Lectures on Modular Functions of One Complex Variable

By

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Notes by

Sunder Lal

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(Revised 1983)

Dedicated to

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Preface To The Revised Edition

Thanks are due to the Editor of the Tata Institute Lecture Notes in Mathematics for the suggestion to reissue my Lecture Notes of 1963, which provided an opportunity to touch up the text with improvements in several respects. In collaboration with Professor S. Raghavan I could expand the Errata from its original four pages to sixteen pages. He took upon himself the task of inserting the necessary modifications in the notes. A truly onerous undertaking! I am indebted to Professor E. Grosswald for valuable proposals in this connection. We have now a substantially revised version which we hope, is much more readable. However, what is perfect in this world! I therefore seek the indulgence of the reader for any possible error in the revised text.

Finally mention should be made of Dr. C.M. Byrne, Mathematics Editor of Springer-Verlag, for her encouraging support at crucial stages of the task and for making me occasionally forget, in a charming manner, that decisions of the publishers are based on prosaic calculations.

It is a pleasure to thank everyone who has been involved in this project.

Heidelberg

November 15, 1983 **H. Maass**

Preface To The First Edition

These are notes of lectures which I gave at the Tata Institute of Fundamental Research in 1962/63. They provide an introduction to the theory of modular functions and modular forms and may be described as elementary, in as much as basic facts from the theory of functions of a complex variable and some properties of the elementary transcendental functions form the only prerequisites. (It must be added that I have counted the Whittaker functions among the elementary transcendental functions). It seemed to me that the investigations of Siegel on discrete groups of motions of the hyperbolic plane with a fundamental region of finite volume form a particularly suitable introduction, since they make possible a simple characterization of groups conjugate to the modular group by a minimal condition.

My thanks are due to Mr. Sunder Lal for his careful preparation of these notes.

Hans Maass

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Horocyclic Groups

1 The Poincaré Model of the Hyperbolic Plane 1

Let $\mathfrak{H}$ denote the upper half-plane $\{\tau = x + iy, x, y \text{ real and } y > 0\}$. It is well-known that any conformal mapping of $\mathfrak{H}$ onto itself is given by

$$\tau \to \tau^* = x^* + iy^* = S < \tau >= \frac{a\tau + b}{c\tau + d}, \tag{1}$$

where $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix with a, b, c, d real and determinant $|S|$ equal to 1. All such matrices form a group under matrix multiplication and we denote this group by Ω. For S_1, S_2 in Ω, we clearly have $(S_1 S_2) < \tau >= S_1 < S_2 < \tau >>$. It is obvious that two elements S_1 and S_2 of Ω define the same mapping of $\mathfrak{H}$ if and only if $S_1 = \pm S_2$. The domain $\mathfrak{H}$ together with Ω can be looked upon as a model for the hyperbolic plane. *The hyperbolic straight lines* in this plane are defined by the segments of the circles (Here and in the following, circles include straight lines) orthogonal to the real axis, which lie in the upper half-plane. On $\mathfrak{H}$, we have a metric form

$$ds^2 = \frac{d\tau d\overline{\tau}}{y^2} = \frac{dx^2 + dy^2}{y^2}. \tag{2}$$

Since

$$d\tau^* = (c\tau + d)^{-2} d\tau \quad \text{and} \quad y^* = |c\tau + d|^{-2} y, \tag{3}$$

the metric form (2) is left invariant by the transformations of Ω. We shall show that with respect to this metric form, the hyperbolic straight

line joining any two points of $\mathfrak{H}$ is the path of shortest distance joining the two points. It is sufficient to prove this assertion for the points i and
2 $iy_0 (y_0 \geq 1)$, since for any two points τ_1 and τ_2 in $\mathfrak{H}$, there exists an element $S \in \Omega$ mapping τ_1, τ_2 respectively to i and iy_0 (with a suitable $y_0 \geq 1$) and further any transformation from Ω maps hyperbolic straight lines to hyperbolic straight lines, leaving the metric form invariant. Let $\tau = \tau(t) = x(t) + iy(t)$ for $a \leq t \leq b$ be a parametric representation of a continuously differentiable curve joining i and iy_0. Then

$$s = \int_a^b \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)} dt \geq \int_a^b \frac{|\dot{y}(t)|}{y(t)} dt \geq \left| \int_a^b \frac{\dot{y}(t)}{y(t)} dt \right| . \tag{4}$$

It $\tau(t)$ were a curve of minimum length, equality must hold everywhere in (4), because, otherwise, the curve $\tau = iy(t)$ for $a \leq t \leq b$, which also joins i and iy_0 would be of shorter length; or the given curve $\tau = \tau(t)$ would contain at least one double point, but this is impossible. This shows, because of the continuity of $\dot{x}(t)$, that $\dot{x}(t)$ and therefore $x(t)$ also vanishes identically and finally that $y(t)$ is monotonically increasing. Thus the curve of minimal length between i and iy_0 must be the hyperbolic straight line joining them, therefore we can choose $t = y$ as a parameter and obtain

$$s = \int_a^b \frac{\dot{y}(t)}{y(t)} dt = \int_1^{y_0} \frac{dy}{y} = log y_0 = \log((i, iy_0, 0, \infty)), \tag{5}$$

where (z_1, z_2, z_3, z_4) generally denotes the cross ratio of the four points z_1, z_2, z_3 and z_4 in the extended complex plane defined by $(z_1, z_2, z_3, z_4) = \dfrac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$. If one of the four points say $z_4 = \infty$, we take $\dfrac{z_1 - z_4}{z_2 - z_4}$
3 $= 1$. Hence, for any two points τ_1 and τ_2 of $\mathfrak{H}$, we obtain from (5) the following formula for the shortest distance:

$$s = \log((\tau_1, \tau_2, \sigma_1, \sigma_2)) = \rho(\tau_1, \tau_2), \tag{6}$$

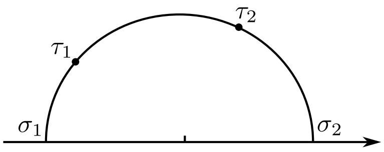

Figure 1.1:

with σ_1, σ_2 representing the points of intersection of the hyperbolic line joining τ_1, τ_2 and the real axis. It can be proved easily that $\mathfrak{H}$ provided with the distance formula (6) is a metric space and this metric on $\mathfrak{H}$ is left invariant by elements of Ω. Moreover, it is well known that this metric defines on $\mathfrak{H}$ the same topology as the usual topology i.e. the topology induced from complex numbers. On $\mathfrak{H}$, we have the measure

$$d\omega = \frac{dxdy}{y^2}, \tag{7}$$

which is invariant under transformations of Ω in view of the formula

$$\frac{\partial(x^*, y^*)}{\partial(x, y)} = \left|\frac{d\tau^*}{d\tau}\right|^2 = {y^*}^2/y^2.$$

We shall prove that the area $\Im(A, B, C)$ with respect to the measure in (7) of a hyperbolic triangle (A, B, C) i.e. a triangle with hyperbolic straight lines as edges with α, β and γ as angles (see figure 1.2 is $\pi - \alpha - \beta - \gamma$. In order to find $\Im(A, B, C)$ it is sufficient to find the area of the triangles (A, B, ∞), (A, C, ∞) and (B, C, ∞), because

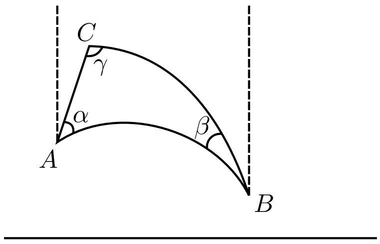

Figure 1.2:

$\Im(A,B,C) = \Im(A,B,\infty) - \Im(A,C,\infty) - \Im(B,C,\infty)$. If (A,B,∞) is
4 a hyperbolic triangle, with the angles α, β and 0 (see figure 1.3, then $\Im(A,B,\infty) = \pi - \alpha - \beta$; indeed,

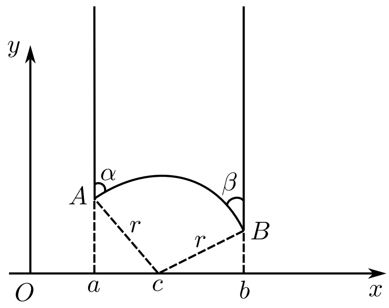

Figure 1.3:

$$\begin{aligned}
\Im(A,B,\infty) &= \iint\limits_{A,B,\infty} \frac{dxdy}{y^2} \\
&= \int_a^b dx \int_{\sqrt{r^2-(x-c)}}^{\infty} y^{-2}dy \\
&= \int_a^b \frac{dx}{\sqrt{r^2-(x-c)^2}} = \arcsin\left(\frac{b-c}{r}\right) - \arcsin\left(\frac{a-c}{r}\right) \\
&= \left(\frac{\pi}{2} - \beta\right) - \left(\alpha - \frac{\pi}{2}\right) = \pi - \alpha - \beta.
\end{aligned}$$

Let $\mathfrak{L}$ denote the unit disc $\{z = u + iv, u, v$ real and $|z| < 1\}$. The mapping

$$\tau \to z = \frac{\tau - i}{\tau + i} = A < \tau > \text{with} A = \left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right)$$

maps $\mathfrak{H}$ conformally onto $\mathfrak{L}$. If $\tau^* = S < \tau >$ is mapped to z^* in $\mathfrak{L}$, then

$$z^* = A < \tau >^* = AS < \tau > = ASA^{-1} < z > = L < z >,$$

with $L = ASA^{-1} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1}$, where $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

This shows that L is a matrix of the type $\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix}$ with

$$\alpha = \frac{a + d + (c - b)i}{2}, \quad \beta = \frac{a - d + (c + b)i}{2}$$

and $\Omega_0 = \{\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix} | \alpha\bar{\alpha} - \beta\bar{\beta} = 1|\} = A\Omega A^{-1}$ is the group of all conformal mapping of $\mathfrak{L}$ onto itself. Therefore we can look upon $\mathfrak{L}$ together with Ω_0 as a model for the hyperbolic plane. The metric form and measure on $\mathfrak{L}$ are given by

$$ds^2 = \frac{4(du^2 + dv^2)}{(l - u^2 - v^2)^2}, \quad d\omega = \frac{4dudv}{(l - u^2 - v^2)^2} \tag{8}$$

The hyperbolic straight lines in $\mathfrak{L}$ are segments of the circles orthog- **5**
onal to the unit circle. On $\mathfrak{L}$, the metric is given by

$$\delta(z_l, z_2) = \rho(\tau_l, \tau_2), \quad \text{where} \quad A < \tau_i >= z_i, \quad i = 1, 2. \tag{9}$$

Definition. *A hyperbolic circle of radius r and centre τ_0 in the hyperbolic plane is the set of those points which are at a hyperbolic distance r from τ_0.*

If $\delta(z, 0) =$ constant, then $|z| =$ constant and so a hyperbolic circle in $\mathfrak{L}$ with centre 0 is some Euclidean circle with the same centre. In general, a hyperbolic circle in $\mathfrak{L}$ and therefore in $\mathfrak{H}$ is some Euclidean circle but their centres need not be the same. We shall find the Euclidean centre and radius of a hyperbolic circle in terms of the hyperbolic centre and radius. For the sake of simplicity, we take the hyperbolic circle U of radius ρ with the centre ih lying on the imaginary axis. It can be proved easily that in this case the Euclidean centre also lies on the imaginary axis. Let im and r be the Euclidean centre and radius of U respectively. Then (see Figure 1.4

$$\rho = \log(y_2/h) = \log(h/y_1),$$

where iy_1 and iy_2 are the points of intersection of U with the imaginary axis. Thus

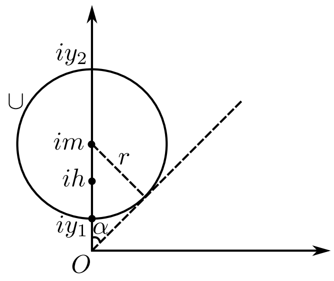

Figure 1.4:

$$y_1 = he^{-\rho}, \quad y_2 = he^{\rho}$$

implying that

$$m = \frac{1}{2}(y_1 + y_2) = h \cosh \rho \tag{10}$$

$$r = \frac{1}{2}(y_2 - y_1) = h \sinh \rho \tag{11}$$

Consequently, the circle U is represented by

$$|\tau - ih \cosh \rho| \; h \sinh \rho.$$

6 Further, we have

$$\sin \alpha = \frac{r}{m} = \tanh \rho, \tag{12}$$

which is independent of h showing that

$$|\frac{x}{y}| = \tan \alpha = \sinh \rho$$

is the locus of those points at a distance ρ from the line $x = 0$. In particular, if the centre of the circle U is i, then its equation is

$$|\tau - i \cosh \rho| = \sinh \rho, \quad \text{with} \quad \rho = \rho(\tau, i)$$

$$\text{or } x^2 + (y - \cosh \rho)^2 = \sinh^2 \rho \Longrightarrow \cosh \rho = \frac{x^2 + y^2 + 1}{2y}$$

Moreover, $|z|^2 = \left|\frac{\tau - i}{\tau + i}\right|^2 = \frac{x^2 + y^2 + 1 - 2y}{x^2 + y^2 + 1 + 2y} = \frac{\cosh\rho - 1}{\cosh\rho + 1} = \tau h^2 \frac{\rho}{2}$. So the hyperbolic polar coordinates can be introduced by

$$\frac{\tau - i}{\tau + i} = \tanh\frac{\rho}{2} \cdot e^{i0} \tag{13}$$

We have already defined a metric on $\mathfrak{L}$, namely $\delta(z_1, z_2) = \rho(\tau_l, \tau_2)$, where $z_i = A < \tau_i >$, $i = 1, 2$. In particular, we have

$$\delta = \delta(z, 0) = \rho(\tau, i), \quad \delta^* = \delta(z^*, 0) = \rho(\tau^*, i),$$

where $z^* = \frac{\overline{\alpha}z + \overline{\beta}}{\beta z + \alpha} = L < z >$. This gives an interesting geometric interpretation for the expression $|\beta z + \alpha|$. We have seen that $l - |z|^2 = 1 - \tanh^2\delta/2 = \cosh^{-2}\delta/2$, from which it can be shown easily that

$$l - |z^*|^2 = \frac{l - |z|^2}{|\beta z + \alpha|^2}.$$

Therefore **7**

$$\frac{\cosh\delta^*/2}{\cosh\delta/2} = |\beta z + \alpha|.$$

Because of the invariance of $\delta(z_l, z_2)$ by Ω_0, $|\beta z + \alpha| = 1$ if and only if

$$\delta(z, L^{-1} < 0 >) = \delta(z^*, 0) = \delta(z, 0)$$

i.e. z has the same hyperbolic distance from 0 and $L^{-1} < 0 >= -\overline{\beta}/\overline{\alpha}$.

2 Discontinuous groups of motions

Let X be a topological space and G a group acting on X i.e.,

i) gx for $g \in G$ and $x \in X$ belongs to X and is uniquely defined,

ii) $g_l(g_2x) = (g_lg_2)x$ for $g_l, g_2 \in G$ and $x \in X$ and

iii) $ex = x$, where e is the unit element of G.

Definition. *Two points x_l, x_2 of X are said to be equivalent with respect to G if there exists an element g in G such that $g(x_l) = x_2$. It is obvious that the set of points $\{gx|g \in G\}$ is a complete set of equivalent points.*

Definition. *The group G is said to act discontinuously on X, if no set of equivalent points has a limit point in X. Here, we consider $g_l x$ and $g_2 x$ with $g_l \neq g_2$ as different elements of the set of points equivalent to x even if $g_l x = g_2 x$.*

In particular, if G is a topological group, then G can be considered as acting on itself. We say that G is *discrete*, if G acts discountinuously on itself.

8 **Lemma 1.** *A topological group G is discrete if and only if there exists a neighbourhood of the unit element containing only a finite number of elements.*

Proof. For the sake of simplicity, we assume that G satisfies the first axiom of countability.

(i) Let every neighbourhood of the unit element e in G contain infinitely many elements. We can then choose a sequence $\{g_n\}$ in G converging to e with $g_n \neq g_{n+1}$ for all $n \geq 1$. Hence, for every x in G, the sequence $\{g_n x\}$ converges to x as n tends to infinity i.e. the set $\{gx|g \in G\}$ of elements in G equivalent to x has x as a limit point, implying that G is not discrete.

(ii) Let G be not discrete, so that G contains a subset $\{gx|g \in G\}$ of elements equivalent to x in G having a limit point, say b. Thus, we can find in G a sequence $\{g_n x\}$ converging to b as n tends to infinity, with the property that $g_n \neq g_{n+1}$ for every $n \geq 1$. Then clearly $\{g_n x b^{-l}\}$ converges to the unit element e and so does the sequence $\{b x^{-l} g_{n+1}^{-1}\}$. This leads us to a (non-trivial) infinite sequence $\{g_n g_{n+1}^{-1}\}$ converging to e. It follows that every neighbourhood of e contains infinitely many elements. Our lemma is proved.

In the sequel, we take X to be the hyperbolic plane and G to be Ω or Ω_0 according as $X = \mathfrak{H}$ or $\mathfrak{L}$. The action of G on X in either case has been defined already in §1.

∎

Theorem 1. *A subgroup Γ of G acts discontinuously on X if and only if Γ is discrete.*

Proof. We work with $X = \mathfrak{L}$ for the proof. **9**

(i) Let Γ act discontinuously on $\mathfrak{L}$. Then we shall show that Γ is discrete. Let, if possible, Γ be not discrete. Then there exists a sequence $\left\{S_n = \begin{bmatrix} \bar{\alpha}_n & \bar{\beta}_n \\ \beta_n & \alpha_n \end{bmatrix}\right\}$ in Γ such that $S_n \neq S_{n+1}$ and $S_n \to E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as $n \to \infty$. This implies that

$$\alpha_n \to 1 \text{ and } \beta_n \to 0 \text{ i.e. } S_n < 0 >= \bar{\beta}_n / \alpha_n \to 0 \text{ as } n \to \infty.$$

Thus the set of equivalent points $\{S < 0 > | S \in \Gamma\}$ has a limit point, contradicting the discontinuous action of Γ. Hence Γ is necessarily discrete.

(ii) Let now Γ be discrete. If Γ does not act discontinuously on $\mathfrak{L}$, then there exists a point z in $\mathfrak{L}$ such that the set $\{S < z > | S \in \Gamma\}$ has a limit point say z^*. Therefore we can find a sequence $\left\{S_n = \begin{bmatrix} \bar{\alpha}_n & \bar{\beta}_n \\ \beta_n & \alpha_n \end{bmatrix}\right\}$ in Γ with $S_n \neq S_{n+1}$ such that $S_n < z > \to z^*$ as $n = \infty$. Let $z_n = S_n < z >= \dfrac{\bar{\alpha}_n z + \bar{\beta}_n}{\beta_n z + \alpha_z}$. Then

$$1 - |z_n|^2 = \frac{1 - |z|^2}{|\beta_n z + \alpha_n|^2} \to 1 - |z^*|^2$$

$$\implies |\beta_n z + \alpha_n| \to \sqrt{\frac{1 - |z|^2}{l - |z^*|^2}} \text{ as } n \to \infty.$$

Since $\left|\dfrac{\beta_n}{\alpha_n}\right| < 1, |\beta_n z + \alpha_n| = |\alpha_n| \left|\frac{\beta_n}{\alpha_n} z + 1\right| \geq |\alpha_n|(1 - |z|)$. This shows that $|\alpha_n|$ and therefore $|\beta_n|$ is bounded. Therefore we can find a subsequence $\{S_{k_n}\}$ of $\{S_n\}$ such that $\{S_{k_n} S^{-1}_{k_{n+1}}\} \to E$ as

$n \to \infty$. But this is contrary to the discreteness of Γ; therefore, Γ acts discontinuously on $\mathfrak{L}$.

□

10 **Definition .** *A conformal transformation of the hyperbolic plane with the corresponding matrix $S \neq \pm E$ is said to be hyperbolic or elliptic or parabolic according as the two fixed points of the transformation are distinct and lie on the boundary of the hyperbolic plane or the two fixed points are inverse points with respect to the boundary circle of the hyperbolic plane or the two fixed points coincide.*

It can be proved easily that if the determinant $|S|$ of S equals 1, then

$$\rho^2(S) = (\sigma(S))^2 \begin{cases} > 4 & \text{if } S \text{ is hyperbolic,} \\ < 4 & \text{if } S \text{ is elliptic,} \\ = 4 & \text{if } S \text{ is parabolic,} \end{cases} \tag{1}$$

where $\sigma(S)$ denotes the trace of S.

By a *hyperbolic group* of transformations of the hyperbolic plane we mean a group consisting wholly of hyperbolic transformations except for E and possibly $-E$ as well. We have the following remarkable theorem for this hype of groups.

Theorem 2. *If a subgroup Γ of Ω is a non-commutative hyperbolic group, then Γ acts discontinuously on the hyperbolic plane X.*

Proof. We shall take the upper half-plane $\mathfrak{H}$ as a model for X. Now Γ contains a hyperbolic element $S \neq \pm E$ with fixed points $\omega, \omega' \neq \omega$. For V in Ω defined by $V < \tau >= (\tau - \omega)(\tau - \omega')^{-l}$, let $S^* = VSV^{-1}$ and $\Gamma^* = V\Gamma V^{-l}$. Then $\sigma(S^*) = \sigma(S)$ so that S^* is again hyperbolic and further Γ^* is also a hyperbolic group, which is discrete if and only if Γ is discrete. Thus passing over to Γ^* if necessary, we may assume that
11 S has already 0 and ∞ as its fixed points, so that $S = \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}$ with $\ell \neq \pm 1$.

If possible, let Γ no act discontinuously on $\mathfrak{H}$. then by theorem 1, Γ is not discrete and therefore contains a sequence $\{T_n\}$ of elements $T_n \neq \pm E$

converging to E. We will show under the given circumstances, that all but finitely many T_n are diagonal. For $T_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, consider the commutators $C_m = ST_mS^{-1}T_m^{-1}$ and $D_m = SC_mS^{-1}C_m^{-1}$. If is easily checked that

$$\begin{aligned} C_m &= \begin{pmatrix} ad - bc\ell^2 & ab(\ell^2 - 1) \\ cd(\ell^{-2} - 1) & ad - bd\ell^{-2} \end{pmatrix} \\ &= \begin{pmatrix} 1 + bc(1 - \ell^2) & ab(\ell^2 - 1) \\ cd(\ell^{-2} - 1) & 1 + bc(1 - \ell^{-2}) \end{pmatrix} \end{aligned}$$

Since Γ is hyperbolic, the element C_m is hyperbolic if $C_m \neq \pm E$. In any case,

$$\sigma^2(C_m) = (2 + 2bc - bc(\ell^2 + \ell^{-2}))^2 = (2 - bc(\ell - \ell^{-1})^2)^2 \geq 4.$$

Similarly, we have

$$\sigma^2(D_m) = (2 - ab(\ell^2 - 1)cd(\ell^{-2} - 1)(\ell - \ell^{-1})^2)^2 = (2 + abcd(\ell - \ell^{-1})^4)^2 \geq 4.$$

Since $\{T_n\}$ converges to $E, \{C_n\}$ converges to E so that $1 + bc(1 - \ell^2)$ tends to 1. Since $\ell \neq \pm 1$ is fixed, this means that bc tends to 0. We claim that $bc = 0$ for all large enough m. In fact, if bc is positive and sufficiently small, we have a contradiction from $\sigma^2(C_m) = (2 - bc(\ell - \ell^{-1})^2)^2 < 4$. Since $ad - bc = 1$, ad tends to 1 and hence disregarding **12** finitely many m, we can suppose that $ad > 0$, so that $abcd$ has the same sign as bc. Let now $bc < 0$, if possible. Then considering D_m instead of C_m, we see that $abcd$ tends to 0 as m tends to ∞. But from $bc < 0$, we obtain that $abcd < 0$ and $|abcd|$ is small for all large m, so that $\sigma^2(D_m) < 4$, a contradiction again. Thus, for all large m, $bc = 0$ and therefore $ad = 1$. Now $\sigma(C_m) = 2ad = 2$ and since Γ is hyperbolic, this means that $C_m = E$, for all large m i.e. $ab = cd = 0$. Since $ad = 1$, we have $b = c = 0$ i.e. T_m is diagonal, for all large m, as required to be proved.

Dropping finitely many T_m, we have in Γ, a sequence $\{T_n\}$ of diagonal matrices $T_n = \begin{pmatrix} \ell_n & 0 \\ 0 & 1/\ell_n \end{pmatrix}$ with $\ell_n \neq \pm 1$. Let now $F = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be an arbitrary element of Γ. To the sequence $\{T_nFT_n^{-1}F^{-1}\}$ (of commutators) converging to E (as $n \to \infty$), we apply the same arguments as to $\{C_m\}$ above. Then we can conclude that necessarily $q = r = 0$ and F is diagonal. In other words, Γ consists entirely of diagonal matrices and is hence commutative, a contradiction. Thus Γ necessarily acts discontinuously on X. □

3 Fundamental domain

Let Γ be a discrete group of motions of the hyperbolic plane.

Definition. *A point-set $\mathfrak{F}$ of the hyperbolic plane is called a* fundamental domain *for Γ if*

1) *$\mathfrak{F}$ contains at least one point from each set of equivalent points with respect to Γ and*

2) *If $z \in \mathfrak{F} \cap \mathfrak{F}_S$ ($\mathfrak{F}_S$ = Image of $\mathfrak{F}$ by S), then z is a boundary point of $\mathfrak{F}$ and $\mathfrak{F}_S$ provided $S \neq \pm E$.*

13 In the following, we shall give a construction of a fundamental domain for Γ by geometrical methods. First of all, we observe that Γ is countable. For, the number of matrices $S = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix}$ with $\alpha\bar{\alpha} \leq n$ (n a natural number) is finte, because if the number of such matrices were infinite, then $\alpha\bar{\alpha} \leq n$ and $\beta\bar{\beta} = \alpha\bar{\alpha} - l < n$ will imply that Γ is not discrete. Since every element $S \in \Gamma$ different from $\pm E$ can have atmost one fixed point in $\mathfrak{L}$ and Γ is countable, there exists a point $\zeta \in \mathfrak{L}$ which is not a fixed point of any element S of Γ different from $\pm E$. In the following, we set $S < \zeta >= \zeta_S$. We shall show that the set $\mathfrak{F}$ defined by

$$\mathfrak{F} = \{z | \delta(z, \zeta) \leq \delta(z, \zeta_S) \text{ for } S \in \Gamma, z \in \mathfrak{L}\} \tag{1}$$

is a fundamental domain for Γ. It is obvious that $\mathfrak{F}$ consists of all points z, whose distance from ζ is not greater than the distance from the equivalent points ζ_S. Obviously, we have

$$\begin{aligned} \mathfrak{F}_T &= \{T < z > | \delta(z, \zeta) \text{ for } S \in \Gamma, z \in \mathfrak{L}\} \\ &= \{z | \delta(T^{-l} < z >, \zeta) \leq \delta(T^{-l} < z >, \zeta_S) \text{ for } S \in \Gamma, z \in \mathfrak{L}\} \end{aligned}$$

$$= \{z|\delta(z,\zeta_T) \leq \delta(z,\zeta_{TS}) \text{ for } S \in \Gamma, z \in \mathfrak{L}\}$$
$$= \{z|\delta(z,\zeta_T) \leq \delta(z,\zeta_S) \text{ for } S \in \Gamma, z \in \mathfrak{L}\}.$$

We now claim that

$$\mathfrak{L} = \bigcup_{\pm T \in \Gamma} \mathfrak{F}_T.$$

Indeed, for an arbitrary $z \in \mathfrak{L}$ the set $\{\delta(z,\zeta_S)|S \in \Gamma\}$ has a minimum because of the discreteness of Γ, i.e. $\delta(z,\zeta_T) \leq \delta(z,\zeta_S)$ for some T in Γ and for all S in Γ; therefore, $z \in \mathfrak{F}_T$. In particular, $\mathfrak{F}$ contains atleast one **14**
point from each set of equivalent points. In order to prove the second condition for $\mathfrak{F}$ to be a fundamental domain, we proceed as follows. Let g_S for $S \neq \pm E$ denote the locus of points $z \in \mathfrak{L}$ such that

$$\delta(z,\zeta) = \delta(z,\zeta_S).$$

This is the equation of an orthogonal circle, which for $\zeta = 0$ as shown in §1 can be represented by $|-\beta z + \bar{\alpha}| = 1$ if $S = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix}$. The line g_S decomposes the hyperbolic plane into two parts. We denote by $\mathfrak{L}_S$ the closed half plane which contains ζ. In fact,

$$\mathfrak{L}_S = \{z \in \mathfrak{L}|\delta(z,\zeta) \leq \delta(z,\zeta_S)\}.$$

Therefore $\mathfrak{F} = \bigcap\limits_{\pm S \in \Gamma} \mathfrak{L}_S$. In particular, since each $\mathfrak{L}_S$ is a convex set and an arbitrary intersection of convex sets is convex, $\mathfrak{F}$ is a convex set. Moreover, it can be verified easily that the boundary of $\mathfrak{F}$ in $\mathfrak{L}$ consists of some segments of the hyperbolic straight lines g_T for $T \in \Gamma$. In other words, z is a boundary point of $\mathfrak{F}$ if and only if $\delta(z,\zeta) \leq \delta(z,\zeta_S)$ for $S \in \Gamma$ and equality holds atleast for one $S \neq \pm E$. Thus if $z \in \mathfrak{F} \cap \mathfrak{F}_T$, then necessarily, we have $\delta(z,\zeta) \leq \delta(z,\zeta_S)$ and $\delta(z,\zeta_T) \leq \delta(z,\zeta_S)$ for $S \in \Gamma$. Taking $S = T$ in the first inequality and $S = E$ in the second, we obtain that

$$z \in \mathfrak{F} \cap \mathfrak{F}_T \Longrightarrow \delta(z,\zeta) = \delta(z,\zeta_T)$$

showing that z is a boundary point of $\mathfrak{F}$ and $\mathfrak{F}_T$. Hence the set defined in (1) is a fundamental domain for Γ. Moreover, we shall now prove that only finitely many lines g_T for T in Γ, constituting the boundary of $\mathfrak{F}$, intersect a given compact set K. Let K be contained in the circle $U = \{z|\delta(z,\zeta) \leq R, z \in \mathfrak{L}\}$. If $z \in g_T \cap U$, then we must have **15**

$$R \geq (z,\zeta) \geq \frac{1}{2}\delta(\zeta,\zeta_T)$$
$$\text{or} \delta(\zeta,\zeta_T) \leq 2R.$$

But Γ is discontinuous and therefore only finitely many T exist for which $\delta(\zeta,\zeta_T) \leq 2R$. We thus conclude that $\mathfrak{F}$ as defined in (1) is a convex set bounded by a countable number of hyperbolic lines, only a finite number of which intersect a given compact set. We call such a fundamental domain $\mathfrak{F}$ a *normal fundamental domain* with the centre ζ. The area of $\mathfrak{F}$ is given by

$$\Im(\mathfrak{F}) = \lim_{\rho\to\infty} 4 \int\limits_{\substack{\delta(z,\zeta)\leq\rho \\ z\in\mathfrak{F}}} \frac{dudv}{(l-u^2-v^2)^2}.$$

This integral may be an improper integral and can have an infinite value. If the value is infinite, then the fundamental domain has limit points on the boundary of the hyperbolic plane. If the fundamental domain is compact, its area is finite; the converse is not true in general. However, we have the following

Theorem 3. *If Γ is a discrete group of motions of the hyperbolic plane containing no parabolic motions and having a normal fundamental domain $\mathfrak{F}$ with finite area, then $\mathfrak{F}$ is compact.*

Proof. Throughout the proof, the hyperbolic plane will be represented by $\mathfrak{L}$. We decompose the boundary of $\mathfrak{F}$ in $\mathfrak{L}$ into connected components, the number of which will be atmost countable. The proof is given in six steps.

16 1) If there exists one connected component which is a closed curve, then this cannot have double points, in view of $\mathfrak{F}$ being convex. So this connected component represents a polygon. Since $\mathfrak{F}$ is convex and therefore simply connected, $\mathfrak{F}$ is in the interior of this polygon and is obviously compact. We exclude this case from our considerations. Starting from a boundary point of $\mathfrak{F}$ in $\mathfrak{L}$ (which exists by the construction of $\mathfrak{F}$), if we move along the boundary of $\mathfrak{F}$ in one of the two possible directions, then the following two possibilities can occur:

(a) we come across only a finite number of vertices i.e. there exists a last edge reaching the boundary of $\mathfrak{L}$,

(b) we meet an infinite number of vertices.

The case in which we come back to the starting point has already been discussed above. Let

$$\ldots\ldots, z_{-1}, z_0, z_1, z_2, \ldots\ldots$$

denote the vertices of one of the connected components of $\mathfrak{F}$. Here, if the sequence terminates on the right with z_r, then through z_r pass two edges of $\mathfrak{F}$, one joining the point z_{r-1} to z_r and the other reaching the boundary of $\mathfrak{L}$. If z_{r+1} is the point of intersection of the boundary of $\mathfrak{L}$ with the edge through z_r reaching the boundary of $\mathfrak{L}$, we define z_{r+1} to be a vertex of the fundamental domain $\mathfrak{F}$. Similarly, if the sequence terminates on the left with z_{-r}, we define the point z_{-r-1} to be a vertex of $\mathfrak{F}$. The triangle Δ_k with the vertices ζ (the centre of $\mathfrak{F}$), z_k and z_{k+1} is contained in $\mathfrak{F}$, since $\mathfrak{F}$ is a convex set.

Let α_k, β_k and γ (as in figure 1.5 be the angles of Δ_k. Then **17**

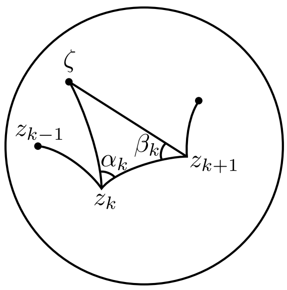

Figure 1.5:

$$\Im(\Delta_k) = \pi - \alpha_k - \beta_k - \gamma_k.$$

If z_k respectively z_{k+1} belongs to the boundary of $\mathfrak{L}$, then α_k respectively β_k is equal to 0. Let $\omega_k = \alpha_k + \beta_{k-1}$.

Then, because of the convexity of $\mathfrak{F}$, we have

$$0 < \omega_k < \pi. \tag{2}$$

Summing the areas of the triangles Δ_k for $p \leq k \leq q$, we obtain

$$\Im_{p,q} = \sum_{k=p}^{q} \Im(\Delta_k) = \sum_{k=p}^{q} (\pi - \alpha_k - \beta_k - \gamma_k)$$

$$\text{or } \Im_{p,q} + \sum_{k=p}^{q} \gamma_k = \pi - \alpha_p - \beta_q + \sum_{k=p+1}^{q} (\pi - \omega_k). \tag{3}$$

But

$$\Im_{p,q} \leq \Im(\mathfrak{F}) < \infty, \quad A_{p,q} := \sum_{k=p}^{q} \gamma_k \leq 2\pi, \tag{4}$$

and therefore, in case (b), the sequences $\Im_{p,q}$ and $A_{p,q}$ converge when $p \to -\infty$ and $q \to \infty$. Since $\alpha_p, \beta_q < \pi$ from (2), the series $\sum_{k=p+1}^{q} (\pi - \omega_k)$ converges when $p \to -\infty$ and $q \to \infty$. The convergence of the sequence $\Im_{p,q}$ implies the convergence of α_p when $p \to -\infty$ and β_p when $q \to \infty$. Let us write

$$\lim_{p \to -\infty} \alpha_p = \alpha_{-\infty}, \quad \lim_{q \to \infty} \beta_q = \beta_{\infty}. \tag{5}$$

In the case of the possibility (a), we choose p to be minimal and
18 q to be maximal and then $\alpha_p = \beta_q = 0$. We shall now show that
the limiting values of α_p and β_q in (5) satisfy the inequalities

$$\alpha_{-\infty} \leq \frac{\pi}{2}, \quad \beta_{\infty} \leq \frac{\pi}{2}. \tag{6}$$

Let $r_k = \delta(z_k, \zeta)$. Then if there exists no extremal p (respectively q), we have

$$r_k \to \infty, \text{ for } k \to -\infty(\text{respectively } k \to \infty).$$

For, only a finite number of edges $\widehat{z_k z_{k+1}}$ ($\widehat{z_k z_{k+1}}$) denoting the hyperbolic line joining z_k and z_{k+1}) can meet a given compact set. We must have

$$r_k < r_{k+1} \text{ for infinitely many } k, \tag{7}$$

since, otherwise, the sequence r_k will be bounded. We now prove that, in the triangle Δ_k for which $r_k < r_{r+1}$, opposite to the greater side we have the greater angle (as in Figure 1.6 i.e.

$$\beta_k < \alpha_k. \tag{8}$$

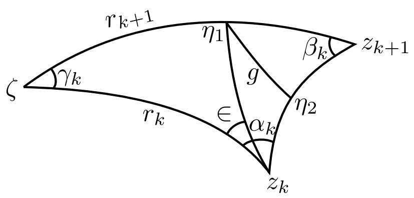

Figure 1.6:

Let g denote the perpendicular bisector of $\widehat{z_k z_{k+1}}$ (in the sense of hyperbolic geometry). Since $z_k < z_{k+1}$, it follows that η_1, the point of intersection of g and the line through ζ and z_{k+1} lies between ζ and z_{k+1}. This means that the angle $\in$ (see Figure 1.6 is positive. **19**
It is obvious that the triangles $< z_k \eta_1 \eta_2 >$ and $< \eta_l \eta_2 z_{k+1} >$ are congruent in the hyperbolic sense. Therefore we have $\beta_k = \alpha_k - \in$, which proves our assertion (8). But $\Im(\Delta_k) + \gamma_k = \pi - \alpha_k - \beta_k > 0$; therefore, from (8), we conclude that

$$\beta_k < \frac{\pi}{2} \text{ for infinitely many } k. \tag{9}$$

Our assertion $\beta_\infty \leq \dfrac{\pi}{2}$ in (6) now follows trivially. Similarly, it can be proved that $\alpha_{-\infty} \leq \dfrac{\pi}{2}$.

