Lectures on Potential Theory

By

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Part I

Introduction And Topological Preliminaries

Introduction

1

The potential theory has been studied very much, especially after the researches of Gauss in 1840, where he studied important problems and methods which gave yet remained partly as basic ideas of modern researchs in this field. For about thirty years many refinements of the classical theory were given; later the axiomatic treatments starting from different particular aspects of the classical theory. About half a dozen of such axiomatic approaches to potential theory, parts of which are not yet published with details, exist. It would be necessary to compare these different approaches and to study the equivalence or otherwise of them. 1

In the following we shall develop some results of such axiomatic theories principally some convergence theorems; they may be used as fundamental tools and applied to classical case as we shall indicate sometimes. We do not presuppose anything of even classical theory.

A survey of the different developments of the potential theory has been given by M. Brelot (Annales de 1'Institut Fourier t4, 1952-54) with a historical view point and with a rather large bibliography. We shall complete it with indication one some recent developments of the theory.

We shall begin with some topological preliminaries that are necessary for our development 2

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Lemma 1. *Let E be a connected topological space. f be a lower semi-continuous function on E such that f cannot have a minimum (respectively a minimum <0) at any point of E without being a constant in some neighbourhood of that point. Then if f attains its lower bound (resp. , supposed to be <0), then $f=$ constant.*

Proof. Consider the set of the points of E where f attains its lower bound and the set of points at which the function is different from its lower bound. These two are disjoint open sets of E where union is E. The former set is non-empty by hypothesis. E being connected this is the whole of E. Hence f is a constant. □

Lemma 2. *Let E be a connected compact topological space, and ω an open set different from E. Let f be a real valued lower semi-continuous function defined on ω and further satisfy the condition that it cannot have a minimum <0 at any point without being a constant in some neighbourhood of the point. If at every point of the boundary $\partial\omega$ of ω, $\lim . \inf . f \geq 0$, then $f \geq 0$.*

Proof. Define a new function F which is equal to f on ω and zero in the complement of ω. The function F is lower semi-continuous, and cannot have a minimum <0 in E without being constant in some neighbourhood of that point. Assume that $F<0$ at some point of E. Then the
3 lower bound of F in E is less than zero and is attained in E. Hence by Lemma 1, F is a constant; this is a contradiction. □

Therefore $F \geq 0$. It may be observed that in the course of the proof only the sequential compactness of E has been made use of.

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Lemma 3 (Choquet)**.** *Let E be a topological space satisfying the second axiom of countability (i.e. possessing a countable base of open sets).*

Let $(f_i)_{i\in I}$ be a family of real valued (finite or not) functions on E. Denote

$$f_I(x) = \inf_{i\in I} . f_i(x)$$

It is possible to extract a countable subset I_o of I such that if g is any lower semi-continuous function on E with $g \leq f_{I_o}$, then $g \leq f_I$.

Proof. It is sufficient to consider functions with values in $[-1, 1]$. (The general case may be deduced from this one with the aid of a transformation of the form $\dfrac{x}{1 + |x|}$ on the real line). □

Let $\omega_1, \omega_2, \ldots\ldots$ be a sequence of open sets, forming a base for open sets of E, with the condition the each ω appears in the sequence infinitely many times. By this arrangement, it is possible to choose same ω with arbitrarily large index. For every n choose i_n satisfying the following inequality

$$\underset{y\in\omega_n}{\text{Inf}} . f_{i_n}(y) - \underset{y\in\omega_n}{\text{Inf}} . f_I(y) < \frac{1}{n} \tag{1}$$

We have a sequence $I_o = (i_n)$ a subset of I. We shall prove that this choice of I_o fulfills the required conditions. 4

Suppose g is a lower semi-continuous function on E with $g \leq f_{i_n}$ for every i_n, we want to show that $g \leq f_I$. For any $x \in E$ and $\varepsilon > 0$, there exists an open neighbourhood N of x such that for every y belonging to $N, g(y) > g(x) - \varepsilon/2$. There exists a ω_p containing x, at every point of which the above inequality holds good, this choice being made in such a way that $1/p < \varepsilon/2$. Hence,

$$\inf_{y\in\omega_p} .g(y) \geq g(x) - \varepsilon/2$$

i.e.,

$$g(x) - \inf_{y\in\omega_p} .g(y) \leq \varepsilon/2 \tag{2}$$

Further we have the inequality

$$\inf_{y\in\omega_p} .g(y) - \inf_{y\in\omega_p} . f_{i_p}(y) \leq 0 \tag{3}$$

With this choice of p the inequality (1) takes the form

$$\inf_{y\in\omega_p} . f_{ip}(y) - \inf_{y\in\omega_p} . f_I(y) < \frac{1}{p} < \varepsilon/2 \tag{1'}$$

We get on adding (1′), (2) and (3)

$$g(x) - \inf_{y\in\omega_p} . f_I(y) \leq \varepsilon$$

ε being arbitrary, it follows that $g(x) \leq f_I(x)$ i.e. the lemma.

Remark. We shall denote the $\lim . \inf . \varphi$ for a function φ at every point by $\hat{\varphi}$ (called its lower semi-continuous regularisation). Making use of this notation Lemma 3 can be put in the form:

If f_I is the $\inf_{i\in I} . f_i$ at each point of a topological space (with countable base for open sets) of a family of real valued functions (finite or not) $\{f_i\}_{i\in I}$, then it is possible to choose a countable subset $I_o \subset I$ such that $\hat{f}_{Io} = \hat{f}_I$.

Part II

General Capacities of Choquet and Capacitability

Chapter 1

Capacitability

1 True capacity and capacitability

Let us give some notions and results of the theory of capacity, developed 6
farther and under somewhat more general conditions by G. Choquet. It was inspired by classical capacity but is is now a general basis tool in analysis. We introduce here some new terms characterizing notions that will be used often. Let E be a Hausdorff space.

Definition 1. *A real valued set function φ (finite or not) defined on the class of all subsets of E will be called a true capacity if it satisfies the following conditions:*

(i) *φ is an increasing function.*

(ii) *For any increasing sequence $\{A_n\}$ of subsets of E*

$$\varphi(\cup A_n) \textit{ or } \varphi(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \varphi(A_n) \textit{ or } \sup \varphi(A_n)$$

(iii) *For any decreasing sequence $\{K_n\}$ of compact subsets of E*

$$\varphi(\cap K_n) \textit{ or } \varphi(\lim_{n\to\infty} K_n) = \lim_{n\to\infty} \varphi(K_n) \textit{ or } \inf \varphi(K_n)$$

A first example of true capacity would be an outer measure induced *by a positive*[1] *measure on a locally compact Hausdorff space. (See any book on measure theory).*

As for the notations we shall follow G. Choquet. Consequently

7 $K-$set will stand for compact set, $K_{\sigma\delta}$- set for a set which is countable union of K-sets and $K_{\sigma\delta}$ - sets for a set which is countable intersection of K_σ-sets.

Definition 2. *A K-analytic set of E is one which is the continuous image of a $K_{\sigma\delta}$ set contained in a compact Hausdorff space.*

Any class of subsets of E, closed under countable unions and intersections is called a Borelian field. In particular the smallest Borelian field containing the compact sets of E is called the K- Borelian field, the elements of this family being termed K-Borel sets. It is noted that a K-Borelian set is a K-analytic set and that in a complete separable metric space any borelian set (in the ordinary sense) is homeomorphic to a K-Borelian set, therefore is a K-analytic set.

Definition 3. *Any set A of E is said to be φ- capacitable if*

$$\varphi(A) = \sup\{\varphi(K) : \textit{ for compact sets } K \subset A\}.$$

In order to prove that K-analytic sets contained in compact sets are capacitable with respect to any true capacity, we need some steps; we first examine how an inverse image transforms a true capacity.

Lemma 1. *Let f be a continuous map of a Hausdorff space F into another Hausdorff space E. Let φ be a true capacity on E. The set function χ defined on F by $\chi(A) = \varphi[f(A)]$ is a true capacity on F; and the image $f(A)$ of a χ-capacitable set A of F is φ- capacitable.*

8 *Proof.* χ is an increasing set function. If $\{A_n\}$ is an increasing sequence of sets of F, $f(A_n)$ is increasing, therefore

$$\chi(\cup A_n) = \varphi[f(\cup A_n)] = \varphi[\cup f(A_n)]$$

[1]positive measure (or set function) will mean measure (or set function) ≥ 0. We understand positive in the sense ≥ 0, but in order to avoid any trouble, we shall write the symbol ≥ 0 instead of the word.

$$= \lim_{n\to\infty} \varphi[f(A_n)]$$
$$= \lim_{n\to\infty} \chi(A_n)$$

□

Finally if K_n is any decreasing sequence of compact sets of F with elementary topological considerations $f(\cap K_n) = \cap f(K_n)$ and therefore

$$\chi(\cap K_n) = \varphi[f(\cap K_n)] = \varphi[\cap f(K_n)] = \inf .\varphi(f(K_n)) = \inf .\chi(K_n).$$

In order to complete the lemma we have to show that the image of a χ-capacitable set by f is φ-capacitable. Let A be a χ- capacitable set and suppose $\chi(A) > -\infty$. For any $\lambda < \chi(A)$, there exists a compact set $K \subset A$ such that $\chi(K) > \lambda$. Then $f(K) \subset f(A)$ and further $\varphi[f(K)] > \lambda$ whereas $\lambda < \varphi[f(A)]$. This being true for every $\lambda < \varphi(f(A))$ it follows that $f(A)$ is φ- capacitable.

Lemma 2. *Every $K_{\sigma\delta}$ set of E is capacitable for any true capacity φ on E.*

Proof. Let A be a $K_{\sigma\delta}$ set of E. By definition

$$A = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_j^i$$

where A_j^i are compact sets of E. Without loss of generality we can 9
assume that A_j^i are increasing with j for every index i. Now for $\lambda < \varphi(A)$ (if $\varphi(A) > -\infty$) we are in search of a compact set K contained in A such that $\varphi(K) > \lambda$. Then the theorem will be a consequence of the increasing property of φ. □

To this end consider the increasing sequence $A_n^1 \cap A$ whose limit is A. By the condition (ii) of the definition of φ,

$$\varphi(A) = \lim_{n\to\infty} \varphi(A_n^1 \cap A)$$

Hence there exists an integer p_1 such that $\varphi(A_n^1 \cap A) > \lambda$. Next consider the increasing sequence $A_{p1}^1 \cap A_n^2 \cap A$ which increases to $A_{p1}^1 \cap A$.

The same argument enables us to find an integer p_2 such that $\varphi(A^1_{p1} \cap A^2_{p2} \cap A) > \lambda$. It is easy to see that proceeding on the same lines, we get by induction a sequence $\{p_j\}$ of integers satisfying for every j,

$$\varphi(A^1_{p1} \cap \ldots\ldots \cap A^j_{pj} \cap A) > \lambda$$

Put $B_i = A^1_{p^1} \cap \ldots\ldots \cap A^i_p$. The sets B_i are compact, decreasing and $\cap B_i \subset A$.

$\varphi(\cap_i B_i) = \lim\limits_i \varphi(B_i) \geq \lambda$. Hence the lemma is proved.

10 **Theorem 1.** *In a Hausdorff space E any K-analytic set A contained in some compact set of E is capacitable for every true capacity φ on E.*

Proof. We may assume for simplicity that the space E itself is compact. By definition there exists a $K_{\sigma\delta}$ set B in a compact space F and a map f onto A defined and continuous on B.

Let Γ be graph of f. We shall prove first that Γ is a $K_{\sigma\delta}$ set of $F \times E$. Since f is continuous and E is Hausdorff, it is known that Γ is a closed subset of $B \times E$. Γ is the intersection of a closed C of $F \times E$ with $B \times E$. On the other hand B being a $K_{\sigma\delta}$ set of F, $B \times E$ is a $K_{\sigma\delta}$ set of $F \times E$, as E is compact. Hence Γ is itself a $K_{\sigma\delta}$ set. For any $H \subset F \times E$, $\varphi(\text{proj}_E H)$ defines a true capacity χ on $F \times E$ (Lemma 1); Γ is χ-capacitable (Lemma 2); therefore its projection A on E is φ-capacitable (Lemma 1). □

Remark. The theorem is true when A is supposed to be contained just in a K_σ set, say $\cup K_n$. It is enough to remark that $A \cap K_n \subset K_n$, therefore capacitable and $A \cap K_n \to A$.

Chapter 2

Weak and Strong Capacity

2 Weak capacity

In many applications it is not easy to verify the three conditions of a **11**
true capacity, especially the second condition concerning the limit of an increasing sequence of sets. We shall hence examine general cases with stronger conditions which are more easily seen to be fulfilled, and also other useful "capacities" which are not true ones. We chiefly consider positive set functions which are the most useful and the only ones we shall need later.

Definition 4. *A real (finite or not) set function $\varphi \geq 0$ defined on the family of all compact subsets of a Hausdorff space is said to be a weak capacity if*

(1) $\varphi(K)$ *is increasing with* K

(2) *For any two compact sets* K_1 *and* K_2

$$\varphi(K_1 \cup K_2) \leq \varphi(K_1) + \varphi(K_2)$$

(3) φ *is continuous on the right for any compact set* K *with* $\varphi(K) < +\infty$, *that is to say for any* $\varepsilon > 0$, *there exists an open set* $\omega \supset K$, *such that for any compact set* $K' \subset \omega$ *but containing* $K, \varphi(K') \leq \varphi(K) + \varepsilon$.

Without using the condition (2), we shall derive the following notions of "inner and outer φ-capacities" respectively in the manner analogous to the derivation of the inner and outer measures from content.

12 We define the inner capacity φ_* and outer capacity $\varphi*$ of any set A as

$$\varphi_*(A) = \sup .\{\varphi(K) : \text{ Compact sets } K \subset A\}$$
$$\varphi_*(A) = \inf .\{\varphi_*(\omega) : \text{ Open sets } \omega \supset A\}.$$

When they are equal, the common value may be called capacity and denoted $\varphi(A)$. Note that for open sets φ_* and φ^* are equal.

Moreover it is true even of any compact set. For, $\varphi_*(K) = \varphi(K)$ and it is enough to show that $\varphi(K) \geq \varphi^*(K)$. For any $\varepsilon > 0$, the right continuity of φ implies the existence of an open set $\omega_1 \supset K$ such that for any compact set $K' \subset \omega_1$ but containing K, the inequality $\varphi(K') \leq \varphi(K) + \varepsilon/2$. Hence $\varphi_*(\omega_1) \leq \varphi(K) + \varepsilon/2$ and $\varphi^*(K) \leq \varphi_*(\omega_1) \leq \varphi(K) + \varepsilon/2$. This being true of arbitrary $\varepsilon > 0, \varphi^*(K) \leq \varphi(K)$.

Let us observe that the right continuity implies the condition (3) of the definition of the true capacity. This comes from the fact that if $\{K_n\}$ is a decreasing sequence of compact sets ($\cap K_n = K$) and ω any open set containing K, K_n is contained in ω for n large enough. Further the two conditions [right continuity and (3) of Def.1] are equivalent if the underlying space is a locally compact metric one.

3 Properties of Inner and Outer capacities

13 **Proposition 1.** *For a weak capacity $\varphi, \varphi_*(\omega)$ is an increasing function of ω and for any increasing sequence of open sets ω_n*

$$\varphi_*\left(\lim_{n\to\infty} \omega_n\right) = \lim_{n\to\infty} \varphi_*(\omega_n)$$

Proof. The increasing nature φ_* is a consequence of the same property of φ. If $a < \varphi_*(\cup\omega_n)$, there exists a compact set K contained in $\cup\omega_n$ such that $\varphi(K) > a$. We can find a positive integer n_o such that $K \subset \omega_n$ for $n \geq n_o$. Hence $\varphi_*(\omega_n)$ >a for $n \geq n_o$. Hence it follows that $\varphi_*(\cup\omega_n) = \lim \varphi_*(\omega_n)$. ◻

Proposition 2. *φ_* is countably subadditive on open sets.*

Proof. First let us prove that φ_* is subadditive on open sets. Let ω_1 and ω_2 be any two open sets. For any $\lambda < \varphi_*(\omega_1 \cup \omega_2)$ we can find a compact set $K \subset \omega_1 \cup \omega_2$ such that $\varphi(K) > \lambda$. Now $K \cap C\omega_1$ and $K \cap C\omega_2$ do not intersect. They are contained in disjoint open subsets of K. The complements relative to K of these two open sets give two open sets give two compact sets K_1 and $K_2, K_1 \subset \omega_1 K_2 \subset \omega_2$ and $K_1 \cup K_2 = K$.

$$\varphi(K_1 \cup K_2) > \lambda$$
$$\varphi(K_1) + \varphi(K_2) > \lambda \text{ by subadditivity of } \varphi$$

A fortiori $\varphi_*(\omega_1) + \varphi_*(\omega_2)\lambda$

This being true of any $\lambda < \varphi_*(\omega_1 \cup \omega_2)$ we have **14**

$$\varphi_*(\omega_1) + \varphi_*(\omega_2) \geq \varphi_*(\omega_1 \cup \omega_2).$$

Induction on n and repeated application of the same inequality gives finite subadditivity. The countable subadditivity is then a consequence of Proposition 1. □

Proposition 3. *The outer capacity φ^* is countably subadditive.*

Proof. Let A_n be any sequence of sets. If $\varphi^*(A_n) = +\infty$ for any n the proposition is immediate. We assume $\varphi^*(A_n) < +\infty$ for every n. Given $\epsilon > 0$, we can find open sets $\omega_n \supset A_n$ such that for every n

$$\varphi^*(\omega_n) \leq \varphi^*(A_n) + \frac{\varepsilon}{2^n}.$$
$$\varphi^*\left(\bigcup_1^\infty A_n\right) \leq \varphi^*\left(\bigcup_1^\infty \omega_n\right) \leq \sum_1^\infty \varphi^*(A_n) + \varepsilon.$$

ε arbitrary, the proposition follows. □

4 Strong Subadditivity

In order to get a true capacity we shall introduce further restrictions and first study the following notion of strong subadditivity.

Definition 5. *Areal valued (+∞ being a possible value) set function $\varphi > -\infty$ defined on a class Phi of subsets of any fixed set, closed under finite unions and intersections, is said to be strongly subadditive if for any two A and B in Φ*

$$\varphi(A \cup B) + \varphi(A \cap B) \leq \varphi(A) + \varphi(B).$$

15 **Proposition 4.** *For a set function $\varphi > -\infty$ the increasing property and the strong subadditivity are together equivalent to either of the following inequalities:*

(i) *For three sets A_1, A_2 and X in Φ,*

$$\varphi(X) + \varphi(X \cup A_1 \cup A_2) \leq \varphi(X \cup A_1) + \varphi(X \cup A_2).$$

(ii) *For three sets a, A and K in Φ, such that $a \subset A$,*

$$\varphi(A \cup K) + \varphi(a) \leq \varphi(a \cup K) + \varphi(A).$$

In fact, the strong subadditivity for the sets $X \cup A_1$ and $X \cup A_2$ gives the inequality

$$\varphi(X \cup A_1 \cup A_2) + \varphi[X \cup (A_1 \cap A_2)] \leq \varphi(X \cup A_1) + \varphi(X \cup A_2)$$

But $\varphi(X)\varphi[X \cup (A_1 \cap A_2)]$ by increasing property of φ so that (i) follows.

Conversely, the substitution in (i) : $X = A \cap B, A_1 = A, A_2 = B$ gives the strong subadditivity.

The conditions (i) implies (ii) by taking in (i) $X = a\, A_1 = A, A_2 = K$ and the converse is seen by substituting $a = X, K = A_2$ and $A = A_1 \cup X$ in the inequality (ii).

Remark. Let ψ be any finite real valued set function defined on the class Φ we have considered.

16 Define

$$V_1(X, A_1) = \psi(X) - \psi(X \cup A_1)$$

$$V_2(X; A_1, A_2) = V_1(X, A_1) - V_1(X \cup A_2; A_1)$$
$$\cdots\cdots\cdots\cdots$$
$$V_{n+1}(X; A_1, A_2, \ldots A_{n+1}) = V_n(X; A_1, \ldots, A_n) - V_n(X \cup A_{n+1}, \ldots, A_n),$$

one may prove that $V_n \leq 0$ implies $V_p \leq 0$ for any $p \leq n$ and that the assumption that χ is increasing and strongly subadditive is equivalent to $V_2 \leq 0$. The cases where $V_n \leq 0$ for any n or for any $n \leq N$ are discussed by Choquet.

Proposition 5. *If $\{A_i\}$ and $\{a_i\}$ are two finite families of sets of Φ with $a_i \subset A_i$ for every i, then for any real valued set function $\varphi > -\infty$ defined on the class Φ, increasing and strongly sub- additive,*

$$\varphi(\cup A_i) + \sum \varphi(a_i) \leq \varphi(\cup a_i) + \sum \varphi(A_i)$$

Proof. According to the proposition 4, (*ii*) :

$$\varphi(A_1 \cup A_2) + \varphi(a_1) \leq \varphi(a_1 \cup A_2) + \varphi(A_1)$$

and
$$\varphi(a_1 \cup A_2) + \varphi(a_2) \leq \varphi(a_1 \cup a_2) + \varphi(A_2)$$

hence we get required inequality when the families consist of two elements each. [It is true even when $\varphi(a_1 \cup A_2) = +\infty$]. The general case 17
follows by induction. □

5 Strong capacity

Definition 6. *A weak capacity which is strongly subadditive on compact sets is called a strong capacity.*

Proposition 6. *If φ is a strong capacity then φ_* is strongly subadditive on open sets and φ^* is strongly subadditive.*

Proof. Let ω_1 and ω_2 be two open sets ; α and β two real numbers, $\alpha < \varphi_*(\omega_1 \cup \omega_2)$ and $\beta < \varphi_*(\omega_1 \cap \omega_2)$. We may find a compact set K contained in $\omega_1 \cup \omega_2$ with decomposition into compact sets K_1 and K_2 such that $K = K_1 \cup K_2, K_1 \subset \omega_1, K_2 \subset \omega_2, \varphi(K) > \alpha$ and $\varphi(K_1 \cap K_2) > \beta$. In fact we start with a compact set K' contained in $\omega_1 \cup \omega_2$ and first get the decomposition $K_1^{'}$ and $K_2^{'}$ as in (3). We enlarge $K_1^{'}$ and $K_2^{'}$ by union with a compact set C contained in $\omega_1 \cap \omega_2$ such that $\varphi(C) > \beta$. Now $K_1^{'} \cup C, K_2^{'} \cup C$ and their union fulfill our requirement. From

$$\varphi(K) + \varphi(K_1 \cap K_2) \leq \varphi(K_1') + \varphi(K_2)$$

we deduce

$$\alpha + \beta \leq \varphi_*(\omega_1) + \varphi_*(\omega_2)$$

and $$\varphi_*(\omega_1 \cup \omega_2) + \varphi_*(\omega_1 \cap \omega_2) \leq \varphi_*(\omega_1) + \varphi_*(\omega_2).$$ □

18 Now let A_1 and A_2 be any two sets. If $\varphi^*(A_1)$ and $\varphi^*(A_2)$ finite, let us take arbitrary $\lambda_i > \varphi^*(A_i)$. We can find open sets ω_1 such that $\varphi_*(\omega_i) < \lambda_i$ $(i = 1, 2), \omega_i \supset A_i$

$$\begin{aligned}\varphi^*(A_1 \cup A_2) + \varphi^*(A_1 \cap A_2) &\leq \varphi^*(\omega_1 \cup \omega_2) + \varphi^*(\omega_1 \cap \omega_2) \\ &\leq \varphi^*(\omega_1) + \varphi^*(\omega_2) \\ &\leq \lambda_1 + \lambda_2\end{aligned}$$

Now the proposition follows.

From the Proposition 5, we deduce

Corollary. *If $\{A_i\}$ and $\{a_i\}$ for $i = 1$ to n be two finite family of sets such that $a_i \subset A_i$ for every i, then any strong capacity satisfies:*

$$\varphi^*\left(\bigcup_{i=1}^{n} A_i\right) + \sum_{i=1}^{n} \varphi^*(a_i) \leq \varphi^*\left(\bigcup_{i=1}^{n} a_i\right) + \sum_{i=1}^{n} \varphi^*(A_i)$$

6 Fundamental Theorem

Theorem 2. *If φ is a strong capacity on E, φ^* is a true capacity; and a set is φ^*- capacitable if and only if the inner and outer capacities of the set are equal.*

Proof. The latter assertion is obvious. We have only to prove that φ^* is a true capacity. □

19 The properties (i) and (iii) of the true capacity have already been seen for φ^*. Hence it remains to prove that for any increasing sequence of sets $A_n, \lim .\varphi^*(A_n) = \varphi^*(\cup A_n)$. We know the property in the case of open sets. (Proposition 1). Let us now consider the general case. If $\varphi^*(A_n)$ in $+\infty$ for any n the result is obvious. We assume $\varphi(A_n)$ to be finite for every n. For $\varepsilon > 0$, we introduce open sets $\omega_n \supset A_n$ such that

$$\varphi^*(\omega_n) \leq \varphi^*(A_n) + \frac{\varepsilon}{2^n}$$

for every n. We have, by the Proposition 5:

$$\varphi^*\left(\bigcup_{i=1}^{n}\right) + \sum_{i=1}^{n} \varphi^*(A_i) \leq \left(\bigcup_{i=1}^{n} A_i\right) + \sum_{i=1}^{n} \varphi^*(\omega_i)$$

$$\varphi^*(A_n) \geq \varphi^*\left(\bigcup_{i=1}^{n} \omega_i\right) - \sum_{i=1}^{n} [\varphi^*(\omega_i) - \varphi^*(A_i)]$$

$$\geq \varphi^*\left(\bigcup_{i=1}^{n} \omega_i\right) - \sum_{i=1}^{n} \frac{\varepsilon}{2^n}.$$

Hence,

$$\lim_{n\to\infty} \varphi^*(A_n) \geq \varphi^*\left(\bigcup_{i=1}^{\infty} \omega_i\right) - \varepsilon$$

$$\geq^* \left(\bigcup_{i=1}^{\infty} A_i\right) - \varepsilon.$$

This inequality is true for any arbitrary $\varepsilon > 0$, we get $\lim\limits_{n\to\infty} \varphi^*(A_n) \geq$ $\varphi^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right)$. The other way inequality being always true we get the re- 20
quired result.

Remark. If we suppose that φ is just $> -\infty$ (instead of ≥ 0) and increasing, continuous to the right and strongly subadditive, we may define as well inner and outer capacities, and the previous fundamental theorem holds good. The same proofs carry over.

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Part III

Potentials with Kernels - Convergence Theorems

Chapter 1

Preliminaries on Measures, Kernels and Potentiels

1 Radon Measures

Let us recall some definitions and properties of Radon measures on a **21**
locally compact Hausdorff space E. For the proofs we refer to N. Bourbaki (Integration).

Let $\mathcal{K}(E)$ be the real vector space of real valued (finite) continuous functions on E with compact support (closure of the set where the function is $\neq 0$). $\mathcal{K}(E, K)$ stands for the subspace of $\mathcal{K}(E)$ the elements of which have support in the compact set K. The space $\mathcal{K}(E, K)$ is provided with the topology of uniform convergence on K.

We shall consider (unless the contrary is mentioned) only *positive measures* called briefly measures; we recall their definition.

Definition 1. *A positive Radon Measures on E is a positive linear functional on* $\mathcal{K}(E)$*.*

A linear combination of measures $\alpha\mu_1 + \beta\mu_2$ is the functional $\alpha\mu_1(f) + \beta\mu_2(f)$. The order of two measures is defined by $\mu_1 \leq \mu_2 \iff \mu_2 = \mu_1 +$ positive measure.

For any f in $\mathcal{K}(E)$, the value $\mu(f)$ of the functional is called the integral of f and is also denoted by $\int f d\mu$. This positive linear functional **22** is obviously continuous on every $\mathcal{K}(E, K)$.

The set $\mathfrak{m}^+$ of the positive Radon Measures is *provided* with the coarsest topology such that, for every function f in $\mathcal{K}(E)$, the mapping $\mu \to \int f d\mu$ is continuous. This topology is called the "topology of vague convergence " or simply " *vague topology*". This may otherwise be described as the topology of simple convergence on $\mathfrak{m}^+$.

Proposition 1. *The set E^1 of unit measures (or Dirac measures) ε_x at the points x in E is a subset of $\mathfrak{m}^+$ which is homeomorphic to E.*

Proposition 2 (Compactness)**.** *A set of $\mu_i \in \mathfrak{m}^+$ such that for any $f \geq 0$ belonging to $\mathcal{K}(E)$, $\int f d\mu_i$, is bounded, is relatively compact in $\mathfrak{m}^+$ for vague topology.*

Proposition 3. *The function $\int f d\mu$ is continuous on $\mathcal{K}(E, K) \times \mathfrak{m}^+$ with the product topology.*

Consider an element (f_o, μ_o) in the product space. Let φ be in $\mathcal{K}(E)$ such that $\varphi = 1$ on K. Then,

$$\left| \int f_o d\mu_o - \int f d\mu \right| \leq \left| \int f_o d\mu_o - \int f_o d\mu \right| + \int f_o d\mu - \int f d\mu \Big|$$

Given $\varepsilon > 0$, there exists a neighbourhood V of f_o in $\mathcal{K}(E, K)$ where $|f - f_o| <' \varepsilon\varphi$ and a neighbourhood W of $\mu_o n \mathfrak{m}^+$ where $\int \varphi d\mu \leq \int \varphi d\mu_o +$ ε and $| \int f_o d\mu_o - \int f_o d\mu| < \varepsilon$. Hence $f \in V, \mu \in W$ implies $| \int f_o d\mu_o -$ $\int f d\mu| < \varepsilon$. The proposition is an immediate consequence.

2 Radon Integrals of functions (for positive measures)

23 Let $\psi \geq 0$ be a lower semi-continuous function defined on. E The integral of χ with respect to the Radon measure μ is by definition

$$\int \chi d\mu = \sup \int f d\mu \, (f \in \mathcal{K}(E), f \leq \psi)$$

An important property is for any increasing directed set of such ψ

$$\sup \int \psi_i d\mu = \int \sup \psi_i d\mu$$

For any finite upper semi-continuous function $\varphi \geq 0$, with $\varphi = 0$ outside a compact set, the integral is defined to be

$$\int \varphi d\mu = \mathrm{Inf}\,.\int f d\mu\,(f \in \mathcal{K}(E),\, f \geq \varphi)$$

We now define for any valued function $g \geq 0$ on E the upper and lower integrals denoted respectively by $\overline{\int}_g d\mu$ and $\underline{\int} g d\mu$

$$\begin{aligned} \overline{\int} g d\mu &= \inf \int_X d\mu\,(\psi \text{ lower semi-continuous}, \psi \geq g) \\ \underline{\int} g d\mu &= \sup \int \varphi d\mu\,(\varphi \text{finite upper semi-continuous}, 0 \leq \varphi \leq \\ &\quad g, \varphi = 0 \text{ outside a compact set}) \end{aligned}$$

It is easily seen that $\underline{\int} g d\mu \leq \overline{\int} g d\mu$

Recall the fundamental property for a sequence $g_n \geq 0$

$$\overline{\int} \liminf g_n d\mu \leq \liminf \overline{\int} g_n d\mu \text{ (Fatou's Lemma)}$$

Definition 2. *A function $g \geq 0$ is said to be $\mu-$ integrable (resp. in the large sense) if the upper and the lower $\mu-$ integrals of g are equal and finite (resp. only equal); the common value is written as $\int g d\mu$ and it is by definition the μ integral of g (finite or not).* **24**

For any function g, utilizing the classical decomposition $g = g^+ - g^-$, we define $\int g d\mu = \int g^+ d\mu - \int g^- d\mu$ whenever the difference and its terms have meaning (g is said to be μ - integrable).

