Lectures on
On Mean Periodic Functions

By
J.P. Kahane

Tata Institute of Fundamental Research, Bombay
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Notes by
P.K. Raman

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Contents

Lecture 1

Introduction

1 Scope of the lectures

In the course of these lectures, we shall consider mean periodic functions of one variable. They are a generalization of periodic functions - a generalization carried out on a basis different from that of the generalization to almost periodic functions. This generalization enables us to consider questions about periodic functions such as Fourier-series, harmonic analysis, and later on, the problems of uniqueness, approximation and quasi-analyticity, as problems on mean periodic functions. For instance, the problems posed by S. Mandelbrojt (Mandelbrojt 1) can be considered as problems about mean periodic functions. In the two final lectures we shall consider mean periodic functions of several variables. 1

2 Definition of mean periodic functions

We consider three equivalent definitions of mean periodic functions. A periodic function $f(x)$, defined on the real line R and with period a, satisfies the equation $f(x) - f(x-a) = 0$. This can be written $\int f(x-y)d\mu(y) = 0$, where $d\mu(x)$ is a measure which is the difference of the Dirac measures at 0 and a. For our generalization, we consider a continuous complex valued solution of the integral equation

$$\int f(x-y)d\mu(y) = 0 \tag{1}$$

where $d\mu(y)$ is a measure with compact support, $\mu \neq 0$. [The support
2 of a continuous function $f(x)$ is the smallest closed set outside which $f(x)$ vanishes. [The support of a measure $d\mu$ is the smallest closed set outside of which the integral of any continuous function $f(x)$ (integral with respect to $d\mu$) is zero.]

Definition 1. Mean periodic functions are continuous complex-valued solutions of homogeneous integral equations of convolution type (1).

The introduction of mean periodic functions as solutions of (1) is due to J. Delsarte (Delsarte 1). His definition differs slightly from ours in that he takes $d\mu(y) = K(y)\,dy$, where $K(y)$ is bounded. When $K(y) = 1$ on $(0, a)$, and zero otherwise, the solutions of (1) are a-periodic functions, whose mean value is zero. This is the reason why Delsarte calls a function whose mean is zero, in the sense of (1), a "mean-periodic" function.

We study the equation (1), for a given $d\mu$.

(a) The solutions of (1) form a vector space over the complex numbers C. If a sequence $\{f_n\}$ of solutions of (1) converges uniformly on every compact set, to f, then f is again a solution of (1). For, (1) can be written as

$$\int f(y)d\nu_x(y) = 0, d\nu_x(y) = d\mu(x-y) \tag{1'}$$

and

$$\int f_n(y)d\nu_x(y) = 0 \Longrightarrow \int f(y)d\nu(y) = 0.$$

With f, each translate $f_\alpha(x) = f(x-\alpha)$ is also a solution of (1). Thus it is natural to consider the topological vector space $\mathscr{C} = \mathscr{C}(R)$ over C all complex-valued continuous functions f with the topology of compact convergence (uniform convergence on each compact). The solutions of (1) form a closed subspace of $\mathscr{C}$, invariant under translations.

3 (b) Solutions of (1) which are of a certain special type are easy to characterize: viz. those of the form $f(x) = e^{i\lambda x}$ where λ is a complex

number. For these solutions we have

$$\int e^{-i\lambda y} d\mu(y) = 0. \tag{2}$$

Other solutions of a similar type are of the form $f(x) = P(x)e^{i\lambda x}$, where $P(x)$ is a polynomial. For $P(x)e^{i\lambda x}$ to be a solution, we must have $\int P(x-y)e^{-i\lambda y} d\mu(y) = 0$ for all x. Considering this for $n+1$ different values of $x(n = \deg P)$, we find that, for $P(x)e^{i\lambda x}$ to be a solution of (1), it is necessary and sufficient that

$$\int y^m e^{-i\lambda y} d\mu(y) = 0 \qquad (0 \leq m \leq n = \deg P) \tag{3}$$

(a system of $n+1$ equations).

It is easy to solve (2) and (3) if one considers $M(w)$, the Fourier transform of $d\mu$: $M(w) = \int e^{-iwx} d\mu(x)$. $M(w)$ is an entire function. In order that $e^{i\lambda x}$ be a solution of (1) it is necessary and sufficient that $M(\lambda) = 0$. In order that $P(x)e^{i\lambda x}$ be a solution of (1) it is necessary and sufficient that $M(\lambda) = M^{(1)}(\lambda) = \cdots = M^{(n)}(\lambda) = 0$ where $M^{(j)}(w) = (-i^j \int x^j e^{-iwx} d\mu(x)$. These conditions follow from (2) and from (3). Thus the study of the solutions of (1) reduces to the study of the zeros of a certain entire function $M(w)$.

We call "simple solutions" the solutions $e^{i\lambda x}$ of (2) and $P(x)e^{i\lambda x}$ of (3). In French, the products $P(x)e^{i\lambda x}$ of a polynomial and an exponential are called "exponentielles-polynômos"; the linear combinations of $e^{i\lambda x}$ are often called "polynômes exponential" i.e. exponential polynomials; **4**
to avoid confusion, we shall use the term "polynomial-exponentials" for the translation of the word "exponentials polynomes" - then, the English order of the term is better than the French one -; if $P(x)$ is a monomial, $P(x)e^{i\lambda x}$ will be called "monomial exponential". We use this terminology later. Linear combinations of simple solutions and their limits are again solutions of (1). We are thus led to another natural definition of mean-periodic functions (as a generalization of periodic functions).

Definition 2. Mean-periodic functions are limits in $\mathscr{C}$ of linear combinations of polynomial exponentials $P(x)e^{i\lambda x}$ which are orthogonal in the sense of (3) to a measure with compact support.

A third natural definition occurs if we consider the closed linearsub-space of $\mathscr{C}$ spanned by $f \in \mathscr{C}$ and its translates: call this space $\tau(f)$.

Definition 3. f is mean-periodic if $\tau(f) \neq \mathscr{C}$.

This intrinsic definition is due to L. Schwartz (Schwartz 1). In order that f be a solution of (1) it is necessary that $\tau(f) \neq \mathscr{C}$. Thus (1) implies (3). Also (2) implies (1). We prove the equivalence later. We take (3) as the definition of mean-periodic functions since it is intrinsic and allows us to pose the problems of harmonic analysis and synthesis in the greater generality.

3 Problems considered in the sequel

We consider the following problems in the sequel.

(1) Equivalence of the definitions (1), (2) and (3).

5 (2) Harmonic analysis and synthesis.

Definition 3 permits us to consider this problem and its solutions allows us to prove (1).

(3) Spectrum of a function; the relation between the spectrum and the properties of the function-for example, uniqueness and quasi-analyticity when the spectrum has sufficient gaps.

(4) Relations between mean-periodic functions and almost periodic functions.

There are mean-periodic functions which are not almost periodic. For example e^x is mean-periodic since $e^{x+1} - e.e^x = 0$. Being unbounded it is not almost periodic. There are almost periodic functions which are not mean periodic. Let $f(x) = \sum a_n e^{i\lambda_n x}$ be an almost periodic function. We can take for $\{\lambda_n\}$ a sequence which has a finite limit point. Then $f(x)$ cannot be mean-periodic, as every $e^{i\lambda_n x}(a_n \neq 0)$ belongs to $\tau(f)$ and no function $M(w) \neq 0$ (Fourier transform of $d\mu$) can vanish on $\{\lambda_n\}$.

($4a$) Are bounded mean-periodic functions almost periodic?

($4b$) What are the properties of $\{\lambda_n\}$ in order that the almost periodic functions with spectrum $\{\lambda_n\}$ be mean-periodic?

(5) Given a set $\left\{P(x)e^{i\lambda x}\right\}$, is it total in $\mathscr{C}$?

A set is total in $\mathscr{C}$ if its closed span is $\mathscr{C}$. If it is not, is it possible that $\left\{P(x)e^{i\lambda x}\right\}$ is total in $\mathscr{C}(I)$, where I is an interval? For sets of the type $\{e^{i\lambda_n x}\}$ this is the problem posed by Paley and Wiener (Paley-Wiener).

(6) If $f \in \mathscr{C}(I)$, is it possible to extend f to a mean-periodic function? **6**

Problem (3) is related to questions in (Mandelbrojt 1) while problem (5) is related to the work of Paley-Wiener, Mandelbrojt, Levinson and Schwartz. Problems (5) and (6) can be posed analogously for analytic functions in an open set of the plane. This would give a new interpretation of some classical results on Dirichlet series and give some new results too.

Lecture 2

Some Preliminaries

1 Topological vector spaces

Let E be a vector space over the field C of complex numbers. E is a topological vector space when a topology is given on E, such that addition and scalar multiplication are continuous from $E \times E$ and $E \times C$ to E. We will confine ourselves only to those topological vector spaces which are locally convex and separated. The topology of a locally convex separated vector space E is specified by a family $\{p_i\}_{i \in I}$ of semi-norms such that for every $x \in E, x \neq 0$, there is an $i \in I$ with $p_i(x) \neq 0$. The dual E' of E is the set of all continuous linear functionals on E. If $x' \in E'$ we write $x'(x) = \langle x, x' \rangle$. Then E and E' are in duality in the sense that **7**

D_1) $\langle x, x' \rangle = 0$ for every $x \in E$ implies $x' = 0$.

D_2) $\langle x, x' \rangle = 0$ for every $x \in E'$ implies $x = 0$.

D_1) is the statement that x' is the zero functional and D_2) is given by the theorem of Hahn-Banach. (Bourbaki, Chap.*II* and *III*).

Condition of F. Riesz. Let F be a closed subspace of E generated by $\{x_i\}_{i \in I}$. In order that $x \in E$ belong to F it is necessary and sufficient that $< x_i, x' >= 0$ for every $i \in I$ should imply $\langle x, x' \rangle = 0$.

The necessity is obvious and the sufficiency is a consequence of the Hahn-Banach theorem.

We recall some examples of classical vector spaces. First come the very well-known Banach spaces $L^p(R^n)$ and $L^p(I^n), 1 \leq p \leq \infty$. Let K **8** be a compact subset of R^n.$\mathscr{C}(K)$ is the space of continuous (complex-valued) functions on K with the topology of uniform convergence. It is a Banach space and its dual $\mathscr{C}'(K)$ is the space of Radon measures on K.

Let Ω be an open subset of R^n.$\mathscr{C}(\Omega)$ is the space of continuous functions on Ω with the compact convergence topology (i.e., uniform convergence on every compact set of Ω). It is an $\mathscr{H}$ -space (a Frechet space, i.e., locally convex, metrisable and complete). The dual $\mathscr{C}'(\Omega)$ is the space of Radon measures with compact support in Ω. The duality is denoted by $\int f d\mu$.

$D(R^n)$ is the space of C^∞-functions (indefinitely differentiable functions) with compact support. It is an $\mathscr{L}\mathscr{H}$-space (inductive limit of $\mathscr{H}$-spaces). Its dual $D'(R^n)$ is the space of distributions.

$\mathscr{E}(R^n)$ is the space of C^∞-functions. It is an $\mathscr{H}$- space and its dual $\mathscr{E}'(R^n)$ is the space of distributions with compact support. (The support of a distribution is the smallest closed set such that $< f, T >= T.f$ vanishes whenever f vanishes on this closed set.)

These spaces were introduced by L. Schwartz(Schwartz 4).

2 Basis in a topological vector space

Let E be a locally convex separated vector space over C. A set $\{x_i\}_{i\in I}$ of elements of E is a *total* set in E if for every $x \in E$ there are sums $\sum\limits_{i=1}^{N} \xi_N^i x_i \longrightarrow x$ as $N \to \infty$ ($\xi_N^i \in C$).

A set $\{x_i\}_{i\in I}$ of elements of E is *free* if $0 = \lim\limits_{N\to\infty} \sum \xi_N^i x_i$ implies $\lim\limits_{N\to\infty} \xi_N^i = 0$ for every $i \in I$. This implies that if $x = \lim\limits_{N\to\infty} \sum \xi_N^i x_i \lim\limits_{N\to\infty} \sum \eta_M^i x_i$ then $\xi = \lim\limits_{N\to\infty} \xi_N^i = \lim\limits_{M\to\infty} \eta_M^i$.

9 A set $\{x_i\}_{i\in I}$ of elements of I is a *basis* if it is total and free.

Remark. In order that $\{x_i\}_{i\in I}$ be a basis of E it is necessary and sufficient that for every $x \in E$ we have $x = \lim\limits_{N\to\infty} \sum \xi_N^i x_i$ and for each i, $\lim\limits_{N\to\infty} \xi_N^i = \xi^i$, the ξ^i being uniquely determined. Then the ξ^i are called the components of x with respect to the basis.

3 Problems of harmonic analysis and synthesis

Let E be a topological vector space of functions defined on an abelian group G; $\tau(f)$ the closed subspace spanned by the translates of f. There may be in E subspaces which are closed, invariant under translations, of finite dimension ≥ 1, and not representable as a sum of two such subspaces (the subspaces generated in $\mathscr{C}$ by $e^{i\lambda x}$ or by $x^p e^{i\lambda x}$, $p = 0, 1, \ldots n$, are of this type); we shall call such subspaces "simple subspaces".

The problem of harmonic analysis can be formulated as the study of simple subspaces contained in $\tau(f)$. The *problem of spectral synthesis* is this : Is it possible to consider f as the limit of finite sums, $\sum f_n$, of f_n belonging to simple subspaces contained in $\tau(f)$? Practically, we know a priori a type of simple subspaces, and we ask whether analysis and synthesis are possible with only these simple subspaces; if it is possible, we know perfectly the structure of $\tau(f)$, and we can recognize whether the only simple subspaces are those we know already. The problems 10
of analysis and synthesis are usually solved by means of the theory of duality. We give here a well-known example for illustration.

Let G be the one-dimensional torus T (circle) and let $E = \mathscr{C}(T)$. Then we can write the equation:

$$a_n = \frac{1}{2\pi}\int_T f(x)e^{-inx}d$$

as $$a_n e^{iny} = \frac{1}{2\pi}\int_T f(x)e^{-inx(x-y)}dx = \frac{1}{2\pi}\int_T f(x+y)e^{-inx}dx$$

If $a_n \neq 0$ then e^{iny} is the limit of linear combinations of translates of f, as the following calculations, taking the uniform continuity of $f(x)e^{inx}$ into account, show:

$$\begin{aligned}\Big| a_n e^{iny} - \frac{1}{2\pi}\sum (x_{i+1}-x_i) f(x_i+y)e^{-inx_i}\Big| \\ \leq \sum \frac{1}{2\pi}\left|\int_{x_i}^{x_{i+1}}\left[f(x+y)e^{-inx} - f(x_i+y)e^{-inx_i}\right]dx\right| < \varepsilon.\end{aligned}$$

Conversely if $e^{inx} \in \tau(f), a_n \neq 0$. For, given an $\varepsilon > 0$ and $\varepsilon < 1$ we have $\left|e^{iny} - \sum .\alpha_i f(x_i + y)\right| < \varepsilon$ for all y and

$$\left|1 - \left(\sum \alpha_i e^{inx_i}\right) a_n\right| \leq \frac{1}{2\pi} \int_T \left|e^{iny} - \alpha_i f(x_i + y)\right| dy.$$

This gives $a_n \neq 0$. So in order that $e^{inx} \in \tau(f)$, or again, for $\tau(e^{inx}) \subset \tau(f)$, it is necessary and sufficient that $a_n \neq 0$. Thus to specify the "simple subspaces" of $\tau(f)$[viz. $\tau(e^{inx})$] we define the *spectrum* $S(f)$ of f to be the set of integers λ_n such that $e^{i\lambda_n x} \in \tau(f)$. Then $\lambda_n \in S(f)$ if and only if a $\lambda_n \neq 0$. On the other hand the answer to the problem
11 of spectral synthesis is given by the theorem of Fejer. (Titchmarsch p. 414) (Zygmund 1).

A second method for solving the problem of harmonic analysis is to apply the condition of Riesz : $e^{inx} \in \tau(f)$ if and if every linear functional vanishing over the translates of f also vanishes at e^{inx}. In other words $f(x + y)d\mu(-x) = 0$ for every y implies $\int e^{inx} d\mu(-x) = 0$. But if $f \sim \sum a_n e^{inx}$ and $d\mu \sim \sum b_n e^{inx}$ are the Fourier developments of f and $d\mu$ then $\int f(x+y)d\mu(x) \sim \sum a_n b_n e^{iny}$ and $\int e^{inx} d\mu(-x) = b_n$. Then, "$a_n b_n = 0$ for every n implies $b_n = 0$" follows from $a_n \neq 0$. The converse is easy.

We can have such a theory for $G = Z$, the group of integers or $G = R$ and E the space $\mathscr{C}$ or $L^p, 1 \leq p \leq \infty$. Obviously we cannot have harmonic synthesis for L^∞ with the strong (normed) topology. For example in the case of $G = T$, the circle, this will imply that every function in $L^\infty(T)$ is continuous. But for the space $L^\infty(Z)$ or $L^\infty(R)$ with the weak topology [for this notion cf. (Bourbaki chap. IV)] the problem of harmonic analysis and synthesis is not solved. For example in $L^\infty(Z)$ let $f = \{f_n\}$ and define the spectrum $S(f)$ of f by $\lambda \in S(f) \Longleftrightarrow \{e^{i\lambda n}\} \in \tau(f)$. If for every $g \in L^1(Z)$, such that $\sum g_n f_{n+m} = 0$ for every m we have $\sum g_n e^{i\lambda n} = 0$, then $\{e^{i\lambda n}\} \in \tau(f)$. Conversely if, for every $\lambda \in S(f), \sum g_n e^{i\lambda n} = 0$, does it follow that $\sum g_n b_n = 0$? This is a problem about absolutely convergent Fourier series.

Lecture 3

Preliminaries (Continued)

1 Fourier transforms of distributions with compact support and the theorem of Paley-Wiener

Let T be a distribution (in particular a measure) with compact support. **12**
We call the *segment of support* of T the smallest closed interval $[a, b]$ containing the support of T. Let $d\mu$ be a measure with compact support. Its Fourier-transform $\mathscr{C}(d\mu) = M(w) = \int e^{-ixw} d\mu(x)$ is an entire function. Writing $w = u + iv$ we have:

$$\left|M(w)\right| < e^{bv} \int \left|d\mu\right|, v \geq 0;$$
$$\left|M(w)\right| < e^{av} \int \left|d\mu\right|, v \leq 0$$

where $[a, b]$ is the segment of support of $d\mu$. Thus $M(w)$ is an entire function of exponential type bounded on the real line, i.e.,:

$$\left|M(w)\right| \leq Ke^{c|w|}; M(u) = 0(1) \tag{1}$$

Moreover

$$M(iv) = 0(e^{bv}), v > 0 \; M(iv) = 0(e^{av}), v < 0. \tag{2}$$

We shall try to find, whether the condition (1) is sufficient in order that an entire function $M(w)$ be the Fourier-transform of a measure $d\mu$

with compact support, and whether condition (2) is sufficient to prove that the support of $d\mu$ is contained in $[a, b]$. In fact, (1) is not sufficient, and we must replace it by a slightly stronger condition.

Theorem of Paley-Wiener. Let $M(w)$ be an entire function of exponential type satisfying the condition:

$$\left|M(w)\right| \leq Ke^{c|w|}; M(u) = 0\left(\frac{1}{|u|^2}\right) \qquad (1')$$

13 *Then $M(w)$ is the Fourier transform $\mathscr{C}(d\mu)$ of a measure with compact support.*

The proof runs in three parts.

(a) Theorem of Phragmen-Lindel of: let $\varphi(z)$ be a function, holomorphic for $\left|\arg\ z\right| \leq \alpha < \dfrac{\pi}{2}$; we suppose that $\varphi(z) = 0(e^{\varepsilon|z|})$ for every $\varepsilon > 0$ and that $|\varphi(z)| \leq B$ on $|\arg\ z| = \alpha$. Then $|\varphi(z)| \leq B$, when $|\arg\ z| \leq \alpha$

We take $\varphi(z)e^{-\varepsilon' z}$ and to this we apply the principle of Maximum Modulus to get the result. (Titchmarsh 5.62)

(b) The result (a) is translated into a theorem for a function $\psi(w)$ holomorphic in $0 \leq\ \arg\ w \leq \dfrac{\pi}{2}$.

Let $\psi(w)$ be holomorphic in this domain, bounded on the boundary, and let $(*)\,\psi(w) = 0(e^{\eta|w|^{2-\varepsilon}})$ for some $\varepsilon > 0$ and for every $\eta > 0$. Then $\psi(w) = 0(1)$ when $0 \leq \arg w \leq \dfrac{\pi}{2}$.

We take $\varphi(z) = \psi(w), z = w^{2-\varepsilon}$ and apply (a). We can replace $(*)$ by $\psi(w) = 0(e^{|w|^{\alpha}})$ for some $\alpha < 2$.

(c) **Proof of the theorem**: by $(1')$, the function $w^2 M(w)e^{icw}$ satisfies the conditions of (b) in $0 \leq \arg\ w \leq \dfrac{\pi}{2}$ and $\dfrac{\pi}{2} \leq \arg w \leq \pi$ and the function $w^2 M(w)e^{-icw}$ satisfies the same conditions in the other quadrants and so we have $M(w) = 0(\dfrac{e^{c|v|}\,{}^2}{|w|})$. As $M(u) = 0(\dfrac{2}{u^2})$, $M(u)$ admits a co-Fourier transform (Conjugate Fourier Transform), $f(x)$ given by

$f(x) = \dfrac{1}{2\pi}\int M(u)e^{ixu}du. M(w)e^{ixw}$ is an entire function. We apply Cauchy's theorem to this function along the rectangular contour 14
with sides $-R \leq u \leq R; R + iv, 0 \leq v \leq v_o; u + iv_o, -R \leq u \leq R; -R + iv, 0 \leq v \leq v_o$. Letting $R \to \infty$ we have the equation $\dfrac{1}{2\pi}\int M(u)e^{ixu}du = \dfrac{1}{2\pi}\int M(u)e^{ixu}du, w = u+iv_o, \int = \int\limits_{-\infty}^{\infty}$ Therefore $f(x) = \dfrac{1}{2\pi}\int M(w)e^{ixu}du$. Suppose $x > c$. Then there exists c' with $x > c' > c'$,

$$\left|M(w)e^{ixw}\right| K\frac{e^{c|v|}}{|w|^2}e^{-c'v} < K'\frac{e^{c|v|-c'v}}{1+u^2}.$$

This is true for every $v = v_0 > 0$. Allowing $v_0 \to \infty$ we find $f(x) = 0$. In a similar manner, if $x < -c$ we have $f(x) = 0$. This means that the support of $f(x) \subset [-c, c]$, Then $M(w) = \int f(x)e^{-ixw}dx = \int e^{-ixw}d\mu(x), d\mu(x) = f(x)dx$.

2 Refinements and various forms of the theorem of Paley-Wiener

The second part of the theorem of Paley-Wiener consists in proving that the condition (2) (which is merely a condition on the behaviour of $M(w)$ on the imaginary axis) is sufficient to know that the segment of support of $d\mu$ is $[a, b]$. This second part is due to Polya and Plancherel. (Plancherel).

Theorem. *We suppose that the entire function $M(w)$ satisfies the conditions:*

$$M(w) \leq Ke^{c|w|} \text{ and } M(u) = 0\left(\frac{1}{|u|^2}\right) \tag{1'}$$

$$M(iv) = 0(e^{bv}), v > 0; M(iv) = 0(e^{av}), v < 0. \tag{2'}$$

Then $M(w) = \mathscr{C}(d\mu)$ and the segment of support of $d\mu$ is contained is $[a, b]$.

Proof. We have seen, from the first part of Paley-Wiener theorem, that 15
(1′) gives us that $M(w) = \mathscr{C}(d\mu)$, with $d\mu = f(x)dx$, and $f(x) = 0$ for $|x| > c$. Condition (2′) gives $\left|M(w)e^{ibw}\right| = 0(1)$ on the positive part of the imaginary axis. Then it is easy to see that $f(x) = 0$ for $x \geq b$ following the same line of argument as in part (c) of the first part of the theorem of Paley-Wiener. In a similar fashion we prove that $f(x) = 0$ for $x < a$. ◻

Now we study $d\mu = f(x)dx$, where $f \in \mathscr{D}(R)$ and the segment of support of f is $[a, b]$. Then $M(w) = \mathscr{C}(d\mu)$ satisfies:

(1′′) $M(w)$ is an entire function of exponential type with $M(u) = 0(\frac{1}{|u|^n})$ for every integer n;

(2′′) $\lim\limits_{v\to\infty} \sup \dfrac{\log|M(iv)|}{v} = b$, $\lim\limits_{v\to\infty} \inf \dfrac{\log|M(iv)|}{v} = a$.

For $M(w) = \dfrac{1}{2\pi} f(x)e^{-ixw}dx = \dfrac{1}{2\pi}\dfrac{1}{(iw)^n}\int f^{(n)}(w)e^{-ixw}dx$ for every n.

So we have (1′′). Conversely if we have 1′′), then $M(w) = \mathscr{C}(f)$ with $f \in \mathscr{D}(R)$, for we have already $f(x) = \dfrac{1}{2\pi}\int M(u)e^{ixu}du$ and 1′′) gives us that $f \in \mathscr{D}(R)$.

(2′′) implies (2′) with a' and b' in (2′), $a' < a < b < b'$, and so we have the segment of support of f actually $[a, b]$. Hence

Theorem. *Conditions* (1′′) *and* (2′′) *are necessary and sufficient in order in order that* $M(w) = \mathscr{C}(f)$ *with* $f \in \mathscr{D}(R)$ *and support of* $f = [a, b]$.

Now we generalize the theorem to distributions. Let $\tau \in \mathscr{E}'(R)$
16 be a distribution with the segment of support $[a, b]$. Its Fourier transform is $T(w) = \langle\tau, e^{-ixw}\rangle$. We know that τ is a finite linear combination of derivatives of measures (Schwartz 4, Chap.*III*, theorem 26) $\tau = \sum \dfrac{d^n}{dx^n}(d\mu_n)$. So $T(w)$ can be written as:

$$T(w) =< \sum \frac{d^n}{dx^n} d\mu_n, e^{ixw} >=< d\mu_n, \pm\frac{d^n}{dx^n} e^{-ixw} >= \Sigma w^n i^n M_n(w)$$

Thus we have $T(w)$ satisfying the following conditions:

(1‴) $T(w)$ is an entire function of exponential type with $\left|T(u)\right| = 0(|u|^N)$ for some N.

(2‴) $\limsup\limits_{v\to\infty} \dfrac{\log\left|T(iv)\right|}{v} = b$, $\lim\inf\limits_{v\to-\infty} \dfrac{\log|T(iv)|}{v} = a$,

Conversely let $T(w)$ satisfy conditions (1‴) and (2‴). From (1‴) we can write $T(w) = P(w)M(w)$, where $P(w)$ is a polynomial and $M(w)$ satisfies (1′). This implies that $M(w) = \mathscr{C}(d\mu)$ and $T(w) = \mathscr{C}(\sum\limits_{o}^{N} a_n \dfrac{d^n}{dx^n} d\mu) = \mathscr{C}(\tau)$. $\tau \in \mathscr{E}'(R)$. We find as before that (2′) implies that the segment of support of τ is $[a, b]$.

Theorem. *Conditions* (1‴) *and* (2‴) *are necessary and sufficient in order that* $T(w) = \mathscr{C}(\tau)$ *with* $\tau \in \mathscr{C}'(R)$ *and support of* $\tau = [a, b]$.

The theorem of Paley-Wiener gives an easy characterisation of the Fourier transforms of the $f \in \mathscr{D}(R)$ or $T \in \mathscr{E}'(R)$. The original form of the theorem of Paley-Wiener states that a necessary and sufficient condition in order a function be the Fourier transform of a function $\in \mathscr{E}' \bigcap L^2$, is that it should be of exponential type, and $\in L^2$ on the real axis. This last statement results from the preceding one, concerning the Fourier forms of $T \in \mathscr{E}'$, and from the invariance of L^2 under Fourier **17**
transformation.

3 Theorem of Hadamard

We recall the classical theorem of Hadamard (Titchmarsch), *Let* $M(w)$ *be an entire function of exponential type with zeros*(other than 0) $\{\lambda_n\}$. *Then* $M(w) = Ke^{aw}w^k \prod \left(1 - \dfrac{w}{n}\right) e^{w/\lambda_n}$ *and* $\sum \dfrac{1}{|\lambda_n|^2} < \infty$

Later on we shall give more properties of functions of exponential type.

4 Convolution product and its simple properties

Let f and g be two functions (integrable, continuous of C^∞ functions) with segment of support $[a, b]$ and $[c, d]$. The convolution of f and g is the function $h = f * g$ defined by

$$h(y) = \int f(y - x)g(x)dx = \int f(x)g(y - x)dx.$$

The convolution product is commutative ($f * g = g * f$), associative ($(f * g) * k = f * (g * k)$) and distributive with respect to addition.

For $y < a + c, h(y) = 0$ and for $y > b + d, h(y) = 0$. So the segment of support of $f * g$ is contained in the segment of support of $f+$ segment of support of g.

$$\text{If } \ell \in \mathscr{C}, \int h(y)\ell(y)dy = \iint \ell(x + x')f(x')g(x)dxdx'.$$

This allows us to define the convolution of two measures with compact, support by means of duality. The convolution of two measures $d\mu_1, d\mu_2 \in \mathscr{C}'$ is a measure $d\nu \in \mathscr{C}', d\nu = d\mu_1 * d\mu_2$ defined by

$$\langle \ell, d\nu \rangle = \iint \ell(x + y)d\mu_1(x)d\mu_2(y)$$

18 for every $\ell \in \mathscr{C}$. The convolution product is again associative, commutative and distributive with respect to addition. We have segment of support of $d\nu \subset$ segment of support of $d\mu_1+$ segment of support of $d\mu_2$. Indeed if the support on $\ell \subset (-\infty, a+c)$ or if the support of $\ell \subset (b+d, \infty)$ then $\int \ell d\nu = 0$.

We have the same definition, by duality, for two distributions with compact support, $S = T_1 * T_2$ is the distribution defined by $\langle S, \ell \rangle = \langle \ell(x+y), T_{1x}.\ T_{2y} \rangle$ for every $\ell \in \mathscr{E}$, $T_{1x}.\ T_{1y}$ being the Cartesian product of T_1 and T_2. (Schwartz 4 Chap. IV).

By considering the duality between distributions (respectively measures) with non-compact support and $\mathscr{D}(R)$ (respectively the space of continuous functions with compact support), we can define in the same way the convolution of a distribution T_1 (respectively measure $d\mu_1$) and a distributive T_2 (respectively measure $d\mu_2$) with compact support and

$S = T_1 * T_2$ (respectively $d\nu = d\mu_1 * d\mu_2$) will be in general a distribution (respectively measure) with non-compact support. For example

$\delta_a * f = f(x-a) = f_a$; $\dfrac{d}{dx} * f = f'$ etc. (Schwartz 4, Chap. VI).

The convolution of several distributions (or measures), all but one of which has a compact support, is associative and commutative.

If $d\mu_1, d\mu_2, T_1, T_2$ have compact supports, we have immediately

$$\mathscr{C}(d\mu_1 * d\mu_2) = \mathscr{C}(d\mu_1)\mathscr{C}(d\mu_2), \mathscr{C}(T_1 * T_2) = \mathscr{C}(T_1)\mathscr{C}(T_2)$$
$$e^{i\lambda x}T_1 * e^{i\lambda x}T_2 = e^{i\lambda x}(T_1 * T_2)$$

($\mathscr{C} = \langle T_x, e^{i\lambda x}\rangle$ = Fourier transform of T). We can extend these equalities to more general cases. For example, the first holds if $d\mu_1$ has a compact support and $\int \left|d\mu_2\right| < \infty$; the third holds whenever T_1 has a compact support.

The convolution by a measure with compact support transforms a continuous function into a continuous function, and also (see "L' integration dans les groupes topologiques" by Andre Weil) a function which is locally $\in L^p$ into a function which is locally $L^p (p \geq 1)$. **19**

Lecture 4

Harmonic analysis for mean periodic functions on the real line

1 Equivalence of definitions I and III

The study of the problem of harmonic analysis and synthesis for mean periodic functions will allow us to prove the equivalence of our definitions of mean periodic functions. We take definition III as the basic definition, viz., f is mean periodic if $\tau(f) \neq \mathscr{C}$. By the condition of Riesz $\tau(f) \neq \mathscr{C}$ if and only if there exists a $d\mu \in \mathscr{C}'$, $d\mu \neq 0$, orthogonal to f_y for every y. Writing this in an equivalent form as **20**

$f * d\mu = 0, d\mu \neq 0$ we see that definitions I and III are equivalent.

2 Carleman transform and spectrum of a function

As preliminary to the study of harmonic analysis and synthesis we introduce the Carleman transform of a mean periodic function

We put

$$\begin{array}{lll} f^+(x) = 0, & f^-(x) = f(x) & \text{for } x < 0 \\ f^+(x) = f(x), & f^-(x) = 0 & \text{for } x \geq 0 \end{array}$$

From (1) we can define $g(y) = f^- * d\mu = -f^+ * d\mu$.

Lemma. *Segment of support of support of $g \subset$ segment of support of $d\mu$.*

Let $[a, b]$ be the segment of support of $d\mu$. Then

$$g(y) = -\int_o^\infty f(x)d\mu(y-x), \quad \text{and} \quad y < a \text{ implies} \quad g(y) = 0.$$
$$g(y) = \int_{-\infty}^o f(x)d\mu(y-x), \quad \text{and} \quad y < b \text{ implies} \quad g(y) = 0$$

Let $M(w) = \int e^{-ixw}d\mu(x)$ and $G(w) = \int e^{-ixw}g(x)dx$.

21 **Definition.** We define the Carleman transform of a mean periodic function f to be the meromorphic function $F(w) = G(w)/M(w)$.

This definition is independent of the measure $d\mu$. In fact suppose $f * d\mu = 0$ and $f * d\mu_i = 0$. Let $g_1 = f^- * d\mu_1$. We have $g_1 * d\mu = f^- * d\mu_1 * d\mu = f^- * d\mu * d\mu_1 = g * d\mu_1$ and so $G(w)M(w) = G(w)M_1(w)$.

The original definition of Carleman for functions which are not very rapidly increasing at infinity cannot be applied to every mean periodic function. But we shall see later that his definition coincides with ours in the special case that $|f(x)| = 0(e^{a|x|})$. (Lecture VI).

Is the quotient of two entire functions of exponential type the Carleman transform of a mean periodic function? Let us look at this question.

If $F(w) = G(w)/M(w)$, $G(w)$ and $M(w)$ two functions of exponential type, in order that $F(w)$ be the Carleman transform of a function $f(x)$ it is necessary that if $a_G, b_G; a_M, b_M$ are the constants of condition (2′) (lecture 3) for G and $M, a_M \leq a_G \leq b_G \leq b_M$. Examples may be constructed to show that this condition is not sufficient even to assert that $F(w)$ is the Carleman transform of a mean periodic distribution. (A mean periodic distribution is defined by the intrinsic property $\tau(T) \neq \mathscr{D}'$ (lecture V 3).

We now proceed to relate the "simple solutions" to the poles of the Carleman transform.

Lemma.

$$f_1(x) = e^{-i\lambda x} f(x) \Rightarrow F_1(w) = F(w + \lambda).$$

Set $$d\mu_1 = e^{-i\lambda x}d\mu, \; f_1(x) = e^{-i\lambda x} f(x).$$

22 Then $g_1(x) = e^{-i\lambda x}g(x)$. For $f * d\mu = 0 \Rightarrow fe^{-i\lambda x} * e^{-i\lambda x}d\mu = 0$. So $f_1 * d\mu_1 = 0$ and

$$\begin{aligned} g_1(x) &= f_1^- * d\mu_1 = e^{-i\lambda x}f^-(x) * e^{-i\lambda x}d\mu(x) = e^{-i\lambda x}(f^- * d\mu) \\ &= e^{-i\lambda x}g(x). \end{aligned}$$

Thus we have $G_1(w) = G(w + \lambda)$ and so $F_1(w) = F(w + \lambda)$.

Now $e^{i\lambda x} \in \tau(f) \Leftrightarrow (f * d\mu = 0 \Rightarrow e^{i\lambda x} * d\mu = 0 = M(\lambda))$.

$P(x)e^{i\lambda x} \in \tau(f) \Leftrightarrow P(x) \in \tau(f_1)$, $f_1 = e^{-i\lambda x}f(x)$.

$P(x) \in \tau(f_1) \Leftrightarrow P(x + y) \in \tau(f_1)$ for every y.

Hence $P(x) \in \tau(f_1) \Leftrightarrow x^p \in \tau(f_1)$, $p = 0, 1, 2, \ldots\ldots n$ = degree of P.

Now $x^p \in \tau(f_1)$, $p = 0, 1, \ldots\ldots, n \Leftrightarrow$ for every $d\mu_1$ with $f_1 * d\mu_1 = 0$ we have

$$M_1(0) = M_1^{(1)}(0) = \ldots\ldots = M_1^{(n)}(0) = 0(M_1 = \mathscr{C}(d\mu_1))$$

This proves

Lemma.

$$\begin{gathered} P(x)e^{i\lambda x} \in \tau(f) \Leftrightarrow \textit{for every } d\mu \textit{ with } f * d\mu = 0, \\ M(\lambda) = M^{(1)} = \ldots\ldots = M^{(n)}(\lambda) = 0 \\ (M = \mathscr{C}(d\mu), n = \textit{degree of } P) \end{gathered}$$

We shall prove that in this case λ is a pole of order $\geq n$ (= degree of P) of $F(w)$. 23

Theorem. *$P(x)e^{i\lambda x} \in \tau(f) \Leftrightarrow \lambda$ is a pole of order $\geq n$ (degree of P) of $F(w)$.*

First we prove that λ is a pole. Without loss of generality we take $\lambda = 0$ (replacing f by f_1, if necessary). Suppose $M(0) = 0$.

