partial z}$$
$$= (z-\overline{z})((z-\overline{z})\frac{\partial}{\partial \overline{z}})'\frac{\partial}{\partial z} + \alpha(z-\overline{z})\frac{\partial}{\partial \overline{z}}) - \beta(z-\overline{z})\frac{\partial}{\partial \overline{z}}$$

which is the first part of (279). The proof of the other part is exactly similar.

We now show that $\Omega_{\alpha\beta}$ annihilates the Eisenstein series (270). A formula proof only requires that it annihilates each term of (270) separately, in other words that (271) is true. Let $A = (a_{\mu\nu})$ be a square matrix the elements of which belong to a commutative ring, and $A_{\mu\nu}$ the algebraic minor corresponding to $a_{\mu\nu}$. With $\tilde{A} = (A_{\nu\mu})$ we have in general $A\tilde{A} = |A|E$.

Then

$$\frac{\partial}{\partial z}|z|^{-\alpha} = (e_{\mu\nu}\frac{\partial}{\partial z_{\mu\nu}}|z|^{-\alpha}) = -\alpha|z|^{-\alpha-1}(e_{\mu\nu}\frac{\partial}{\partial z_{\mu\nu}}|z|)$$
$$= -\alpha|z|^{-\alpha-1}(z_{\mu\nu}) = -\alpha|z|^{-\alpha-1}\tilde{z}$$

216 $\frac{\partial}{\partial \overline{z}}|\overline{z}|^{-\beta}$ can be similarly computed and we have

$$\left.\begin{aligned} \partial/_{\partial z}|z|^{-\alpha} &= -\alpha|z|_{\tilde{z}}^{-\alpha-1} \\ \partial/_{\partial \overline{z}}|z|^{-\beta} &= -\beta|\overline{z}|^{-\beta-1}\tilde{z} \end{aligned}\right\} \tag{281}$$

Since from (279)

$(z-\overline{z})^{-1}\Omega_{\alpha\beta} = ((z-\overline{z})\partial/_{\partial\overline{z}})'\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial\overline{z}} - \beta\frac{\partial}{\partial z}$ it follows by means of (281) that

$$(z-\overline{z})^{-1}\Omega_{\alpha\beta}|z|^{-\alpha}|\overline{z}|^{-\beta} = -\alpha\beta|z|^{-\alpha}|\overline{z}|^{-\beta-1}\tilde{z} + \alpha\beta|z|^{-\alpha-1}|\overline{z}|^{-\beta}\tilde{z} - \alpha((z-\overline{z})\frac{\partial}{\partial\overline{z}})'|z|^{-\alpha-1}\tilde{z}|\overline{z}|^{-\beta}.$$

The last term on the right can be rewritten as

$-\alpha((z-\overline{z})\frac{\partial}{\partial\overline{z}}|z|^{-\beta})'|z|^{-\alpha-1}\tilde{z}$ with the terms outside the parenthesis not depending on $\overline{z}$. Using (281) this is easily seen to be equal to $\alpha\beta|z|^{-\alpha-1}|\overline{z}|^{-\beta-1}\overline{z}(z-\overline{z})\tilde{z}$

Thus

$$(z-\overline{z})^{-1}\Omega_{\alpha\beta}|z|^{-\alpha}|\overline{z}|^{-\beta} = -\alpha\beta|z|^{-\alpha}|\overline{z}|^{-\beta-1}\overline{z} + \alpha\beta|z|^{-\alpha-1}|\overline{z}|^{-\beta}\tilde{z} + \alpha\beta|z|^{-\alpha-1}|\overline{z}|^{-\beta-1}\tilde{Z}(z-\overline{z})\tilde{z}$$

and the right side is now easily seen to vanish, there by establishing **217**
(271). Since $((z-\overline{z})\frac{\partial}{\partial\overline{z}})'\frac{\partial}{\partial z}$ is the transpose of $((z-\overline{z})\frac{\partial}{\partial z})'\frac{\partial}{\partial\overline{z}}$ we verify without difficulty that

$$\tilde{\Omega}_{\alpha\beta} = (z-\overline{z})((z-\overline{z})^{-1}\Omega_{\alpha\beta})'$$

and this says that $\Omega_{\alpha\beta}$ and $\tilde{\Omega}_{\alpha\beta}$ annihilate the same functions as we wanted them to do.

We have still to prove the invariance of the manifold formed by the solutions of

$$\Omega_{\alpha\beta}\mathscr{F}(z,\overline{z}) = o \qquad (272)'$$

under the transformations

$$f(z,\overline{z}) \to f^{*}(z,\overline{z}) = |cz+D|^{-\alpha}|c\overline{z}+D|^{-\beta}f(M<z>M<\overline{z}>) \qquad (273)'$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{S}$. The proof is rather long and mainly consists in securing the following operator identity:

$$|cz+D|^{-\alpha}|c\overline{z}+D|^{-\beta}\Omega^{*}_{\alpha\beta}|cz+D|^{\alpha}|c\overline{z}+D|^{\beta} = (zc'+D')^{-1}((cz+D)\Omega'_{\alpha\beta})' \qquad (282)$$

where $\Omega^*_{\alpha\beta}$ is the operator which results from $\Omega_{\alpha\beta}$ on replacing z, $\overline{z}$ by $Z^* = M < Z >$, $\overline{Z} = M < \overline{z} >$ respectively.

Assuming (282) for a moment, to obtain our main result, we argue as follows:-

We have to show that, under (272), $\Omega_{\alpha\beta}\mathfrak{f}^*(z, \overline{z} = o$

This is equivalent to showing that the right side of (282), or equivalently, the left side annihilates $\mathfrak{f}^*(z, \overline{z}$.

218 The left side of (282) applied to $f^*(z, \overline{z}$ gives

$$|cz + D|^{-\alpha}|c\overline{z} + D|^{-\beta}\Omega^*_{\alpha\beta} f(z^*, \overline{z}^*)$$

and this clearly vanishes under (272). This proves the desired result. We have then only to establish (282).

By means of the general rule relating to any three square matrices M_1, M_2, M_3 which are such that the elements of M_1. commute with those of M_2, viz.

$$\left.\begin{array}{r}(M_1M_2)' = M_2'M_2', (M_1(M_2M_3)')' = M_2(M_1M_3')' \\ \sigma(M_1M_2) = \sigma(M_2M_1)\end{array}\right\} \tag{283}$$

It is possible to reduce the proof of (282) to showing that

$$\begin{cases}((cz + D)\frac{\partial}{\partial z})'|cz + D|^{\alpha} = |cz + D|^{\alpha}((cz + D)\frac{\partial}{\partial z})' + \alpha|cz + D|^{\alpha}c' \\ ((cz + D)\frac{\partial}{\partial z})'|c\overline{z} + D|^{\beta} = |c\overline{z} + D|^{\beta}((c\overline{z} + D)\frac{\partial}{\partial z})' + \beta|c\overline{z} + D|^{\beta}c'\end{cases} \tag{284}$$

It suffices of course to establish the first part of (284) and that too under the assumption $|C| \neq 0$ as in the alternative case, the corresponding $(C, D)'s$ form a sub-manifold of lower dimension as stated earlier.

In this case, (284) can be reduced to its special form corresponding to $C = E, D = o$ viz.

$$(z\frac{\partial}{\partial z})'|z|^{\alpha}(z\frac{\partial}{\partial z}) + \alpha|z|^{\alpha}E$$

or equivalently

$$(z\frac{\partial}{\partial z})'|z|^{\alpha}(z\frac{\partial}{\partial z}) + \alpha|z|^{\alpha}E$$

219 by a substitution of the form $Z \to Z + S$ with an appropriate symmetric matrix S, and the last relation is immediate by a proof analogous to that of (281).

To deduce (282) as a consequence of (283) and (284) one need first to determine the transformation properties of the operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ relative to symplectic substitutions. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ denote a symplectic matrix and let

$$z^* = (AZ + B)(cz + D)^{-1}, \bar{z}^* = (A\bar{Z} + B)(c\bar{z} + D)^{-1}$$

From (122) we deduce that

$$\alpha z^* = (zc' + D')^{-1} dz (cz + D)^{-1}, \alpha \bar{z}^* = (\bar{z}c' + D')^{-1} d\bar{z}(c\bar{z} + D)^{-1}$$

If $f = f(z, \bar{z})$ is an arbitrary function of z, $\bar{z}$ its total differential αf can be represented in the form

$$df = \sigma(dz \frac{\partial}{\partial z} f) + \sigma(d\bar{z} \frac{\partial}{\partial \bar{z}} f) \tag{285}$$

on the one hand, and on the other,

$$\begin{aligned} df &= \sigma(dz^* \frac{\partial}{\partial z^*} f) + \sigma(d\bar{z} \frac{\partial}{\partial \bar{z}^*}, f) \\ &= \sigma\Big\{dz(cz + D)^{-1}(\frac{\partial}{\partial z} f)(zc' + D')^{-1}\Big\} \\ &\qquad + \sigma\Big\{d\bar{z}(c\bar{z} + D)^{-1}(\frac{\partial}{\partial \bar{z}^*} f)(\bar{z}c' + D')^{-1}\Big\} \end{aligned}$$

Comparing the last with (285) one deduce that

$$\begin{aligned} \frac{\partial}{\partial z} f &= (cz + D)^{-1}(\frac{\partial}{\partial z^*} f)(zc' + D')^{-1} \\ \frac{\partial}{\partial \bar{z}} f &= (c\bar{z} + D)^{-1}(\frac{\partial}{\partial \bar{z}^*} f)(\bar{z}c' + D')^{-1} \end{aligned}$$

Consequently we have the operator identities

$$\left.\begin{aligned} \frac{\partial}{\partial z^*} &= (cz+D)((cz+D)(\frac{\partial}{\partial z})' \\ \frac{\partial}{\partial \overline{z}^*} &= (c\overline{z}+D)((c\overline{z}+D)(\frac{\partial}{\partial \overline{z}})' \end{aligned}\right\} \tag{286}$$

220 With these preliminaries we take up the proof of (282). For convenience we shall denote the product $|cz+D|^{\alpha}|c\overline{z}+D|^{\beta}$ by φ. Then by means of (283), (284), (286) and (122) we have

$$\Omega^*_{\alpha\beta}\alpha = \left\{(z^*-\overline{z}^*)(z^*-\overline{z}^*)(\frac{\partial}{\partial\overline{z}^*})'\frac{\partial}{\partial Z^*} + \alpha(z^*-\overline{z}^*)\frac{\partial}{\partial\overline{z}^*} - \beta z^*-\overline{z}^*)\frac{\partial}{\partial\overline{z}^*}\right\}$$

$$\begin{aligned}
=&(zc'+D')^{-1}(z-\overline{z})(c\overline{z}+D)^{-1}\left\{(z-\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})'\right\}((cz+D)\frac{\partial}{\partial\overline{z}})'\alpha \\
&+\left\{\alpha(zc'+D')^{-1}(z,\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})' - \beta(\overline{z}c' \right. \\
&\qquad\qquad\qquad\qquad \left. + D'^{-1}(z-\overline{z})((cz+D)\frac{\partial}{\partial\overline{z}})\right\}\alpha \\
=&|cz+D|^{\alpha}(zc'+D')^{-1}(z,\overline{z})(c\overline{z}+D)^{-1} \\
&\qquad\qquad \left\{(z,\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})'\right\}|c\overline{z}+D|^{\beta}((cz+D)\frac{\partial}{\partial z}) \\
&+\alpha|cz+D|^{\alpha}(zc'+D')^{-1}(z,\overline{z})(c\overline{z}+D)^{-1}\left\{(z,\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})'\right\}|c\overline{z}+D|^{\beta}c' \\
&+\alpha\varphi(zc'+D')^{-1}(z,\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})' + \alpha\beta\varphi(zc'+D')^{-1}(z,\overline{z})c' \\
&-\beta\varphi(zc'+D')^{-1}(z,\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})' \\
&-\alpha\beta\varphi(\overline{z}c'+D')^{-1}(z,\overline{z})c' \\
&=\varphi(zc'+D')^{-1}(z,\overline{z})(c\overline{z}+D)^{-1}\left\{(z-\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})'\right\}((cz+D)\frac{\partial}{\partial\overline{z}})' \\
&+\beta\varphi(zc'+D')^{-1}(z,\overline{z})(c\overline{z}+D)(c(z-\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})' \\
&+\alpha\varphi(zc'+D')^{-1}(z,\overline{z})(c\overline{z}+D)\left\{(c(z-\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})'\right\} \\
&+\alpha\beta\varphi(zc'+D')^{-1}(z,\overline{z})((c\overline{z}+D)^{-1}c(z,\overline{z})c' \\
&+\alpha\psi(zc'+D')^{-1}(z,\overline{z})((c\overline{z}+D)\frac{\partial}{\partial\overline{z}})' + \alpha\beta\varphi(zc'+D')^{-1}(z-\overline{z})c' + .. \\
&-\beta\varphi(\overline{z}C'+D')^{-1}(z-\overline{z})((cz+D)\frac{\partial}{\partial z})' - \alpha\beta\varphi(\overline{z}C'+D')^{-1}(z-\overline{z})C'
\end{aligned}$$

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Collecting the like terms together it is seen that the terms involving $\alpha\beta\varphi$ together and to zero while those involving φ, $\alpha\varphi$ and $\beta\varphi$ by themselves, all survive. These terms ultimately turn out to be respectively

$$(ZC' + D')^{-1}(Z - \overline{Z})((z - \overline{Z})\frac{\partial}{\partial z})'((CZ + D)\frac{\partial}{\partial z})',$$

$$\alpha\varphi(ZC' + D')^{-1}(z - \overline{Z})((CZ + D)\frac{\partial}{\partial z})' \text{ and}$$

$$-\beta\varphi(ZC' + D')^{-1}(Z - \overline{Z})((CZ + D)\frac{\partial}{\partial z})'$$

Thus

$$\Omega^*_{\alpha\beta}\varphi = \varphi(zc' + D'^{-1}, (z - \overline{z})\left\{((z - \overline{z})\frac{\partial}{\partial \overline{z}})' + \alpha((cz + D)\frac{\partial}{\partial \overline{z}})' - \beta((cz + D)\frac{\partial}{\partial \overline{z}})'\right\}$$

It then follows that

$$\left\{cz - \overline{z})^{-1}(zc' + D', \zeta_{\alpha\beta}\varphi\right\}' = \varphi(cz + D)\left\{((z - \overline{z})\frac{\partial}{\partial z}\right\}' +$$

$$+\alpha\varphi(cz + D)\frac{\partial}{\partial \overline{z}} - \beta\varphi(cz + D)\frac{\partial}{\partial z}$$

$$= \varphi(cz + D)\left\{((z\overline{z})'\frac{\partial}{\partial z})'\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial z} - \beta\frac{\partial}{\partial z}\right\}'$$

$$= \varphi(cz + D)\left\{(z - \overline{z})^{-1}\Omega_{\alpha\beta}\right\}'$$

Consequently

$$\left[(cz + D)^{-1}\left\{(z - \overline{z})^{-1}(zc' + D')\Omega^*_{\alpha\beta}\varphi\right\}'\right]' = \varphi(z - \overline{z})^{-1}\Omega_{\alpha\beta}$$

or

$$\left[(cz + D)^{-1}\left\{(zc' + D')\Omega^*_{\alpha\beta}\varphi\right\}'\right]' = \varphi\Omega_{\alpha\beta}$$

Hence finally 222

$$\varphi^{-1}\Omega^*_{\alpha\beta}\varphi = (zc' + D')^{-1}\left((cz + D)\Omega'_{\alpha\beta}\right)'$$

and this precisely is the assertion of (282). The proof is now complete.

Let us introduce the variables X, Y as $x = \frac{1}{2}(z + \overline{z})$ and $\mathcal{Y} = \frac{1}{2\ell}(Z, \overline{Z}$ and define $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ as in (275) with $Z = (z_\mu)$ replaced by $X = (x_{\mu\nu})$ and $\mathcal{Y} = (y_{\mu\nu})$ respectively. One then obtains the transformation formulae

$$\left.\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{2}(\frac{\partial}{\partial x} - \frac{\partial}{\partial \mathcal{Y}}), \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{\partial}{\partial \mathcal{Y}}),\\ \frac{\partial}{\partial z_{\mu\nu}} &= \frac{1}{2}(\frac{\partial}{\partial x_{\mu\nu}} - \ell\frac{\partial}{\partial y_{\mu\nu}}), \frac{\partial}{\partial z_{\mu\nu}} = \frac{1}{2}(\frac{\partial}{\partial x_{\mu\nu}} + \ell\frac{\partial}{\partial y_{\mu\nu}})\end{aligned}\right\} \tag{287}$$

Consider the operator Δ defined by

$$\Delta = -\sigma(z, \overline{z})\Big((z, \overline{z})\frac{\partial}{\partial \overline{z}}\Big)' \frac{\partial}{\partial z} \tag{288}$$

In terms of X and Y, Δ takes the form

$$\Delta = -\sigma\Big\{(\mathcal{Y}(y\frac{\partial}{\partial \overline{x}})' \frac{\partial}{\partial x} + \mathcal{Y}(\mathcal{Y}\frac{\partial}{\partial \mathcal{Y}})' \frac{\partial}{\partial \mathcal{Y}})\Big\} \tag{289}$$

This is immediate from the relations

$$-\sigma(z - \overline{z})\Big((z - \overline{z})\frac{\partial}{\partial \overline{z}}\Big)' \frac{\partial}{\partial z} = \sigma\mathcal{Y}\Big(\mathcal{Y}(\frac{\partial}{\partial x} + \mathcal{Y}\frac{\partial}{\partial y})\Big)'\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial \mathcal{Y}})\Big)$$

$$= \sigma\Big\{\mathcal{Y}(\mathcal{Y}\frac{\partial}{\partial x})' \frac{\partial}{\partial x} + \mathcal{Y}(\mathcal{Y}\frac{\partial}{\partial y})' \frac{\partial}{\partial y}\Big\} + \sigma\Big\{\mathcal{Y}(\mathcal{Y}\frac{\partial}{\partial y})' \frac{\partial}{\partial x} - \mathcal{Y}(\mathcal{Y}\frac{\partial}{\partial x})\frac{\partial}{\partial y}\Big\}$$

and the fact that

$S = (\mathcal{Y}\frac{\partial}{\partial \mathcal{Y}})' \frac{\partial}{\partial x}(\mathcal{Y}\frac{\partial}{\partial x})' \frac{\partial}{\partial y}$ is a skew symmetric matrix so that $\sigma(\mathcal{Y}S) = o$

223 One interesting fact about Δ is its invariance relative to the symplectic substitutions. Let the substitution $Z, \overline{Z} \to Z^*, \overline{Z}^*$ carry Δ into Δ^* where $Z^* = (AZ + B)(CL + D)$ and $\overline{Z}^* = (A\overline{Z} + B)(C\overline{Z} + D)^{-1}$ with $\begin{pmatrix} AB \\ CD \end{pmatrix} \in S$. We observe from (122) that

$$Z^* - \overline{Z}^* = (ZC^1 + D')^{-1}(Z, \overline{Z})(C, \overline{Z} + D)^{-1}$$

$$= (\bar{Z}C' + D)^{-1}(Z, \bar{Z})(CZ + D)^{-1} \qquad (122)'$$

Consequently, by means of (286) and (283) we have

$$\begin{aligned}
\Delta^{*}\tilde{f} &= -\sigma(Z^{*} - \bar{Z}^{*})((Z^{*}\bar{Z}^{*})\frac{\partial}{\partial \bar{Z}^{*}})\frac{\partial}{\partial Z^{*}}\tilde{f} \\
&= -\sigma(ZC + D)^{-1}(Z - \bar{Z})(C\bar{Z} + D)^{-1}\Big\{(C\bar{Z} + D')^{-1}(Z - \bar{Z}) \\
&\qquad ((C\bar{Z} + D)\frac{\partial}{\partial \bar{Z}})'\} \times \{(CZ + D)(\frac{\partial}{\partial z}\tilde{f})(ZC' + D')\Big\} \\
&= -\sigma(Z - \bar{Z})(C\bar{Z} + D)^{-1}\Big\{(ZC' + D')^{-1}(Z - \bar{Z})((C\bar{Z} + D)\frac{\partial}{\partial \bar{z}})'\Big\} \\
&\qquad \times \Big\{(CZ + D)(\frac{\partial}{\partial z}\tilde{f})\Big\} \\
&= -\sigma(Z - \bar{Z})((ZC' + D')^{-1}(Z - \bar{Z})\frac{\partial}{\partial Z})'(CZ + D)(\frac{\partial}{\partial z}\tilde{f}) \\
&= -\sigma(Z - \bar{Z})((ZC' + D')^{-1}(Z - \bar{Z})\frac{\partial}{\partial z}\tilde{f}) = \Delta\tilde{f}
\end{aligned}$$

and it is immediate that $\Delta^{*} = \Delta$

We with to identify Δ with the Laplace Beltrami operator associated with the symplectic metric. We may recall that the Laplace Beltrami operator in a Riemannian space with co-ordinate systems $\chi^{1}, \chi^{2} \cdot \chi^{N}$ and the fundamental metric form $ds^{2} = \sum_{\mu,\nu} g_{\mu,\nu} d\chi^{\mu} d\chi^{\nu}$ is given by $\sum_{\mu,\nu} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi^{\mu}}\left(\sqrt{g} g^{\mu\nu} \frac{\partial}{\partial \chi^{\nu}}\right)$ where $g = g(g_{\mu\nu})$ and $(g_{\mu\nu})(g^{\mu\nu}) = E$. With **224**
respect to a geodesic co-ordinate system at a given point, this operator takes in this point the simple form

$$\sum_{\mu,\nu} g^{\mu,\nu} \frac{\partial^{2}}{\partial \lambda^{N} \partial \chi^{\nu}} \qquad (290)$$

and can be easily computed. The Laplace Beltrami operator in any Riemannian space is invariant relative to the movements of the space. In particular the Laplace Beltrami operator associated with the symplectic metric is invariant under symplectic movements. As this has been shown to be true of Δ also, to prove the equivalence of the two, it suffices to verify that they define the same operator at a special point. We choose this special point to be $Z = (E = \overline{Z})$. At this point let us introduce the co-ordinate system $W, \overline{W}$ by $W = (Z-1E)(Z+1E)^{-1}$, $\overline{W} = (\overline{Z}+iE)(\overline{Z}-(E)^{-1}$ which is a geodesic one relative to the symplectic metric. Analogous (286) we have

$$\frac{\partial}{\partial Z} = \frac{-1}{Z}(W-E)\Big((W-E)\frac{\partial}{\partial W}\Big)' \frac{\partial}{\partial \overline{Z}} = \frac{1}{2}(\overline{W}-E)\Big((\overline{W}-E)\frac{\partial}{\partial \overline{W}}\Big)' \quad (291)$$

It terms of W, $\overline{W}$ we have

$$\Delta = \sigma(E - W\overline{W})\Big((E - W\overline{W})\frac{\partial}{\partial \overline{W}}\Big)' \frac{\partial}{\partial W}$$

The special point $Z = iE = -\overline{Z}$ transforms into $W = 0 = \overline{W}$ and to compute the Laplace Beltrami operator at this point, one needs only to determine the coefficients of the fundamental metric form

$$ds^2 = 4\sigma\Big(dW(E - W\overline{W})^{-1} d\overline{W}(E - W\overline{W})^{-1}\Big)$$

225 at this point. Then in (290) one obtains the same operator as Δ. We proceed to consider modular forms associated with arbitrary sub-groups of the symplectic group. Let G be a group of symplectic substitutions and $\nu(M)$ a *multiplicator system* of $G(M \in G)\alpha, \beta$ being arbitrary complex numbers, a function $\mathfrak{f}(Z, \overline{Z})$ is said to be a modular form of the type $(G, \alpha, \beta\vartheta)$

if

(i) $\mathfrak{f}(Z, \overline{Z})$ is regular in the domain where $Z + \overline{Z}$ is real and $\dfrac{1}{2i}(Z - \overline{Z})$ is real and positive

(ii) $\Omega_{\alpha\beta}\mathfrak{f}(Z, \overline{Z}) = 0$

(iii)
$$\mathfrak{f}(Z, \overline{Z}|M = \nu(M)\mathfrak{f}(Z, \overline{Z}), M \in G, \quad (292)$$

Some restriction on the behaviour of $\mathfrak{f}(Z,\overline{Z})$ on the boundary of $\mathscr{Y}$ is perhaps necessary but we assume none. We shall denote by $\{G,\alpha,\beta,\nu\}$ the linear space of all modular forms of the type (G,α,β,ν). It is an easy consequence of our definition that for every symplectic substitution M,

$$\{G,\alpha,\beta,\nu\}|M = \{M^{-1}GM\alpha,\beta,\nu^*\} \tag{293}$$

with an appropriate multiplicator system ν^* depending on M, where the left side just means the set of all $\mathfrak{f}(Z,\overline{Z})|M$ for $\mathfrak{f}(Z,\overline{Z}) \in \{G,\alpha,\beta,\nu\}$

We would like to treat the question, whether it is possible to set up a correspondence between $\{G,\alpha,\beta,\nu\}$ and $\{G,\alpha\pm 1,\beta\mp 1,\nu\}$. Such a correspondence, we shall see, will be defined by certain differential 226
operators. Consider the Eisenstein series (270), viz.

$$g(Z,\overline{Z},\alpha,\beta) = \sum_{C,D} h(C,D)|CZ+D1^{-\alpha}|C\overline{Z}+D|^{-\beta} \tag{270$'$}$$

and introduce differential operators M_α, N_β with the properties

$$\left.\begin{aligned} M_\alpha g(Z,\overline{Z},\alpha,\beta) &= \varepsilon_n(\alpha) g(Z,\overline{Z},\alpha+1,\beta-1) \\ N_\beta g(Z,\overline{Z},\alpha,\beta) &= \varepsilon_n(\beta) g(Z,\overline{Z},\alpha+1,\beta-1) \end{aligned}\right\} \tag{294}$$

where

$$\varepsilon_n(\lambda) = \lambda(\lambda-\frac{1}{2})\cdots(\lambda-\frac{n-1}{2}). \tag{295}$$