Permutation of Integration. We shall use the particular case of a lower semi-continuous function $\psi(x, y) \geq 0$ on $E \times E$ for which

$$\int d\nu(y) \int \psi(x, y) d\mu(x) = \int \mu(x) \int \psi(x, y) d\nu(y)$$

Inner and Outer measure - measurablility. The inner and outer measures of a set α are defined to be the lower and upper integrals (respectively) of the characteristic function φ_α of φ. If the two integrals are equal and finite, α is said to be μ- integrable, of measure $\mu(\alpha)$, the common value. Then the intersection with any compact set is also $\mu-$ integrable and $\mu(\alpha) = \sup .\mu(K)$ for all compact sets K contained in α. Note that an open set of measure zero is more directly characterised by the property that $\int f d\mu = 0$ for any $f \in \mathcal{K}(E)$ with support in ω. The support S_μ of a measure μ is defined to be the closed set complement of the largest open set whose measure is zero. We say also that a measure μ is supported by a set α (which may not be closed) if the outer measure of $C\alpha$ is zero.

25 A set α is $\mu-$ measurable if $\alpha \cap K$ is $\mu-$ integrable for every compact set K. A set, $\mu-$ measurable for every μ is defined to be a measurable set [Examples : closed and open sets].

A real valued function f may be defined as μ measurable if the sets where $f \geq \alpha$, $f \leq \alpha$ are $\mu-$ measurable. [Examples : semi - continuous functions]. A μ measurable function is $\mu-$ integrable if and only if $\int |f| d\mu < +\infty$.

We recall the Lusin's property of μ -measurable functions (taken as definition by Bourbaki, Integration Ch. IV) : for any compact set K and any $\varepsilon > 0$, there exists a compact set $K_1 \subset K$ such that $\mu(K - K_1) \leq \varepsilon$ and that the restriction of f on K_1 is continuous.

Restriction of Measure. The restriction of a measure μ on a $\mu-$ measurable set α may be defined as the measure μ_α determined by the functional $\int f\varphi_\alpha d\mu$ [$f \in \mathcal{K}(E), \varphi_\alpha$] being the characteristic function of $\alpha. f\varphi_\alpha$ is $\mu-$ integrable]. For any $\mu-$ integrable function f it is seen that $f\varphi_\alpha$ is $\mu-$ integrable, that f is μ_α integrable and that $\int f\varphi_\alpha d\mu$ denoted also by $\int_\alpha f d\mu$ is equal to $\int f d\mu_\alpha$. Note that when $\mu(C\alpha) = 0, \mu_\alpha \leq \mu$.

Suppose now that α is a compact set and μ a measure with support contained in α. We may associate a measure μ' on the subspace α such that if f is finite and continuous on α, and F its continuation by zero, $\mu'(f) = \int F d\mu$. Conversely, for any measure μ' on α, there exists a unique measure μ on E such that $\mu(C\alpha) = 0$ and its associated measure (by the above method) is μ' on α.

26 For such μ and μ' and any function f on α (continued arbitrarily to E) both $d\mu$ and $d\mu'$ integrals exist and are equal. Hence we do not distinguish between μ and μ' and refer to them as measure on α.

3 Kernels and Potentials

A real valued function $G(x, y) \geq 0$ on $E \times E$, integrable in y in the large sense for every x as regards and Radon measure ≥ 0 on E is called a kernel on E.

Definition 3. *The potential of a measure μ in $\mathfrak{m}^+$ with respect to the kernel $G(x, y)$ is defined as $G\mu(x) = \int G(x, y)d\mu(y)$.*

In order to develop a large theory of potentials, it is easier to assume that the function $G\mu(x)$ on E is lower semi-continuous for every μ. We shall even suppose more.

Theorem 1. *The potential $G\mu(x)$ is lower semi-continuous on $E \times \mathfrak{m}^+$ if and only if the kernel $G(x, y)$ is lower semi-continuous on $E \times E$.*

Proof. E is homeomorphic to the subset E^1 of $\mathfrak{m}^+$ formed by the Dirac measures at the points of E. The potential for the Dirac measure ε_y at y in E is $G(x, y)$. Hence the necessary part follows. □

Conversely, the lower semi-continuous function $G(x, y)$ is the upper envelope of an increasing directed (filtrante) family $\{G_i(x, y)\}_{i \in I}$ of finite continuous function on $E \times E$ with compact supports. Now $y \to G_i(x, y)$ is an element of some $\mathcal{K}(E, K)$ which is a continuous function of x. Hence by Proposition 3, $\int G_i(x, y)d\mu(y)$ is a finite continuous function on $E \times \mathfrak{m}^+$.

The limit $\int G(x, y)d\mu(y)$ according to the directed set is also the up- **27**
per envelop which is lower semi-continuous in $E \times \mathfrak{m}^+$.

We shall only use from now lower semi-continuous kernels.

Let us introduce a fundamental tool in our theory of potentials.

Lemma 1. *Let $G(x, y)$ be a lower semi-continuous kernel on E. Let μ be a Radon Measures with compact support K in E and $G\mu(x)$ finite on K. Then for any $\varepsilon > 0$, there exists a compact set $K' \subset K$ such that the restriction μ' of μ to K' satisfies:*

(i) $$\int d\mu - \int d\mu' < \varepsilon$$

and (ii) the restriction of $G\mu'(x)$ to K' is continuous.

Proof. $G\mu(x)$ being lower semi - continuous, the application of Lusin's property (§2) provides a compact set K' contained in K such that if μ' is the restriction of μ to K' then $0 \leq \int d\mu - \int d\mu' < \varepsilon$ and $G\mu(x)$ restricted to K' is continuous. Now $\mu'' = \mu - \mu'$ being a positive Radon measure, $G\mu''(x)$ is lower semi-continuous on E and further,

$$G\mu(x) = G\mu'(x) + G\mu''(x).$$

The above being a decomposition of the continuous function $G\mu(x)$ into sum of two lower semi-continuous functions, the individual members of the right hand side are themselves continuous. This completes the proof. □

The above lemma will be useful in the development of the conver-
28 gence theorems. This lemma enables us to discard a set of small $\mu-$ measure, and consider continuous functions in the complement, instead of lower semi-continuous functions. Here we shall prefer this approach to the other ones of taking limits of continuous potentials.

Chapter 2

Negligible Sets and Regular Kernels

4 Definitions and Fundamental Lemmas

In the convergence that we are going to discuss, we shall meet with 29
exceptional set of points which may be ignored in some sense, such as a set of measure zero in measure theory. We wish to introduce a notion of smallness to describe these sets, without any appeal to the capacity theory at the first instance. We remind that the kernel $G(x, y)$ is always supposed to be lower semi-continuous ≥ 0.

Definition 4. *A compact set K in E is said to be G-negligible if for any Radon measure $\mu \neq 0$ on K, the potential $G\mu(x)$ is unbounded on K.*

It is obvious that finite union of G-negligible compact sets is G-negligible (whereas it may not be true of the intersection). Compact subsets of G-negligible sets are not necessarily G-negligible. In order to avoid such a disturbing possibility we shall introduce a restriction on the kernel.

Definition 5. *A kernel $G(x, y)$ is said to be ‘ regular ’ (Choquet) if it satisfies that following continuity principle : for any measure $\mu \in \mathfrak{m}^+$ with compact support K, if the restriction to K of $G\mu(x)$ is finite and continuous, then $G\mu(x)$ is finite and continuous on the whole space.*

Immediately follows from Lemma 1 the

Lemma 2. *If G is regular, for any non-zero measure $\mu \in \mathfrak{m}^+$ with compact support K such that $G\mu(x)$ is finite on K, there exists a non-zero*
30 *measure $\mu' \leq \mu$ with compact support in a subset of K, such that $G\mu'(x)$ is finite and continuous on the whole space.*

Consequently for regular kernels, any compact subset of a G-negligible compact set is also G-negligible. This enables us to give a consistent definition of general G-negligible sets.

Definition 6. *For a regular kernel G, a set A is defined to be G-negligible if every compact set contained in A is negligible.*

An equivalent definition for the G-negligibility in the case of a relatively compact set A is that for any non-zero measure supported by A (i.e. for which complement of A has outer measure zero), $G\mu(x)$ is unbounded on A.

Fundamental Lemma (3). If G is regular, on every compact set K which is not G-negligible, there exists a positive measure $\mu \neq 0$, such that $G\mu(x)$ is finite and continuous on E.[1]

Proposition 4. *G being regular, given a (positive) measure μ on a compact set K such that $G\mu(x)$ is finite on K, there exists an increasing sequence μ_n of measures on K with the following properties:*

(i) $\mu_n \leq \mu$ for all n

(ii) $(\mu_n - \mu)(E) \to 0$ as $n \to \infty$. This implies $G\mu_n \to G\mu$.

31 (iii) $G\mu_n$ is finite continuous on E.

Proof. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers tending to zero. There exists a measure $\nu_1 \leq \mu$, $\int d\mu - \int_1 d\nu_1 < \varepsilon_1$ and $G\nu_1$ finite

[1]In the classical case, it is a generally ignored result of de la Vallee Poussin (Le Potentiel Logarithmique : Paris et Louvain' 49): a consequence of already developed theory. By means of the Lusin property, Choquet introduced this and Lemmas 1 and 2 at the beginning, as powerful tools.

to continuous on E. Let $\mu = \nu_1 + \mu'_1$. A similar argument applied to μ'_1 gives a measure $\nu_2 \leq \mu'_1$ and such that $\int d\mu'_1 - \int d\nu_2 < \varepsilon_2$ and $G\nu_2$ finite and continuous on E. Then induction assumption on n and a similar argument for the passage from n^{th} to $(n+1)th$ stage provide a sequence of measures $\{\nu_i\}$. The sequence $\{\mu_n\}$ of measures defined by $\mu_n = \sum_{i=1}^{n} \nu_i$ answers to our call. The first two result from the choice of μ'_n s and imply $\mu_n \to \mu$ vaguely. Because of lower semi- continuity of $G\mu(x)$ in $E \times \mathfrak{m}^+ : \lim ._n \inf .G\mu_n(x) \geq G(\mu)(x)$ for x in E. But $G\mu_n \geq G\mu$. Hence $G\mu_n(x) \to G\mu(x)$. □

5 Associated kernel and Energy (Choquet)

Definition 7. *The associated kernel G^* of a kernel G on E is defined to be $G^*(x, y) = G(y, x)$.*

Definition 8. Energy *The G- energy of a measure μ is the integral $\int G\mu(x)d\mu(x)$.*

It is noted immediately that G-energy and G^*-energy of a measure are equal.

An immediate consequence of this remark is the

Proposition 5. *If K is a compact set which is not G-negligible, there exists a measure $\mu \neq 0$ on K for which G-energy is finite.* **32**

Proposition 6. *If G is a regular kernel on E and $\mu \neq 0$ on K with finite G-energy then K is not a G-negligible set.*

Proof. Since the energy is finite, $G\mu(x)$ is finite valued almost everywhere (relative to the measure μ) on K. It is possible to find (See lemma 1) a compact set $K' \subset K$ for which the restriction of μ' of μ is $\neq 0$ and the restriction of $G\mu'$ finite and continuous ; K' is not G-negligible. Now, K is not G-negligible follows because G is regular. □

Corollary . *For regular kernels G and G^*, the negligible sets are the same.*

6 Examples of Regular kernels

A. Newtonian kernel . Let us consider for simplicity the space R^3 and the kernel $G(x,y) = \dfrac{1}{|x-y|}$. Let $\mu \geq 0$ be a measure on a compact set K in R^3. We shall prove that if the function $G\mu(x)$ restricted to K is finite continuous at a point $x_\circ$ in K, then $G\mu(x)$ is itself continuous at $x_\circ$ as a point of the whole space (property of Evans Vasilesco). Since $G\mu(x)$ is finite and continuous in C_K, the regularity follows.

In fact for every point x and a projection y on $K(|x-y| \min .)$ we have for any $z \in K$,.

$$\left|y-z\right| \leq \left|x-y\right| + \left|x-z\right| \leq 2\left|x-z\right|$$

Hence for any measure ν on K,

$$\int \frac{d\nu(z)}{|x-z|} \leq 2 \int \frac{d\nu(z)}{\left|y-z\right|}$$

33 Let V be a spherical domain of centre $x_\circ, \mu_V$ and μ_{CV} be the restrictions of μ to $V \cap K$ and $CV \cap K$. Since $G\mu(x_\circ)$ is finite there is no mass at $x_\circ$, and $C\mu_V(x_\circ) < \varepsilon$ if V is sufficiently small. Because of the hypothesis of continuity and because of the obvious continuity in V° of $G\mu_{CV}$, the restriction of $G\mu_V$ to K is continuous at $x_\circ$ and smaller than 2ε is another smaller spherical domain V_1. Hence $G\mu_V < 4\varepsilon$ in V_1.

$$\begin{aligned} \left|G\mu(x) - G\mu(x_\circ)\right| &\leq \left|G\mu_{CV}(x) - G\mu_{CV}(x_\circ)\right| + \left|G\mu_V(x)\right| \\ &\qquad - \left|G\mu_V(x_\circ)\right| \quad (x \in V_1) \\ &\leq \left|G\mu_{CV}(x) - G\mu_{CV}(x_\circ)\right| + 5 \in \\ &\leq 6 \in \text{ for } \left|x - x_\circ\right| \text{ sufficiently small.} \end{aligned}$$

This completes the proof.

Such a proof can be immediately extended in R^n $(n \geq 3)$ with the kernel $\left|x-y\right|^{2-n}$; and to a bounded region A of the plane with the kernel $\log \dfrac{D}{|x-y|}$ ($D >$ diameter of A). It is even obvious to make extension in metric space and much more general kernels, for example $|x-y|^{-\alpha}(\alpha >$

0) . In fact many such examples are particular cases of the following one.

B. **General example of regular kernel**

Definition 9. Maximum principle *A kernel G is said to satisfy the maximum principle if for every measure $\mu \geq 0$ with compact support K, $G\mu(x) \sup_{y \in K}. G\mu(y)$ for every $x \in E$*

That the Newtonian and more general kernels in the Euclidean spare 34
or in a metric space satisfy the maximum principle was proved by Maria and Frostman. In what follows we shall use a Weaker condition.

Definition 10. *A Kernel G is said to satisfy a* weak maximum principle *if there exists a $\lambda > 0$ such that for every measure $\mu \geq 0$ with compact support $S\mu$, $G\mu(x) \leq .\lambda \sup_{y \in S} G\mu(y)$ for every $x \in E$.*

Theorem 2 (Similar to a Choquet's theorem)**.** *Let G be a continuous kernel, finite for $x \neq y$. If G satisfies the weak maximum principle in E or only* locally *(that is to say in an open neighbourhood of each point), then G is regula.*

Proof. We suppose that, the restriction of G to the support K (compact) of μ is finite continuous at $x_\circ$ in K, and we shall prove that $G\mu$ is continuous at $x_\circ$. It is enough to consider $x_\circ$ on the boundary of K. □

Suppose first $G(x_\circ, x_\circ) = +\infty$, then the mass of $x_\circ$. If V is an open neighbourhood of $x_\circ$, let us introduce the restriction μ_V and μ_{CV} of the measure μ to V and CV respectively. $G\mu_{CV}$ is finite and continuous in V because of the finiteness and continuity of G in the complement of the diagonal. Thus $G\mu_V$ has a restriction to $V \cap K$ which is continuous at $x_\circ$. If V is small enough, $G\mu_V(x_\circ) < \varepsilon$ and $G\mu_V(x) < 2\varepsilon$ for $x \in V_1 \cap K$ where
V_1 is a compact neighbourhood (small enough) of $x_\circ, \subset V$. Therefore 35
$G\mu_{V_1} < 2\epsilon$ on $V_1 \cap K$. Applying the weak principle to V, we get $G\mu_V < 2\varepsilon\lambda$ in V.

Now

$$|G\mu(x) - G\mu(x_\circ)| = |G\mu_{CV_1}(x_0) - G\mu_{CV_1}(x)| + |G_{\mu_{V_1}}(x)| + |G\mu_{V_1}(x_\circ)|.$$

On the right the third term is less than ε, the second is less then $2\lambda\varepsilon$ in V. For a neighbourhood $V_\circ$ of $x_\circ$ we know a λ which remains valid for any $V \subset V_\circ$. Then for any ε we determine V in $V_\circ$, then a compact V_1 so that the second member will be arbitrarily small if $|x - x_\circ|$ is small enough. Suppose now $G(x_\circ, x_\circ) \neq \infty; \mu.$ has a mass m at $x_\circ$ whose potential is finite continuous. The potential due to the other masses is finite continuous at $x_\circ$ as in the previous case. Hence the same property for $G\mu$.

Chapter 3

Convergence theorems (with exceptional G*-negligible sets)[1]

7 Case of a compact space

We shall study now some convergence theorems for potentials with general kernels, first on a compact space E, and then on a locally compact space. We shall see that the limits of potentials are functions which differ from potentials only on sets, rare enough according to the previous notion of negligibility or to a certain capacity zero that we shall introduce for a second group of theorems. **36**

Definition 11. *If G is a regular kernel, a property P is said to hold nearly G-everywhere if the set of points of E at which P does not hold is a G-negligible set.*

Theorem 3. *Let E be a compact space. G a lower semi - continuous kernel* ≥ 0*, with* G^1 *regular on E. If* μ_n *is a sequence of measures in* $\mathfrak{M}^+$ *converging vaguely to* μ *then*

$$\lim_{n} . \inf_{n\to\infty} .G\mu_n = C\mu \text{ nearly } G^* - \text{everywhere}$$

[1]This Chapter improves and develops the paper [6] of the bibliography

Proof. The lower semi-continuity of the kernel gives

$$\lim_n . \inf_{n\to\infty} .G\mu \geq G\mu$$

37 Let K be a compact set in the set of points A of E at which strict inequality holds good. We assert that K is G^*- negligible. Otherwise due to regularity of G^*, there exists a positive measures $\nu \neq 0$ on K for which the G^*-potential is finite and continuous on E (lemma 3). Since μ_n tends to μ in the vague topology,

$$\int G^*\nu d\mu_n \to \int G^*\nu d\mu = \int G\mu d\nu.$$

On the other hand by Fatou's lemma,

$$\int \lim_n . \inf .G\mu_n d\nu \leq . \lim_n \inf \int G\mu_n .d\nu$$
$$= \lim_n . \inf . \int G\mu_n .d\nu = \lim_n . \inf . \int G^*\nu d\mu_n$$

therefore

$$\int \lim_n . \inf \mu_n d\nu = \int G\mu d\nu$$

This contradicts the assumption that $\lim_n . \inf .G\mu_n$ for points of K. Hence A is a G^*-negligible set. The proof is complete. □

Theorem 4. *Let E be a compact space and G a kernel with* G^* *regular. If* $\{\mu_n\}$ *is a sequence of measures in* $\mathfrak{M}^+$ *with* $\mu_n(E)$ *bounded further* $G\mu_n(x)$ *converges to a function* $f(x)$ *then there exists a measure* $\mu \geq 0$ *(limits of a subsequence of* μ_n'*s) such that* $G\mu_n(x) = f(x)$ *nearly* G^**-everywhere.*

Proof. Since $\mu_n(E)$ are bounded the set $\{\mu_n\}$ is relatively compact for the vague topology in $\mathfrak{M}^+$. Hence we can choose a subsequence $\{\mu_{\alpha_n}\}$ of $\{\mu_n\}$ which converges vaguely to a measure $\mu \in \mathfrak{M}^+$. An application of
38 theorem 1 above shows that $G\mu(x) = f(x)$ nearly G^*-everywhere. □

Theorem 5. *We again assume E to be compact and* G^* *regular. If* $\{\mu_i\}_{i\in I}$ *is a family of measures in* $\mathfrak{M}^+$ *with total mass* $\mu_i(E)$ *bounded for* $i \in I$

such that the family $\{G\mu\}_{i\in I}$ is a directed set for the increasing order, then the upper envelope φ of $\{G\mu_i\}_{i\in I}$ is equal nearly G^*-everywhere to the potential of a measure (limits of μ_i according to a filter on I finer than the filter $\mathscr{F}$ of sections with the order induced by the ordered family $\{G\mu_i\}_{i\in I}$.

Proof. For any measure ν

$$\int G\mu_i d\nu = \int G^*\nu d\mu_i$$

The first integral tends for $\int \varphi d\mu$ according to the filter $\mathscr{F}$. □

Let us introduce a filter finer than this filter for which $\{\mu_i\}$ converge to a measure $\mu \geq 0$. (This is possible because of the relative compactness of the set $\{\mu_i\}_{i\in I}$ for which $\mu_i(E)$ are bounded). Then if $G^*\nu$ is finite continuous the second integral tends to $\int G^*\nu d\mu$, or what is the same to $\int G\mu d\nu$.

Suppose that $\varphi \neq G\mu$ on a non G^*- negligible set and therefore on a non G^*-negligible compact set K. We may choose for ν, a measure on $K, \neq 0$ such that $G^*\nu$ is everywhere finite continuous. Then our equality

$$\int \varphi d\nu = \int G\mu d\nu \text{ gives a contradiction.}$$

Theorem 6. *Let the compact space E possess a countable space and G be a kernel with G^* regular. If $\{\mu_i\}_{i\in I}$ is any family of measures ≥ 0 on E such that $\{G\mu_i\}_{i\in I}$ is bounded for all $i \in I$, then the lower envelope $\inf_i.$ $G\mu_i$ is equal to a potential of a measure $\mu \leq 0$, nearly G^*-everywhere.* 39

More precisely there exists a sequence $\mu_{\alpha_n} \to \mu$ such that $\{G\mu_{\alpha_n}\}$ is decreasing and $G\mu \leq \inf_i.\ G\mu_i \leq \inf_{\alpha_n}$, these three functions being equal G^*-nearly everywhere.

Proof. In virtue of Lemma 1 (1.3), there exists a sequence $\{G\mu_{\alpha_n}\}$ $\{\alpha_n\}$ being a countable subset of I) of the family of such that for any lower semi continuous function g on E such that $g \leq \inf_{\alpha_n}.\ G\mu_{\alpha_n}$ implies $g \leq \inf_{i\in I} G\mu_i$, we can choose in the family a decreasing sequence

$\left\{G\mu_{\alpha'_n}\right\} G\mu_{\alpha'_n} \leq G\mu_{\alpha_n}$ therefore with the same property. As $\mu'_{\alpha_n}(E)$ is bounded for every α'_n there exists a subsequence $\{\beta_n\}$ such that $\{\mu_{\beta_n}\}$ converges to a measure $\mu \geq 0$. By Theorem 3, the potential $G\mu = \lim_{\beta_n} . \inf .G\mu_{\beta_n}$ G^*- nearly everywhere. The fact that $\{G\mu_{\alpha'_n}\}$ has been choosen to be decreasing gives

$$G\mu = \inf_n .G\mu_{\alpha_n} \text{ nearly} G^* \text{ - everywhere}$$

But $G\mu \leq \inf_i G\mu_i$; therefore the three functions of the inequality $G\mu \leq \inf_i .G\mu_1^i \leq \inf_n .G\mu_{\alpha_n}$ are equal G^*- nearly everywhere. □

8 Extension of the converge theorems for locally compact space

40 The previous convergence theorems allow extension to the locally compact spaces with slightly more restrictions on the measures. In the sequence of theorems that follows E denote *a locally compact space*, G a lower-semi continuous kernel with the associated kernel G^* regular on E.

We shall say that a family $\{\mu_i\}_{i \in I}$ is G^*-admissible if for any measure $\nu \geq 0$ whose support is compact and such that $G^*\nu$ is finite continuous, $\int_{C_k} G^*\nu d\mu_i \to 0$ uniformly with respect to $i \in I$, when $K \to E$ according to the directed family of the compact sets K of E.

General extension

The theorems 3, 4, 5 and 6 of (3 §7) are valid with (μ_i) or (μ_n) on E and with the supplementary conditions that

1) the family (μ_i) or (μ_n) is G^*-admissible

2) in theorems 4, 5 and 6 $\{\mu_i(K)\}$ or $\{\mu_n(K)\}$ are supposed to be bounded for every compact set K.

It will be sufficient to given in detail the extension of Theorem 3.

Theorem 3′. G^* being regular. Let (μ_n) be a sequence of G^*-admissible measure ≥ 0 converging vaguely to the measure $\mu \geq 0$. Then $\lim_m . \inf /G\mu_n = G\mu, G^*$- nearly everywhere.

Proof. As in the case of compact space we assume that $\lim_{n}.\inf .G\mu_n > G\mu$ on a compact set K_1 which is not G^* - negligible ; we will arrive at a contradiction as follows. □ 41

There exists on K_1 a positive measure $\nu \neq 0$ such that $G^*\nu$ is finite and continuous on the whole space. We shall prove that $\int G^*\nu d\mu_n \to \int G^*\nu d\nu$, and a contradiction follows exactly as in the proof of theorem 3. But in general $G^*\nu$ does not have compact support. We may introduce for every compact set K a function φ_K which is finite continuous, zero outside a compact set, satisfying $0 \geq \varphi_K \geq G^*\nu$ everywhere and $\varphi_K = G^*\nu$ on K. The existence of such a function φ_K results from the normality of the space.

Let us first observe that $\int G^*\nu d\mu_n$ is bounded: For any compact set K' it is equal to the sum $\int_{C'_K} .G^*\nu d\mu_n + \int_{K'} G^*\nu d\nu_n$. The first integral is less than $\varepsilon(>0)$ for a suitable K' ; the second is bounded because $\mu_n(K')$ is bounded. If

$$\int G^*\nu d\mu_n \leq \lambda,\ \text{then}\ \int \varphi_K d\mu_n \leq \lambda \text{ and the limit } \int \varphi_K d\mu$$

is also $\leq \lambda$. Therefore $\int G^* d\mu$ (the upper bound of the integrals of all such φ_K) is $\leq \lambda$.

Now

$$\Big|\int G^*\nu d\mu_n - \int G^*\nu d\mu\Big| \leq \Big|\int (G^*\nu - \varphi_K) d\mu_n\Big| + \Big|\int \varphi_K d\mu_n - \int \varphi_K d\mu\Big| + \Big|\int (G^*\nu - \varphi_K) d\mu\Big|$$

The first integral on the right, is $\leq \int_{C_K} G^*\mu_n$ which is smaller than $\varepsilon/3$ for all n and a suitable K_1 (say $K_\circ$); the third one is $\geq \int_{C_K} G^*_\nu d\mu$ which is $< \epsilon/3$ for a suitable K_1 say $K^1_\circ$, because $\int G^*\nu.d\mu$ is finite. 42
Taking for K the union $K_\circ \cup K^1_\circ$, we maintain these inequalities, with such a fixed K the second integral on the right is $< \varepsilon/3$ for N sufficiently large; therefore the left member is smaller than ε for $n > N$. Hence as $n \to \infty$, $\int G^*\nu, d\mu_n \to G^*.d\mu$. Hence we get a contradiction as in the case of the compact space E.

Remark. It would be interesting to have suitable criteria for G^*- admissibility. Let us indicate two important ones.

(i) Supports of μ_n's are contained in a fixed compact set;

(ii) $G^*(x, y) \to 0$ uniformly in y on any compact set, when extends to the Alexndroff point of E.

Chapter 4

G-capacity[1]

9 G-capacity and G-capacity measures

For any lower semi-continuous kernel $G \geq 0$ on a locally compact Haus- **43**
dorff space E, we shall define G and G^{*1} -capacities which will be used to characterise exceptional sets in other and similar convergence theorems.

Definition 12. *The G- capacity of any compact set K is defined by G-cap* $(K) = \sup .\{\mu(K) : S\mu \subset K, G\mu \leq 1 \ \text{on} \ E\}$.

Lemma 4. *Let us consider measures $\mu_i (i \in I)$ on a compact set K, such that $\mu_i(K)$ is bounded, and μ_i converges (in vague topology) to μ according to a filter on I and satisfy $G\mu_i \leq 1$. Then $\mu_i(K) \to \mu(K)$ and $G\mu \leq 1$.*

For if $f \in \mathscr{K}(E), \int f d\mu_i \to \int f d\mu$ and if $0 \leq f(y) \leq G(x, y)$ for a fixed $x, \int f d\mu_1 \leq 1 \Rightarrow \int f d\mu \leq 1$. Hence follows the inequality $\int G(x, y) d\mu(y) \leq 1$.

Proposition 7. *If for a compact set K, G - cap. (K) is finite, there exists measures μ, called G-capacitary measures, such that $G\mu \leq 1$ and $\mu(K) = G - cap.(K)$.*

[1] In the chapter 4 and 5 we develop some of the ideas of Choquet [8, 9]. This chapter also contains some new results like theorems 8, 10 and proposition 12.

Proof. It is possible to find a sequence of measures μ_n on K such that $G\mu_n \leq 1$ and $\mu_n(K) \to G$-cap (K). Since $\mu_n(K)$ is bounded, there exists
44 a subsequence μ_{n_p} which converges to a measures μ on K. The previous lemma asserts that this μ satisfies the required conditions. □

Remark. The set of G-capacity measures of K is a convex and compact set.

Theorem 7. *The function G-cap. is a weak capacity on the compact sets of E.*

Proof. The G-cap. is obviously an increasing function. It is subadditive. Let K_1 and K_2 be a compact sets, μ any measure on $K_1 \cup K_2$ such that $G\mu \leq 1$; then

$$\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2) \leq G - \text{cap.}(K_1) + G - \text{cap.}(K_2)$$

and
$$G - \text{cap.}(K_1 \cup K_2) \leq G - \text{cap.}(K_1) + G - \text{cap.}(K_2)$$

Finally let us prove that G- cap. is continuous to the right on compact sets. Assuming that the contrary is true, we start from a compact set K with G-cap. $(K) < +\infty$ and we can find, for a suitable $\varepsilon > 0$ and any open set ω containing K, a compact set K_ω such that

$$K \subset K_\omega \subset \omega \text{ and } G - \text{cap.}(K_\omega) > G - \text{cap.}(K) + \varepsilon.$$

There exists on K_ω a measure μ_ω such that $\mu_\omega(K_\omega) = G - cap(K_\omega) + \varepsilon$ and $G\mu_\omega(x) \leq 1$. We may restrict ourselves to open sets ω contained in a compact neighbourhood K_1 of K. The family of such open sets is
45 directed for decreasing order. We take a filter, finer than the filter of sections, according to which μ_ω converges vaguely to a measure on K_1. Using a functions $f \in \mathscr{K}(E)$ whose support S_f is in CK, we see that $\int f d\mu_\omega = 0$ when $\omega \cap S_f = \phi$ therefore $\int f d\mu = 0$, that is $S_\mu \subset K$. By lemma 4, $\mu(K) = G - cap(K) + \varepsilon$, also $G\mu \leq 1$, and hence $\mu(K) \leq G -$ cap.(K). This is contradiction. The proof is complete. □

Proposition 8. *Let α be the set where a potential $G\mu \geq \lambda > 0$. Then the inner G^*-capacity of α is* $\dfrac{\mu(E)}{\lambda}$.