If $G(0) \neq 0$ there is nothing to prove. Suppose $G(0) = 0$. Then we can define

$$d\mu_1(x) = \int_{-\infty}^{x} d\mu(t) - \int_{x}^{\infty} d\mu(t)$$

$$g_1(x) = \int_{-\infty}^{x} g(t)dt = -\int_{x}^{\infty} g(t)dt$$
$$M_1(w) = \int e^{-ixw} d\mu_1(x) = \frac{-iM(w)}{w}$$
$$G_1(w) = \int e^{-ixw} g_1(x)dx = \frac{-iG(w)}{w}$$

Moreover we have $g_1(x) = \int\limits_{-\infty}^{0} f^-(t)d\mu_1(x-t)$ because the derivatives are the same, and the functions are 0 when $x \to +\infty$. For a similar reason (with $x \to -\infty$) we have $g_1(x) = -\int f^+(t)d\mu_1(x-t)$. Thus we have $f * d\mu_1 = 0$ and so $M_1(0) = 0$. If $G_1(0) \neq 0$, then $F(w)$ has a pole at the origin. If $G_1(0) = 0$ we iterate this method and finally arrive at $M_m(w) = -iM_{m-1}(w)/w, G_m(w) = -iG_{m-1}(w)/w$ with $M_m(0) = 0$ and $G_m(0) \neq 0$. Thus we see that λ is a pole.

To see that λ is a pole of order $\geq n$ (degree of P) by the above lemma, we have merely to replace the condition $M(0) = 0$ by $M(0) = M^{(1)}(0) = \ldots.\ M^{(n)}(0) = 0$ in the above construction.

24 The above construction gives us the following corollary.

Corollary. *Suppose $M(\lambda) = 0$ and λ is not a pole of $F(w)$. Then $M(w)/_{(w-\lambda)} = \mathscr{C}(p_\lambda)$ with $f * p_\lambda = 0$ and segment of support of $p_\lambda \subset$ segment of support of $d\mu$.*

Definition. The spectrum $S(f)$ of a mean periodic function $f(x)$ is defined to be the set of poles of $F(x)$, each counted with its order of multiplicity.

3 The problem of harmonic synthesis

It will turn out that the only "simple subspaces" in $\mathscr{C}$ are generated by the translates of a polynomial exponential. In order to answer the problem of harmonic synthesis, we shall try to prove that f can be approximated by sums of polynomial exponentials belonging to $\tau(f)$.

Lemma. *Suppose $d\nu \in \mathscr{C}'$ and f mean periodic. Then $\varphi = f * d\nu$ is also mean periodic.*

This results from $\varphi * d\mu = f * d\nu * d\mu = 0$.

We shall study the spectrum of φ. Since $d\nu$ has compact support, for $|x| \geq x_0, x_0$ sufficiently large, $\varphi^- = f^- * d\nu$ coincide. Thus we can write $\varphi^- = f^- * d\nu + h, h(x)$ being a function with compact support. Let Φ, N, H be the Fourier transforms of $\varphi, d\nu$ and h. We have the following equations:

$$\begin{aligned} \varphi^- * d\mu &= (f^- * d\nu * d\mu) + (h * d\mu) \\ \Phi(w)M(w) &= G(w)N(w) + H(w)M(w) \\ \Phi(w) &= F(w)N(w) + H(w) \end{aligned}$$

Thus we have the following lemma : **25**

Lemma. *The spectrum of $\varphi = f * d\nu$ is the set of poles of $F(w)N(w)$.*

Now our problem requires for solution the result that $P_n(x)e^{i\lambda x} * d\nu = 0$ for every $P_n(x)e^{i\lambda x} \in \tau(f)$ implies $f * d\nu = 0$. This is just the reformulation of the problem using the condition of Riesz. Now $P_n(x)e^{i\lambda x} * d\nu = 0$ for every $P_n(x)e^{i\lambda x} \in \tau(f)$ implies $N(w)$ vanishes on the spectrum of f. Thus, $F(w)N(w)$ is an entire function. By the above lemma, the solution of the problem of synthesis will follow if we prove that a mean periodic function f whose spectrum is void is zero.

Lecture 5

Harmonic synthesis for mean-periodic functions on the real line

1 Solutions of the problem of harmonic synthesis

We give the solution of the problem of harmonic synthesis by proving 26
that if the spectrum of f is void then f is identically zero.

Suppose that $(1) f * d\mu = 0$ and the spectrum of is void. First we prove that $(1') f * x d\mu = 0$. We shall suppose that $d\mu = p dx$ where p is sufficiently differentiable (say, N-times differentiable). This can be done by convoluting $d\mu$, if necessary, with a suitable differentiable function. In this case, we have

$$\mathscr{C}(d\mu) = M(w) = 0\left(\frac{1}{{}_{u}N}\right) \text{ for some } N \geq 2.$$

The corollary of a theorem in 2 of lecture IV, gives us the following lemma:

Lemma. *Suppose $M(\lambda) = 0$ and $F(w)$ has no pole at λ. Then $M(w)/(w-\lambda) = \mathscr{C}(p_\lambda)$, with $f * p_\lambda = 0$ and segment of support of $p_\lambda \subset$ segment of support of $d\mu$.*

Now by the theorem of Hadamard, we have

$$M(w) = \mathscr{C}(p) = Ke^{aw}w^k \prod \left(1 - \frac{w}{\lambda_n}\right) e^{\frac{w}{\lambda n}} \text{ and } \sum \frac{1}{|\lambda n|^2}$$

$$\frac{M'(w)}{M(w)} = a + \frac{k}{w} + \sum \left(\frac{1}{w-\lambda} + \frac{1}{\lambda n}\right)$$

27 Since $M'(w) = \mathscr{C}(-ixp)$ and $M(w)/(w-\lambda) = \mathscr{C}(p_\lambda)$, we have

$$\mathscr{C}(-ixp) = a\mathscr{C}(p) + k\mathscr{C}(p_0) + \sum_n (\mathscr{C}(p_{\lambda n}) + \frac{1}{\lambda n}\mathscr{C}(p)). \qquad (*)$$

From the equation $(*)$ we want to have the equation

$$-ixp = a\ p + k\ p_0 + \sum_n (p_{\lambda_n} + \frac{1}{\lambda_n} p) \qquad (**)$$

and the equation

$$f * (-ixp) = a\ f * p + k\ f * p_0 + \sum_n (f * p_{\lambda_n} + \frac{1}{\lambda_n} f * p) = 0. \quad (***)$$

To pass from $(*)$ to $(**)$, it is sufficient to have convergence in L_1 for the summation in $(*)$ and to pass form $(**)$ to $(***)$ it is sufficient to have uniform convergence in $(**)$, since each term in it has support in the same interval.

We write

$$X_n(w) = \frac{M(w)}{w-\lambda_n} + \frac{M(w)}{\lambda n} = \frac{wM(w)}{\lambda_n(w-\lambda_n)} = \frac{w^2 M(w)}{\lambda_n(w-\lambda_n)w}$$

$$|w-\lambda_n| > \frac{\lambda_n}{2} \Rightarrow |X_n| < 2|w\ M(w)|/|\lambda_n|^2 \qquad (1)$$

$$\left|\frac{\lambda_n}{2}\right| > |w-\lambda_n| > 1 \Rightarrow |X_n| < 2|w^2 M(w)|/|\lambda_n|^2 \qquad (2)$$

$$|w-\lambda| < 1 \Rightarrow \left|\frac{M(w)}{w-\lambda_n}\right| \leq | \sup_{|w'-w|=2} \left|\frac{M(w')}{w'-\lambda_n}\right| \leq | \sup_{|w'-w|=2} |M(w')|$$

$$\Rightarrow |X_n| \leq \frac{4|w|^2}{|\lambda_n|^2} \sup_{|w'-w|=2} |M(w')| for |\lambda_n| > x_0. \qquad (3)$$

Since $\left|M(w) < K_i e^{a|v|}/|w|^N\right|$ for some N, we have, in each of the above three cases $\left|X(u)\right| < K/\left|u\right|^{N-2}\left|\lambda_n\right|^2$. Now we have a uniform majorization of each term in the summation of (*) so that the sum is ab- 28
solutely convergent and we also have $\left|P_{\lambda_n} + P/\lambda_n\right| = 0(\frac{1}{|\lambda_n|^2}2)$. Thus we have $(***)$, i.e., $f * xp = 0$. By iterating this process, we have $f * h(x)p(x) = 0$ for every polynomial $h(x)$. This implies that $\int f(y-x)e^{iux}p(x)dx = 0$, since on the support of $p(x)$ one can approach e^{iux} uniformly by the polynomials $h(x)$. In other words $\mathscr{C}(f(y-x)p(x)) = 0$ for every y. By the uniqueness theorem on Fourier transform, we have $f(y-x)p(x) = 0$ for every y. Thus $f(x) = 0$, which gives the

Lemma. *If the spectrum of f is void, $f \equiv 0$.*

In the last lecture we have seen that this lemma gives us the following theorem:

Theorem. *Suppose $\tau(f) \neq \mathscr{C}$. f belongs to the closed subspaces spanned by the "polynomial exponentials" in $\tau(f)$. Thus $\tau(f)$ is the closed span of "polynomial exponentials" contained in it.*

In this theorem we have a solution of the problem of harmonic analysis and synthesis of mean periodic functions.

2 Equivalence of all the definitions of mean periodic functions

The above theorem enables us to prove the equivalence of our definitions stated in the introduction. Using the condition of Riesz we have already proved the equivalence of definition I and III (lecture 4, 1). We have seen in the introduction that definition Implies definition I. Our last theorem has just gives us the result that definition III implies definition II.

Theorem. *Definitions I, II, III, of the lecture 1are equivalent.*

The equivalence of definitions I and III allows us to define the mean 29
period of a mean periodic function.

Definition. The mean period of a mean periodic function is defined as the infimum of the lengths of the segment of support of measures $d\mu$ orthogonal to $\tau(f)$.

Remark. This definition implies that if $f = 0$ on a segment $(a, a+\ell+\varepsilon)$, $\varepsilon > 0$, then $f \equiv 0$. Let ℓ be the mean-period of f, and suppose $f = 0$ on $(0, \ell + \varepsilon)$: then $g^{+} \equiv 0$, and hence $F \equiv 0$, and so $f \equiv 0$. This means that a mean-periodic function cannot be zero on any interval of length larger than its mean-period, if it is not the zero-function. We shall see better results of this type (lecture 9).

In the next lecture, using definition II, we shall develop various equivalent forms of this definition.

3 Mean periodic C^{∞} - functions and mean-periodic distributions

Just as one defines mean periodic functions by the intrinsic property $\tau(f) \neq \mathscr{C}$, for $f \in \mathscr{C}$, one can define mean periodic C^{∞}- functions f or distributions T by $\tau(f) \neq \mathscr{E}$, $f \in \mathscr{E}$ or $\tau(T) \neq \mathscr{D}'$, $T \in \mathscr{D}'$. (Here $\tau(f)$) is the span of the translates of f, *in the space considered*; for example, if $f \in \mathscr{E}$, "$\tau(f)$" in $\mathscr{C}$ is not "$\tau(f)$" in "$\mathscr{E}$", which again is not "$\tau(f)$ in $\mathscr{D}$", but, as $\mathscr{E} \subset \mathscr{C} \subset \mathscr{D}'$, "$\tau(f)$ in $\mathscr{E}$" is dense in "$\tau(f)$ in $\mathscr{C}$", which is dense in "$\tau(f)$ in $\mathscr{D}$". We can have definitions similar to definitions I and II, by replacing $d\mu$ with $T \in \mathscr{E}'$ when $f \in \mathscr{E}$ and with $\varphi \in \mathscr{D}$ when $T \in \mathscr{D}'$. It is possible to develop the whole theorem in particular the equivalence
30 of the definitions, by considering harmonic analysis and synthesis. In obtaining the criterion for simple subspaces the same reasoning applies, with $T \in \mathscr{E}'$ or $\varphi \in \mathscr{D}$ playing the role of $d\mu$. The proof of the theorem on synthesis will-depend on the lemma in 1. The same proof holds if one replaces $M(w)$ by $\mathscr{C}(T)$ with $T \in \mathscr{E}'$ or $\mathscr{C}(\varphi)$ with $\varphi \in \mathscr{D}$. In either case we get $\mathscr{C}(f_y(-x)p(x)) \equiv 0(\mathscr{C}(T_y p(x)) \equiv 0)$ for every y, which gives us by the uniqueness theorem that $f \equiv 0(T \equiv 0)$.

If a mean periodic function (distribution) is also a C^{∞} - function, it is a mean periodic C^{∞}- function. In other words $M.P.\mathscr{E} \equiv (M.P.\mathscr{C}) \cap \mathscr{E} = (M.P.\mathscr{D}) \cap \mathscr{E}$. For if $f \in \mathscr{E}$ be such that f is a mean periodic distribution then there exists $\varphi \in \mathscr{D}$ with $f * \varphi = 0$. But φ is again a distribution with compact support; so $f \in M.P.\mathscr{E}$. In the same way, if $f \in (M.P.\mathscr{C}) \cap \mathscr{E}$, $f \in M.P.\mathscr{E}$. Without confusion we can say that a function is mean periodic and a C^∞- function or a mean periodic C^∞-function. We have also $M.P.\mathscr{C} = (M.P.\mathscr{D}) \cap \mathscr{C}$.

4 Other extensions

By the method we used in lecture 4 and 5, it is possible to develop the theory of harmonic analysis and synthesis for functions which are "mean-periodic on a half-line", i.e., continuous functions on $[0, \infty]$ such that their negative translates $(f_\alpha(x) = f(x - \alpha), \alpha < 0)$, restricted to $[0, \infty]$, are not a total set in the space of the continuous functions on $[0, \infty]$, with the topology of compact convergence (see Koosis (2)).

Another method seems to be necessary in order to study mean-periodic functions on R^n (see Malgrange, Ehrenpreis, and lect. 23 and 24) or mean-periodic functions (sequence) or Z^n (whose theory was given, quite recently, by Lefranc). 31

Lecture 6

Mean period and fourier series

1 Mean period

The equivalence of the definitions of mean periodic functions allows us **32**
to define the mean period of a function, or again, the mean period related to its spectrum Λ, in different ways.

Definition II can be put in a different form. Let $\mathscr{C}_\Lambda$ be the closed subspace spanned by $\left\{x^p e^{i\lambda x}\right\}(\lambda)_{p+1} \in \Lambda$ (We use the notation $(\lambda)_{p+1} \in \Lambda$ if $\lambda \in \Lambda$, at least $p+1$ times). We have either $\mathscr{C}_\Lambda = \mathscr{C}$ or $\mathscr{C}_\Lambda \neq \mathscr{C}$. Definition II means: f is mean periodic if $f \in \mathscr{C}_\Lambda$ and $\mathscr{C}_\Lambda \neq \mathscr{C}$. Then $S(f) \subset \Lambda; S(f)$ being the spectrum of f.

We recall (lect. 2, 1) that given a closed interval I $\mathscr{C}(I)$ is the space of continuous functions on I, with the topology of uniform convergence; $\mathscr{C}'(I)$ is the space of measure with support in I. We define $\mathscr{C}_\Lambda(I)$ as the closed subspace generated by $\left\{x^p e^{i\lambda x}\right\}(\lambda)_{p+1} \in \Lambda$.

Definition of mean period

1. ***Mean period of a mean periodic function.***

We recall this definition from 2, lecture 5. The mean period of a mean periodic function is defined to be the infimum of the lengths of the segments of support of $d\mu \perp \tau(f)$.

2. ***Mean period related to Λ.***

33 (a) It is the common mean period of functions with spectrum Λ (for which $\tau(f) = \mathscr{C}_\Lambda$). Now we use the fact that $\mathscr{C}_\Lambda(I) \neq \mathscr{C}(I)$ is equivalent to saying that there exists a measure $d\mu$ orthogonal to all $x^p e^{i\lambda x}$, with support of $d\mu \subset I$.

(b) The mean period, L, is the infimum of I where I is such that $\mathscr{C}_\Lambda(I) \neq \mathscr{C}(I)$.

(c) The mean period, L, is the supremum of I where I is such that $\mathscr{C}_\Lambda(I) = \mathscr{C}(I)$.

(d) The mean period, L, is the infimum of lengths of segment of support of measure $d\mu$ orthogonal to $x^n e^{i\lambda x}, n = 0, \ldots, p, (\lambda)_{p+1} \in \Lambda$.

(e) The mean-period related to Λ is the infimum of the L' such that there exists an entire function $M(w)$ of exponential type with $M(w) = 0(e^{\frac{1}{2}L'|w|}), M(\Lambda) = 0, M(u) = 0(1)(|u| \to \infty)$.

(f) Same definition, with $M(u) = 0(|u|^N)$ for one $N = N(M(u))$, instead of $M(u) = 0(1)$.

(g) Same definition, with $M(u) = 0(|u|^{-n})$ for each n, instead of $M(u) = 0(|u|^N)$.

The equivalence between (d) and (e) is a simple consequence of the Paley-Wiener theorem (lect. 3). The equivalence between (e) and (f) is obvious, because multiplication by a polynomial does not affect the exponential type of an entire function. The equivalence between (f)
34 and (g) depends on the following remark: given $\varepsilon > 0$, the function $\Phi = (w) \prod_1^\infty \frac{\sin \alpha_n w}{\alpha_n w}$, with $\sum_1^\infty \alpha_n = \sum_1^\infty |\alpha_n| < \varepsilon$, satisfies $\Phi(w) = 0(e^{\varepsilon|w|})$ and $\Phi(u) = 0(|u|^{-n})$ for each n. Then, if $M(w)$ satisfies the conditions in (e), $M(w)\Phi(w)$ satisfies those in (g) (with $L' + \varepsilon$ instead of L').

The equivalence between (e) (f) (g) shows that the mean-period related to $\wedge$ would be the same, when, defined either from the C-function or from the distribution of spectrum $\wedge$.

Half of the mean-period has been called by Schwartz "radius of totality" associated with Λ (Schwartz 2) (because of the definition (c)). In (Paley-Wiener), (Levinson), (Schwartz 2), problems of closure of a set of exponentials over an interval are considered, which naturally lead to the calculation of the mean period related with a sequence $\wedge$.

In this direction, the simplest facts are the following.

Theorem. *Let* L *be the mean-period of* Λ. *If* $\sum\limits_{\lambda \in \Lambda} \dfrac{1}{|\lambda|} < \infty$ *then* $L = 0$, *Further if* $\lambda_n - D_n = 0(1)$ $(n \to \pm\infty), \lambda_{-n} = -\lambda_n (n = 0, 1, \ldots,)$, *and* $\Lambda = \{\lambda_n\}$, *then* $L = 2\pi D$.

Proof of the first part: Let $\Lambda = \Lambda_1 \cup \Lambda_2, \Lambda_1$ finite, and $\sum\limits_{\lambda \in \Lambda_2} \dfrac{1}{|\lambda|} < \varepsilon$. As $(\sin 2u)^2 < \dfrac{4u^2}{1 + u^2}$, the function

$$M(w) = \sum_{\lambda \in \Lambda_1} \left(1 - \frac{w^2}{2}\right) \sum_{\lambda \in \Lambda_2} \left(1 - \frac{w^2}{2}\right) \left(\frac{\sin 2w/|\lambda|^2}{2w/|\lambda|}\right)$$

is $0(|u|^N)$ on the real axis when N is large enough of exponential type 35
$< 4\varepsilon$, and $M(\lambda) = 0$ for $\lambda \in \Lambda$. According to definition $(f), L = 0$.

For the second part, the proof depends on an estimate of $\prod\limits_1^\infty (1 - \dfrac{w^2}{\lambda_n^2}) = M_1(w)$. We shall see (lect. 10) that $M_1(w) = 0(e^{\pi D|w|})$. Moreover, a careful calculation shows $M_1(u) = 0(|u|^N)$ for N large enough (Paley-Wiener p. 93 - 94) According to definition $(f), L \geq 2\pi D$. From the Jensen formula we have $L \leq 23\pi D$ (lect. 11).

In lect. 12 we shall prove $L \geq 2\pi D_{\max}$, where $D_{\max}$ is the maximum density of Polya of the sequence Λ (see appendix 1). This inequality, together with the above theorem, led L. Schwartz to the following hypothesis; if $\Lambda = \{\lambda_n\}$ is real, symmetric $(\lambda_{-n} = -\lambda_n)$ and has a density $\lim \dfrac{n}{\lambda n} = D$, then $L = 2\pi D$. In (Kahane 1, p.57) this equality is given as a consequence of the following statement: each analytic function $f \in \mathscr{C}_\Lambda(I)$ is continuable into an analytic function $f \in \mathscr{C}(I)$. This last statement was not proved, and it is not at all equivalent to the known results about continuation of analytic functions (lect. 16, 17). In fact, Kahane's statement as well as the Schwartz hypothesis are false, since we can construct a real symmetric sequence $\wedge$ of density zero, whose mean-period is infinitely (see appendix 2).

2 Fourier series of a mean periodic function

36 We shall study the Carleman transform $F(w)$ of f in order to define the Fourier series of f.

If $\left|f(x)\right| < e^{a|x|}$, then we define with Carleman (Carleman)

$$F^{+}(w) = \int\limits_{-\infty}^{0} f(x)e^{-ixw}dx, F^{-}(w) = -\int\limits_{0}^{\infty} f(x)e^{-ixw}dx.$$

When $w = u + iv, F^{+}$ is defined and holomorphic in $v > a$ and F^{-} is defined and holomorphic in $v < -a$. We can write

$$F^{+}(u+iv) = \int\limits_{\infty}^{0} f(x)e^{vx}e^{-iux}dx = \mathscr{C}(e^{vx}f^{-})v \text{ fixed } v > \varepsilon$$

$$F^{-}(u+iv) = \mathscr{C}(-e^{vx}f^{+}), v \text{ fixed } v < -a$$

Suppose f is mean periodic and let $g = -f^{+} * d\mu = f^{-} * d\mu$. Then (see lect. 3, 4), $e^{vx}f^{-} * e^{vx}d\mu = e^{vx}g$; $M(u+iv) = \mathscr{C}(e^{vx}d\mu)$; $G(u+iv) = \mathscr{C}(e^{vx}g)$; $F^{+}(u+iv)M(u+iv) = G(u+iv), v > a$; $F^{-}(u+iv)M(u+iv) = G(u+iv), v < -a$, when $v > a$, we have $F^{+} = F$ and when $v < -a, F^{-} = F$. Hence the definition of $F(w)$ is consistent with the definition of Carleman. We shall use this interpretation of one definition of the Carleman transform to define the Fourier series of f.

Suppose $\varphi(x) = \sum Ax^{p}e^{i\lambda x}$, with $A = A(\lambda, p, \varphi)$ and the sum a finite sum. Then we have the following equation:

$$\Phi^{+}(w) = \sum A \int\limits_{-\infty}^{0} x^{p}e^{-ix(w-\lambda)}dx = \sum Ap!/(i(w-\lambda))^{p+1}$$

Suppose f is the given mean-periodic function; there is a sequence of finite sums $\varphi \in \tau(f)$ tending to f in $\mathscr{C}$ (lect. 5 1). Let $d\mu$ be a measure with the segment of support $(-\ell, 0)$ such that $f * d\mu = 0, \varphi * d\mu = 0$, **37**
since $\varphi \to f$ in $\mathscr{C}, \varphi^- \to f^-$ in $\mathscr{C}$ and $\varphi^- * d\mu \to g$ in $\mathscr{C}$. Hence $\Phi(w)M(w) \to F(w)M(w)$.

Therefore $(*) A(\lambda, p, \varphi) \to A(\lambda, p)$ where the term containing $(w-\lambda)^{-p-1}$ in the development of $F(w)$ is $A(\lambda, p)p!/i^{p+1}(w-\lambda)^{p+1}$.

Definition. We denote the polar part in the expansion of $F(w)$ by $C(f) = \sum A(\lambda, p)p!/i^{p+1}(w-\lambda)^{p+1}$. The *Fourier series of f is defined to be the formal sum $\mathscr{I}(f)$ having an expansion of the following form*

$$f \sim \mathscr{I}(f) = \sum_{(\lambda)p+1 \in \Lambda} A(\lambda, p) x^p e^{i\lambda x},$$

where $A(\lambda, p)$ is the term obtained from the polar part $C(f)$ of the Carleman transform $F(w)$ of f.

From $(*)$, taking into account the remark at the end of lect. 2 2, we have:

Theorem. *Suppose $\tau(f) \neq \mathscr{C}$. The $x^p e^{i\lambda x} \in \tau(f)$ form a basis of $\tau(f)$. Each $f \in \tau(f)$ admits a Fourier development which is the formal sum $\mathscr{I}(f) = \sum_{\lambda^{p+1} \in \Lambda} A(\lambda, p) x^p e^{i\lambda x}$, with respect to this basis and the formal Carleman transform of this sum is the polar part $C(f)$ of $F(w), C(f) = \sum_i \dfrac{A(\lambda, p)p!}{i^{p+1}(w-\lambda)^{p+1}}$.*

Corollary. *Each simple subspace of $\mathscr{C}$ is generated by a finite number of monomial exponentials $x^p e^{i\lambda x}, p = 0, 1, \ldots n$*

Remark. The function $f \in \tau(f) \neq \mathscr{C}$ *is uniquely determined by its Fourier series $\mathscr{I}(f)$. In other words, if all the $A(\lambda, p)$ are zero, then* **38**
$f \equiv 0$.

In (Kahane 1), $F(w)$ is called a "Fourier-transform" of f; it is not a good definition. It is possible to define a kind of Fourier-transform in the following manner. Consider the operation D^p_λ defined for functions analytic in the complex plane by $\langle D^p_\lambda, \varphi(w)\rangle = \varphi^{(p)}(\lambda)$. The Fourier transform $\mathscr{C}(f)$ can be defined as a formal linear combination of D^p_λ such that $\mathscr{I}(f) = \dfrac{1}{2\pi}\langle \mathscr{C}(f), e^{iwx}\rangle$ and $C(f)(w) = \dfrac{1}{2\pi i}\langle \mathscr{C}(f)(w'), \dfrac{1}{w-w'}\rangle$; for example, if the spectrum is simple,

$$\mathscr{C}(f) = 2\pi \sum A(\lambda) D^0_\lambda, \mathscr{I}(f) = \sum A(\lambda) e^{i\lambda x}, C(f) = \sum \frac{A(\lambda)}{i(w-\lambda)}$$

Let us find the relation between the Fourier series of f and $f * d\nu, d\nu \in \mathscr{C}'$. $C(f * d\nu)$ = polar part of $C(f)\mathscr{C}(d\nu)$, for if $\varphi \to f$ in $\mathscr{C}$, $\varphi * d\nu \to f * d\nu$ and $\mathscr{C}(f * d\nu) = \mathscr{C}(f)\mathscr{C}(d\nu) + N$. Therefore $\mathscr{I}(f * d\nu) = \sum A(\lambda, p)(x^p e^{i\lambda x} * d\nu) = \mathscr{I}(f) * d\nu$.

The definition of Fourier series of a mean periodic C^∞−functions or distributions is given in the same fashion. Comparing $\mathscr{I}(T)$ and $\mathscr{I}(T * \dfrac{d}{dx})$ we have:

Theorem. *The derivative of a mean periodic distribution T, is mean periodic and its Fourier series is obtained by deriving $\mathscr{I}(T)$ formally.*

We have a similar theorem for the primitive of a distribution.

39 *The primitive T_1 of a mean-periodic distribution T is mean-periodic, and its Fourier series is obtained by taking a* formal primitive of $\mathscr{I}(T)$.

For, there exists $\varphi \in \mathscr{D}, \varphi \neq 0$, such that $T * \varphi = 0$; then $T_1 * \varphi' = 0$, and T_1 is mean periodic; the property of its Fourier series depends on the precedent theorem.

Lecture 7

Bounded mean periodic functions and their connection with almost periodic functions

Let f be a bounded mean periodic function, i.e., $|f| < M$, $f \in \mathscr{C}, \tau(f) \neq$ 40
$\mathscr{C}$. In this case the Carleman transform is $F(w) = \int\limits_{-\infty}^{0} f(x)e^{-ixw}dx, w =$
$u + iv, v > 0$, and $F(w) = -\int_0^{\infty} f(x)e^{-ixw}dx, v < 0$. As $\int\limits_{-\infty}^{0} e^{vx}dx = \dfrac{1}{v}$
for $v > 0$, and $\int\limits_{C}^{-\infty} e^{vx}dx = \dfrac{1}{v}$ for $v < 0$, we have in both cases $|F(w)| <$
$\dfrac{M}{|V|}$. This implies the spectrum is real. Taking the polar development of $F(w)$ in λ (that gives the principal part when w tends to λ), we see 1^0) the spectrum is simple 2^0) the Fourier coefficients $A(\lambda)$ are bounded: $|A(\lambda)| < M$. Thus we have proved:

Theorem. *A bounded periodic function has its spectrum real and simple and its Fourier coefficients are bounded.*

Suppose $|f| < M$, f'' continuous and $|f''| < M''$; f'' is again mean-periodic, and $f \sim \sum A(\lambda)e^{i\lambda x} \Rightarrow f'' \sim -\sum A(\lambda)\lambda^2 e^{i\lambda x}$ and f'' is again mean periodic. Thus $|\lambda^2 A(\lambda)| < M''$. Since $\lambda \in S(f)$ are the zeros of an entire function of exponential type, $\sum \dfrac{1}{|\lambda^2|} < \infty$. Thus the Fourier series of f, being majorized by $\sum \dfrac{1}{|\lambda^2|}$, is absolutely convergent and so f is an almost periodic function.

We recall the various definitions of almost periodic functions and find their connection with bounded mean periodic functions.

41 ***The definition of Bohr.***

We consider functions $f \in \mathscr{C}(R)$.

Definition (Bohr). Given $\varepsilon > 0$, the number τ is an *almost period* (corresponding to ε) of f if $\left|f_{\tau - f}\right| < \varepsilon$. A set is *relatively dense* if there exists a length $\ell > 0$, such that in any interval of length ℓ, there is at least one point of the set.

A function is *almost periodic* (in the sense of Bohr) if

(a) for every $\varepsilon > 0$, the almost periods of f form a relatively dense set. This is equivalent to either of the following:

(b) The translates of f form a relatively compact set of L.

(c) f is the uniform limit in R of a sequence of trigonometric polynomials $\sum a_n e^{i\lambda_n x}$, λ_n real (see for example (Besicovitch)).

Properties.

(1) $a(\lambda) = \lim\limits_{T\to\infty} \dfrac{1}{2T} \int\limits_{-T}^{T} f(x) e^{-i\lambda x} dx$ exists.

(2) $\sum \left|a(\lambda)\right|^2 = \lim\limits_{T\to\infty} \dfrac{1}{2T} \int\limits_{-T}^{T} \left|f(x)\right|^2 dx$ (Perceval's relation).

Here only a countable number of a $(\lambda) \neq 0$.

In order to find the analogue of the Riesz-Fischer theorem Besicovitch introduced the following norm. (Besicovitch).

$$D(f) = \lim_{T\to\infty} \sup \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx, D(f, z) = D(f - z).$$

It may happen that $D(f) = 0$ without $f = 0$.

42 **Definition of Besicovitch.** A bounded function $f \in \mathscr{C}(R)$ is *almost periodic* (in the sense of Besicovitch) if it belongs to the closed subspace spanned by $\{e^{i\lambda x}\}, \lambda \in R$, the closure being taken in the metric $D(f)$. Then $f \sim \sum a(\lambda)e^{i\lambda x}$.

Besicovitch showed that if $\sum \left|a(\lambda)\right|^2 < \infty$, then there exists a function $f \sim \sum a(\lambda)e^{i\lambda x}$.

Definition of Schwartz. Consider the space $\mathscr{B}$ of C^∞- functions all of whose derivatives are bounded. (Schwartz 3). A function $f \in \mathscr{B}$ is $\mathscr{B}$-*almost periodic* if the set of its translates from a relatively compact set in $\mathscr{B}$. These functions $f \in \mathscr{B}$ are such that they are almost periodic in the sense of Bohr and all their derivatives $f^{(p)}$ are also almost periodic in the sense of Bohr.

Let $\mathscr{B}'$ be the dual of $\mathscr{B}$. $\mathscr{B}'$ is the space of distributions which are finite sums of derivatives (in the sense of distributions) of bounded functions. A distribution $T \in \mathscr{B}'$ is defined to be $\mathscr{B}'$- almost periodic if it satisfies either the definition (b) or (c) of Bohr in the space $\mathscr{B}'$. There is a simple connection between the classes $M.P.$, $\mathscr{B}A.P.$ and $\mathscr{B}'A.P.$ which consist respectively of mean-periodic distributions, $\mathscr{B}$- almost periodic functions and $\mathscr{B}'$ - almost periodic distributions.

Theorem. *$M.P. \cap \mathscr{B} = M.P. \cap \mathscr{B}.A.P.$ and $M.P. \cap \mathscr{B}' = M.P. \cap \mathscr{B}'A.P.$.*

The first part results from the fact that f is almost periodic (Bohr) 43
whenever $f \in \mathscr{C}$, f and f" bounded; moreover, we see that $f \in M.P \cap \mathscr{B} \Rightarrow f \sim \sum a(\lambda)e^{i\lambda x}$, $a(\lambda) = 0(\left|\lambda\right|^{-n})$ for every $n > 0$. The second part results from:

$$f \in M.P. \cap \mathscr{B}' \Rightarrow f \sim \sum a(\lambda)e^{i\lambda x}, a(\lambda) = 0(\left|\lambda\right|^{N}) \text{ for one} N.$$

The corresponding result for Bohr almost periodic functions is the following:

Theorem. *A uniformly continuous bounded mean periodic function is almost periodic (in the sense of Bohr).*

We have seen that if f has a bounded second derivative, f is almost periodic. In other cases we regularize f with the help of suitable functions. Let $\triangle_\varepsilon$ be the conical function defined by $\triangle_\varepsilon(x) = \sup(0, \dfrac{\varepsilon - |x|}{\varepsilon^2})$. $\triangle_\varepsilon(x)$ has its support in $(-\varepsilon, \varepsilon)$ and $\int_{-\varepsilon}^{\varepsilon} \triangle_\varepsilon(x)dx = 1$. Then $f * \triangle_\varepsilon * \triangle_\varepsilon$ is a bounded mean periodic function having a bounded second derivative and so almost periodic. Since f is uniformly continuous, $f * \triangle_\varepsilon * \triangle_\varepsilon \to f$ uniformly when $\varepsilon \to 0$. Hence by the definition (c) of Bohr f is almost periodic in the sense of Bohr.

A natural question is to ask whether every bounded mean periodic function is uniformly continuous. In fact, that this is not true in general is seen from the following example.

We take the Fejer Kernel $K_\nu(x) = (\nu \sin^2 \dfrac{\nu x}{2} / (\dfrac{\nu x}{2})^2)$. It is possible to choose a increasing sequence μ_n such that the function $\sum\limits_{n=1}^{\infty} \dfrac{1}{\mu_n} K_{\mu_n}(\dfrac{x}{2^n} - \pi)$

44 is bounded, and the sum uniformly continuous on each compact set. The spectrum of the Fejer Kernel $K_{\mu_n}(\dfrac{x}{2^n})$ consists of $(2\mu_n - 1)$ points between $-\mu_n/2^n$ and $\mu_n/2^n$.The spectrum of $e^{i\lambda n^x} K_{\mu_n}(\dfrac{x}{2^n} - \pi)$ consists of $(2\mu_n - 1)$ points between $\lambda_n - \dfrac{\mu_n}{2^n}$ and $\lambda_n + \dfrac{\mu_n}{n}$. Now it is possible to choose λ_n satisfying the following conditions:

(1) Each term of the series $\sum\limits_{1}^{\infty} e^{i\lambda_n x} \dfrac{1}{\mu_n} K_{\mu_n}(\dfrac{x}{2^n} - \pi)$ consists of function whose spectra do not overlap.

(2) Denote by $\{\nu_n\}$ the spectrum of this sum, $\sum \dfrac{1}{\nu_n} < \infty$

Then the function $f(x) = \sum\limits_{1}^{\infty} e^{i\lambda_n x} \dfrac{1}{\mu_n} K_{\mu_n}\left(\dfrac{x}{2^n} - \pi\right)$ is not uniformly continuous in R, but is bounded and (see theorem), lect. 6, 1) has mean-period zero.

This example show that there are bounded mean periodic functions which are not almost periodic in the sense of Bohr. Now one may ask for the relation between bounded mean periodic functions and almost periodic functions of Besicovitch.

We shall prove that, if f is a bounded mean periodic function then $\sum \left|\Lambda(\lambda)\right|^2 < \infty$. In other words, its Fourier coefficients are those of an almost periodic function in the sense of Besicovitch.

Let $f\varepsilon = f * \Delta_\varepsilon$, $f \sim \sum A(\lambda)e^{i\lambda x}$.

Now $\mathscr{C}(\Delta_\varepsilon) = \dfrac{\sin^2 \frac{\varepsilon x}{2}}{\left(\frac{\varepsilon x}{2}\right)^2} = \delta_\varepsilon(x)$; $f \sim \sum A(\lambda)\delta_\varepsilon(\lambda)e^{i\lambda x}$.

Since $|\Lambda(\lambda)| \leq M$ and $\sum \dfrac{1}{|\lambda|^2} < \infty$, we have

$$\sum |A(\lambda)\delta_\varepsilon(\lambda)|^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f_\varepsilon|^2 \leq M^2.$$

When $\varepsilon \to 0$, $\delta_\varepsilon(\lambda) \to 1$ and so

Theorem. *If f is a bounded mean-periodic function,* **45**

$$f \sim \sum A(\lambda)e^{i\lambda x}, |f| \leq M, \text{ then } \sum |A(\lambda)|^2 \leq M^2$$

One has also the summation formula of Fejer, viz.,

$$\sum_{|\lambda|<T} \left(1 - \frac{|\lambda|}{T}\right) A(\lambda)e^{i\lambda x} = f * \frac{1}{2\pi} \frac{T \sin^2 \frac{Tx}{2}}{\left(\frac{Tx}{2}\right)^2} \to f$$

uniformly if f is uniformly continuous and other wise it is uniformly convergent on every compact set.