The truth of (294) for all $g(Z,\overline{Z})$ or equivalently for all coefficients $h(C,D)$ in (270)′ clearly requires that

$$M_\alpha|CZ+D|^{-\alpha}|C\overline{Z}+D|^{-\beta} = \varepsilon_n(\alpha)|CZ+D|^{-\alpha-1}|C\overline{Z}+D|^{-\beta+1},$$

and as in earlier contexts, it suffices to suffices to require this for (C,D) with $|C|\neq 0$. Then by a replacement of the form $Z\to Z+S$ with a suitable symmetric matrix S, the above can be reduced, assuming that M_α is invariant for such a replacement, to

$$M_\alpha|Z|^{-\alpha}|\overline{Z}|^{-\beta} = \varepsilon_n(\alpha)|Z|^{-\alpha-1}|\overline{Z}|^{-\beta+1}$$

If further we are sure that in M_α only $\dfrac{\partial}{\partial Z_{\mu\nu}}$ appears explicitly and it is independent of $\dfrac{\partial}{\partial Z_{\mu\nu}}$ as will be the case, then the last relation simplifies further into

$$M_\alpha |Z|^{-\alpha} = \varepsilon_n(\alpha)|Z|^{-\alpha-1}|\overline{Z}| \tag{296}$$

227 It is now our task to construct M_α with these required properties, viz:-

(i) M_α depends only on $\dfrac{\partial}{\partial Z_{\mu\nu}}$ and does not involve $\dfrac{\partial}{\partial \overline{Z}_{\mu\nu}}$

(ii) M_α is invariant relative to a replacement of Z by $Z + S$ with an arbitrary symmetric matrix S. (297)

(iii) M_α satisfies (296)

For arbitrary integers $I \leq L_1 < L_2 < \cdots < l_h \leq n, 1 \leq k_1 < k_2 < k_h \leq n$ we get

$$\left.\begin{aligned} \begin{pmatrix} l_1, l_2 \cdots l_h \\ k_1, k_2 \cdots k_h \end{pmatrix}_z &= |Z_{i_\mu k\nu}| \\ \begin{bmatrix} l_1 l_2 \cdots l_h \\ k_1 k_2 \cdots k_h \end{bmatrix}_z &= |e_{i_\mu k k_\nu} \frac{\partial}{\partial z_{\nu_\mu} l_\nu}| \end{aligned}\right\} \tag{298}$$

Also let

$$s_h(Z - \overline{z}, \frac{\partial}{\partial z}) = \sum_{\substack{1 \leq l_1 < \cdots < l_h \leq n \\ 1 \leq k_1 < \cdots < k_h \leq n}} \begin{pmatrix} l_1 l_2 \cdots l_h \\ k_1 k_2 \cdots k_h \end{pmatrix}_{z-\overline{z}} \begin{bmatrix} l_i l_2 \cdots l_l \\ k_1 k_2 \cdots k_h \end{bmatrix}_z \tag{299}$$

for $h = 1, 2, \ldots n$ and $s_o(Z - \overline{z}, \dfrac{\partial}{\partial z}) = 1$

Finally we introduce

$$M_\alpha \sum_{h=0}^{n} \frac{\varepsilon_n(\alpha)}{\varepsilon_h(\alpha)} s_h(Z - \overline{Z}, \frac{\partial}{\partial z})(\varepsilon_o(\alpha) = 1) \tag{300}$$

That M_α satisfies the first two conditions in (297) is a matter of easy verification. Besides, in the case $n = 1$, M_α is identical with K_α

introduced earlier. The proof that M_α satisfies (296) is a consequence of the following result

$$\begin{bmatrix} l_1 l_2 \cdots l_h \\ k_1 k_2 \cdots k_h \end{bmatrix}_Z |Z|^{-\alpha} = (-1)^h \varepsilon_h |\alpha| |Z|^{-\alpha-1} \overline{\begin{pmatrix} l_1 l_2 \cdots l_k \\ k_1 k_2 \ldots k_h \end{pmatrix}_Z} \tag{301}$$

where $\overline{\begin{pmatrix} l_1 l_2 \cdots l_h \\ k_1 k_2 \cdots k_l \end{pmatrix}_Z}$ denotes the algebraic minor of $\begin{pmatrix} l_1 l_2 \cdots l_h \\ k_1 k_2 \cdot k_h \end{pmatrix}_Z$. **228**

This minor differs from the determinant of the submatrix which arises from Z on cancelling the rows and column corresponding to the indices $l_1, l_2, \ldots l_h$ and $k_1, k_2, \ldots k_h$ respectively by the sign $(-1)^{l_1+l_2+k_1+k_2\cdots k_h}$. In the case $h = n$, we define the algebraic minor to be 1. The relation (301) can be proved by resorting to induction of $\dagger$ but we leave it here for fear of the length of such a proof. Certainly a simpler proof of (301) will be desirable.

Assuming (301) we proceed to establish (296). It is immediate from (301) that

$$s_h(Z, \bar{Z}, \frac{\partial}{\partial z})|Z|^{-\alpha} = (-1)^h \varepsilon_h(\alpha) |Z|^{d-1} \sum_{1 \le i_1 < l_h \le n} D_{l_1 l_2 \ldots l_h}$$

where

$$D_{l_i l_2 \cdots l_h} = \sum_{1 \le k_1 < k_2 < \cdots k_h} \begin{pmatrix} l_1, l_2 \ldots l_h \\ k_1 k_2 \cdots k_h \end{pmatrix}_{Z - \bar{Z}} \overline{\begin{pmatrix} l_1, l_2 \ldots l_h \\ k_1 k_2 \cdots k_h \end{pmatrix}}$$

By a standard development of a determinant (Laplace's decomposition theorem) it is clear that $D_{l_1, l_2 \cdots l_2}$ is the determinant of the n- rowed matrix which arises from Z on replacing its rows with the indices $l_1, l_2, \ldots l_2$ by the corresponding rows of $Z - \bar{Z}$. We split up this resulting matrix into a sum of matrices each row of which is upto a constant sign **229**
a row of either Z or $\bar{Z}$, and the summands which result are 2^h is number.

Let $\Delta_{g_1 g_2 \ldots g_r}$ denote the determinant of the matrix which arises from Z on replacing its rows with the indices $g_{\prime\prime} g_2, \ldots g_n$ by the corresponding rows of $\bar{Z}$. For a given $\Delta_{g_1 g_2 \ldots g_n}$ to appear in the above decomposition of $D_{i_1, i_2, \ldots i_h}$ it is necessary and sufficient that the set $(f, r_2 \ldots g_r)$ is a subset of the set of indices $(l_1 l_2 \ldots l_0)$ so that a fixed determinant $\Delta_{g_1 g_2 \ldots g_r}$ occurs in the above decomposition of $D_{l_1 l_2 \ldots l_h}$ for a fixed h exactly $\binom{n-r}{h-r}$ times, and the sign which it takes is $(-i)^r$. Consequently we have

$$\sum_{1\leq i_1<\cdots<l_h\leq n} D_{i_1, i_2, \ldots u_h} = \sum_{r=0}^{h} \sum_{1\leq r_1<} (-1)^r \binom{n-n}{h-r} \Delta_{g_1 g_2, \ldots g}$$
$$= \sum_{r=0}^{h} (-1)^r \binom{n-n}{h-r} A_n$$

with

$$A_n = \sum_{1\leq g_1<\cdots<g_n\leq n} \Delta_{y, g_2} \varphi r > 0 \text{ and } A_o = |Z|$$

Clearly $A_n = |\overline{Z}|$. Then

$$M_\alpha |Z|^{-\alpha} = \sum_{h=0}^{n} \frac{\varepsilon_n(\alpha)}{\varepsilon_h(\alpha)} s_h(Z - \overline{Z}, \frac{\partial}{\partial z}) |Z|^{-\alpha}$$
$$= \varepsilon_n(\alpha)|Z|^{-\alpha-1} \sum_{h=0}^{n} (-1)^h \sum_{h=0}^{h} (-1)^h \binom{n-n}{h-r} A_n$$
$$= \varepsilon_r(a)|Z|^{-\alpha-1} \sum_{r=0}^{n} \left(\sum_{h=n}^{n} (-1)^{h-r} \binom{n-n}{h-r} \right) A_n$$

230 The sum within the parenthesis can be rewritten as $\sum_{h=0}^{r-n} (-1)^h \binom{n-1}{h}$ and this latter sum is equal to 0 or 1 according as $n-r > 0$ or $n-r = 0$ 0. The above now reduces to

$$M_\alpha |Z|^{-\alpha} = \varepsilon_n(\alpha)|Z|^{-\alpha-1} A_n = \varepsilon_n(\alpha)|Z|^{-\alpha-l} |\overline{Z}|$$

as was desired. The deduction of the first half of (294) at this stage is similar to the deduction from (274) of the corresponding result for (270).

We have still to introduce the operator N_β. We introduce the operator λ by the requirement

$$\lambda \mathfrak{f}(z, \overline{z}) = \mathfrak{f}(-\overline{z} - z) \tag{302}$$

and set $N_\beta = \lambda M_\beta \lambda$. It is immediate that for $n = 1$ we have $N_\beta = -\wedge_\beta$ while we had $M_\alpha = K_\alpha$ in this case. It is easily seen that N_β satisfies the second half of (294) in view of M_α satisfying the first half. In general it can be conjectured that

$$(N_{\beta-1} M_\alpha - \varepsilon_n(\alpha)\varepsilon_n(\beta-1))\{G, \alpha, \beta, \nu\} = 0$$
$$(M_{\alpha-1} N_\beta - \varepsilon_n(\beta)\varepsilon_n(\alpha-1))\{G, \alpha, \beta, \nu\} = 0 \tag{303}$$

$$M_\alpha\{G, \alpha, \beta, \nu\} \subset \{G, \alpha+1, \beta-1, \nu\}$$
$$N_\beta\{G, \alpha, \beta, \nu\} \subset \{G, \alpha+1, \beta-1, \nu\}$$

This has been proved however only in the case $n = 1$ and 2 and the proof makes use of certain operator identities. **231**

We turn new to a different problem, viz. that of finding a Fourier development for modular function defined in (292). Net much progress has so has been recorded in this direction. We assume that the given group G contains symplectic substitutions of the form $\begin{pmatrix} E & O \\ 0 & E \end{pmatrix}$ (S Symmetric), say, as is the case with $\mathcal{M}$. If $\mathfrak{f}(z, \overline{z}) \in \{G, \alpha, \beta, \nu\}$ and $\mathfrak{f}(z, \overline{z}) = \sum_T \alpha(y, <$ $\pi T) \in^{2\pi i \sigma(T\chi)}$, the summation for T being over all semi-integral matrices, the hypothesis, viz. $\Omega_{\alpha,\beta}\mathfrak{f} = 0$ implies that $\Omega_{\alpha\beta}$ annihilates each term of the above sum.

Thus $\Omega_{\alpha\beta} 0 \alpha^{z2\pi T} e^{\pi i \sigma(T\chi)}$ which on replacing $2\pi T$ by T gives $\Omega_{\alpha\beta} e^{i\sigma(T\chi)} a(y, T) T 0$. This represents a system of differential equations-here we can consider T to be an arbitrary real symmetric matrix-for the functions $a(y, T)$, and we have to determine all the solutions of this system which are regular in $y > 0$. Only in the case $n = 2$ some real progress has been achieved and the main result in this case is that linear space $\{\alpha\beta T\}$ of the solutions is finite dimensional for T such that $|T| \neq 0$. Before we give a detailed account of the results in this case it will be useful to make the following remarks on a special parametric representation of the matrices $y^{(z)} > 0$

Every 2 X 2 positive matrix y has a parametric representation of the form

$$Y = \sqrt{|y|} \begin{pmatrix} (x^2 + y^2) y^{-1} \, xy^{-1} \\ xy^{-1} g^1 \end{pmatrix} \tag{304}$$

232 and the requirement $y > 0$ is equivalent with $y > 0$. Let $Z = X + iy$.Or the surface $|y| = L$, the correspondence $y \to y$ is $y - 1$. This surface can be considered as a Riemannian surface, as in $y > 0$ we have the fundamental metric form $ds^2 = \sigma(y^{-1}\tau y)^2$. Relative to this metric, the mappings $y \to \mathcal{U}y\mathcal{U}, \mathcal{U}$ unimodular, are movements (and the quadratic form ds^2 is invariant relative to these movements). The space $|y| = 1$ is nothing else than the hyperbolic plane in this metric. For if $\omega = \sqrt{|y|}$ it can be shown that

$$ds^2 = Z\left(\frac{d\omega^2}{\omega^2} + \frac{dx^2 + dy^2}{y^2}\right) \tag{305}$$

On $|y| = 1$ we have $d\omega = 0$ and then in (305) one has clearly the fundamental metric form for the hyperbolic plane. If $\mathcal{U}$ is a proper unimodular matrix $(\varphi|\mathcal{U}| = 1)$ then $y \to \mathcal{U}y\mathcal{U}'$ and $Z \to \mathcal{U} < Z >$ are representations of the same hyperbolic movement. We shall have more to say one the determinantal surface $|y| = 1$ later.

For the special case $n = 2$ we are considering, we state the following specific results. Introduce x, y by (304) and set $\mathcal{U} = (\tau(y0))^2 - 4|y^\tau|$

Case i $T = 0$.

233 In this case the solutions $a(y, 0)$ can be written as

$$a(y, 0) = \varphi(x, y)|y|^{\frac{1}{2}(1\alpha+\beta)} + C_1|y|^{\frac{3}{2}-d-\beta} + 0_2 \tag{306}$$

where $\varphi(x, y)$ represents an arbitrary regular solution in $y > 0$ of the wave equation

$$y^2(\varphi_{\chi\chi} + \varphi_{yy}) - (\alpha + \beta - 1)(\alpha + \beta - 2)\varphi = 0$$

and C_1, C_2 are arbitrary constants. This is the general solution for $\alpha+\beta \neq \frac{3}{2}, 2, 1$. The singular cases $\alpha+\beta = \frac{3}{2}, 2, 1$ are those at which at least two of the three exponents $\frac{1}{2}(1 - \alpha - \beta)$, $\frac{3}{2} - \alpha - \beta$, 0 occurring in (306) are equal and require some modifications. We exclude these cases here.

Case ii : $T \geq 0$, rank $\tau = T$.

Here we shall have

$$a(y, T) = \varphi(u)i|y|^{\frac{3}{2}-\alpha-\beta} + \psi(u) \tag{307}$$

where $\varphi(u)$ and $\psi(u)$, denote confluent hypergeometric functions satisfying the differential equations

$$\mathcal{U}\varphi^{''} + (3-\alpha-\beta)\varphi' + (\alpha-\beta-\mathcal{U})\varphi = 0$$
$$\mathcal{U}\psi^{''} + (\alpha+\beta)\psi' + (\alpha-\beta-u)\psi = 0$$

We have two independent solutions each for $\mathcal{U}\vartheta$ and the dimension of $\{\alpha, \beta T\}$ is consequently equal to 4 in this case. Here again (307) is the general solution for $\alpha+\beta \neq \dfrac{3}{2}$ and we exclude the exceptional **234**
case from our considerations. We remark that the condition $T \geq 0$ is irrelevant here. We can also take $T \geq 0$ and in general,

$$\{\alpha, \beta, T\} = \{\beta, \alpha, -T\}.$$

Case iii $T > 0$

This is the first, general case we are dealing with in so far as the solutions $a(y, T)$ here depend on functions of more than one variable (unlike the φ and ψ of the earlier case). We have in this case

$$a(y, T) = \sum_{\nu-0}^{\infty} g_x(u)\vartheta^\nu, (|\nu| < u^2) \tag{308}$$

the functions $g_\nu(\mathcal{U})$ are defined recursively by

$$4(\nu+1)^2\mu g_{\nu+1} + \mathcal{U}g_\nu^{''} + 2(2\nu+\alpha+\beta)g_\nu^{'} + (2\alpha-2\beta-u)g_\nu = 1\mathcal{U} \geq 0$$

and

$$g_o(\nu) = u^{1-\alpha-\beta}\psi(\mathcal{U}), \psi(\mathcal{U}) = \frac{1}{u}\varphi(u)$$
$$\varphi^{''} = \left(1 + \frac{2(\beta-\alpha)}{\mathcal{U}} + \frac{(\alpha+\beta+1)(\alpha+\beta-2)}{\mathcal{U}^2}\right)\varphi.$$

Every possible function $g_\nu(\mathcal{U})$ leads to a series (308) converging in the whole domain $y > 0$ so that the dimension of $\{\alpha, \beta, T\}$ is easily seen to be 3. As in the earlier case we can also consider T with $-T > 0$

Case iv : *rank* $T = 2$, T *indefinite*

This is the last case and we have here

$$a(y,T) = \sum_{\nu=0}^{\infty} h_\nu(\alpha) u^\nu, (u^2 < \nu) \tag{309}$$

235 The functions $h_\nu(\vartheta)$ are recursively defined by

$$(\nu+2)(u+1)h_{\nu+2} + 4\nu h_\nu^{''} + 4(\alpha+\beta+\nu)h_\nu^1 - \ell_\nu = 0, \nu \geq 0$$

and

$$8\nu^2 h_o^{''} + 4(\alpha+3\beta+3_o)\vartheta p_o^{''}(4(\alpha+\beta)^3 + 2(\alpha+\beta-1) - 2\nu)h_o^{'} +$$

$$-(\alpha+\beta)h_o = (\alpha,\beta)h_i$$

$$8\nu h_i^{'} + (\alpha-\beta)r_j = (\beta-\alpha)h_o$$

Every permissible h_o, h_1 leads to a series (309) which converges in $y > 0$ and the dimension of $\{\alpha, \beta, T\}$ is four.

It remains to study these functions $a(y,T)$. These are generalisations of the confluent hypergeometric functions of 2 variables. We can also obtain them in the following way. We develop the Eisenstein series. $\sum |CZ+D|^{-\alpha}|C\bar{Z}+D|^{-\beta}$ as a power series,

$$\sum |CZ+D|^{-\alpha}|C\bar{Z}+D|^{-\beta} = \sum_T a(y, 2\pi T) \in^{k\pi i\sigma(T\chi)} m$$

If we compute the coefficients by means of the Poisson summation formula, we obtain representations of $a(y, 2\pi T)$ by integrals. It is desirable to give a characterisation of these special functions $a(y 2\pi T) \in \{\alpha, \beta 2\pi T\}$ and this has to be done by studying the behaviour of these functions in neighbourhood of the boundary of the domain $y > 0$.

236 For a detailed account of some of the topics treated in this section we refer to :-

H. Maass, Die Differentialgleichungen in der Theorie der Siegelachen Modul functionen, Math. Ann. 126 (1553), 44-68.

Chapter 16

Closed differential forms

Our aim in this section is to generalise to the case of non-analytic forms **237**
some of the concepts associated with the analytic forms of Grenzkreis group of the first kind considered by Petersson, and study their applications to the space of symplectic geometry. Let G denote a Grenzkreis group of the first kind. With respect to a multiplicator system ϑ, let $< G, \mathfrak{R}, \vartheta >$ denote the space of all analytic forms of G of weight $\mathfrak{R}$-analytic in the usual sense, viz. that the only singularities if any, are poles in the local uniformising variables of the Riemann surface $\mathfrak{f}$ defined by G (We can assume that $\mathfrak{f}$ is a closed Riemann surface). In conjunction with $< G, \mathfrak{R}, \vartheta >$ we have got to consider also the space $< G, 2-\mathfrak{R}, 1/\vartheta >$, a sort of *adjoint* to the first. For $r \geq 2$ the weight of $g(z) \in < G, 2-\mathfrak{R}, 1/\vartheta >$ is zero or negative so that its dimension (= - weight) is positive or zero. Since it is known that an automorphic form of zero or positive dimension which is everywhere regular vanishes identically, it follows that in this case $g(z)$ is uniquely determined by its principal parts if all the principal parts are zero $g(z)$ itself is identically zero. The question now arises whether there always exists an automorphic form of a given weight $2-\mathfrak{R}$ with arbitrarily given principal parts. While this is true for $\mathfrak{R} = 2$ in the case of the Gaussian sphere-a Riemann surface of genus zero-this is not unreservedly true either if $\mathfrak{R} > 2$ or if the genus is strictly positive. Nevertheless it is possible to give a necessary and sufficient condition for the given principal parts to

238 satisfy, in order that they should be all the principal parts of an automorphic form of weight $\mathfrak{K}$ of G. We state this condition presently. For $f(z) \in \langle G, \mathfrak{K}, \mu$ and $g(z) \in \langle G, 2-\mathfrak{K}, 1/\mu\rangle$ introduce

$$\omega(\mathfrak{f}, g) = \mathfrak{f}(z)y(z)dz \tag{310}$$

which is a closed meromorphic differential form on the Riemann surface defined by G. That $\omega(\mathfrak{f}, g)$ is invariant relative to G is immediate if we observe that the product $\mathfrak{f}(z)g(z)$ is an automorphic form of weight 2 (the sum of the weights of $\mathfrak{f}$ and g) while dz behaves as an automorphic form of weight -2, and the multiplicator systems for $\mathfrak{f}, g$ are ν and $1/\vartheta$ respectively. If $\mathfrak{f}$ denotes a fundamental domain of G in $\mathscr{Y}_1$, by integrating $\omega(\mathfrak{f}, g)$ over the boundary of $\mathfrak{f}$ it is easily seen that

$$\sum_{Z\in\mathfrak{f}} \text{Residue } \omega(\mathfrak{f}, g) = 0 \tag{311}$$

On the left side of (311) we face only a finite sum as we assume that the Riemann surface is closed and there is only a finite number of singularities in the fundamental domain $\mathfrak{f}$. For $\mathfrak{K} > 2$, as $\mathfrak{f}(z)$ runs over a basis of *everywhere regular* form in $\langle G, \mathfrak{K}, \mu$ the left side of (311) depends only on the principal parts of $g(z) \in \langle G, 2-\mathfrak{K}. 1/\mu\langle$ and the conditions (311) themselves are called the *principal part conditions.* These are necessary and sufficient for the existence of a form $g(z)$ with assigned principal parts.

We now wish to generalise the notion of $\omega(\mathfrak{f}, g)$ to the case of the
239 linear spaces $\{G, \alpha, \beta, \mu\}$ of non analytic automorphic forms of degree n, and in particular we have to settle when two such spaces can be considered adjoint like the spaces $\langle G, \mathfrak{K}, \mu\rangle$ and $\langle G, 2-\mathfrak{K}, 1/\mu\rangle$. The two important properties of $\omega(\mathfrak{f}, g)\langle$ are

(i) it is completely invariant relative to G

(ii) it is closed.

The last condition is trivial in the earlier case of analytic forms and is not so in the present case.

First we have to introduce the concept of the dual differential forms in the ring of exterior differential forms. Consider a Riemannian manifold R in which local complex coordinates $\mathscr{Y}' = (Z_1, Z_2, \ldots, Z_n)$, $\overline{\mathfrak{z}} = (\overline{z}, \overline{z}_z, \ldots, \overline{z}_n)$ are introduced such that a neighbourhood of each point is mapped pseudo conformally onto a neighbourhood of the origin of the corresponding coordinate space. We consider $\mathfrak{z}$ and $\overline{\mathfrak{z}}$ as formally independent complex variables, and the above means that if

$$\mathfrak{z}^{*'} = (z_1^* z_2^*, \ldots, z_n^*), \overline{\mathfrak{z}}^{*'} = (\overline{z_1^*}, \overline{z_2^*}, \ldots, \overline{z_n^*})$$

is any other system of complex coordinates at the given point then we have

$$\mathfrak{z}^* = \mathfrak{f}(\mathfrak{z}), \overline{\mathfrak{z}^*} = \overline{\mathfrak{f}}(\overline{\mathfrak{z}}) \tag{312}$$

with the elements of the columns $\mathfrak{f}(\mathfrak{z})$, $\overline{\mathfrak{f}\mathfrak{z}}$ representing regular functions of $(z_1, z_2, \ldots, z_n)$, $(\overline{z_1}, \overline{z_2}, \ldots, \overline{z_n})$ respectively, which vanish at $\mathfrak{z}\overline{\mathfrak{z}} = 0$. In the local co-ordinate system the metric fundamental form ds^2 is a Hermitian form and let us assume

$$ds^2 = d\mathfrak{z}' G d\overline{\mathfrak{z}} \tag{313}$$

with a Hermitian metric G.