Let us introduce a compact set $K \subset \alpha$ and a measure ν on K such that $G^*\nu \leq 1$

$$\nu(K) \leq \int \frac{G\mu}{\lambda} = \frac{1}{\lambda}\int G^*\nu d\mu \leq \frac{\mu(E)}{\lambda}$$

Therefore G^*-cap $(K) \leq \frac{\mu(E)}{\lambda}$ and inner G^*-cap $(\alpha) \leq \dfrac{\mu(E)}{\lambda}$

Proposition 9. *The inner capacity φ_* induced by the weak capacity $\varphi(K) = G - cap(K)$ on E is subadditive on measurable subsets of E.*

Proof. Let A and B be two measurable sets. Let K be a compact set contained in $A \cup B$ such that $\varphi(K) > a$ for a choice $a < \varphi_*(A \cup B)$. Let μ be a measure on K such that $\mu(K) > a$ and $G\mu \leq 1$. For $\epsilon > 0$, we can find compact sets K, K'', respectively contained in $K \cap A$ and $K \cap B$ and such that $\mu(K \cap A) \leq \mu(K') + \varepsilon/2$ and $\mu(K \cap B) \leq \mu(K'') + \varepsilon/2$. By considering the restriction of μ to K' and K'' we obtain $\mu(K') \leq \varphi_*(K \cap A)$ and $\mu(K'') \leq \varphi_*(K \cap B)$. Now we have

$$\begin{aligned} a < \mu(K) &\leq (K \cap A) + \mu(K \cap B) \\ &\leq \mu(K') + \mu(K'') + \varepsilon \\ &\leq \varphi_*(K \cap A) + \varphi_*(K \cap B) + \varepsilon \\ &\leq \varphi_*(A) + \varphi_*(B) + \varepsilon \end{aligned}$$

$\varepsilon > 0$ being arbitrary and a being any number $< \varphi_*(A \cup B)$ the subadditivity follows. **46** ◻

10 G-capacity and G-negligible sets

A G-negligible compact set has G-capacity zero as can be easily seen. The converse is true G satisfies the weak maximum principle or if E is compact and G is regular. For if a compact set K of capacity zero is not G-negligible, we may find (Lemma 1) a measure $\mu \neq 0$ on K such that the restriction to $S\mu$ of $G\mu$ is finite and continuous. The hypothesis implies that $G\mu$ is bounded in the whole space and hence $G - cap(K) \neq 0$, which is a contradiction. Hence we deduce:

Proposition 10. *For a regular kernel G on E, G-negligible sets and sets of* inner *G-capacity zero are the same if either E is compact or G in addition satisfies weak maximum principle.*

11 *G*-capacity and strong sub-additivity

In order to make the $G-$capacity strongly subadditive we need some more conditions on the kernel. This leads us to define two new principles.

Definition 13. Principle of equilibrium (for open sets) -G-satisfies *this principle if for any relatively compact open set ω, there exists a measure μ on $\bar{\omega}$ such that $G\mu \leq 1$ on E and $G\mu = 1$ on ω. Without the conditions $S\mu\bar{\omega}$ and $G\mu \leq 1$ everywhere we shall speak of the " weak principle of equilibrium".*

47 **Definition 14.** Weak Domination Principle -*G satisfies this principle if for any two measures μ and ν, with compact supports and bounded G-potentials, the condition $G\nu \geq G\mu$ on $S\mu$ implies $G\nu \geq G\mu$ everywhere.*

Proposition 11. *Let μ_1 and μ_2 be two measures with compact supports. Let G^* satisfy the the weak principle of equilibrium. Then $G\mu_1 \geq G\mu_2$ implies $\mu_1(E) \geq \mu_2(E)$.*

Proof. Let us introduce a relatively compact open set ω containing $S\mu_1 \cup S\mu_2$. There exists a measure ν such that $G^*\nu = 1$ on ω. Now,

$$\int G\mu_1 d\nu \geq \int G\mu_2 d\nu$$

Therefore
$$\int G^*\nu d\nu_1 \geq \int G^*\nu d\mu_2$$

and hence the result. □

Theorem 8. (of strong subadditivity) .

Let G satisfy the principle of equilibrium and weak domination principle, further let G^* have the weak principle of equilibrium. Then the G-cap. is strongly subadditive.

Proof. Let φ be the G-capacity. For any two compact sets K_1 and K_2 on has to verify the condition,

$$\varphi(K_1 \cup K_2) + \varphi(K_1 \cap K_2) \leq \varphi(K_1) + \varphi(K_2).$$

□

Since φ is continuous to the right, we may introduce relatively compact open sets $\omega_i (i = 1, 2)$ such that $K_i \subset \omega_i$ and $\varphi(\overline{\omega}_i) \leq \varphi(K_i) + \varepsilon$. By the equilibrium principle, there exists on $\overline{\omega}_i$ measure μ_i such that $G\mu_i \leq 1$ everywhere and $G\mu_i = 1$ on ω; hence $\mu_i(\overline{\omega}_i) \leq \varphi(\overline{\omega}_i)$. Let ν_1 and ν_2 be two measures on $K_1 \cup K_2$ and $K_1 \cap K_2$ respectively such that $G\nu_i \leq 1$. Now, $G\mu_i = 1$ on K_i therefore $G\mu_1 \geq G\nu_1$ on K_1 and $G\mu_2 \geq G\nu_2$ on K_2. As G satisfies the weak domination principle, $G\mu_2 \geq G\nu_2$ everywhere. Hence on $K_1, G\mu_1 + G\mu_2 \geq G\nu_1 + G\nu_2$. The same inequality holds good on K_2 by a similar argument. Now applying the weak domination principle of G once again, the last inequality is true everywhere as it holds on $K_1 \cup K_2$. This in turn gives, (Prop.11), 48

$$\mu_1(E) + \mu_2(E) \geq \nu_1(E) + \nu_2(E)$$

Hence

$$\varphi(K_1) + \varphi(K_2) + 2\varphi \geq \varphi(K_1 \cup K_2) + \varepsilon(K_1 \cap K_2).$$

This inequality being true for any arbitrary $\varepsilon > 0$, the strong subadditivity follows.

12 G-polar sets

Definition 15. *A relatively compact set α of E is called a G-polar set (with respect to the kernel G) if there exists a measure μ (called associated measure) with compact support such that $G\mu = +\infty$ on α.*

Any set is called G-polar, if its intersection with every compact set is G-polar. 49

Theorem 9. *A relatively compact G-polar set has outer G^*-capacity zero.*

Let α be such a set and μ an associated measure. Let $\omega_n = \{x : G\mu > n\}$. By proposition 8, we know that the inner G^*-cap. $(\omega_n) \leq \frac{1}{n}\mu(E)$. It follows that outer G^*-cap. (α) is zero.

The converse of this theorem is not true without some additional hypothesis.

Definition 16. *A kernel G satisfies* complete equilibrium principle *if for every relatively compact open set ω, there exists a measure $\mu \geq 0$, on the boundary $\partial\omega$ with the property $G\mu \leq 1$ everywhere, $G\mu = 1$ on ω.*

Proposition 12. *Let G be a kernel satisfying complete equilibrium principle and further $G(x, y)$ be a continuous function in y, for every x, in the complement of the set $\{x\}$. Then for any relatively compact open set ω, there exists on $\partial\omega$ a measure μ satisfying $G\mu \leq 1$ everywhere, $G\mu = 1$ on ω and $\mu(\partial\omega) \leq$ inner G-cap. (ω). Moreover, if G satisfies the (weak) domination principle and G^* the weak equilibrium principle we may replace the latter inequality by the equality.*

Let us consider open sets $\omega \underset{i}{\subset} \overline{\omega} \underset{i}{\subset} \omega$ and μ_i on $\partial\omega_i$ such that $G\mu_i \leq 1$, $G\mu_i = 1$ on ω_i. The set of such ω_i is directed for increasing order and by taking a suitable filter finer than the filter of sections,we get a vague limit μ of μ_i. As $\mu_i(E) \leq G$-cap $(\overline{\omega}_i) \leq$ inner G-cap (ω), we deduce from Lemma 4, $\mu(E) \leq$ inner G-cap. (ω), $G\mu \leq 1$. Moreover,
50 by arguments similar to the one we have used before, $S\mu \subset \partial\omega$ and if $x \in \omega$, $\int G(x, y) d\mu_i(y) \to \int G(x, y) d\mu(y)$, therefore $G\mu = 1$ on ω. If $\lambda <$ inner G-cap, (ω), we can find a compact set $K \subset \omega$ and a measure ν on K such that $G\mu \leq 1$ and $\nu(E) \geq \lambda$. If $\omega_i \supset K$ then $G\mu_1 \geq G\nu$ on K and therefore (by domination principle of G) everywhere. Now follows by Prop.11, $\mu_1(E) \geq \nu(E) \geq \lambda$, and the final assertion.

Theorem 10. *Let G be a kernel satisfying the complete equilibrium principle and $G(x, y)$ for any x a continuous function of y outside $\{x\}$. Then a relatively compact set α with outer G-cap. $(\alpha) = 0$ is a G-polar set.*

Let $\{\omega_n\}$ be a decreasing sequence of relatively compact open sets containing α, such that inner G-cap. $(\omega_n) < \dfrac{1}{n^2}$. There exists on the boundary $\partial\omega_n$ a measure μ_n such that $G\mu_n = 1$ on ω_n and $\mu_n(E) \leq \dfrac{1}{n^2}$. Now $\sum \mu_n$ defines a measure [by the condition $(\sum \mu_n)(f) = \sum \mu_n(f)$ for any f in $\mathscr{K}(E)$]. It is easily seen that $\sum \mu_n$ has compact support which is contained in $\bar{\omega}_1$ and that $\sum G\mu_n = G(\sum \mu_n)$. Hence the G-potential of this measure equals $+\infty$ on α

Corollary. *If the kernel G is finite and continuous with respect to each variable when $x \neq y$, and if G and G^* satisfy complete principle of equilibrium, then the G and G^* polar sets and the sets of outer G and G^*-capacity zero are all the same.*

Chapter 5

Second Group of Convergence Theorems

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We start with a general kernel G on a locally compact space E. We shall **51**
introduce more restrictions than in Chapter 3 to get better precision.

Reduction Method of Choquet

We shall prove an analogue of Lusin's property. The device involved is an important one. This concerns in throwing out an open set of small capacity, in the complement of which the restriction of the potential due to a measure is finite and continuous.

Decomposition lemma (5). Let E be a locally compact space, μ a positive measure on a compact set K and G a regular kernel finite and continuous in the complement of the diagonal in $E \times E$. Then for any numbers $\varepsilon > 0$ and $\eta > 0$, μ can be expressed as the sum of two measures $\pi \geq 0$ and $\nu \geq 0$ such that

(i) $\pi(E) < \eta$

(ii) there exists an open set ω such that the inner G^*-cap $(\omega) < \varepsilon$ and that the restriction of $C\nu$ to $C\omega$ is continuous and $\leq \dfrac{\mu(E)}{\varepsilon}$.

Proof. Let ω be the set of points x where $G\mu(x) > \dfrac{\mu(E)}{\varepsilon}$; G^*-cap$(\omega) \leq \varepsilon$ (Prop 8). There exists a compact set $K_1 \subset \omega \cap K$ such that $\mu(\omega) -$
52 $\mu(K_1) < \eta/2$. On the other hand since $G\mu$ is bounded on $C\omega$ there exists a measure $\mu' \leq \mu$, with compact support K_2 contained in $C\omega$ such that $G\mu'$ is finite and continuous every where and $\mu(E) - \mu'(E) < \eta/2$. □

Now $\nu = \mu' + \mu_{K_1} \leq \mu$ and $\pi = \mu \ldots \nu$ fulfill the requirements of the lemma.

Fundamental lemma (6). The hypothesis being the same on E, G and μ (as in the previous lemma) for any $a > 0$, it is possible to find an open set ω of G^*- cap. < a and such that $G\mu$ restricted to $C\omega$ is bounded and continuous.

Proof. Let $\varepsilon_n > 0$, and $\eta_n > 0$ be two sequences of numbers such that $\sum \varepsilon_n < a, \eta_n \to 0$ and $\sum \dfrac{\eta_{n-1}}{\varepsilon n} < +\infty$. Applying the decomposition lemma to μ we get two measures π_1 and ν_1 and open set ω_1 such that $\pi_1(E) < \eta_1, \mu = \nu_1 + \pi_1, G^*$-cap. $(\omega_1) < \varepsilon_1$ and the restriction of $G\nu_1$ to $C\omega_1$ is less than $\mu(E)/\varepsilon_1$ and continuous. Taking the decomposition of π_1, we get ν_2, π_2, ω_2 satisfying $\pi_1 = \nu_2 + \pi_2 \pi_2(E) < \eta_2, G^* - cap(\omega_2) < \varepsilon_2$. Repetition of the process gives at the n^{th} stage

$$\pi_n = \nu_{n+1} + \pi_{n+1}\pi_{n+1}(E) < \eta_{n+1}, G^* - \text{ cap.}(\omega_{n+1}) < \eta_{n+1}$$

and $G\nu_{n+1}$ restricted to $C\omega_{n+1}$ is continuous and $< \dfrac{\pi_n(E)}{\varepsilon_{n+1}} < \dfrac{\eta_n}{\varepsilon_{n+1}}$ we assert that $\omega = \bigcup \omega_i$ answers to our need. Firstly

$$G^* - \text{cap.}\left(\bigcup \omega_i\right) \leq \sum G^* - \text{cap.}(\omega_i) < \sum \varepsilon_n < a.$$

□

53 And $\sum\limits_1^n \nu_p \to \mu$ in the strong sense $[(\mu - \sum\limits_1^n \nu_p)(E) \to 0]$, hence as in proposition 4, $G \sum \nu_n \to G\mu$. Finally, because $G\mu_n \leq \dfrac{\eta_n}{\varepsilon_{n+1}}$ on $C\omega_n \supset C\omega$; $\sum\limits_1^n G\nu_p$ converges uniformly and is bounded on $C\omega$; the sum is bounded and continuous on $C\omega$.

14 Convergence Theorems

Definition 17. *A property is said to hold G-quasi everywhere if the exceptional set where in the property does not hold good is of outer G-cap. zero.*

Theorem 11. *Let E be a locally compact space, G a kernel finite continuous in the complement of the diagonal in $E \times E$ and further G and G^* be regular. Let μ_n be a sequence of positive measures on a compact set K tending to a measure μ vaguely. Then* $\liminf .G\mu_n = G\mu$, *G^*-quasi everywhere.*

Proof. We may, find for any real number $a > 0$, open sets ω, ω_n such that G^*-cap. $(\omega) < \dfrac{a}{2}$, G^*--cap. $(\omega_n) < \dfrac{a}{2^{n+1}}$ and the potentials $G\mu, G\mu_n$ are finite and continuous respectively on $C\omega, C\omega_n$. If $\Omega = \bigcup\limits_n \omega_n \cup \omega$, we have G^*-cap. $(\omega) < a, G\mu_n, G\mu$ are finite and continuous on $C\Omega$ □

Because of the hypothesis of the continuity of G, at any point of $CK, G\mu_n \to G\mu$. Moreover, the lower semi-continuity implies $\lim ._n \inf .G\mu_n \geq G\mu$ everywhere. The points where the strict inequality $(\lim_n . \inf G\mu_n > G\mu)$ holds good are characterised by the property that there exists integers p and q such that $G\mu_n \geq G\mu + \dfrac{1}{q}$ for $n \geq p$. In other words the set of "exceptional points" is 54

$$\left\{x : \lim_n . \inf .G\mu_n(x) > G\mu(x)\right\} = \bigcup_{p,q\in\mathbb{Z}^+} \left\{x : G\mu_n(x) \geq G\mu(x) + \frac{1}{q} \text{ for } n \geq p\right\}.$$

Let $A_{p,q} = \{x : G\mu_n(x) \geq G\mu(x) + \dfrac{1}{q} \text{ for } n \geq p\} \cap C\Omega$. The sets $A_{p,q}$ being closed and contained in K is compact. We assert G^*-cap. $(A_{p,q}) = 0$. If not, there exist a non-zero measure ν on $A_{p,q}$ such that $G^*\nu$ is finite and continuous. This follows because G^* is regular (Lemma 2). Now

$$\int G\mu_n d\nu = \int G^*\nu d\mu_n$$

and the right hand side tends to $\int G^*\nu d\mu = \int G\mu d\nu$ as n tends to infinity. By the nature of definition of $A_{p,q}$

$$\int G\mu_n d\nu \geq \int G\mu d\nu + \frac{1}{q}\nu(E).$$

Hence
$$\int G\mu d\nu \leq \int G\mu d\nu + \frac{1}{q}\nu(E).$$

This is impossible because $\nu(E) \neq 0$. Hence G^*-cap. $(A_{p,q}) = 0$ for all p, q in Z^+. Hence it follows that in $C\Omega$ the set of exceptional points has outer G^*-capacity zero. The above procedure is valid for arbitrary $a > 0$ and the the Ω got from it. Now it is immediately verified that $\lim . \inf . G\mu_n = G\mu$ on E G^*-quasi everywhere.

The following two theorems follow on a line similar to the one used in Chapter 3 in an analogous situation.

55 **Theorem 12.** *E and G satisfy the same conditions as in Theorem 11. Let $\{\mu_n\}$ be a sequence of positive measures on a compact set K such that $\{\mu_n(E)\}$ is a bounded sequence and $G\mu_n(x)$ tends to $\varphi(x)$ pointwise, G^*-quasi-everywhere. Then φ is equal G^*-quasi -everywhere to the potential of a measure μ which is the vague limit of a subsequence of the given sequence μ.*

Theorem 13. *G being the same as int Theorem 11, in addition let E satisfy the second axiom of countability. Suppose $\{\mu_i\}_{i \in I}$ is any family of measures ≥ 0, on a compact set K, such that $\{\mu_i(E)\}$ is a bounded family and $G\mu_i$ is directed for the natural decreasing order. Then the lower envelope of $G\mu_i$ is equal to the potential of a measure on K, G^*-quasi-everywhere (this latter measure is the vague limit of a suitable subsequence from the given family).*

Remark . The proof of the above convergence theorems (of Chapters 3 and 5), are similar to those introduced in [6], but simpler; this being rendered possible by the lemmas based on Lusin's property (or a similar one with capacity). There is another way of using continuous potentials to get analogous convergence theorems with the aid of functional analysis (see [1], [2]).

Chapter 6

An Application to The Balayage Principle[1]

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Definition 18. *Let α be any subset of a locally compact space E. Let G* **56**
be a kernel on E and $G\mu$ a potential. For any function φ on α such that $0 \leq \varphi \leq G\mu$, let R^{α}_{φ} denote the lower envelope in E of all potentials $G\mu_i$ which majorise φ on α.

It is immediate that $R^{\alpha}_{G\mu} \leq G\mu$ everywhere and $R^{\alpha}_{G\mu} = G\mu$ on α.

Definition 19. *G is said to satisfy the principle of lower envelope if the infimum of two arbitrary potentials is also a potential.*

Theorem 14. *Suppose*

(i) *E is a compact space with a countable base for open sets.*

(ii) *G satisfies principle of lower envelope*

(iii) *G^* is regular*

[1]Similar but perhaps more interesting developments were given later in a note "Remarks sur le balayage". Bull. Soc. royale des. Sc. Liege, 30^{e} ame. 1961, *p*.210.

(iv) *there exists a* G^**-potential* > 0 *and bounded (for instance, this is satisfied if* $G^* > 0$ *and* E *is not* G^**-negligible.) then*

a) $R^\alpha_\varphi[\alpha, \varphi, \mu$ *as in Def. 18*] *and*

57 *b) if in addition* G *is regular and finite and continuous in the complement of the diagonal in* $E \times E$, *then* R^α_φ *is equal to a potential (which is again* $\leq R^\alpha_\varphi$*)* G^**-quasi-everywhere.*

Proof. Consider the family $\{\mu_i\}$ of measures such that $G\mu_i \geq \varphi$ on α; this family is non-empty by hypothesis. The lower envelope of $G\mu_i$ is not changed by keeping only those μ_i such that $G\mu_i \leq G\mu$ (μ is in the family) and that we shall suppose now. □

The $G\mu_i$ form a directed set for the natural decreasing order (by principle of lower envelope). Let $G^*\nu$ be a potential satisfying $0 < G^*\nu < L$ (L finite). From the inequality $G\mu_i \leq G\mu$ we get $\int G^*\nu d\mu_i \leq \int G^*\nu d\mu$ and therefore

$$(\inf .G^*\nu)\mu_i(E) \leq L\mu(E).$$

This shows that $\{\mu_i(E)\}$ is bounded. Applying the convergence theorems 5 (13), we deduce immediately (a) [$resp.$ (b)]

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With some more restriction on the kernel. we shall deduce a strong principle of balayage (sweeping out process) from a weaker one.

Definition 20. *If for two measures* μ_1 *and* μ_2, $G\mu_1 = G\mu_2$ G^**-quasi everywhere implies* $\mu_1 = \mu_2, G$ *is said to satisfy the* principle of uniqueness.

If G satisfies principles of uniqueness and lower envelope then "$G\mu_1 \leq G\mu_2$ G^*-quasi everywhere "implies" $G\mu_1 \leq G\mu_2$ everywhere.

Definition 21. (Weak Principle of balayage). *It is the hypothesis that*
58 *there exists a base* $\{\omega_i\}$ *of open sets with the following property: for any measure* μ *on may compact set* $K \subset \omega_i$, *there exists another measure* μ' *such that* $\mu'(\omega_i) = 0, G\mu' \leq G\mu$ *everywhere and* $G\mu' = G\mu$ *on* $C\omega_i$.

Theorem 15. *In addition to the hypotheses of Theorem 14 (b) let G satisfy the principle of uniqueness. Given any set α and a measure μ there is a* unique *measure μ_0 such that $R^{\alpha}_{G\mu} = G\mu_0 G^*$-quasi everywhere. This measure μ_0 is such that $G\mu_0 \leq G\mu$ everywhere and $G\mu_0 = G\mu$ on α G^*-quasi everywhere; and further if G satisfies the weak balayage principle then $\mu_0(C\bar{\alpha}) = 0$ (i.e. support of $\mu_0 \subset \bar{\alpha}$).*

Proof. We start with measures μ_i such that $G\mu_i \leq G\mu, G\mu_i = G\mu$ on α and arrive at a measure μ_0 such that $G\mu_0 = R^{\alpha}_{R\mu} G^*$-quasi everywhere; and this last property determines the measure μ_0 uniquely. □

Now assume that G satisfies the weak balayage principle. In order to prove that the support of μ_0 is contained in $\bar{\alpha}$ we shall realise μ_0 as the vague limit of a sequence of measures whose supports are outside some neighbourhood of (arbitrary) point $x \in C\bar{\alpha}$. Let V_x be a neighbourhood of x disjoint from an open set containing $\bar{\alpha}$. There exists W of the given base (of open sets) with $x \in W \subset V_x$. Let K be a compact neighbourhood of x contained in W. Now $\mu_i = (\mu_i)_K + (\mu_i)_{CK}$. By the weak balayage principle, there exists a measure μ'_i on CW such that $G\mu'_i \leq G(\mu_i)_K$ everywhere with equality holding on CW. This shows the measures of the form $\mu''_i = \mu'_i + (\mu_i)_{CK}$ is a subfamily of $\{\mu_i\}$ and inf. $G\mu'' = \inf .G\mu_i = R^{\alpha}_{G\mu}$.

Consider μ''_{i_1} and μ''_{i_2}. $\inf(G\mu''_{i_1}, G\mu''_{i_2})$ is a potential $G\mu_{i_3}$. By taking 59
the corresponding μ''_{i_3}, we see that $G\mu''_{i_3} \leq \inf .(G\mu''_{i_2}, G\mu''_{i_2})$. We conclude (by Theorem 13) that the lower envelope $R^{\alpha}_{G\mu}$ of the family $G\mu''_i$ is equal G^*-quasi everywhere to a potential $G\mu_1$ where μ_1 is the vague limit of a suitable sequence from μ''_i. If $\overset{\circ}{K}$ is the interior of K, since $\mu''_i(\overset{\circ}{K}) = 0$ for every i, $\mu_1(\overset{\circ}{K}) = 0$. Now $G\mu_1 = G\mu_0$ G^*-quasi everywhere. Now it is immediately seen that the support of μ_0 is contained in $\bar{\alpha}$.

Remark. With various restrictions on α, μ, G we could study the case of a locally compact space. It would be interesting to find conditions to give exceptional sets as polar ones.

We mention that all the principles that we studied in this part are satisfied in the classical case. (Green's kernel in Green's space, for example in a bounded euclidean domain: see [5]. The relations between such principles are now being discussed [14]); a good base is the complete study of a space containing a finite number of points [10].

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Part IV

Axiomatic Theory of Harmonic and Superharmonic Functions - Potentials

Chapter 1

Generalised Harmonic Functions

1 The Fundamental axioms

The following theory of harmonic and superharmonic functions derives 61
its inspiration from the earlier axiomatic theories of Tauts [5] and mainly from Doob's theory [3]. Doob wished to include the study of certain differential equations, not only of elliptic type but also of parabolic type and his principal scope was the interpretation of the behaviour of the solutions at the boundary by means of the notions of probability theory. With other, partly weaker partly stronger hypothesis we want to follow more closely but much farther the classical potential theory, first as far as a kernel may be avoided. We do not examine the probabilistic interpretations.

Fundamental space. We shall consider a connected locally compact (but not compact) Hausdorff space Ω. We introduce an Alexandroff point to get the compactification $\overline{\Omega}$.

Harmonic Functions. To each open set ω of Ω is assigned a real vector space of real valued continuous functions, called the harmonic functions in ω, defined on ω. The vector space satisfy the following local axioms:

Axiom 1. (i) If ω_0 is an open subset of ω, the restriction to ω_0 of any harmonic function in ω is harmonic in ω_0.

62 (ii) If u is a function defined in an open set ω, and harmonic in an open neighbourhood of every point of ω, u harmonic in ω.

In order to state the second axiom we need the following Definition of regularity of an open set.

Definition 1. *An open set ω in Ω is called regular if (i) it is relatively compact in Ω (i.e., its closure in the topology of $\overline{\Omega}$ is in Ω) (ii) for any finite continuous function f on the boundary $\partial\omega$ of ω there exists an unique harmonic function H_f^ω (briefly H_f) on ω such that H_f tends to f at each point of the boundary and (iii) for such a function $f \geq 0, H_f \geq 0$.*

Axiom 2. The second axiom states that there exists a base of regular domains for the open sets of the topology of Ω.

The axiom 2 implies that the space is locally connected. Note that if an open set ω is regular, any connected component δ of ω is also regular and $H_f^\omega = H_f^\delta$ in δ. That is the result of the possible finite continuous extension in Ω of a finite continuous function on the boundary of δ.

Definition 2. *Harmonic measure: Let ω be any regular open set. For any point x of $\omega, H_f^\omega(x)$ is a positive linear functional on the space of finite continuous functions defined on $\partial\omega$. In other words, H_f^ω defines a positive Radon measure on $\partial\omega$, denoted by ρ_x^ω or $d\,\rho_x^\omega$, called the harmonic measure relative to ω and x; so that we may write $H_f^\omega(x) = \int f\,d\,\rho_x^\omega$.*

If δ is the component of ω containing $x \in \omega, d\,\rho_x^\omega$ and $d\,\rho_x^\delta$ considered as measure on Ω (see Part III) are identical.

63 **Axiom 3.** Any family of harmonic functions defined on a domain ω and directed for natural increasing order (ordered increasing directed family) has an upper envelope (or a pointwise limit following the corresponding filter) which is $+\infty$ everywhere in ω or harmonic in ω.

In case the space Ω satisfies second axiom of countability[1] Axiom 3 is equivalent to a similar one with increasing sequences replacing any arbitrary family. It is an immediate consequence of the topological lemma of Choquet [Part I, Lemma 3].

2 Examples

(a) The classical harmonic functions on an open set of $\mathbb{R}^n$ satisfy the three axioms: Axiom 1 is satisfied because of the local character of the Laplacian operator. Axioms 2 and 3 follow as consequences of the properties of Poisson's integral.

(b) More generally in any euclidean domain Ω, we shall consider the functions with continuous second order derivatives (partial) which satisfy the elliptic equation

$$\sum a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum b_i \frac{\partial u}{\partial x_i} + Cu = 0$$

where $\sum a_{ik} X_i X_k$ is a positive definite quadratic form. All the coefficients and their first derivatives are supposed to be continuous and to satisfy the condition of Lipshitz locally; $C \leq 0$ [2]

Axiom 1 is satisfied because of the local character of integrals. Axioms 2 and 3 result from a local integral representation similar to the Poisson integral.

Various generalisations are possible, for instance, using varieties instead of R^n. Other examples would be desirable, particularly in connection with the general potential theory with kernels. **64**

3 First consequences

Proposition 1. *(Form A). Any function $u \geq 0$, harmonic in a domain is everywhere greater than zero or everywhere equal to zero in ω.*

[1]Unnecessary restriction (Constantinescu-Cornea: See Add. Chapter)

[2]This condition on C is not necessary (See Herve Thesis; add. chap.)

For, the sequence $\{nu\}$ has a limit which is everywhere $+\infty$ or harmonic in ω. Therefore if $u = 0$ at some point $u = 0$ everywhere.

This property is equivalent to

(A′) a harmonic function u cannot have a minimum zero at a point without being zero in some neighbourhood of it.

Note that (A) is a consequence of axiom 3 alone, even restricted to sequences.

We have immediately from (A),

Corollary 1. *On any regular open set ω there exists a harmonic function greater than $k > 0$, for instance $\int d\rho_x^\omega$.*

Corollary 2. *Consider the regular open sets ω containing a point x_0 in E ordered by inclusion. From this directed family we shall use the filter $\mathscr{F}$ of sections. Then $\int d\rho_{x_o}^\omega \underset{\mathscr{F}}{\longrightarrow} 1$ and $\int d\rho_{x_0}^\omega \underset{\mathscr{F}}{\longrightarrow}$ to the unit mass at x_o vaguely.*

For if h is harmonic > 0 in the neighbourhood $\int h(y) d\rho_{x_o}^\omega(y) = h(x_o)$ then $\int d\rho_{x_o}^\omega \underset{\mathscr{F}}{\longrightarrow} 1$.

65 **Theorem 1.** *If axioms 1 and 2 are satisfied, Axiom 3 is equivalent to the following property of the harmonic measure: for any regular domain ω (or only for these of a base) the summability relative to $d\rho_x^\omega$ independent of x in ω and for such a summable of on $\partial\omega$ $\int f\, d_x^\omega$ is finite continuous in ω and harmonic.*

Let us assume axioms 1, 2 and 3. Let ω be a regular domain and ψ a lower bounded and lower semi continuous function on $\partial\omega$: $\int \psi\, d\rho_x^\omega$ for $x \in \omega$ is the supremum of $\int \theta\, d\rho_x^\omega$ as θ ranges through the continuous functions on $\partial\omega$ such that $\theta \leq \psi$; therefore $\int \psi d\rho_x^\omega$ is either harmonic or $+\infty$.

Let now f be any function on $\partial\omega$, introducing the functions ψ which are lower bounded and lower semi continuous and which are such that $\psi \geq f$. We know that $\overline{\int} f\, d\rho_x^\omega$ is defined and is equal to inf . $\int \psi\, d\rho_x^\omega$ for these ψ. Hence $\overline{\int} f d\rho_x^\omega$: is either harmonic or identically $+\infty$ or $-\infty$. Similar conclusions can be derived as regards $\underline{\int} f d\rho_x^\omega$, which is $\leq \overline{\int} f d\rho_x^\omega$

: now with the aid of the property (A) we deduce the given properties of $d\,\rho_x^\omega$ (the summability - $d\,\rho_x^\omega$ signifies that $\int$ and *underline*$\int$ are equal and finite).

Conversely let us assume the properties of $d\,\rho_x^\omega$ and the first and second axioms. Suppose $\{f_j\}_{i\in I}$ is any family of harmonic functions on a domain δ, directed for increasing order, then we want to prove that the (pointwise) supremum of this family is $+\infty$ everywhere or a harmonic function. By considering $f_i - f_{i_o}$, we may suppose for the proof that all 66
$f_i \geq 0$.

