# Lectures on

# Partial Differential Equations and Representations of Semi-groups

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Tata Institute of Fundamental Research, Bombay
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# Partial Differential Equations
# and Representations of Semi-groups


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# Introduction

Different kinds of problems can be put about partial differential equations.

a) **Local problems**, i.e. problems of regularity of solutions when we know the degree of regularity of the coefficients and the second member.

b) **Boundary value problems**. These problems generally have a physical origin. As an example we have the first boundary value problem – the famous Dirichlet problem – for the Laplacian. We have a bounded domain $\Omega$ with a smooth boundary in $R^n$; we are given a function $g$ in $\Omega$ and a function $h$ on the boundary of $\Omega$. The problem is to find a function $f$ in $\overline{\Omega}$ such that $\Delta f = g$ in $\Omega$ and $f = h$ on the boundary of $\Omega$. $\left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)$.Another problem is Neumann's problem for $\Delta$: find $f$ such that $\Delta f = g$ in $\Omega$ and $\frac{\partial f}{\partial n} = h$ on the boundary. We can also consider the problem wherein $f$ is prescribed on a part of the boundary and $\frac{\partial f}{\partial n}$ on the rest of the boundary. Under suitable assumption on $g$ and $h$ these problems have one and only one solution. The problems with a physical origin are usually well–posed.

c) **Mixed problems or initial and boundary value problems**. Let $\Omega$ be a bounded domain with a smooth boundary $\mathcal{S}$. We consider the problem of heat conduction in $\Omega$. From a physical point of view, it is clear that the knowledge of the temperature in $\overline{\Omega}$ at time

0 and that of the temperature at the boundary at every time $t > 0$ should completely determine the temperature in $\overline{\Omega}$ at any time $t$. The corresponding problem is this: given a function $u_0(x)$ in $\overline{\Omega}$ and a function $h(x,t)$, $t \geq 0$, $x \in S$, find a function $u(x,t)$ such that

i) $\dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t)$

ii) $u(x,0) = u_0(x)$ (initial condition)

iii) $u(x,t) = h(x,t)$ for every $t > 0$ and $x \in S$ (boundary condition)

Another problem of this type arises when we know the initial temperature of the body and the amount of heat that flows across the boundary at every subsequence moment. The problem is to find $u(x,t)$ such that

i) $\dfrac{\partial u(x,t)}{\partial t} = \Delta u(x,t)$

ii) $u(x,0) = u_0(x)$

iii) $\dfrac{\partial u(x,t)}{\partial n} = h(x,t)$ $(t > 0, x \in S)$.

We shall formulate these problems, or rather weaker versions of these, in the framework of spaces of distributions and solve them.

# Part I

# Mixed Problems in Partial Differential Equations

# Lecture 1

# Vector valued Distributions

**Notations**. $\mathscr{D}$ denotes the space of $C^\infty$ functions with compact supports 1
on $R^N$.

$\mathscr{E}$ denotes the space of all $C^\infty$ functions on $R^N$. $\mathscr{S}$ denotes the space of 'rapidly decreasing' functions on $R^N$. All these spaces are provided with their usual topologies. (See "Theorie des distributions" by L. Schwartz, Vol. 1 and 2) We denote by $\mathscr{D}'$, $\mathscr{E}'$ and $\mathscr{S}'$ the strong duals of $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{S}$ respectively. $\mathscr{D}'$, $\mathscr{E}'$ and $\mathscr{S}'$ are respectively the space of distributions on $R^N$, the space of distributions with compact support on $R^N$ and the space of 'tempered' distributions on $R^N$.

**Definition of a vector valued distribution:**

Let $E$ be a locally convex Hausdorff topological vector space. We will refer to such a space as $ELC$.

**Definition 1.1.** *A linear continuous map from $\mathscr{D}$ to $E$ is defined to be an $E$-valued distribution or a distribution with values in $E$.*

**Remark .** The space of $E$-valued distributions depends only on the bounded sets of $E$.

*Proof.* Since $\mathscr{D}$ is bornological (Theorie des distributions, Tome 1, p. 71) a linear map from $\mathscr{D}$ to $E$ is continuous if and only if it takes bounded sets of $\mathscr{D}$ into bounded sets of $E$. Hence the space of $E$-valued distributions depends only on the bounded sets of $E$. In particular if we replace the topology of $E$ by the weakened topology, due to the identity between the bounded sets in the initial topology and the weakened

2 topology, we have the space of $E$-valued distributions to be the same algebraically in the above two cases. □

We denote by $\mathscr{D}'(E)$ the space of $E$-valued distributions. *Topology of* $\mathscr{D}'(E)$. On $\mathscr{D}'(E)$ we put the topology of uniform convergence on bounded sets of $\mathscr{D}$. Since bounded sets of $\mathscr{D}$ are relatively compact, the topology that we introduce is the same as the topology of uniform convergence on compact sets of $\mathscr{D}$.

**Examples of vector valued distributions:**

Let $T$ be a distribution on $R^N$ and $\vec{e}$ a fixed vector of $E$. $T\vec{e}$ defined by $T\vec{e}(\varphi) = T(\varphi)\vec{e}$ for every $\varphi \in \mathscr{D}$ is an $E$-valued distribution. $T\vec{e}$ maps the whole of $\mathscr{D}$ either into a one-dimensional subspace of $E$ or into zero according as $\vec{e} \neq 0, T \neq 0$ or one of the quantities $e$ and $T$ is zero.

The map $(T, \vec{e}) \to T\vec{e}$ of $\mathscr{D}'xE$ into $\mathscr{D}'(E)$ is a bilinear map and hence induces a linear map $i : \mathscr{D}' \otimes E \to \mathscr{D}'(E)$. This map '$i$' is an injection. For, let $\{\vec{e}_\nu\}$ be a basis of $E$. Any element of $\mathscr{D}' \otimes E$ can be written as $\sum T_\nu \otimes \vec{e}_\nu$, $T_\nu \in \mathscr{D}'$. Now, $i(\sum T_\nu \otimes \vec{e}_\nu) = \sum T_\nu \vec{e}_\nu$. Hence if $i(\sum T_\nu \otimes \vec{e}_\nu) = 0$, we have $\sum T_\nu \vec{e}_\nu = 0$, or $\sum T_\nu(\varphi)\vec{e}_\nu = 0$ for every $\varphi \in \mathscr{D}$. The linear independence of the $\vec{e}_\nu$'s gives $T_\nu(\varphi) = 0$ for every $\varphi \in \mathscr{D}$. Hence $\sum T_\nu \otimes \vec{e}_\nu = 0$, which proves that '$i'$ is an injection.

It is easy to see that the image of $\mathscr{D}' \otimes E$ under this injection is the space of continuous linear maps from $\mathscr{D}$ into $E$ which are of finite rank, that is to say, which map $\mathscr{D}$ into a finite dimensional subspace of $E$. When $E$ is finite dimensional, every $E$-valued distribution is of finite
3 rank and so $\mathscr{D}'(E)$ can be identified algebraically with $\mathscr{D}' \otimes E$. When $E$ is finite dimensional by choosing a basis $\vec{e}_1, \ldots, \vec{e}_m$ of $E$ we see that any $\vec{T} \in \mathscr{D}'(E)$ can be written as

$$\vec{T} = T_1\vec{e}_1 + T_2\vec{e}_2 + \cdots + T_m\vec{e}_m,$$

where $T_1, \ldots, T_m$ are uniquely determined scalar distributions. Instead of giving $\vec{T}$ it suffices to give the $m$-scalar distributions $T_1, T_2, \ldots, T_m$.

Now we give an example of a distribution which can have infinite rank.

If $f$ is a complex valued continuous function we know that $f$ defines a distribution, also denoted by $f$, in the following way:

$$f(\varphi) = \int_{R^N} f(x)\varphi(x)\,dx \quad \text{for every} \quad \varphi \in \mathscr{D}.$$

We shall now define an analogous vector valued distribution. Let $E$ be a complete $ELC$. If $\varphi \in \mathscr{D}$ the function $\vec{f}\varphi$ defined by $\vec{f}\varphi(x) = \varphi(x)f(x)$ is an $E$-valued continuous function with compact support. $E$ being complete, the integral $\int_{R^N} f(x)\varphi(x)\,dx$ (for the definition of this integral, see Bourbaki, Integration, Chap.III § 4) is an element of $E$.

The map $\varphi \to \vec{f}(\varphi) = \int_{R^N} \vec{f}(x)\varphi(x)\,dx$, which is evidently linear, is an $E$-valued distribution. We have to prove the continuity of the map $\varphi \to \vec{f}(\varphi)$ of $\mathscr{D}$ in $E$. Suppose $\{\varphi_n\}$ is a sequence of functions all having their supports in a fixed compact set $K$ and tending uniformly to 0, together with all their partial derivatives. Let $V(K)$ be the volume of the compact set $K$. Then

$$\int_{R^N} f(x)\varphi(x)\,dx \in mV(K)\vec{f}(K)$$

(Bourbaki, Integration, Chap. III, § 4).

where $m = \sup_{x\in K} |\varphi(x)|$ and $\widehat{\vec{f}(K)}$ is the convex, closed envelope of the 4

compact set $\vec{f}(K)$. Since $E$ is complete $\widehat{\vec{f}(K)}$ is compact.

Hence

$$\int_{R^N} \vec{f}(x)\varphi_n(x)\,dx \in m_n V(K)\widehat{\vec{f}(K)},$$

where $m_n = \sup_{x\in K} |\varphi_n(x)|$. If $\varphi_n$ tend to 0 uniformly on $K$, we have $\vec{f}(\varphi_n) \to 0$ in $E$. This proves the continuity of the map $\varphi \to \vec{f}(\varphi)$.

**The identity distribution**. The identity map of $\mathscr{D}$ into $\mathscr{D}$ is a continuous linear map of $\mathscr{D}$ into $\mathscr{D}$, hence it is a $\mathscr{D}$-valued distribution.

**Definition 1.2.** *An $E$-valued distribution $\vec{T} : \mathscr{D} \to E$ is said to be of order $m$, $m$ an integer $\geq 0$, if $\vec{T}$ can be extended into a continuous map from $\mathscr{D}^m$ to $E$. (For the definition and the topology of $\mathscr{D}^m$ refer to "Theorie des distributions", vol. 1).*

One knows that a scalar distribution is locally of finite order. But the analogous result is in general false for vector valued distributions. For example, the identity distribution (example (2.3))is of infinite order in every open subset. In fact, if $i : \mathscr{D} \to \mathscr{D}$ is of finite order, every $E$-valued distribution, $E$ being a complete $ELC$, will be of finite order. For, let $f : \mathscr{D} \to E$ be an $E$-valued distribution and $\tilde{i} : \mathscr{D}^m \to \mathscr{D}$ be the extension of $i$ into a continuous linear map of $\mathscr{D}^m$ in $\mathscr{D}$. Then $\tilde{f} = f \circ \tilde{i} : \mathscr{D}^m \to E$ is an extension of $f$ into a continuous linear map of $\mathscr{D}^m$ into $E$. But given any open set $\Omega$ there exists a distribution on $\Omega$ with values in $C$ (field of complex numbers) which is of infinite order.

5 Suppose $E$ and $F$ are two locally convex Hausdorff spaces such that $E$ is a subspace of $F$ with a finer topology. It may happen that an $E$-valued distribution $\vec{T}$ which is of infinite order becomes a distribution of finite order considered as a distribution with values in $F$. For example take $E = \mathscr{D}$ and $F = \mathscr{D}'$. The identity distribution of $\mathscr{D}$ with values in $\mathscr{D}$ is of infinite order. But considered as a $\mathscr{D}'$-valued distribution it is given by the indefinitely differentiable $\mathscr{D}'$-valued function $\vec{f}$ defined as $\vec{f}(a) = \delta_a, \delta_a$ being the Dirac distribution at $'a'$. It is easily seen that $\vec{f}$ is a $C^\infty$, $\mathscr{D}'$-valued function. We now verify that the distribution given by $\vec{f}$ is the same as the $\mathscr{D}'$-valued distribution '$i$'.

The distribution defined by $\vec{f}$ maps any $\varphi \in \mathscr{D}$ into the element $\vec{f}(\varphi) = \int\limits_{R^N} f(a)\varphi(a)\,da$ of $\mathscr{D}'$. Now, I claim $\vec{f}(\varphi)$ is the same as the element $i(\varphi)$ of $\mathscr{D}'$. $i(\varphi)$, considered as an element of $\mathscr{D}'$ maps any $\psi \in \mathscr{D}$ into the element $\varphi(\psi) = \int\limits_{R^N} \varphi(a)\psi(a)\,da$ of $C$. To show that $\vec{f}(\varphi) = i(\varphi)$ we have to show merely

$$\langle \vec{f}(\varphi), \psi \rangle = \varphi(\psi) \quad \text{for every} \quad \psi \in \mathscr{D}.$$

Now,

$$
\begin{aligned}
\langle \overrightarrow{f}(\varphi), \psi \rangle &= \langle \int_{R^N} \overrightarrow{f}(a)\varphi(a)\, da, \psi \rangle = \int_{R^N} \langle \overrightarrow{f}(a)\varphi(a), \psi \rangle\, da \\
&= \int_{R^N} \langle \delta_a \psi(a), \psi \rangle\, da = \int_{R^N} \varphi(a) \langle \delta_a, \psi \rangle\, da \\
&= \int_{R^N} \psi(a)\psi(a)\, da \\
&= \varphi(\psi).
\end{aligned}
$$

# Lecture 2

# Vector valued Distributions (Contd.)

Let $E$ and $F$ be two locally convex Hausdorff spaces and $u : E \to F$ 6
be a continuous linear map. If $\overrightarrow{T} : \mathscr{D} \to E$ is an $E$-valued distribution $u \circ \overrightarrow{T} : \mathscr{D} \to F$ is an $F$-valued distribution. $u \circ \overrightarrow{T}$ is called the image of $\overrightarrow{T}$ by $u$. The distribution $u \circ \overrightarrow{T}$ has at least as simple properties as $\overrightarrow{T}$. For example, if $\overrightarrow{T}$ is of finite order, $u \circ \overrightarrow{T}$ is of finite order. The support of $u \circ \overrightarrow{T}$ is contained in that of $\overrightarrow{T}$. In particular, if $\overrightarrow{T}$ has a compact support, $u \circ \overrightarrow{T}$ has a compact support. If $\overrightarrow{T}$ is given by a function $\overrightarrow{f} : R^N \to E$, then $u \circ \overrightarrow{T}$ is given by the function $u \circ \overrightarrow{f} : R^N \to F$. This follows immediately from the equality

$$u\left(\int\limits_{R^N} \overrightarrow{f}(x)\varphi(x)\,dx\right) = \int\limits_{R^N} u(\overrightarrow{f}(x))\varphi(x)\,dx.$$

Every distribution $\overrightarrow{T} : \mathscr{D} \to E$ is a continuous image of the identity distribution for $\overrightarrow{T} = \overrightarrow{T} \circ I$. In this sense the identity distribution is the worst possible distribution. Suppose $\overrightarrow{T} \in \mathscr{D}'(E)$ and $\overleftarrow{e}' \in E', E'$ denoting the dual of $E$. Write for $\varphi \in \mathscr{D}$, $\langle \varphi | \overrightarrow{T} | = \overrightarrow{T}(\varphi)$ and $\langle \varphi | \overrightarrow{T} | \overleftarrow{e}' \rangle = \overleftarrow{e}'(\overrightarrow{T}(\varphi))$. The mapping $(\varphi, \overrightarrow{T}, \overleftarrow{e}') \to \langle \varphi | \overrightarrow{T} | \overleftarrow{e}' \rangle$ is a trilinear form on $\mathscr{D} \times \mathscr{D}'(E) \times E'$. $\overleftarrow{e}' \circ \overrightarrow{T}$ is a scalar distribution and we denote this distribution by $|\overrightarrow{T} | \overleftarrow{e}' \rangle$.

**Proposition 2.1.** *If $\overrightarrow{T}$ is an $E$-valued distribution, the map from $E'$ to $\mathscr{D}'$ which takes $\overleftarrow{e}'$ into $|\overrightarrow{T}|\overleftarrow{e}'\rangle$ is the transpose of the map $\overrightarrow{T} : \mathscr{D} \to E$.*

7 *Proof.* We have

$$\begin{aligned} |\overrightarrow{T}|\overrightarrow{e}'\rangle(\varphi) &= \langle\varphi|\overrightarrow{T}|\overleftarrow{e}'\rangle \quad \text{for every} \quad \varphi \in \mathscr{D} \\ &= \langle \overrightarrow{T}(\varphi), \overleftarrow{e}'\rangle_{E,E'} \\ &= \langle \varphi, \overset{t}{\overrightarrow{T}}\overleftarrow{e}'\rangle. \end{aligned}$$

This proves the proposition. ◻

Let $E$ be a locally convex Hausdorff topological vector space, we denote by $E'_c$ the dual of $E$ endowed with the topology of uniform convergence on convex, compact, stable subsets of $E$. By Mackey's theorem (Bourbaki *EVT*, Tome 2, Chapter IV, Theorem 2), the dual of $E'_c$ is identical with $E$. Moreover if $E$ and $F$ are *ELC* and $u : E \to F$ is a continuous linear map, the transpose ${}^t u : F'_c \to E'_c$ is continuous, because $u$ maps convex, compact, stable subsets of $E$ into convex, compact, stable subsets of $F$.

Now, suppose $\overrightarrow{T} : \mathscr{D} \to E$ is an $E$-valued distribution ${}^t\overrightarrow{T} : E'_c \to \mathscr{D}'$ is a continuous map for $\mathscr{D}' = \mathscr{D}'_c$. Conversely, we have

**Proposition 2.2.** *If $u : E'_c \to \mathscr{D}'$ is a continuous linear map, it is the transpose of a uniquely determined $E$-valued distribution.*

*Proof.* Let $u : E'_c \to \mathscr{D}'$ be a continuous linear map. Then ${}^t u : (E'_c)'_c \leftarrow (\mathscr{D}')'_c$ is a continuous linear map. But $(\mathscr{D}')'_c = \mathscr{D}$, and $(E'_c)'_c$ has a topology finer than the topology of $E$. $[(E'_c)'_c$ is algebraically the same as $E$ by Mackey's theorem]. To prove that the topology of $(E'_c)'_c$ is finer than that of $E$, we first remark that the initial topology on $E$ is the topology of uniform convergence on equicontinuous subsets of $E'$. To prove our assertion, we have only to show that any equicontinuous
8 subset of $E'$ is contained in a convex, compact, stable subset of $E'_c$. Let $A$ be any equicontinuous subset of $E'$. Let $\widehat{A}$ be the convex, weakly closed stable envelope of $A$. $\widehat{A}$ is then weakly compact and equicontinuous. But on equicontinuous subsets the topology of compact convergence and the

weak topology coincide. Hence $\widehat{\mathrm{A}}$ is a compact subset of $E'_c$. Since the topology of $(E'_c)'_c$ is finer than that of $E$, $t_u : E \leftarrow \mathscr{D}$ is also continuous. This proves our proposition, as ${}^t(t_u) = u$. □

The above proposition shows that the vector spaces $\mathscr{L}(\mathscr{D}, E)$ and $\mathscr{L}(E'_c, \mathscr{D}')$ are algebraically isomorphic.

# Lecture 3

# Spaces of distributions $\mathscr{H}$

**Definition 3.1.** *A space of distributions $\mathscr{H}$ is, by definition, an ELC which is contained in $\mathscr{D}'$ as a linear subspace with a finer topology: that is to say, the injection $i : \mathscr{H} \to \mathscr{D}'$ is continuous.* 9

**Definition 3.2.** *A space of distributions $\mathscr{H}$ is said to be normal if $\mathscr{D}$ is contained in $\mathscr{H}$ with a finer topology and $\mathscr{D}$ is dense in $\mathscr{H}$.*

**Proposition 3.1.** *If $\mathscr{H}$ is a normal space of distributions,*

1) *the space $\mathscr{H}'_c$ is a normal space of distributions, and*

2) *the space $\mathscr{H}'_\delta$ (the dual of $\mathscr{H}$ with the topology of uniform convergence of bounded sets of $\mathscr{H}$) is a space of distributions.*

*Proof.* The space $\mathscr{H}'$ is first of all a subspace of $\mathscr{D}'$. In fact, if $T \in \mathscr{H}'$ and $\tilde{T} = T|\mathscr{D}$ ($T$ restricted to $\mathscr{D}$), since the topology of $\mathscr{D}$ is finer than the topology induced by $\mathscr{H}$, $\tilde{T}$ is a continuous linear functional on $\mathscr{D}$. Hence $\tilde{T} \in \mathscr{D}'$. The mapping $T \to \tilde{T}$ is an injection, for if $\tilde{T} = 0$, we have $T = 0$ because $\mathscr{D}$ is dense in $\mathscr{H}$. Thus we see that $\mathscr{H}'$ is a subspace of $\mathscr{D}'$.

Since the injections $\mathscr{D} \to \mathscr{H} \to \mathscr{D}'$ are continuous, we have by transposition $\mathscr{D}'_c \leftarrow \mathscr{H}'_c \leftarrow (\mathscr{D}')'_c$ are continuous. But $\mathscr{D}'_c = \mathscr{D}'$ and $(\mathscr{D}')'_c = \mathscr{D}$. Hence $\mathscr{D}'_c \leftarrow \mathscr{H}'_c \leftarrow \mathscr{D}$ are continuous. Also since $\mathscr{H} \to \mathscr{D}'$ is an injection, $\mathscr{D}$ is dense in $\mathscr{H}'$ if we take the weak topology $\sigma(\mathscr{H}', \mathscr{H})$. Now $\mathscr{H}'_c$ and $\mathscr{H}'$ with the topology $\sigma(\mathscr{H}', \mathscr{H})$ have the same dual (Mackey's Theorem). It follows therefore that the linear

subspace $\mathscr{D}$ which is dense in $\mathscr{H}'$ in the topology $\sigma(\mathscr{H}', \mathscr{H})$ is also
10 dense in the topology of $\mathscr{H}_c'$. (See Bourbaki, EVT IV, § 2, no. 3, Cor. 1). This completes the proof of (1).

The proof of (2) is, in fact, trivial. The continuity of the injections $\mathscr{D} \to \mathscr{H} \to \mathscr{D}'$ gives the continuity of the injections $\mathscr{D}_\delta' \leftarrow \mathscr{H}_\delta' \leftarrow (\mathscr{D}')_\delta'$. But $\mathscr{D}_\delta' = \mathscr{D}'$ and $(\mathscr{D}')_\delta' = \mathscr{D}$. □

**The space $\mathscr{H}(E)$:**

Let $\mathscr{H}$ be a space of distributions. Let $E$ be an $ELC$.

**Definition 3.3.** *The space $\mathscr{H}(E)$ consists of all $E$-valued distributions $\vec{T}$ which have the following property. ${}^t\vec{T} : E_c' \to \mathscr{D}'$ maps actually $E_c'$ into $\mathscr{H}$ and is a continuous map of $E_c'$ into $\mathscr{H}$. We have $\mathscr{H}(E) \approx \mathscr{L}(E_c', \mathscr{H})$.*

**Definition 3.4.** *Let $\mathscr{H}$ be any linear subspaces of $\mathscr{D}'$. We say that an $E$-valued distribution $\vec{T}$ belongs scalarly to $\mathscr{H}$ if ${}^t\vec{T} : E_c' \to \mathscr{D}'$ actually maps $E'$ into $\mathscr{H}$. In other words, for every $\overleftarrow{e}' \in E'$, we have $\langle \vec{T}, \overleftarrow{e}' \rangle \in \mathscr{H}$.*

**Definition 3.5.** *We say that a space of distributions $\mathscr{H}$ has the $\mathcal{E}$- property, if for every locally convex, Hausdorff, complete vector space any $E$-valued distribution $\vec{T}$ which scalarly belongs to $\mathscr{H}$ belongs to $\mathscr{H}(E)$.*

**Proposition 3.2.** *If $\mathscr{H}$ has the $\mathcal{E}$-property, every subspace of $\mathscr{H}$ with the induced topology has also the $\mathcal{E}$-property.*

*Proof.* Let $\mathscr{K}$ be a linear subspace of $\mathscr{H}$ with the induced topology.Let $\vec{T}$ be an $E$-valued distribution, with $E$ a complete $ELC$, satisfying $\langle \vec{T},$
11 $\overleftarrow{e}' \rangle \in \mathscr{K}$ for every $\overleftarrow{e}' \in E'$. We have to show that $\vec{T} \in \mathscr{K}(E)$. In other words, we have to show that ${}^t\vec{T} : E_c' \to \mathscr{D}'$ takes $E_c'$ into $\mathscr{K}$ and is continuous. Now $\langle \vec{T}, \overleftarrow{e}' \rangle \in \mathscr{K}$ for every $\overleftarrow{e}'$ merely means that ${}^t\vec{T}(\overleftarrow{e}') \in \mathscr{K}$ for every $\overleftarrow{e}' \in E'$. Hence ${}^t\vec{T} : E_c' \to \mathscr{D}'$ maps $E_c'$ into $\mathscr{K}$. Since $\mathscr{K} \subset \mathscr{H}$ and $\mathscr{H}$ has the $\mathcal{E}$-property ${}^t\vec{T} : E_c' \to \mathscr{H}$ is continuous. Now, the topology of $\mathscr{K}$ is the induced topology and ${}^t\vec{T}(E_c') \in \mathscr{K}$. Hence ${}^t\vec{T} : E_c' \to \mathscr{K}$ is continuous. □

**Proposition 3.3.** *Suppose $\mathscr{H}$ satisfies the following conditions*

*(1) $\mathscr{H}$ is a normal space of distributions.*

*(2) $\mathscr{H}$ has a fundamental system of $\mathscr{D}'$-closed neighbourhood of* $0$ *in $\mathscr{H}$, that is to say, $\mathscr{H}$ has a fundamental system of neighbourhoods of* $0$ *which are closed in the topology induced from $\mathscr{D}'$.*

*(3) The bounded sets of $\mathscr{H}$ are relatively compact. Then $\mathscr{H}$ has the $\mathcal{E}$-property.*

*Proof.* Let $E$ be any complete $ELC$ and let $\vec{T}$ be any $E$-valued distribution scalarly belonging to $\mathscr{H}$, that is to say, ${}^t\vec{T} : E'_c \to \mathscr{D}'$ maps $E'_c$ into $\mathscr{H}$. Since ${}^t\vec{T} : E'_c \to \mathscr{D}'$ is continuous, ${}^t\vec{T} : E'_c \to \mathscr{H}_{\mathscr{D}'}$ is continuous, where $\mathscr{H}_{\mathscr{D}'}$ is the space $\mathscr{H}$ with the topology induced by $\mathscr{D}'$. The topology on $\mathscr{H}$ is finer than the topology induced by $\mathscr{D}'$. To prove the $\mathcal{E}$-property we have to show that ${}^t\vec{T} : E'_c \to \mathscr{H}$ is continuous. According to (2), if we prove that ${}^t\vec{T}^{-1}(W)$ is a neighbourhood of 0 in $E'_c$ for any convex, stable neighbourhood $W$ of 0 in $\mathscr{H}$ which is $\mathscr{D}'$ closed, it will follow that ${}^t\vec{T} : E'_c \to \mathscr{H}$ is continuous. Since $W$ is $\mathscr{D}'$-closed and
since ${}^t\vec{T} : E'_c \to \mathscr{H}_{\mathscr{D}'}$ is continuous, ${}^t\vec{T}^{-1}(W)$ is closed in $E'_c$. ${}^t\vec{T}^{-1}(W)$ **12**
is a convex, stable set of $E'_c$. Since $W$ is absorbing, ${}^t\vec{T}^{-1}(W)$ is also absorbing. Since ${}^t\vec{T}^{-1}(W)$ is a convex closed set in $E'_c$ it is also closed in $E'$ with the weak topology and since it is convex, stable, absorbing and weakly closed, it is a neighbourhood of 0 in the strong topology on $E'$, that is in $E'_\delta$. Hence ${}^t\vec{T} : E'_\delta \to \mathscr{H}$ is continuous. The injection $\mathscr{H} \to \mathscr{D}'$ is continuous. Hence the transpose $\mathscr{D} \to \mathscr{H}'_c$ is continuous and the image is dense in $\mathscr{H}'_c$. Because of (3) we have $\mathscr{H}'_\delta = \mathscr{H}'_c$. Also $\vec{T} : \mathscr{H}'_\delta \to (E'_\delta)'_\delta$ is continuous. Let $E''$ be the bidual of $E$. The topology $\mathcal{E}$ of uniform convergence on equicontinuous subsets of $E'$ is coarser than the topology of $(E'_\delta)'_\delta$. Hence $\vec{T} : \mathscr{H}'_\delta \to E''_{\mathcal{E}}$ is continuous. Hence the composite $\mathscr{D} \to \mathscr{H}'_\delta \to E''_{\mathcal{E}}$ is continuous. The image of $\mathscr{D}$ by the composite is contained in $E$ and on $E$, $E''_{\mathcal{E}}$ induces on $E$ the same topology as the initial topology of $E$. Since the image of $\mathscr{D}$ is dense in $\mathscr{H}'_\delta$, the image of $\mathscr{H}'_\delta$ in $E''_{\mathcal{E}}$ is contained in the closure of $E$ in $E''_{\mathcal{E}}$. But

$E$ being complete we have $\overrightarrow{T} : \mathscr{H}'_\delta \to E''_\mathcal{E}$ maps $\mathscr{H}'_\delta$ in $E$. The topology of $E$ being the one induced by $E''_\mathcal{E}$, we have $\overrightarrow{T} : \mathscr{H}'_\delta = \mathscr{H}'_c \to E$ continuous. Hence ${}^t\overrightarrow{T} : E'_c \to (\mathscr{H}'_c)'_c$ is continuous. But $(\mathscr{H}'_c)'_c$ is the same as $\mathscr{H}$ with a finer topology. Therefore ${}^t\overrightarrow{T} : E'_c \to \mathscr{H}$ is continuous. □

This proves our proposition.

# Lecture 4

# The $\mathcal{E}$-product of two locally convex Hausdorff spaces

Let $L$ and $M$ be two locally convex Hausdorff vector spaces. We shall define a space $L\mathcal{E}M$. 13

**Definition 4.1.** *$L\mathcal{E}M$ is the set of bilinear forms on $L'_c \times M'_c$ hypocontinuous with respect to the equicontinuous subsets of $L'$ and $M'$. (For the definition of hypocontinuity, see Bourbaki, E.V.T., Chap. III, § 4). $L\mathcal{E}M$ is a linear space. We put on $L\mathcal{E}M$ the topology of uniform convergence on products of equicontinuous subsets of $L'$ and $M'$.*

Let $\mathscr{E} \in L\mathcal{E}M, l' \in L'$ and $m' \in M'$. Write $\mathscr{E}(l', m') = \langle l'|\mathscr{E}|m'\rangle$ For any fixed $m' \in M'$ the mapping $l' \to \langle l'|\mathscr{E}|m'\rangle$ is a continuous linear form on $L'_c$ and hence defines an element of $L$ which we denote by $|\mathscr{E}|m'\rangle$. The mapping $m' \to |\mathscr{E}|m'\rangle$ is a continuous, linear map of $M'_c$ in $L$. That it is linear is trivial. To show that it is continuous we have to only show that if $m' \to 0$ $|\mathscr{E}|m'\rangle \to 0$ in $L$. Now $|\mathscr{E}|m'\rangle \to 0$ if for $l'$ lying in an equicontinuous subset of $L'$ we have $\langle l'|\mathscr{E}|m'\rangle \to 0$ uniformly. But this is half of the hypocontinuity assumption on $\mathscr{E}$. Hence $m' \to |\mathscr{E}|m'\rangle$ is a continuous linear map of $M'_c$ into $L$. Thus $\mathscr{E}$ defines an element of $\mathscr{L}(M'_c, L)$. Similarly, using the other half of the hypocontinuity hypothesis, we can show that $\mathscr{E}$ determines an element of $\mathscr{L}(L'_c, M)$. In fact this is nothing but the transpose of the linear map $M'_c \to L$ that corresponds to $\mathscr{E}$.

Let us denote by $\mathscr{E}'$ the continuous linear map of $M'_c$ into $L$ that 14

corresponds to $\mathscr{E}$. Then $\mathscr{E}'(m')$ is that element of $L$ which satisfies $\mathscr{E}\langle l', m'\rangle = \langle l|\mathscr{E}|m'\rangle = \langle \mathscr{E}(m'), l'\rangle L, L'$. Conversely suppose that $\eta : M'_c \to L$ is a continuous linear map.
Then the bilinear map $\eta : L'_c \times M'_c \to C$ defined by $\eta(l', m') = \langle \eta(m'), l'\rangle$ is hypocontinuous with respect to the products of equicontinuous subsets of $L'_c$ and $M'_c$. First we show that if $l'$ lies in an equicontinuous subset of $L'$ and $m' \to 0, \langle \eta(m'), l'\rangle \to 0$ uniformly. Since $\eta$ is continuous, $\eta(m') \to 0$ in $L$ and hence $\langle \eta(m'), l'\rangle \to 0$ uniformly if $l'$ lies in an equicontinuous subset of $L'$. The transpose $t_\eta : L'_c \to M$ is also continuous for $(M'_c)'_c$ is finer than $M$. This gives the other half of the hypocontinuity, namely, if $m'$ lies in an equicontinuous set of $M'$ and $l' \to 0$ in $L'$,

$$\langle \eta(m'), l'\rangle = \langle m', t_\eta(l')\rangle \to 0$$

uniformly.

Thus we see that $L\mathcal{E}M \approx \mathscr{L}(M'_c, L)$ algebraically. Similarly we have $L\mathcal{E}M \approx \mathscr{L}(L'_c, M)$ algebraically.

**Topologies on $\mathscr{L}(M'_c; L)$ and $\mathscr{L}(L'_c; M)$.**

On both these spaces we put the $\mathcal{E}$-topology which we define below.

**Definition 4.2.** *Let $E$ and $F$ be two ELC. The $\mathcal{E}$-topology on the space $\mathscr{L}(E'_c, F)$ is the topology of uniform convergence on equicontinuous subsets of $E'$.*

**Proposition 4.1.** *The algebraic isomorphisms between the three spaces $L\mathcal{E}M, \mathscr{L}_{\mathcal{E}}(L'_c, M)$ and $\mathscr{L}_{\mathcal{E}}(M'_c, L)$ are topological isomorphisms.*

15 *We shall prove the isomorphism $L\mathcal{E}M \approx \mathscr{L}_{\mathcal{E}}(M'_c, L)$ is topological, the other case being similar to this.*

*Let $\mathscr{E} \in L\mathcal{E}M$ and $\mathscr{E}'$ the corresponding element in $\mathscr{L}_{\mathcal{E}}(M'_c, L)$. We show $\mathscr{E} \to 0$ in $L\mathcal{E}M \iff \mathscr{E}' \to 0$ in $\mathscr{L}_{\mathcal{E}}(M'_c, L)$. Now, $\mathscr{E} \to 0$ in $L\mathcal{E}M$ if and only if for $l' \in P, m' \in Q, P$ and $Q$ being arbitrary equicontinuous sets of $L'$ and $M'$ respectively, we have $\mathscr{E}(l', m') \to 0$ uniformly. $\mathscr{E}'$ tends to 0, if and only if for $m'$ in any equicontinuous subset, say $R$ of $M'$, $\mathscr{E}'(m') \to 0$ in $L$ or for $l'$ in any equicontinuous set $S$ of $L'$, $\langle \mathscr{E}'(m'), l'\rangle \to 0$ uniformly. This is precisely equivalent to $\mathscr{E}(l', m') \to 0$ uniformly for $(l', m') \in P \times Q, P$ and $Q$ any equicontinuous subsets of $L'$*

*and $M'$ respectively. Hence $\mathscr{E} \to 0$ in $L\mathcal{E}M \Longleftrightarrow \mathscr{E}' \to 0$ in $\mathscr{L}_{\mathcal{E}}(M'_c, L)$. This proves proposition 4.1.*

**Examples of the $\mathcal{E}$-product of a space of distributions and an $ELC$.**

1) $\mathscr{D}'\mathcal{E}E \approx \mathscr{L}_{\mathcal{E}}(E'_c, \mathscr{D}') \approx \mathscr{L}_{\mathcal{E}}(\mathscr{D}; E)$ (by proposition 4.1).