2) Let m denote the maximal q if it exists, and otherwise, set $m = \infty$; let n denote the minimal p if in exists and otherwise set $n = -\infty$. From (3), it follows that

$$\sum_{k=n}^{m} \gamma_k + \sum_{k=n}^{m} \Im(\Delta_k) = \pi - \alpha_n - \beta_m + \sum_{k=n+1}^{m} (\pi - \omega_k). \tag{10}$$

If we sum both sides of (10) over all the connected components of the boundary of $\mathfrak{F}$, then we still have

$$\sum_k \gamma_k \leq 2\pi \text{ and } \sum_k \Im(\Delta_k) \leq \Im(\mathfrak{F}) < \infty. \tag{11}$$

But $\pi - \alpha_n - \beta_m \geq 0$, therefore the series $\sum_k (\pi - \omega_k)$ of positive terms is convergent and for a given $\in > 0$, we have

$$0 < (\pi - \omega_k) < \in \text{ for almost all } k. \tag{12}$$

3) We have defined the set $\mathfrak{F}$ by means of the inequalities

$$\delta(z, \zeta) \leq \delta(z, \zeta_S) \text{ for } S \in \Gamma.$$

If equality holds precisely for one $S \neq \pm E$ i.e. if z lies exactly on one perpendicular bisector, then z is a boundary point of $\mathfrak{F}$ and not
20 a vertex. Since the edge $\widehat{z_k z_{k+1}}$ of $\mathfrak{F}$ is a perpendicular bisector of $\widehat{\zeta\zeta_A}$ for some $A \in \Gamma$, we have, for any point z on this edge different from z_k and z_{k+1},

$$\delta(z, \zeta) = \delta(z, \zeta_A) \text{ and } \delta(z, \zeta) < \delta(z, \zeta_S) \text{ for } S \neq \pm A, \pm E,$$

or $\quad \delta(z, \zeta_A) = \delta(z, \zeta_{A^{-1}A}) \text{ and } \delta(z, \zeta_A) \leq \delta(z, \zeta_{SA}) \text{ for } S \neq \pm A^{-1}, \pm E.$

This shows that z is a boundary point of $\mathfrak{F}_A$ and not a vertex. Moreover z_k and z_{k+1} are necessarily vertices of $\mathfrak{F}_A$. We now assert that a vertex of $\mathfrak{F}$, say z_0, can be a vertex of only finitely many images of $\mathfrak{F}$ by elements of Γ. Let, if possible, z_0 be a vertex of $\mathfrak{F}_{A_i}(i = 1, 2, \ldots)$ with $A_i \in \Gamma$. Then we have

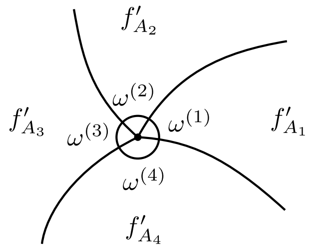

Figure 1.7:

$$\begin{aligned}
\delta(z_0, \zeta) &= \delta(z_0, \zeta_{A_1}) \\
\delta(z_0, \zeta_{A_l}) &= \delta(z_0, \zeta_{A_2}) \\
&\cdots\cdots\cdots\cdots\cdots \\
\delta(z_0, \zeta_{A_i}) &= \delta(z_0, \zeta_{A_{i+1}})
\end{aligned}$$

and so on. Thus we obtain that $\delta(z_0, \zeta) = \delta(z_0, \zeta_{A_i})$ for $i = 1, 2, \ldots$ i.e. the points ζ_{A_i} lie on the hyperbolic circle with the radius $\delta(z_0, \zeta)$. This implies, because of the discreteness of Γ, that there can exist only finitely many such A_i, proving our assertion above. Let $\mathfrak{F}_{A_i}, i = 1, 2, \ldots, r (r \geq 3)$ be all the images of $\mathfrak{F}$ with z_0 as a vertex and let $\omega^{(i)}$ denote the angle of $\mathfrak{F}_{A_i}$ at z_0. Then

$$\omega^{(1)} + \omega^{(2)} + \cdots + \omega^{(r)} = 2\pi \tag{13}$$

Since $A_i^{-l} < z_0 >$ will be some vertex of $\mathfrak{F}$, say $z_j \omega^{(i)}$ must coincide with ω_j.

We assume in the rest of the proof that $-E \in \Gamma$, without loss of generality and denote $\Gamma/\{\pm E\}$ by $\overline{\Gamma}$.

4) Let $B_i < z_0 > (i = 1, 2, \ldots, S)$ with $B_i \in \Gamma$ be the complete set of different vertices of $\mathfrak{F}$ which are equivalent with z_0. Denote by Γ_0 the subgroup consisting of those transformations of Γ which **21**

leave z_0 fixed. Since $A_i^{-l} < z_0 >$ with A_i as defined above is a vertex of $\mathfrak{F}$, we have $A_i^{-l} < z_0 >= B_j < z_0 >$ i.e. $A_i B_j < z_0 >= z_0$ for some j with $1 \leq j \leq s$ and therefore $A_i B_j = L$ for some $L \in \Gamma_0$. The number of distinct transformations $A_i^{-1} = \pm B_j L^{-1}$ is ℓs, where ℓ is the order of $\overline{\Gamma}_0 := \Gamma_0/\{\pm E\}$; for, if $\pm B_j L_1^{-1} = B_h L_2^{-1}$ for $L_k \in \Gamma_0 (k = 1, 2)$, then $B_j < z_0 >= B_h < z_0 >$ and this is possible only if $j = h$, implying in turn that $\pm L_1 = L_2$. This shows that $r = \ell s$ and ℓ is finite; therefore $\overline{\Gamma}_0$ is a cyclic group. Let $\overline{\Gamma}_0$ be generated by the rotation L_0 of angle $2\pi/\ell$. Then

$$\{\pm A_i^{-1} | i = 1, 2, \ldots, r\} = \{\pm B_j L_0^{-t} | j = 1, 2, \ldots, s; t = 0, 1, \ldots, \ell - l\}.$$

Since the fundamental domain $\mathfrak{F}$ has the angle $\omega^{(j)}$ at the vertex $B_j < z_0 >= B_j L_0^{-t} < z_0 >$, the image $\mathfrak{F}_{A_i} = \mathfrak{F}_{L_0^t B_j^{-1}}$ has the same angle $\omega^{(j)}$ at z_0. Therefore, from (13), it follows that

$$\omega^{(1)} + \omega^{(2)} + \cdots + \omega^{(s)} = \frac{2\pi}{\ell} (s\ell \geq 3) \tag{14}$$

But, by (12), $\omega_k > \dfrac{2\pi}{3}$ for almost all k; therefore, equation (13) can be satisfied only by a finite number of systems ω_k. Thus there exist only finitely many classes of equivalent vertices of $\mathfrak{F}$, since each such class corresponds uniquely to a subsystem of $\{\omega_k\}$ satisfying (14) with some ℓ and further any two such subsystems are disjoint. We conclude that every connected component of the boundary of $\mathfrak{F}$ has finitely many vertices. From (10) with $\alpha_n = \beta_m = 0$ now, we obtain that the right hand side of (10) is at least π. But the left hand side of (10) when summed over all con-
22 nected component is finite. Therefore it follows that the number of connected components of $\mathfrak{F}$ is finite.

5) We shall now show that no arc of the boundary of $\mathfrak{L}$ can be contained in the boundary of $\mathfrak{F}$. Without loss of generality, we can assume that $\zeta = 0$. Let us suppose that the arc $\widehat{AB}$ with the angle $\gamma > 0$ belongs to the boundary of $\mathfrak{F}$. Then due to the convexity of $\mathfrak{F}$, the whole sector belongs to $\mathfrak{F}$. But this is impossible, since the

area of this sector is infinite while $\mathfrak{F}$ has finite area. For the same reason, $\mathfrak{F}$ can at most have only finitely many improper vertices i.e. vertices on the boundary of $\mathfrak{L}$.

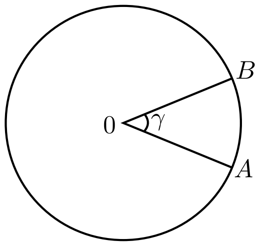

Figure 1.8:

6) We prove next that $\mathfrak{F}$ does not have any improper vertex. Here, for the first time, we shall make use of the assumption on Γ that it does not contain parabolic transformations. Let z_0 be an improper vertex of $\mathfrak{F}$ so that $|z_0| = 1$. Since $\mathfrak{F}$ does not contain any arc of the boundary of $\mathfrak{L}$. two edges of $\mathfrak{F}$, say g_0 and f_0, touch each other at z_0. Let $\mathfrak{F}_{A_1}$ be an image of $\mathfrak{F}$ by $A_1 \in \Gamma$, which has the edge g_0 in common with $\mathfrak{F}$. By the same argument as above, $\mathfrak{F}_{A_1}$ must have two edges g_0 and g_1 touching at z_0. Proceeding in this way, we obtain $\ldots\ldots, \mathfrak{F}_{A_{-2}}, \mathfrak{F}_{A_{-1}}, \mathfrak{F}_{A_0}, \mathfrak{F}_{A_1}, \mathfrak{F}_{A_2}, \ldots\ldots$ with $A_0 = E$ and $A_i \in \Gamma$ having z_0 as the common improper vertex. Obviously $A_r^{-1} < z_0 >$ is an improper vertex of $\mathfrak{F}$. But $\mathfrak{F}$ has only finitely many vertieces; therefore, there exist two integers p and q **23**
with $p \neq q$ such that

$$A_p^{-1} < z_0 >= A_q^{-1} < z_0 > \Longrightarrow C < z_0 >= z_0 \text{ with } C = A_p A_q^{-1} \neq \pm E.$$

Since, by assumption, Γ does not contain any parabolic transformation, the transformation C which has a fixed point z_0 on $|z| = 1$, ought to be hyperbolic. Let z_1 be the other fixed point of C. Let, further, $\tau = T < z >$ be a transformation which maps the unit disc onto the upper-half plane and the points z_0, z_1 to $\infty, 0$ respectively. Then $C = T^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} T$, with $\lambda \neq 0, \pm 1$. Clearly, the cyclic group generated by TCT^{-1} maps the set $\mathfrak{g} = \{\tau =$

$x + iy, |x| \leq y; \tau \in \mathfrak{H}\}$ onto itself. Let $\mathfrak{F}_0$ be the fundamental domain in $T^{-1} < \mathfrak{g} >$ of the cyclic group generated by C, say, with $\lambda^2 > 1$, such that (see Figure 1.9

$$T < \mathfrak{F}_0 >= \{\tau | \tau = x + iy, 1 \leq |\tau| \leq \lambda^2, |x| \leq y\}.$$

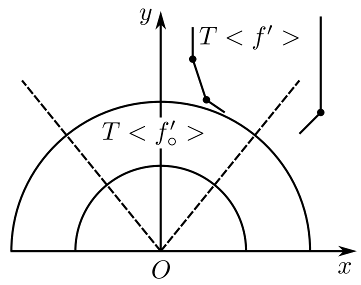

Figure 1.9:

Then $\mathfrak{F}_0$ is a compact set in the hyperbolic plane and therefore there exists a constant M such that $\delta(z^*, \zeta) \leq M$ for $z^* \in \mathfrak{F}_0$.

Since $T < \mathfrak{F} >$ has ∞ as a boundary point, there exists a point $z \in \mathfrak{F}$ such that $\delta(z, \zeta) > M$ and $T < z >\in y$. This implies that $C^p < z >$ belongs to $\mathfrak{F}_0$ for some integer p and therefore

$$\delta(C^p < z >, \zeta) = \delta(z, \zeta_{C^{-p}}) \leq M$$

But $M < \delta(z, \zeta) \leq \delta(z, \zeta_{C^{-p}})$, since z belongs to $\mathfrak{F}$; therefore, our
24 supposition that z_0 is an improper vertex of $\mathfrak{F}$ is untenable. Thus $\mathfrak{F}$ is bounded by only finitely many edges, has no improper vertices and is therefore a polygon. Hence $\mathfrak{F}$ is compact and the proof of theorem 3 is complete.

Concerning the existence of improper vertices of a fundamental domain for a discrete group of transformations of the hyperbolic plane, we prove the following

□

Theorem 4. *Let Γ be a discrete group of transformations of the hyperbolic plane and $\mathfrak{F}$ a normal fundamental domain for Γ with $\Im(\mathfrak{F}) < \infty$. Then $\mathfrak{F}$ has at least one improper vertex if and only if Γ contains parabolic elements.*

Proof. If Γ does not contain a parabolic element, then by theorem 3, $\mathfrak{F}$ does not have improper vertices. Conversely, we shall show that if $\mathfrak{F}$ does not have improper vertices (i.e. $\mathfrak{F}$ is compact), then Γ does not contain parabolic elements.

If possible, let Γ contains a parabolic element P. Then we assert the existence of a sequence $\{z^{(k)}\}$ such that $\delta(z^{(k)}, P < z^{(k)} >) \to 0$ as $k \to \infty$. Indeed, we can assume that ∞ is the only fixed point of P in $\mathfrak{H}$ i.e. P is defined by $\tau \to \tau + \mu$ for some real number μ, then $\rho(\tau, \tau + \mu) < \mu/y < \in$ for sufficiently large y and for arbitrary $\in > 0$. Let $S_k \in \Gamma$ be so determined that $S_k < z^{(k)} >$ belongs to $\mathfrak{F}$. Then the sequence $\{\delta(S_k < z^{(k)} >, \zeta)\}$ is bounded. Therefore there exists a subsequence of $\{S_k < z^{(k)} >\}$ convergent in $\mathfrak{H}$. Denoting this subsequence again by $\{S_k < z^{(k)} >\}$, let z^* be its limit. It follows immediately that 25

$$\delta(S_k < z^{(k)} >, S_k P < z^{(k)} >) \to 0 \Longrightarrow S_k P < z^{(k)} > \to z^*$$
$$\Longrightarrow \delta(S_k P < z^{(k)} >, z^*) \to 0 \Longrightarrow \delta(z^{(k)}, P^{-1} S_k^{-1} < z^* >) \to 0 \text{ as } k \to \infty.$$

But since $\delta(S_k < z^{(k)} >, z^*) \to 0$ and since further

$$\delta(S_k^{-1} < z^* >, P^{-1} S_k^{-1} < z^* >) \leq \delta(S_k^{-1} < z^* >, z^{(k)})$$
$$+ \delta(z^{(k)}, P^{-1} S_k^{-1} < z^* >), \text{ we see that}$$
$$\delta(S_k^{-1} < z^* >, P^{-1} S_k^{-1} < z^* >) \to 0 \text{ as } k \to \infty.$$

Thus $\delta(S_k P S_k^{-1} < z^* >, z^*) \to 0$ as $k \to \infty$ and the discreteness of Γ implies that $S_k P S_k^{-1} < z^* > = z^*$ for sufficiently large k i.e. P has $S_k^{-1} < z^* >$ as a fixed point. But this is impossible, since $S_k^{-1} < z^* >$ belongs to $\mathfrak{H}$ while P is a parabolic transformation. Therefore Γ can not contain any parabolic element.

The improper vertices of $\mathfrak{F}$ are nothing but fixed points of parabolic transformations of Γ; hereafter, we shall call them *parabolic cusps*.

In the following, we shall assume that Γ is a discrete group of transformations of the hyperbolic plane having a normal fundamental domain with centre ζ and $\Im(\mathfrak{F}) < \infty$ i.e. $\mathfrak{F}$ is bounded by finitely many hyperbolic straight lines and has finitely many parabolic cusps. Let k be an edge of $\mathfrak{F}(\subset \mathfrak{L})$ joining z_0 and z_1 (see Figure 1.10. Let k be the perpendicular bisector of $\widehat{\zeta\zeta_A}$.

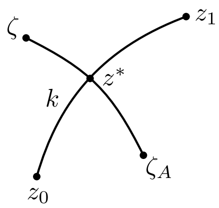

Figure 1.10:

Then k is an edge of $\mathfrak{F}_A$ and therefore $A^{-1} < k >$ is an edge of $\mathfrak{F}$.
26 If $A^{-1} < k >= k$, then A^{-1} permutes z_0 and z_l, since A^{-1} preserves the orientation. It follows now that $A^2 = \pm E$. Since A^{-1} permutes ζ and ζ_A also, the two lines $\widehat{\zeta\zeta_A}$ and $\widehat{z_0 z_1}$ are mapped onto themselves and therefore the point of intersection z^* of these two lines must be a fixed point of A. Hence A is an elliptic transformation of order 2. We introduce z^* as a vertex of $\mathfrak{F}$. Then the two edges $\widehat{z_0 z^*}$ and $\widehat{z^* z_1}$ are permuted by A. If $A^{-1} < k > \neq k$, then we have an edge of $\mathfrak{F}$ different from k but equivalent to k by Γ. proceeding in this way, we obtain that $\mathfrak{F}$ is a closed convex polygon bounded by a finite number of hyperbolic straight lines $k_i, k_i^*(i = 1, 2, \ldots t)$ such that k_i and k_i^* are pairwise equivalent under Γ i.e. there exist elements $A_i \in \Gamma$ such that $A_i < k_i^* >= k_i(i = 1, 2, \ldots, t)$. We shall call the transformations A_i *the boundary substitutions of* $\mathfrak{F}$. For a given $\mathfrak{F}$, if $\{A_i | i = 1, 2, \ldots, t\}$ is a set of its boundary substitutions, then the set $\{A_i^{\pm 1}, i = 1, 2, \ldots, t\}$ is uniquely determined by $\mathfrak{F}$.

We shall see that the set of boundary substitutions of the fundamental domain $\mathfrak{F}$ generate the group Γ. First, we prove the following □

Lemma 2. *Let $\mathfrak{F}$ be a normal fundamental domain of a discrete group Γ of transformations of the hyperbolic plane. Then any compact set K in the hyperbolic plane has non-empty intersection with only a finite number of images of $\mathfrak{F}$ under elements of* Γ.

Proof. Without loss of generality, we can assume that K is a disc with centre ζ and radius ρ. If possible, let $\mathfrak{F}_{S_i} \cap K \neq \phi$ for an infinity of distinct $S_i (\in \Gamma), i = 1, 2, \ldots$ Picking $z^{(i)}$ in $\mathfrak{F}_{S_i} \cap K$, we have $z^{(i)}_{S_i^{-1}} \in \mathfrak{F}$ and

$$\delta(z^{(i)}, \zeta) \leq \rho \Longrightarrow \rho > \delta\left(z^{(i)}_{S_i^{-1}}, \zeta_{S_i^{-1}}\right) \geq \delta\left(z^{(i)}_{S_i^{-1}}, \zeta\right).$$

27

Therefore

$$\delta(\zeta, \zeta_{S_i}) \leq \delta(\zeta, z^{(i)}) + \delta(z^{(i)}, \zeta_{S_i}) \leq \rho + \rho = 2\rho, \text{ for } i = 1, 2, 3, \ldots$$

which is impossible from the discreteness of Γ. The proof of the lemma is complete. ◻

Theorem 5. *The set of the boundary substitutions of a normal fundamental domain for a discrete group* Γ *of transformations of the hyperbolic plane generates* Γ.

Proof. Let $\mathfrak{F}$ be a normal fundamental domain for Γ with the centre ζ. Let S be an arbitrary element of Γ. Then by Lemma 2, the hyperbolic straight line $\widehat{\zeta\zeta_S}$ intersects only a finite number of images of $\mathfrak{F}$ under elements of Γ. Let $\mathfrak{F}_{B_0}, \mathfrak{F}_{B_1}, \ldots, \mathfrak{F}_{B_n}$ be the images of $\mathfrak{F}$, which intersect $\widehat{\zeta\zeta_S}$ and are so arranged that $\mathfrak{F}_{B_{i-1}}$ and $\mathfrak{F}_{B_i}$ have an edge, say s_i, in common. Then $B_0 = \pm E$ and $B_n = \pm S$. Obviously, $B_{i-1}^{-1}(s_i)$ is an edge of $\mathfrak{F}$ and it is the perpendicular bisector of $\widehat{\zeta\zeta}_{B_{i-1}^{-1}B_i}$; therefore, we must have $B_{i-1}^{-1} B_i = \pm A_{ri}^{\pm 1}$, where A_{r_i} is a boundary substitution of $\mathfrak{F}$. It is now immediate that $s = \pm B_0 B_1 B_1^{-1} B_2 B_2^{-1} B_3 \ldots B_{n-1}^{-1} B_n = \pm A_{r_1}^{\pm 1} \ldots A_{r_n}^{\pm 1}$ and our theorem is proved.

Let $d(U)$ denote the Euclidean diameter of an arbitrary point set U. We now show that for $\in > 0$, $d(\mathfrak{F}_A) \leq \in$ for almost all A in Γ. By Lemma 2, only finitely many images of $\mathfrak{F}$ intersect the circle $\{z \| z| \leq h <$ **28**
$1\}$. Let $\mathfrak{F}_A$ with A in Γ be outside the disc $|z| \leq h$.

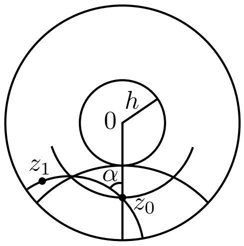

Figure 1.11:

Choose $z_0 \in \mathfrak{F}_A$ so that $|z_0| = \inf_{z \in \mathfrak{F}_A}(|z|)$. Then the hyperbolic tangent to $|z| = h$ at the point of intersection of $|z| = h$ and the line joining 0 and z_0 is perpendicular to this line. We claim that $\mathfrak{F}_A$ lies in the lens domain bounded by the hyperbolic tangent mentioned above and the unit circle, which is also defined by

$$\left| z - \frac{z_0}{|z_0|} \frac{1+h^2}{2h} \right| \leq \frac{1-h^2}{2h}, |z| \leq 1.$$

If possible, let $z_1 \in \mathfrak{F}_A$ lie outside the above lens domain; then the line $\widehat{z_0 z_1}$ is in $\mathfrak{F}_A$, because $\mathfrak{F}_A$ is a convex set. But the angle α (see Figure 1.11 is $< \pi/2$; therefore, there exists a point z^* on the hyperbolic line $\widehat{z_0 z_1}$ such that $|z^*| < |z_0|$, which contradicts the minimality of $|z_0|$. Hence $\mathfrak{F}_A$ lies in the above lens domain. It can be seen easily that the diameter of this domain is $2(1-h^2)/(1+h^2)$. Therefore it follows that

$$d(\mathfrak{F}_A) < 2(1-h^2)/(1+h^2) \to 0 \text{ as } h \to 1.$$

We now assert that a discrete group Γ with a normal fundamental domain of finite area is a *Grenzkreis group of the first kind* in the sense
29 of Petersson [3]. In order to prove this assertion, we have to show that, given a point z^* with $|z^*| = 1$, there exists a sequence of points z_n in $\mathfrak{L}$ and a sequence $\{L_n\}$ of pairwise distinct transformations $L_n \in \Gamma$ such that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} L_n < z_n >= z^*.$$

For given $\in> 0, n-1$ elements $S_1, S_2, S_3, \ldots\ldots, S_{n-1}$ in Γ and a point z^* with $|z^*| = 1$, we can find S in Γ with $S \neq S_i, i = 1, 2, \ldots, n-1$

and a point z in $|z| < 1$ such that

$$|z - z^*| < \in, |z_S - z^*| < 2 \in .$$

In order to prove this, we consider the point set

$$U_\in = \{z||z - z^*| < \in, |z| < 1\}.$$

Since $\Im(U_\in) = \infty$, infinitely many images $\mathfrak{F}_T$ will intersect $U_\in$. Therefore, we have $U_\in \cap \mathfrak{F}_T \neq \emptyset$ for infinitely many T and $d(\mathfrak{F}_T) < \in$, for almost all T.

Let T_1 and T_2 be two elements of Γ satisfying the above two conditions and the additional condition

$$T_2 \neq S_i T_1 \text{ for } i = 1, 2, \ldots, n - 1.$$

We choose a point $z \in \mathfrak{F}_{T_1} \cap U_\in$ and set $S = T_2 T_1^{-l}$ so that $S < z > \in \mathfrak{F}_{T_2}$. Let z' be a point in $U_\in \cap \mathfrak{F}_{T_2}$. Then $|z' - z^*| < \in$ and $|z' - z_S| < \in$. This shows that

$$|z^* - z_S| \leq |z^* - z'| + |z' - z_S| \leq 2 \in .$$

But since z belongs to $U_\in$, $|z - z^*| < \in$; therefore z chosen above is **30**
a required point. Let $\in = \dfrac{1}{n}, z = z_n$ and $S = S_n$. Then obviously the sequences $\{z_n\}$ and $\{S_n < z_n >\}$ converge to z^* as $n \to \infty$. Moreover, by choice, the elements of the sequence $\{S_n\}$ are pairwise distinct. This completely proves our assertion that Γ is a Grenzkreis group of the first kind.

We mention only the validity of the converse fo the above statement, namely: a Grenzkreis group of the first kind has a normal fundamental domain $\mathfrak{F}$ with finite area. This assertion amounts essentially to the statement that $\mathfrak{F}$ is bounded by a finite number of line segments which lie on hyperbolic straight lines. This was proved by M. Heins, W. Fenchel and J. Nielsen (jointly), L. Greenberg, A. Marden (cf. the article of L. Greenberg in "Discrete Groups and Automorphic Functions", edited by W.J. Harvey, Academic Press 1977, and the cited literature).

Following Rankin [6], we call a discrete group with a normal fundamental domain of finite area a *horocyclic group*. The boundary of a normal fundamental domain for a discrete group can be described by means of the so-called *isometric circles of the group* discussed by Ford [1]. ◻

4 Riemann surfaces

Let $\mathfrak{F}$ be a closed normal fundamental domain for a horocyclic group Γ.
31 Let ζ be the centre of $\mathfrak{F}$. By joining ζ with the vertices of $\mathfrak{F}$, we obtain a decomposition of $\mathfrak{F}$ into an even number of triangles. If we identify the equivalent edges of $\mathfrak{F}$, we get a closed orientable polyhedron $\mathscr{R}$. Let p denote the topological genus of $\mathscr{R}$. If e, k and d denote respectively the number of vertices, the number of edges and the number of triangles of $\mathscr{R}$, then the Euler-Poincare characteristic formula states that

$$e - k + d = 2 - 2p.$$

Let $\mathfrak{g}_1, \mathfrak{g}_2, \ldots \mathfrak{g}_\sigma$ be the different classes of parabolic cusps and $\mathfrak{n}_1$, $\mathfrak{n}_2, \ldots, \mathfrak{n}_{e_1}$ be the classes of the proper cusps of $\mathfrak{F}$. Let $z_{i1}, z_{i2}, \ldots, z_{ir_i}$ be all the vertices of $\mathfrak{F}$ in the class $\mathfrak{n}_i$ and $\omega_i^{(1)}, \omega_i^{(2)}, \ldots, \omega_i^{(r_i)}$ be the angles of $\mathfrak{F}$ at the vertices z_{ij}, $j = 1, 2, \ldots, r_i$. Then

$$\omega_i^{(1)} + \omega_i^{(2)} + \cdots + \omega_i^{(r_i)} = 2\pi/\ell_i \quad (i = 1, 2, \ldots e_1)$$

for some natural number ℓ_i. We have already proved that z_{ij} for $j = 1, 2, \ldots, r_i$ are fixed points of elliptic transformations in case $\ell_i > 1$ and the group of transformations which leave z_{ij} fixed is of order ℓ_i. Since, for a fixed i, the two points z_{ij} and $z_{ij'}$ for $j \neq j'$ are equivalent, the groups leaving z_{ij} and $z_{ij'}$ fixed are conjugate subgroups of Γ. Let e_0 be the number of classes of proper cusps of $\mathfrak{F}$ for which $\ell_i > 1$. We so choose our notation that

$$\ell_i > 1 \text{ for } i = 1, 2, \ldots, e_0.$$

It is obvious that the sum of the angles of all the triangles of $\mathscr{R}$ is
32 given by

$$2\pi + \sum_{i=1}^{e_1} \{\omega_i^{(1)} + \omega_i^{(2)} + \cdots + \omega_i^{(r_i)}\} = \pi d - \Im(\mathfrak{F}).$$

and consequently

$$1 + \sum_{i=1}^{e_1} 1/\ell_i = \frac{1}{2}d - \frac{1}{2\pi}\mathfrak{I}(\mathfrak{F}).$$

Obviously, we have $e = \sigma + e_1 + 1$. Since each edge belongs exactly to two triangles, we have $3d = 2k$. As a result, $\sigma + e_1 + 2p = \frac{1}{2}d + 1$ and therefore

$$\sigma + e_1 + 2p = 2 + \sum_{i=1}^{e_1} 1/\ell_i + \frac{1}{2\pi}\mathfrak{I}(\mathfrak{F});$$

$$\text{i.e.}\quad \frac{1}{2\pi}\mathfrak{I}(\mathfrak{F}) = 2p - 2 + \sigma + e_1 - \sum_{i=1}^{e_1} 1/\ell_i$$

$$= 2p - 2 + \sigma + \sum_{i=1}^{e_0}(1 - 1/\ell_i), \tag{1}$$

since $\ell_i = 1$ for $e_0 < i \leq e_1$.

It can be shown easily that the right hand side of equation (1) has a positive minimum equal to $1/42$. This minimum is attained only for one set of values, namely, $p = 0$, $\sigma = 0$, $e_0 = 3$, $\ell_1 = 2$, $\ell_2 = 3$ and $\ell_3 = 7$. It can be proved that there exists a group for which a fundamental domain has area $\pi/21$. (For the proof, see [2], page 621). Thus, in general,

$$\mathfrak{I}(\mathfrak{F}) \geq \pi/21,$$

where $\mathfrak{F}$ is a fundamental domain for a horocyclic group. For $\sigma > 0$, the right hand side of (1) has the minimum $1/6$ and therefore, in this case, $\mathfrak{I}(\mathfrak{F}) \geq \pi/3$. This minimum is again attained for only one set of values given by $p = 0$, $\sigma = 1$, $e_0 = 2$, $\ell_1 = 2$ and $\ell_2 = 3$. We shall see later that **33**
this set of values is realised for 'the modular group'.

We now prove that the area of a normal fundamental domain for a horocyclic group does not depend on the choice of its centre ζ. Assume that $\mathfrak{g}$ is a fundamental domain for Γ bounded by only a finite number of segments of hyperbolic straight lines. Then we conclude that

$$z \in \mathfrak{F} \Longrightarrow z_A \in \mathfrak{g} \text{ for some } A \in \Gamma \Longrightarrow z \in \mathfrak{g}_{a^{-1}} \cap \mathfrak{F}$$

or

$$\bigcup_{\pm A \in \Gamma} (\mathfrak{g}_{A^{-1}} \cap \mathfrak{F}) = \mathfrak{F}.$$

The sets $\mathfrak{g}_{A^{-1}} \cap \mathfrak{F}$ and $\mathfrak{g}_{B^{-1}} \cap \mathfrak{F}$ intersect on their common boundary for $B \neq \pm A$. For, $z \in (\mathfrak{g}_{A^{-1}} \cap \mathfrak{F}) \cap (\mathfrak{g}_{B^{-1}} \cap \mathfrak{F})$ implies that z_A and z_B belong to the boundary of $\mathscr{G}$ i.e. z is a boundary point of $\mathfrak{g}_{A^{-1}}$ and $\mathfrak{g}_{B^{-1}}$ and therefore of the sets $\mathfrak{g}_{A^{-1}} \cap \mathfrak{F}$ and $\mathfrak{g}_{B^{-1}} \cap \mathfrak{F}$. Since the sets $\mathfrak{g}_{A^{-1}} \cap \mathfrak{F}$ for $A \in \Gamma$ are measurable and the intersection of two such distinct sets is a set of measure zero, we have

$$\mathfrak{I}(\mathfrak{F}) = \sum_{\pm A \in \Gamma} \mathfrak{I}(\mathfrak{g}_{A^{-1}} \cap \mathfrak{F}) = \sum_{\pm A \in \Gamma} \mathfrak{I}(\mathfrak{G} \cap \mathfrak{F}_A) = \mathfrak{I}(\mathfrak{G})$$

and our assertion is completely proved.

It can be shown that the polyhedron $\mathscr{R}$ is topologically equivalent to the space obtained by adding to the quotient-space $\mathfrak{H}/\Gamma$ the equivalence classes of parabolic cusps which are sufficient to compactify the space $\mathfrak{g}/\Gamma$, provided the neighbourhoods of the cusps are defined in a suitable way.

34 In the following, we shall speak of $\overline{\mathfrak{H}} = \mathfrak{H} \cup$ {all parabolic cusps of Γ} as a covering surface of $\mathscr{R}$. Let $\mathfrak{g} \in \mathscr{R}$; we say that $\mathfrak{g}$ is *the trace point* of $\tau \in \mathfrak{g}$ if τ belongs to $\overline{\mathfrak{H}}$.

Definition. *An equivalence class $\mathfrak{g} \in \mathscr{R}$ of parabolic cusps is called a logarithmic branch point.*

Thus the number of logarithmic branch points of $\mathscr{R}$ is σ.

Definition. *Let $\mathfrak{g} \in \mathscr{R}$ be the trace point of $\tau_0 \in \mathfrak{H}$. Then, for $-E \in \Gamma, \overline{\Gamma}_0 = \Gamma_0 | \{\pm E\}$ where $\Gamma_0 = \{S | S < \tau_0 >= \tau_0, S \in \Gamma\}$, is a cyclic group of order ℓ uniquely determined by $\mathfrak{g}$. We shall call $\mathfrak{g}$ a branch point of $\mathscr{R}$ of order $\ell - 1$ or a point $\tau_0 \in \mathfrak{g}$ a point of ramification index $\ell - 1$ if $\ell > 1$. If $\ell = 1$, we shall call $\mathfrak{g}$ a reqular point of $\mathscr{R}$.*

We have already observed that there exist only finitely many branch points of $\mathscr{R}$. Therefore, given a branch point $\mathfrak{g}$ of $\mathscr{R}$, there exists a neighbourhood of $\mathfrak{g}$ which does not contain any other branch point of $\mathscr{R}$.

We shall now introduce some *local uniformising parameters* or, as we shall usually say, *local coordinates* on $\mathscr{R}$ which define an analytic structure on $\mathscr{R}$ for which it is a Riemann surface. We shall call $\mathscr{R}$ the *Riemann surface associated to the group* Γ. A *local coordinate at a point* $\mathfrak{g}_0$ of $\mathscr{R}$ is a function $t(\mathfrak{g})$ such that

1) $t(\mathfrak{g})$ maps topologically an open neighbourhood U_0 of $\mathfrak{g}_0$ onto an open neighbourhood of 0 in the complex t-plane,

2) for another point $\mathcal{G}$ of U_0, $t - t(\mathcal{G})$ is a local coordinate at $\mathcal{G}$, and

3) if $s = s(\mathfrak{g})$ is another local coordinate at $\mathfrak{g}_0$, then in a neighbourhood of $\mathfrak{g}_0$, the function s can be expressed as a convergent power series **35**
$$s = c_1 t + c_2 t^2 + \ldots \text{ with } c_1 \neq 0$$
and conversely, every such convergent power series defines a local coordinate at $\mathfrak{g}_0$.

Let $g(\mathfrak{g})$ be a function defined in a neighbourhood of $\mathfrak{g}_0$ such that
$$g(\mathfrak{g}) = \sum_{r=k}^{\infty} c_r t^r, c_k \neq 0,$$
where $t = t(\mathfrak{g})$ is a local coordinate at $\mathfrak{g}_0$. Then $g(\mathfrak{g})$ is said to be of *degree* k at $\mathfrak{g}_0$. It can be verified that k does not depend upon the choice of a local coordinate at $\mathfrak{g}_0$. If $k \geq 0$, then $g(\mathfrak{g})$ is said to be *regular at* $\mathfrak{g}_0$. We call $\mathfrak{g}_0$ a *zero of order* k of $g(\mathfrak{g})$ when $k > 0$ and a *pole of order* $|k|$ when $k < 0$.

In what follows, by a domain we shall always understand an open connected set.