Instead of $\mathscr{C}$, let us consider the space E^2, consisting of the functions which are locally $\in L^2$.E^2- mean-periodic functions are defined, as usual, by the condition $\tau(f) \neq E^2$ ($\tau(f)$ closed *in* E^2). E^2– bounded functions are defined by $\int_x^{x+1} |f|^2 = 0(1)$ uniformly with respect to x.

The problem is to study the functions which are E^2-mean-periodic and E^2-bounded. In particular 1°) is it possible to get the Parceval and Riesz-Fischer theorems ? 2°) is it possible to have the summation formula of Fejér? 3° what are their connections with Besicovitch almost-periodic functions and with Stepanoff almost-periodic functions (which are analogous to Bohr a.p. functions, with the norm $\sup_x(\int_x^{x+1}|f|^2)^{\frac{1}{2}}$ instead of $\sup_x|f(x)|$)? For more details about this type of questions, see (Kahane, 2).

Lecture 8

Approximation by Dirichlet's polynomials and some problems of closure

1 Dirichlet polynomials and approximation on an interval

A Dirichlet polynomial is a finite sum of the form $\sum a(\lambda)e^{\lambda z}, \lambda \in C, z \in$ 46
C. Before studying the approximation by such polynomials in a domain Ω of the complex plane, we study the same problem in the space $\mathscr{C}(I)$.

Let $\mathscr{C}_\Lambda(I)$ be the closed subspace of $\mathscr{C}(I)$ spanned by $\left\{e^{i\lambda x}\right\}_{\lambda\in\Lambda}$ (if Λ is simple) or $\left\{x^p e^{i\lambda x}\right\}(\lambda)^{p+1} \in \Lambda$ (let us recall that $(\lambda)^{p+1} \in \Lambda$ means $\lambda \in \Lambda$, at least $p+1$ times) and let $\mathscr{C}_\Lambda(I) \neq \mathscr{C}(I)$. We have the following theorem:

Theorem. *Suppose $\mathscr{C}_\Lambda(I) \neq \mathscr{C}(I)$. Then $\left\{x^p e^{i\lambda x}\right\}(\lambda)^{p+1} \in \Lambda$ form a basis of $\mathscr{C}_\Lambda(I)$ and each function in $\mathscr{C}(I)$ is characterised by its development.*

For, suppose $I = [0,1] f \in \mathscr{C}_\Lambda(I)$, and let $d\mu$ be a measure $\not\equiv 0$, with support in I, such that $\int_I x^p e^{i\lambda x} d\mu(x) = 0 (\lambda)^{p+1} \in \Lambda$. Let us put $f^* = f$ on I, $f^* = 0$ outside I, and $g- = f^* * d\mu$ on $I, g = 0$ outside I. Defining $F(w) = \dfrac{\mathscr{C}(g)}{\mathscr{C}(d\mu)}$, we get the same relation as in lect. 6 §2 between the polar part of $F(w)$ and the Fourier expansion of f, when

f is a finite linear combination of $x^p e^{i\lambda x}$; then, taking a limit, we prove
47 *the existence of a Fourier expansion $S(f)$ for all $f \in \mathscr{C}_\Lambda(I)$, with the same relation as in lecture 6, §2, between $S(f)$ and $F(w)$; hence $S(f) \equiv 0 \Longrightarrow F \equiv 0 \Longrightarrow g \equiv 0 \Longrightarrow f^{**}d\mu = 0$ on I. The same holds if we take $I = [-\ell, 0]$. Using these remarks, it is easy to see that $S(f) \equiv 0 \Longrightarrow f^* * d\mu \equiv 0 \Longrightarrow f^* \equiv 0 \Longrightarrow f \equiv 0$, and the theorem is proved.*

We consider the same kind of problem in the complex plane; to this end we first describe space analogous to $\mathscr{C}_\Lambda(I)$ and obtain a theorem of closure.

2 Runge's theorem

Let Ω be an open set in C and $\mathscr{H}(\Omega)$ be the space of holomorphic functions in Ω with the compact convergence topology. $\mathscr{H}(\Omega)$ is an $\mathcal{F}-$space. Let $\mathscr{H}'(\Omega)$ be its dual. Since $\mathscr{H}(\Omega)$ is a closed subspace of $\mathscr{C}(\Omega)$, by the Hahn - Banach theorem $\mathscr{H}'(\Omega) \subset \mathscr{C}'\Omega$. Every vector of $\mathscr{H}'(\Omega)$ defines an equivalence class of measures in $\mathscr{C}'(\Omega)$, which are merely the extensions of this vector (by the Hahn - Banach theorem) to a linear functional in $\mathscr{C}(\Omega)$. In other words, $d\mu_1 \sim d\mu_2$ if for every $f \in \mathscr{H}(\Omega)$, $\int_\Omega f d\mu_1 = \int_\Omega f d\mu_2$. Thus the dual $\mathscr{H}'(\Omega)$ of $\mathscr{H}(\Omega)$ is the quotient of $\mathscr{C}'(\Omega)$ by the subspace orthogonal to $\mathscr{H}(\Omega)$.

Runge's theorem. *Suppose Ω is connected (but Ω need not be connected). In $\mathscr{H}(\Omega)$ the set $\{z^p\}$, $p = 0, 1, \ldots$ form a total set.*

In other words, by the condition of Riesz, $\int z^p d\mu = 0$ for $p = 0, 1, \ldots$ implies $\int f(z) d\mu = 0$ for every $f \in \mathscr{H}(\Omega)$.

48 *Proof.* Let support of $d\mu' \subset \Omega' \subset \bar{\Omega}' \subset \Omega(\Omega'$ open and connected) and let $|z_o|$ be large enough. Then $\dfrac{1}{z - z_o} = -\dfrac{1}{z_o}(1 + \dfrac{z}{z_o} + \cdots)$, the series in the right hand side being convergent in Ω'. Thus $\dfrac{1}{z - z_o}$ belongs to the closed span of the monomials in $\mathscr{C}(\Omega')$. Moreover, $\dfrac{1}{(z - z_o)^{n+1}}$ belongs to this closed span ; then $\int \dfrac{d\mu(z)}{(z - z_o)^{n+1}} = 0 (n = 0, 1, \ldots)$. But

$\varphi(\zeta) = \int \dfrac{d\mu(z)}{z-\zeta}$ is holomorphic outside Ω' (because $[\Omega'$ is connected) ; from $\varphi^{(n)}(z_o) = n! \int \dfrac{d\mu(z)}{(z-z_o)^{n+1}} = 0 (n = 0, 1, \ldots)$ results $\varphi \equiv 0$. Let C be closed rectifiable curve around Ω' in $\Omega - \Omega'$. Then by Cauchy's theorem

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_C \varphi(\zeta) f(\zeta) d\zeta = -\int d\mu(z) \frac{1}{2\pi i} \int_C f(\zeta) \frac{d\zeta}{\zeta - z} = \\ &= -\int f(z) d\mu(z) \end{aligned}$$

Remarks about $\mathscr{H}'(\Omega)$. □

Remarks 1. To each $d\mu \in \mathscr{C}'(\Omega)$ corresponds a $\varphi(\zeta) = \int \dfrac{d\mu(\zeta)}{z-\zeta}$ which is holomorphic outside the support K of $d\mu$, and vanishing at infinity. $d\mu_1 \sim d\mu_2$ if and only if $\varphi_1(\zeta) = \varphi_2(\zeta)$, and the duality between $\mathscr{H}(\Omega)$ and $\mathscr{H}'(\Omega)$ can be defined by $\int f d\mu = \int_C f(z)\varphi(z)dz$, $f \in \mathscr{H}(\Omega), C$: curve surrounding K, and contained in Ω. Then it is convenient to represent the elements of $\mathscr{H}'(\Omega)$ as the functions vanishing at infinity and holomorphic outside a compact subset of Ω.

Remarks 2. Let Ω be connected and $0 \in \Omega$. Suppose $\varphi(z) = \sum\limits_0^\infty \dfrac{a_n}{z^{n+1}}$.
Then $\dfrac{1}{2\pi i} \int_c f(z)\varphi(z)dz = \sum\limits_0^\infty \dfrac{a_n}{n} f^{(n)}(0)$. Thus one can represent the lin- 49
ear functional on $\mathscr{H}(\Omega)$ as differential operators of infinite order with constant coefficients. In general, the relation between the coefficients a_n and Ω is not simple. It is simple when Ω is a circle around origin and of radius R. Then $\sum \dfrac{a_n}{n!} f^{(n)}(0)$ is a linear functional if and only if $\limsup |a_n|^{1/n} < R$.

We give without proof an extension of Runge's theorem due to Mergelyan and Lavrentie (Mergelyan). Let K be a compact set of the complex plane and let $\mathscr{H}(K)$ be the space of continuous functions on K which are holomorphic in the interior of K with the topology of Uniform Convergence.

Mergelyan's theorem. *In order that the set $z^p, p = 0, 1, \ldots$ be total in $\mathscr{H}(K)$ it is necessary and sufficient that the complement of K should consist of one region (i.e., "K does not divide the plane ").*

3 Problems of Closure in the Complex Plane

In terms of the duality between $\mathscr{H}(\Omega)$ and $\mathscr{H}'(\Omega)$ we obtain a condition of closure.

Let $\mathscr{H}_\Lambda(\Omega)$ be the closed span of $\{e^{\lambda z}\}_{\lambda \in \Lambda}$ (or $\{z^p e^{\lambda z}\}(\lambda)^{p+1} \in \Lambda$) in $\mathscr{H}(\Omega)$. Then, by the condition of Riesz, we have

$$\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega) \Longleftrightarrow \left[\int e^{\lambda z} d\mu = 0, \lambda \in \Lambda \Longrightarrow \int f d\mu = 0, f \in \mathscr{H}(\Omega)\right]$$

Let $\varphi(\zeta) = \int \dfrac{d\mu(z)}{(z-\zeta)}, \zeta \notin$ support of $d\mu$, then let $\Phi(w) = \int e^{wz} d\mu(z)$. We may call $\Phi(w)$ the transform of $d\mu$. Then condition $\int e^{\lambda z} d\mu(z) = 0, \lambda \in \Lambda$ is merely $\Phi(\Lambda) = 0$ and if this implies $\int f d\mu = 0, f \in \mathscr{H}(\Omega)$,
50 by our duality, $\varphi(\zeta) \equiv 0$ and $\Phi(w) \equiv 0$. Conversely if $[\Phi(\Lambda) = 0 \Longrightarrow \Phi(w) \equiv 0]$, does it follow that $\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega)$, or again $\varphi(\zeta) \equiv 0$? one can get the answer using Runge's theorem but we prefer to deduce it from a relation between Φ and φ, which is interesting in itself.

We have the following equations:

$$\frac{1}{2\pi i} \int_C \varphi(\zeta) e^{w\zeta d\zeta} = \frac{1}{2\pi i} \int\limits_\Omega \int_C \frac{e^{w\zeta}}{z-\zeta} d\zeta d\mu(z) = \int e^{wz} d\mu(z)$$

$$\Phi(w) = \frac{1}{2\pi i} \int_C \varphi(\zeta) e^{w\zeta} d\zeta. \tag{1}$$

If $\varphi(z) = \dfrac{1}{z-\zeta}$, we have $\Phi(w) e^{\zeta w}$; but when Re $(z-\zeta) \geq 0$, $\dfrac{1}{z-\zeta} = \int_0^\infty e^{-u(z-\zeta)} du$. Heuristically we can have a formula, reciprocal to formula (1) in the following form

$$\int_0^\infty \Phi(u) e^{-uz} du = \varphi_o(z) \tag{2}$$

$$\varphi_o(z) = \varphi(z) \tag{3}$$

For the proof of (3), let $\varphi(z)$ be regular outside a compact set $K \subset \Omega$ and vanishing at infinity. Then (1) gives

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{a_n}{zn+1}, |a_n| < R^n \Longrightarrow \Phi(w) = \sum \frac{a_n}{n!} w^n$$

and therefore $|\Phi(w)| < e^{R|w|}$. Conversely let $|\Phi(w)| < e^{R|w|}$. Then (2) has a meaning for Re $z > R$ and it is easily seen that

$$\int_0^\infty \sum \frac{a_n}{n!} u^n e^{-uz} du = \sum \frac{a_n}{z^{n+1}} = \varphi_o(z)$$

Definition. $\varphi_o(z) = \int_0^\infty e^{-uz}\Phi(u)du$ is defined to be the *Laplace transform* of the entire function $\Phi(w)$ of exponential type. 51

Now if $\Phi(w) \equiv 0$, then $\varphi(z) \equiv 0$ and thus we have the following closure theorem.

Theorem. $\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega) \Longleftrightarrow [\Phi(w) = \int e^{wz} d\mu(z)]$,
$d\mu \in \mathscr{H}'(\Omega), \Phi(\Lambda) = 0 \Longrightarrow \Phi \equiv 0$.

Thus the problem of closure is related to the problem of the distribution of the zeros of an entire function of exponential type.

Definition. Let $\Phi(w)$ be an entire function of exponential *type*. The *type* of Φ is the lower bound of τ such that $\Phi(w) = 0(e^{\tau|w|})$. The *type* $h(\theta)$ of Φ in the direction θ is defined by :

$$h(\theta) = \limsup_{r\to\infty} \frac{\log \Phi(re^i)}{r}$$

we shall see in the next lecture how the formula (1) helps us to find $h(\theta)$.

Lecture 9

Laplace - Borel transform and conjugate diagram of an entire function of exponential type

1 Conjugate diagram and Laplace - Borel transform

We recall the formulae: $\varphi(z)$ is holomorphic on C and vanishes at infinity 52
; $\Phi(w)$ is an entire function of exponential type b.

$$\Phi(w) = \frac{1}{2\pi i}\int_C e^{wz}\varphi(z)dz \tag{1}$$

$$\varphi(z) = \int_0^\infty \Phi(u)e^{-uz}dz; Rez > b. \tag{2}$$

If (1) and (2) hold, then

$$\varphi_o(z) = \varphi(z) \text{ for } Rez > b. \tag{3}$$

$h(\theta) = \limsup \dfrac{\log|\Phi(re^{i\theta})|}{r}$ = the type of $\Phi(w)$ in the direction θ.

Interpretation of (1):

$$\Phi(re^{i\theta}) \leq e^{rk_c^{(\theta)}} \times \frac{1}{2\pi} \int \varphi(z)dz$$

where $k_c(\theta) = \sup\limits_{z \in c} Re(ze^{i\theta})$.

Interpretation of $k_c(\theta)$: the closed convex hull of C is the intersection of the half planes $x\cos\theta - y\sin\theta \leq k_c(\theta)$. Let $k(\theta) = \inf k_c(\theta)$. Then the intersection of the planes $x\cos\theta - y\sin \leq k(\theta)$, is the smallest convex set outside of which $\varphi(z)$ is holomorphic. By abuse of language we call this set "the convex hull of the singularities of φ".

53 Evidently $h(\theta) \leq k_c(\theta)$ and so $h(\theta) \leq k(\theta)$.

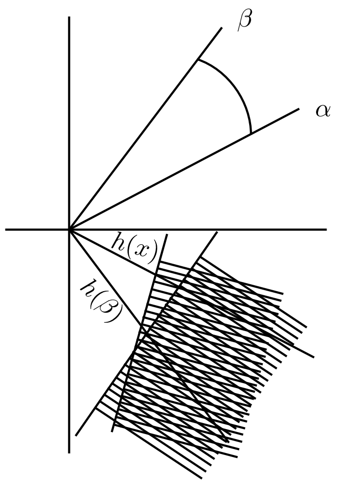
Interpretation of (2): we consider the following equation

$$\varphi_\alpha(z) = \int\limits_0^{\infty e^{i\alpha}} \Phi(w)e^{-wz}dw, w = re^{i\chi}. \tag{2_α}$$

We have $\varphi_\alpha(z)$ holomorphic for Re $ze^{i\alpha} > h(\alpha)$. In the same manner we have the equation:

$$\varphi_\beta(z) = \int\limits_0^{\infty e^{i\beta}} \Phi(w)e^{-wz}dw, w = re^{i\beta} \tag{2_β}$$

We have $\varphi_\beta(Z)$ holomorphic for Re $ze^{i\beta} > h(\beta)$. Suppose $\alpha \not\equiv \beta($mod $2\pi)$. Then the intersection of the half-planes in which φ_α and φ_β are holomorphic is non-empty; moreover for any point in the intersection of these halfplanes we have $\varphi_\alpha(z) = \varphi_\beta(z)$. For this it is sufficient to show that $\int_C \Phi(w)e^{-wz}dw \to 0$ as $R \to \infty$, where C is the smaller are joining $Re^{i\beta}$ and $Re^{i\alpha}$. Since $w^2\Phi(w)e^{-wz}$ is bounded on the lines $(0, \infty e^{i\beta})$ and $(0, \infty e^{i\alpha})$ it is sufficient to apply the theorem of Phragmen Lindelof.



Thus we have $\varphi(z)$ defined and holomorphic outside of every half plane $x\cos\alpha - y\sin\alpha \leq h(\alpha)$. Hence $k(\theta) \leq h(\theta)$ and we have the following theorem:

Theorem. *The equations* **54**

$$\Phi(w) = \frac{1}{2\pi i}\int_C e^{wz}\varphi(z)dz \tag{1}$$

$$\varphi_\alpha(z) = \int_0^{\infty e^{i\alpha}} \Phi(w)e^{-wz}dz \tag{2}$$

allow us to associate to each function φ, holomorphic at infinity and vanishing there, an entire function Φ of exponential type and conversely. If S is the "convex hull of the singularities" of $\varphi(z)$ and $h(\theta)$ the type of $\Phi(w)$ in the direction θ, then S is the intersection of the half-planes $x\cos\theta - y\sin\theta \leq h(\theta)(0 \leq \theta \leq 2\pi)$.

Definition. $\varphi(z)$ is defined to be the *Laplace-Boral* transform of $\Phi(w)$ and S is defined to be the *conjugate diagram* of $\Phi(w)$.

The notion of the conjugate diagram is due to G. Polya.

2 Basis and Fourier development in $\mathscr{H}_\Lambda(\Omega)$

We suppose for simplicity Λ to be simple and we consider the Fourier development of functions in $\mathscr{H}_\Lambda(\Omega)$. The formal development will be established if we prove that $\{e^{\lambda z}\}$ form a basis in $\mathscr{H}_\Lambda(\Omega)$.

Theorem 1. *Suppose Ω is connected and $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$. Then $\{e^{\lambda z}\}_{\lambda\in\Lambda}$ form a basis in $\mathscr{H}_\Lambda(\Omega)$.*

Since $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$, there exists a measure $d\mu \in \mathscr{H}'\Omega$ with $\int e^{\lambda z}d\mu(z) \neq 0, \lambda \in \Lambda$. Since Ω is connected a closed rectifiable cure C can be found with the support of $d\mu$ in its interior. Then we have the following equations:

$$\Phi(w) = \int e^{wz}d\mu(z) = \frac{1}{2\pi i}\int_C e^{wz}\varphi(z)dz$$

We have trivially, $\left\{e^{\lambda z}\right\}_{\lambda\in\Lambda}$ total in $\mathscr{H}_\Lambda(\Omega)$. To show that $\left\{e^{\lambda z}\right\}$ is free 55
it is sufficient to show that there exists a measure which is orthogonal to all $e^{\lambda z}$ except one, say λ_1. For this it is sufficient to suppose that $\lambda_1 = \theta \in \Lambda$. We have $\Phi(0) = 0$, *i.e.* $\int_0 \varphi(z)dz = 0$. Let φ_1 be the primitive of φ which vanishes at infinity. Since $\int_C \varphi(z)dz = 0$, we have $\Phi(w) = -\dfrac{1}{2\pi i} w \int_C e^{wz}\varphi_1(z)dz$. If $\dfrac{\Phi(w)}{w} \neq 0$, we have a measure $d\mu$, given by $\varphi_1(z)$ not orthogonal to $i = e^{oz}$. In the contrary case we iterate the process.

Corollary. *If $\Omega' \supset \Omega, \Omega'$ open and Ω connected and if $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$ then $\{e^{\lambda z}\}_{\lambda\in\Lambda}$ form a basis of $\mathscr{H}_\Lambda(\Omega')$.*

Remark 1. We cannot extend the result to the case where Ω is not connected. For let $\Lambda = Z$ and Ω consist of two disjoint circular domains around πi and $-\pi i$. Here $1 \in$ span of $\{e^{nz}\}n \neq 0$.

Remark 2. Let $\Phi(w) = \dfrac{1}{2\pi i}\int_C e^{wz}\varphi(Z)dz$ and $\Phi(\Lambda) = 0$. Then for each $\lambda \in \Lambda$, we can find $\varphi_\Lambda(z)$ holomorphic outside the same convex domain as $\varphi(z)$ and satisfying the equation

$$\frac{\Phi(w)}{w-\lambda} = \frac{1}{2\pi i}\int_C e^{wz}\varphi_\lambda(z)dz.$$

Theorem 2. *Suppose Ω is convex and $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$. Then every $f \in \mathscr{H}_\Lambda(\Omega)$ is uniquely defined by its development. In other words if all the coefficients in the development of f are zero, the function is identically zero.*

56 *Proof.* Let $g \in \mathscr{H}_\Lambda(\Omega)$ and $g(z) \sim \sum 0e^{\Lambda z}$. Let $\Phi(w)$ and $\varphi(z)$ be determined as before with $\Phi(\Lambda) = 0$. We have the equation

$$\Phi_\lambda(w) = \Phi(w)/(w-\lambda) = \frac{1}{2\pi i}\int_c \varphi_\lambda(z)e^{wz}dz.$$

By our hypothesis, we have for every $\lambda \in \Lambda$, $\int_c \varphi_\lambda(z)g(z)dz = 0$. $\square$

Now $\Phi(w) = w^p e^{aw} \prod_{\lambda \in \Lambda} \left(1 - \dfrac{w}{\lambda}\right) e^{w/\lambda}$.

$$\Phi'(w) = a\Phi + \frac{p\Phi_o(w)}{w} + \sum_\lambda \left(\frac{\Phi}{\lambda} + \Phi_\lambda\right).$$

We set $X_\lambda(w) = \Phi(w)/\lambda + \Phi_\lambda(w)$. In the same way as we have proved in the case of mean periodic functions (Lecture 5, §1) we have the following inequalities:

$$\begin{aligned}
\left|X_\lambda\,(w)\right| &< \frac{2}{\left|\lambda\right|^2} \left|w\,\Phi\,(w)\right| \text{ when } \left|w - \lambda\right| > \frac{\left|\lambda\right|}{2}\\
\left|X_\lambda\,(w)\right| &< \frac{2}{\left|\lambda\right|^2} \left|w\,\Phi\,(w)\right| \text{ when } 1 \leq \left|w - \lambda\right| \frac{|\lambda|}{2}\\
\left|X_\lambda\,(w)\right| &< \frac{K}{\left|\lambda\right|^2} \sup_{|w-w'|=2} \left|w'^2 \Phi(w')\right| \text{ when } \left|w - \lambda\right| < 1.
\end{aligned}$$

Therefore we have the following inequality

$$\left|X_\lambda\,(w)\right| < \frac{K}{\left|\lambda\right|^2} e^{h(\theta)\left|w\right| + \varepsilon\left|w\right|}$$

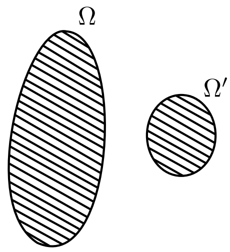
uniformly in w. Since Ω is convex, we can take a path C in Ω around the conjugate diagram of Φ and by taking the Laplace-Borel transform along this path we have $\left|\varphi_\lambda(z) + \varphi(z)/\lambda\right| < K/\left|\lambda\right|^2$ uniformly in z. Therefore we have the following implication:

$$\int_c g(z)\left[a\varphi + p\varphi_o + \sum(\varphi_\lambda + \varphi/_\lambda)\right] dz = 0 \Longrightarrow \int_c g(z) z\varphi(z) dz = 0.$$

As the same holds if we replace $\varphi(z)$ by $\varphi_\lambda(z)$, we have $zg(z) \in \mathscr{H}_\Lambda(\Omega)$ and $zg(z) \sim \sum oe^{\lambda z}$. Therefore if $g(z) \sim \sum oe^{\lambda z} \in \mathscr{H}_\Lambda(\Omega) g(z)$ $P(z) \in \mathscr{H}_\Lambda(\Omega)$ for every polynomial $P(z)$. If $g(z) \not\equiv 0$. We can suppose that $g(z)$ has only a finite number of zeros in $\Omega-$ if necessary we can 57
replace Ω by a smaller domain containing $C-$ so that every holomorphic function with these zeros is in $\mathscr{H}_\Lambda(\Omega)$, and $\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega)$, contrary to the hypothesis that $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$.

Remark. Suppose $\Omega' \subset \Omega, \Omega$ convex and $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$. Then one cannot assert that every $f \in \mathscr{H}_\Lambda(\Omega')$ is determined by its development. As an example, let Λ = set of integers and Ω be a convex set containing πi and $-\pi i$. Let Ω_i be disjoint from Ω, and contained in $Rez > 0, \left|Imz\right| < \pi$. We take $\Omega' = \Omega \cup \Omega_1$, $f = 0$ in Ω, f analytic $\not\equiv 0$ in Ω_1.

Let us put $Z = e^z$, $f(z) = F(Z)$. F is zero on the set $\log\Omega$; and analytic $\not\equiv 0$ in $\log\Omega_1$. By the theorem of Runge, F can be approximated by polynomials in $\log\Omega_1$, and then $f \in \mathscr{H}_\Lambda(\Omega_1)$. As $\{e^{nz}\}n \in N$ is a basis in $\mathscr{H}_\Lambda(\Omega)$, the development of f has zero coefficients, what ever be f in Ω_1.



Problems.

1) It is possible to get a statement like the above theorem if the open convex set Ω is replaced by a closed convex set?

2) Also, is a similar statement possible on replacing the open convex set Ω by an open connected set ?

Lecture 10

Canonical products and their conjugate diagrams

1 Canonical products and location of conjugate diagram

Given a sequence $\Lambda = \{\lambda_n\}, \left|\lambda_n\right| \to \infty$, it is always possible to construct **58**
an entire function with λ_n as zeros. We suppose Λ to be symmetric, $\Lambda = \{\pm\lambda_n\}, \left|\lambda_{n+1}\right| \geq \left|\lambda_n\right| (n = 1, 2, \ldots), \sum\limits_1^\infty \dfrac{1}{|\lambda_n|^2} < \infty$, and (to avoid complications), $|\lambda_1| > 0$. Then the simplest of these functions is the canonical product

$$C(w) = \prod_1^\infty \left(1 - \frac{w^2}{\lambda_n 2}\right).$$

For example, if $\lambda_n = n, C(w) = \dfrac{\sin \pi w}{\pi w}$. With convenient hypothesis about the λ_n, we shall see that $C(w)$ is an entire function of exponential type, and be able to locate its conjugate diagram.

We first recall the following definitions (Mandelbrojt 2).

$$n(r) = \sum_{|\lambda n| < r} 1 : \text{ distribution function of } \Lambda.$$

$$D(r) = n(r)/r : \text{ density function of } \Lambda.$$

$$D^{\cdot} = \limsup_{r \to \infty} D(r) : \text{ upper density of } \Lambda.$$

$$D. = \lim\inf_{r\to\infty} D(r): \text{ lower density of } \Lambda.$$

We have $D., = \lim\sum \frac{n}{|\lambda n|} \geq \lim\inf \frac{n}{|\lambda n|} = D.$. When $D^{\cdot} = D = D,$. we define D to be the density of Λ.

$\bar{D}(r) = \frac{1}{r}\int_o^r D(t)dt$: Mean density function of Λ.

We define the mean upper density $\bar{D}^{\cdot}$, mean lower density $\bar{D}.$, and the mean density $\bar{D}$ in the same way. We have $D. \leq \bar{D}. \leq \bar{D} \leq D^{\cdot}$, and one can prove $D^{\cdot} < e\bar{D}^{\cdot}$ (Mandelbrojt 3).

59 **Calculation of Mandelbrojt (Mandelbrojt** 3**) . .**

We make the hypothesis that $\bar{D}^{\cdot} < \infty$, and we majorise the type of $C(w)$, in order to have a location of the conjugate diagram.

$$\left|C(w)\right| \leq \prod\left(1 + \frac{r^2}{|\lambda|^2}\right) = \varphi(r), \log\varphi(r) = \int_o^\infty \log\left(1 + \frac{r^2}{|\lambda_n|}\right) dn(\lambda).$$

To calculate $\varphi(r)$ we integrate by parts

$$\int_o^x \log(1 + r^2|\lambda^2|)dn(\lambda) = \int_o^x \frac{n(\lambda)}{r^2 + \lambda^2}\frac{n(\lambda)}{\lambda}d\lambda + \left[n(\lambda)\log\left(1 + \frac{r^2}{\lambda 2}\right)\right]_o^x.$$

Since $\bar{D}^{\cdot} < \infty$, we cannot have $\frac{n(X)}{X} \to \infty$. Therefore there exists a sequence $X_n \to \infty$, such that $\frac{n(X_n)}{x_n^2} \to 0$. Since $n(0) = 0$ and

$$n(x_n)\log\left(1 + \frac{r^2}{x_n^2}\right) \to 0 \text{ as } x_n \to \infty$$

we have $$\log\varphi(r) = \int_o^\infty \frac{2r^2}{r^2 + \lambda^2}d(\lambda)D\lambda.$$

$$\int_o^x \frac{2r^2}{r^2 + \lambda^2}D(\lambda)d\lambda = \int_o^x \frac{4r^2}{(r^2 + \lambda^2)^2}\lambda\bar{D}(\lambda)d\lambda + \left[\lambda\bar{D}(\lambda)\frac{2r^2}{r^2 + \lambda^2}\right]_o^x.$$

So,

$$\log\varphi(r) = \int_o^\infty \frac{4r^2\lambda^2}{(r^2 + \lambda^2)^2}\bar{D}(\lambda)d\lambda = r\int_o^\infty \frac{4t^2}{(1 + t^2)^2}\bar{D}(t_r)dt.$$

$$\limsup_{r\to\infty} \frac{\log \varphi(r)}{r} \leq \bar{D^{\cdot}} \int_0^\infty \frac{4t^2}{(1+t^2)^2} dt.$$

We have $\pi = \int\limits_o^\infty \frac{4t^2}{(1+t^2)^2} dt$. This gives the relation

$$\underline{\underline{h(\theta) \leq \pi \bar{D^{\cdot}}}} \qquad (M)$$

Geometrically, this signifies that the conjugate diagram is situated in a circle with centre at origin and or radius $\pi \bar{D^{\cdot}}$.

Calculation of Carlson (Bernstein, note *III*).

We now suppose that we have one of the following equivalent relations

$$[D^{\cdot} = D_{.} = D = 1, \arg \lambda_n \to 0 \Longleftrightarrow \frac{\lambda n}{n} \to 1.]$$

Under these conditions the conjugate diagram is particularly simple. **60**
Suppose $\lambda_n = n$. Then $C(w) = \dfrac{sin\pi w}{\pi w}$ is of type π in the upper and lower half planes. Also we have $h(0) = h(\pi) = 0$. Since the conjugate diagram is convex, it is the segment between πi and $-\pi i$.

Now,

$$\frac{\pi w C(w)}{\sin \pi w} = \prod \frac{(1 - w^2/\lambda_n^2)}{(1 - w^2/n^2)} = \prod \left(1 + \frac{w^2/n^2 - w^2/\lambda_n^2}{1 - w^2/n^2}\right).$$

Let $0 < \alpha < \dfrac{\pi}{4}$, and $w = re^i\theta, \alpha < |\theta| < \dfrac{\pi}{2}$. In this case, we have the inequality $|w^2 - n^2| > n^2 \sin 2\alpha$. Since $(\lambda_n^2 - n^2)/\lambda_n^2 \to 0$, we have the following inequalities:

$$\left|\frac{\pi w C(w)}{\sin \pi w}\right| \leq \prod_1^\infty \left|1 + \frac{w^2(\lambda_n^2 - n^2)/\lambda_n^2}{n^2 - w^2}\right| = \prod_1^\infty X_n$$

$$\prod_1^{N-1} X_n \prod_N^\infty (1 + \frac{\varepsilon r^2}{n^2 \sin 2\alpha})$$

$$\prod_1^{N-1} X_n |\sin(i\pi r \sqrt{\varepsilon/\sin 2\alpha})|$$

This gives us $\limsup_{r\to\infty} 1/r \log \dfrac{C(w)}{\sin \pi w} \leq 0$; from which we have

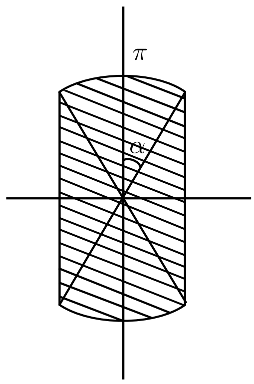
$$\limsup_{r\to\infty} \frac{\log |C(w)|}{r} \leq \pi |\sin\theta|, \theta \neq 0, \pi (\mod 2\pi)$$

By taking the function $(\sin \pi w)/\pi w C(w)$ we have the same calculation with the roles of λ_n and n interchanged. Since $\dfrac{n}{\lambda_n} \to 1$, we can replace λ_n by n for large values of n in majorising $(\sin \pi w)/\pi w C(w)$ and get the reverse inequality. Thus we get

(c) $\limsup_{r\to\infty} |\dfrac{\log C(w)}{r}| = \pi |\sin\theta|, w = re^{i\theta}, \theta \neq 0, \pi (\mod 2\pi)$. This result shows us that if the density of Λ is D, then the conjugate diagram of $C(w)$ is the segment joining $i\pi D$ and $-i\pi D$

61 We can relax slightly the hypothesis in the calculation of Carlson, viz., make the hypothesis that $D = 1$ and $\limsup |arg \lambda_n| \leq \alpha \leq \dfrac{\pi}{2}$. Then, for $n > N, |arg \lambda_n| < \alpha + \varepsilon = \alpha$. Taking $0 < argw =< \frac{\pi}{2} - \alpha'$ we have

$$\prod_N^\infty \left|1 - \frac{w^2}{\lambda_{n^2}}\right| < \prod_N^\infty \left| - 1 \frac{w^2}{\lambda_n^2 e^{-2i\alpha'}}\right|.$$



Hence $h(\theta) \leq \pi sin(\pi + \alpha)$ if $0 < \theta < \pi/2 - \alpha'$.In this case the conjugate diagram turns out to be contained in the portion of the disc $|z| \leq \pi$, containing 0 and bounded by the lines $x = \pm\pi \sin\alpha$.

2 Theorems of Jensen and Carleman

We now recall well - known formulae which will be used in the problems of closure and quasi- analyticity among others.

Jensen's formula. *Let $F(w)$ be a function meromorphic inside and continuous on a circle of radius R and center 0, and let $F(w)$ be without zero or pole at 0 and on $|w| = R$. Denote by $n_1(r)$ the number of zero in $|z| < r$ and by $n_2(r)$ the number of poles in $|z| < r$. Then we have the following relation:*

$$\int_0^R \left(\frac{n_1(r)}{r} - \frac{n_2(r)}{r}\right) dr = \frac{1}{2\pi} \int_0^{2\pi} log \frac{F(Re^{i\theta})}{F(0)} d$$

Carleman's formula. *Let $F(w)$ be meromorphic in the half plane $u \geq 0 (w = u + iv)$ with zeros at $r_k e^{i\theta_k}$ and poles at $\varrho_j e^{i\alpha_j}$ and with no zero or pole on $u = 0$. Taking a contour D consisting of a part of the imaginary axis and a semi - circle of radius R and center 0 in the half plane $u \geq 0$,* **62**
we have the following relation:

$$\begin{aligned} \sum \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \cos\theta_k - \left(\frac{1}{\rho_j} - \frac{\rho_j}{R^2}\right) \cos\alpha_j \\ = \frac{1}{\pi R} \int\limits_{\pi/2}^{\pi/2} \log |F(Re^{i\theta})| \cos\theta d\theta \\ + \frac{1}{2\pi} \int_0^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) \log |F(iy)F(-iy)| dy + 0(1) \end{aligned}$$

the summation being on $r_k e^{i\theta k}, \rho_j e^{i\alpha_j}$ inside D.
For the proofs Cf. Titchmarsh.

Lecture 11

Application of Jensen's and Carleman's formulae to closure theorems

1 Application of Jensen's formula

The formula of Jensen allows us to have a condition of totality of a set **63**
$\{e^{\lambda z}\}$ in an open set Ω. The relation

$$[\Phi(w) = \int\limits_{\Omega}^{e^{wz}} d\mu(z), \Phi(\Lambda) = 0, d\mu \in \mathscr{H}'(\Omega)] \Rightarrow \Phi = 0,$$

implies that $\{e^{\lambda z}\}$ is total in Ω. Let us see what happens when $\Phi \not\equiv 0$.

If the convex closure of Ω is the intersection of the half planes $x\cos\theta - y\sin\theta \leq k(\theta)$, then $h(\theta) = \limsup\limits_{r\to\infty} \log\dfrac{|\Phi(re^{i\theta})|}{r} \leq k(\theta) - 2\varepsilon, \varepsilon > 0$. We can find r such that $R > r$ implies $\log|\Phi(Re^{i\theta})| < (h(\theta) + \varepsilon)R$. Suppose $0 \notin \Lambda$, and denote by $n_1(r)$ the distribution function of $\{|\lambda|\}$. Taking $F(w) = \Phi(w)$ in Jensen's formula, we have the following:

$$\int_0^R \frac{n_1(r)}{r} dr \leq \frac{1}{2\pi}\int_0^{2\pi} (h(\theta) + \varepsilon) R d\theta + 0(1).$$

Since $h(\theta) \leq k(\theta) - 2\varepsilon$ we have the following relation:

$$\frac{1}{R}\int_o^R \frac{n_1(r)}{r}dr = \bar{D}^{\cdot}(R) < \frac{1}{2\pi}\int_o^{2\pi} k(\theta)d\theta - \varepsilon + 0(\frac{1}{R})$$

In order that $\Phi \equiv 0$ it is sufficient that this condition is not satisfied. In other words, if $\Phi(\Lambda) = 0, 0 \notin \Lambda$, in order that $\Phi \equiv 0$ it is sufficient to have the inequality

$$\bar{D}^{\cdot}_1 \geq \frac{1}{2\pi}\int_o^{2\pi} k(\theta)d\theta.$$

In this formula, $\bar{D}^{\cdot}_1$ is the mean upper density of the sequence $|\lambda_n|$. If Λ is symmetric, $\Lambda = \{\pm\lambda_n\}$, the formula becomes

$$2\bar{D}^{\cdot} \leq \frac{1}{2\pi}\int_o^{2\pi} k(\theta)d\theta \; (*)$$

64 In particular, we have the following result:

Theorem. *If Λ is symmetric and $C(w) = \prod(1 - \frac{w^2}{\lambda_n^2})$ is of exponential type zero on the real line, the conjugate diagram of $C(w)$ reduces to the segment $[-i\alpha, i\alpha], \alpha = \pi\bar{D}$.*

To prove it, we take $\Omega)[-i\alpha, i\alpha]$ such that $k(\theta) = \alpha\sin\theta| + \varepsilon$ If $\alpha < \pi\bar{D}^{\cdot}$, the formula $(*)$ will be satisfied for ε sufficiently small and $\{e^{+i\lambda_n z}\}$ will be total in $\mathscr{H}_\Lambda(\Omega)$, which is false. Therefore $\alpha \geq \pi\bar{D}^{\cdot}$. As we know (Lecture 10, §1) $\alpha \leq \pi\bar{D}$, we have $\alpha = \pi\bar{D}^{\cdot}$.