   But we have $\mathscr{D}'(E) \approx \mathscr{L}_{\mathcal{E}}(\mathscr{D}; E)$ topologically for in $\mathscr{D}$, considered as the dual of $\mathscr{D}'$, bounded sets are equicontinuous.

2) $\mathscr{S}'\mathcal{E}E \approx \mathscr{L}_{\mathcal{E}}(\mathscr{S}; E) \approx \mathscr{L}(E'_c, \mathscr{S}')$ and $\mathscr{S}'(E) \approx \mathscr{S}'\mathcal{E}E$ for $\mathscr{S}'(E) \approx \mathscr{L}_{\mathcal{E}}(\mathscr{S}; E)$ topologically since $\mathscr{S}', \mathscr{S}$ are Montel spaces.

3) $\mathscr{O}_M\mathcal{E}E \approx \mathscr{L}_{\mathcal{E}}(E'_c; \mathscr{O}_M)$.

   (For the definition of the spaces $\mathscr{S}, \mathscr{S}', \mathscr{O}_M, \mathscr{O}_c, \mathscr{O}'_M, \mathscr{O}'_c$ refer: Theorie des distributions, Tome ii).

4) $\mathscr{E}'(E) \approx \mathscr{L}_{\mathcal{E}}(\mathscr{E}; E)$ and $\mathscr{E}'\mathcal{E}E \approx \mathscr{L}_{\mathcal{E}}(\mathscr{E}; E) \approx \mathscr{L}_{\mathcal{E}}(E'_c, \mathscr{E}')$.

**Covariance property of the $\mathcal{E}$-product.** **16**

Let $u : L_1 \to L_2$ and $v : M_1 \to M_2$ be continuous linear maps, $L_1, M_1, L_2$ and $M_2$ being locally convex Hausdorff topological vector spaces. We shall now see how with $u$ and $v$ a continuous linear map, say $u\mathcal{E}v : L_1\mathcal{E}M_1 \to L_2\mathcal{E}M_2$ can be associated. There are, in fact, three ways of defining this map according as we consider the three forms of writing the $\mathcal{E}$-product, namely $L_1\mathcal{E}M_1, \mathscr{L}_{\mathcal{E}}(M'_c, L_1)$ and $\mathscr{L}_{\mathcal{E}}(L'_{1c}, M_1)$.

**Definition 4.3 1.** *Let $(l'_2, m'_2) \in L'_2 \times M'_2$ and $\mathscr{E} \in L_1\mathcal{E}M_1$. Let $(u\mathcal{E}v)(\mathscr{E})$ be the bilinear form $\eta$ defined by*

$$\eta(l'_2, m'_2) = \mathscr{E}({}^t u(l'_2), {}^t v(m'_2)).$$

*We now prove that $\eta \in L_2\mathcal{E}M_2$. For this we have to only prove hypocontinuity of $\eta$ with respect to the equicontinuous subsets of $L'_{2c}$ and $M'_{2c}$. Suppose $l'_2$ lies in an equicontinuous subset of $L'_2$. Then there exists a neighbourhood $U_2$ of 0 in $L_2$ such that $|\langle l_2, l'_2\rangle| < 1$ for $l_2 \in U_2$. Since $u$ is continuous, $U_1 = u^{-1}(U_2)$ is a neighbourhood of 0 in $L_1$ and for any $l_1$ in $U_1$ we have $|\langle l_1, t_u(l'_2)\rangle| = |\langle u(l_1), l'_2\rangle|$. But $u(l_1) = l_2 \in U_2$.*

*Hence $|\langle l_1, t_u(l_2')\rangle| < l$ for every $l_1 \in U_1$. Hence the set $\{{}^t u(l_2') | l_2' \in$ an equicontinuous subset of $L_2'\}$ is an equicontinuous subset of $L_1'$. Let $m_2' \to 0$ in $M_{2c}'$. Then since $t_v : M_{2c}' \to M_{1c}'$ is continuous, ${}^t v(m_2')^{\to 0}$ in $M_{1c}'$. Hence $\mathscr{E}^{({}^t u(l_2'); {}^t v(m_2'))\to 0}$ uniformly if $l_2'$ lies in an equicontinuous subset of $L_2'$ and $m_2' \to 0$ in $M_{2c}'$. Similarly we prove the other part of the hypocontinuity.*

17 *Thus $\eta \in L_2 \mathcal{E} M_2$. The mapping $\mathscr{E} \to \eta$ is denoted by $u\mathcal{E}v$.*

**Definition 4.3 2.** *Let $\mathscr{E} \in \mathscr{L}_{\mathcal{E}}(M_{1c}', L_1)$. Let $u : L_1 \to L_2$ and $v : M_1 \to M_2$ be continuous linear maps. Then $t_v \overset{M_{1c}'}{\underset{M_{2c}'}{\uparrow}}$ and $\overset{L_1}{\underset{L_2}{\downarrow}} u$ are continuous linear maps. $\mathscr{E} : M_{1c}' \to L_1$ is continuous linear. The composite $\eta = u \circ \mathscr{E} \circ t_v : M_{2c}' \to L_2$ is a continuous linear map and hence $\eta \in \mathscr{L}(M_{2c}', L_2)$. With $\mathscr{E} \in \mathscr{L}_{\mathcal{E}}(M_{1c}', L_1)$ we associate the element $\eta \in \mathscr{L}_{\mathcal{E}}(M_{2c}', L_2)$.*

**Definition 4.3 3.** *Let $\mathscr{E} \in \mathscr{L}_{\mathcal{E}}(L_{1c}', M_1)$. $t_u : L_{1c}' \leftarrow L_{2c}', v : M_1 \to M_2$ continuous. Hence the composite $\eta = v \circ \mathscr{E} \circ t_u : L_{2c}' \to M_2$ is a continuous linear map. With $\mathscr{E} \in \mathscr{L}_{\mathcal{E}}(L_{1c}', M_1)$ we associate the element $\eta \in \mathscr{L}_{\mathcal{E}}(L_{2c}', M_2)$.*

**Proposition 4.2.** *The above three definitions give one and the same element $\eta$ of $L_2 \mathcal{E} M_2 \approx \mathscr{L}_{\mathcal{E}}(M_{2c}', L_2) \approx \mathscr{L}_{\mathcal{E}}(L_{2c}', M_2)$.*

*Proof.* Let us, for the sake of clarity, denote the elements got from definitions 4.3 (1), 4.3 (2) and 4.3 (3) by $\eta_1, \eta_2$ and $\eta_3$. Our assertion will be proved if we show

$$\eta_1(l_2', m_2') = \langle \eta_2(m_2'), l_2' \rangle_{L_2, L_2'} = \langle \eta_3(l_2'), m_2' \rangle_{M_2, M_2'}.$$

$$\text{Now,} \quad \eta_1(l_2', m_2') = \mathscr{E}^{({}^t u(l_2')' \, {}^t v(m_2'))} \tag{i}$$

$$\begin{aligned} \langle \eta_2(m_2'), l_2' \rangle_{L_2, L_2'} &= \langle u \circ \mathscr{E} \circ {}^t v(m_2'), l_2' \rangle_{L_2, L_2'} \\ &= \langle \mathscr{E}^t v(m_2')' \, {}^t u(l_2') \rangle_{L_1, L_1'} \\ &= \mathscr{E}({}^t u(l_2')' \, {}^t v(m_2')^{)} \end{aligned} \tag{ii}$$

$$\text{Similarly} \quad \langle \eta_3(l_2'), m_2' \rangle_{M_2, M_2'} = \mathscr{E}^{({}^t u(l_2')' \, {}^t v(m_2'))} \qquad \text{(iii)}$$

18 A comparison of (i), (ii), and (iii) gives the required proposition. ◻

**Proposition 4.3.** *If $u : L_1 \to L_2$ and $v : M_1 \to M_2$ are injections $u\mathcal{E}v : L_1\mathcal{E}M_1 \to L_2\mathcal{E}M_2$ is also an injection.*

*Proof.* We have to show that $(u\mathcal{E}v)\mathscr{E}_1 = 0 \Rightarrow \mathscr{E}_1 = 0$. Now $(u\mathcal{E}v)\mathscr{E}_1 = \langle {}^t u(l_2') | \mathscr{E}_1 | {}^t v(m_2') \rangle$. Since $u$ and $v$ are injections, $t_u(L_2')$ and $t_v(M_2')$ are dense in $L_1'$ and $M_1$. Hence $\mathscr{E}_1$ which is a separately continuous bilinear form, is zero on the product $t_u(L_2') \times t(M_2')$ where $t_u(L_2')$ and $t_v(M_2')$ are dense subspaces of $L_1'$ and $M_1'$. Hence $\mathscr{E}_1 = 0$. (See Bourbaki, EVT, Chap. III, § 4, No. 3).

In fact, one can even show that if $u : L_1 \to L_2$ and $v : M_1 \to M_2$ are monomorphisms, $u\mathcal{E}v : L_1\mathcal{E}M_1 \to L_2\mathcal{E}M_2$ is a monomorphism. That is to say if we assume that $u : L_1 \to u(L_1)$ is a topological isomorphism with the topology induced on $u(L_1)$ by $L_2$ and $v : M_1 \to v(M_1)$ is a topological isomorphism with the topology induced by $M_2$, then $u\mathcal{E}v : L_1\mathcal{E}M_1 \to u\mathcal{E}v(L_1\mathcal{E}M_1)$ is a topological isomorphism with the induced topology.

Also we have $L \otimes M \subset L\mathcal{E}M$. The topology on $L \otimes M$ induced by $L\mathcal{E}M$ is called the $\mathcal{E}$-topology and provided with this topology $L \otimes M$ is denoted by $L \underset{\mathcal{E}}{\otimes} M$ ◻

**Proposition 4.4.** *If $L$ and $M$ are complete, $L\mathcal{E}M$ is complete.*

*Proof.* Let $(\mathscr{E}_j)$ be a Cauchy filter on $L\mathcal{E}M$. This Cauchy filter gives rise to a Cauchy filter, which also we denote by $(\mathscr{E}_j)$, on $\mathscr{L}_{\mathcal{E}}(L_c', M)$. Since $M$ is complete it follows that there exists a linear map $\mathscr{E} : L_c' \to M$ such that $\mathscr{E}_j$ converges to $\mathscr{E}$ uniformly on every equicontinuous subset of $L'$. Similarly $(\mathscr{E}_j)$ defines a Cauchy filter on $\mathscr{L}_{\mathcal{E}}(M_c', L)$ which also we de- 19
note by $(\mathscr{E}_j)$ and this defines a linear map $\mathscr{E}' : M_c' \to L$. Also trivially $\mathscr{E}$ and $\mathscr{E}'$ are transposes of each other. Now, every equicontinuous subset of $L'$ is contained in a compact (for $L_c'$), convex, equicontinuous subset of $L'$. Since the restriction of $\mathscr{E}$ to every equicontinuous subset of $L'$ is continuous, it follows that the image of every equicontinuous subset of $L'$ by $\mathscr{E}$ is contained in a compact convex subset of $M$. This proves

that $\mathscr{E}' : M'_c \to L$ is continuous. Similarly $\mathscr{E} : L'_c \to M$ is continuous. Consequently $\mathscr{E} \in L\mathcal{E}M$. □

# Lecture 5

# The Approximation Property

**Definition 5.1.** *We say that an ELC L has the approximation property* 20
*if $L' \otimes L$ is dense in $\mathscr{L}_c(L, L)$.*

*Trivially if the identity map $I : L \to L$ is adherent to $L' \otimes L$ in $\mathscr{L}_c(L, L)$, L has the approximation property. In fact, if I is adherent to $L' \otimes L$ in $\mathscr{L}_c(L, L)$, we have $L' \otimes M$ dense in $\mathscr{L}_c(L, M)$ for every ELC M.*

*The spaces $\mathscr{D}, \mathscr{D}', \mathscr{E}, \mathscr{E}', \mathscr{D}^m, L', (\mathscr{D}')^{(o)}, \mathscr{S}, \mathscr{S}', \mathscr{O}_M$ and $\mathscr{O}'_c$ have all the approximation property. It is not known whether $\mathscr{D}'^m$ with the strong topology has the approximation property or not.*

**Proposition 5.1.** *If L or M has the approximation property, we have $L \otimes M$ dense in $L\mathcal{E}M$. If for every ELCM, $L \otimes M$ is dense in $L\mathcal{E}M$, then L has the approximation property.*

*Assume that L has the approximation property. We have to prove that continuous linear maps from $M'_c \to L$ of finite rank are dense in $\mathscr{L}_{\mathcal{E}}(M'_c, L)$. Since L has the approximation property, we can find a filter $(v_j)$ of maps of finite rank of L into L converging to the identity map in $\mathscr{L}_c(L; L)$, i.e., the filter $(v_j)$ converges uniformly to I on compact discs of L (disc = convex, stable subset). Let $u \in \mathscr{L}_{\mathcal{E}}(M'_c L) = L\mathcal{E}M$. Since every equicontinuous subset of $M'$ is contained in a compact disc of $M'_c$, the filter $(v_j \circ u)$ converges uniformly on equicontinuous subsets of $M'_c$ to u. Also $v_j \circ u$ are maps of finite rank of $M'_c$ in L. Hence the required result follows.*

*Conversely, suppose $L \otimes M$ is dense in $L\mathcal{E}M$ for every ELCM. Tak-* 21 *ing for $M$ the space $L'_c$, we have $L \otimes L'_c$ dense in $L\mathcal{E}L'_c$. The topology of $(L'_c)'_c$ is finer than that of $L$, though algebraically they are the same. Hence $I : (L'_c)'_c \to L$ is continuous and hence it can be approximated by continuous maps of finite rank of $L$ in $L$ on equicontinuous subsets of $(L'_c)'$. Any convex, compact stable subset of $L$ is equicontinuous in $(L'_c)'$. This proves our proposition.*

As pointed out in the previous lecture, we have

(1) $\mathscr{D}'(E) \approx \mathscr{D}'\mathcal{E}E \approx \mathscr{L}_\delta(\mathscr{D}, E) \approx \mathscr{L}_\mathcal{E}(E'_c, \mathscr{D}')$

(2) $\mathscr{S}'(E) \approx \mathscr{S}'\mathcal{E}E \approx \mathscr{L}_\delta(\mathscr{S}, E) \approx \mathscr{L}_\mathcal{E}(E'_c, \mathscr{S}')$, and

(3) $\mathscr{E}'(E) \approx \mathscr{E}'\mathcal{E}E \approx \mathscr{L}_\delta(\mathscr{E}, E) \approx \mathscr{L}_\mathcal{E}(E'_c, \mathscr{E}')$.

**Definition 5.2.** *Let $\overrightarrow{T}$ be an $E$-valued distribution. The support of $\overrightarrow{T}$ is, by definition, the smallest closed set $\Omega \subset R^n$ such that if $\varphi$ is any $C^\infty$ function with compact support whose support is contained in the complement of $\Omega$, we have $\overrightarrow{T}(\varphi) = 0$.*

**Remark.** An element of $\mathscr{E}'(E)$ need not have compact support. In fact, the identity map $I : \mathscr{E} \to \mathscr{E}$ is an element of $\mathscr{E}'(\mathscr{E})$. It does not have a compact support. For, if it had a compact support $K$, every continuous image of $I$ will have its support in $K$. In particular, every scalar-valued distribution with compact support, being a continuous image of $I$, will have its support in $K$, a fixed compact set, which is absurd.

However, if an $E$-valued distribution $\overrightarrow{T}$ has a compact support, $\overrightarrow{T} \in \mathscr{E}'(E)$. In fact, ${}^t\overrightarrow{T} : E'_c \to \mathscr{D}'$ maps $E'_c$ into $\mathscr{E}'$.

22 **Proposition 5.3.** *If $E$ has a neighbourhood of* 0 *which does not contain any straight line, then every element of $\mathscr{E}'(E)$ has a compact support.*

*Proof.* Let $V$ be a neighbourhood of 0 in $E$ not containing any straight line and $\overrightarrow{T} : \mathscr{E} \to E$ a continuous linear map. Since $\overrightarrow{T}$ is continuous, $\exists$ an integer $m \geq 0$, a compact set $K$ and an $\varepsilon > 0$ such that for every $\varphi \in \mathscr{E}$ with $\sup\limits_{|p| \leq m, x \in K} |D^p\varphi(x)| \leq \varepsilon$ we have $\overrightarrow{T}(\varphi) \in V$. Let $\psi \in \mathscr{D}$

with support in the complement of $K$. Then $\sup\limits_{|p|\leq m, x\in K} |D^p \lambda\psi(x)| = 0$ and hence $\lambda \overrightarrow{T}(\psi) \in V$ for every $\lambda$. Since $V$ does not contain any straight line, $\overrightarrow{T}(\psi) = 0$ or the support of $\overrightarrow{T}$ is contained in $K$. Therefore $\overrightarrow{T}$ has a compact support. □

**Corollary.** *$E$ having a neighbourhood of* 0 *not containing any straight line is equivalent to saying that there exists a continuous semi-norm on $E$ which is a norm.*

*If $E$ is a normed space, then any $\overrightarrow{T} \in \mathscr{E}'(E)$ has a compact support.*

**Definition 5.3.** *The space of $E$-valued distributions with compact support is denoted by $\tilde{\mathcal{E}}'(E)$.*

*We have $\tilde{\mathscr{E}}'(E) \subset \mathscr{E}'(E)$ algebraically.*

**Definition 5.4.** *$\tilde{\mathscr{E}}^m(E)$ is the space of $m$-times continuously differentiable functions from $R^n$ to $E$.*

**Proposition 5.4.** *We have the algebraic inclusion $\mathscr{E}^m(E) \subset \tilde{\mathscr{E}}^m(E)$.*

*Proof.* Let $\overrightarrow{T}$ be a continuous linear map $\mathscr{E}'^m_c \to E$. Let $\overrightarrow{f}$ be an $E$-valued function defined as follows: $\overrightarrow{f}(a) = T(\delta_a)$. Now the map $a \to \delta_a$ **23**
is an $m$-times continuously differentiable function of $R^n$ with values in $\mathscr{E}'^m_c$ and $T$ is a continuous linear map. Hence $\overrightarrow{f}$ is an $m$-times continuously differentiable function. We show that $\overrightarrow{T}$ is the distribution defined by the function $\overrightarrow{f}$. We have

$$\begin{aligned}\langle \overrightarrow{f}, \overleftarrow{e}'\rangle(a) = \langle \overrightarrow{f}(a), \overleftarrow{e}'\rangle = \langle \overrightarrow{T}(\delta_a), \overleftarrow{e}'\rangle = \delta_a(\langle \overrightarrow{T}, \overleftarrow{e}'\rangle) \\ = \langle \overrightarrow{T}, \overleftarrow{e}'\rangle(a) \quad \text{(as a function)}\end{aligned}$$

□

This proves our assertion.

**Proposition 5.5.** *If $E$ is complete, $\tilde{\mathscr{E}}^m(E) = \mathscr{E}^m(E)$.*

*Proof.* If we prove that $\tilde{\mathscr{E}}^m(E) \subset \mathscr{E}^m(E)$. we are through, because of proposition 5.4. Let $\overrightarrow{f} \in \tilde{\mathscr{E}}^m(E)$. Since $E$ is complete, $\overrightarrow{f}$ can be used to define a distribution $\overrightarrow{f}(\varphi) = \int\limits_{R^n} \overrightarrow{f}(x)\varphi(x)\,dx$. It is evident that the above distribution $\overrightarrow{f}$ scalarly belongs to $\mathscr{E}^m$. Now $\mathscr{E}^m$ does not satisfy the $\mathcal{E}$-property. We cannot immediately conclude that $\overrightarrow{f} \in \mathscr{E}^m(E)$. We have to prove that the map $E'_c \to \mathscr{E}^m$ defined by $\overrightarrow{f}$ is continuous. Suppose $\overleftarrow{e}' \to 0$ in $E'_c$, we have to show that $\langle \overrightarrow{f}, \overleftarrow{e}' \rangle \to 0$ in $\mathscr{E}^m$, i.e., $D^p\langle \overrightarrow{f}, \overleftarrow{e}' \rangle \to 0$ uniformly on every compact subset $K \subset R^n$ for $|p| \leq m$. But $D^p\langle \overrightarrow{f}, \overleftarrow{e}' \rangle = \langle D^p f, \overleftarrow{e}' \rangle$ for $|p| \leq m$. For each $p$ with $|p| \leq m$, the set of values $D^p \overrightarrow{f}(x)$, $x \in K$ is a compact set in $E$. Since $E$ is complete the convex, stable, closed envelope of the set $\{D^p \overrightarrow{f}(x)\}_{x \in K}$ is compact and so, $\langle D^p \overrightarrow{f}, \overleftarrow{e}' \rangle \to 0, |p| \leq m$ uniformly for $x \in K$. □

**Characterization of** $\mathscr{E}^m(E)$. In the general case, one sees that $\mathscr{E}^m(E)$ is the set of all $m$-times continuously differentiable functions $\overrightarrow{f}$ satisfying the following conditions: For each $p$ with $|p| \leq m$, and for each compact set $K \subset R^n$, the convex, stable, closed envelope of $D^p \overrightarrow{f}(K)$ is compact in $E$.

**24** Let $E$ be a complete $ELC$. Let $\mathfrak{H}$ denote the space of holomorphic functions on $R^{2n}$ provided with the canonical complex structure. We put on $\mathfrak{H}$ the topology induced by that of $\mathscr{E}^\circ$. Let $\mathfrak{H}(E)$ denote the space $\mathfrak{H}\varepsilon E$.

**Definition 5.5.** *Any element $\overrightarrow{f} \in \mathfrak{H}(E)$ is called a holomorphic function with values in $E$.*

**Proposition 5.7.** *Let $\overrightarrow{f}(z)$ be an $E$-valued function such that for every $\overleftarrow{e}'$, the function $\varphi_{\mathfrak{H}}$ defined by $\varphi_{\overleftarrow{e}'}(z) = \langle \overrightarrow{f}(z), \overleftarrow{e}' \rangle$ is in $\mathfrak{H}$. Then $\overrightarrow{f} \in \mathfrak{H}(E)$ and we have a formula similar to the formula of Cauchy:*

$$\overrightarrow{f}(z) = \frac{1}{2\pi i} \int \frac{\overrightarrow{f}(\zeta)}{\zeta - z} d\zeta.$$

*Proof.* Since $\mathfrak{H}$ has the $\mathcal{E}$-property, if $\overrightarrow{f}$ belongs scalarly to $\mathfrak{H}$, $\overrightarrow{f}$ belongs to $\mathfrak{H}(E)$. This proves the first part. To prove the second part we see that for every $\overleftarrow{e}' \in E'$, $\varphi_{\overleftarrow{e}'}$ being a scalar-valued holomorphic function, we have

$$\varphi_{\overleftarrow{e}'}(z) = \langle \overrightarrow{f}(z), \overleftarrow{e}' \rangle = \frac{1}{2\pi i} \int \frac{\varphi_{\overleftarrow{e}'}(\zeta)}{\zeta - z} \, d\zeta$$

$$= \frac{1}{2\pi i} \int \frac{\langle \overrightarrow{f}(\zeta), \overleftarrow{e}' \rangle}{\zeta - z} d\zeta.$$

That is to say, $$\langle \overrightarrow{f}(z), \overleftarrow{e}' \rangle = \frac{1}{2\pi i} \int \frac{\langle \overrightarrow{f}(\zeta), \overleftarrow{e}' \rangle}{\zeta - z} d\zeta$$

$$= \frac{1}{2\pi i} \int \left\langle \frac{\overrightarrow{f}(\zeta)}{\zeta - z}, \overleftarrow{e}' \right\rangle d\zeta$$

$$= \left\langle \frac{1}{2\pi i} \int \frac{\overrightarrow{f}(\zeta)}{\zeta - z} \, d\zeta, \overleftarrow{e}' \right\rangle.$$

Hence $\overrightarrow{f}(z) = \frac{1}{2\pi i} \int \frac{\overrightarrow{f}(\zeta)}{\zeta - z} \, d\zeta$.

□

# Lecture 6

# Operations on Vector valued Distributions

$E$ will always denote a complete E L C. 25

**Differentiation of vector valued distributions.**

Let $\overrightarrow{T} \in \mathscr{D}'(E)$. Let $D^p$ be the operator $\dfrac{\partial^{p_1+\cdots+p_n}}{\partial x_1^{p_1}\ldots\partial x_n^{p_n}}$ where $p$ is the $n$-tuple $(p_1, p_2, \ldots, p_n)$. $D^p\overrightarrow{T}$ is defined to be the map, which maps $\varphi \in \mathscr{D}$ into $(-1)^{|p|}\overrightarrow{T}(D^p\varphi) \in E$ It is easily seen that $D^p\overrightarrow{T}$ is an $E$-valued distribution.

We have $\langle D^p\overrightarrow{T}, \overleftarrow{e'}\rangle = D^p\langle\overrightarrow{T}, \overleftarrow{e'}\rangle$. This follows from the very definition of $D^p$.

**Scalar product**.

Let $\mathscr{H}$ be a normal space of distributions. Then $\mathscr{H}_c'$ is a normal space of distributions (Proposition 4.1). Let us denote by S.T the scalar product of any element $S \in \mathscr{H}$ and any element $T \in \mathscr{H}'$. Let now $\overrightarrow{S} \in \mathscr{H}(E)$ and $T \in \mathscr{H}'$.

**Definition 6.1.** *$\overrightarrow{S} \in \mathscr{H}(E)$. Hence $\overrightarrow{S}$ can be considered as a continuous linear map of $\mathscr{H}_c'$ in $E$. Hence if $T \in \mathscr{H}_c'$, $\overrightarrow{S}(T) \in E$. We define $\overrightarrow{S}\,{}_{\mathrm{i}}T$ to be the element $\overrightarrow{S}(T)$ of $E$.*

**Definition 6.2.** *Let $\overrightarrow{S} \in \mathscr{H}(E)$. Then ${}^t\overrightarrow{S} : E_c' \to \mathscr{H}$ is a continuous linear map. We have agreed to denote by $\langle\overrightarrow{S}, \overleftarrow{e'}\rangle$ the element ${}^t\overrightarrow{S}(\overleftarrow{e'})$*

*of $\mathscr{H}$. Let $T \in \mathscr{H}'$. Then $\overrightarrow{S}\,\dot{_2}T$ is defined to be that element of E which satisfies*

$$\begin{aligned}\langle \overrightarrow{S}_2.T, \overleftarrow{e'}\rangle_E, E' &= \langle {}^t\overrightarrow{S}(\overleftarrow{e}')'^T\rangle_{\mathscr{H},\mathscr{H}'}\\ &= {}^t\overrightarrow{S}(\overleftarrow{e}')^{\cdot T}\\ &= \langle \overrightarrow{S}, \overleftarrow{e}'\rangle \cdot T\end{aligned}$$

26 **Definition 6.3.** *Let $\overrightarrow{S} \in \mathscr{H} \subset E$ and $T \in \mathscr{H}'$. Then $T : \mathscr{H} \to C$ and $I : E \to E$ are continuous linear maps. Hence $T \in I : \mathscr{H} \subset E \to C \subset E = E$ is a continuous linear map (Definitions 4.3 (1), 4.3 (2) and 4.3 (3)). We define $\overrightarrow{S}_3.T$ to be the element $T \in I(\overrightarrow{S})$ of E.*

**Proposition 6.1.** *The elements $\overrightarrow{S}\,\dot{_1}T$, $\overrightarrow{S}\,\dot{_2}T$ and $\overrightarrow{S}\,\dot{_3}\,T$ of E are all equal.*

$$\begin{aligned}\textit{Now}\quad \langle \overrightarrow{S}_1.T, \overleftarrow{e}'\rangle_{E,E'} &= \langle \overrightarrow{S}(T), \overleftarrow{e}'\rangle = \langle T, {}^t\overrightarrow{S}(\overleftarrow{e}')\rangle_{\mathscr{H}',\mathscr{H}}\\ &= \langle \overrightarrow{S}_2.T, \overleftarrow{e}'\rangle_{E,E'}.\end{aligned}$$

This proves $\overrightarrow{S}_1.T = \overrightarrow{S}_2 \cdot T$.
Also $\langle \overrightarrow{S}_3 \cdot T, \overleftarrow{e}'\rangle_{E,E'} = \langle T \in I(\overrightarrow{S}), \overleftarrow{e}'\rangle_{E,E'}$.
Proposition 4.2 gives $T \in I(\overrightarrow{S})$, considered as an element of $C \subset E$ or as an element of $\mathscr{L}_\varepsilon(C_C; E)$, is the same as the composite of the maps ${}^tT : C \to \mathscr{H}_c', \mathscr{H}_c' \xrightarrow{\overrightarrow{S}} E$ and $I : E \to E$.
Hence $\langle T \in I(\overrightarrow{S}), \overleftarrow{e}'\rangle_{E,E'} = \langle I \circ \overrightarrow{S} \circ t_T(1), \overleftarrow{e}'\rangle_{E,E'}$.
Also $\overrightarrow{S} \circ t_T(1) = \overrightarrow{S}(T)$. Hence we have

$$\begin{aligned}\langle I \circ \overrightarrow{S} \circ t_T(1), \overleftarrow{e}'\rangle_{E,E'} &= \langle \overrightarrow{S}(T), \overleftarrow{e}'\rangle\\ &= \langle \overrightarrow{S}_1.T, \overleftarrow{e}'\rangle.\end{aligned}$$

Hence $\overrightarrow{S}\,\dot{_3}T = \overrightarrow{S}\,\dot{_1}T$.
As example, we see in the following situations we can define a scalar product:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1) | $\overrightarrow{T} \in \mathscr{D}'(E), \varphi \in \mathscr{D}$ | ; | | 2) | $\overrightarrow{T} \in \mathscr{D}_c'^m(E), \varphi \in \mathscr{D}^m$; |
| 3) | $\overrightarrow{T} \in \mathscr{S}'(E), \varphi \in \mathscr{S}$ | ; | | 4) | $\overrightarrow{T} \in \mathscr{D}(E), \varphi \in \mathscr{D}'$; |
| 5) | $\overrightarrow{\varphi} \in \mathscr{D}(E), T \in \mathscr{D}'$ | ; | and | 6) | $\overrightarrow{\varphi} \in \mathscr{D}^m(E), T \in \mathscr{D}'^m$. |

**Properties of the scalar product:** 27

**Proposition 6.2.** *If $T$ belongs to an equicontinuous subset of $\mathscr{H}'$ and $\vec{S}$ tends to* 0 *in $\mathscr{H}(E)$, $\vec{S} \cdot T$ tends to zero uniformly in $E$.*

*Proof.* Now, $\vec{S} \in \mathscr{L}_\varepsilon(\mathscr{H}'_c E)$, the $\varepsilon$-topology being the topology of uniform convergence on equicontinuous sets of $\mathscr{H}'$. Hence when $T$ lies in an equicontinuous subset of $\mathscr{H}'$ and $\vec{S}$ tends to 0 in $\mathscr{H}(E)$, $\vec{S}(T)$ tends to 0 uniformly in $E$. But $\vec{S} \cdot T = \vec{S}(T)$. Hence $\vec{S} \cdot T$ tends to 0 in $E$ uniformly with respect to $T$ in an equicontinuous subset of $\mathscr{H}'$, when $\vec{S}$ tends to 0 in $\mathscr{H}(E)$. ◻

**Proposition 6.3.** *If $\vec{S}$ lies in a compact set of $\mathscr{H}(E)$ and $T$ tends to* 0 *in $\mathscr{H}'_c$ and if $\mathscr{H}$ is complete, then $\vec{S} \cdot T$ tends uniformly to* 0 *in $E$.*

*Proof.* Let $K$ be any compact subset of $\mathscr{H}(E) = \mathscr{L}_\varepsilon(E'_c \mathscr{H})$. If $A$ is any equicontinuous subset of $E'_c$, $\underset{\vec{S} \in K}{U} S(A)$ lies in a compact subset of $\mathscr{H}$. If $\vec{S} \in K$ and $\overleftarrow{e}'$ lies in an equicontinuous subset of $E'$, we have to show that $\langle \vec{S}(T), \overleftarrow{e}' \rangle$ tends to 0 uniformly as $T \to 0$ in $\mathscr{H}'_c$. We have

$$\langle \vec{S}, \overleftarrow{e}' \rangle \cdot T = \langle \vec{S}(T), \overleftarrow{e}' \rangle.$$

That is to say $\langle \vec{S}(T), \overleftarrow{e}' \rangle = T(\langle \vec{S}, \overleftarrow{e}' \rangle)$. From what has been said above $\langle \vec{S}, \overleftarrow{e}' \rangle$ lies in a compact subset of $\mathscr{H}$. Since $\mathscr{H}$ is complete, the convex stable envelope of a compact set is compact. Hence if $\vec{S}$ lies in a compact set of $\mathscr{H}(E)$ and $\overleftarrow{e}'$ lies in an equicontinuous subset of $E'$, $\langle \vec{S}, \overleftarrow{e}' \rangle$ lies in a compact disc of $\mathscr{H}$. Hence if $T \to 0$ in $\mathscr{H}'_c$, $T(\langle \vec{S}, \overleftarrow{e}' \rangle) \to 0$ uniformly. This proves proposition 6.3. ◻

**Proposition 6.4.** *If $\vec{S}$ belongs to a bounded subset of $\mathscr{H}(E)$ and $T$ tends to* 0 *in the strong dual $\mathscr{H}'_\delta$ then $\vec{S} \cdot T$ tends to* 0 *uniformly in $E$.* 28

*Proof.* It suffices to show that when $\overleftarrow{e}'$ lies in an equicontinuous subset of $E'$, $\langle \vec{S} \cdot T, \overleftarrow{e}' \rangle$ tends to 0 uniformly. Let $\vec{S}$ remain in the bounded set $B$ of $\mathscr{H}(E)$. Now $\mathscr{H}(E) \approx \mathscr{L}_\varepsilon(E'_c, \mathscr{H})$. If $B$ is bounded in $\mathscr{H}(E)$, then $\bigcup_{\vec{S}\in B} \overrightarrow{S}(H)$ is a bounded set of $\mathscr{H}$, whatever be the equicontinuous subset $H$ of $E'_c$. We have $\langle \overrightarrow{S} \cdot T, \overleftarrow{e}' \rangle = \langle \overrightarrow{S}, \overleftarrow{e}' \rangle . T = T(\langle \overrightarrow{S}, \overleftarrow{e}' \rangle)$. When $\overleftarrow{e}'$ lies in an equicontinuous subset of $E'$ and $\overrightarrow{S}$ lies in a bounded set $B$ of $\mathscr{H}(E)$ we have seen that $\langle \overrightarrow{S}, \overleftarrow{e}' \rangle$ lies in a bounded set of $\mathscr{H}$. Since $T \to 0$ in $\mathscr{H}'_\delta$ we have $T(\langle \overrightarrow{S}, \overleftarrow{e}' \rangle) \to 0$ uniformly with respect to $\overrightarrow{S}$ in a bounded set of $\mathscr{H}(E)$ and $\overleftarrow{e}'$ in an equicontinuous subset of $E'$. This proves proposition 6.4. □

Combining propositions 6.2 and 6.4, we get the following

**Proposition 6.5.** *The mapping $(\overrightarrow{S}, T) \to \overrightarrow{S} \cdot T$ of $\mathscr{H}(E) \times \mathscr{H}'_\delta \to E$ is a bilinear map hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and equicontinuous subsets of $\mathscr{H}'$.*

**Proposition 6.6.** *For any element of the form $S\overrightarrow{e}$ in $\mathscr{H}(E), S \in \mathscr{H}$, $\overrightarrow{e} \in E$, we have $S\overrightarrow{e} \cdot T = S \cdot T\overrightarrow{e}$.*

*Proof.* To prove this we have only to verify that

$$\langle S\overrightarrow{e} \cdot T, \overleftarrow{e}' \rangle_{E,E'} = \langle S \cdot T\overrightarrow{e}, \overleftarrow{e}' \rangle_{E,E'}$$

for every $\overleftarrow{e}' \in E'$. Now

$$\langle S\overrightarrow{e} \cdot T, \overleftarrow{e}' \rangle = \langle S\overrightarrow{e}(T), \overleftarrow{e}' \rangle = \langle S \cdot T\overrightarrow{e}, \overleftarrow{e}' \rangle.$$

**29** When $\mathscr{H}$ satisfies the approximation property, we have a characterization for the scalar product that we have introduced. □

**Proposition 6.7.** *Let $\mathscr{H}$ satisfy the approximation property and $E$ be a complete ELC. The bilinear map that we have defined is the only bilinear map which is separately continuous and which satisfies $U\overrightarrow{e} \cdot T = (U \cdot T)\overrightarrow{e}$, for every $U \in \mathscr{H}$, $\overrightarrow{e} \in E$ and $T \in \mathscr{H}'_c$.*

*Proof.* We have already seen that the bilinear map defined by us is separately continuous and satisfies $U\overrightarrow{e} \cdot T = (U \cdot T)\overrightarrow{e}$.