Definition. *Let $\mathcal{G}^*$ be a domain in $\overline{\mathfrak{H}}$. A function $f(\tau)$ defined in $\mathcal{G}^*$ is said to be an automorphic function with respect to a horocyclic group* Γ, *if*
$$f(\tau_S) = f(\tau) \text{ for } S \in \Gamma, \text{ whenever } \tau \text{ and } \tau_S \text{ are in } \mathcal{G}^*.$$

Now $\mathscr{G} = \{\mathfrak{g}|\mathfrak{g} \in \mathscr{R}, \mathfrak{g}$ is the trace point of some τ in $\mathscr{G}^*\}$ is a domain in $\mathscr{R}$ and the function $g(\mathfrak{g})$ defined by $g(\mathfrak{g}) = f(\tau)$, where $\mathfrak{g}$ is the trace point of τ, is well-defined in $\mathscr{G}$. Conversely, let $\mathscr{G}$ be a domain in $\mathscr{R}$, 36
$\mathscr{G}^* = \{\tau|\tau \in \overline{\mathfrak{H}}$, the trace point of τ belongs to $\mathscr{G}\}$ and $\mathscr{G}_0^*$ a connected component of $\mathscr{G}^*$. If $g(\mathfrak{g})$ is a function defined in $\mathscr{G}$, then the function $f(\tau)$ defined by $f(\tau) = g(\mathfrak{g})$, where $\mathfrak{g}$ is the trace point of τ, is an automorphic function defined in $\mathscr{G}_0^*$ with respect to Γ.

We now describe a suitable system of local coordinates at various points of $\mathscr{R}$.

1. Let $\mathfrak{g}_0 \in \mathscr{R}$ be a branch point of order $\ell - 1$. Let τ_0 be a point in $\mathfrak{H}$ with the trace point $\mathfrak{g}_0$. Then the subgroup $\overline{\Gamma}_0 \subset \overline{\Gamma}(= \Gamma$ modulo $\{\pm E\}$, if $-E \in \Gamma)$ consisting of those transformations of $\overline{\Gamma}$ which leave τ_0 fixed is a cyclic group of order ℓ. Let L be a generator of $\overline{\Gamma}_0$, which we can take to be a rotation through an angle $2\pi/\ell$. Defining z by $z = (\tau - \tau_0)/(\tau - \overline{\tau}_0)$, the transformation $\tau \to \tau_L$ corresponds to the mapping $z \to e^{2\pi i/\ell} z$. Actually, we have, for $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,
$$\frac{L < \tau > - \tau_0}{L < \tau > - \overline{\tau}_0} = \frac{L < \tau > - L < \tau_0 >}{L < \tau > - L < \overline{\tau}_0 >} = \frac{c\overline{\tau}_0 + d}{c\tau_0 + d} \cdot \frac{\tau - \tau_0}{\tau - \overline{\tau}_0} = \mu \frac{\tau - \tau_0}{\tau - \overline{\tau}_0},$$
where $|\mu| = 1$. If $U = \{\tau \in \mathfrak{H} \| z| < \in\}$ is an open disc with centre τ_0 and with $\in$ small enough to ensure that the equivalence of τ_1 and τ_2 in U under $\overline{\Gamma}$ implies already their equivalence under $\overline{\Gamma}_0$, then, only for $\tau = \tau_0$, a point τ in U is uniquely determined by its trace point . For $\tau \neq \tau_0$ in U, there exist exactly ℓ different points $\tau_1, \tau_2, \ldots, \tau_\ell$ with the same trace point $\mathfrak{g}$ as τ. But precisely one of these ℓ points belongs to $U_0 = \{\tau \in U \| z| < \in, 0 \leq arg z < 2\pi/\ell\}$. This $\tau \to \mathfrak{g}$ is a $1 - 1$ mapping of U_0 onto a neighbourhood of $\mathfrak{g}_0$ in $\mathscr{R}$. Moreover, $z \to t = z^\ell$ is a 1-1 mapping of U_0 onto the
37 open disc $\{t \| t| < \in^\ell\}$. Finally, $\mathfrak{g} \to \tau \to t$ is a 1-1 mapping, via U_0, of a neigbourhood of $\mathfrak{g}_0$ on $\mathscr{R}$ onto a neighbourhood of 0 in the complex plane. Let us assume that there exists an analytic structure on $\mathscr{R}$ such that t is regular at $\mathfrak{g}_0$ with respect to this analytic structure. Then
$$t = c_0 + c_1 s + c_2 s^2 + \cdots,$$

where $s = s(\mathfrak{g})$ is a local coordinate at $\mathfrak{g}_0$. But $t(\mathfrak{g}_0) = s(\mathfrak{g}_0) = 0$ and $s \to t$ is *a* 1-1 map, therefore $c_0 = 0$ and $c_1 \neq 0$. This shows that t is necessarily a local coordinate at $\mathfrak{g}_0$.

2. Let $\mathfrak{g}_0$ be a logarithmic branch point of $\mathscr{R}$, τ_0 a parabolic cusp with trace point $\mathfrak{g}_0$ and A a transformation of $\mathfrak{H}$ onto itself which maps τ_0 to ∞. Thus the group $A\Gamma A^{-1}$ has ∞ as a parabolic cusp. The subgroup $\overline{\Gamma}_0 \subset \Gamma/\{\pm E\}$, consisting of those transformations leaving τ_0 fixed is an infinite cyclic group generated by some transformation, say P. Setting $\tau_* = A < \tau >$, the transformation $\tau \to \tau_P$ corresponds to a translation $\tau^* \to \tau^* + \mu$, where we can assume that $\mu > 0$. We introduce $U_0 = \{\tau | \tau^* = x^* + iy^*, 0 \leq x^* < \mu, y^* > m\}$ and conclude, as in the preceding case, that, for large enough m, the mapping $\mathfrak{g} \to \tau \to t = e^{2\pi i A<\tau>/\mu} (\tau \in U_0)$ for $\mathfrak{g} \neq \mathfrak{g}_0$, together with $\mathfrak{g}_0 \to \tau_0 \to 0$ gives a 1-1 mapping of a neighbourhood of $\mathfrak{g}_0$ onto a neighbourhood of 0 in the complex plane. The assumption about t being regular at $\mathfrak{g}_0$ with respect to a given analytic structure on $\mathscr{R}$ implies again that $t(\mathfrak{g})$ is necessarily a local coordinate at $\mathfrak{g}_0$.

It can be checked that the local coordinate system on $\mathscr{R}$ that we have defined in (i) and (ii) above has all the properties required of a local coordinate system on $\mathscr{R}$. **38**

Let $f(\mathfrak{g})$ be a meromorphic function defined in a neighbourhood of $\mathfrak{g}_0$ and of degree k. Then $f(\mathfrak{g})$ has a power-series expansion

$$f(\mathfrak{g}) = \sum_{n=k}^{\infty} c_n t^n \text{ with } c_k \neq 0,$$

where $t = ((\tau - \tau_0)/(\tau - \overline{\tau}_0))^{\ell}$ or $e^{2\pi i A<\tau>/\mu}$ according as $\mathfrak{g}_0$ is a branch point of order $\ell - 1$ or a logarithmic branch point.

5 Meromorphic functions and Differentials

In this section, Γ will denote a horocyclic group and $\mathscr{R}$ the Riemann surface associated to Γ.

Definition. *A function $f(\mathfrak{g})$ defined in a domain $\mathcal{G}$ of $\mathscr{R}$ is said to be a meromorphic function on $\mathcal{G}$ if, for every point $\mathfrak{g}_0 \in \mathcal{G}$, the function has a power series expansion, in terms of a local coordinate at $\mathfrak{g}_0$, with only a finite number of negative exponents.*

If $f(\mathfrak{g})$ is defined on the whole of $\mathscr{R}$, then it will have at most a finite number of poles because $\mathscr{R}$ is compact; otherwise, $f(\mathfrak{g})$ can have an infinite number of poles in the domain of its definition. The set of all meromorphic functions on $\mathscr{R}$ forms a field. A meromorphic function
39 $f(\mathfrak{g})$ on $\mathscr{R}$ gives rise to an automorphic function $f(\tau)$ on $\mathfrak{H}$ for the group Γ with f meromorphic on $\mathfrak{H}$ and having a Fourier expansion

$$f(\tau) = \sum_{n-k}^{\infty} c_n e^{2\pi \text{ in } A<\tau>/\mu}, c_k \neq 0.$$

at a parabolic cusp τ_0 of Γ, where A denotes a transformation of $\mathfrak{H}$ on to itself mapping the cusp τ_0 to ∞. Conversely, it is obvious that every such automorphic function on $\mathfrak{H}$ defines a meromorphic function on $\mathscr{R}$. In the following, we shall denote by $v_{\mathfrak{g}}(f)$ the degree of the meromorphic function $f(\mathfrak{g})$ at the point $\mathfrak{g}$ belonging to the domain of definition of f. We shall call the product $\prod\limits_{\mathfrak{g}} \mathfrak{g}^{v_{\mathfrak{g}}(f)}$ *the divisor of f* and the sum $\sum\limits_{\mathfrak{g}} v_{\mathfrak{g}}(f)$ *the degree of f on $\mathscr{R}$* and denote them by (f) and $v(f)$ respectively. Since $f(\mathfrak{g})$ can have only a finite number of zeros and poles on $\mathscr{R}$, the product (f) is a finite product and the sum $v(f)$ is finite.

Definition. *Let $\mathcal{G}$ be a domain in $\mathscr{R}$. A meromorphic differential or simply a differential ω on $\mathcal{G}$ is an assignment, for every local coordinate t in $\mathcal{G}$, of a meromorphic function ω_t defined in the domain of definition of t such that the following condition is satisfied:*

It t and s are two local coordinates defined in $\mathcal{G}$ with U_t and U_s as their domain of definition, then

$$\omega_s \frac{ds}{dt} = \omega_t \text{ in } U_t \cap U_s.$$

Two differentials ω and ω^* on $\mathcal{G}$ are equal, whenever $\omega_t = \omega_t^*$ for every local coordinate t defined in $\mathcal{G}$. Using the condition in the definition

of a differential, it can be shown, by means of the principle of analytic continuation, that two differentials ω and ω^* are equal, if $\omega_t = \omega_t^*$ holds for some one local coordinate t. If $f(\mathfrak{g})$ is a meromorphic function de- **40**
fined on $\mathscr{G}$, then it defines on $\mathscr{G}$ a meromorphic differential df for which $(df)_t = \dfrac{df}{dt}$. If ω is a differential and $f(\mathfrak{g})$ a meromorphic function on $\mathscr{G}$, then by f_ω we shall understand the differential which assigns the meromorphic function $f\omega_t$ to the local variable t defined in $\mathscr{G}$. It follows that if ω is a differential defined in $\mathscr{G}$, then $\omega = \omega_t dt$; for,

$$(\omega_t dt)_t = \omega_t \frac{dt}{dt} = \omega_t.$$

Thus if t is a local coordinate at a point $\mathfrak{g}_0$ of $\mathscr{G}$, then

$$\omega = \omega_t dt = \left(\sum_{n=k}^{\infty} c_n t^n\right) dt.$$

We define the *degree of* ω at $\mathfrak{g}_0$ to be the least k for which $c_k \neq 0$ and denote it by $v_{\mathfrak{g}_0}(\omega)$. It can be seen easily that the degree of a differential at a point does not depend upon the choice of the local coordinate at the point. We say that $\mathfrak{g}_0$ is a *zero or a pole of* ω, if it is a zero or a pole of ω_t. If ω is a meromorphic differential on $\mathscr{R}$, we define the *divisor* (ω) *and degree* $v(\omega)$ in the same way as we defined the divisor and degree of a meromorphic function on $\mathscr{R}$ i.e. $(\omega) = \prod_{\mathfrak{g}} \mathfrak{g}^{v_{\mathfrak{g}}(\omega)}$ and $v(\omega) = \sum_{\mathfrak{g}} v_{\mathfrak{g}}(\omega)$. We shall call the coefficient $c_{-1}(c_{-1} = 0$ if $k \geq 0)$ in the expansion of ω at $\mathfrak{g}_0$ mentioned above as the *residue* of ω at $\mathfrak{g}_0$ and write

$$\operatorname{res}_{\mathfrak{g}_0} \omega = c_{-1}.$$

The independence of the residue of ω at $\mathfrak{g}_0$ from the choice of t follows from the integral representation for the residue given by **41**

$$c_{-1} = \frac{1}{2\pi i} \oint \omega_t \, dt,$$

where the integral is over a closed curve winding round 0 exactly once and contained in the domain of definition of t.

Let ω be a differential on $\mathscr{R}$. We say that

1) ω is of *the first kind*, if $v_{\mathfrak{g}}(\omega) \geq 0$ for every $\mathfrak{g} \in \mathscr{R}$,

2) ω is of *the second kind*, if ω has at least one pole and the residues at the various poles vanish, and

3) ω is of *the third kind*, if ω is not any of the above two kinds i.e. ω has at least one pole with a non-zero residue.

Let $\mathfrak{g}_0$ be the trace point of $\tau_0 \in \mathfrak{H}$ and let further, $\mathfrak{g}_0$ be a regular point. Then $\tau - \tau_0$ and $S < \tau > -S < \tau_0 >$ for $S \in \Gamma$ are local coordinates at $\mathfrak{g}_0$. If ω is a differential on $\mathscr{R}$, then

$$\omega_{(\tau-\tau_0)} = \omega_{(S<\tau>-S<\tau_0>)}(c\tau + d)^{-2}, \text{ where } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For, by definition of ω, we have

$$\begin{aligned} \omega_{(\tau-\tau_0)} &= \omega_{(S<\tau>-S<\tau_0>)} \frac{d(S < \tau > -S < \tau_0 >)}{d(\tau - \tau_0)} \\ &= \omega_{(S<\tau>-S<\tau_0>)}(c\tau + d)^{-2}. \end{aligned}$$

Moreover, if τ_1 is another point such that the trace point $\mathfrak{g}_1$ of τ_1 lies 'sufficiently near' $\mathfrak{g}_0$, then $\mathfrak{g}_1$ is a regular point and $\tau - \tau_1$ is a local
42 coordinate at $\mathfrak{g}_1$. Therefore, we have

$$\omega_{(\tau-\tau_0)} = \omega_{\tau-\tau_1} \frac{d(\tau - \tau_1)}{dt} dt = \omega_{(\tau-\tau_1)}$$

showing that $\omega_{(\tau-\tau_0)}$ is independent of the choice of τ_0 in a neighbourhood of τ_0. But if τ_1 is not 'sufficiently near' τ_0, then we can join them by a curve which does not pass through the parabolic and elliptic fixed points of Γ; it can then be seen easily that $\omega_{(\tau-\tau_1)}$ can be obtained by analytic continuation of $\omega_{(\tau-\tau_0)}$ along the curve joining τ_0 and τ_1 mentioned above. Hence we obtain a uniquely determined meromorphic function $f(\tau)$ given by $f(\tau) = \omega_{(\tau-\tau_0)}$, which is defined on $\mathfrak{H}$ but for a set D consisting of all elliption fixed points of Γ. The definition of $f(\tau)$ does not depend upon the choice of τ_0, as has already been proved above. Since

$$\omega_{(S<\tau>-S<\tau_0>)}(c\tau + d)^{-2} = \omega_{(\tau-\tau_0)} \text{ for } S \in \Gamma,$$

we obtain that

$$f(S < \tau >)(c\tau + d)^{-2} = f(\tau). \tag{1}$$

By $v_{\mathfrak{g}}(f)$ we shall denote the degree of $f(\tau)$ at the point $\mathfrak{g}$. Obviously

$$v_{\mathfrak{g}}(f) = v_{\mathfrak{g}}(\omega_t) = v_{\mathfrak{g}}(\omega)(\mathfrak{g} \text{ regular}).$$

We shall now discuss the behaviour of $f(\tau)$ at the points of D. Since the branch points of $\mathscr{R}$ do not have a limit point, D is a discrete set and therefore $f(\tau)$ has isolated singularities at the points of D. We shall show that $f(\tau)$ can have at most a pole at any point of D. Let $\mathfrak{g}_0$ be a branch point of $\mathscr{R}$, of order $\ell - 1 > 0$, which is the trace point of $\tau_0 \in \mathfrak{H}$. Then $t = ((\tau - \tau_0)/(\tau - \overline{\tau}_0))^{\ell}$ is a local coordinate at $\mathfrak{g}_0$. Let $\tau_1 \in \mathfrak{H}$ be **43**
such that its trace point $\mathfrak{g}_1$ is a regular point and lies in the domain of definition of t. By the definition of ω, we have

$$\begin{aligned} \omega_t dt &= \omega_{(\tau-\tau_1)} \frac{d(\tau - \tau_1)}{dt} dt = \omega_{(\tau-\tau_1)} \frac{d\tau}{dt} dt \\ &= f(\tau)\frac{d\tau}{dt} dt = \left(\sum_{n=k}^{\infty} c_n t^n\right) dt. \end{aligned}$$

with $c_k \neq 0$ if $v_{\mathfrak{g}_0}(\omega) = k$. But $\dfrac{dt}{d\tau} = 2i\ell y_0 t^{1-1/\ell}(\tau - \overline{\tau}_0)^{-2}(\tau_0 = x_0 + iy + 0)$; therefore

$$\begin{aligned} (\tau - \overline{\tau}_0)^2 f(\tau) &= 2i\ell y_0 \sum_{n=k}^{\infty} c_n t^{n+1-1/\ell}, c_k \neq 0 \\ &= 2i\ell y_0 \sum_{n=k}^{\infty} c_n \left(\frac{\tau - \tau_0}{\tau - \overline{\tau}_0}\right)^{\ell(n+1)-1} \end{aligned} \tag{2}$$

This shows that $f(\tau)$ is meromorphic at the point τ_0 and our assertion is established. We define the *degree* of $f(\tau)$ at the point $\mathfrak{g}_0$ to be the least exponent which actually appears in the t-power series (2). Thus we see immediately that

$$v_{\mathfrak{g}_0}(f) = v_{\mathfrak{g}_0}(\omega) + 1 - 1/\ell(\mathfrak{g}_0 \text{ elliptic}).$$

Let $\mathfrak{g}_0$ be a logarithmic branch point of $\mathscr{R}$ and τ_0 a point in the class of $\mathfrak{g}_0$. We have seen already that $t = e^{2\pi i A<\tau>/\mu}$, where $A = \left(\begin{smallmatrix} a_0 & a_3 \\ a_1 & a_2 \end{smallmatrix}\right)$ with $|A| = 1$ is a transformation of $\mathfrak{H}$ onto itself mapping τ_0 to ∞ and μ is a real number > 0, is a local coordinate at $\mathfrak{g}_0$. Let $\tau_1 \in \mathfrak{H}$ be such that the trace point $\mathfrak{g}_1$ of τ_1 is a regular point lying in the domain of definition of t. Then, as in the preceding case, we get

$$\omega_t dt = \omega_{(\tau-\tau_1)} \frac{d\tau}{dt} dt = f(\tau) \frac{d\tau}{dt} dt = \left(\sum_{n=k}^{\infty} c_n t^n \right) dt$$

44 with $c_k \neq 0$ if $v_{\mathfrak{g}_0}(\omega) = k$. But $\dfrac{dt}{d\tau} = \dfrac{2\pi i}{\mu}(a_1\tau + a_2)^{-2} t$; therefore we obtain that

$$(a_1\tau + a_2)^2 f(\tau) = \frac{2\tau i}{\mu} \sum_{n=k}^{\infty} c_n t^{n+1}. \tag{1}$$

We define the *degree* of $f(\tau)$ at $\mathfrak{g}_0$ to be the least exponent which actually appears in the t-power series (3) and obtain

$$v_{\mathfrak{g}_0}(f) = v_{\mathfrak{g}_0}(\omega) + 1 (\mathfrak{g}_0 \text{ parabolic}).$$

It can be verified that the degree of $f(\tau)$ at $\mathfrak{g}_0$ does not depend upon the choice of τ_0 in the class of $\mathfrak{g}_0$.

The above discussion shows that given a meromorphic differential ω on $\mathscr{R}$ we can associate with it a meromorphic function $f(\tau)$ on $\mathfrak{H}$ with the following properties:

1) $f(S < \tau >)(c\tau + d)^{-2} = f(\tau)$ for $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and

2) $(a_1\tau + a_2)^2 f(\tau) = \sum_{n=k}^{\infty} c_n e^{2\pi i n A<\tau>/\mu}$ at any parabolic cusp of Γ, mapped to ∞ by $A = \left(\begin{smallmatrix} a_0 & a_3 \\ a_1 & a_2 \end{smallmatrix}\right) \in \Omega$.

We shall call such a meromorphic function of $\mathfrak{H}$ a *meromorphic automorphic form of weight 2 for* Γ. Conversely, it is obvious that given on automorphic form of weight 2, there exists a differential on $\mathscr{R}$ of which the associated meromorphic function on $\mathfrak{H}$ is the given automorphic form. Later on, we shall speak of the series describing the behaviour of $f(\tau)$ at a parabolic cusp of Γ, as the Fourier expansion of

45 $f(\tau)$ at this cusp. We shall call $f(\tau)$ an *integral automorphic form* or simply an *automorphic form* of weight 2 for Γ, if $f(\tau)$ is regular in $\mathfrak{H}$ and further, if no term with a negative exponent occurs in the Fourier expansions of $f(\tau)$ at the parabolic cusps of Γ. If ω is a differential of the first kind on $\mathscr{R}$, then the associated function $f(\tau)$ on $\mathfrak{H}$ has the following two characteristic properties:

1) $f(\tau)$ is an integral automorphic form of weight 2 for Γ and

2) the constant term in the Fourier expansions of $f(\tau)$ at the parabolic cusps of Γ, vanishes.

We call such an automorphic form a *cusp form a weight 2 for* Γ. Conversely, it is easy to verify that if $f(\tau)$ is a cusp form of weight 2, then the associated differential on $\mathscr{R}$ is a differential of the first kind.

Let $(f) = \prod_{\mathfrak{g}\in\mathscr{R}} \mathfrak{g}^{v_{\mathfrak{g}}(f)}$ denote the divisor of a meromorphic automorphic form $f(\tau)$ of weight 2 for Γ. Then we have proved above that

$$(f) = (\omega)\prod_{r=1}^{\sigma}\mathfrak{g}_1\prod_{s=1}^{e_0}\mathfrak{n}_s^{1-1/\ell_s} \tag{4}$$

where ω is the associated differential on $\mathfrak{R}$; $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_\sigma$ are the logarithmic branch points of $\mathfrak{R}$ and $\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_{e_0}$ are the branch points of finite positive order given by $\ell_1 - 1, \ell_2 - 1, \ldots, \ell_{e_0} - 1$ respectively. Similarly, if $\sum_{\mathfrak{g}} v_{\mathfrak{g}}(f)$ denotes the degree $v(f)$ of $f(\tau)$, then

$$v(f) = v(\omega) + \sigma + \sum_{j=1}^{e_0}(1 - 1/\ell_j) \tag{5}$$

The complete construction of all meromorphic functions and differentials on a compact Riemann surface has been given by H. Petersson in a series of papers with the help of the so-called 'Poincare series'.

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The Modular Group and its Subgroups

1 The Modular Group

In § 4 of chapter 1, we observed that if Γ is a hyperbolic horocyclic group of motions of the hyperbolic plane, then $\mathscr{G}$ is an unbranched covering surface of the Riemann surface $\mathscr{R}$ associated to Γ, since Γ does not have any elliptic or parabolic fixed points. Moreover, if $\mathfrak{F}$ is a normal fundamental domain in $\mathscr{G}$ for Γ, then, by (1) of chapter 1, § 4, 47

$$0 < \frac{1}{2\pi}\Im(\mathfrak{F}) = 2p - 2,$$

where p is the genus of $\mathscr{R}$. This shows that $p > 1$ and therefore, a closed Riemann surface of genus 1 cannot have $\mathscr{G}$ as an unbranched covering surface. Thus horocyclic groups whose associated Riemann surface is of genus $p \leq 1$, must have parabolic or elliptic fixed points. On the one hand, the study of such groups is of importance from the point of view of applications in the Theory of Numbers; on the other hand, it is naturally preferable to have an unbranched covering surface for the Riemann surface when the study of the Riemann surface is of foremost importance. The latter object can be achieved by means of the uniformisation theory for Riemann surfaces. A principal result of this theory states that all closed Riemann surfaces of genus $p > 1$ are associated with hyperbolic horocyclic groups of motions of the hyperbolic plane and in the same way, all closed Riemann surfaces of genus 1 are

associated with the fixed pointfree discrete groups of Euclidean motions
48 of the whole complex z-plane with compact fundamental domain. In the following, we shall examine the latter case more closely. Let B be a discrete group of motions of the whole complex z-plane generated by two real-independent translations $z \to z + \omega$ and $z \to z + \omega'$. Let $\mathfrak{M}$ be the module generated by ω and ω' over the ring of rational integers. It is obvious that $\mathfrak{M}$ is isomorphic to B. We can assume, without loss of generality, that $\tau = \dfrac{\omega'}{\omega}$ has positive imaginary part, so that τ belongs to $\mathscr{G}$. A fundamental domain for B is given by

$$\mathfrak{F} = \{z|z = r\omega + r'\omega' \text{ with } 0 \leq r, r' \leq 1\}.$$

If we identify the equivalent edges of $\mathfrak{F}$, we obtain an orientable polyhedron $\mathscr{R}$. Let $\mathfrak{g}_0 \in \mathscr{R}$ be the trace-point of z_0; then the function $t = t(\mathfrak{g}) = z - z_0$ is introduced as a local coordinate at $\mathfrak{g}_0$. Provided with the analytic structure defined by this local coordinate system, $\mathscr{R}$ becomes a Riemann surface of genus 1, with the whole z-plane as an unbranched covering surface. We shall call the complex numbers ω and ω' *periods* corresponding to the Riemann surface $\mathscr{R}$ and $\mathfrak{M}$ the *period module* associated to $\mathscr{R}$. We define two Riemann surfaces $\mathscr{R}$ and $\mathscr{R}^*$ of genus p (not necessarily 1) to be *conformally equivalent* if

1) there exists a topological mapping σ of $\mathscr{R}$ onto $\mathscr{R}^*$ and

2) if $t = t(\mathfrak{g})$ is a local coordinate at a point $\mathfrak{g}_0$ and $t^* = t^*(\mathfrak{g}^*)$ is a local coordinate at the point $\mathfrak{g}_0^* = \sigma(\mathfrak{g}_0)$ of $\mathscr{R}^*$, then a neighbourhood of 0 in the t-plane is mapped conformally onto a neighbourhood of 0 in the t^*-plane by

$$t = t(\mathfrak{g}) \to \mathfrak{g} \to \sigma(\mathfrak{g}) = \mathfrak{g}^* \to t^*(\mathrm{t}^*) = t^*,$$

i.e. in a neighbourhood of 0, we have

$$t^* = c_1 t + c_2 t^2 + \ldots \text{ with } c_l \neq 0.$$

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For the conformal equivalence of Riemann surfaces of genus 1, we prove the following

Theorem 6. *Two Riemann surfaces $\mathscr{R}$ and $\mathscr{R}^*$ of genus 1 with $\mathfrak{M}$ and $\mathfrak{M}^*$ as their respective period modules are conformally equivalent if and only if there exists a complex number $k \neq 0$ such that $\mathfrak{M}^* = k\mathfrak{M} = \{k\alpha | \alpha \in \mathfrak{M}\}$.*

Proof. If $\mathfrak{M}^* = k\mathfrak{M}$ with a $k \neq 0$, then $z \mod \mathfrak{M} \to z^* \mod \mathfrak{M}^*$ uniquely defined by $z \to z^* = kz$ is a conformal mapping of $\mathscr{R}$ onto $\mathscr{R}^*$. Let $\mathscr{R}$ and $\mathscr{R}^*$ be conformally equivalent; then we shall show that $\mathfrak{M}^* = k\mathfrak{M}$ for some complex number $k \neq 0$. Let φ (respectively φ^*) be the trace mapping from the complex z-plane to $\mathscr{R}$ (respectively $\mathscr{R}^*$) i.e.

$$\varphi(z) = \mathfrak{g} \text{ for } z \in \mathfrak{g} \text{ in } \mathscr{R}, \text{ and}$$
$$\varphi^*(z^*) = \mathfrak{g}^* \text{ for } z^* \in \mathfrak{g}^* \text{ in } \mathscr{R}^*.$$

□

Since the trace mapping φ is locally a topological mapping, it follows that an arc W in the z-plane is uniquely fixed by its starting point and its image in $\mathscr{R}$ by φ and to every arc W_0 in $\mathscr{R}$, there corresponds an arc W in the z-plane such that $\varphi(W) = W_0$, where the initial point of W is a given point with the initial point of W_0 as its trace point. Let $\mathfrak{g}_0$ and $\mathfrak{g}_0^* = \sigma(\mathfrak{g}_0)$, where σ denotes the conformal mapping of $\mathscr{R}$ onto $\mathscr{R}^*$, be corresponding points of $\mathscr{R}$ and $\mathscr{R}^*$. We choose two points z_0 and z_0^* in the z-plane such that $z_0 \in \mathfrak{g}_0$ and $z_0^* \in \mathfrak{g}_0^*$. Let z be any arbitrary point of the z-plane and W an arc joining z_0 and z. Then the arc $\sigma\varphi(W)$ in $\mathscr{R}^*$, having $\mathfrak{g}_0^*$ as its initial point, uniquely determines an arc W^* in **50**
the z-plane with the initial point z_0^*. The end point z^* of W^* is uniquely determined because an arc W is closed in the z-plane if and only if $\varphi(W)$ is homotopic to zero in $\mathscr{R}$ and the latter property is preserved by a topological mapping. We define a $1-1$ mapping ψ from the z-plane onto itself by $z^* = \psi(z)$. Since $\psi = (\varphi^*)^{-1} \cdot \sigma \cdot \varphi$ locally, ψ is a conformal mapping of the z-plane onto itself. Therefore necessarily we have

$z^* = kz + c$ for some complex numbers k and c with $k \neq 0$. But $\varphi(W)$ is closed or open if and only if $\sigma\varphi(W)$ is closed or open; therefore

$$z - z_0 \equiv 0 (\mod \mathfrak{M}) \Leftrightarrow z^* - z_0^* \equiv 0 (\mod \mathfrak{M}^*)$$

with $z^* - z_0^* = k(z - z_0)$. This proves our theorem.

From the above theorem, it follows that if $\{\omega^*, \omega^{*'}\}$ is a basis of $\mathfrak{M}^*$, then the conformal equivalence of $\mathscr{R}$ and $\mathfrak{R}^*$ implies that $\{\omega^*/k, \omega^{*'}/k\}$ is a basis of $\mathscr{M}$ i.e. there exists an integral matrix $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant $|S| = \pm 1$ such that $\omega^{*'}/k = a\omega' + b\omega, \omega^*/k = c\omega' + d\omega$ where $\{\omega, \omega'\}$ is a basis of $\mathfrak{M}$. This shows that if $\tau^* = \omega^{*'}/\omega$ and $\tau = \omega'/\omega$, then

$$\tau^* = S < \tau >= \frac{a\tau + b}{c\tau + d}.$$

We can assume without loss of generality that τ as well as τ^* has positive imaginary part. Therefore $|S|$ has necessarily to be equal to 1. Consider the differential $d(z/\omega)$ on $\mathscr{R}$, where ω is as above. It is a
51 differential of the first kind and its integral along any closed curve on $\mathscr{R}$ has a value which is a linear combination of 1 and τ. We shall call τ a *normed period* of $\mathscr{R}$. So we have proved the following

Theorem 7. *Two Riemann surfaces $\mathscr{R}$ and $\mathscr{R}^*$ of genus 1 are conformally equivalent if and only if their normed periods are equivalent under the group*

$$\Gamma = \{\begin{matrix} a & b \\ c & d \end{matrix} | ad - bc = 1; a, b, c, d \text{ integral}\}.$$

We shall call the group Γ defined in theorem 7 *the modular group* and, unless otherwise stated, denote it always by Γ. It is obvious that the group Γ acts discontinuously on $\mathscr{G}$. The above discussion shows that to every point of the quotient space $\mathscr{G}/\Gamma$ corresponds a class of conformally equivalent Riemann surfaces of genus 1 and conversely, to every such class is associated uniquely a point of the space $\mathscr{G}/\Gamma$ represented by a normed period of some element of the class.

In the sequel, we shall adhere to the following notation:

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } V = U^{-1}T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

In order to find a normal fundamental domain $\mathfrak{F}$ for Γ, we proceed as follows. A simple consideration shows that $iy_0 (y_0 > 1)$ is not a fixed point for Γ. The perpendicular bisectors of the lines joining the points $U < iy_0 >= iy_0 + 1, U^{-1} < iy_0 >= iy_0 - 1$ and $T < iy_0 >= i/y_0$ to

the point iy_0 are given by the equations $x = \frac{1}{2}, x = -\frac{1}{2}$ and $x^2 + y^2 = 1$ respectively, where $\tau = x + iy$. Therefore the construction of $\mathfrak{F}$ in chapter 1, § 3, shows that $\mathfrak{F}$ must be contained in the hyperbolic triangle $\mathfrak{F}_0 = \{\tau \big| |x| \leq \frac{1}{2}, x^2 + y^2 \geq 1\}$ bounded by the hyperbolic lines $x = \pm\frac{1}{2}$ and $x^2 + y^2 = 1$. Obviously $\Im(\mathfrak{F}_0) = \pi/3$. Moreover, $\Im(\mathfrak{F}) \geq \pi/3$, 52
because the group Γ contains a parabolic transformation and the area of a fundamental domain of a discrete group of motions of the hyperbolic plane, which contains parabolic transformations, is, as proved in ch. 1, § 4, at least equal to $\pi/3$. Since $\mathfrak{F} \subset \mathfrak{F}_0$, we have $\Im(\mathfrak{F}) = \Im(\mathfrak{F}_0) = \pi/3$. But both $\mathfrak{F}$ and $\mathfrak{F}_0$ are closed sets, therefore $\mathfrak{F} = \mathfrak{F}_0$, proving that a normal fundamental domain of the group Γ is given by

$$\mathfrak{F} = \{\tau | \tau = x + iy, |x| \leq \frac{1}{2}, x^2 + y^2 \geq 1, y > 0\} \tag{1}$$

We shall show that Γ *cannot be a proper subgroup of a discrete group of motions of the hyperbolic plane* i.e. Γ is a maximal discrete subgroup of Ω. If possible, let Γ be properly contained in a discrete group Γ^*. We choose the centre iy_0 of $\mathfrak{F}$, the normal fundamental domain of Γ constructed above, in such a way that iy_0 is not a fixed point for Γ^* also. Then the normal fundamental domain $\mathfrak{F}^*$ of Γ^* with the centre iy_0 is contained in $\mathfrak{F}$. But $\mathfrak{F}^*$ has a parabolic cusp because Γ^* contains Γ and therefore at least one parabolic transformation; thus $\Im(\mathfrak{F}^*) \geq \pi/3$ and as in the preceding case, $\mathfrak{F} = \mathfrak{F}^*$. Since the boundary substitutions of $\mathfrak{F}$ and $\mathfrak{F}^*$ are the same, the groups Γ and Γ^* are generated by the same set of transformations in view of theorem 5. Hence we must have $\Gamma = \Gamma^*$.

The normal fundamental domain $\mathfrak{F}$ of Γ defined in (1) has 3 inequivalent fixed points, namely, the points $i, \rho = e^{2\pi i/3}$ and ∞. The two fixed points i and ρ are elliptic fixed points, because the subgroups of Γ which leave them fixed are generated respectively by T and V. Since $T^2 = V^3 = -E$, these points correspond to the branch points of order 1 and 2 respectively on the Riemann surface associated to Γ or, in other 53
words, i and ρ are the elliptic fixed points of ramification index 1 and 2 respectively. From equation (1) of chapter 1, § 4, we see that the associated Riemann surface of the group Γ is of genus zero.

Remark. Any horocyclic group Γ with a normal fundamental domain $\mathfrak{F}$ of volume $\Im(F) = \pi/3$ and with $\sigma > 0$ is conjugate to the modular group.

Proof. In the notation of ch. 1, § 4 we have $\sigma = 1, e_0 = 2, \ell_1 = 2, \ell_2 = 3$. Without loss of generality, we may assume that the fixed points $\omega_1^{(1)}$ and $\omega_2^{(1)}$ coincide with i and $\rho = e^{2\pi i/3}$ respectively (replacing Γ by a conjugate group, if necessary). Then $T, V \in \Gamma$. But T and V generate the modular group which is a maximal discrete group, as we have seen. Thus Γ is identical with the modular group. $\square$

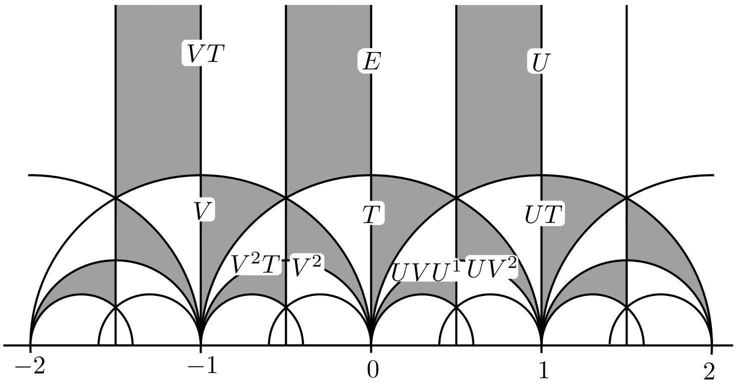

Figure 2.12:

54 **Theorem 8.** *The transformations* $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $W = -V = -U^{-1}T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ *generate the modular group* Γ. *They satisfy the relations*

$$T^4 = W^3 = E, \quad WT^2 = T^2W$$

and these are the defining relations for the group.