Let ω be a regular domain of the base $\mathscr{B}$ such that $\bar{\omega} \subset \delta$. Then,

$$f_i(x) = \int f_i(y)\, d\,\rho_x^\omega(y) \text{ for every } x \text{ in } \omega.$$

Taking the limits of both the sides following the corresponding filter, (i.e.the supremum), we get

$$\lim.f_i(x) = \int \lim.f_i(y)\, d\,\rho_x^\omega(y)$$

If the right handside is $+\infty$ for some x in ω, then it is $+\infty$ for every x in ω; if it is finite for some x in ω, it is finite everywhere in ω. It follows immediately that the set of points of δ at which $\lim f_i(x)$ takes the value $+\infty$ and the set of points at which the limit is finite are two disjoint open sets of δ and consequently one of then is empty. If limit $f_i(x)$ is finite on δ, it is $d\rho_x^\omega$ summable and $\int \lim f_i(y) d\,\rho_x^\omega(y)$ is finite and continuous in any regular domain ω of δ such that $\omega \subset \bar{\omega} \subset \delta$. Therefore limit $f_i(x)$ is continuous everywhere in δ; then $\int \lim f_i(y)\, d\,\rho_x^\omega(y)$ being harmonic in any ω, $\lim\ f_i(x)$ is harmonic in δ itself.

Corollary. *On $\partial\omega$, the sets of harmonic measure zero are independent of x in ω.*

This is an immediate consequence of (A) and of the harmonicity of $\overline{\int}\varphi_e\, d\,\rho_x^\omega$, φ_e being the characteristic function of the set e contained in $\partial\omega$.

Proposition 2. *Let ω be a regular domain. Any neighbourhood of any point on the boundary $\partial\omega$ has non-zero measure as regards $d\,\rho_x^\omega$.* 67

Suppose the contrary is true of a neighbourhood N of x_o in $\partial\omega$. There exists a finite continuous function f on $\partial\omega$ which is equal to 1 at x_o and zero outside N. Now the harmonic function $H_f = \int f d\rho_x^\omega$ is zero in ω and must tend to 1 at x_o. This is clearly an impossibility.

Proposition 3. *Let f be a function on the boundary $\partial\omega$ of a regular open set ω. If f is bounded above, the function $\int f\, d\rho_x^\omega$ satisfies for any x_o in $\partial\omega$*

$$\Lambda = \lim_{x\in\omega, x\to x_o} . \sup \int f\, d\rho_x^\omega \leq \lim_{y\in\partial\omega, y\to x_o} . \sup . f(y) = \lambda$$

Proof. If $\lambda < +\infty$, let us choose $\lambda_1 > \lambda$, there exists then a neighbourhood U of x_o in which $f(y) < \lambda_1$; since f is bounded above, we can find a finite continuous function $F \geq f$, such that $F \leq \lambda_1$ in a neighbourhood of x_o. Then

$$\overline{\int} f\, d\rho_x^\omega = \int F\, d\rho_x^\omega = H_F^\omega(x)$$

and the last integral tends to $F(x_o)$ as $x \to x_o (x \in \omega)$. We conclude therefore $\Lambda \leq \lambda_1$ and hence $\Lambda \leq \lambda$. □

We have immediately a corresponding result for $\lim . \inf . \underline{\int}$.

4 The case where the constants are harmonic

68 **Proposition 4.** *In this case $\int d\rho_x^\omega = 1$. Now for any harmonic function u in an open set $\omega, u \geq inf f$ [*$\lim . \inf$ *u at the boundary points].*

If this were not true, u would attain its minimum k (finite and smaller than the right hand side) at a point $y_o \in \omega$. In the connected component δ of ω containing the point $y_o, u - k$ would be zero; u would be equal to k on δ although the lim . inf at any boundary point of δ is $> K$. This is a contradiction.

We have therefore a minimum principle and a similar maximum principle.

An important case where the constants are harmonic is the following one.

5 h-harmonic functions

Definition 3. *Let us observe that if h is a finite and continuous function > 0 in Ω then the quotients u/h of all harmonic functions u in Ω satisfy the three axioms with the same regular open sets. Moreover if h is harmonic, the new family of functions contains constants. These new functions are called h-harmonic functions. The new harmonic measure $d\,\rho'^{\omega}_x$ is such that the h-harmonic function in ω taking continuous boundary values $f(x)$ is $\int f(y)\,d\,\rho'^{\omega}_x(y)$ but also*

$$\frac{1}{h(x)}\int h(y)\,f(y)\,d\,\rho^{\omega}_x(y);$$

that is

$$d\,\rho'^{\omega}_x(y) = \frac{h(y)}{h(x)}\,d\,\rho^{\omega}_x(y).$$

Chapter 2

Superharmonic and Hyperharmonic Functions

6

Definition 4. *Let the space Ω satisfy the fundamental axioms. A function v defined on an open set ω_o of Ω is a hyperharmonic function if it satisfies* 69

(i) v is lower semi continuous

(ii) $v > -\infty$

(iii) for any regular open set (or domain) $\omega \subset \bar{\omega} \subset \omega_o$,

$v(x) \geq \int v d\, \rho_x^\omega$ *for every x in ω.*

A function u such that $-u$ is hyperharmonic, is called hyperharmonic.

First properties. On any open set ω_o of Ω, the hyperharmonic functions satisfy:

(1) If v_1 and v_2 are hyperharmonic functions then $\lambda_1 v_1, \lambda_2\, v_2$ and $\lambda_1\, v_1 + \lambda_2\, v_2$ are again hyperharmonic functions for any finite $\lambda_1 > 0, \lambda_2 > 0$. The same is true of $\inf(v_1, v_2)$.

(2) If $(v_i)_{i\in I}$ is any family of hyperharmonic functions directed for increasing order, then the upper envelope of this family is again hyperharmonic. (This is a consequence of the possibility of interchanging the operations of taking supremum and integration for $d\,\rho_x^{\omega}$).

(3)

70 **Theorem 2.** *Any hyperharmonic function on a domain ω, taking the value $+\infty$ on any open subset of ω is identically $+\infty$ in ω.*

Consider the set of points A of ω in whose neighbourhood $v = +\infty$. A is a non-empty open subset of ω. Suppose this is not the whole of ω. There exists a non empty connected component A_1 of A and a boundary point x_o of A_1 in ω. For any regular domain $\omega_1 \subset \bar{\omega}_1 \subset \omega$ containing the point x_o,

$$v(x_o) \geq \int v\,d\,\rho_x^{\omega}$$

We may choose one such ω_1, not containing A_1; then there exists a point of the boundary of ω_1 in A_1 and so $v = +\infty$ in a neighbourhood of that point. Any non-empty open set of the boundary of ω_1 has non-zero $d\,\rho_{x_o}^{\omega_1}$- measure. It follows that $v(x_o) = +\infty$. Further the summability being independent of any particular point of ω_1, $v(x) = +\infty$ for every $x \in \omega_1$. Hence x_o is in A. This is a contradiction. Therefore $v = +\infty$ everywhere in ω.

This leads us to the following definition:

Definition 5. Superharmonic functions:

A hyperharmonic function on an open set ω_o, which takes finite values at least at one point of each of the components of ω_o is called a superharmonic function. A function u such that $-u$ is superharmonic is called subharmonic.

71 A superharmonic function is $d\,\rho_x^{\omega_1}$ - summable for every x belonging to a regular open set ω_1 contained in ω_o with $\bar{\omega}_1 \subset \omega_o$.

Definition 6. Hyper and super h-harmonic functions. *If h is a finite continuous function > 0 on Ω, we can consider the hyper h-harmonic* *functions and the corresponding condition is,*

$$v(x) \geq \int v \, d\,\rho_x'^{\omega} = \int v(y) \frac{h(y)}{h(x)} \, d\,\rho_x^{\omega}$$

We see that the new functions are the quotients by $h(x)$ of the hyper and superharmonic functions.

8 Minimum principle

Theorem 3 (i)**.** *If v is a hyperharmonic function ≥ 0 in a domain, then $v = 0$ everywhere or $v > 0$ everywhere. An equivalent form is the following:*

If a hyperharmonic function on an open set has minimum zero at a point x then it is zero in some neighbourhood of x.

Let us prove first form. Suppose $v > 0$ at a point: then $v > 0$ in a neighbourhood of the point. Then nv tends to a hyperharmonic function which is $+\infty$ in an open set and hence everywhere. Therefore $v > 0$ everywhere in the domain.

Theorem 3 (ii)**.** *Suppose in an open set there exists a harmonic*[1] *func-* 72
tion $h > \varepsilon > 0$ (as it is the case with regular open sets). Any hyperharmonic function v in ω which has at every point on the boundary a limits inferior ≥ 0 is itself ≥ 0 everywhere on ω.

Proof. It is enough to consider a domain ω. $\frac{v}{h}$ continued by zero on the boundary points of ω is $> -\infty$ and lower semi continuous on closure of ω. If v were not ≥ 0 in ω, let $k < 0$ be the infimum of $\frac{v}{h}$ in $\bar{\omega}$. k is attained at some point x_o in ω. Now, in ω $\frac{v}{h} - k$ is hyper-h-harmonic ≥ 0 and equal to zero at x_o. Consequently $\frac{v}{h} - k = 0$ or $v = kh$ in ω, which contradicts the fact that $\liminf v \geq 0$ at any boundary point. □

In case, the constants are harmonic, the hyperharmonic function v satisfies:

$$v \geq \inf[\lim.\inf \ v \text{ at any boundary point }].$$

[1] or even only superharmonic (constantinescu-Cornea-Loeb.see add.chapter)

We have already proved that this property holds for harmonic functions (cl.Ch.I.3).

9 Local criterion

Theorem 4. *The condition* (iii) *of the definition of hyperharmonicity of a function v on an open set ω_o can be replaced by a weaker (locally) one viz: for any x_o in ω_o there exists for every v a base regular neighbourhoods ω' such that $\overline{\omega} \subset \omega_o$ and $v(x_o) \geq \int v \, d\rho^{\omega'}_{x_o}$.*

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We shall name the functions characterised by the condition of the above theorem [besides (i) and (ii) of Def. 4] as N - functions on $\omega_\circ$. These functions have properties similar to those of hyperharmonic functions; some of which we shall see before proving the theorem.

1) If v is a N -function ≥ 0 in an open set ω, nv and $\lim\ nv$ are also N - functions in ω.

2) If a N-functions is $+\infty$ in a domain $\omega \subset \omega_\circ$ it is $+\infty$ at every boundary point of ω in ω.

   These two properties follow with arguments similar to those in §6.

3) If $v \geq 0$ in a domain $\omega, v > 0$ or $v = 0$ everywhere in ω.

   Infact the set δ where $v > 0$ is open. Suppose it is a non-empty set different from ω. There exists a connected component δ_1 of δ which is non-empty; $\lim .nv$ is a N-function and is $+\infty$ on δ_1. Therefore on $\partial\delta_1 \cap \omega$ (non-empty); that implies $v > 0$ on this set.[2] This is a contradiction.

4) Let ω be an open set such that there exists a harmonic function $h > \varepsilon > 0$ on ω. If v is a N -function on ω, $\dfrac{v}{h}$ is an analogous function but with the measure $d\rho'^{\omega}_x$ instead of $d\rho^{\omega}_x$, as in Theorem 3(ii) it follows
74 that, lim, inf $.v \geq 0$ at all the boundary points of ω_o implies $v \geq 0$ in ω.

[2]In a shorter way, if $x_o \in \partial\delta_1 \cap \omega$, the hypothesis imply directly, for a suitable $\omega' v(x_o) \geq \int vd\rho^{\omega'}_{x_o} > 0$.

Remark . The property 4) can be proved in this way also. We may suppose ω_o to be the whole space Ω and constants to be harmonic. We have to prove that, if w satisfies axioms for an N - function the condition $\lim . \inf . w \geq 0$ at the Alexandroff point $\mathscr{O}$ of Ω implies $w \geq 0$. If $\inf . w$ were equal to $k < 0$, we introduce $w_1 = w - k \geq 0$ which is equal to zero at a point x_o in Ω. For any x_1 where $w_1 = 0$ there exists a regular neighbourhood ω such that $\int w_1 \, d\rho^{\omega}_{x_1} \leq 0$; this implies $w_1 = 0$ on $\partial\omega$. Let us consider the family Φ of open neighbourhoods δ_i of x_o in Ω such that $\lim\limits_{x \in \delta_i} . \inf . w_1(x) = 0$ at any boundary point. There is a largest one, the union δ_o (because in $\overline{\Omega}, \partial\delta_o \subset \cup_i \partial\delta_i$). The point $\mathscr{O}$ is not obviously in $\partial\delta_o$. There exists a point $z \in \partial\delta_o$ different from $\mathscr{O}$ and x_o. Now there is a regular neighbourhood δ' of z, such that $w_1 = 0$ on $\partial\delta'$ and $\delta' \cup \delta_o$ belongs to $w\,\Phi$; this contradicts the maximality of $\delta_\circ$. Hence the result.

Proof of the theorem. Given a regular open set $\omega \subset \overline{\omega} \subset \omega_o$, let us consider $V = v - \int \theta \, d\rho^{\omega}_x$ where θ is any finite continuous function such that $\theta \leq v$ on the boundary $\partial\omega$ of ω. V satisfies all the three axioms of N -functions in ω and $\lim . \inf . V \geq 0$ at all boundary points of ω. Therefore $V \geq 0$. We conclude that $v(x) \geq \int v \, d\rho^{\omega}_x$.

10 Elementary case of "balayage"

Theorem 5. *If v is a hyperharmonic function in ω_o and if ω is a regular open set with $\omega \subset \overline{\omega} \subset \omega_o$, the function E^{ω}_v which is equal to v outside ω and to $\int v \, d\rho^{\omega}_x$ in ω is hyperharmonic in ω_o. [Similar result holds by replacing "hyperharmonic" by "superharmonic"].* 75

The local conditions are fulfilled. The lower semi-continuity of the new function follows from the proposition 3 (no.3) applied to $\int (\underline{-v}) d\rho^{\omega}_x$. To any x, let us associate the regular neighbourhoods δ (with $\overline{\delta} \subset \omega_o$) such that $\overline{\delta} \subset C\overline{\omega}$ if $x \in C\overline{\omega}$ or $\overline{\delta} \subset \omega$ of $x \in \omega$. They will be used in the condition (iii) and E^{ω}_v satisfies the conditions of the local criterion.

Another proof (without using the local criterion). Consider any ω' regular ($\overline{\omega}' \subset \omega_o$) and any finite continuous function θ on $\partial\omega'$ satisfying $\theta \leq E^{\omega}_v \leq v$.

First $\int \theta \, d\rho_x^{\omega'} \leq \int v \, d\rho_x^{\omega'} \leq v$

Using prop.3, we see that in $\omega \cap \omega'$, $\int \theta \, d\rho_x^{\omega'} - \int v \, d\rho_x^{\omega}$ has at any boundary point a lim . sup ≤ 0, therefore is ≤ 0. Hence we have in ω'

$$\int \theta \, d\rho_x^{\omega'} \leq E_v^{\omega}$$

therefore

$$\int E_v^{\omega} \, d\rho_x^{\omega'} \leq E_v^{\omega}.$$

Chapter 3

Nearly Hyper or Superharmonic Functions - Reduced Functions

11

Definition 7. *Let $\mathscr{B}$ be a fixed base of regular domains of Ω. A real functions v is said to be a $\mathscr{B}$- nearly hyperharmonic function or a $S_{\mathscr{B}}$-function if* 76

(i) *v is locally bounded below*

(ii) *for every $\omega \in \mathscr{B}, v(x) \geq \overline{\int} v \, d\rho_x^{\omega}$ (for every $x \in \omega$)*

When $\mathscr{B}$ is the family of all regular domains, we say that v is a *nearly hyper-harmonic* function, or a S - function.

Note that a $S_{\mathscr{B}}$-function is not generally a nearly hyperharmonic function (as it is true when the function is lower semi- continuous, because it is in this case hyperharmonic according to Theorem 4). For instance let us start from the classical harmonic functions in R^2 and consider the function equal to zero except on the boundary of a disc ω where it is equal to 1. This function is a $S_{\mathscr{B}}$-function for any base $\mathscr{B}$ of discs which does not contain ω, but is obviously not a $S_{\mathscr{B}_1}$ function for $\mathscr{B}_1 = \mathscr{B} \cup \{\omega\}$, it is not a nearly hyperharmonic function.

The notion depends on $\mathscr{B}$, has no local character; there is no local criterion as in Theorem 4.

77 We define naturally the $S_{\mathscr{B}}$-functions in any open set of Ω as $S_{\mathscr{B}}$ in any component considered as a space. The importance of this notion is seen from the following two theorems. The first one is immediate.

Theorem 6. *The lower envelope of any set of $S_{\mathscr{B}}$- functions that are locally uniformly bounded below is a $S_{\mathscr{B}}$ - function.*

The second theorem needs the following remark. If ω_1 and ω_2 are two regular domains of the given base such that $\overline{\omega}_1 \subset \omega_2$ then the function $w(x) = \overline{\int} v\, d\rho^{\omega_2}$ $(x \in \omega_2)$ is equal in ω_1 to $\int wd\rho_x^{\omega_1}$ which is $\leq \overline{\int} v\, d\rho_x^{\omega_1}$.

Theorem 7. *If v is an $S_{\mathscr{B}}$ - function, the regularised function $\hat{v}(x)$ defined at every point x as* $\lim . \inf\limits_{y \to x} v(y)$ *is hyperharmonic and*

$$\hat{v}(x) = \sup_{\substack{\omega \ni x \\ \omega \in \mathscr{B}}} \overline{\int} v\, d\rho_x^{\omega} = \lim_{\mathscr{F}} . \overline{\int} v\, d\rho_x^{\omega}$$

(Limit according to the filter $\mathscr{F}$ of sections of the directed decreasing family of $\omega \in \mathscr{B}$ and ω containing x, ordered by inclusion.)

Proof. By the definition of the function $\hat{v}(x)$, it is lower semicontinuous and $-\infty < \hat{v}(x) \leq v(x)$. Now $\overline{\int} v\, d\rho_x^{\omega}$ is continuous of x in $\omega, \hat{v}(x) \geq \overline{\int} v\, d\rho_x^{\omega} \geq \overline{\int} \hat{v}\, d\rho_x^{\omega}$. Therefore $\hat{v}(x)$ is hyperharmonic and also $\hat{v}(x) \geq \sup\limits_{\substack{\omega \in \mathscr{B} \\ \omega \ni x}} \overline{\int} v\, d\rho_x^{\omega}$. ☐

78 On the other hand, given $\varepsilon > 0$, in a neighbourhood δ of x_o,

$$v(x) > \hat{v}(x_o) - \varepsilon$$

Therefore $\overline{\int} v\, d\rho_x^{\omega} \geq (\hat{v}(x_o) - \varepsilon) \int d\rho_x^{\omega}$ for $\omega \in \mathscr{B}, \omega \subset \delta, x \in \omega$. But we know that $\int d\rho_x^{\omega} \underset{\mathscr{F}}{\to} 1$. Hence the theorem.

12

Properties. Some of the properties of the hyperharmonic functions extend themselves to $S_{\mathscr{B}}$- functions.

a) If V_1 and V_2 are $S_{\mathscr{B}}$- functions then $\lambda_1 V_1, \lambda_1 V_1 + \lambda_2 V_2$ (for $\lambda_1 > 0, \lambda_2 > 0$) and Inf $.(V_1, V_2)$ are $S_{\mathscr{B}}$- functions.

b) If V_n is an increasing sequence of $S_{\mathscr{B}}$-functions then $\lim .v_n$ is an $S_{\mathscr{B}}$- function.

c) If v is a $S_{\mathscr{B}}$- function in a domain ω and $+\infty$ in the neighbourhood of a point, $v = +\infty$ everywhere in ω, (because $\hat{v} = +\infty$).

If a $S_{\mathscr{B}}$ - functions is not equal to $+\infty$ everywhere, we say that it is $\mathscr{B}$ - nearly superharmonic or a $S^*_{\mathscr{B}}$- function. We have immediately the notion of nearly superharmonic or S^*- functions.

d) If there exists a harmonic function $h > \varepsilon > 0$ on an open set ω_o the condition on the $S_{\mathscr{B}}$-function v that $\lim . \inf .v \geq 0$ at all the boundary points of ω_o implies $v \geq 0$. [Follows by considering $\hat{v}$.]

e) If v is a $S_{\mathscr{B}}$ - function in Ω and $\omega \in \mathscr{B}$, the function E^{ω}_v equal to v 79
on $C\omega$ and to $\overline{\int} v \, d\rho^{\omega}_x$ on ω is a $S_{\mathscr{B}}$ - function.

We have to prove that if $\omega' \in \mathscr{B}, E^{\omega}_v \geq \overline{\int} E^{\omega}_v \, d\,\rho^{\omega}_x$ for every x in ω'. As $E^{\omega}_v \leq v$ everywhere and $E^{\omega}_v = v$ on $C\omega$, the required inequality is true on $\omega' \cap C\omega$. Let us introduce on $\partial\omega$ functions f_n with the following properties.

α) $f_n \geq v$, lower semi-continuous, $\{f_n\}$ decreasing and such that $\int f_n \, d\,\rho^{\omega}_x \to \overline{\int} v \, d\,\rho^{\omega}_x$ for any $x \in \omega$.

Again a sequence of functions g_n on $\partial\omega'$ satisfying,

β) g_n a decreasing sequence of lower semi-continuous functions $\geq E_v$ and $\int g_n \, d\,\rho^{\omega'}_x \to \overline{\int} E^{\omega}_v \, d\,\rho^{\omega'}_x$ for any $x \in \omega'$.

On $\partial\omega' \cap \overline{\omega}$, replace g_n by inf $.(g_n, f_n$ or $\int f_n \, d\,\rho^{\omega}_x)$ and thereby get another sequence g'_n satisfying the conditions (β). On $\partial\omega \cap \omega'$ replace f_n by sup $.(f_n, \int g'_n \, d\,\rho^{\omega'}_x)$ and get on $\partial\omega$ functions f'_n fulfilling (α).

Now for any finite continuous functions $\theta \leq g'_n$ on $\partial\omega'$, $\int \theta \, d\rho_x^{\omega'} - \int f'_n \, d\rho_x^{\omega} \leq 0$ on $\omega \cap \omega'$ because at any boundary point the $\lim . \sup . \leq 0$.

Hence

$$\underline{\int} g'_n \, d\rho_x^{\omega} \leq \underline{\int} f'_n \, d\rho_x^{\omega} \text{ in } \omega \cap \omega'$$

then
$$\int E_v \, d\rho_x^{\omega'} \leq \int v \, d\rho_x^{\omega} = E_x^{\omega} \text{ in } \omega \cap \omega'.$$

This completes the proof. [1]

Definition 8. *We call any set - negligible (resp. negligible) in (or in*
80 *any open set ω_o) if its intersection with the boundary of any $\omega \in \mathscr{B}$ (resp. any regular domain) ($\overline{\omega} \subset \omega_o$ in the general case) has a $d\rho_x^{\omega}$ -measure zero. (Then the complimentary set is dense in ω_o).*

We may use $\mathscr{B}$- nearly everywhere (nearly everywhere) in the sense "except on a $\mathscr{B}$-negligible (negligible) set".

Remark.

1. If two hyperharmonic functions are equal $\mathscr{B}$-nearly everywhere they are equal everywhere. Any superharmonic function is finite nearly everywhere.

2. If a hyperharmonic (resp. superharmonic) function is majorised on a $\mathscr{B}$-negligible set we get $\mathscr{B}$- nearly (resp. superharmonic) hyperharmonic function. It would be interesting to compare any $\mathscr{B}$-nearly hyperharmonic function v with $\hat{v}$ which is the greatest hyperharmonic minorant.

13 Applications

Introduction of the reduced function.

[1] For another proof see remark n^o 15 and Seminar on potential theory II

Definition 9. *Let E be a subset of Ω and φ any function ≥ 0 on E. We denote by R^E_φ the lower envelope of all hyperharmonic functions $w \geq 0$ on Ω which majorise φ on E. It is a nearly hyperharmonic function.*

In the case when φ is the trace of a superharmonic function v on Ω, we call R^E_v the reduced function of v relative to E and the regularised function $\hat{R}^E_v$ is called the balayaged function of v relative to E (or extremised function relative to CE).

Immediate Properties: 81

(i) $R^E_{\lambda\varphi} = \lambda R^E_\varphi$ $(\lambda > 0)$; R^E_φ increases with φ and E.

$$R^E_{\varphi_1+\varphi_2} + \varphi_2 \leq R^E_{\varphi_1} + R^E_{\varphi_2}; R^E_\varphi \leq R^{E_1}_\varphi + R^{E_2}_\varphi \ (E_1 \cup E_2 = E)$$

The same properties hold good for $\hat{R}^E_\varphi$.

(ii) $R^E_\varphi \geq \varphi$ on E. If $E_1 \supset E, R^{E_1}_\psi = R^E_\varphi$ where $\psi = R^E_\varphi$.

(iii) The most interesting case comes up when φ is the trace of a superharmonic function $v \geq 0$ in Ω.

$0 \leq \hat{R}^E_\varphi \leq R^E_v \leq v$ everywhere. $R^E_v = v$ on E and $\hat{R}^E_v = v$ on E.

In particular if ω is a regular open set $R^{C\omega}_v = \int v \, d\, \rho^\omega_x$ on ω and $R^{C\omega}_v = \hat{R}^{C\omega}_v$ everywhere. For we know that $E^\omega_v \geq R^{C\omega}_v$. Conversely let ω be superharmonic ≥ 0 in Ω and majorise v on $C\omega$. If θ is a continuous function on $\partial\omega, \theta \leq v$, the superharmonic function $w - \int \theta \, d\, \rho^\omega_x$ in ω is ≥ 0 because of its behaviour on the boundary. Hence $R^{C\omega}_v \geq \int v \, d\, \rho^\omega_x$ in ω.

Remark 1. We may as well replace the hyperharmonic functions (motorising φ on E) by $S_{\mathscr{B}}$ - functions and study the lower envelope. But this new function is of little use when it is not identical with the first envelope.

Definition 10. *We define immediately $(R^E_\varphi)_{\omega_o}$ for a domain $\omega_o \subset \Omega$ and $E \subset \omega_o$ by considering ω_o in the place of Ω. For an open set ω_o, we define $(R^E_\varphi)_{\omega_o}$ equal in every component ω_i to $(R^{E\cap\omega_i}_\varphi)_{\omega_i}$.* 82

Chapter 4

Sets of Harmonic and Superharmonic Functions Lattices and Potentials

14 Saturated set of hyperharmonic functions

Definition 11. *A set $\mathscr{U}$ of hyperharmonic functions defined on an open* 83
set $\omega_0 \subset \Omega$ is said to be saturated if it satisfies the following conditions:

(i) *if $u_1, u_2 \in \mathscr{O}$ then inf (u_1, u_2) belongs to $\mathscr{O}$*

(ii) *for any $u \in \mathscr{O}$ and any regular domain $\omega \subset \overline{\omega} \subset \omega_0$, the functions E_u^ω, (equal to u outside ω and to $\int u d\rho_x^\omega$ in ω), again belongs to $\mathscr{O}$.*

The intersection of any set of saturated sets of hyperharmonic functions is again saturated. Hence there exists a least saturated set containing any given set S of hyperharmonic functions; it is called the saturated extension S^* of S.

We have a similar definition and property for the saturated set of hyperharmonic functions.

Theorem 8. *The lower envelope of any saturated set $\mathscr{O}$ of hyperharmonic functions on a domain δ is $+\infty, -\infty$ or harmonic.*

Proof. Let ω be any regular domain with $\bar{\omega} \subset \delta$. $E_u^\omega \leq u$ for every
84 $u \in \mathscr{O}$ and $\inf\limits_{u \in \mathscr{O}} E_u^\omega \leq \inf\limits_{u \in \mathscr{O}} u$. But in ω, E_u^ω form a directed set for decreasing order, of functions which are equal to $+\infty$ or harmonic. Hence the infimum (lower envelope) is either $+\infty$ or $-\infty$ or harmonic in Ω. We see that the sets where the envelope in question is $+\infty$, $-\infty$ or finite are disjoint open set of δ. It follows that the lower envelope is $+\infty$, $-\infty$ or finite (and therefore harmonic in this case) in the domain δ. □

A similar theorem holds good in the case of hyperharmonic functions.

15 Examples

a) Give $E \subset \Omega$ and a function $\varphi \geq 0$ on E, then the set of all hyperharmonic functions ≥ 0 which are $\geq \varphi$ on E form a saturated set on $C\bar{E}$. Hence R^E is equal in every component of $C\bar{E}$ to $+\infty$ or a harmonic function.

b) Let ω be an open set $\subset \Omega$ and f any real (finite or not) valued function on the boundary $\partial\omega$ of ω in the topology of $\bar{\Omega}$ (compactified space).

Definition 12. *$\bar{H}_f^\omega$ is the lower envelope in ω of all hyperharmonic functions on ω satisfying* $\lim . \inf . v \geq_{>-\infty} f$ *at all points of $\partial\omega$.*

Proposition 4. *In any connected component ω_i of ω $\bar{H}_f^{\omega_i} = \bar{H}_f^\omega = +\infty$, $-\infty$ or a harmonic function.*

Theorem 9. *If ω is a regular open set $\bar{H}_f^\omega(x) = \int f d\rho_x^\omega$.*

85 *Proof.* If f is a finite continuous function θ on $\partial\omega$, the behaviour of $\int \theta d\rho_x^\omega$ at the boundary implies $\int \theta \rho_x^\omega \geq \bar{H}_\theta^\omega(X)$. On the other hand, for any v satisfying the boundary condition corresponding to θ, $v - \int \theta d\rho_x^\omega$ has a lim.inf. ≥ 0 at the boundary. Therefore $v \geq \int \theta d\rho_x^\omega$ and $\int \theta d\rho_x^\omega \geq \bar{H}^\omega(x)$ □

Suppose f is a lower semi-continuous function (lower bounded) ψ. At the boundary $\lim . \inf . \int \psi d\rho_x^\omega \geq \psi$, therefore $\int \psi d\rho_x^\omega \geq \bar{H}_\psi^\omega(x)$. We see that for continuous $\theta \leq \psi$, $\overline{H}^\omega_\theta(x) = \int \theta d\rho^\omega_x \leq \overline{H}^\omega_\psi(x)$; this is true of all $\theta \leq \psi$. Hence $\int \psi d\rho^\omega_x \geq \overline{H}^\omega_\psi(x)$.

Now for any function f, introduce the v whose envelope is $\overline{H}^\omega_f$ and $\psi(y) = \lim.\inf.\ V$ at the boundary points y, $v \geq \overline{H}^\omega_\psi \geq \overline{H}^\omega_f$. Therefore $\overline{H}^\omega_f$ is the lower envelope of $\overline{H}^\omega_{f_i}$ for all f_i which are lower semi-continuous, lower bounded and $\geq f$. But $\int f d\rho^\omega_x = \inf_i. \int f_i d\rho^\omega_x$. Hence the theorem.

Proposition 5. *Suppose that for any hyperharmonic function v on ω, the condition* lim. inf. $v \geq 0$ *implies* $v \geq 0$. *Then if $\overline{v}$ is equal to v on $\partial\omega \cap \Omega$ and to zero at the Alexandroff point of Ω. Then $\overline{H}^\omega_v = R^{C\omega}_v$ on ω.*

Let w be any hyperharmonic function in ω satisfying lim. inf. $w \geq \overline{v}$ or the boundary. Inf. (w,v) continued by v is hyperharmonic ≥ 0 in ω and majorises $R^{C\omega}_v$. Conversely any hyperharmonic function $w \geq 0$ in Ω, majorising v on $\partial\omega \cap \Omega$ satisfies in Ω lim. inf. $w \geq \overline{v}$ at any boundary point, therefore is $\geq \overline{H}_v$.

Theorem 10. *Let ω' be an open set $\subset \omega$, f a function on $\partial\omega$ (in $\overline{\Omega}$), F equal to f on $\partial\omega$ and to $\overline{H}^\omega_f$ on Ω. Then $\overline{H}^\omega_f = \overline{H}^\omega_F$ on ω'.* 86

We may suppose ω, ω' connected. First $\overline{H}^\omega_f \overline{H}^{\omega'}_F$. Now if $\overline{H}^\omega_f = -\infty$, the theorem is obvious.