We shall see later (Lecture 12, §1) that if $C(u)$ is bounded on the real axis, one has $\alpha = \pi D^{\cdot} = \pi D^{\cdot} = \pi D$

2 Application of Carleman's formula

We have see the use of Jensen's formula in a theorem of closure involving the behaviour of the sequence Λ in the whole plane. But when we want to use only the behaviour of the sequence Λ in one half plane we can use the formula of Carleman. For simplicity we suppose λ_n are real and positive, and $\bar{D}^{\cdot} < \infty$(if not, $\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega)$ what ever be the open bounded set. Moreover let Ω be contained in a horizontal strip $\alpha \leq v \leq \beta$. Let $\Phi(w) = \int_\Omega e^{wz} d\mu(z)$. We have the closure theorem $\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega)$, when the following relation holds:

$$[\Phi(\lambda) = 0 \text{ for every } \lambda \in \Lambda] \Rightarrow \equiv 0.$$

Let us see what happens when $\Phi \not\equiv 0$. Let $h(\theta)$ be the type of $\Phi(w)$ along θ, then $h(\pi/2) < -\alpha, h(-\pi/2) < \beta$. Applying Carleman's formula we have the following relations:

$$\int_o^R \left(\frac{1}{r} - \frac{r}{R^2}\right) dn(r) = \frac{1}{2\pi} \int_o^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) \log|\Phi(iy)\Phi(-iy)| dy + 0(1)$$

$$\int_o^R \left(\frac{1}{r^2} - \frac{1}{R^2}\right) n(r) dr < \frac{1}{2\pi}(\beta - \alpha) \int_o^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) y dy + 0(1)$$

$$= \frac{\beta - \alpha}{2} \log R + 0(1)$$

$$\frac{1}{\log R} \int_o^R \frac{D(r)}{r} dr < \frac{\beta - \alpha}{2\pi} + 0(1).$$

Suppose $\bar{D}_\cdot < \infty$; then **65**

$$\int_o^R \frac{dn(r)}{r} = \int_o^R \frac{D(r)}{r} + 0(1) = \int_o^R \frac{\bar{D}(r)}{r} dr + 0(1).$$

Definition.

$$\hat{D}_\cdot = \limsup_{R\to\infty} \frac{1}{\log R} \int_o^R \frac{dn(r)}{r} = \limsup_{R\to\infty} \frac{1}{\log R} \int_o^R (\frac{D(r)}{r} dr$$
$$= \limsup_{R\to\infty} \frac{1}{\log R} \int_o^R \frac{\bar{D}(r)}{r} dr$$

is defined to be the *logarithmic upper density of* Λ

We have $\hat{D}_\cdot \leq \bar{D}_\cdot \leq D_\cdot$, since $\int_o^R \frac{D(r)}{r} dr \leq \log R(\bar{D}_\cdot + \varepsilon)$. In order that $\Phi \equiv 0$ it is sufficient to have $\hat{D}_\cdot \geq \dfrac{\beta - \alpha}{2\pi}$. Thus we have the closure theorem.

Theorem.

$$\hat{D}^{\cdot} \geq \frac{\beta - \alpha}{2} \Rightarrow \mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega).$$

The constants occurring in the above inequality are the best possible. To see this we take $\Lambda = N = (1, 2, \ldots)$ and Ω a strip of width greater than 2π. We do not have closure in this case and $\overline{D}^{\cdot} = 1 < (\beta - \alpha)/2\pi$.

When Λ and Ω are given, either Jensen's or Carleman's formula can be applied to prove $\mathscr{H}_\Lambda(\Omega) = \mathscr{H}(\Omega)$. Roughly specking, Carleman's formula is better if Ω is "flat" enough, or if the part of Λ which is in some half-plane is "scarce". If we try to prove $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$ the methods of the lecture 10 have be applied.

3 Theorem of closure on a compact set

66 Let K be a compact set which does not divide the plane, and $\mathscr{H}(K)$ be the space of functions, continuous on K and holomorphic in the interior of K; the polynomials form a total set in $\mathscr{H}(K)$ (Theorem of Mergelyan, Lecture 8). Let $\mathscr{H}(K)$ be the closed span of $\{e^{\lambda z}\}_{\lambda \in \Lambda}$ in $\mathscr{H}(K)$. We want to find when $\mathscr{H}_\Lambda(K) = \mathscr{H}(K)$. Let

$$\Phi(w) = \int_K e^{wz} d\mu(z) \tag{*}$$

with $\Phi(\Lambda) = 0$. In order that $\mathscr{H}_\Lambda(K) = \mathscr{H}(K)$ it is sufficient that $(*)$ implies $\Phi \equiv 0$ and $\mathscr{H}_\Lambda(K) \neq \mathscr{H}(K)$ if there exists a $\Phi \not\equiv 0$ satisfying $(*)$. Suppose $\Phi \not\equiv 0$. We have the following majorization:

$$|\Phi(w)| < \max_{z \in K} |e^{zw}| \int_K |d\mu|$$

Now we apply Jensen's formula, as in §1. We have the inequality:

$$R\overline{D}_1(R) < \frac{1}{2\pi} R \int_0^{2\pi} k(\theta) d\theta + 0(1)$$

where $k(\theta) = \max_{Z \in K} Re(ze^{i\theta}) = \max_{Z \in K} (xcos\theta - ysin\theta)$, and $\overline{D}_1(R)$ is the function of mean density of the sequence $\{|\lambda|\}_{\lambda \in \Lambda}$(then, if $\Lambda = \{\pm\lambda_n\}$, $\overline{D}_1(R) = 2\overline{D}(R)$, $\overline{D}(R)$ being the function of mean density of the sequence $\{|\lambda_n|\}$.

Theorem. *In order that* $\{e^{\lambda z}\}_{\lambda\in\Lambda}$ *be total in* $\mathscr{H}(K)$ *it is sufficient that* $\limsup\limits_{R}(R\bar{D}_1(R) - \frac{1}{2\pi}R\int_0^{2\pi}k(\theta)d\theta) = \infty.$

In particular, let $K = [-\pi, \pi]$. Then $\mathscr{H}_\Lambda(K) = \mathscr{H}(K) = \mathscr{C}(K)$ when the following condition is satisfied:

$$\limsup(R\bar{D}(R) - 2R) = \infty \tag{1}$$

The result is a very precise one. For example, take $\Lambda = \pm n = Z - \{0\}$. 67
Then $R\bar{D}_1(R) = 2R - \log R + 0(1)$; the condition (1) is not satisfied, and it is easily seen that $\mathscr{C}_\Lambda(K) \neq \mathscr{C}(K)$. If we add one element $\alpha \neq 0$ to Λ, we add $\log R + 0(1)$ or $R\bar{D}_1(R)$; the condition (1) is not yet satisfied, and in fact $\mathscr{C}_{\Lambda+\{\alpha\}}(K) \neq \mathscr{C}(K)$ (this is very easy to see if $\alpha = 0$, because the functions take equal values at π and $-\pi$, and still holds for $\alpha \neq 0$). But if we add two elements $\alpha, \beta \neq 0$ to Λ (1) is satisfied, and $\mathscr{C}_{\Lambda+\{\alpha\}+\{\beta\}}(K) = \mathscr{C}(K)$.

4 Theorem of Muntz

We shall see what happens if we try to apply Carleman's formula in the case of a line segment. We can get a finer result by having a condition of totality in an infinite interval. For simplicity we can take the half- line $L = (-\infty, 0)$ Let $\mathscr{C}_0(L)$ be the space of functions, continuous on L and vanishing at infinity. We take Λ to be a sequence of positive numbers. In order that $\{e^{\lambda z}\}$ be non-total it is necessary and sufficient that there exists a measure $d\mu$ orthogonal to $\{e^{\lambda z}\}$ and not orthogonal to $\mathscr{C}_o(L)$. Then $\Phi \equiv 0$ where $\Phi(w) = \int\limits_{-\infty}^{0} e^{wz}d\mu(z)$ with $\Phi(\Lambda) = 0$.For $u \geq 0$ we have $|\Phi(w)| < \int |d\mu|$. Applying Carleman's formula we have $\int_o^R \dfrac{D(r)}{r}dr = 0(1)$, where $D(r)$ is the function of density of Λ. As $rD(r) \nearrow$ we have

$$\int\limits_R^\infty \frac{RD(R)}{r^2}dr, < \int\limits_R^\infty \frac{D(r)}{r}dr = 0(1)$$

then $D = 0$. Then $\int_0^R \dfrac{dN(r)}{r} = D(R) + \int_o^R \dfrac{D(r)}{r} dr = 0(1)$. This means
68 $\sum\limits_{\lambda \in \Lambda} \dfrac{1}{\lambda} < \infty$. Therefore $\{e^{\lambda z}\}$ is total whenever $\sum \dfrac{1}{\lambda} = \infty$. Conversely, if $\sum \dfrac{1}{\lambda} < \infty$, we do not have totality. This is proved by taking a function $\Phi(w)$ of exponential type with $\Phi(\Lambda) = 0$ and small at infinity. For the construction of $\Phi(w)$, we take

$$\Phi(w) = \prod_{\lambda \in \Lambda} \frac{\sin \pi w/\lambda}{\pi w/\lambda}, |\Phi(u)| = \prod_1^3 \prod_4^\infty = 0\left(\frac{1}{u^2}\right) 0(1) \text{ and}$$

$\Phi(w)$ is of exponential type (we have already seen such a constructed in Lecture 6, §1.). Thus $\Phi(w)$ is the Fourier transform of a measure, $\Phi(w) \neq 0$ and $\Phi(\Lambda) = 0$.

Thus we have the following closure theorem.

Theorem. *In order that* $\{e^{\lambda z}\}_{\lambda \in \Lambda, \lambda > 0}$ *be total in* $\mathscr{C}_0(0, \infty)$ *it is necessary and sufficient that* $\sum\limits_{\lambda \in \Lambda} \frac{1}{\lambda} = \infty$.

This theorem gives us, by making the transformation $e^z = x, e^{\lambda z} = x^\lambda$ the theorem of Muntz, viz., $\{x^\lambda\}$ is total in $\mathscr{C}_o[0, 1]$ if and only if $\sum \dfrac{1}{\lambda} = \infty$.($\mathscr{C}_o[0, 1]$ being the subspace of $\mathscr{C}[0, 1]$ consisting of the f vanishing at zero).

The proof of the above theorem gives also the following result:

In each closed interval $I\{\mathscr{C}_\Lambda(I) = \mathscr{C}(I), \Lambda$ positive $\} \Leftrightarrow \sum\limits_{\lambda \varepsilon \Lambda} \frac{1}{\lambda} = \infty$ and also $\{\mathscr{C}_\Lambda(I) = \mathscr{C}(I), \Lambda$ negative $\} \Leftrightarrow \sum\limits_{\lambda \in \Lambda} \frac{1}{\lambda} = \infty$.

We shall complete the last result in Lecture 16 §1.

Lecture 12

LEVINSON'S THEOREM Problem of Continuation

1 Levinson's theorem on entire functions of exponential type and its application to the problem of closure

Levinson's theorem *Let $\Phi(w)$ be an entire function of exponential type, having $(-ik, ik)$ as its conjugate diagram and satisfying the following condition* 69

$$\int_1^\infty \log|\Phi(u)\Phi(-u)|\frac{du}{u^2} < \infty \tag{a}$$

Let $r_k e^{i\theta_k}$ be the zeros of $\Phi(w)$ with $N_r(r)$ the distribution of zeros of $\Phi(w)$ in the right half - plane and $N_\ell(r)$ the distribution of zeros of $\Phi(w)$ in the left half-plane. Under these conditions, the following relations hold:

$$\sum\{|\sin\theta_k|/r_k\} < \infty \tag{1}$$

$$\lim_{r\to\infty}\frac{N_r(r)}{r} = D = \lim_{r\to\infty}\frac{N_\ell(r)}{r}, D = k/\pi \tag{2}$$

The first part is proved by applying Carleman's formula, but the second part is more involved. (Levinson, Chap. III).

Let us consider the problem of closure of $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ in an interval I if length $|I|$. We want to find conditions on $|I|$ and in order that $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$

be total in $\mathscr{C}(I)$; in other words, we want to find conditions on $|I|$ and Λ in order that the following relation holds; $\{\Phi(w)$ is an entire function of exponential type $\leq \frac{|I|}{2}$ with $\Phi(u)$ bounded and $\Phi(\Lambda) = 0\} \Rightarrow \Phi \equiv 0$. We have already studied this problem by means of Jensen's and Carleman's formulae. In order to find new conditions on Λ we first define the *maximum density* of Polya.

70 **Definition.** Given a sequence Λ, consider sequences Λ' having density D' and $\Lambda' \supset \Lambda$. If the set of Λ' is empty we define the maximum density, D_{Max}, of Λ to be ∞. Otherwise, $D_{Max} = \inf\limits_{\Lambda' \supset \Lambda}$ (density D' of Λ')

Now the zeros of Φ form a sequence $\Lambda' \supset \Lambda$. Denoting by $\Lambda^+ and \Lambda^-$ the set of $\lambda \in \Lambda$ in the right half plane and left half plane respectively, we have by Levinson's theorem that $\Lambda'^+ and \Lambda'^-$ have the same density $D, D \geq D_{Max}$ of Λ^+ and $D \geq D_{Max}$ of Λ^-. Moreover, $D = \frac{k}{\pi}, k$ being the type of Φ. Thus we have the following closure theorem:

Theorem (Levison).

$$\{|I| < 2\pi D_{Max} \text{ of } \Lambda^+\} \text{ or } \{|I| < 2\pi D_{Max} \text{ of } \Lambda^-\} \Rightarrow \mathscr{C}_\lambda(I) = \mathscr{C}(I).$$

We have a similar theorem for the spaces $\mathscr{D}(I) or \mathscr{E}$. Theorems of this type apply to sequences which do not have a density. For example, take the sequence formed of

$$\{10^N, 10^N + 1, 10^N + 2, \ldots, 10^N + 10^{N-1}\} \text{ for } N = 10^{10^n}, n = 1, 2, \ldots$$

It has $D_{Max} = 1$ but has no density and even the upper density is very small $(D = \frac{1}{11})$. One can easily calculate the maximum density of Polya by the following formula:

$$D_{Max} \text{ of } \Lambda = \limsup_{k \nearrow 1} \left[\limsup_{r \to \infty} \left\{ \frac{n(kr) - n(r)}{kr - r} \right\} \right]$$

We shall give a proof of this result in appendix I.

The above theorem of closure gives immediately that the mean period of a mean periodic function $f, L \geq 2\pi D_{Max}$ of Λ^+ and $L \geq 2\pi$ D_{Max} of Λ^- where Λ is the spectrum of f.

71 The theorem about the supports of the convolution of two distributions can be deduced from Levinson's theorem as follows:

If T is a distribution with segment of support I, the density of zeros of its Fourier transform $\mathscr{C}(I)$ (either in Re $z > 0$ or Re $z < 0$) is $|I|/2\pi$. It is sufficient to show this when $I = [-k, k]$: then $\mathscr{C}(T)$ satisfies the hypothesis of Levinson's theorem.

Theorem on supports. $T = T_1 * T_2 \Rightarrow I = I_1 + I_2$.

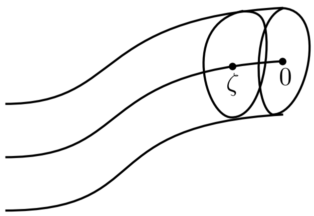
Since we know that $I \subset I_1 + I_2$, it is sufficient to show that $|I| = |I_1| + |I_2|$. This results from the fact that the density of zeros of $\mathscr{C}(T) = \mathscr{C}(T_1)\mathscr{C}(T_2)$ is the sum of the density of zeros of $\mathscr{C}(T_1)$ and that of $\mathscr{C}(T_2)$.

We shall see another application of Levinson's theorem, to a problem of quasi - analyticity, in lecture 19.

2 Problem of continuation - Description of the problem

Consider a function $f \in \mathscr{H}_\Lambda(\Omega)$ (or $\mathscr{C}_\Lambda(I), \mathscr{D}'(I)$ etc. We suppose $\mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$ ($\mathscr{C}_\Lambda(I) \neq \mathscr{C}(I) etc$). Then the natural question is to ask whether one can continue it beyond $\Omega(orI)$. More precisely, we have to give conditions on Λ and Ω so that every $f \in \mathscr{H}_\Lambda(\Omega)$ (or $\mathscr{C}_\Lambda(I)$ etc.) is analytically continuable into a domain $G \supset \Omega$(or into R); then we have to give properties of f in $G(orR)$ in terms of its properties in $\Omega(orI)$ (See foot note p.71@)

First, let Ω_0 be an open set, Ω_ζ the translate of Ω by the translation carrying 0 into ζ; suppose $f \in \mathscr{H}_\Lambda(\Omega_o) \neq \mathscr{H}(\Omega_o)$, and f is analytically continuable into a domain generated by a chain of translates of Ω_0. i.e., $f \in \mathscr{H}(\Omega_\zeta)$, for every ζ belonging to a curve C with origin in O.

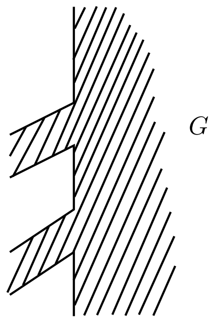


We shall show that $f \in \mathscr{H}_\Lambda(\Omega)$. For this, it is sufficient to prove that 72 if $\int_\Omega e^{\lambda z} d\mu(z) = 0$ for every $\lambda \in \Lambda$ and if we set

$$g(\zeta) = \int\limits_{K\subset\Omega} f(z-\zeta) d\mu(z), \zeta \in C,$$

then $g(\zeta) \equiv 0$. But $g(\zeta)$ is analytic in a neighbourhood of C and zero in a neighbourhood of 0. So $g(\zeta) \equiv 0$.

Suppose now Ω is the right half plane. $\Omega = \{u \geq 0\}$. Let $f \in \mathscr{H}_\Lambda(\Omega) \neq \mathscr{H}(\Omega)$. For example, f can be a Dirichlet series $f = \sum a(\lambda)e^{\lambda z}(\lambda$ negative). In any case $f \sim \sum a(\lambda)e^{\lambda z}$. Let G be the domain formed by the right half-plane and parallel strips projecting into the left half- plane. (see figure).



Suppose $f \in \mathscr{H}(G)$, i.e., f is continuable in G. Every bounded subset of G can be translated in G until it is in Ω. Then, if $d\mu$ is a measure with compact support in G, orthogonal to $\{e^{\lambda z}\}_{\lambda\in\Lambda}$, its support is in Ω_z which can be related to an $\Omega_o \subset \Omega$ by a chain of translates $\subset G$. Under these conditions, we have just seen that $f \in \mathscr{H}_\Lambda(\Omega_o)$ and $f \in \mathscr{H}(G) \Rightarrow f \in \mathscr{H}_\Lambda(\Omega_\zeta)$, i.e., f orthogonal to $d\mu$. Then, $f \in \mathscr{H}_\Lambda(G)$. This proves the existence of a sequence $\sum\limits_o^N a_N(\lambda)e^{\lambda z} \to f$ in $\mathscr{H}(G)$. In the case of Dirichlet series, when Ω is the half-plane of convergence, this is called ultra convergence. By means of a conformal mapping, we get a result about ultra convergence of Taylor's series; corresponding to the case when the strips in G are horizontal, we obtain a classical result about ultra convergence in a star domain (that is usually obtained by the method of Mittag Leffler, which gives a summation process).

73 We now give an analogous result on the line.

Definition. A class $K\{M_n\}$ of C^∞ functions on the line is defined to be the class of f satisfying the condition that $f \in K\{M_n\}$ if and only if for every $n, |f^{(n)}(x)| < kM_n$ on a closed interval J, k being constant $k(J)$ depending on J and f;

2. $K\{M_n\}$ is said to be quasi - analytic if the only function of the class all of whose derivatives vanish at the origin is the zero function (see lecture 19, §1).

Suppose further f can be continued on the line in such a manner that $f \in K\{M_n\}$. Then f is mean periodic with spectrum Λ. For let $d\mu$ be a measure with $\int e^{i\lambda x} d\mu(x) = 0$ for every $\lambda \in \Lambda$. Let $g(\xi) = \int_I f(x+\xi) d\mu(x)$, where I is the support of $d\mu$. Now $f \in K\{M_n\} \Rightarrow g \in K\{M_n\}$. But $g^{(n)}(0) = 0$ for all n. So $g = 0$. This means that f and all its translates are orthogonal to $d\mu$, i.e., $\tau(f) \neq \mathscr{C}$. This gives that if $f \in \mathscr{C}_\lambda(I)$ and $f \in \mathscr{C}(R)$, in order that $f \in \mathscr{C}_\Lambda(R)$ it is sufficient that $f \in K\{M_n\}$ quasi - analytic. This kind of result was first given by S. Mandelbrojt (Mandelbrojt 1).

@ These problems are considered in the lectures 13, 15 and 16 for the complex plane, and in the lectures 17 and 18 for the line. We give now some very easy results.

Lecture 13

A method of continuation

1 Principle of continuation

Give $f \in \mathscr{H}_\Lambda(\Omega)$, first we shall find out a method of continuation of f to a point Z. Suppose that it is possible to find a measure $d\mu = d\mu_Z$ with support in Ω such that $d\mu - \delta_z$ is orthogonal to $e^{\lambda z}$ for every $\lambda \in \Lambda$, where δ_z is the Dirac measure at Z. In other words, the following relation is given **74**

$$e^{\lambda z} = \int^{e^{\lambda z}} d\mu(z). \tag{1}$$

We can try to replace $e^{\lambda z}$ by $f(z)$ in (1), i.e. we can try to get

$$f(Z) = \int f(z) d\mu(z). \tag{2}$$

This is certainly possible when $Z \in \Lambda$, since $f \in \mathscr{H}_\Lambda(\Omega)$. When $Z \notin \Omega$ this gives a means of defining f at Z. More precisely, if f can be approximated by $\sum a(\lambda)e^{\lambda z}$ in Ω, then formally we have the following relations

$$f(Z) - \sum a(\lambda)e^{\lambda Z} = \int_K [f(z) - \sum a(\lambda)e^{\lambda z}] d\mu_Z(z);$$

$$|f(z) - \sum a(\lambda)e^{\lambda z}| < \varepsilon \int |d\mu_z|$$

Hence the majorization does not depend on Z but on $d\mu_z$. If Z belongs to a continuum $\mathscr{C}$ and if it is possible to find a measure $d\mu_z$ with support in $K \subset \Omega, K$ independent of Z, such that $\int |d\mu_z| < B$ uniformly in Z, then f can be approximated on $\mathscr{C}$ by the same sums $\sum a(\lambda)e^{\lambda z}$ as on K and $\sup\limits_{Z\in\mathscr{C}} |f(Z)| \leq \text{Const} \max\limits_{z\in K} |f(z)|$. To find the measure $d\mu$ is exactly the same thing as to find its Fourier transform. In other words, we seek an entire function $M(w) = M_Z(w)$ of exponential type with conjugate diagram in Ω such that $-M(w) + e^{wZ}$ vanishes for $w \in \Lambda$

75 Moreover, we want $M(w)$ to be the type $M(w) = \dfrac{1}{2\pi i} \int_C \varphi_Z^{(z)} e^{wz} dZ$, C being a curve in Ω fixed for all Z, and $\varphi_Z(z)$ uniformly bounded on C when Z belongs to a given continuum. According to formulae (1) and (2) in lecture 9, this last condition is satisfied whenever $M_Z(w)$ admits a uniform majorization

$$|M_Z(w)| < \frac{K}{1+r^2} e^{rk(\theta)} (w = re^{i\theta}) \tag{3}$$

and C is the frontier for the convex set defined by

$$X \cos\theta - y \sin\theta \leq k(\theta) \tag{4}$$

Then, (2) can be written as

$$f(Z) = \frac{1}{2\pi i} \int_C \varphi_Z(z) f(z) dz \tag{5}$$

Let us remark that (3) need not be required for every θ; if C can be defined from (4) with $\theta \in S, S$ being a given subset of $[0, 2n]$, it is sufficient to have (3) when $\theta \varepsilon S$, and to assume that $M_Z(w)$ is of exponential type; in particular, if C is a convex polygon, S can be taken as a discrete set; in this case, we shall say *"S is associated with C"*.

Thus the principle of continuation can be formulated as follows: Let $D(w)$ be an entire function of exponential type, vanishing Λ, with its conjugate diagram J contained in Ω. Let C be a convex polygon contained in Ω, and containing J in its interior, and let S be a set associated with C. Let $Z \in G$, and suppose that, for each Z, there exists a meromorphic function $A(w) = A_Z(w)$, with polar part $\sum\limits_{\lambda\in\Lambda} \dfrac{e^{\lambda Z}}{D'(\lambda)(w-\lambda)}$ uniformly bounded when $\theta \in S$, and also when $|w| = R_j \lim R_j = \infty$. Then C and $M_Z(w) = D(w)A_Z(w)$ satisfy the condition above. If we put

$$\varphi_z(w) = \int_0^{\infty^{i\alpha}} D(w)A_Z(w)e^{-wz}dw \tag{6}$$

and if $f \in \mathscr{H}_\Lambda(\Omega)$, (5) defines a continuation of f in G, in such a manner that 1^0) on G f is a uniform limit of the same Dirichlet polynomials $\sum a(\lambda)e^{\lambda z}(\lambda \in \Lambda)$ as on C (in particular, if G is an open set, f is analytic on G, if G is an open set and $G \cap \Omega \neq \phi$, (5) provides an analytic continuation of f from Ω into G). 76

$$2^0) \text{ on } G, |f(Z)| < K \sup_{Z, \in C} |f(z)|, K \text{ independent of } Z. \tag{7}$$

Let us remark that, if Λ is the sequence of the negative integers, our principle of continuation gives the same result as the Cauchy formula, translated after a change of variable $\zeta = e^{-z}$. Thus, (5) is a kind of generalization of the Cauchy formula.

2 Application of the principle of continuation

We have a solution of the problem of continuation of we can construct the function $D(w)$ and $A(w)$ satisfying the above principle. We shall apply this principle directly in simple cases and with a little modification in other cases, for example for $\mathscr{C}_\Lambda(K), K \subset R$; then we shall obtain, for certain real sequence Λ, the analytic continuation of $f \in \mathscr{C}_\Lambda(K)$; this forms the main result in the thesis of L. Schwartz (Schwartz 1 (see lecture 16)). In certain cases, we shall take $A(w)$ having an integral representation of the form $A(w) = \lim\limits_{j\to\infty} \int_{C_j} \dfrac{e^{w'Z}dw'}{D(w')(w-w')}$, C_j being certain closed curves (see lecture 15).

We first apply this principle in the simplest way. We suppose Λ is a negative sequence,viz. every element of Λ is a negative number $\Lambda = \{\lambda\}_{\lambda<0}$. The required $D(w)$ is provided by the canonical product $D(w) = C(w) = \prod\limits_{\lambda\in\Lambda}(1 - \frac{w^2}{\lambda^2})$. This begin achieved, to construct $A(w)$,

we make some more assumptions on Λ in such a way that the polar
77 part of $A(w)$ is normally convergent outside the union of small circles $U_\lambda = \{|w - \lambda| < \rho\}$, $A(w)$ being uniformly majorized outside these small circles.

I. Let Λ possess a density D.

We have $D(w) = C(w) = \prod(1 - \frac{w^2}{\lambda^2})$ and the conjugate diagram of $C(w)$ is the segment joining $i\pi D$ and $-i\pi D$. We take Ω to be a small domain containing this segment so that f is analytic in Ω. Suppose $A(w)$ is of the form

$$A(w) = \sum \frac{e^{\lambda Z}}{C'(\lambda)(w - \lambda)} \tag{1}$$

In order that this series is normally convergent when $Z = X + iY$ varies outside $U_\lambda = \{\left|w - \lambda\right| < \zeta\}$, it is sufficient to have $\exp(\lambda X - \log|C'(\lambda)|) < e^{\lambda\varepsilon}$ or again the following relation

$$X > \delta + \varepsilon, \varepsilon > 0, \text{ where } \delta = \limsup \frac{\log|C'(\lambda)|}{\lambda}.$$

Definition. When Λ possesses a density $D, \delta = \limsup \frac{\log|C'(\lambda)|}{\lambda}$ is defined as the *" index of condensation"* of Λ.

A sequence Λ is called "regular" if $\lim_{\lambda' \neq \lambda} \sup |\lambda' - \lambda| > 0$. If Λ is a regular sequence it can be proved (V. Bernstein, note II) that $\delta = 0$. When $\delta = 0$. we have the

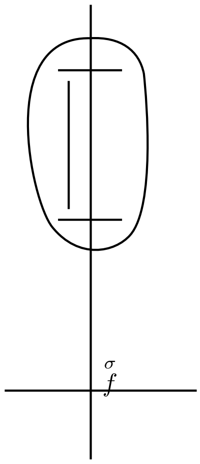
Theorem. *Let Λ be a negative sequence with density D and let $f \in \mathscr{H}_\Lambda(\Omega)$, $\Omega \supset [-i\pi D, i\pi D]$. When $\delta = 0$, every function $f \in \mathscr{H}_\Lambda(\Omega)$ can be continued analytically in the right half plane $x \geq 0$ into a sum of convergent Dirichlet's series $\sum a(\lambda)e^{\lambda z}$*

As a consequence of this theorem we obtain a classical result on Dirichlet's series. Let $f(z) = \sum a(\lambda)e^{\lambda z}$ be a Dirichlet's series where Λ is a sequence of positive number having ∞ as the sole limit point. If $f(z)$
78 is convergent for $ReZ = x_o$, it is also convergent for $ReZ = x > x_o$. Thus one can define the abscissa of convergence, σ_a, as an ordinate through $x = \sigma_a$ to the right of which $\sum a(\lambda)e^{\lambda z}$ is convergent and to the left of which $\sum a(\lambda)e^{\lambda z}$ is not convergent

We have the following corollary:

Corollary. *The Dirichlet's series* $\sum a(\lambda)e^{\lambda z}$ *with exponents extracted from a sequence having a density D and index of condensation zero, admits at least one singularity on every segment of length* $2\pi D$ *on its abscissa of convergence. If* $D = 0$ *and if* Λ *is a sequence of integers, the change of variables* $\zeta = e^{-z}$ *gives the Fabry theorem, viz., the circle of convergence is the natural boundary of a gap series.*

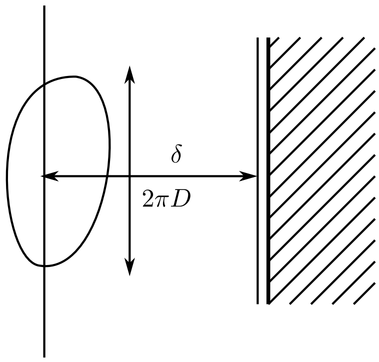
Proof of the Corollary. Indeed if there is no singularity, Ω can be taken very narrow containing this segment and the Dirichlet's series is ultra-convergent in Ω (lecture 12, 2). Taking a segment of length $2\pi D$ in Ω parallel to this segment and to the left of the abscissa of convergence σ_f we see by the above theorem that the Dirichlet's series is convergent to the left of σ_f, which is impossible.



Let σ_k ("abscissa of holomorphy") be the infimum of the σ such that $f(z) = \sum a(\lambda)e^{\lambda z}$ is analytically continuable for $\mathrm{Re} z > \sigma$. When $\delta \neq 0$ and if σ_h and σ_c are the abscissa of holomorphy and convergence of Dirichlet's series $\sum a(\lambda)e^{\lambda z}$, applying as above the result about ultra-convergence and the condition that it converges for $x > \sigma_h + \delta + \varepsilon$ we obtain the following result of V. Bernstein.

Theorem. $\sigma_c - \sigma_h \leq \delta$*, and every segment of length* $2\pi D$ *on* $Re\, z = \sigma_h$ 79
contains at least one singularity.

Suppose now $f(z)$ is an entire function, $f(z) = \sum_{\lambda \in \Lambda} a(\lambda)e^{-\lambda z}$. The inequality (7) allows us to compare the order of magnitude of f in the whole plane and in the strip, when $x = Rez \to -\infty$. Let $\mathscr{Y}$ be a horizontal strip of width $2\pi D + \varepsilon$, $M(x) = \sup\limits_{Rez = x} |f(z)|$

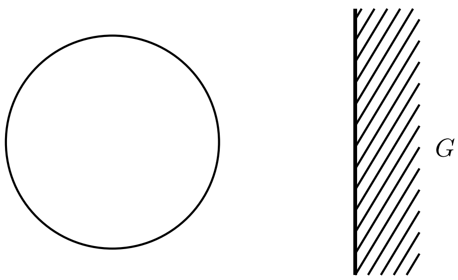


$M_{\mathscr{Y}}(x) \sup\limits_{Rez=x, z\in\mathscr{Y}} |f(z)|$. We have immediately

Theorem. *$M(x) \leq KM_{\mathscr{Y}}(x-\delta-\varepsilon'), K$ depending only on Λ, ε and $\varepsilon'(\varepsilon > 0, \varepsilon' > 0$ width of $\mathscr{Y} = 2\pi D + \varepsilon)$*

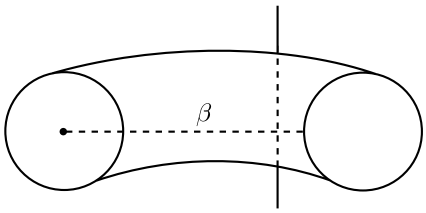
We obviously get the same result on supposing $\mathscr{Y}$ to be a curvilinear strip.

II. Let the sequence Λ possess a finite mean upper density $\overline{D^{\cdot}}$. If Λ is regular and $\left|\lambda' - \lambda\right| > h > 0$, it has been proved by S. Mandelbrojt that $\limsup \frac{\log|C'(\lambda)|}{\lambda} = \delta < B$, where $B = B(\overline{D^{\cdot}}, h)$ is a constant depending on $\overline{D^{\cdot}}$. and h (Mandelbrojt 2, 3).



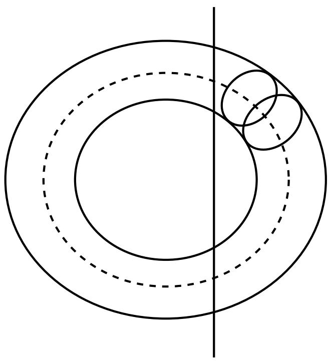
We know that the conjugate diagram of $C(w)$ is contained in a circle of radius $\pi\overline{D^{\cdot}}$. We suppose Ω to be a circular with radius $\pi\overline{D^{\cdot}} + \varepsilon$ around 0 and let $f \in \mathscr{H}_\Lambda(\Omega)$. We take $A(w)$ given by (1) as in the part. $A(w)$ is
80 normally convergent when $X > \delta + \varepsilon$. Thus we have a continuation of

f into the half plane G to the right of Ω at a distance $B(\bar{D}^{\cdot}, h)$ from its center, and in G, f is represented by a convergent Dirichlet's series. The result proved in the last lecture about the continuation by means of translates of Ω permits us to find again the following result of S. Mandelbrojt (Mandelbrojt 2, 3) which generalizes a theorem of A. Ostrowski's.



Theorem. *If σ_c is the abscissa of convergence of f as a Dirichlet's series, then f cannot be analytically continued into the domain generated by circular discs Ω_ζ of radius $\pi\bar{D}^{\cdot} + \varepsilon, \varepsilon > 0$ whose centers ζ belong to a continuum $\mathscr{C}$ which has at least one point ζ with $Re\zeta > \sigma$ and at least one point $\zeta' \in \mathscr{C}$ with $Re\zeta' < \sigma_c - B(\bar{D}^{\cdot}, h)$.*

Now, if we continue f with ζ running along a closed path, we come back to the original function (see fig.), according to theorem 2, lecture 9, §2.



It is natural to ask: Is it possible to have such a figure with singularities inside? In other words, can we have continuation of $f(Z)$ along a

chain of translates Ω_ζ whose union forms an annular region with singu-
81 larities of $f(Z)$ in the interior of the bounding curves? We shall consider this question in a later lecture. (Lect. 16)

By the same argument as in I, we get a result about the order of magnitude of f in the plane the whole plane and in a strip. Here $\mathscr{Y}$ is a horizontal strip, of width $2\pi\overline{D}^{\cdot} + \varepsilon$.

Theorem. *$M(x) \geq KM_{\mathscr{Y}}(X - \underline{B} - \varepsilon')$, K depending only on Λ, ε and $\varepsilon (\varepsilon > 0, \varepsilon' > 0)$ width $\mathscr{Y} = 2\pi\overline{D}^{\cdot} + \varepsilon)$*

Analogous result hold for curvilinear strips.