Proof. Since U and T are the boundary substitutions of the normal fundamental domain $\mathfrak{F}$ for Γ given in (1), the transformations U and T and therefore $W = -V = -U^{-1}T$ and T generate Γ. $\square$

It can be easily verified that the generators satisfy the given relations. Let $R = E$ be an arbitrary relation in Γ. Without loss of generality, we can assume that

$$R \equiv T^{v_1} W^{u_1} T^{v_2} W^{u_2} T^{v_2} \cdots W^{u_n} = E.$$

With the help of the given relations and cyclic permutations of the factors, we transform this relation into a reduced relation of the type

$$M_n = T^{e_1} W^{e_1} T^{e_2} W^{e_2} \cdots T^{e_n} W^{e_n} = T^{2e_0}$$

with $e_0 = 0$ or 1 and $e_i = \pm 1 (i = 1, 2, \ldots, n)$. We shall show that necessarily $n = 0$ i.e. any given relation is a consequence of the relations mentioned in the theorem, and this will complete the proof. We prove, by induction on n, that if $M_n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$, then $a_n, b_n, c_n, d_n \geq 0$ for $n \geq 1$ and moreover b_n and c_n are not simultaneously zero. When $n = 1$,

$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

according as $e_1 = 1$ or -1. Let us assume that the assertion is true for M_n. Then **55**

$$\begin{aligned} M_{n+1} &= M_n T^{e_{n+1}} W^{e_{n+1}} \\ &= \begin{pmatrix} a_n + b_n & b_n \\ c_n + d_n & d_n \end{pmatrix} \text{ or } \begin{pmatrix} a_n & a_n + b_n \\ c_n & c_n + d_n \end{pmatrix} \end{aligned}$$

according as $e_{n+1} = 1$ or -1. This shows that the relation $M_n = E$ is satisfied if and only if $n = 0$ and consequently $e_0 = 0$.

Finally, we shall mention the use of the modular group in the reduction theory of positive definite binary quadratic forms. Throughout our discussion, we shall consider two-rowed real position symmetric matrices as associated to positive definite binary quadratic forms. Let $A = \left(\begin{smallmatrix} a_0 & a_1 \\ a_1 & a_2 \end{smallmatrix}\right)$ be a real symmetric matrix. Then A is positive ($A > 0$) if and only if $a_0 > 0$ and $a_0 a_2 - a_1^2 > 0$.

Definition . *Two positive symmetric matrices A and B are said to be equivalent if there exists an integral matrix S with $|S| = \pm 1$ such that*

$B = SAS'$ where S' is the transpose of S. We say that A and B are properly equivalent if $|S| = 1$. If $A = \left(\begin{smallmatrix} a_0 & a_1 \\ a_1 & a_2 \end{smallmatrix}\right) > 0$, then the polynomial

$$(1\ \xi)A\begin{pmatrix}1\\ \xi\end{pmatrix} = a_0 + 2a_1\xi + a_2\xi^2$$

has complex conjugate zeros. Let the two zeros be $-\tau$ and $-\overline{\tau}$, such that τ belongs to $\mathscr{G}$. Then

$$a_0 + 2a_1\xi + a_2\xi^2 = a_2(\xi + \tau)(\xi + \overline{\tau}).$$

We shall say that the point $\tau \in \mathscr{G}$ obtained in this way is associated
56 *to the matrix A. Obviously*

$$2a_1/a_2 = \tau + \overline{\tau} = 2x, a_0/a_2 = \tau\overline{\tau} = x^2 + y^2 (\tau = x + iy).$$

If $w = \sqrt{|A|} > 0$, then

$$\begin{aligned} w^2 = a_0a_2 - a_1^2 &= a_2^2(a_0/a_2 - a_1^2/a_2^2) \\ &= a_2^2(x^2 + y^2 - x^2) = a_2^2y^2. \end{aligned}$$

This shows that the matrix A has the representation

$$A = \frac{w}{y}\begin{pmatrix} x^2 + y^2 & x \\ x & 1 \end{pmatrix}.$$

If $B = \left(\begin{smallmatrix} b_0 & b_1 \\ b_1 & b_2 \end{smallmatrix}\right)$ is another positive symmetric matrix equivalent to A i.e. $B = SAS'$, where S is some integral matrix of determinant ± 1, then $|B| = |A|$ and therefore

$$B = \frac{w}{y^*}\begin{pmatrix} x^{*2} + y^{*2} & x^* \\ x^* & 1 \end{pmatrix}$$

with some $\tau^* = x^* + iy^*$ such that $-\tau^*$ and $-\overline{\tau}^*$ are the zeros of the polynomial

$$(1\xi)B\begin{pmatrix}1\\ \xi\end{pmatrix} = b_0 + 2b_1\xi + b_2\xi^2.$$

But with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *we have*

$$\begin{aligned}
(1\ \xi)B\begin{pmatrix}1\\ \xi\end{pmatrix} &= (1\ \xi)SAS'\begin{pmatrix}1\\ \xi\end{pmatrix}\\
&= (1\ \xi)\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a_0 & a_1\\ a_1 & a_2\end{pmatrix}\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}1\\ \xi\end{pmatrix}\\
&= (a + c\xi b + \xi d)\begin{pmatrix}a_0 & a_1\\ a_1 & a_2\end{pmatrix}\begin{pmatrix}a & + & c\xi\\ b & + & d\xi\end{pmatrix}\\
&= (a + c\xi)^2 a_0 + 2a_1(b + d\xi)(a + c\xi) + a_2(b + d\xi)^2\\
&= a_2\{(a + c\xi)^2\tau\overline{\tau} + (\tau + \overline{\tau})(b + d\xi)(a + c\xi) + (b + d\xi)^2\}\\
&= a_2\{(a + c\xi)\tau + (b + d\xi)\}\{(a + c\xi)\overline{\tau} + (b + d\xi)\}\\
&= a_2\{\xi(c\tau + d) + (a\tau + b)\}\{\xi(c\overline{\tau} + d) + a\overline{\tau} + b\}\\
&= a_2|c\tau + d|^2(\xi + S < \tau >)(\xi + S < \overline{\tau} >);
\end{aligned}$$

therefore **57**

$$\tau^* = S < \tau > \text{ for } |S| = 1.$$
$$\tau^* = S(\overline{\tau}) \text{ for } |S| = -1.$$

We shall say that a positive symmetric matrix A is reduced when the point $\tau \in \mathscr{G}$ *associated to A belongs to the fundamental domain* $\mathfrak{F}$, *of the modular group given in* (1). *We have proved that in an equivalence class of properly equivalent matrices there always exists a reduced matrix and this matrix is uniquely determined if the associated point* τ *belongs to the interior of* $\mathfrak{F}$.

2 Subgroups of the Modular Group

In general, here and in the following, we shall consider those subgroups Γ^* of Γ which contain $-E$ and are of finite index in Γ. We shall denote the index $(\Gamma : \Gamma^*)$ of Γ^* in Γ by μ. Let $S_1, S_2, \ldots, S_\mu$ be a complete system of representatives of the right cosets of Γ by Γ^*, so that

$$\Gamma = \bigcup_{i=1}^{\mu} \Gamma^* S_i.$$

58 If $\mathfrak{F}$ is a normal fundamental domain for Γ, we shall show that $\mathfrak{F}^*$ given by

$$\mathfrak{F}^* = \bigcup_{i=1}^{\mu} \mathfrak{F}_{S_i}$$

is a fundamental domain for γ^*. Since

$$\bigcup_{L\in\Gamma^*} \mathfrak{F}_L^* = \bigcup_{L\in\Gamma^*} \bigcup_{i=1}^{\mu} \mathfrak{F}_{LS_i} = \bigcup_{L\in\Gamma} \mathfrak{F}_L = \mathscr{G},$$

$\mathfrak{F}^*$ contains atleast one point from each set of equivalent points with respect to Γ^*. In order to prove that $\mathfrak{F}^*$ is a fundamental domain for Γ^*, it remains to show that if τ belongs to $\mathfrak{F}^* \cap \mathfrak{F}_L^*$ for some $L \in \Gamma^*$, $L \neq \pm E$, then τ belongs to the boundary of $\mathfrak{F}^*$. Our assumption $\tau \in \mathfrak{F}^* \cap \mathfrak{F}_L^*$ for some L in Γ^* implies that τ is in $\mathfrak{F}_{S_i} \cap \mathfrak{F}_{LS_j}$ for some i, j with $1 \leq i, j \leq \mu$. If $\mathfrak{F}_{LS_j} = \mathfrak{F}_{S_h}$ for some h so that $LS_j = \pm S_h$, we obtain $j = h$ and $L = \pm E$, contradicting our assumption. Therefore $\mathfrak{F}_{LS_j} \neq \mathfrak{F}_{S_h}$ for $1 \leq h \leq \mu$. Now it is obvious that the interior points of $\mathfrak{F}_{LS_j}$ are exterior points of $\mathfrak{F}^*$; consequently, τ is a boundary point of $\mathfrak{F}^*$ and therefore of $\mathfrak{F}_L^*$. Hence $\mathfrak{F}^*$ is a fundamental domain for the group Γ^*. Conversely, if the set $\mathfrak{F}^* = \bigcup\limits_{i=1}^{t} \mathfrak{F}_{S_i}$, where $S_i \in \Gamma$, is a fundamental domain for Γ^*, then it can be easily proved that $t = \mu$ and $\{S_1, S_2, \ldots, S_\mu\}$ is a complete set of coset representatives of Γ modulo Γ^*.

It is obvious that the parabolic cusps of Γ^* are the same as those of Γ, namely the rational points on the real axis and ∞. Let $s_1, s_2, \ldots, s_\sigma$
59 be a complete system of inequivalent parabolic cusps of Γ^*. There exist transformations A_i in Γ such that

$$A_i^{-1} < \infty >= s_i, i = 1, 2, \ldots, \sigma.$$

Consider the group $A_i\Gamma^* A_i^{-1}$; it has ∞ as a fixed point and therefore contains U^r for some integer r. Let $N_i > 0$ be so determined that U^{N_i} is the least positive power of U belonging to the group $A_i\Gamma^* A_i^{-1}$. Then the transformations $-E$ and U^{N_i} generate the group contained in $A_i\Gamma^* A_i^{-1}$ which leaves ∞ fixed. The integer N_i determined above does not depend upon the choice of the cusp s_i in the class of s_i. If $s'_i = L < s_i >$ for $L \in \Gamma^*$ and $B_i^{-1} < \infty >= s'_i$ for some $B_i \in \Gamma$, then

$$\begin{aligned} & s_i = A_i^{-1} < \infty >= L^{-1}B_i^{-1} < \infty >\Longrightarrow A_iL^{-1}B_i^{-1} < \infty >= \infty \Longrightarrow \\ \Longrightarrow \quad & A_iL^{-1}B_i^{-1} = \pm U^r \text{ for some integral } r \Longrightarrow \\ \Longrightarrow \quad & A_i\Gamma^*A_i^{-1} = U^rB_i\Gamma^*B_i^{-1}U^{-r} \Longrightarrow U^{N_i} \in B_i\Gamma^*B_i^{-1}. \end{aligned}$$

This shows that if $U^{N'_i}$ is the least positive power of U belonging to $B_i\Gamma^*B_i^{-1}$, then $N'_i \leq N_i$. Similarly we get $N_i \leq N'_i$, which proves that $N'_i = N_i$. We shall call the integer N_i the *width of the cusp sector* at the cusp s_i. We shall now construct a fundamental domain for Γ^* which shows a connection between μ and the widths of cusp sectors at the various cusps of Γ^*. As we have seen above, it is sufficient to give a coset decomposition of Γ modulo Γ^* which indicates the desired connection. We shall show that $\bigcup\limits_{i=1}^{\sigma}\bigcup\limits_{r=0}^{N_i-1}\Gamma^*A_i^{-1}U^r$, where $A_i \in \Gamma$ and $N_i > 0$ as determined above, is a coset decomposition of Γ modulo Γ^*. If $S \in \Gamma$, then, for some i with $1 \leq i \leq \mu$ and $L \in \Gamma^*$, we have **60**

$$S < \infty >= L < s_i >= LA_i^{-1} < \infty >\Longrightarrow S = \pm LA_i^{-1}U^t \text{ for some } t.$$

Let $t = aN_i + r$ with $0 \leq r < N_i$, then

$$S = \pm L(A_i^{-1}U^{N_i}A_i)^aA_i^{-1}U^r \in \Gamma^*A_i^{-1}U^r.$$

Hence we obtain that

$$\Gamma = \bigcup_{i=1}^{\sigma}\bigcup_{r=0}^{N_i-1}\Gamma^*A_i^{-1}U^r. \tag{1}$$

Moreover, if S is a common element of $\Gamma^*A_i^{-1}U^r$ and $\Gamma^*A_j^{-1}U^s$ with $1 \leq i, j \leq \sigma, 0 \leq s < N_j$ and $0 \leq r < N_i$, then $S < \infty >$ is equivalent to both s_i and s_j with respect to Γ^*. Because of the choice of the cusps, this is possible only if $i = j$, therefore $A_i^{-1}U^{r-s}A_i$ belongs to Γ^* showing that $r - s \equiv 0 (\text{mod } N_i)$. But $0 \leq r, s < N_i$; therefore $s = r$ and this completely proves that the decomposition of Γ given in (1) is a coset decomposition of Γ modulo Γ^*. Hence

$$\mathfrak{F}^* = \bigcup_{i=1}^{\sigma} \left(\bigcup_{r=0}^{N_i-1} \mathfrak{F}_{U^r} \right)_{A_i^{-1}} \tag{2}$$

is a fundamental domain for Γ^*, which we shall use in the sequel. If $\mathfrak{F}$ is the fundamental domain for Γ given by 1 of the previous section, the $\left(\bigcup\limits_{r=0}^{N_i-1} \mathfrak{F}_{U^r} \right)_{A_i^{-1}}$ is a cusp sector at the cusp s_i and the width of this sector is nothing but the ordinary width of $\bigcup\limits_{r=0}^{N_i-1} \mathfrak{F}_{U^r}$, namely N_i. In particular, we obtain that

$$(\Gamma : \Gamma^*) = \mu = N_1 + N_2 + \cdots + N_\sigma. \tag{3}$$

61 Obviously, the elliptic fixed points of Γ^* are either equivalent to i or $\mathfrak{e} = e^{2\pi i/3}$ with respect to Γ; therefore an elliptic fixed point of Γ^* is either of ramification index 1 or 2. Let e_1 (respectively e_2) denote the number of elliptic fixed points of Γ^* of ramification index 1 (respectively 2). Since $\mathfrak{I}(\mathfrak{F}^*) = \dfrac{\mu\pi}{3}$, we see from formula (1) in chapter 1, § 4 that the genus p of the Riemann surface associated to the group Γ^* is given by

$$p = \frac{\mu}{12} + 1 - \frac{\sigma}{2} - \frac{e_1}{4} - \frac{e_2}{3}.$$

Let us further assume that Γ^* is a normal subgroup of Γ. If τ_1 and τ_2 are two points of $\mathscr{G}$ equivalent with respect to Γ, then the subgroups Γ_1 and Γ_2 of Γ^* which leave respectively τ_1 and τ_2 fixed, are conjugate subgroups in Γ. Let $\tau_2 = A < \tau_1 >$ with $A \in \Gamma$. Then the group $A^{-1}\Gamma_2 A \subset A^{-1}\Gamma^* A = \Gamma^*$ leaves τ_1 fixed, therefore $A^{-1}\Gamma_2 A \subset \Gamma_1$. Similarly $A\Gamma_1 A^{-1} \subset \Gamma_2$. Hence $A^{-1}\Gamma_2 A = \Gamma_1$ and therefore the fixed points of Γ^* which are equivalent with respect to Γ are of the same type. In particular, all the widths of the cusp sectors at various parabolic cusps of Γ^* are equal and we obtain from 3

$$\mu = N\sigma, \text{ if } N_i = N \text{ for } i = 1, 2, \ldots, \sigma.$$

Moreover, $e_1 = N$ or 0 (respectively $e_2 = N$ or 0) according as i (respectively ρ) is a fixed point of Γ^* or not. Thus we obtain the following table the genus of a normal subgroup Γ^* of Γ:

p	Fixed points of Γ^*
$1 - \dfrac{\mu}{2}\left(\dfrac{1}{N} + 1\right)$	i, ρ, ∞
$1 - \dfrac{\mu}{2}\left(\dfrac{1}{N} + \dfrac{1}{2}\right)$	ρ, ∞
$1 - \dfrac{\mu}{2}\left(\dfrac{1}{N} + \dfrac{1}{3}\right)$	i, ∞
$1 - \dfrac{\mu}{2}\left(\dfrac{1}{N} - \dfrac{1}{6}\right)$	∞

(4)

In the above table, μ is the index of Γ^* in Γ and N is the width of the cusp sector at any parabolic cusp of Γ^*. **62**

Let N be a natural number. Then the set of matrices $S \in \Gamma$ with

$$S = \begin{pmatrix} a & b \\ c & b \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } N)$$

form a group which we shall denote by $\Gamma[N]$. In the following, we shall determine the index $\mu(N)$ of $\Gamma[N]$ in Γ. It is obvious that two matrices A and B of Γ belong to the same coset of Γ modulo $\Gamma[N]$ if and only if

$$A\Gamma[N] = B\Gamma[N] \Longleftrightarrow A^{-1}B \in \Gamma[N] \Longleftrightarrow A \equiv B(\text{mod } N).$$

This means that $\mu(N)$ is the number of matrices S of Γ which are incongruent modulo N. We assert that $\mu(N)$ is also the number of integral matrices $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ which are incongruent modulo N and for which the determinant $|S| = ad - bc \equiv 1(\text{mod } N)$. In order to prove this assertion, we have to show that for any matrix $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with **63**
$ab - bc \equiv 1(\text{mod } N)$, there exists a matrix S_1 in with $S_1 \equiv S(\text{mod } N)$. Since $ad - bc \equiv 1(\text{mod } N)$, it follows that $(c, d, N) = 1$. Let $(d, N) = q$; then $(c, q) = 1$ and there exists an integer s such that

$$s \equiv d/p(\text{mod } N/q) \text{ and } (s, c) = 1.$$

This shows that $d' = sq = d + rN$ is such that $(d', c) = 1$ and $\begin{pmatrix} a & b \\ c & d' \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\text{mod } N)$. Let $ad' - bc = 1 + wN$. Consider the matrix $\begin{pmatrix} a+xN & b+yN \\ c & d' \end{pmatrix}$ where x and y are integers so determined that $xd' - yc = w$; such integers x and y exist, because $(c, d') = 1$. It is obvious that $\begin{pmatrix} a+xN & b+yN \\ c & d' \end{pmatrix}$ is a desired matrix $S_1 \equiv S(\text{mod } N)$.

The function $\mu(N)$ is a multiplicative function of N i.e.

$$\mu(N_1\ N_2) = \mu(N_1)\mu(N_2) \text{ for } (N_1, N_2) = 1.$$

For the proof, we observe that a solution $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the matrix congruences

$$S \equiv S_i(\text{mod } N_i)\ (i = 1, 2)$$

exists and is uniquely determined modulo N_1N_2, since $(N_1, N_2) = 1$. Further, $|S_i| \equiv 1(\text{mod } N_i)$ for $i = 1, 2$ imply that $|S| \equiv 1(\text{mod } N_1N_2)$ and vice versa. The assertion is now a consequence of

$$S \text{ mod } N_1N_2 \leftrightarrow S_i \text{mod } N_i\ (i = 1, 2).$$

Thus, in order to evaluate $\mu(N)$, it is sufficient to determine its value of $N = p^\alpha$, where p is a prime number.

Let $\mu_k(p^\alpha)$, for $0 \leq k \leq \alpha$, denote the number of solutions of

$$ad - bc \equiv 1(\text{mod } p^\alpha), (a, p^\alpha) = p^k,$$

64 which are distinct modulo p^α.

1) $\underline{k = 0}$. The congruence $ad \equiv 1+bc(\text{mod } p^\alpha)$ will have a unique solution for d modulo p^α when b and c are given arbitrarily modulo p^α. But a modulo p^α can be any one of the $\varphi(p^\alpha)$ prime residue classes modulo p^α; therefore $\mu_0(p^\alpha) = p^{2\alpha}\varphi(p^\alpha)$.

2) $\underline{k \geq 1}$. Let, first of all, a be fixed. Then the congruence

$$ad \equiv 1 + bc(\text{mod } p^\alpha),\ (a, p^\alpha) = p^k$$

will have a solution for d, if and only if $l + bc \equiv 0(\text{mod } p^k)$. For a given b, the congruence $bc \equiv -1(\text{mod } p^k)$ will have a unique solution for $c$mod p^k and therefore $p^{\alpha-k}$ solutions modulo p^α. But, for

b, we can take any of the $\varphi(p^\alpha)$ prime residue classes modulo p^α; therefore, the number of solutions modulo p^α of $bc \equiv -1 (\bmod\ p^k)$ is $p^{\alpha-k}\varphi(p^\alpha)$. For fixed a, b and c, the congruence $ad \equiv 1 + bc (\bmod\ p^\alpha)$ determines d uniquely modulo $p^{\alpha-k}$. This means that, for fixed a, b and c, the congruence $ad \equiv 1 + bc(\bmod\ p^\alpha)$ has p^k solutions d modulo p^α and therefore, for fixed a, it has $\varphi(p^\alpha)\ p^\alpha$ solutions. Since, for a fixed k, the integer a with $(a, p^\alpha) = p^k$ has $\varphi(p^{(\alpha-k)})$ distinct values modulo p^α, we see that

$$\mu_k(p^\alpha) = \varphi(p^{\alpha-k})\varphi(p^\alpha)p^\alpha \text{ for } k \geq 1.$$

The cases 1) and 2) above together give

$$\begin{aligned}
\mu(p^\alpha) &= \mu_0(p^\alpha) + \sum_{k=1}^{\alpha} \mu_k(p^\alpha)\\
&= \phi(p^\alpha)p^{2\alpha} + \sum_{k=1}^{\alpha} \varphi(p^{\alpha-k})\varphi(p^\alpha)p^\alpha\\
&= p^\alpha\ \varphi\ (p^\alpha)\{p^\alpha + (p^{\alpha-1} - p^{\alpha-2}) + (p^{\alpha-2} - p^{\alpha-3})\\
&\qquad + \cdots + p - 1 + 1\}\\
&= p^{3\alpha}(1 - p^{-2}).
\end{aligned}$$

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Hence we obtain that

$$\mu(N) = N^3 \prod_{p|N} (1 - p^{-2})(p \text{ a prime number } > 0). \tag{5}$$

Obviously, the group $\Gamma[N]$ does not contain $-E$ for $N > 2$. Let $\Gamma^*[N]$ denote the group generated by $-E$ and $\Gamma[N]$, so that

$$\Gamma^*[N] = \{S | S \in \Gamma, S \equiv \pm E (\bmod\ N)\}.$$

The index $\mu^*(N) = (\Gamma : \Gamma^*[N])$ is given by

$$\mu^*(N) = \begin{cases} \mu(N) = 6, & \text{for } N = 2\\ \frac{1}{2}\mu(N) = \frac{1}{2}N^3 & \prod_{p|N}(1 - p^{-2}), \text{ for } N > 2. \end{cases} \tag{6}$$

We call $\Gamma^*[N]$ *the principal congruence subgroup of level* N. It is a normal subgroup of Γ. Obviously, N is the width of the cusp sector at any parabolic cusp of $\Gamma^*[N]$ and therefore $\mu^*(N)/N$ is the number of inequivalent parabolic cusps of $\Gamma^*[N]$. Since, for $N > 1$, $\Gamma^*[N]$ contains neither $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) n$ or $V = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, i and ρ are not fixed points of $\Gamma^*[N]$. Together with (4), this shows that the genus $p(N)$, of $\Gamma^*[N]$, is given by

$$p(N) = 1 + \frac{\mu^*(N)}{2}\left(\frac{1}{6} - \frac{1}{N}\right)$$

Finally, we obtain by (5),

$$p(N) = \begin{cases} 0, & \text{for } N = 1, 2, 3, 4, 5. \\ 1 + \frac{N^2(N-6)}{24} \prod\limits_{q|N}(1 - q^{-2}), & \end{cases} \tag{7}$$

66 where q runs over positive prime divisors of N.

A subgroup Γ^* of Γ is called a *congruence subgroup,* if Γ^* contains a principal congruence subgroup of level N for some $N \geq 1$. The following remarkable theorem of Fricke-Wohlfahrt enables us to associate with Γ^* a uniquely determined principal congruence subgroup.

Theorem 9. *Let $N_1, N_2, \ldots, N_\sigma$ be the widths of the cusp sectors of a complete system of inequivalent parabolic cusps of a congruence subgroup Γ^* and $\overline{N}$ the least common multiple of $N_1, N_2, \ldots, N_\sigma$. Then $\Gamma^*[N] \subset \Gamma^*$ if and only if $\overline{N}$ divides N.*

Proof. Let us assume that $\Gamma^*[N] \subset \Gamma^*$. Let $s_1, s_2, \ldots, s_\sigma$ be a complete system of inequivalent parabolic cusps of Γ^* and let $A_i^{-1} < \infty >= s_i$, $A_i \in \Gamma$ for $i = 1, 2, \ldots, \sigma$. Then N_i is the least natural number with the property that U^{N_i} belongs to $A_i \Gamma^* A_i^{-1}$. But $\Gamma^*[N] \subset \Gamma^*$; therefore

$$U^N \in \Gamma^*[N] = A_i \Gamma^*[N] A_i^{-1} \subset A_i \Gamma^* A_i^{-1}.$$

This implies that N_i divides N for $i = 1, 2, \ldots, \sigma$ and therefore $\overline{N}$ divides N. $\square$

Let N be a natural number divisible by $\overline{N}$. In order to prove that $\Gamma^*[N] \subset \Gamma^*$, it is sufficient to prove that $\Gamma[\overline{N}] \subset \Gamma^*$, since $-E \in \Gamma^*$ and $\Gamma^*[N] \subset \Gamma^*[\overline{N}]$. Let $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an arbitrary element of $\Gamma[\overline{N}]$.

67 i) For any matrix A in Γ and any integer g, we claim that

$$S_1 = A^{-1}\begin{pmatrix} 1 & g\overline{N} \\ 0 & 1 \end{pmatrix} A \in \Gamma^*.$$

In fact, since $A^{-1} < \infty >$ is equivalent to s_j with respect to Γ^* for some j with $l \leq j \leq \sigma$ and since N_j divides $\overline{N}$, we see that $\begin{pmatrix} 1 & g\overline{N} \\ 0 & 1 \end{pmatrix} \in A\Gamma^* A^{-1}$, which implies our claim.

ii) By the definition of Γ^*, there exists a natural number n such that $\Gamma^*[n] \subset \Gamma^*$. Since $(c, d) = 1$ and further $\overline{N}$ divides d, we have $(c\overline{N}, d) = 1$. Therefore, we can find an integer g such that $d' = d + gc\overline{N}$ is coprime to n. But $\begin{pmatrix} 1 & g\overline{N} \\ 0 & 1 \end{pmatrix} \in \Gamma^*$ in view of (i) with $A = E$. It follows that $S_2 = S\begin{pmatrix} 1 & g\overline{N} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ag\overline{N}+b \\ c & d' \end{pmatrix} \in \Gamma^* \iff S \in \Gamma^*$. Hence we may assume, in the sequel, that $(d, n) = 1$ already.

iii) Consider the matrix

$$S_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} 1 & h\overline{N} \\ 0 & 1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ g\overline{N} & 1 \end{pmatrix} = \begin{pmatrix} * & b+dh\overline{N} \\ c+dg\overline{N} & d \end{pmatrix}$$

where g and h are arbitrary integers. Since $S \in \Gamma[N]$, we have $b = b_1\overline{N}, c = c_1\overline{N}$ with integral b_1, c_1. Now there exist integers g, h such that n divides both $c_1 + dg$ and $b_1 + dh$. Thus n divides both b_3 and c_3; moreover, $d_3 = d$ and so $(d_3, n) = 1$, by (ii). Applying (i) with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, if follows that $\begin{pmatrix} 1 & 0 \\ -g\overline{N} & 1 \end{pmatrix}$ and therefore also $\begin{pmatrix} 1 & 0 \\ g\overline{N} & 1 \end{pmatrix}$ is in Γ^*. For the matrix S, we could have thus assumed already, as in (ii) and we *do* indeed *assume* in the sequel that $b \equiv c \equiv 0 (\mathrm{mod}\ n)$ and $(d, n) = 1$.

We now complete the proof of theorem 9, using steps (i)-(iii) above. 68
Applying (i) with $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ shows that $\begin{pmatrix} 1+g\overline{N} & g\overline{N} \\ -g\overline{N} & 1-g\overline{N} \end{pmatrix} \in \Gamma^*$ for any integer g. Since $S \in \Gamma[\overline{N}]$, we have $a \equiv 1 (\mathrm{mod}\ \overline{N})$ i.e. $a = 1 + g\overline{N}$ for an integer g. It follows that $\begin{pmatrix} a & a-1 \\ 1-a & 2-a \end{pmatrix} \in \Gamma^*$. Moreover, $d \equiv 1 (\mathrm{mod}\ \overline{N})$ and hence all the three matrices on the right hand side of

$$\begin{pmatrix} a & ad-1 \\ 1-ad & d(2-ad) \end{pmatrix} = S_4 = \begin{pmatrix} 1 & 0 \\ 1-d & 1 \end{pmatrix}\begin{pmatrix} a & a-1 \\ 1-a & 2-a \end{pmatrix}\begin{pmatrix} 1 & d-1 \\ 0 & 1 \end{pmatrix}$$

are in Γ^*, implying that $S_4 \in \Gamma^*$. Now

$$SS_4^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d(2-ad) & 1-ad \\ ad-1 & a \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \equiv E(\mathrm{mod}\ n),$$

since $b \equiv c \equiv 0(\mathrm{mod}\ n)$ and consequently $ad \equiv 1(\mathrm{mod}\ n)$. Thus $SS_4^{-1} \in \Gamma^*[n] \subset \Gamma^*$ which together with $S_4 \in \Gamma^*$ implies that $S \in \Gamma^*$, proving the theorem.

This theorem shows $\Gamma^*[\overline{N}]$ with $\overline{N}$ as defined above is the maximal principal congruence subgroup contained in Γ^*. We shall call this uniquely determined number $\overline{N}$ as *the level of the congruence subgroup* Γ^*. With the help of the above theorem, we shall show later that there are subgroups of finite index in Γ which are *not* congruence subgroups.

In what follows, unless otherwise stated, $\mathfrak{F}$ will denote the normal fundamental domain of Γ given by (1) of the previous section. The
69 congruence subgroups $\Gamma_0[N]$ and $\Gamma^0[N]$ (N a natural number) defined by

$$\Gamma_0[N] = \{S | S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, c \equiv 0(\mathrm{mod}\ N)\}$$
$$\Gamma^0[N] = T\Gamma_0[N]T^{-1}$$

are of some importance. Obviously $\Gamma_0[N]$ contains $\Gamma[N]$ and therefore

$$(\Gamma : \Gamma_0[N]) = (\Gamma : \Gamma[N])/(\Gamma_0[N] : \Gamma[N]).$$

But the index of $\Gamma[N]$ in $\Gamma_0[N]$ is equal to the number of integral quadruples a, b, c and d incongruent modulo N with $ad \equiv 1(\mathrm{mod}\ N)$ and $c \equiv 0(\mathrm{mod}\ N)$; therefore $(\Gamma_0 N : \Gamma[N]) = N\varphi(N)$. Using (5), we obtain

$$(\Gamma : \Gamma_0[N]) = \frac{\mu(N)}{N\varphi(N)} = N \prod_{q|N}\left(1 + \frac{1}{q}\right),$$

where q runs over positive prime divisors of N. It is obvious that the group $\Gamma^0[N]$ defined above consists of the matrices $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $b \equiv 0(\mathrm{mod}\ N)$ and further, $(\Gamma : \Gamma^0[N]) = (\Gamma : \Gamma_0[N])$. Since

$$U^r \in \Gamma_0[N] \Leftrightarrow r \equiv 0(\mathrm{mod}\ 1), U^r \in \Gamma^0[N] \Leftrightarrow r \equiv 0(\mathrm{mod}\ N),$$

the width of the cusp sector at the inequivalent cusps ∞ and 0 for $\Gamma_0[N]$ is 1 and N respectively and for $\Gamma^0[N]$ the respective widths are N and 1. If $N = q$ is a prime number, then

$$\mathfrak{F}^* = \left\{ \bigcup_{j=0}^{q-l} \mathfrak{F}_{U^j} \right\} \bigcup \mathfrak{F}_T,$$

which consists of the two cusp sectors at the inequivalent cusps ∞ and 0 for $\Gamma^0[q]$, is a fundamental domain of $\Gamma^0[q]$. This is a consequence of relations (2) and (3), since there are only two inequivalent parabolic cusps for $\Gamma^0[q]$. **70**

We now consider some special subgroups of Γ. Let K be the commutator subgroup of Γ. In order to prove $(\Gamma : K) = 12$, we represent an arbitrary element S in Γ as $S = W^{a_1} T^{b_1} W^{a_2} T^{b_2} \cdots W^{a_r} T^{b_r}$. According to the defining relations for the generators W, T of Γ given in theorem 8, the sums $e_1(S) = a_1 + a_2 + \cdots + a_r$ (respectively $e_2(S) = b_1 + b_2 + \cdots + b_r$) are uniquely determined modulo 3 (respectively modulo 4). If S is a commutator, it is obvious that $e_1(S) \equiv 0 (\bmod 3)$ and $e_2(S) \equiv 0 (\bmod 4)$. Thus this is true also for an arbitrary element of K, since K is generated by commutators. Since $e_1(W) \equiv 1 (\bmod 3)$ and $e_2(T^2) \equiv 2 (\bmod 4)$, it follows that W and T^2 are not in K but we have

$$(TK)^4 = (WK)^3 = K \text{ and } WTK = TWK.$$

This proves that Γ/K is an abelian group of order 12 and the group K^* generated by K and $T^2 (= -E)$ is a normal subgroup of index 6 in Γ. Since $UK = T^3 W^{-1} K$ is an element of order 12 in Γ/K, we get $\Gamma = \bigcup\limits_{r=0}^{5} K^* U^r$. This proves that $\mathfrak{F}^* = \bigcup\limits_{r=0}^{5} \mathfrak{F}_{U^r}$ (see figure 2.13 is a fundamental domain for K^*. Since the normal subgroup K^* does not contain both T and V, neither i nor ρ is a fixed point of K^*. The width of the cusp sector at the parabolic cusp of $\mathfrak{F}^*$ is 6; therefore, from (4), we see that the genus of K^* is 1. Since $TU^{-3} \equiv V^3 T^{-2} \equiv E (\bmod K^*)$, it is obvious that the transformations

$$A_1 = U^6, A_2 = TU^{-3}, A_3 = UTU^{-4}, A_4 = U^2 TU^{-5}$$

belong to K^*. They are the boundary substitutions of $\mathfrak{F}^*$; therefore together with $-E$ they generate K^*. 71

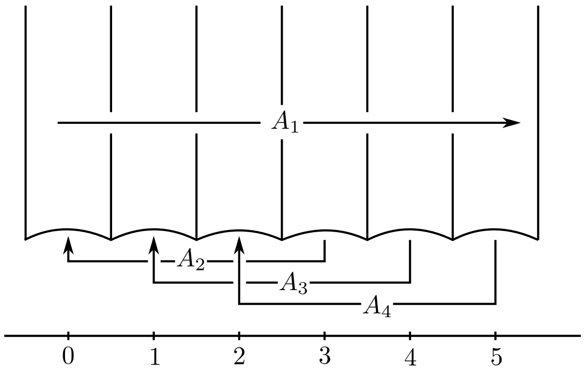

Figure 2.13:

We shall now show that the group K^* is a congruence subgroup of level 6. Let Γ_0 denote the group generated by $\Gamma^*[6]$ and K^*. It can be verified that $\Gamma_0/\Gamma^*[6]$ is an abelian group and moreover,

$$\begin{aligned} A_3^2 &\equiv A_2^4 (\mathrm{mod}\ \Gamma^*[6]) \\ A_2^5 A_3 &\equiv A_4 (\mathrm{mod}\ \Gamma^*[6]) \\ A_1 \equiv A_2^6 &\equiv E (\mathrm{mod}\ \Gamma^*[6]). \end{aligned}$$

72 showing that, for $0 \leq k < 6$ and $0 \leq \ell < 2$, $A_2^k A_3^\ell$ represent the cosets of Γ_0 modulo $\Gamma^*[6]$. Thus $(\Gamma_0 : \Gamma^*[6]) \leq 12$ and therefore

$$6 = (\Gamma : K^*) \geq (\Gamma : \Gamma_0) = \frac{(\Gamma : \Gamma^*[6])}{(\Gamma_0 : \Gamma^*[6])} \geq \frac{72}{12} = 6,$$

which proves that $(\Gamma : K^*) = (\Gamma : \Gamma_0)$ or $\Gamma_0 = K^*$ and $\Gamma^*[6] \subset K^*$. The commutator subgroup is a particular example of the so-called 'cycloid subgroup' of Γ. In general, a subgroup Z of Γ is said to be a *cycloid subgroup*, if the fundamental domain of Z has only one cusp sector i.e. $\sigma = 1$. If μ is the index of Z in Γ, then $\bigcup_{i=0}^{\mu-1} \mathfrak{F}_{U^i}$ is a fundamental domain for Z clearly. Petersson has constructed an infinite number of cycloid subgroups of Γ and proved that exactly 1667868 are congruence subgroups thus showing that there are infinitely many subgroups of Γ which are not congruence subgroups. He has further proved that all subgroups of index ≤ 6 in Γ are congruence subgroups.