Suppose there exists two bilinear maps, say $\mu_1$ and $\mu_2$ of $\mathscr{H}(E) \times \mathscr{H}'_c \longrightarrow E$ which are separately continuous and satisfy $\mu_1(S\overrightarrow{e}, T) =$

$\langle S,T\rangle\vec{e}$ and $\mu_2(S\vec{e},T) = \langle S,T\rangle\vec{e}$ for every $S \in \mathscr{H}$, $\vec{e} \in E$ and $T \in \mathscr{H}'$. Since $\mathscr{H} \otimes E$ is dense in $\mathscr{H}(E)$, the equality of $\mu_1$ and $\mu_2$ on $(\mathscr{H} \otimes E) \times \mathscr{H}_c'$ and the separate continuity of $\mu_1$ and $\mu_2$ give $\mu_1 = \mu_2$ on $\mathscr{H}(E) \times \mathscr{H}_c'$. □

# Lecture 7

# Multiplicative product of a vector valued distribution and a scalar valued distribution

Let $\mathscr{H}, K$ and $\mathscr{L}$ be three spaces of distributions on $R^n$. **30**

**Definition 7.1.** *A bilinear map $U$ of $\mathscr{H} \times \mathscr{K} \to \mathscr{L}$ which is separately continuous and which coincides with the multiplication of functions on $\mathscr{D} \times \mathscr{D}$ is called a multiplication between the elements of $\mathscr{H}$ and the elements of $\mathscr{K}$ with values in $\mathscr{L}$.*

*If $S \in \mathscr{H}, T \in \mathscr{K}$, we write $SUT$ for $U(S,T)$.*

**Theorem 7.1.** *Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{L}$ be any three locally convex spaces. Let $E$ be a complete ELC. Let $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ be a bilinear map hypocontinuous with respect to the bounded sets of $\mathscr{H}$ and $\mathscr{K}$. Then we can define a bilinear map $\tilde{U} : \mathscr{H}(E) \times \mathscr{K} \to \mathscr{L}(E)$ which is separately continuous and which satisfies $S\vec{e}\tilde{U}T = (SUT)\vec{e}$ for every $S \in \mathscr{H}, \vec{e} \in E$ and $T \in \mathscr{K}$; moreover it is hypocontinuous with respect to the bounded sets of $\mathscr{H}(E)$ and $\mathscr{K}$.*

*If $\mathscr{H}$ satisfies the approximations property, the bilinear map that we define is the only bilinear map which is separately continuous and which satisfies $S\vec{e}\tilde{U}T = (SUT)\vec{e}$ for every $S \in \mathscr{H}, \vec{e} \in E$ and $T \in \mathscr{K}$.*

*Proof.* Let $T$ be any element of $\mathscr{K}$. Then it defines a continuous linear map $m_T : \mathscr{H} \to \mathscr{L}$ as follows: $m_T(S) = SUT$ for every $S \in \mathscr{H}$. Let $I : E \to E$ be the identify map of $E$ in $E$. The map $m_T \in I : \mathscr{H}(E) \to \mathscr{L}(E)$ (for the definition of $m_T \in I$ refer to Lecture 4) is a continuous linear map. We define $\vec{S}\tilde{U}T$ to be $m_T \in I(\vec{S})$ for every $\vec{S} \in \mathscr{H}(E)$. We shall prove that $\tilde{U}$ thus defined has all the properties mentioned in the theorem. ◻

**31** We have $\mathscr{H}(E) \approx \mathscr{L}_\varepsilon(E'_c, \mathscr{H}) \approx \mathscr{L}_\varepsilon(\mathscr{H}'_c, E)$. From definitions 4.3 (1), 4.3 (2) and 4.3 (3), we have the following results.

(i) Considered as an element of $\mathscr{L}_\varepsilon(E'_c, \mathscr{L})$, $m_T \in I(\vec{S}) = \vec{S}\tilde{U}T$ is the same as the composite of the maps

$$t_I : E'_c \to E'_c, \vec{S} : E'_c \to \mathscr{H}, m_T : \mathscr{H} \to \mathscr{L}$$

where $t_T$ is the transpose of the identity mapping of $E$ in $E$; in other words, $t_T$ is the identity mapping of $E'_c$.

(ii) Considered as an element of $\mathscr{L}_\varepsilon(\mathscr{L}'_c, E), m_T \in I(\vec{S})$ is the same as the composite of the maps

$$t_{m_T} : \mathscr{L}'_c \to \mathscr{H}'_c, \vec{S} : \mathscr{H}'_c \to E, I : E \to E,$$

$\vec{S}$ being considered as an element of $\mathscr{L}_\varepsilon(\mathscr{H}'_c, E)$.

First we shall show that $\tilde{U}$ is hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{K}$. Let $\vec{S}$ remain in a bounded set of $\mathscr{H}(E)$ and $T \to 0$ in $\mathscr{K}$. To show that $\vec{S}\tilde{U}T$ tends to 0 in $\mathscr{L}(E) = \mathscr{L}_\varepsilon(E'_c, \mathscr{L})$, we have to prove that if $\overleftarrow{e}'$ lies in an equicontinuous set of $E'$, $\vec{S}UT(\overleftarrow{e}') \to 0$ uniformly in $\mathscr{L}$. Since $\vec{S}$ lies in a bounded set $B$, $\bigcup_{\vec{S}\varepsilon B, \overleftarrow{e}'\varepsilon H} \langle \vec{S}, \overleftarrow{e}'\rangle$, $H$ being an equicontinuous subset of $E'$, is bounded in $\mathscr{H}$. Hence $\langle \vec{S}, \overleftarrow{e}'\rangle UT$ tends to 0 uniformly in $\mathscr{L}$. But one sees easily that $\langle \vec{S}, \overleftarrow{e}'\rangle UT = \vec{S}\tilde{U}T(\overleftarrow{e}')$. Hence $\vec{S}UT(\overleftarrow{e}') \to 0$ uniformly in $\mathscr{L}$. In other words, $\vec{S}UT$ tends to in $\mathscr{L}(E)$ uniformly when $\vec{S}$ remains in a bounded set of $\mathscr{H}(E)$ and $T \to 0$ in $\mathscr{K}$.

Now suppose $\overrightarrow{S} \to 0$ in $\mathscr{H}(E)$ and $T$ remains bounded in $\mathscr{K}$. Then for $\overleftarrow{e}'$ lying in an equicontinuous set $H$ of $E'$, $\langle \overrightarrow{S}, \overleftarrow{e}' \rangle \to 0$ uniformly in $\mathscr{H}$ and hence when $T$ lies in a bounded set of $\mathscr{K}$, $\langle \overrightarrow{S}, \overleftarrow{e}' \rangle UT = \overrightarrow{S}\tilde{U}T(\overleftarrow{e}') \to 0$ uniformly in $\mathscr{L}$. Hence $\overrightarrow{S}\tilde{U}T \to 0$ in $\mathscr{L}(E)$ uniformly. 32

It is trivially seen that $S\overrightarrow{e}\tilde{U}T = (SUT)\overrightarrow{e}$ for every $S \in \mathscr{H}, \overrightarrow{e} \in E, T \in \mathscr{K}$.

Now suppose $\mathscr{H}$ satisfies the approximation property. Then $\tilde{U} : \mathscr{H}(E) \times \mathscr{K} \to \mathscr{L}(E)$ is the only bilinear map separately continuous and satisfying $S\overrightarrow{e}\tilde{U}T = (SUT)\overrightarrow{e}$ for $S \in \mathscr{H}, \overrightarrow{e} \in E$ and $T \in \mathscr{K}$. For if $U'$ is another such bilinear map, we have $\tilde{U}|(\mathscr{H} \otimes E) \times \mathscr{K} = U'|(\mathscr{H} \otimes E) \times \mathscr{K}$. Since $\mathscr{H} \otimes E$ is dense in $\mathscr{H}(E)$, we have $\tilde{U} = U'$ from the separate continuity of both $\tilde{U}$ and $U'$. This proves our theorem.

**Proposition 7.2.** *Let $\mathscr{H}$ and $\mathscr{L}$ be normal spaces of distributions and $\mathscr{K}$ a locally convex Hausdorff topological vector space. Let $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ be a bilinear map hypocontinuous with respect to bounded subsets of $\mathscr{H}$ and $\mathscr{K}$. For each $T \in \mathscr{K}$ let $m_T : \mathscr{H} \to \mathscr{L}$ be the mapping defined by $m_T(S) = SUT$. Then $t_{m_T} : \mathscr{L}'_c \to \mathscr{H}'_c$ is linear and continuous. Let $\alpha \in \mathscr{L}'_c$. Let us denote by $\dot{\mathscr{L}}$ the scalar product between $\mathscr{L}$ and $\mathscr{L}'_c$ and by $\dot{\mathscr{H}}$ the scalar product between $\mathscr{H}$ and $\mathscr{H}'_c$. Let us denote by the same symbols the extensions to $\mathscr{L}(E)$ and $\mathscr{L}'_c$ and to $\mathscr{H}(E)$ and $\mathscr{H}'_c$. Then*

$$\overrightarrow{S}\,\dot{\mathscr{H}}\,T\alpha = (\overrightarrow{S}\tilde{U}T)\,\dot{\mathscr{L}}\,\alpha, \quad \textit{where} \quad T\alpha = t_{m_T}(\alpha).$$

*Proof.* We have to only verify that for every $\overleftarrow{e}' \in E'$, $\langle \overrightarrow{S}_{\mathscr{H}} \cdot T\alpha, \overleftarrow{e}' \rangle = \langle (\overrightarrow{S}\tilde{U}T)_{\mathscr{L}} \cdot \alpha, \overleftarrow{e}' \rangle$.

$$\text{we have} \quad \begin{aligned} \langle \overrightarrow{S}_{\mathscr{H}} \cdot T\alpha, \overleftarrow{e}' \rangle &= \langle \overrightarrow{S}, \overleftarrow{e}' \rangle_{\mathscr{H}} \cdot T\alpha = \langle \overrightarrow{S}, \overleftarrow{e}' \rangle_{\mathscr{H}} \cdot t_{m_T}(\alpha) \\ &= m_T \langle \overrightarrow{S}, \overleftarrow{e}' \rangle_{\mathscr{L}} \cdot \alpha = (\langle \overrightarrow{S}, \overleftarrow{e}' \rangle UT)_{\mathscr{L}} \cdot \alpha \\ &= \langle \overrightarrow{S}\tilde{U}T, \overleftarrow{e}' \rangle_{\mathscr{L}} \cdot \alpha = \langle \overrightarrow{S}\tilde{U}T_{\mathscr{L}} \cdot \alpha, \overleftarrow{e}' \rangle. \end{aligned}$$

□

33 **Examples of multiplicative products.**

1) The multiplicative product $\alpha T$ defined for $\alpha \in \mathscr{E}$ and $T \in \mathscr{D}'$ as $\alpha T(\varphi) = T(\alpha\varphi)$ for every $\varphi \in \mathscr{D}$ is a bilinear map $\mathscr{E} \times \mathscr{D}' \to \mathscr{D}'$ which is hypocontinuous with respect to bounded subsets of $\mathscr{E}$ and $\mathscr{D}'$. (Ref: Theorie des distributions, Tome 1, pp. 117, Chap. V, § 2, Théorème 3). If $E$ is any complete $ELC$ we can define bilinear maps, hypocontinuous with respect to bounded sets, as explained in Theorem 7.1 in the following cases:

$$\text{(a)} \quad \mathscr{D}'(E) \times \mathscr{E} \to \mathscr{D}'(E) \quad \text{and} \quad \text{(b)} \quad \mathscr{D}' \times \mathscr{E}(E) \to \mathscr{D}'(E).$$

2) If $\alpha \in \mathscr{O}_M$ and $T \in \mathscr{S}'$ the multiplicative product $\alpha T \in \mathscr{S}'$. The mapping $(\alpha, T) \to \alpha T$ of $\mathscr{O}_M \times \mathscr{S}' \to \mathscr{S}'$ is hypocontinuous with respect to the bounded subsets of $\mathscr{O}_M$ and of $\mathscr{S}'$. If $E$ is any complete $ELC$, as explained in theorem 7.1, we get a bilinear map which is hypocontinuous with respect to the bounded subsets, in the following cases:

   (a) $\mathscr{S}'(E) \times \mathscr{O}_M \to \mathscr{S}'(E)$

   (b) $\mathscr{S}' \times \mathscr{O}_M(E) \to \mathscr{S}'(E)$.

**The Convolution product.**

If $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ is a separately continuous bilinear map of $\mathscr{H} \times \mathscr{K}$ in $\mathscr{L}$, where $\mathscr{H}, \mathscr{K}$ and $\mathscr{L}$ are three locally convex Hausdorff spaces and if $E$ is a complete $ELC$ we can define a bilinear map $\tilde{U} : \mathscr{H}(E) \times \mathscr{K} \to \mathscr{L}(E)$ as explained in theorem 7.1. We take any fixed $T \in \mathscr{K}$ and consider the continuous linear map $m_T \in I : \mathscr{H}(E) \to \mathscr{L}(E)$, where $m_T : \mathscr{H} \to \mathscr{L}$ is the continuous linear map $S \to SUT$ and $I : E \to E$ is the identity map. This can be applied to the product of convolution. We get then products of convolution of certain vector valued distributions by certain scalar valued distributions.

34 For example, we can define convolution in the following cases:

(1) $\mathscr{S}'(E) \times \mathscr{O}_c \to \mathscr{S}'(E)$.

(2) $\mathscr{S}' \times \mathscr{O}_c(E) \to \mathscr{S}'(E)$.

**Regularization:**

**Definition 7.2.** *The mapping* $(T,\alpha) \to T^{*}\alpha$ *of* $\mathscr{D}' \times \mathscr{D} \to \mathscr{E}$ *is called the regularization.*

*If* $E$ *is a complete ELC we can get a bilinear map* $\mathscr{D}'(E) \times \mathscr{D} \to \mathscr{E}(E)$ *as explained already. This bilinear map is called the regularization in the case of vector valued distributions.*

*As in the scalar case, we have* $\overrightarrow{T}^{*}\alpha(x) = \overrightarrow{T}_{\xi}.\alpha(x-\xi)$ *where* $\cdot$ *denotes the extension of the multiplicative product. In fact, this is proved by forming the scalar product with any* $\overleftarrow{e}' \in E'$.

*Let* $\mathscr{E}^{\circ}$ *and* $\mathscr{D}^{\circ}$ *denote the space of continuous functions and the space of continuous functions with compact support with their usual topologies. For* $f \in \mathscr{E}^{\circ}$ *and* $g \in \mathscr{D}^{\circ}$ *we have* $f * g \in \mathscr{E}^{\circ}$ *and we have the formula*

$$f * g(x) = \int_{R^n} f(x-\xi)g(\xi)\,d\xi \tag{1}$$

*Now suppose* $E$ *is a complete ELC. We have a convolution* $\mathscr{E}^{\circ}(E) \times \mathscr{D}^{\circ} \to \mathscr{E}^{\circ}(E)$*. Suppose we take* $\overrightarrow{f} \in \mathscr{E}^{\circ}(E)$ *and* $g \in \mathscr{D}^{\circ}$*, we have a formula similar to (1), namely*

$$\overrightarrow{f} * g(x) = \int_{R^n} \overrightarrow{f}(x-\xi)\,d\xi.$$

*Proof.* We have, for every $\overleftarrow{e}' \in E'$

$$\begin{aligned}\langle \overrightarrow{f} * g(x), \overleftarrow{e}' \rangle &= \langle \overrightarrow{f} * g, \overleftarrow{e}' \rangle(x)\\ &= \langle \overrightarrow{f}, \overleftarrow{e}' \rangle * g(x).\end{aligned}$$

$\langle \overrightarrow{f}, \overleftarrow{e}' \rangle * g$ is the convolution of the function $\langle \overrightarrow{f}, \overleftarrow{e}' \rangle \in \mathscr{E}^{\circ}$ and $g\varepsilon\mathscr{D}^{\circ}$. **35**
We have, therefore,

$$\begin{aligned}\langle \overrightarrow{f}, \overleftarrow{e}' \rangle * g(x) &= \int_{R^n} \langle \overrightarrow{f}, \overleftarrow{e}' \rangle(x-\xi)g(\xi)\,d\xi\\ &= \int_{R^n} \langle \overrightarrow{f}(x-\xi), \overleftarrow{e}' \rangle g(\xi)\,d\xi\end{aligned}$$

$$= \int_{R^n} \langle \overrightarrow{f}(x-\xi)g(\xi), \overleftarrow{e}' \rangle \, d\xi$$

$$= \left\langle \int_{R^n} \overrightarrow{f}(x-\xi)g(\xi)\, d\xi, \overleftarrow{e}' \right\rangle.$$

Hence $\langle \overrightarrow{f} * g(x), \overleftarrow{e} \rangle = \left\langle \int_{R^n} \overrightarrow{f}(x-\xi)g(\xi)\, d\xi, \overleftarrow{e}' \right\rangle$.

This gives $\overrightarrow{f} * g(x) = \int_{R^n} \overrightarrow{f}(x-\xi)g(\xi)\, d\xi$. □

# Lecture 8

# Fourier Transform of a vector valued distribution

One knows (Theorie des distributions, tome 2, § 6, Chap. VII) that the Fourier transform $\mathscr{F}$ is a topological isomorphism of $\mathscr{S}'$ on $\mathscr{S}'$, and the inverse of $\mathscr{F}$ which is also a topological isomorphism is called the conjugate of $\mathscr{F}$ and is denoted by $\overline{\mathscr{F}}$. **36**

**Definition 8.1.** *Let $E$ be a complete $ELC$. Let $I$ be the identity map of $E$. The continuous linear map $\mathscr{F} \in I$ of $\mathscr{S}'(E)$ in $\mathscr{S}'(E)$ is called the Fourier transform of $\mathscr{S}'(E)$ into itself.*

*Using the fact that $\overline{\mathscr{F}}$ is the inverse of $\mathscr{F}$ and that $(\mathscr{F} \in I)(\overline{\mathscr{F}} \in I) = \mathscr{F}\overline{\mathscr{F}} \in I$ we see that $\mathscr{F} \in I$ $\mathscr{S}'(E) \to \mathscr{S}'(E)$ is in fact an automorphism. We agree to denote the automorphism also by $\mathscr{F}$.*

**Proposition 8.1.** *For $\overrightarrow{T} \in \mathscr{S}'(E)$ and $\varphi \in \mathscr{S}$ we have*

$$\mathscr{F}\overrightarrow{T}(\varphi) = \overrightarrow{T}(\mathscr{F}\varphi).$$

*Proof.* The Fourier transform of any $T \in \mathscr{S}'$ is defined exactly by the equality $\mathscr{F}T(\varphi) = T(\mathscr{F}\varphi)$, for every $\varphi \in \mathscr{S}$. To prove $\mathscr{F}\overrightarrow{T}(\varphi) = \overrightarrow{T}(\mathscr{F}\varphi)$ we have to only form the scalar product with any $\overleftarrow{e}' \in E'$ and apply the equality $\mathscr{F}T(\varphi) = T(\mathscr{F}\varphi)$, for any $T \in \mathscr{S}', \varphi \in \mathscr{S}$. In fact,

$$\langle \mathscr{F}\overrightarrow{T}(\varphi), \overleftarrow{e}' \rangle = \langle \mathscr{F}\overrightarrow{T}, \overleftarrow{e}' \rangle(\varphi) = \mathscr{F}\langle \overrightarrow{T}, \overleftarrow{e}' \rangle(\varphi)$$

$$= \langle \overrightarrow{T}, \overleftarrow{e}' \rangle (\mathscr{F}\varphi) = \langle \overrightarrow{T}(\mathscr{F}\varphi), \overleftarrow{e}' \rangle.$$

This proves $\mathscr{F}\overrightarrow{T}(\varphi) = \overrightarrow{T}(\mathscr{F}\varphi)$. □

**Proposition 8.2.** *For every $\overrightarrow{T} \in \mathscr{S}'(E)$ and $s \in \mathscr{O}_M$ we have $\mathscr{F}(\overrightarrow{T} \cdot s) = \mathscr{F}(\overrightarrow{T}) * \mathscr{F}s$ where $\overrightarrow{T} \cdot s$ denotes the multiplicative product of $\overrightarrow{T} \in \mathscr{S}'(E)$ and $s \in \mathscr{O}_M$.*

37 *Proof.* One knows (Theorie des distributions, tome 2) that the multiplicative product $T \cdot s$ of any $T \in \mathscr{S}'$ and $s \in \mathscr{O}_M$, is an element of $\mathscr{S}'$. Hence for any $\overrightarrow{T} \in \mathscr{S}'(E)$ and $s \in \mathscr{O}_M$, $\overrightarrow{T} \cdot s \in \mathscr{S}'(E)$. Also one knows the validity of the equality $\mathscr{F}(T \cdot s) = \mathscr{F}(T) * \mathscr{F}s$ for any $T \in \mathscr{S}'$ and $s \in \mathscr{O}_M$. Now, for any $\overleftarrow{e}' \in E'$,

$$\begin{aligned}
\langle \mathscr{F}(\overrightarrow{T} \cdot s), \overleftarrow{e}' \rangle &= \mathscr{F}\langle \overrightarrow{T} \cdot s, \overleftarrow{e}' \rangle \\
&= \mathscr{F}(\langle \overrightarrow{T}, \overleftarrow{e}' \rangle \cdot s) \\
&= \mathscr{F}\langle \overrightarrow{T}, \overleftarrow{e}' \rangle * \mathscr{F}s \\
&= \langle \mathscr{F}\overrightarrow{T}, \overleftarrow{e}' \rangle * \mathscr{F}s \\
&= \langle \mathscr{F}\overrightarrow{T} * \mathscr{F}s, \overleftarrow{e}' \rangle.
\end{aligned}$$

This gives the equality $\mathscr{F}(\overrightarrow{T}.s) = \mathscr{F}(\overrightarrow{T}) * \mathscr{F}s$.

We know that if $f$ is a continuous function with compact support the Fourier transform $\mathscr{F}f$ is given by

$$\mathscr{F}f(x) = \int_{R^n} f(\xi) e^{-2\pi i \xi . x} d\xi.$$

□

**Proposition 8.3.** *If $\overrightarrow{f}$ is a continuous function with compact support having values in E the Fourier transform $\mathscr{F}(\overrightarrow{f})$ is given by $\{\mathscr{F}(\overrightarrow{f})\}(x) = \int_{R^n} \overrightarrow{f}(\xi) e^{-2\pi i \xi . x} d\xi$, E being a complete ELC.*

$$\textit{In fact,} \qquad \langle \mathscr{F}\overrightarrow{f}(x), \overleftarrow{e}' \rangle = \mathscr{F}\langle \overrightarrow{f}(x), \overleftarrow{e}' \rangle$$

$$= \mathscr{F}\langle \overrightarrow{f}, \overleftarrow{e}'\rangle(x).$$

*When $\overrightarrow{f}$ is a continuous function with values in E and having a compact support, for every $\overleftarrow{e}' \in E', \langle \overrightarrow{f}, \overleftarrow{e}'\rangle$ is a continuous complex valued function with compact support. Hence*

$$\mathscr{F}\langle \overrightarrow{f}, \overleftarrow{e}'\rangle(x) = \int\limits_{R^n} \langle \overrightarrow{f}, \overleftarrow{e}'\rangle(\xi) e^{-2\pi i\xi.x}\, d\xi.$$

*That is to say* 38

$$\langle \mathscr{F}\overrightarrow{f}(x), \overleftarrow{e}'\rangle = \int\limits_{R^n} \langle \overrightarrow{f}(\xi), \overleftarrow{e}'\rangle e^{-2\pi i\xi.x}\, d\xi$$

$$= \left\langle \int\limits_{R^n} \overrightarrow{f}(\xi) e^{-2\pi i\xi.x}\, d\xi, \overleftarrow{e}' \right\rangle.$$

*Hence*

$$\mathscr{F}\overrightarrow{f}(x) = \int\limits_{R^n} \overrightarrow{f}(\xi) e^{-2\pi i\xi.x}\, d\xi.$$

**Laplace Transform:**

Let $t$ be a real variable. Let $\mathscr{D}'_+$ denote the space of distributions with supports contained in the half-line $[\circ, \infty)$. On the space $\mathscr{D}'_+$ we take the topology induced by that of $\mathscr{D}'$.

**Definition 8.2.** *Let $T \in \mathscr{D}'_+$. We say that $T$ has a Laplace transform if there exists a real number $\xi_\circ$ such that for $\xi > \xi_\circ$ we have $e^{-\xi t} T \in \mathscr{S}'$.*

There exist distributions $T \in \mathscr{D}'_+$ such that there exists no real $\xi_\circ$ with $e^{-\xi t} T \in \mathscr{S}'$ for $\xi > \xi_\circ$. For example, let $\alpha(t)$ be a $C^\infty$ function with support in $[0, \infty)$ which is equal to 1 for $t \geq 1$. Let $T$ be the distribution defined by the function $e^{t^2} \cdot \alpha(t)$. This distribution has no Laplace transform.

Now, let $T \in \mathscr{D}'_+$. We have the following three possibilities:

1 There exists no real $\xi$ such that $e^{-\xi t} T \in \mathscr{S}'$.

2 For every real $\xi$ we have $e^{-\xi t}T \in \mathscr{S}'$.

3 There exists at least one real number '$a$' such that $e^{-at}T \in \mathscr{S}'$ and at least one real number '$b$' such that $e^{-bt}T \notin \mathscr{S}'$.

**Proposition 8.4.** *In case (3) there exists a real number $\xi_\circ$ such that for $\xi > \xi_\circ$ we have $e^{-\xi t}T \in \mathscr{S}'$ and for $\xi < \xi_\circ$ we have $e^{-\xi t}T \notin \mathscr{S}'$. Moreover, for $\xi > \xi_\circ$ we have actually $e^{-\xi t}T \in \mathscr{O}'_c$.*

39 *Proof.* If $\alpha$ is any real number such that $e^{-\alpha t}T \in \mathscr{S}'$ we have, for any $\beta > \alpha, e^{-\beta t}T \in \mathscr{S}'$. In fact, $e^{-\beta t}T = (e^{-\alpha t}T) \cdot e^{-(\beta-\alpha)t}.\tilde{\alpha}(t)$ where $\tilde{\alpha}(t)$ is a $C^\infty$-function with $\tilde{\alpha}(t) = 1$ for $t \geq 0$ and $\tilde{\alpha}(t) = 0$ for $t \leq -1$ and $0 \leq \tilde{\alpha}(t) \leq 1$. The function $e^{-(\beta-\alpha)t} \cdot \tilde{\alpha}(t)$ is an element of $\mathscr{S}$ and hence $e^{-\beta t}T$ which is the product of $e^{-\alpha t}T$ and $e^{(\beta-\alpha)t}\tilde{\alpha}(t)$ is an element of $\mathscr{S}'$. If we put all the real numbers $r$ such that $e^{-rt}T \notin \mathscr{S}'$ into a class $L$ and all the real numbers $\alpha$ such that $e^{-\alpha t}T \in \mathscr{S}'$ into another class $R$, from our assumptions, it follows that $L$ and $R$ are non-empty. From what we have proved above, it follows that the classes $L$ and $R$ determine a real number $\xi_\circ$ having the properties mentioned in the proposition. □

As for the other part, we will in fact, prove that if $p$ is a complex number with $Rlp > \xi_\circ$ $e^{-pt}T \in \mathscr{O}'_c$. We have, $e^{-pt}T = e^{-\xi' t}T.e^{-(p-\xi')t}\alpha(t)$ where $\alpha(t)$ is a real valued $C^\infty-$ function satisfying $\alpha(t) = 1$ for $t \geq 0$ and $\alpha(t) = 0$ for $t \leq -1$ and $0 \leq \alpha(t) \leq 1$. Let $Rlp = \xi > \xi_\circ$. Choose $\xi'$ real such that $\xi > \xi' > \xi_\circ$. Then $e^{-\xi' t}T \in \mathscr{S}'$ and $e^{-(p-\xi')t}\alpha(t) \in \mathscr{S}$ if $Rlp > \xi'$. Hence $e^{-pt}T \in \mathscr{O}'_c$.

**Definition 8.3.** *In case (1) we say that $T$ has no Laplace transform at all. In case (2) the whole of the complex plane is defined to be the domain of definition of the Laplace transform of $T$. In case (3) the half plane $Rlp > \xi_\circ$ is defined to be the domain of definition of the Laplace transform of $T$.*

In case (2) we can show for every complex number $p$ we have $e^{-pt}T \in \mathscr{O}'_c$. In fact, if $Rlp = \xi$, we choose a real $\mu$ such that $\xi > \mu$.
40 We then find a real $\xi'$ with $\xi > \xi' > \mu$. Then $e^{-pt}T = e^{-\xi' t}T.e^{-(p-\xi')}\alpha(t)$ where $\alpha(t)$ is a $C^\infty$-function satisfying $\alpha(t) = 1$ for $t \geq 0$ and $\alpha(t) = 0$ for $t \leq -1$ and $0 \leq \alpha(t) \leq 1$. We have $e^{-\xi' t}T \in \mathscr{S}'$ and $e^{-(p-\xi')t}\alpha(t) \in \mathscr{S}$. Hence $e^{-pt}T \in \mathscr{O}'_c$.

**Proposition 8.5.** *Let there exist a real $\xi_\circ$ such that for $Rlp > \xi_\circ$ we have $e^{-pt}T \in \mathscr{S}'$. Then the mapping $p \to e^{-pt}T$ is a holomorphic function of the complex variable $p$ with values in $\mathscr{O}'_c$, for $Rlp > \xi_\circ$.*

*Proof.* Let $Rlp = \xi > \xi_\circ$. Let $\xi'$ be such that $\xi > \xi' > \xi_\circ$. We have $e^{-pt}T = e^{-\xi' t}T.e^{-(p-\xi')t}.\alpha(t)$ where $\alpha(t)$ is a $C^\infty$-function which satisfies $\alpha(t) = 1$ for $t \geq 0$ and $\alpha(t) = 0$ for $t \leq -1$ and $0 \leq \alpha(t) \leq 1$ for every $t$. We have $e^{-\xi' t}T \in \mathscr{S}'$ and it is a fixed element of $\mathscr{S}'$. The function $p \to e^{-(p-\xi')t}\alpha(t)$ is a holomorphic function for $Rlp > \xi'$, with values in $\mathscr{S}$. In fact, the derivative is $-te^{-(p-\xi')t}\alpha(t)$. That is to say

$$\lim_{h\to 0}\left\{\frac{e^{-(p-\xi'+h)t} - e^{-(p-\xi')t}}{h} + te^{-(p-\xi')t}\right\}\alpha(t) = 0$$

in $\mathscr{S}$. □

The map $\gamma_{s'} : \mathscr{S} \to \mathscr{O}'_c$ given by $\gamma_{s'}(s) = s', s'$ is a continuous linear map for every fixed $s' \in \mathscr{S}$. Now $e^{-pt}T$ is a fixed element of $\mathscr{S}'$ and the function $p \to e^{-(p-\xi')t}.\alpha(t)$ is a holomorphic function of the complex variable $p$ with values in $\mathscr{S}$ and hence the composite of the linear map $\gamma_{e^{-\xi' t}T}$ and the above holomorphic function is a holomorphic function. This proves our assertion.

**Definition 8.4.** *Suppose $T \in \mathscr{D}'_+$ has a Laplace transform. The Laplace transform is then defined to be the function $p \to F(p) = \int e^{-pt}T dt$, which is defined in the domain of definition of the Laplace transform of $T$.* **41**

In fact for $p$ in the domain of definition we have seen that $e^{-pt}T \in \mathscr{O}'_c$. Now $1 \in (\mathscr{O}'_c)'_c$ and the integral $\int e^{-pt}T dt$ is nothing but the scalar product $1.e^{-pt}T$ by definition. In the domain of definition of the Laplace transform, we have $p \to F(p)$ a holomorphic function of the complex variable $p$.

**Properties of the Laplace transform:**

**Proposition 8.6.** *Let $T \in \mathscr{D}'_+$ have a Laplace transform. Let the half plane of existence of the Laplace transform of $T$ be $Rlp > \xi_\circ$, in case* *the whole plane is not the domain of existence of the Laplace transform. For any fixed $\xi > \xi_\circ$ the mapping $\eta \to F(\xi + i\eta)$ defines an element of $\mathscr{O}_M$. In case the whole of the complex plane is the domain of existence of the Laplace transform, for every real $\xi$, $\eta \to F(\xi + i\eta)$ is an element of $\mathscr{O}_M$.*

*Proof.* For any fixed $\xi > \xi_\circ$ the function $\eta \to F(\xi + i\eta)$ is the Fourier transform of the distribution $e^{-\xi t}T$ in $\mathscr{O}'_c$ and hence is an element of $\mathscr{O}_M$. In the second case for any fixed real $\xi, \eta \to F(\xi + i\eta)$ is the Fourier transform of $e^{-\xi t}T \in \mathscr{O}'_c$ and hence is an element of $\mathscr{O}_M$.