If $\overline{H}^\omega_f = +\infty$, any hyperharmonic function on ω', whose lim. inf. at the boundary is $F_{>-\infty}$ forms with the continuation $+\infty$, a hyperharmonic function in ω satisfying lim inf $\geq f$ at the $\geq f_{>-\infty}$ at the boundary. Hence $\overline{H}^{\omega'}_F = +\infty$. Now suppose $\overline{H}^\omega_f$ finite. Let v be a hyperharmonic function on ω' satisfying the boundary condition lim. inf. $v^1 \geq F$; the function inf. $(v^1, \overline{H}^\omega_f)$ in ω', continued by $\overline{H}^\omega_f$ is a hyperharmonic function V in ω. Now let v be any hyperharmonic function in ω satisfying the boundary condition lim. inf $v \geq f$. Let us study at the boundary of ω the function $U = V + v - \overline{H}^\omega_f$; by considering the sets where $V = v'$(on ω) and where $V = \overline{H}^\omega_f$(on ω), we conclude lim. inf. $U \geq f$ therefore $U \geq \overline{H}^\omega_f$. Hence $V \geq \overline{H}^\omega_f$, $v' \geq \overline{H}^\omega_f$ and finally $\overline{H}^{\omega'}_f \geq \overline{H}^\omega_f$.

Remark. Form this general theorem we may deduce, a shorter proof of property $(e)n^0 12$.

16 Harmonic minorants and majorants

Let $\mathscr{U}$, $\mathscr{H}$ be any two set of hypoharmonic and hyperharmonic functions respectively on an open set $\omega \subset \Omega$ such that for any $u \in \mathscr{U}$ and $v \in \mathscr{H}$, we have $u \leq v$. From this we deduce that for any $u \in \mathscr{U}^*$ and any $v \in \mathscr{H}^*$, $u \leq v$ or $\sup_{u \in \mathscr{U}} .u_* \inf_{v \in \mathscr{H}} .v_*$

87 For the set of $u \in \mathscr{U}^*$ which are $\leq$ one fixed $v \in \mathscr{H}$ form a saturated set containing $\mathscr{U}$ and is therefore identical to $\mathscr{U}^*$. Now the functions of $\mathscr{H}^*$ which are $\geq$ one fixed $u \in \mathscr{U}^*$ form a saturated set identical to $\mathscr{H}^*$ (by the same argument).

Particular case. Given $\mathscr{H}$, Let $\mathscr{U}$ be the set of all hypoharmonic functions which are $\leq$ every $v \in \mathscr{H}$, then

a) $\mathscr{U} = \mathscr{U}^*$ and does not change when $\mathscr{H}$ is replaced by $\mathscr{H}^*$

b) $V = \inf_{v \in \mathscr{H}^*} .v$ is equal to $+\infty$, $-\infty$ or harmonic in any connected component of ω.

If $V < +\infty$, it is the greatest hypoharmonic minorant of $\mathscr{H} or \mathscr{H}^*$. V is finite if and only if there exists a superharmonic function in $\mathscr{H}$ and a subharmonic minorant of $\mathscr{H}$; the V is the greatest harmonic minorant of $\mathscr{H} or of \mathscr{H}^*$.

Remark. If v is superharmonic, and u subharmonic, the condition $u \leq v$ implies that the greatest harmonic minorant of v majorises the smallest harmonic majorant of u.

17 Lattice on harmonic functions

Theorem 11. *The set of harmonic functions ≥ 0 on an open set ω is a lattice for the natural order(and even a complete lattice).*

(A partially ordered set will be called a upper semi lattice (moreover complete) if the set of two elements (resp. . in addition any nonvoid upper bounded set) has a smallest majorant; and a complete lower semi
88 lattice if the set of two elements and any nonvoid lower bounded set has

a greatest minorant. A lattice satisfying both the conditions is called a complete lattice).

We have only to note that if u_1, u_2 are harmonic functions ≥ 0, u_1+u_2 and 0 are harmonic majorant and minorant of u_1 and u_2 respectively.

Corollary. *The vector space of functions on ω which can be expressed as differences of two harmonic functions ≥ 0 is for the natural order, a Riesz space and further a complete lattice. (See Bourbaki Integration, Chap. II).*

18 Lattice on superharmonic functions

Proposition 6. *The set of superharmonic functions ≥ 0 on ω is an upper semi-lattice for the natural order (and further a complete lattice).*

Give two superharmonic functions $v_1 \geq 0$, $v_2 \geq 0$ then $v_1 + v_2$ is a superharmonic majorant. Given a set $\mathscr{H}$ of superharmonic functions v, suppose that the set W of superharmonic majorants w is non-empty. The lower envelope of W is nearly superharmonic functions w_0. We deduce now that $\hat{w}_0 \geq v$ from the inequality $w_0 \geq$ any $v \in \mathscr{H}$. Then w_0 is the smallest superharmonic majorant of $\mathscr{H}$. The passage to lattice-property is well known.

Definition 13. Specific order*: Let us call specific order for super harmonic functions on ω, the order $<$ in which $v_1 < v_2$ signifies $v_2 = v_1+$ (superharmonic functions ≥ 0); this implies the natural order.*

Theorem 12. *The set of superharmonic functions ≥ 0, on ω is an upper* 89
semi-lattice for the specific order (and further a complete lattice).

If v_1, v_2 are superharmonic functions ≥ 0, then $v_1 + v_2$ (respectively 0) is a specific majorant (minorant). We have to prove that if some v_i are superharmonic and have a common specific majorant (we consider all the specific majorants w) there is a smallest one.

Let $W = \inf_w . \; w$ (infimum in the natural sense). If $w = v_i + \omega'_i$ (ω'_i superharmonic ≥ 0), then let $W_i = \inf_w \omega'_i \geq 0$.

Then $W = v_i + W_i'$; W and W_i' are nearly superharmonic functions and $\hat{W} = v_i + \hat{W}_i > v_i$. Therefore $\hat{W}$ is a specific majorant of the w_i : $\hat{W} = W$.

We have to see that any fixed specific majorant w_0 is $> W$. Let us follow R.M. Herve and prove first that the functions U, equal to $w_0 - W$ where it is defined and equal to $+\infty$ where $w_0 = W = +\infty$, is a nearly superharmonic functions ≥ 0. For any regular domain ω, we have to see that

$$U(x) \geq \overline{\int} U d\rho_x^\omega (x \in \omega) \text{ or } w_0(x) - \overline{\int} U d\rho_x^\omega \geq W(x)$$

Let us consider on $\partial\omega$ any function ψ, which is dp_x^ω summable, lower bounded, lower semi continuous and $\geq -U$. Note that in ω,
90 $f(x) = \int \psi d\rho_x^\omega$ has a lim. inf at the boundary which is $\geq -U$. It will be enough to prove that $w_0 + f \geq W$. Observe that $w_0 + f$ in ω has a limit inferior $\geq W$ at the boundary; therefore the function α_ε $(\varepsilon > o)$ equal to inf. $(w_0 + f + \varepsilon w_0, W)$ in ω and continued by W is superharmonic. By a similar argument the function β_ε equal to $((w_0)^i + f + \varepsilon w_0, w_i')$ and continued by W_i' is superharmonic. Moreover $\beta_\varepsilon \geq 0$ (see Theorem 3(ii)).

But $\alpha_\varepsilon = v_i + \beta_\varepsilon$, therefore $\alpha_\varepsilon > v_i$, $\alpha_\varepsilon \geq W$, $w_0 + f + \varepsilon w_0 \geq W$ in ω, then also $w_o + f \geq W$.

Now U is a nearly superharmonic function; $w_0 = W + U$ nearly everywhere $w_0 = W + \hat{U}$, $w_0 > W$.

19 Vector space of differences of superharmonic function ≥ 0

Let us consider now the pairs (u, v) of superharmonic functions ≥ 0 on ω. We define an equivalence relation, in the set of pairs (u, v), denoted by $(u_1, v_1) \sim (u_2, v_2)$ by the condition

$u_1 + v_2 = u_2 + v_1$ which is equivalent to say that $u_1 - v_1 = u_2 - v_2$ nearly everywhere; every difference being defined nearly everywhere.

We denote the equivalence class containing (u, v) by $[u, v]$ and we say the function $u - v$, defined nearly everywhere, is associated to $[u, v]$. These equivalence classes form a vector space S over the real number field, if we introduce the obvious operations corresponding to the usual operations for associated functions, nearly everywhere.

More precisely, 91

$$\lambda[u, v] = [\lambda u, \lambda v] \text{ if } \lambda \text{ is real and } \geq 0 \ (0.\infty \text{ being } 0)$$
$$\lambda[u, v] = [-\lambda u, -\lambda v] \text{ if } \lambda \text{ is real and } < 0$$
$$[u_1, v_1] + [u_2, v_2] = [u_1 + u_2, v_1 + v_2]$$

(A) The *natural* order on S, denoted $[u_1, v_1] \geq [u_2, v_2]$, is defined by $u_1 + v_2 \geq u_2 + v_1$ or $u_1 - v_1 \geq u_2 - v_2$ nearly everywhere.

The corresponding "positive cone" is the set of $[u, v]$ such that $u - v \geq 0$ nearly everywhere; it is not the set of the $[u, 0]$.

Let us observe that, in the ordinary sense, nearly everywhere

$$\sup(u_1 - v_1, u_2 - v_2) = u_1 + u_2 - \inf(u_2 + v_1, u_1 + v_2).$$

Therefore, there is for the natural order in S a sup $([u_1, v_1], [u_2, v_2])$ and an associated function is sup $(u_1 - v_1, u_2 - v_2)$ nearly everywhere. (Note that if $v_1 = v_2 = 0$, it is different form the supremum which exists in the subset of the $[u, 0]$).

Similar result hold for the infimums. In general we see, with this natural order that if $X, Y \in S$ and X', Y' are any associated functions then, $X'^{+}, X'^{-}, |X'|$, sup (X', Y'), inf (X', Y') (lofineun nearly everywhere in the ordinary sense) are associated to $X^{+}, X^{-}, |X|$, sup (X, Y), and inf (X, Y) respectively. Hence

Proposition 7. *The vector space S with the natural order is a Riesz space.* 92

(B) More important is the specific order $(>)$ *defined by $[u_1, v_1] > [u_2, v_2]$ if and only if $[u_1, v_1]$ - $[u_2, v_2] = [w, 0]$ for some $w \geq 0$ (superharmonic). or equivalently,*

$$u_1 - v_1 = u_2 - v_2 + w \text{ nearly everywhere}.$$

This is the order corresponding to the choice of the "positive cone" S^{+} of the $[u, 0]$.

From the general theory of order vector spaces [Bourbaki Integration Ch, II], we obtain the,

Theorem 13. *The vector space S with the specific order is a Riesz space, (and actually a complete lattice, i.e. is "complement reticule").*

Note that the correspondence $[u, 0] \leftrightarrow u$ is an isomorphism which allows the identification of these notations. For the sake of shortness we may use $[u, v]$ and $u - v$ equivalently.

20 Potentials

Any superharmonic function which has a harmonic minorant possesses a greatest harmonic minorant as well.

Proposition 8. *If v is a superharmonic function ≥ 0, R_v^{CE}, $R_v^{C\overset{0}{E}}$ tend to the greatest harmonic minorant of v, following the filter of sections of*
93 *the increasing directed family of the relatively compact sets E of Ω*

In fact R_v^{CE} is harmonic in $\overset{\circ}{E}$ and tends to a harmonic minorant of v. If there exist harmonic minorants > 0, for any such function u, we have $u \leq R_v^{C\overset{\circ}{E}}$ on $\overset{\circ}{E}$ (since any superharmonic function in Ω which is $\geq v$ outside $\overset{\circ}{E}$ must be $\geq u$ in $\overset{\circ}{E}$ because of the behaviour of $v - u$ at the boundary of $\overset{0}{E}$ and by Theorem 3(ii)).

Remark. If $v_1 = v_2$ outside a compact set, the harmonic minorants of v_1 and v_2 in Ω are the same.

Definition 14. *A superharmonic function $v \geq 0$ in Ω is called a positive potential, briefly a potential, if its greatest harmonic minorant is zero.*

Same definition for a potential in an open set $\omega \subset \Omega$.

Immediate Properties.

If v is a potential, $\lambda v (\lambda > 0)$ is also a potential. Any superharmonic function w such that $0 \leq w \leq v$ is a potential. The infimum and the sum of two potentials v_1 and v_2 are also potentials (the latter follows because of $R_{v_1+v_2}^{CE} \geq R_{v_1}^{CE} + R_{v_2}^{CE}$).

Proposition 9. *Suppose v is hyperharmonic in an open set $\omega \subset \Omega$, and satisfies*

(i) *at any point of* $\Omega \cap \partial\omega$, lim. inf. $v \geq 0$.

(ii) *There exists a potential V in Ω such that $v \geq -V$ in Ω. Then $v \geq 0$.*

For, inf. $(v, 0)$ continued by 0 is a superharmonic function v_1 in Ω and $v_1 \geq -V$ in Ω. Therefore v_1 majorises the smallest harmonic 94
majorant of $-V$.

Difference of Potentials. If X' and Y' are equal nearly everywhere to different of potentials (in other words if $X', Y' \in S$) it is the same for X'^{+}, X'^{-}, X', sup (X', Y') and inf (X', Y').

In the space S, the subset consisting of $[u, v]$ with potentials u, v, is a subspace S' and is closed for the operations sup and inf (of two elements), $|*|$, $(*)^{+}$ and $(*)^{-}$; and is hence a Riesz space for both the orders. Similar results hold if we consider only the finite continuous functions.

21 Existence of a Potentials > 0: Consequences

The case where there exists no such potential in a domain is rather trivial; it is easy to prove, in this case, that the superharmonic functions ≥ 0 are all harmonic and proportional.

Suppose there exists a potential $V > 0$ (i.e. there exist non-harmonic superharmonic functions > 0 on Ω)

(i) On any relatively compact open set $\omega \subset \Omega$ there exists a harmonic function $> \varepsilon > 0$: $\hat{R}_v^{C\omega} > 0$ is one

(ii) **Proposition 10.** *If E is a relatively compact set, for any superharmonic functions $v \geq 0, \hat{R}_v^E$ is a potential.*

It is obvious if v is bounded on $\bar{E}$, because for a suitable $\lambda > 0$, $\lambda V > v$ on E and

$$\lambda V \geq R_{\lambda V}^E \geq R_v^E \geq \hat{R}_v^E.$$

In the general case, $v = w$ (potential) $+h$ (harmonic ≥ 0).

$$\hat{R}_v^E \leq \hat{R}_h^E + \hat{R}_w^E.$$

95 Where $\hat{R}_h^E$ is a potential according to particular case and $\hat{R}_w^E$ is a potential because it is $\leq w$.

As a complement, we take a regular domain ω, and $E \subset \overline{E} \subset \omega$; then $(\hat{R}_v^E)_\omega \to 0$ at the boundary.

Let us introduce an open set ω_1 such that $\overline{E} \subset \omega_1 \subset \overline{\omega}_1 \subset \omega(\hat{R}_v^{\omega_1}) - \int(\hat{R}_v^{\omega_1})d\rho_x^\omega$ is in ω a superharmonic function $w \geq 0$ tending to zero at the boundary of ω; therefore λw, for a suitable $\lambda > 0$, is $\geq v$ on $\overline{E}$. Hence $\lambda w \leq (R_v^E)\omega$.

(iii) **Proposition 11.** *There exists a finite continuous potential $V_0 > 0$.*

We first construct a locally bounded potential > 0; we introduce two open sets ω', ω'' such that $\overline{\omega}' \subset \omega'' \subset \overline{\omega}'' \subset \Omega\, V'' = \hat{R}_V^{C\omega''}$ is a potential > 0 harmonic in $\omega'' V' = \overline{R}_{V''}^{\omega}$ is a potential > 0, harmonic in $C\omega'$ and bounded on any compact set of Ω. We use a finite covering of $\overline{\omega}'$ with regular domains ω_i and replace V' by $\int V' d\rho_x^{\omega_1}$ in ω_1 to get V_1'. Successively replacing V_i' by $\int V_i' d\rho_x^{i+1}$ in ω_{i+1} (and this process is finite) we finally get a continuous potential $V_0 > 0$.

(iv) **Proposition 12.** *Give any domain ω, there exists a finite continuous potential > 0 which is not harmonic in ω.*

Let α be an open set such that $\alpha : \overline{\alpha} \subset \overline{\omega}$. The potential $\hat{R}_0^\alpha$ is equal to V_0 in α and hence > 0 in ω. By considering a covering of $\overline{\alpha}$ by means of regular domains δ_i we deduce as in Prop.11 a finite continuous potential $w \leq V_0$ such that $w = \hat{R}_{V_0}^\alpha$ outside $\cap\delta_i$. Hence
96 $w > 0$ and harmonic outside $\cup\delta_i$. Moreover w is not harmonic in ω, for otherwise w would be harmonic in $\Omega(\leq V)$ and therefore identically zero.

Extension. *Give a countable union of disjoint domains ω_i, there exists a finite continuous potential $w > 0$, which is not harmonic in all ω_i.*

If V_i is a potential corresponding to ω_i (for each i) got by the previous argument, $\sum \lambda_i V_i$ where $\lambda_i > 0$ and $\sum \lambda_i < +\infty$ serves the purpose.

Theorem 14 (Continuation Theorem)**.** *Suppose there exists a potential $V > 0$ in Ω. Let V be a superharmonic function, $v \geq 0$ in a regular domain δ. For any open set $\omega \subset \overline{\omega} \subset \delta$, there exists a potential in Ω which is equal to v in ω, upto a harmonic function in ω.*

Proof. We know that $(\hat{R}^{\omega}_{v})\delta$ is superharmonic ≥ 0, harmonic in $\delta - \overline{\omega}$ equal to v in ω and tends to zero at the boundary of δ. ◻

Let us introduce and open set ω_1 such that $\omega \subset \omega_1 \subset \overline{\omega}_1 \subset \delta$ and a finite continuous potential $V_0 > 0$ in ω which is not harmonic in δ. $V_0 - H^{\delta}_{V_0}$ is superharmonic > 0 in Ω and tends to zero at the boundary. For a suitable $\lambda > 0$, $\lambda(V_0 - H^{\delta}_{V_0}) \geq (\hat{R}^{\omega}_{v})\delta$ on $\partial\omega_1$. Hence the inequality holds on $\delta - \overline{\omega}_1$.

We deduce that the function equal to λV_0 in $C\delta$ and to $\lambda H^{\delta_{V_0}} + (\hat{R}^{\omega}_{v})\delta$ in δ is a potential and meets the requirements of the theorem.

Extension. *R.M. Herve has proved this theorem independently for any open set ω by a different method. An adaptation of the previous proof* 97
(using Theorem 9) may also be given.

Theorem 15 (Approximation Theorem (R.M. Herve))**.** *If there exists a potential V0, any finite continuous function f on a compact set K can be uniformly approximated on K by the difference of two finite continuous potentials on ω.*

Proof. Consider all such differences X. If V_0 is a fixed finite continuous potential V_0 0, the quotients X/V_0 form a real vector space of finite continuous functions. Moreover this vector space contains constants and X/V_0 as well. ◻

In order to apply the theorem of Stone on approximation and obtain the approximation of f/V_0 by means of some X/V_0 on K(and then the approximation of f by X) we should verify that X/V_0 separate points of K. Let $x_0 \neq y_0$ in K and a regular domain x_0 and y_0. Let W be a finite continuous potential non harmonic in ; if $\dfrac{W(x_0)}{V_0(x_0)} = \dfrac{W(y_0)}{V_0(y_0)}$ we replace W by $W_1 = Wd_x$ in. We get a finite continuous potential such that $\frac{W_1(x_0)}{V(x_0)} \neq \frac{W_1(y_0)}{V(y_0)}$.

Chapter 5

General Dirichlet Problem

22 Fundamental envelopes

The Perron - Wiener method, introduced in 1924 to solve and generalize the classical Dirichlet problem was further deeply studied and then systematically extended to various ideal boundaries but, using chiefly the classical harmonic functions. With the axiomatisation of these functions, we have to develop the method in the general set-up. **98**

Definition 15. *A saturated set* $\sum$ *if hyperharmonic functions on* Ω *will be called* additively, *(resp.* completely*) saturated, if sum of any two elements of* $\sum$ *belongs to* $\sum$, *(resp. any linear combination with coefficients* > 0*) and every hyperharmonic majorant of an element of* $\sum$ *belongs to* $\sum$.

Definition 16. *Let* $\mathscr{L}$ *be a set of filters* $\mathscr{F}$ *on* Ω *such that one of them has any adherent point in* Ω. $\mathscr{L}$ *and a set* $\sum_0$ *of hyperharmonic functions in* Ω *are said to be* associated, *if for any* $v \in \sum_0$, *the condition*

$$\liminf_{\mathscr{F}} .v \geq 0 \textit{ for every } \mathscr{F} \in \mathscr{L}, \textit{implies} v \geq 0$$

Examples. The set $\sum_1$ of all hyperharmonic functions is completely saturated. Again the set $\sum_2$ of all hyperharmonic functions such that each is bounded below is completely saturated.

1) If Ω_0 is a relatively compact domain in the fundamental space Ω, the 99
intersection of Ω_0 with the neighbourhoods of the boundary points of Ω_0 forms a set of filters which is associated to $\sum_1$ or $\sum_2$ considered in the space Ω_0 if there exists a harmonic function h on Ω_0, $h > \varepsilon > 0$. This is satisfied, for instance, when Ω_0 is a regular domain or if there exists a potential > 0 on Ω. (See Theorem 3(*ii*)).

2) In a bounded euclidean domain (more generally in a "Green space") let us consider the Green lines (gradient curves of the Green functions $G_{x_0}(y)$ issued from the pole x_0. The sets on a regular line (where $\inf . G_{x_0} = 0$) where $G_{x_0} < \varepsilon$ forms for all $\varepsilon > 0$ a base of a filter. All such filters form a set associated to $\sum_2$. See [2].

Theorem 16. *Let f be a real valued function (finite or not) on the set $\mathscr{L}$ of filters $\mathscr{F}$ associated to an additively saturated set $\sum$ of hyperharmonic functions. The function v of $\sum$ satisfying,*

$$\liminf_{\mathscr{F}} .v \geq \underset{>-\infty}{f(\mathscr{F})}$$

for every $\mathscr{F} \in \mathscr{L}$, form a saturated set whose lower envelope $\overline{\mathscr{H}}_f$ is $+\infty$, $-\infty$ or harmonic. Define $\underline{\mathscr{H}_f} = -\overline{\mathscr{H}}_{-f}$, then $\underline{\mathscr{H}}_f \leq \overline{\mathscr{H}}_f$.

Proof. Any E_v^ω (see $n^0 10$) for a regular domain ω has the same $\liminf_{\mathscr{F}}$
100 as v. Therefore the set of v under consideration is a saturated one and we may apply the Theorem 8. It remains to see the inequality. ◻

Let v and w be any two functions in $\sum$ satisfying respectively the conditions $\liminf_{\mathscr{F}} v \geq \underset{>-\infty}{f(\mathscr{F})}$, $\lim . \inf .w \geq \underset{>-\infty}{f(\mathscr{F})}$

Now

$$\liminf_{\mathscr{F}} (v + w) \geq \liminf_{\mathscr{F}} v + \liminf_{\mathscr{F}} .w \geq 0$$

As $v + w \in \sum, v + w \geq 0$.

As v and w are $> -\infty$, $v \geq -w$, $\inf_v . \; v \geq \sup_w(-w) = -\inf_w$ or $\overline{\mathscr{H}}_f \geq -\overline{\mathscr{H}}_{-f}$.

23 Properties of $\overline{\mathscr{H}}_f$

From now we suppose essentially $\overline{\mathscr{H}}_f$ is defined by means of a completely saturated set $\sum$ and of an associated set $\mathscr{L}$.

Proposition 13. (i) *$\overline{\mathscr{H}}_f$ is an increasing positively homogeneous function of f*

(ii) *If $\overline{\mathscr{H}}_f + \overline{\mathscr{H}}_g$ has a meaning (at a point and hence every where), then it is $\geq \overline{\mathscr{H}}_{f+g}$ where $f + g$ is arbitrarily chosen when ever it is not defined.*

(Basic property) Theorem 17. *Let f_n be an increasing sequence of real functions on $\mathscr{L}$ converging to f and further $\overline{\mathscr{H}}_{f_n} > -\infty$ for every n. Then $\overline{\mathscr{H}}_{f_n} \to \overline{\mathscr{H}}_f$.*

The theorem is obviously true if $\lim \overline{\mathscr{H}}_{f_n} = +\infty$. Hence we assume $\lim. \overline{\mathscr{H}}_{f_n} < +\infty$; and it is enough to show that $\overline{\lim_n} \overline{\mathscr{H}}_{f_n} \geq \overline{\mathscr{H}}_f$. Let x_0 101
be any point of Ω. For an arbitrary $\varepsilon > 0$ choose ε_n such that $\sum \varepsilon_n = \varepsilon$ Let v_n be a hyperharmonic function such that $\liminf_{\mathscr{F}} .v_n \geq \underset{-\infty}{f(\mathscr{F})}$ and $v_n(x_0) \leq \overline{\mathscr{H}}_{f_n} + \varepsilon_n$ for every n. Define $W = \lim \overline{\mathscr{H}}_{f_n} + \sum_{n=1}^{\infty} (v_n - \overline{\mathscr{H}}_{f_n})$. This hyperharmonic function majorises any v_n, therefore belongs to $\sum$ and satisfies $\liminf_{\mathscr{F}} .W \geq f_n(\mathscr{F})$ then $\liminf_{\mathscr{F}} .W \geq \underset{>-\infty}{f(\mathscr{F})}$ we conclude $W \geq \overline{\mathscr{H}}_f$ and $W(x_0) \leq \lim \overline{\mathscr{H}}_{f_n}(x_0) + \varepsilon$. Therefore $\overline{\mathscr{H}}_f(x_0) \leq \lim_n .\overline{\mathscr{H}}_{f_n}(x_0)$.

Negligible sets: Definition 17. *A subset $\alpha \subset \mathscr{L}$ is said to be negligible if $\overline{\mathscr{H}}_{\varphi\alpha} = 0$ for the characteristic function φ_α. Subsets of negligible sets are negligible and any countable union of negligible sets is again negligible. The term "almost everywhere on $\mathscr{L}$" is used as equivalence of "except on a negligible set".*

Applications

(i) If $f = 0$ almost everywhere, then $\overline{\mathscr{H}}_f = 0$ (consider first theorem where f is $+\infty$ on a negligible set and zero else where).

(ii) If $\overline{\mathscr{H}}_f$ and $\underline{\mathscr{H}}_f$ ar finite, the set where f is $\pm\infty$ is negligible. (Consider $f + (-f)$ and use Prop. 13(*ii*)).

(iii) If $f_1 = f_2$ almost everywhere, then $\overline{\mathscr{H}}_{f_1} = \overline{\mathscr{H}}_{f_2}$.

(Consider first $f_1 = +\infty$ on a negligible set and $= f_2$ elsewhere, use prop. 13(*ii*)).

24 Resolutivity

102 **Definition 18.** *If $\overline{\mathscr{H}}_f$ and $\underline{\mathscr{H}}_f$ are equal at a point, they are equal everywhere. In case they are equal finite, therefore harmonic, f is said to be resolutive and the common envelope $\mathscr{H}_f$ is called the* generalised solution*.*

First Properties

1) If f is resolutive for any constant $\lambda \neq 0$, λf is resolutive and $\mathscr{H}_{\lambda f} = \lambda \mathscr{H}_f$.

2) If f_n is an increasing sequence of resolutive functions, $\lim . f_n = f$ is resolutive if $\mathscr{H}_{f_n}(x_0)$ is bounded at some point x_0. For $\mathscr{H}_{f_n} \leq \underline{\mathscr{H}}_f \leq \overline{\mathscr{H}}_f$ and $\overline{\mathscr{H}}_{f_n} \to \overline{\mathscr{H}}_f$.

3) If f_n are finite valued resolutive functions converging uniformly to f and if $\overline{\mathscr{H}}_1$ is finite, then f is resolutive. For any $\varepsilon > 0$,

$$f_n - \varepsilon \leq f \leq f_n + \varepsilon \text{ for } n \geq N(\varepsilon).$$

Then

$$\underline{\mathscr{H}}_{f_n} + \underline{\mathscr{H}}_{-\varepsilon} \leq \underline{\mathscr{H}}_f \leq \overline{\mathscr{H}}_f \leq \overline{\mathscr{H}}_{f_n} + \overline{\mathscr{H}}_{\varepsilon}.$$

or

$$\underline{\mathscr{H}}_{f_n} - \varepsilon \overline{\mathscr{H}}_1 \leq \underline{\mathscr{H}}_f \leq \overline{\mathscr{H}}_f \leq \overline{\mathscr{H}}_{f_n} + \varepsilon \mathscr{H}_1.$$

4) If f is resolutive and $f = f_1$ almost everywhere, then f_1 is resolutive and $\mathscr{H}_{f_1} = \mathscr{H}_f$. For, if f is resolutive the set where f is infinite is negligible. The property follows as a consequence of (ii) and (iii) of §23.

103 **Equivalence classes of resolutive functions**.

It is desirable to have the resolutive functions f and $\mathscr{H}_f$ as summable functions and the corresponding integral in some suitable sense.

We shall say that two resolutive functions f_1 and f_2 are equivalent ($f_1 \sim f_2$) if $f_1 = f_2$ almost everywhere. Let $\overline{f}$ denote the class containing f.

The set Γ of equivalence classes of resolutive functions is a real vector space with obvious addition and scalar multiplication. $\mathscr{H}_f$ is the same for every functions in the same class. Hence every point of Ω defines a linear functional on Γ, namely, the value of $\mathscr{H}_f(x_0)$ for the respective classes. We may also introduce a natural order in Γ; an equivalence class $\overline{f}$ is $\geq$ another class $\overline{g}$ if any function in $\overline{f}$ is almost everywhere greater than or equal to any function in $\overline{g}$. In order to see whether sup $(f, g) \in \Gamma$, we have to study sup (f, g). We only prove:

Basic Lemma 1. If f and g are resolutive, $\underline{\mathscr{H}}_{\sup.(f,g)} = \overline{\mathscr{H}}_{\sup.(f,g)}$ (both the sides are $= +\infty$ or harmonic and equal). Assume $\underline{\mathscr{H}}_{\sup(f,g)} < +\infty$, otherwise the Lemma is obvious. Observe that $\sup(\mathscr{H}_f, \mathscr{H}_g) \leq \underline{\mathscr{H}}_{\sup.(f,g)}$. There exists a least harmonic majorant h_0 for $\sup(\mathscr{H}_f, \mathscr{H}_g)$, hence $h_0 \leq \underline{\mathscr{H}}_{\sup.(f,g)}$. For any $\epsilon > 0$ and a point $x_0 \in \Omega$ make the choice of v and w such that for every $\mathscr{F}$,

$$\text{(i)} \liminf_{\mathscr{F}} .v \geq \underset{>-\infty}{f(\mathscr{F})} \qquad \text{(ii)} \liminf_{\mathscr{F}} .w \geq \underset{>-\infty}{fg(\mathscr{F})}$$

$$v(x_0) \leq \mathscr{H}_f(x_0) + \frac{\in}{2} \; w(x_0) \leq \mathscr{H}_g(x_0) + \frac{\in}{2}.$$

Then $w_1 = h_0 + (v - \mathscr{H}_f) + (w - \mathscr{H}_g)$ is hyperharmonic $\geq v$ and w, 104
is in $\sum$ and

$$\liminf_{\mathscr{F}} .w_1 \geq \sup_{>-\infty}(f(\mathscr{F}), g(\mathscr{F}))$$

Therefore $w_1 \geq \overline{\mathscr{H}}_{\sup.(f,g)}\ \varepsilon + h_0(x_0) \geq \overline{\mathscr{H}}_{\sup.(f,g)}(x_0)$ then $h_0(x_0) \geq \overline{\mathscr{H}}_{\sup(f,g)}(x_0)$ and

$$\underline{\mathscr{H}}_{\sup.(f,g)}(x_0) \geq \overline{\mathscr{H}}_{\sup(f,g)^{x_0}}$$

Consequence. Proposition 14. *From the proof we see that if $\mathscr{H}_f$ and $\mathscr{H}_g$ have a common superharmonic majorant, this function majorises $\overline{\mathscr{H}}_{\sup.(f,g)}$.*

Corollaries. If f and g are resolutive, then sup $.(f, g)$ is resolutive if and only if $\mathscr{H}_f, \mathscr{H}_g$ have a common superharmonic majorant.