In the following, we shall determine all congruence subgroups of Γ of level 2. Let Γ^* denote such a subgroup. Then, by theorem 9, Γ^* contains $\Gamma^*[2] = \Gamma[2]$. Since the index of $\Gamma[2]$ in Γ is 6, the index of Γ^* in Γ is either 6 or 3 or 2. This shows that $(\Gamma^* : \Gamma[2]) = 1, 2, 3$ according as $(\Gamma : \Gamma^*) = 6, 3, 2$.

Let $(\Gamma^* : \Gamma[2]) = 1$ i.e. $\Gamma^* = \Gamma[2]$. Since U^2 belongs to $\Gamma[2]$, the width of the cusp sector at the cusp ∞ is 2. But $\Gamma[2]$ is a normal subgroup of Γ and therefore the width of the cusp sector at any cusp of $\Gamma[2]$ is 2 and there are three inequivalent cusps of $\Gamma[2]$, for example 0,1 and ∞. It is obvious that

$$\mathfrak{F}^* = (\mathfrak{F} \cup \mathfrak{F}_U) \cup (\mathfrak{F}_T \cup \mathfrak{F}_{TU}) \cup (\mathfrak{F}_{UT} \cup \mathfrak{F}_{UTU})$$

is a fundamental domain for $\Gamma[2]$, because it consists of three cusp **73**
sectors at the inequivalent cusps of $\Gamma[2]$.

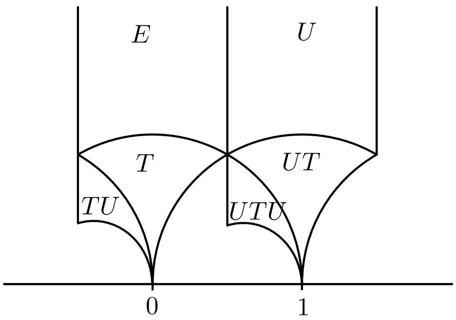

Figure 2.14:

It now follows that

$$\Gamma = \Gamma[2] \cup \Gamma[2]U \cup \Gamma[2]T \cup \Gamma[2]TU \cup \Gamma[2]UT \cup \Gamma[2]UTU.$$

If we replace some parts of $\mathfrak{F}^*$ in figure 2.14 by suitable equivalent parts, then we obtain a fundamental domain as shown in figure 2.15.

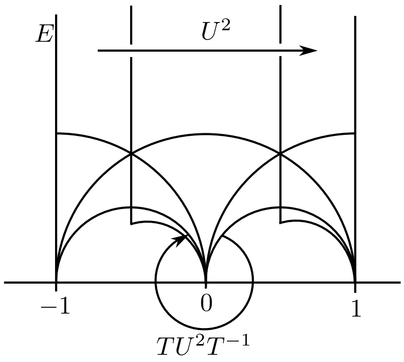

Figure 2.15:

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It is obvious that U^2 and TU^2T^{-1} are in $\Gamma[2]$; they are the boundary substitutions of the fundamental domain in figure 2.15. Therefore U^2 and TU^2T^{-1} together with $-E$ generate $\Gamma[2]$.

Let $(\Gamma^* : \Gamma[2]) = 3$. Then $\Gamma^*/\Gamma[2]$ is a Sylow subgroup of order 3 in $\Gamma/\Gamma[2]$. But $\Gamma^*/\Gamma[2]$ is of index 2 in $\Gamma/\Gamma[2]$; therefore it is a normal subgroup. Hence $\Gamma^*/\Gamma[2]$ is uniquely determined and therefore Γ^* is a uniquely determined normal subgroup of Γ. We shall denote it by N_2. The group $N_2/\Gamma[2]$ is generated by any element of order 3. Since V does not belong to $\Gamma[2]$ and V^3 belongs to $\Gamma[2]$, we have

$$N_2 = \Gamma[2] \cup \Gamma[2]V \cup \Gamma[2]V^2.$$

Moreover U does not belong to N_2; because, if U belongs to N_2,
75 then $T = UV$ belongs to N_2 implying that $\Gamma = N_2$, a contradiction to $(\Gamma : N_2) = 2$. Therefore

$$\Gamma = N_2 \bigcup N_2 U.$$

and $\mathfrak{F}^* = \mathfrak{F} \bigcup \mathfrak{F}_U$ (see figure 2.16 is a fundamental domain for N_2 with the boundary substitutions U^2 and UVU^{-1}.

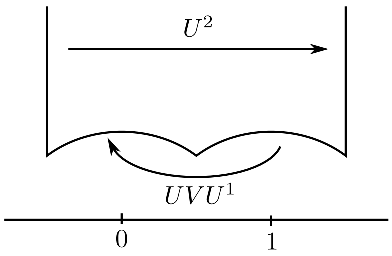

Figure 2.16:

Let $(\Gamma^* : \Gamma[2]) = 2$. Then $\Gamma^*/\Gamma[2]$ is a Sylow subgroup of order 2 and there are three conjugate Sylow subgroups of order 2 in $\Gamma/\Gamma[2]$. One of these subgroups is the so-called *theta-group*

$$\Gamma_\vartheta = \Gamma[2] \bigcup \Gamma[2]T.$$

The group Γ_ϑ consists of the matrices $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ with either $b \equiv c \equiv 0(\text{mod } 2)$ or $a \equiv d \equiv 0(\text{mod } 2)$ and it is not a normal subgroup **76**
of Γ. The width of the cusp sector at ∞ is 2, because U^2 is the least positive power of U belonging to Γ_ϑ. Since $(\Gamma : \Gamma_\vartheta) = 3$, we obtain from (3) that Γ_ϑ has only one cusp, say 1 (inequivalent to ∞), the width of the cusp sector at which is 1. It is obvious that

$$\mathfrak{F}^* = (\mathfrak{F} \bigcup \mathfrak{F}_U) \bigcup \mathfrak{F}_{UT}$$

is a fundamental domain for Γ_ϑ, since it consists of cusp sectors at the two inequivalent cusps of Γ_ϑ. Therefore $\Gamma = \Gamma_\vartheta \bigcup \Gamma_\vartheta U \bigcup \Gamma_\vartheta UT$ is a coset decomposition of Γ modulo Γ_ϑ. If we replace a part of the above mentioned fundamental domain by a suitable equivalent part, we obtain a fundamental domain $\mathfrak{F}^*$ given by

$$\mathfrak{F}^* = \{\tau | \tau = x + iy, |x| \leq 1, |\tau| \geq 1\} \qquad \text{(see figure 2.17)}$$

with the boundary substitutions T and U^2.

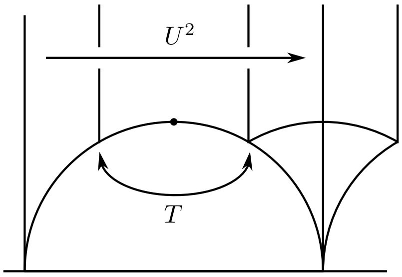

Figure 2.17:

77 The above discussion shows that there exist only 5 congruence subgroups of Γ of level 2.

We shall now construct infinitely many subgroup of finite index in Γ, which are not congruence subgroups. We consider configurations I and II as shown in figure 2.18 and which formed by images of $\mathfrak{F}$ or images of parts of $\mathfrak{F}$ under Γ.

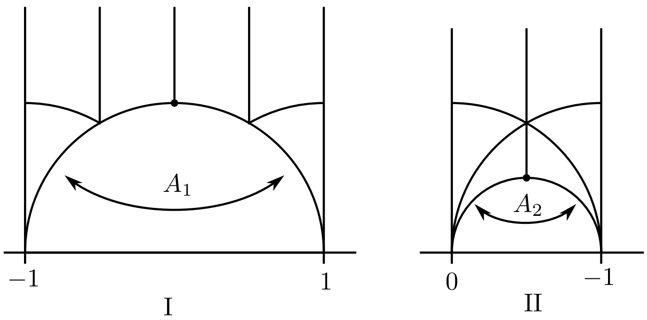

Figure 2.18:

We take a copies of the configuration I and b copies of the configuration II and arrange them in some order so that the resulting figure is a connected domain say $\mathfrak{F}^*$ (see figure 2.19 for $a = 3$, $b = 2$), which has i as a boundary point.

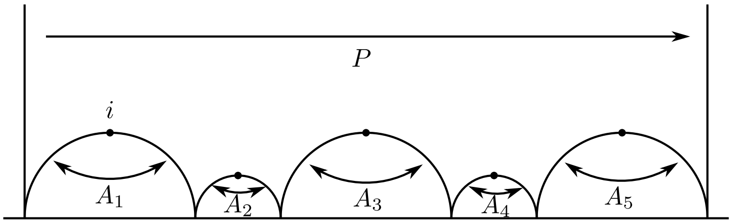

Figure 2.19:

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The transformations $A_1, A_2, \ldots, A_r (r = a + b)$ are elliptic transformations of order 2, because they are conjugates of T in Γ. Any circular arc of the boundary of $\mathfrak{F}^*$ is mapped onto itself by one of these elliptic transformations. The vertical edges of $\mathfrak{F}^*$ are equivalent under $P = U^{2a+b}$. Let Γ^* denote the subgroup of Γ generated by $P, A_1, \ldots, A_r$. We shall show that $\mathfrak{F}^*$ is a fundamental domain for Γ^* and that for a suitable choice of a and b, Γ^* is not a congruence subgroup. From the fact that $\mathfrak{F}^*$ is a fundamental domain for Γ^* (to be proved later), it is obvious that Γ^* has only two inequivalent parabolic cusps, for example ∞ and 1, and the elliptic fixed points of Γ^* are equivalent to i under Γ. Moreover, the widths of the cusp sectors at ∞ and 1 are given by $N_\infty = 2a + b$, $N_1 = a + 2b$ respectively. Using (3), we get $\mu = (\Gamma : \Gamma^*) = N_\infty + N_1 = 3(a + b)$. Let us suppose that Γ^* is a congruence subgroup of level N. Let us assume that $p = a + b$ is a prime number and $a, b \geq 2$. Since $\Gamma \supset \Gamma^* \supset \Gamma[N]$, μ divides $\mu(N)$ and therefore p divides

$$\mu(N) = N^3 \prod_{q|N} (1 - q^{-2}) = N^* \prod_{q|N} (q^2 - 1)(N|N^*|N^3, q \text{ a prime number}).$$

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But N, by theorem 9, is the least common multiple of $N_\infty = p + a$ and $N_1 = p + b$, therefore p does not divide N. Thus p divides $(q^2 - 1)$ for at least one prime q dividing N. Since $p \geq 5$, $q \pm 1$ is even and p divides $\frac{1}{2}(q \pm 1)$. Obviously, q divides either $p + a$ or $p + b$, because q divides N which is the least common multiple of $p + a$ and $p + b$. But $a, b \geq 2$ and $a + b = p$, therefore $q \leq 2p - 2$. This shows that p divides

$\dfrac{1}{2}(q \pm 1)$ which is different from zero and strictly less than p. Hence our supposition that Γ^* is a congruence subgroup is false when $a + b = p$ is a prime number and $a, b \geq 2$. But there are infinitely many pairs of natural numbers (a, b) with the above property; therefore our assertion that there exist infinitely many subgroups of Γ of finite index which are not congruence subgroups is proved.

We shall prove now that $\mathfrak{F}^*$ is a fundamental domain for Γ^*. In order to prove that $\mathfrak{F}^*$ contains atleast one point from each set of equivalent points under Γ^*, it is sufficient to prove that $\mathscr{G} = \bigcup_{\pm S \in \Gamma^*} \mathfrak{F}_S$. Let D' and D'' denote the two triangles into which the imaginary axis splits the fundamental domain $\mathfrak{F}$ for Γ. Let $D_0, D_1, \ldots, D_n$ be a chain of triangles such that D_i is equivalent to D' and D'' under Γ and the triangles D_i and $D_{i+1} (i = 0, 1, \ldots, n-1)$ have an edge in common. Then we shall show by induction on n that D_n is contained in $\mathfrak{F}^*_S$ for some $S \in \Gamma^*$, provided D_0 belongs to some image of $\mathfrak{F}^*$ under Γ^*. Let us assume that
80 the triangle D_t belongs to $\mathfrak{F}^*_{S_t}$ for $S_t \in \Gamma^*$ for some t with $0 \leq t < n$. Then $S_t^{-1} < D_t >$ is contained in $\mathfrak{F}^*$. Since the triangles $S_t^{-1} < D_t >$ and $S_t^{-1} < D_{t+1} >$ have an edge in common, the triangle $S_t^{-1} < D_{t+1} >$ is either contained in $\mathfrak{F}^*$ or in $\mathfrak{F}^*_{R_t}$, where R_t is any one of the boundary substitutions $p^{\pm 1}, A_1, A_2, \ldots A_r$ of $\mathfrak{F}^*$. In any case, D_{t+1} is contained in $\mathfrak{F}^*_{S_{t+1}}$ with $S_{t+1} = S_t$ or $S_t R_t$. Since, by assumption, D_0 belongs to some image of $\mathfrak{F}^*$ under Γ^*, it follows, by induction, that D_n belongs to $\mathfrak{F}^*_{S_n}$ for some S_n belonging to Γ^*. This proves that an arbitrary triangle D equivalent to D' or D'' under Γ is contained in some image of $\mathfrak{F}^*$ under Γ^*, because we can always find a chain $D_0, D_1, \ldots, D_n$ with the above properties and with the additional property that D_0 is contained in $\mathfrak{F}^*$ and $D_n = D$. Our assertion now follows easily.

Denote, in general, the set of interior points of a given point set $\mathfrak{M}$ by ${}^i\mathfrak{M}$. In order to prove that $\mathfrak{F}^*$ is a fundamental domain for Γ^*, it remains to show that ${}^i\mathfrak{F}^* \bigcap {}^i\mathfrak{F}^*_S \neq \emptyset$ for some S in Γ^* implies $S = \pm E$. The domain $\mathfrak{F}^*_{p^k}$ is obviously bounded by two vertical lines and circular arcs say $b_{k\ell}$. We choose our notation in such a way that the arc $b_{k\ell}$ is mapped by $p^k A_\ell P^{-k}(\ell = 1, 2, \ldots, r)$ onto itself. Consider the domain

$$\hat{\mathfrak{F}} = \bigcup_{k=-\infty}^{\infty} \mathfrak{F}^*_{p^k}.$$

It is bounded by the circular arcs $b_{k\ell}(-\infty < k < \infty, \ell = 1, 2, \ldots, r)$. Let $H_{k\ell}$ denote that closed part of the hyperbolic plane which is bounded by $b_{k\ell}$ and which does not contain $\hat{\mathfrak{F}}$. Obviously $H_{k\ell} \bigcap H_{k'\ell'} = \emptyset$ if $(k, \ell) \neq (k', \ell')$. With the help of the relations $A^2_\ell = -E(\ell = 1, 2, \ldots, r)$, 81
any element S of Γ^* can be written in the form

$$S = \pm p^{k_1} A_{\ell_1} P^{k_2} A_{\ell_2} \cdots P^{k_n} A_{\ell_n} P^k,$$

where $k_1, k_2, \ldots, k_n$, k are arbitrary integers. Since $A^2_\ell = -E$, we can assume without loss of generality that if $k_i = 0$, then $\ell_{i-1} \neq \ell_i$ for $i = 2, 3, \ldots, n$. Let $A_{k\ell} = P^k A_\ell P^{-k}$. Then $A^2_{k\ell} = -E$ for all k and ℓ and moreover, S can be written in the form

$$S = \pm A_{t_1\ell_1} A_{t_2\ell_2} \cdots A_{t_n\ell_n} P^t, \tag{8}$$

where $t_1 = k_1, t_2 = k_1 + k_2, \ldots, t_n = k_1 + k_2 + \cdots + k_n$ and $t = k_1 + k_2 + \cdots + k_n + k$. We shall now prove by induction on n that $\mathfrak{F}_{A_{t_1\ell_1} A_{t_2\ell_2} \ldots A_{t_n\ell_n}}$ is contained in $H_{t_1\ell_1}$. For $n = 1$, the assertion is trivial. Let us assume that $n > 1$ and $\hat{\mathfrak{F}}_{A_{t_2\ell_2} A_{t_3\ell_3} \ldots A_{t_n\ell_n}}$ is contained in $H_{t_2\ell_2}$. Since $(t_1, \ell_1) \neq (t_2, \ell_2)$, the substitution $A_{t_1\ell_1}$ maps $H_{t_2\ell_2}$ into $H_{t_1\ell_1}$. Therefore, it follows that $\hat{\mathfrak{F}}_{A_{t_1\ell_1} A_{t_2\ell_2} \ldots A_{t_n\ell_n}}$ is indeed contained in $H_{t_1\ell_1}$. Let ${}^i\mathfrak{F}^* \bigcap {}^i\mathfrak{F}^*_S \neq \emptyset$ where S has the form (8) above. In case $n \geq 1$, we conclude that

$${}^i\mathfrak{F}^* \bigcap {}^i\mathfrak{F}^*_S \subset {}^i\hat{\mathfrak{F}} \bigcap {}^i\hat{\mathfrak{F}}_{A_{t_1\ell_1} A_{t_2\ell_2} \ldots A_{t_n\ell_n}} \subset {}^i\hat{\mathfrak{F}} \bigcap H_{t_1\ell_1} = \emptyset$$

in contradiction with our assumption. Therefore $n = 0$. But then ${}^i\mathfrak{F}^* \bigcap {}^i\mathfrak{F}^*_{p^t} \neq \emptyset$ implying $t = 0$ i.e. $S = \pm E$. Hence $\mathfrak{F}^*$ is a fundamental domain for Γ^*.

3 Excursion into Function Theory

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In this Section, we prove some results involving function theory on Riemann surfaces to be used in the sequel. Until the end of this section, Γ

will denote a horocyclic group, $\mathfrak{F}$ a normal fundamental domain for Γ in $\mathscr{G}$ and $\mathscr{R}$ the Riemann surface associated to Γ.

Theorem 10. *If ω is a meromorphic differential on $\mathscr{R}$, then*

$$\sum_{\mathfrak{g}\in\mathscr{R}} res_{\mathfrak{g}}\omega = 0.$$

Proof. Since ω can have only finitely many poles on $\mathscr{R}$, the sum mentioned in the statement of the theorem is a finite sum. Let $f(\tau)$ denote the meromorphic automorphic form of weight 2 for Γ associated with the differential ω in chapter 1 § 5, so that $\omega = f(\tau)d\tau$. Let $\mathfrak{p}$ be the set consisting of those boundary points of $\mathfrak{F}$ which are either poles of ω or proper or improper vertices of $\mathfrak{F}$. For every point $\tau_0 \in \mathfrak{p}$, we find a hyperbolic disc C_{τ_0} with τ_0 as centre and satisfying the conditions

1. $S < C_{\tau_0} >= C_{\tau_1}$ if $S < \tau_0 >= \tau_1$,
2. $C_{\tau_0} \bigcap C_{\tau_1} = \emptyset$ if $\tau_0 \neq \tau_1$, and
3. ω is regular on the boundary and interior of C_{τ_0} with the possible exception of τ_0.

□

If is obvious that we can find a set of discs C_{τ_0} satisfying conditions 1., 2. and 3. mentioned above. If τ_0 is an improper vertex of $\mathfrak{F}$, i.e. a parabolic cusp of Γ, and lies on the real axis, then for C_{τ_0} we take a disc
83 touching the real axis at τ_0. In particular, if we map τ_0 to ∞, then C_{τ_0} will be mapped into the domain $y \geq c$ for some $c > 0$. Let D denote the domain obtained from $\mathfrak{F}$ by removing the discs C_{τ_0} for $\tau_0 \in \mathfrak{p}$ i.e. $D = \mathfrak{F}\backslash \bigcup_{\tau_0\in\mathfrak{p}} C_{\tau_0}$. Then D is bounded by hyperbolic lines, which are pairwise equivalent, and hyperbolic circular arcs. Let ∂D denote the boundary of D oriented in the positive direction. Then

$$\frac{1}{2\pi i}\int_{\partial D} f(\tau)d\tau = \sum_{\tau_0\in D} \mathrm{res}_{\tau_0} f(\tau)d\tau = \sum_{\mathfrak{g}\in\mathscr{R}^*} \mathrm{res}_{\mathfrak{g}}\omega, \qquad (1)$$

where $\mathscr{R}^*$ is the set obtained from $\mathscr{R}$ by removing the trace-points of all $\tau_0 \in \mathfrak{p}$. If s_1 and s_2 are two equivalent edges of D, then there exists an element A in Γ such that $A < s_1 >= s_2$. Since $f(\tau_A)d\tau_A = f(\tau)d\tau$ and $A < s_1 >$ is oriented in the negative direction, it follows that $\int\limits_{s_1+s_2} f(\tau)d\tau = 0$ and therefore

$$\frac{1}{2\pi i}\int\limits_{\partial D} f(\tau)d\tau = \sum_{\tau_0\in\mathfrak{p}} \frac{1}{2\pi i}\int\limits_{b_{\tau_0}} f(\tau)d\tau, \tag{2}$$

where b_{τ_0} is the circular arc contained in both $\mathfrak{F}$ and the boundary of C_{τ_0}. We decompose $\mathfrak{p}$ into a complete system of equivalent points $\mathfrak{g}_i = \{\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_{k_i}}\}$, $i = 1, 2, \ldots, r$. Then $\mathfrak{g}_i$ can be interpreted as a point of $\mathscr{R}$ and obviously we have

$$\mathscr{R} = \mathscr{R}^* \bigcup \{\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_r\}. \tag{3}$$

Let $\mathfrak{g} = \{\tau_1, \tau_2, \ldots, \tau_k\}$ be any one of the points $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_r$. Then the contribution to the right hand side of (2) from the points for which $\mathfrak{g}$ is given by **84**

$$\sum_{\tau_j\in\mathfrak{g}} \frac{1}{2\pi i}\int\limits_{b_{\tau_j}} f(\tau)d\tau.$$

Since $\mathfrak{g}$ consists of equivalent points, there exist transformations A_j in Γ such that $A_j < \tau_j >= \tau_1$ for $j = 1, 2, \ldots, k$ and there exists a neighbourhood of τ_1 in $\bigcup_{i=1}^k \mathfrak{F}_{A_j}$ consisting of a complete sector of inequivalent points under Γ. We shall now discuss the two cases, namely when $\mathfrak{g}$ is a branch point of $\mathscr{R}$ of finite order or not, separately.

1. Let $\mathfrak{g}$ be a branch point of $\mathscr{R}$ of order $\ell - 1 \geq 0$. Then $t = z^\ell = ((\tau - \tau_1)/(\tau - \overline{\tau}_1))^\ell$ is a local coordinate at $\mathfrak{g}$ and the sector of inequivalent points mentioned above can be described by

   $$|z| \leq \in \text{ and } 0 \leq \arg z < \frac{2\pi}{\ell}.$$

   In terms of the local coordinate t at $\mathfrak{g}$, it is given by $|t| \leq \in^\ell$ but oriented in the negative direction, where $\in$ is a suitable positive real number. Therefore we obtain

$$\begin{aligned}\sum_{\tau_j \in \mathfrak{g}} \frac{1}{2\pi i} \int\limits_{b_{\tau_j}} f(\tau) d\tau &= \sum_{j=1}^{k} \frac{1}{2\pi i} \int\limits_{b_{\tau_j}} f(\tau_{A_j}) d_{\tau_{A_j}} \\ &= \frac{1}{2\pi i} \int\limits_{\substack{|z|=\epsilon \\ 0 \leq \arg z < 2\pi/\ell}} f(\tau) d\tau = -\frac{1}{2\pi i} \oint\limits_{|t|=\epsilon^{\ell}} \omega \\ &= -\mathrm{res}_{\mathfrak{g}} \omega.\end{aligned}$$

2. Let $\mathfrak{g}$ be a logarithmic branch point. Then τ_1 is a parabolic fixed point of Γ. Let A denote a real matrix of determinant 1 which maps τ_1 to ∞. Then $t = e^{2\pi i A < \tau > / \mu}$ with a suitable $\mu > 0$ is a local
85 coordinate at $\mathfrak{g}$. Let $\tau^* = A < \tau >= x^* + iy^*$. The sector of inequivalent points, constituting a neighbourhood of τ_1 in $\bigcup_{j=1}^{k} \mathfrak{F}_{A_j}$ mentioned above, is described by

$$0 \leq x^* < \mu \text{ and } y^* \geq c$$

and in terms of the local coordinate it is described by $|t| \leq e^{-2\pi c/\mu}$, where c is some positive real number. As above, we obtain that

$$\sum_{j=1}^{k} \frac{1}{2\pi i} \int\limits_{b_{\tau_j}} f(\tau) d\tau = -\mathrm{res}_{\mathfrak{g}} \omega.$$

Thus, from (1) and (2), we get

$$\sum_{\mathfrak{g}_0 \in \mathscr{R}^*} \mathrm{res}_{\mathfrak{g}_0} \omega = -\sum_{i=1}^{r} \mathrm{res}_{\mathfrak{g}_i} \omega$$

and because of (3), we have

$$0 = \sum_{\mathfrak{g}_0 \in \mathscr{R}^*} \mathrm{res}_{\mathfrak{g}_0} \omega + \sum_{i=1}^{r} \mathrm{res}_{\mathfrak{g}_i} \omega = \sum_{\mathfrak{g} \in \mathscr{R}} \mathrm{res}_{\mathfrak{g}} \omega,$$

which is the statement of our theorem.

If f is a non-constant meromorphic function on $\mathscr{R}$ and if t is a local coordinate at a point $\mathfrak{g}_0$ of $\mathscr{R}$, then

$$f = \sum_{n=k}^{\infty} c_n t^n$$

where $c_k \neq 0$, if the degree of f at $\mathfrak{g}_0$ is k.

It follows trivially that

$$\frac{df}{f} = \left\{\frac{k}{t} + b_0 + b_1 t + \dots\right\} dt$$

and therefore $\operatorname{res}_{\mathfrak{g}_0} \dfrac{df}{f} = k$. Hence, from theorem 10, we obtain the following 86

Theorem 11. *The degree of a non-constant meromorphic function on $\mathscr{R}$ is zero or equivalently a non-constant meromorphic function has the same number of zeros and poles on $\mathscr{R}$ counted with their multiplicity.*

We call a point $\mathfrak{g}$ of $\mathscr{R}$ a c-place (for a complex number c) of a function f on $\mathscr{R}$ if the function $f - c$ has a zero at $\mathfrak{g}$. Since the functions f and $f - c$ have the same number of poles on $\mathscr{R}$, from theorem 11 it follows that a non-constant meromorphic function takes every value equally often. We shall say that a non-constant meromorphic function on $\mathscr{R}$ *is of order n* if it takes every value n times. Consequently, we have

Theorem 12. *A regular function on $\mathscr{R}$ which has no poles is a constant.*

4 The Elliptic Modular Functions

It is well-known that the field of elliptic functions (i.e. doubly periodic meromorphic functions, say with primitive periods ω, ω' for which $\tau = \dfrac{\omega'}{\omega}$ has positive imaginary part) consists precisely of all the rational functions of $\mathscr{P}$ and $\mathscr{P}'$ with complex coefficients where is Weierstrass' function given by

$$\mathscr{P}(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left\{\frac{1}{(z - m\omega' - n)^2} - \frac{1}{(m\omega' + n\omega)^2}\right\}$$

The elliptic function $\mathscr{P}(z)$ satisfies the differential equation

$$\begin{aligned}(\mathscr{P}'(z))^2 &= 4(\mathscr{P}(z))^3 - g_2\mathscr{P}(z) - g_3\\ &= 4(\mathscr{P}(z) - e_1)(\mathscr{P}(z) - e_2)(\mathscr{P}(z) - e_3)\end{aligned}$$

87 with $e_1 = \mathscr{P}(\omega'/2)$, $e_2 = \mathscr{P}(\omega/2)$, $e_3 = \mathscr{P}((\omega + \omega')/2)$, $g_2 = 60G_4(\omega', \omega)$ and $g_3 = 140G_6(\omega', \omega)$ where, in general,

$$G_k(\omega', \omega) = \sum_{(m,n)\neq(0,0)} (m\omega' + n\omega)^{-k}.$$

The series $G_k(\omega', \omega)$ converges absolutely for $k > 2$ and obviously vanishes identically when k is an odd integer. The elliptic function $\mathscr{P}(z)$ is of order 2, because it has exactly one pole of order 2 in a period parallelogram. Therefore $\mathscr{P}(z)$ takes every value twice in a period parallelogram. Since $\mathscr{P}'(\omega'/2) = 0, \mathscr{P}(z)$ takes the value e_1 at $\omega'/2$ twice and therefore e_1 is different from e_2 and e_3. Similarly, it is proved that $e_2 \neq e_3$. This shows that the cubic equation

$$4t^3 - g_2t - g_3 = 0$$

has distinct roots, which implies that its discriminant

$$\Delta_0(\omega', \omega) = 16(e_1 - e_2)^2(e_2 - e_3)(e_1 - e_3)^2 = g_2^3 - 27g_3^2 \neq 0.$$

It is obvious that the functions $G_k(\omega', \omega)$ and $\Delta_0(\omega', \omega)$ are homogeneous functions of ω', ω and therefore

$$G_k(\omega', \omega) = \omega^{-k}G_k(\tau), \Delta_0(\omega', \omega) = \omega^{-12}\Delta_0(\tau),$$

where $\tau = \omega'/\omega$ belongs to $\mathscr{G}$. Consequently, we have

$$\Delta_0(\tau) = 2^4 \cdot 3^3 \cdot 5^2\{2^2 \cdot 5G_4^3(\tau) - 7^2G_6^2(T)\}. \tag{1}$$

In what follows, Γ will denote the modular group and $\mathfrak{F}$ the fundamental domain of Γ given in (1) of §1. The transformations, which map the pair of primitive periods ω' and ω to another pair of primitive periods, generating the same lattice as ω' and ω are given by

$$\omega' \longrightarrow a\omega' + b\omega, \omega \longrightarrow c\omega' + d\omega,$$

88 where $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to Γ if the sign of $Im(\omega'/\omega)$ is preserved; or equivalently by

$$\tau \longrightarrow S < \tau > \text{ and } \omega \longrightarrow \omega(c\tau + d).$$

Obviously these transformations leave the functions $G_k(\omega', \omega)$ and $\Delta_0(\omega', \omega)$ invariant; therefore, the effect of these transformations on $G_k(\tau)$ and $\Delta_0(\tau)$ is given by

$$(c\tau + d)^{-k} G_k(S < \tau >) = G_k(\tau), (c\tau + d)^{-12} \Delta_0(S < \tau >) = \Delta_0(\tau).$$

As we shall see, the functions $G_k(\tau)$ and $\Delta_0(\tau)$ are meromorphic modular forms of weight k and 12 respectively for the group Γ. By a *meromorphic modular form of weight k (k a natural number) for the group* Γ we mean a function $f(\tau)$ satisfying the following conditions:

1. $f(\tau)$ is a meromorphic function in $\mathscr{G}$,
2. $(c\tau + d)^{-k} f(S < \tau >) = f(\tau)$ for every $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ, and
3. at the parabolic cusp ∞ of Γ, $f(\tau)$ has the Fourier expansion

$$f(\tau) = \sum_{n=k}^{\infty} c_n e^{2\pi i n\tau}$$

with only finitely many negative exponents.

We call $f(\tau)$ an *integral modular form* or simply a modular form of weight k for Γ if $f(\tau)$ is regular in $\mathscr{G}$ and if, in the Fourier expansion of $f(\tau)$ at the cusp ∞, no term with negative exponent occurs. If $f(\tau)$ is a modular form and the constant term in the Fourier expansion of $f(\tau)$ 89
at the parabolic cusp vanishes, then we call $f(\tau)$ *a cusp form*. We shall now show that $G_k(\tau)$ is a modular form of weight k and $\Delta_0(\tau)$ is a cusp form of weight 12 for the modular group Γ. The uniform convergence of the so-called *Eisenstein series* $G_k(\tau) = \sum_{(m,n)\neq(0,0)} (m\tau + n)^{-k}$, can be deduced from

Lemma 3. *Let* (c, d) *be two real numbers and let* $y \geq \in > 0$, $|x| \leq \ell$. *Then there exists a number* $\delta = \delta(\ell, \in) > 0$ *such that*

$$|c\tau + d| \geq \delta |ci + d| \quad (\tau = x + iy).$$

Proof. If $c = d = 0$, we have nothing to prove. Therefore we assume that at least one of c or d is non-zero. We can obviously assume further that $c^2 + d^2 = 1$, for homogeneity. Now the proof takes an indirect course. Assume there exist sequences $\{c_n\}$, $\{d_n\}$ and $\{\tau_n = x_n + iy_n\}$ such that $c_n^2 + d_n^2 = 1$, $|x_n| \leq \ell$, $y_n \geq \in > 0$ and $\lim_{n\to\infty} |c_n\tau_n + d_n| = 0$. This shows that

$$(c_n x_n + d_n)^2 + c_n^2 y_n^2 \to 0 \text{ as } n \to \infty,$$

and so $|c_n y_n| \to 0$. Thus $y_n \geqq \in$ implies $c_n \to 0$ and now $d_n \to 0$ as a consequence of $|c_n x_n + d_n| \to 0$, $|x_n| \leqq \ell$ in contradiction with $c_n^2 + d_n^2 = 1$. This proves the lemma. $\square$

Since the integral

$$\iint\limits_{x^2+y^2\geq 1} (x^2 + y^2)^{-k/2} dxdy = \int\limits_1^\infty \int\limits_0^{2\pi} \frac{drd\theta}{r^{k-1}} = \frac{2\pi}{k-2}$$

converges for $k > 2$, it follows that the series $G_k(i)$ is convergent for $k > 2$. Therefore by using the above lemma, we see that the series
90 $G_k(\tau)$, for $k > 2$, converges absolutely and uniformly in the domain $|x| \leq \ell, y \geq \in > 0$. This proves that $G_k(\tau)$ is a regular function in $\mathscr{G}$. Moreover, $G_k(\tau + 1) = G_k(\tau)$ and $G_k(\tau)$ has Fourier expansion

$$G_k(\tau) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n\tau}.$$

In order to calculate the coefficients c_n, we consider the function

$$f(\tau) = \sum_{n=-\infty}^{\infty} (\tau + n)^{-k},$$

which obviously is regular in $\mathscr{G}$. Since $f(\tau + 1) = f(\tau)$, we have

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau},$$

where
$$a_n = \int_0^1 f(\tau) e^{-2\pi i n\tau} dx$$
$$= \sum_{m=-\infty}^{\infty} \int_0^1 (\tau + m)^{-k} e^{-2\pi i n\tau} dx$$
$$= e^{2\pi n y} \int_{-\infty}^{\infty} (x + iy)^{-k} e^{-2\pi i n x} dx.$$

By deforming the path of integration suitably in the z-place (with Re $z = x$), it can be proved easily that

$$a_n = \begin{cases} 0 & \text{for } n < 0 \\ -2\pi i & e^{2\pi n y} \operatorname{res}_{z=-iy} (z + iy)^{-k} e^{-2\pi i n z} \text{ for } n \geq 0 \end{cases}$$

i.e. 91

$$a_n = \begin{cases} 0 \text{ for } n < 0 \\ \frac{(-2\pi i)^k}{(k-1)!} n^{k-1} \text{ for } n \geq 0. \end{cases}$$

Thus we have

$$f(\tau) = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n \tau}.$$

We now consider the series $G_k(\tau)$ only for even integral k, as we know already that $G_k(\tau) \equiv 0$ for odd integers k. By definition,

$$G_k(\tau) = \sum_{(m,n)\neq(0,0)} (m\tau + n)^{-k} = 2\sum_{n=1}^{\infty} n^{-k} + 2\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} (m\tau + n)^{-k}$$
$$= 2\sum_{n=1}^{\infty} n^{-k} + 2\sum_{m=1}^{\infty} f(m\tau),$$

because $m\tau$ belongs to $\mathscr{G}$ with τ. If $\zeta(s)$ denotes the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} (\text{for Re } s > 1)$$

then

$$\begin{aligned} G_k(\tau) &= 2\zeta(k) + 2\sum_{m=1}^{\infty} f(m\tau) \\ &= 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1}e^{2\pi i n m\tau}. \end{aligned}$$

92 Collecting terms for which $\ell = mn$, we obtain

$$\begin{aligned} G_k(\tau) &= 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!}\sum_{\ell=1}^{\infty}\left(\sum_{\substack{d|\ell \\ d>0}}^{\infty} d^{k-1}\right)e^{2\pi i\ell\tau} \\ &= 2\pi(k) + 2\frac{(-2\pi i)}{(k-1)!}\sum_{n=1}^{\infty} d_{k-1}(n)e^{2\pi i n\tau}, \end{aligned}$$

where $d_k(n) = \sum_{\substack{d|n \\ d>0}} d^k$.