Let $T \in \mathscr{D}'_+$. When $T$ has a Laplace transform, the domain of existence is usually denoted by $Rlp > a$ where either a is a real number $\xi_\circ$ or stands for the symbol $-\infty$. If $S \in \mathscr{D}'_+$ has $F(p)$ as its Laplace transform in $Rlp > a$ we write $S \underset{\xi>a}{\sqsupset} F(p)$. □

42 **Proposition 8.7.** *Let $S$ and $T \in \mathscr{D}'_+$ and $S \underset{\xi>a}{\sqsupset} F(p)$ and $T \underset{\xi>b}{\sqsupset} G(p)$. Then $S * T \underset{\xi>Max(a,b)}{\sqsupset} F(p)G(p)$.*

*Proof.* For $Rlp = \xi > Max(a,b)$ we have $e^{-pt}S \in \mathscr{O}'_c$ and $e^{-pt}T \in \mathscr{O}'_c$. Hence for $Rlp > Max(a,b)$ $S * T$ has a Laplace transform. If $H(p)$ denotes the Laplace transform of $S * T$, for any fixed $\xi > Max(a,b)$ we have

$$\begin{aligned} H(\xi + i\eta) &= \mathscr{F}_\eta(e^{-\xi t}S * T) \\ &= \mathscr{F}_\eta(e^{-\xi t}S)\mathscr{F}_\eta(e^{-\xi t}T) \\ &= F(\xi + i\eta)G(\xi + i\eta). \end{aligned}$$

Hence $$H(p) = F(p)G(p).$$

□

**Corollary.** *$T \in \mathscr{D}'_+, T \underset{\xi>a}{\sqsupset} F(p)$ implies $T' \underset{\xi>a}{\sqsupset} p\, F(p)$. This follows from the fact that for the distribution $\delta'_\circ$ the domain of existence of the Laplace transform is the whole of the complex plane and that the Laplace transform of $\delta'_\circ$ is precisely p.*

*In fact* $\delta'_\circ * T \underset{\xi>a}{\sqsupset} p\, F(p)$. *But* $\delta'_\circ * T = T^1$.

**Proposition 8.8.** *Let $T \in \mathscr{D}'_+$ have a Laplace transform. Let the domain of definition be $Rlp >$ '$a$' where either $a$ is a real number or stands for the symbol '$-\infty$'.*

If '$a$' is real, for any $\varepsilon > 0$, in the half plane $Rlp \geq a + \varepsilon$ we have a uniform majorisation $|F(p)| \leq A(|p|^2 + 1)^k, F(p)$ being the Laplace transform of $T$. If '$a$' stands for the symbol '$-\infty$', for any real $\xi$ we have $|F(p)| \leq A(|p|^2 + 1)^k$ in the half plane $Rlp \geq \xi$. We admit this proposition.

**Proposition 8.9.** *Suppose $F(p)$ is a holomorphic function of the complex variable $p$ defined in some half plane, say $\xi > \xi_\circ$.*

*Suppose for every $\varepsilon > 0$ we have a uniform majorisation for $F$ in the* 43
*half plane $\xi \geq \xi_\circ + \varepsilon$, of the type $|F(p)| \leq A(|p|^2 + 1)^k$, then $F(p)$ is the Laplace transform of some distribution $T \in \mathscr{D}'_+$.*

*Proof.* Let $p = \xi + i\eta$ with $\xi \geq \xi_\circ + \varepsilon$ and $k < -1$. Let

$$f_\xi(t) = \frac{1}{2\pi i} \int\limits_{\xi - i\infty}^{\xi + i\infty} e^{pt} F(p)\, dp.$$

This integral certainly exists. We have, in fact,

$$|f_\xi(t)| \leq \frac{1}{2\pi} A e^{\xi t} \int \frac{|dp|}{1 + |p|^2} = \frac{A}{2\pi} e^{\xi t}.\pi = \frac{A}{2} e^{\xi t}.$$

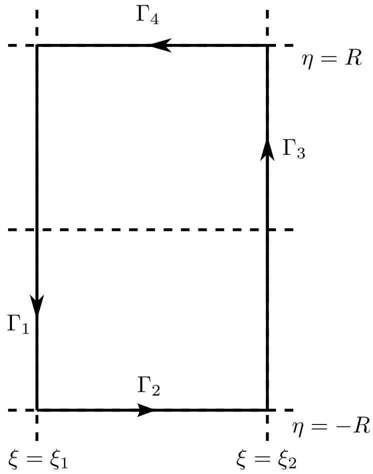
□

The function $f_\xi(t)$ does not depend on $\xi$ as long as $\xi \geq \xi_\circ + \varepsilon$. For, if we take the rectangle $\Gamma$ bounded by the lines $\xi = \xi_1, \xi = \xi_2$ and $\eta = R$ and $\eta = -R$ the integral

$$\frac{1}{2\pi i} \int\limits_\Gamma e^{pt} F(p) dp = 0 \quad \text{by Cauchy's theorem.}$$

Now

$$\int\limits_\Gamma e^{pt} F(p)\, dp = \sum_{i=1}^{4} \int\limits_{\Gamma_i} e^{pt} F(P)\, dp$$




where $\Gamma_1$ is the portion of the line $\xi = \xi_1$ lying between the lines $\eta = \mathcal{R}$, $\eta = -\mathcal{R}$; $\Gamma_2$ is the portion of the line $\eta = -\mathcal{R}$ lying between the lines $\xi = \xi_1$ and $\xi = \xi_2$; $\Gamma_3$ is the line $\xi = \xi_2$ lying between the lines $\eta = -\mathcal{R}$ and $\eta = \mathcal{R}$ and so on.

Now

$$\lim_{R\to\infty} \int_{\Gamma_2} |e^{pt}F(p)|\,|dp| = 0 \quad \text{and} \quad \lim_{R\to\infty} \int_{\Gamma_4} |e^{pt}F(p)|\,|dp| = 0.$$

Hence, we have $\frac{1}{2\pi i}\left\{\int\limits_{\xi_1 - i\infty}^{\xi + i\infty} e^{pt}F(p)\,dp + \int\limits_{\xi_2 + i\infty}^{\xi_2 - i\infty} e^{pt}F(p)\,dp\right\} = 0.$

44 In other words, $f_{\xi_1}(t) - f_{\xi_2}(t) = 0$. Hence $f_{\xi_1}(t) = f_{\xi_2}(t)$. We now show that if $t < 0$, $f(t) = 0$, where $f(t) = f_\xi(t)$ for any $\xi \geq \xi_\circ + \varepsilon$. In fact, $|f(t)| < \frac{A}{2}e^{\xi t}$. Allowing $\xi \to \infty$ we have

$$|f(t)| \leq 0.$$

Now, we have
$$f(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{\xi t}F(\xi + i\eta)e^{i\eta t}\, id\eta (\xi \geq \xi_\circ + \varepsilon)$$
$$= \frac{1}{2\pi}e^{\xi t}\int_{-\infty}^{\infty} F(\xi + i\eta)e^{i\eta t}\,d\eta.$$

Hence, $\quad 2\pi e^{-\xi t} f(t) = \bar{\mathscr{F}}_\eta(F(\xi + i\eta))$. Hence

$$F(\xi + i\eta) = \bar{\mathscr{F}}_\eta(e^{-\xi t}.2\pi f(t)).$$

Hence $2\pi e^{-\xi t} f(t)$ has for its $\eta$-Fourier transform the function $F(\xi + i\eta)$, for any $\xi \geq \xi_\circ + \varepsilon, \varepsilon > 0$. Hence $f(t) \underset{\xi > \xi_\circ}{\sqsupset} F(p)$. Obviously $f(t)$ defines a distribution in $\mathscr{D}'_+$ for $f(t) = 0$, for $t < 0$.

Now suppose $F$ satisfies $|F(p)| \leq A(1 + |p|^2)^k$ in $\xi \geq \xi_\circ + \varepsilon$ for any $\varepsilon > 0$, $k$ being an integer. The above is equivalent to assuming

$$|F(p)| \leq A'|p|^{k'}$$

where $k'$ is some integer and $A'$ is some constant, for $Rlp \geq \xi_\circ + \varepsilon$. Now $\frac{F(p)}{p^{k'+2}}$ is holomorphic in $Rlp > Max(0, \xi_\circ) = \nu$ (say), and for any $\varepsilon > 0$, we have

$$\left|\frac{F(p)}{p^{k'+2}}\right| \leq \frac{A''}{(1 + |p|^2)} \quad \text{for} \quad Rlp \geq \nu + \varepsilon,$$

$A''$ being some constant. Hence by what we have proved, there exists a distribution $T' \in \mathscr{D}'_+$ such that

$$T' \underset{\xi > \nu}{\sqsupset} \frac{F(p)}{p^{k'+2}}.$$

Hence $T = \underbrace{(\delta' * \cdots * \delta')}_{k'+2 \text{ times}} * T' \underset{\xi > \nu}{\sqsupset} F(p)$, and $(\delta' * \cdots * \delta') * T' \in \mathscr{D}'_+$. Thus 45
we see that there exists a distribution $T. \in \mathscr{D}'_+$ which has $F(p)$ as its Laplace transform in the domain of definition of its Laplace transform.

That $T$ is unique follows from the fact that the Fourier transform $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is an isomorphism onto.

# Lecture 9

# The Laplace transform of vector valued distributions

Let $E$ be a complete $ELC$. Let $\overrightarrow{T} \in \mathscr{D}'_+(E)$. As in the case of scalar distributions we have the following three possibilities: 46

1) There exists no real numbers $\mathscr{E}$ such that $e^{-\mathscr{E}t}\overrightarrow{T} \in \mathscr{S}'(E)$.

2) For every real $\mathscr{E}$ we have $e^{-\mathscr{E}t}\overrightarrow{T} \in \mathscr{S}'(E)$.

3) There exist two real numbers $\mathscr{E}'_\circ$ and $\mathscr{E}_1$ such that we have $e^{-\mathscr{E}'_\circ t}$ $\overrightarrow{T} \in \mathscr{S}'(E)$ and $e^{-\mathscr{E}_1 t}\overrightarrow{T} \notin \mathscr{S}'(E)$.

**Proposition 9.1.** *In case (3) there exists a real number $\mathscr{E}_\circ$ such that for $\mathscr{E} > \mathscr{E}_\circ$ we have $e^{-\mathscr{E}t}\overrightarrow{T} \in \mathscr{S}'(E)$ and for $\mathscr{E} < \mathscr{E}_\circ$ we have $e^{-\mathscr{E}t}\overrightarrow{T} \notin \mathscr{S}'(E)$.*

*Proof.* This will follow immediately if we show that $e^{-\mu t}\overrightarrow{T} \in \mathscr{S}'(E), \mu$ real, and $\nu > \mu$ imply $e^{-\nu t}\overrightarrow{T} \in \mathscr{S}'(E)$.

$$Now, \quad e^{-\nu t}\overrightarrow{T} = e^{-\mu t}\overrightarrow{T} \cdot e^{-(\nu-\mu)t}\alpha(t),$$

where $\alpha(t)$ is a $C^\infty$-function which is 1 for $t \geq 0$, which is 0 for $t \leq -1$, and which satisfies $0 \leq \alpha(t) \leq 1$. Now $e^{-\mu t}\overrightarrow{T} \in \mathscr{S}'(E)$ and $e^{-(\nu-\mu)t}\alpha(t) \in \mathscr{S}$ and hence $e^{-\nu t}\overrightarrow{T} \in \mathscr{O}'_c(E) \subset \mathscr{S}'(E)$. Thus, we have, in case (3), a real number $\mathscr{E}_\circ$ such that for $\mathscr{E} > \mathscr{E}_\circ$ we have

$e^{-\mathscr{E}t}\overrightarrow{T} \in \mathscr{S}'(E)$ and for $\mathscr{E} < \mathscr{E}_\circ$, $e^{-\mathscr{E}t}\overrightarrow{T} \notin \mathscr{S}'(E)$. As in the case of scalar distributions, we say in case (1) the distribution $\overrightarrow{T}$ has no Laplace transform and in cases (2) and (3) it has a Laplace transform. In case (2) the whole of the complex plane is defined to be the domain of existence of the Laplace transform of $\overrightarrow{T}$ and case (3) the half-plane $Rlp > \mathscr{E}_\circ$ is defined to be the domain of existence of the Laplace transform of $\overrightarrow{T}$. □

47 **Proposition 9.2.** *In case (2) we have $e^{-pt}\overrightarrow{T} \in \mathscr{O}'_c(E)$ for every $p$, and in case (3) we have $e^{-pt}\overrightarrow{T} \in \mathscr{O}'_c(E)$ for every $p$ satisfying $Rlp > \mathscr{E}_\circ$.*

*Proof.* **Case(2)**. For every $\overleftarrow{e}' \in E'$, $\langle e^{-pt}\overrightarrow{T}, \overleftarrow{e}'\rangle \in \mathscr{S}'$ and hence $\langle e^{-pt}\overrightarrow{T}, \overleftarrow{e}'\rangle \in \mathscr{O}'_c$ for every $p$, from what we have seen in lecture 8.

**Case(3)**. For every $\overleftarrow{e}' \in E'$ and $Rlp > \mathscr{E}_\circ$ we have $\langle e^{-pt}\overrightarrow{T}, \overleftarrow{e}'\rangle \in \mathscr{S}'$ and hence $\langle e^{-pt}\overrightarrow{T}, \overleftarrow{e}'\rangle \in \mathscr{O}'_c$ for every $p$ such that $Rlp > \mathscr{E}_\circ$. Now, since $\mathscr{O}'_c$ satisfies the $\mathcal{E}$-property we have the required result. □

**Proposition 9.3.** *The function $p \to e^{-pt}\overrightarrow{T}$ is a holomorphic function with values in $\mathscr{O}'_c(E)$ in case (2) and a holomorphic function with values in $\mathscr{O}'_c(E)$, in the half plane $Rlp > \mathscr{E}_\circ$ in case (3).*

*Proof.* Similar to the proof of proposition 8.5. □

**Definition 9.1.** *Let $\overrightarrow{T} \in \mathscr{D}'_+(E)$ have a Laplace Transform. We denote the domain of existence of the Laplace Transform of $\overrightarrow{T}$ by $Rlp > a$ where either 'a' is a certain real number $\mathscr{E}_\circ$ or stands for the symbol '$-\infty$'. The function $\overrightarrow{F}(p)$ defined in $Rlp > \mathscr{E}_\circ$ by*

$$\overrightarrow{F}(p) = \int_0^\infty e^{-pt}\overrightarrow{T}\,dt = 1.e^{-pt}\overrightarrow{T}$$

with values in $E$ is called the Laplace transform of $\overrightarrow{T}$. The scalar product $1.e^{-pt}\overrightarrow{T}$ is the scalar product of $1 \in (\mathscr{O}'_c)'_c$ and $e^{-pt}\overrightarrow{T} \in \mathscr{O}'_c(E)$. The function $p \to \overrightarrow{F}(p)$ is a holomorphic function of the complex variable with values in $E$.

**Proposition 9.4.** *If $\overrightarrow{F}(p)$ is the Laplace Transform of $\overrightarrow{T} \in \mathscr{D}'_+(E)$, the domain of existence being $Rlp >: a :$ and if $U \in \mathscr{D}'_+$ has $G(p)$ as its Laplace transform, with $Rlp >: b :$ as the domain of existence $\overrightarrow{T} * U$ has, in $Rlp > Max(a,b) = \alpha$ the function $\overrightarrow{F}(p)\ G(p)$ as its Laplace transform.*

*Proof.* Now $\langle e^{-pt}\overrightarrow{T} * U, \overleftarrow{e}\rangle = e^{-pt}\{\langle \overrightarrow{T}, \overleftarrow{e}'\rangle * U\}$ for every $\overleftarrow{e}' \in E'$. **48**
Hence, for every $\overleftarrow{e}' \in E'$, $\langle e^{-pt}\overrightarrow{T} * U, \overleftarrow{e}\rangle = (e^{-pt}\langle \overrightarrow{T}, \overleftarrow{e}'\rangle) * e^{-pt}U$. For $Rlp > \alpha$ we have $e^{-pt}\langle \overrightarrow{T}, \overleftarrow{e}\rangle \in \mathscr{O}'_c$ and $e^{-pt}U \in \mathscr{O}'_c$ and hence $e^{-pt}\langle \overrightarrow{T}, \overleftarrow{e}'\rangle * e^{-pt}U \in \mathscr{O}'_c$.

Hence $\langle e^{-pt}\overrightarrow{T} * U, \overleftarrow{e}'\rangle \in \mathscr{O}'_c$ for $Rlp > \alpha$ and this for every $\overleftarrow{e}' \in E'$. Since $\mathscr{O}'_c$ has the $\mathcal{E}$-property, we have $e^{-pt}\overrightarrow{T} * U \in \mathscr{O}'_c(E) \subset \mathscr{S}'(E)$. Also the Laplace transform of $\langle \overrightarrow{T}, \overleftarrow{e}'\rangle * U$ is the same as $\langle \overrightarrow{F}(p), \overleftarrow{e}'\rangle \cdot G(p)$. This proves that

$$\overrightarrow{T} * U \underset{Rlp>\alpha}{\sqsupset} \overrightarrow{F}(p)G(p).$$

□

We shall be only interested in the case when $E$ is a Banach space. We shall now study the properties of the Laplace transform in this particular case. **Notation**. In what follows $E$ is a Banach space.

**Proposition 9.5.** *$\overrightarrow{T} \in \mathscr{D}'_+(E)$ is a distribution having $\overrightarrow{F}(p)$ as its Laplace transform with $Rlp > a$ as the domain of existence. Then if $a$ is a real number, given any $\varepsilon > 0$ in the half plane $Rlp \geq a + \varepsilon$ we have a uniform majorisation $\| \overrightarrow{F}(p) \| \leq A(1 + |p|^2)^k$. If $a$ stands for the symbol $-\infty$ then for any real number $r$ we have a majorisation $\| \overrightarrow{F}(p) \| \leq A(1 + |p|^2)^k$ in $Rlp \geq r, A$ being a constant $> 0$.*

*Proof.* First we prove that the following two statements are equivalent:

1) There exists an integer $k$ such that

$$\| \overrightarrow{F}(p) \| \leq A(1 + |p|^2)^k$$

2) For every sequence of complex numbers $p_1, p_2, p_3, \ldots$ such that $|p_n| \to \infty$ and for every sequence of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ such that $\alpha_n |p_n|^k$ tends to 0 for every integer $k$, the sequence $\| \overrightarrow{F}(p_n) \| \alpha_n$ is bounded.

**49** Trivially (1) implies (2). We have to only prove (2) implies (1). Suppose (2) is satisfied and (1) is not satisfied. Given any integer $n$ we can find a $p_n$ such that

$$\| \overrightarrow{F}(p_n) \| \geq A(1 + |p_n|^2)^n.$$

Take for $\{\alpha_n\}$ the sequence $\frac{1}{|p_n|} n/2$. Then obviously for every integer $k$, the sequence $\alpha_n |p_n|^k$ tends to 0. Hence by (2) we should have $\frac{\| \overrightarrow{F}(p_n) \|}{|p_n|^{n/2}}$ bounded. But $\frac{\| \overrightarrow{F}(p_n) \|}{|p_n|^{n/2}} \geq \frac{A(1+|p_n|^2)^n}{|p_n|^{n/2}}$. Obviously this sequence is not bounded. Hence (2) has to imply (1).

Let $\overrightarrow{T} \underset{\mathscr{E}>a'}{\sqsupset} \overrightarrow{F}(p)$. Then $\langle \overrightarrow{T}, \overleftarrow{e}' \rangle \underset{\mathscr{E}>a'}{\sqsupset} \langle \overrightarrow{F}(p), \overleftarrow{e}' \rangle$ for every fixed $\overleftarrow{e}' \in E'$. Hence there exists an integer $\mu'_e$ such that $|\langle \overrightarrow{F}(p), \overleftarrow{e}' \rangle| \leq A_{\overleftarrow{e}'} (1 + |p|^2)^{\mu \overleftarrow{e}'}$ where $A_{\overleftarrow{e}'}$ is a constant $> 0$, uniformly in $Rlp \geq a + \varepsilon$ if $a$ is real or in $R1p \geq r, r$ any real number if $a$ stands for the symbol '$-\infty$'.

Now suppose $\{p_n\}$ is any sequence of complex numbers with $|p_n| \to \infty$ and suppose $\{\alpha_n\}$ is any sequence of real numbers with $\alpha_n |p_n|^k \to 0$ for every integer $k$

We have $\qquad |\langle \overrightarrow{F}(p_n)\alpha_n, \overleftarrow{e}' | \leq A_{\overleftarrow{e}'} (1 + |p_n|^2)^{\mu_{\overleftarrow{e}'}} |\alpha_n|$.

Now as $n \to \infty, |\alpha_n| (1 + |p_n|^2)^{\mu_{\overleftarrow{e}'}} \to 0$, and hence $\langle \overrightarrow{F}(p_n)\alpha_n, \overleftarrow{e}' \rangle$ is a bounded sequence of points. Hence the set of points $\overrightarrow{F}(p_n)\alpha_n$ is a weakly bounded set in $E$, and hence a strongly bounded set. Hence $\| \overrightarrow{F}(p_n) \| |\alpha_n|$ is bounded.

This completes the proof of proposition 9.5. □

**Proposition 9.6.** *If $\overrightarrow{F}(p)$ is a holomorphic function of the complex variable $p$ in $Rlp \geq \mathscr{E}$ with values in $E$ and if we have a majorisation*

$$\| \overrightarrow{F}(p) \| \leq A(1 + |p|^2)^k \quad \text{in} \quad Rlp \geq \mathscr{E},$$

50 *$\overrightarrow{F}(p)$ is the Laplace transform of a certain unique distribution $\overrightarrow{T} \in \mathscr{D}'_+(E)$ in their common domain of definition.*

*Proof.* Similar to the scalar case. □

# Lecture 10

# Partial Differential Equations - Weak boundary value problems

**Heat conduction equation:** 51

Let $\Omega$ be a bounded domain in $R^n$ with a smooth boundary $\mathcal{S}$. The heat conduction problem in $\Omega$ with Neumann's boundary condition is the following. We are given smooth functions $F(x,t)[x \in \Omega, 0 < t < \infty], H(x,t)(x \in \Omega, 0 < t < \infty)$ and $U_\circ(x)(x \in \Omega)$. The problem is to find a smooth function $U(x,t)$ continuous in $\overline{\Omega} X\ (0,\infty)$ and differentiable in $\Omega X ]0,\infty)$ such that

i) $\frac{\partial U(x,t)}{\partial t} - \Delta U(x,t) = F(x,t)\,(t > 0, x \in \Omega)$[$\Delta$ is the Laplacian in $R^n$];

ii) for each fixed $t > 0, U(x,t)$ and $H(x,t)$ have the same normal derivative at every point of the boundary;

iii) $U(x,\circ) = U_\circ(x)$.

**Remark concerning condition (ii):** Actually one is given initially a function $h(x,t), x \in S$ and it is required that $U(x,t)$ satisfy the condition

$ii')$ $\frac{\partial(x,t)}{\partial n_x} = h(x,t)$ for $x \in S$ and every $t$.

We shall, however, assume that there exists a smooth function $H(x,t)$, $x \in \Omega$, such that $\frac{\partial H(x,t)}{\partial n} = h(x,t), x \in S$.

Putting $U - H = u$, we are led to the following homogeneous problem: Given $f(x,t)$ and $u_\circ(x)$ which are sufficiently differentiable find $u(x,t)$ continuous in $\bar{\Omega} X(0,\infty)$ and differentiable in $\bar{\Omega} X(0,\infty)$ such that

52 i) $\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t), t > 0$;

ii) for each fixed $t > 0, u(x,t)$ has vanishing normal derivative at the boundary;

iii) $u(x,t) = u_\circ(x)$.

In the frame-work of Hilbert spaces, the above problem, in a weaker formulation, can be posed as follows. Consider the Hilbert space $H^1(\Omega)$ and the associated space $N$ (See Lions:**"On Elliptic Partial Differential Equations"**, Tata Inst. of Fundamental Research, Bombay, Lec. 6).

**Problem 1.** Given a continuous function $F(t)(t > 0)$ with values in $L^2(\Omega)$ and a function $u_\circ \in N$, find a function $u$, with values in $L^2(\Omega)$, once continuously differentiable in $t > 0$ such that for $t > 0$, $\frac{\partial u}{\partial t} - \Delta u = F$ and such that $u(t) \to u_\circ$ in $N$ as $t \to 0$.

We shall transform this problem in the following way. Define $\tilde{u}(t)$ (with values in $N$) by $\tilde{u}(t) = u(t)$ for $t > 0$ and $\tilde{u}(t) = 0, t < 0$. Consider $\tilde{u}$ as an element of $\mathscr{D}'_+(t,N)$ (space of distributions with values in $N$ and with supports in $(0,\infty]$. Similarly define $F \in \mathscr{D}'_+(t,L^2)$. Since we require $u(0) = u_\circ$, we must have $\frac{\partial \tilde{u}}{\partial t} = \left(\frac{\partial u}{\partial t}\right)^{\sim} + \delta_t u_\circ$, and problem 1 reduces to

**Problem 1′.** To find $\tilde{u} \in \mathscr{D}'_+(t,N)$ with $\tilde{u}$ once continuously differentiable in $t > 0$ and $= 0$ for $t < 0$ and such that

$$(*) \qquad \frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} = \delta u_\circ + \tilde{F}.$$

Finally we may abandon the requirement that $\tilde{u}$ be differentiable in $t > 0$ and replace the right hand member of $(*)$ by an arbitrary element of $\mathscr{D}'_+(t,L^2)$. We then have

53 **Problem 2.** Given $T$ in $\mathscr{D}'_+(t,L^2)$ find $u$ in $\mathscr{D}'_+(t,N)$ such that

$$\frac{\partial u}{\partial t} - \Delta u = T.$$

We shall treat only problem 2. We shall show that the problem admits of a unique solution. But what we would have solved will only be a problem much weaker than the original problem we posed. To solve the original problem completely one has to show that $u \in \mathscr{D}'_+(t, N)$ that we have found is a differentiable function in $t > 0$ and also one has to prove the regularity properties of $u(t, x)$ as a function of $x$.

# Lecture 11

# Weak boundary value problems

We first formulate the generalized weak "boundary value problem". **54**
**Problem 11.1.** $Q$ is a Banach space and $Q'$ is its strong dual. $V$ is a Hilbert space satisfying $V \subset Q, V \subset Q'$ with continuous injections and $a(u,v)$ is a continuous sesquilinear form on $V$. Let $V$ be dense in $Q$. Then we can find a subspace $N$ of $V$ and a continuous linear map $A : N \to Q'$ such that $\langle Au, \overline{v} \rangle = a(u,v)$. (See Lions: **"On Elliptic Partial Differential Equations"**, Tata Institute of Fundamental Research, Lecture 5). Let $\vec{g} \in \mathscr{D}'_+(t, Q')$. We look for $\vec{u} \in \mathscr{D}'_+(t, N)$ such that $\frac{d\vec{u}}{dt} + A\vec{u} = \vec{g}$.

**Theorem 11.1.** *We follow the above notations. Assume that*

1) $\vec{g}$ *has a Laplace transform:* $\vec{g} \sqsupset \vec{G}(p)$ *for* $Rlp > a$.

2) $\langle u, \overline{u} \rangle V \geq 0$ *for* $u \in V$, *where* $\langle , \rangle_V$ *denote the scalar product in* $V$.

3) *There exists an* $\alpha > 0$ *and a real* $\mathscr{E}_1$ *such that*

$$a_1(u,u) + \mathscr{E}\langle u, \overline{u} \rangle \geq \alpha \parallel u \parallel_v^2 \quad for \quad \mathscr{E} \geq \mathscr{E}_1$$

*where* $a_1(u,v)$ *is the real part of* $a(u,v)$ *and* $\parallel \quad \parallel_V$ *denotes the norm in* $V$.

*Then there exists a unique $\vec{u} \in \mathscr{D}'_+(t, N)$ such that $\vec{u}$ has a Laplace transform and satisfies $\frac{d\vec{u}}{dt} + A\vec{u} = \vec{g}$.*

*Proof.* $\vec{G}(p)$ is a holomorphic function with values in $Q'$. For every fixed $p$ such that $Rlp > \gamma = Max(\mathscr{E}_1, a)$ consider the equation

$$(p + A)\vec{U}(p) = \vec{G}(p) \tag{1}$$

55 Since $a_1(u, v) + \mathscr{E}\langle u, \overline{u}\rangle \geq \| u \|_v^2$ for $\mathscr{E} \geq \mathscr{E}_1$ with $\alpha > 0$, it follows (See Lions: **"On Elliptic Partial Differential Equations"**, Tata Institute of Fundamental Research, Lecture 5) that the operator $(p + A) : N \to Q'$ is an isomorphism for $Rlp > \gamma$. Hence there exists a unique $\vec{U}(p) \in N$ such that (1) is valid. In fact, $\vec{U}(p) = (p + A)^{-1}\vec{G}(p)$ for every fixed $p$ with $Rlp > \gamma$. □

We shall next show that $\vec{U}(P)$ is a holomorphic function with values in $N$. For this we need the following general

**Lemma.** *Suppose $N$ and $Q'$ are two Banach spaces and $L(P)$ a holomorphic function in a domain $\Omega$ (in $C$) with values in $\mathscr{L}(N, Q')$. Assume $L(P)$ has a continuous inverse $L^{-1}(P)$ at every point $p$ of $\Omega$. Then the function $p \to L^{-1}(p)$ is a holomorphic function in $\Omega$ with values in $\mathscr{L}(Q', N)$.*

*Proof.* For $p, p + h \in \Omega$ we have

$$L^{-1}(p + h) - L^{-1}(p) = L^{-1}(p + h)\{L(P) - L(P + h)\} L^{-1}(P).$$

The norm of $L^{-1}(p + h)$ remains bounded as $h \to 0$. For if not, we can find a sequence $\{h_n\}n = 1, 2, \ldots$ of complex numbers tending to zero, and a sequence $X_n$ of element in $Q'$ such that

$$L^{-1}(p + h_n)X_n = \lambda_n \geq n \quad \text{and} \quad \| X_n \| = 1 \quad \text{in} \quad Q'.$$

Let $Y_n = \frac{X_n}{\lambda_n}$. Then $\| Y_n \|_{Q'} \to 0$ and $\| L^{-1}(p + h_n)Y_n \|_V = 1$. Let $Z_n = L^{-1}(p + h_n)Y_n$. Then $\| Z_n \| = 1$. Now $\| L(p + h_n)Z_n \|_{Q'} = \| Y_n \|_{Q'}$ and this tends to 0 as $n \to \infty$. Due to the continuity of $p \to L(p)$ we have

$\lim_{n\to\infty} \parallel L(p+h_n)-L(p) \parallel = 0$. Hence $\lim_{n\to\infty} \parallel L(p)Z_n \parallel_{Q'} = 0$. But $L(p)$ is invertible and $\parallel Z_n \parallel = 1$. This cannot happen. Hence $\parallel L^{-1}(p+h) \parallel$ is
56 bounded as $h \to 0$. Hence $\lim_{h\to\circ} \parallel L^{-1}(p+h) - L^{-1}(p) \parallel = 0$. This proves the continuity of $p \to L^{-1}(p)$ for $p$ in $Rlp > \gamma$. □

We now prove the differentiability of the function $p \to L^{-1}(p)$ in $\Omega$. For $p$ and $(p+h) \in \Omega$, we have

$$\frac{L^{-1}(p+h) - L^{-1}(p)}{h} = L^{-1}(p+h)\frac{L(p) - L(p+h)}{h}L^{-1}(p).$$

Now, $\lim_{h\to 0} L^{-1}(p+h) = L^{-1}(p)$. Hence

$$\lim_{h\to 0} \frac{L^{-1}(p+h) - L^{-1}(p)}{h} = L^{-1}(p)L'(p)L^{-1}(p),$$

where $L'(p)$ is the derivative of $L(p)$.

Continuing with proof of theorem 11.1, we first remark that the function $p \to p + A$ is a holomorphic function in $Rlp > \gamma$ with values in $\mathscr{L}(N, Q')$. From $\vec{U}(p) = (p+A)^{-1}\vec{G}(p)$ and the holomorphic nature of $\vec{G}(p)$ and $(p+A)^{-1}$ in $Rlp > \gamma$, it follows that $\vec{U}(p)$ is a holomorphic function of $p$ with values in $N$ for $Rlp > \gamma$.

The next step in proving our theorem is to show that $\vec{U}(p)$ is the Laplace transform of a well determined distribution $\vec{u} \in \mathscr{D}'_+(t, N)$ and that this $\vec{u}$ satisfies the equation $(\frac{d}{dt} + A)\vec{u} = \vec{g}$. For any $f \in Q'$ we have $v \to \langle f, \overline{v}\rangle_{Q',Q}$ to be a continuous linear functional on $V$. Hence

$$\langle f, \overline{v}\rangle = \langle \tilde{J}f, v\rangle_V \quad \text{where} \quad \tilde{J} : Q' \to V$$

is some fixed continuous linear map. Let

$$\alpha_p(u, v) = a(u, v) + p\langle u, \overline{v}\rangle.$$

The map $v \to a(u, v)$ is also a continuous linear map of $V$ in $V$. Hence there exists a fixed continuous linear map $\tilde{K} : V \to V$ such that

$$a(u, v) = \langle \tilde{K}u, v\rangle_V.$$

$$\begin{aligned}\text{Then}\quad \alpha_p(u,v) &= \langle \tilde{K}u, v\rangle_V + p\langle u, \overline{v}\rangle \\ &= \langle \tilde{K}u, v\rangle_V + p\langle \tilde{J}u, v\rangle_V \quad (\text{since} \quad V \subset Q') \\ &= \langle (\tilde{K} + p\tilde{J})u, v\rangle_V.\end{aligned}$$

57 For each fixed $p$ in $Rlp > \gamma$ the element $\overrightarrow{U}(p)$ of $N$ is nothing but that element of $N$ satisfying

$$\alpha_p(\overrightarrow{U}(p), v) = \langle \overrightarrow{G}(p), \overline{v}\rangle_{Q',Q}.$$

Hence $\overrightarrow{U}(p)$ is that element of $N$ satisfying

$$\langle (\tilde{K} + p\tilde{J})\overrightarrow{U}(p), v\rangle_V = \langle \tilde{J}\overrightarrow{G}(p), v\rangle_V.$$

$$\text{Hence} \qquad (\tilde{K} + p\tilde{J})\overrightarrow{U}(p) = \tilde{J}\overrightarrow{G}(p).$$

If we show that for $Rlp > \gamma$ the operator $\tilde{K} + p\tilde{J}$ is invertible, we will get

$$\overrightarrow{U}(p) = (\tilde{K} + p\tilde{J})^{-1}\tilde{J}\overrightarrow{G}(p).$$

Now $\qquad \left|\langle (\tilde{K} + p\tilde{J})u, u\rangle_V\right| = |a(u,u) + p\langle u, \overline{u}\rangle|.$
We have $Rl\{a(u,u) + p\langle u, \overline{u}\rangle\} = a_1(u,u) + \mathscr{E}\langle u, \overline{u}\rangle$ where $\mathscr{E} = Rlp$. (This follows from the assumption that $\langle u, \overline{u}\rangle$ is real). Hence

$$\begin{aligned}|a(u,u) + p\langle u, \overline{u}\rangle| &\geq a_1(u,u) + \mathscr{E}\langle u, \overline{u}\rangle \\ &\geq \alpha \parallel u \parallel_V^2 \quad \text{for} \quad \mathscr{E} > \nu. (\text{ with } \alpha > 0).\end{aligned}$$

Hence $\parallel (\tilde{K} + p\tilde{J})u \parallel_V \parallel u \parallel_V \geq \alpha \parallel u \parallel_V^2$. Hence

$$\parallel (\tilde{K} + p\tilde{J})u \parallel_V \geq \alpha \parallel u \parallel_V \text{ with } \alpha > 0.$$

This implies, since we are in a Hilbert space, that the operator $\tilde{K} + p\tilde{J}$ is invertible and that $\parallel (\tilde{K} + p\tilde{J})^{-1} \parallel \leq \frac{1}{\alpha}$ for $Rlp > \nu$. Hence we have $\overrightarrow{U}(p) = (\tilde{K} + p\tilde{J})^{-1}\tilde{J}\overrightarrow{G}(p)$.

$$\begin{aligned}\parallel \overrightarrow{U}(p) \parallel &\leq \parallel (\tilde{K} + p\tilde{J})^{-1} \parallel \parallel \tilde{J} \parallel \parallel \overrightarrow{G}(p) \parallel \\ &\leq \frac{1}{\alpha}\beta \parallel \overrightarrow{G}(p) \parallel\end{aligned}$$

58 where $\beta$ is some constant. Since $\overrightarrow{G}(p)$ has a uniform polynomial majorisation in any half plane $Rlp \geq \nu + \varepsilon, \varepsilon > 0$, the same is true of $\overrightarrow{U}(p)$. Hence $\overrightarrow{U}(p)$ is the Laplace transform of a certain distribution $\overrightarrow{u} \in \mathscr{D}'_+(t, N)$, which is unique.