In the ring of exterior differential forms let us introduce **240**

$$\begin{aligned}
[d\mathfrak{z}] &= dz_1 dz_2 \ldots dz_n,\\
[d\overline{\mathfrak{z}}] &= d\overline{z}_1 d\overline{z}_2 \ldots d\overline{z}_n,\\
\omega_\nu &= (-1)^{\nu-1} dz_1 \ldots dz_{\nu-1} dz_{\nu+1} \ldots dz_n\\
\overline{\omega}_\nu &= (-1)^{\nu-1} d\overline{z}_1 \ldots d\overline{z}_{\nu-1} \alpha \overline{z}_{\nu+1} \ldots d\overline{z}_n
\end{aligned} \tag{314}$$

Then

$$[d\mathfrak{z}] = dz_2 \omega_2; [d\overline{\mathfrak{z}}] = d\overline{z}, \overline{\omega}_\nu \tag{315}$$

Denote with $\mathscr{W}$, $\overline{\mathscr{W}}$ the columns with the elements ω_ν, $\overline{\omega}_\nu$ respectively, i,e.

$$\mathscr{W}' = (\omega_1, \omega_2, \ldots \omega_n); \quad \overline{\mathscr{W}}' = (\overline{\omega}_1, \overline{\omega}_2, \ldots \overline{\omega}_n) \tag{316}$$

Consider a differential form θ fo degree 1. It can be represented as

$$\theta = \mathscr{Y}' d\mathfrak{z} + \mathscr{U}' d\overline{\mathfrak{z}} \tag{317}$$

where $\mathscr{Y}$, $\mathscr{U}$ are columns with functions of $\mathfrak{z}\overline{\mathfrak{z}}$ as elements. We also introduce the dual form

$$\tilde{\theta} = |G|(\mathscr{Y}'\overline{G}^{-1}\mathscr{W}[d\mathfrak{z}] + \mathscr{U}'G^{-1}\mathscr{W}[d\overline{\mathfrak{z}}]) \tag{318}$$

We shall show that $\tilde{\theta}$ is uniquely defined by θ or what is the same, it does not depend on the local coordinate system. We have only to prove that $\tilde{\theta}$ is invariant relative to the pseudo conformal mappings of the kind (313), in the $\mathfrak{z}^*$, $\overline{\mathfrak{z}^*}$ system θ have the representation

$$\theta = \mathscr{Y}^{*'} d\mathfrak{z}^* + \mathscr{U}^{*'} d\overline{\mathfrak{z}^*}$$

241 From (313) we have, with a certain non singular matrix T,

$$d\mathfrak{z}^* = T d\mathfrak{z}, T d\overline{\mathfrak{z}^*} = \overline{T} d\overline{\mathfrak{z}}$$

and then

$$\mathscr{Y} = T'\mathscr{Y}^* \text{ and } \mathscr{U} = \overline{T}'\mathscr{U}^*$$

A simple computation shows that

$$T'\mathscr{W}^* = |T|\mathscr{W}, \overline{T}'\overline{\mathscr{W}}^* = |\overline{T}|\overline{\mathscr{W}};$$
$$[T'\mathfrak{z}^*] = |T|[d\mathfrak{z}], [d\overline{\mathfrak{z}^*}] = |\overline{T}|[d\overline{\mathfrak{z}}]$$

where the elements with a star denote the corresponding elements in the new coordinate system $\mathfrak{z}^*$, $\overline{\mathfrak{z}^*}$. Besides, we also have

$$ds^2 = d\mathfrak{z}' G d\overline{\mathfrak{z}} = d\mathfrak{z}^{*'} G^* d\overline{\mathfrak{z}^*}$$

and it is easy to deduce that

$$T' G^* \overline{T} = G$$

The invariance of (318) relative to the transformations (313) is now immediate. If

$$\frac{\partial}{\partial \mathfrak{z}}, (\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}), \frac{\partial}{\partial \overline{\mathfrak{z}}}, (\frac{\partial}{\partial \overline{z_1}}, \frac{\partial}{\partial \overline{z_2}}, \cdots, \frac{\partial}{\partial \overline{z_n}}) \tag{319}$$

we have

$$
\begin{aligned}
d\tilde{\theta} &= d\bar{\mathfrak{z}}' \frac{\partial}{\partial \bar{\mathfrak{z}}} |G| \mathscr{Y}' \bar{G}^{-1} \bar{\mathscr{W}} [d\mathfrak{z}] + d\mathfrak{z}' \frac{\partial}{\partial z} |G| \mathscr{U} G^{-1} \mathscr{W} [d\bar{\mathfrak{z}}] \\
&= \frac{\partial}{\partial \bar{\mathfrak{z}}}' |G| G^{-1} \mathscr{Y} [d\bar{\mathfrak{z}}][d\mathfrak{z}] + \frac{\partial}{\partial \mathfrak{z}} |G| G^{-1} \mathscr{U} [d\mathfrak{z}][d\bar{\mathfrak{z}}] \\
&= \{(-\mathscr{U} \frac{\partial}{\partial \bar{\mathfrak{z}}}' |G| G^{-1}) \mathscr{Y} + \frac{\partial}{\partial \mathfrak{z}}' |G| G^{-1} \mathscr{U} \} [d\mathfrak{z}][d\bar{\mathfrak{z}}] \qquad (320)
\end{aligned}
$$

We therefore infer that the condition for the form $\tilde{\theta}$ to be closed is that **242**

$$
(-1)^n \frac{\partial}{\partial \bar{\mathfrak{z}}}' |G| G^{-1} \mathscr{Y} + \frac{\partial}{\partial \mathfrak{z}}' |G| G^{-1} \mathscr{U} = 0 \qquad (321)
$$

We now compute the invariant volume element in the metric (312). Let $z_\nu = r_2 + iy_\nu$, $\bar{z}_\nu = r_\nu + iy_\nu (\nu = 1.2 \cdots n)$ and $\mathfrak{z} = \varepsilon + i\mathscr{Y}$, $\bar{\mathfrak{z}} = \mathcal{E} + i\mathscr{Y}$. Then $dz_\nu d\bar{z}_\nu = -zidx_\nu dy$ so that $[d\mathfrak{z}][d\bar{\mathfrak{z}}]$ and $[d\mathcal{E}][d\mathscr{Y}]$ differ only by a constant factor. Using the transformation formulae for these differential forms for a change of coordinate systems and the transformation properties of $|G|$ it is easily seen that $|G|[d\mathfrak{z}][d\bar{\mathfrak{z}}]$ is invariant for the coordinate transformations. This is therefore true of $|G|[d\mathcal{E}][d\mathscr{Y}]$ and this is precisely then the invariant volume element, viz.

$$
d\mathscr{U} = |G|[d\mathcal{E}][d\mathscr{Y}] \qquad (322)
$$

We wish to interpret these results in the space $\mathscr{Y}$ of symplectic geometry. The fundamental metric form here is

$$
ds^2 = \sigma(dzy^{-1}d\bar{z}y^{-1}), z = z + iy, \bar{z} = x - iy
$$

If we set $|y|y^{-1} = (y_{\mu\nu})$, then $y_{\mu\nu}$ is just the algebraic minor of $\mathcal{Y}_{\mu\nu}$ in $y = (\dagger_{\mu\nu})$ and we have

$$
\begin{aligned}
d\mathscr{S}^2 &= |y|^{-2} \sum_{\substack{\mu,\nu \\ \varrho,\sigma}} dz_{\mu\nu} y_{\nu\varrho} d\bar{z}_{\varrho\sigma} y_{\sigma\mu} \\
&= |y|^{-2} \sum_{\substack{\mu\leq\nu \\ \varrho,\sigma}} \frac{1}{e_{\mu\nu}} dz_{\mu\nu} y_{\nu\varrho} d\bar{z}_{\varrho\sigma} y_{\sigma\mu}
\end{aligned}
$$

$$= |y|^{-2} \sum_{\substack{\mu\le\nu\\ \varrho\le\sigma}} \frac{1}{2e_{\mu\nu}e_{\varrho\sigma}}(y_{\mu\sigma}y_{\nu\varrho} + y_{\nu\sigma}y_{\varrho\sigma})dz_{\mu\nu}d\overline{z}_{\varrho\sigma}$$

$$= \sum_{\mu\le\nu,\varrho\le\sigma} \mathcal{G}_{\mu\nu,\varrho\sigma}dz_{\mu\nu}d\overline{z}_{\varrho\sigma} \tag{323}$$

with

$$\mathcal{G}_{\mu\nu,\varrho\sigma} = \frac{1}{2e_{\mu\nu}e_{\varrho\sigma}}|y|^{-2}(y_{\mu\sigma}y_{\nu\varrho} + y_{\nu\sigma}y_{\varrho\sigma}) \tag{324}$$

If $G = (\mathcal{G}_{\mu\nu,\varrho\sigma})$ and $G^{-1} = (\mathcal{G}^{\mu\nu,\varrho\sigma})$, we find from the relation $GG^{-1} = E$ that

$$\mathcal{G}^{\mu\nu,\varrho\sigma} = \frac{1}{2}(\mathcal{Y}_{\mu\varrho}\mathcal{Y}_{\nu\sigma} + \mathcal{Y}_{\mu\sigma}\mathcal{Y}_{\nu\varrho}) \tag{325}$$

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The transformations we have to consider are the symplectic substitutions

$$Z^* = (AZ + B)(CZ + D)^{-1}, \overline{Z}^* = (A\overline{Z} + B)(C\overline{Z} + D)^{-1}$$

and these are pseudo conformal mappings of $\mathscr{Y}$ in the sense of (313). Let us introduce as in the earlier case,

$$[dz] = \prod_{\mu\le\nu} dz_{\mu\nu}, [d\overline{z}] = \prod_{\mu\le\nu} d\overline{z}_{\mu\nu}$$

with the lexicographical order of the factors and

$$\omega_{\mu\nu} = \pm \prod_{\substack{\varrho\le\sigma\\ (\varrho,\sigma)\ne(\mu,\nu)}} dz_{\varrho\sigma} = \omega_{\nu\mu} \qquad \text{for } \mu \le \nu,$$

$$\overline{\omega}_{\mu\nu} = \pm \prod_{\substack{\varrho\le\sigma\\ (\varrho,\sigma)\ne(\mu,\nu)}} d\overline{z}_{\varrho\sigma} = \overline{\omega}_{\nu\mu}$$

244 the ambiguous sign being fixed in each by stipulating that

$$[dz] = dz_{\mu\nu}\omega_{\mu\nu}; [d\overline{z}] = d\overline{z}_{\mu\nu}\overline{\omega}_{\mu\nu} \tag{326}$$

In the place of $\mathscr{W}, \overline{\mathscr{W}}$ we introduce

$$\Omega = (e_{\mu\nu}\omega_{\mu\nu}, \overline{\Omega} = (e_{\mu\nu}\overline{\omega}_{\mu\nu}. \tag{327}$$

As before we start with a differential form of degree 1 given by

$$\theta = \sigma(pdz) + \sigma(\theta d\overline{z}) \tag{328}$$

where p, θ are $n-$rowed symmetric with functions as elements. We have to compute the dual form $\tilde{\theta}$.

In the earlier notation,

$$\tilde{\theta} = (\mathcal{Y}'\overline{G}^{-1}\overline{\mathscr{W}}[d\mathfrak{z}] + \mathscr{U}'G^{-1}\mathscr{W}[d\overline{\mathfrak{z}}]). \tag{(318)$'$}$$

$|G|$ can be fixed as follows. We know that the invariant volume element in symplectic geometry is

$$d\vartheta = |y|^{-n-1}[dx][dy]$$

From (321) it is also given by

$$d\vartheta = |G|[d\mathcal{E}][d\mathscr{Y}].$$

A comparison of the two shows that $|G|$ is equal to a constant multiple fo $|Y|^{-n-1}$ and this constant factor can be assumed to be 1 for the purposes of $\tilde{\theta}$. Then $|G| = |Y|^{-n-1}$. We now compute the product $\mathscr{Y}'\tilde{\theta}^{-1}\overline{\mathscr{Y}}$. **245**

$$\begin{aligned}\mathcal{Y}'\overline{G}^{-1}\overline{\mathscr{W}} &= \sum_{\substack{\mu\le\nu\\ \varrho\le\sigma}} \frac{1}{e_{\mu\nu}} P_{\mu\nu}\frac{1}{2}(\mathcal{Y}_{\mu\varrho}\mathcal{Y}_{\nu\sigma} + \mathcal{Y}_{\mu\varrho}\mathcal{Y}_{\nu\varrho})\overline{\omega}_{\varrho\sigma}\\ &= \sum_{\substack{\mu\le\nu\\ \varrho\le\sigma}} P_{\mu\nu}\frac{1}{2}(\mathcal{Y}_{\mu\varrho}\mathcal{Y}_{\nu\sigma} + \mathcal{Y}_{\mu\varrho}\mathcal{Y}_{\nu\varrho})\overline{\omega}_{\varrho\sigma}\\ &= \sum_{\mu,\nu;\varrho,\sigma} e_{\mu\nu}P_{\mu\nu}\mathcal{Y}_{\mu\varrho}\mathcal{Y}_{\nu\sigma}\overline{\omega}_{\varrho\sigma} = \sigma(ypy\overline{\Omega})\end{aligned}$$

The other product $\mathscr{U}'G^{-1}\mathscr{W}$ similarly reduces to $\sigma(y\theta y\overline{\Omega})$ and from (318) we have

$$\tilde{\theta} = |Y|^{-n-1}(\sigma(ypy\overline{\Omega})[dz] + \sigma(y\theta y\Omega)[d\overline{z}]). \tag{329}$$

Then form (320) we will have for the differential of $\tilde{\theta}$,

$$d\tilde{\theta} = \{(-1)^{\frac{n(n+1)}{2}}\sigma(\frac{\partial}{\partial z}ypy|Y|^{-n-1}) + \sigma(\frac{\partial}{\partial z}y\theta y|Y|^{-n-1})\}[dz][d\overline{z}]. \tag{330}$$

We now specialise θ by appropriate choices of P and Q and investigate in detail the cases in which $d\tilde{\theta}$ vanishes.

Let $\{\alpha, \beta\}$ denote the linear space of all regular functions $\mathfrak{f}(z, \overline{z})$ in $\mathscr{Y}$ which satisfy the differential equation (272). Let $\alpha, \beta; \alpha', \beta'$ be complex numbers, arbitrary for the present, and choose

$$\mathfrak{f} = \mathfrak{f}(z, \overline{z}) \in \{\alpha, \beta\}, \mathcal{G} = \mathcal{G}(z, \overline{z}) \in \{\alpha', \beta'\} \tag{331}$$

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We then set

$$p = \mathcal{E}|Y|^{y} y^{-1} \mathcal{G} K_{\alpha} \mathfrak{f}, Q = |Y|^{y} y^{-1} \mathcal{F} \Lambda_{\beta} \mathcal{G} \tag{332}$$

where we assume

$$\mathcal{E}^2 = 1, y = \alpha + \alpha' = \beta + \beta' \tag{333}$$

and the differential operators K_{α} and Λ_{β} are defined by (276). With this P and Q, (328) defines θ, and we set

$$\omega(\mathfrak{f}, g) = \theta \tag{334}$$

We shall study the behaviour of this differential form relative to the symplectic substitutions

$$z^* = (az + B)(cz + B)^{-1}, \overline{z}^* = (a\overline{z} + B)(c\overline{z} + B)^{-1}.$$

The replacement of $z, \overline{z}$ by $z^*, \overline{z}^*$ in any operator or function shall in general be denoted by putting a star ($*$) over it and in particular

$$\mathfrak{f}^* = \mathfrak{f}(z^*, \overline{z}^*), \mathcal{G}^* = \mathcal{G}(z^*, \overline{z}^*)$$

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is any symplectic metric, we also introduce

$$\mathfrak{f}(z, \overline{z})|M = |cz + D|^{-\alpha} |C\overline{z} + D|^{-\beta} \mathfrak{f}(z^*, \overline{z}^*)$$
$$\mathcal{G}(z, \overline{z})|M = |cz + D|^{-\alpha'} |C\overline{z} + D|^{-\beta'} \mathcal{G}(z^*, \overline{z}^*)$$

and for any differential form $\mu(z, \overline{z})$,

$$\mu(z, \overline{z})|M = \mu(z,^{*} \overline{z}^{*}) \qquad (335)$$

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By means of (286), (122)′ and (284) we have

$$(z^{*} - \overline{z}^{*})\frac{\partial}{\partial z^{*}}\mathfrak{f} = (\overline{z}C' + D')^{-1}(z, \overline{z})((cz + D)\frac{\partial}{\partial z})'$$
$$|ez + D^{\alpha}||c\overline{z} + D|^{\beta}(\mathfrak{f}|M)$$
$$= |ez + D^{\alpha}||c\overline{z} + D|^{\beta}(\overline{z}C' + D')^{-1}(z, \overline{z})((cz + D)$$
$$\{\alpha C' + ((cz + D)\frac{\partial}{\partial z})'\}(\mathfrak{f}|M)$$
$$= |ez + D^{\alpha}||c\overline{z} + D|^{\beta}(\overline{z}C' + D')^{-1}[\{\alpha(zc' + D')$$
$$- \alpha(\overline{z}c' + D')\}(\mathfrak{f}|M) + (z - \overline{z}\frac{\partial}{\partial z}\mathfrak{f}|M)(zc' + D')]$$

and

$$K_{\alpha}^{*}\mathfrak{f}^{*} = |Cz + D^{\alpha}||c\overline{z} + D|^{\beta}(\overline{z}C' + D')^{-1}(K_{\alpha}\mathfrak{f}|M)(zc' + D')$$

Further,

$$dz^{*} = (zc' + D')^{-1}dz(zc + D)^{-1}$$
$$Y{*}^{-1} = (cz + D)Y^{-1}(\overline{z}c' + D')$$
$$|Y *^{\nu}| = |Y|^{y}|cz + D|^{-y}|C\overline{z} + D|^{-y}$$

and this yields

$$|Y^{*}|^{y} y *^{-1} g^{*}(K_{\alpha}^{*}\mathfrak{f}^{*})dz^{*} = (cz + D)|Y|^{y}Y^{-1}(g|M)(k_{\alpha}\mathfrak{f}|M)dz(cz + D).$$

In our notation (355) the left side is just

$$(|Y|^{y}Y^{-1}\mathcal{G}(K_{\alpha}\mathfrak{f})dz)|M = (\frac{1}{\mathcal{E}}pdz)|M$$

Hence we deduce that 248

$$\sigma(\frac{1}{\mathcal{E}}pdz)|M = \sigma\{(cz+D)|y|^{\gamma}y^{-1}(g|M)(K - \alpha\mathfrak{f}|M)dz(cz+D)^{-1}$$
$$= \sigma\{|y|^{\gamma}y^{-1}(giM)(K_{\alpha}\mathfrak{f}|M)dz\}$$

One proves analogously that

$$\sigma(Qd\overline{z}) = \sigma\mathfrak{f}|Y|^{\gamma}y^{-1}\{|M)(\Lambda_{\beta}\mathcal{G}|M|d\overline{z})\}$$

and then it is immediate that

$$\omega(\mathfrak{f},\mathcal{G})|M = \omega(\mathfrak{f}|M.\mathcal{G}|M). \tag{336}$$

This is naturally true of the dual form $\tilde{\omega}(\mathfrak{f},\mathcal{G})$ also so that

$$\tilde{\omega}(\mathfrak{f},\mathcal{G})|M = \tilde{\omega}(\mathfrak{f}|M.\mathcal{G}|M). \tag{337}$$

Let us now compute $d\tilde{\omega}(\mathfrak{f},\mathcal{G})$ form (330). In doing this we shall need the following identity

$$((z-\overline{z})\frac{\partial}{\partial\overline{z}})' = \{((z-\overline{z})\frac{\partial}{\partial\overline{z}})'W\}y - \frac{1}{2}\sigma(yw)E - \frac{1}{2}w'y \tag{338}$$

where $W = (\omega_{\mu\nu})$ denote an arbitrary matrix its elements all functions. In fact, since

$$e_{\varrho\mu}\frac{\partial Y_{\nu,\mu}}{\partial\overline{z}_{\varrho,\mu}} = -\frac{1}{4i}(\partial_{\varrho\tau}\partial_{\mu\nu} + \partial_{\varrho\nu\partial_{\mu\tau}})$$

we have

$$((z-\overline{z})\frac{\partial}{\partial\overline{z}})'(wy) = (\sum_{\varphi}(z_{\nu\varrho} - \overline{z}_{\nu\varrho})e_{\varrho\mu}\frac{\partial}{\partial\overline{z}_{\varrho,\mu}}(\sum_{\tau}\omega_{\mu\tau}\mathcal{Y}_{\tau\nu})$$

$$= (\sum_{\varrho\sigma\tau}(z_{\nu\varrho} - \overline{z}_{\sigma\varrho})e_{\varrho\mu}\frac{\partial}{\partial\overline{z}_{\varrho,\mu}}(\sum_{\tau}\omega_{\mu\tau}\mathcal{Y}_{\tau\nu})$$
$$= [((z-\overline{z})\frac{\partial}{\partial\overline{z}})'W]y - \frac{1}{4i}(\sum_{\varrho\sigma\tau}(z_{\varrho\sigma} - \overline{z}_{\varrho\sigma})\omega_{\sigma\tau}(\delta_{\varrho\tau}\delta_{\mu\nu} + \delta_{\mu\tau}\delta_{\varrho\nu})$$
$$= [((z-\overline{z})\frac{\partial}{\partial\overline{z}})'W]y - \frac{1}{4i}\sigma((z-\overline{z})W)\mathcal{E}\frac{1}{4i}W'(z,\overline{z})$$

249 which is just another form of (338).

Similarly one obtains the analogous identity, viz.

$$((z-\bar{z})\frac{\partial}{\partial z})'(Wy) = [(z-\bar{z})\frac{\partial}{\partial z})'Wy]y + \frac{1}{2}\sigma(yW)E + \frac{1}{2}W'Y \qquad (339)$$

Finally one also has

$$\left.\begin{aligned} \mathcal{E}yP|y|^{-n-1} &= \alpha|y|^{\mu-n-1}y\mathfrak{f}\mathcal{G} + |y|^{\mu-n-1}\mathcal{G}(z-\bar{z})(\frac{\partial}{\partial z}\mathfrak{f})y \\ y\theta|y|^{-n-1} &= \beta|y|^{\mu-n-1}y\mathfrak{f}\mathcal{G} + |y|^{\mu-n-1}\mathcal{F}(z-\bar{z})(\frac{\partial}{\partial z}\mathcal{G})y \end{aligned}\right\} \qquad (340)$$

Using (280) and (338-340) we can now state that

$$\varepsilon\frac{\partial}{\partial\bar{z}}yP|y|^{-n-1} =$$

$$=\alpha\left\{-\frac{y-n-1}{2\ell}|y|^{y-n-1}y^{-1} + |y|^{y-n-1}\frac{\partial}{\partial\bar{z}}\right\}yfg+$$

$$+\left\{-\frac{y-n-1}{2\ell}|y|^{y-n-1}y^{-1} + |y|^{y-n-1}\frac{\partial}{\partial\bar{z}}\right\}g.(z-\bar{z})(\frac{\partial}{\partial z}f)y$$

$$= \frac{d\ell}{2}(y-n-1)|y|^{(}y-n-1)fgE - (y-n-1)|y|^{(}y-n-1)g(\frac{\partial}{\partial z}f)+$$

$$-\frac{d\ell}{2}|y|^{(y-n-1)}\left\{-\frac{n+1}{2}E + \left((z-\bar{z})\frac{\partial}{\partial\bar{z}}\right)\right\}fg+$$

$$+|y|^{(y-n-1)}(\frac{\partial}{\partial\bar{z}}g)(z-\bar{z})(\frac{\partial}{\partial\bar{z}}f)y$$

$$+g|y|^{(y-n-1)}\left\{-\frac{n+1}{2}E + \left((z-\bar{z})\frac{\partial}{\partial\bar{z}}\right)'\right\}(\frac{\partial}{\partial z}f)y$$

$$= \frac{di}{2}(y-\frac{n+1}{2})|Y|^{y-n-1}\mathfrak{f}\mathcal{G}E - (y-\frac{n}{2})|y|^{yn-1}\mathcal{G}(\frac{\partial}{\partial z}\mathfrak{f})+$$

$$= \frac{di}{2}(y-\frac{n+1}{2})|Y|^{y-n-1}\mathfrak{f}\mathcal{G}E - (y-\frac{n}{2})|y|^{yn-1}\mathcal{G}(\frac{\partial}{\partial z}\mathfrak{f})+$$

$$\alpha|y|^{yn-1}\mathcal{G}(\frac{\partial}{\varrho z}\mathfrak{f})y + \alpha|Y|^{y-n-1}\mathfrak{f}(\frac{\partial}{\partial z}\mathcal{G})Y+$$

$$2i|y|^{y-n-1}(\frac{\partial}{\partial z}\mathcal{G})y(\frac{\partial}{\partial z}\mathscr{G})Y$$

$$+\mathcal{G}|y|^{y-n-1}[\{((Z-\bar{Z},\frac{\partial}{\partial\bar{z}})'\frac{\partial}{\partial z}\mathfrak{f}\}y - \frac{1}{2}\sigma(y\frac{\partial}{\partial z}\mathfrak{f})E]$$

$$= \frac{\alpha i}{2}(y - \frac{n+1}{2})|y|^{y-n-1}\mathfrak{f}\mathcal{G}E - (y - \frac{n}{2} - \beta)|y|^{y-n-1}\mathcal{G}(\frac{\partial}{\partial z}\mathfrak{f})y +$$
$$+ \alpha|y|^{y-n-1}\mathfrak{f}(\frac{\partial}{\partial z}\mathcal{G})y + 2i|y|^{yn-1}(\frac{\partial}{\partial \overline{z}})y(\frac{\partial}{\partial z}\mathfrak{f})y +$$
$$- \frac{1}{2}|y|^{y-n-1}\mathcal{G}\sigma(y\frac{\partial}{\partial z}\mathfrak{f})E \quad (341)$$

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In stating the last relation we have used the fact that

$$((z - \overline{z})\frac{\partial}{\partial \overline{z}}\mathfrak{f})\frac{\partial}{\partial z}\mathfrak{f} = -\alpha\frac{\partial}{\partial \overline{z}}\mathfrak{f} + \beta\frac{\partial}{\partial z}\mathfrak{f}$$

which is easily proved.