Lecture 14

Lemmas concerning minimum modulus of canonical products

We need certain lemmas to find minorization of $\left|C(w)\right|$ **82**

Lemma 1. *Suppose* $\bar{D}^{\cdot} < \infty$. *Let* $C(w) = \prod\left(1 - \dfrac{w^2}{\lambda^2}\right)$. *Given* $\varepsilon > 0$, *there exists an infinity of* $R_j \nearrow \infty$ *such that for* $|w| = R_j$ *we have* $\left|C(w)\right| > e^{-\varepsilon R_j}$.

Proof. We have $\left|C(w)\right| \geq \left|1 - \dfrac{r^2}{|\lambda^2|}\right|$, where $|w| = r$. We apply Carleman's formula to the function $C^*(w) = \prod\left(1 - \dfrac{w^2}{|\lambda^2|}\right)$, in the upper half plane $v \geq 0$. Then we have the following relation:

$$\int_{-R}^{R} \log\left|C^*(u)\right| \frac{du}{u^2} = o \tag{1}$$

This relation implies that there exists an infinity of $R_j \nearrow \infty$ such that $\log\left|C^*(R_j)\right| > -\varepsilon R_j$; *i.e* for $|w| = R_j |C(w)| \geq |C^*(R_j)| > e^{-\varepsilon R_j}$. ◻

Lemma 2 (H. Cartan)**.** *Suppose* $\delta > 0$ *and* $M_1, \ldots, M_n$ *are* n *given points in the complex plane. Then we can find* m *discs,* $m \leq n$ *the sum of* *whose radii is* $2n\delta$ *in such a manner that if* M *is any point outside these discs the product of the distances* $MM_1, \ldots, MM_n > \left(\frac{n\delta}{e}\right)^n$.

Proof. Let k_1 be the largest integer such that there exists a disc of radius $k_1\delta$ containing at least k_1 of the points M_j, and let C_1 be such a disc. Obviously C_1 contains exactly k_1 points M_j. Let us remove these k_1 points, and define in the same manner from the remaining points (if there exist any) a disc C_2 of radius k_2^δ, and so on: we get a finite number
83 of discs, say $C_1, \ldots, C_m$, of radii $k_1\delta, \ldots, k_m\delta$, and $k_1 + \cdots + k_m = n$. Now, let $\Gamma_1, \ldots, \Gamma_m$ be discs concentric with $C_1 \ldots C_m$ and of twice the radii. From the construction of $C_1, \ldots, C_m$, it follows that, if a disc $\int(M, k\delta)$ of center M and radius $k\delta$ contains at least k points M_j, then it contains a point $M_q \in C_q$ with $k_q \geq k$; thus $M \in \Gamma_q$. □

Suppose now $M \notin \cup\Gamma_j$; then, ever $\int(M, k\delta)$ contains at most $k-1$ points M_j. Let the distances MM_j be that $MM_1 \leq MM_2 \cdots MM_n$. Then $MM_1 > \delta, MM_2 > 2\delta, \ldots, MM_n > n\delta$, and

$$MM_1 \cdot MM_2 \cdots MM_n > n!\delta^n > \left(\frac{n\delta}{e}\right)^n$$

that completes the proof.

Definitions. Given a finite set of points $\{M_j\}$ and $\delta > 0$, we call $\{\Gamma_j\}$ a system of "Cartan discs relative to $\{M_j\}$ and δ".

Let Λ be a sequence of points in the complex plane without finite points of accumulation, and $\delta > 0$. By a system of "Cartan discs relative to Λ and δ", we shall mean the union of the system of Cartan discs relative to Λ and the annulus $2^n - 1 \leq |z| < 2^{n+1} - 1 (n = 0, 1, \ldots,)$

Lemma 3. *Suppose the symmetrical sequence* Λ *has a density* D*. Given* $\varepsilon > 0$ *and* $\delta > 0$ *we have* $|C(w)| = \prod |1 - w^2/\lambda^2| > e^{-\varepsilon}|w|$ *for* $|w|$ *sufficiently large, and* w *varying outside Cartan discs of a system relative to* Λ *and* δ.

Proof. Write $C(w) = \prod_1 \prod_2$ where $\prod_1 = \prod_{r/\gamma < |\lambda| < \gamma r} (1 - w^2/\lambda^2), \gamma > 1$ to be chosen later to be near 1. Denoting by $n(r)$ the distribution function of $\{|\lambda|\}$, we have the following relations:

$$\log|\prod_2| > \left\{\int_o^{r/\gamma} + \int_{r\gamma}^{\infty}\right\}\left[\log\left|1 - \frac{r^2}{\lambda^2}\right| dn(\lambda)\right] (r = |w|)$$

$$= n\left(\frac{r}{\gamma}\right)\log(\gamma^2 - 1) - n(\gamma r)\log\left(1 - \frac{1}{\gamma 2}\right)$$

$$+ r\int_0^{1/\gamma} \frac{2}{1 - u^2} D(ru)du - \int_\gamma^\infty \frac{2}{u^2 - 1} D(ru)du$$

$$\int_o^{1/\gamma} \frac{2}{u^2 - 1} D(ru)du < \log\frac{\gamma + 1}{\gamma - 1}(D + 0(1)) \quad (r \to \infty)$$

and $$\int_\gamma^\infty \frac{2}{u^2 - 1} D(ru)du < \log\frac{\gamma + 1}{\gamma - 1}(D + 0(1)) \quad (r \to \infty)$$

$$n\left(\frac{r}{\gamma}\right)\log(\gamma^2 - 1) - n(\gamma r)\log\left(1 - \frac{1}{\gamma^2}\right) > 2\log\gamma . n(\gamma r) > 0$$

if $\gamma^2 < 2$. Then $\log\left|\prod_2\right| > -\varepsilon/2$ if $\gamma^2 < 2$, and r large enough. Now **84** $\prod_1$ is a product of N terms, $N < 2D_r(\gamma - 1/\gamma)$, if r is large enough. Let $\prod_{1,n}$ be the product of those terms, whose zeros are in use in the annulus $2^n - 1 \leq |z| \leq 2^{n+1} - 1$; $\prod_1$ can be written as $\prod_{1,n} . \prod_{1,n+1}$ for a convenient n. Take w outside any Cartan disc and suppose (if necessary, by changing the sign of λ) $|w - \lambda| \leq |w + \lambda|$; then

$$\left|\prod_{1,n}\right| > \prod_{1,n}\left|1 - \frac{w}{\lambda}\right| > \frac{1}{(\gamma r)^M}\prod_{1,n}\left|w - \lambda\right| > \frac{1}{(\gamma r)^M}\left(\frac{M\delta}{e}\right) M$$

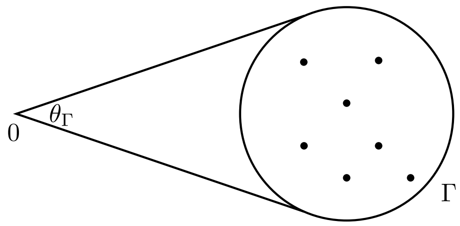
where M is the number of terms of $\prod_{1,n}$; $M \leq N < 2Dr(\gamma - \dfrac{1}{\gamma})$

$$\log\left|\prod_{1,n}\right| > M\log\frac{m}{\gamma er} = r\frac{M}{r}\left(\frac{M}{r}.\frac{\delta}{\gamma^e}\right) > -\frac{\varepsilon}{4}r,$$

if γ is chosen near enough to 1. That completes the proof. □

Lemma 4. *Suppose* $\sum \dfrac{1}{\lambda} < \infty$ *and* $C(w) = \prod(1 - \dfrac{w^2}{\lambda^2})$. *For almost all* θ, *we have* $\lim\limits_{r\to\infty} \log \dfrac{|C(re^{i\theta})|}{r} = 0$.

Proof. We choose the set of " Cartan discs", Γ, as in the previous lemma. Let θ_Γ be the angle which a circle Γ subtends at origin. Since the radius of the circle Γ are of the from $2n\delta$ with precisely n points λ in side Γ, we have $\theta_\Gamma \sim \sum\limits_{\lambda\in\Gamma} \dfrac{1}{|\lambda|}$ and $\sum |\theta_\Gamma| < \infty$ □



85 If we consider only those circles far away from the origin, we have $\sum |\theta_\Gamma| < \varepsilon$. Then outside of these angles θ, by the previous lemma, we have $|C(w)| > e^{-\varepsilon}|w|$. Thus we have $\lim \dfrac{\log |C(w)|}{r}$=0 almost everywhere.

One can prove, in the same manner, the following lemma.

Lemma 5. *If* $D = 0$, *then* $\lim \dfrac{\log |C(w)|}{r} = 0$, *when* $w \to \infty$, *w vary ing outside of Cartan discs associated with* Λ.

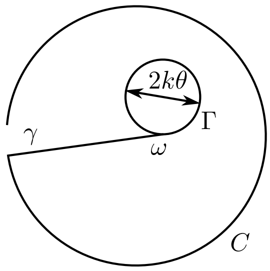
Lemma 6. *Let* $\Phi(w)$ *and* $\psi(w)$ *be two functions of exponential type. Suppose* $|\Phi(w)| < |\psi(w)| < e^{|v|}$ *outside of discs* Γ, *each of which is of radius* $2k\varepsilon(k = k(\Gamma)$*: an integer) and contains k zeros of* ψ. *Then, for sufficiently small* ε, $|\Phi(w)| < e^{|v|}$.

Proof. Let w belong to the frontier of a Γ. Applying Jensen's formula to ψ on a circle C of center w and radius r, we get

$$\log |\psi(w)| < \frac{1}{2\pi} \int_0^{2\pi} \log |\psi(w) + re^{i\theta}| d\theta - \log \frac{r^k}{(2k\varepsilon)^k}$$

$$< |v| + \frac{2}{\pi} r - r\frac{k}{r} \log \frac{r}{2k\varepsilon} < |v| - r$$

for $r = k$ and ε small. Applying Cauchy's theorem for $\Phi(w)$ on C,



$$|\Phi(w)| < e^{\max|v|-r} \text{ for } w' \in C. \text{ So } |\Phi(w)| < c^{|v|}$$

□

Lemma 7. *If Λ has a density D and if arg $\lambda \to 0$, then* $\lim \frac{\log|C(w)|}{r} = \pi D|\sin\theta|$ *for $\theta \not\equiv 0(\mod \pi)$. (Cf. Method of Carlson, lecture 10§1)*

Lemma 8. *If λ_n are real, $|C(re^{i\theta})| > 1$ if $|\theta \pm \frac{\pi}{2}| < \frac{\pi}{4}$. The proof is obvious.*

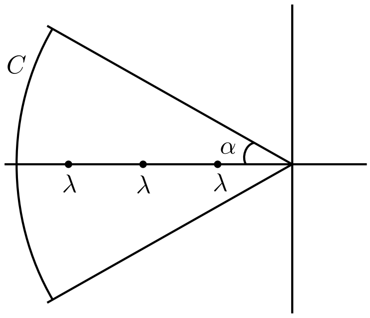
Lecture 15

Continuation Theorems

1 Theorems about negative sequences, or sequences contained in a salient angle

In Lecture 13, we have seen how the principle of continuation can be 86
applied when the sequence Λ consists of only negative numbers. We had to suppose that the canonical product $C(w) = \prod(1 - \dfrac{w^2}{\lambda^2})(\lambda \in \Lambda)$ satisfies the condition that $\dfrac{|\log C'(\lambda)|}{\lambda}$ is bounded, and that Λ possesses a density or, at least, a, mean upper density. Then, for a convenient Ω, each $f \in \mathscr{H}_\Lambda(\Omega)$ can be "continued" in a certain half-plane $x > x_o$. It is not a true continuation when Ω and the half plane have no common points. Is it possible to consider the function, defined in the half plane, as an analytic continuation of f? This question will be solved by the use of result about minimum moduls of $C(w)$, and a representation of $A_z(w)$ as a contour integral.

Suppose we fix arg Z to satisfy the condition that $|\arg Z| < \dfrac{\pi}{2} - \alpha, \alpha > 0$. Consider the contour C, formed of the lines $[r\,e^{i(\pi-\alpha)},\ r\,e^{i(\pi+\alpha)}, 0 \leq r \leq R]$ and the circular are $[Re^{i(\pi+\theta)}|\theta| \leq \alpha]$, and taken in the positive direction. Let w be outside the angle made by this contour C. By Cauchy's formula, we have the following equation:



$$\frac{1}{2\pi i}\int_C \frac{e^{w'Z}dw'}{C(w')(w-w')} = \sum_{(\lambda \text{ inside } C)} \frac{e^{\lambda Z}}{C'(\lambda)(w-\lambda)} = A_C(w)$$

87 We now apply Lemma 1 to prove that the integral over the circular arc of C tends to zero when R tends to infinity, along a convenient sequence $\{R_j\}$. We suppose that α is chosen so that $|Z|\cos(\arg Z+\theta) < -2 \in < 0$, for $|\theta-\pi| < \alpha$. Then, by lemma 1, there exists a sequence $R_j \nearrow \infty$ such that $|C(w')| > e^{-\in R_j}$ for $|w'| = R_j$. But $|e^{w'Z}| < e^{-2\in R_j}$ and hence the integral on the sector of radius R_j tends to zero when $R_j \to \infty$

Set
$$A^+(w) = \frac{1}{2\pi i}\int_0^{\infty e^{i(\pi-\alpha)}} \frac{e^{w'Z}dw'}{C(w')(w-w')};$$
$$A^+(w) = \frac{1}{2\pi i}\int_0^{\infty e^{i(\pi+\alpha)}} \frac{e^{w'Z}dw'}{C(w')(w-w')}.$$

If A^+ and A^- exist, then $\lim\limits_{R_j\to\infty} A_C(w) = A(w) = A^+(w) - A^-(w)$. When $X\cos\alpha \pm Y\sin\alpha + \frac{1}{R}\log|C(Re^{i\alpha})| > \in > 0$ for large values of R and $Z = X + iY, A^+(w)$ (resp. $A^-(w)$) defines a function analytic outside the cone defined by the contour C and uniformly bounded in Z.

When w lies inside the cone defined by C, i,e., when $|\arg w - \pi| < \alpha$ the same formula for $A(w)$ will hold if we adjoin to the contour C a small cut of the path of integration and a small circle encircling w on which $\int \dfrac{e^{w'Z}dw'}{C(w')(w-w')} = -2\pi i\frac{e^{wZ}}{C(w)}$. Such factors are uniformly bounded for

$|w| = R_j, |\arg w - \pi| < \alpha$. Thus we have $A(w)$ satisfying our requirement of lecture 13; §1) it is uniformly bounded in Z when w varies outside the cone generated by C or when $|w| = R_j, |\arg w - \pi| < \alpha$; 2) it has polar part $\sum\limits_{\lambda} e^{\lambda z}/C'(\lambda)(w-\lambda)$, and 3) it is analytic everywhere except at these poles) wherever Z satisfies the following conditions:

a) for a sequence of values of $R_j \nearrow \infty$, and for $|\theta - \pi| \leq \alpha$ we have $|Z| \cos(\arg Z + \theta) - \frac{1}{R_j} \log |C(R_j e^{i\theta})| < - \in < 0$; **88**

b) for large values of R, we have

$$X \cos \alpha \pm Y \sin \alpha + \frac{1}{R} \log |C(Re^{i(\pi \pm \alpha)})| > \in > 0, (Z = X + iY).$$

We will be able to find the variability of Z satisfying these conditions by making some more assumptions on Λ.

1. Λ possesses a density

By the lemma of Carlson, if $\theta \not\equiv 0(\mod \pi)$, we have $\lim \frac{\log |C(Re^{i\theta})|}{R} = \pi D |\sin \theta|$. Using this and lemma 3, we are led to consider only condition a) to determine the variability of Z. Moreover, by the continuation developed in Lecture 13 for negative sequence we have the following theorem:

Theorem. *Suppose Λ is a sequence of negative numbers having a density D and Ω an open set containing the segment $(-i\pi D, i\pi D)$. Then it is possible to continue every function $f \in \mathscr{H}_\Lambda(\Omega)$ to be analytic in the right half-plane $x \geq 0$. Moreover for $|\arg Z| < \beta < \dfrac{\pi}{2}$ the continuation yields us a function which is bounded and uniformly approximated by linear combination of $e^{\lambda z}, \lambda \in \Lambda$. If Λ has a finite index of condensation δ, the continuation yields us a Dirichlet's series convergent for $x \geq \delta$*

The same statement holds for a sequence Λ without density, if we replace D by D_{max}. For there exists $\Lambda' \supset \Lambda, \Lambda'$ having a density, and density $\Lambda' = D_{max} \Lambda$ (see Appendix 1)

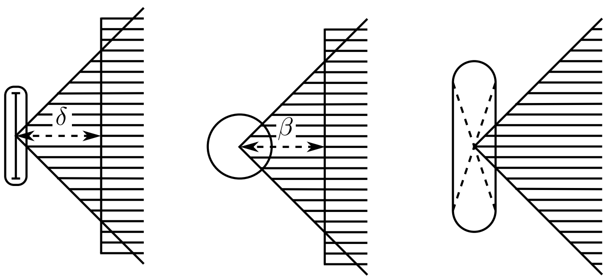
Corollary. *The sum of a Dirichlet's series whose sequence of exponents possesses a maximum density D_{max}, admits at least one singularity on* *every segment of length larger then $2\pi D_{max}$ on its abscissa of holomorphy.*

2. Λ has a finite mean upper density $\bar{D}$.

89 In this case, by lemma 8, $|C(re^{i\theta})| > 1$ for $|\theta \pm \frac{pi}{2}| < \frac{\pi}{4}$. So, we are obliged to take into consideration both the conditions a) and b) to determine the variability of Z. As before the continuation developed for this case in Lecture 13, gives us a similar theorem for continuation of $f \in \mathscr{H}_\Lambda(\Omega)$, where Ω is an open set the containing the circle $|z| \leq \pi \bar{D}$, into that portion of the right half plane contained between the lines $\theta = \pm\frac{\pi}{4}$. The index of condensation δ is replaced by the constant B.

3. Λ is contained in a salient angle

Suppose Λ is a sequence contained in a salient angular region around the negative real axis, i.e., Λ consists of points λ of the form $re^{i(\pi+\theta)}$, $|\theta| \leq \beta$. Then using $|C(w)| < (1 + \frac{|w|^2}{|\lambda|^2})$ and $|C(u)| < \prod |1 - \frac{u^2e^{2i\beta}}{|\lambda|^2}\Big|$ for locating the conjugate diagram and using $|C(w)| > \prod |1 - \frac{w^2e^{2i\beta}}{|\lambda_2|^2 i\beta}\Big|$ If $\frac{\pi}{2} < \arg w < \pi - \beta$ for having the minorization of $|C(w)|$ we can prove similar theorems taking $\alpha > \beta$

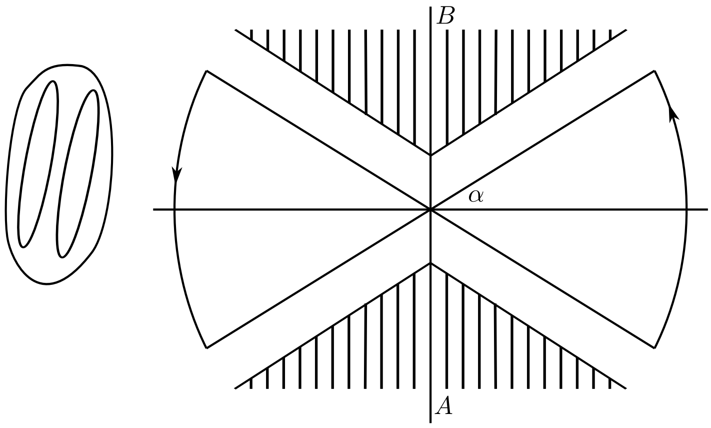


Case 1 Case 2 Case 3

2 Theorems for symmetric sequences

90 First let Λ be a real symmetric sequence having a density D. The canon-

ical product $C(w) = \prod_{\lambda\in\Lambda}(1 - \frac{w^2}{\lambda^2})$ has for its conjugate diagram the segment $(-i\pi D, i\pi D\lambda)^{\in\Lambda}$. Let Ω be a domain containing this segment. The conjugate diagrams of $C(w)e^{wc}$ and $C(w)e^{-wc}$, $c > 0$, are two segments parallel to the given segment and on either side of it.



For sufficiently small $c > 0$, they are contained in Ω. Now we take $D(w) = C(w)Ch(cw)[2Ch(cw) = e^{wc} + e^{-wc}]$ in order that we get a simultaneous majorization of $e^{w'Z}/D(w')$ $(w - w')$ on two circular arcs of the contour of integration situated symmetrically.

We take the contour C consisting of the lines $re^{i\alpha}$ and $re^{-i\alpha}$, $-R \leq r \leq R$ and circular arcs $Re^{i\theta}$ and $-Re^{i\theta}$, $-\alpha \leq \theta \leq \alpha$. Let w be a point with $|w - w'| > 1$ for any point w' in the interior of this contour. By Cauchy's theorem we have

$$A_C(w) = \frac{1}{2\pi i}\int_C \frac{e^{w'Z}dw'}{D(w')(w-w')} = \sum_{\lambda \mathrm{inside} C} e^{\lambda Z}/D'(\lambda)(w-\lambda).$$

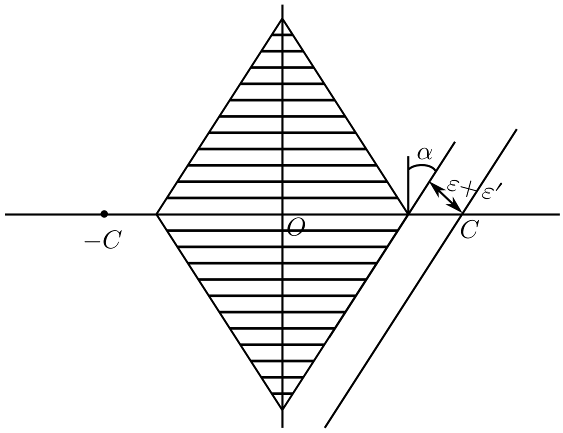
Now, by Lemma 1, we have $|C(w)| > e^{-\in R_j}$ on an infinity of circles of radii $1R_j \nearrow \infty$ and so taking into account the majorization

$$\left|e^{w'z}/D(w')\right| < e^{r(R\left|\cos(\varphi+\theta)\right|+\in-C\left|\cos\theta\right|)}$$

where $Z = Re^{i\varphi}$, the integral on the circular arcs of C vanish when $R_j \nearrow \infty$ if the following condition is satisfied:

$$\Big\{R\cos(\varphi+\theta) + \in -C\Big|\cos\theta\Big| < - \in' \ \mathrm{f}or\Big|\theta\Big| \le \alpha, \Big|\theta - \pi\Big| \le \alpha. \qquad (*)$$

91 If the condition $(*)$ is realized we can proceed along the same line of argument as on negative sequence by considering the positive and negative parts of Λ separately and we will have $A(w)$ bounded and analytic inside the angles A and B as indicated in the figure,



In order to satisfy the condition $(*)$, the region of variability of Z for a given α is a rhombus with diagonals $(-c', c')$ and $(-id, id)$ with $c' = d\tan\alpha = c - \dfrac{\in + \in'}{\cos\alpha}$. Since α is arbitrary, the continuation is possible along a band around the imaginary axis.

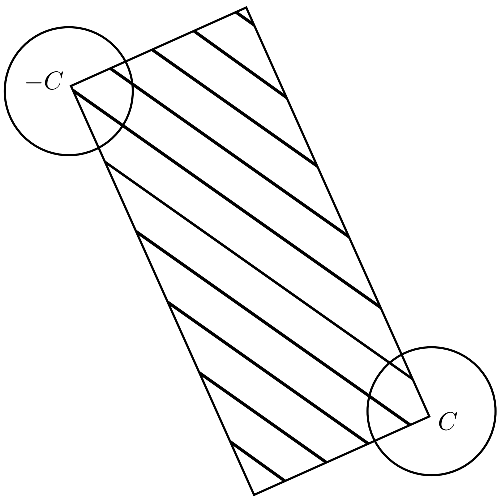
Such a problem of continuation was first considered by $A.F.$ Leontiev (Leontiev). Our method is different from that of Leontiev. The same proof holds when Λ is not necessarily real but accumulates near the real axis, viz, arg $(\pm\lambda_n) \to 0$ when $\lambda_n \to \infty$. Thus we have the following theorem.

Theorem. *Let Λ be a symmetric sequence accumulating near the real axis and possessing a density D. Let Ω be an open domain containing the segment $(-i\pi D, i\pi D)$. Then every function $f \in \mathscr{H}_\Lambda(\Omega)$ cam be analytically continued into a vertical band $\mathscr{B}$, (which may be degenerate* *into a half-plane or the whole plane) such that on each segment of length larger than $2\pi D$ on the boundary of $\mathscr{B}$ there is at least one singularity of the function.*

The last part of the theorem results from the relation $f \in \mathscr{H}_\Lambda(\Omega_\zeta)$ for **92**
every translate Ω_ζ of Ω, such that f is analytic along a chain of translates joining Ω_ζ to Ω (Lecture 12, §2). Moreover, one can prove $f \in \mathscr{H}_\Omega(\mathscr{B})$ (Kahane 1, p. 98).

Suppose now that Λ is a symmetric sequence accumulating near the real axis and having a mean upper density $\overline{D}$·. The above method can be applied, if we take $\alpha > \frac{\pi}{2}$, by using lemma 8 instead of lemma 7, Lecture 14. This leads to the following result.

Theorem. *Suppose Λ is symmetric, accumulates near the real axis, and possesses a mean upper density $\overline{D}$·t. Take as Ω an open set containing the disc $|z| \leq \pi\overline{D}$·, and let Ω_c and Ω_{-c} be its translates by c and $-c$, $\left|\arg c\right| < \frac{\pi}{4}$. Then every $f \in \mathscr{H}_\Lambda(\Omega_c \cup \Omega_{-c})$ can be continued in a rectangle whose sides make angles $\frac{\pi}{4} \pmod{\frac{\pi}{2}}$ with the real axis, and having c and $-c$ as vertices.*



Lecture 16

Further Theorems of Continuation

1 Theorem of Schwartz

We have proved in Lecture 11, §4 that if Λ is a negative sequence, $\Lambda = \{-\lambda_n\}$ and $\sum \dfrac{1}{\lambda_n} = \infty$, then $\mathscr{C}_\Lambda(I) = \mathscr{C}(I)$. Conversely if $\sum \dfrac{1}{\lambda_n} < \infty$, then $\mathscr{C}_\Lambda(I) \neq \mathscr{C}(I)$. Let us recall that the function 93

$$D(w) = \prod_1^N \left(1 + \frac{w}{\lambda_n}\right) \prod_{N+1}^\infty \frac{\sin \pi w/\lambda_n}{w/\lambda_n}$$

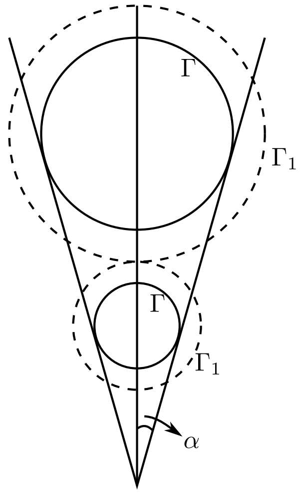
satisfies $D(\Lambda) = 0, D(u) = 0(\dfrac{1}{u^2})$ and $\left|D(w)\right| < Ke^{h|v|}$, where $h = \pi \sum\limits_{N+1}^\infty \dfrac{1}{\lambda_n}$.

Suppose $f \in \mathscr{C}_\Lambda(I), \Lambda = \{-\lambda_n\}$ and $\sum \dfrac{1}{\lambda_n} < \infty$. Then we shall prove that f can be continued to a half-plane at the right, above and below I.

We take $I = (-h, h)$ and $A_Z(w) = \sum e^{-\lambda Z}/D'(\lambda)(w - \lambda)$, where $D(\lambda) = 0$. The idea is still to define $A_Z(w)$ as an integral. To do this, we take a set of "Cartan circles" Γ constructed for a given $\in > 0$ and Λ (see Lecture 14). Let w lie on a circle Γ_1 concentric to circle Γ and of radius equal to

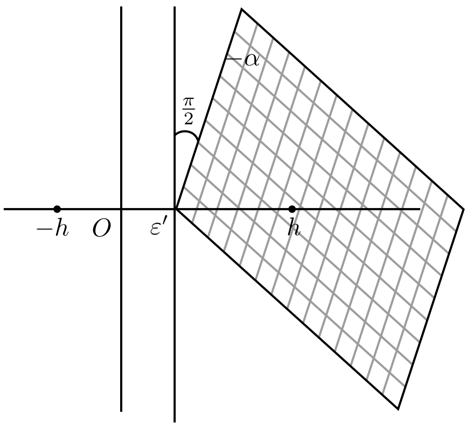
$\in$ + radius of Γ. Consider the following expression:

$$A_Z(w) = \frac{1}{2\pi i}\int \frac{e^{w'Zdw'}}{D(w')(w-w')}, w\varepsilon \text{ one } \Gamma_1$$



We have

$$\left|e^{w'Z}\right| = e^{rR\cos(\varphi+\theta)}, Z = Re^{i\varphi}, w' = re^{i\theta}.$$



When Z varies in an angle $\left|\arg(Z- \in')\right| << \dfrac{\pi}{2}(\in'> 0)$, one can choose $\in'$ $-$ and then the Γ's so that $A_Z(w)$ is uniformly bounded in Z on $\cup\Gamma_1$. Indeed if the "Cartan circles" Γ are contained in an angle $\left|\arg w - \dfrac{\pi}{2}\right| < \alpha$ and if $\left|\arg(Z- \in')\right| \leq \pi/2 - \alpha$ we have a uniform majorization. Also by construction, $\alpha \to 0$ when $\in\to 0$. Consider $w^2 M_Z(w) = A_Z(w)D(w)w^2$. Outside the circles Γ_1 we have $\left|w^2 M_Z(w)\right| < K_1\left|w^2 D(w)\right|$. If $\in$ is small enough, it follows, by lemma 6, lecture 14, that $\left|w^2 M_Z(w)\right| < K_2 e^{h|v|}$ and also $\left|M_Z(w)\right| < K_2 e^{h|v|}$ uniformly in Z. Thus $M_Z(w) = A_z(w)D(w)$ is the Fourier transform of a measure with support in $(-h, h)$. This enables us to define $f(Z) = \int_{-h}^{h} f(z)d\mu_Z(z)$. Moreover, $\int \left|d\mu_Z\right|$ is bounded when $\left|\arg(Z- \in')\right| \leq \dfrac{\pi}{2} - \alpha$. So $f(Z)$ is uniformly approximated in this angle by the same linear combinations of $e^{-\lambda_n Z}$ which approximate f uniformly on $(-h, h)$. 94

Theorem. *If Λ is an imaginary sequence $\{i\lambda n\}$ such that $\lambda_n > 0$, $\sum \dfrac{1}{\lambda_n} < \infty$ every $f \in \mathscr{C}_\Lambda(I)$ is an analytic function, continuable into each right half-plane $Rez \geq x_o$, x_o in the interior of I. In each angle, $\left|\arg(z-x_o)\right| < \beta < \dfrac{\pi}{2}(x_o$ in the interior of $I)$, $\left|f(z)\right| < K \sup_{x\in I}\left|f(x)\right|$, K depending only on the angle.*

Problem. It would be interesting to have a half-plane $Rez \leq x_o$ instead of the angle, in the last statement. This is certainly possible when Λ is regular enough. Then, for a mean-periodic function f of spectrum Λ we would have

$$\left|M(x)\right| = \sup_{Rez=x}\left|f(z)\right| = K \sup_{|\xi|<\eta}\left|f(x+\xi)\right|$$

a relation between the order of magnitude on the whole plane and on the line.

2 Interpolation Results

Suppose $f \in \mathscr{H}_\Lambda(\Omega_c \cup \Omega_{-c})$. We have seen that if Λ is real and with finite $\underline{D}^{\cdot}$, Ω_c and Ω_{-c} being circles with centres $-c$ and c and radii larger than 95

$\pi\bar{D}$, we can continue f into a rectangle with vertices at c and $-c$.Results of this type will give us some indications about the domain of existence of the continued function.

Suppose $D(w)$ is an entire function of exponential type with $D(\Lambda) = 0$ and let Ω be an open set containing the conjugate diagram of D. We take two translates Ω_c and Ω_{-c} of Ω. We try to construct a measure $d\nu = d\nu_{c,Z}(z)$ with support in $\Omega_{-c} \cup \Omega_c$ with the aid of the measure μ such that we have $e^{\lambda Z} = \int e^{\lambda z} d\nu$ whenever $\lambda \in \Lambda$. Then we can define $f(Z) = \int f(z) d\nu_{c,Z}(z)$. We take $d\nu = d\mu_Z * (\delta_c - \delta_{-c})$, where δ is the Dirac measure. $\mathscr{C}(d\nu) = M(w)(e^{-wc}) = 2M(w)Sh(cw)$. If we want the conjugate diagram of $M(w)$ in Ω we should expect $M(w)$ to have the same majorization as $D(w)$. Let $M(w) = D(w)A(w)$. We suppose first that $D(w)$ and $Sh(cw)$ have no common zeros. We construct $A(w)$ bounded in circles of radius R_j and on certain directions $\arg w = \theta \in E$. Moreover, we want $2D(\lambda)Sh(c\lambda)A(\lambda) = e^{\lambda Z}$ for each $\lambda \in \Lambda$. For that we take the following form of $A(w)$:

$$A(w) = \frac{e^{wZ}}{D(w)Sh(cw)} - \sum_{-\infty}^{\infty} \frac{e^{\mu_n Z}}{D(\mu_n)Sh'(\mu_n c)(w - \mu_n)}, \mu_n = \frac{i\pi n}{c}.$$

Let E be the set of points in $[0, 2\pi]$ where $\liminf_{r\to\infty} \frac{\log\left|D(re^{i\theta})\right|}{r} > \in > 0$.

In order to have the conjugate diagram of $M(w)$ in Ω, it is sufficient to have E dense in $[0, 2\pi]$, $\arg(i/c) \in E$ and $Z \in [-c, c]$.

96 Here we give some results without details where this method is applicable.

Theorem. *Let $C(w)$ be an entire function of exponential type with $C(\Lambda) = 0$ and the conjugate diagram of $C(w)$ contained in Ω. Let the set $E(\in)$ of points θ for which $\liminf_{r\to\infty} \frac{\log\left|C(re^{i\theta})\right|}{r} > - \in$ be dense in $(0, 2\pi)$ for every $\in > 0$. Then every function $f \in \mathscr{H}_\Lambda(\Omega_{-c} \cup \Omega_c)$ can be continued analytically along the segment $(-c, c)$.*

We take $D(w) = C(w)\sin(\in w)Sh(\in w)$ and apply the above method.

As a corollary, we get the following result: if $G = \bigcup_{\zeta\in y} \Omega_\zeta$ and if $f \in \mathscr{H}_\Lambda(G)$, f is analytically continuable in the convex closure of $\mathscr{Y}$.

A more interesting result is the following, which is more difficult to prove.

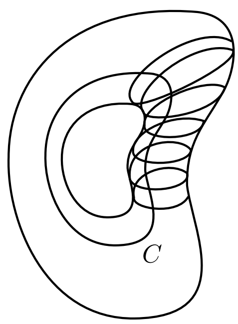
Theorem. *Suppose Λ and Ω satisfy the hypothesis of the last theorem. Let $\mathscr{Y}$ be a connected set containing 0, and $G = \bigcup_{\zeta\in\mathscr{Y}} \Omega_\zeta$. If $f \in \mathscr{H}_\Lambda(\Omega)$ and $f \in \mathscr{H}(G)$, f is analytically continuable in the convex closure of $\mathscr{Y}$.*

From this theorem one can deduce that every analytic mean-periodic function on the line is analytically continuable in a horizontal strip (perhaps degenerated into a half-plane or the whole plane), such that every segment of length L (mean period related to Λ) on the frontier of the strip, contains at least one singularity of f.

For the proofs, see (Kahane 1, p. 100-104).

3 Theorems of Leontiev

In (Leontiev) a method is given for the answer to the problem raised in **97**
lecture 13, viz. whether continuation along a chain of translates Ω_ζ of Ω, with ζ in a closed curve $\mathscr{Y}$ containing 0, of a function $f \in \mathscr{H}_\Lambda(\Omega)$, implies that f is analytically continuable in the interior of $\mathscr{Y}$. The answer is affirmative when Ω is conveniently related to Λ. We sketch the method of Leontiev.



Let $C(w) = \prod_1^\infty \left(1 - \dfrac{w^2}{\lambda_n^2}\right) = \mathscr{C}(d\alpha)$ and let

$$C_k(w) = \prod_{n=k}^\infty \left(1 - \frac{w}{\lambda_n^2}\right) = \mathscr{C}(d\alpha_k).$$

Heuristically $C_k(w) \to 1$ when $k \to \infty$ and $d\alpha_k \to$ Dirac measure.

More precisely, $\left|C_k(w)\right| < Ke^{\pi|w|(\overline{D}^{\cdot}+\in)}$ and $C_k(w) \to 1$ on every compact set. Then, if Γ is a curve around the disc $|z| < \pi(\overline{D}^{\cdot} + \in)$ the Laplace-Borel transform of $C_k(w)$, denoted by $\varphi_k(z)$, tends uniformly on Γ to $\dfrac{1}{z}$. Suppose $\Omega \supset \Gamma$, and $f \in \mathscr{H}_\Lambda(\Omega)$; then the Dirichlet polynomial $f_k(z) = \dfrac{1}{2\pi i} \int_\Gamma f(z+\zeta)\varphi_k(\zeta)d\zeta$ tends uniformly to $f(z)$ when $\left|z\right|$ is small enough.