Example. 1) if f is resolutive, then f^+ is resolutive if and only if $\mathscr{H}_f$ has a superharmonic majorant ≥ 0.

105 2) If f, g are resolutive and ≥ 0 sup. (f, g) and inf. (f, g) are resolutive. Therefore the set of the resolutive functions ≥ 0, the set of their equivalence classes are lattices for the natural order.

If we want a subspace of Γ which would contain the class of the ordinary sup. of two functions or only which would be a Riesz space for the natural order, and f belonging to such a class would be majorised by a resolutive function ≥ 0; therefore f^+ would be resolutive. Hence any subspace under consideration would be contained in the subspace of the equivalence classes of the absolutely resolutive functions defined as follows.

25 Absolute Resolutivity

Definition 19. *A real function f on $\mathscr{L}$ is said to be absolutely resolutive if f^+ and f^- are both resolutive. It is equivalent to say that f equals the difference of two resolutive functions ≥ 0 almost everywhere or at every point where the difference has sense.*

Now the finite absolutely resolutive functions and the equivalence classes of absolutely resolutive functions contain resp. the ordinary sup. of two functions and the corresponding classes; and anyone of these classes contains finite absolutely resolutive functions.

We have the largest subspace of Γ we wished to have. More over, we give the following interpretation as a Daniell Integral of $\mathscr{H}_f$ for an absolutely resolutive functions f.

Starting with the Riesz space of the finite absolutely resolutive func-
106 tions α and the increasing linear functional $\mathscr{H}_\alpha(x_0)$, we see that they satisfy the Daniell condition, viz : α_n decreasing to zero implies $\mathscr{H}_{\alpha_n}(x_0) \to 0$ (x_0 fixed in Ω). We define the corresponding Daniell integral by continuation of the functional as follows: (for fixed $x_0 \in \Omega$) if

$\psi = \lim .\alpha_n$ (α_n increasing), (we see that $\lim .\mathscr{H}_{\alpha_n}(x_0)$ is the same for all sequences α_n with limit ψ, and we denote $I(\psi) = \lim. \mathscr{H}_{\alpha_n}(x_0)$. In the same way if $\varphi = \lim .\alpha_n$ (α_n decreasing) we denote $I(\varphi) = \lim .\mathscr{H}_{\alpha_n}(x_0)$. We have immediately $I(\psi) = \overline{\mathscr{H}}_\psi(x_0), I(\varphi) = \underline{\mathscr{H}}_\varphi(x_\circ)$. Now for any function $f(\mathscr{F})$, the superior and the inferior Daniell integrals are defined as

$$\overline{I}(f) = \inf_{\psi\geq f}, I(\psi),\ \underline{I}(f) = \sup_{\varphi\leq f}.I(\varphi)$$

It is obvious that $\underline{I}(f) \leq \underline{\mathscr{H}}_f(x_0) \leq \overline{\mathscr{H}}_f(x_0) \leq \overline{I}(f)$. f is said to be I-summable if $\underline{I}(f) = \overline{I}(f)$ (finite); in this case, it is well known that f^+ and $\overline{f}$ are also I- summable and therefore resolutive.

Then f is absolutely resolutive and $I(f) = \mathscr{H}_f(x_0)$.

Conversely, suppose f is resolutive and ≥ 0. Define f_n as a function equal to f where f is finite and to n where f is infinite; f_n is resolutive, i. e. a function of type α, therefore an I-summable function and $I(f_n) = \mathscr{H}_{f_n}(x_0)$. As f_n is increasing we get

$$I(f_n) = \mathscr{H}_{f_n}(x_0) \to \mathscr{H}_f(x_0) \text{ (finite)}.$$

But the limit of I- summable functions f_n with $I(f_n)$ bounded is **107**
again I-summable and $I(f) = \lim .I(f_n) = \mathscr{H}_f(x_0)$.

We may now consider any absolutely resolutive function f and conclude,

Theorem 18. *For any $x_0 \in \Omega$, the absolutely resolutive functions $f(\mathcal{F})$ are the summable functions for a certain Daniell integral I (positive linear functional) (or a certain abstract positive measure) on $\mathscr{L}$ and $I(f) = \mathscr{H}_f(x_0)$.*

Corollary. *For a set $e \in \mathscr{L}$ and the characteristic function φ_e the condition $\overline{I}(\varphi_e) = 0$ is equivalent to $\overline{\mathscr{H}}_{\varphi_e} = 0$, which does not depend on x_0.*

Proposition 15. *Any resolutive function is absolutely resolutive in the following cases*

(i) $\sum$ *is the set of all hyperharmonic functions such that every element has a harmonic minorant ≤ 0.*

(ii) $\sum$ *is the set of all lower bounded hyperharmonic functions and moreover* $\overline{\mathscr{H}}_1$ *is finite.*

In fact in both the cases, for any resolutive function f, $\mathscr{H}_f$ has a superharmonic majorant ≥ 0.

Remark 1. For any Daniell integral (positive linear functional) let us start from all finite summable functions and apply the Daniell continuation with the same values for the integrals. We get same superior and
108 inferior integrals and the set of summable functions remains unchanged.

Therefore a Daniell integral is completely defined by the class of the summable functions or of the finite summable functions and the value of the corresponding integrals. On the other hand if we start with some Riesz-space of finite summable functions the new summable functions are summable in the old sense but the converse is not true.

As for the I-integral of Theorem 18, it is therefore interesting to note that we may start with all bounded I-summable functions (i. e. absolutely resolutive functions) and then get all summable functions by means of the Daniell continuation, if we suppose that there exists a resolutive function φ which is bounded and > 0.

In fact if $f \geq 0$ is resolutive, $\psi_n = \inf .(f, n\varphi)$ is bounded resolutive and $\mathscr{H}_{\psi_n} \leq \mathscr{H}_f$. Hence the limit of ψ_n, i.e., f is summable in the sense of the integral we obtain by continuation.

Remark 2. The case where constants are resolutive is important, because in this case any I-summable function is I-measurable.

A particular case is the one where the constants are harmonic in Ω. Therefore, if there exists in some theory, a harmonic function $h > 0$, it seems better to use the h-harmonic and h-super-harmonic functions and to study the Dirichlet problems for these functions.

Chapter 6

Some Examples of Dirichlet Problem

26

Some well known Dirichlet problems are particular cases of the following one: 109

Let Ω be dense in a space $\varepsilon \neq \Omega$. $\mathscr{L}_1$ be the sets of the intersections with Ω of the neighbourhoods of the points of the boundary $B = \varepsilon - \Omega$. (We may also consider a subset $\mathscr{L}_1'$ corresponding to s subset B' of B). A function on $\mathscr{L}_1$ (or $\mathscr{L}_1'$) is considered as well a function on B (or B'), because of the obvious one-one correspondence. Let $\sum_1$ be the set of all hyperharmonic functions on Ω (we may also consider a subset $\sum_2$ as in $n^\circ 22$). Let us suppose $\sum_1$ (or $\sum_2$) and $\mathscr{L}_1$ (or $\mathscr{L}_1'$) are associated. Then Theorem 18 holds but we have to study further in every case the resolutive and absolutely resolutive functions, the integral representation (for which it is for instance interesting to find sets of resolutive functions from which the Daniell continuation gives the general integral) and finally the behaviour at the boundary of the generalised solution.
First example(with regular domain ω)

Proposition 16. *ω is considered as the fundamental space. $\bar{\omega} = \varepsilon$.*

The envelope $\overline{\mathscr{H}}_f$ corresponding to the previous $\mathscr{L}_1$ and $\sum_1$ (obvi-

110 ously associated) has been already introduced ($n^0 15$) denoted by $\overline{H}\omega_f$ and seen to be equal to $\overline{\int} f d\rho_x^{\omega}$. Therefore the resolutive functions are the $d\rho_x^{\omega}$- summable functions (hence absolutely resolutive) and for these function $\mathscr{H}_f(x) = \int f d\rho_x^{\omega}$. The Daniell integral of Theorem 18 is the Radon integral for the harmonic measures $d\rho_x^{\omega}$.

27 Second Example (with any relatively compact domain ω)

Theorem 19. *We suppose the existence of a potential* > 0 *(besides axioms 1, 2 & 3). We consider* $\overline{\omega}$ *as the space* ε *and the corresponding* $\mathscr{L}_1$ *and* $\sum_1$ *that are associated. The corresponding envelope* $\overline{\mathscr{H}_f}$ *has been studied and denoted by* $\overline{H}_f^{\omega}$ *and we may write also* $\underline{H}_f$, H_f^{ω} *instead of* $\underline{\mathscr{H}}_f$ *and* $\mathscr{H}_f$. *(We suppress index* ω *for simplicity).*

Now the finite continuous functions θ on ω are resolutive [1] (hence absolutely) and $\mathscr{H}_\theta(x)$ for such a θ defines a Radon integral that we denote $\int \theta d\mu_x . d\mu_x$ is called the harmonic measure (identical to $d\rho_x^{\omega}$ if ω is regular). The $d\mu_x$ summability and the sets of $d\mu_x$-measure are independent of $x \in \omega$. Any resolutive function is absolutely resolutive. $\overline{\int} f d\mu_x \leq \overline{H}_f^{\omega}(x)$ and hence resolutive functions f are summable and $H_f^{\omega}(x) = \int f d\rho_x^{\omega}$.

Moreover if Ω has a countable base for open sets, $\overline{\int} f d\mu_x^{\omega} = \overline{H}_f^{\omega}(x)$

111 and the $d\mu_x$-summable function are resolutive (the Daniell integral of Theorem 18 is the $d\mu_x$- integral).

Let V_1 be a finite continuous potential > 0 in Ω. As $\overline{H}_{V_1}^{\omega} \leq V_1$, $\limsup\limits_{x \to y}$. $\overline{H}_{V_1}^{\omega}(x)$ at any boundary point y is $\leq V_1(y)$, therefore $\overline{H}_{V_1}^{\omega} \leq \underline{H}_{V_1}^{\omega}$ on ω ; hence the equality. The same holds good for the difference of such V_1. Now any finite continuous function f on $\partial\omega$ can be approximated by differences ω of such potentials (Theorem 15)

$$w - \varepsilon \leq f \leq w + \varepsilon$$

Then $\quad H_w - \varepsilon \overline{H}_1 \leq \underline{H}_f \leq \overline{H}_f \leq H_w + \varepsilon \overline{H}_1.$

[1] Result and proof due to R.M.Herve.

Hence the resolutivity of f.

The properties concerning the $d\mu_x$-summability, the $d\mu_x$ measure zero are immediate consequences of the definitions of the Radon integral ($\overline{\int}$ and $\underline{\int}$) and of axiom 3. The resolutivity implies absolute resolutivity, because of the fact that the envelopes $\overline{H}_f$ and $\underline{H}_f$ remain the same by changing $\sum_1$ into $\sum_2$. (See Prop. 15 (iii)).

For any f on $\partial\omega$, we remark finally (see the proof of Theorem 9) that if v is any superharmonic function satisfying $\psi' = \lim. \inf .v \geq \underset{>-\infty}{f}$, at every boundary point, $\overline{H}^{\omega}_f \leq \overline{H}^{\omega}_{\psi} \leq v$, therefore $\overline{H}_f = \operatorname{Inf} \overline{H}_{\psi}$ for all ψ lower semi-continuous $\geq f$ and $> -\infty$. On the other hand if θ is a continuous function $\leq$ such a ψ, $H_\theta = \int \theta d\mu_x \leq \underline{H}_{\psi}$.

Hence $\overline{\int} f d\mu_x \leq \int \psi d\mu_x \leq \underline{H}_{\psi}$, therefore $\overline{\int} f d\mu_x \leq \overline{H}_f$. 112

Suppose now Ω has a countable base for open sets, therefore is metrizable, or only that on the subspace $\partial\omega$ any open set is a K_σ set. Then any lower semi-continuous function $\psi > -\infty$, is the limit of an increasing sequence of finite continuous functions $\theta_n, H_{\theta_n} \to \overline{H}_{\psi}$. Hence $\int \theta_n d\mu_x \to \int \psi d\mu_x$ therefore $\overline{H}_{\psi} = \int \psi d\mu_x$ and using a previous remark $\overline{H}_f(x) = \overline{\int} f d\mu_x$.

In the last hypothesis, the Daniell continuation from H_θ for finite continuous functions θ gives the general representation of Theorem 18.

28 Extension to a relatively compact open set ω

Hypothesis regarding Ω is assumed to be the same as in $n^0 27$.

We consider for any f on $\partial\omega$, the envelope $\overline{H}^{\omega}_f$ already introduced and the other one equal to $-\overline{H}^{\omega}_{-f}$. We recall that for any connected component ω_i of ω, $\overline{H}^{\omega_i}_f = \overline{H}^{\omega}_f$ on ω_i. (We may suppress the indices ω, ω_i when there is on ambiguity.) The resolutivity and absolute resolutivity are defined by the same condition for all ω_i, and the corresponding generalised solution is denoted by H^{ω}_f.

For a finite continuous function θ on $\partial\omega$ and any fixed $x \in \omega$, $H_\theta(x)$ is a Radon integral $\int \theta d\mu^{\omega}_x$. The measure $d\mu^{\omega}_x$, when considered in Ω is identical to the corresponding one on the component containing x. For

113 any f on $\partial\omega$, $\overline{\int} f d\mu_x^\omega \leq \overline{H}_f^\omega(x)$ and the equality holds good at least when Ω has a countable base for open sets.

Negligible sets e on $\partial\omega$ are defined by the condition $\overline{H}^\omega_{\varphi_e} = 0$ on ω. The existence of a superharmonic function ≥ 0 which tends to $+\infty$ at every point of $e < \partial\omega$, implies that e is negligible; the converse is true when ω is a countable union of domains.

A weaker condition is $\underline{H}^\omega_{\varphi_e} = 0$. This implies that a bounded harmonic function which tends to zero at any boundary point outside such an e is identically zero.

Proposition 17 (Variation of ω)**.** *Let F be a continuous function on $\partial\omega$. Given $\varepsilon > 0$ and a compact set $K \subset \omega$, there exists a compact set K_1 satisfying $K \subset K_1 \subset \omega$ and such that for any open set $\omega_1, K_1 \subset \omega_1 \subset \omega$ the inequality $\left|H_F^\omega - H_F^{\omega_1}\right| < \varepsilon$ holds on K.*

In fact, given $\varepsilon' > 0$ we may choose a superharmonic function v and a subharmonic function u (on ω), satisfying

$$\lim.\inf.v \geq F \text{ at all points of } \partial\omega, v - H_F^\omega < \varepsilon' \text{ on } K$$
$$\text{and} \qquad \lim.\sup.u \leq F \text{ at all points of } \partial\omega, H_F^\omega - u < \varepsilon' \text{ on } K.$$

Let h be a harmonic function > 0 on an open set containing $\overline{\omega}$.

Let v_1 and u_1 be respectively the continuations of v and u by their lim. inf and lim. sup. $\partial\omega$. On $\overline{\omega}$, the inequality $v_1 - F + \varepsilon' h > 0$ on
114 $\partial\omega$ implies the same inequality in an open set containing $\partial\omega$. We have a similar property for u. Hence there exists in ω a compact set K_1 containing K such that on $\omega - K_1$,

$$v - F + \varepsilon' h > 0, \qquad u - F - \varepsilon' h < 0.$$

Therefore the condition $K_1 \subset \omega_1 \subset \omega$ implies on ω_1,

$$v \geq H^{\omega_1}_{F-\varepsilon'_h} = H_F^{\omega_1} - \varepsilon' h \text{ and } u \leq H^{\omega_1}_{F+\varepsilon' h} = H_F^{\omega_1} + \varepsilon' h$$

i.e., $\left|H_F^\omega - H_F^{\omega_1}\right| \leq \varepsilon' h + \varepsilon'$

on K and hence the proposition.

Corollary. *For any increasing directed set S of open sets $\omega_1 \subset \omega$ (ordered by inclusion) such that any compact set $K \subset \omega$ is contained in some set of S, $d\mu_x^{\omega_1}$ (for a fixed $x \in \omega$) converges vaguely to $d\mu_x^{\omega}$ according to the corresponding filter.*

29 Behaviour of the generalised solution at the boundary Regular boundary points[1]

Definition 20. *A point $x_0 \in \partial\omega$ (for the open set $\omega \subset \bar{\omega} \subset \Omega$) is said to be regular, if for every finite continuous function θ on $\partial\omega$,*

$$H_\theta^\omega(x) \to \theta(x_0).(x \in \omega, x \to x_0).$$

Proposition 18. *ω is regular if and only if all boundary points are regular.* **115**

Proposition 19. *If f is an upper bounded function on $\partial\omega$, x_0 a regular boundary point then,*

$$\lim_{x\in\omega,}.\sup_{x\to x_0}.\bar{H}_f^\omega(x) \leq \lim_{y\in\partial\omega,}.\sup_{y\to x_0}.f(y).$$

Proof. If the right hand side λ is $< +\infty$, let $\lambda' > \lambda$ and θ a finite continuous function on $\partial\omega$, such that $\theta(x_0) < \lambda'$, $\theta(y) \geq f(y)$ on $\partial\omega$. Then $\bar{H}_f \leq H_\theta, \lim_{x\to x_0}.\sup.\bar{H}_f \leq \theta(x_0) \leq \lambda'$. □

Hence the proposition.

Corollaries

(i) for any bounded and resolutive function f which is continuous at a regular boundary point $x_0, H_f(x) \to f(x_0)$.

(ii) For any superharmonic function $V \geq 0, \hat{R}_V^{CW}(x_0)$ if x_0 is a regular point.

[1]In the note [2] the question was studied as an application of thin sets. A part of the study, as it is developed here, needs only a weaker hypothesis.

In fact $\overline{H}_V^\omega = R_V^{CW}$ in ω (no. 15) and $\lim_{x\in\omega} . \inf_{x\to x_0} .\underline{H}_V(x) \geq V(x_0)$ (Prop. 19)

There fore

$$\lim_{x\to} . \inf_{x_0} .R_V^{C\omega} \geq V(x_0),$$
$$\hat{R}_V^{C\omega} \geq V(x_0).$$

As $\hat{R}_V^{C\omega} \leq V$ the result is true.

Criteria of Regularity (we assume axioms 1, 2, 3 on Ω and the exis-
116 tence of a potential > 0 on Ω).

Theorem 20. *$x_0 \in \partial\omega$ is a regular point if and only if for every finite continuous potential $V, \hat{R}_V^{C\omega}(x_0) \geq V(x_0)$.*

Proof. The necessary part has already been proved. Conversely the inequality implies in particular,

$$\lim_{x\to x_0, x\in\omega} . \inf .R_V^{C\omega} \geq V(x_0)$$

As $V \geq R_V^{C\omega} = H_V^\omega$ in ω, we deduce $H_V(x) \to V(x_0)$ and because of the approximation theorem (15) we conclude $H_\theta(x) \to \theta(x_0)$ for any finite continuous function θ on $\partial\omega$. ◻

Corollary. *If $\omega_1 \subset \omega_2 \subset \overline{\omega}_2 \subset \Omega$, $x_0 \in \partial\omega_1 \cap \partial\omega_2$ the regularity of $x_0 \in \partial\omega_2$ implies the regularity for ω_1 (for $\hat{R}_V^{C\omega_1} \geq \hat{R}_V^{C\omega_2}$)*

Proposition 20 (Local character of Regularity)**.** *If $x_0 \in \partial\omega$ is regular for ω, it is regular for $\omega \cap \delta$ for any open neighbourhood δ of x_0; conversely if x_0 is regular for one such $\omega \cap \delta$, it is regular for ω.*

The first part follows from the last corollary.

As regards the converse we consider the function V' defined in $\overline{\omega}$ as equal to V in $\partial\omega$ and to H_V^ω in ω. V' on $\partial\omega'(\omega' = \omega \cap \delta)$ is resolutive [Th. 10, Ch. IV and and analogous result]. Now $H_V^\omega = H_{V'}^{\omega'}$ on ω' and now it follows [from Cor. 1, Prop. 19] that $H_{V'}^{\omega'} \to V'$ at $x_0 \in \partial\omega$,
117 therefore, $H_V^\omega \to V$ at $x_0 \in \partial\omega$ i.e., the point x_0 is regular on $\partial\omega$.

Theorem 21. *Let V_0 be a potential > 0, finite and continuous at a point $x_0 \in \partial\omega$. For x_0 to be regular point of ω it is necessary and sufficient that for every neighbourhood σ of x_0, $\hat{R}_{V_0}^{\sigma\cap C\omega}(x_0) = V_0(x_0)$.*

Proof. We can suppose σ to be compact. □

Suppose x_0 is a regular point. Let σ be a compact neighbourhood of x_0 and Ω_1 a relatively compact open set such that $(\sigma \cap \overline{\omega}) \subset \Omega_1$. Let $\Omega_1' = \Omega_1 - (\sigma \cap C\omega)$ and g the function equal to V on $\partial(\sigma \cap C\omega_1)$ and zero on $\partial\Omega_1$. Obviously $R_{V_0}^{\sigma\cap C\omega} \geq \overline{H}_g^{\Omega_1'} \geq \underline{H}_g^{\Omega_1'}$ in Ω_1'. Since x_0 is also a regular point of Ω_1', $\lim . \inf . \underline{H}_g^{\Omega_1'}(x) \geq V_0(x_0)$.

$$x \in \Omega_1', x \to x_0$$

Hence
$$\hat{R}_{V_0}^{\sigma\cap C\omega}(x_0) \geq V_0(x_0).$$

Conversely, if possible let x_0 be irregular. Then there exists a finite continuous potential $V > 0$ such that $\hat{R}_V^{C\omega}(x_0) < V(x_0)$. We may choose $\lambda > 0$ such that $\hat{R}_V^{C\omega}(x_0) < \lambda V_0(x_0) < V(x_0)$. For a suitable compact neighbourhood σ of x_0, $\lambda V_0 < V$ on σ.

Therefore, $\hat{R}_{\lambda V_0}^{\sigma\cap C\omega} \leq \hat{R}_V^{\sigma\cap C'\omega}(x_0) < \lambda V_0(x_0)$. Hence $\hat{R}_{V_0}^{\sigma\cap C\omega} V_0(x_0)$. This is clearly a contradiction. The Theorem is proved.

Theorem 22. *Suppose there exists on ω, or only on $\omega \cap \delta_0$ for an open neighbourhood δ_0 of $x_0 \in \delta\omega$, a superharmonic function $w > 0$, which tends to zero when $x \to x_0$. Then x_0 is regular for ω. Then converse is true when ω is a countable union of domains (for instance when Ω has a countable base for open sets).* 118

Proof. Let θ be any finite continuous function on $\partial\omega$, we have to prove that $H_\theta(x) \to \theta(x_0)$ $(x \in \omega)$. We may suppose $\theta \geq 0$ and actually $\theta = 0$ in a neighbourhood of x_0, for, the general case can be deduced from this one. □

Let us introduce a regular domain δ containing x_0, and such that $\overline{\delta} \subset \delta_0$ and $\theta = 0$ on $\partial\omega \cap \overline{\delta}$. Let θ_0 be the continuation of θ by the function H_θ^ω defined on ω and we remark $H_\theta^\omega = \overline{H}_\theta^{\omega\cap\delta}$ (Theorem 10).

We may choose a compact set σ on $\partial\delta\cap\omega$ such that $\sigma' = (\partial\delta\cap\omega)-\sigma$ satisfies $\int\limits_{\sigma'} d\rho^{\delta}_{x_0} < \varepsilon$ (for a $\varepsilon > 0$). With a suitable $\lambda > 0$, the function $w_1 = \lambda w + (\sup.\theta_0)\int\limits_{\sigma'} d\rho^{\delta}_{x}$ is in $\omega\cap\delta$ a superharmonic function > 0 whose lim . inf. at the boundary is $\geq \theta_0$. Hence $w_1 \geq \overline{H}^{\omega\cap\delta}_{\theta_0}$.

On the other hand, $\lim.\sup.(w_1) \leq \varepsilon(\sup.\theta_0)$. The same holds for $\overline{H}^{\omega\cap\delta}_{\theta_0}$ or H^{ω}_{θ}. As ε is arbitrary, $H^{\omega}_{\theta}(x) \to 0$ $(x\in\omega, x\to x_0)$.

119 The converse with the additional hypothesis is an application of prop. 12 (extended) because the set of components ω_i is countable. If V is a finite continuous potential > 0, which is not harmonic in every ω_i, $V - H^{\omega}_{V} > 0$ in ω and tends to zero when $x\in\omega$ tends to x_0.

Corollary. *If $x_0 \in \partial\omega$ is regular for ω, it is exterior or regular to every component ω_i of ω; the converse is true $\{\omega_i\}$ is countable.* [2]

The first part is an immediate consequence of the definition of regularity. As regards the converse we introduce superharmonic functions $w_i > 0$ ($w_i \leq$ a fixed finite continuous potential > 0) such that, in case where $x_0 \in \partial\omega_i, w_i \to 0$ $(x\in\omega_i, x\to x_0)$. Then $\sum \dfrac{1}{n^2} w_i$ is superharmonic > 0 in ω, and tends to zero $(x\in\omega, x\to x_0)$.

We shall study later the set of irregular points as an application of a theory which needs a further axiom.

30 Third Example of Dirichlet Problem

The argument of Theorem 19 can be generalised as follows: We have

(i) a compact and metric space ε (instead of the latter condition we may assume that any open set in $\varepsilon - \Omega$ is a K_σ-set.

(ii) for any superharmonic function v on Ω, the condition for every $y\in\varepsilon-\Omega$, $\lim_{x\to y}.\inf.v(x) \geq 0$ $(x\in\Omega)$ implies $v \geq \Omega$.

120 (This condition is satisfied for instance when there exists a harmonic

[2]This restriction is unnecessary.

function $h > \varepsilon > 0$ on Ω).

We essentially take $\mathscr{L}_1$ and $\sum_1$ we suppose further that the finite continuous functions on $\varepsilon - \Omega$ are resolutive.

Proposition 21. *Under the above conditions, for any finite continuous function θ on $\varepsilon - \Omega$, $\mathscr{H}_\theta(x)$ defines a Radon integral $\int \theta d\mu_x$; then $\overline{\mathscr{H}}_f(x) = \overline{\int} f d\mu_x$. The set of resolutive functions is identical with that of $d\mu_x$-summable functions (which is independent of $x \in \Omega$); hence resolutive functions are absolutely resolutive and for such functions $f, \mathscr{H}_f(x) = \int f d\mu_x$. The Daniell continuation of $\mathscr{H}_\theta(x)$ gives the general Daniell integral of Theorem 18, which is here identical to the previous Radon integral.*

Chapter 7

Reduced Functions and Polar Sets

31

We suppose only the axioms 1, 2 & 3, and go to complete first the properties of the reduced functions, R_v^E for Ω or $(R_v^E)_\omega$ for an open set $\omega \subset \Omega$, relative to the set E and the superharmonic function $v \geq 0$ (on Ω or ω). 121

Lemma 1. *$(R_v^{E\cap\omega})\omega \leq R_v^{E\cap\omega} \leq R_v^E \leq R_v^{E\cup C\omega}$ and the difference between the extreme terms is majorised by $R_v^{C\omega}$ in ω.*

The only non-trivial part is the last result. It is obvious if $\omega = \Omega$. If not, let x_0 be any point in ω. Then for any $\lambda > (R_v^{E\cap\omega})_\omega(x_0)$ and $\lambda_1 > R_v^{C\omega}(x_0)$, there exist superharmonic functions $w \geq 0$ and $w_1 \geq 0$ respectively in ω and Ω such that,

$$w \geq v \text{ on } E\cap\omega \quad \text{and} \quad w(x_0) \leq \lambda$$
$$w_1 \geq v \text{ on } C\omega \quad \text{and} \quad w_1(x_0) \leq \lambda_1.$$

Let us define w' equal to v on $C\omega$ and to $\inf.(w_1 + w, v)$ on ω.This is a superharmonic function ≥ 0, on Ω and this $w' \geq v$ on $E \cap C\omega$; therefore $w' \geq R_v^{E\cup C\omega}$. Hence $R_v^{E\cup C\omega}(x_0) \leq w'(x_o) \leq w'(x_0) \leq \lambda + \lambda_1$. This being true of any $\lambda > (R_v^{E\cap\omega})_\omega(x_0)$ and $\lambda_1 > R_v^{C\omega}(x_0)$, we get

$$R_v^{E\cup C\omega}(x_0) \leq (R_v^{E\cap\omega})_\omega(x_0) + R_v^{C\omega}(x_0)$$

122 Now it is obvious that the inequality is true at all points $x_0 \in \omega$.

Corollary. *If v is a potential, $(R_v^{E\cap\omega})_\omega \to R_v^E$ at any point, according to the directed set $\omega \subset \overline{\omega} \subset \Omega$.*

In fact, $R_v^{C\omega}$ tends to the greatest harmonic minorant of v, i.e. zero.

Theorem 23. *Let $V > 0$ be a finite continuous superharmonic function in Ω and $x_0 \in \Omega$. Then $R_v^E(x_0)$ defines on compact sets E a strong capacity such that for any set E, the corresponding outer capacity is $R_v^E(x_0)$.*

Proof. The case in which there exists no potential > 0 is obvious. Hence we assume the existence of a finite continuous potential $V_0 > 0$.

Let us first prove that $R_V^E(x_0) = \inf .R_V^\omega(x_0)$ for all open sets ω containing E. Let v be a superharmonic function ≥ 0 such that $v \geq V$ on E and $R_V^E(x_0) + \varepsilon \geq v(x_0)$, (for some choice of $\varepsilon > 0$). The inequality $v \geq V(1-\varepsilon)$ holds in an open set $\omega \supset E$, therefore $\dfrac{v}{1-\varepsilon} \geq R_V^\omega$. Then

$$(1-\varepsilon)R_V^\omega(x_0) \leq v(x_0) \leq R_V^E(x_0) + \varepsilon$$

i.e.,
$$R_V^\omega(x_0) - R_V^E(x_o) \leq \varepsilon(1 + V(x_0))$$

Hence the $R_V^E(x_0) = \inf .R_V^\omega(x_0)$ as required. □

Now, suppose E is a compact set. $R_V^E(x_0)$. is an increasing function of E. The above property shows that this function is continuous on the
123 right. We shall show that this set function is strongly subadditive, i.e.,

$$R_V^{E_1\cup E_2} + R_V^{E_1\cap E_2} \leq R_V^{E_1} + R_V^{E_2}$$

We may suppose that V is a potential by taking inf. $(V, \lambda V_o)$ instead of V, where λ is so chosen that $\lambda V_o \geq V$ on $E_1 \cup E_2$. (The in-equality will be unaltered.) The inequality is first of all true for all points of $E_1 \cup E_2$; for instance if $x_o \in E$,

$$R_V^{E_1\cup E_2} = V = R_V^{E_1},\ R_V^{E_1\cap E_2} \leq R_V^{E_2}.$$

If we take two superharmonic functions $v_1 \geq 0$ and $v_2 \geq 0$ such that $v_1 \geq V$ on E_1, $v_2 \geq V$ on E_2 then it is enough to prove that

$$R_V^{E_1 \cup E_2} + R_V^{E_1 \cap E_2} \leq v_1 + v_2.$$

Hence consider in $C(E_1 \cup E_2)$ the function

$$D = v_1 + v_2 - \left(R_V^{E_1 \cup E_2} + R_V^{E_1 \cap E_2}\right).$$

The reduced functions being everywhere upper semi-continuous, we have for $x \in C(E_1 \cup E_2)$ and $x \to y \in \partial(E_1 \cup E_2)$

$$\lim.\sup.R_V^{E_1 \cup E_2} + R_V^{E_1 \cap E_2} \leq R_V^{E_1 \cup E_2}(y) + R_V^{E_1 \cap E_2}(y) \leq R_V^{E_2}(y) + R_V^{E_2}(y)$$

and hence, $\lim.\inf.D \geq 0$.