# Lecture 12

# Topological tensor products

Let $L$ and $M$ be two vector spaces (algebraic) over $K$. Let $L \otimes M$ be the **59**
tensor product of $L$ and $M$ over $K$. Then for any vector space $N$ over $K$ there exists a biunique correspondence between the bilinear maps of $L \times M$ in $N$ and the linear maps of $L \otimes M$ in $N$. In fact to the bilinear map $u$ of $L \times M$ in $N$ corresponds the linear map $\tilde{u} : L \otimes M$ in $N$ which takes $l \otimes m$ into $u(l, m)$. The map $\eta : L \times M \to L \otimes M$ defined by $\eta(l, m) = l \otimes m$ is a bilinear map and is called the canonical bilinear map of $L \times M$ in $L \otimes M$. We have commutativity in the following diagram:

$$\begin{array}{ccc} L \times M & \xrightarrow{\ \eta\ } & L \otimes M \\ & {\scriptstyle u} \searrow \quad \swarrow {\scriptstyle \tilde{u}} & \\ & N & \end{array}$$

Let now $L$ and $M$ be two locally convex Hausdorff vector spaces over $C$. Let $L \otimes M$ be the algebraic tensor product of $L$ and $M$ over $C$.

**Theorem 12.1.** *There exists a unique locally convex, Hausdorff topology on $L \otimes M$ such that under the usual correspondence between bilinear maps of $L \times M$ in an ELC $N$ and the linear maps of $L \otimes M$ in $N$, the continuous bilinear maps of $L \times M$ in $N$ precisely correspond to the continuous linear maps of $L \otimes M$ provided with this topology in $N$. Moreover under the biunique correspondence between bilinear maps of $L \times M$ in $C$ and linear maps of $L \otimes M$ in $C$, equicontinuous sets of bilinear maps of $L \times M$ in $C$ correspond to equicontinuous sets of linear* *maps of $L\underset{\pi}{\otimes}M$ in $C$ ($L\underset{\pi}{\otimes}M$ being the tensor product provided with the above topology) and conversely.*

60 *Proof.* Assuming the existence of at least one such topology, we will prove the uniqueness. Let $\pi_1$ and $\pi_2$ be two such topologies. Take for $N$ the space $L\underset{\pi_1}{\otimes}M$. The identity map of $L\underset{\pi_1}{\otimes}M \to N$ is continuous and hence the bilinear map which corresponds to this, that is to say, $\eta : L \times M \to N$ is continuous. Now that $\eta : L \times M \to L\underset{\pi_2}{\otimes}M$ is continuous, we have $i : L\underset{\pi_2}{\otimes}M \to L\underset{\pi_2}{\otimes}M$ continuous. Hence $\pi_2$ is finer than $\pi_1$. Interchanging the roles of $\pi_1$ and $\pi_2$ we see that $\pi_1$ is finer than $\pi_2$. Hence $\pi_1 = \pi_2$. □

We now go to the proof of the existence of one such topology. Let $\mathscr{B}(L, M)$ denote the set of continuous bilinear forms on $L \times M$. The spaces $\mathscr{B}(L, M)$ and $L \otimes M$ are in duality with respect to the scalar product which is got by restricting the scalar product between $\mathscr{L}(L, M; C)$ and $L \otimes M$ where $\mathscr{L}(L, M; C)$ is the set of all bilinear maps of $L \times M$ in $C$. Now the duality between $\mathscr{B}(L, M)$ and $L \otimes M$ allows us to define a topology on $L \otimes M$, namely the topology of uniform convergence on equicontinuous subsets of $\mathscr{B}(L, M)$ which is a locally convex, Hausdorff topology. This topology on $L \otimes M$ we denote by $\pi$ and provided with this topology the space $L \otimes M$ is denoted by $L\underset{\pi}{\otimes}M$. We shall prove that $\pi$ is a topology having all the properties mentioned in the theorem. We shall show first that under the biunique correspondence of bilinear maps of $L \times M$ in $C$ and linear maps of $L\underset{\pi}{\otimes}M$ in $C$, equicontinuous sets of bilinear maps precisely correspond to equicontinuous linear maps. Let $H \subset \mathscr{L}(L, M; C)$ be any equicontinuous set. Then $H \subset \mathscr{B}(L, M)$ trivially. Let $\tilde{H}$ be the corresponding subset of $\mathscr{L}(L \otimes M, C)$ (the set of all
61 linear maps of $L \otimes M$ in $C$). Let $\Gamma(H)$ be the stable envelope of $H$ in $\mathscr{L}(L, M; C)$. $\Gamma(H)$ is an equicontinuous set and hence $\Gamma(H) \subset \mathscr{B}L, M)$. Let $W = \Gamma(H)^\circ$ be the polar of $\Gamma(H)$ with respect to the duality between $\mathscr{B}(L, M)$ and $L \otimes M$.

Since $\Gamma(H)$ is stable,

$$W = \{\gamma \in L \otimes M / |\langle r, \gamma\rangle| \leq 1, \text{ for every } r \in \Gamma(H)\}.$$

Now $\tilde{h}(\gamma) = |\langle h, \gamma\rangle|$ where $\tilde{h} \in \tilde{H}$ and $h$ is the corresponding bilinear map. For $h \in H$ we have $|\langle h, \gamma\rangle| \leq 1$. Hence $|\tilde{h}(v)| \leq 1$ for $v \in W$. Hence $\tilde{H}$ is an equicontinuous set of linear maps of $L \underset{\pi}{\otimes} M$ in $C$ (as $W$ is a neighbourhood of 0 in $L \underset{\pi}{\otimes} M$).

Now suppose $\tilde{H}$ is any equicontinuous set of linear maps of $L \underset{\pi}{\otimes} M$ in $C$. Let $H$ be the corresponding set of bilinear maps of $L \times M$ in $C$. Given any neighbourhood $N$ of 0 in $L \otimes M$ we show that there exist neighbourhoods $U$ and $V$ of the zero elements in $L$ and $M$ respectively such that the set $U \otimes V \subset N$. (The set $U \otimes V$ is, by definition, the set of all elements of the form $u \otimes v$, $u \in U$, $v \in V$). Now $N \supset W^{\circ}$ where $W$ is an equicontinuous subset of $\mathscr{B}(L, M)$. We can find stable neighbourhoods $U$ and $V$ of the zero elements such that $|w(u, v)| \leq 1$ $w \in W$, $u \in U$ and $v \in V$. The pair $(U, V)$ does what we need, for if $u \in U$ and $v \in V$ we have $|\langle w, u \otimes v\rangle| = |w(u, v)| \leq 1$. Hence $u \otimes v \in W^{\circ} \subset N$. Hence $U \otimes V \subset N$. (This incidentally proves the continuity of the map $\eta : L \times M \to L \underset{\pi}{\otimes} M$).

Now for any $h \in H$ and $(u, v) \in U \times V$ we have

$$|h(u, v)| = |\tilde{h}(u \otimes v)| \leq 1,$$

since $U \otimes V \subset N$. Hence $H$ is an equicontinuous set of bilinear maps of $L \times M$ in $C$.

Now we prove that the continuous bilinear maps of $L \times M$ in any *ELC* $N$, precisely correspond to the continuous linear maps of $L \underset{\pi}{\otimes} M$ in $N$. For this, we need the following **62**

**Lemma 12.1.** *Let $\varphi$ be any continuous bilinear map of $L \times M$ in $N$. With each $\mu \in N'$ we associate the bilinear form $\langle \varphi, \mu\rangle$ defined by $\langle \varphi, \mu\rangle$ $(l, m) = \langle \varphi(l, m), \mu\rangle$. The bilinear form $\langle \varphi, \mu\rangle$ is an element of $\mathscr{B}(L, M)$, (trivially). If $W'$ is any equicontinuous subset of $N'$, the set $\underset{w' \in W'}{\cup} \langle \varphi, w'\rangle$ is an equicontinuous subset of $\mathscr{B}(L, M)$.*

*Proof.* Since $W'$ is an equicontinuous subset of $N'$ we can find a neighbourhood $\Gamma$ of 0 in $N$ such that

$$|\langle r, w'\rangle| \leq 1 \quad \text{for every} \quad r \in \Gamma \quad \text{and} \quad w' \in W'$$

Since $\varphi : L \times M \to N$ is continuous, there exist neighbourhoods $U$ and $V$ of the zero elements of $L$ and $M$ respectively such that $\varphi(U \times V) \subset \Gamma$. For $(u, v) \in U \times V$ we have

$$|\langle\varphi, w'\rangle(u, v)| = |\langle\varphi(u, v), w'\rangle| = |\langle r, w'\rangle| \leq 1 \quad \text{with} \quad r \in \Gamma.$$

Hence $\underset{w' \in W'}{\cup}\{\langle\varphi, w'\rangle\}$ is an equicontinuous subset of $\mathscr{B}(L, M)$.

Now, let $\varphi : L \times M \to N$ be bilinear and continuous. For any equicontinuous set $W'$ of $N'$ we have $\underset{w' \in W'}{\cup}\{\langle\varphi, w'\rangle\}$ is an equicontinuous subset of $\mathscr{B}(L, M)$. Hence if $\alpha \to 0$ in $L \underset{\pi}{\otimes} M$,

$$\langle\langle\varphi, w'\rangle, \alpha\rangle \underset{\mathscr{B}(L,M), L \otimes M}{\to 0}$$

uniformly for $w' \in W'$. If $\tilde{\varphi} : L \underset{\pi}{\otimes} M \to N$ is the corresponding linear map, we have $\langle\tilde{\varphi}(\alpha), w'\rangle_{N,N'} = \langle\langle\varphi, w'\rangle, \underset{\mathscr{B}, L \times M}{\alpha}\rangle$ and this tends to 0 uniformly when $w' \in W'$. ◻

63 Hence $\tilde{\varphi}(\alpha) \to 0$ in $N$ when $\alpha \to 0$ in $L \underset{\pi}{\otimes} M$. Hence $\tilde{\varphi}$ is continuous.

Conversely, suppose $\tilde{\varphi} : L \underset{\pi}{\otimes} M \to N$ is continuous. Let $\varphi$ be the corresponding bilinear map of $L \times M \to N$. We have $\varphi = \tilde{\varphi} \circ \eta$. As has been shown already, $\eta$ is continuous. Hence $\varphi$ is continuous.

**Corollary 1.** *$\pi$ is the strongest (finest) locally convex Hausdorff topology on $L \otimes M$ such that $\eta : L \times M \to L \otimes M$ is continuous. In fact $\pi$ is a locally convex Hausdorff topology on $L \otimes M$ such that $\eta : L \times M \to L \underset{\pi}{\otimes} M$ is continuous. Let $\pi'$ be any locally convex Hausdorff topology such that $\eta : L \times \to L \underset{\pi'}{\otimes} M$ is continuous. Then $i : L \underset{\pi}{\otimes} M \to L \underset{\pi'}{\otimes} M$ is continuous and hence $\pi$ is finer than $\pi'$.*

**Corollary 2.** *The topology $\pi$ is finer than the topology $\varphi$.*

In fact, with the topology $\varepsilon, \eta : L \times M \to L \underset{\varepsilon}{\otimes} M$ is continuous, Hence $\pi$ is **finer than $\varepsilon$.**

**Proposition 12.1.** *Let $u : L_1 \to L_2$ and $v : M_1 \to M_2$ be continuous linear maps where $L_1, L_2, M_1$ and $M_2$ are locally convex, Hausdorff, topological vector spaces. Then $u \otimes v = L_1 \underset{\pi}{\otimes} M_1 \to L_2 \underset{\pi}{\otimes} M_2$ is a continuous linear map.*

*Proof.* To show this, it suffices to prove that the bilinear map $u \times v : L_1 \times M_1 \to L_2 \underset{\pi}{\otimes} M_2$ defined by $u \times v(l_1, m_1) = u(l_1) \otimes v(m_1)$ is continuous. If $l_1 \to 0$ in $L_1$ and $m_1 \to 0$ in $M_1$, we have $u(l_1)$ and $v(m_1)$ tending to 0 in $L_2$ and $M_2$. Hence $u(l_1) \otimes v(m_1) \to 0$ in $L_2 \underset{\pi}{\otimes} M_2$ since the canonical map $L_2 \times M_2 \to L_2 \underset{\pi}{\otimes} M_2$ is continuous. Hence $u \times v : L_1 \times M_1 \to L_2 \underset{\pi}{\otimes} M_2$ is continuous. ◻

**Corollary.** *If $u : L_1 \to L_2$ and $v : M_1 \to M_2$ are continuous injections, $u \otimes v : L_1 \underset{\pi}{\otimes} M_1 \to L_2 \underset{\pi}{\otimes} M_2$ is a continuous injection.* **64**

*This is an immediate consequence of the fact that $C$ is a field and of proposition 12.1*

**Remarks:**

1. Though $u \otimes v : L_1 \underset{\pi}{\otimes} M_1 \to L_2 \underset{\pi}{\otimes} M_2$ is a continuous injection, the extension of $u \otimes v$ by continuity from $L_1 \underset{\pi}{\otimes} M_1 \to L_2 \otimes M_2$ (the completions) is a continuous linear map, but not necessarily an injection.

2. Let $\mathscr{H}$ be a complete space of distributions and $E$ a complete $ELC$. The space $\mathscr{H}(E)$ is complete. The map $i : \mathscr{H} \underset{\pi}{\otimes} E \to \mathscr{H} \underset{\varepsilon}{\otimes} E$ given by $i(\alpha) = \alpha$ is a continuous linear map which extends itself into a continuous linear map $\hat{i} : \mathscr{H} \underset{\pi}{\hat{\otimes}} E \to \mathscr{H}(E)$. $\hat{i}$ is in general not an injection.

3. Let $L, M$ and $N$ be three $ELC$s over $C$. The canonical isomorphism of $(L \otimes M) \otimes N$ with $L \otimes (M \otimes N)$ is a topological isomorphism of $(L \underset{\pi}{\otimes} M) \underset{\pi}{\otimes} N$ with $L \underset{\pi}{\otimes} (M \underset{\pi}{\otimes} N)$. By using trilinear maps we can introduce a locally convex, Hausdorff topology $\pi$ on $L \otimes M \otimes N$ such that the canonical biunique correspondence between trilinear

maps of $L \times M \times N$ and linear maps of $L \otimes M \otimes N$ in any *ELC* $F$ takes the continuous trilinear maps precisely into the continuous linear maps $(L \otimes M \otimes N)_\Pi$ in $F$. The canonical isomorphism of $L \otimes (M \otimes N)$ with $(L \otimes M \otimes N)$ is a topological isomorphism of $L \underset{\pi}{\otimes} (M \underset{\pi}{\otimes} N)$ with $(L \otimes M \otimes N)_\pi$. That is to say, we have

$$(L \otimes M \otimes N)_\pi \approx L \underset{\pi}{\otimes} (M \underset{\pi}{\otimes} N) \approx (L \underset{\pi}{\otimes} M) \underset{\pi}{\otimes} N$$

as topological vector spaces.

# Lecture 13

# Topological tensor products (contd.)

We shall give an intrinsic characterisation of the topology on the tensor product defined already. Let $L$ and $M$ be two $ELC$s over $C$. Let $U$ and $V$ be neighbourhoods of the zero elements in $L$ and $M$. Let $\Gamma(U \otimes V)$ be the convex, stable envelope of the set $U \otimes V$ in $L \otimes M$, $U \otimes V$ being the set of points $(u \otimes v,\ u \in U,\ v \in V)$. The set $\Gamma(U \otimes V)$ is an absorbing set in $L \otimes M$. In fact any element of $L \otimes M$ is of the form $\sum l_\nu \otimes m_\nu,\ l_\nu \in L$ and $m_\nu \in M$, the sum written being a finite sum. $U$ absorbs $l_\nu$ and $V$ absorbs $m_\nu$. Hence $U \otimes V$ absorbs each of the elements $l_\nu \otimes m_\nu$. Hence the convex, stable, envelope $\Gamma(U \otimes V)$ absorbs any finite sum of the elements of the form $l_\nu \otimes m_\nu$. We can take the sets $\Gamma(U \otimes V)$ for a fundamental system of neighbourhoods of 0 in a certain locally convex topology on $L \otimes M$. This topology is precisely the topology $\pi$. Let us denote the topology just defined by $\tau$. 65

**Proposition 13.1.** *The topologies $\pi$ and $\tau$ are identical.*

*Proof.* If we show that the topology $\tau$ is the finest locally convex topology on $L \otimes M$ such that the canonical bilinear map $\eta : L \times M \to L \otimes M$ is continuous, we are through. Obviously $\eta$ is continuous, for if we take any neighbourhood of 0 in $L \otimes M$ it contains a set of the form $\Gamma(U \otimes V)$, $U$ a neighbourhood of 0 in $L$ and $V$ a neighbourhood of 0 in $M$. Now $\eta^{-1}(\Gamma(U \otimes V) \supset U \times V$ and this is a neighbourhood of $(0, 0)$ in $L \times M$.

66 Suppose $\theta$ is any topology on $L \otimes M$ such that $L \times M \xrightarrow{\eta} L\underset{\theta}{\otimes}M$ is continuous and $\theta$ is locally convex. Any neighbourhood of 0 in $L\underset{\theta}{\otimes}M$ contains a convex, stable neighbourhood $W$ of 0. If we show that $W$ is a neighbourhood of 0 even for $\tau$ we are through. Since $\eta : L \times M \to L\underset{\theta}{\otimes}M$ is continuous, there exist neighbourhoods $U$ and $V$ of the zero elements of $L$ and $M$ such that $U \otimes V \subset W$. Since $W$ is convex and stable $\Gamma(U \otimes V) \subset W$. Hence $W$ is a neighbourhood of 0 in $\tau$. □

**Seminorms.** Let $U$ and $V$ be convex, stable neighbourhoods of the zero elements in $L$ and $M$ respectively. Let $p$ be the seminorm associated with $U$ and $q$ the seminorm associated with $V$. Let $r$ be the seminorm associated with $\Gamma(U \otimes V)$. It is possible to prove that $r$ is the same as the tensor product of the two seminorms $p$ and $q$ defined as follows: $p \otimes q(\mathscr{E}) = \operatorname{Inf} \sum_{\nu} p(x_\nu) q(y_\nu)$ where $\mathscr{E} = \sum x_\nu \otimes y_\nu$ is a way of expressing $\mathscr{E}$ as the sum of a finite number of elements of the type $x_\nu \otimes y_\nu, x_\nu \in L, Y_\nu \in M$. Also, $m$ if for a pair of elements $x, y; x \in L$ and $y \in M$ we have $\mathscr{E} = x \otimes y$, we can show that $r(\mathscr{E}) = p(x)q(y)$. If $L$ and $M$ are normed spaces, $L \otimes M$ is a normal space with the tensor product of the norms on $L$ and $M$ as the norm. That is to say,

$$\| \mathscr{E} \| = \operatorname{Inf} \sum \| x_\nu \| \| y_\nu \| .$$

Suppose $L$ and $M$ are both Frechet spaces. One can show then that $L\hat{\underset{\pi}{\otimes}}M$ is also Frechet.

*Proof.* Let $L$ and $M$ be Frechet spaces. In order to show that $L\hat{\underset{\pi}{\otimes}}M$ is a Frechet space it suffices to show that in $L\underset{\pi}{\otimes}M$ we have a countable base for the neighbourhoods of 0. $L$ and $M$ being metrizable, we have
67 countable bases for the neighbourhoods of the zero elements in the case of the two spaces $L$ and $M$. If $U_n$ and $V_n$ are countable bases for neighbourhoods of the zero elements in $L$ and $M$ respectively, $\Gamma(U_n \otimes V_n)$ is a fundamental system of neighbourhoods of 0 in $L\underset{\pi}{\otimes}M$. Hence $L\underset{\pi}{\otimes}M$ is metrizable and hence its completion is a complete metric space and as it is locally convex, it is a Frechet space.

In what follows immediately $L$ and $M$ stand for two Frechet spaces. □

**Proposition 13.2.** *Any $\mathscr{E} \in L\hat{\otimes}M$ can be written as a convergent infinite series $\sum \lambda_\nu x_\nu \otimes y_\nu$ in which the $x_\nu$s and $y_\nu$s can be so chosen as to converge to 0 as $\nu \to \infty$ in $L$ and $M$ respectively and $\sum_\nu |\lambda_\nu| < \infty$ Also if $K$ is compact in $L\hat{\underset{\pi}{\otimes}}M$, any $\mathscr{E} \in K$ can be written as $\sum_\nu \lambda_\nu x_\nu \otimes y_\nu$ with $\sum_\nu |\lambda_\nu| < \infty$ and $x_\nu \in A, y_\nu \in B$ where $A$ and $B$ are compact subsets of $L$ and $M$ respectively. That is to say $K \subset \overline{\Gamma(A \otimes B)}$. For the proof, refer to Memoirs of the American Mathematical Society, No. 16, Products Tensoriels Topologique et Espaces Nucleaires, by Alexandre Grothendieck, p. 51.*

**Definition 13.1.** *An ELC $E$ is said to be nuclear if for* **every** *ELC, $F$ the $\pi$ and $\varepsilon$ topologies coincide on $E \otimes F$.*

*We now give a criterion for a locally convex Hausdorff space $L$ to be nuclear.*

**Criterion 13.1.** *An ELC $E$ is nuclear if and only if for every Banach space $B$, we have $L\underset{\pi}{\otimes}B = L\underset{\varepsilon}{\otimes}B$.*

*We shall give another criterion, which is, to some extent, better than the above criterion. Before giving the criterion, we introduce certain notions needed to state the criterion.*

**Definition 13.2.** *Let $N$ be a complete ELC. Let $L$ be an ELC. A linear continuous map $u : L \to N$ is called nuclear if it can be written as* **68**

$$u = \sum_\nu \lambda_\nu (l'_\nu \otimes m_\nu)$$

*with the $l'_\nu$s contained in an equicontinuous subset of $L'$ and $m_\nu$ contained in a bounded subset of $N$ and $\sum_\nu |\lambda_\nu| < \infty$.*

*To see that the definition makes sense, we consider the expression $\mu = \sum_\nu \lambda_\nu (l'_\nu \otimes m_\nu)$ with the $l'_\nu$s lying in an equicontinuous subset of $L'$ and the $m_\nu$ lying in a bounded set of $N$ and $\sum_\nu |\lambda_\nu| < \infty$. Let $\mu(l)$ for any* *l be defined as* $\sum_\nu \lambda_\nu(l'_\nu(l).m_\nu)$. *We shall show that* $l \to \mu(l) \in \hat{N} = N$ *is a continuous map. Let* $V$ *be any neighbourhood of* $0$ *in* $N$. *Since* $m_\nu s$ *lie in a bounded set, there exists* $\alpha > 0$ *such that* $m_\nu \in \alpha V$ *for every* $\nu$. *Since the* $l'_\nu$ *lie in an equicontinuous set, given any* $\epsilon > 0$, *we can find a neighbourhood of* $W_\epsilon$ *of* $0$ *in* $L$ *such that* $|\langle l, l'_\nu\rangle| \leq \epsilon$ *for any* $l \in W_\epsilon$. *Hence*

$$\sum \lambda_\nu \langle l, l'_\nu\rangle m_\nu \in \wedge \in \alpha\overline{\Gamma(V)}$$

*where* $\sum|\lambda_\nu| < \wedge$ *and* $\overline{\Gamma(V)}$ *is the convex, stable, closed envelope of* $V$. *But* $V$ *itself can be chosen to be a disked neighbourhood of* $0$ *in* $N$. *Hence* $\sum \lambda_\nu\langle l, l'_\nu\rangle m_\nu \in \wedge \in \alpha V$. *Choose* $\epsilon = \frac{1}{\wedge\alpha}$. *Then* $\sum_\nu \lambda_\nu\langle l, l'_\nu\rangle m_\nu \in V$. *This proves the continuity of* $\mu$. *That* $\mu$ *is linear is obvious. If* $u : L \to N$ *is a nuclear map, we can express* $u$ *as* $\sum_\nu \lambda_\nu(l'_\nu \otimes m_\nu)$ *with* $m_\nu \to 0$ *in* $N$ *and* $l'_\nu$ *lying in an equicontinuous subset of* $L'$. *In fact, since* $\sum_\nu |\lambda_\nu|$ *converges we can find a divergent sequence of real numbers* $\{r_\nu\}$, *diverging to* $+\infty$ *such that the series* $\sum_\nu \lambda_\nu r_\nu$ *still converges absolutely. Then* $\sum_\nu \lambda_\nu(l'_\nu \otimes m_\nu) = \sum \lambda_\nu r_\nu(l_\nu \otimes \frac{m_\nu}{r_\nu})$. *Since the* $m_\nu s$ *lie in a bounded set,* $\frac{m_\nu}{r_\nu} \to 0$ *as* $\nu \to \infty$ *and* $\sum|\lambda_\nu r_\nu| < \infty$.

69 **Criterion 13.2.** *Let* $U$ *be any disked neighbourhood of* $0$ *in* $L$. *With* $U$ *we associate a seminorm* $p$. *This seminorm gives a certain equivalence relation in* $L$. $x \sim y$ *if and only if* $p(x) = p(y)$. *We put on* $L$ *the coarsest topology under which the seminorm* $p$ *is continuous. Then we take the quotient space* $L_U$ *under the equivalence relation defined with the help of the seminorm* $p$. *Let* $\hat{L}_U$ *be the completion of* $L_U$. $\hat{L}_U$ *is a Banach space.* $L$ *is Nuclear if and only if the canonical map* $L \to \hat{L}_U$ *is a Nuclear map for every disked neighbourhood* $U$. *(A. Grothendieck: Espaces Nucleares, Memoirs of the Amer. Math. Soc., No. 16, p. 34). We shall now prove the following*

**Proposition 13.3.** *If for every disked neighbourhood* $U$ *of* $0$ *the canonical map* $L \to \hat{L}_U$ *is a nuclear map, any continuous linear map* $u : L \to B$, $B$ *being any Banach space is nuclear.*

*Proof.* There exists a disked neighbourhood $U$ of $0$ in $L$ such that $u(U) \subset \Gamma$ where $\Gamma$ is the unit ball of $B$. The continuous linear map $u$

can be factored into the canonical map $L \to L_U$ and a continuous linear map of $\hat{L}_U \to B$. Since $L \to \hat{L}_U$ is nuclear, our assertion follows. □

**Theorem 13.1.** *A nuclear space has the approximation property.*

**Theorem 13.2.** *In a nuclear complete space L all the bounded sets are relatively compact.*

**Theorem 13.3.** *A nuclear Banach space is finite dimensional.*

We admit Criterion 13.2, and Theorem 13.1. Theorem 13.3 is an immediate consequence of Theorem 13.2. We prove here Theorem 13.2.

*Proof.* Let $B$ be a bonded set. Without loss of generality, $B$ can be assumed to be disked, for the convex, stable envelope of a bounded set is bounded . Since $L$ is complete, to show that $B$ is *relatively* compact, **70**
it suffices to show that $B$ is precompact. For $B$ to be precompact, it is necessary and sufficient that for any disked neighbourhood of 0 say $U$ the image of $B$ in $L_U$ is precompact. Now $L \to \hat{L}_U$ is a nuclear map and hence a compact map. Hence the image of $B$ in $\hat{L}_U$ is relatively compact. This completes the proof of Theorem 13.2. □

**Theorem 13.4.** *If $L$ and $M$ are both nuclear, $L \underset{\pi}{\otimes} M = L \underset{\varepsilon}{\otimes} M$ is nuclear and $L \times M$ is nuclear.*

The proof is trivial in the case of $L \underset{\pi}{\otimes} M$.

**Theorem 13.5.** *$L$ is nuclear if and only if $\hat{L}$ is nuclear.*

**Theorem 13.6.** *If $L$ is a nuclear Frechet space, the strong dual $L'$ is a nuclear space.*

**Theorem 13.7.** *If $L$ and $M$ are Frechet nuclear spaces, $\mathscr{L}_\delta(L, M)$ is a nuclear space.*

**Theorem 13.8.** *If $L$ is a nuclear space, any subspace of $L$ with the induced topology is a nuclear space. If $H$ is any closed subspace of $L$, the quotient space $L/H$ is also nuclear.*

**Examples of nuclear spaces.** $\mathscr{D}'_+, \mathscr{D}, \mathscr{D}', \mathscr{E}, \mathscr{E}', \mathscr{S}, \mathscr{S}', \mathscr{O}_M, \mathscr{O}'_M, \mathscr{O}'_c, \mathscr{O}_c$ are nuclear spaces. $\mathscr{E}^m$ and $L^p$ are not nuclear. In fact in $\mathscr{E}^m$ and $L^p$ bounded sets are not relatively compact.

For the proofs of all these results, refer to Espaces Nucleaires by A. Grothendieck, Memoirs of Amer. Math. Soc., No.16.

# Lecture 14

# Multiplication of vector valued distributions

Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{L}$ be three $ELCs$. Let $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ be a bilinear 71
map which is hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$. Let $E, F, G$ be three Banach spaces and $B : E \times F \to G$ be a continuous bilinear map. We ask the question whether it will be possible to define a bilinear map:

$\mathscr{H}(E) \times \mathscr{K}(F) \to \mathscr{L}(G)$ say $\underset{B}{U}$ such that is satisfies the following conditions:

1) $\underset{B}{U}$ is hypocontinuous with respect to bounded sets of $\mathscr{H}(E)$ and $\mathscr{K}(F)$.

2) For decomposed elements, that is for elements of the type $S\vec{e}$ and $T\vec{f}$ of $\mathscr{H}(E)$ and $\mathscr{K}(F)$ we have $S\vec{e}\underset{B}{U}T\vec{f} = (SUT)B(\vec{e}, \vec{f})$ with $\vec{e} \in E, \vec{f} \in F, S \in \mathscr{H}$ and $T \in \mathscr{K}$.

In general it will not be possible to define such a map. We shall give here, without proof, a certain example in which such a map $\underset{B}{U}$ cannot be defined. Let $\mathscr{D}^{\circ}$ be the space of continuous functions with compact support on $R^N$. Let $\mathscr{D}^{\circ\prime}_{\delta}$ be the strong dual of $\mathscr{D}^{\circ}$, that is to say $\mathscr{D}^{\circ\prime}_{\delta}$ is the space of measures on $R^N$. Let $\vec{\mu} \in \mathscr{D}^{\circ\prime}_{\delta}(E)$ and $\vec{\varphi} \in \mathscr{D}^{\circ}(F)$ where $E$ and $F$ are two Banach spaces. Let $B : E \times F \to G$ be a continuous

bilinear map. The duality between $\mathscr{D}^{\circ'}_{\delta}$ and $\mathscr{D}^{\circ}$ gives a bilinear map of $\mathscr{D}^{\circ}_{\delta} x \mathscr{D}^{\circ}$ in $C$ hypocontinuous with respect to the bounded sets. But a bilinear map $\underset{B}{U} : \mathscr{D}^{\circ'}(E) \times \mathscr{D}^{\circ}(F) \to G$ cannot be defined to satisfy the
72 conditions (1) and (2). We shall now prove the following

**Theorem 14.1.** *Let $\mathscr{H}, \mathscr{K}, \mathscr{L}$ be three locally convex separated complete vector spaces all of which are nuclear. Let the strong duals of the three spaces be also nuclear. Let $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ be a bilinear map hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$. Let $E, F$ and $G$ be three Banach spaces with a continuous bilinear map $B : E \times F \to G$. Then there exists one and only one bilinear map $\underset{B}{U} : \mathscr{H}(E) \times \mathscr{K}(F) \to \mathscr{L}(G)$ which satisfies*

1) *$S\vec{e} \underset{B}{U} T\vec{f} = (SUT)R(\vec{e}, \vec{f})$ for every $\vec{e} \in E, \vec{f} \in F, S \in \mathscr{H}$ and $T \in \mathscr{K}$.*

2) *$(\vec{S}, \vec{T}) \to \vec{S} \underset{B}{U} \vec{T}$ is separately continuous in $\vec{S}$ and $\vec{T}$. Moreover $\underset{B}{U}$ has the following supplementary properties.*

3) *$\underset{B}{U}$ is hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{K}(F)$.*

4)

$$\vec{S} \underset{B}{U} \vec{T} = (I_{\mathscr{L}} \varepsilon \tilde{B})(U_{\vec{S}}^{\otimes} I_F)(\vec{T}) = (I_{\mathscr{L}} \varepsilon \tilde{B})(I_E \otimes U_{\vec{T}})(\vec{S})$$

*where $I_{\mathscr{L}}, I_E$ and $I_F$ are the identity mappings of $\mathscr{L}, E$ and $F$ respectively, and $U_{\vec{S}}, U_{\vec{T}}$ and $\tilde{B}$ are defined as follows. $U_{\vec{S}} : \mathscr{K} \to \mathscr{L}(E)$ on any $T \in \mathscr{K}, U_{\vec{S}}(T) = \vec{S}UT$ (for the definition of $\vec{S}UT$ refer to lecture 7). $U_{\vec{T}} : \mathscr{H} \to \mathscr{L}(E)$, on any $S \in \mathscr{H}, U_{\vec{T}}(S) = SU\vec{T}$ $B$ being a continuous bilinear map of $E \times F$ in $G$, $B$ gives rise to a continuous linear map $B'$ of $E \underset{\pi}{\otimes} F$ in $G$ which in its turn can be extended to a continuous linear map of $E \underset{\pi}{\hat{\otimes}} F$ in $G$. $\tilde{B}$ denotes this extended map.*

*Proof.* First, assuming the existence of a bilinear map satisfying (1) and (2), we shall prove the uniqueness of the map. Suppose there are two bilinear maps $\underset{B}{U}^1$ and $\underset{B}{U}^2$ satisfying (1) and (2). $\mathscr{H}$ and $\mathscr{K}$ being nuclear, 73
they have the approximation property and $\mathscr{H}\underset{\pi}{\otimes}E = \mathscr{H}\underset{\varepsilon}{\otimes}E$. Therefore $\mathscr{H}\underset{\pi}{\hat{\otimes}}E = \mathscr{H}(E), \mathscr{K}\underset{\pi}{\hat{\otimes}}F = \mathscr{K}(F)$, because $\mathscr{H}(E)$ and $\mathscr{K}(F)$ are complete. Hence the sets $\mathscr{H}\otimes E$ and $\mathscr{K}\otimes F$ are dense in $\mathscr{H}(E)$ and $\mathscr{K}(F)$. On the product of the sets $(\mathscr{H}\otimes E)\times(\mathscr{K}\otimes F)$ the bilinear maps that we define are well determined because of (1). Now the separate continuity of both $\underset{I}{U}^1$ and $\underset{B}{U}^2$ gives $\underset{B}{U}^1 = \underset{I}{U}^2$ on the whole of $\mathscr{H}(E)\times\mathscr{K}(F)$.