If we suppose also $f \in \mathscr{H}(G), G = \bigcup\limits_{\zeta \in \mathscr{Y}} \Omega_\zeta$, then $f_k(z)$ is defined by the same formula in a neighbourhood of $\mathscr{Y}$, so it is bounded and tends uniformly to $f(z)$ on $\mathscr{Y}$. Hence

98 **Theorem.** *Suppose Λ is a symmetrical sequence $\{\pm\lambda_n\}$ such that $\left\{\left|\lambda_n\right|\right\}$ has a mean upper density $\overline{D}^{\cdot}$, Ω is an open set containing $|z| \leq \pi\overline{D}^{\cdot}$, $\mathscr{Y}$ is a closed curve passing through* 0, $G = \bigcup\limits_{\zeta \in \mathscr{Y}} \Omega_\zeta$, *and H is the interior of $\mathscr{Y}$. Then, every $f \in \mathscr{H}_\Lambda(\Omega)$ which is analytically continuable in G can be continued in H as a function $\in \mathscr{H}_\Lambda(H)$.*

Corollary. *If $\dfrac{n}{\lambda_n} \to 0$, the domain of existence of every $f \in \mathscr{H}_\Lambda(H)$ is simply-connected.*

One can prove more, viz. the following result of Leontiev (stated partly by G. Polya);

Theorem. *If $\dfrac{n}{\lambda_n} \to 0$, the domain of existence of every $f \in \mathscr{H}_\Lambda(\Omega)$ is convex.*

For the proof, see (Leontiev).

Lecture 17

Continuation on the line

1 Some definitions about positive sequences $\{\lambda_n\} = \Lambda$

We give a few definitions with notations. We have already defined (lecture 10) $n(r)D(r), D^{\cdot}, D_{\cdot}, \overline{D}(r), \overline{D}^{\cdot}, \overline{D}_{\cdot}$. We define the following expres- 99
sions to make list complete.

$$\hat{D}(r) = \frac{1}{\log r}\int_o^r \frac{dn(t)}{t}; \hat{D}^{\cdot} = \limsup_r \hat{D}(r); \hat{D}_{\cdot} = \liminf_r \hat{D}(r).$$

Taking $\lambda_n \nearrow$ we shall consider $\limsup(\lambda_{n+1}-\lambda_n)$ and $\liminf(\lambda_{n+1}-\lambda_n)$.

The maximum density $D_{\max}$ (minimum density $D_{\min}$) is the lower bound (upper bound) of densities of sequences containing (contained in) and having a density. They are given by the following formulas (the first one is proved in appendix 1; the second one follows from the remark at the end.

$$D_{\max} = \lim_{\zeta\to 1-0}\limsup_{r\to\infty}\frac{n(r)-n(\xi r)}{r-\xi r}$$
$$D_{\min} = \lim_{\zeta\to 1-0}\lim_{r\to\infty}\inf\frac{n(r)-n(\xi r}{r-\xi r}$$

We call a sequence *regular* if $\liminf(\lambda_{n+1}-\lambda_n) > 0$, and a sequence with density D *well distributed* if $\lambda_n - nD = 0(1)$. (In general, we have only $\lambda_n - nD = o(n)$).

We define the *upper distribution density* $\triangle^{\cdot}$ (lower distribution density $\triangle.$) to be the lower bound (upper bound) of densities of well distributed sequences containing (contained in) Λ. We have the following formulae to calculate them:

$$\triangle^{\cdot} = \lim_{h\to\infty} \lim_{r\to\infty} \sup \frac{n(r+h)-n(r)}{h}$$
$$\triangle. = \lim_{h\to\infty} \lim_{r\to\infty} \inf \frac{n(r+h)-n(r)}{h}$$

100 **Proof (for $\triangle^{\cdot}$):** set $\triangle_h = \lim\limits_{r\to\infty} \sup \dfrac{n(r+h)-n(r)}{h}$. For sufficiently small $\in > 0$ there exists a well distributed sequence $\Lambda^* \supset \Lambda$ and having a density $D_h + \in$. Therefore $\triangle^{\cdot} \leq \lim\limits_{h\to\infty} \inf \triangle_h$. On the other hand, if Λ^* has a density D^*, we have $n^*(r+h) - n^*(r) = hD^* + 0(1)$ and $D^* > \triangle_h - o(1)$. So $\triangle^{\cdot} \geq \lim\limits_{h\to\infty} \sup \triangle_h$. Therefore $\triangle^* = \lim\limits_{h\to\infty} \triangle_h$.

Now $\triangle^{\cdot} < \infty$ means that for an arbitrary given interval of length h, there are only a bounded number of λ_n in it. $\triangle. < \infty$ implies that the sequence is relatively dense in the sense of Bohr, i.e. there exists a well distributed sequence contained in it. We have the following relations between these densities:

$$\frac{1}{\lim\sup(\lambda_{n+1}-\lambda_n)} \leq \triangle. \leq D_{\min} \leq D. \leq \bar{D}. \leq \hat{D}. \leq \hat{D}^{\cdot} \leq \hat{\bar{D}}^{\cdot} \leq D^{\cdot}$$
$$D^{\cdot} \leq D_{\max} \leq \triangle^{\cdot} \leq \frac{1}{\lim\inf(\lambda_{n+1}-\lambda_n)}$$

One can prove $\quad D^{\cdot} \leq e\bar{D}^{\cdot}$ (Mandelbrojt2, p.53).

If $\bar{D}^{\cdot} = \bar{D}.$, $D. = D^{\cdot}$ and so $D_{\min} = D_{\max}$.

If $\Lambda_1 + \Lambda_2 = \Lambda, \Lambda$ having a density, $D(\Lambda) = D_{\max}\Lambda_1 * D_{\min}\Lambda_2$.

The definitions of well-distributed sequences, upper and lower distribution densities can be immediately translated in case of real non-symmetric sequences. We shall make use of these notions in the next paragraph.

2 A problem of continuation on the line

101 Suppose Λ to be real, and take the space $\mathscr{E}_\Lambda(I)$. We are interested in

finding conditions on Λ and I in order that every $f \in \mathscr{E}_\Lambda(I)$ is a restriction of an $f \in \mathscr{E}_\Lambda$. Moreover, if *on* I f belongs to a specified class of C^∞-functions, we are interested to know the related properties of its continuation on the line.

It can be natural to expect that if $\left|I\right| > 2\pi D_{\max}$ and if $f \in \mathscr{C}_\Lambda(I)$, then f has a continuation so that $f \in \mathscr{C}_\Lambda$. We shall give an example at the end of this lecture which shows that this is not true even when $f \in \mathscr{E}_\Lambda(I)$.

Suppose Λ is a real regular sequence and let $\left|I\right| > 2\pi\triangle^{\cdot}$. Can every C^∞-function $\in \mathscr{C}_\Lambda(I)$ be continued into a C^∞ - function in $\mathscr{C}_\Lambda$?

Definition. The class $C_I\{M_n\}$ for a given sequence $\{M_n\}$ and a given interval I is defined to be the set of C^∞- functions on I which verify the conditions $\left|f^{(n)}\right| < KM_n$ where K depends only on f. We define $C\{M_n\} = C_{(-\infty,\infty)}\{M_n\}$.

Theorem. *Suppose* $f \in \mathscr{C}_\Lambda(I) \cap C_I\{M_n\}$, Λ *real and regular,* $M_n \nearrow$ $|I| > 2\pi\triangle^{\cdot}$. *Then* f *can be continued into a function belonging to* $\mathscr{C}_\Lambda \cap C\{M_{n+p}\}$ *for an integer* $p = p(\Lambda, I)$.

It is sufficient to prove theorem for well distributed sequences, since we can find $a\Lambda' \supset \Lambda$ with $|I| > 2\pi\triangle^{\cdot}$. We suppose I to be symmetric with respect to the origin.

Consider the canonical product

$$C(w) = (w - \lambda_o)\prod_{n=1}^{\infty}\left(1 - \frac{w}{\lambda_n}\right)\left(1 - \frac{w}{\lambda_n}\right)$$

we use the following result of B. Levin about $C(w)$. (Levin 1 and 2; **102**
Mandelbrojt 3). $C(w)$ is of type Π, $|C(u)| < K(1 + |u|^N)$, and $\left|C'(\lambda_n)\right| > K'/(1 + |\lambda_n|^N)$ where K, K' and N are dependent on the given sequence.

$C(w)$ is not the transform of a measure but of a distribution. We want to construct a distribution whose transform $M(w)$ has the same type as $C(w)$ and satisfies the equation $M(w) - e^{iwX} = 0$ whenever $W \in \Lambda$. In order to construct $M(w)$, we take $M(w) = C(w)A(w), A(w)$ having a polar part $\approx \sum\{e^{(i\lambda_n X)}/C'(\lambda_n)\ (w - \lambda_n)\}$. To assure normal convergence we take $M(w) = w^q C(w)\sum\limits_n \dfrac{e^{i\lambda_n X}}{\lambda_n^q C'(\lambda_n)(w - \lambda_n)}$. This series is normally convergent outside circles of radius $\in$ around the $\lambda'_n s$ for $q > N+2$. We want to have a uniform majorization for $M_X(w)$. For this take a strip around the real axis: in this strip we have (Phragmen Lindelof $\left|C(w)\right| < K'|1+w|^N$; then in the strip minus these circles $\left|M_x(w)\right| < K''\left|w\right|^q\left|1+w\right|^N$ and (Cauchy) $\left|M_X(u)\right| < \dfrac{K''}{1+u^2}(1+u^p)$ for p even, $p \geq N+q+2$. Thus $M_X(u) = \mathscr{C}(T_X), T_x = d\mu_x + \dfrac{d^p}{dx^p}d\nu_x$ has support in I, and $\int_I \left|d\mu_x\right|$ and $\int_I \left|d\nu_x\right|$ are uniformly bounded.

Let us consider $f \in \mathscr{E}_I(\Lambda)$; we can continue f at the point x by the formula

$$f(X) = \langle f_j, T_x\rangle = \int_I (f d\mu_x + f^{(p)} d\nu_x).$$

By our standard argument, the continued function belongs to $\mathscr{E}_\Lambda$. More over, if $M_n \nearrow$ and $f \in \mathscr{C}_I\{M_n\}$, then the continued function $\in \mathscr{C}\{M_{n+p}\}$.

For a refinement of this result, see (Kahane 2).

103 In this type of result $\triangle^{\cdot}$ is the good density to consider. Actually we cannot get more if we replace $\triangle^{\cdot}$ by $D_{\max}$. In order to show this we construct an example of Λ and $f \in \mathscr{C}_\Lambda(I)$, $f a$ C^∞- function such that $f \notin \mathscr{C}_\Lambda$ and Λ has $\triangle^{\cdot} = 1$ and $D. = D_{\max} = 0$.

We take Λ to be a sequence of integers which are situated in such a manner that $2n$ of them are in an interval I_n, the intervals I_n being disjoint and having lengths which increase indefinitely. To do this we take $p_n = n^k, k > 1$ and the sequence Λ is the union of the sets $\{p_n - n, p_n - (n-1), \ldots, p_n + n\}$. It is verified easily that $D = 0$ and $\triangle^{\cdot} = 1$. We shall construct a function $f \in \mathscr{C}_\Lambda(I), I = [-\pi+\varepsilon, \pi- \in]$ and not continuable in $\mathscr{E}_\Lambda$, nor even in $\mathscr{D}'_\Lambda$.

Let $\alpha(x) = \sum\limits_{-\infty}^{\infty} C_j e^{ijx}$. We can choose C_j in such a manner that $C_o = 1$ and $\alpha(x)$ vanishes on I. By the theorem of Denjoy Carleman we can have $\alpha(x)$ to be not in any quasi-analytic class (lecture 19, §2), but in the class $C\{n^{\alpha n}\}$ for $\alpha > 1$. Then $\left|\alpha^{(k)}(x)\right| < k^{\alpha k}$. As $\alpha^{(k)}(x) = (i)^k \sum C_j j^k e^{ijx} je, \left|C_j\right| < \dfrac{|\alpha(k)|}{j^K}$. Hence $\left|C_j\right| < \min\limits_n \dfrac{n^{\alpha^n}}{|j|^n} < e^{-j^\alpha}, \alpha' < 1$.

Also $\sum\limits_{|j|\geq n} |C_j| < e^{-n^\beta}, \beta < 1$. This majorizes $S_n(x) = \sum\limits_{-n}^{n} C_j e^{ijx}$ on I. Let $f(x) = \sum\limits_{n=1}^{\infty} a_n e^{ip} n^x S_n(x)$. If $\sum |a_n| e^{-n^\beta} < \infty$, f is continuous on I. We can take a_n such that $\sum a_n p_n^m e^{-n^\beta} < \infty$ for every m and a_n increasing more rapidly than any polynomial in p_n. On account of the construction of a_n, $f(x)$ is infinitely differentiable in I, but cannot be continued either in $\mathscr{E}_\Lambda$ or in $\mathscr{D}'_\Lambda$ (because a_n would be the Fourier coefficient of order p_n).

Lecture 18

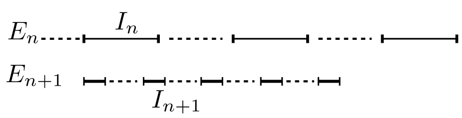
Continuation on the Line and Banach -Szidon Sequences

We have seen in the last lecture that if $f \in \mathscr{C}_\Lambda(I), |I| > 2\pi\triangle$ and if 104
f belongs to a class of C^∞- functions, then f can be continued to a function $\in \mathscr{C}_\Lambda$. The answer without the assumption of differentiability is not known completely. But conditions on Λ can be given to have the answer in particular cases. Suppose Λ is real, symmetric and very lacunary in the sense that $\lambda_{n+1}|.\lambda_n >> 3$. Then we shall prove that every function $f \in \mathscr{C}_\Lambda(I)$ is continuable into $\mathscr{C}_\Lambda$. We take a trigonometric polynomial $P(x) = \sum\limits_{n_o} r_n \cos(\lambda_n x + \varphi_n), \left|r_n\right| \geq 0$.

For λ_n large enough the set E_n is defined as the set of points x satisfying the following conditions:

$$\pi/2 - \alpha + 2k\pi < \lambda_n x + \varphi_n < \pi/2 + \alpha + 2k\pi (k = .. - 1, 0, 1, \ldots)$$

The connected components of E_n are intervals of length $\dfrac{\pi - 2\alpha}{\lambda_n}$.



If n_o is large enough, E_{n_o} has at least one connected component entirely in I, say I_{n_o}. Assuming $\dfrac{\lambda_{n+1}}{\lambda_n} \geq 1 + 2\dfrac{\pi + 2\alpha}{\pi - 2\alpha}$, there is at least one connected component of E_{n+1}, say I_{n+1}, inside a connected component I_n of E_n (see fig.) ; we define, in that way, I_{n_o+1}. $I_{n_o+2}, \ldots$, Take $X \in \cap I_n$; thus

$$P(X) > \sin\alpha \sum_{n_o}^{N} r_n, \tag{1}$$

Let $\Lambda' = \Lambda - \{\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_{n_o-1}\}$, $\left|f\right|_I = \max_{x\in I}\left|f(x)\right|$ and $\|f\| =$
105 $\sum r_n$, if $f = \sum\limits_{n_o}^{\infty} r_n \cos(\lambda_n x + \varphi_n)$, $\sum r_n = \sum\left|r_n\right| < \infty$; they are respectively norms in two Banach spaces, say $\mathscr{C}(I)$ and $A_{\Lambda'}$. (1) proves the equivalence of these norms for the polynomials in $\mathscr{C}_{\Lambda'}(I)$: $\sin\alpha\|P\| < |P|_I \leq \|P\|$. As the polynomials P form a dense subset in $\mathscr{C}_{\Lambda'}(I)$ and in $A_{\Lambda'}$, we can identify $\mathscr{C}'_{\Lambda}(I)$ and A'_{Λ}: every $f \in \mathscr{C}_{\Lambda'}(I)$ can be expressed as an element of $A_{\Lambda'}$. The same result holds by adding a finite number of terms to Λ' (because

$$f \in \mathscr{C}_{\Lambda}(I) \Leftrightarrow f = f_1 + f_2,\ f_1 = \sum_{1}^{n_0-1} r_n \cos(\lambda_n x + \varphi_n),\ f \in \mathscr{C}_{\Lambda'}(I)).$$

Thus we have prove the following proposition:

Proposition. *Suppose* $\Lambda = \{\pm\lambda_n\}$ *is a real symmetric sequence which is lacunary in the sense that* $\lambda_{n+1}/\lambda_n >> 3$. *Then every function* $f \in \mathscr{C}(I)$ *is the sum of an absolutely convergent Fourier-series and* f *is continuable to* $\mathscr{C}_{\Lambda}$.

A result of this type was considered by Szidon (Zygmund 1, chap. VI §6.4) who has proved that a bounded periodic function whose spectrum Λ is lacunary in the sense of Hadamard, viz. $\lambda_{n+1}/\lambda_n >> 1$, has an absolutely convergent Fourier series.

Definition. A sequence Λ is said to be a Szidon sequence if for every interval I, $f \in \mathscr{C}(I) \Rightarrow f = \sum a(\lambda)e^{i\lambda x}$ and $\sum\left|a(\lambda)\right| < \infty$.

We shall make a brief study of Szidon sequences and give alternate definitions of them by a simple lemma on Banach spaces.

106 Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two Banach spaces with norms $\|\ \|_1$ and $\|\ \|_2$ and

let $\mathscr{E}_1$ be an algebraic subspace of $\mathscr{E}_2$. Moreover, for every $x \in \mathscr{E}_1$, let $\|x\|_1 \geq \|x\|_2$. Then $\mathscr{E}'_2$ is imbedded in $\mathscr{E}'_1$ by $\langle x, x'\rangle_1 = \langle x, x'\rangle_2 \mathscr{E}'_2 \subset \mathscr{E}'$.

By a theorem of Banach, if $\mathscr{E}_1 = \mathscr{E}_2$ then the norms are equivalent and $\mathscr{E}'_1 = \mathscr{E}'_2$. Conversely, if $\mathscr{E}'_1 = \mathscr{E}'_2$, then is it true that $\mathscr{E}_1 = \mathscr{E}_2$? But $\|x\|_1 = \sup\limits_{\|x'\| \leq 1} \left|\langle x, x'\rangle\right|, \|x\|_2 = \sup_{\|x'\| \leq 1} \left|\langle x, x'\rangle\right|$. Since $\mathscr{E}'_1 = \mathscr{E}'_2$ and since $\|x'\|_2 \geq K^{-1}\|x'\|_1, K > 0$, we have $\|x\|_1 < K \sup\limits_{\|x'\|} \leq \left|\langle x, x'\rangle\right| = K\|x\|_2$. This means that $\mathscr{E}_1$ is closed in $\mathscr{E}_2$. In order that $\mathscr{E}_1 = \mathscr{E}_2$ it is necessary and sufficient that $\mathscr{E}_1$ is dense in $\mathscr{E}_2$. That proves

Lemma. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two Banach spaces with norms $\| \ \|_1$ and $\| \ \|_2$; suppose $\mathscr{E}_1$ is an algebraic subspace of $\mathscr{E}_2, \mathscr{E}_1$ is dense in $\mathscr{E}_2$ and $\|x\|_1 \leq \|x\|_2$ for every $x \in \mathscr{E}'_1$. If $\mathscr{E}'_1 = \mathscr{E}'_2$, then $\mathscr{E}_1 = \mathscr{E}_2$.*

Definition. We define a sequence Λ to be a *Banach sequence* if for every interval I and every sequence $b(\lambda) \to 0$ as $\lambda (\in \Lambda) \to \infty$ there exists a function $f \in L^1(I)$ such that $b(\lambda) = \int_I f e^{-i\lambda x} dx \,(\lambda \in \Lambda)$.

We prove a theorem relating Banach sequences and Szidon sequences, by applying the lemma.

Let $\mathscr{E}_2$ be the space of sequences $\{a(\lambda)\}$ converging to zero endowed with the norm, $\|\{a(\lambda)\}\|_2 = \sup\limits_{\lambda} \left|a(\lambda)\right|$.

Let $\mathscr{E}_1$ be the space of sequences $\{b(\lambda)\}$ with $b(\lambda) = \int_I f e^{-i\lambda x} dx$, where $\int_I \left|f(x)\right| dx < \infty$. The norm in $\mathscr{E}_1$ is taken as $\|\{b(\lambda)\}\|_1 = \inf \int_I |f|$ where the infimum is taken over all f for which $b(\lambda) = \int_I e^{-i\lambda x} f(x) dx$. $\mathscr{E}_1$ is isomorphic to $L^1(I)/H$, where H is the subspace of $L^1(I)$ orthogonal to $\{e^{i\lambda x}\}$.

$\mathscr{E}'_2$ is the space of absolutely convergent sequences $\{C(\lambda)\}$, with the bilinear form on $\mathscr{E}_2 x \mathscr{E}'_2$ as $\langle a(\lambda), C(\lambda)\rangle = \sum a(\lambda) C(\lambda)$. The norm in $\mathscr{E}'_2$ is given by $\|C(\lambda)\|_2 = \sum C(\lambda) . \mathscr{E}'_2$ is isomorphic to A_Λ.

$\mathscr{E}'_1$ is isomorphic to the subspace $\mathscr{C}_\Lambda(I)$ of $L^\infty(I)$ spanned by $\{e^{i\lambda x}\}$, with the norm of $\{d(\lambda)\} = \|\varphi\|_{L^\infty(I)}$ where $\varphi \sim \sum d(\lambda) e^{i\lambda x}$. The bilinear form giving the duality is : $\langle b(\lambda), d(\lambda)\rangle = \int_I f\varphi$, where $b(\lambda) = \int_I f(x) e^{-i\lambda x} dx$.

Now the sequence Λ is a Banach sequence if and only if $\mathscr{E}_1 = \mathscr{E}_2$, **107**
and a Szidon sequence if and only if $\mathscr{E}'_1 = \mathscr{E}'_2$. A Banach sequence Λ is

a Szidon sequence. In order to see that a Szidon s equence be a Banach sequence, it is sufficient to prove that $\mathscr{E}_1$ is dense in $\mathscr{E}_2$. This is so since every linear functional on $\mathscr{E}_2$ which is zero on $\mathscr{E}_1$ is zero. We can apply again our lemma, taking $(\mathscr{E}'_1, \mathscr{E}'_2)$ instead of $(\mathscr{E}'_2, \mathscr{E}'_1)$.

$\mathscr{E}''_2$ is the space of bounded sequences $\{e(\lambda)\}$ with $\langle C(\lambda), e(\lambda)\rangle = \sum c(\lambda)e(\lambda)$ and the norm $\|e(\lambda)\| = \sup\left|e(\lambda)\right|$.$\mathscr{E}''_1$ is isomorphic to the quotient space $M(I)/K$ where $M(I)$ is the space of measures with support in I and K is the subspace of it orthogonal to $\{e^{i\lambda x}\}$. $\{f(\lambda)\} \in \mathscr{E}''_1$ when $f(\lambda) = \int_I e^{i\lambda x} d\mu, d\mu \in M(I)/K$ and $\|b(\lambda)\| = \inf\limits_{\mu} \int |d\mu|$. When $\mathscr{E}_1 = \mathscr{E}_2, \mathscr{E}''_1 = \mathscr{E}''_2$. Moreover $\mathscr{E}'_2$ is dense in $\mathscr{E}'_1$ (because obviously A_Λ is dense in $\mathscr{C}_\Lambda(I)$); then $\mathscr{E}'_1 = \mathscr{E}'_2 \Rightarrow \mathscr{E}''_1 = \mathscr{E}''_2$. We get the following theorem:

Theorem. *The definitions of Szidon sequences and Banach sequences are equivalent, and equivalent to the following: for every integral I and for every bounded sequence* $\{e(\lambda)\}(\lambda \in \Lambda)$, *there exists a measure* $d\mu$ *with support in I, such that* $e(\lambda) = \int_I e^{-i\lambda x} d\mu(x) (\lambda \in \Lambda)$.

We have given sufficient conditions for a sequence $\{\pm\lambda_n\}$ to be a Banach-Szidon sequence. Actually, Zygmund has shown (Zygmund 2) that the condition $\dfrac{\lambda_{n+1}}{\lambda_n} >> 1$ is sufficient. Necessary conditions involve arithmetical properties of λ_n, as it appears from the following proposition (proved in (Kahane 2)).

108 Let $\xi_1, \ldots, \xi_p$ be a real numbers. Consider $n_1\xi_1 + \cdots + n_p\xi_p, n_j$ being integers satisfying the inequality $|n_1| + \cdots + |n_p| \leq s$. If Λ is a Banach-Szidon sequence, there exists a constant $A = A(\lambda)$ such that among the set of $n_1\xi_1 + \cdots + n_p\xi_p$ there are only $[A_p \log(1 + s)]$ of them in Λ.

Examples. Take $p = 1$. We cannot have more than $0(\log s)$ points of Λ in an arithmetical progression containing s terms.

Let $s = 2$, and $p = 1, 2, \ldots$ Whatever be the sequence $\{\xi_j\}$, we cannot have in Λ more than $0(p)$ points among the p^2 points $\xi_j + \xi_k$, $|j| \leq p, |k| \leq p$.

The study of Banach-Szidon sequences is interesting in itself, but can give only a partial answer to the following problem:

Problem. Give a relation, between Λ and I, such that every $f \in \mathscr{C}_\Lambda(I)$ is the restriction on I of a function $\in \mathscr{C}_\Lambda$.

This problem becomes easier and has pretty good solutions if we replace $\mathscr{C}$ by the space of functions which belong to L^2 on every interval, with the topology of the convergence in L^2 on every interval ((Paley - Wiener), (Kahane 2)).

Lecture 19

Quasi-analytic classes of functions (Quasi-analyticity D and I)

1 Quasi-analyticity and mean periodicity

We shall see presently that problems of quasi-analyticity appear in a natural way as problems on mean periodic functions. **109**

1. Consider the following problem. Given a set E consider the closed span $\tau_E(f)$ of translates $f_y, y \in E$, of f. Find the conditions on f in order that $\tau_E(f) = \tau(f)$. In other words, the problem, in a restricted sense, is to find a relation between E and the spectrum $S(f)$ in order that $\tau_E(f) = \tau(f)$.

2. On the other hand, the above problem suggests the following one. Let f be a C^∞- function and let $\delta(f)$ be the closed span of the derivatives of f. (When E is not a discrete set, $\delta(f)$ is a subset of $\tau_E(f)$). Find conditions about f so that $\delta(f) = \tau(f)$. Here we require a condition involving a class of C^∞- functions, i.e. a condition on the class and the spectrum $S(f)$, for example conditions of the type $\{K\{M_n\}, S(f)\}$, $f \in K\{M_n\}$.

Definition. The class $K\{M_n\}$ is defined to be the class of C^∞- functions f such that on every interval I, $\left|f^{(n)}\right|_I < K_I M_n$, $(n = 0, 1, \ldots)$, where $\{M_n\}$

is a given sequence and K_I a constant depending on I and f. $C\{M_n\}$ is defined as a class of C^∞- functions which satisfy the relation $\left|f^{(n)}\right| < KM_n$ on the real line, K depending only on f.

The above two problems are closely related to quasi-analyticity.

110 1. In order that $\tau_E(f) = \tau(f)$ it is necessary and sufficient (condition of Riesz) that every measure $d\mu$ orthogonal to $\tau_E(f)$ be orthogonal to $\tau(f)$. Or again, $g(y) = \int f(x+y)d\mu(-x) = 0$ for every $y \in E \Rightarrow g \equiv 0$. As $f \in \mathscr{C}_\Lambda$ implies $g \in \mathscr{C}_\Lambda$, we have an answer to this problem if we have a relation $R\{E, \Lambda\}$ between E and Λ such that $\{g \in \mathscr{C}_\Lambda, g(y) = 0, y \in E\} \Rightarrow g \equiv 0$. This is nothing but a problem of uniqueness. We will be mainly interested in the case when E is an interval.

   Definition. A class of functions is called an I-quasi-analytic class if each function of the class is defined by its values on I, I being an interval.

2. By the condition of Riesz, denoting by

$$g(y) = \inf f(x+y)d\mu(-x),$$

   we have $$\delta(f) = \tau(f) \Leftrightarrow [g^{(n)}(0) = 0 \ \forall n \Rightarrow g \equiv 0].$$

   When f has spectrum $\Lambda, g \in \mathscr{C}_\Lambda$. Moreover if $f \in K\{M_n\}$ then $g \in K\{M_n\}$ so that $g \in K\{M_n\} \cap \mathscr{C}_\Lambda$. Thus we have an answer to this problem if we have a relation $\{\{M_n\}, \Lambda\}$ such that $\{g \in K\{M_n\} \cap \mathscr{C}_\Lambda, g^{(n)}(0) = 0 \forall n\} \Rightarrow g \equiv 0$. The same is true in replacing $K\{M_n\}$ by $C\{N_n\}$.

   Definition. A class of C^∞ - functions is defined as a D-quasi-analytic class if the only function g of the class all of whose derivatives vanish at the origin is the zero function.

In Lecture 5 §2, we proved a result equivalent to the following: $\mathscr{C}_\Lambda$ is an I-quasi-analytic class if $|I| >$ mean-period related to Λ. We give here a far stronger result.

111 **Theorem.** *Let Λ be a sequence of complex numbers such that $\mathscr{C}_\Lambda \neq \mathscr{C}$, and Λ^* and Λ^- the parts of Λ respectively to the right and to the left of the imaginary axis. $\mathscr{C}_\Lambda$ is a quasi - analytic class whenever*

$$|I| > 2\pi D_{\min}(\Lambda^+) \text{ or } |I| > 2\pi D_{\min}(\Lambda^-).$$

Proof. We use the notations of Lecture 4. $f \in \mathscr{C}_\Lambda$, $f * d\mu = 0$, $f^{-1} * d\mu = g$, $F(w) = \dfrac{G(w)}{M(w)}$, $S(f)$ = spectrum of f. Denoting by $\sum(d\mu)$ and $\sum(g)$ the null-sets of $M(w)$ and $G(w)$, we have $S(f) = \sum(d\mu) - \sum(g) \cap \sum(d\mu)$. Let L_μ and L_g be the lengths of the segments of supports of $d\mu$ and g. According to Levinson's theorem, $L = 2\pi$ density $\sum \pm(d\mu)$ ($\sum \pm$ is the part of $\sum$ to the right (respectively to the left) of the imaginary axis). Now we use the result that if S_1 and S_2 are disjoint sequences, and $S = S_1 \cup S_2$ has a density, density $S = D_{\max} S_1 + D_{\min} S_2$. Thus $L = 2\pi D_{\min} S^{\pm}(f) + 2\pi D_{\max}(S^{\pm}(g) \cap S^{\pm}(d\mu)); L_\mu \leq 2\pi D_{\min} S^{\pm}(f) + L_g$. If $f = 0$ on $(-1, 0)$, then $L_g \leq L - 1$. Hence

$$D_{\min} S^+(f) \geq (L_\mu - L_g)/2\pi \geq 1/2\pi.$$

So if $1 > 2\pi D_{\min} S^+(f)$ or $1 > 2\pi D_{\min} S^-(f)$, $f \equiv 0$. As a translation of f does not change $S(f)$, the theorem is proved. □

The above theorem is stated in Levinson (Levinson, chap. II), when Λ is a sequence of integers, i.e. f is a periodic function; then the proof does not require the Carleman transform of f.

2 Theorem of Denjoy-Carleman

We first study D-quasi-analytic classes in which no condition on Λ is involved. We shall prove a classical theorem of Denjoy-Carleman about D-quasi-analytic classes. It is a local property and so can be stated for an interval.

Definition. Let $C_I\{M_n\}$ be $\in$ class of C^∞- functions satisfying the condi- 112
tions $\left|f^{(n)}(x)\right| < KM_n$ on an interval $I, 0 \in I, K$ depending only on f. We say that it is a quasi-analytic class if $f^{(n)}(0) = 0$ for every $n \Rightarrow f \equiv 0$.

Given a sequence $\{M_n\}$ let $\{M_n^C\}$ be the largest sequence which satisfies $M_n^c \leq M_n$ and $M_{n+1}^c/M_n^c \nearrow$.

Theorem of Denjoy-Carleman. $C_I\{M_n\}$ is a quasi- analytic class $\Leftrightarrow$ $\sum\limits_1^\infty \dfrac{M_n^c}{M_{n+1}^c} = \infty$.

A variant of this theorem (stated by Denjoy without proof) is that $C_I\{M_n\}$ is a quasi-analytic class $\Leftrightarrow$ $\sum\limits_1^\infty \dfrac{1}{(M_n^c)^1/n} = \infty$. The equivalence of these two follows from the inequality:

$$\sum \in_n < \sum (\in_1 \cdots \in_n)^{1/n} < e \sum \in_n \text{ for } \in_n \searrow 0.$$

(Cf. Hary-Littlewood-Polya: Inequalities, or S. Mandelbrojt 2).

Interpretation of M_n. Consider in the plane the points $(n, \log M_n)$ and construct the polygon of Newton on these points. It is a convex curve $m(u)$. The ordinate at n intersecting this curve gives $\log M_n^c$. The sequence $\{\log M_n^c\}$ is called the convex regularized sequence corresponding to $\{\log M_n\}$. (Mandelbrojt 2).

Proof of Theorem. We can suppose $I \supset [0,1], K = M_o = 1$ and $M_n = M_n^c$. Consider $\varphi(x) = f(x) f(1-x)$ in $0 \leq x \leq 1$ and $\varphi(x) = 0$ elsewhere. Since $\log M_n$ is convex, M_n is the $\max\limits_{i \leq n}(M_i M_{n-i})$. Thus we have the following majorization for $\varphi^{(n)}(x)$:

$$\left|\varphi^{(n)}(x)\right| \leq M M_n + \tbinom{n}{1} M_1 M_{n-1} + \cdots + M_n M_o \leq 2^n M_n$$

and φ is null, with all its derivatives, at the origin. Consider $\Phi(w) =$
113 $\tau(\varphi) = \int_0^1 \varphi(x) e^{-ixw} dw$. Integrating by parts, we have

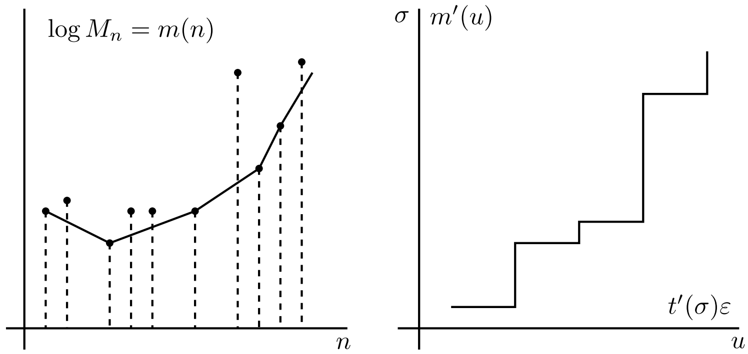
$$\Phi(w) = \int_o^1 \varphi^{(p)}(x) \frac{e^{-ixw}}{(iw)^p} dx \quad \text{and} \quad \left|\Phi(w)\right| < 2^p M_p/|w|P.$$

Now we introduce the function $T(r) = \sup\limits_p r^P/M_p$. Then $\left|\Phi(w)\right| < \dfrac{1}{T(r/2)}$. We apply Carleman's formula for $\Phi(w)$ in the upper or lower half plane and get $\int^R \dfrac{\log\{\Phi(u)\Phi(-u)\}}{u^2} du$ cannot tend to $-\infty$. Since

$\left|\Phi(w)\right| < \dfrac{1}{T(r/2)'}$, we have $\int^{\infty} \dfrac{\log T(r)}{r^2} dr < \infty$. Therefore $\int^{\infty} \dfrac{\log T(r)}{r^2}$ $dr = \infty$ implies quasi-analyticity (Ostrowski's form).

We have $\log T(e^{\sigma}) = t(\sigma) = \max\limits_{n}(n\sigma - \log M_n) = \max\limits_{u}(\sigma u - m(u))$. Moreover the relation between $t(\sigma)$ and $m(u)$ is reciprocal. Indeed $m(u) = \max(u\sigma - t(\sigma))$. Also the derivatives $t'(\sigma)$ and $m'(u)$ are inverse functions in a sense made precise by the graph. First we suppose $m(u) \to \infty$ when $u \to \infty$. Now

$$\int^{X} e^{-\sigma} t(\sigma) d\sigma = -e^{X} t(X) + \int^{X} e^{-\sigma} t'(\sigma) d\sigma$$



Thus we have $\int^{\infty} e^{-\sigma} t(\sigma) d\sigma = \infty \Leftrightarrow \int^{\infty} e^{-\sigma} t'(\sigma) d\sigma = \infty$. **114**
Since $\sigma = m'(u)$ and $u = t'(\sigma)$, we have the following relations:

$$\int^{T} e^{-m'(u)} u dm'(u) = -e^{-m'(T)} T + \int_{\infty}^{T} e^{-m'(u)} du$$

$$\int^{\infty} e^{-\sigma} t'(\sigma) d\sigma \quad = \infty \Leftrightarrow \int^{\infty} e^{-m'(u)} du = \infty.$$

Between $n \leq u \leq n+1, m'(u) = \log M_{n+1} - \log M_n$ and $-m'(u) = \log \dfrac{M_n}{M_{n+1}}$.

$$\int^{\infty} e^{-m'(u)} du = \infty \Leftrightarrow \sum \frac{M_n}{M_{n+1}} = \infty$$

$$\int^{\infty} \frac{\log T(r)}{r^2} dr = \infty \Leftrightarrow \int^{\infty} e^{-\sigma} t(\sigma) d\sigma = \infty.$$

Thus, finally we have the relation

$$\int^{\infty} \frac{\log T(r)}{r^2} dr = \infty \Leftrightarrow \int^{\infty} \frac{M_n^C}{M_{n+1}^{C+1}} = \infty.$$

This relation still holds when $m'(u)$ is bounded.

We have proved that if $\sum M_n^C / M_{n+1}^C = \infty, C_I\{M_n\}$ is a quasi-analytic class. Suppose $\sum M_n^C / M_{n+1}^C < \infty$. Then we construct a C^∞- function with compact support which is $\not\equiv 0$ and which satisfies the conditions $\left|f^{(n)}\right| < M_n^c$ and $f^{(n)}(0) = 0$ for every n. To construct f it is convenient to construct its transform.

Take $F(w) = \left(\dfrac{\sin \in w}{\in w}\right)^2 \prod\limits_1^\infty \dfrac{\sin \alpha_j w}{\alpha_j w}$. It is an entire function of exponential type if $\sum \alpha_j < \infty, \alpha_j > 0$. We can majorise $\left|\dfrac{\sin \alpha_j u}{\alpha_j u}\right|$ by 1 for $j \geq N$ and we have $\left|F(u)\right| < \left(\dfrac{\sin \in u}{\in u}\right)^2 \prod\limits_1^N \dfrac{1}{|\alpha_j u|}$.