Hence our assertion that $G_k(\tau)$ for $k > 2$ is a modular form of weight k for Γ is proved.

The values of the Riemann zeta function $\zeta(s)$ for $s = k(k$ an even integer) are given by the formula

$$\zeta(k) = (-1)^{\frac{k}{2}-1}\frac{(2\pi)^k B_k}{2\cdot k!},$$

where B_k denotes the k-th Bernoulli number. The complete sequence $B_1, B_2, \ldots$ of Bernoulli number is given by the formal equations

$$(B+1)^n - B^n = 0 \quad (n > 1)$$

where, in the binomial expansion on the left hand side, B^k is to be replaced by B_k. The proof of the above formula can be found in [6], where the Bernoulli numbers are also calculated. We mention here the values of some B_k:

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{-1}{30},$$

$$B_{10} = \frac{5}{66}, \quad B_{12} = \frac{-691}{2730}, \quad B_{14} = \frac{7}{6}.$$

93 Using the above formula for the values of $\zeta(k)$, we obtain that

$$G_k(\tau) = (-1)^{\frac{k}{2}-1} \frac{(2\pi)^k B_k}{k!} \left(1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} d_{k-1}(n) e^{2\pi i n\tau}\right). \tag{2}$$

Since the product of two modular forms of weight k_1 and k_2 is a modular form of weight (k_1+k_2), it follows by (i) that $\Delta_0(\tau)$ is a modular form of weight 12 for Γ. Let $t = e^{2\pi i\tau}$. Then from (2) we get

$$G_4(\tau) \equiv \frac{\pi^4}{3^2 \cdot 5}(1 + 2^4 \cdot 3 \cdot 5t)(\text{mod } t^2)$$

and

$$G_6(\tau) \equiv \frac{2\pi^6}{3^5 \cdot 5 \cdot 7}(1 - 2^3 \cdot 3^2 \cdot 7)(\text{mod } t^2).$$

Therefore equation (1) shows that

$$\Delta_0(\tau) \equiv (2\pi)^{12} t (\text{mod } t^2)$$

or

$$\Delta_0(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} c_n e^{2\pi i n\tau}$$

with suitable numbers c_n and in particular $c_1 = 1$. Hence $\Delta_0(\tau)$ is a cusp form of weight 12 for Γ. Moreover, this property determines $\Delta_0(\tau)$ uniquely upto a constant factor. For if $f(\tau)$ is a cusp form of weight 12 for Γ, then the function $f(\tau)/\Delta_0(\tau)$ which is invariant under Γ and which has no singularities in $\mathscr{G}$ (due to the non-vanishing of Δ_0 in $\mathscr{G}$) is constant, by theorem 12. We shall now show that the normed modular form

$$\Delta(\tau) = (2\pi)^{-12} \Delta_0(\tau)$$

has integral coefficients in its Fourier expansion at the parabolic cusp. 94
Let $\mathfrak{M}$ be the module of power series in $t = e^{2\pi i\tau}$ with integral coefficients. Let us set

$$P(t) = \sum_{n=1}^{\infty} d_3(n) t^n, Q(t) = \sum_{n=1}^{\infty} d_5(n) t^n.$$

Then by (1) and (2),

$$\Delta(\tau) = (2\pi)^{-12}\Delta_0(\tau) = 2^{-6}\cdot 3^{-3}\{(1+2^4\cdot 3\cdot 5P(t))^3-(1-2^3\cdot 3^2\cdot 2Q(t))^2\}.$$

But $P(t) \equiv Q(t) \equiv 0(\text{mod } \mathfrak{M})$; therefore

$$\Delta(\tau) \equiv \frac{1}{12}\{5P(t) + 7Q(t)\}(\text{mod } \mathfrak{M}).$$

Thus $\Delta(\tau)$ has all the coefficients in its Fourier expansion at the parabolic cusp integral if and only if

$$5d_3(n) + 7d_5(n) \equiv 0(\text{mod } 12) \text{ for every } n \geq 1.$$

But all these congruences evidently hold since $5d^3 + 7d^5 \equiv 5d^3(1 - d^2) \equiv 0(\text{mod } 12)$ for every integer d.

As an object of special importance in the field of meromorphic functions on the Riemann surface $\mathscr{R}$ associated to Γ, we introduce the function

$$J(\tau) = g_2^3/(g_2^3 - 27g_3^2) = 2^6\cdot 3^3\cdot 5^3 G_4^3(\tau)/\Delta_0(\tau).$$

Since $J(\tau)$ is a quotient of two modular forms of the same weight for Γ, it is invariant under the group Γ. Thus $J(\tau)$ is *an elliptic modular function* i.e. a meromorphic function on $\overline{\mathscr{G}}/\Gamma$ where $\overline{\mathscr{G}}$ arises from $\mathscr{G}$ on adding the (set of) parabolic fixed points of Γ (i.e the set of rational numbers). Clearly $J(\tau)$ has a simple pole at the parabolic cusp ∞ of Γ, since

$$J(\tau) \equiv 1/(1728t)(\text{mod } t^0).$$

95 Since $\Delta_0(\tau) \neq 0$ for τ in $\mathscr{G}$, $J(\tau)$ is regular in $\mathscr{G}$ and consequently takes every value exactly once on the Riemann surface $\mathscr{R}$ attached to the group Γ. We now show that $J(i) = 1$ and $J(\rho) = 0$ with $\rho = e^{2\pi i/3}$. Since $\rho^3 = 1$ and $\rho^2 + \rho + 1 = 0$, we have

$$\begin{aligned} G_4(\rho) = \rho \sum_{(m,n)\neq(0,0)} (m\rho^2 + n\rho)^{-4} &= \rho \sum_{(m,n)\neq(0,0)} ((m-n)\rho + m)^{-4} \\ &= \rho G_4(\rho). \end{aligned}$$

Similarly

$$G_6(i) = \sum_{(m,n)\neq(0,0)} (mi+n)^{-6} = i^{-6} \sum_{(m,n)\neq(0,0)} (m-in)^{-6} = -G_6(i).$$

Thus $G_4(\rho) = G_6(i) = 0$, showing that $J(i) = 1$ and $J(\rho) = 0$. Using the fact that $\Delta(\tau)$ has integral coefficients in the Fourier series at ∞, it can be proved easily that

$$1728J(\tau) = \frac{1}{t} + a_0 + a_1 t + \cdots \quad t = e^{2\pi i\tau},$$

where a_n are integers.

Theorem 13. *A meromorphic modular function $f(\tau)$ for the modular group Γ is a rational function of $J(\tau)$ with complex coefficients. If, in addition, $f(\tau)$ is regular in $\mathscr{G}$, then $f(\tau)$ is a polynomial in $J(\tau)$.*

Proof. The function $J(\tau)$ maps the Riemann surface $\mathscr{R}$ associated to the group Γ onto the J-sphere and this correspondence between $\mathscr{R}$ and the J-sphere is one-one. Moreover, if $\tau_0 \in \mathscr{G}$, then obviously $J(\tau) - J(\tau_0)$ is a local coordinate at the trace point of τ_0 and if $\tau_0 = \infty$, then $1/J(\tau)$ is a local coordinate at the logarithmic branch point of $\mathscr{R}$. Therefore if **96**
$f(\tau)$ is a meromorphic modular function for Γ, then $f(\tau) = g(J(\tau))$ is a meromorphic function on the J-sphere and therefore a rational function of $J(\tau)$ with complex coefficients. If $f(\tau)$ is regular in $\mathscr{G}$, then the only possible pole of $f(\tau)$ is at $\tau = \infty$ and therefore $g(J)$ can have only one pole at infinity on the J-sphere. Hence $f(\tau) = g(J)$ must be a polynomial in $J(\tau)$. □

In the following, we shall denote by $[\Gamma, k]$ the linear space (over the complex number field) of (integral) modular forms of weight $k(k > 0, k$ integral) for the group Γ. We shall calculate the dimension of this (linear) space and indeed find a basis of $[\Gamma, k]$ which consists of power-products of $G_4(\tau)$ and $G_6(\tau)$. If k is an odd integer, then the dimension of $[\Gamma, k]$ is zero, since any $f(\tau)$ in $[\Gamma, k]$ vanishes identically as evident from $(c\tau+d)^{-k} f(S<\tau>) = f(\tau)$ for $S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In order to prove that the dimension of $[\Gamma, 2]$ is zero, we proceed as follows. Let us suppose that $f(\tau)$ belongs to $[\Gamma, 2]$. Consider the function

$$g(\tau) = \int_i^\tau f(\tau) d\tau.$$

Since $\mathscr{G}$ is simply connected, $g(\tau)$ does not depend upon the path of integration between i and τ. It is obvious that

$$g(S < \tau >) = g(\tau) + C_S \text{ for } S \in \Gamma,$$

where C_S is a constant depending on S. Taking S to be an elliptic transformation and τ its fixed point, we see immediately that $C_S = 0$ and in particular $C_T = C_V = 0$. But T and V generate Γ and $C_{RS} = C_R + C_S$
97 for R and S belonging to Γ; therefore $C_S = 0$ for all S in Γ. In particular, $C(U) = 0$, which implies that $g(\tau + 1) = g(\tau)$, forcing the constant term in the Fourier series of $f(\tau)$ at ∞ to be zero and showing that $g(\tau)$ is regular at ∞. Now the function $g(\tau)$, which is invariant under Γ and which is regular in $\mathscr{G}$ and at ∞, must be constant, by theorem 12. Hence $f(\tau) \equiv 0$, which proves that the dimension of $[\Gamma, 2]$ is zero.

Let us now consider $[\Gamma, k]$ for even integral $k > 2$. For any f in $[\Gamma, k]$, we first find the expansion at a point τ_0 in $\mathscr{G}$ of ramification index $\ell - 1 \geq 0$. Since the function $J'(\tau)$ is a meromorphic modular form of weight 2 for Γ, according to Chapter 1 § 5, we obtain that

$$(\tau - \overline{\tau}_0)^2 J'(\tau) = \sum_n a_n t^{n-1/\ell} \text{ with } t = ((\tau - \tau_0)/(\tau - \overline{\tau}_0))^\ell.$$

Consider the function $f(\tau)(J'(\tau))^{1-k/2}$. It is a meromorphic modular form of weight 2 for Γ; therefore, as above,

$$(\tau - \overline{\tau}_0)^2 f(\tau)(J'(\tau))^{1-k/2} = \sum_n b_n t^{n-1/\ell},$$

which implies that

$$(\tau - \overline{\tau}_0)^k f(\tau) = \{\sum_n b_n t^{n-1/\ell}\}\{\sum_n a_n t^{n-1/\ell}\}^{(k/2)-1}$$

$$= \sum_n c_n t^{n-k/2\ell}.$$

Since $f(\tau)$ is regular at τ_0, we must have $n \geq k/2\ell$ in the above expansion of $f(\tau)$ at the point τ_0. As the functions $f(\tau)J'(\tau)^{1-k/2}$ and $J'(\tau)$ have a well-defined degree at the trace-point of τ_0, the modular form $f(\tau)$ also has a well-defined degree (possibly fractional) at τ_0 measured in terms of a local coordinate at τ_0. We call τ_0 an *'unavoidable zero'* of $f(\tau)$, if $k/2\ell$ is not an integer. If $v(\tau_0)$ denote the multiplicity of the zero at the point τ_0 in $\mathscr{G}$, then for $\tau_0 = i$ (respectively ρ) with $\ell = 2$ (respectively 3), we have **98**

$$v(i) \geq \frac{1}{2} \text{ if } k \equiv 2 (\text{mod } 4)$$

$$v(\rho) \geq \begin{cases} \frac{1}{3} \text{ if } k \equiv -2 (\text{mod } 6) \\ \frac{2}{3} \text{ if } k \equiv 2 (\text{mod } 6). \end{cases}$$

The sum of the multiplicities of all zeros of $f(\tau)$ in the fundamental domain $\mathfrak{F}$ of Γ i.e. the degree $v(f)$ is $k/12$. For, the function $(f(\tau))^{12}/(\Delta(\tau))^k$ is a modular function for the modular group and therefore has as many zeros as poles, so that $12v(f) = k$. In particular, it follows that the degree of $G_4(\tau)$ is $\dfrac{1}{3}$. But we have proved that $v(\rho) \geq \dfrac{1}{3}$; therefore $G_4(\tau)$ does not vanish at any point of $\mathscr{G}$ inequivalent to ρ which is a zero of multiplicity $\dfrac{1}{3}$. Similarly, it follows that $G_6(\tau)$ has a zero of multiplicity $\dfrac{1}{2}$ at i and has no zero inequivalent to i. Let $\dfrac{k}{12} = g + \dfrac{a}{3} + \dfrac{b}{2}$ where $g \geq 0$, $0 \leq a < 3$, $0 \leq b < 2$ and g, a, b are integers uniquely determined by k. Then $k \equiv 2b (\text{mod } 4)$ and $k \equiv -2a (\text{mod } 6)$. Therefore $f(\tau)$ has unavoidable zeros of order $\dfrac{b}{2}$ at i and $\dfrac{a}{3}$ at ρ. From the above discussion, it follows that $h(\tau) = f(\tau)G_4^{-a}(\tau)G_6^{-b}(\tau)$ belongs to $[\Gamma, 12g]$ and $h(\tau)\Delta^{-g}(\tau)$ is a modular function invariant under Γ, which has a pole of multiplicity at most g at the parabolic cusp ∞ of Γ and no other singularity. Thus by theorem 13,

$$\frac{h(\tau)}{\Delta^g(\tau)} = a_0 + a_1 J(\tau) + \cdots + a_g (J(\tau))^g \implies$$

$$h(\tau) = \{a_0 + a_1 J(\tau) + \cdots + a_g (J(\tau))^g\} \Delta^g(\tau).$$

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Conversely, $\{a_0 + a_1 J(\tau) + \cdots + a_g (J(\tau))^g\} \Delta^g(\tau)$ for all choices of complex numbers $a_0, a_1, \ldots, a_g$ belongs to $[\Gamma, 12g]$. This shows that the dimension of the space $[\Gamma, 12g]$ is equal to $g+1$. But the mapping $f(\tau) \to h(\tau)$ from $[\Gamma, k]$ to $[\Gamma, 12g]$ is linear and one-one, therefore we obtain that the dimension of $[\Gamma, k]$ is also $g+1$. Since $J^r(\tau)\Delta^g(\tau)$ for $0 \leq r \leq g$ coincides but for a constant factor with $G_4^{3r}(\tau)\ \{20G_4^3(\tau) - 49G_6^2(\tau)\}^{g-r}$, it follows that

$$f(\tau) = \{a_0 + a_1 J(\tau) + \cdots + a_g J(\tau)^g\} \Delta^g(\tau) G_4^a(\tau) G_6^b(\tau)$$

can be written as a sum of power-products of $G_4(\tau)$ and $G_6(\tau)$ i.e.

$$f(\tau) = \sum_{p,q} c_{pq} G_4^p(\tau) G_6^q(\tau)$$

where c_{pq} are complex numbers and the sum is taken over all integral $p, q \geq 0$ with $4p + 6q = k$. But $\dfrac{k}{12} = g + \dfrac{a}{3} + \dfrac{b}{2}$; therefore $\dfrac{k}{12} = \dfrac{p}{3} + \dfrac{q}{2}$, which implies that $p = a + 3r$, $q = b + 2s$ with integral $r, s \geq 0$ and consequently $r + s = g$. This shows that there are $g+1$ solutions of the equation $4p + 6q = k$. Hence the products $G_4^p(\tau)G_6^q(\tau)$ with $4p + 6q = k$ form a basis for $[\Gamma, k]$. Let $[x]$ denote the greatest integer $\leq x$. Then, obviously $\left[\dfrac{k}{12}\right] = g$, except when $a = 2$ and $b = 1$ i.e. when $k \equiv$
100 2(mod 12) and, in that case, $\left[\dfrac{k}{12}\right] = g + 1$. Hence we obtain that

$$\text{dimension of } [\Gamma, k] = \begin{cases} \left[\frac{k}{12}\right] + 1, & \text{if } k \not\equiv 2 \pmod{12} \\ \left[\frac{k}{12}\right], & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$

We summarise that results proved above in the following

Theorem 14. *The power products $G_4^p(\tau)G_6^q(\tau)$, where p, q are non-negative integers with $4p + 6q = k$ form a basis of the space $[\Gamma, k]$ for* *any non-negative even integer k. The dimension of the space $[\Gamma, k]$ for such k is given by*

$$\left[\frac{k}{12}\right] + 1, \ \textit{for } k \not\equiv 2 (\textit{mod } 12)$$

$$\textit{and} \qquad \left[\frac{k}{12}\right], \qquad \textit{for } k \equiv 2 (\textit{mod } 12).$$

In particular, $[\Gamma, k]$ has dimension 1 for $k = 4, 6, 8, 10$. Therefore $G_8(\tau) = cG_4^2(\tau)$ with some constant $c \neq 0$. By considering the normalized Eisenstein series i.e. a suitable constant multiple of $G_k(\tau)$ and having 1 as the constant term in the Fourier expansion at the parabolic cusp for Γ, we obtain

$$1 + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} d_7(n) e^{2\pi i n\tau} = \{1 + 2^4 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} d_3(n) e^{2\pi i n\tau}\}^2.$$

Comparing the coefficients on both sides, we obtain with the help of $d_k(mn) = d_k(m) d_k(n)$ for $(m, n) = 1$ the following interesting relation for $n = p$, a prime number,

$$p^3(p^4 - 1) = 2^3 \cdot 3 \cdot 5 \cdot \sum_{\substack{a+b=p \\ a,b \geq k}} d_3(ab),$$

which is not true in general when n is not a prime number. **101**

The Eisenstein series $G_2(\tau) = \sum\limits_{(m,n)\neq(0,0)} (m\tau + n)^{-2}$ is not absolutely convergent, because otherwise it would represent a non-vanishing modular form of weight 2 for Γ, which contradicts the fact that dimension of $[\Gamma, 2]$ is zero. But it can be proved that this series is conditionally convergent. By using the transformation properties of this series, Hurwitz constructed a modular form of weight 12 for Γ in the form of an infinite product, which vanishes at the parabolic cusp of Γ and therefore coincides with the modular form $\Delta(\tau)$. In the following, we shall prove this fact by using a method of Hecke. Consider the series

$$G_2(\tau, s) = \sum_{(m,n)\neq(0,0)} (m\tau + n)^{-2} |m\tau + n|^{-s},$$

where s is a complex number. It is obvious that the series is absolutely convergent for $Re(s) > 0$. Further, for $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ, we have

$$G_2(S < \tau >, s)(c\tau + d)^{-2}|c\tau + d|^{-s} = G_2(\tau, s).$$

In order to prove that the series $G_2(\tau, s)$ can be continued analytically to the left of the half-plane $Re(s) > 0$, we find the Fourier expansion of $G_2(\tau, s)$. Let

$$f(\tau, s) = \int_{n=-\infty}^{\infty} (\tau + n)^{-2}|\tau + n|^{-s}$$

and

$$\varphi(\tau, u) = f(\tau + u, s),$$

102 where u is a real variable and τ belongs to $\mathscr{G}$. Obviously $\varphi(\tau, u)$ is a periodic function of u and we can write

$$\varphi(\tau, u) = \sum_{r=-\infty}^{\infty} a_r(\tau, s)e^{2\pi iru},$$

where the Fourier coefficients $a_r(\tau, s)$ are given by

$$\begin{aligned} a_r(\tau, s) &= \int_0^1 \varphi(\tau, u)e^{-2\pi iru} du. \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 (\tau + u + n)^{-2}|\tau + u + n|^{-s}e^{-2\pi iru} du \\ &= \int_{-\infty}^{\infty} (\tau + u)^{-2}|\tau + u|^{-s}e^{-2\pi iru} du. \end{aligned}$$

Rearranging the series for $G_2(\tau, s)$, we obtain that

$$G_2(\tau, s) = 2\zeta(2 + s) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} (m\tau + n)^{-2}|m\tau + n|^{-s}$$

$$= 2\zeta(2+s) + 2\sum_{m=1}^{\infty} f(m\tau, s),$$

because with τ, $m\tau$ also belongs to $\mathscr{G}$ for $m \geq 1$. But $f(m\tau, s) = \varphi(m\tau, 0) = \sum\limits_{r=-\infty}^{\infty} a_r(m\tau, s)$ and from the integral representation of $a_r(m\tau, s)$ it follows that $a_r(m\tau, s) = m^{-1-s} a_{rm}(\tau, s)$; therefore

$$G_2(\tau, s) = 2\zeta(2+s) + 2\sum_{m=1}^{\infty} m^{-1-s} \sum_{r=-\infty}^{\infty} a_{rm}(\tau, s)$$

$$= 2\zeta(2+s) + 2\zeta(1+s) a_0(\tau, s) + 2\sum_{n\neq 0} \left\{ \sum_{\substack{m|n\\ m>0}} m^{-1-s} \right\} a_n(\tau, s).$$

In order to calculate $a_n(\tau, s)$, we proceed as follows. With the help of the substitution $u^2 + 1 = v^{-1}$, we obtain that 103

$$\gamma(\alpha) = \int_0^\infty \frac{du}{(u^2+1)^\alpha} = \frac{1}{2}\int_0^1 v^{\alpha - \frac{3}{2}} (1-v)^{-\frac{1}{2}} dv = \frac{1}{2} \frac{\Gamma\left(\alpha - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha)}.$$

Since $a_0(\tau, s)$ does not depend upon the real part of τ, we have

$$a_0(\tau, s) = \int_{-\infty}^{\infty} (iy+u)^{-2} |(iy+u)|^{-s} du = y^{-1-s} \int_{-\infty}^{\infty} (i+u)^{-2} (u^2+1)^{-\frac{s}{2}} du$$

$$= y^{-1-s} \int_{-\infty}^{\infty} \frac{u^2 - 1}{(u^2+1)^{2+\frac{s}{2}}} du = \frac{2}{y^{1+s}} \{\gamma(1 + \frac{s}{2}) - 2\gamma(2 + \frac{s}{2})\}$$

$$= \frac{1}{y^{1+s}} \left\{ \frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{s}{2}+1)} - 2\frac{\Gamma(\frac{s+3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{s}{2}+2)} \right\}$$

$$= \frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{1}{2})}{y^{1+s}\Gamma(\frac{s}{2}+1)} \left(1 - 2\frac{s+1}{s+2}\right) = \frac{-s\Gamma(\frac{s+1}{2})\Gamma(\frac{1}{2})}{2y^{1+s}\Gamma(\frac{s}{2}+2)}.$$

This shows that $\zeta(1+s)\, a_0(\tau, s)$ is regular in the half plane $Res > -1$, because $s\zeta(s+1) \to 1$ as $s \to 0$. Moreover

$$\{\zeta(1+s) a_0(\tau, s)\}_{s=0} = -\frac{\pi}{2y}.$$

For $n \neq 0$,

$$a_n(\tau, s) = \int\limits_{-\infty}^{\infty} (\tau + u)^{-2-\frac{s}{2}} (\overline{\tau} + u)^{-\frac{s}{2}} e^{-2\pi i n u} du.$$

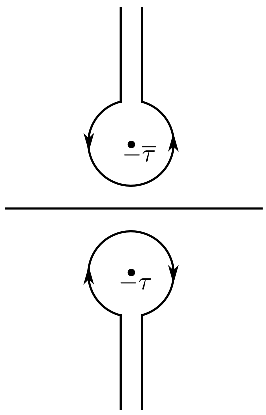
We choose the branches of the multiple-valued functions in the integrand as follows:

$$(\tau + u)^{-\alpha} = e^{-\alpha \log(\tau+u)}, (\overline{\tau} + u)^{-\beta} = e^{-\beta \log(\overline{\tau}+u)},$$

104 where

$$\log(\tau + u) = \log|\tau + u| + i \arg(\tau + u), \frac{-\pi}{2} \leq \arg(\tau + u) < \frac{3\pi}{2},$$
$$\log(\overline{\tau} + u) = \log|\overline{\tau} + u| + i \arg(\overline{\tau} + u), \frac{-3\pi}{2} < \arg(\overline{\tau} + u) \leq \frac{\pi}{2},$$

so that, in case u is real, $\arg(\tau + u) + \arg(\overline{\tau} + u) = 0$. We denote in the u-plane by C_1 the contour described by $Re(\tau + u) = 0$, $Im(\tau + u) \leq -\dfrac{3}{2}y$ and the circle $|\tau+u| = \dfrac{1}{2}y$ with the negative orientation. We denote by C_2 the reflection of C_1 on the axis $Imu = 0$. Under the assumption $Res > 0$, the integral representing $a_n(\tau, s)$ can be converted into a contour integral over C_1 or C_2 according as $n > 0$ or $n < 0$.



We decompose $a_n(\tau, s)$ into two parts $b_n(\tau, s)$ and $c_n(\tau, s)$ such that $a_n(\tau, s) = b_n(\tau, s) + c_n(\tau, s)$ with

$$b_n(\tau, s) = \begin{cases} \oint\limits_{|\tau+u|=\frac{1}{2}y} (\tau + u)^{-2-\frac{s}{2}}(\overline{\tau} + u)^{-\frac{s}{2}} e^{-2\pi inu} du \text{ for } n > 0 \\ \oint\limits_{|\overline{\tau}+u|=\frac{1}{2}y} (\tau + u)^{-2-\frac{s}{2}}(\overline{\tau} + u)^{-\frac{s}{2}} e^{-2\pi inu} du \text{ for } n < 0 \end{cases}$$

and

$$c_n(\tau, s) = \begin{cases} -2\sin\frac{\pi s}{2} e^{2\pi in\tau} \int\limits_{\frac{1}{2}y}^{\infty} t^{-2-\frac{s}{2}}(t + 2y)^{-\frac{s}{2}} e^{-2\pi nt} dt \text{ for } n > 0 \\ -2\sin\frac{\pi s}{2} e^{2\pi in\overline{\tau}} \int\limits_{\frac{1}{2}y}^{\infty} (t + 2y)^{-2-\frac{s}{2}} t^{-\frac{s}{2}} e^{2\pi nt} dt \text{ for } n < 0. \end{cases}$$

In any case, $b_n(\tau, s)$ and $c_n(\tau, s)$ are entire functions of s. Moreover, given a compact set K in the s-plane, there exists a positive constant $C = C(y, K)$ such that **105**

$$|b_n(\tau, s)| < Ce^{-\pi|n|y}, |c_n(\tau, s)| < Ce^{-2\pi|n|y}$$

for $s \in K$ and therefore

$$|a_n(\tau, s)| < 2Ce^{-\pi|n|y}$$

This shows that, if $|Res| < \sigma_0$ for $s \in K$, then

$$4C \sum_{n\neq 0} \left\{ \sum_{\substack{m|n \\ m>0}} m^{\sigma_0 - 1} \right\} e^{-\pi|n|y}$$

is a convergent majorant for

$$G_2(\tau, s) - 2\zeta(2 + s) - 2\zeta(1 + s)a_0(\tau, s).$$

Hence $G_2(\tau, s) - 2\zeta(2 + s) - 2\zeta(1 + s)a_0(\tau, s)$ is an entire function of s and therefore $G_2(\tau, s)$ is regular in the half-plane $Res > -1$, in view of our having already proved that $2\zeta(l+s)a_0(\tau,s)$ is regular for $Res > -1$. It follows immediately from the integral representation of $c_n(\tau,s), b_n(\tau,s)$ that

$$c_n(\tau,0)=0 \text{ for every } n\neq 0, b_n(\tau,0)=0 \text{ for } n<0$$

and

$$\begin{aligned} b_n(\tau,0) &= -2\pi i \operatorname{res}_{u=-\tau}(u+\tau)^{-2}e^{-2\pi i n u} \\ &= -4\pi^2 n e^{2\pi i n\tau} \text{ for } n>0. \end{aligned}$$

106 Substituting the values of $a_n(\tau,0)$ in (3), we obtain that

$$\begin{aligned} G_2(\tau)=G_2(\tau,0) &= 2\zeta(2)-\frac{\pi}{y}-8\pi^2\sum_{n=1}^{\infty}\left(\sum_{mn,m>0} m\right)e^{2\pi i n\tau} \\ &= \frac{-2\pi i}{\tau-\overline{\tau}}+\frac{\tau^2}{3}\left\{1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right\}, \end{aligned}$$

where $q=e^{2\pi i\tau}$, because

$$\sum_{n=1}^{\infty}\left(\sum_{\substack{m|n\\ m>0}} m\right)q^n=\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} mq^{mr}=\sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}.$$

Consider the analytic function

$$f(\tau)=G_2(\tau)+\frac{2\pi i}{\tau-\overline{\tau}}=\frac{\pi^2}{3}\left(1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right).$$

For $S=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ in Γ, it satisfies the transformation formula

$$f(S<\tau>)(c\tau+d)^{-2}=f(\tau)-\frac{2\pi i c}{c\tau+d},$$

because $G_2(S<\tau>)(c\tau+d)^{-2}=G_2(\tau)$ and

$$\frac{2\pi i}{S<\tau>-S<\overline{\tau}>}(c\tau+d)^{-2}-\frac{2\pi i}{\tau-\overline{\tau}}=\frac{2\pi i}{\tau-\overline{\tau}}\left(\frac{c\overline{\tau}+d}{c\tau+d}-1\right)=\frac{-2\pi i c}{c\tau+d}.$$

In the following, we take for $\log z$ the principal branch i.e. with $\log z$ real for positive real values of z. Let us set

$$g(\tau) = 2\pi i\tau + 24\sum_{n=1}^{\infty} \log(1 - q^n).$$

Then $g'(\tau) = \dfrac{-6}{\pi i} f(\tau)$ and the transformation formula for $f(\tau)$ implies that 107

$$\begin{aligned} dg(S < \tau >) &= dg(\tau) + \frac{12c}{c\tau + d} d\tau \\ &= dg(\tau) + 12d(\log(c\tau + d)), \end{aligned}$$

and therefore

$$g(S < \tau >) - g(\tau) - 12\log(c\tau + d) = C(S),$$

where $C(S)$ is a constant depending on S. This shows that

$$h(\tau) = e^{g(\tau)} = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$$

satisfies the transformation formula

$h(S < \tau >)(c\tau + d)^{-12} = \mathscr{C}(S)h(\tau)$ with $\mathscr{C}(S) = e^{C(S)}$. By iteration, $\mathscr{C}(S_1S_2) = \mathscr{C}(S_1)\mathscr{C}(S_2)$ and $\mathscr{C}(S) = \mathscr{C}(-S)$. But $h(\tau + 1) = h(\tau)$ and $h(i) \neq 0$; therefore $\mathscr{C}(U) = \mathscr{C}(T) = 1$ showing that $\mathscr{C}(S) = 1$ for every S in Γ. Thus $h(\tau)$ is a modular form of weight 12 for Γ, which vanishes at the parabolic cusp. Hence it follows that $h(\tau) = c\Delta(\tau)$ with some constant c. But the coefficient of $e^{2\pi i\tau}$ in the Fourier expansion of $h(\tau)$ at ∞ is 1, therefore $c = 1$ and we obtain that

$$\Delta(\tau) = e^{2\pi i\tau}\prod_{n=1}^{\infty}(1 - e^{2\pi in\tau})^{24}.$$

The exact value of the constant $C(S)$ occurring in the transformation formula for $g(\tau)$ has been computed by Rademacher in [5].

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Modular forms of Real Dimension

1 Modular forms and Partial Fraction Series

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As the study of 'the theta-series associated with a quadratic form' amply makes it clear, we need to consider modular forms of nonintegral weight as well, if we wish to apply the theory of modular forms to number-theoretic problems. Keeping in mind such an objective, we consider, a little more generally, modular forms of arbitrary real weight for a horocyclic group Γ. Unless otherwise stated, the horocyclic groups under consideration will contain $-E$.

Before defining a modular form of real weight for a horocyclic group Γ, we prove, for the sake of completeness, the transformation formula for the theta-series, which shows that, in general, the theta-series is not a modular form of integral weight.

$$\text{Let} \qquad Q[x] = \sum_{k,\ell=1}^{m} q_{k\ell} x_k x_\ell (q_{k\ell} = q_{\ell k})$$

be a real positive definite quadratic form in m variables. Corresponding to Q, we define the theta-series

$$\vartheta(\tau, Q) = \sum_{g} e^{\pi i \tau Q[g]}$$

where g runs over all m-rowed columns with integral coefficients. The series $\vartheta(\tau, Q)$ obviously converges absolutely and uniformly in any compact set of $\mathscr{G}$ and therefore represents a regular function of τ in $\mathscr{G}$. If

110 the matrix Q associated to the quadratic form $Q[x]$, is integral i.e. $q_{k\ell}$ is an integer for $k, \ell = 1, 2, \ldots, m$, and if, further, the determinant $|Q| = 1$, then $\vartheta(\tau, Q)$ satisfies the transformation formula

$$\vartheta\left(-\frac{1}{\tau}, Q\right) = (-i\tau)^{m/2}\vartheta(\tau, Q),$$

where $(-i\tau)^{\alpha} = e^{\alpha \log(-i\tau)}$ abd $\log(-i\tau)$ is positive for $x = 0$. In order to prove this, we consider the function

$$f(x, Q) = \sum_{g} e^{-\pi Q[g+x]}$$

with $x' = (x_1, x_2, \ldots, x_m)$ and $Q[x]$ a real positive definite quadratic form. Since $f(x, Q)$ is a periodic function, it has a Fourier expansion

$$f(x, Q) = \sum_{g} a(g) e^{2\pi i g' x}$$

where the coefficients $a(g)$ are given by

$$\begin{aligned} a(g) &= \int_0^1 \cdots \int_0^1 f(x, Q) e^{-2\pi i g' x} dx \quad (dx = dx_1 dx_2 \ldots dx_n) \\ &= \sum_{n} \int_0^1 \cdots \int_0^1 e^{-\pi Q[h+x] - 2\pi i g' x} dx \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\pi Q[x] - 2\pi i g' x} dx \\ &= e^{-\pi Q^{-1}}[g] \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\pi Q[x + iQ^{-1} g]} dx. \end{aligned}$$

If R is a real matrix such that $Q = R'R$, let $R'^{-1} g = RQ^{-1} g = (a_t)$. Then with the help of the substitution $y = Rx$, we obtain that

$$a(g) = |Q|^{-\frac{1}{2}} e^{-\pi Q^{-1}[g]} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\pi (y + iR'^{-1} g)(y + iR'^{-1} g)} dy$$

$$= |Q|^{-\frac{1}{2}} e^{-\pi Q^{-1}[g]} \prod_{r=1}^{m} \int_{-\infty}^{\infty} e^{-\pi(y_r+ia_r)^2} dy_r$$

$$= |Q|^{-\frac{1}{2}} e^{-\pi Q^{-1}[g]} \left\{ \int_{-\infty}^{\infty} e^{-\pi y^2} dy \right\}^m$$

$$= |Q|^{-\frac{1}{2}} e^{-\pi Q^{-1}}[g]$$

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Substituting the values of $a(g)$ in the Fourier expansion of $f(x, Q)$ we get, for $x = 0$,

$$f(0, Q) = \sum_g e^{-\pi Q[g]} = \sum_g a(g) = |Q|^{-\frac{1}{2}} \sum_g e^{-\pi Q^{-1}[g]}.$$

If t is a positive real number, then replacing Q by tQ in the above relation, we see immediately that

$$\sum_g e^{-\pi t Q[g]} = (t)^{-\frac{m}{2}} |Q|^{-\frac{1}{2}} \sum_g e^{-\frac{\pi}{t} Q^{-1}[g]}.$$

Our assertion follows at once, on replacing t by $-i\tau$ (i.e. essentially invoking the principle of analytic continuation). Finally, we assume that Q is integral and $|Q| = 1$ so that on the right hand side of the last relation we can replace g by Qg. We state the result proved above in the following

Theorem 15. *Let Q be an integral symmetric positive matrix of m rows and determinant 1. Let $Q[x]$ be the quadratic form associated with Q. Then the theta series* 112

$$\vartheta(\tau, Q) = \sum_g e^{\pi i \tau Q[g]}$$

satisfies the transformation formulae:

$$\vartheta\left(-\frac{1}{\tau}, Q\right) = (-i\tau)^{\frac{m}{2}} \vartheta(\tau, Q), \vartheta(\tau + 2, Q) = \vartheta(\tau, Q).$$

Theorem 15 shows that the function $\vartheta^8(\tau, Q)$ behaves like a modular form of weight $4m$ with respect to the substitutions T and U^2, which generate Γ_ϑ, the theta group. Indeed, $\vartheta^8(\tau, Q)$ is a modular form of weight $4m$ for Γ_ϑ and therefore

$$\vartheta^8(S<\tau>, Q) = (c\tau + d)^{4m}\vartheta^8(\tau, Q), \text{ for } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\vartheta.$$

Thus

$$\vartheta(S<\tau>, Q) = v(S)(c\tau + d)^{\frac{m}{2}}\vartheta(\tau, Q),$$

where $v(S)$ is a certain 8-th root of unity uniquely determined on fixing a branch of the multiple-valued function $(c\tau + d)^{\frac{m}{2}}$. This shows that in order to apply the theory of modular forms to theta-series for odd m, we require the notion of a modular form of semi-integral weight with multipliers. In particular, when $Q[x] = x^2$,

$$\vartheta(\tau) = \vartheta(\tau, Q) = \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2}.$$

We shall call the multiplier system v of this theta-series, the *theta multiplier system*.