We have similar lines the dual formula also, viz. $\frac{\partial}{\partial z}yQ|y|^{y-n-1}$

$$= (\beta'(y - \frac{n+1}{2})|y|^{y-n-1}\mathfrak{f}\mathcal{G}E + (y - \frac{n}{2} - y)|y|^{y-n-1}\mathfrak{f}(\frac{\partial}{\partial \overline{z}}\mathcal{G})y +)$$
$$-\beta'\mathcal{G}|y|^{y-n-1}((\frac{\partial}{\partial z}\mathfrak{f})y + 2i|y|^{y-n-1}((\frac{\partial}{\partial z}\mathfrak{f})y((\frac{\partial}{\partial \overline{z}})y$$
$$+\frac{1}{2}|y|^{y-n-1}\mathfrak{f}\sigma(y(\frac{\partial}{\partial \overline{z}}\mathcal{G})E \quad (342)$$

251 In view of our assumption (333), setting $\mathcal{E} = -(-1)\frac{n(n+1)}{2}$ we now obtain that

$$d\tilde{\omega}(\mathfrak{f}, \mathcal{G}) = \frac{in}{2}(\beta' - \alpha)(y - \frac{n+1}{2})|y|^{y-n-1}\mathfrak{f}\mathcal{G}[dz][d\overline{Z}] \quad (343)$$

In particular we shall have

$$d\tilde{\omega}(\mathfrak{f}\mathscr{G}) = 0 \quad (344)$$

in the two cases

$$y = \frac{n+1}{2}\alpha' = \frac{n+1}{2} - \alpha, \beta' = \frac{n+1}{2} - \beta \quad (345)$$

and

$$y = \alpha + \beta, \alpha' = \beta, \beta' = \alpha \quad (346)$$

We shall look into these cases a little more closely.

Let $\mathfrak{f} \in \{G, \alpha, \beta, \mu\}$, $\mathcal{G} \in \{G, \alpha', \beta', \mu'\}$ where we can assume $u' = 1/\nu$. From (336) and (337) it is seen that the differential forms $\omega(\mathfrak{f}, g)$ and $\tilde{\omega}(\mathfrak{f}, g)$ are invariant relative to the substitutions of G. With regard to the case (346) we can now make the following remark, viz. if α, β are real and $|\mu| = 1$, the linear spaces $\{G, \alpha, \beta, \mu\}$ and $\{G; \beta, \alpha, 1/\mu\}$ are mapped onto each other by the involution

$$\mathfrak{f}(z, \overline{Z}) = \mathcal{G}(z, \overline{z}) = \overline{\mathfrak{f}}(\overline{z}, z).$$

Indeed, if $\mathfrak{f}(z, \overline{z}) \in \{G, \alpha, \beta, \mu\}$ then $\Omega_{\alpha\beta}\mathfrak{f}(z, \overline{z}) = 0$ so that, taking complex conjugates, $\Omega_{\beta\alpha}\mathcal{G}(z, \overline{z}) = 0$ On the other hand, if $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in G$, then

$$\begin{aligned} &\mathfrak{f}(M\langle \overline{z} \rangle)|cz + D|^{-\alpha}|c\overline{z} + D|^{-\beta} = \vartheta(M)\mathfrak{f}(z\overline{z}), \\ &\qquad \overline{\mathfrak{f}}(M\langle \overline{z} \rangle, M\langle \overline{z} \rangle)|C\overline{z} + D|^{-\alpha}|C\overline{z} + D|^{-\beta} = \overline{\mu}(M)\overline{\mathfrak{f}}(\overline{z}, z). \end{aligned}$$

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This means that $\mathcal{G}(\overline{z}, z)|M = \frac{1}{\mu}(M)\mathcal{G}(z, \overline{z})$ and our assertion is proved. We can now state if α, β are real and $|\mu| = 1$ then

$$\left.\begin{aligned} \tilde{\omega}(f(z, \overline{z}), \overline{\mathcal{G}}(\overline{z}, z))|M = \tilde{\omega}(f(z, \overline{z}), \overline{\mathcal{G}}(\overline{z}), z) \\ d\tilde{\omega}(f(z, \overline{z}), \overline{\mathcal{G}}(\overline{z}, z)) \end{aligned}\right\} \tag{347}$$

for all $f(z, \overline{z})$, $\mathcal{G}(z, \overline{z}) \in \{G, d\beta, \mu\}$.

Applications of (347) have so far been made only in the case when $n = 1$. The next case of interest will be when $n = 2$. In this case, an explicit expression for $\tilde{\omega}(\mathfrak{f}, \mathcal{G})$ can be given as follows. Let $\mathscr{L}_q$ denote the intersection of the fundamental domain $\mathfrak{f}$ of the modular group of degree n with the domain defined by $|y| \leq q$. In either of the cases (345), (346) whence $d\tilde{\omega} = 0$, we will have

$$\int_{\mathfrak{K}q} \omega(\mathfrak{f}, \mathcal{G}) = 0$$

where R_q denotes the boundary of $\mathscr{Q}_q$. In view of the invariance properties of $\omega(\mathfrak{f}, \mathcal{G})$ set out earlier, this reduces to

$$\int_{\mathfrak{f}q} \omega(\mathfrak{f}, \mathcal{G}) = 0 \tag{348}$$

253 where $\mathfrak{f}_q$ denotes that part of R_q which lies on $|y| = q$. While this is all true for any n, in the case of $n = 2$, we have the parametric representation (304) for matrices $y > 0$, viz.

$$y = u \begin{bmatrix} (x^2 + y^2)y^{-1} xy^{-1} \\ xy^{-1} y^{-1} \end{bmatrix}, u > 0, y > 0; u = \sqrt{|y|}$$

One 7_q of course we shall have $du = 0$ and then it can be shown that

$$\begin{aligned}\tilde{\omega}(f, g) = 4u^{2\nu-2}\Big\{ & i(x^2 + y^2)y^{-1} \frac{\partial(fg)}{\partial x_1} + iy^{-1} \frac{\partial(fg)}{\partial x_{12}} + iy^{-1} \frac{\partial(fg)}{\partial x_{22}} \\ & + g\frac{\partial f}{\partial u} - f\frac{\partial g}{\partial u} + z(\alpha - \beta)u^{-1} fg \Big\} dx_u dx_{12} y^{-2} dx dy \end{aligned} \tag{349}$$

The explicit computation of the integral on the left side of (348) appears to be possible only by a detailed knowledge if the Fourier expansions of the forms f and g.

The following reference pertain to the subject matter of this section:-

1. **H. Mass** Über eine neue Art von nichtanalytischen automorphen Funktionen and die Bestimming Dirichlet scher Reihen durch Funktionalgleichungen, Math. Ann. 121 (1949), 141 - 193.

2. **H. Petersson** , Zur analytischen Theorie der Grenzkeris gruppen, Teil I bis TeilV, Math. Ann 115 (1938), 23 - 67, 175-204, 518-
254 572, 670-709, Math. Zeit 44(1939), 127-155.

3. **H. Petersson** Konstruktion der Modulformen and der zu gewissen Grenzkreisgruppen gehorigen automorphen Formen von positiver reeller Dimension and die vollstandinge Bestimmning ihrer Fourie - koeffizienten, Sitz. Ber Akad. Wiss. Heidelberg 1950, Nr. 8

Chapter 17

Differential equations concerning angular characters of positive quadratic forms

The notion of angular characters was first introduced by Hecke in the study of algebraic number fields. These are functions u defined on the non zero elements of a give algebraic number field K such that 255

1. $u(\alpha) = u(r\alpha\varepsilon)$ for rational numbers $r \neq 0$ and units $\in$ of K

2. For a given $\alpha \in K$, $\alpha \neq 0$, the set of values $u(\alpha)$ (for the different $u's$) and the norm N_α of α determine the principal ideal (α) of α

It is possible to realise the angular characters u (α) with the variable α as solution of a certain eigen value problem and this enables us to carry out certain explicit analytic computations with $u(\alpha)$. Analogous considerations can be developed for positive quadratic forms. Here we ask for functions $u(y)$ defined on the space of positive matrices $y = y^{(n)}$ which are invariant relative to the transformations $y \to ry[u]$ where r is a positive real number and u, an unimodular matrix, such that the determinant $|y|$ and the set of all values $u(y)$ for a given y determine uniquely

the class of matrices $y[u]$ which are equivalent with y. It can be expected that suitable functions $u(y)$ appear again as solutions of an eigen value problem. The use of such angular characters will be particularly felt in determining a set of Dirichlet series equivalent with a give modular form of degree n in analogy with a theory of Hecke. A satisfactory treat-
256 ment of the theory of angular characters for positive quadratic forms has so far been possible only in the case $n = 2$. We now make our above statements precise.

Consider a space of n real coordinates $y_\nu > 0, \nu = 1.2\ldots n$, and the linear differential operators Ω in this space,

$$\Omega = \sum C_{\nu_1 \nu_1} \ldots \nu_n \frac{\partial^{\nu_1}}{\partial y_2^{\nu_2}} \cdots \frac{\partial^{\nu_n}}{\partial y_n^{\nu_n}} \tag{350}$$

We shall denote by ψ the set of all such operators Ω which are invariant relative to the group of mappings

$$\mathscr{Y}_\nu \to \mathscr{Y}_\nu^* = a_\nu y_\nu, \nu = 1, 2 \ldots n$$

where there $a'_\nu \mathscr{S}$ are arbitrary positive real numbers. Given any algebraic number field $K-$ for simplicity we assume K to be totally real - we can associate with K a space of n real coordinates as follows. Let the dimension of K over the field of rationals be n, and for $\alpha \in K$, let $\alpha^{(1)}, \alpha^{(2)}, \ldots \alpha^{(n)}$ denote the conjugates of α. We then set $\mathscr{Y}_\nu = |\alpha^{(\nu)}|, \nu = 1, 2 \ldots n, |\alpha^{(\nu)}|$ denoting the absolute value of $\alpha^{(\nu)}$.

The angular characters of K are functions $u(\alpha) = u(\alpha^{(1)}, \ldots \alpha^{(\nu)})$ of n real variables which depend only on the absolute values $|\alpha^{(1)}|, |\alpha^{(2)}|, \ldots |\alpha^{(n)}|$ of the variables, and which satisfy the following conditions:-

1. u is an eigen function of every differential operator $\Omega \in \mathscr{P}$
2. u is invariant relative to the mappings $\mathscr{Y}_\nu \to r\mathscr{Y}_\nu$ where r is an arbitrary positive real number.
3. u is invariant relative to the unit group of K

257 We shall look into these defining properties of u a little more closely.

Clearly all the operators $\mathscr{Y}_\nu \dfrac{\partial}{\partial y_\nu}, \nu = 1, 2 \ldots n$ lie in ψ, and it is easily seen that these operators actually generate ψ, meaning that any operator in ψ is a polynomial of the type

$$\sigma_{\nu_1 \nu_2 .. \nu_n} a_{\nu_1 \nu_2 \ldots \nu_n} (\mathscr{Y}_1 \frac{\partial}{\partial y_1})^{\nu_1} \ldots (\mathscr{Y}_n \frac{\partial}{\partial y_n})^{\nu_n}$$

with constant coefficient $a_{\nu_1 \nu_2 \ldots \nu_n}$. Then the first condition on u reduces to the requirement that u is an eigen function of each $\mathscr{Y}_\nu \dfrac{\partial}{\partial y_\nu}, \nu = 1, 2 \ldots n$, in other words that

$$(\mathscr{Y}_\nu \frac{\partial}{\partial y_\nu} + \lambda_\nu) u = 0, \nu = 1, 2 \ldots n$$

with appropriate $\lambda'_\nu s$. Hence we conclude that

$$u = C \prod_{\nu=1}^{n} \mathscr{Y}_\nu^{-\lambda_\nu} \tag{351}$$

The second condition on u requires that u is homogeneous of degree 0 in $Y_1, Y_2, \ldots Y_n$ and then

$$\sum_{\nu=1}^{n} y_\nu \frac{\partial}{\partial Y_\nu} u = 0 \tag{352}$$

form which it follows that $\sum_{\nu=1}^{n} \lambda_\nu = 0$

We now come to the last condition on u. It is well known that any unit of K is a power product of $n - 1$ fundamental units $\mathscr{E}_1, \mathscr{E}_2 \ldots \mathscr{E}_{n-1}$ multiplied by a certain root of unity. Hence the last condition only amounts to requiring that u is invariant under the mappings $Y_\nu \to Y_\nu |\mathscr{E}_\mu^{(\nu)}|, \mu = 1, 2 \ldots n - 1$. Then

$$\prod_{\nu=s}^{n} (y_\nu |\mathscr{E}_\mu^{(\nu)}|)^{\lambda_\nu} = \prod_{\nu=s}^{n} y_\nu^{-\lambda_\nu} \text{ so that}$$

$$\prod_{\nu=1}^{n} |\mathscr{E}_K^{(\nu)}|^{-\lambda_\nu} = 1, \mu = 1, 2, \ldots n - 1$$

Together with (352) this can be written as 258

$$\sum_{\nu=1}^{n} \lambda_\nu = 0$$

$$\sum_{\nu=1}^{n} \lambda_\nu \log|\varepsilon_\mu^{(n)}| = 2\Re_\mu \pi i\} i\mu = 1, 2, \ldots n-1$$

For a given set of integers $\Re_\mu$ these n equations define uniquely the n eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ and hence also u by (351). Varying systems of $\Re' s$ determine the different $u' s$.

We now generalise these considerations for positive quadratic forms or what is the same for positive matrices. Here ψ will stand for the set of all linear differential operators Ω on the space of positive matrices $Y^{(n)}$ which are invariant relative to the group of mappings $T \to Y[R], R$ being an arbitrary non singular matrix. As in the earlier case we can show that the operators $\sigma(Y\dfrac{\partial}{\partial Y})^{\Re} \in \psi, \Re = 1, 2 \ldots n$ and they generate ψ where as usual, $\dfrac{\partial}{\partial y} = (e_{\mu,\nu}\dfrac{\partial}{\partial y_{\mu,\nu}})$. Let $Y^* = Y[R], R$ non singular. For an arbitrary function $f(y)$ we have for the total differential,

$$df = \sigma(dy\frac{\partial}{\partial y}f) = \sigma(dy^*\frac{\partial}{\partial y^*}f)$$

Also $dy^* = (dy)[R]$ so that

$$\sigma(dy\frac{\partial}{\partial y}f) = df = \sigma(dy^*\frac{\partial}{\partial y^*}f) = \sigma(dyR\frac{\partial}{\partial y^*}fR')$$

and consequently,

$$\frac{\partial}{\partial y} = R\frac{\partial}{\partial y^*}R'\frac{\partial}{\partial y^*} = \frac{\partial}{\partial y}[R^{-1}]$$

Then

$$(Y^*\frac{\partial}{\partial y^*})^* R'(Y\frac{\partial}{\partial y})^R R^{-1}, \Re = 1, 2, \ldots n, \tag{353}$$

259 and

$$\sigma(Y^*\frac{\partial}{\partial y^*})^{\mathfrak{R}} = \sigma(Y\frac{\partial}{\partial y})^{\mathfrak{R}} \tag{354}$$

This proves that $\sigma(Y\frac{\partial}{\partial y})^{\mathfrak{R}} \in \psi, \mathfrak{R} = 1, 2 \dots n$. Consider now any operator $\Omega \in \psi$. As a linear operator it is of the form

$$\Omega = \Omega(y, \frac{\partial}{\partial y}) = \sum_{R_1} C_{R_1}(y) \prod_{\mu,\nu=1}^{n} \frac{\partial^{\mu\nu}}{\partial y_{\mu\nu}^{s\mu\nu}} \varrho_{\mu\nu\geq 0}\text{integral},$$

and $R_1 = (\varrho_{\mu\nu}) = R_1'$.

Since Ω is invariant relative to the mapping $Y \to y^* = y[R], R$ non-singular, we have

$$\Omega(Y[R], \frac{\partial}{\partial y}[R^{-1}]) = \Omega(y, \frac{\partial}{\partial y}), |R| \neq 0 \tag{355}$$

Let Y_0 by an arbitrary point an let V be such that $Y_0[V] = E$. Let u be an arbitrary orthogonal matrix ($u'u = E$) and set $R = Vu$. Then

$$Y_0[R] = (Y_o[V])[u] = E. \tag{356}$$

Let $f(y)$ be an arbitrary function and define

$$g(y) = \Omega(y\frac{\partial}{\partial y})f(y) \tag{357}$$

With $Y^* = y[V]$ we have $\frac{\partial}{\partial y^*} = \frac{\partial}{\partial y}[V^{-1}]$ so that

$$\frac{\partial}{\partial y}[R^{-1}] = (\frac{\partial}{\partial y}[V^{-1}])[u] = \frac{\partial}{\partial y^*}[u]$$

Then from (355), (356) and (357) we have

$$\begin{aligned} g(y_0) &= \{\Omega(y_0, \frac{\partial}{\partial y})f(y)\}_{y=y_o} \\ &= \{\Omega(E, \frac{\partial}{\partial y^*}[u])f(y^*[V^{-1}])\}_{y^*=E} \end{aligned} \tag{358}$$

Since this is true for all orthogonal matrices u we conclude that the right side of (358) is independent of u, in other words,

$$\Omega(E, \frac{\partial}{\partial y^*}[u]) = \Omega(E, \frac{\partial}{\partial y^*}),$$

or equivalently,

$$\Omega(E, \frac{\partial}{\partial y}[u]) = \Omega(E, \frac{\partial}{\partial y}) \tag{359}$$

Since the ring generated by the differential operators $\dfrac{\partial}{\partial y_{\mu\nu}}$ is isomorphic with the polynomial ring generated by $Y_{\mu\nu}$ under the isomorphism $Y_{\mu\nu} \leftrightarrow e_{\mu\nu}\dfrac{\partial}{\partial y_{\mu\nu}}$, the truth of (359) for every orthogonal matrix u is equivalent with the relation

$$\omega(E, Y[u]) = \omega[E, y]$$

for these $u's$. In particular, choosing u such that $Y[u] = \Lambda = (\delta_{\mu\nu},)$ we shall have

$$\Omega(E, \Lambda) = \Omega(E, Y) \tag{360}$$

The left side is a symmetric function of the $\lambda'_\nu s, \nu = 1, 2 \ldots n$ in fact symmetric polynomial in the $\lambda'_\nu s$ so that it is a polynomial of the 'power sums' $\sum_{\nu=1}^{n} \lambda_\nu^{\Re} = \sigma(\Lambda)^{\Re}), \Re = 1, 2 \ldots.$ Thus $\Omega(E, y) = \Omega(E, \Lambda) = P(G(\Lambda) \ldots \sigma(\Lambda)^n)$, a polynomial with constant coefficients. Since u is orthogonal and $y[u] = \Lambda$ we have $\sigma(\Lambda) = \sigma(y[u]) = \sigma(y)$, and then

$$\Omega(E, Y) = P(\sigma(y), \sigma(y)^2, \ldots \sigma(y)^n), \tag{361}$$

261 where P is a polynomial with constant coefficients.

Then form (358),

$$\begin{aligned} g(y_0) &= \{\Omega(E, \frac{\partial}{\partial y^*) f(y^*[V^{-1}]}\}_{y^*=E} \\ &= \{P(\sigma(\frac{\partial}{\partial y^*)}, \sigma(\frac{\partial}{\partial y^*})^2, \ldots \sigma(\frac{\partial}{\partial y^*})^n s(y^*[V^{-1}]\}_{y^*=E} \end{aligned}$$

Comparing the two operators $\sigma(y^*\frac{\partial}{\partial y^*})^{\mathfrak{K}}$ and $\sigma(\frac{\partial}{\partial y^*})^{\mathfrak{K}}$, and in particular their highest degree terms at the point $y^* = E$, we can write

$$g(y_0) = \{P(\sigma(y^*\frac{\partial}{\partial y^*}), \ldots \sigma(y^*\frac{\partial}{\partial y^*})^n f(y^*[V^{-1}]\}_{y^*=E}$$
$$+\{\Omega^*(y^*\frac{\partial}{\partial y^*}) f(y^*[V^{-1}])\}_{y^*-E}$$

where the degree of Ω^* in $\frac{\partial}{\partial y^*}$ is smaller than that of Ω in $\frac{\partial}{\partial y^*}$. Due to the invariance of the operators $\sigma(Y\frac{\partial}{\partial y})^{\mathfrak{K}}$ for the mappings $y \to Y^*$, the above gives

$$g(y_0) = \{P(\sigma(y\frac{\partial}{\partial y}), \ldots \sigma(y\frac{\partial}{\partial y})^n f(y[V]\}_{y=y_0}$$
$$+\{\Omega_1(y\frac{\partial}{\partial y}) f(y)\}_{y=y_0}$$

with Ω_1 having the same degree in $\frac{\partial}{\partial y}$ as Ω^* in $\frac{\partial}{\partial y^*}$ and since y_0 is arbitrary by choice,

$$g(Y) = P(\sigma(y\frac{\partial}{\partial y}), \ldots \sigma(y\frac{\partial}{\partial y})^n f(y) + \Omega_1(y\frac{\partial}{\partial y}) f(y).$$

Since $g(y) = \Omega(y, \frac{\partial}{\partial y}) f(y)$ and the above is true for every function 262
$f(y)$ it is immediate that

$$\Omega(y, \frac{\partial}{\partial y}) = P(\sigma(y\frac{\partial}{\partial y}), \sigma(y\frac{\partial}{\partial y})^2 \ldots \sigma(y\frac{\partial}{\partial y})^n)_+ \Omega_1(y\frac{\partial}{\partial y})$$

where the degree of Ω_1 in $\frac{\partial}{\partial y}$ is less than that Ω in $\frac{\partial}{\partial y}$) and P is a polynomial with constant coefficients. By resorting to induction on the degree of Ω in $\frac{\partial}{\partial y}$ we conclude that $\Omega(\frac{\partial}{\partial y})$ is a polynomial in $\sigma(y\frac{\partial}{\partial y})^{\mathfrak{K}}$, $\mathscr{R} = , \ldots n$, and this was what we set out to prove.

We now define the angular characters as functions $u(y)$ which satisfy the following requirements.

(1) u is an eigen function of each operator $\Omega \in \psi$

(2) u is homogeneous of degree o in y, in other words, $u(ry) = u(y)$ for every real $r > 0$,

(3) u is invariant relative to the group of mappings $y \to y[u]$ where u is an arbitrary unimodular matrix.

(4) u is square integrable over a fundamental domain of the group of mappings $y \to y[u]$, u unimodular, in the determinant surface $|y| = 1$ with a certain invariant volume element $d\vartheta_1$.

Let us examine the consequences of these defining properties of u. Since we have shown that ψ is generated by the elements $\sigma(y\dfrac{\partial}{\partial y})^{\Re}, \Re = 1, 2 \ldots n$, the first condition on u can be replaced by a system of differential equations

$$(\sigma(y\frac{\partial}{\partial y})^{\Re} + \eta_{\Re})u(y) = 0 k = 1, 2 \ldots n \tag{362}$$

263 The second condition is then nothing else but the requirement that $\eta_1 = 0$. The third condition means that u is a kind of automorphic function, and it is a pertinent question as to the existence of non-trivial solutions of (362) which satisfy the second and third condition above. We shall subsequently show that such non trivial solutions u always exist. But these special functions u fail to be angular characters in that they do not need the fourth requirement.

The equation in (362) corresponding to $\Re = 2$ is of particular interest, as in this case $\sigma(y\dfrac{\partial}{\partial y})^{\Re} = \sigma(y\dfrac{\partial}{\partial y})^2$ can be proved to be the *Laplace Beltrami* operator of the space $y > 0$, considered as a Riemannian space relative to the metric

$$dr^s = \sigma(y^{-1}dy)^2 \tag{363}$$

We prove this as follows. The Laplace - Beltrami operator is invariant relative to the transformations $y \to y[R]$, R non-singular which are movements of the space $y > 0$, and the operator $\sigma(y\dfrac{\partial}{\partial y})^2$ is also invariant relative to these transformations as is seen from (354). In view of the transitivity of the group of movements then, it suffices to show that the two operators are equal at the special point $\gamma = E$. At this point we choose the coordinate system

$$x = (-y + E)(y + E)^{-1} \tag{364}$$

The substitution $y \to x$ is a $1-1$ mapping of the space $y > 0$ onto the space $E - x'x > 0, x' = x$, and a simple computation shows that

$$ds^2 = 4\sigma((E - xx')^{-1}dx)^2 \tag{365}$$

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It is then immediate that our coordinate system at E is a geodesic one. Clearly the point $y = E$ corresponds to $x = 0$. By means of the relations

$$\begin{aligned} \frac{\partial}{\partial y} &= -\frac{1}{2}(E + x)((E + x)\frac{\partial}{\partial x})' \qquad (366)\\ \frac{\partial}{\partial x}x &= \frac{n+1}{2}E + (x\frac{\partial}{\partial x})' \\ \frac{\partial}{\partial x}(x\frac{\partial}{\partial x}) &= (x\frac{\partial}{\partial x}\frac{\partial}{\partial x})' + \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\sigma(\frac{\partial}{\partial x})E \end{aligned}$$

which are easily proved, we obtain that at $x = 0$

$$(y\frac{\partial}{\partial y})^2 = \frac{1}{4}\left\{\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\sigma\frac{\partial}{\partial x}E - \frac{n+1}{2}\frac{\partial}{\partial x}\right\},$$

and then

$$\begin{aligned} \sigma(y\frac{\partial}{\partial y})^2 &= \frac{1}{4}\sigma\frac{\partial}{\partial x})^2 = \sum_{\mu,\nu} e_{\mu\nu}^2 \frac{\partial^2}{\partial x_{\mu\nu}^2} \\ &= \frac{1}{4}\sum_{\nu}\frac{\partial^2}{\partial x_{\mu\nu}^2} + \frac{1}{8}\sum_{\mu<\nu}\frac{\partial^2}{\partial x_{\mu\nu}^2}. \end{aligned}$$

On the other hand, at $x = 0$, we have

$$ds^2 = 4\sigma(dx)^2 = 4\sum_{\nu} dx_{\nu\nu}^2 + 8\sum_{\mu<\nu} dx_{\mu\nu}^2$$

and we can then conclude from (290) that $\sigma(y\frac{\partial}{\partial y})^2$ is identical with the Laplace - Beltrami operator at $x = 0$ and hence also at any other point.