We write $F(w) = \mathscr{C}(f), f(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} du$, since $F(u)$ is rapidly decreasing. Also we have the following relations:

$$f^{(n)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)(iu)^n e^{ixu} du$$

$$\left|f^{(n)}(x)\right| < (\alpha_1 \cdots \alpha_n)^{-1} \int_{-\infty}^{\infty} \left(\frac{\sin \in u}{\in u}\right)^2 du.$$

We take $M_n^c = (\alpha_1 \cdots \alpha_n)^{-1}$, i.e. $\alpha_n = M_n^C / M_{n+1}^{C-\infty}$.

Thus our construction is complete.

Lecture 20

New Quasi-analytic classes of functions

We study quasi-analyticity of $\mathscr{C}_\Lambda$ with conditions involving Λ. First we recall a result of S. Mandelbrojt about periodic functions (Mandelbrojt 1). 115

$$\left.\begin{aligned} f \sim \sum (a_n \cos \lambda_n x + b_n \sin \lambda_n x) \\ \sum 1/\lambda_n^\sigma < \infty, 0 < \sigma < 1 \\ \int_o^\alpha |f| < Ke^{-(\alpha^{-\rho})}, \alpha < \alpha_o, \rho > \frac{\sigma}{1-\sigma} \end{aligned}\right\} \Rightarrow f \equiv 0$$

This signifies that if the spectrum is very lacunary, then it is not possible to have $\int_o^\alpha \cdots$ very small. This leads us to find conditions involving $\{\lambda_n\}$ and $I(\alpha)$ such that $\int_o^\alpha |f| < I(\alpha) \Rightarrow f \equiv 0$. Such functions form a class $I(\alpha)$.

Definition. Given a positive function $I(\alpha)$ ($\alpha > 0, I(\alpha) \nearrow$), a class of C^∞ - functions is defined to be a $I(\alpha)$ quasi-analytic class when the only function of the class which satisfies the conditions $\int_o^\alpha |f| < I(\alpha)(\alpha \to 0)$ is the zero function.

There is a connection between $I(\alpha)$ quasi-analyticity and D quasi-analyticity. Indeed if $f \in C_I\{M_n\}, 0 \in I$ and $f^{(n)}(0) = 0$, for $n = 0, 1, \ldots,$

then by Taylor's formula

$$f(x) = \frac{x^n}{n!} f^{(n)}(\theta x),$$

$0 \leq \theta \leq 1$ and so $\left|f(x)\right| \leq \dfrac{x^n}{n!} M_n$ which gives that $\int_o^\alpha \left|f(x)\right| dx < \min\limits_n \dfrac{\alpha^{n+1}}{(n+1)!}! M_n$. So quasi-analyticity $I(\alpha)$ implies quasi-analyticity D, when $I(\alpha) = \min\limits_n \left\{\dfrac{\alpha^{n+1} M_n}{(n+1)!}\right\}$. This function is similar to the function
116 $T(r)$ introduced in the last lecture, but it seems not possible to obtain the Denjoy-Carleman theorem by considering $I(\alpha)$.

Our problem will be to define a relation between Λ and $I(\alpha)$ such that $\mathscr{E}_\Lambda$ is an $I(\alpha)$ quasi-analytic class. First we formulate the method we shall use in this lecture (other methods will be explained in the next one). Suppose $f \in \mathscr{E}_\Lambda \neq \mathscr{E}$ and the mean period corresponding to Λ is zero; for convenience, suppose $0 \notin \Lambda$. Then for every $\alpha > 0$ it is possible to find a measure $d\mu_\alpha$ with support in $[-\alpha/2, \alpha/2]$ such that $f * d\mu_\alpha = 0$, and we can assume the conditions: $d\mu_\alpha = \mu'_\alpha dx, \mu'_\alpha \in L^\infty$, and $\int d\mu_\alpha = M_\alpha(0) = 1$. Let $g = -f * d\mu_\alpha$. Then (notations of Lecture 4)$G_\alpha(w) = F(w) M_\alpha(w)$ and $G_\alpha(0) = F(0)$. Suppose we have $\int_o^\alpha |f(x)| < I(\alpha)$. Then $||g_x||_\infty < I(\alpha)||\mu'_\alpha||_\infty$. As $\left|G_\alpha(0)\right| \leq \alpha ||g_\alpha||$, we have $\left|F(0)\right| < \alpha I(\alpha)||\mu'_\alpha||_\infty$. Suppose that, for an infinity of $\alpha \to 0$, we can choose $d\mu_\alpha$ in such a manner that $\alpha I(\alpha)||\mu'_\alpha||_\infty \to 0$. Then $F(0) \equiv 0$. This being true for any $f \in \mathscr{C}_\Lambda$, we take primitives of f instead of $f(x)$ and $F'(0) = 0$ etc. Thus $F \equiv 0$ and $f \equiv 0$. Thus we are able to formulate our condition as follows:

Suppose that to each $\alpha > 0$ we associate $\mu'_\alpha \in L^\infty$ with support in $[-\dfrac{\alpha}{2}, \dfrac{\alpha}{2}]$ such that $\int \mu'_\alpha = 1$ and $\int e^{i\lambda x} \mu'_\alpha dx = 0$ for every $\lambda \in \Lambda$. If $\liminf_{\alpha \to 0} \alpha I(\alpha)||\mu'_\alpha||_\infty = 0$, then $\mathscr{C}_\Lambda$ is an $I(\alpha)$ quasi-analytic class.

We shall use this condition in the following form.

117 **Lemma.** *Suppose that to each $\alpha > 0$ we associate an entire function $M_\alpha(w)$ of exponential type $\leq \dfrac{\alpha}{2}$, such that $M_\alpha(u) \in L^1, M_\alpha(0) = 1$,*

$M_\alpha(\Lambda) = 0$. *If* $\liminf\limits_{\alpha\to o} \alpha I(\alpha) \int |M_\alpha(u)| = 0$, *then* $\mathscr{C}_\Lambda$ *is an* $I(\alpha)$*- quasi-analytic class.*

We apply the above condition in the case when Λ is a real symmetric sequence, $\Lambda = \{\pm\lambda_n\}$ such that $\sum \dfrac{1}{\lambda_n} < \infty$. To construct the function $M_\alpha(w)$ we take the canonical product $C(w) = \prod\limits_1^\infty \left(1 - \dfrac{w^2}{\lambda_n^2}\right)$ and take $M_\alpha(w) = C(w) \prod\limits_o^\infty \left(\dfrac{\sin \alpha_j w}{\alpha_j^w}\right)$ with $4 \sum \alpha_j = \alpha$. we take this additional factor since $C(w)$ does not behave well real axis. $M_\alpha(w)$ is a function of exponential type $\dfrac{\alpha}{2}$, since $C(w)$ is of type 0. Now we have to construct α_n in such a manner that $M_\alpha(u) \in L^1$. To do this we first try to majorise $M_\alpha(w)$. Suppose $\lambda_n \leq u \leq \lambda_{n+1}$. We have the following calculations:

$$\begin{aligned} \left|M_\alpha(u)\right| &= \prod_1^n \left(\frac{u^2}{\lambda_m^2} - 1\right) \prod_{n+1}^\infty \left(1 - \frac{u^2}{\lambda_m^2}\right) \prod_o^\infty \left(\frac{\sin \alpha_j u}{\alpha_j u}\right)^2 \\ &< \frac{u^{2n}}{\lambda_1^2 \cdots \lambda_n^2} \, \frac{1}{\alpha_1^2 \cdots \alpha_n^2 u^{2n}} \min\left(1, \frac{1}{\alpha_o^2 u^2}\right) \end{aligned}$$

(we already used such a majorization, for $\prod\limits_o^\infty \dfrac{\sin \alpha_j u}{\alpha_j u}$, in Lect. 19, §2).

$$\int_o^\infty \left|M_\alpha(u)\right| du = \int_o^1 + \int_1^\infty < \left(1 + \frac{1}{\alpha_o^2}\right) (\max_n(\lambda_1\alpha_1, \ldots, \lambda_n\alpha_n))^{-2}.$$

This majorization is not useful if $\sum \dfrac{1}{\lambda_n} = \infty$, because the second member is ∞ (if not, we would have $(\alpha_1 \cdots \alpha_n)^{1/n} > \dfrac{K}{(\lambda_1 \cdots \lambda_n)}^{1/n} > \dfrac{K}{\lambda_n}$, and the equiconvergence of $\sum \alpha_n$ and $\sum (\alpha_1 \cdots \alpha_n)^{1/n}$, which we stated in Lect. 19 §2, leads to a contradiction). To get a result, we must suppose $\sum \dfrac{1}{\lambda_n} < \infty$. Choose a sequence $\{l_n\}, l_n \to \infty$, such that $\sum_1^\infty \frac{l_n}{\lambda_n} < \frac{l}{8}$ and take $\alpha_o = \alpha/8, \alpha_j = l_j \alpha/\lambda_j$. Then $4 \sum_o^\infty \alpha_j < \infty$ and $\max\limits_n(\lambda_1\alpha_1 \cdots$ **118**
$\lambda_n\alpha_n) = \max\limits_n(l_1 \cdots l_n \alpha^n)$. This is finite since $l_n \to \infty$. (The expression

$(\max_n(l_1\cdots l_n\alpha^n))^{-2}$ is of the same form as $\max r^n/M_n$ which we have seen already). Thus we have

$$\int_{-\infty}^{\infty}|M_\alpha| < \text{Const}\,.(\max_n(l_1\cdots l_n\alpha^n))^{-2}\alpha^{-2};$$

$M_\alpha(w)=0$ on Λ and $M_\alpha(w)$ is of type $\leq\alpha$. Now the condition $\liminf\limits_{\alpha\to 0}$ $\alpha I(\alpha)\int M_\alpha(u)=0$ follows from $\liminf\limits_{\alpha\to 0} I(\alpha)/\alpha\min\limits_n(l_1\cdots l_n\alpha^n)^2=0$. By changing the first l_j if necessary and by replacing l_j by $kl_j(k>1)$ for sufficiently large j we have the following condition for $\mathscr{C}_\Lambda$ to be $I(\alpha)$ quasi-analytic.

Theorem. *Suppose* $\Lambda=\{\pm\lambda_n\}, 0<\lambda_1<\lambda_2\cdots\sum\limits_1^\infty\dfrac{1_n}{\lambda_n}$ *with* $1_n\nearrow\infty$ *and* $\liminf\limits_{\alpha\to 0}\dfrac{I(\alpha)}{\alpha\min\limits_n((l_1\cdots l_n\alpha^n)^{-2}}<\infty$. *Then* $\mathscr{C}_\Lambda$ *is an* $I(\alpha)$ *quasi-analytic class.*

When $\{\lambda_n\}$ is not a sequence of real numbers, but symmetric and $\sum\dfrac{1}{|\lambda_n|}<\infty$, the same method can be used with the additional hypothesis that $\left|\lambda_j\right|/j^{1+\epsilon}\nearrow$ so as to have a good majorization of $\prod\left|1-u^2/\lambda_j^2\right|$ and in this case $\left|\prod(1+\dfrac{u^2}{|\lambda_j|^2})\right|<k\max(k_2u)^{2n}/|\lambda_1\cdots\lambda_n|^2$ (see (Kahane 1)). The condition $|\lambda_j|/j^{1+\epsilon}\nearrow$ is a condition of regularity.

Our condition of $I(\alpha)$-quasi-analyticity gives S. Mandelbrojt's theorem. Suppose $\sum\dfrac{1}{\lambda_n^\tau}<\infty, 0<\sigma<1$. Take $1_n=\lambda_n^{1-\sigma}$. Thus $\sum\limits_{1_n>n\frac{1-\sigma}{\sigma}}1/1^{\sigma(1-\sigma)}<\infty$. Take $\lambda_n\nearrow$ and so $1_n\nearrow, 1_n^{\frac{\sigma}{1-\sigma}}/n\to\infty$ and

$$\max_n\frac{r^n}{l_1\cdots l_n}<\max_n\frac{((\frac{\sigma}{r^{1-\sigma}})^n)^{1-\sigma)/\sigma}}{(n!)^{(1-\sigma)/\sigma}}<e^{\frac{(1-\sigma)}{r^o}r^{\sigma/1-\sigma}}$$

119 Taking $\alpha=1/r$ we have quasi-analyticity whenever $I(\alpha)<\alpha^2e^{\frac{1-\sigma}{\sigma}}$

$\alpha^{\sigma/1-\delta}$.

Again, we can derive a condition of D-quasi-analyticity for $C\{M_n\} \cap \mathscr{C}_\Lambda$. We have seen that $C_I\{M_n\} \cap \mathscr{C}_\Lambda$ is D-quasi-analytic with $I(\alpha) = \min_n \dfrac{\alpha^{n+1}}{(n+1)!} M_n$. Thus it is sufficient that $\min_n(\alpha^{2n} M_{2n}/(2n+1)!) < \min_n (l_1 \cdots l_n \alpha^n)^2$ for an infinity of $\alpha, \alpha \to 0$. This means that the reverse inequality does not hold for $\alpha > \alpha_o$. In other words, taking $\alpha = 1/r, \max_n(r^{2n}(M_{2n}/(2n+1)!)^{-1}) \leq \max_n r^{2n}(l_1 \cdots l_n)^{-2}$ does not hold for $r > r_o$. Now we use the following lemma.

Lemma. *Let $\{A_n\}$ and $\{B_n\}$ be two positive sequences with $B_{n+1}/B_n \nearrow$. Then*

$$\{B_n \leq A_n, (n > n_o)\} \Longleftrightarrow \{\max_n r^n/B_n \geq \max_n r^n/A_n\}(r > r_o).$$

For the proof Cf. (Mandelbrojt 3, p.7 and p. 18).

Taking $A_n = \dfrac{M_{2n}}{(2n+1)!}, B_n = (l_1 \cdots l_n)^2$ our condition becomes: $\dfrac{M_{2n}}{(2n+1)!} \geq (l_1 \cdots l_n)^2$ does not hold for $n > n_o$ (whatever we choose n_o). Replacing M_n by kM_n, we get:

Theorem. *We make the same assumptions about $\Lambda, \lambda_n, 1_n$ as in the above theorem; I is an arbitrary interval.*

$$\text{If } \liminf_{n\to\infty} \frac{M_{2n}}{(2n+1)!(1_1 \cdots 1_n)^2} > 0 \ \mathscr{C}_\Lambda \cap C_I\{M_n\}$$

is a D-quasi-analytic class.

Lecture 21

Applications of the Formula of Jensen and Carleman to quasi-analytic Classes D and $I(\alpha)$

1 Principle of the method

We give here an alternative method for obtaining conditions of quasi-analyticity. This method is applicable to bounded mean-periodic functions (for quasi-analyticity $I(\alpha)$ and for mean periodic functions belonging to a class $C\{M_n\}$ (for quasi-analyticity D). Moreover, it is easily seen that the method can be applied to those classes of bounded functions on the real line whose Carleman transforms (in the classical sense) are meromorphic, either on the whole plane or at the right (left) of the imaginary axis (for example, almost periodic functions whose spectrum have no finite point of accumulation on the line (resp. on a half line)). **120**

Following a method of B. Levin, we take $I(\alpha) = e^{-\alpha r(\alpha)}$ where $r(\alpha)$ is a decreasing function of α which is ∞ on $[0, \alpha_\infty]$ and continuous with values from ∞ to r_o on $(\alpha_\infty, \alpha_o], 0 \leq \alpha_\infty < \alpha_o$. We denote $l_g/\alpha(r)$ the inverse function, defined on $[r_o, \infty)$, such that $\alpha(\infty) = \alpha_\infty$.

Let now $\int_o^\alpha |f| \leq I(\alpha)$ $(0 < \alpha < \alpha_o)$; if $\alpha_\infty \neq 0$, this means $f = 0$ on

$[0, \alpha_\infty]$. Let $F(w)$ be the Carleman transform of f(Lect. 6, §3).

$$F(w) = \int\limits_{-\infty}^{o} f(x) e^{ixw} dx \text{ for } v > 0$$

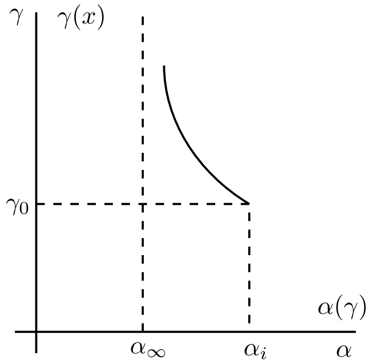
$$F(w) = -\int\limits_{o}^{\infty} f(x) e^{-xw} dx \quad \text{for } v < 0.$$

121 We have the following majorizations:

$$|F(w)| = |\int\limits_{o}^{\infty} f(x)\, e^{-ixw}\, dx| \leq |\int\limits_{o}^{\alpha}| + |\int\limits_{\alpha}^{\infty}|$$

$$\leq I(\alpha) + K\frac{e^{-\alpha|v|}}{|v|}\ v < 0$$

$$|\, F(w)\, | \leq \int\limits_{-\infty}^{o} |\, f(x)\, |\, e^{xv}|\, dx < \frac{K}{|V|},\ v > 0.$$



For a given w we can choose α as we want and we take $\alpha = \alpha(|v|)$, $|v| = r(\alpha)$. Thus the above majorization for $v < 0$ reduces to the following one:

$$|F(w)| < \left(1 + \frac{K}{|v|}\right) e^{-\alpha r(\alpha)} \leq \left(1 + \frac{K}{|v|}\right) e^{-|v|\alpha(|w|)}.$$

This relation permits us to have the following lemma:

Lemma 1. *The set of bounded functions in* $\mathscr{C}_\Lambda \neq \mathscr{C}$ *is an* $I(\alpha)$ *quasi-analytic class, with* $I(\alpha) = e^{-\alpha r(\alpha)}$ *as soon as the following relation is satisfied:*

$$\left.\begin{aligned} \log|F(w)| < -\log|v|, v > 0\ (w = u + iv = re^{i\theta}) \\ \log|F(w)| < -\log\frac{|v|}{|1+v|} - |v|\alpha(r), v < 0 \\ F(w)\ \text{ meromorphic with simple poles at } \Lambda. \end{aligned}\right\} \Rightarrow F \equiv 0$$

Let now $f \in C\{M_n\} \cap \mathscr{C}_\Lambda, \mathscr{C}_\Gamma \neq \mathscr{C}$ and $f^{(n)}(0) = 0$ for every n.

Then $F(w)$ satisfies the following relations:

$$F(w) = \int_{-\infty}^{o} f^{(n)}(x)\frac{e^{-ixw}}{(iw)^n}\,dx \quad v > 0$$

$$|F(w)| < \frac{K\,M_n}{|w|^n}\int_{-\infty}^{o} e^{vx}\,dx = \frac{K\,M_n}{|v|\,r^n} \quad v > 0$$

$$|F(w)| < \frac{K\,M_n}{|v|r^n} \qquad v < 0$$

We have seen in Lecture 7 (since f is bounded) that $F(w)$ is a meromorphic function with real simple poles at Λ and from the above inequalities we have $|F(w)| < \inf_n \dfrac{K\,M_n}{|v|\,r^n}$.

Thus we have the following Lemma: **122**

Lemma 2. *The class* $\mathscr{C}_\Lambda \cap C\{M_n\}, \mathscr{C}_\Lambda \neq \mathscr{C}$ *is a D-quasi-analytic class as soon as the following relation is satisfied:*

$$\left.\begin{aligned} &F(w) \textit{ if meromorphic with simple poles at } \Lambda \\ &\log|F(w)| < -\log|v| - S(r) \\ &\textit{with } S(r) = \max_n(n\log r - \log M_n) \end{aligned}\right\} \Rightarrow F \equiv 0$$

2 Application of Jensen's and Carleman's formulae

Application of Jensen's formula: Here we assume $\Lambda = \{\pm\lambda_n\}$. Definitions of $n(r), \overline{D}(r), \overline{D}\cdot$ are given in Lect. 10. §1.

We suppose that $F(w)$ satisfies the majorizations of Lemma 1 and that $F(w) \not\equiv 0$. First, Jensen's formula gives the following majorization: (Lecture 10.§2)

$$-2 \int_o^r \frac{n(t)}{t}\, dt \leq \frac{1}{2\pi} \int_o^{2\pi} \log |F(re^{i\theta})| \, d\theta - \log \,|\, F(0) \,|$$

Division of $F(w)$ by w^p will not alter the conditions of Lemma 1 and so we can take $F(0) \neq 0$. Using the majorizations of Lemma 1 in the above inequality, we have

$$-2\, r\, \overline{D}\,(r) < -\frac{1}{\pi}\, r\alpha(r) - \frac{1}{2}\, \log\, r + 0(1)$$

and so $\alpha(r) < 2\pi\overline{D}(r)$ for sufficiently large r.

This relation gives us the following theorem:

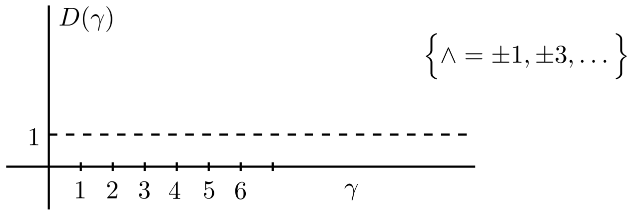
Theorem 1. *The set of bounded functions of $\mathscr{C}_\Lambda \neq \mathscr{C}$ is an $\underline{I}(\alpha)$ quasi-analytic class, with $I(\alpha) = e^{-\alpha r(\alpha)}$ as soon as $\alpha(r) \geq 2\pi\overline{D}(r)$ for an infinity of values of $r \to \infty$.*

123 **Remarks.** 1) If $\alpha(\infty) > 2\pi\overline{D}$, the above theorem gives that the set of bounded functions of $\mathscr{C}_\Lambda \neq \mathscr{C}$ is a quasi-analytic class I. $|I| = 1$, whenever $1 > 2\pi\overline{D}$.. This result is contained in Levinson's theorem (Lecture 19.§1).

2) If $\overline{D}(r) \leq \overline{D}.$ for an infinity of values of $r \to \infty$ it is sufficient to have $\alpha(\infty) = 2\pi\overline{D}.$ to apply the above theorem and thus, in this case, we get a precision of Levinson's theorem.

3) The above Remark (2) applies to the case of odd integers where we have $\overline{D}. = \dfrac{1}{2}$, then also the case of even integers. But if we add ± 1 to Λ the result ceases to apply. Indeed, there exists a function

$f(x) = \sin x + \sum\limits_{-\infty}^{\infty} C_n\, e^{2\sin x}$ which vanishes on $[0, \pi]$, $f(x) \not\equiv 0$. Adding $\{\pm 1\}$ to Λ, we add $\dfrac{\log r}{r}$ to $\bar{D}(r)$. Thus we cannot replace $\bar{D}(r)$ by $\bar{D}(r) + \dfrac{\log r}{r}$ in the above theorem. In this sense the above theorem is a precise one for sequences Λ whose distribution is nearly the same as that of integers.



Let now $F(w)$ satisfy the conditions of Lemma 2 and let us see what happens when we suppose $F(w) \not\equiv 0$. Applying Jensen's formula and using the conditions of Lemma 2, we obtain, as before, the following inequality:

$$-2\, r\, \bar{D}(r) < -S(r) - \log\, r + 0(1)$$

where $$S(r) = \max_n\, (n \log r - \log M_n).$$

As $$r\, \bar{D}(r) = N(r) \log\, r - \int\limits_o^r \frac{dN(t)}{t} = \log \frac{r^n}{\lambda_1 \cdots \lambda_n}$$

if $\lambda_n < r < \lambda n + 1$, the last inequality can be written **124**

$$\max_n \frac{r^{2n+1}}{\lambda_1^2 \cdots \lambda_n^2} > K\, \max \frac{r^{2n+1}}{M_{2n+1}} \quad (K > 0).$$

Using the lemma stated at the end of last lecture, the above inequality $M_{2n+1} > K|\lambda_1^2 \cdots \lambda_n^2|$ where K is a constant. Thus in order to have $F(w) \equiv 0$ it is sufficient to have the reverse inequality for an infinity of values of n, which gives the following theorem:

Theorem 2. *$\mathscr{C}_\Lambda \cap C\{M_n\}$ is a D -quasi-analytic class as soon as the following condition is satisfied:*

$$\liminf_{n\to\infty} \frac{M_{2n+1}}{\lambda_1^2 \cdots \lambda_n^2} = 0.$$

Remark. The above condition is simpler and more precise than the condition of D - quasi-analyticity obtained in the last lecture. However, this condition is applicable only when Λ is real and it involves $C\{M_n\}$ instead of $C_I\{M_n\}$. But the latter inconvenience can be suppressed whenever $\triangle = 0$, by the use of the theorem of continuation of Lecture 17.

Using a result proved in the next lecture, we can obtain that the above condition is also necessary for D-quasi-analyticity, when Λ is sufficiently lacunary.

Application of Carleman's formula: We assume now nothing about the negative part of Λ; the functions $n^+(r), D^+(r)$ are related to the positive part of Λ.

Suppose $F(w)$ satisfies the condition of Lemma 7 and let $F(w) \not\equiv 0$.

125 The application of Carleman's formula in the right half-plane gives

$$\int^r (1/t - t/r^2)\, dn^+(t) > \frac{1}{2\pi} \int^r (1/v^2 - 1/r^2) v\alpha(v)\, dv + 0(1).$$

Now, if $D^+(r)$ is bounded, integrating by parts, this reduces ∞:

$$\int^r \frac{2\pi\, D^+(t) - \alpha(t)}{t}\, dt > 0(1).$$

Thus we have the following theorem:

Theorem 3. *The class of bounded functions of $\mathscr{C}_\Lambda \neq \mathscr{C}$ is an $I(\alpha)$ quasi-analytic class, with $I(\alpha) = e^{-\alpha r(\alpha)}$ when $D^{+\cdot} < \infty$ and*

$$\limsup_{r\to\infty} \int^r \frac{\alpha(t) - 2\pi D^+(t)}{t}\, dt = \infty.$$

Suppose now that $F(w)$ satisfies the conditions of Lemma 2 and $F(w) \not\equiv 0$. Set $w' = ie^{-i\pi a} w^a$, with $\frac{1}{2} \leq a \leq 1$. We take $F_1(w') = F(w)$. $F_1(w')$ is meromorphic in the right half-plane having poles only on the line $\arg w' = \frac{\pi}{2} - \pi a$ and the distribution of these poles is $N_1(\rho) = N^+(\rho^{\frac{1}{a}})$. Moreover we have

$$\log |F_1(w')| < -\log |\rho^{\frac{1}{a}} \sin \frac{2\theta' - \pi}{2a}| - S(\beta^{\frac{1}{a}}).$$

$(w' = \rho\, e^{i\theta'})$. Carleman's formula applied to $F_1(w' + 1)$ in the right half-plane gives us the following inequality:

$$\limsup_{\rho \to \infty} \int^{\rho} (S(\tau^{\frac{1}{a}}) - \pi \sin \pi a N^+(\tau^{\frac{1}{a}}) \frac{d\tau}{\tau^2} < \infty$$

This relation gives us the following theorem:

Theorem 4. *$\mathscr{C}_\Lambda \cap C\{M_n\}$ is a D - quasi - analytic class when*

$$N^+(r) = 0(r^a) \text{ and } \limsup_{r \to \infty} \int^{r} \frac{S(t) - \pi \sin \pi a N^+(t)}{t^{1+a}} dt = \infty$$

with $S(r) = \max_n (n \log r - \log M_n)$, $\frac{1}{2} \leq a \leq 1$.

Remarks. 1) For $a = 1$, we get the condition of Denjoy-Carleman. **126**

2) If $\int^{\infty} \frac{N^+(t)}{t^{3/2}} dt < \infty$, $\mathscr{C}_\Lambda \cap C\{M_n\}$ is D-quasi-analytic if $C\{\sqrt{M_n}\}$ is D-quasi - analytic.

When $F(w)$ is not meromorphic in a right or left half-plane we cannot apply either Carleman's or Jensel's formula. Partial results in this direction can be got by applying a formula due to Mandelbrojt and MacLane (Kahane 1).

Lecture 22

Reciprocal theorems about quasi-analyticity D and $I(\alpha)$

In the Lectures 20 and 21, we gave sufficient conditions in order that $\mathscr{C}_\Lambda$ **127**
should be an $I(\alpha)$ quasi-analytic class, or $\mathscr{C}_\Lambda \cap C_I\{M_n\}$ resp $\mathscr{C}_\Lambda \cap C\{M_n\}$ a D-quasi-analytic class. We stated that if Λ is sufficiently lacunary, some of these conditions are necessary. In order to know whether our sufficiency conditions are good (i.e., whether it is not possible to relax them very much), and, if possible, to find necessary and sufficient conditions, we shall construct a function $f \in \mathscr{C}_\Lambda$ "as small as possible" near the origin. Actually, we want first to have $f^{(n)}(0) = 0 (n = 0, 1, \ldots)$ and $\sup_x\{|f^{(n)}(x)|\}$ increasing as slowly as possible.

We saw in the Lecture 21 that the smallness at infinity of $F(w)$, the Carleman transform of f, is related to the smallness of f near the origin. Therefore, it is natural to take $F(w)$ "as small as possible" at infinity. Suppose $\Lambda = \{\pm\lambda_n\}$ to be symmetric and real. Then the Carleman transform of every $f \in \mathscr{C}_\Lambda$ is the product of $F_o(w) = \prod(1 - \dfrac{w^2}{\lambda_n^2})^{-1}$ by an entire function. With convenient hypothesis on Λ we shall construct $f_o \in \mathscr{C}_\Lambda$, whose Carleman transform is $F_o(w)$. It is natural to expect that f_0 is the function we want.

We suppose $\Lambda = \{\pm\lambda_n\}$ to be symmetric, real and lacunary in the same that $\dfrac{\lambda n + 1}{\lambda n} > K > 1$. We denote by $\sum(\dfrac{A_k}{w - \lambda_k} - \dfrac{A_k}{+\lambda_k})$ the polar

128 part of $R_0(w) = \prod(1 - \frac{w^2}{\lambda_j^2})^{-1}$; then $A_k = -\frac{\lambda k}{2} \prod_{j \neq k}(1 - \frac{\lambda_k}{\lambda_j^2})^{-1}$. We define $f_0(x) = 2 \sum A_k \sin \lambda_k x$; indeed, if $\sum |A_k| < \infty$ (and the following calculation proves that it is realised), $F_o(w)$ is the Carleman transform of f_o. We now try to get a majorization for $|f_0^{(n)}(x)|$.

$$
\begin{aligned}
|f_o^{(n)}| &< 2 \sum_1^\infty |A_k| \lambda_k^n \\
|A_k| &= \frac{\lambda_k}{2} \frac{\lambda_1^2 \cdots \lambda_k^2}{\lambda_k^{2k}} \prod_{j=1}^{k-1} |\frac{\lambda_j^2}{\lambda_k^2} - 1|^{-1} | \prod_{j=k+1}^{\infty} \left(1 - |\frac{\lambda_k^2}{\lambda_j^2}\right)^{-1} \\
|A_k| \lambda_k^{2k} &< C \lambda_1^2 \cdots \lambda_{k-1}^2 | \lambda_k^3 \\
|f_o^{(2n-1)}| &< 2C \sum_{k=1}^{\infty} \lambda_1^2 \cdots \lambda_k^2 \, \lambda_k^{(2(n-k)} \\
&= 2C \, \lambda_1^2 \cdots \lambda_n^2 \left(\sum_1^{n-1} \frac{\lambda_k^{2(n-k)}}{\lambda_{k+1}^2 \cdots \lambda_n^2} \right) + 1 + \sum_{n+1}^{\infty} \frac{\lambda_{n+1}^2 \cdots | \lambda_k^2}{\lambda_k^{2(k-n)}} \\
&< C_1 \, \lambda_1^2 \cdots \lambda_n^2
\end{aligned}
$$

and $|f_0^{(2n)}| < C_1 \, \lambda_n^2 \cdots \lambda_1^2 \, \lambda_{n+1}$ by a similar calculation. We take $M_{2n-1} = \lambda_1^2 \cdots \lambda_n^2$ and $M_{2n} = \lambda_1^2 \cdots \lambda_n^2 \, \lambda_{n+1}$. Then we have $f_o \in C\{M_n\}$.

Moreover $f_0^{(n)}(0)(n = 0, 1, \ldots)$. For, if N were the first integer such that $f_0^{(N)}(0) \neq 0$, we would have, for $v > 0$.

$$
F_0(w) = \frac{f_o^{(N)}(0)}{(iw)^N} = \frac{f_0^{(N+1)}(0)}{(iw)^{N+1}} + \int_{-\infty}^{0} f_0^{(N+2)}(x) \, (ix)^{-N-1} e^{-ixw} dx
$$

and $\lim\limits_{v \to \infty} (-v)^N F(-iv) = f^{(N)}(0) \neq 0$; since $\prod(1 + \frac{v^2}{\lambda_k^2})$ increases more rapidly than any polynomial, this is impossible.

In Lecture 21, we found that $\liminf\limits_{n \to \infty} \dfrac{M_{2n}}{\lambda_1^2 \cdots \lambda_n^2 \lambda_{n+1}} = 0$ is a sufficient condition for the D-quasi-analyticity of $\mathscr{C}_\Lambda \cap C\{M_n\}$. The properties of f gives us the following result:

Theorem 1. *If $\Lambda = \{\pm\lambda_n\}$ is real, symmetric and lacunary in the sense that $\dfrac{\lambda_{n+1}}{\lambda_n} > K > 1$, necessary and sufficient condition in order that $\mathscr{C}_\Lambda \cap C\{M_n\}$ should be D-quasi-analytic is that either* 129

1) $\liminf\limits_{n\to\infty} \dfrac{M_{2n}}{\lambda_1^2 \cdots \lambda_n^2 \lambda_{n+1}} = 0$, *or*

2) $\mathscr{C}_\Lambda \cap C\{M_n\}$ *does contain the function f_o whose Carleman transform is* $F_o(w) = \prod(1 - \dfrac{w^2}{\lambda_n^2})^{-1}$.

Using Taylor's formula as in Lecture 20, § 1, we have

$$\int\limits_o^\alpha |\ f_o\ | < C_1 \int\limits_o^\alpha \frac{x^{2n-1}}{(2n-1)!}\lambda_1^2 \cdots \lambda_n^2 \, dx = C_1 \frac{\alpha^{2n}}{(2n)!} \lambda_1^2 \cdots \lambda_n^2$$

Hence

$$\int\limits_o^\alpha |f_o| < C_1 \min_n \left(\frac{\alpha}{(2n)!} \lambda_1^2 \cdots \lambda_n^2\right) \tag{1}$$

We saw (Lect. 20) that $\mathscr{C}_\Lambda$ is an $I(\alpha)$ quasi-analytic class if $\sum \dfrac{l_n}{\lambda} < \infty$ and

$$\liminf_{\alpha\to\infty} \frac{I(\alpha)}{\alpha \min(l_1 \cdots l_n \alpha^n)^2} < \infty \tag{2}$$

(1) shows that this condition cannot be very much relaxed; for example, we cannot replace $\sum \dfrac{l_n}{\lambda} < \infty$ by $l_n = \dfrac{\lambda_{n+1}}{2n}$.

If λ_{n+1}/λ_n is bounded, the condition is invariant by a change of $\{l_n\}$ into $\{l_{n+p}\}$, and also into $\{l_{n+p}\}$; then, if it holds with $I(\alpha)$ it still holds with $\alpha^{-2p}\ I(\alpha)$. That is no longer true if we assume

$$\sum \frac{\lambda_n}{\lambda_{n+1}} < \infty \tag{3}$$

For, we can take in (2) $l_n = l_{n-1}$; but, according to (1), we cannot $l_n = \lambda_{n+1}$. This remark will lead us tu show that f_0 is "the smallest" function $\in \mathscr{C}_\Lambda$ near the origin, in the sense that

$$f \in \mathscr{C}_\Lambda, \liminf_{\alpha \to 0} \int\limits_o^\alpha |f| \Big/ \int\limits_o^\alpha |f_o| < \infty \tag{4}$$

130 implies $f = K\ f_o, K$ constant.

We suppose (3) and (4). Since Λ is a sequence of Banach - Szidon, f is bounded. Let $F(w)$ be the Carleman transform of f; we have $|F((u+iv)| < \dfrac{K}{|v|}$ (Lect. 7); hence

$$|F(w)\ (w - \lambda_n)\ (w - \lambda_{n+1})| < K(\lambda_{n+1} - \lambda_n)$$

on the circle of diameter $(\lambda_n, \lambda_{n+1})$, and $F(w)$ is uniformly bounded outside the discs of radius 1 around the $\pm\lambda_n$. Suppose, f/f_o is not a constant. Then $A(w) = F(w)/F_o(w)$ is an entire function which is not a constant. Since F_o^{-1} is of exponential type zero, and F is bounded outside our discs, $A(w)$ is also of exponential type zero; therefore, it has at least one zero w_1. Let $F(w) = (w - w_1)\ F_1(w)$. Without restriction, we can suppose $f(0) = 0$; then the differential equation $f = if_1' + w_1 f_1$ has one solution such that $f_1'(0) = f_1(0) = 0$ and, using the formulae of Lecture 6, § 2, we see that $F_1(w)$ is the Carleman transform of f_1. Now, using the trivial inequality $\int\limits_o^\alpha |f_1| < \alpha \int\limits_o^\alpha |f_1'|$, we get, for α small enough,

$$\int\limits_o^\alpha |f| > \int\limits_o^\alpha |f_1'| - |w_1| \int\limits_o^\alpha |f_1| > \frac{1}{2} \int\limits_o^\alpha |f_1'| > \frac{1}{2\alpha} \int\limits_o^\alpha |f_1|$$

This inequality, together with (4), shows that f_1/f_o is not a consistent. We can iterate our argument and get $f_2 \in \mathscr{C}_\Lambda$, $f_3 \in \mathscr{C}_\Lambda$, $f_3 \not\equiv 0$

$$\int\limits_o^\alpha |f| > \frac{1}{2\alpha} \int\limits_o^\alpha |f_1| > \frac{1}{4\alpha^2} \int\limits_o^\alpha |f_2| > \frac{1}{8\alpha^3} \int\limits_o^\alpha |f_3|$$

Taking into account (1) and (4), we get

$$\liminf_{\alpha \to o} \Big(\int\limits_o^\alpha \Big|f_3\Big| \Big/ \min_n \Big(\frac{\alpha^{2n-3}}{(2n)!}\ \lambda_1^2 \cdots \lambda_n^2 \Big) \Big) = 0,$$

which is in contradiction with the fact that $\mathscr{C}_\Lambda$ is an $I(\alpha)$ quasi-analytic class whenever $\liminf_{\alpha\to o} \dfrac{I(\alpha)}{\alpha \min(\lambda_1 \cdots \lambda_n \alpha^{n+1})^2} = 0$. We conclude that f/f_o must be a constant. We express the result in the following way. **131**

Theorem 2. *If $\Lambda = \{\pm\lambda_n\}$ is real, symmetric and very lacunary in the sense that $\sum \dfrac{\lambda_n}{\lambda_{n+1}} < \infty$, a necessary and sufficient condition in order that $\mathscr{C}_\Lambda$ should be an $I(\alpha)$ quasi-analytic class is*

$$\liminf_{\alpha\to\infty} \left(I(\alpha) \Big/ \int_o^\alpha |f_o| \right) = 0,\ f_o$$

being defined in Theorem 1. Every time $f \in \mathscr{C}_\Lambda$ and

$$\liminf_{\alpha\to o} \Big(\int_o^\alpha |f| / \int_o^\infty |f_o| \Big) < \infty,\ f/f_o$$

is a constant.