Now we shall prove the existence of a bilinear map satisfying (1) and (2).

Given a bilinear map $U : \mathscr{H}\times\mathscr{K}\to\mathscr{L}$ hypocontinuous with respect to bounded sets we have already seen how to define a bilinear map $\mathscr{H}(E)x\mathscr{K}\to\mathscr{L}(E)$ hypocontinuous with respect to bounded sets satisfying certain consistency conditions (see lecture 7). Each $\vec{S}\in\mathscr{H}(E) = \mathscr{H}\underset{\pi}{\hat{\otimes}}E$ defines a continuous linear map $U_{\vec{S}} : \mathscr{K}\to\mathscr{L}(E) = \mathscr{L}\underset{\pi}{\hat{\otimes}}E$ as follows: $U_{\vec{S}}(T) = \vec{S}UT$. $U_{\vec{S}}$ gives rise to a continuous linear map $U_{\vec{S}}\otimes I_F : \mathscr{K}\underset{\pi}{\otimes}F\to(\mathscr{L}\underset{\Pi}{\hat{\otimes}}E)\otimes F$, which can be extended by continuity into a linear map, which also we denote by $U_{\vec{S}}\otimes I_F$ of $\mathscr{K}\underset{\pi}{\hat{\otimes}}F\to(\mathscr{L}\underset{\pi}{\hat{\otimes}}E)\underset{\pi}{\hat{\otimes}}F$; that is to say, $U_{\vec{S}}\otimes I_F : \mathscr{K}\underset{\pi}{\hat{\otimes}}F\to\mathscr{L}\underset{\pi}{\hat{\otimes}}(E\underset{\pi}{\hat{\otimes}}F)$ is a continuous linear map. Since $\mathscr{L}$ is nuclear, we have

$$\mathscr{L}\hat{\otimes}(E\underset{\pi}{\hat{\otimes}}F) = \mathscr{L}(E\underset{\pi}{\hat{\otimes}}F).$$

$U_{\vec{S}}\otimes I_F : \mathscr{K}(F)\to\mathscr{L}(E\underset{\pi}{\hat{\otimes}}F)$ is a continuous linear map. Now $\tilde{B} : E\underset{\pi}{\hat{\otimes}}F\to G$ is a continuous linear map. Hence the map $I_{\mathscr{L}}\varepsilon\tilde{B}$ is a continuous linear map of $\mathscr{L}(E\underset{\pi}{\hat{\otimes}}F)\to\mathscr{L}(G)$. We define $\vec{S}\underset{B}{U}\vec{T}$ to be the element $(I_{\mathscr{L}}\varepsilon\tilde{B})((U_{\vec{S}}\otimes I_F)(\vec{T}))$. We shall now show that $\underset{B}{U} : \mathscr{H}(E)\times\mathscr{K}(F)\to\mathscr{L}(G)$, which is trivially bilinear, has the fol- 74
lowing properties:

(i) If $S\in\mathscr{H}, T\in\mathscr{K}, \vec{e}\in E$ and $\vec{f}\in F, S\vec{e}\underset{B}{U}T\vec{f} = (SUT)$

$B(\vec{e}, \vec{f})$:

(ii) If $\vec{S}$ remains in a bounded set in $\mathscr{H}(E)$, and $\vec{T} \to 0$ in $\mathscr{K}(F)$, then $\vec{S} \underset{B}{U} \vec{T}$ tends to 0 uniformly in $\mathscr{L}(G)$; and

(iii) If $\vec{S} \to 0$ in $\mathscr{H}(E)$ and $\vec{T}$ is fixed in $\mathscr{K}(F)$, then $\vec{S} \underset{B}{U} \vec{T} \to 0$ in $\mathscr{L}(G)$.

We remark that proving (ii) and (iii) is more than proving separate continuity.

**Proof of (i).** Let $\vec{T} = T\vec{f}, T \in \mathscr{K}$ and $\vec{f} \in F$. Then for any $\vec{S} \in \mathscr{H}(E), (U_{\vec{S}} \otimes I_F)(\vec{T})$ is nothing but $(\vec{S}UT) \otimes \vec{f}$. If $\vec{S} = S\vec{e}$, we see that

$$\vec{S}UT = (S\vec{e})UT = (SUT)\vec{e}.$$

Hence $\quad (U_{\vec{S}} \otimes I_F)(\vec{T}) = (SUT)\vec{e} \otimes \vec{f} \quad$ if $\quad \vec{S} = S\vec{e}, \vec{T} = T\vec{f}$.

$$\text{Hence } (I_{\mathscr{L}} \varepsilon \tilde{B})((U_{\vec{S}} \otimes I_F)(\vec{T})) = (SUT)\tilde{B}(\vec{e} \otimes \vec{f})$$
$$= (SUT)B(\vec{e}, \vec{f}).$$

**Proof of (ii).** Let $S$ remain in a bounded set of $\mathscr{H}(E)$ and $\vec{T} \to 0$ in $\mathscr{K}(F)$. We know that if $T \to 0$ in $\mathscr{K}$ and $\vec{S}$ remains in a bounded set of $\mathscr{H}(E)$, $\vec{S}UT \to 0$ in $\mathscr{L}(E)$ uniformly, that is to say $U_{\vec{S}} : \mathscr{K} \to \mathscr{L}(E)$ is an equicontinuous set of linear maps when $\vec{S}$ lies in a bounded set of $\mathscr{H}(E)$. Hence the set of operators $U_{\vec{S}} \otimes I_F : \mathscr{K} \underset{\pi}{\hat{\otimes}} F \to \mathscr{L}(E) \underset{\pi}{\hat{\otimes}} F = \mathscr{L}(E \underset{\pi}{\hat{\otimes}} F)$, $\vec{S}$ in a bounded set of $\mathscr{H}(E)$, is an equicontinuous set. Hence if $\vec{T} \to 0$ in $\mathscr{K}(F), (U_{\vec{S}} \otimes I_F)(\vec{T}) \to 0$ uniformly. $I_{\mathscr{L}} \varepsilon \tilde{B}$ is a fixed, continuous, linear map of $\mathscr{L}(E \underset{\pi}{\hat{\otimes}} F)$ in $\mathscr{L}(G)$ and hence $(I_{\mathscr{L}} \varepsilon \tilde{B})((U_{\vec{S}} \otimes I_F)(\vec{T})) \to 0$ uniformly in $\mathscr{L}(G)$ when $\vec{S}$ remains in a bounded set of $\mathscr{H}(E)$.

75 **Proof of (iii).** First we show that if $T$ remains in a bounded set $A$ of

$\mathscr{K}$ and $\vec{f}$ remains in a fixed ball, say $\| \vec{v} \| \leq \alpha$ of $F$ and if $\vec{S} \to 0$ in $\mathscr{H}(E)$, $\vec{S} \underset{B}{U} T \vec{f}$ tends to 0 in $\mathscr{L}(G)$ uniformly. If we show that $(U_{\vec{S}} \otimes I_F) T \vec{f} \to 0$ in $\mathscr{L}(E \underset{\pi}{\hat{\otimes}} F)$ uniformly, the above result will follow. Now $(U_{\vec{S}} \otimes I_F)(T \vec{f}) = (\vec{S} UT) \otimes \vec{f}$. $\vec{S} UT \to 0$ in $\mathscr{L}(E)$ uniformly if $\vec{S} \to 0$ in $\mathscr{H}(E)$ because $T$ remains bounded in $\mathscr{K}$ (theorem 7.1), and since $\vec{f}$ remains in a bounded sets, $(\vec{S} UT) \otimes \vec{f} \to 0$ uniformly as $\vec{S} \to 0$ in $\mathscr{H}(E)$.

Now suppose that $\vec{T}$ lies in the closure of the convex, stable, envelope $\Gamma(A, B_\alpha)$ of the product of a bounded set $A$ in $\mathscr{K}$ and of a bounded set $B_\alpha$ in $F$, the closure being taken in the sense of $\mathscr{K}(F)$. Let $\vec{S} \to 0$ in $\mathscr{H}(E)$. We shall show that $\vec{S} \underset{B}{U} \vec{T} \to 0$. Since, convex, stable, closed neighbourhoods form a fundamental system of neighbourhoods of 0 in $\mathscr{L}(G)$, it is sufficient to prove that $W$ being a closed disc of $\mathscr{L}(G)$ which is a neighbourhood of 0, there exists a neighbourhood $N$ of 0 in $\mathscr{H}(E)$ such that $\vec{S} \in N$ implies that $\vec{S} \underset{B}{U} \vec{T} \in W$. Any $\vec{T} \in \overline{\Gamma(A, B_\alpha)}$ can be got as $\lim \vec{T}_j$, $\vec{T}_j$ being a filter of sets in $\Gamma(A, B_\alpha)$. To any neighbourhood $W$ of 0 in $\mathscr{L}(G)$ there corresponds a neighbourhood $N$ of 0 in $\mathscr{H}(E)$ such that $\vec{S} \underset{B}{U} \vec{T} \in W$ for every $\vec{S} \in N$ and $\vec{T} \in A \otimes B_\alpha$. Since $W$ is convex, stable we have $\vec{S} \underset{B}{U} \vec{T}_j \in W$ for any $\vec{T}_j \in \Gamma(A, B_\alpha)$. Now if $\vec{T} = \lim, \vec{T}_j$ with $\vec{T}_j \in \Gamma(A, B_\alpha)$, we have $\vec{S} \underset{B}{U} \vec{T}_j \to \vec{S} \underset{B}{U} \vec{T}$. Hence $\vec{S} \underset{B}{U} \vec{T} \in \overline{W}$. But since $W$ is closed, $W = \overline{W}$. Hence $\vec{S} \underset{B}{U} \vec{T} \in W$.

Now, suppose $\vec{T}$ is an element of $\mathscr{K}(F)$. Since $\mathscr{K}$ is nuclear, bounded sets in $\mathscr{K}$ are relatively compact. Hence $\mathscr{K}'_\delta = \mathscr{K}'_c$. We have assumed that $\mathscr{K}'_\delta$ is nuclear. $\mathscr{K}(F) = \mathscr{L}_\varepsilon(\mathscr{K}'_c, F) = \mathscr{L}_\varepsilon(\mathscr{K}'_\delta, F)$. If **76**
$\vec{T} \in \mathscr{K}(F)$, since $\mathscr{K}'_\delta$ is nuclear, and $F$ a Banach space, we see that $\vec{T}$ is a nuclear map. Hence $\vec{T} = \sum_n \lambda_n k_n \otimes \vec{f}_n, k_n \in \mathscr{K} = (\mathscr{K}'_\delta)'$ with $\sum_n |\lambda_n| < \infty$, $k_n$ being an equicontinuous set of $\mathscr{K}$ considered as the dual of $\mathscr{K}'_c$ and $\vec{f}_n$ lying in a bounded set of $F$. This just means that $\vec{T} \in$

$\overline{\Gamma(A, B)}$ with $A$ and $B$ bounded sets in $\mathscr{K}$ and $F$ respectively, closure in $\mathscr{L}(\mathscr{K}'_\delta, F) = \mathscr{K}(F)$. Hence $\vec{S}\, U\, \vec{T} \to 0$ as $\vec{S} \to 0$ in $\mathscr{H}(E)$.

Thus we have seen that the bilinear map $\underset{B}{U} : \mathscr{H}(E) \times \mathscr{K}(F) \to \mathscr{L}(G)$ satisfies (i), (ii) and (iii). In particular, $\underset{B}{U}$ is separately continuous. Now, let $\underset{B}{U'}$ be defined by $\vec{S}\, \underset{B}{U'}\, \vec{T} = (I_{\mathscr{L}} \varepsilon \tilde{B})(I_E \otimes U_{\vec{T}})(\vec{S})$. $\underset{B}{U'}$ can be proved to satisfy

$(i)'$ $S\, \vec{e}\, \underset{B}{U'}\, T\, \vec{f} = (SUT)B(\vec{e}, \vec{f})$ when $S \in \mathscr{H}, T \in \mathscr{K}, \vec{e} \in E$ and $\vec{f} \in F$.

$(ii)'$ When $\vec{T}$ remains in a bounded subset of $\mathscr{K}(F)$ and $\vec{S} \to 0$ in $\mathscr{H}(E)$, $\vec{S}\, \underset{B}{U'}\, \vec{T} \to 0$ uniformly in $\mathscr{L}(G)$.

$(iii)'$ If $\vec{T} \to 0$ in $\mathscr{K}(F)$ and $\vec{S}$ is a fixed element of $\mathscr{H}(E)$, the $\vec{S}\, \underset{B}{U'}\, \vec{T} \to 0$ in $\mathscr{L}(G)$.

In particular, $\underset{B}{U'}$ is separately continuous.

Thus $\underset{B}{U}$ and $\underset{B}{U'}$ are separately continuous bilinear maps satisfying the condition (1) of the theorem. But from the uniqueness of such a map which we have proved already, it follows that $\underset{B}{U} = \underset{B}{U'}$.

Now, if we combine (ii), (iii), $(ii)'$, and $(iii)'$ we see that $\underset{B}{U} = \underset{B}{U'}$ satisfies also the conditions (3) and (4) stipulated in the theorem. □

77 **A particular case of the multiplicative product.**

Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{L}$ be three nuclear spaces with nuclear strong duals. Let $E$ be a Banach algebra. If $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ is a bilinear map hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$, by taking for $B$ the multiplication in $E$ we get bilinear map $\underset{B}{U} : \mathscr{H}(E) \times \mathscr{K}(E) \to \mathscr{L}(E)$ which is hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{K}(E)$.

# Lecture 15

# Operations on vector valued distributions (contd.)

Suppose $\mathscr{H}$ is a nuclear, barrelled space with a nuclear strong dual $\mathscr{H}'_\delta$. **78**
Let $\mathscr{H}$ and $\mathscr{H}'_\delta$ be assumed to be complete. Let $E, F$ and $G$ be three Banach spaces and $B$ a continuous bilinear map of $E \times F$ in $G$. Then we can get a bilinear map of $\mathscr{H}(E) \times \mathscr{H}'(F)$ in $G$ hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{H}'(F)$. Since $\mathscr{H}$ is barrelled, the scalar product defining the duality between $\mathscr{H}$ and $\mathscr{H}'$ is a bilinear map of $\mathscr{H} \times \mathscr{H}'$ into $C$ hypocontinuous with respect to the bounded sets of $\mathscr{H}$ and $\mathscr{H}'_\delta$. Let us denote the scalar product by the symbol '.'. $B$ being a continuous bilinear map of $E \times F$ in $G$, we get, as explained in the previous lecture, a bilinear map $\underset{B}{.} : \mathscr{H}(E) \times \mathscr{H}'(F) \to C(G) = G$, hypocontinuous with respect to the bounded sets of $\mathscr{H}(E)$ and $\mathscr{H}'(F)$.

Now, $\mathscr{D}$ and $\mathscr{S}$ are barrelled spaces. Hence if $B : E \times F \to G$ is a continuous bilinear map, we get a separately continuous bilinear map in each of the following cases, which is further hypocontinuous with respect to the bounded sets

1) $\mathscr{D}(E) \times \mathscr{D}'(F) \to G$

2) $\mathscr{S}(E) \times \mathscr{S}'(F) \to G$

If $\vec{\varphi} \in \mathscr{D}(E)$ and $\vec{T} \in \mathscr{D}'(F)$ we denote by $\vec{T} \underset{B}{\cdot} \vec{\varphi}$ the image in $G$ of the element $(\vec{\varphi}, \vec{T})$ by the bilinear map in case (1). We use similar notation in case (2) also.

79 For the Banach spaces $E, F$ and $G$ we can take $E, E'$ and $C$ and for $B$ the canonical bilinear form on $E \times E'$. Then when $\mathscr{H}$ and $\mathscr{H}'$ are complete nuclear spaces with $\mathscr{H}$ barrelled, we get a bilinear form on $\mathscr{H}(E) \times \mathscr{H}'(E)$ hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{H}'(E')$. If $\vec{\varphi} \in \mathscr{H}(E)$ and $\vec{T} \in \mathscr{H}'(E')$ the image of $(\vec{\varphi}, \vec{T})$ in $C$ by the above bilinear map is denoted by $\langle \vec{T}., \vec{\varphi} \rangle$.

Since $\mathscr{H}$ is nuclear and $\mathscr{H}$ and $E$ are complete, we have $\mathscr{H}(E) = \mathscr{H} \hat{\underset{\pi}{\otimes}} E$. Let $u \in (\mathscr{H}(E))'$. $u$ is a linear form on $\mathscr{H} \hat{\underset{\pi}{\otimes}} E$ which is continuous. Now $\mathscr{H} \otimes E$ is dense in $\mathscr{H} \hat{\underset{\pi}{\otimes}} E$, therefore the space of continuous linear forms on $\mathscr{H} \hat{\underset{\pi}{\otimes}} E$ is the same space of continuous linear forms on $\mathscr{H} \underset{\pi}{\otimes} E$. The definition of the $\pi$-topology hence gives $(\mathscr{H}(E))' =$ space of continuous bilinear forms on $\mathscr{H} \times E$ algebraically. Then $u$ can be considered also as a continuous bilinear form on $\mathscr{H} \times E$. For any fixed $h \in \mathscr{H}, \vec{e} \to u(h, \vec{e})$ is a continuous linear map of $E$ in $C$; hence it defines an element $u_h$ of $E'$. The mapping $h \to u_h$ of $\mathscr{H}$ in $E'$, with the structure of a Banach space on it, is a continuous linear map. Conversely, suppose that $\mu : \mathscr{H} \to E'$ is a continuous linear map. Then $\tilde{\mu}$ defined by $\tilde{\mu}(h, \vec{e}) = \langle \mu(h), \vec{e} \rangle$ for every $\vec{e} \in E$ and $h \in \mathscr{H}$ is a continuous bilinear form on $\mathscr{H} \times E$. Hence we have $(\mathscr{H}(E))' = \mathscr{L}(\mathscr{H}, E')$ algebraically. Now, consider the space $\mathscr{H}'(E')$. We have seen that we can define one and only one bilinear form on $\mathscr{H}(E) \times \mathscr{H}'(E')$ hypocontinuous with respect to the bounded sets satisfying

$$\langle T\vec{e}., H'\overleftarrow{e}' \rangle = \langle T.H' \rangle_{\mathscr{H}, \mathscr{H}'} \langle \vec{e}, \overleftarrow{e}' \rangle_{E,E'}.$$

If $\vec{H}' \in (\mathscr{H}(E))'$ and $\vec{\varphi} \in \mathscr{H}(E)$ the scalar product defining the duality between $\mathscr{H}(E)$ and $(\mathscr{H}(E))'$ satisfies

$$\langle H\vec{e}, H'\overleftarrow{e}' \rangle = \langle H, H' \rangle_{\mathscr{H}, \mathscr{H}'} \langle \vec{e}, \overleftarrow{e}' \rangle_{E,E'}.$$

80 Also $(\mathscr{H}(E))' = \mathscr{L}(\mathscr{H}, E')$ and $\mathscr{H}'(E') = \mathscr{L}((\mathscr{H}')'_c, E')$. $(\mathscr{H}')'_c$ is algebraically the same as $\mathscr{H}$ but, in general, has a topology finer than that of $\mathscr{H}$. In the case of barrelled spaces $(\mathscr{H}'_c)' = \mathscr{H}$ topologically.It

follows from Theorem 14.1 that the scalar product defining the duality between $\mathscr{H}(E)$ and $(\mathscr{H}(E))'$ is the same as the bilinear form that is defined on $\mathscr{H}(E) \times \mathscr{H}'(E')$ by the process described in Theorem 14.1

**The convolution of two vector valued distributions.**

Let $E, F$ and $G$ be three Banach spaces and $B : E \times F \to G$ a continuous bilinear map. Let $\vec{S} \in \mathscr{S}'(E)$ and $\vec{T} \in \mathscr{O}'_c(F)$. The convolution operation between the elements of $\mathscr{S}'$ and the elements of $\mathscr{O}'_c$ satisfies the conditions stipulated in Theorem 14.1 and the spaces $\mathscr{S}', \mathscr{S}, \mathscr{O}'_c, \mathscr{O}_c$ are all nuclear complete spaces. (See: Memoirs of the Amer. Math. Soc., No. 16, Products Tensoriels Topologique et Espaces Nucleaires, by A. Grothendieck). Hence, as explained in Theorem 14.1 we can define a bilinear map $\underset{B}{*} : \mathscr{S}'(E) \times \mathscr{O}'_c(F) \to \mathscr{S}'(G)$ which is hypocontinuous with respect to the bounded subsets of $\mathscr{S}'(E)$ and $\mathscr{O}'_c(F)$. We call $\vec{S} \underset{B}{*} \vec{T}$ the convolution of $\vec{S}$ and $\vec{T}$ under $B$.

We know that $S \in \mathscr{D}'_+$ and $T \in \mathscr{D}'_+$ implies $S * T \in \mathscr{D}'_+$, where $\mathscr{D}'_+$ is the space of distributions $\in \mathscr{D}'$ with supports bounded on the left (it is the dual of $\mathscr{D}_-$). The map $(S, T) \in S * T$ of $\mathscr{D}'_+ \times \mathscr{D}'_+$ in $\mathscr{D}'_+$ also satisfies the conditions stipulated in Theorem 14.1. Hence we get a bilinear map $\underset{B}{*} : \mathscr{D}'_+(E) \times \mathscr{D}'_+(F) \to \mathscr{D}'_+(G)$ hypocontinuous with respect to the bounded sets. If $E$ is a Banach Algebra, by taking $E =$ **81**
$F = G = E$ and $B =$ the multiplication in $E$, we get a bilinear map $\mathscr{D}'_+(E) \times \mathscr{D}'_+(E) \to \mathscr{D}'_+(E)$ hypocontinuous with respect to the bounded subsets of $\mathscr{D}'_+(E)$ and $\mathscr{D}'_+(E)$.

Now suppose $\vec{S} \in \mathscr{S}'(E)$ and $\vec{T} \in \mathscr{O}'_c(F)$. Then $\vec{S} \underset{B}{*} \vec{T} \in \mathscr{S}'(G)$, where $B : E \times F \to G$ is a bilinear continuous map and $E, F$ and $G$ are three Banach spaces. The Fourier transforms of $\vec{S}$ and $\vec{T}$ satisfy $\mathscr{H}(\vec{S}) \in \mathscr{S}'(E)$ and $\mathscr{H}(\vec{T}) \in \mathscr{O}_M(F)$. Since $\mathscr{O}_M$ and $\mathscr{O}'_M$ are nuclear and complete (refer to: Memories of the Amer. Math. Soc., No16, Espaces Nucleaire, by A. Grothendieck), using the product between the elements of $\mathscr{S}'$ and the elements of $\mathscr{O}_M$ and using the bilinear map $B$ we define a product $(B)$ between the elements of $\mathscr{S}'(E)$ and $\mathscr{O}_M(F)$ as explained in Theorem 14.1. Then we have $\mathscr{H}(\vec{S} \underset{B}{*} \vec{T}) =$

$\mathscr{H}(\overrightarrow{S})(B)\mathscr{H}(\overrightarrow{T})$. This follows from the separate continuity of the operations on the two sides and from the equality
$\mathscr{H}(S\overrightarrow{e} * T\overrightarrow{f}) = \mathscr{H}(S\overrightarrow{e})_{(B)}\mathscr{H}(Tf)$ for $S \in \mathscr{S}', T \in \mathscr{O}'_c, \overrightarrow{e} \in E$ and $\overrightarrow{f} \in F$ and from the fact that $\mathscr{S}' \otimes E$ and $\mathscr{O}'_c \otimes F$ are dense in $\mathscr{S}'(E)$ and $\mathscr{O}'_c(F)$ respectively.

In the case in which $\overrightarrow{T} \in \mathscr{D}'(F)$ is a continuous function $\overrightarrow{g}$ with values in $F$, $\overrightarrow{\varphi} \underset{B}{.} \overrightarrow{T}$ for every $\overrightarrow{\varphi} \in \mathscr{D}(E)$ can be expressed as an integral. We will show that

$$\overrightarrow{\varphi} \underset{B}{.} \overrightarrow{T} = \int\limits_{R^n} B(\overrightarrow{\varphi}(x), \overrightarrow{g}(x)dx.$$

we know that $\underset{B}{.} : \mathscr{D}'(F)\times\mathscr{D}(E) \to G$ is a separately continuous function. Hence $\underset{B}{.} : \mathcal{E}^\circ(F) \times \mathscr{D}(E) \to G$ is also separately continuous. Also the mapping

$$(\overrightarrow{g}, \overrightarrow{\varphi}) \to \int\limits_{R^n} B(\overrightarrow{\varphi}(x), \overrightarrow{g}(x)dx$$

82 is a separately continuous map of $\mathcal{E}^\circ(F) \times \mathscr{D}(E) \to G$. Also for every $\varphi \in \mathscr{D}, g \in \mathcal{E}^\circ, \overrightarrow{e} \in E$ and $\overrightarrow{f} \in F$, we have

$$\begin{aligned}
\int\limits_{R^n} B(\varphi\overrightarrow{e}(x), g\overrightarrow{f}(x))dx &= \int\limits_{R^n} B(\varphi(x)\overrightarrow{e}, g(x)\overrightarrow{f})dx \\
&= \int\limits_{R^n} \varphi(x)g(x)B(\overrightarrow{e}, \overrightarrow{f})dx \\
&= \left\{\int\limits_{R^n} \varphi(x)g(x)dx\right\}.B(\overrightarrow{e}, \overrightarrow{f}).
\end{aligned}$$

The mappings

$$\underset{B}{.} : \mathcal{E}^\circ(F) \times \mathscr{D}(E) \to G \text{ and } (\overrightarrow{g}, \overrightarrow{\varphi}) \to \int\limits_{R^n} B(\overrightarrow{\varphi}(x), \overrightarrow{g}(x))dx$$

are separately continuous and agree on the decomposed elements. Since $\mathcal{E}^\circ$ and $\mathscr{D}$ have the approximation property, we see that the two maps are identical. Also if $f \in \mathscr{D}^\circ$ and $g \in \mathcal{E}^\circ$ the convolution $f * g$ is an element of $\mathcal{E}^\circ$ and is given by $\int\limits_{R^n} f(x-\mathscr{E})g(\mathscr{E})d\mathscr{E}$. Now suppose $\vec{f} \in \mathscr{D}^\circ(E)$ and $\vec{g} \in \mathcal{E}^\circ(F)$. Then we shall prove that $\vec{f} \underset{B}{*} \vec{g} \in \mathcal{E}^\circ(G)$ and is given by the formula $\vec{f} \underset{B}{*} \vec{g}(x) = \int\limits_{R^n} B(\vec{f}(x-\mathscr{E}), \vec{g}(\mathscr{E}))d\mathscr{E}$. The maps $\underset{B}{*} : \mathscr{D}^\circ(E) \times \mathcal{E}^\circ(F) \to \mathscr{D}'(G)$ and $(\vec{f}, \vec{g}) \to \int\limits_{R^n} B(f(x-\mathscr{E}), g(\mathscr{E}))d\mathscr{E}$ of $\mathscr{D}^\circ(E) \times \mathcal{E}^\circ(F) \to \mathcal{E}^\circ(G)$ are separately continuous and agree on the decomposed vectors, for if $f \in \mathscr{D}^\circ, g \in \mathcal{E}^\circ$ and $\vec{e} \in E$ and $\vec{v} \in F$. We have

$$
\begin{aligned}
f\vec{e} \underset{B}{*} g\vec{v}(x) &= f * g(x) B(\vec{e}, \vec{v}) \\
&= \left( \int f(x-\mathscr{E})g(\mathscr{E})d\mathscr{E} \right) B(\vec{e}, \vec{v}) \\
&= \int B(f(x-\mathscr{E})\vec{e}, g(\mathscr{E})\vec{v})d\mathscr{E} \\
&= \int B(f\vec{e}(x-\mathscr{E}), g\vec{v}(\mathscr{E}))d\mathscr{E}.
\end{aligned}
$$

Now since $\mathscr{D}^\circ$ and $\mathcal{E}^\circ$ have the approximation property $\mathscr{D}^\circ \otimes E$ and $\mathcal{E}^\circ \otimes F$ are dense in $\mathscr{D}^\circ(E)$ and $\mathcal{E}^\circ(F)$, hence we deduce the conclusion.

# Lecture 16

# Weak boundary value problems

Let $E, F$ and $G$ be three Banach spaces and let $B : E \times F \to G$ be a continuous bilinear map. 83

**Proposition 16.1.** *Let $\vec{S} \in \mathscr{D}'_+(E)$ and $\vec{T} \in \mathscr{D}'_+(F)$ be two distributions such that $\vec{S} \underset{Rl\ p>a}{\sqsupset} \vec{S}(p)$ and $\vec{T} \underset{Rl\ p>a}{\sqsupset} \vec{T}(p)$. Then $\vec{S} \underset{B}{*} \vec{T}$ has a Laplace transform for $Rlp > a$ and the Laplace transform is precisely $B(\vec{S}(p), \vec{T}(p))$.*

*Proof.* For $Rl\ p > a, e^{-pt}\vec{S} \in \mathscr{S}'(E)$ and $e^{-pt}\vec{T} \in \mathscr{S}'(F)$. In fact, we know that more is true. For $Rl\ p > a, e^{-pt}\vec{S} \in \mathscr{O}'_c(E)$ and $e^{-pt}\vec{T} \in \mathscr{O}'_c(F)$. We have $e^{-pt}\vec{S} \underset{B}{*} e^{-pt}\vec{T} = e^{-pt}(\vec{S} \underset{B}{*} \vec{T})$. The convolutions $\mathscr{D}'_+ \times \mathscr{D}'_+ \to \mathscr{D}'_+$ and $\mathscr{O}'_c \times \mathscr{O}'_c \to \mathscr{O}'_c$ coincide on the elements on which both are defined. Hence $e^{-pt}(\vec{S} \underset{B}{*} \vec{T}) = e^{-pt}\vec{S} \underset{B}{*} e^{-pt}\vec{T} \in \mathscr{O}'_c(G)$ for $Rl\ p > a$. Hence $\vec{S} \underset{B}{*} \vec{T}$ has a Laplace transform for $Rl\ p > a$ and the Laplace transform is the function $B(\vec{S}(p), \vec{T}(p))$.

We shall consider an application of the above theory to a problem in differential equations.

Let $Q'$ be a Banach space and $N$ a Hilbert space with $N \subset Q'$ with a continuous injection. Let $A : N \to Q'$ be a continuous linear operator.

Let $\vec{f} \in \mathscr{D}'_+(Q')$. To find, if possible, $\vec{u} \in \mathscr{D}'_+(N)$ such that $(\frac{d}{dt}+A)\vec{u} = \vec{f}$. This problem can be restated as follows: To find $\vec{u} \in \mathscr{D}'_+(N)$ such that

$$(\delta'_t I + \delta_t A) * \vec{u} = \vec{f}$$

where $I : N \to Q'$ is the injection. We have $\delta'_t I$ and $\delta_t A$ in $\mathscr{D}'_+(\mathscr{L}(N, Q'))$. □

84 We use the properties of the convolution operation to get new operators.

Let $\mathscr{H}, \mathscr{K}$ and $\mathscr{L}$ be three nuclear complete spaces with nuclear, complete strong duals. Let $B : E \times F \to G$ be a continuous bilinear map, $E, F$ and $G$ being three Banach spaces. Let $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ be a bilinear map hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$. As explained in Theorem 14.1 we can define a bilinear map $\underset{B}{U} : \mathscr{H}(E) \times \mathscr{K}(F) \to \mathscr{L}(G)$ hypocontinuous with respect to the bounded subsets.

Let $S\vec{e} \in \mathscr{H}(E)$ with $S \in \mathscr{H}, \vec{e} \in E$ and $\vec{T} \in \mathscr{K}(F)$. We shall find an expression for $S\vec{e}\underset{B}{U}\vec{T}$. For each $\vec{e} \in E$, let $B_{\vec{e}} : F \to G$ be the continuous linear map $B_{\vec{e}}(\vec{f}) = B(\vec{e}, \vec{f})$. The continuous linear map $B_{\vec{e}} : F \to G$ allows us to define a continuous linear map $I_{\mathscr{L}}\varepsilon B_{\vec{e}}$ of $\mathscr{L}(G)$. This linear map also we denote by $B_{\vec{e}}$. Using the bilinear map $U : \mathscr{H} \times \mathscr{K} \to \mathscr{L}$, hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$, we can define a bilinear map $\tilde{U} : \mathscr{H} \times \mathscr{K}(F) \to \mathscr{L}(F)$ hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}(F)$ as explained in Theorem 7.1. We have then $S\vec{e}\underset{B}{U}\vec{T} = B_{\vec{e}}(S\tilde{U}\vec{T})$. In fact, if $\vec{T}$ is of the form $T\vec{f}, T \in \mathscr{K}, \vec{f} \in F$ we have

$$\begin{aligned} B_{\vec{e}}(S\tilde{U}\vec{T}) &= B_{\vec{e}}((SUT)\vec{f}) = (I_{\mathscr{L}}\varepsilon B_{\vec{e}})((SUT)\vec{f}) \\ &= (SUT).B_{\vec{e}}(\vec{f}) = (SUT)B(\vec{e}, \vec{f}), \end{aligned}$$

and $S\vec{e}\underset{B}{U}T\vec{f} = SUTB(\vec{e}, \vec{f})$. Hence from the approximation property for $\mathscr{K}$ it follows that $S\vec{e}\underset{B}{U}\vec{T} = B_{\vec{e}}(S\tilde{U}T)$ for any $\vec{T} \in \mathscr{K}(F)$.