113 Let r be a real number. We define

$$(c\tau + d)^r = e^{r\log(c\tau+d)} \text{ for real } (c, d) \neq (0, 0) \text{ and } \tau \in \mathscr{G}$$

with $\log z = \log|z| + i\arg z$, where $\log|z|$ is real and $-\pi < \arg z \leq \pi$. Obviously

$$\arg(c\tau + d) = \begin{cases} \arg\left(\tau + \frac{d}{c}\right) + \frac{\pi}{2}(\operatorname{sgn} c - 1) & \text{for } c \neq 0 \\ \frac{\pi}{2}(1 - \operatorname{sgn} d) & \text{for } c = 0. \end{cases}$$

Let $\underline{M} = (m_1, m_2)$ be a pair of real numbers distinct from $(0, 0)$ and let $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a real matrix with determinant 1. Then we have

$$(m_1 S<\tau> + m_2)(c\tau + d) = (m_1'\tau + m_2'),$$

with $(m_1', m_2') = \underline{M}S$. Therefore it follows that

$$\log(m_1 S<\tau> + m_2) = \log(m_1'\tau + m_2') - \log(c\tau + d) + 2\pi i w(\underline{M}, S)$$

where $w(\underline{M}, S)$ is an integer depending on $\underline{M}$ and S. We can now conclude that

$$w(\underline{M}, S) = \frac{1}{2\pi}\{\arg(m_1 S < \tau > + m_2) - \arg(m_1'\tau + m_2') + \arg(c\tau + d)\}.$$

Obviously $|w(\underline{M}, S)| \leq \dfrac{3}{2}$. But $w(\underline{M}, S)$ is an integer, and therefore $|w(\underline{M}, S)| \leq 1$. Let $M = \left(\begin{smallmatrix} m_0 & m_3 \\ m_1 & m_2 \end{smallmatrix}\right)$ denote a real matrix of determinant 1, with $\underline{M}$ as second row. Defining $\underline{w(M, S) = w(\underline{M}, S)}$, we shall now compute $w(M, S)$ explicitly for $M = \left(\begin{smallmatrix} m_0 & m_3 \\ m_1 & m_2 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $MS = \left(\begin{smallmatrix} * & * \\ m_1' & m_2' \end{smallmatrix}\right)$ with $|M| = |S| = 1$. Here and in the sequel, $\underline{M}$ will always denote the second row of the matrix M. 114

1. $\underline{m_1 c m_1' \neq 0}$. By the definition of $w(M, S)$,

$$\begin{aligned} 2\pi w(M, S) &= \arg(S < \tau > + \frac{m_2}{m_1}) - \arg(\tau + \frac{m_2'}{m_1'}) + \arg(\tau + \frac{d}{c}) \\ &\quad + \frac{\pi}{2}(\operatorname{sgn} m_1 - 1) - \frac{\pi}{2}(\operatorname{sgn} m_1' - 1) + \frac{\pi}{2}(\operatorname{sgn} c - 1) \end{aligned}$$

   Let $x = \operatorname{Re}\tau$ be fixed and $y = \operatorname{Im}\tau \to \infty$. Then

$$\begin{aligned} & S < \tau > + \frac{m_2}{m_1} \to \frac{a}{c} + \frac{m_2}{m_1} = \frac{m_1'}{cm_1}, \\ & \arg(S < \tau > + \frac{m_2}{m_1}) \to -\frac{\pi}{2}(\operatorname{sgn} m_1 c m_1' - 1), \\ & \arg(\tau + \frac{m_2'}{m_1'}) \to \frac{\pi}{2} \text{ and } \arg(\tau + \frac{d}{c}) \to \frac{\pi}{2}. \end{aligned}$$

   This shows that

$$4w(M, S) = -\operatorname{sgn} m_1 c m_1' + \operatorname{sgn} m_1 - \operatorname{sgn} m_1' + \operatorname{sgn} c.$$

2. $\underline{cm_1 \neq 0}$, $\underline{m_1' = 0}$. Obviously, $m_1 = -cm_2'$ and

$$2\pi w(M, S) = \arg(S < \tau > + \frac{m_2}{m_1}) + \arg(\tau + \frac{d}{c}) + \frac{\pi}{2}(\operatorname{sgn} m_2' - 1)$$

$$+\frac{\pi}{2}(\operatorname{sgn} m_1 - 1) + \frac{\pi}{2}(\operatorname{sgn} c - 1).$$

Let us take $x = -\dfrac{d}{c}$. Then we see immediately that

$$S < \tau > + \frac{m_2}{m_1} = \frac{i}{c^2 y} + \frac{a}{c} + \frac{m_2}{m_1} = \frac{i}{c^2 y}, \quad \tau + \frac{d}{c} = iy.$$

and

$$\begin{aligned} 4w(M, S) &= \operatorname{sgn} m_1 - 1 + \operatorname{sgn} m_2' + \operatorname{sgn} c \\ &= -(1 - \operatorname{sgn} c)(1 - \operatorname{sgn} m_1). \end{aligned}$$

3. $\underline{cm_1' \neq 0}$, $\underline{m_1 = 0}$. It is obvious that $m_1' = m_2 c$ and

$$\begin{aligned} 2\pi w(M, S) = &- \arg(\tau + \frac{m_2'}{m_1'}) + \arg(\tau + \frac{d}{c}) - \frac{\tau}{2}(\operatorname{sgn} m_2 - 1) \\ &- \frac{\pi}{2}(\operatorname{sgn} m_1' - 1) + \frac{\pi}{2}(\operatorname{sgn} c - 1) \end{aligned}$$

115 Letting y tend to ∞ for a fixed x, we obtain

$$\begin{aligned} 4w(M, S) &= 1 + \operatorname{sgn} c - \operatorname{sgn} m_2 - \operatorname{sgn} m_1' \\ &= (1 + \operatorname{sgn} c)(1 - \operatorname{sgn} m_2). \end{aligned}$$

4. $\underline{c = 0}$, $\underline{m_1 m_1' \neq 0}$. We obtain immediately that $ad = 1$, $S < \tau >= a^2\tau + ab$ and

$$\begin{aligned} 2\pi w(M, S) = &\arg(S < \tau > + \frac{m_2}{m_1}) - \arg(\tau + \frac{m_2'}{m_1'}) + \frac{\pi}{2}(\operatorname{sgn} m_1 - 1) \\ &- \frac{\pi}{2}(\operatorname{sgn} m_1' - 1) - \frac{\pi}{2}(\operatorname{sgn} d - 1). \end{aligned}$$

or $$\begin{aligned} 4w(M, S) &= 1 + \operatorname{sgn} m_1 - \operatorname{sgn} m_1' - \operatorname{sgn} d \\ &= (1 - \operatorname{sgn} a)(1 + \operatorname{sgn} m_1) \end{aligned}$$

5. $\underline{c = m = m'_1 = 0}$. We have $m'_2 = m_2 d$, $ad = 1$ and

$$\begin{aligned} 2\pi w(M,S) &= \frac{-\pi}{2}(\operatorname{sgn} m_2 - 1) + \frac{\pi}{2}(\operatorname{sgn} m'_2 - 1) - \frac{\pi}{2}(\operatorname{sgn} d - 1) \\ \text{or}\quad 4w(M,S) &= 1 - \operatorname{sgn} d - \operatorname{sgn} m_2 + \operatorname{sgn} m'_2 \\ &= (1 - \operatorname{sgn} a)(1 - \operatorname{sgn} m_2) \end{aligned}$$

We collect the results above in

Theorem 16. *Let $M = \left(\begin{smallmatrix} * & * \\ m_1 & m_2 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be two real matrices with determinant* 1 *and (m'_1, m'_2) the second row of the matrix MS. Then*

$$w(M,S) = \begin{cases} \frac{1}{4}\{\operatorname{sgn} c + \operatorname{sgn} m_1 - \operatorname{sgn} m'_1 - \operatorname{sgn}(m_1 c m'_1)\}, & \text{if } m_1 c m'_1 \neq 0 \\ -\frac{1}{4}(1 - \operatorname{sgn} c)(1 - \operatorname{sgn} m_1), & \text{if } c m_1 \neq 0, m'_1 = 0 \\ \frac{1}{4}(1 + \operatorname{sgn} c)(1 - \operatorname{sgn} m_2), & \text{if } c m'_1 \neq 0, m_1 = 0 \\ \frac{1}{4}(1 - \operatorname{sgn} a)(1 + \operatorname{sgn} m_1), & \text{if } m_1 m'_1 \neq 0, c = 0 \\ \frac{1}{4}(1 - \operatorname{sgn} a)(1 - \operatorname{sgn} m_2), & \text{if } c = m_1 = m'_1 = 0 \end{cases}$$

With the help of the summand system $w(M,S)$, we form the factor 116
system $\sigma^{(r)}(M,S)$ for an arbitrary real number r, by defining

$$\sigma(M,S) = \sigma^{(r)}(M,S) = e^{2\pi i r w(M,S)}.$$

It is immediate from the definition that

$$(m_1 S < \tau > + m_2)^r = \sigma(M,S) \frac{(m'_1 \tau + m'_2)^r}{(c\tau + d)^r}.$$

If S_1 and S_2 are two real two-rowed matrices with determinant 1, then from the relation $S_1 < S_2 < \tau >>= S_1 S_2 < \tau >$, we have

$$\sigma(M, S_1 S_2)\ \ \sigma(S_1, S_2) = \sigma(M S_1, S_2)\ \ \sigma(M, S_1).$$

In particular, we get from theorem 16 that

$$\sigma(S, S^{-1}) = \sigma(S^{-1}, S) \text{ and } \sigma(E,S) = \sigma(S,E) = 1.$$

Since the value of $w(M,S)$ does not depend on the first row of M or the second column of S, we have

$$w(U^{\xi} M, S U^{\eta}) = w(M,S), \sigma(U^{\xi} M, S U^{\eta}) = \sigma(M,S),$$

where $U^{\xi} = \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right)$ for any real number ξ.

Let Γ be a horocyclic group and let $f(\tau) \not\equiv 0$ be a function with the transformation property

$$f(S < \tau >) = v(S)(c\tau + d)^r f(\tau) \text{ for } S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

It follows immediately from the transformation property of $f(\tau)$ that

$$v(-E)(-1)^r = 1, \text{ i.e. } v(-E) = e^{-\pi ir}. \tag{1}$$

117 If $S = S_1 S_2$ with $S_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right) \in \Gamma (j = 1, 2)$, then

$$\begin{aligned} f(S < \tau >) = f(S_1 S_2 < \tau >) &= v(S_1)(c_1 S_2 < \tau > + d_1)^r f(S_2 < \tau >) \\ &= v(S_1)v(S_2)\sigma(S_1, S_2)(c\tau + d)^r f(\tau), \end{aligned}$$

where $\overline{S} = (c, d)$. Since $f(\tau) \not\equiv 0$, we have

$$v(S_1 S_2) = \sigma(S_1, S_2)v(S_1)v(S_2). \tag{2}$$

We shall call a system of numbers $v(S)$ defined for all $S \in \Gamma$ a *multiplier system* for the group Γ and real weight r, when $|v(s)| = 1$ and $v(S)$ satisfies equations (1) and (2). If $v_1(S)$ (respectively $v_2(S)$) is a multiplier system for the group Γ and weight r_1 (respectively r_2) then $v_1 v_2(S) := v_1(S)v_2(S)$ is a multiplier system for Γ and weight $r_1 + r_2$, because

$$\sigma^{(r_1)}(S_1, S_2)\sigma^{(r_2)}(S_1, S_2) = \sigma^{(r_1+r_2)}(S_1, S_2).$$

Moreover, when r is an even integer, $v(S)$ is an *even abelian character* of Γ i.e. $S \to v(S)$ is a homomorphism of Γ into the multiplicative group of complex numbers of absolute value 1 such that $v(-S) = v(S)$. This shows that if $v_1(S)$ and $v_2(S)$ are two multiplier system for Γ and the same weight r, then $v_0(S) := \dfrac{v_1}{v_2}(S)$ is a multiplier system for Γ and weight 0; therefore $v_0(S)$ is an abelian character of Γ. Hence we obtain all multiplier systems for Γ and weight r, from a fixed multiplier system v_1, in the form $v_1 v_0$, when v_0 runs over the set of even abelian characters of Γ. But the group of even abelian characters of Γ is isomorphic to the group Γ / K^*, where K^* denotes the group generated by the commutator subgroup of Γ and $-E$. Thus the number of distinct multiplier
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systems for any real weight r and group Γ is equal to the order of the group Γ/K^* provided there exists *one* multiplier system for the weight r and otherwise, it is 0. In particular, if Γ is the modular group, then the number of distinct multiplier systems for Γ and any real weight is 6, since $(\Gamma : K^*) = 6$ as already proved in Chapter 2, § 1 and since at least one multiplier system for an arbitrary real weight will be shown to exist (See proof of theorem 19 below).

Let us further assume that the above-mentioned function $f(\tau)$ is regular in $\mathscr{G}$. We shall now examine the behaviour of $f(\tau)$ at the fixed points of Γ. Let ρ be a parabolic cusp of Γ and $A = \left(\begin{smallmatrix} a_0 & a_3 \\ a_1 & a_2 \end{smallmatrix}\right)$ a real matrix of determinant 1 such that $A < \rho >= \infty$. Let N denote the least positive real number with the property that

$$H = A^{-1}U^N A \in \Gamma, H = \begin{pmatrix} h_0 & h_3 \\ h_1 & h_2 \end{pmatrix}.$$

Obviously, the subgroup of Γ which leaves ρ fixed is generated by H and $-E$. We set

$$g(\tau) = (a_1\tau + a_2)^r f(\tau).$$

Using the transformation property of $f(\tau)$, we obtain that

$$\begin{aligned} g(H < \tau >) &= (a_1 H < \tau > +a_2)^r v(H)(h_1\tau + h_2)^r f(\tau) \\ &= \sigma(A, H)v(H)(a_1\tau + a_2)^r f(\tau) \\ &= e^{2\pi i\kappa} g(\tau), \end{aligned}$$

$$\text{where} \qquad e^{2\pi i\kappa} = \sigma(A, H)v(H), 0 \leq \kappa < 1. \tag{3}$$

If we replace τ by $A^{-1} < \tau >$, then 119

$$g(A^{-1}U^N < \tau >) = e^{2\pi i\kappa} g(A^{-1} < \tau >)$$

and therefore the function

$$h(\tau) = g(A^{-1} < \tau >)e^{-2\pi i\kappa\tau/N}$$

is a periodic function of τ, of period N. Hence

$$h(\tau) = P(e^{2\pi i\tau/N})$$

$$\text{or}(a_1\tau + a_2)^r f(\tau) = e^{2\pi i\kappa A<\tau>/N} P(e^{2\pi i A<\tau>/N}),$$

where $P(z)$ is a convergent Laurent series in z. If we assume that $P(z)$ does not contain negative powers of z, then

$$(a_1\tau + a_2)^r f(\tau) = \sum_{n+\kappa\geq 0} c_{n+\kappa} e^{2\pi i(n+\kappa)A<\tau>/N}. \tag{4}$$

Let τ_0 be an elliptic fixed point of Γ. Since the transformation

$$z = \frac{\tau - \tau_0}{\tau - \overline{\tau}_0} = A < \tau >, \quad A = \begin{pmatrix} 1 & -\tau_0 \\ 1 & -\overline{\tau}_0 \end{pmatrix}$$

maps the complex conjugate fixed points τ_0 and $\overline{\tau_0}$ to the points 0 and ∞ respectively, the group $A\Gamma A^{-1}$ has the elliptic fixed point pair 0 and ∞. Therefore for some real number ϑ, the matrix $\left(\begin{smallmatrix} e^{i\zeta} & 0 \\ 0 & e^{-i\vartheta} \end{smallmatrix}\right)$ belongs to $A\Gamma A^{-1}$ and the set of all real numbers ϑ, such that $\left(\begin{smallmatrix} e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} \end{smallmatrix}\right)$ belongs to $A\Gamma A^{-1}$, is a discrete module containing π. If ϑ_0 is the least positive number in
120 this discrete module, then $\pi = \vartheta_0\ell$ for some integer $\ell > 0$. Let us set

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A^{-1} \begin{pmatrix} e^{\pi i/\ell} & 0 \\ 0 & e^{-\pi i/\ell} \end{pmatrix} A.$$

Then the order of L is 2ℓ i.e. $L^\ell = -E$ and the group of transformations of Γ which leave τ_0 fixed is generated by L. From the definition of L, we obtain that $a - \tau_0 c = e^{\pi i/\ell}$. For

$$AL = \begin{pmatrix} a - \tau_0 c & b - \tau_0 d \\ a - \overline{\tau}_0 c & b - \overline{\tau}_0 d \end{pmatrix} = \begin{pmatrix} e^{\pi i/\ell} & 0 \\ 0 & e^{-\pi i/\ell} \end{pmatrix} A = \begin{pmatrix} e^{\pi i/\ell} & -e^{\pi i/\ell_{\tau_0}} \\ e^{-\pi i/\ell} & -e^{-\pi i/\ell_{\overline{\tau}_0}}. \end{pmatrix}$$

But $a + d = e^{\pi i/\ell} + e^{-\pi i/\ell}$ and therefore

$$(c\tau_0 + d) = (a + d) - (a - \tau_0 c) = e^{-\pi i/\ell}.$$

If $\varphi(\tau) = (\tau - \overline{\tau}_0)^r$, then

$$\varphi(L < \tau >) = (L < \tau > - L < \overline{\tau}_0 >)^r = \left(\frac{\tau - \overline{\tau}_0}{(c\tau + d)(c\overline{\tau}_0 + d)}\right)^r$$

$$= \gamma_r(L)(c\tau + d)^{-r}\varphi(\tau),$$

with a certain constant $\gamma_r(L)$ depending on L. Putting $\tau = \tau_0$ in the above relation, we see immediately that

$$\gamma_r(L) = (c\tau_0 + d)^r = e^{-\pi ir/\ell}$$

Writing

$$g(\tau) = (\tau - \overline{\tau}_0)^r f(\tau)$$

and using the transformation property of $\varphi(\tau)$ and $f(\tau)$, we obtain

$$g(L < \tau >) = \varphi(L < \tau >) f(L < \tau >) = e^{-\pi ir/\ell} v(L) g(\tau).$$

But $L^\ell = -E$ and $g(\tau) \not\equiv 0$; therefore applying the mapping $\tau \to L <$ **121**
$\tau >$ successively ℓ times, we get

$$(v(L))^\ell = e^{\pi ir}$$
$$\text{or } v(L) = e^{\pi ir/\ell} e^{2\pi ia_0/\ell} (0 \le a_0 < \ell).$$

This implies that

$$g(L < \tau >) = e^{2\pi ia_0/\ell}$$
$$\text{i.e. } g\left(A^{-1} \begin{pmatrix} e^{\pi i/\ell} & 0 \\ 0 & e^{-\pi i/\ell} \end{pmatrix} < z >\right) = e^{2\pi ia_0/\ell} g(A^{-1} < z >)$$
$$(\text{with } z = A < \tau >).$$

Thus the function

$$h(z) = z^{-a_0} g(A^{-1} < z >),$$

which is invariant under the transformation $z \to e^{2\pi i/\ell}$, has a power-series expansion in terms of the local coordinate $t = ((\tau - \tau_0)/(\tau - \overline{\tau}_0))^\ell = z^\ell$ at the point τ_0. We may now conclude that

$$g(A^{-1} < z >) = \sum_{\ell n + a_0 \ge 0} c_{n + a_0/\ell} \ t^{n + a_0/\ell}$$
$$\text{i.e}(\tau - \overline{\tau}_0)^r f(\tau) = \sum_{n + a_0/\ell \ge 0} c_{n + a_0/\ell} \ t^{n + a_0/\ell} \ \text{ with } \ t = ((\tau - \tau_0)/(\tau - \overline{\tau}_0))^\ell. \tag{5}$$

Definition. *A function $f(\tau)$ is said to be an automorphic form of weight r (r a real number) for a horocyclic group Γ and the multiplier system $v(S)$ if*

122 1) *$f(\tau)$ is regular in $\mathscr{G}$,*

2) *$f(S<\tau>) = v(S)(c\tau+d)^{-r} f(\tau)$, for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ, and*

3) *at every parabolic cusp $A^{-1} < \infty >$ of Γ, $f(\tau)$ has a Fourier expansion given by* (4) *i.e.*

$$(a_1\tau + a_2)^r f(\tau) = \sum_{n+\kappa \geq 0} c_{n+\kappa} e^{2\pi i(n+\kappa)A<\tau>/N},$$

where $A = \begin{pmatrix} a_0 & a_3 \\ a_1 & a_2 \end{pmatrix}$ is a real matrix of determinant 1. In particular, if, for the horocyclic group Γ, we take a subgroup Γ_0 of finite index in the modular group then, in condition 3), the matrix A already belongs to the modular group, since every parabolic cusp of the subgroup Γ_0 can be obtained in this way. We have shown above that if τ_0 is an elliptic fixed point of Γ, then $(\tau - \overline{\tau}_0)^r f(\tau)$ has a power series expansion given by (5) *at the point τ_0.*

In the following, we shall *confine ourselves to the subgroups of finite index in the modular group*. As in chapter 2, § 1, Γ will denote the modular group and the subgroups Γ_0 under consideration will be assumed to contain $-E$. The set of all automorphic forms of weight r for the group Γ_0 and the multiplier system $v(S)$ forms a vector space over the complex number field. We shall denote this vector space by $[\Gamma_0, r, v]$.

We shall now show that the power series expansions of $f(\tau)$ at equivalent points are of the same type, so that the degree of $f(\tau)$ at any point on the Riemann surface associated with Γ_0 is well-defined.

123 **Case 1.** Let $\rho = A^{-1} < \infty >, \rho^* = B^{-1} < \infty > = L^{-1}A^{-1} < \infty >, L \in \Gamma_0$, be two equivalent parabolic cusps of Γ_0. Then $B^{-1} = \pm L^{-1}A^{-1}U^{-k}$ for some integer k. Since $-E$ belongs to Γ Γ_0, we can assume that $B = U^k AL$. Let $A = \begin{pmatrix} * & * \\ a_1 & a_2 \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ b_1 & b_2 \end{pmatrix}$ and $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Replacing τ by $L < \tau >$ in (4), we obtain that

$$(a_1 L < \tau > + a_2)^r f(L < \tau >) = \sum_{n+\kappa \geq 0} c_{n+\kappa} \; e^{2\pi i(n+\kappa)AL<\tau>/N}$$

$$\text{or } (a_1 L < \tau > + a_2)^r v(L)(c\tau + d)^r f(\tau) = \sum_{n+\kappa \geq 0} c_{n+\kappa} \quad e^{2\pi i (n+\kappa) U^{-\kappa} B < \tau > /N}$$

$$\text{or } \sigma(A, L) v(L)(b_1\tau + b_2)^r f(\tau) = \sum_{n+\kappa \geq 0} (c_{n+\kappa} \quad e^{-2\pi(n+\kappa)k/N})\ e^{2\pi i(n+\kappa) B < \tau > /N}$$

$$\text{or } (b_1\tau + b_2)^r f(\tau) = \sum_{n+\kappa \geq 0} c'_{n+\kappa} \quad e^{2\pi i (n+\kappa) B < \tau > /N}$$

$$\text{with } c'_{n+\kappa} = \frac{c_{n+\kappa} e^{-2\pi i (n+\kappa) k/N}}{\sigma(A, L) v(L)}.$$

Case 2. Let τ_0 and τ_0^* be two equivalent points of $\mathscr{G}$. Then $\tau_0^* = S^{-1} < \tau_0 >$ for some $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ_0. If we replace τ by $S < \tau >$ in (5), we obtain that

$$(S < \tau > - \overline{\tau}_0)^r f(S < \tau >) = v(S)(c\tau + d)^r \left(\frac{\tau - \overline{\tau}_0^*}{(c\tau + d)(c\overline{\tau}_0^* + d)} \right)^r f(\tau)$$

$$= \sum_{\ell n + a_0 \geq 0} c_{n + a_0/\ell} \left(\frac{\tau - \tau_0^*}{\tau - \overline{\tau}_0^*} \cdot \frac{c\overline{\tau}_0^* + d}{c\tau_0^* + d} \right)^{\ell n + a_0}.$$

Thus 124

$$(\tau - \overline{\tau}_0^*)^r f(\tau) = \sum_{n + a_0/\ell \geq 0} c^*_{n + a_0/\ell} \left(\frac{\tau - \tau_0^*}{\tau - \overline{\tau}_0^*} \right)^{\ell n + a_0}$$

$$\text{with } c^*_{n + a_0/\ell} = (c\overline{\tau}_0^* + d)^r c_{n + a_0/\ell} \left(\frac{c\overline{\tau}_0^* + d}{c\tau_0^* + d} \right)^{\ell n + a_0} \gamma(S, \tau_0)$$

where $\gamma(S, \tau_0)$ is a complex number of absolute value 1. Hence our assertion is completely proved.

We define the degree $v_{\mathfrak{g}}(f)$ of $f(\tau)$ at the point $\mathfrak{g}$ of $\mathscr{R}_0$, the Riemann surface associated to Γ_0, to be the least index $n + \kappa$ (respectively $n + a_0/\ell$) such that $c_{n+\kappa} \neq 0$ (respectively $c_{n + a_0/\ell \neq 0}$) according as $\mathfrak{g}$ is an equivalence class of parabolic cusps or Γ_0 or not. Obviously, the degree of $f(\tau)$ at any point $\mathfrak{g}_0$ of $\mathscr{R}_0$ is the multiplicity of the zero of $f(\tau)$ at $\tau_0 \in \mathfrak{g}_0$ measured in terms of the local coordinate. Thus the total degree of $f(\tau)$ i.e. the sum of the degrees of $f(\tau)$ at all points of $\mathscr{R}_0$ is equal to the 'number of zeros' of $f(\tau)$. It can be proved easily that the number κ

defined by (3) does not depend upon the choice of the cusp $A^{-1} < \infty >$ in its equivalence class. We shall call the multiplier system v at a given cusp $A^{-1} < \infty >$ *unramified* (respectively *ramified*) according as the number κ defined above is zero (or not).

Let $f(\tau)$ be an element of $[\Gamma_0, r, v]$. Then, at a parabolic cusp $\rho = A^{-1} < \infty >$ with $A = \left(\begin{smallmatrix} a_0 & a_3 \\ a_1 & a_2 \end{smallmatrix}\right) \in \Gamma$, the form $f(\tau)$ has a Fourier expansion:

$$\left.\begin{matrix} f(\tau) \\ (\tau-\rho)^r f(\tau) \end{matrix}\right\} = \sum_{n+\kappa\geq 0} b_{n+\kappa} \quad e^{2\pi i(n+\kappa)A<\tau>/N} \quad \begin{matrix} \text{for } \rho = \infty \quad A = E \\ \text{for } \rho \neq \infty, \end{matrix}$$

125 because, if $\rho \neq \infty$, then $(\tau-\rho)^r = (\tau + a_2/a_1)^r$ is a constant multiple of $(a_1\tau + a_2)^r$. We define the number

$$C(\rho) = \begin{cases} 0, & \text{if } v \text{ is ramified at } \rho. \\ b_0, & \text{if } v \text{ is unramified at } \rho. \end{cases}$$

The complex number $C(\rho)$ defined above does not depend upon the choice of A, because if B is another element in Γ such that $B^{-1} < \infty >= \rho$, then $A = \pm U^k B$ and this does not affect the coefficient b_0. We associate with $f(\tau)$ the *partial fraction series*

$$G(\tau) = C(\infty) + \sum_{\rho\neq\infty} C(\rho)(\tau-\rho)^{-r}.$$

We shall prove that the series $G(\tau)$ converges absolutely and uniformly in every domain $|x| \leq c$, $y \geq \in > 0$ for $r > 2$ and belongs to $[\Gamma_0, r, v]$. In order to find the contributions of the various cusps to the partial fraction series $G(\tau)$, we have to consider only those cusps at which the multiplier system is unramified. In that case, the contribution of the cusp $\rho = A^{-1} < \infty >$ is the first term of the series

$$f(\tau) = (a_1\tau + a_2)^{-r} \sum_{n=0}^{\infty} c_n e^{2\pi i n A<\tau>/N}, \text{ with } c_n = c_n(A).$$

Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ_0 and let $M = AL = \left(\begin{smallmatrix} m_0 & m_3 \\ m_1 & m_2 \end{smallmatrix}\right)$. Then replacing τ by $L < \tau >$ in the above series for $f(\tau)$, we obtain that

$$f(\tau) = \frac{1}{(a_1 L < \tau > + a_2)^r v(L)(c\tau + d)^r} \sum_{n=0}^{\infty} c_n e^{2\pi i n AL<\tau>/N}$$

$$= \frac{1}{\sigma(A,L)v(L)(m_1\tau+m_2)^r}\sum_{n=0} c_n e^{2\pi i n M<\tau>/N}$$

This shows that the contribution of the cusp $L^{-1}A^{-1} < \infty >= L^{-1} <$ **126**
$\rho >$ to the series $G(\tau)$ is

$$\frac{c_0(A)}{\sigma(A,L)v(L)(m_1\tau+m_2)^r}.$$

Let $\rho_k = A_k^{-1} < \infty >$, $k = 1, 2, \ldots, \sigma_0$ be a complete system of inequivalent parabolic cusps of Γ_0 at which the multiplier system is unramified. Then

$$G(\tau) = \sum_{k=1}^{\sigma_0} c_0(A_k) G(\tau, A_k)$$

$$\text{with } G(\tau, A_k) = \sum_{L^{-1}<\rho_k>} \frac{1}{\sigma(A_k,L)v(L)(m_1+m_2)^r}$$

where (m_1, m_2) is the second row of the matrix $A_k L$ and the sum runs over those elements L of Γ_0 which give rise to distinct cusps in the equivalence class of ρ_k. Obviously $L_1^{-1} < \rho_k >= L_2^{-1} < \rho_k >$ for L_i in $\Gamma_0 (i = 1, 2)$ if and only if $L_2 L_1^{-1}$ belongs to the group Z_k generated by $-E$ and $A_k^{-1} U^{N_k} A_k$, where N_k is the least positive real number such that $A_k^{-1} U^{N_k} A_k$ belongs to Γ_0. This shows that in the summation of the so-called 'Eisenstein series' $G(\tau, A_k), L$ runs over a complete representative system of the right cosets of Γ_0 modulo Z_k. Thus $G(\tau)$ is a finite linear combination of Eisenstein series, which converge absolutely and uniformly in every domain $|x| \leq c, y \geq \epsilon > 0$ for $r > 2$ and therefore $G(\tau)$ is regular in $\mathscr{G}$. In order to prove this statement, we have to use the same argument as for $G_k(\tau)$ in chapter 2, § 4. Further, if $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to Γ_0, then

$$\begin{aligned}
&(c\tau+d)^{-r} G(S < \tau >, A_k)\\
&\qquad = \sum_{L^{-1}<\rho>} \frac{1}{\sigma(A_k,L)} \frac{1}{(c\tau+d)^r v(L)(m_1 S < \tau > + m_2)^r}\\
&\qquad = V(S) \sum_{L^{-1}<\rho>} \frac{1}{\sigma(A_k,L)v(S)v(L)\sigma(A_k L, S)(m_1'\tau + m_2')^r}
\end{aligned}$$

127 where (m_1', m_2') is the second row of the matrix $A_k LS$. But

$$\sigma(A_k, L)\sigma(A_k L, S)v(S)v(L) = \sigma(A_k, LS)v(LS)$$

and LS for a fixed S in Γ_0 where L runs over a complete representative system of right cosets of Γ_0 modulo Z_k; therefore

$$(c\tau + d)^{-r} G(S < \tau >, A_k) = v(S)G(\tau, A_k),$$

showing that

$$G(S < \tau >) = v(S)(c\tau + d)^r G(\tau) \text{ for } S \in \Gamma_0.$$

In order to prove that $G(\tau)$ belongs to $[\Gamma_0, r, v]$, it remains to show that $G(\tau)$ does not have negative exponents in its Fourier expansion at various cusps of Γ_0. This follows immediately from

Lemma 4. *Let $\rho_1, \rho_2, \rho_3, \ldots$ be a sequence of distinct real numbers and $c_0, c_1, c_2, \ldots\ldots$ a sequence of complex numbers. If the series*

$$g(\tau) = c_0 + \sum_{n=1}^{\infty} c_n(\tau - \rho_n)^{-r}$$

converges absolutely at a point of $\mathscr{G}$, then

1) *the series $g(\tau)$ converges absolutely and uniformly in every domain $|x| \leq c$, $y \geq \in > 0$ and therefore $g(\tau)$ is regular in $\mathscr{G}$, and*

2) $$\lim_{y\to\infty}(A^{-1} < \tau > -\rho)^r g(A^{-1} < \tau >) = \begin{cases} c_n & \text{for } \rho = \rho_n \\ 0 & \text{for } \rho \neq \infty, \rho_1, \rho_2, \ldots \end{cases}$$

128 $$\lim_{y\to\infty} g(\tau) = \lim_{y\to\infty} g(A^{-1} < \tau >) \qquad = c_0 \text{ for } \rho = \infty,$$

uniformly in a given domain $|x| \leq c$. Here A is a real matrix of determinant 1 such that $\rho = A^{-1} < \infty >$.

Proof. (i) Let $g(\tau)$ be absolutely convergent at the point $\tau_0 = x_0 + iy_0$ of $\mathscr{G}$. Let $\tau = x + iy$ be in $\mathscr{G}$ such that $|x| \leq c$, $y \geq \in > 0$. Then for $\tau' = x' + iy' = \dfrac{\tau - x_0}{y_0}$, we have $|x'| \leq c'$, $y' \geq \in' > 0$ for certain c'

and $\in' > 0$ depending on c and $\in$. By lemma 3 (chapter 2, § 4), it follows that

$$|\tau' + \frac{x_0 - \rho_n}{y_0}| \geq \delta |i + \frac{x_0 - \rho_n}{y_0}| = \frac{\delta}{y_0} |\tau_0 - \rho_n| \Longrightarrow$$
$$|\tau - \rho_n| \geq \delta |\tau - \rho_n|$$

for some $\delta = \delta(\in, c) > 0$. From here follows immediately assertion 1) of the lemma.

(ii) If $\rho = \infty$, then $A^{-1} = \left(\begin{smallmatrix} \lambda & a \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ and $g(A^{-1} < \tau >)$ is a series of the same type as $g(\tau)$. Therefore, it suffices to consider the case $A = E$. But

$$\lim_{y \to \infty} g(\tau) = c_0, \text{ uniformly in } |x| \leq c$$

follows directly from $\lim\limits_{y \to \infty} (\tau - \rho_n)^{-r} = 0$ for all n and the uniform convergence of the series $g(\tau)$.

Let us now assume that $\rho \neq \infty$. The case $\rho = \rho_n$ for some n is reduced to the case $\rho \neq \infty, \rho_1, \rho_2, \ldots$, if we replace the series $g(\tau)$ by the series $g(\tau) - c_n(\tau - \rho_n)^{-r}$, which satisfies the requirements of lemma 4. Let $A = \left(\begin{smallmatrix} a_0 & a_3 \\ a_1 & a_2 \end{smallmatrix}\right)$ and $\rho = A^{-1} < \infty >= -\dfrac{a_2}{a_1} \neq \rho_1, \rho_2, \ldots$ After some computation, we get **129**

$$A^{-1} < \tau > - \rho = -1/(a_1^2(\tau - A < \tau >)),$$
$$\frac{A^{-1} < \tau > - \rho}{A^{-1} < \tau > - \rho_n} = \frac{1}{a_1^2(\rho_n - \rho)(\tau - A < \tau_n >)}$$

Therefore

$$(A^{-1} < \tau > - \rho)^r g(A^{-1} < \tau >)$$
$$= a_1^{-2r} \left\{ \frac{c_0 \in_0}{(\tau - A < \infty >)^r} + \sum_{n=1}^{\infty} \frac{c_n (\rho_n - \rho)^{-r} \in_n}{(\tau - A < \rho_n >)^r} \right\}$$

with certain $\in_n$ of absolute value 1 for $n \geq 0$. This is just a series of the type described in lemma 4 but now without constant term.

Thus, as we have proved already

$$\lim_{y\to\infty}(A^{-1} < \tau > -\rho)^r g(A^{-1} < \tau >) = 0 \text{ uniformly in } |x| \leq c.$$

This completes the proof of lemma 4.