Now in $C(E_1 \cup E_2), D \geq -2V$, hence by Prop. 9(ii) $n^\circ.20), D \geq 0$. This proves the strong subadditivity.

We shall see that for any open set $\omega \subset \Omega$, **124**

$$R_V^\omega = \sup_{K \subset \omega}.R_V^K \quad (K \text{ compact}),$$

which implies that $R_V^\omega(x_o)$ is the corresponding inner capacity of ω ·, and finally that $R_V^E(x_o)$ is the outer capacity for any set E.

In fact, $\hat{R}_V^K \leq V; \hat{R}_V^K$ tends, according to the directed set of $K \subset \omega$, to a superharmonic function ≥ 0, which is equal to V on ω, therefore majorises R_V^ω. We conclude,

$$\sup_{K \subset \omega}.R_V^K = \sup_{K \subset \omega}.\hat{R}_V^K = R_V^\omega = \hat{R}_V^\omega.$$

32 Polar sets

Definition 21. *A set E is said to be polar in the open set $\omega_\circ \subset \Omega$, if there exists in $\omega_\circ$ a superharmonic function $v \geq 0$ (or equivalently a potential), called associated to E, equal to $+\infty$ at least on $E \cap \omega_o$.*

The term "quasi-everywhere in $\omega_\circ$" "will mean except for a set polar in $\omega_\circ$".

The property of a set being polar in $\omega_\circ$ is equivalent to its being polar in every component of $\omega_\circ$.

A set will be said to be polar from inside if every compact subset is polar.

125 **First Properties of Polar sets in Ω:**

(i) For any open set ω, relatively compact, a polar set E on $\partial\omega$ is negligible for ω, because $\overline{H}^{\omega}_{\varphi_E}$ is majorised by any function associated to E.

CE is therefore dense (for a polar set E); any hyperharmonic function is determined by its values outside a polar set E (at $x \in E$, $v(x) \lim \int v d\rho^{\omega}_{x}$ according to the filter $\mathscr{F}_x$ of section of the decreasing directed set of all regular domains containing x).

A function will be said to quasi-hyperharmonic or quasi superharmonic if it is quasi-everywhere equal to a hyper (resp. super) harmonic function; in both the cases the latter function is unique.

(ii) Any countable union of polar sets is polar.

We choose each E_n an associated function v_n such that $\int v_n d\rho^{\omega_o}_{x_o} = \dfrac{1}{n^2}$($\omega_o$ regular, $x_o \in \omega_o$), then $\sum v_n$ is associated to $\cup E_n$.

(iii) If E is a closed set polar in Ω, then CE is connected.

If not, $CE = \omega_1\omega_2, \omega_1 \neq \phi, \omega_2 \neq \phi, \omega_1 \cap \omega_2 = \phi$ and ω_1 and ω_2 open sets. Let w be an associated function (to E). By changing the value of w into $+\infty$ on ω_2 we get a hyper harmonic function w'(w' satisfies the local criterion). $w' = +\infty$ on ω_2 and it is not identically $+\infty$, this is a contradiction.

(iv) Let E be a closed polar set in Ω. A superharmonic function v in $\Omega - E$, supposed to be lower bounded on every compact subset of Ω, can be uniquely continued on E to a superharmonic function.

126 Let w be associated to E. Define the sequence v_n of superharmonic functions as : $v_n = +\infty$ on E and $v_n = v + (1/n)w$ in CE uniformly lower bounded on every compact set. Then inf. v_n is a nearly superharmonic function V, and $V = v$ at any point where w is finite. We deduce, $\int V d\rho_x^\omega = \int v d\rho_x^\omega$ for any ω regular, $\overline{\omega} \subset C(E)$. Hence $\hat{V} = v$ on E, v is the continuation needed.
Consequence. If h is harmonic in $\Omega - E$ and bounded on every compact set of Ω, there exists a unique harmonic continuation of h on Ω.

33 Criterion

Theorem 24. *If E is a polar set in Ω, and w a superharmonic function > 0 in Ω, then $\hat{R}_w^E = 0$ everywhere and $R_w^E = 0$ q.e. Conversely, if for one superharmonic function $w > 0$, $\hat{R}_w^E$=0 every where, or if there exists $x_o \in \Omega$ such that $\hat{R}_w^E(x_o) = 0$ then E is a polar set.*

Suppose E is a polar set, v an associated function and x_o such that $v(x_o) < +\infty$. Then $\lambda v \geq w$ on E for any $\lambda > 0$, therefore $R_w^E(x_o) = 0$. Hence $R_w^E = 0$ q.e., therefore on a dense set. It follows $\hat{R}_w^E = 0$.

Conversely, suppose $R_w^E(x_o) = 0$. There exists superharmonic function $v_n > 0$ such that $v_n \geq w$ on E and $v_n(x_o) < \dfrac{1}{n^2}$. Then $\sum v_n$ is superharmonic > 0 and $= +\infty$ on E.

If we suppose only $\hat{R}_w^E = 0$ at a point, therefore everywhere, R_w^E 127
is a nearly superharmonic function whose regularised function is zero. Therefore $\int R_w^E d\rho_x^\omega = 0$ for any regular domain ω we deduce that $R_w^E = 0$ nearly everywhere, therefore at least at one point.

Corollary . *For any set F and a polar set E, and any superharmonic function $v \geq 0$, $\hat{R}_V^{E\cup F} = \hat{R}_v^F$.*

Theorem 25 (Local character Theorem)**.** *Suppose there exists a potential > 0 in Ω. Let E be a set which is locally polar (i.e. for some neighbourhood δ of every point $E \cap \delta$ is polar in δ). Then E is a polar set in Ω.*

Proof. Let us suppose that V_o is a continuous potential > 0.

a) Suppose first $E \subset \overline{E} \subset \delta$ where δ is a regular domain, and E polar in δ. If v is a superharmonic function in δ, associated to E, we know (Theorem 14) that there exists in Ω a potential which is equal to v on an open set containing $\overline{E}$ upto a harmonic function. This shows E is polar in Ω.

b) Let us suppose that E is relatively compact. To every $x \in \Omega$, we associate a neighbourhood δ such that $E \cap \delta$ is polar in δ. Let us introduce an open set δ'' and a regular domain δ' such that $\delta'' \subset \overline{\delta''} \subset \delta' \subset \delta$. $E \cap \delta''$ is polar in δ' therefore in Ω. We cover $\overline{E}$ by a finite union of such δ'' that we call δ''_i. The sets $\delta''_i \cap E$ are polar in Ω, and so also is E.

c) (general case) We consider E as the union of two sets E_1 and E_2 whose exteriors are not empty. Suppose $x_o \notin \overline{E}_1$. For any relatively
128 compact set $\omega \ni x_o, E \cap \overline{\omega}$ is polar in $\Omega[(b)]$, therefore $E_1 \cap \omega$ is polar in ω and $\hat{R}^{E_1 \cap \omega}_{V_o} = 0$; $R^{E_1 \cap \omega}_{V_o}(x_o) = 0$. As $R^{E_1 \cap \omega}_{V_o} \to R^{E_1}_{V_o}$ according to the directed family of the considered sets ω (lemma 1, cor. 1) we conclude $R^{E}_{V_o}(x_o) = 0$. Now the theorem follows immediately.

□

Proposition 22. *A K-analytic set E_o (in the sense of Choquet) which is polar from inside is polar.*

Suppose E_o non-empty and $V_o > 0$ a continuous potential in Ω. For any compact set $K, E_o \cap K$ is a K-analytic set, the capacity $R^{E}_{V_o}(x_o)$ is equal to zero for any compact set $E \subset E_o \cap K (x_o \notin E_o \cap K)$ therefore also for $E_o \cap K$; as $E_o \cap K$ is polar for any K, E_o is polar.

Chapter 8

Convergence Theorems

34 Domination Principle

We suppose only the axioms 1, 2 and 3 on the space Ω. 129

Definition 22. *Let $v \geq 0$ be a superharmonic function in Ω.*

(a) the support of v is the complement of the largest open set in which v is harmonic;

(b) the best harmonic minorant of v in an open set $\omega \subset \Omega$ is $R_v^{C\omega}$ or $\bar{H}_v^{\omega}$.

We shall now introduce a new axiom and prove some equivalent forms of the same.
Axiom D (form α) : If v is a potential in Ω, which is locally bounded in Ω and harmonic in a domain $\delta \subset \bar{\delta} \subset \Omega$; any other potential which is $\geq v$ on $C\delta$ is $\geq v$ on δ as well.
(Equivalent forms) (β): For any relatively compact open set $\omega \subset \Omega$, and any superharmonic function ≥ 0 which is locally bounded in Ω, the best harmonic minorant in ω is equal to the greatest one.

(α) (Domination Principle): If v is any locally bounded potential ≥ 0, for any superharmonic function $w \geq 0$, the fact that $w \geq v$ on the support of v, implies $w \geq v$ everywhere.

Let us assume the existence of a potential > 0 (otherwise α, β and γ are trivial). Let us prove the equivalences. First of all β and γ imply α.

130 Let us suppose (α) and prove (β). We may assume in the hypothesis of (β) that v is a potential. Let us introduce a relatively compact open set ω_o in Ω such that $\overline{\omega} \subset \omega_o$ and observe that $\hat{R}_v^{\omega_o}$ is a potential with the same harmonic majorant as v in ω. We may also suppose ω to be a domain because of the properties of the minorants involved.

We observe that the greatest harmonic minorant of v in ω is inf. $R_v^{C\delta}$ for all $\delta \subset \overline{\delta} \subset \omega$, and its continuation by v is a nearly superharmonic function $V \leq v$. Let w be any superharmonic function ≥ 0 in Ω, and $w \geq v$ on $C\omega$. Then $w_1 = \inf(v, w)$ is superharmonic ≥ 0 and $\leq v$. w_1 is a potential $\geq \hat{V}$ on $C\omega$, therefore $R_v^{C\omega} \geq \hat{V}$ on ω. Hence $R_v^{C\omega} = V$ on ω.

Let us suppose α) and β) and prove γ). Let v be a locally bounded potential > 0 with support S and w a superharmonic function ≥ 0 and $w \geq v$ on S. Consider any superharmonic function $w_1 \geq 0$ with $w_1 \geq v$ on $C\omega$ where $\omega \subset \overline{\omega} \subset \Omega$. Then $w + \inf .(w_1, v) \geq v$ on $S \cup C\omega$; therefore $w + \inf .(w_1, v) \geq R_v^{S \cup C\omega}$ and $w + R_v^{c\omega} \geq R_v^{S \cup C\omega}$. By β), the second member $= v$ in $\omega \cap CS$. Now $R_v^{C\omega} \to 0$ according to the directed set of $\omega \subset \overline{\omega} \subset \Omega$. Hence $w \geq v$.

Remark 1. If we suppress in $\alpha), \beta), \gamma)$ the hypothesis that v is locally bounded, these properties still remain equivalent with the same arguments, but they are no longer true in the classical case.

Remark 2. When the constant 1 is superharmonic, D) implies that for
131 any locally bounded potential $v, v \leq \sup .v$ on S (the support of v). (Maximum principle).

Local Character of (D). Using the continuation theorem (14, n^o 21) we see that whenever there exists a potential > 0 on Ω, axiom D for Ω, (in the form β) implies the same property for any regular subspace or even any subspace and therefore for a neighbourhood of each point. The converse seems true. We know the proof only in the case when Ω has a countable base for open sets. This converse has been proved as a consequence of a further study by R.M. Herve.

35 First application of axiom D

Theorem 26. *Let v be a superharmonic function in Ω, ≥ 0, locally bounded and with support S. We suppose the axioms 1, 2, 3 and D on Ω. For any point y on the boundary of S,*

$$\underset{x \in CS,\, x \to y}{\lim . \sup .} v(x) = \underset{x \in \partial S,\, x \to y.}{\lim . \sup .} v(x)$$

Therefore v is continuous in Ω at y, if the restriction of v to S is continuous at y.

Proof. We shall use the following remark (that is true independent of D). □

Let v be a superharmonic function $v(x_o)$ finite (at the point x_o). Consider the regular domains ω containing x_o, and the set $e \subset \partial\omega$ where $v > v(x_o) + \varepsilon$. Then $\int \varphi_e d\rho^{\omega}_{x_o} \to 0$ according to the filter corresponding to the directed set of these ω (φ_e being the characteristic function of e). This is an immediate consequence of $\int v d\rho^{\omega}_{x_o} \to v(x_o)$. 132

Observe now that in Theorem 26, the inequality

$$\underset{x \in CS,\, x \to y}{\lim . \sup} v(x) \geq \underset{x \in \partial S,\, x \to y}{\lim . \sup .} v(x)$$

is obvious. Suppose strict inequality holds good, introduce a number λ in between the numbers. Let δ be the open set of CS where $v > \lambda$, and δ' the intersection of δ with a regular domain $\omega \ni y$. In δ', $v = \overline{H}^{\delta'}_{v}$ according to $D(\beta)$. Let us decompose v as $v = \varphi_1 + \varphi_2$ where $\varphi_1 = v$ on $C\delta, 0$ on δ and $\varphi_2 = 0$ or $C\delta$ and v and δ. Now, $v = \overline{H}^{\delta'}_{v} \leq \overline{H}^{\delta'}_{\varphi_1} + \overline{H}^{\delta'}_{\varphi_2}$ and $\varphi_1 \leq \lambda$ on $\partial\delta'$ for ω small enough. Given $\lambda' > \lambda$, $\overline{H}^{\delta'}_{\varphi_1} \leq \lambda'$ when ω is small enough (because this envelope is majorised by a harmonic function which would be defined in the neighbourhood of y and equal to $\dfrac{\lambda + \lambda'}{2}$ at y). On the other hand, we have in δ'

$$\overline{H}^{\delta'}_{\varphi_2} \leq \overline{H}^{\omega}_{\varphi_2} = \int \varphi_2 d\rho^{\omega}_{x}$$

This integral for $x = y$ tends to 0 according to the considered directed set of the ω. Therefore $\lim_{x\in\delta' x\to y} . \sup . \overline{H}^{\delta'}_{\varphi_2} x)$ is arbitrarily small for a suitable ω. We conclude that $\lim_{x\in CS, x\to y} . \sup .. v(x) \leq \lambda'$ therefore $\leq \lambda$. This is a contradictional. Hence the theorem.

133 **Remark 1.** As a particular case, if $S = \{y\}, v$ is continuous at y.

Remark 2. We want to emphasize that only with the axioms 1, 2, 3 and, the axiom D implies that any locally bounded potential whose restriction on the support is continuous, is itself continuous on Ω. It is remarkable that when Ω satisfies the second axiom of countability the converse is true (due to R.M. Herve).

36 Convergence theorems

Theorem 27 (convergence of sequences)**.** *We suppose 1, 2, 3 and D and a countable base for open sets on Ω. Let v_n be fa decreasing sequence of superharmonic functions ≥ 0. The limit v which is nearly superharmonic is different form the regularised function $\hat{v}$ on a polar set (i.e., v is quasi-superharmonic).*

Proof. We may suppose the existence of a finite continuous potential $V_o > 0$ (otherwise, it is trivial). Let us suppose first v_1 to be upper bounded and $\epsilon > 0$. Let α be the set of points x where $v(x) - \hat{v}(x) > \varepsilon$. It is enough to prove that α is polar, for, that will imply $\{x : v(x) \underline{>} \hat{v}(x)\}$ is polar. As α is a borelian set in the compact metrizable space $\overline{\Omega}, \alpha$ is a K-analytic set, and we have only to see that every compact subset K of α is polar (cf. Prop. 22). ◻

We introduce an open set ω such that $K \subset \omega \subset \bar{\omega} \subset \Omega$ then, $v_n(x) \geq H^{\omega-K}_{v_n}(x) \geq H^{\omega-K}_{v}(x)$ in $\omega - K$. Hence $v(x) \geq H^{\omega-k}_{v}(x); v(x) \geq$
134 $H^{\omega-K}_{v}(x)$ $H^{-K}_{v}(x)$ in $\omega - K$. According to axiom D (form), $H^{\omega-K}_{\hat{v}}$ is the greatest harmonic minorant of $\hat{v}$ in $\omega - K$, therefore (by the above inequality) equal to $H^{\omega-K}_{v}$. Hence $H^{\omega-K}_{v-\hat{v}} = 0$.

Now $v - \hat{v} > \varepsilon$ on K. Let us choose $\lambda > 0$ such that $\lambda V_o < \varepsilon$ on K and define φ on $\partial(\omega - K)$ such that $\varphi = 0$ on $\partial\omega$ and $\varphi = \lambda V_o$ on K. Then

$$(R^K_{\lambda V_o})_\omega = H^{\omega-K}_\varphi \leq H^{\omega-K}_{v-\hat{v}} \text{on } \omega - K.$$

Hence $(R^K_{V_o})_\omega = 0$ on $\omega - K; K$ is polar in ω, therefore in Ω.

In the general case, for every positive integer p, we introduce $v^p_n = \inf .(v_n, pV_o)$ which is locally bounded and decreases as n increases. Form the particular case we deduce, $w_p = \hat{w}_p$ quasi everywhere, if $w_p = \lim\limits_{n\to\infty} .v^p_n$, for every p. Now $v = \lim\limits_{n\to\infty} .v_n = \lim\limits_{n\to\infty} .w_p$.

Therefore quasi-everywhere in $\Omega, v = \lim\limits_{p\to\infty} .\hat{w}_p$; the limit on the right hand side is superharmonic ($\leq v_1$).

Corollary. *With the same hypothesis, let $v_n \geq 0$ be a sequence of hyperharmonic functions in Ω. Then* $\inf .v_n, \lim\limits_{n\to\infty} . \inf .v_n$ *are quasi-superharmonic or equal everywhere to $+\infty$ (and nearly hyperharmonic).*

In fact the function $w^p_n == inf.(v_n, v_{n+1}, \ldots, v_{n+p})$ is hyperharmonic, decreasing as p increases, and the limit as $p \to \infty$ is a function w_n which is nearly hyperharmonic, and quasi-superharmonic or equal to $+\infty$. Now we observe that w_n is increasing and the corollary follows.

Theorem 28 (General Lower Envelope)**.** *With the same hypothesis on* **135**
Ω (Axioms 1, 2, 3 and D and second axiom of countability) let us consider a family of superharmonic functions $v_i \geq 0$. The lower envelope $v = \inf\limits_i .v_i$, which is a nearly superharmonic function, equals $\hat{v}$ quasi-everywhere in Ω.

Proof. We know by the topological lemma of Choquet that there exists a sequence v_{α_n} in the family such that, the lower regularised functions of $\inf .v_{\alpha_n}$ and $\inf .v_i$ are the same. Then

$$\inf \hat{.} v_{\alpha_n} = \inf \hat{.} \inf .v_i \leq \inf .v_{\alpha_n}.$$

□

Hence the theorem.

37 Application to reduced functions

Remark. The axioms 1, 2 and 3 the first axiom of countability for Ω (there exists at each point a countable base of neighbourhoods) imply for any polar set E and a point $x_o \in CE$, an associated function finite at x_o.

For, if U_n is a countable base of neighbourhoods of x_o, there is for $E \cap CU_n$ an associated function v_n such that $v_n < \dfrac{1}{n^2}$, $\sum v_n$ satisfies the required condition.

Proposition 23 (Properties of the reduced function)**.** *Let Ω satisfy the axioms 1, 2, 3 and D and second axiom of countability. Let $E \subset \Omega$ and $\varphi \geq 0$, a function on E majorised by a superharmonic function $V \geq 0$ in Ω.*

(i) $\hat{R}^E_\varphi = R^E_\varphi$ quasi-everywhere

136 (ii) on $CE, R^E_\varphi = \hat{R}^E_\varphi$

(iii) $\hat{R}^E_\varphi$ is the smallest superharmonic function ≥ 0 which is $\geq \varphi$ quasi-everywhere on E. $\hat{R}^E_V = Vq.e$ on E.

In fact if a superharmonic function $v \geq 0$ in Ω satisfies $v \geq \varphi$ on E except on a polar set α. Let us introduce w associated to α. Then $v + \varepsilon w \geq \varphi$ on E, therefore $\geq R^E_\varphi$ everywhere. Now if we take as v the function $\hat{R}^E_{\varphi_E}$ and choose w finite at a point $x_o \in CE$, we have $\hat{R}^E_\varphi(x_o) \geq R^E_\varphi(x_o)$, hence the property ($ii$).

On the other hand for any $v, v \geq R^E_\varphi$ at any point where w is finite, i.e. quasi-everywhere. Hence $v \geq \hat{R}^E_\varphi$ quasi-everywhere.

Chapter 9

Thins Sets - Some Applications[1]

38

To start with we suppose the axioms 1, 2 and 3 and the existence of a potential $V > 0$ on Ω. Without the last condition the following definition would not allow the existence of thinsets. 137

Definition 23. *A set $E \subset \Omega$ is said to be thin at a point $x_\circ \notin E$, if $x_\circ \notin \overline{E}$ or if $x_\circ \in \overline{E}$ and if there exists a superharmonic function $v \geq 0$ on Ω such that $\liminf\limits_{x\in E, x\to x_\circ} v(x) > v(x_\circ)$.($v$ is said to be associated to E and $x_\circ$.)*

A set $E \subset \Omega$ is said to be thin at $x_\circ \in E$, if $\{x_\circ\}$ is polar and $E - \{x_\circ\}$ is thin at $x_\circ$.

It is therefore obvious that for any superharmonic function function $v \geq 0$ in Ω, if a set E is not thin at $x_\circ$, then $x_\circ \in \overline{E}$ and $\underset{x\in E, x\to x_\circ}{\lim, \inf}, vv(x_\circ)$. A finite union of thin sets at $x_\circ$ is thin at $x_\circ$.

Proposition 24 (Local character Proposition)**.** *A set E is thin at $x_\circ$ if and only if for every domain ω containing $x_\circ, E \cap \omega$ is thin at $x_\circ$ in the subspace ω.*

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[1]This Chapter develops partly, with some improvements, a note published in collaboration with R.M. Herve [2].

This is an immediate consequence of the continuation theorem (14) and Theorem 25.

Theorem 29 (General ciriterion)**.** *Let v be a superharmonic* [1] *function*
138 > 0 *in* Ω, *finite and continuous at the point* $x_\circ \in \Omega$. *In order that a set* $E \not\ni x_\circ$ *is thin at* $x_\circ$, *it is necessary and sufficient that there exists a neighbourhood* σ *of* $x_\circ$ *such that* $R_v^{E\cap\sigma}(x_\circ) < v(x_\circ)$.

Proof. If E is thin, the property is obvious when $x_\circ \notin \bar{E}$; if $x_\circ \in \bar{E}$, then there exists a superharmonic function $w > 0$ such that

$$\liminf_{x \in E,\, x \to x_\circ} w(x) > w(x_\circ)$$

Choose $\lambda > 0$ such that $\lambda v(x_\circ)$ lies strictly in between the two members of the above inequality; then a neighbourhood σ of $x_\circ$ such that on $E \cap \sigma$, $w(x) > \lambda v(x)$. We deduce $w \geq R_v^{E\cap\sigma}$. Hence $R_v^{E\cap\sigma}(x_\circ) \leq \dfrac{w(x_\circ)}{\lambda} < v(x_\circ)$. ◻

Conversely, suppose $R_v^{E\cap\sigma}(x_\circ)$ for a neighbourhood σ. Then there exists a superharmonic function $w > 0$ such that $w \geq v$ on $E \cap \sigma$, and $w(x_\circ) < v(x_\circ)$. Therefore if $x_\circ \in \bar{E}$,

$$\liminf_{x \in E,\, x \to x_\circ} w(x) \geq v(x_\circ) > w(x_\circ).$$

Proposition 25 (Application Proposition)**.** *For an open set* $\omega \subset \bar{\omega} \subset \Omega$, *if* $C\omega$ *is thin at a boundary point* $x_\circ \in \sigma\omega$, *then* $x_\circ$ *is irregular for* ω.

Proof. $\{x_\circ\}$ is polar and $C\omega - \{x_\circ\} = E$ is thin at $x_\circ$; for any function v of Theorem 29, and a neighbourhood σ of $x_\circ$, we have $\hat{R}_v^{C\omega\cap\sigma-\{x_\circ\}} < v(x_\circ)$. By Theorem 24 (corollary), $\hat{R}^{C\omega\cap\sigma-\{x_\circ\}} = \hat{R}_v^{C\omega\cap\sigma}$; therefore $\hat{R}_v^{\omega\cap\sigma}(x_\circ) <$
139 $v(x_\circ)$. Now from Theorem 21, it follows that $x_\circ$ is not a regular point for ω. ◻

[1]Superharmonic is unnecessary. Same proof.

39 Fine topology

Theorem 30. *The complements of the sets $E \not\ni x_\circ$ that are thin at $x_\circ$, form for all $x_\circ \in \Omega$ the filter of neighbourhoods for a topology on Ω. This topology is called the 'fine topology' on Ω. Among the topologies on the set Ω which are finer (in the large sense) than the topology of the given space Ω, Φ be the coarsest for which all superharmonic functions $v \geq 0$ are continuous. Then Φ and the fine topology are identical.*

Proof. If V is a Φ-neighbourhood of a point $x_\circ$ in Ω, it contains an open set (of the initial topology) or is the intersection of such an open set with some sets of type $\alpha = \{x : v < \beta\}$ where v is a superharmonic function ≥ 0 with $v(x_\circ) < \beta$. On $C\alpha, v \geq \beta$; hence the sets $C\alpha$ and their union are thin at $x_\circ$, therefore also CV. □

Conversely V be a set such that $x_\circ \in V, CV$ is thin at $x_\circ$. If $x_\circ$ is in the interior (in the initial sense) of V, V is also a Φ-neighbourhood of $x_\circ$ if $x_\circ$ is not in the interior (in the initial sense) of V, there exists a superharmonic function $v \geq 0$ such that

$$v(x_\circ) < \beta < \underset{vx \to x_\circ, x \in CV}{\lim . \inf .}.$$

For a suitable ordinary neighbourhood δ of $x_\circ, v \geq \beta$ on $CV \cap \delta$. 140
Therefore, $\delta \cap \{x : v < \beta\} \cap V$ and V is a Φ -neighbourhood of $x_\circ$. The various topological notions that are associated with Φ- topology are called 'fine' : fine closure, fine limit, etc.; (the name was given by Cartan in the classical case).

Proposition 26. *Let E be a closed set thin at $x_\circ \in \partial E$ and ω an open neighbourhood of $x_\circ$. For any superharmonic function $v \geq 0$ in $\delta' = \omega \cap CE$ there exists a fine limit at $x_\circ$.*

Proof. $\{x_\circ\}$ is polar and the proposition is obvious if $x_\circ$ is isolated on E (see n°. 32 iv). □

If not choose a superharmonic function $w \geq 0$ on Ω (associated to E and $x_\circ$), such that

$$\lim . \inf . w - w(x_\circ) = d > 0$$

$$x \in E - \{x_\circ\}$$
$$x \to x_\circ$$

On a fine neighbourhood V of $x_\circ$, $w \leq w(x_\circ) + \varepsilon$ $(0 < \varepsilon < d)$.

If $v(x) \to +\infty$ $(x \in V \cap \delta', x \to x_\circ)$, v on ω' has the fine limit $+\infty$ at $x_\circ$. If not, let $\lambda = \lim\limits_{x \to x_\circ, x \in V \cap \delta'} . \inf . v(x) < +\infty$.

Let us compare

$$\lim_{\substack{y \in \delta E - x_\circ \\ y \to x_\circ}} . \inf . (\lim_{x \to y, x \in \delta'} . \inf . (v+kw)) \geq kw(x_\circ) + d = k_1$$

$$\lim_{x \in \delta', x \to x_\circ} . \inf . (v+kw) \leq \lim_{\substack{x \in \delta' \cap V \\ x \to x_\circ}} . \inf . (v+kw) \leq \lambda + k(w(x_\circ) + \varepsilon) = k_2$$

141 We shall choose k_1 and k_2 such that $k_1 > k_2$, then a finite continuous potential $V_\circ$ such that $k_1 > V_\circ(x_\circ) > k_2$.

Now the function equal to inf .$(v+kw, V_\circ)$ on δ', and to $V_\circ$ on $E - \{x_\circ\}$ is superharmonic in a suitable open neighbourhood $\delta_1 \subset \delta$ of $x_\circ$, outside $x_\circ$. This function V_1 may be therefore continued at $x_\circ$ in order to become superharmonic in δ_1. Therefore on $\delta_1 - \{x_\circ\}, V_1$ has a finite limit (at $x_\circ$), equal to $\lim\limits_{x \neq x_\circ, x \to x_\circ} . \inf . V_1 < V_\circ(x_\circ)$. We deduce that $v + kw$ on δ' has the same fine limit at $x_\circ$. As w has a fine limit, the same holds for v. This theory may be developed further, as in the classical case, using first only a countable base of open sets in Ω (thanks to some results of R.M.Herve). We will only give the following important theorem, with more restrictions, using the convergence theorem.

40 Further development

with axioms 1, 2, 3 D a countable base in Ω and a potential > 0.

Theorem 31. *The subset of a set E where E is thin is a polar set.*

Let $V_\circ$ be a finite continuous potential > 0 and $\{\omega_i\}$ a countable base of Ω. If E is thin at $x_\circ \in E$, $\{x_\circ\}$ is polar, $E - \{x_\circ\}$ is thin and there exists an $\omega_i \ni x_\circ$ such that

$$\hat{R}^{E \cap \omega_i}_{V_\circ}(x_\circ) = \hat{R}^{E \cap \omega_i - \{x_\circ\}}_{V_\circ}(x_\circ) = R^{E \cap \omega_i - \{x_\circ\}}_{V_\circ}(x_\circ) < V_\circ(x_\circ)$$

142 Hence $x_\circ \in \{x : \hat{R}_{V_\circ}^{E\cap\omega_i}\}$ the intersection of this set with $E \cap \omega_i$ is polar (n^o.37). Therefore the set of all $x_\circ$ is polar.

Corollary. *A polar set is characterized as a set e which is thin at any point of* Ω *or at every point of e.*

Application to the Dirichlet problem for $\omega \subset \bar{\omega} \subset \Omega$

Theorem 32. *With the previous hypothesis, the regularity of a point* $x_\circ \in \partial\omega$ *is equivalent to the non-thinness of* $C\omega$ *at* $x_\circ$*. Then follow: (i) the set of the irregular boundary points is polar* [2]*, (ii) any bounded harmonic function on* ω*, which tends to 0 at every regular boundary point, is equal to zero, (iii) there exists for any finite continuous function* θ *on* $\partial\omega$*, a unique bounded harmonic function on* ω *which tends to* $\theta(x)$ *at every regular boundary point x (it is* H_θ^ω*).*

Proof. If $\{x_\circ\}$ is not polar, $C\omega$ is not thin at $x_\circ$, further we have by using a finite continuous superharmonic function $V > 0$, for every neighbourhood σ of $x_\circ$,

$$\hat{R}_V^{C\omega\cap\sigma}(x_\circ) = R_v^{\sigma\cap C\omega}(x_\circ) = V(x_\circ)$$

Therefore $x_\circ$ is regular (Theorem 21).