$I$ and $A$ are two fixed elements of the vector space $\mathscr{L}(N, Q')$. $\delta'_t I$
85 and $\delta_t A$ are distributions with values in $\mathscr{L}(N, Q')$. $\delta_t A * \vec{u}$ is, by the

formula $S \overrightarrow{e} \underset{B}{U} \overrightarrow{T} = B_{\overrightarrow{e}}(S\tilde{U}\overrightarrow{T})$, the same as $\underset{A}{B}(\delta_t * \overrightarrow{u})$ where $\underset{A}{B}$ is the mapping $I_{\mathscr{D}'_+} \varepsilon \Gamma_A, \Gamma_A : N \to Q'$ given by $\Gamma_A.n = An$ for every $n \in N$. Hence $\underset{A}{B}(\delta_t * \overrightarrow{u}) = A\overrightarrow{u}$. Similarly $\delta'_t I * \overrightarrow{u} = \frac{d}{dt}\overrightarrow{u}$. Thus we see that the equation $(\frac{d}{dt} + A)\overrightarrow{u} = \overrightarrow{f}$ is the same as $(\delta'_t I + \delta_t A) * \overrightarrow{u} = \overrightarrow{f}$. What we do first is to look for $\overrightarrow{G} \in \mathscr{D}'_+(t, \mathscr{L}(Q', N)$ such that

$$(\delta'_t I + \delta_t A) * \overrightarrow{G} = \delta_t I_Q,$$
$$\text{and } \overrightarrow{G} * (\delta'_t I + \delta_t A) = \delta_t I_N,$$

where $I_{Q'}$ and $I_N$ are the identity mappings of $Q'$ and $N$. Suppose such a $\overrightarrow{G}$ has been found out. Then our contention is: $\overrightarrow{u} = \overrightarrow{G} * \overrightarrow{f}$ is the only solution of the equation $(\frac{d}{dt} + A)\overrightarrow{u} = \overrightarrow{f}$. To prove this fact, we use the following associativity property of the convolution. Let $*$ denote the convolution of $\mathscr{D}'_+ \times \mathscr{D}'_+ \to \mathscr{D}'_+$. We have for $U, V$ and $W \in \mathscr{D}'_+$ the following relation: $(U * V) * W = U * (V * W)$. Let $B_1 : L \times M \to P$ and $B_2 : P \times N \to Z$ be bilinear continuous maps with $L, M, P, N$ and $Z$ Banach spaces. Let $\alpha_1 : L \times Q \to Z$ and $\alpha_2 : M \times N \to Q$ be bilinear continuous maps with $L, M, N, Q, Z$ Banach spaces. Let $B_{1(l)} : M \to P$ be the linear map $B_{1(l)}(m) = B_1(l, m)$. Let $B_{2, B_{1(l)}(m)} : N \to Z$ be the linear map $B_{2, B_{1(l)}(m)}(n) = B_2(B_{1(l)}(m), n)$. Let $\mu : L \times M \times N \to Z$ be the trilinear map defined by $\mu(l, m, n) = B_{2, B_{1(l)}(m)}(n)$. Similarly we can associate with $\alpha_2$ and $\alpha_1$ a trilinear continuous map, say $\gamma : L \times M \times N \to Z$. If we assume that $\mu = \nu$ we have the following equality: For any **86**
$\overrightarrow{S} \in \mathscr{D}'_+(L)$, $\overrightarrow{T} \in \mathscr{D}'_+(M)$ and $\overrightarrow{U} \in \mathscr{D}'_+(N)$,

$$(\overrightarrow{S} \underset{B_1}{*} \overrightarrow{T}) \underset{B_2}{*} \overrightarrow{U} = \overrightarrow{S} \underset{\alpha_1}{*} (\overrightarrow{T} \underset{\alpha_2}{*} \overrightarrow{U}).$$

The proof is, in fact, trivial. The convolution $\mathscr{D}'_+ \times \mathscr{D}'_+ \to \mathscr{D}'_+$ satisfies the associativity. Hence from the fact that $\mu = \nu$, we get

$$(S\overrightarrow{e} \underset{B_1}{*} T\overrightarrow{f}) \underset{B_2}{*} U\overrightarrow{g} = (S * T * U)\mu(\overrightarrow{e}, \overrightarrow{f}, \overrightarrow{g})$$
$$= (S * T * U)\nu(\overrightarrow{e}, \overrightarrow{f}, \overrightarrow{g})$$

$$\text{and } S\vec{e} \underset{\alpha_1}{*} (T\vec{f} \underset{\alpha_2}{*} U\vec{g}) = (S * T * U)\nu(\vec{e}, \vec{f}, \vec{g})$$

for any $S, T, U \in \mathscr{D}'_+$ and $\vec{e} \in L$, $\vec{f} \in M$ and $\vec{g} \in N$.
Now, since $\mathscr{D}'_+$ has the approximation property, we get the required associativity formula. Now, $\vec{u} = \vec{G} * \vec{f}$ is a solution of $(\delta'_t I + \delta_t A) * \vec{u} = \vec{f}$. For $(\delta'_t I + \delta_t A) * \vec{G} * \vec{f} = \delta_t I_{Q'} * \vec{f} = I_{Q'}(\delta_t * \vec{f}) = I\vec{f} = \vec{f}$. Hence $\vec{u} = \vec{G} * \vec{f}$ is a solution. Suppose $\vec{v}$ is any solution of

$$(\delta'_t I + \delta_t A) * \vec{u} = \vec{f}.$$

We have
$$\vec{G} * \{(\delta'_t I + \delta_t A) * \vec{v}\} = \vec{G} * \vec{f}.$$

$$\begin{aligned}
\text{But } \vec{G} * \{(\delta'_t I + \delta_t A) * \vec{v}\} &= \{\vec{G} * (\delta'_t I + \delta_t A)\} * \vec{v} \\
&= \delta_t I_N * \vec{v} = v.
\end{aligned}$$

$$\text{Hence } \vec{v} = \vec{G} * \vec{f}.$$

Thus $\vec{G} * \vec{f}$ is the only solution of $(\delta'_t I + \delta_t A) * \vec{u} = \vec{f}$. In applying
**87** the associativity formula, we should take note of the following fact: The obvious bilinear maps $\mathscr{L}(N, Q') \times N \to Q'$, $\mathscr{L}(Q', N) \times Q' \to N$ and $\mathscr{L}(Q', N) \times \mathscr{L}(N, Q') \to \mathscr{L}(N, N)$ and $\mathscr{L}(N, N) \times N \to N$ satisfy the condition which enables us to conclude that the trilinear maps $\mu$ and $\nu$ corresponding to these are the same.

Hence the problem $(\frac{d}{dt} + A)\vec{u} = \vec{f}$ will have one and only one solution $\vec{u} \in \mathscr{D}'_t(t, N)$, if we can find a $\vec{G} \in \mathscr{D}'_+(t, \mathscr{L}(Q', N))$ such that

$$\begin{aligned}
(\delta'_t I + \delta_t A) * \vec{G} &= \delta_t I_{Q'} \\
\text{and } \vec{G} * (\delta'_t I + \delta_t A) &= \delta_t I_N.
\end{aligned}$$

We, in fact, look for a $\vec{G}$ having a Laplace transform. If at all such a $\vec{G}$ exists, it will satisfy

$$(pI + A)\vec{G}(p) = I_{Q'} \quad \text{and} \quad \vec{G}(p)(pI + A) = I_N.$$

Hence, if only we assume that for $Rl\ p > \mathscr{E}_\circ$ ($\mathscr{E}_\circ$ some real number), the operator $(p + A)$ is invertible and that the inverse is majorised uniformly in the half plane $Rl\ p \geq \mathscr{E}_\circ + \varepsilon$ for any $\varepsilon > 0)$ there exists a $\overrightarrow{G}(p)$ which is the Laplace transform of a unique $\overrightarrow{G} \in \mathscr{D}'_+(t, \mathscr{L}(Q', N))$ satisfying $(\delta'_t I + \delta_t A) * \overrightarrow{G} = \delta_t I_{Q'}, \overrightarrow{G} * (\delta'_t I + \delta_t A) = \delta_t I_N$. Then the problem $(\frac{d}{dt} + A)\overrightarrow{u} = \overrightarrow{f}$ has one and only one solution, namely

$$\overrightarrow{u} = \overrightarrow{G} * \overrightarrow{f}.$$

# Part II

# REPRESENTATIONS OF SEMI-GROUPS

# Lecture 17

# Representations of semi-groups

**Definition 17.1.** *A semi-group is a set $G_+$ with a binary associative law of composition, having an identity element. That is to say, there is defined a mapping $(x,y) \to x.y$ of $G_+ \times G_+$ in $G_+$ satisfying the following conditions:* 88

1) $x.(y.z) = (x.y).z$ *for every* $x, y, z \in G_+$.

2) *There exists an element $e \in G_+$ such that $e.x = x$ and $x.e = x$ for every $x \in G_+$.*

**Definition 17.2.** *A topological semi-group $G_+$ is a Hausdorff topological space with a semi-group structure such that the mapping $(x,y) \to x.y$ of $G_+ \times G_+ \to G_+$ is continuous. In this section, we deal only with locally compact semi-groups.*

**Definition 17.3.** *A measure $\mu$ on $G_+$ is said to be summable if $\int\limits_{G_+} |d\mu| < \infty$. One knows that, if $\mu$ is a summable measure, for any continuous bounded complex valued function $f$ the integral $\int\limits_{G_+} f(x)d\mu(x)$ can be defined. We agree to denote this by $\mu(f)$.*

**Definition 17.4.** *Let $\mu$ and $\nu$ be summable measures on $G_+$. The direct image of the measure $\mu \otimes \nu$ on $G_+ \times G_+$ by the mapping $(x,y) \to x.y$,* *which trivially exists, is defined to be the convolution of the measures $\mu$ and $\nu$ and is denoted by $\mu * \nu$.*

*If $\varphi$ is any continuous bounded function on $G_+$, the function $\psi : G_+ \times G_+ \to C$ defined by $\psi(s,t) = \varphi(st)$ is a continuous bounded function. Hence the integral*

$$\int_{G_+\times G_+} \psi d(\mu\otimes\nu) = \int_{G_+}\int_{G_+} \psi(s,t)d\mu(s)d\nu(t)$$

89 *has a meaning. We have the equality $\mu * \nu(\varphi) = \int_{G_+}\int_{G_+} \varphi(st)d\mu(s)d\nu(t)$.*

**Definition 17.5.** *For an integrable (or summable) measure $\mu$, $\int_{G_+} |d\mu|$ is defined to be the norm of $\mu$ and is denoted by $\| \mu \|$.*

*One knows that if $\mu$ and $\nu$ are summable measures on $G_+, \mu * \nu$ is also summable and that $\| \mu * \nu \| \leq \| \mu \| \| \nu \|$ (Refer to Elements de Mathematique, Integration, by N. Bourbaki).*

**Strict convergence.**

**Definition 17.6.** *A sequence of measures $\{\mu_j\}$ is said to strictly converge to $0$ if $\mu_j(\varphi) \to 0$ for every fixed $\varphi$, continuous with compact support and if there exists a positive real number $\varepsilon(K)$ corresponding to each compact set K such that $\int_{\complement K} |d\mu_j| \leq \varepsilon(K)$ independent of j, ($\complement K$ is the complement of $K$), with $\varepsilon(K) \to 0$ according as the 'filtrant set' of compact subsets of $G_+$ ordered by inclusion, that is to say, given any $\varepsilon > 0$ there exists a compact K such that for any compact set $\Gamma$ of $G_+$ with $\Gamma \supset K$ we have $\varepsilon(\Gamma) < \varepsilon$. In fact, it is sufficient if there exists a compact set K such that $\varepsilon(K) < \varepsilon$.*

**Lemma 17.1.** *If $\{\mu_j\}$ is a sequence of measures converging strictly to $0$, for any fixed continuous bounded function $\varphi, \mu_j(\varphi) \to 0$.*

*Proof.* Let $\alpha$ be any continuous function with $0 \leq \alpha(x) \leq 1$ for every $x \in G_+$ and with compact support. We have $\mu_j(\varphi) = \mu_j(\alpha\varphi) + \mu_j((1-\alpha)\varphi)$. We may assume $\varphi \not\equiv 0$. Since $\mu_j \to 0$ strictly, given any $\varepsilon > 0$ we can find a compact set $K$ such that $\int_{\complement K} |d\mu_j| < \frac{\varepsilon}{2\|\varphi\|}$ where $\| \varphi \| = \sup_{x\in G_+} |\varphi(x)|$.

For $\alpha$, choose a continuous function with compact support which is 1 on $K$ and with $0 \leq \alpha(x) \leq 1$. Then $\alpha.\varphi$ is a continuous function with a compact support. Hence we can find a $j(\varepsilon)$ such that for $j \geq j(\varepsilon)$ we have $|\mu_j(\alpha\varphi)| < \frac{\varepsilon}{2}$. We have $\mu_j((1-\alpha)\varphi) = \int\limits_{\complement K} (1-\alpha)\varphi d\mu_j$, since 90
$1 - \alpha = 0$ on $K$. Hence

$$|\mu_j((1-\alpha)\varphi)| \leq \| \varphi \| \int\limits_{\complement K} |d\mu_j| \leq \frac{\| \varphi \| .\varepsilon}{2 \| \varphi \|} \text{ for } j \geq j(\varepsilon).$$

Hence $|\mu_j(\varphi)| \leq |\mu_j(\alpha\varphi)| + |\mu_j((1-\alpha)\varphi)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $j \geq j(\varepsilon)$. □

**Lemma 17.2.** *If $\{\mu_j\}, \{\nu_j\}$ are two sequences of measures strictly converging to $\mu$ and $\nu$ respectively, the sequence $\{\mu_j * \nu_j\}$ strictly converges to $\mu * \nu$.*

*We shall first show that if $\{\mu_j\}$ and $\{\nu_j\}$ tend to 0 strictly, the sequence $\{\mu_j * \nu_j\}$ tends to 0 strictly. If $\varphi$ is a function in $\mathscr{C}(G_+ \times G_+)$ of the type $\varphi(x,y) = \psi(x)\eta(y)$ where $\psi \in \mathscr{C}(G_+), \eta \in \mathscr{C}(G_+)$ we have*

$$(\mu_j \otimes \nu_j)(\varphi) = \mu_j(\psi).\nu_j(\eta).$$

*($\mathscr{C}(G_+)$ denotes the set of complex valued functions on $G_+$ with compact support). Hence for a $\varphi$ of the above mentioned form, $\mu_j \otimes \nu_j)(\varphi) \to 0$. The linear combinations of elements of the form $\varphi(x,y) = \psi(x)\eta(y)$ form a dense subset of $\mathscr{C}(G_+ \times G_+)$. Since $\{\mu_j\}$ and $\{\nu_j\}$ are strictly convergent sequences of measures, $\{\mu_j\}$ and $\{\nu_j\}$ are bounded sequences of measures and hence $\{\mu_j \otimes \nu_j\}$ is a bounded sequence of measures on $G_+ \times G_+$. Hence the set consisting of the elements $\{\mu_j \otimes \nu_j\}$ is an equicontinuous set in the dual of $\mathscr{C}(G_+ \times G_+)$. On this set the topology of simple convergence on a dense subspace of $\mathscr{C}(G_+ \times G_+)$ by Ascoli's Theorem. Hence for every $\varphi \in \mathscr{C}(G_+ \times G_+)$ we have $\mu_j \otimes \nu_j(\varphi) \to 0$ as $j \to \infty$.*

We shall prove the strict convergence of $\mu_j \otimes \nu_j$ to 0. We have already proved the 'vague' convergence of $\mu_j \otimes \nu_j$ to 0, that is to say, for every fixed $\varphi \in \mathscr{C}(G_+ \times G_+), \mu_j \otimes \nu_j(\varphi)$ tends to 0. To prove the strict convergence of $\{\mu_j \otimes \nu_j\}$ it is sufficient to prove that there exist constants

$\varepsilon(H\times K)$ for compact sets of the form $H\times K$, $H$ and $K$ being compact **91**
in $G_+$ such that given any $\varepsilon>0$ there exists a compact set $H\times K$ with $\varepsilon(H\times K)<\varepsilon$ and $\int\limits_{[(H\times K)} |d\mu_j\otimes\nu_j| < \varepsilon(H\times K)$. Now

$$\begin{aligned}\int\limits_{[(H\times K)} |d\mu_j\otimes\nu_j| &\le \int\limits_{G_+\times[K} |d\mu_j\otimes\nu_j| + \int\limits_{[H\times G_+} |d\mu_j\otimes\nu_j|\\ &\le A\int\limits_{[K}|d\nu_j| + B\int\limits_{[H}|d\mu_j|\\ &\le A\varepsilon(K)+B\varepsilon(H),\end{aligned}$$

where $A\ge\int\limits_{G_+}|d\mu_j|$ and $B\ge\int\limits_{G_+}|d\nu_j|$. (Such real numbers $A$ and $B$ exist). If we choose $K$ and $H$ in such a way that $\varepsilon(K)<\frac{\varepsilon}{2A}$ and $\varepsilon(H)<\frac{\varepsilon}{2B}$ we have

$$\int\limits_{[H\times K} |d\mu_j\otimes\nu_j| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus $\varepsilon(H\times K)=A\varepsilon(K)+B\varepsilon(H)$ are constants satisfying

$$\int\limits_{[(H\times K)} |d\mu_j\otimes\nu_j| \le \varepsilon(H\times K)$$

and given any $\varepsilon>0$ there exists $H\times K$ such that $\varepsilon(H\times K)<\varepsilon$. We have thus proved that $\mu_j\otimes\nu_j\to 0$ strictly.

Let $u:E\to F$ be a continuous map of a locally compact space $E$ into a locally compact space $F$. Let $\{\lambda_j\}$ be a sequence of summable measures strictly converging to 0 on $E$. The direct image of the $\lambda_j's$ by $u$ converges to 0 strictly. First of all $u\lambda_j\to 0$ vaguely. For, if $\varphi$ is any continuous function with compact support on $F, u*\varphi$ is a continuous bounded function on $E$ and $\lambda_j(u*\varphi)\to 0$. Hence $u\lambda_j(\varphi)=\lambda_j(u*\varphi)\to 0$. It is sufficient to prove the existence of constants $\varepsilon(H)$ for compact sets $H$ of the form $H=u(K)$, $K$ being compact in $E$ such that given any
**92** $\varepsilon>0$ there exists an $\varepsilon(H)$ with $\varepsilon(H)<\varepsilon$ and $\int\limits_{[H}|du\lambda_j|\le\varepsilon(H)$. Now

$$\int\limits_{[H}|du\lambda_j| \le \int\limits_{\overset{-1}{u}([H)}|d\lambda_j| \le \int\limits_{[K}|d\lambda_j| \le \varepsilon(K).$$

Given any $\varepsilon > 0$ we know that there exists a compact set $K$ in $E$ such that $\varepsilon(K) < \varepsilon$. We have already proved that if $\mu_j$ and $\nu_j$ tend to 0 strictly $\mu_j \otimes \nu_j$ tends to 0 strictly. $\mu_j * \nu_j$ being the direct image of $\mu_j \otimes \nu_j$ by the map $(x, y) \to x.y$ of $G_+ \times G_+$ in $G_+$ we have $\mu_j * \nu_j \to 0$ strictly.

Now suppose $\mu_j \to \mu$ and $\nu_j \to \nu$ strictly. Then $\mu_j - \mu$ and $\nu_j - \nu$ are sequences of measures strictly converging to 0. Hence $(\mu_j - \mu) * (\nu_j - \nu)$ converges to 0 strictly. We have

$$\mu_j * \nu_j - \mu * \nu = (\mu_j - \mu) * (\nu_j - \nu) + \mu * (\nu_j - \nu) + (\mu_j - \mu) * \nu$$

Now $(\mu_j - \mu) * (\nu_j - \nu) \to 0$ strictly. To complete the proof of the lemma we have only to prove the following: if $\Gamma_j$ is a sequence of measures strictly converging to zero and $\mu$ is a fixed summable measure, then $\mu * \Gamma_j$ and $\Gamma_j * \mu$ tend to zero strictly. For this it is enough to prove that $\mu \otimes \Gamma_j$ and $\Gamma_j \otimes \mu$ tend to zero strictly. (see the general considerations given above). This is proved the same way we proved that $\mu_j \otimes \nu_j \to 0$ strictly if $\mu_j \to 0$ and $\nu_j \to 0$ strictly, using the following fact: if $\mu$ is a summable measure, then $\int_{\complement K} |d\mu| \to 0$ following the filtered set of compact subsets $K$.

Let $\lambda$, $\mu$ and $\nu$ be three summable measures on $G_+$. Then $\lambda * \mu * \nu$ is defined to be the direct image of the measure $\lambda \otimes \mu \otimes \nu$ by the map $(x, y, z) \to x.y.z$ of $G_+ \times G_+ \times G_+$ in $G_+$. It is easily seen that

$$\lambda * (\mu * \nu) = \lambda * \mu * \nu = (\lambda * \mu) * \nu.$$

Also we have $\delta_x * \delta_y = \delta_{xy}$ and $\delta_e * \mu = \mu * \delta_e = \mu$ where $\delta_x$ is the unit **93**
mass (Dirac measure) at $x$ and $e$ is the identity element of $G_+$.

**Representation of semi-groups.**

**Definition 17.6.** *Let $G_+$ denote a locally compact semi-group. Let $E$ be a complete ELC. A* **representation** *of $G_+$ in $E$ is a map $U : G_+ \to \mathscr{L}_s(E, E)$ satisfying the following conditions:*

*(i) $U(x.y) = U(x) \circ U(y), U(e) = I$ the identity map of $E$;*

*(ii) $U : G_+ \to \mathscr{L}_s(E, E)$ is continuous, and*

*(iii) For every compact set $K$ in $G_+$ the set of operators $\{U(k), k \in K\}$ is an equicontinuous set of linear maps of $E$ in $E$.*

*Property (iii) is called the property of local equicontinuity. We shall consider here only representations satisfying the following stronger condition of global equicontinuity.*
*(iii)′ The set of operators $\{U(x), x \in G_+\}$ is an equicontinuous set of linear maps of $E$ in $E$.*

*Let $\mathscr{M}_{G_+}$ denote the set of all summable measures on $G_+$. It is an algebra under the operations of addition and convolution.*

**Lemma 17.3.** *The representation $U : G_+ \to \mathscr{L}_s(E,E)$ can be extended into a map, which also we denote by $U$, of $\mathscr{M}_{G_+}$ in $\mathscr{L}_s(E,E)$. When $\mu = \delta_x$, the unit mass at $x$, $U(\mu)$ will be $U(x)$.*

*Proof.* Since $\{U(x), x \in G_+\}$ is an equicontinuous set of operators, it is also a bounded set in $\mathscr{L}_s(E,E)$. Since $\mu$ is summable, the integral $\int\limits_{G_+} U(x)d\mu(x)$ exists and is an element of $(\mathscr{L}_s(E,E))^\wedge$, the completion
94 of $\mathscr{L}_s(E,E)$. (Refer to Integration, by N. Bourbaki). Since $E$ is complete, $(\mathscr{L}_s(E,E))^\wedge \subset \wedge_s(E,E)$ where $\wedge_s(E,E)$ is the space of all linear maps of $E$ in $E$ with the topology of simple convergence. Now the set of elements $\{U(x)/x \in G_+\}$ is an equicontinuous subset $\mathscr{U}$ of $\mathscr{L}_s(E,E)$. Let $\hat{\mathscr{U}}$ be the convex stable closed envelope of $\mathscr{U}$ in $\wedge_s(E,E)$. $\hat{\mathscr{U}}$ is also equicontinuous and hence $\hat{\mathscr{U}} \subset \mathscr{L}(E,E)$. We have

$$U(\mu) = \int\limits_{G_+} U(x)d\mu(x) \in \hat{\mathscr{U}} \int\limits_{G_+} d|\mu| \subset \mathscr{L}_s(E,E).$$

Hence $U(\mu) \in \mathscr{L}_s(E,E)$ for every $\mu \in \mathscr{M}_{G_+}$. Trivially $U(\delta_x) = U(x)$.

If $\vec{e} \in E$, we have

$$U(\mu)\vec{e} = \int\limits_{G_+} U(x)\vec{e}\, d\mu(x),$$

since the map $U \to U\vec{e}$ is continuous.

For any fixed $\theta \in \mathscr{L}_s(E,E)$, the maps $\Gamma \to \Gamma \circ \theta$ and $\Gamma \to \theta \circ \Gamma$ of $\mathscr{L}_s(E,E)$ in $\mathscr{L}_s(E,E)$ are continuous linear maps. Hence

$$U(\mu) \circ \theta = \int_{G_+} (U(x) \circ \theta) d\mu(x)$$

$$\text{and } \theta \circ U(\mu) = \int_{G_+} (\theta \circ U(x)) d\mu(x)$$

for every fixed $\theta \in \mathscr{L}(E,E)$. □

# Lecture 18

# Representations of semigroups (contd.)

**Lemma 18.1.** *If $\mu$ and $\nu$ are summable measures, $\mu * \nu$ is a summable measure and $U(\mu * \nu) = U(\mu) \circ U(\nu)$.* 95

*Proof.* We have already remarked that $\mu * \nu$ is summable and that $\int |d\mu * \nu| = \| \mu * \nu \| \leq \| \mu \| \| \nu \| = \int |d\mu| . \int |d\nu|$. We shall now prove that $U(\mu * \nu) = U(\mu) \circ U(\nu)$. For any bounded continuous function $\varphi$ with values in $C$ we know that $\int \int \varphi(st) d\mu(s) d\nu(t) = \mu * \nu(\varphi)$. We shall show that this formula is true for any vector valued continuous, bounded function.

Let $F$ be an $ELC$ and $\hat{F}$ its completion. Let $\overrightarrow{\varphi}$ be an $F$-valued function which is continuous and bounded. Considered as an $\hat{F}$-valued function also $\overrightarrow{\varphi}$ is continuous and bounded. Let $\overleftarrow{f}' \in (\hat{F})'$. Then $\langle \overrightarrow{\varphi}, \overleftarrow{f}' \rangle$ is a continuous bounded function with values in $C$. Hence $\mu * \nu \langle \overrightarrow{\varphi}, \overleftarrow{f}' \rangle = \int \int \langle \overrightarrow{\varphi}(st), f' \rangle \, d\mu(s) d\nu(t)$. Since $\overrightarrow{f} \to \langle \overrightarrow{f}, \overleftarrow{f}' \rangle$ is a continuous linear map of $\hat{F}$ in $C$, we have $(\mu * \nu) \langle \overrightarrow{\varphi}, \overleftarrow{f}' \rangle = \langle (\mu * \nu)(\overrightarrow{\varphi}), \overleftarrow{f}' \rangle$ and $\int \int \langle \overrightarrow{\varphi}(st), \overleftarrow{f}' \rangle d\mu(s) d\nu(t) = \langle \int \int \overrightarrow{\varphi}(st) d\mu(s) d\nu(t), \overleftarrow{f}' \rangle$. Hence

$$\langle (\mu * \nu)(\overrightarrow{\varphi}), \overleftarrow{f}' \rangle = \langle \int \int \overrightarrow{\varphi}(st) d\mu(s) d\nu(t), \overleftarrow{f}' \rangle$$

Therefore

$$(\mu * \nu)(\overrightarrow{\varphi}) = \int \int \overrightarrow{\varphi}(st) d\mu(s) d\nu(t) \tag{1}$$

If it so happens that $(\mu * \nu)(\overrightarrow{\varphi})$ is in $F$ itself, we have $\int\int \overrightarrow{\varphi}(st)d\mu(s)d\nu(t) \in F$ because of equality (1). □

Since $\{U(x), x \in G_+\}$ is an equicontinuous set of linear maps, it is a bounded set, and hence $\int\int U(st)d\mu(s)d\nu(t) = (\mu * \nu)(U)$. But for any summable measure $\lambda$ we have defined $U(\lambda)$ to be $\int U(x)d\lambda(x)$
96 or as $\lambda(U)$. Hence $(\mu * \nu)(U) = U(\mu * \nu) = \int\int U(st)d\mu(s)d\nu(t)$. To evaluate the double integral $\int\int U(st)d\mu(s)d\nu(t)$ we use Fubini's Theorem for integrals of vector valued functions. We want only the following form of Fubini's Theorem. If $\overrightarrow{\varphi}(s,t)$ is a continuous, bounded $F$-valued function on $G_+ \times G_+$ and if $\mu$ and $\nu$ are summable measures

$$\int\limits_{G_+ \times G_+}\int \overrightarrow{\varphi}(s,t)d\mu(s)d\nu(t) = \int\limits_{G_+} d\mu(s)\int\limits_{G_+} \overrightarrow{\varphi}(s,t)d\nu(t) = \int\limits_{G_+} d\nu(t)\int\limits_{G_+} \overrightarrow{\varphi}(s,t)d\mu(s).$$

Now, the integrals $\int\limits_{G_+} \overrightarrow{\varphi}(s,t)d\nu(t)$ and $\int \overrightarrow{\varphi}(s,t)d\mu(s)$ exist for all $s$ and $t \in G_+$ and are continuous functions of $s$ and $t$ respectively. By forming the scalar product with any $\overleftarrow{f}' \in (\hat{F})'$ and applying the theorem of Fubini for scalar valued function we get

$$\int\limits_{G_+ \times G_+}\int \overrightarrow{\varphi}(s,t)d\mu(s)d\nu(t) = \int\limits_{G_+} d\mu(s)\int\limits_{G_+} \overrightarrow{\varphi}(s,t)d\nu(t) = \int\limits_{G_+} d\nu(t)\int\limits_{G_+} \overrightarrow{\varphi}(s,t)d\mu(s)$$

Applying this form of Fubini's Theorem we get

$$\int\limits_{B_+ \times G_+}\int U(st)d\mu(s)d\nu(t) = \int\limits_{G_+} d\mu(s)\int\limits_{G_+} U(st)d\nu(t) = \int\limits_{G_+} d\mu(s)\int\limits_{G_+} U(s)\circ U(t)d\nu(t).$$

Since $V \to U(s) \circ V$ for a fixed $s \in G_+$ is a linear continuous map of $\mathscr{L}_s(E,E)$ in itself, we have

$$\int\limits_{G_+ \times} \int\limits_{G_+} U(st)d\mu(s)d\nu(t) = \int\limits_{G_+} d\mu(s)\left\{U(s) \circ \int\limits_{G_+} U(t)d\nu(t)\right\}$$
$$= \int\limits_{G_+} d\mu(s)U(s) \circ U(\nu)$$
$$= \left\{\int\limits_{G_+} d\mu(s)U(s)\right\} \circ U(\nu)$$
$$\text{since} \quad V \to V \circ U(\nu)$$

is a continuous linear map of $\mathscr{L}_s(E,E)$ into itself.

Hence $\int\limits_{G_+ \times} \int\limits_{G_+} U(st)d\mu(s)d\nu(t) = U(\mu) \circ U(\nu)$ **97**

Hence $U(\mu * \nu) = U(\mu) \circ U(\nu)$.

Suppose $E$ is a Banach space and each $U(x)$ satisfies $\| U(x) \| \leq 1$. Then $\| U(\mu) \| \leq \int |d\mu|$ for any summable measure $\mu$. This follows immediately from the definition $U(\mu) = \int\limits_{G_+} U(x)d\mu(x)$.

**Proposition 18.1.** *Let E be a complete ELC. If* $\int |d\mu| \to 0$*, then* $U(\mu) \to 0$ *in* $\mathscr{L}_\delta(E,E)$.

*Proof.* We have $U(\mu)\vec{e} = \int\limits_{G_+} U(x)\vec{e}\, d\mu(x)$ for every $\vec{e} \in E$. Suppose we take vectors $\vec{e}$ in a bounded set $B$ of $E$. Since the set $\{U(x), x \in G_+\}$ is an equicontinuous set of linear maps of $E$ in $E$, the set $\Gamma = \bigcup\limits_{\substack{x \in G_+ \\ \vec{e} \in B}} \{U(x)\vec{e}\}$ is a bounded set of $E$. Now $U(\mu)\vec{e} \in \hat{\Gamma} \int\limits_{G_+} |d\mu|$ where $\hat{\Gamma}$ is the convex, closed, stable envelope of $\Gamma$. $\hat{\Gamma}$ is bounded since $\Gamma$ is. If $\int |d\mu| \to 0$ we see that for $\vec{e} \in B$, $U(\mu)\vec{e} \to 0$ uniformly in $E$. Hence our proposition. ◻

**Proposition 18.2.** *If* $\{\mu_j\}$ *is a sequence of measures tending to* 0 *strictly,* $\{U(\mu_j)\}$ *tends to* 0 *in* $\mathscr{L}_s(E,E)$.

*Proof.* We have to prove that for every fixed $\vec{e} \in E$, $U(\mu_j)\vec{e} \to 0$ in $E$. But $\int_{G_+} U(x)\vec{e}\, d\mu_j(x) = U(\mu_j)\vec{e}$. Now $U(x)\vec{e}$ is a bounded vector valued function of $G_+$ in $E$; this follows from the fact that $\{U(x), x \in G_+\}$ is an equicontinuous set of operators. Hence the proposition is proved if we prove the following more general proposition.

For any continuous bounded $E$ valued function $\vec{\varphi}(x)$ on $G_+$ we have $\int \vec{\varphi}(x) d\mu_j(x) \to 0$ if $\mu_j \to 0$ strictly. Assume first, that $\vec{\varphi}$ is a continuous function with values in $E$ having a compact support. Then
98 $\vec{\varphi} \in \mathscr{D}^\circ(E)$ and $\mu_j \in \mathscr{D}_c^{\circ'}$. The integral $\int \vec{\varphi}(x) d\mu_j(x)$ is nothing but the product $\vec{\varphi}\,.\,\mu_j$ extending the scalar product defining the duality between $\mathscr{D}^\circ$ and $\mathscr{D}_c^{\circ'}$. This product is hypocontinuous with respect to compact subsets of $\mathscr{D}^\circ(E)$ and compact subsets of $\mathscr{D}'^\circ_c$. Hence if $\vec{\varphi} \in \mathscr{D}^\circ(E)$ is a fixed element and $\mu_j \to 0$ in $\mathscr{D}_c'^\circ$ we have $\vec{\varphi}.\mu_j \to 0$. Our assumption is that $\mu_j \to 0$ strictly. If we prove that $\mu_j \to 0$ strictly implies $\mu_j \to 0$ in $\mathscr{D}_c'^\circ$ we are through. When $\mu_j \to 0$ strictly $\{\mu_j\}$ is a bounded set of measures and hence $\{\mu_j\}$ is an equicontinuous set of measures. Hence the topology of compact convergence and the topology of simple convergence induce the topology on the set $\{\mu_j\}$. Hence $\mu_j \to 0$ in $\mathscr{D}_c^{\circ'}$.

Now we go to the case of a continuous, bounded function $\vec{\varphi}(x)$ with values in $E$.