We are also interested in the determinant surface $|y| = 1$ and we can introduce the Laplace - Beltrami operator Δ_1 in this surface. We wish
265 to determine the relationship between Δ and Δ_1. We first observe that in (363).

$$dr^2 = d^z \log |y| \tag{367}$$

$$\begin{aligned} For, -d^2 log|y| &= -d(d \log |y|) = -d(\frac{1}{|y|} d|y|) \\ &= d(\frac{1}{|y|}, \sigma(\frac{\partial}{\partial y}|y|) = -d(\sigma(dyy^{-1})) \\ &= -d(\sigma(ydy^{-1})) = \sigma((dy^{-1})dy) \\ &= -\sigma(-y^{-1}dyy^{-1dy}) = \sigma(y^{-1}dy)^z \\ &= ds^2 \end{aligned}$$

The relationship between Δ and Δ_1 is now provided by the following general lemma, viz.

Lemma 25. *Let R be a domain in the space if n real coordinates $y_1, y_2, \ldots y_n$, and $x = x(y_1, y_2, \ldots y_n)$, a positive homogeneous function on $\Re$ of degree $\Re > 0$. Denote by M the surface defined by $x(y_1, y_2, \ldots y_n) = 1$. Let R, M be considered as Riemann space relative to the metric $ds^2 = d^2\varphi$ where $\varphi = -\log x$, and let Δ, Δ_1 denote the Laplace Beltromi operator in R and M respectively. If $h_1 = h_1(y_1, y_2, \ldots y_n)$ is an arbitrary function in M and*

$$h = h_1(\frac{y_1}{n\sqrt{x}}, \frac{y_2}{\Re\sqrt{x}}, \ldots \frac{y_n}{\Re\sqrt{x}})$$

266 *the homogeneous function of degree 0 in R which extends h_1 then we have*

$$\Delta_1 h_1 = \Delta h \tag{368}$$

Proof. We chose in R a special coding system $(x_1, x_2, \ldots x_n)$ as follows. Let $y_\mu = \psi_\mu(x_1, x_2, \ldots x_{n-1})(1 \leq \mu \leq n)$ be a parametric representation of the surface M so that in particular, rank $(\dfrac{\partial\varphi_\mu}{\partial x_\nu}) = n-1$. Then the equations

$$y_\mu = y_\mu(x_1, x_2, \ldots x_{n-1})x_n, i \le \mu \le n \tag{369}$$

define the desired coordinate system $(x_1, x_2, \ldots x_n)$. To prove our assertion we need only show that the square matrix $(\dfrac{\partial\varphi_\mu}{\partial x_\nu})$ has the rank n Since $\dfrac{\partial\varphi_\mu}{\partial x_\nu} = \dfrac{\partial\varphi_\mu}{\partial x_\nu}\times_n$ for $\nu < n$ and $\dfrac{\partial\varphi_\mu}{\partial x_\nu} = \varphi\mu, \varphi = 1, 2 \ldots n$, it suffices to show that the homogeneous linear equations

$$\sum_{\nu=1}^{n-1} \xi_\nu \frac{\partial\varphi_\mu}{\partial x_\nu} \times n + \xi_n\varphi_\mu = 01 \le \mu \le n \tag{370}$$

admit of only the trivial solution. Since $x(\varphi_1, \varphi_2 \ldots \varphi_n) = 1$ identically in $x_1, x_2, \ldots x_{n-1}$, we have

$$\sum_{\mu=1}^{n} \frac{\partial X}{\partial\varphi_\nu}(\varphi)\frac{\partial\varphi_\mu}{\partial x_\nu} = 0, 1 \le \nu < n \tag{371}$$

□

Besides,

$$\sum_{\mu=1}^{n} \frac{\partial\lambda}{\partial\varphi_\nu}(\varphi)\varphi_\mu = \Re\lambda(\varphi) > 0 \tag{372}$$

by a standard result on homogeneous functions. Multiplying both sides of (371) by ξxn and of (372) by ξ and adding, we obtain in view of (370) **267**
that $\xi\Re\lambda(\varphi) > o$ whence it follows that $\xi_n = o$ Since we know that rank $(\dfrac{\partial\varphi_\mu}{\partial x_\nu}) = n - 1,^{\mu=1,2,\ldots n}_{\nu=1,2,\ldots n-1,}$ we conclude from (370) that $\xi_o = o$ for all $\nu = 1, 2, \ldots n$. We have now shown that (369) gives a parametric representation of the whole space R in terms of that in $|\nu|$. Let us compute fundamental metric form ds^2 in this special coordinate system. We have

$$ds^2 = d^2\varphi = d(\sum_{\nu=1}^{n} \frac{\partial\varphi}{\partial x_\nu} dx_\nu) = \sum_{\mu,\nu=1}^{n} \frac{\partial^2\varphi}{\partial x_\mu}\partial_{x_\nu}^{dx_\mu} dx_\nu$$

$$= \sum_{\mu,\nu} dx_\mu dx_\nu \tag{373}$$

with $g_{\mu\nu} = \dfrac{\partial^z \varphi}{\partial x_\mu \, _{\partial x_\nu}}$ $(1 \le \mu, \nu \le n)$

Also

$$\frac{\partial \varphi}{\partial x_\nu} = \frac{\partial(-\log \chi}{\partial x_\nu} = -\frac{1}{\chi}\frac{\partial(\chi(y)}{\partial x_\nu} = -\frac{1}{2}\sum_{\mu=1}^{n}\frac{\partial \chi(y)}{\partial y_\mu}\frac{\partial y_\mu}{\partial x_\nu}$$

$$= -\frac{1}{\chi}\sum_{\mu=1}^{n}\frac{\partial \chi(y)}{\partial y_\mu}\frac{\partial \varphi_\mu}{\partial x_\nu} x_n \tag{374}$$

Since χ is homogeneous of degree $\Re$ in $y_1, y_2, \ldots y_n$, $\dfrac{\partial \chi}{\partial y_\mu}$ is homogeneous of degree $\Re - 1$ in $y_1, y_2, \ldots y_n$ Also, all the $y's$ are linear functions in χ_n and it then follows from (374) that $\dfrac{\partial \varphi}{\partial x_\nu}$ is a homogeneous function of degree o in x_n for $\nu < n$. In other words, for $\nu < n$, $\dfrac{\partial \varphi}{\partial \chi_\nu}$ is independent of x_n and consequently

$$g_{n\nu} = g_{\nu n} = o \tag{375}$$

for these values of ν.

268 Also,

$$\frac{\partial \varphi}{\partial x_n} = -\frac{1}{\chi}\sum_{\mu=1}^{n}\frac{\partial \chi(y)}{\partial y_\mu}\frac{\partial y_\mu}{\partial x_n}$$

$$= -\frac{1}{2}\sum_{\mu}\frac{\partial \chi_{(y)}}{\partial y_\mu}\frac{y_\mu}{x_n} = -\frac{-\Re}{x_n}$$

so that

$$g_{nn} = \frac{\partial^2 \varphi}{\partial x^2} = \frac{\Re}{x_n^2}. \tag{376}$$

Then from (373), ds^2 is given by

$$ds^2 = \sum_{\mu,\nu=1}^{n-1} g_{\mu\nu} dx_\mu dx_\nu + \frac{\Re}{x_n^2} dx_n^2. \tag{377}$$

On N of course we have $x_n = 1$ so that $dx_n = o$, and then

$$ds^2 = \sum_{\mu,\nu=1}^{n-1} g_{\mu\nu} dx_\mu dx_\nu \tag{378}$$

gives the metric form.

Let $(g^{\mu\nu})$ $(\mu, \nu < n)$ denote the inverse of the matrix $(g_{\mu\nu})$ and let g denote the determinant of $(g_{\mu\nu})$. Since $\frac{\partial\varphi}{\partial x_\nu}$ is independent of x_n for $\nu < n$ it follows that $g_{\mu\nu}(\mu, \nu < n)$ and consequently g are independent of x_n. Then by the definition of $\triangle$, using (375) and (376), we have

$$\begin{aligned}
\triangle &= \frac{1}{\sqrt{g g_n} n} \sum_{\mu,\nu=1}^{n} \frac{\partial}{\partial x_\mu}(\sqrt{g g_{nn}} g^{\mu\nu} \frac{\partial}{\partial x_\nu}) \frac{1}{\sqrt{g g_{nn}}} \frac{\partial}{\partial x_n} \sqrt{g g_{nn}} \frac{x_n^2}{\Re} \frac{\partial}{\partial x_n} \\
&= \frac{1}{\sqrt{g}} \sum_{\mu,\nu=1}^{n-1} \frac{\partial}{\partial x_\mu}(\sqrt{g} g^{\mu\nu} \frac{\partial}{\partial x_\nu}) + \frac{1}{\Re} x_n \frac{\partial}{\partial x_n} \chi_n \frac{\partial}{\partial x_n} \\
&= \triangle_1 + \frac{1}{\Re}(x_n \frac{\partial}{\partial x_n})^2
\end{aligned} \tag{379}$$

Since $h(y_1, y_2, \ldots y_n) = h_1\left(\frac{y_1}{\sqrt[\Re]{\chi}}, \ldots \frac{y_n}{\sqrt[\Re]{\chi}}\right)$ by assumption and **269**

$$\sqrt[\Re]{\chi} = \sqrt[\Re]{\chi(\varphi . x_n)} = x_n \sqrt[\Re]{\chi(\varphi)} = x_n \tag{380}$$

in an obvious notation, we obtain

$$h(y_1, y_2, \ldots y_n) = h_1(\varphi_1, \varphi_2 \cdots \varphi_n)$$

and $\frac{\partial h}{\partial x_n} = \frac{\partial h_1}{\partial x_n} = o$. It now follows from (379) that $\triangle h = \triangle_1 h_1$ and Lemma 25 is proved. Finally we remark that if $d\vartheta, d\vartheta_1$ denote the invariant volume elements of R and H relative to the metrics (377) and (378) respectively, then

$$d\vartheta = \sqrt{g g_{nn}} \prod_{\nu=1}^{n} dx_\nu = \frac{\sqrt{\Re}}{x_n} \sqrt{g} \prod_{\nu=1}^{n} dx_\nu,$$

$$d\vartheta_1 = \sqrt{g}\prod_{\nu=1}^{n-1} dx_\nu$$

so that $d\vartheta$ and $d\vartheta_1$ are connected by the relation,

$$d\vartheta = \sqrt{\mathfrak{K}}x_n^{-1}dx_n d\vartheta_1 \tag{381}$$

We now interpret (368) and (381) in the particular space $y > o$. In view of (367) we can state

Lemma 26. *Let $\triangle$ denote the Laplace-Beltrami operator of the space $y > o$ relative to the metric $ds^2 = \sigma(y^{-1}dy)^2$ and $\triangle_1$ the Laplace-Beltrami operator of the determinant surface $|y| = 1$ relative to the induced metric. If $h_1(y)$ is an arbitrary function on this surface and $h(y)$ the homogeneous function of degree 0 on the whole space $y > 0$ defined*
270 *by $h(Y) = h_1(|Y|^{1/n}Y)$ then we have $\triangle h = \triangle_1 h_1$.*

This is how (368) reads in the space $y > 0$. We now take up (381). The fundamental metric in the space of real matrices $y = y^{(n)}$ was given by $ds^2 = \sigma(y^{-1}dy)^2$, and the $L - 1$ transformations of this space onto itself which leave the metric invariant are given by $y \to y^* = y[R]$ with an arbitrary real non singular matrix R. From (138) we know that

$$\frac{\partial(y^*}{\partial(y)} = |R|^{n+1} = |y^*|^{\frac{n+1}{2}}|y|^{-\frac{n+1}{2}}$$

and then then $|y|^{-\dfrac{n+1}{2}}[dy]$ is left invariant by the transformations $y \to y^*$. It follows that the invariant volume element $d\vartheta$ in this case is given by

$$d\vartheta = \sqrt{C}|y|^{-\frac{n+1}{2}}[dy]$$

with an arbitrary constant C. We fix C by requiring $d\vartheta$ in the standard form, viz. $d\vartheta = \sqrt{g}\pi_{p\leq\nu}dy_{\mu\nu}$ in the usual notation. Then $g = c|y|^{(n+1)}$ and in particular at $y = E$ we have $g = C$. But at this point

$$ds^2 = \sigma(dy)^2 = \sum_{\mu=1}^{n} dy_{\mu\mu}^2 + 2\sum_{\mu<\nu} dy_{\mu\nu}^2$$

so that the matrix (g_{rs}) whose determinant is g is diagonal with η of the diagonal elements 1 and the rest $\frac{\eta(n-1}{2}$ as 2. Thus, at $Y = E$, $g = z^{n(n-1)/2}$ and this is then the value for c Thus

$$d\vartheta = 2^{n(n-1)/4}|y|^{-\frac{n+1}{2}}[dy] \tag{382}$$

From (381) we have $\chi_n^{-1} dx_n d\vartheta_1 = \frac{1}{\sqrt{\Re}} d\vartheta$ where χ_n is given by (380), viz. $\chi_n = \sqrt[\Re]{\chi(y)}$. In the present case, $\chi(y) = |y|$ and $\Re = n$ so that $x_n = n\sqrt{|y|} = y$ (say), and then from (381) and (382),

$$y^{-1} dy d\vartheta_1 = \frac{1}{\sqrt{n}} 2^{n(n-1)/4}|y|^{-\frac{n+1}{2}}[dy] = \frac{1}{\sqrt{n}} d\vartheta. \tag{383}$$

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We can now compute the volume of the fundamental domain of the unimodular group acting on the determinant surface $|y| = 1$. If $\Re$ denotes the space of all reduced matrices, this volume is given by $V_n = \int\limits_{y_1 \in \Re} d\vartheta_1$. Now

$$\begin{aligned} V_n \Lambda(\frac{n(n+1}{2} &= \int\limits_o^\infty e^{-y} y^{\frac{n(n+1}{2}-1} dy \int\limits_{\substack{y_1 \in \Re \\ |y_1|=1}} d\vartheta_1 \\ &= \int\limits_{y \in \Re} e^{\sqrt[n]{|y|}} |y|^{\frac{n+1}{2}} y^{-1} dy d\vartheta_1 \\ &= \frac{1}{\sqrt{n}} 2^{n(n-1)/4} \int\limits_{y \in \Re} e^{-\sqrt[n]{|y|}} [dy]. \end{aligned}$$

The last integral has already been explicitly computed in (149) and putting it in, we have

$$V_n = \frac{n+1}{2} \sqrt{n} 2^{n(n-1)/4} \vartheta_n \tag{384}$$

In particular we have shown that the volume of the fundamental domain is finite. We can therefore state that if $u(y)$ is homogeneous of degree 0, is invariant relative to the modular group and satisfies the equations (362) and if further $\mathcal{U}$ is bounded in the fundamental domain, then $\mathcal{U}$ is square integrable and is hence an angular character.

272 We return now to the question of existence of solutions of (362) and examine how far they meet the other three requirements for an angular character. We first observe that if $U(y)$ is a solution of (362) with the set of eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$, then $\mathcal{U}$ is a homogeneous function of degree $-\lambda_1$ so that $|Y|^{\lambda_1|n}\mathcal{U}(y)$ is homogeneous of degree 0. We show presently that for every solution $\mathcal{U}(y)$ of (362), $|Y|^s\mathcal{U}(y)$ (for an arbitrary complex constant s) is also a solution with certain eigen values $\lambda_1^*, \lambda_2^*, \ldots \lambda_n^*$ depending on s and then in particular $|y|^{\lambda_1|n}\mathcal{U}(y)$ will be a solution of (362) with $\lambda_1^* = o$. In other words, given any $\mathcal{U}(y)$ which satisfies the first condition for an angular character, we can determine one which, besides the first, fulfills the second condition too.

That $|Y|^s\mathcal{U}(y)$ is a solution of (362) is an immediate consequence of the following operator identity,

$$(Y\frac{\partial}{\partial y})^{\Re}|y|^s = \sum_{\nu=0}^{\Re} s^{\nu}\binom{\Re}{\nu}|y|^s(y\frac{\partial}{\partial y})^{\Re-\nu} \tag{385}$$

where we remark that $|Y|$ is to be treated as an operator. For $\Re = 1$ it is directly verified that

$$(y\frac{\partial}{\partial y}|y|^s = |y|^s y\frac{\partial}{\partial y} + s|y|^s E,$$

and then (385) is proved by induction on $\Re$. One also obtains the analogous identity

$$((y\frac{\partial}{\partial y})')^{\Re}|y|^s = \sum_{\nu=0}^{\Re} s^{\nu}\binom{\Re}{\nu}|y|^s((y\frac{\partial}{\partial y})')^{\Re-\nu} \tag{386}$$

and this we need later.

273 We now show that (362) has non trivial solutions which are invariant relative to the unimodular substitutions $y \to y[\mathcal{U}]$. We decompose y, $\dfrac{\partial}{\partial y}$ as

$$y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}, \quad \frac{\partial}{\partial y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_3} \\ \frac{1}{2}\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_2} \end{pmatrix}$$

with $y_1 = y_1^r$, $o < r < n$, and prove by induction on $\Re$ that

$$(y\frac{\partial}{\partial y})^{\Re}|y_1|^{-s} = (-1)^{\Re}s(s-\frac{n-r}{2})^{\Re-1}|y_1|^{-s}\begin{pmatrix} E & o \\ y_3'y_1^{-1} & o \end{pmatrix}. \tag{387}$$

Indeed,

$$\begin{aligned}(y\frac{\partial}{\partial y})|y_1|^{-s} &= -s|y_1|^{-s-1}y\frac{\partial}{\partial y}|y_1| \\ &= -s|y_1|^{-s-1}y\begin{pmatrix} |y_1|y_1^{-1} & o \\ o & o \end{pmatrix} \\ &= -s|y_1|^{-s}\begin{pmatrix} E & o \\ y_3'Y_1^{-1} & o \end{pmatrix}\end{aligned}$$

and then we have only to assume (387) for a particular value of $\Re \geq 1$ and prove it for the next higher value.

Now

$$(y\frac{\partial}{\partial y})^{\Re+1}|y_1|^{-s} = (-1)^{\Re}s(s-\frac{n-r}{2})^{\Re-1}y\frac{\partial}{\partial y}|y_1|^{-s}\begin{pmatrix} \in & 0 \\ y_3'y_1^{-1} & 0 \end{pmatrix} \tag{388}$$

by the induction assumption. Also,

$$\frac{\partial}{\partial y}|y_1|^{-s}\begin{pmatrix} E & o \\ y_3'y_1^{-1} & o \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_3} \\ \frac{1}{2}\frac{\partial}{\partial y_3}, & \frac{\partial}{\partial y_2} \end{pmatrix}\begin{pmatrix} |y_1|^{-s}E & o \\ |y_1|^{-s}y_3'y \quad {}_1^{-1} & o \end{pmatrix} =$$

$$\begin{aligned} &= \begin{pmatrix} -s|y_1|^{-s}y_1^{-1} + \frac{1}{2}|y_1|^{-s}(\frac{\partial}{\partial y_3}y_3')y_1^{-1} & o \\ o & o \end{pmatrix} \\ &= |y_1|^{-s}\begin{pmatrix} -sy_1^{-1} + \frac{1}{2}(n-r)Ey_1^{-1} & o \\ o & o \end{pmatrix} \\ &= -(s-\frac{n-r}{2})|y_1|^{-s}\begin{pmatrix} y_1^{-1} & o \\ o & o \end{pmatrix}. \end{aligned}$$

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Putting this in (388) we have

$$(y\frac{\partial}{\partial y})^{\Re+1}|y_1|^{-s} = (-1)^{\Re+1}s(s-\frac{n-r}{2})^{\Re}|y_1|^{-s}|y_2|^{-s}y\begin{pmatrix} y_1^{-1} & o \\ o & o \end{pmatrix}$$

$$= (-1)^{\Re+1} s(s - \frac{n-r}{2})^{\Re} |y_1|^{-s} \begin{pmatrix} E, & o \\ y_3' y_1^{-1} & o \end{pmatrix}$$

and the proof of (387) is now complete.

It follows from (387) that

$$\left\{\sigma(y\frac{\partial}{\partial y})^{\Re} + \lambda_{\Re}\right\} |y_1|^{-s} = o, \lambda_{\Re} = rs(\frac{n-r}{2} - s)^{\Re-1}; \Re \geq 1. \qquad (389)$$

In other words we have shown that $|y_1|^{-s}$ is a solution of (362) belonging to the set of eigen values.

$$\eta_{\Re} = rs(\frac{n-r}{2} - s)^{\Re-1}, \Re = 1, 2, \ldots n. \qquad (390)$$

But itself $|y_1|^{-s}$ is not invariant relative to the unimodular substitutions; in fact if $\mathcal{U} = (QR)$ is any unimodular matrix with $Q = Q^{(n,r)}$,
275 then due to a transformation $y \to y[\mathcal{U}]$ of y, $y_1 = y\begin{bmatrix} E^{(r)} \\ o \end{bmatrix}$ goes over into $(y[\mathcal{U}]\begin{bmatrix} E^{(r)} \\ o \end{bmatrix} = y[Q]$, and $|y_2|^{-s}$ goes into $|y[Q]|^{-s}$. However it is easily seen from (389) that $|y[Q]|^{-s}$ also satisfies the same equation with the same set of eigen values and then

$$\mathscr{E}(y, s) = \sum_{Q} |y[Q]|^{-s} \qquad (391)$$

where Q runs through a complete representative system of the classes $\{Q^{n,r}\}$ of primitive matrices is again a solution of (362) belonging to the above system of eigen values (390). Of course, we have $\mathscr{E}(y[\mathcal{U}], s) = \mathscr{E}(y, s)$ for any unimodular matrix $\mathcal{U}$; in other words $\mathscr{E}(y, s)$ is a solution of (362), which is invariant relative to the unimodular transformations.

The function $\mathscr{E}(y, s)$ is a generalisation of the usual Epstein's zeta function. It follows from (387) that

$$\left\{(y\frac{\partial}{\partial y})^2 + (s - \frac{n-r}{2})(y\frac{\partial}{\partial y})\right\} |y_1|^{-s} = o$$

and hence we can conclude that

$$\left\{(y\frac{\partial}{\partial y})^2 + (s - \frac{n-r}{2})(y\frac{\partial}{\partial y})\right\} \mathscr{E}(y, s) = o \qquad (392)$$

It is possible to characterise $\mathscr{E}(y, s)$ as a solution of (392), whose Fourier coefficients (in a certain sense) have specified properties.

Finally we prove that for every solution $\mathcal{U}(y)$ of (362), $\mathcal{U}^*(Y) = \mathcal{U}(Y^{-1})$ also satisfies a differential equation of the same kind,

$$\left\{\sigma(y\frac{\partial}{\partial y})^{\Re} + \lambda^*_{\Re}\right\}\mathcal{U}^*(y) = o, \Re = 1, 2, \ldots n,$$

where $\lambda^*_{\Re}, \Re = 1, 2, \ldots n$ are uniquely determined by the eigen value s $\lambda_1, \lambda_2, \ldots \lambda_n$ of $\mathcal{U}$. We need a few operator identities as preliminaries. Let $y\frac{\partial}{\partial y} = \triangle = (\omega_{\mu\nu})$. We prove first that 276

$$\omega_{\mu\nu}\omega_{x\lambda} = \omega_{\mu\lambda}\omega_{\mu\nu} + \frac{1}{2}\delta_{\nu\mu} - \frac{1}{2}\delta_{\lambda\mu}\omega_{\mu\nu} \tag{393}$$

Since $e_{\mu\nu}\frac{\partial y_{x\lambda}}{\partial y_{\mu\nu}} = \frac{1}{2}(\delta_{\mu\nu,x\lambda} + \delta_{\mu,\nu,\lambda\nu}), \delta_{\mu\nu,x\lambda} = \delta_{\mu x}\delta_{\nu\lambda}$

We have

$$\begin{aligned}
\omega_{\mu\nu}\omega_{x\lambda} &= (\sum_{\rho=1}^{n} y_{\mu\rho}e_{\rho\nu}\frac{\partial}{\partial y_{\rho\nu}})(\sum_{\sigma=1}^{n} y_{x\sigma}e_{\sigma\lambda}\frac{\partial}{\partial}y_{\sigma\lambda})\\
&= \sum_{\rho,\sigma=1}^{n} y_{\mu\rho}e_{\rho\nu}\frac{\partial}{\partial y_{\rho\nu}}y_{x\sigma}e_{\sigma\lambda}\frac{\partial}{\partial y_{\sigma\lambda}}\\
&= \sum_{\rho,\sigma=1}^{n} y_{\mu\rho}y_{x\sigma}e_{\rho\nu}\frac{\partial}{\partial y_{\sigma\lambda}}\frac{1}{2}\sum_{\rho,\sigma=1}^{n} y_{\mu\rho}y_{\mu\rho}(\delta_{\rho\nu,x\sigma} + \delta_{\rho\nu,\sigma})e_{\sigma\lambda}\frac{\partial}{\partial y_{\sigma\lambda}}\\
&= \sum_{\rho,\sigma=1}^{n} y_{x\sigma}e_{\sigma\lambda}\frac{\partial}{\partial}y_{\sigma\lambda}y_{\mu\rho}e_{\rho\nu}\frac{\partial}{\partial}y_{\rho\nu}\frac{1}{2}\sum_{\rho,\sigma=1}^{n} y_{\mu\rho}(\delta_{\rho,\nu x\sigma} + \delta_{\rho\nu,\sigma x})\\
&\qquad e_{\sigma\lambda}\frac{\partial}{\partial y_{\sigma\lambda}} - \frac{1}{2}\sum_{\rho,\sigma=1}^{n} y_{x\sigma}(\delta_{\sigma\lambda,\mu\rho} + \delta_{\sigma\lambda,\rho\mu}e_{\rho\nu}\frac{\partial}{\partial}y_{\rho\nu}\\
&= \omega_{x\lambda}\omega_{\mu\nu} + \frac{1}{2}y_{\mu x}e_{\nu\lambda}e_{\nu\lambda}\frac{\partial}{\partial y_{\nu\lambda}} + \frac{1}{2}\delta_{\nu\lambda}\omega_{\mu\nu}\\
&\qquad + \frac{1}{2}y_{x\mu}e_{\lambda\nu}\frac{\partial}{\partial y_{\lambda\nu}} - \frac{1}{2}\delta_{\lambda\mu}\omega_{x\nu}
\end{aligned}$$

$$= \omega_{x\lambda}\omega_{\mu\nu}t\frac{1}{2}\delta_{\nu x}\omega_{\mu\lambda} - \frac{1}{2}\delta_{\lambda\mu}\omega_{\mu\nu},$$

277 and (393) is established.