Remark. It is easy to extend this result. Indeed, if $f \in \mathscr{C}_\Lambda$ and $\liminf_{\alpha\to o} \left(\alpha^p \int_o^\alpha |f| \Big/ \int_o^\alpha |f_o|\right) < \infty$, f is a linear combination (with constant coefficients) of $f_o, f_o^1, \ldots, f_o^{(p)}$. It means not only that f_o is "the smallest" function $\in \mathscr{C}_\Lambda$ near the origin, but also the linear combinations of f_o and its derivatives are "smaller" than any other $f \in \mathscr{C}_\Lambda$.

Lecture 23

Mean Periodic Functions of Several Variables

In the case of several variables instead of considering continuous function, we consider C^{∞}-functions. It would also be possible to consider distributions. 132

We consider the space $\mathscr{E} = \mathscr{E}(R^n)$, $f \in \mathscr{E}$, $f(x) = f(x_1, \ldots, x_n)$, $x = (x_1, \ldots, x_n) \in R^n$. The monomial exponentials are of the form $x_1^{i_1} \cdots x_n^{i_n}$ $e^{i(\lambda_1 x_1 + \cdots + \lambda_n x_n)}$ and the polynomial exponentials are of the form $P(x_1, \ldots, x_n)$ $e^{i(\lambda_1 x_1 + \cdots + \lambda_n x_n)}$.

A function f is mean periodic if $\tau(f) \neq \mathscr{E}(R^n)$. An equivalent definition is that f is mean periodic if there exists a $\mu \in \mathscr{E}'$, $\mu \neq 0$, $\mu * f = 0$.

Problem 1. If f is mean periodic, can we assert that f belongs to the span of monomial exponentials belonging to $\tau(f)$?

The answer is in the negative (Ehrenpreis 1). In R^2, let $f = x_1 + x_2$. The monomial exponentials in $\tau(f)$ are constants.

Problem 2. Let $f \in \tau \subset \mathscr{E}, \tau$ a closed subspace invariant under translations. Is $f \in$ span of polynomial exponentials $\in \tau$?

Solution is not known.

Problem 3. Suppose $f * \mu = 0, \mu \in \mathscr{E}'(R_n)$. Is $f \in$ span of polynomial exponentials Q with $Q * \mu = 0$?

This is answered in the affirmative (Malgrange 1).

Let $V(\mu)$ denote the set of linear combinations of polynomial exponentials verifying the equation $Q * \mu = 0$.

133 Let $\check{\nu}$ be the distribution symmetric to the distribution ν. If $\check{\nu} * Q = 0$ for every polynomial exponential Q verifying the equation $Q * \mu = 0$ and $f * \mu = 0$, then is $\check{\nu} * f = 0$? This problem is put in the following form:

$$\{(\check{\nu} \in (V(\mu)^{\perp}), f * \mu = 0\} \Rightarrow \nu * f = 0.$$

Theorem 1. $f * \mu = 0 \Rightarrow f \in \overline{V(\mu)}$. *This theorem follows from parts* a) *and* e) *of the following theorem (Malgrange* 1*).*

Theorem 2. *The following conditions are equivalent:*

a) $\check{\nu} \in [V(\mu)]^{\perp}$

b) Every exponential polynomial Q satisfying the equation $Q * \mu = 0$ also satisfies the equation $Q * \nu = 0$. In other words, $V(\mu) \subset V(\nu)$.

c) $\mathscr{C}(\mu)/\mathscr{C}(\nu)$ is an entire function.

d) $\mathscr{C}(\mu)/\mathscr{C}(\nu)$ is an entire function of exponential type.

e) $\nu \in \mu * \mathscr{E}' =$ closed ideal generated by μ.

We recall that $F(w)$ is of exponential type if $|F(w)| < A\, e^{B|w|}, |w| = |w_1| + \cdots + |w_n|$. We prove the theorem in four steps, inserting the lemmas and propositions which we require in the four steps, inserting the lemmas and propositions which we require in the proof of each step.

1. $a) \iff b)$. Indeed $\check{\nu} \in [V(\mu)]^{\perp} \Rightarrow \check{\nu}.Q = 0$ for every exponential polynomial Q satisfying $Q * \mu = 0$, and so $\check{\nu} \in [V(\mu)]^{\perp} \iff \check{\nu} * \tau_a Q = 0$ for every translate $\tau_a Q$ of Q.

   In other words, $\check{\nu} \in [V(\mu)]^{\perp} \iff Q * \nu = 0$ for every exponential polynomial Q satisfying $Q * \mu = 0$.

2. $b \iff c$). Let $\mathscr{C}(\mu) = M(w); w = (w_1, \ldots, w_n)$. According to Schwartz's theory of the Fourier transforms of distributions, b) is equivalent to the relation: $\{\mathscr{C}(\mu)\mathscr{C}(Q) = 0 \Rightarrow \mathscr{C}(\nu)\mathscr{C}(Q) = 0\}$. So $c) \to b)$.

Now $\mathscr{C}(Q)$ is a distribution with support at λ and we denote it by **134**
D_λ. b) gives us that $m.D_\lambda = 0 \Rightarrow N.D_\lambda = 0$, where $M.D_\lambda$ is the product of the distributions M and D_λ and so also $N.D_\lambda$. To prove that $b) \Rightarrow c)$ we use the following lemma:

Lemma. *Let M and N be analytic at the origin. In order that N/M be analytic at the origin it is necessary and sufficient that for every distribution D with support at origin $M.D = 0 \Rightarrow N.D = 0$.*

Necessity is obvious. To prove the sufficiency, consider the ring $\mathscr{A}$ of formal power-series $\sum a_{i_1,\cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$. It is a topological ring with the simple convergence of the coefficients. It is locally convex and its dual $\mathscr{A}' = \mathscr{E}'_o$ is the space of distributions with support at the origin. The scalar product is given by

$$\langle D, \sum \cdots \rangle = \sum a_{i_1 \cdots i_n} \langle D, x_1^i \cdots x_n^{i_n} \rangle.$$

We use the following result proved in (Cartan X, p.7).

$$\frac{N}{M} \text{ is analytic at } 0 \iff \frac{N}{M} \in \mathscr{A}.$$

Thus we have to prove that $N \in M\mathscr{A}$, the ideal generated by M. We use the fact that this ideal is closed (Cartan XI, p.7). Then it is sufficient to prove that $D \in (M\mathscr{A})^\perp \Rightarrow \langle D, N \rangle = 0$. But $\langle D, MK \rangle = \langle M.D, K \rangle$, so that $D \in (M\mathscr{A}) \iff M.D = 0$. As we assume $M.D = 0 \Rightarrow N.D = 0$, and $\langle D, N \rangle = \langle N.D, 1 \rangle$, we have proved $N \in M\mathscr{A}$, then N/M is analytic at the origin.

Since b) gives that $M.D_\lambda = 0 \Rightarrow N.D_\Lambda = 0$ we have, by the above lemma, N/M analytic at every point λ.

3. $c) \iff d$) is a consequence of the following theorem: (Malgrange) (Ehrenpreis 1).

Theorem of Lindelof-Malgrange-Ehrenpreis: If F and G are func- **135** *tions of exponential type and F/G is entire, then F/G is also a function of exponential type*.

In this proof, L. Ehrenpreis used a theorem on minimum modules. On the other hand, Malgrange's proof is directly inspired from that of Lindelof.

We have to return to analytic functions of one variable.

Suppose $F(w) = w^P e^{aw} \prod_1^\infty (1 - w/\lambda_j) e^{w/\lambda_j}$. This form of $F(w)$ need not imply that $F(w)$ is of exponential type. For example, the inverse of the classical Γ- function is not of exponential type. Lindelof gave a characterization of entire functions of exponential type. Let the zeros of $F(w)$ be arranged in such a fashion that $|\lambda_{j+1}| \geq |\lambda_j|$. Then $F(w)$ is of exponential type if and only if $\dfrac{n}{|\lambda_n|} = 0(1)$ and $\sum\limits_{|\lambda_i|<|K} 1/\lambda_i = 0(1)$ for all K. Using this characterization, one proves in the case of one variable that $F(w)/G(w)$ is of exponential type. Malgrange gave a refinement of this characterization.

Proposition 1. *Let $F(w)$ be an entire function with $F(0) \neq 0$ and $|F(w)| < A\, e^{B|w|}$ and let λ_n be the zeros of $F(w)$. Then*

$$a)\ \frac{n}{\lambda_n|} < \frac{C}{|F(0)|},\ b)\ |a + \sum_1^n \frac{1}{\lambda_j}| < \frac{D}{|F(0)|^2};$$
$$C = C(A, B), D = D(A, B).$$

Conversely, if $\dfrac{n}{\lambda_n} < e$ and $|a + \sum\limits_1^p \dfrac{1}{\lambda_j}| < D$ for every n and p, then

$$F(w) = e^{aw} \prod_1^\infty \left(1 - \frac{w}{\lambda_j}\right) e^{w/\lambda_j}$$

is an entire function and

$$|F(w)| = |F(w)/F(0)| < A\, e^{B|w|}, A = A(c, D) \text{ and } B = B(C, D).$$

136 For the first part of the proposition, a) follows easily from Jensen's formula: b) is more involved. The second part follows from some calculations (Malgrange).

Proposition 2. *Let* $|F(w)| < A\ e^{B|w|}$ *with* $F(0) = 1$ *and let* $|G(w)| < A'\ e^{B'|w|}$ *with* $G(0) = 1$. *Suppose* $H(w) = F(w)/G(w)$ *is an entire function. Then* $|H(w)| < A''e^{B''|w|}$ *with* A'' *and* B'' *depending only on* A, A', B *and* B'.

Proposition 2 is an immediate consequence of Proposition 1.

Now the proof of Lindelof theorem for several variables follows easily from Proposition 2.

Suppose $|F(w_1 \cdots w_n)| < A\ e^{B(|w_1|+\cdots|w_n|)}$,

$$|G(w_1, \ldots, w_n)| < A'\ e^{B'(|w_1|+\cdots+|w_n|)} \text{ and } H(w) = F(w)/G(w)$$

is an entire function. We fix $w_1 \cdots w_n$ such that $|w| = |w_1| + \cdots + |w_n| = 1$ and take $F_w(\delta) = F(\delta w_1, \ldots, \delta w_n), G_w(\delta) = G(\delta w_1, \ldots, \delta w_n)$. Then by Proposition 2, $|H_w(\delta)| < A''e^{B''|\delta|}$.

Hence, for any $w = (w_1, \ldots, w_n)$, we have $|H(w)| < A''e^{B''|w|}$ (because $H(w) = H_{w/|w|}(|w|)$).

Lecture 24

Mean Periodic Functions of Several Variables (Continuation)

We complete the proof of the main theorem of the last lecture. $4, d) \iff e$. 137

d) $\mathscr{C}(\nu)/\mathscr{C}(\mu)$ is an entire function of exponential type.

e) $\nu \in \overline{\mu * \mathscr{E}'}$, the closed ideal generated by μ in $\mathscr{E}'$.

Now $e) \iff c)$ is evident and since $c) \iff d), e) \Rightarrow d)$. The proof of $d) \Rightarrow e)$ is involved.

Let $\mathbb{Q}\mathscr{Y}'$ be the space of Fourier-transforms of distributions in $\mathscr{E}'$. Let $M, N \in \mathbb{Q}\mathscr{Y}'$ and suppose N/M is of exponential type. Moreover suppose it is possible to show that $\dfrac{\partial M}{\partial w_1} N \in \mathscr{C}(\overline{\mu * \mathscr{E}'})$. Then we can prove that $d) \Rightarrow e)$, because we can iterate the same process with each variable and get $\dfrac{\partial M}{\partial w_2}\dfrac{\partial M}{\partial w_1} N \in \mathscr{C}(\overline{\mu * \mathscr{E}'})$ etc., and finally $(P(x)\mu) * \nu \in \overline{\mu * \mathscr{E}'}$, for every polynomial $P(x)$. Using the result that if $\mu \neq 0$ it is possible to find a sequence $\{P(x)\}$ and $\lambda \in R^n$ such that $P(x)\mu \to \delta_\lambda$ we find that $\delta_\lambda * \nu \in \overline{\mu * \mathscr{E}'}$, i.e. $\nu \in \overline{\mu * \mathscr{E}'}$. Now we can write $\dfrac{\partial M}{\partial w_1} N = \dfrac{1}{M}\dfrac{\partial M}{\partial w_1} NM$. To prove that $\dfrac{\partial M}{\partial w_1} N \in \mathscr{C}(\overline{\mu * \mathscr{E}'})$ it is sufficient to prove the following conjecture (true in the case of one variable).

Conjecture. $\dfrac{1}{M}\dfrac{\partial M}{\partial w_1} NM \in \mathbb{Q}\mathscr{Y}'$, when $M.N \in \mathbb{Q}\mathscr{Y}'$ and $\dfrac{N}{M}$ is entire. Unfortunately we are not able to prove this conjecture. But to prove that $d) \Rightarrow e)$ it is sufficient (as indicated by Malgrange) to prove only the following proposition:

138 **Proposition 1.** *$P, R \in \mathbb{Q}\mathscr{Y}'$ and P/Q entire implies that* $\dfrac{1}{Q}\dfrac{\partial Q}{\partial w_1}\ P\ Q^2$ $(0, w_2, \ldots, w_n) \in \mathbb{Q}\mathscr{Y}'$.

The idea of the proof is to majorise $\dfrac{1}{Q}\dfrac{\partial Q}{\partial w_1}$ in the same way as $\dfrac{M'(w)}{M(w)}$ in Lecture 5, §1, and to use Proposition 1 of the last lecture (B.Malgrange).

Definition. $H(\mu)$ is the set of distribution τ such that $\tau * \nu \in \mu * \mathscr{E}'$, for every ν with $\mathscr{C}(\nu)/\mathscr{C}(\mu)$ entire.

Theorem. $H(\mu)$ is dense in $\mathscr{E}'$.

Suppose this theorem is proved. Allowing $\tau \to \delta$ we have $\nu = \lim\limits_{\substack{\tau \in H(\mu) \\ \tau \to \delta}} \tau * \nu$ and $\nu \in \mu * \mathscr{E}'$. Thus we have $d) \Rightarrow e)$.

Definition. Let $\sigma \in \mathscr{E}'$. $H(\mu, \sigma)$ is the set of distributions such that $\tau * \sigma \in H(\mu)$.

We say that $H(\mu, \sigma) \in (\delta_p)$(i.e. has the property δ_p) if there exists a set of distributions $\sigma_1, \ldots, \sigma_r$ such that $\sigma_1 *, \ldots * \sigma_r \in H(\mu, \sigma)$ with $\mathscr{C}(\sigma_j)$ depending only on p variables.

Proposition 2. *Let $0 \leq p \leq n$. If $H(\mu, \sigma) \in (\delta_p)$, then $H(\mu, \sigma)$is dense in $\mathscr{E}'$.*

This will prove our theorem, for $H(\mu) = H(\mu, \delta) \in (\delta_n)$. We prove Proposition 2 by induction. For this we need the following proposition:

Proposition 3. *Let $\tau \in \mathscr{E}'$ be such that $\mathscr{C}(\tau)$ depends only on $p+1$ variables. Suppose $\mathscr{C}(\tau) = T = T(w_1, \ldots, w_{p+1})$. If*

$$H(\mu, \sigma * \tau) \in (\delta_p), \text{ then } H(\mu, \sigma * x_1 \tau) \in (\delta_p).$$

Proof. $(\sigma_1 \ * \cdots * \ \sigma_r \ * \tau * \sigma * \nu) \in \mu * \mathscr{E}'$ gives, by taking the Fourier transform that $(S_1 \cdots S_r \ T \ S \ N/M) \in \mathbb{Q}\mathscr{Y}'$. We take $Q = T$ and $P = T \ S_1 \cdots S_r S \, N/M$ in Proposition 1, and get **139**

$$\frac{\partial T}{\partial w_1} \ T^2(0, w, \cdots, w) S_1 \cdots S_r \ S \ \frac{N}{M} \in \mathbb{Q}\mathscr{Y}'.$$

Now $T^2(0, w_2, \ldots, w_r)$ is the Fourier transform of a distribution and depends only on p variables. So $H(\mu, \sigma * x_1 \tau) \in (\delta_p)$. □

Proof of Proposition 2. $H(\mu, \sigma) \in (\delta_o)$ means that $\delta \in H(\mu, \sigma)$ and so, since $H(\mu, \sigma)$ is an ideal in $\mathscr{E}'$, it is dense in $\mathscr{E}'$. Suppose that the proposition is true for p. Let $H(\mu, \sigma) \in (\delta_{p+1})$. Then there exist $\mathscr{C}(\sigma_j)$ depending on $(p+1)$ variables such that $\sigma_1 * \cdots * \sigma_r * \sigma_r * \nu \in \mu * \mathscr{E}'$, for every ν satisfying the condition $\tau(\nu)/\mathscr{C}(\mu)$ entire. Therefore $H(\mu, \sigma_1 * \cdots * \sigma_r * \sigma) = \mathscr{E}'$ and so $\in (\delta_p)$. Applying successively Proposition 3, we get that $H(\mu, P_1 \sigma_1 * \cdots * P_r \sigma_r * \sigma) \in (\delta_p)$, whatever be the polynomials P_r depending only on those x_j for which the w_j occurs in $\mathscr{C}(\sigma_r)$. According to the hypothesis, $H(\mu, P_1 \sigma_1 * \ldots * P_r \sigma_r * \sigma)$ is dense in $\mathscr{E}'$; this means that $P_1 \sigma_1 * \cdots * P_r \sigma_r \in \overline{H(\mu, \sigma)}$. It is possible to choose $P_1, \cdots, P_r$ in such a manner that $P_j \sigma_r \to \delta_{\lambda_j}$, $j = 1, \ldots, r$. Therefore

$$\delta_{\lambda_1} * \cdots * \delta_{\lambda_r} \in \overline{H(\mu, \sigma)} \text{ and so } \overline{H(\mu, \sigma)} = \mathscr{E}'.$$

That achieves the proof of the theorem.

We give a few complements when μ is a partial differential operator with constant coefficients. Solutions of $\mu * f = 0$ are the solutions of the homogeneous equation $D * f = 0$. Consider $\mathscr{E}(\Omega), \Omega$ an set of R^n.

Theorem. *Let Ω be an open convex set in R^n and let $D * f = 0$. Then $f \in$ span in $\mathscr{E}(\Omega)$ of the polynomial exponential Q satisfying the equation $D * Q = 0$.* **140**

Indeed, $\mathscr{C}(D)$ is a polynomial $P(w_1, \cdots, w_n)$ and $\mathscr{C}(\nu) = N(w_1, \ldots, w_n)$. We can suppose (perhaps after a change of variables) $P(w_1, \ldots, w_n) = w_1^m + A_1 w^{m-1} + \cdots + A_m, A_1, A_2, \ldots, A_m$ being functions of $w_2, \ldots,$

w_n. Then, by Cartan's lemma (Lect. 14, Lemma 2)

$$\left|\frac{N}{P}(w_1, \ldots, w_n)\right| < \sup_{|\zeta| \leq 2c} \left|N(w_1 + \zeta, w_2, \ldots, w_n)\right|$$

so that $\mathscr{C}(\nu)\Big/\mathscr{C}(D) \in \mathbb{Q}\mathscr{Y}'$ and hence $\nu = D * \mu, \mu \in \mathscr{E}'$. Moreover, the support of μ is in the convex closure of the support of ν, which is in Ω. Then $\nu * f = \mu * D * f = 0$ and $\langle \nu, f \rangle = 0$.

Problem 1. Is it possible to replace Ω convex by Ω connected or simply connected?

Theorem. $\{\Omega$ *convex, open and* $D * f = 0\} \Rightarrow f$ $\in$span of polynomials P satisfying $D * P = 0$ is valid if and only if the irreducible factors of $\mathscr{C}(D)$ are all zero at 0.

For example, when $D = \triangle$, the Laplacian, every harmonic function is the limit of polynomials. We indicate the idea of the Proof (Malgrange). From the fact that $\mathscr{C}(\nu)\Big/\mathscr{C}(D)$ is holomorphic at 0, we will have $\mathscr{C}(\nu)\Big/\mathscr{C}(D)$ entire. Indeed if $R = \mathscr{C}(D) = R_1 \ldots R_r$ and $V_j = \{w | R_j(w) = 0\}$, the polar manifold of $\mathscr{C}(V)/\mathscr{C}(D)$ is the union of V_j. If $0 \notin \cup V_j$ and if $\mathscr{C}(\nu)/\mathscr{C}(D)$ is holomorphic at 0, the polar manifold is void.

141 We conclude with the consideration of analytic mean periodic functions. Instead of $\mathscr{E}(R^n)$ we take $\mathscr{H}(C^n)$ with the topology of compact convergence in R^{2n}. The dual $\mathscr{H}'$ is a quotient space of the space of measures.

Let $\mathscr{C}(\mu) = \langle \mu, e^{w_1 z_1 + \cdots + w_n z_n} \rangle, \mu \in \mathscr{H}'$. $\mathscr{C}(\mu)$ is of exponential type.

Theorem (Malgrange - Ehrenpreis). *If* $M(w) = M(w_1 \cdots w_n)$ *is an entire function of exponential type, then* $M(w) = \mathscr{C}(\mu), \mu \in \mathscr{H}'$.

Proof. Let $M(w) = \sum a_{i_1, \ldots, i_n} w_1^{i_1} \cdots w_n^{i_n}$. $|M(w)| < A\ e^{B|w|}$ with $|w| = |w_1| + \cdots + |w_n|$ gives the following majorization:

$$|a_{i_1 \cdots i_n}| < \text{Const}\,.\frac{B^{i_1 + \cdots + i_n}}{i_1! \cdots i_n!}.$$

□

Consider the linear form $\langle D, f\rangle$, $f \in \mathscr{H}$ defined by

$$\langle D, f\rangle = a_{i_1\cdots i_n}\frac{\partial^{i_1+\cdots+i_n}}{\partial Z_1^{i_1}\cdots\partial z_n^{i_n}}\ f(0).$$

It is a continuous linear functional and so $\in \mathscr{H}'$. Moreover,

$$M(w) = \langle D, e^{w_1 z_1+\cdots+w_n z_n}\rangle.$$

Now it is possible to extend D to a measure such that $M(w) = \mathscr{C}(\mu)$. Then it is easy to prove (Malgrange):

Theorem. *Let $f \in \mathscr{H}(C^n), \mu \in \mathscr{H}'(C^n)$. If $f * \mu = 0$, then f belongs to the span of polynomial exponentials satisfying the same equation.*

These considerations suggest the following problem. Consider $f * \mu = 0$ as a class of partial differential equations of infinite order. Then

$$\mu * f = \sum a_{i_1\cdots i_n}\frac{\partial^{i_1+\cdots+i_n}}{\partial z_1^{i_1}\cdots\partial z_n^{i_n}}\ f(z) = 0$$

gives a homogeneous partial differential equation. This cannot be de- **142**
fined for every $z \in \Omega$ when $f \in \mathscr{H}(\Omega)$. The only case when $\mu * f$ is defined in the same open set as f is that when the Fourier transform of the operator is of type of zero. In that case, we say that μ is of minimal type.

Problem 2. When μ is of minimal type, $f \in \mathscr{H}(\Omega)$ and $\mu * f = 0$, will this imply that $f \in$ span of exponential polynomials satisfying $\mu * Q = 0$ in its domain of existence?

In the case of one variable we know, by the method of A.F.Leontiev, that the domain of existence is convex and $f \in \mathscr{H}_\Lambda(\Omega)$.

APPENDIX I
On the maximum density of Polya

143 Let Λ be a positive sequence and let $n(r) = \sum\limits_{\lambda<r} 1, \lambda \in \Lambda$.

Definition. $D_{\max}\Lambda = \inf_{\Lambda'\supset\Lambda}\{D'(\Lambda')\}, \Lambda'$ being a sequence having density D'.

We have the following relation:

$$D' \geq D_{\max} \Longleftrightarrow \{n'(R) - n'(r) \geq n(R) - n(r), R > r, \lim_{R\to\infty} \frac{n'(R)}{R} = D'\}.$$

1. An inequality for $D_{\max}$.

For $k > 1$, $\dfrac{n'(kr) - n'(r)}{kr - r} \geq \dfrac{n(kr) - n(r)}{kr - r}$. As the first member $\to D'$ when $r \to \infty$, we have for every $k > 1, D_{\max} \geq \limsup\limits_{r\to\infty} \dfrac{n(kr) - n(r)}{kr - r}$. Therefore we have following inequality:

$$D_{\max} \geq \limsup_{k\searrow 1} \limsup_{r\to\infty} \frac{n(kr) - n(r)}{kr - r} = \limsup_{k\searrow 1} \varphi(k) \qquad (*)$$

where $\varphi(k) = \limsup\limits_{r\to\infty} \dfrac{n(kr) - n(r)}{kr - r}$.

2. Study of $\varphi(k)$

Using the fact that $p + p'/q + q'$ lies between p/q and p'/q' we have

$$\frac{n(kr) - n(r)}{kr - r} \leq \sup\left\{\frac{n(\sqrt{kr}) - n(r)}{\sqrt{kr} - r}, \frac{n(kr) - n(\sqrt{kr})}{kr - \sqrt{kr}}\right\} \text{ and so } \varphi(k) \leq \varphi(\sqrt{k}).$$

Therefore we take a $k > 1$ and consider the quantity $\triangle$ defined by

$$\triangle = \lim_{p\to\infty} \varphi(k_p) \text{ where } k_p = k^{2^{-p}}. \qquad (**)$$

$\triangle$ exists because $\varphi(k_p)$ is monotone when $p \to \infty$.

3 Construction of $\Lambda^* \supset \Lambda, \Lambda^*$ having density $\triangle$.

Suppose $\in_1, \in_2, \ldots, \in_q, \ldots$ is a sequence of positive numbers $\to 0$. In what follows we use the following notation: $k^i_q = k^{i2^{-q}}$ and $k_q =$
144 $k^1_q = k^{2^{-q}}$. Then $k^{i+1}_q = k_q . k^i_q$. Using the definition of $\varphi(k)$ we are able to determine a sequence of even integers $\{i_q\}$ with $i_{q+1} > 2i_q$ in such a manner that the following conditions are satisfied:

$$\text{for} \quad i \geq i_1, \frac{n(k^{i+1}) - n(k^i)}{k^{i+1} - k^i} < \varphi(k) + \in_1, (k^{i+1} - k^i)^{-1} < \in_1 \tag{a}$$

$$\text{for} \quad i \geq i_q, \frac{n(k^{i+1}_q) - n(k^i_q)}{k^{i+1}_q - k^i_q} < \varphi(k_q) + \in_q, (k^{i+1}_q - k^i_q)^{-1} < \in_q \tag{b}$$

The set of segments (k^i_q, k^{i+1}_q) for $i = i_q, \ldots, (\frac{1}{2} i_{q+1} - 1), q = 1, 2, \ldots$ are disjoint and cover the semi-axis $[k^{i_1}, \infty]$.

In order to determine Λ^* we define $n * (r)$ taking only integral values and satisfying the following conditions:

$$n * (R) - n * (r) \geq n(R) - n(r), R \geq r \tag{1}$$

$$\frac{n * (k^{i+1}_q) - n * (k^i_q) + 0(1)}{k^{i+1}_q - k^i_q} = \varphi(k_q) + \in_q, 0 \leq 0(1) \leq 1 \tag{2}$$

whenever $i_q \leq i < \frac{1}{2} i_{q+1}$. Indeed (1) is equivalent to $\Lambda^* \supset \Lambda$ and (2) will show us how many points we must add to Λ in order to get Λ^*. Now in view of the inequalities (a) and (b) above, the definition of Λ^* (or again that of $n * (r)$) satisfying conditions (1) and (2) can always be achieved.

The density of $\Lambda *$ on an interval (a, b) being $\dfrac{n * (b) - n * (a)}{b - 1}$, we set $D^*[k^i_q, k^{i+1}_q) - \dfrac{n * (k^{i+1}_q) - n * (k^i_q)}{k^{i+1} - k^i_q}$ and we have, by (2)

$$\varphi(k_q) \leq D * [k^i_q, k^{i+1}_q) \leq \varphi(k_q) + \in_q . \tag{***}$$

Consider the density of Λ^* on an interval $[k^{ip}_p, k^i_q)$ with $i_q \leq i < \frac{1}{2} i_{q+1}$ and $q > p$. It lies between the lower and upper bound of densities on the intervals $[k^i_s, k^{i+1}_s)$ situated to the right of k^{ip}_p. But by $(* * *)$ all 145 these densities differ from Δ by a quantity which tends to zero with $1/p$. Finally, consider the density of Λ^* on an interval $[k_p^{ip}, r)$ with $k_q^i \leq r < k_q^{i+1}$ and $i_q \leq i < \dfrac{1}{2} i_{q+1}$. Let this density be denoted by $X_p(r)$. We have

$$X_p(k_q^i)\frac{k_q^i - k_p^{ip}}{k_q^{i+1} - k_p^{ip}} \leq X_p(r) \leq X_p(k_q^{i+1})\frac{k_q^{i+1} - k_p^{ip}}{k_q^i - k_p^{ip}}$$

since $$\frac{n^*(k_q^i) - n^*(k_p^{ip})}{k_q^{i+1} - k_p^{ip}} \leq \frac{n^*(r) - n^*(k_p^{ip})}{r - k_p^{ip}} \leq \frac{n^*(k_q^{i+1}) - n^*(k_p^{ip})}{k_q^i - k_p^{ip}}.$$

When $r \to \infty$, we have $\limsup X_p(r) < \Delta + \epsilon'_p, \liminf X_p(r) > \Delta - \epsilon'_p$, with $\lim\limits_{p\to\infty} \epsilon'_p = 0$. Therefore Λ^* has density Δ.

From 1, 2, 3, we get the following result:

$$D_{\max}(\Lambda) = \Delta = \limsup_{k \searrow 1} \varphi(k)$$

and there exists a sequence $\Lambda^* \supset \Lambda$, *having a density equal to* $D_{\max}(\Lambda)$.

4. A new expression for $D_{\max}$.

We use two simple inequalities on $\varphi(k^{1-\alpha}) - \varphi(k)$, α being small enough:

1^o) as $(k-1), \varphi(k)$ is an increasing function,

$$\varphi(k) \geq \varphi(k^{1-\alpha})\frac{k^{1-\alpha} - 1}{k - 1} \leq \varphi(k^{1-\alpha})(1 - k^{\alpha})$$

2^o) from the equality

$$n(kr) - n(r) = n(kr) - n(k^{\alpha} r) + n(k^{\alpha} r) - n(r)$$

we get $$\varphi(k) \leq \frac{k - k^{\alpha}}{k - 1}\varphi(k^{1-\alpha}) + \frac{k^{\alpha} - 1}{k - 1}\varphi(k^{\alpha}) \leq \varphi(k^{1-\alpha}) + \alpha\varphi(k^{\alpha})$$

and $$-\alpha\Delta \leq \varphi(k^{1-\alpha}) - \varphi(k) \leq k\,\alpha\,\Delta. \qquad (*)$$

146 Suppose now $\liminf\limits_{k \searrow 1} \varphi(k) = \Delta'$. There exists a sequence $k'_n \searrow 1$

such that $\varphi(k'_n) \to \Delta'$. From the sequence $\left\{\dfrac{\log\log k'_n}{\log 2}\right\}$ one can extract a sequence which is convergent modulo 1; we can suppose the sequence itself convergent mod. 1, viz. $\dfrac{\log\log k'_n}{\log 2} = -q_n + h + \in_n$ $(\in_n \to 0)$. Defining $k = \exp(2^h)$ and $1 + \alpha_n = 2^{\in_n}$, we have $k'_n = k_{q_n}^{1+\alpha_n}$. Using the inequality ($\otimes$), we have $\lim\limits_{n\to\infty}(\varphi(k'_n) - \varphi(k_{q_n})) = 0$, then $\Delta' = \Delta$. That proves the following equality:

$$D_{\max}(\Lambda) = \lim_{k\searrow 1} \varphi(k) = \lim_{k\searrow 1} \limsup_{r\to\infty} \frac{n(kr) - n(r)}{kr - r}$$

APPENDIX II
Construction of a sequence with density zero and Mean-period infinity

147 We shall use the following fact, which is a simple consequence of Carleman's theorem: If $F(w)$ is an entire function of exponential type bounded on the real axis, then

$$\lim_{R\to\infty}\int_1^R \log|F(u)F(-u)|\frac{du}{u^2} \tag{*}$$

exists and is finite (see for example, (Levinson) p.26). We shall construct the sequence Λ in such a manner that no entire function of exponential type, vanishing on Λ, can satisfy this condition; thus the mean-period of Λ is infinity.

Let $\{\mu_k\}$ be a rapidly increasing sequence of positive numbers (we shall specify later), and $\{\nu_k\}$ a sequence of integers with $\nu_k = \in_k \mu_k = o(\mu_k)(k = 1, 2, \ldots)$. We take as Λ the sequence $\{\mu_k\}$, each μ_k counted ν_k times. Let F be, if possible, an entire function of exponential type bounded on the real axis; then, either $F(w) + F(-w)$ or $\dfrac{F(w)}{w}$ is even, and, by means of a trivial change, we can suppose

$$F(w) = \prod_1^\infty \left(1 - \frac{w^2}{\mu_j^2}\right)_j^{\nu_j} \prod_1^\infty \left(1 - \frac{w^2}{w_j^2}\right).$$

Whatever be k, we have (with $r = |w|$)

$$|\left(1 - \frac{w^2}{\mu_k^2}\right)^{-\nu_k} F(w)| \leq \prod_1^\infty \left(1 + \frac{r^2}{\mu_j^2}\right) \prod_1^\infty \left(1 + \frac{r^2}{|w_j|^2}\right) < e^{Br}$$

148 The last inequality holds if B is large enough, according to the calculation of Mandelbrojt (Lecture 10, §1); B does not depend on k. Thus

$$\log|F(\mu_k + t)| < B(\mu_k + t) + \nu_k \log\frac{|2\mu_k\, t + t^2|}{\mu_k^2}.$$

Set $t_k = \dfrac{1}{2}\mu_k \exp\left(-\dfrac{2B}{\in_k}\right)(= o(\mu_k))$ and $0 < t < t_k$; then

$$\nu_k \log \frac{2\mu_k t + t^2}{\mu_k^2} \sim \nu_k \log \frac{2t}{\mu_k} < -\frac{2B}{\mu_k}\nu_k = -2B\mu_k.$$

Hence

$$\log|F(\mu_k + t)| < (\frac{1}{2} + o(1))\nu_k \log \frac{2t}{\mu_k} \quad (0 < t < t_k)$$

if $k > k_o$ large enough

$$\begin{aligned}\int_o^{t_k} \log|F(\mu_k + t)|\frac{dt}{(\mu_k + t)^2} &< \frac{1}{4}\frac{\nu_k}{k}\int_o^{t_k} \log \frac{2t}{\mu_k}\frac{dt}{\mu_k}\\ &= \frac{1}{4} \in_k \frac{t_k}{\mu_k} \log \frac{2t_k}{e\mu_k}\\ &< -\frac{B}{4}\exp\left(-\frac{2B}{\in_k}\right).\end{aligned}$$

Now we shall choose $\{\in_k\}$ in such a manner that $\int_1^\infty \dfrac{\log^- F(u)}{u^2}du = -\infty$; as $\log^+ F(u)$ is bounded, we cannot have $(*)$ finite, and the existence of F leads to a contradiction. It is sufficient to take $\in_k = (\log\ (\log)k)^{-1}$; then

$$\int_1^\infty \frac{\log F(u)}{u^2}du$$
$$< \sum_{k_o}^\infty \log^- |F(\mu + t)|\frac{dt}{(\mu_k + t)^2} < -\frac{B}{4}\sum_k^\infty \exp\left(-\frac{2B}{\in_k}\right) = -\infty$$

(The first inequality holds if $\mu_{k+1} > \mu_k + t_k$, what is true if μ_k is rapidly increasing).

And now we can choose $\{\mu_k\}$ as rapidly increasing as we want; **149**
in particular, we can choose it in order that the density of Λ is zero, i.e.$\sum_1^k \nu_k = o(\mu_k)$ (for example, $\nu_k = \in_k \mu_k = 2^k$). Thus Λ has density zero, and mean period infinity.

We have constructed Λ as a sequence of multiple points. Now we can replace the point μ_k counted ν_k times, by ν_k points near enough to μ_k, and the result still holds.

It would be interesting to know if such an example can be constructed with a "regular" Λ; for example, for a sequence of distinct integers.

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