To show that $\int \vec{\varphi}(x) d\mu_j(x) \to 0$ we have to show that given any convex neighbourhood $V$ of 0 in $E$, there exists a $j_V$ such that for $j \geq j_V$ we have $\int \vec{\varphi}(x) d\mu_j(x) \in V$. Let $B$ be the set of values $\{\vec{\varphi}(x)\}$. $B$ is bounded and hence $\widehat{B}$ also. Hence there exists an $\varepsilon > 0$ such that $\varepsilon\widehat{B} \subset \frac{V}{2}$. Let $K$ be a compact subset of $G_+$ such that the $\varepsilon(K)$ that corresponds to $K$ is less than $\varepsilon$. Let $\alpha$ be a continuous function equal to 1 or $K$ and with compact support, satisfying $0 \leq \alpha(x) \leq 1$ for every $x \in G_+$. Then $\mu_j(\vec{\varphi}) = \mu_j(\alpha\vec{\varphi}) + \mu_j((1-\alpha)\vec{\varphi})$, i.e., $\int_{G_+} \vec{\varphi}(x) d\mu_j(x) = \int_{G_+} \alpha(x)\vec{\varphi}(x) d\mu_j(x) + \int_{G_+} (1-\alpha(x))\vec{\varphi}(x) d\mu_j(x)$. Since $\alpha(x)\vec{\varphi}(x)$ is a continuous function with compact support, we have a $j_V$ such that for $j \geq j_V$,

$$\int_{G_+} \alpha(x)\vec{\varphi}(x) d\mu_j(x) \in \frac{V}{2}.$$

Also $\int_{G_+} (1-\alpha(x))\overrightarrow{\varphi}(x)d\mu_j(x) = \int_{\complement K} (1-\alpha(x))\overrightarrow{\varphi}(x)d\mu_j(x)$ **99**

$$\in \widehat{B}\int_{\complement K} |d\mu_j|$$

$$\in \varepsilon\widehat{B} \subset \frac{V}{2}.$$

Hence $\int_{G_+} \overrightarrow{\varphi}(x)d\mu_j(x) \in V$. This proves our proposition. □

# Lecture 19

# Representations of semi-groups (contd.)

Let , from now on, $G_+$ denote a closed convex cone in $R^n$, containing 100
the origin. We shall assume that $G_+$ is the closure of its interior. $G_+$ is a topological semi-groups.

**Definition 19.1.** *A distribution $T$ on $R^n$ is said to be $G_+$ summable if $T$ has its support in $G_+$ and $T = \sum D^p \mu_p$ where $\mu_p$ are summable measures on $R^n$. The space of $G_+$ summable distributions will be denoted by $\mathscr{D}'_{L^1}(G_+)$.*

*If $T \in \mathscr{D}'_{L^1}(G_+)$ and $T' \in \mathscr{D}'_{L^1}(G_+)$ then $T * T' \in \mathscr{D}'_{L^1}(G_+)$. If $T = \sum D^p \mu_p, T' = \sum D^q \nu_q$ then $T * T' = \sum D^{p+q} \mu_p * \nu_q$. If $\alpha \in \mathscr{D}(R^n)$ with support in $G_+$ and $T \in \mathscr{D}'_{L^1}(G_+)$ then $\alpha * T$ has* **its support in** *$G_+$ and $\alpha * T dx$ is a summable measure* **in** *$G_+$.*

**Definition 19.2.** *A sequence $\{T_j\}$ of $G_+$ summable distributions is said to converge to zero strictly if $T_j = \sum_{|p| \leq m} D^p \mu_p, j$ with $m$ independent of $j$, and $\{\mu_{p,j}\}$ is, for every $p$, a sequence of summable measures tending to zero strictly.*

*If $\{T_j\}$ and $\{S_j\}$ are two sequences of summable distributions strictly tending to zero, then $\{T_j * S_j\}$ tends to zero strictly.*

*We have seen how one can define $U(\mu)$ for $\mu \in \mathscr{M}_{G_+}$. Now we shall see how one can define $U(T)$ for $T \in \mathscr{D}'_{L^1}(G_+)$. However $U(T)$ will* *not, in general, be defined on the whole of $E$. But the domain $E_T$ of $U(T)$ will be a dense subspace of $E$.*

**Definition 19.3.** *Let $\mathscr{H}$ be the filter having for a base the sets $A_\varepsilon$, for every $\varepsilon > 0$, defined as follows:*

$$A_\varepsilon = \left\{ \begin{array}{l} \alpha|\alpha \in (R^n), \textit{Support of } \alpha \subset G_+, \alpha \geq 0, \\ 1-\varepsilon \leq \int \alpha \leq 1+\varepsilon \textit{ and support of } \alpha \textit{ contained} \\ \textit{in a } \varepsilon\textit{-neighbourhood of } 0. \end{array} \right\}$$

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**Definition 19.4.** *Let $T \in \mathscr{D}'_{L^1}(G_+)$. Let $E_T$ be the set of elements $x$ of $E$ such that $\lim\limits_{\mathscr{F}} U(\alpha * T)x$ exists in $E$ and for $x \in E_T$ define $U(T)x$ to be $\lim\limits_{\mathscr{F}} U(\alpha * T)x$. ($\lim\limits_{\mathscr{F}}$ denotes the limit as $\alpha \to \delta$ following the filter $\mathscr{F}$. $U(\alpha * T)x$ has a meaning for every $x \in E$, since $\alpha * T$ has its support in $G_+$ and defines a summable measure in $G_+$).*

**Definition 19.5.** *We define $\mathscr{D}(G_+)$ to be the subspace of functions in $\mathscr{D}(R^n)$ whose supports are contained in $G_+$ and $\mathscr{D}_{L^1}(G_+)$ to be the space of $C^\infty$ functions on $R^n$ with supports in $G_+$ and with summable derivatives of all orders.*

**Proposition 19.1.** *For $T \in \mathscr{D}'_{L^1}(G_+), \rho \in \mathscr{D}_{L^1}(G_+)$ and $x \in E_T$ we have $U(\rho * T) = U(\rho) \circ U(T)x$.*

*Proof.* $\rho$ being a summable function with support in $G_+ \rho(x)dx$ is a summable measure, with support in $G_+$. We denote this measure also by $\rho$. $U(\rho)$ and $U(\alpha * T)$ are continuous linear operators in $E$ and we have

$$U(\rho) \circ U(\alpha * T)x = U(\rho * \alpha * T)x$$

Now, let $\alpha \to \delta$ following the filter $\mathscr{F}$. Since $U(\rho)$ is a continuous map of $E$ in $E$. We have

$\lim\limits_{\mathscr{F}} U(\rho) \circ U(\alpha * T)x = U(\rho) \circ U(T)x$ for every $x \in E_T$. Also $\rho * \alpha * T = \alpha * \rho * T$ (commutativity), and as $\alpha \to \delta$ following $\mathscr{F}, \alpha * \rho * T \to \rho * T$ strictly (Lemma 17.2). Hence $\lim\limits_{\mathscr{F}} U(\rho * (\alpha * T))x = U(\rho * T)x$. Thus $U(\rho * T)x = U(\rho) \circ U(T)x$ for every $x \in E_T$. □

102 **Proposition 19.1′.** Let $S, T \in \mathscr{D}'_{L^1}(G_+)$, $x \in E_T$. Then $x \in E_{S*T}$ if and only if $U(T).x \in E_S$ and if it is so

$$U(S * T)x = U(S) \circ U(T)x.$$

*Proof.* If $S$ is an integrable distribution and $\alpha \in \mathscr{D}(G_+)$. Then $\alpha * S \in \mathscr{D}_{L^1}(G_+)$. Take for $\rho$ the element $\alpha * S$ of $\mathscr{D}_{L^1}(G_+)$ in the previous proposition. Since $U(\alpha * S * T)x = U(\alpha * S) \circ U(T)x$, whatever be $x \in E_T$, we see that if any one of $\lim_{\mathscr{F}} U(\alpha * S * T)x$ and $\lim_{\mathscr{F}} U(\alpha * S)U(T)x$ exists, the other also exists and we have the equality of the two limits. Thus if $x \in E_T$ we have $x \in E_{S*T}$ if and only if $U(T)x \in E_S$ and then $U(S * T)x = U(S)U(T)x$. □

**Corollary 1.** *If $\varphi \in \mathscr{D}_{L^1}(G_+)$, the element $U(\varphi)x \in E_T$ for every $T \in \mathscr{D}'_{L^1}(G_+)$. Moreover $U(T)U(\varphi)x = U(T * \varphi)x$.*

**Corollary 2.** $\bigcap_{T \in \mathscr{D}'_{L^1}} E_T(G_+)$ *is dense in $E$. In fact, if $\varphi \in \mathscr{D}(G_+)$ tends to $\delta_e = \delta$ following the filter $\mathscr{F}$, $U(\varphi)x \to Ix = x$.*
*In particular, we have also $E_T$ dense in $E$ for every $T \in \mathscr{D}'_{L^1}(G_+)$.*

**Proposition 19.2.** *The mapping $(T, x) \to U(T)x$ is a closed mapping. That is to say, if $\{T_j\}$ is a sequence of summable distributions with supports in $G_+$ and tending strictly to $T$ and if $\{x_j\}$ is a sequence of elements of $E$ tending to $x$ and if $U(T_j)x_j$ has a meaning for each $j$ and if $\lim U(T_j)x_j = y$ in $E$, then $U(T)x$ has a meaning and $U(T)x = y$. (The word mapping is not used here in the usual sense. $U(T)x$ need not be defined for every $x$).*

*Proof.* We shall, in fact, prove a result somewhat stronger than the one that we have stated. Even when $U(T_j)x_j \to y$ weakly in $E$, we shall show that $x \in E_T$ and that $U(T)x = y$. For $\alpha \in \mathscr{D}(G_+)$, $U(\alpha * T_j)x_j =$ 103
$U(\alpha) \circ U(T_j)x_j$ from Proposition 19.1. For $\alpha$ fixed in $\mathscr{D}(G_+)$, $\alpha * T_j \to \alpha * T$ in the sense of strict convergence of measures. Hence $U(\alpha * T_j)$ remains in an equicontinuous set of linear maps of $E$ in $E$ and tends to $U(\alpha * T)$ in $\mathscr{L}_s(E, E)$.

If $\{V_j\}$ is a sequence of elements lying in an equicontinuous set of linear maps of $E$ in $E$ and if $V_j \to V$ in $\mathscr{L}_s(E, E)$ and if $x_j \to x \in E$, the sequence $V_j x_j \to Vx$ in $E$. In fact,

$$Vx - V_j x_j = (V - V_j)x + V_j(x - x_j)$$

Since $[V_j]$ is equicontinuous and $x - x_j \to 0$ in $E$, $V_j(x - x_j) \to 0$ in $E$. ($[V_j]$ denotes the set of the linear maps $V_j$). Since $V - V_j \to 0$ in $\mathscr{L}_s(E, E)$, for every fixed $x \in E$, $(V - V_j)x \to 0$ in $E$.

Taking for $V_j$ the sequence $U(\alpha * T_j)$ we see that $U(\alpha * T_j)x_j \to U(\alpha * T)x$ in $E$ as $j \to \infty$. Since $U(\alpha)$ is a continuous linear map of $E$ in $E$ it is also weakly continuous. Hence if $U(T_j)x_j \to y$ weakly in $E$, $U(\alpha) \circ U(T_j)x_j \to U(\alpha)y$ in $E$ weakly. But $U(\alpha) \circ U(T_j)x_j \to U(\alpha * T)x$ strongly in $E$. Hence we must have $U(\alpha) \circ U(T_j)x_j \to U(\alpha)y$ strongly in $E$ and $U(\alpha)y = U(\alpha * T)x$. If $\alpha \to \delta$ following the filter $\mathscr{F}$, $U(\alpha)y \to y$ in $E$. Hence $\lim\limits_{\mathscr{F}} U(\alpha * T)x$ exists and is equal to $y$. That is to say, $U(T)x$ has a meaning and $y = U(T)x$. □

**Corollaries:**

1) For each $T \in \mathscr{D}'_{L^1}(G_+)$, $U(T)$ is a closed operator. For, if $x_j \to x$ in $E$ and $U(T)x_j \to y$ in $E$, choosing $T_j = T$ in the above proposition, we see that $U(T)x$ has a meaning and $U(T)x = y$.

104 2) If $x$ is an element of $E$ such that

$$\text{weak}\lim_{\mathscr{F}} U(\alpha * T)x = y$$

exists, then $x$ belongs to the domain of $U(T)$ and $U(T)x = y$ and $y = \lim\limits_{\mathscr{F}} U(\alpha * T)x$ in $E$.

3) If $S_j$ is a filter of summable distributions strictly converging to $\delta$ and if $U(S_j * T)x$ is defined for every $j$ and if $U(S_j * T)x \to y$ weakly in $E$, then $x \in E_T$ and $\lim U(S_j * T) = U(T)x$.

4) For defining the operator corresponding to $T \in \mathscr{D}'_{L^1}$, even if we choose for the filter $\mathscr{F}'$ a filter finer than the filter $\mathscr{F}$ used above

and require the limit to exist only weakly we will get the same operator $U(T)$, i.e., the domain of $U(T)$ will not be enlarged. For if $\mathscr{F}' = \varphi_i$ is a finer filter, then $\varphi_i * T$ will tend to $T$ strictly. If weak $\lim_{\mathscr{F}} U(\varphi_i * T)x = y$ exists, then by the proposition $\lim U(\alpha_j * T)x$ exists and is equal to $y$.

We thus see that the definition for $U(T)$ we have chosen is the most general one; it gives the largest possible domain of definition for $U(T)$.

# Lecture 20

# Representations of semi-groups (contd.)

Let $X$ be a tangent vector to the cone $G_+$ at the origin. If for any $\varphi \in$ 105
$\mathscr{D}(G_+)$ define $X(\varphi)$ = derivative of $\varphi$ at 0 along the direction $X$, $X$ can be considered as a distribution. It is an element of $\mathscr{D}'_{L^1}(G_+)$. In fact $X$ has a compact support, the point '0'. The operator $U(X)$ is called the infinitesimal generator corresponding to the tangent vector $X$ at 0. The directional derivative $X(\varphi)$ is by definition

$$\lim_{t\to\circ} \frac{\varphi(tX) - \varphi(0)}{t}.$$

Hence $$X(\varphi) = \lim_{t\to\circ} \tfrac{\delta_{tX}-\delta_\circ}{t}(\varphi).$$

**Proposition 20.1.** *$U(X)x$ exists if and only if*

$$\lim_{t\to\circ} \frac{U(\delta_{tX}) - I}{t} x \quad \textit{exists, and}$$
$$\lim_{t\to\circ} \frac{U(\delta_{tX}) - I}{t} x = U(X)x.$$

*Proof.* Assume that $\lim\limits_{t\to\circ} U \frac{\delta_{tX}-\delta_\circ}{t} x$ exists. Now $\frac{\delta_{tX}-\delta_\circ}{t} \to x$ strictly as $t \to 0$. By Proposition 19.2, we see that $U(X)x$ exists and is equal to $\lim\limits_{t\to\circ} \frac{U(\delta_{tX})-I}{t} x$.

Conversely, suppose that $U(X)x$ has a meaning. We have

$$\lim_{t\to\circ}\frac{\delta(tX)-\delta(\circ)}{t}=\lim_{t\to\circ}(X*\mu_t)$$

where $\mu_t$ is a measure, concentrated on the line segment joining the vectors 0 and $tX$, which is homogeneous and gives to the segment a total mass 1. If $U(X)x$ exists, by Proposition 19.1', we have, as $U(\mu_t)U(X)x$
106 exists, $U(\mu_t*X)x=U(\mu_t)U(X)x$. But due to commutativity $\mu_t*X=X*\mu_t$. Hence $U(X*\mu_t)x=U(\mu_t)U(X)x$. But since $\mu_t\to\delta$ strictly, we have

$$\begin{aligned}\lim_{t\to\circ}U(X*\mu_t)x&=U(\delta)U(X)x\\&=U(X)x.\end{aligned}$$

Hence, if $U(X)x$ exists, $\lim_{t\to\circ}U(X*\mu_t)x$ exists. In other words, $\lim_{t\to\circ}\frac{U(\delta(tX)-I}{t}x$ exists and is equal to $U(X)x$. □

**Proposition 20.2.** *Let $x\in E$. The following four properties are equivalent:*

*i)* $U(T)x$ *exists for every* $T\in\mathcal{E}'^1_{G_+}$.

*ii)* $U(T)x$ *exists for every* $T\in\mathcal{E}'^1_\circ$, *i.e.* $U(X)x$ *exists for every* $X\in R^n$.

*iii) The function* $U(\hat{s})\times(s\to U(s)X)$ *belongs to* $\mathcal{E}^1_s(E)$. *[We shall say that a function on a cone is once continuously differentiable if it is once continuously differentiable in the interior and the derivatives have a continuous extension to the cone].*

*iv) The function* $U(\hat{s})x$ *has weak derivatives at the origin in every direction along the cone.*

*Proof.* Evidently i) implies ii). ii) implies i) since $T\in\mathcal{E}'^1_{G_+}$ can be written as $T=\sum\mu_i*\frac{\partial}{\partial x_i}$, where $\mu_i$ are summable measures with supports in $G_+$ (Whitney regularity) and we any apply Proposition 19.1'. ii) implies iii): Let $X_1,\ldots,X_n$ be independent vectors tangential to the cone. If $s_\circ$ is an interior point of the cone, using the method of proof of Proposition 20.1, we see that $U(\hat{s})x$ is differentiable in the direction $X_i(i = 1, \ldots, n)$ at $s_\circ$ and that the derivative is equal to $U(s_\circ)U(X_i)x$. $U(X_i)x$ being a fixed vector in $E$, the function $s \to U(s)U(X_i)x$ is a continuous function on the cone. Since the $X_i$ form a base for $R^n$, it follows that $U(\hat{s})x \in \mathcal{E}_s^1(E)$. iii) implies iv), as is easily seen. iv) implies ii) by Proposition 19.2. □

# Lecture 21

# Remarks on the representation of non-commutative Lie semi-groups

Let $G$ be a Lie group of dimension $n$ and $G_+$ a sub semi-group of $G$ such that 107

i) $G_+$ is closed in $G$;

ii) the unit element of $G$ belongs to $G_+$.

iii) $G_+$ is the closure of its interior.

We call $G_+$ a Lie semi-group.

We denote by $\mathscr{D}'_{L^1}(G_+)$ the set of distributions $T$ on $G$ with support in $G_+$ which are of the form

$$T = \sum_p D_p * \mu_p \quad \text{(finite sum)}$$

where $D_p$ are distributions[1] in $G$ with support at the unit element $e$, and $\mu_p$ are summable measures on $G$. A sequence $T_j \in \mathscr{D}'_{L^1}(G_+)$ is said

[1]By a distribution we mean a continuous linear form on the space of indefinitely differentiable *functions* on $G$ with compact supports.

to converge to zero strictly if $T_j = \sum_p D_p * \mu_{p,j}$, where $D_p$ are fixed distributions with support at $e$ of order at most $m$, and if for every $p$ the measures $\mu_{p,j}$ converge to zero strictly.

Let $U$ be a representation of $G_+$, by equicontinuous operators in $E$. For $T \in \mathscr{D}'_{L^1}(G_+)$ we shall define a linear operator $U(T)$ on $E$. Let $\mathscr{F} = \{\alpha_j\}$ be a filter of $C^\infty$ $n$-forms, ($n$ = dimension of $G$) of the second kind (odd type) on $G$ such that

i) $\{\alpha_j\}$ have compact supports contained in $G_+$;

ii) $\alpha_j \geq 0$, for every $j$;

108 iii) The support of $\alpha_j$ tends uniformly to the unit element of $G$;

iv) $\int_G \alpha_j \to 1$.

By definition the domain $E_T$ of $U(T)$ will consist of those $x$ in $E$ for which $\lim_{\mathscr{F}} U(\alpha * T)x$ exists and $U(T)x$ is defined to be this limit. (Note the order in which $\alpha$ and $T$ enter in the convolution).

**Remark.** If $\mathscr{F}'$ is another filter having the same properties as $\mathscr{F}$ and $U'(T)$ the corresponding operator, it will follow from the results to be indicated later, that $U(T)$ and $U'(T)$ have the same domain of definition and are equal on their common domain of definition.

Proposition 19.1 is true also in the non-abelian case, if $\rho$ is a $C^\infty$ form with compact support. To uphold this proposition in the non-abelian case we have to prove the following: if $D$ is a distribution with support at the origin and $\mu$ a summable measure, then $(\rho*\alpha)*(D*\mu) \to \rho * D * \mu$ strictly as $\alpha \to \delta$ following $\mathscr{F}$. For this, it is sufficient to prove that $\rho * \alpha * D \to \rho * D$ strictly. But this follows from the separate continuity of the convolution map $\mathscr{D}^n \times \mathcal{E}^1 \to \mathscr{D}^n$.

Proposition 19.1' is also true, if $S$ has compact support. Proposition 19.2 is also true; proof is the same.

Suppose $x$ is an element of $E$ such that $\lim_{\mathscr{F}} U(T * \alpha)x = y$ exists $(T \in \mathscr{D}'_{L^1}(G_+))$. Since $T * \alpha \to T$ strictly we see, using Proposition 19.2 that $x$ belongs to the domain of $U(T)$ and $U(T)x = y$. Thus the definition for $U(T)$ we have given using convolution on the left by $\alpha$ gives a domain of definition for $U(T)$ which is larger than the domain we would have obtained if we chose to convolve on the right by $\alpha$.

# Appendix Representations of the semi-group of positive reals. Hille-Yosida Theorem for complete locally convex spaces

In this section we take for $G_+$ the (additive) semi-group of positive real 109
numbers.

Let $U$ be a representation (equicontinuous) of $G_+$ in $E$. The linear operator $U(-\delta')$ ($\delta'$ = first derivative of the Dirac measure) is called the infinitesimal generator of the representation. We shall show that every complex number $p$ with $Rl\ p > 0$ is in the resolvent set of the infinitesimal generator. We have $pI - U(-\delta') = U(p\delta + \delta')$. We have further, in $R$, $(p\delta+\delta') * Y(t)e^{-pt} = Y(t)e^{-pt} * (p\delta+\delta') = \delta, Y(t)$ denoting the Heaviside function. Now for $Rl\ p > 0, e^{-pt}$ is a summable measure in $G_+$ and hence $U(e^{pt})$ is a continuous linear operator in $E$ (see Lemma 17.3). Since $U(\delta) = I$, using Proposition $1'$, we see that $U(e^{-pt})$ is the inverse of $pI - U(-\delta')$.

We shall now prove an equicontinuity property of the resolvent operators of the infinitesimal generator $A$ of the one-parameter semi-group

$U(t)$. Since

$$(\delta + \frac{\delta'}{p}) * pe^{-pt} = \delta(p > 0),$$

as before we see, using Proposition 1′, that $U(pe^{-pt}) = (I - \frac{A}{p})^{-1}$. But as $\int\limits_{\circ}^{\infty} pe^{-pt}dt = 1$ we see that (see Lemma 17.3) $U(pe^{-pt})$ belongs to the convex closed stable envelope $\hat{\mathscr{U}}$ of the set $\mathscr{U} = \{U(t), t \geq 0\}$. In a similar way, we see that $(I - \frac{A}{p})^{-m}(p > 0, m = 1, 2, \ldots) \in \hat{\mathscr{U}}$. Hence the set of operators $\{(I - \frac{A}{p})^{-m}\}$, as $p$ runs through strictly positive numbers and $m$ through positive integers, is equicontinuous (with $\mathscr{U}$).

110 We shall now show that the equicontinuity condition we proved for the resolvent operators of the infinitesimal generator of a one-parameter semi-group is also sufficient to ensure that a densely defined linear operator in $E$ be the infinitesimal generator of a one-parameter semi-group. The problem here is to define the exponential of such an operator. Before going into this problem we shall first consider the question of defining the exponential of a continuous linear operator.

**The exponential of a continuous linear operator.**

Let $E$ be a complete $ELC$ and $T$ a continuous linear operator of $E$ into itself. We try to define $\exp tT$ as a continuous linear operator by means of the series[2]

$$(\exp tT)x = \sum_{k=\circ}^{\infty} \frac{(tT)^k x}{k!} \ (x \in E) \ (t \geq 0).$$

The series will converge for every $x \in E$ and represent a continuous linear operator of $E$ into itself, at least if $T$ and its iterates $T^k (k = 2, 3, \ldots)$ are equicontinuous. Actually the series $\sum\limits_{k=\circ}^{\infty} \frac{(tT)^k x}{k!}$ will converge at every $x \in E$ if the set $\{T, T^2, \ldots\}$ is weakly bounded. For then if $q$ is any

[2] If $E$ is a Banach space, the series $\sum \frac{(tT)^k}{k!_T}$ converges in the uniform topology for *any* continuous linear operator $T$ of $E$ into $E$.

continuous semi-norm on $E$, we have

$$\sum_{k=0}^{m}\frac{q((tT)^k x)}{k!}=\sum_{k=0}^{m}\frac{t^k q(T^k x)}{k!}\leq C\sum_{k=0}^{\infty}\frac{t^k}{k!},$$

$C$ being a positive constant, so that the series $\sum_{k=0}^{\infty}\frac{q((tT)^k x)}{k!}$ is convergent. Since $E$ is complete, it follows that $\sum_{k=0}^{\infty}\frac{(tT)^k x}{k!}$ is convergent in $E$. To show, under the hypothesis that the set $\{T, T^2, \ldots\}$ is equicontinuous, that $x \to (\exp T)x$ is a continuous operator, it is sufficient to show that the operators $B_n = \sum_{k=0}^{n}\frac{(tT)^k}{n!}$ are equicontinuous since the pointwise limit **111**
of a sequence of equicontinuous linear operators is a continuous linear operator. To show this we use the following criterion for equicontinuity which will also be used later. Let $\{B_\alpha\}$ be a family of linear operators of $E$ into $E$; in order that $\{B_\alpha\}$ be equicontinuous, it is necessary and sufficient that the following condition be satisfied: for every continuous semi-norm $q$ on $E$ there exists a continuous semi-norm $p$ on $E$ and a strictly positive number $\underline{a}$ such that

$$q(B_\alpha(x)) \leq ap(x), \quad \text{for every} \quad \alpha \quad \text{and} \quad x \in E.$$

(see Espaces Vectoriels Topologiques, Ch. II, by N. Bourbaki). To prove that the above $B_n$ are equicontinuous, let $q$ be a continuous semi-norm on $E$. Since $\{T^k\}_{k=0,1,\ldots}$ are equicontinuous there exists a continuous semi-norm $p$ and $a > 0$ such that $q(T^k x) \leq a\, p(x)_k$ $(k = 0, 1, \ldots, x \in E)$. Now

$$q(B_n x) = q\Big(\sum_{k=0}^{n}\frac{(tT)^k x}{k!}\Big) \leq \sum_{k=0}^{n}\frac{t^k q(T^k x)}{k!}$$
$$\leq (a.\exp t)p(x)$$

which shows that $B_n$ are equicontinuous. One proves the same way the following results.

i) Writing $T_t x = \exp(tT)x$, the map $t \to \exp tTx$ is a continuous function from $(0, \infty)$ in $E$, for every $x \in X$, and

(a) $\lim\limits_{h\downarrow\circ}\frac{T_h x-x}{h}$ exists for every $x\in E$ and in equal to $Tx$;

(b) $\lim\limits_{h\to\circ}\frac{T_{t+h}x-T_t x}{h}$ exists for $t>0$ and $x\in E$ and equal to $T_t T x = T T_t x$.

ii) Let $T$ and $S$ be two continuous linear operators such that $ST = TS$ and such that $\{T^k\}$ and $\{S^k\}$ are equicontinuous. Then

$$\sum_{k=\circ}^{n}\frac{t^k(T+S)^k}{k!}$$

converges pointwise to a continuous linear operator $\exp(t(T+S) = \exp tT.\exp tS = \exp tS.\exp tT$ and

$$\lim_{h\downarrow\circ}\frac{\exp h(T+S)x-x}{h} = (T+S)x, x\in E$$

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$$\begin{aligned}\lim_{h\to\circ}&\frac{\exp(h+t)(T+S)x-\exp t(T+S)x}{h}\\ &= \exp t(S+T)\quad (S+T)x\\ &= (S+T)\exp t(S+T)x\quad (x\in E).\end{aligned}$$

We now prove the

**Theorem (Hille-Yosida).** Let $E$ be a complete $ELC$. Suppose that $A$ is a densely defined linear operator on $E$ such that for every strictly positive $p, (I-\frac{A}{p})^{-1}$ exists and such that the family $\mathscr{F}$ of operators $\{(I-\frac{A}{p})^{-m}\}$ ($p$ strictly positive, $m=1,2,\ldots$) is equicontinuous. Then there exists a uniquely determined representation $T(t)(t\geq 0)$, which is equicontinuous with $\mathscr{F}$, whose infinitesimal generator is $A$.

*Proof.* We follow Yosida's method of proof.

Writing $J_\lambda = (I-\lambda^{-1}A)(\lambda>0)$ we have evidently: $AJ_\lambda x = \lambda(J_\lambda - I)x, x\in E$ and $AJ_\lambda x = J_\lambda Ax = \lambda(J_\lambda - I)x$, for $x\in\mathscr{D}(A)$, where $\mathscr{D}(A)$ denotes the domain of $A$. We shall prove that $J_\lambda x\to x$, as $\lambda\to\infty$, for every $x\in E$. If $x\in\mathscr{D}(A), J_\lambda x - x = \lambda^{-1}J_\lambda(Ax)$ and hence $J_\lambda x - x\to 0$ as

$\lambda \to \infty$, as the set $\{J_\lambda(Ax)\}$ is bounded. Since $\mathscr{D}(A)$ is dense in $E$ and $\{J_\lambda\}_{\lambda>0}$ is equicontinuous, it follows that $J_\lambda x \to x$ for every $x \in E$.

Set

$$T_t^{(\lambda)} = \exp(tAJ_\lambda) = \exp(t\lambda(J\lambda - I)) = \exp(-\lambda t)\exp(\lambda t J_\lambda).$$

It is easily seen, using for example the criterion for equicontinuity used earlier, that the operators $\{T_t^{(\lambda)}\}(\lambda > 0, t \geq 0)$ are equicontinuous with $\mathscr{F}$. We remark that $J_\lambda J_\mu = J_\mu J_\lambda, \lambda, \mu > 0$. We now prove that as $\lambda \to \infty, T_t^{(\lambda)}$ converges in the topology of simple convergence, to a continuous linear operator $T_t$ and for fixed $x$, $T_t^{(\lambda)} x \to T_t x$ uniformly when 113
$t$ lies in a compact set. To prove this, let $q$ be a continuous semi-norm on $E$. Since $\{T_t^{(\lambda)}\}$ are equicontinuous there exist a continuous semi-norm $p$ and $a > 0$ such that $q(T_t^{(\lambda)} x) \leq$ a $p(x)$ for $\lambda > 0, t \geq 0$ and every $x \in E$. For $\lambda, \mu > 0$ and $x \in \mathscr{D}(A)$

$$\begin{aligned} q(T_t^{(\lambda)}(x) - T_t^{(\mu)} x) &= q\left[\int_0^t \frac{d}{ds}\{T_{t-s}^{(\mu)} T_s^{(\lambda)} x\} ds\right] \\ &= q\left[\int_0^t T_{t-s}^{(\mu)} T_s^{(\lambda)} (AJ_\lambda - AJ_\mu) x\right] \\ &\leq ta^2 p\left[(J_\lambda A - J_\mu A)x\right], \end{aligned}$$

and $(J_\lambda A - J_\mu A)x \to 0$, as $\lambda, \mu \to \infty$ as $x \in \mathscr{D}(A)$. So $\lim\limits_{\lambda,\mu\to\infty} q(T_t^{(\lambda)}(x) - T_t^{(\mu)} x) = 0$ uniformly when $t$ lies in a compact set. Since $\mathscr{D}(A)$ is dense in $E$ and the set of operators $\{T_t^{(\lambda)}\}$ is equicontinuous, we see that $\lim\limits_{\lambda\to\infty} T_t^{(\lambda)} x \equiv T_t x$ exists for every $x \in E$ and $t$ uniformly in any compact set that the set of operators $\{T_t\}_t \geq 0$ is equicontinuous with $\mathscr{F}$. From the uniform convergence, $t \to T_t$ is a continuous map of $t \geq 0$. To prove that $T_{t+s} = T_t T_s$, let $q$ be a continuous semi-norm and let $\underline{a}$ and $p$ have the same meaning as before. Then, using $T_{t+s}^{(\lambda)} = T_t^{(\lambda)} T_s^{(\lambda)}$,

$$q((T_{t+s} - T_t T_s)x) \leq q(T_{t+s} - T_{t+s}^{(\lambda)} x) + q(T_{t+s}^{(\lambda)} x - T_t^{(\lambda)} T_s^{(\lambda)} x)$$

$$+ q(T_t^{(\lambda)} T_s^{(\lambda)} - T_t^{(\lambda)} T_s x) + q(T_t^{(\lambda)} T_s x - T_t T_s x)$$
$$\leq q(T_{t+s} x - T_{t+s}^{(\lambda)} x) + ap(T_s^{(\lambda)} x - T_s x)$$
$$+ q\left[(T_t^{(\lambda)} - T_t)(T_s x)\right] \to 0.$$

Since $q(T_{t+s}x - T_t T_s x) = 0$ for every continuous semi-norm $q$, we must have $T_{t+s} = T_t T_s$.

114 Let $A'$ be the infinitesimal generator of the semi-group $T_t$. We have to show that $A' = A$. To show this, it is sufficient to show that $A'$ is an extension of $A$ (i.e., $x \in \mathscr{D}(A)$ implies $x \in \mathscr{D}(A')$ and $Ax = A'x$). For, $t \to T_t$ being an equicontinuous representation, $(I - \lambda^{-1} A') : \mathscr{D}(A') \to E$ is a bijection for $\lambda > 0$ and by hypothesis $(I - \lambda^{-1} A) : \mathscr{D}(A) \to E$ is a bijection, for $\lambda > 0$, so that $\mathscr{D}(A) = \mathscr{D}(A')$. To prove that $A'$ is an extension of $A'$ let $x \in \mathscr{D}(A)$. Then $T_s^{(\lambda)} A J_\lambda x \to T_s A I x$. For if $q$ is a continuous semi-norm, we have

$$q(T_s Ax - T_s^{(\lambda)} A J_\lambda x) \leq q(T_s Ax - T_s^{(\lambda)} Ax) + q(T_s^{(\lambda)} Ax - T_s^{(\lambda)} A J_\lambda x)$$
$$\leq q\left[(T_s - T_s^{(\lambda)})(Ax)\right] + ap(Ax - J_\lambda Ax)$$
$$\to 0, \quad \text{as} \quad \lambda \to \infty, \quad (\text{since} \quad J_\lambda Ax \to Ax).$$

$$\text{Now, } T_t x - x = \lim_{\lambda\to\infty} T_t^{(\lambda)} x - x$$
$$= \lim_{\lambda\to\infty} \int_0^t T_s^{(\lambda)} A J_\lambda x \, ds$$
$$= \int_0^t \lim_{\lambda\to\infty} T_s^{(\lambda)} A J_\lambda x$$
$$= \int_0^t T_s Ax$$

so that $\lim_{t\downarrow\circ} \frac{T_t x - x}{t}$ exists and equal to $Ax$, i.e., if $x \in \mathscr{D}(A)$, then $x \in \mathscr{D}(A')$ and $A'x = Ax$.

The uniqueness of $T_t$ follows from the following fact, which is proved the same way as in the case of Banach spaces: If $t \to T_t$ is a representation (equicontinuous) and $A$ is the infinitesimal generator of $T_t$ then

$$T_t x = \lim_{\lambda \to \infty} \exp(tAJ_\lambda)x, \quad \text{for every} \quad x \in E.$$

□

**Remarks.** (i) In a Banach space the condition of the theorem reads: there exists a constant $M > 0$ such that

$$\| (\lambda I - A)^{-m} \| \leq M/\lambda m \quad (\text{for} \quad m = 1, 2, \ldots, \lambda > 0)$$

(ii) For the proof of the theorem it is sufficient to assume that $E$ is quasi-complete.