If $\{x_\circ\}$ is polar,

$$\hat{R}_V^{c\omega\cap\sigma}(x_\circ) = \hat{R}_V^{c\sigma\cap\omega-\{x_\circ\}}(x_\circ) = \hat{R}_V^{c\sigma\cap\omega-\{x_\circ\}}$$

□

The equality of these members with $V(x_\circ)$ for every σ implies (Theorems 21 and 29) both the regularity of $x_\circ$ and the non-thinness of $C\omega$ at $x_\circ$.

Remark. Theorem 32 gives therefore, a criterion of non-thinness of a closed set E at a point $x_\circ \in \partial E$. It is interesting to prove it directly. (See bibliography) 143

[2]when ω is connected, the hypothesis of a countable base of open sets is unnecessary (quite different proof by R. willin, see add. bibliography).

We recall now that a subset e of the boundary of ω is defined to be a negligible set if $\overline{H}^{\omega}_{\varphi_e} = 0$. ($\varphi_\varepsilon$ characteristic function of e), or as a set of $d\mu^{\omega}_{x}$- measure zero for any $x \in \omega$.

Lemma. *If for an open set $\delta, \delta \cap \omega = \delta \cap \omega_1$ (ω and ω_1 being two relatively compact open sets) the negligible sets of $\partial\omega \cap \delta$ or $\partial\omega_1 \cap \delta$ are the same relative to ω or ω_1.*

Consider the non-trivial case, $\omega \subset \omega_1$ and $e \subset \partial\omega \cap \delta$ negligible for ω. Then $\overline{H}^{\omega_1}_{\varphi_e} = \overline{H}^{\omega}_{\psi}$ on ω for a suitable ψ equal to φ_e on $\partial\omega \cap \delta$. Now if we change ψ into zero on $e, \overline{H}^{\omega}_{\psi}$ does not change. We deduce that $\overline{H}^{\omega_i}_{\varphi_e}$ tends to zero at every regular point of the boundary of ω_1. Hence $\overline{H}^{\omega_1}_{\varphi_e} = 0$.

Theorem 33. *With the same hypothesis, the boundary points of $\omega \subset \overline{\omega} \subset \Omega$ where ω is thin form a negligible set (for ω).*

Proof. Let us use a finite continuous superharmonic function $V > 0$. □

If ω is thin at $x_\circ$, there exists an open neighbourhood σ containing $x_\circ$, such that the potential V_1 defined by $\hat{R}^{\omega\cap\sigma}_{V}$ or $R^{\omega\cap\sigma}_{V}$ is smaller than $V(x_\circ)$ at the point $x_\circ$. Now on $\omega \cap \sigma\, V$ and V_1 are equal, have the same greatest harmonic minorant therefore (by axiom D) the same best harmonic minorant, i.e.,

$$H^{\omega\cap\sigma}_{V} = H^{\omega\cap\sigma}_{V_1} \text{ or } H^{\omega\cap\sigma}_{V-V_1} = 0$$

144 Hence the set $\{x \in \partial\omega \cap \sigma : V_1(x) < V(x)\}$ is negligible for $\omega \cap \sigma$, and therefore for ω. Now we can cover $\partial\omega$ by finite number of such σ, hence we deduce that the subset of $\partial\omega$ where ω is thin is negligible for ω.

Chapter 10

Generalisation of the Riesz -Martin Integral Representation

41

We shall give only a brief survey without detailed proofs. **145**

For the developments see [1] Seminaire II.

Choquet's theorem on extremal elements

On a vector space $\mathscr{L}$, a point x of a convex subset A is said to be an extremal element of A, if there exists no open segment containing x on A, defined by $\lambda x_1 + \mu x_2, x_1, x_2 \in A$, $0 < \lambda < 1$ and $\lambda + \mu = 1$ for all λ and μ. Let ${}^*A$ denote the extremal elements of A.

If $\mathscr{L}$ is locally convex Hausdorff and A is a convex compact subset, the centre of gravity of a positive Radon measure μ on A may be defined as follows: when μ is defined by a finite system S of point masses ≥ 0 (m_i at z_i; with $\sum m_i = 1$), the centre of gravity G_S is the point $\sum m_i z_i$. For any μ with $\mu(A) = 1$, there exists an ultra-filter on the sets of the previous S, such that it converges vaguely to μ. Then G_S converges to a point G in A; this point G is independent of the filter and is called the centre of gravity of μ. This is a consequence of the fact that for any continuous linear real functional $f(z), G$ satisfies $f(G) = \int f(z) d\mu(z)$.

Theorem 34 (Choquet)**.** *On a real, ordered, locally convex vector space $\mathscr{L}$ consider the convex cone $\mathscr{C}$ of positive elements. We suppose $\mathscr{C}$ has*
146 *a compact base B (intersection of all rays of $\mathscr{C}$ with an affine manifold which does not pass through the origin).*

(i) *Suppose $\mathscr{C}$ is a lattice; then given any point $x \in B$, there exists at most one unitary measure $\mu \geq 0$ on $B, (\mu(B) = 1)$ supported by $*B$ and whose centre of gravity is x.*

(ii) *Suppose B is metrizable; then there exists at least one such μ. Integral representation of the harmonic functions ≥ 0 (generalised Martin representation).*

Lemma. *For the set $H^+_{x_\circ}$ of the harmonic functions $u \geq 0$ on Ω. satisfying $u(x_\circ) = 1$ (for fixed $x_\circ \in \Omega$), the property of compactness for the topology of the uniform convergence on every compact subset is equivalent to the following "Harnack axiom".*

For every $x_1 \in \Omega$, $\dfrac{u(x)}{u(x_1)} \to 1 (x \to x_1)$ uniformly for all functions of the set $H^{'+}$ of the harmonic functions > 0 on Ω. (equivalent form : the functions of $H^+_{x_1}$ are equicontinuous at x_1 for every $x_1 \in \Omega$).

Theorem 35. *Suppose $H^{'+}$ (if non-empty) satisfies the Harnack axiom. Let H be the vector space of difference of harmonic functions ≥ 0 with the topology of uniform convergence on compact set of Ω.*

(i) Given any function u of H^+ (the set of harmonic functions ≥ 0 on Ω) on the compact set $H^+_{x_\circ}$, there exists at most one Radon measure $\mu \geq 0$, supported by ${}^*H^+_{x_\circ}$ (set of extremal elements of $H^+_{x_\circ} \subset H$) such that

147 $u(x) = \int w(x) d\mu(w)$ ($x \in \Omega$; w variable on the base $H^+_{x_\circ}$ of the cone H^+).

(ii) If Ω is a countable union of compact sets, there exists one such measure. Moreover, for any $\mu \geq 0$ on $H^+_{x_\circ}$, $\int w(x) d\mu(w) \in H^+$.

Obviously H is locally convex, H^+ is convex and further H^+ is known to be a lattice for the natural order; with the hypothesis $H^+_{x_\circ}$ is metrizable. We may therefore use the centre of gravity and the linear functional $u(x)$ of u (for fixed x).

If we want such a representation for every domain of Ω, we have to suppose Harnack axiom for every domain. We emphasize that if axioms 1 and 2 are assumed, the axiom 3 together with the Harnack axiom for all domains of Ω is equivalent to the

Axiom $3'$ [1] There exists a base of domains of Ω, such that for each one of them the set of harmonic functions > 0 is non-empty and further each satisfies the Harnack axiom (see another form in Sem *II*).

42 Integral representation of the superharmonic functions ≥ 0

Let us use the locally convex vector space $S\,(n^o.19)$ of the equivalence classes corresponding to the differences of superharmonic functions ≥ 0, the specific order and the positive cone S^+ (whose elements may be identified with superharmonic functions ≥ 0). We know that S^+ is a lattice.

We introduce the set $S^+_{x_\circ,\omega_\circ}$ of the superharmonic functions of S^+ satisfying the condition $\int v d\rho^{\omega_\circ}_{x_\circ} = 1$ for a regular domain $\omega_\circ$ and $x_\circ \in \omega_\circ$. It is a base of the convex cone S^+.

In order to apply Choquet's Theorem, we need a topology in S for **148**
which $S^+_{x_\circ,\omega_\circ}$ becomes compact and metrizable.

Proposition 27. *For the elements $[u, v]$ of S, $\left|\int u d\rho^\omega_x - \int v d\rho^\omega_x\right|$ for a fixed regular domain ω and a fixed point $x \in \omega$ is a seminorm. All these semi-norms define on S a topology $\mathscr{C}$ which is locally convex, Hausdorff and compatible with the specific order.*

We remark that when $(3')$ is satisfied, $\mathscr{C}$ on H^+ is identical with the topology of uniform convergence on compact sets.

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[1] Actually the Harnack axiom is a consequence of axiom 1, 2, 3 and 1, 2, 3 1, 2, 3 (Mokobodski - Loeb-B/ Walsh; see add. Chapter).

Definition. *A regular open set is said to be completely de terminating if for two superharmonic functions, v_1 and $v_2 \geq 0$ on Ω, harmonic on ω, the condition $v_1 = v_2$ on $C\omega$ implies $v_1 = v_2$ on ω.*

We shall introduce a new axiom.

Axiom 4. There exists on Ω a base of completely determinating (regular) domains. In case Ω has a countable base, axiom 4 implies the existence of a countable base of completely determinating regular domains.

It does not seem impossible that axiom 4 which is satisfied (even for all regular domains) and used in some proofs of classical theory, is a consequence of axioms 1, 2 and 3′.

Proposition 28. *Suppose axioms 1, 2 and 3′ and 4 and a countable base for open sets on Ω. For the topology $\mathscr{C}$ (introduced above) the base $S^+_{x_\circ,\omega_\circ}$ of the cone S^+ is metrizable and compact.*

Now with the set ${}^*S^+_{x_\circ,\omega_\circ}$ of the extremal points of $S^+_{x_\circ,\omega_\circ}$, we may use
149 Choquet's theorem, the linear functional $\int v d\rho^\omega_x$ of v and the property $v \int d\rho^\omega_x \underset{\mathcal{F}}{\longrightarrow} v(x)$ (fixed $v \in S^+$; fixed x; ω variable according to filter $\mathscr{F}$ of Theorem 7.)

Theorem 36. *With the same hypothesis on Ω (as in Prop. 28) and with the topology $\mathscr{C}$, there exists for every $v \in S^+$ an unique Radon measure $\mu \geq 0$ on $S^+_{x_0,\omega_0}$, supported by ${}^*S^+_{x_0,\omega_0}$ and such that $v(x) = \int w(x) d\mu(w)$ ($w \in S^+_{x_0,\omega_0}$).*

Note also that for any μ on $S^+_{x_0,\omega_0}$, $\int w(x) d\mu(w) \in S^+$.

Decomposition of this representation.

Let P^+ denote the set of potentials on $\Omega.P^+_{x_0,\omega_0}$ the subset in $S^+_{x_0,\omega_0}$ and ${}^*P^+_{x_0,\omega_0}$ the set of external elements of $P^+_{x_0,\omega_0}$. Then $H^+_{x_0} \cap P^+_{x_0,\omega_0} = \phi$. $S^+_{x_0,\omega_0} =^* H^+_{x_0,\omega_0} \cup^* P^+_{x_0,\omega_0} =^* S^+_{x_0,\omega_0} \cap P^+_{x_0,\omega_0}$. When P^+ is not empty, $S^+ = \overline{P}+, S^+_{x_0,\omega_0} = \overline{P}^+_{x_0,\omega_0}, {}^*S^+_{x_0,\omega_0} \subset^* \overline{P}^+_{x_0,\omega_0}$. The sets considered with starts are G_δ-sets in $S^+_{x_0,\omega_0}$.

Theorem 37. *With hypothesis as in Theorem 36, for any $v \in S^+$,*

$$v(x) = \int w(x) d\mu_1(w) + \int w(x) d\mu_2(w)$$

where μ_1 and μ_2 are unique Radon measures ≥ 0 on $S^+_{x_0,\omega_0}$, with supports ${}^*H^+_{x_0,\omega_0}$ and ${}^*P^+_{x_0,\omega_0}$ respectively. **150**

The first integral is the greatest harmonic minorant of v and the second one is a potential.

Classical case: In the case of Green space, for instance a bounded the domain of the Euclidean space, the extreme potentials appears easily as the normalised Green function; $g_y(x) = \dfrac{G_y(x)}{\int G_y(x) d\overset{\omega_0}{\rho}(x)}$ (Green function $G_y(x)$ with pole y). The correspondence $y \to g_y(x)$ defines a homomorphism between Ω and ${}^*P^+_{x_0,\omega_0}$ (this set having topology $\mathscr{C}$)

The measure μ_2 of Theorem 37 may be therefore considered as a measure on Ω and $\int w(x) d\mu_2(w)$ is equal to $\int G_y(x) d\nu(y)$ with another measure $\nu \geq 0$ on Ω. So we get the classical Riesz representation of potentials.

On the other hand ${}^*H^+_{x_0}$ is a subset of the compact $H^+_{x_0} \cap {}^*\overline{P}^+{}_{x_0,\omega_0}$ which is subset of $S^+_{x_0,\omega_0}$ and is a definition of the classical Martin boundary [2] Δ (upto some homeomorphism). It is measure on the compact set Δ, supported by the set of the minimal points and $\int w(x) d\mu_1(w)$ gives the Martin representation of the greatest harmonic minorant of v.

43 Further Theory-Kernel

R.M. Herve has succeeded in avoiding axiom 4 in the integral representation with the help of a different topology on S. On the other hand, she **151**
remarked that (with $1, 2, 3'$ countable base, pot > 0), the extreme potentials have a point support and the importance of the "case of proportionality" where all potentials with same point support are proportional. Such a proportionality allows the use of a Green-type function. This is

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a kernel and the study becomes a particular case of the theory initiated in Part III.

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and see additional chapter dedicated to the further developments.

Chapter 11

Additional Chapter and Bibliography for Part IV (2$^{\text{nd}}$ edition)

After the publication of these lectures, very many researches were made on the field of Part *IV* and are mostly developed or mentioned in 2 courses (Brelot, Montreal [21], Bauer, Hamburg [4]) and in a summary of Constantinescu [27]. Let us give a short survey with mention of the latest results. **153**

Mokobodski proved (see [21]) that $1, 2, 3 \Rightarrow 3'$ with a restriction of countable base open sets, that was cancelled by Loeb-Walsh [55]. [1] He proved also, even in large conditions that the points where proportionality does not hold form a polar set. [still unpublished]. Mrs. Herve [46] brought to *IV* many complements: first a topology on the space $S^+ - S^+$, which allows only with axioms 1, 2, 3, countable base, $pot > 0$, to find a compact metrizable base of S^+ and whose introduction was simplified by Mokobodski [62]; she discussed a balayage theory and equivalences of D, and by adding proportionality and a new axiom developed a theory of adjoint harmonic functions which axiomatizes the adjoint elliptic equations; she studied the linear elliptic equation of second order, even

[1] See improved hypothesis in Constantinescu [31]

later [49] with discontinuous coefficients, showing that the solution in a suitable sense considered by specialists like Stampacchiea satisfies the axioms of Part IV. Boboc-Constantinescu-Cornea [$7 \to 12, 25 \to 34$] brought also many improvements or complements, even in more gen-
154 eral conditions we shall mention later: axiom 3 expressed equivalently with sequences [32], the existence of a superharmonic function > 0 implies (with $1, 2, 3$) that Ω is a countable union of compact sets [34]; weaker hypothesis for the properties of R_v^e; example where countable base, proportionality, axiom D are not satisfied. Similar complements or improvements (for instance, cancellation of a countable base or of D) were often given in paper like [24], [51] or in further developments or larger axiomatice we shall review now.

Gowrisankaran [40], [43] introduced and used an extension of classical Naim's "thinness at boundary", first in a general abstract setting. The harmonic function which are extreme points on some base of S^+ are called minimal and form on some fixed base β of S^+ a set Δ_1 (minimal boundary). A set e is said to be thin at $X \in \Delta_1$, if $R_X^e \neq X$ and the complementary sets of thin sets at X form a filter $\mathcal{F}_X$ according to which limits are called "fine". Only with $1, 2, 3$, countable base and pot > 0, Gowrisankaran proved that $\dfrac{u}{v}$ (quotient of two superharmonic function > 0) have a finite fine limit at all points of Δ_1, except on a set of μ_v measure zero (measure corresponding to v in the integral representation) (Extension of a Doob's result in classical case). Note that, by adding the "proportionality" Brolet [still unpublished] proved that there is a unique topology on $\Omega \cup \Delta_1$, giving the fine topology on Ω, and for every $X \in \Delta_1$, neighbourhoods interesting Ω according to $\mathcal{F}_X$; this topology induces on Δ_1 the discrete topology and may be called fine topology
155 on $\Omega \cup \Delta_1$ denoted $\overset{v}{\Omega}$. Still with "proportionality"', Gowrisankaran considered on the base β of S^+, the set. of minimal potentials; it is homeomorphic to Ω and has a compact closure which gives a "Martin space" $\hat{\Omega}$ where Ω is dense and a "Martin boundary" $\Delta = \hat{\Omega} - \Omega$ where Δ_1 is the "minimal part". Dirichlet problems may be studied with adding D for h-harmonic function with Δ and $\hat{\Omega}$ or Δ_1 and $\overset{v}{\Omega}$ (same resolutivity and same solutions) or without D and proportionality, with $\Delta_1, \mathcal{F}_X$ and

boundary condition only μ_h a.e. By studying $\frac{u}{v}$, the condition $u > 0$ may be weakened [41] as Doob did in in classical case. Gowrisankaran extended also the Naim's comparison of all compact boundaries which may allow a Dirichlet problem (unpublished). In [42], he studied the doubly harmonic and superharmonic functions $f(x, y)$ (x and y resp. In harmonic spaces). The uniform integrability, introduced by Doob in potential theory, was used by Brelot [16], Gowrisankaran, Mrs Lamer Na im [58] in order to characterize harmonic functions as solution of various Dirichlet problems. Still with axioms 1, 2, 3, countable base, potential > 0, and constants harmonic, Mrs Lumer-Naim studied the complex harmonic functions $f = u + iv$, such that for every f, the L^p_j-norms are uniformly with respect bounded with respect to $j(1 \leq p \leq +\infty$, fixed p) These norms are relative to the harmonic measure $d\rho^{w_j}_{x_0}$ (relatively compact open sets $w_j \ni x_0$). These functions have finite fine limits μ_1 a.e on Δ_1. Similarly study for subharmonic functions. Extension by changing the norm $|f|^p$ in $\Phi(f), \Phi$ positive convex increasing. Various applications extending classical results.

Brelot [17], [13], then Fuglede [37], [38] discussed axiomatically the first idea of thinness.[2]. Brelot extended [15] the functional Keldych's characterization of the ordinary Dirichlet problem and [16] various results on fine topology; he compared [15], [16] both the inner and minimal thinnesses and studied [20] $\hat{R}^e_w$ (w superharmonic > 0) for decreasing sets e (that includes the capacities for decreasing set). thanks to the fine topology; that led to the study of R^{Ω}_{φ} for decreasing fine upper semicontinuous functions. Doob, inspired by the fine topology, developed [35] [3], a general theory of "small" sets such that in large conditions, a union of open sets in equal to a countable subunion upto a "small" set. Important applications were given to potential theory and corresponding probabilistic questions. 156

Loeb [54] compared two sheaves of harmonic functions on the same

[2]see a survey of the notions of thinness in Brelot [22] and the courses (Bombay $TIFR$1966) on topologies and boundaries in potential theory

[3]In [20] [35] see proof in various condition of a thinness of Getoor [44] originally given in probability theory and that choquet (ann IF) discussed axiomatically. It says that for a measure which does not charge polar sets, there exists a fine closed support.

space and studied the case where 1 is superharmonic. A continuation was developed by Loeb-Walsh [57] with the use of compactify boundaries. Constantinescu-Corna had studied compactifications of Ω and corresponding Dirichlet problems. They also studied [40] the correspondence of two harmonic spaces Ω, Ω' (continuous application $\varphi : \Omega \to \Omega'$) such that if u' is harmonic in $\omega' \subset \Omega', u' \circ \varphi$ is harmonic
157 in $\varphi^{-1}(\omega')$. Sibony [66] developed independently the same idea in order to extend the results of Constantinescu-Cornea-Doob on the analytic corresponding between hyperbola Riemann surfaces.

A de la Pradella [53] extended the classical theory of a Dirichlet problem for compact sets and studied the property of quasi analyticity (i.e. a harmonic function is zero in a domain when it is zero in a neighbourhood of a point). Using the theory of adjoint harmonic functions, he proved that the quasi-analyticity of these functions is equivalent to an approximation property relative to the first sheaf (approximation on the boundary of any relatively compact domain ω of any real finite continuous function by a linear combination of potentials with point support on a fixed neighbourhood of point of ω.

Let us emphasize two important feature of this axiomatic theory, first the bridge built by P.Meyer [59] with semi-groups and Markov processes, which allows to integrate nearly the theory (with 1, 2, 3, countable base, pot > 0) in the Hunt's frame (see also the recent books of Meyer [60], [61]) for the general correspondence between potential theory and Markov processes). Secondly the role of nuclear spaces that Loeb-*B* Walsh set in evidence [56] and which implies $3'$ from 1, 2, 3, countable base (the harmonic function on any open set form a space which is provided with the topology of uniform convergence on compact sets and then becomes a Frechet nuclear space). Both these featuring may be extended to more general axiomatics we will consider now.

158 In order to include the heat equation and other parabolic equations in the application of an axiomatic theory, as already did, Beuer [1, 2, 4] changed the axioms as follows: Same Hausdorff space connectedness is not required (trivial extension); if local connectedness and no compactness were not supposed, they would be con-connectedness and no

compactness were not supposed, they would be con-sequence of the following axioms. Same axiom 1. Same definition of regular open set (but with non-empty boundary) same axiom 2 (base of regular open sets); axiom 3 becomes weaker; for an increasing directed set of harmonic function on an open set ω, the limit is harmonic when $\mathcal{J}$ is superbounded (form K_1) when the limit is finite everywhere (form K_2) or when it is finite on a dense set (form K_D used by Doob in a metrizable space). As that does not allow a necessary minimum principle, a non-local separation axiom is added. Hyperharmonic functions may be introduced and for a harmonic function $h > 0$, also h-harmonic and hyperharmonic functions. Now "axiom T" means that a harmonic function $h > 0$ exists on Ω and that the h-hyperharmonic functions separate Ω. A stronger form T [4] means the same for positive h-hyperharmonic functions. A similar form that Bauer used in his final version [4] is the existence of an $h > 0$ and for any $x, y, x \neq y$ of two hyperharmonic functions $u_1, u_2 > 0$ such that $u_1(x)u_2(y) \neq u_1(y)u_2(x)\,(\circ.\infty = 0)$. Bauer used often and definitively (in [4]) the existence of a countable base of open sets. In this case and with K_D and T^*, there is an equivalent form of this **159**
axiomatic setting which is more easily comparable to Part IV (see Brelot [16]). For some questions, the existence of potential which is > 0 at any given point is also supposed (and gives the so called strong harmonic spaces [4]). Note that even without a countable base a countable base, all these axioms of Bauer are satisfied in the Brelot's axiomatic (with $1, 2, 3$ and existence of 2 non proportional harmonic functions in Ω). Now applications are possible to the heat equation and also to a large class of parabolic equations (Guber [45]). Little by little, with more or less strong hypothesis, more of the results of Brelot-Mrs. Herve's theory without D were extended, adapted even completed; the first steps of the ordinary Dirichlet problems are similar [5] and a more general problem for non-relatively compact open sets was studied later (Bauer [5]); an integral representation is much more difficult by lacking of a compact base of S^+, but Mokobodski [still unpublished] succeeded to overcome the difficulties of using the most refined results of Choquet on extreme

[4]For a summary of the last changes and improvements of the Bauer's theory, see [6].

[5] See complements in [52]

elements. A precise convergence theorem for superharmonic functions is not possible without D. But one may use a weak general theorem for any family of real fictions u_i in any topology logical space, when all u_i belong to a family Φ of functions ≥ 0 for which a suitable corresponding notion of weak thinness is defined; the exceptional set where $\widehat{\inf} u_i \neq \inf u_i$ is a countable union of sets which are weakly thin at every
160 point (Brelot [16]); and in the strongest Bauer's theory the weak thinness of $e \notin x_0$ is identical to the thinness. Let us emphasize another feature; i.e. the notion of absorbing set. (even without the last axiom of strong harmonicity). Such a closed set e is characterized by the existence of v hyperharmonic ≥ 0 which is zero exactly on the set; or by the property of supporting the harmonic measure $d\rho_{x_0}^{\omega_0}$ of any regular open ω_0 ($x_0 \in e, x_0 \in \omega_0$). This allows to give a deeper comparison of both Brelot and Bauer's axiomatics. Finally, for the strongest Bauer's theory the property of nuclearity (even without the last axiom) and the embedding in a Hunt type have been realized (resp. Bauer [4], Boboc-Constantinescu-Corna [10]). This nuclearity was essentially used in the important thesis of Hinrichsen [50] which generalizes the fundamental Cauchy formula of function theory in the Bauer's frame.

A slight extension of the Bauer's theory was made by Boboc Constaninescu-Corna [8], [9] in order to show that weak hyperothesis which offer some new applications, are sufficient to get various important results on the specific order (lattice properties and applications), on reduce functions and balayage. In differs from Bauer's theory essentially by changing the separation axiom (which furnished the minimum principle) into a weaker condition (close to this principle): the existence of a harmonic positive function in a neighbourhood of any fixed point and the covering property of Ω by a family of "M.P open sets" for which a suitable minimum principle holds.

161 The question arises of determining or characterizing all sheaves satisfying such systems of axioms. Striking results are already in Bony [14] α) for a domain of $\mathbb{R}^n$ and a sheaf invariant by translation. With axiom $1, 2$ and constants harmonic, there exists a) in case α, an open set Ω_0 dense in Ω and a pre-elliptic operator A (quadratic form ≥ 0) in Ω_0 with finite continuous coefficients, such that $Au = 0 \Leftrightarrow u$ harmonic locally;

axiom K_1 is even satisfied and the case of R^2 may be depend. b) in case (β), there exists a pre-elliptic operator A with constant coefficients and "$Au = o$ (in the sense of distributions) is equivalent to harmonicity"; as example of detailed study: "axion 3 $\Leftrightarrow$ A elliptic". A book on relation of these axiomatics and operators is under preparation [13].

By deeping these previous axiomatics, Mokobodski and D. Sibony [63] first considered the converse problem of determining a sheaf of harmonic functions for which the superharmonic functions form a given cone. More precisely, they start from a convex cone *Gamma* of lower semi-continuous bounded functions (on a locally compact space whose relatively compact open sets ω have a non-void boundary) which separate Ω, and suppose essentially that Γ is *maximal* (with respect to the inclusion order) among similar cones whose functions satisfy the minimum principle (i.e. $u \geq$ constant λ on $\partial\omega \Rightarrow u \geq \lambda$ on ω) (this maximality is satisfied for the cone of superharmonic functions in large axiomatics where constants are harmonic). Under some conditions, one may define a sheaf of harmonic functions (i.e. satisfying axioms $1, 2$ and even K_1) such that any bounded superharmonic function on an open set **162**
ω is equal to a function of the cone in any $\omega' \subset \overline{\omega'} \subset \overline{\omega}$ upto a harmonic function. In further very important researches [64] after completing the Choquet theory of "adapted cones" which is an essential tool, they start from a convex cone of continuous functions ≥ 0 (even bounded) and develop a theory generalizing Hunt's theory and where the given functions are potentials.

Let us finally mention the extension of the Poison integral to the semi-simple Lie Groups by Füztenberg [39] (see a lecture by Delzart [36]) and essay by Monna [65] of an axiomatic for order elements which are more general that functions.

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[59] P.A. Mayer Ann. I.F. 13/2, 1963

[60] - Probabilities and Potentials (Blaisdell pub. co. Boston 1966, French version, Hermann Paris 1966)

[61] - Processes de Markov, Lecture Notes, 26, Springer 1967

[62] G.Mokobodski Ann. I.F. 15/1, 1965

[63] Mokobodski- sibony Ann. I.F. 17/1, 1967

[64] Mokobodski-Sibny C R Ac.Sc.Pris 263(1966) A, p. 126 164
166 (1967) A, p.(1967) A, p. 264, Development to appear in Ann I.F

[65] A.F.Monna Nieuw Archief voor Wiskunde (3) XIV, pp 213-221, 1966

[66] D.Sibony C R Ac Sc.Paris 260,1965 Imporved development to appear Ann I.F.

Complemetry Ganaraly Bibliography on Potentional theory (revised in 2nd edition). **167**

Classical potential theory is being studied further (see for instance the treatise of Landkoff 1966). Various aspects of, modern potential theory are to be seen in the "colloque detheorie du potential" Paris-Orsay (1964) (Anaales T.F 15/2, 1965) and in the Seminaire du potential and Sem Choquet. A general bibliography is in the third edition of the French course : Brelot "Elements de la theorie classique du potential" Paris 1965.

Let us give here only short indications on modern potential theory, for other directions then those studied in these lectures.

i) Study and use of original Martin's boundary and similar boundaries

M Parreau	Ann I.F. 3 (1951-52)
M Brelot	J. of Math. 35 (1956)
L.Naim	Ann I.F. 7 (1957)
J.L Doob	Ann I.F. 9 (1959)
Brelot and Doob	Ann I.F. 13 1963
Dynkin	Ann I.F. 15/1, 1965
Constantinescu -Cornea	Ideale RAnder Riemannscher Flacher (Ergeb 32, Springer 1963)

ii) Relations with function theory

Doob	J. of Math 5,1961 Anales I.F. 15/2, 1965
Kuramochi	J.Fac Sc. Hokkaido Univ., esp 16,17 **168**
Constantinescu-Cornea	(see i))
Borelot	Symposim of Erevan (1965)

iii) Essential connections with semi-groups and Markov processes. Wide filed open chiefly by Doob (see previous bibl). Fundamental synthesis of Hunt (III J.) of Math. 1 and 2. 1957-58. See further researches in Meyer (previous bibl) in the treatise of Dynkin, in the reports of the Berkeley (1965) and Loutraki (1966) colloq., in Deny (Ann I.F.12 (1962), 15/1, (1965)) and Lion (Ann T. F 16/2 (1966)).

iv) Axiomatistion of Dirichlet integral (by Seurling and Deny) and Dirichlet spaces of Deny.

Beurling -Deny	Acta Math 99 (1958)
Beuriling	Symposium on Banach Algebras (Stanford 1958)
Deny	Sem.Potential (all volumes); Sem. Bourbaki 12, 1959-60
Thomas	Sem. pot 9 (1964-65)

v) Axiomatic approch to the Dirchlet problem with the "Silov boundary"

H. Bauer	Ann. I. F 11 (1961)
Further reserches by	
Rogalski	C R Ac. Sc. Paris 263 A, p. 664 and 726 and Seminaire Choquet

vi) Bessel potentials by Aronszajn, Smith Ann I. F. 11, 15/1,17/2,18/2.

169 vii) Relations with harmonic analysis, theory of games,ergodic theory. See papers of Carleson, Herz, Fuglede, Meyer (Ann. I.F. 15/1, 1965).

viii) Relations with partial differential equations. See Mrs. Herve's papers (additional bibl); on the Martin boundary for such equations see: S Ito J. Math Soc:, japan 16, 1964

G. Wildenhain Potential -theorie linearer elliptische Differential gleichungen beliebiger Ordnung (Dentscha Akod. Berlin 1967).