We use (393) to show that for any operator matrix $A = (\alpha_{\mu\nu})$,

$$\sigma(\triangle^{\Re}A) = \sum_{\nu\nu_2\cdots\nu_{\Re+1}} \omega_{\nu_2\nu_3}\cdots\omega_{\nu_{g-1}\nu_g}\omega_{\nu_1\nu_2}\omega_{\nu_g\nu_{g+1}}\cdots\omega_{\nu_\Re\nu_{\Re+1}}\alpha_{\nu_{\Re+1}\nu_1}$$
$$- \frac{1}{2}\sigma(\Lambda^{g-2})(\Lambda^{\Re+g+1}A) + \frac{n}{2}\sigma(\Lambda^{\Re-1}A) \tag{394}$$

The proof is by induction on j. For $j = 2$ (394) is obviously true. Assume then $j \geq 2$. By means of (393), the induction assumption gives that

$$\sigma(\Lambda^{\Re}A) = \sum_{\nu_1,nu_2\cdots\nu_{\Re+1}} \omega_{\nu_1\nu_2\cdots}\omega_{\nu_\Re\nu_{\Re+1}}\alpha_{\nu_{\Re+1}\nu_1}$$
$$= \sum_{\nu_1,\nu_2,\ldots\nu_{\Re+1}} \omega_{\nu_2\nu_3}\cdots\omega_{\nu_g\nu_{g+1}}\omega_{\nu_1,\nu_2}\omega_{\nu_{g+1}\nu_{g+2}}\cdots\omega_{\nu_\Re\nu_{\Re+1}}\alpha_{\nu_{\Re+1}\nu_1}$$
$$+ \frac{1}{2}\sum_{\nu_1,\nu_2,\ldots\nu_{\Re+1}} \omega_{\nu_2\nu_3}\cdots\omega_{\nu_{g+1}\nu_g}\delta_{nu_2\nu_g}\omega_{\nu_1\nu_{g+1}}\omega_{\nu_{g+1}}$$
$$\omega_{\nu_{g+1}\nu_{g+2}}\cdots\omega_{\nu_\Re\nu_{\Re+1}}\alpha_{\nu_{\Re+1}\nu_1}$$
$$- \frac{1}{2}\sum_{\nu_1,\nu_2,\ldots\nu_{\Re+1}} \omega_{\nu_2\nu_3}\cdots\omega_{\nu_{g-1}\nu_{g+1}\nu_1}\omega_{\nu_g\nu_2}$$
$$\omega_{\nu_{g+1}\nu_{g+2}}\cdots\omega_{\nu_\Re\nu_{\Re+1}}\alpha_{\Re+1\nu_1}$$
$$- \frac{1}{2}\sigma(\Lambda^{g-2})\sigma(\Lambda^{\Re-g+1}A) + \frac{n}{2}\sigma(\Lambda^{\Re-1}\Lambda)$$
$$= \sum_{\nu_1,\nu_2,\ldots\nu_{\Re+1}} \omega_{\nu_2\nu_3}\cdots\omega_{\nu_g\nu_{g+1}}\omega_{\nu_1\nu_2}\omega_{\nu_{g+1}\nu_{g+2}}\cdots\omega_{\nu_\Re\nu_{\Re+1}}\alpha_{\nu_{\Re+1}\nu_1}$$
$$- \frac{1}{2}\sigma(\Lambda^{g-1}\sigma(\Lambda^{\Re-g}\Lambda) + \frac{n}{2}\sigma(\Lambda^{\Re-1}A)$$

278 and this proves (394). Putting $j = k + 1$ in (394) we obtain that

$$\sigma(\Lambda^{\Re}A) = \sum_{\nu_1,\nu_2,\ldots\nu_{\Re+1}} \omega_{\nu_2\nu_3}\cdots\omega_{\nu_\Re\nu_{\Re+1}}\omega_{\nu_1\nu_2}\alpha_{\nu_{\Re+1}\nu_1}$$

$$\frac{1}{2}\sigma(\Lambda^{\mathfrak{K}-1})\sigma(A) + \frac{n}{2}\sigma(\Lambda^{\mathfrak{K}-1}A)$$
$$= \sigma(\Lambda^{\mathfrak{K}-1}(\Lambda' A')) - \frac{1}{2}\sigma(\Lambda^{\mathfrak{K}-1})\sigma(A) + \frac{n}{2}\sigma(\Lambda^{\mathfrak{K}-1}A).$$

Choosing for A the special operator matrix $((\Lambda')^{\ell})'$ this yields,

$$\sigma(\Lambda^{\mathfrak{K}}((\Lambda')^{\ell})' = \sigma(\Lambda^{\mathfrak{K}-1}((\Lambda')^{\ell+1})')$$
$$- \frac{1}{2}\sigma(\Lambda^{\mathfrak{K}-1})\sigma(\Lambda')^{\ell}\frac{n}{2}\sigma(\Lambda^{\mathfrak{K}-1}((\Lambda')^{\ell})' \qquad (395)$$

As a consequence of (395) we can conclude by induction on $\mathfrak{K}$ that

$$\sigma(\Lambda^{\mathfrak{K}} = \sigma(\Lambda')^{\mathfrak{K}} + \sum_{\substack{\nu_1\nu_2\cdots\nu_r \\ \nu_1+\cdots+\nu_r<\mathfrak{K}}} e^{\mathfrak{K}}_{\nu_1\nu_2..\nu_r}\sigma(\Lambda')^{\nu_s}\sigma(\Lambda')^{\nu_2}\cdots\sigma(\Lambda')^{\nu_r} \qquad (396)$$

Inverting this system of equations one also has

$$\sigma(\Lambda')^{\mathfrak{K}} = \sigma(\Lambda^{\mathfrak{K}}) + \sum_{\substack{\nu_1\nu_2\cdots\nu_n \\ \nu_1+\cdots+\nu_r<\mathfrak{K}}} d^{\mathfrak{K}}_{\nu_1\nu_2..\nu_n}\sigma(\Lambda)^{\nu_1}\sigma(\Lambda)^{\nu_2}\cdots\sigma(\Lambda)^{\nu_r} \qquad (397)$$

with certain constant coefficients $d^{k}_{\nu_1\nu_2\ldots\nu_r}$. **279**

Let now $y^* = y^{-1}$ and $\mathcal{U}^*(y) = \mathcal{U}(y^*)$. Then $dy^* = -y^{-1}dyy^{-1}$, and for any function φ we have

$$d\varphi = \sigma(dy\frac{\partial}{\partial y}\varphi) = \sigma(dy^*\frac{\partial}{\partial y^*}\varphi) = -\sigma(dyy^{-1}(\frac{\partial}{\partial y^*}\psi)y^{-1})$$

and consequently

$$\frac{\partial}{\partial y}\varphi = -y^{-1}(\frac{\partial}{\partial y^*}\varphi)y^{-1}, \text{ on } y\frac{\partial}{\partial y} = -(y^*\frac{\partial}{\partial y^*})'.$$

Replacing y by $y*$ in (397) we can state now that

$$\sigma(y\frac{\partial}{\partial y})^{\mathfrak{K}}\mathcal{U}^*(y) = (-1)^{\mathfrak{K}}\sigma((y^*\frac{\partial}{\partial y^*}))^{\mathfrak{K}}\mathcal{U}(y^*)$$
$$= (-1)^{\mathfrak{K}}\left\{\sigma(\chi^*\frac{\partial}{\partial y^*})^{\mathfrak{K}} + \sum_{\substack{\nu_1\nu_2\cdots\nu_r \\ \nu_1+\nu_r<\mathfrak{K}}} d^{\mathfrak{K}}_{\nu_1\nu_2\cdots\nu_r}\sigma(y^*\frac{\partial}{\partial y^*})^{\nu_1}\cdots\sigma(y^*\frac{\partial}{\partial y^*})^{\nu_r}\right\}\mathcal{U}(y^*)$$

$$= (-1)^{\Re}\left\{-\lambda_{\Re} + \sum_{\substack{\nu_1\nu_2 1\cdots\nu_r\\ \nu_1+\cdots+\nu_r<\Re}} (-1)^r d^{\Re}_{\nu_1\nu_2\cdots\nu_r}\lambda_{\nu_1}\lambda_{\nu_2}\cdots\lambda_{\nu_r}\right\}\mathcal{U}(y^*)$$
$$= -\lambda^*_{\Re}\mathcal{U}^*(y)$$

with

$$\lambda^*_{\Re}(-1)^{\Re}\left\{\sum_{\substack{\nu_1\nu_2\cdots\nu_r\\ \nu_1+\cdots+\nu_r<\Re}} (-1)^r d^{\Re}_{\nu_1\nu_2\cdots\nu_n}\lambda_{\nu_1}\lambda_{\nu_2}\cdots\lambda_{\nu_r}\right\} \tag{398}$$

A detailed calculation shows that

$$\sigma(\Lambda') = \sigma(\Lambda), \sigma(\Lambda')^2 = \sigma(\Lambda)^2$$
$$\sigma(\Lambda')^3 = \sigma(\Lambda)^3 - \frac{n}{2}\sigma(\Lambda)^2 + \frac{1}{2}(\sigma(\Lambda))^2,$$
$$\sigma(\Lambda')^4 = \sigma(\Lambda)^4 - n\sigma(\Lambda)^3 + \frac{1}{2}\sigma(\Lambda)\sigma(\Lambda)^2 + \frac{1}{2}$$
$$+ \frac{1}{2}\sigma(\Lambda)^2\sigma(\Lambda)\frac{n^2}{4}\sigma(\Lambda)^2 - \frac{n}{4}(\sigma(\Lambda))^2$$

280 and it follows then that

$$\lambda^*_1 = \lambda_1, \lambda^*_2, \lambda^*_3 = -\lambda_3 + \frac{n}{2}\lambda_2 + \frac{1}{2}\lambda_1^2,$$
$$\lambda^*_4 = \lambda_4 - n\lambda_3 - \lambda_1\lambda_2 + \frac{n^2}{4}\lambda_2 + \frac{n}{4}\lambda_1^2.$$

It would be of interest to determine the cases when we shall have $\lambda^*_\nu = \lambda_\nu$ for each ν.

When all is said and done, the question remains still open whether angular characters according to our definition actually exist. We only know that if we are given one angular character we can obtain others from it; for instance, if $\mathcal{U}(y)$ is angular character then so is $\mathcal{U}^*(y) = \mathcal{U}(y^{-1})$. For $n > 2$ the existence question is still a problem. For $n = 2$ however, the angular characters are completely determined and they are just solutions of the wave equations relative to the modular group.

Let us introduce the operators $H_{\Re}$ by

$$H_{\Re} = \frac{1}{2}(\sigma(y\frac{\partial}{\partial y})^{\Re} + \sigma((y\frac{\partial}{\partial y}i)^{\Re}) \tag{399}$$

and observe in view of (397) that 281

$$H_{\Re} = \sigma(y\frac{\partial}{\partial y})^{\Re} + \frac{1}{2}\sum_{\substack{\nu_1\nu_2\cdots\nu_r \\ \nu_1+\cdots+\nu_r<\Re}} d_{\nu_1\nu_2\cdots\nu_r}\sigma(y\frac{\partial}{\partial y})^{\nu_1}\cdots\sigma(y\frac{\partial}{\partial y})^{\nu_r}$$

for $\Re = 1, 2\cdots n$.

It is easily seen that by means of these n equations, $\sigma(y\dfrac{\partial}{\partial y})^{\Re}, \Re = 1.2.\cdots n$ can be expressed in terms of the $H'_{\Re}s$.

It follows therefore that $H_{\Re}, \Re = 1.2\cdots n$ also generate the space of invariant (linear) differential operators as $\sigma(y\dfrac{\partial}{\partial y})^{\Re} - \Re = 1.2\cdots n$ do. The angular characters $\mathcal{U}(y)$ are then eigen functions of these operators $H_{\Re}, \Re = 1.2\cdots n$, and in place of (362) we have an equivalent system of differential equations for $\mathcal{U}(y)$, viz.

$$(H_{\Re} + \mu_{\Re}\mathcal{U}(y) = o, \Re = 1.2, \ldots n \tag{400}$$

with certain eigen values $\mu_{\Re}(\Re = 1.2\cdots n)$ which are uniquely determined by the eigen values $\lambda_{\Re}$ of $\mathcal{U}(y)$ and conversely. We shall give an interesting integral formula (407) involving these operators.

First we have to generalise the method of partial integration. Let $A = (a_{\mu\nu})$, $B = (\ell_{\mu\nu})$ be arbitrary matrices whose elements are all functions of the elements of y We know that $d\vartheta = 2^{n(n-1)/4}|y|^{-\dfrac{n+1}{2}}[dy]$ is the volume element in the space of matrices $y > o$, invariant relative to the transformations $y \to y[R]$, $|R| \neq 0$. Let

$$\omega_{\mu\nu} = \pm\prod_{\rho,\sigma\neq\mu,\nu} dy_{\rho\sigma},$$

the ambiguous sign being determined by the requirement that

$$dy_{\mu\nu}\omega_{\mu\nu} = [dY] \text{and let}$$
$$\Omega = 2^{n(n-1)/4}|Y|^{-\frac{n+1}{2}}Y(e_{\mu\nu}\omega_{\mu\nu}),$$

a matrix of differential forms of degree $\frac{n(n+1)}{2} - 1$. We wish to prove 282 that

$$\int\limits_{\mathcal{G}} B\{Y\frac{\partial}{\partial Y}A\}d\vartheta = -\int\limits_{\mathcal{G}}\{Y(\frac{\partial}{\partial Y})'B'\}'Ad\vartheta + \int\limits_{\ell\mathcal{G}} B\Omega A \tag{401}$$

where $\mathcal{G}$ denotes a compact domain with a given orientation and *bg* is its boundary, assumed to be piece wise smooth. Now

$$\begin{aligned} & B(y\frac{\partial}{\partial Y}A) + ((Y\frac{\partial}{\partial Y})'B')A = \\ & = (\sum_{\rho\sigma\tau})\ell_{\mu\rho}y_{\rho\sigma}e_{\sigma\tau}(\frac{\partial}{\partial y}_{\sigma\tau}a_{\tau\nu})) + (\sum_{\rho\sigma\tau}e_{\rho\tau}(\frac{\partial}{\partial Y}_{\rho\tau}\ell_{\mu\sigma})a_{\tau\nu}) \\ & = (\sum_{\rho\sigma\tau}y_{\rho\sigma}e_{\rho\sigma\tau}(\ell_{\mu\sigma}\frac{\partial}{\partial Y}_{\rho\tau}a_{\tau\nu} + a_{\tau\nu}\frac{\partial}{\partial Y}_{\rho\tau}\ell_{\mu\sigma})) \end{aligned} \tag{402}$$

Also

$$\begin{aligned} d(B\Omega A) &= d(B2^{n(n+1)}/4)|Y|^{-\frac{n+1}{2}}Y(e_{\mu\nu\omega_{\mu\nu}})A) \\ &= 2^{n(n+1)/4}d(\sum_{\rho\sigma\tau}\ell_{\mu\nu}|Y|^{-\frac{n+1}{2}}y_{\rho\sigma}r_{\sigma\tau\omega_{\sigma\tau}a_{\mu\nu}}) \\ &= 2^{n(n+1)/4}(\sum_{\rho\sigma\tau}e_{\sigma\tau}\frac{\partial}{\partial y}_{\sigma\tau}|Y|^{-\frac{n+1}{2}}y_{\rho\sigma})\omega_{\sigma\tau}) \\ &= 2^{n(n+1)/4}(\sum_{\rho\sigma\tau}e_{\sigma\tau}\frac{\partial}{\partial y}_{\sigma\tau}(\ell_{\mu\rho}a_{\tau\nu}|Y|^{-\frac{n+1}{2}}y_{\rho\sigma})\omega_{\sigma\tau})[dY] \\ &= (\sum_{\rho\sigma\tau}y_{\rho\sigma}e_{\sigma\tau}\frac{\partial}{\partial y_{\sigma\tau}}(\ell_{\mu\rho}a_{\tau\nu})d\vartheta + \\ &\qquad + 2^{n(n-1)/4}(\sum_{\rho\sigma\tau}e_{\sigma\tau}\ell_{\mu\rho}a_{\tau\nu}\frac{\partial}{\partial y}_{\sigma\tau}(|Y|^{-\frac{n+1}{2}y_{\rho\sigma}})[dy] \end{aligned} \tag{403}$$

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We shall show that $\sum_\sigma e_{\sigma\tau}\frac{\partial}{\partial y}_{\sigma\tau}(|Y|^{-\frac{n+1}{2}y_{\rho\sigma}}) = 0$, and then the last term on the right side of (403) vanishes. Let $Y_{\mu\nu}$ denoted the algebraic minors of the elements of Y so that $|Y|Y^{-1} = (Y_{\mu\nu})$

Then

$$\sum_{\sigma} e_{\sigma\tau} \frac{\partial}{\partial y_{\sigma\tau}} (|Y|^{-\frac{n+1}{2}} y_{\rho\sigma}) =$$
$$= \frac{n+1}{2} |Y|^{-\frac{n+1}{2}} \sum_{\sigma} Y_{\sigma\tau} y_{\rho\sigma} + \sum_{\sigma} |Y|^{-\frac{n+1}{2}} \delta_{\rho\tau} e_{\sigma\tau}$$
$$= \frac{n+1}{2} |Y|^{-\frac{n+1}{2}} \delta_{\rho\tau} + \frac{n+1}{2} |Y|^{\frac{n+1}{2}} \delta_{\rho\tau} = 0$$

as desired. (403) now reduces, in view of (402), to

$$d(B\Omega A) = (\sum_{\rho\sigma\tau} y_{\rho\sigma} e_{\rho\sigma} \frac{\partial}{\partial_{\sigma\tau}} (\ell_{\mu\rho} a_{\tau\nu})) d\vartheta$$
$$= (B(Y \frac{\partial}{\partial Y} A) + ((Y \frac{\partial}{\partial Y})' B', A) d\vartheta$$

and then, applying Stoke's formula by which

$$\int_{\mathcal{G}} d\omega = \int_{\ell\mathcal{G}} \omega$$

for an arbitrary exterior differential from, ω we have

$$\int_{\mathcal{G}} B(Y \frac{\partial}{\partial Y} A) d\vartheta = -\int_{\mathcal{G}} ((Y \frac{\partial}{\partial Y})' B)' A d\vartheta + \int_{\ell\mathcal{G}} B\Omega A.$$

More generally we can show that **284**

$$\int_{\mathscr{G}} \left\{ (y \frac{\partial}{\partial y})^{\mathfrak{K}} A' \right\} d\vartheta = (-1)^{\mathfrak{K}} \int_{\mathscr{G}} \left\{ (y \frac{\partial}{\partial y})')^{\mathfrak{K}} B' \right\} A d\vartheta + \sum_{\vartheta=0}^{\mathfrak{K}-1} (-1)^{\nu}$$
$$\int_{\ell\mathscr{G}} \left\{ ((y \frac{\partial}{\partial y})^{\nu} B' \right\} \Omega \left\{ Y (\frac{\partial}{\partial y})^{\mathfrak{K}-1-\nu} \right\} \qquad (404)$$

For $\mathfrak{K} = 1$, (404) reduces to (401) proved above and then (404) is established by restoring to introduction on $\mathfrak{K}$

Setting $A = \varphi E$ and $B = \psi E$ in (404) where φ and ψ are arbitrary functions, and taking the trace of both sides, we obtain that

$$\int_{\mathcal{G}} \psi\sigma(y\frac{\partial}{\partial y})^{\mathfrak{R}}\varphi d\vartheta = (-1)^{\mathfrak{R}} = \int_{\mathcal{G}} \psi\sigma(y\frac{\partial}{\partial y})')^{\mathfrak{R}}\psi d\vartheta + \sum_{\nu=0}^{\mathfrak{R}=1}(-1)^{\nu} \int_{\ell\mathcal{G}} \sigma\left\{((y\frac{\partial}{\partial})^{\nu})\right\}\Omega\left\{(y\frac{\partial}{\partial y})^{\mathfrak{R}-1-\nu}\varphi\right\} \tag{405}$$

Interchanging φ and ψ and replacing ν by $\mathfrak{R}-1-\nu$ in (405) we obtain by transposition that

$$\int_{\mathcal{G}} \psi\sigma((y\frac{\partial}{\partial y})')^{\mathfrak{R}}\psi d\vartheta = (-1)^{\mathfrak{R}} = \int_{\mathcal{G}} \psi\sigma(y\frac{\partial}{\partial y})^{\mathfrak{R}}\psi d\vartheta + \sum_{\nu=0}^{\mathfrak{R}}(-1)^{\nu} \int_{\ell\mathcal{G}} \sigma\left\{(y\frac{\partial}{\partial})^{\nu}\right\}\Omega\left\{((y\frac{\partial}{\partial y})')^{\mathfrak{R}-1-\nu}\right\}\varphi \tag{406}$$

It follows by addition, from (405) and (406) that

$$\int_{\mathcal{G}} \psi H_{\mathfrak{R}}\psi d\vartheta = (-1)^{\mathfrak{R}} \int_{\mathcal{G}} \varphi H_{\mathfrak{R}}\psi d\vartheta + \frac{1}{2}\sum_{\nu=0}^{\mathfrak{R}-1}(-1)^{\nu} \int_{\ell\mathcal{G}} \sigma\left\{((y\frac{\partial}{\partial y})')^{\nu}\psi\right\}\Omega\left\{(y\frac{\partial}{\partial y})^{\mathfrak{R}-1-\nu}\varphi\right\} + \frac{1}{2}\Sigma_{\nu=0}^{\mathfrak{R}-1}(-1)^{\nu} \int_{\ell G} \sigma\left\{(\frac{\partial}{\partial y})^{\nu}\psi\right\}'\Omega\left\{((y\frac{\partial}{\partial y})')^{\mathfrak{R}-1-\nu}\varphi\right\} \tag{407}$$

285 where $H_{\mathfrak{R}}$ is defined in (399).

By means of the formula (407) one can prove the orthogonal relations for angular characters. For we require a special representation for the scalar product of the angular characters given below. Let u, $\tilde{u}$, be two angular characters and introduce the scalar product $(u, \tilde{u})$ by

$$(u, \tilde{u}) = \int_{y_i \in \mathfrak{R}|y_1|=1} u(r_1)\tilde{u}(y_1)d\vartheta_1 \tag{408}$$

Then,

$$
\begin{aligned}
(u,\tilde{u}) &= \int\limits_{\substack{y_1\in\mathfrak{K}\\ |y_1|=1}} u(y_1)\tilde{u}(y_1)d\vartheta_1 \int\limits_1^e y^{-1}dy, y=|y|^{1/n}\\
&= \int\limits_{\substack{y\in\mathfrak{K}\\ i\leq|y|\leq e^n}} u(y)\tilde{u}y^{-1}dyd\vartheta_1\\
&= \int\limits_{\substack{y\in\mathfrak{K}\\ i\leq|y|\leq e^n}} u(y)\tilde{u}(y)d\vartheta
\end{aligned}
$$

We now take for $\mathscr{G}$ in (407) the domain $\{y\in\mathfrak{K}, 1, |y|\leq e^n\}$.

This domain is not compact and this presents some difficulty. We have to approximate this by compact domains $\mathscr{G}i=1,2,\ldots$, apply (407) to each $\mathscr{G}_i$ and resort to a limiting process. Of course all this needs 286 justification.

A detailed account of some of the results quoted in this section one finds in the following references.

1. E. Hecke , Eine neue Art von Zetafunktionen und ihre Beziehungen Zur Verteilung der Primzahlen, math.Zeit. 1 (1918), 357-376.

2. W. Roelcke , Über die Wellengleichung bei Grenzkreisgruppen erster Art, Act. Math (in print).

Chapter 18

The Dirichlet series corresponding to modular forms of degree *n*

We wish to determine a set of Dirichlet series which is equivalent with a given modular form of degree n. We need as pre liminaries the followings the following lemma and a host of other operator identities. **287**

Lemma 27. *Let* $\mathfrak{f}(Z) = \sum_{T geO} a(T)e^{2\pi i\sigma(TZ)}$ *(T semi integral) (98) be a modular from of degree n and weight* $\mathfrak{K} \equiv (2)$. *Then*

$$|a(T)| \leq \vartheta_o |T|^{\mathfrak{K}} \tag{409}$$

for $T > o$ *with a certain constant* ϑ_o

Proof. In view of (95) both sides of (409) are invariant relative to a modular substitution $T \to T[u]$.Hence we need prove (409) only for $T > o$ which are further reduced. Clearly $a(T)$ has the integral representation,

$$a(T) = \int\limits_{\mathcal{H}} f(\times + iT^{-1})e^{2\pi i\sigma(T\times i^{-1})}{}_{[d\times]}$$

as in (pp.64) and then

$$|a(T)| \leq \max_{\times\varepsilon\mathcal{H}} |f(\times + iT^{-1})|e^{2\pi n}$$

Hence to infer (401) we need only prove that

$$|f(\times + iT^{-1})| \leq \mathscr{C}_1 |T|^{\mathfrak{K}}$$

and this again is ensured from (49) as by assumption T is reduced pro-
288 vided we show that

$$|\mathfrak{f}(\times + iT^{-1})| \leq \mathscr{C}_2 (t_{11} t_{22} \ldots t_{nn})^{\mathfrak{K}} \tag{410}$$

where we assume $T = (t_{\mu\nu})$ □

We now prove (410). Let $Z \in \mathscr{Y}_n$ and determine $M \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$ such that $Z_1 = M < Z > \in \mathfrak{f}_n$ Let $N \begin{pmatrix} S & -E \\ E & D \end{pmatrix} \in M_n$ and $Z_a N^{-1} < Z >= (-Z + S)^{-1}$.

Then $Z_1 = MN < Z_o >$ and $MN = \begin{pmatrix} * & * \\ cs + d & -\varepsilon \end{pmatrix}$ by the property of modular forms we have $\mathfrak{f}(Z_o) = | - z + s|^{\mathfrak{K}} \mathfrak{f}(z)$, and $\mathfrak{f}(z_1) = |(cs + D)z_o - C_o| f(z_o)$ and consequently

$$\mathfrak{f}(z) = | - z + s|^{-\mathfrak{K}} |(cs + D)z_o - C| \mathfrak{f}(z_1) \tag{411}$$

Choose $S = (\mathscr{S}_{\mu\nu})$ as an integral symmetric matrix with $\pm \mathscr{S}_{\mu\nu} \leq n$ and $|cs + D| \neq 0$. Such a choice is clearly possible as $|CS + D|$ is a polynomial of degree almost n. Not vanishing identically in the elements $\mathscr{S}_{\mu\nu}$ of S. Since C, S, D are all integral matrices, we have then in particular

$$||CS + D|| \geq 1. \tag{412}$$

Let Y, Y_o denote the imaginary parts of Z, Z_o. Since $||Z + S|| \geq |Y|$ for any real symmetric matrix S and $|Y_o| = |Y|| - Z + S|^{-2}$ for $z_o = (-Z + S)^{-1}$
289 as is seen from (72) we conclude from (411) - (412) that

$$\begin{aligned} |f(Z)| &\leq \mathscr{C}_3 || - z + s||^{\mathfrak{K}} ||z_o - (cs + D)^{-1} c||^{-\mathfrak{K}} \\ &\leq \mathscr{C}_3 || - z + s||^{-\mathfrak{K}} |y_o|^{-\mathfrak{K}} \\ &= \mathscr{C}_3 || - Z + S||^{\mathfrak{K}} |y|^{-\mathfrak{K}} | \end{aligned} \tag{413}$$

If now elements of X are reduced modulo 1 as in (410) and $Y = T^{-1}, T > 0$, and semi integral, then the elements of $-X$ and of $B = -X + S$ are bounded , and

$$(-Z + S)y^{-1} = (BT - iE), Z = \times + iY \tag{414}$$

The elements of $(BT - iE)$ are of the form $\sum \mathscr{C}_{\mu\nu} t_{\rho\nu} - i\rho_{\mu\nu}$ and the absolute value of $\{\sum \mathscr{C}_{\mu\nu} t_{\rho\nu} - i\rho_{\mu\nu}\}$ is at most equal to $\mathscr{C}_4 t_{\nu\nu}$ as $t_{\nu\nu} \geq 1$. Hence $\|BT - iE\| \leq \mathscr{C}_5 t_{11} t_{12} \ldots t_{nn}$ and (410) now follows as a consequence of (413) and (414).

Let us split up the series on the right of 98′ according to the rank of T and write

$$\mathfrak{f}(iy) = \sum_{r=0}^{n} \mathfrak{f}_r(y),\ f_n(y) = \sum_{\substack{t \geq 0 \\ ran\mathfrak{R} T = r}} a(T) e^{2\pi\sigma(TY)}.$$

Set $Y = yY_1, y > 0$ and $|Y_1| = 1$, and introduce the integral

$$R(\mathscr{S}, Y_1) = \int_{o}^{\infty} f_n(yY_1) y^{n\mathscr{S}-1} dy \tag{415}$$

$R(\mathscr{S}, Y_1)$ is a function on the determinant surface $|Y_1| = 1$ and is invariant relative to the unimodular substitutions, viz.

$$R(\mathscr{S}, Y_1[u]) = R(\mathscr{S}, Y_1)$$

for any unimodular matrix u. Let $u(y)$ be any angular character and let **290**

$$\xi(\mathscr{S}, u) = \int_{\substack{Y_1 \varepsilon \mathfrak{R} \\ |Y_1|=1}} R(\mathscr{S}, Y_1) u(Y_1) d\vartheta_1 \tag{416}$$

where as usual, $\mathfrak{R}$ is the space of reduced matrices.

Since $y^{-1} dy d\vartheta_1 = \frac{1}{\sqrt{n}} d\vartheta$ from (383), we have

$$\xi(\mathscr{S}, u) = \int_{\substack{Y_1 \varepsilon \mathfrak{R} \\ |Y_1|=1}} \int_{y=0}^{\infty} \mathfrak{f}_n(Y) u(Y) |Y|^{n-1} y dy d\vartheta_1$$

$$= \frac{1}{\sqrt{n}} \int\limits_{Y \varepsilon \Re} \mathfrak{f}_n(Y) u(Y) |Y|^{\mathscr{S}} d\vartheta$$

In this we substitute for $\mathfrak{f}_n(Y)$ its representation by a series and integrate termwise. All these and the subsequent transformations can be justified if $u(Y)$ is bounded and the real part of s is sufficiently large, and we assume this to be the case. Then

$$\xi(\mathscr{S}, u) = \frac{1}{\sqrt{n}} \sum_{T > o} a(T) \int\limits_{Y \varepsilon \Re} e^{-2\pi\sigma(TY)} u(Y) |Y|^{\mathscr{S}} d\vartheta \tag{417}$$

In the right side (417) we have a sum of the type $\sum\limits_{T>o} F(T)$ where T runs over all semi integral matrices.

If { T } denotes the class of all matrices $T[u]$ for a given $T(u)$ (-unimodular) then in general we have

$$\sum_{T} > 0 F(T) = \sum_{\{T\}} \frac{1}{\varepsilon(T)} \sum_{u} FF(T[u])$$

where $\in (T)$ denotes the number of units of T, i.e., of unimodular ma-
291 trices u with $T[u] = T$ this number is finite as $T > 0$ and u runs over all unimodular matrices while $\{T\}$ runs over all distinct classes of positive semi integral matrices T. Since by assumption $\Re \equiv 0(2)$ we have from (95) that $a(T[u]) = a(T)$ and then from (417),

$$\begin{aligned}
\xi(\mathscr{S}, u) &= \frac{1}{\sqrt{n}} \sum_{\{T\}} \frac{a(T)}{\varepsilon(T)} \sum_{u} \int\limits_{Y \varepsilon \Re} e^{-2\pi\sigma(T[u]Y)} u(Y) |Y|^{\mathscr{S}} d\vartheta \\
&= \frac{1}{\sqrt{n}} \sum_{\{T\}} \frac{a(T)}{\varepsilon(T)} \sum_{u} \int\limits_{Y \varepsilon \Re} e^{-2\pi\sigma(T[u']Y)} u(Y[u']) |Y[u']|^{\mathscr{S}} d\vartheta \\
&= \frac{1}{\sqrt{n}} \sum_{\{T\}} \frac{a(T)}{\varepsilon(T)} \int\limits_{Y > 0} e^{-2\pi\sigma(TY)} u(Y) |Y|^{\mathscr{S}} d\vartheta
\end{aligned}$$

The substitution $Y[R] \to Y$ where $T = RR'$ then yields

$$\xi(\mathscr{S}, u) = \frac{2}{\sqrt{n}} \sum_{\{T\}} \frac{a(T)}{\varepsilon(T)} |T|^{\mathscr{S}} \int\limits_{Y > o} e^{-2\pi\sigma(Y)}{}_{u_1(Y)|Y|^{\mathscr{S}} d\vartheta} \tag{418}$$

where

$$u_1(Y) = u(Y[R^{-1}])$$

Where

$$W(\mathscr{S}, u_1) = \int\limits_{Y>o} e^{-2\pi\sigma(Y)}{}_{u_1(Y)|Y|^{\mathscr{S}}d\vartheta} \tag{419}$$

that gives

$$\xi(\mathscr{S}, u) = \frac{2}{\sqrt{n}} \sum_{\{T\}} \frac{a(T)}{\varepsilon(T)} |T|^{\mathscr{S}} W(\mathscr{S}, u_1) \tag{420}$$

We may remark that u_1 is a solution of (362) belonging to the same set of eigen values as u and if u is bounded so is u_1. We have to compute $W(\mathscr{S}, u_1)$ and before we do this we need some preparations. For an arbitrary function matrix $A = (a_{\mu\nu})$ we prove that 292

$$(Y\frac{\partial}{\partial Y})'AY = (Y(Y\frac{\partial}{\partial Y})'A)')' \to \frac{1}{2}A'Y + \frac{1}{2}\sigma(AY)E \tag{421}$$

Indeed we know that

$$e_{\rho\mu}\frac{\partial y}{\partial y_{\rho\mu}} = \frac{1}{2}(\delta_{\rho\tau}\delta_{\mu\nu} + \delta_{\rho\nu}\delta_{\mu\tau})$$

and then

$$\begin{aligned}
(Y\frac{\partial}{\partial Y})'AY &= (\sum_{\rho} y_{\nu\rho}e_{\rho\mu}\frac{\partial}{\partial y_{\rho\mu}})(\sum_{\tau} a_{\mu\tau}y_{\tau\nu}) \\
&= (\sum_{\rho\sigma\tau}) y_{\sigma\rho}e_{\sigma\mu}\frac{\partial}{\partial y_{\rho\mu}} a_{\sigma\tau}y_{\tau\nu}) \\
&= (\sum_{\rho\sigma\tau} y_{\mu\tau}y_{\sigma\rho}e_{\rho\nu}\frac{\partial}{\partial y_{\rho\nu}} a_{\sigma\tau})' + \frac{1}{2}(\sum_{\rho\sigma\tau} y_{\sigma\rho}a_{\sigma\tau}\delta_{\rho\tau}\delta_{\mu\nu}) \\
&\qquad + \frac{1}{2}(\sum_{\rho\sigma\tau} y_{\sigma\rho}a_{\sigma\tau}\delta_{\rho\tau}\delta_{\mu\nu}) \\
&= (Y((Y\frac{\partial}{\partial Y})'A)')' + \frac{1}{2}\sigma(AY)E + \frac{1}{2}A'Y.
\end{aligned}$$

By induction on $\mathfrak{R}$ and by means of (421) one easily proves now that

$$(Y\frac{\partial}{\partial Y})'Y^{\mathfrak{R}} = \frac{\mathfrak{R}}{2}Y^{\mathfrak{R}} + \frac{1}{2}\sum_{\nu=1}^{\mathfrak{R}}\sigma(Y^{\nu})Y^{\mathfrak{R}-\nu}(\mathfrak{R} \geq 1) \tag{422}$$

293 A direct computation yields that

$$(Y\frac{\partial}{\partial Y})'\sigma(Y^{\mathfrak{R}}) = \mathfrak{R}Y^{\mathfrak{R}}(\mathfrak{R} > 0) \tag{423}$$

and

$$(Y\frac{\partial}{\partial Y})'e^{-2\pi\sigma(Y)}{}_{=-2\pi e}{}^{-2\pi\sigma(Y)}{}_{Y} \tag{424}$$

By repeated applications of (422) – (424) we obtain that

$$((Y\frac{\partial}{\partial Y})')^{\mathfrak{R}}e^{-2\pi\sigma(Y)}{}_{(-2\pi)^{\mathfrak{R}}Y^{\mathfrak{R}}} + e^{-2\pi\sigma(Y)} \sum_{\nu_1+\cdots+\nu_r+\nu<\mathfrak{R}} a^{\mathfrak{R}}_{\nu_1\nu_2\ldots\nu_r\nu}\nu\sigma(Y^{\nu}1)\ldots\sigma(Y^{\nu r})Y^{\nu} \tag{425}$$

with certain constant coefficients $a^{\mathfrak{R}}{}_{\nu_1\nu_2\ldots\nu_r\nu}$. Taking traces, we have $\sigma((Y\frac{\partial}{\partial Y})')^{\mathfrak{R}}e^{-2\pi\sigma(Y)} = e^{-2\pi\sigma(Y)}\{(-2\pi)^{\mathfrak{R}} + \sum_{\nu_1+\cdots+\nu_{\mathscr{S}}<\mathfrak{R}}$ $mathscrC^{\mathfrak{R}}_{\nu_1\nu_2\ldots\nu_{\mathscr{S}}}\sigma(Y_1^{\nu})\ldots\sigma(Y^{\nu_{\mathscr{S}}})\}$ and this is easily generalised (by inductions on ρ) to the following result, viz.

$$\begin{aligned}&\sigma((Y\frac{\partial}{\partial Y})')^{\nu_\rho}\sigma((Y\frac{\partial}{\partial Y})')^{\nu_{\rho-1}}\ldots\sigma((Y\frac{\partial}{\partial Y})')^{\nu_1}e^{-2\pi\sigma(Y)}\\ =&e^{2\pi\sigma(Y)}\left\{(-2\pi)^{\nu_1+\cdots+\nu_\rho}{}_{\sigma(Y^{\nu_1})}\ldots\sigma(Y^{\nu_\rho}) + \mathscr{C}_{\nu_1\nu_2\ldots\nu_\rho}(Y)\right\}\end{aligned} \tag{426}$$

with

$$\mathscr{C}^{(Y)}_{\nu_1\nu_2\ldots\nu_\rho} = \sum_{\substack{\mu_1\ldots\mu_r\\ \mu_1+\cdots+\mu_r<\nu_1+\cdots+\nu_\rho}} C^{\nu_1\nu_2\ldots\nu_\rho}_{\mu\mu_2\ldots\mu_r}\sigma(Y^{\mu_1})\sigma(Y_2^{\mu})\ldots\sigma(Y_r^{\mu})$$

294 The interesting fact is that the terms on the right side of (426) are all homogeneous (of different degrees) and the first term is of the highest degree, viz $\nu_1 + \nu_2 + \cdots + \nu_\rho$ while the degrees of the other terms

are strictly less. The $\mathscr{C}^{\Re}_{\nu_1\nu_2\ldots\nu_r}{}'\mathscr{S}$ and $e^{\nu_1\ldots\nu_\rho}_{\mu_1\ldots\mu_r}{}'\mathscr{S}$ occurring above are all constants.

We now return to the integral formula (405) and specialise that functions φ and ψ occurring there as $\varphi = u_1(Y), \psi = \mathfrak{f}(Y)|Y|^{\mathscr{S}}$ where $u_1(Y)$ is a bounded solution of the differential equation system (362) and f is an arbitrary function at our disposal. For $\mathscr{G}$ we now take the whole domain $Y > 0$. By a suitable choice of $f(Y)$ we can obtain the integral over the whole domain as a limit of integrals over appropriate subdomains $\mathscr{G}$ such that in the limit, the integrals over the boundary of $\mathscr{G}$ vanish. If $\lambda_1 \ldots \lambda_n$ are the eigenvalues of $u_1(Y)$, using (383) we obtain- from (405) for $\Re \geq n$ that

$$
\begin{aligned}
&-\lambda_{\Re} \int\limits_{Y>o} \mathfrak{f}(Y|Y|^{\mathscr{S}} u_1(Y) d\vartheta) = \\
&= (-1)^{\Re} \int\limits_{Y>o} u_1(Y)\sigma((Y\frac{\partial}{\partial Y})')^{\Re}|Y|^{\mathscr{S}} f(Y)d\vartheta \\
&= (-1)^{\Re} \sum_{\nu=o}^{\Re} \mathscr{S}^{\nu}\tbinom{\Re}{\nu} \int\limits_{Y>o} \{\sigma((Y\frac{\partial}{\partial Y})')^{\Re-\nu}\mathfrak{f}(Y)\}|Y|^{\mathscr{S}} u_1(Y)d\vartheta \\
&= (-1)^{\Re} \sum_{\nu=o}^{\Re} \mathscr{S}^{\Re-\nu}\tbinom{\Re}{\nu} \int\limits_{Y>o} \{\sigma((Y\frac{\partial}{\partial Y})')^{\Re-\nu}\mathfrak{f}(Y)\}|Y|^{\mathscr{S}} u_1(Y)d\vartheta \qquad (427)
\end{aligned}
$$

Finally by induction on ρ we obtain by means of (427) that for in- **295**
dices $\Re_1, \Re_2, \ldots, \Re_\rho \geq n$

$$
\lambda_{\Re_1}\lambda_{\Re_2}\ldots\lambda_{\Re_\rho} \int\limits_{y>o} \mathfrak{f}(Y)|Y|^{\mathscr{S}} u_1(Y)d\vartheta =
$$

$$
(-1)^{\Re_1,\Re_2+\cdots+\Re_\rho+\rho} \sum_{\nu_1=0}^{\Re_1}\sum_{\nu_2=0}^{\Re_2}\cdots\sum_{\nu_\rho=0}^{\Re_\rho}[*]
$$

where

$$
[*] = \mathscr{S}^{\Re_1+\Re_2+\cdots+\Re_\rho-\nu_1-\nu_2\ldots\nu_\rho} \Big\{ \tbinom{\Re_1}{\nu_1}\tbinom{\Re_2}{\nu_2}\ldots\tbinom{\Re_\rho}{\nu_\rho}
$$

$$\times \int\limits_{Y>o} \{\sigma((Y\frac{\partial}{\partial Y})')^{\nu_\rho}\sigma((Y\frac{\partial}{\partial Y})')^{\nu_{\rho-1}}\ldots\sigma((Y\frac{\partial}{\partial Y})')^{\nu_1}\mathfrak{f}(Y)\}|Y|^{\mathscr{S}}$$
$$\times u_1(Y)d\vartheta\} \qquad (428)$$

choosing in particular $\mathfrak{f}(Y) = e^{-2\pi\sigma(Y)}$ and assuming that the real part of $\mathscr{S}$ is sufficiently large, all the above steps can be justified, and using (426) we finally obtain that

$$\lambda_{\Re_1}\lambda_{\Re_2}\cdots\lambda_{\Re_\rho}e^{2\pi\sigma(Y)}{}_{|Y|^{\Re}u_1(Y)d\vartheta}$$
$$= (-1)^{\Re_1+\Re_2+\cdots+\Re_\rho-nu_1-\nu_2-\cdots-\nu_\rho}\sum_{\nu_1=0}\sum_{\nu_2=0}^{K_2}\cdots\sum_{\substack{\nu\\ \rho}=0}^{\Re_\rho}$$
$$\binom{\Re_1}{\nu_1}\binom{\Re_2}{\nu_2}\cdots\binom{\Re_\rho}{\nu_\rho}\mathscr{S}^{\Re_1+\cdots+\Re_\rho-\nu_1\ldots\nu_\rho[*]}$$

where

$$[*] = (-2\pi)^{\nu_1+\nu_2+\cdots+\nu_\rho}\int\limits_{Y>o} e^{-2\pi\sigma(Y)}{}_{\sigma(Y^{\nu_1})}\ldots\sigma(Y^{\nu_\rho})|Y|^{\mathscr{S}}u_1(Y)0$$
$$+\sum_{\substack{\mu_1\ldots\mu_r\\ \mu_1+\cdots+\mu_r<\nu_1+\cdots\nu_\rho}} C^{\nu_1\nu_2\ldots\nu_\rho}_{\mu_1\mu_2\ldots\mu_2}\int\limits_{Y>0} e^{-2\pi\sigma(Y)}{}_{\sigma(Y^{\mu_1})}\ldots$$
$$\sigma(Y^{\nu_r})|Y|^{\Re}u_1(y)d\vartheta \qquad (429)$$

296

Introduce again y and Y_1 in place of Y by

$$Y = yY_1, |Y| = y^n, y > o$$

and denotes as usual by $d\vartheta_1$ the invariant volume element of the determinant surface $Y_1 > 0|Y_1| = 1$. Then from (383)

$$d\vartheta = \sqrt{n}y^{-1}dyd\vartheta_1.$$

In the sequel we consider only homogeneous functions $u_1(Y)$ of degree 0 i.e., we assume $\lambda_1 = 0$. In (429), after the above substitutions, the integral in y is a y integral. Carrying out this integration over y we obtain that

$$\int\limits_{Y>o} e^{-2\pi\sigma(Y)}\sigma(Y^{\nu_1})\dots(\sigma(Y^{\nu_\rho})|Y|^{\mathscr{S}} u_1(y)d\vartheta$$
$$= \sqrt{n}y(n\mathscr{S}+\nu_1+\cdots+\nu_\rho)(2\pi)^{n\mathscr{S}-\nu_1\cdots\nu_\rho}$$
$$\times\int\limits_{Y_1>0}(\sigma(Y_1))^{-n\mathscr{S}-\nu_1\cdots\nu\nu_\rho}\sigma(Y_1^{\nu_1})\dots\sigma(Y_1^{\nu_\rho})u_1(y_1)d\vartheta_1$$

Let

$$\mathcal{J}_{\nu_1\nu_2\dots\nu_\rho}(\mathscr{S},u_1)=\frac{\Lambda(n\mathscr{S}+\nu_1+\cdots+\nu_\rho)}{\Lambda(n\mathscr{S})}$$
$$\times\int\limits_{Y_1>O}u_1(Y_1)(\sigma(Y_1))^{-n\mathscr{S}-\nu_1-\cdots-\nu_\rho}\sigma(Y_1^{\nu_1})\dots\sigma(Y_1^{\nu_\rho})d\nu_1 \qquad (430)$$

and 297

$$\mathcal{J}(\mathscr{S},u_1)=\int\limits_{Y_1>O}u_1(Y_1)(\sigma(Y_1))^{-n\mathscr{S}}d\nu_1 \qquad (431)$$

Then (429) implies that

$$\lambda_{\Re_1}\lambda_{\Re_2}\dots\Re(\mathscr{S},u_1)=(-1)^{\Re_1+\cdots+\Re_\rho+\rho}\sum_{\nu_1=O}^{\Re_1}\sum_{\nu_2=O}^{\Re_2}\cdots$$
$$\cdots\sum_{\nu_\rho=O}^{\Re_\rho}\binom{\Re_1}{\nu_1}\binom{\Re_2}{\nu_2}\cdots\binom{\Re_\rho}{\nu_\rho}\mathscr{S}^{\Re_1}+\cdots+\Re_\rho-\nu_1-\cdots-\nu_\rho\times$$
$$\times\{(-1)^{\nu_1+\cdots+\nu_\rho}\mathcal{J}_{\nu_1\cdots\nu_\rho}(\mathscr{S},u_1)+\sum_{\substack{\mu_1\dots\mu_r\\ \mu_1+\cdots+\mu_r<\nu_1+\cdots+\nu_\rho}}$$
$$C^{\nu_1\nu_2\dots\nu_\rho}_{\mu_1\mu_2\dots\mu_r}(2\pi)^{-\mu_1-\cdots-\mu_r}\mathcal{J}_{\mu_1\dots\mu_r}(\mathscr{S},u_1) \qquad (432)$$

We therefore obtain for the integral corresponding to the indices $(\nu_1,\nu_2\dots\nu_\rho)=(\Re_1,\Re_2,\dots\Re_\rho)$ a representation of the kind

$$\mathcal{J}_{\Re_1\Re_2\cdots\Re_\rho}(\mathscr{S},u_1)=(-1)^\rho\lambda_{\Re_1}\lambda_{\Re_2}\dots\lambda_{\Re_\rho}\mathcal{J}(\mathscr{S},u_1)$$

$$\sum_{\substack{\mu_1\ldots\mu_r\\ \mu_1+\cdots+\mu_r<\Re_1+\ldots+\Re_\rho}} P^{\mu_1\mu_2\ldots\mu_r}_{\Re_1\Re_2\ldots\Re_\rho}(\mathscr{S})\mathcal{J}_{\mu_1\mu_2\ldots\mu_r}(\mathscr{S},u_1) \tag{433}$$

298 where $P^{\mu_1\ldots\mu_r}_{\Re_1\ldots\Re_\rho}$ is either 0 or is a polynomial in $\mathscr{S}$ of degree atmost $\Re_1 + \Re_2 + \cdots + \Re_\rho - \mu_1 - \mu_2 - \cdots \mu_r$. Since the coefficients $C^{\nu_1\nu_2\ldots\nu_\rho}_{\mu_1\mu_2\ldots\mu_r}$ by their very definition, do not depend on the eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$, this is true also of the polynomials $P^{\mu_1\ldots\mu_\rho}_{\Re_1\ldots\Re_\rho}(\mathscr{S})$. The method of induction on $\Re_1 + \Re_2 + \cdots + \Re_\rho$ yields by virtue of (433) that

$$\mathcal{J}_{\Re_1\Re_2\ldots\Re_\rho}(\mathscr{S},u_1) = q_{\Re_1\Re_2\ldots\Re_\rho}(\mathscr{S})\mathcal{J}(\mathscr{S},u_1) \tag{434}$$

where $q_{\Re_1\Re_\rho}(\mathscr{S})$ is again either 0 or is a polynomial in $\mathscr{S}$ of degree at most $\Re_1 + \Re_2 + \cdots + \Re_\rho$. We now show that

$$q_{\Re_1\Re_2\ldots\Re_\rho}(\mathscr{S}) = n^\rho \mathscr{S}^{\Re_1+\cdots+\Re_\rho} + \text{ lower powers of } \mathscr{S}, \tag{435}$$

in other words that $q_{\Re_1\cdots\Re_\rho}(\mathscr{S}$ is of degree exactly $\Re_1 + \cdots + \Re_\rho$ and that the coefficient of the highest degree term is precisely n^ρ. The proof is again by induction on $\Re_1 + \cdots + \Re_\rho$. If $\Re_1 + \cdots + \Re_\rho = O$ whence $\Re_1 = \Re_2 = \cdots = \Re_\rho = O$ we have obviously $q_{\Re_1\ldots\Re_\rho} = n^\rho$, ensuring that the start is good. Assume now that $\Re_1 + \cdots + \Re_\rho > O$ and

$$q_{\mu_1\mu_2\ldots\mu_r}(\mathscr{S}) = n^r \mathscr{S}^{\mu_1+\cdots+\mu_r} + \qquad \text{lower powers of } \mathscr{S}. \tag{436}$$

for

$$\mu_1 + \mu_2 + \cdots \mu_r < \Re_1 + \Re_2 + \cdots + \Re_\rho.$$

If

$$q_{\Re_1\Re_2\ldots\Re_\rho}(\mathscr{S}) = C\mathscr{S}^{\Re_1+\cdots\Re_\rho+} \qquad \text{lower powers of } \mathscr{S} \tag{437}$$

we have to show that $C = n^\rho$.

The application of (434) – (437) in (432) yields a polynomial identity. The power $\mathscr{S}^{\Re_1+\Re_2+\cdots+\Re_\rho}$ appears only on one side of this identity so

299 that the coefficient of this power must necessarily vanish. This requires that

$$\sum_{\substack{\nu_1=O\\ \nu_1+\cdots+\nu_\rho<\Re_1+\cdots+\Re_\rho}}^{\Re_1} \cdots \sum_{\nu_\rho=O}^{\Re_\rho} (-1)^{\nu_1+\cdots+\nu_\rho}\binom{\Re_1}{\nu_1}\cdots\binom{\Re_\rho}{\nu_\rho}n^\rho$$

$$+(-1)^{\mathfrak{R}_1+\cdots+\mathfrak{R}_\rho} C = O$$

Since this equation determines C uniquely and if we substitute C by n^ρ this equation is satisfied - at least one $\mathfrak{R}_\nu$ being greater than 0 - we conclude that $C = n^\rho$.

We observe more over that the coefficient of $\mathscr{S}^{\mathfrak{R}_1+\cdots+\mathfrak{R}_\rho-1}$ in $q_{\mathfrak{R}_1\mathfrak{R}_2\ldots\mathfrak{R}_\rho}(\mathscr{S}$ does not depend on the eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$. While this is evident in case $\mathfrak{R}_1+\cdots+\mathfrak{R}_\rho = 1$, as in this case $q_{\mathfrak{R}_1\ldots\mathfrak{R}_\rho}(\mathscr{S}) = n^\rho \mathscr{S}$, the general result follows by induction on applying (434) in (433).

Now for any positive matrix Y, $|Y|$ is a rational function of the sums of powers of the characteristic roots $\lambda_1, \lambda_2 \cdots \lambda_n$ of Y, and by a standard result on elementary symmetric functions, it is then a rational function of $\sigma(Y^\nu)(=\sum_{\mu=1}^n \lambda_\mu^\nu)$, $\nu = 1, 2, \ldots n$. In particular $\sigma(Y^n)$ can be expressed as a function of $|Y|$ and $\sigma(Y^\nu)$, $\nu < n$. If we take $Y = Y_1$ with $|Y_1| = 1$, then $\sigma(Y_1^n)$ is a function of $\sigma(Y_1^\nu)$, $\nu < n$, and this function can be determined to be of the form

$$\sigma(Y_1^n) = (-1)^{n+1} n + \sum_{\substack{\nu_1\cdots\nu_n<n\\ \nu_1+\cdots+\nu_r=n}} d_{\nu_1\nu_2\cdots\nu_r}\sigma(Y_1^{\nu_1})\cdots\sigma(Y_1^{\nu_1}) \tag{438}$$

with constant coefficients $d_{\nu_1\nu_2\ldots\nu_r}$. **300**

Now by definition

$$\mathcal{J}_n(\mathscr{S}, u_1) = \frac{\Lambda(n\mathscr{S}+n)}{\Lambda(n\mathscr{S})}\int\limits_{Y_1>O} u_1(Y_1)(\sigma(Y_1))^{-n\mathscr{S}-r}\sigma(Y_1^n)dv_1 \tag{439}$$

and using (438) we have

$$\begin{aligned}\mathcal{J}_n(\mathscr{S}, u_1) = (-1)^{n+1} n \frac{\Lambda(n\mathscr{S}+n)}{\Lambda(n\mathscr{S})}\mathcal{J}(\mathscr{S}+1, u_1)\\ + \sum_{\substack{\nu_1\nu_2\ldots\nu_r<n\\ \nu_1+\cdots+\nu_r=n}} d_{\nu_1\nu_2\ldots\nu_r}\mathcal{J}_{\nu_1\nu_2\ldots\nu_r}(\mathscr{S}, u_1)\end{aligned}$$

Hence

$$\frac{\Lambda(n\mathscr{S}+n)}{\Lambda(n\mathscr{S})}\mathcal{J}(\mathscr{S}+1, u_1) = q(\mathscr{S})\mathcal{J}(\mathscr{S}, u_1) \tag{440}$$

with

$$q(\mathscr{S}) = (-1)^{n+1}\frac{1}{n}(q_n(\mathscr{S}) - \sum_{\substack{\nu_1\ldots\nu_r<n\\ \nu_\rho+\cdots+\nu_r=n}} d_{\nu_1\cdots\nu_r} q_{\nu_1\cdots\nu_r}(\mathscr{S} \tag{441}$$

$q(\mathscr{S})$ is polynomial in $\mathscr{S}$ of degree n. The power $\mathscr{S}^n$ appears in $q(\mathscr{S})$ with the coefficient

$$(-1)^{n+1}\frac{1}{n}(n - \sum_{\substack{\nu_1\cdots\nu_r<n\\ \nu_\rho+\cdots+\nu_r=n}} d_{\nu_1\nu_2\ldots\nu_r} \quad n^r)$$

301 and it is immediate from (438) (setting $Y_1 = E$) that this coefficient is exactly equal to 1. Hence $q(\mathscr{S})$ can be factored as

$$q(\mathscr{S}) = (\mathscr{S} - d_1)(\mathscr{S} - d_2)\cdots(\mathscr{S} - d_n) \tag{442}$$

with certain complex constants $d_1, d_2 \ldots d_n$, and from our earlier statements concerning the coefficient of $\mathscr{S}^{\nu_1+\nu_2+\cdots+\nu_\rho-S}$ in $q_{\nu_1\nu_2\ldots\nu_\rho}(\mathscr{S})$ we infer that the coefficient of $\mathscr{S}^{n-1}$ in $q(\mathscr{S})$ viz. $\pm\mathscr{L}_1 + \mathscr{L}_2 + \cdots + d_n$ does not depend on the eigen values $\lambda_1 \ldots \lambda_n$. By means of the functional equation for $\Lambda(\mathscr{S})$, viz. $\Lambda(\mathscr{S} + 1) = \mathscr{S}\Lambda(\mathscr{S}$ we finally obtain from (440) that

$$\mathcal{J}(\mathscr{S} + 1, u_1) = \frac{(\mathscr{S} - \alpha_1)\cdots(\mathscr{S} - \alpha_n)}{n^n\mathscr{S}(\mathscr{S} + \frac{1}{n})\cdots(\mathscr{S} + \frac{n-1}{n})}\mathcal{J}(\mathscr{S}, u_1) \tag{443}$$

We shall use this transformation formula for computing $\Lambda(\mathscr{S}, u_1)$ explicitly, by a method which Huber developed recently for the case $n = 2$. Clearly the function

$$G(\mathscr{S}) = \frac{\Lambda(\mathscr{S} - d_1)\Lambda(\mathscr{S} - d_2)\ldots\Lambda(\mathscr{S} - d_n)}{n^{n\mathscr{S}}\Lambda(\mathscr{S})(\mathscr{S} + \frac{1}{n})\ldots\Lambda(\mathscr{S} + \frac{n-1}{n})} \tag{444}$$

satisfies the same transformation formula as $\Lambda(\mathscr{S}, u_1)$ does in (443) so that, if we set

$$H(\mathscr{S}, u_1) = \frac{\mathcal{J}(\mathscr{S}, u_1)}{G(\mathscr{S})} \tag{445}$$

then $H(\mathscr{S}, u_1)$ is a periodic function of $\mathscr{S}$ and

$$H(\mathscr{S}+1, u_1) = H(\mathscr{S}, u_1) \tag{446}$$

302 For values of $\mathscr{S}$ for which the real part of $\mathscr{S}$ is sufficiently large, $\mathcal{J}(\mathscr{S}, u_1)$ and $G(\mathscr{S})$ are regular functions of $\mathscr{S}$ and then this is true also of $H(\mathscr{S})$. Since $H(\mathscr{S})$ is further periodic, it is an entire function. We wish to show that this entire function is actually a constant depending only upon $u_1(E)$. For this we need some asymptotic formulae.

It is well known that

$$\Lambda(\mathscr{S}-\alpha) \sim \sqrt{2\pi}\mathscr{S}^{\mathscr{S}-\alpha-\frac{1}{2}}e^{-\mathscr{S}} \qquad (\mathscr{S}\to\infty)$$

and it follows that

$$G(\mathscr{S}) \sim n^{-n\mathscr{S}}\mathscr{S}^{-\alpha_1-\alpha_2-\cdots-\alpha_n-\frac{n-1}{2}}(\mathscr{S}\to\infty)I. \tag{447}$$

We shall now determine the value of the sum $\alpha_1+\alpha_2+\cdots+\alpha_n$. We know that this sum is independent of the eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$, in other words it is independent of the function u_1. We can then choose $u_1 \equiv 1$ and obtain by means of Lemma 14 in view of (383) that

$$\begin{aligned}
(2\pi)^{-n\mathscr{S}+\frac{n(n-1)}{4}}\Lambda(\mathscr{S})\Lambda(\mathscr{S}-\frac{1}{2})\cdots\Lambda(\mathscr{S}-\frac{n-1}{2}) &= \int\limits_{Y>O} e^{-2\pi\sigma(Y)}|Y|^{\mathscr{S}}dv\\
&= \sqrt{n}\int\limits_{Y_1>O}\int\limits_{Y>O} e^{-2\pi y\sigma(Y_1)}y^{n\mathscr{S}-1}dy d\vartheta_1\\
&= \sqrt{n}(2\pi)^{-n\mathscr{S}}\Lambda(\mathscr{S},1).
\end{aligned}$$

Hence 303

$$\mathcal{J}(\mathscr{S},1) = (2\pi)^{\frac{n(n-1)}{4}}\frac{\Lambda(\mathscr{S})\Lambda(\mathscr{S}-\frac{1}{2})\cdots\Lambda(\mathscr{S}-\frac{n-1}{2})}{\sqrt{n}\Lambda(n\mathscr{S})}$$

Since by the Gaussian multiplication formula,

$$\Lambda(n\mathscr{S}) = \frac{n^{n\mathscr{S}-\frac{1}{2}}}{(2\pi)^{(n-1)/2}}\Lambda(\mathscr{S})\Lambda(\mathscr{S}_{\frac{1}{n}})\cdots\Lambda(\mathscr{S}-\frac{n-1}{n}) \tag{448}$$

the above gives that

$$\mathcal{J}(\mathscr{S},1) = (2\pi)^{\frac{(n+2)(n-1)}{4}} \frac{\Lambda(\mathscr{S})\Lambda(\mathscr{S}-\frac{1}{2}\cdots\Lambda(\mathscr{S}-\frac{n-1}{2})}{n^{n\mathscr{S}}(\mathscr{S})\Lambda(\mathscr{S}+\frac{1}{2})\cdots\Lambda(\mathscr{S}+\frac{n-1}{n})} \tag{449}$$

and consequently

$$\frac{\mathcal{J}(\mathscr{S}+1.1)}{\mathcal{J}(\mathscr{S},1)} = \frac{\mathscr{S}(\mathscr{S}-\frac{1}{2})\cdots(\mathscr{S}-\frac{n-1}{n})}{n^n\mathscr{S}(\mathscr{S}+\frac{1}{n})\cdots(\mathscr{S}+\frac{n-1}{n})}$$

A comparison with (443) now shows that

$$\alpha_1+\alpha_2+\cdots+\alpha_n = \frac{n(n-1)}{4}. \tag{450}$$

Substituting this in (447) we have

$$G(\mathscr{S}) \sim n^{-n\mathscr{S}}\mathscr{S}^{-n+z)(n-1)/4} \qquad (\mathscr{S}\to\infty) \tag{451}$$

and in particular, the asymptotic value of $G(\mathscr{S})$ is independent of the eigen values $\lambda_1,\lambda_2\ldots\lambda_n$.

Also it follows from (449) that

$$\mathcal{J}(\mathscr{S},1) \sim (2\pi)^{n+2)(n-1)/4} n^{-n\mathscr{S}}\mathscr{S}^{-(n+2)(n-1)/4} \tag{452}$$

304 and against (451) this gives

$$\mathcal{J}(\mathscr{S},1) \sim (2\pi)^{(n+2)(n-1)/4} G(\mathscr{S})(\mathscr{S}\to\infty) \tag{453}$$

It is also clear from (451) that

$$G(\mathscr{S}+\Re) \sim n^{-n\mathscr{S}}G(\Re) \qquad (\Re\to\infty) \tag{454}$$

where we let $\Re$ to tend to ∞ through all rational values. With $\omega(Y) = u_1(Y)(\frac{\sigma(Y)}{n})^{-\mathscr{S}}$ we can now state from (446) that $H(\mathscr{S}+\Re,u_1) = H(\mathscr{S},u_1)$ for every integer $\Re$. If now $\mathscr{S}\to\infty$ through all rational values, say, it is clear in view of (454) that

$$H(\mathscr{S},u_1) = \lim_{\Re\to\infty} H(\mathscr{S}+\Re,u_1) = \lim_{\Re\to\infty}\frac{\mathcal{J}(\mathscr{S}+\Re,u_1)}{G(\mathscr{S}+\Re)}$$

$$= \lim_{\Re\to\infty} \frac{\mathcal{J}(\mathscr{S}+\Re, u_1)}{n^{-n\mathscr{S}} G(\mathscr{S})} \tag{455}$$

Now

$$n^{n\mathscr{S}} \mathcal{J}(\mathscr{S}+\Re, u_1) = n^{n\mathscr{S}} \int\limits_{Y_1>O} \sigma(Y_1)^{-n\mathscr{S}-n\Re} u_1(Y) d\vartheta_1$$

$$= \int\limits_{Y_1>O} u_1(Y) (\frac{\sigma(Y)}{n})^{\mathscr{S}} (\sigma(Y_1))^{-n\Re} d\vartheta_1 = \mathcal{J}(\Re, \omega).$$

Hence we conclude from (455) by means of (454) that

$$H(\mathscr{S}, u_1) = \lim_{\Re\to\infty} \frac{\mathcal{J}(\Re,\omega)}{G(\Re)} = (2\pi)^{(n+2)(n-1)/4} \lim_{\Re\to\infty} \frac{\mathcal{J}(\Re,\omega)}{\mathscr{J}(\Re,1)} \tag{456}$$

We now prove for continuous and bounded functions $\omega(Y)$ that **305**

$$\lim_{\Re\to\infty} \frac{\mathcal{J}(\Re,\omega)}{\mathcal{J}(\Re,1)} = \omega(E) \tag{457}$$

In view of the linearity of $\mathcal{J}(\Re,\omega)$ in ω we can further assume that $\omega(E) = 0$ as in the alternative case we need only argue with the function $\omega(Y) - \omega(E)$ in the place of $\omega(Y)$. Representing $\sigma(Y)$ and $|Y|$ in terms of the characteristic roots of Y one easily sees that $\sigma(Y) \geq n\sqrt{|Y|}$ for $Y > 0$. In particular, $\sigma(Y_1) \geq n$ for $Y_1 > 0$, $|Y_1| = 1$, and the equality is true only for $Y_1 = E$. This implies that if $\sigma(Y_1) \to n$ and $Y_1 \to A$ then $\sigma(A) = n$ and consequently $A = E$. Hence given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that $|\omega(Y_1)| <\in$ for $\sigma(Y_1) < n(1+\delta)$. Let $|\omega(Y_1)| \leq \vartheta$ for all Y_1. (By assumption ω is bounded).

Then,

$$\frac{\mathcal{J}(\Re,\omega)}{\mathcal{J}(\Re,1)} = \frac{\int\limits_{y_1>O} \omega(Y_1)\sigma(Y_1)^{-n\Re} d\vartheta}{\int\limits_{Y_1>O} \sigma(Y_1)^{-n\Re} d\vartheta_1},$$

and hence

$$\left|\frac{\mathcal{J}(\Re,\omega)}{\mathcal{J}(\Re,1)}\right| \leq \varepsilon \frac{\int\limits_{\sigma(Y_1)\leq n(1+\delta)} (\sigma(Y_1)^{-n\Re} d\vartheta_1}{\int\limits_{Y_1>O} (\sigma(Y_1))^{-n\Re} d\vartheta_1} + \vartheta \frac{\int\limits_{\sigma(Y_1)>n(1+\delta)} (\sigma(Y_1))^{-n\Re} d\vartheta_1}{\int\limits_{Y_1>O} (\sigma(Y_1))^{-n\Re} d\vartheta_1} \cdots$$

$$\leq \varepsilon + \vartheta(1+\delta)^{-\frac{n\Re}{2}} n^{-\frac{n\Re}{2}} \frac{\mathcal{J}(\frac{\Re}{2},1)}{\mathcal{J}(\Re,1)}$$

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From (452) we have

$$n^{-\frac{n\Re}{2}} \frac{\mathcal{J}(\frac{\Re}{2},1)}{\mathcal{J}(\Re,1)} \sim 2^{(N+2)(N-1)/4} (\Re \to \infty)$$

and then

$$\left|\frac{\mathcal{J}(\Re,\omega)}{\mathcal{J}(\Re,1)}\right| \sim 2\varepsilon \text{ for } \Re \geq \Re_O(\varepsilon)$$

In other words,

$$\lim_{\Re\to\infty} \frac{\mathcal{J}(\Re,\omega)}{\mathcal{J}(\Re,1)} = O = \omega(E).$$

Applying (457) in (456) we obtain that

$$H(\mathscr{S},u_1) = (2\pi)^{(n+2)(n-1)/4}\omega(E).$$

But $\omega(E) - \mu_1(E)$ and thus

$$H(\mathscr{S},u_1) = (2\pi)^{(n+2)(n-1)/4}u_1(E). \tag{458}$$

After all these, we are in a position to face the integral $W(\mathscr{S},u_1)$ in (419). We have

$$\begin{aligned} W(\mathscr{S},u_1) &= \int\limits_{y>O} e^{-2\pi\sigma(Y)} u_1(Y)|Y|^{\mathscr{S}} d\vartheta \\ &= \sqrt{n}(2\pi)^{-n\mathscr{S}}\Lambda(n\mathscr{S})\mathcal{J}(\mathscr{S},u_1) \\ &= \sqrt{n}(2\pi)^{-n\mathscr{S}}\Lambda G(n\mathscr{S})H(\mathscr{S},u_1)\cdots \\ &= (2\pi)^{-n\mathscr{S}}\Lambda(n\mathscr{S})n^{-n\mathscr{S}} \frac{\Lambda(\mathscr{S}-d_1)\cdots\Lambda(\mathscr{S}-d_n)}{\Lambda(\mathscr{S})\Lambda(\mathscr{S}+\frac{1}{n})\cdots\Lambda(\mathscr{S}+\frac{n-1}{n})} \end{aligned}$$

$$(2n)\frac{(n+2)(n)}{4}$$

$$\sqrt{n}u_1$$

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We know from (418) that $u_1(E) = u(T^{-1}) = u^*(T)$ in the notation in (p. 251). Using again the Gaussian multiplication formula (448) the above gives that

$$W(\mathscr{S}, u_1) = (2\pi)^{-n\mathscr{S}+\frac{n(n-1)}{4}}\Lambda(\mathscr{S}-d_1)\cdots\Lambda(\mathscr{S}-d_n)u^*(T) \qquad (459)$$

Then from (420)

$$\begin{aligned}\xi(\mathscr{S}, u) &= \frac{2}{\sqrt{n}}\sum_{\{T\}}\frac{a(T)}{\varepsilon(T)}|T|^{-\mathscr{S}}W(\mathscr{S}, u_1)\\ &= \frac{2}{\sqrt{n}}(2\pi)^{-n\mathscr{S}+\frac{n(n-1)}{4}}\Lambda(\mathscr{S}-d_1)\cdots\Lambda(\mathscr{S}-dn)D(\mathscr{S}, u) \qquad (460)\end{aligned}$$

where

$$D(\mathscr{S}, u) = \sum_{\{T\}}\frac{a(T)u^*(T)}{\varepsilon(T)}|T|^{-\mathscr{S}}. \qquad (461)$$

The next question is whether the function $D(\mathscr{S}, u)$ defined by the above Dirichlet series for values of $\mathscr{S}$ whose real parts are sufficiently large, can be continued analytically in the whole plane. We have

$$\xi(\mathscr{S}, u) = \frac{1}{\sqrt{n}}\int\limits_{y\in\mathfrak{R}}\mathfrak{f}_n(Y)u(Y)|Y|^{\mathscr{S}}d\vartheta.$$

We can take in the place of $\mathfrak{R}$ a domain which is invariant relative to the transformation $Y \to Y^{-1}$. Since the volume element $d\vartheta$ is also 308
invariant relative to this transformation we can write

$$\begin{aligned}\xi(\mathscr{S}, u) = \frac{1}{\sqrt{n}}&\int\limits_{\substack{Y\in\mathfrak{R}\\|Y|\geq 1}}\mathfrak{f}_n(Y^{-1})|Y|^{-\mathscr{S}}u^*(Y)d\vartheta\\ &+\frac{1}{\sqrt{n}}\int\limits_{\substack{Y\in\mathfrak{R}\\|y|\geq 1}}\mathfrak{f}_n(Y)|Y|^{\mathscr{S}}u(Y)d\vartheta \qquad (462)\end{aligned}$$

By the transformation formula for modular forms we have $\mathfrak{f}(-Z^{-1}) = |Z|^{\mathfrak{R}}\mathfrak{f}(Z)$ and in particular, $\mathfrak{f}(iY^{-1}) = i^{n\mathfrak{R}}|Y|^{\mathfrak{R}}\mathfrak{f}(iY)$. Then,

$$\sum_{r=o}^{n} \mathfrak{f}_r(Y^{-1}) = \mathfrak{f}(iY^{-1}) = i^{n\mathfrak{R}}|Y|^{\mathfrak{R}} \sum_{r=o}^{n} \mathfrak{f}_r(Y)$$

and consequently

$$\mathfrak{f}_n(Y^{-1}) = i^{n\mathfrak{R}}|Y|^{\mathfrak{R}}\mathfrak{f}_n(Y) + \sum_{r=o}^{n-1} (i^{n\mathfrak{R}}|Y|^{\mathfrak{R}}\mathfrak{f}_r(Y) - \mathfrak{f}_r(Y^{-1}))$$

Substituting this in (462) we have

$$\xi(\mathscr{S}, u) = \frac{1}{\sqrt{n}} \int\limits_{\substack{Y\in\mathfrak{R}\\|Y|\geq 1}} \mathfrak{f}_n(Y)\{i^{n\mathfrak{R}}|Y|^{\mathfrak{R}-\mathscr{S}} u^*(Y) + |Y|^{\mathscr{S}} u(Y)\} d\vartheta + \sum_{r=o}^{n-1} \frac{1}{\sqrt{n}}$$
$$\int\limits_{\substack{Y\in A\\|Y|\geq 1}} (i^{n\mathfrak{R}}|Y|^{\mathfrak{R}}\mathfrak{f}_r(Y) - \mathfrak{f}_r(Y^{-1}))|Y|^{-\mathscr{S}} u^*(Y) d\vartheta \qquad (463)$$

Here we have assumed that the order of summation and integration can be inverted. We now compute explicitly the integral corresponding to $r = o$ in the sum on the right side of (463). With $\mathfrak{f}_o(Y) = a_o$, this reduces to

$$\frac{a(o)}{\sqrt{n}} \int\limits_{\substack{Y\in\mathfrak{R}\\|Y|\geq 1}} (i^{n\mathfrak{R}} y^{n(\mathfrak{R}-\mathscr{S})} - y^{-n\mathscr{S}}) u^*(Y_1)\sqrt{n} y^{-1} dy d\vartheta_1$$
$$= a(o) \int\limits_{\substack{Y_1\in\mathfrak{R}\\|Y_1|=1}} u^*(Y_1) d\vartheta_1 \left(\frac{i^{n\mathfrak{R}}}{n(\mathscr{S}-\mathfrak{R})} - \frac{1}{n\mathscr{S}}\right)$$
$$= -\frac{a(o)(1, u^*)}{n}\left(\frac{1}{\mathscr{S}} + \frac{i^{n\mathfrak{R}}}{\mathfrak{R}-\mathscr{S}}\right) \qquad (464)$$

309 where $(1, u^*)$ is the scalar product of the two angular characters introduced in (408).

The first term on the right side of (463) is an entire function of $\mathscr{S}$. In the special cases when $n = 1, 2$, all the other integrals occurring there can be shown to be meromorphic in $\mathscr{S}$ and it will follow that in these cases $\xi(\mathscr{S}, u)$ and consequently $D(\mathscr{S}, u)'$ can be continued analytically in the whole domain. It is quite likely that this is true for arbitrary n too. Further, due to a transformation

$$\mathscr{S} \to \Re - \mathscr{S}, u \to u^*,$$

the first integral on the right side of (463) gets multiplied by a factor $i^{n\Re}$. **310**
For $n = 1, 2$, it is known that all the integrals occurring on the right side of (463) have the above property so that in these cases we have

$$\xi(\Re - \mathscr{S}, u^*) = i^{n\Re}\xi(\mathscr{S}, u) \tag{465}$$

It is to expected that this transformation formula is also valid in the general case and our conjectures are further supported (1) by the fact that one of the integrals computed in (464) (corresponding to $r = 0$) verifies all these properties.

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