Hence our assertion that the series $G(\tau)$ belongs to $[\Gamma_0, r, v]$ follows immediately from lemma 4. Moreover, it is obvious from the above discussion that $f(\tau) - G(\tau)$ is a cusp form i.e. in the Fourier expansion of $f(\tau) - G(\tau)$ at various cusps of Γ_0, the constant term is equal to zero. Thus we have proved the following

□

Theorem 17. *For every automorphic form $f(\tau)$ belonging to $[\Gamma_0, r, v]$,*
130 *$r > 2$, there exists an automorphic form $G(\tau)$ of the same type in the form of a partial fraction series*

$$G(\tau) = C(\infty) + \sum_{\rho\neq\infty} C(\rho)(\tau - \rho)^{-r}$$

where ρ runs over all distinct parabolic cusps of Γ_0 different from ∞, such that $f(\tau) - G(\tau)$ is a cusp form.

With the help of the general theory, we completely characterise some linear spaces of automorphic forms for the theta-group in the following

Theorem 18. *Let v_ϑ be the multiplier system for the theta series $\vartheta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i\tau n^2}$ and Γ_ϑ, the theta group generated by τ and U^2. Then the space $[\Gamma, k/2, v^k]$ is generated by $\vartheta^k(\tau)$ for $k = 1, 2, 3, 4, 5, 6, 7$.*

Proof. We have proved, in chapter 2 § 1, that Γ has two inequivalent parabolic cusps say ∞ and 1. Obviously, the multiplier system v_ϑ is unramified at ∞. We shall prove now that v_ϑ is ramified at the cusp 1. Let $A = TU^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, so that $\infty = A < 1 >$. Since the width of the cusp sector at the cusp 1 is 1, the transformation $H = A^{-1}UA$ belongs to Γ. The transformation $H = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ generates a cyclic subgroup of $\Gamma_\vartheta/\{\pm E\}$ which has 1 as a fixed point. By theorems 15 and 16, we obtain that with respect to the cusp 1

$$e^{2\pi i\kappa} = \sigma(A, H)v_\vartheta(H)$$

$$= \sigma\left((-1,1), \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\right) \sigma((-1,0), U^{-2}) v_\vartheta(T) v_\vartheta(U^{-2})$$
$$= v_\vartheta(T) = e^{\pi i/4}$$

But by assumption $0 \le \kappa < 1$, therefore $\kappa = 1/8$ and if $f(\tau)$ belongs **131**
to $[\Gamma_\vartheta, k/2, v_\vartheta^k]$ for $1 \le k < 8$, then $f(\tau)$ has an unavoidable zero at $\tau = 1$ of multiplicity $k/8$. Since $v_\vartheta^8(S) = 1$ for every $S \in \Gamma_\vartheta$, $f^{24}(\tau)$ belongs to $[\Gamma_\vartheta, 12k, 1]$. But we have already proved that $\Delta^k(\tau)$ belongs to $[\Gamma_\vartheta, 12k, 1]$; therefore, the function $f^{24}(\tau)/\Delta^k(\tau)$ is invariant under the transformations of Γ_ϑ. This implies that $f^{24}(\tau)$ and $\Delta^k(\tau)$ have the same number of zeros on the Riemann surface $\mathscr{R}$ associated to Γ. Since $\Delta(\tau)$ has 3 zeros on the Riemann surface, namely a double zero at ∞ and a simple zero at 1, the number of zeros of $f(\tau)$ is $k/8$. In particular, the number of zeros of $\vartheta(\tau)$ is $1/8$. But $\vartheta(\tau)$ has a zero of multiplicity $1/8$ at 1; it follows that on $\mathscr{R}$, it is the only zero of $\vartheta(\tau)$ and $\vartheta(\tau) \neq 0$ for τ in $\mathscr{G}$. Thus, for $1 \le k < 8$, $\vartheta^{-k}(\tau) f(\tau)$ is invariant under the transformations of Γ_ϑ and has no singularities. Consequently, $\vartheta^{-k}(\tau) f(\tau)$ is constant, which proves our theorem completely.

It is an immediate consequence of the above theorem that the partial fraction series of $\vartheta^k(\tau)$ for $5 \le k < 8$ is a constant multiple of $\vartheta^k(\tau)$. But that constant has to be equal to 1, because the constant terms in the Fourier series of the above-mentioned forms are equal to 1; therefore, from the Fourier series of $G(\tau)$, we obtain the number of representations of an integer as sum of k squares for $5 \le k < 8$. For $1 \le k < 5$, the analogous results could be obtained by using the method of Hecke i.e. by considering the series of the type $G_2(\tau, s)$, which we introduced in chapter 2, § 4 for deducing an infinite product expression for $\Delta(\tau)$.

We shall now determine all the multiplier systems for the modular **132**
group. □

Theorem 19. *For the modular group Γ and for every weight r, there exist exactly 6 multiplier systems.*

Proof. For the proof of the theorem, it is sufficient to prove that for every weight r, there exists a multiplier system as we have already stated above.

Since $\Delta(\tau) \neq 0$ for τ belonging to $\mathscr{G}$, we can define $g(\tau) = \log\Delta(\tau)$ uniquely in $\mathscr{G}$, so that

$$g(\tau) = 2\pi i\tau + 24\sum_{n=1}^{\infty}\log(1 - e^{2\pi in\tau}).$$

We set $\Delta^r(\tau) = e^{rg(\tau)}$. From $\Delta(S < \tau >) = (c\tau + d)^{12}\Delta(\tau)$, follows that

$$\Delta^{r/12}(S < \tau >) = v_0(S)(c\tau + d)^r\Delta^{r/12}(\tau).$$

for $S = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ with a certain multiplier system $v_0(S)$ for the group Γ and the weight r. Hence the theorem is established.

In order to calculate the multiplier systems explicitly, we proceed as follows. Let τ_0 denote any one of the elliptic fixed points $e^{2\pi i/3}, i$ of Γ. Then $V = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0\end{smallmatrix}\right)$ (respectively $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$) generates the subgroup of Γ leaving $e^{2\pi i/3}$ (respectively i) fixed. Let $v(S)$ be a multiplier system for Γ and the weight r. Since

$$V^3 = T^2 = -E, w(T, T) = w(V^2, V) = -1 \text{ and } w(V, V) = 0,$$

we have

$$\begin{aligned} e^{-\pi ir} &= \sigma(V^2, V)\ \ \sigma(V, V)(v(V))^3 = \sigma(T, T)(v(T))^2 \\ &= e^{-2\pi ir}(v(V))^3 = e^{-2\pi ir}(v(T))^2, \end{aligned}$$

133 showing that

$$\begin{aligned} v(V) &= e^{\frac{\pi ir}{3}+\frac{2\pi i}{3}a}, (a = 0, 1, 2) \\ v(T) &= e^{\frac{\pi ir}{2}+\frac{2\pi i}{2}b}, (b = 0, 1). \end{aligned}$$

But the multiplier system v is uniquely determined by $v(V)$ and $v(T)$; therefore the six sets of pairs (a, b) completely determine the multiplier system.

Since $UV = T$, we have obviously

$$\begin{aligned} v(T) = \sigma(U, V)v(U)v(V) &= v(U)v(V) \\ &= e^{2\pi i\kappa}v(V), 0 \leq \kappa < 1, \end{aligned}$$

because $v(U) = e^{2\pi i\kappa}$, $0 \leq \kappa < 1$.

Therefore the following congruence holds:

$$\frac{r}{12} \equiv \kappa + \frac{a}{3} + \frac{b}{2} \pmod 1. \tag{6}$$

It is obvious that for a given r, the integers a, b are uniquely determined by the number κ. Moreover

$$\kappa \equiv \frac{r-h}{12} \pmod 1 \text{ for } h = 4a + 6b.$$

Hence we obtain the following table for the values of h, a and b,

h	0	4	6	8	10	14
a	0	1	0	2	1	2
b	0	0	1	0	1	1

which shows that the multiplier system is uniquely determined by h. For instance, for the multiplier system $v_0(S)$ given in the proof of theorem 19 it can be proved easily that 134

$$v_0(U) = e^{\frac{\pi i r}{6}} \Longrightarrow \kappa \equiv r/12 \pmod 1 \Longrightarrow h = 0 \Longrightarrow a = 0, b = 0,$$

showing that v_0 is the multiplier system for Γ and weight r determined by $h = 0$. We deduce immediately that the theta multiplier system cannot be extended to a multiplier system for the group Γ, because if it were true, then

$$v_\vartheta(U^2) = 1 \Longrightarrow v_\vartheta(U) = \pm 1 \Longrightarrow \kappa = 0 \text{ or } \frac{1}{2}$$

and this contradicts the congruence equation (6) when $r = \frac{1}{2}$.

Before concluding this section, we give an application of the theory of modular forms to the theory of quadratic forms. We shall prove here that an even integral matrix $Q > 0$ with $|Q| = 1$ exists only if $m \equiv 0 \pmod 8$, where m is the number of rows and columns of Q. By theorem 15,

$$\vartheta(\tau, Q) = \sum_g e^{\pi i \tau Q[g]}$$

belongs to $[\Gamma, m/2, v]$ for some multiplier system v. Since $Q[x]$ is an even integer for every integral vector x, it follows immediately that $v(U) = 1$ implying that $\kappa = 0$. Similarly, we obtain that $v(T) = e^{\pi im/4}$ and therefore $b = 0$. The result now follows from the congruence relation 6. Conversely, if $m \equiv 0 (\mod 8)$, then there exist m-rowed even
135 integral matrices, as is well known from the theory of quadratic forms. A nice construction of such forms has been given by E. Witt with the help of the theory of lattices in Abh. Math. Sem. Hamburg 14 (1941). □

2 Poincare Series and Eisenstein Series

In this section, we shall denote, as before, the modular group by Γ and a subgroup of finite index in Γ by Γ_0. Moreover, unless otherwise stated, Γ_0 will be assumed to contain $-E$. We shall construct some special modular forms called 'Eisenstein series' and 'Poincaré series' which not only belong to the space $[\Gamma_0, r, v]$ but also generate the same. The proof of this statement will be completed in the next section. First of all, we show that $[\Gamma_0, r, v]$ is a vector space of finite dimension over the complex number field. Let $f(\tau)$ and $g(\tau)$ be two modular forms belonging to $[\Gamma_0, r, v]$ such that $f(\tau) \not\equiv 0$, $g(\tau) \not\equiv 0$; then $f(\tau)/g(\tau)$ is an automorphic function for the group Γ_0, which is either a constant or has the order zero. In any case, $f(\tau)$ and $g(\tau)$ have the same number of (always inequivalent) zeros. Let ν be the number of zeros of $f(\tau)$ and N_0 the least integer greater than ν. Let

$$f(\tau) = \sum_{n+\kappa\geq 0} c_{n+\kappa} e^{2\pi i(n+\kappa)\tau/N}, (0 \leq \kappa < 1)$$

be the Fourier expansion of $f(\tau)$ at the cusp ∞. Then $f(\tau)$ is determined uniquely by the coefficients $c_{n+\kappa}$ for $n \leq N_0$. Indeed, if $c_{n+\kappa} = 0$ for $n \leq N_0$, then $f(\tau)$ must vanish identically, since the total number ν of zeros of $f(\tau) \not\equiv 0$ is less than N_0. This shows that a modular form $f(\tau)$ belonging to $[\Gamma_0, r, v]$ is uniquely determined by $N_0 + 1$ coefficients in its Fourier series at ∞ and therefore the dimension of $[\Gamma_0, r, v]$ is atmost $N_0 + 1$.

136 We shall now explain the construction of Poincaré series for Γ_0. Let

$f(\tau)$ be a modular form belonging to the space $[\Gamma_0, r, v]$. Then at the cusp $A^{-1} < \infty >$ with $A = \left(\begin{smallmatrix} a_0 & a_3 \\ a_1 & a_2 \end{smallmatrix}\right) \in \Gamma$, $f(\tau)$ has the Fourier expansion

$$(a_1\tau + a_2)^r f(\tau) = \sum_{n+\kappa\geq 0} c_{n+\kappa} e^{2\pi i(n+\kappa)A<\tau>/N}$$

where N is the least natural number so that $H = A^{-1}U^N A$ belongs to Γ_0 and κ is a real number determined by

$$\sigma(A, H)v(H) = e^{2\pi i\kappa}, 0 \leq \kappa < 1.$$

If we replace τ by $L < \tau >$ with $L = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0$, then by using the transformation formula for $f(\tau)$, we obtain

$$f(\tau) = \sum_{n+\kappa\geq 0} c_{n+\kappa} \frac{e^{2\pi i(n+\kappa)M<\tau>/N}}{\sigma(A, L)v(L)(m_1\tau + m_2)^r}$$

where $M = AL = \left(\begin{smallmatrix} m_0 & m_3 \\ m_1 & m_2 \end{smallmatrix}\right)$. We now make use of the *principle of cross summation* i.e. in the above expansion for $f(\tau)$, we put $c_{n+\kappa} = 0$ for all n except for one fixed n for which we put $c_{n+\kappa} = 1$ and then formally $\sum$ over a complete system $\gamma(A, \Gamma_0)$ of transformations $M = AL$ with $L \in \Gamma_0$ such that the second rows (m_1, m_2) of various matrices $M = AL$ are distinct. In other words, we form the function

$$G_r(\tau, v, A, \Gamma_0, n + \kappa) = \sum_{M\in\gamma(A,\Gamma_0)} \frac{e^{2\pi i(n+\kappa)M<\tau>/N}}{\sigma(A, L)v(L)(m_1\tau + m_2)^r}. \qquad (1)$$

This series is a so-called *Poincaré series for the group* Γ_0. We shall now show that the Poincare series $G_r(\tau, v, A, \tau_0, n + \kappa)$ does not depend **137**
upon any special choice of the system $\gamma(A, \Gamma_0)$ and for $r > 2$, it belongs to the space $[\Gamma_0, r, v]$. Let L and L^* be two transformations of Γ_0 such that $M = AL$ and $M^* = AL^*$ have the same second row. Then $M^* = U^k M$ for some integer k, therefore

$$\begin{aligned} L^*L^{-1} = A^{-1}U^k A \in \Gamma_0 &\Longrightarrow N \text{ divides } k \\ &\Longrightarrow L^*L^{-1} = H^\ell \text{ if } k = N\ell \text{ and } H = A^{-1}U^N A. \end{aligned}$$

Consequently, we have

$$e^{2\pi i(n+\kappa)M^{*}<\tau>/N} = e^{2\pi i\kappa\ell} e^{2\pi i(n+\kappa)M<\tau>/N}.$$

Further

$$\begin{aligned} p_\ell &= \sigma(A, H^\ell)v(H^\ell) = \sigma(A, H^\ell)\sigma(H, H^{\ell-1})v(H)v(H^{\ell-1}) \\ &= \sigma(A, H^{\ell-1}) \;\; \sigma(A, H)v(H)v(H^{\ell-1}) = e^{2\pi i\kappa} p_{\ell-1} = e^{2\pi i\ell\kappa}. \end{aligned}$$

But

$$\begin{aligned} \sigma(A, L^*)v(L^*) &= \sigma(A, H^\ell L)v(H^\ell L) \\ &= \frac{\sigma(AH^\ell)\sigma(A, H^\ell)\sigma(H^\ell, L)}{\sigma(H^\ell, L)} v(H^\ell)v(L) \\ &= \sigma(A, L)p_\ell v(L) = e^{2\pi i\ell\kappa}\sigma(A, L)v(L); \end{aligned}$$

therefore, if we take M^* instead of M in $\gamma(A, \Gamma_0)$, the contribution to the series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ remains unchanged, which proves that the series does not depend upon any special choice of the system $\gamma(A, \Gamma_0)$. The series $\sum\limits_{(m_1,m_2)\neq(0,0)} |m_1\tau + m_2|^{-r}$, which converges uniformly in ev-
138 ery domain $|x| \leq c$, $y \geq \in > 0$ for $r > 2$, is a majorant for the series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$. It follows in the same manner as for $v = 1$, $n+\kappa = 0$, that the series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ converges absolutely and uniformly in every domain $|x| \leq c$, $y \geq \in > 0$ for $r > 2$ and therefore represents a regular function in $\mathscr{G}$. In order to examine the behaviour of the series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ under the transformations of the modular group,we define, for any $S \in \Gamma$, the transform v^S of the multiplier system v by

$$v^S(L^*) = v(L)\frac{\sigma(L, S)}{\sigma(S, L^*)} \text{ with } L^* = S^{-1}LS.$$

If $L_i^* = S^{-1}L_iS$ for $L_i \in \Gamma_0 (i = 1, 2)$, then

$$\begin{aligned} & v^S(L_1^*L_2^*) = \sigma(L_1^*, L_2^*)v^S(L_1^*)v^S(L_2^*) \\ & \iff v(L_1L_2)\frac{\sigma(L_1L_2, S)}{\sigma(S, L_1^*L_2^*)} = \sigma(L_1^*, L_2^*)v(L_1)\frac{\sigma(L_1, S)}{\sigma(S, L_1^*)}v(L_2)\frac{\sigma(L_2, S)}{\sigma(S, L_2^*)} \end{aligned}$$

$$\begin{aligned}
&\iff \frac{\sigma(L_1,L_2)\sigma(L_1L_2,S)}{\sigma(S,L_1^*L_2^*)} = \frac{\sigma(L_1^*,L_2^*)\sigma(L_1,S)\sigma(L_2,S)}{\sigma(S,L_1^*)\sigma(S,L_2^*)}\\
&\iff \sigma(L_1,L_2S)\sigma(S,L_1^*)\sigma(S,L_2^*) = \sigma(L_1,S)\sigma(S,L_1^*)\sigma(L_1S,L_2^*)\\
&\iff \sigma(L_1,L_2S)\sigma(S,L_1^*)\sigma(S,L_2^*) = \sigma(L_1,S)\sigma(S,L_1^*)\sigma(L_1S,L_2^*)\\
&\iff \sigma(L_1,L_2S)\sigma(S,L_2^*) = \sigma(S,L_2^*)\sigma(L_1,L_2S).
\end{aligned}$$

Thus v^S is a multiplier system for the group $S^{-1}\Gamma_0 S$ and weight r. In particular, when S belongs to Γ_0,

$$V^S(S^{-1}LS) = \frac{\sigma(S^{-1},LS)\sigma(L,S)}{\sigma(S^{-1},S)}v(L) = \frac{\sigma(L,S)}{\sigma(S,L^*)}v(L),$$

showing that $v^S = v$. It can be verified easily that **139**

$$\begin{aligned}
\gamma(A,\Gamma_0)S &= \Gamma'(A,\Gamma_0) \text{ for } S\in\Gamma_0,\\
\gamma(A,\Gamma_0)S &= \gamma''(AS,S^{-1}\Gamma_0 S) \text{ for } S\in\Gamma,
\end{aligned}$$

where γ' and γ'' are systems of the same type as γ. Let $S = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ be an arbitrary element of Γ and (m_1^*, m_2^*) the second row of the matrix $MS = ALS$. Since

$$\begin{aligned}
\sigma(A,L)\ \ \sigma(AL,S)v(L) &= \frac{\sigma(A,L)\sigma(AL,S)\sigma(S,L^*)}{\sigma(L,S)}v^S(L^*)\\
&= \sigma(A,LS)\sigma(S,L^*)v^S(L^*)\\
&= \sigma(A,S)\sigma(AS,L^*)v^S(L^*),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{G_r(S<\tau>,v,A,\Gamma_0,n+\kappa)}{(c\tau+d)^r}\\
&\quad= \frac{1}{\sigma(A,S)}\sum_{M^*\in\gamma''(AS,S^{-1}\Gamma_0 S)}\frac{e^{2\pi i(n+\kappa)M^*<\tau>/N}}{\sigma(AS,L^*)v^S(L^*)(m_1^*\tau+m_2^*)^r}
\end{aligned}$$

where $M^* = MS = ALS$. This shows that

$$\frac{G_r(S<\tau>,v,A,\Gamma_0,N+\kappa)}{(c\tau+d)^r} = \frac{1}{\sigma(A,S)}G_r(\tau,v^S,AS,S^{-1}\Gamma_0 S,n+\kappa).$$

In particular, when S belongs to Γ_0, we have

$$\frac{G_r(S<\tau>, v, A, \Gamma_0, n+\kappa)}{(c\tau+d)^r} = v(S)\sum_{M^*\in\gamma'(A,\Gamma_0)} \frac{e^{2\pi i(n+\kappa)M^*<\tau>/N}}{\sigma(A, LS)v(LS)(m_1^*\tau+m_2^*)^r} \quad i.e.$$
$$\frac{G_r(S<\tau>, v, A, \Gamma_0, n+\kappa)}{(c\tau+d)^r} = v(S)G_r(\tau, v, A, \Gamma_0, n+\kappa),$$

140 since

$$\sigma(A, L)\sigma(AL, S)v(L) = \frac{\sigma(A, L)\sigma(AL, S)}{\sigma(L, S)v(S)}v(LS)$$
$$= \frac{\sigma(A, LS)v(LS)}{v(S)}.$$

The above discussion shows that for $r > 2$, $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ belongs to $[\Gamma_0, r, v]$ provided the Fourier expansion of $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ at the cusp ∞ contains no terms with a negative exponent. But the latter is obvious from the fact that the majorant $\sum\limits_{(m_1,m_2)\neq(0,0)} |m_1\tau+m_2|^{-r}$ of $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ is bounded uniformly in x when y tends to infinity. Hence we have proved

Theorem 20. *The Poincaré series*

$$G_r(\tau, v, A, \Gamma_0, n+\kappa) = \sum_{M\in\gamma(A,\Gamma_0)} \frac{e^{2\pi i(n+\kappa)M<\tau>/N}}{\sigma(A, L)v(L)(m_1\tau+m_2)^r}$$

represents a modular form of weight r for the group Γ_0 and multiplier system v, provided $r > 2$. Moreover, it is a cusp form, for $n+\kappa > 0$.

If the multiplier system v is unramified at the cusp $A^{-1} < \infty >$ i.e. $\kappa = 0$, then for $n = 0$

$$G_r(\tau, v, A, \Gamma_0, 0) = \sum_{M\in\gamma(A,\Gamma_0)} \frac{1}{\sigma(A, L)v(L)(m_1\tau+m_2)^r}.$$

Since we can assume that with AL, also $-AL$ belongs to $\gamma(A, \Gamma_0)$ and the contribution of both these transformations to $G_r(\tau, v, A, \Gamma_0, 0)$ is the same, we see immediately that

$$G_r(\tau, v, A, \Gamma_0, 0) = 2G(\tau, A),$$

141 where $G(\tau, A)$ is the series introduced in the previous section. We refer to the series $G_r(\tau, v, A, \Gamma_0, 0)$ as the Eisenstein series for the group Γ_0.

In what follows, we shall adhere to the following notation:

$$(f|S)(\tau) := f(S<\tau>)(c\tau + d)^{-r}$$

for real $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $|S| > 0$ and any f defined on $\mathscr{G}$. It is obvious that

$$(f|S_1S_2)(\tau) = \sigma(S_1, S_2)((f|S_1)|S_2)(\tau)$$

for two real matrices S_1 and S_2. If $f(\tau)$ belongs to $[\Gamma_0, r, v]$, then

$$(f|S)(\tau) = v(S)f(\tau) \text{ for } S \in \Gamma_0.$$

If S_1 and S_2 are two transformations belonging to the modular group, then

$$(v^{S_1})^{S_2} = v^{S_1S_2};$$

for, if L belongs to Γ_0; then

$$\begin{aligned}
(v^{S_1})^{S_2}(S_2^{-1}S_1^{-1}LS_1S_2) &= v^{s_1}(S_1^{-1}LS_1)\frac{\sigma(S_1^{-1}LS_1, S_2)}{\sigma(S_2, S_2^{-1}S_1^{-1}LS_1S_2)}\\
&= v(L)\frac{\sigma(L, S_1)\sigma(S_1^{-1}LS_1, S_2)}{\sigma(S_1, S_1^{-1}LS_1)\sigma(S_2, S_2^{-1}S_1^{-1}LS_1S_2)}\\
&= v(L)\frac{\sigma(L, S_1)\sigma(S_1^{-1}, LS_1S_2)\sigma(LS_1, S_2)\sigma(S_1^{-1}, LS_1)}{\sigma(S_1^{-1}, LS_1)\sigma(S_1, S_1^{-1})\sigma(S_1^{-1}S_1S_2, S_2^{-1}S_1^{-1}LS_1S_2)}\\
&= v(L)\frac{\sigma(S_1^{-1}, LS_1S_2)\sigma(S_1, S_2)\sigma(L, S_1S_2)\sigma(S_1^{-1}, S_1S_2)}{\sigma(S_1, S_1^{-1})\sigma(S_1^{-1}, LS_1S_2)\sigma(S_1S_2, S_2^{-1}S_1^{-1}LS_1S_2)}\\
&= v(L)\frac{\sigma(L, S_1S_2)}{\sigma(S_1S_2, S_2^{-1}S_1^{-1}LS_1S_2)} = v^{S_1S_2}(S_2^{-1}S_1^{-1}LS_1S_2).
\end{aligned}$$

With the help of this composition rule for multiplier system, it is 142
easy to see that the mapping

$$f(\tau) \to g(\tau) = (f|A)(\tau) \quad (A \in \Gamma)$$

is a bijection linear transformation from the vector space $[\Gamma_0, r, v]$ to the vector space $[A^{-1}\Gamma_0 A, r, v^A]$. If $f(\tau)$ belongs to $[\Gamma_0, r, v]$, then $(f|A)(\tau)$ belongs to $[A^{-1}\Gamma_0 A, r, v^A]$; indeed, for $L \in \Gamma_0$,

$$((f|A)|A^{-1}LA)(\tau) = \sigma(A^{-1}, LA)(((f|A)|A^{-1})|LA)(\tau)$$

$$
\begin{aligned}
&= \frac{\sigma(A^{-1}, LA)}{\sigma(A, A^{-1})}(f|LA)(\tau)\\
&= \frac{\sigma(A^{-1}, LA)\sigma(L, A)}{\sigma(A, A^{-1})}((f|L)|A)(\tau)\\
&= \frac{\sigma(A^{-1}, LA)\sigma(L, A)}{\sigma(A, A^{-1})}v(L)(f|A)(\tau)\\
&= v^A(A^{-1}LA)(f|A)(\tau).
\end{aligned}
$$

It is, moreover, trivial to see that $(f|A)(\tau)$ is regular in $\mathscr{G}$ and in its Fourier expansion at the various parabolic cusps of $A^{-1}\Gamma_0 A$, no term with a negative exponent occurs. Conversely, if $g(\tau)$ belongs to $[A^{-1}\Gamma_0 A, r, v^A]$, then as above $(g|A^{-1})(\tau)$ belongs to $[\Gamma_0, r, (v^A)^{A^{-1}}]$. but $(v^A)^{A^{-1}} = v^{A}A^{-1} = v$; therefore $(g|A^{-1})(\tau)$ belongs to $[\Gamma_0, r, v]$, proving our assertion above.

The following theorem gives explicitly the dimension of the space $[\Gamma, r, v]$.

Theorem 21. *Let v be a multiplier system for the modular group and*
143 *weight r such that*

$$v(U) = e^{2\pi ik}, v(V) = e^{\frac{\pi ir}{3}+\frac{2\pi ia}{2}}, v(T) = e^{\frac{\pi ir}{2}+\frac{2\pi ib}{2}},$$

where $0 \le \kappa < 1$, $0 \le a < 3$, $0 \le b < 2$ and a, b are integers. Then

$$\text{dimension } [\Gamma, r, v] = \begin{cases} \frac{r}{12} - \kappa - \frac{a}{3} - \frac{b}{2} + 1 & , \text{ for } r - 12\kappa \ge 0 \\ 0 & , \text{ for } r - 12\kappa < 0 \end{cases}$$

Proof. (i) If k is a negative even integer, then the dimension of the space $[\Gamma, k, 1]$ is zero. Assume that k is a negative even integer and the modular form $f(\tau)$ belongs to $[\Gamma, k, 1]$. Then the function $f^2(\tau)G_4^{-k/2}(\tau)$, which is invariant under the transformations of Γ and has no poles, must be a constant. But we have already seen that $G_4(\rho) = 0$ for $\rho = e^{2\pi i/3}$; therefore $f^2(\tau)G_4^{-k/2}(\tau) = 0$, implying that $f(\tau) \equiv 0$. Hence the dimension of the space $[\Gamma, k, 1]$ is zero for negative k.

(ii) Let r^* be the weight and v^* the multiplier system of the modular form $\Delta^\kappa(\tau)$ for the group Γ. Let κ^*, a^* and b^* be the numbers determined by v^* in the sense of theorem 21. Obviously $\kappa^* = \kappa$ and $r^* = 12\kappa$; therefore $\kappa^* = \kappa = \dfrac{r^*}{12}$. But we have proved in the previous section that $\dfrac{r^*}{12} \equiv \kappa^* + \dfrac{a^*}{3} + \dfrac{b^*}{2} \pmod 1$; therefore, it follows that $a^* = b^* = 0$. Consequently, we obtain

$$\frac{r^*}{6} + \frac{a^*}{3} = \frac{r^*}{6} = 2\kappa \equiv \frac{r}{6} + \frac{a}{3} \pmod 1$$
$$\frac{r^*}{4} + \frac{b^*}{2} = \frac{r^*}{4} = 3\kappa \equiv \frac{r}{4} + \frac{b}{2} \pmod 1,$$

showing that v and v^* take the same values for the transformations U, V and T. Since $r \equiv 12\kappa = r^* \pmod 2)$, v and v^* are identical. Let **144**
$f(\tau)$ be a modular form belonging to the space $[\Gamma, r, v]$. Then $f(\tau)$ has an unavoidable zero of multiplicity at least equal to κ at the cusp ∞ and the above discussion shows that $g(\tau) = f(\tau)\Delta^{-\kappa}(\tau)$ belongs to $[\Gamma, r - 12\kappa, 1]$. Conversely, if $g(\tau)$ belongs to $[\Gamma, r - 12\kappa, 1]$, then $g(\tau)\Delta^\kappa(\tau)$ belongs to $[\Gamma, r - 12\kappa, 1]$, then $g(\tau)\Delta^\kappa(\tau)$ belongs to $[\Gamma, r, v]$. Thus it is proved that the dimensions of $[\Gamma, r, v]$ and $[\Gamma, r - 12\kappa, 1]$ are the same. Therefore, if

$$\frac{r - 12\kappa}{12} = g + \frac{a}{3} + \frac{b}{2},$$

where g, a and b are integers such that $0 \leq a < 3$, $0 \leq b < 2$, then by theorem 14 it follows that, for $r - 12\kappa \geq 0$ (due to $r - 12\kappa$ being an even integer),

$$\text{dimension } [\Gamma, r - 12\kappa, 1] = g + 1 = \frac{r - 12\kappa}{12} - \frac{a}{3} - \frac{b}{2} + 1.$$

For $r - 12\kappa < 0$, the dimension of $[\Gamma, r - 12\kappa, 1]$ is zero by part (i) of our proof. Hence the theorem is established.

As in the case of modular forms of integral weight for the modular group, the number of zeros of a modular form $f(\tau)$ belonging to $[\Gamma, r, v]$ is $r/12$. Indeed, the number of zeros of $f(\tau)$ is equal to the sum of the number of zeros of $\Delta^\kappa(\tau)$ and $g(\tau) = f(\tau)\Delta^{-\kappa}(\tau)$. Now we have seen in the course of the proof of theorem 21 that $g(\tau)$ is a modular form the integral weight $r - 12\kappa$ and so the number of zeros of $g(\tau)$ is a modular form of integral weight $r - 12\kappa$ and so the number of zeros of $g(\tau)$ is $\dfrac{r-12\kappa}{12}$. Therefore the number of zeros of $f(\tau)$ is $\kappa + \dfrac{r-12\kappa}{12} = \dfrac{r}{12}$. □

3 Metrisation and Completeness Theorem

The general transformation formula

$$(G_r(\tau, v, A, \Gamma_0, n+\kappa)|A)(\tau) = \frac{1}{\sigma(A,S)} G_r(\tau, v^S, AS, S^{-1}\Gamma_0 S, n+\kappa)$$

145 proved for the Poincaré series defined by (1) of § 2 shows that $G_r(\tau, v, A, \Gamma_0, b+\kappa)$ is a cusp form for $n+\kappa > 0$; in case $n+\kappa = 0$ i.e. $n = 0 = \kappa$, $G_r(\tau, v, A, \Gamma_0, 0)$ does not represent a cusp form, since the constant term in its Fourier expansion at the cusp $A^{-1} < \infty >$ is different from zero. In order to prove that the Poincaré series and Eisenstein series together generate the space $[\Gamma_0, r, v]$, it is sufficient in view of theorem 17 to prove the same for the space $[\Gamma, r, v_0]$ of cusp forms contained in $[\Gamma_0, r, v]$. By using *Petersson's Metrisation Principle*, we shall prove presently that the series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ generate the space $[\Gamma_0, r, v]_0$ for any (fixed and) given A.

Let D be a hyperbolic triangle in $\mathscr{G}$ with proper or improper vertices and $f(\tau)$, $g(\tau)$ two functions, which are continuous in D and are such that the function $f(\tau)\overline{g(\tau)}y^r (\tau = x+iy)$ is bounded in D. Then the integral

$$\chi(D, f, g) = \iint\limits_D f(\tau)\overline{g(\tau)}y^{r-2}dxdy$$

exists. If $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any real matrix of determinant 1, then

$$\chi(D_{S^{-1}}, f|S, g|S) = \chi(D, f, g),$$

where $D_{S^{-1}}$ is the image of D by S^{-1}; indeed, if we replace τ by $S < \tau >$ in the integrand in $\chi(D, f, g)$ and use the invariance of $y^{-2}dxdy$, we have

$$f(S < \tau >)\overline{g(S < \tau >)}\{y/(c\tau+d)(c\overline{\tau}+d)\}^r y^{-2}dxdy$$

$$= (f|S)(\tau)\overline{(g|S)(\tau)}y^{r-2}dxdy,$$

which proves the assertion. In particular, if $f(\tau)$ and $g(\tau)$ belong to $[\Gamma_0, r, v]$, then the integral $\chi(D, f, g)$ exists provided that either $f(\tau)$ of $g(\tau)$ vanishes at an improper vertex of D. Moreover, if S belongs to Γ_0, then obviously **146**

$$\chi(D_{S^{-1}}, f, g) = \chi(D_{S^{-1}}, f|S, g|S) = \chi(D, f, g).$$

Let $\mathfrak{F}_0$ be a fundamental domain for Γ_0, which consists of a finite number of hyperbolic triangles. Further, let at least one of $f(\tau)$ and $g(\tau)$ in $[\Gamma_0, r, v]$, be a cusp form. Then the integral

$$(f,g)_{\Gamma_0} = \frac{1}{\Im(\mathfrak{F})} \iint\limits_{\mathfrak{F}_0} f(\tau)\overline{g(\tau)}y^{r-2}dxdy$$

exists, where $\Im(\mathfrak{F}_0)$ is the hyperbolic area of Γ_0. From the transformation property of $\chi(D, f, g)$ above, it follows that the integral $(f,g)_{\Gamma_0}$ does not depend upon the choice of $\mathfrak{F}_0$. As a matter of fact, $(f,g)_{\Gamma_0}$ is independent of Γ_0 as well, in the sense that if f and g are two modular forms of weight r for two subgroups Γ_0 and Γ_1 of finite index in Γ, then $(f,g)_{\Gamma_0} = (f,g)_{\Gamma_1}$. We consider first the case when one of the groups Γ_1 or Γ_2 contains the other. Let us assume that Γ_1 is contained in Γ_0 and $(\Gamma_0 : \Gamma_1) = \mu$. Let $\Gamma_0 = \bigcup_{i=1}^{\mu} \Gamma_1 A_i$ for some A_i belonging to $\Gamma_0, i = 1, 2, \ldots, \mu$ be a coset decomposition of Γ_0 modulo Γ_1. Then $\mathfrak{F}_1 = \bigcup_{i=1}^{\mu} (\mathfrak{F}_0)_{A_i}$ is a fundamental domain for Γ_1. Since $\Im(\mathfrak{F}_1) = \mu\Im(\mathfrak{F}_0)$ **147**
and $\bigcup(\mathfrak{F}_0)_{A_i}$ for $i = 1, 2, \ldots, \mu$ is a fundamental domain for Γ_0, it follows immediately that

$$\begin{aligned}
(f,g)_{\Gamma_0} &= \frac{1}{\mu}\sum_{i=1}^{\mu} \frac{1}{\Im((\mathfrak{F})_{A_i})} \iint\limits_{(\mathfrak{F}_0)_{A_i}} f(\tau)\overline{g(\tau)}y^{r-2}dx\,dy \\
&= \frac{1}{\mu\Im(\mathfrak{F}_0)} \iint\limits_{\bigcup_i (\mathfrak{F}_0)_{A_i}} f(\tau)\overline{g(\tau)}y^{r-2}dx\,dy \\
&= \frac{1}{\Im(\mathfrak{F}_1)} \iint\limits_{\mathfrak{F}_1} f(\tau)\overline{g(\tau)}y^{r-2}dx\,dy = (f,g)_{\Gamma_1}.
\end{aligned}$$

In the general case, consider the group Γ^* generated by Γ_0 and Γ_1. It is obvious that both Γ_1 and Γ_0 are subgroups of finite index in Γ^* and $f(\tau)$ and $g(\tau)$ are modular forms of weight r for the group Γ^* and the same multiplier system. Hence, by the particular case considered above, we get $(f,g)_{\Gamma_0} = (f,g)_{\Gamma^*} = (f,g)_{\Gamma_1}$.

In the following, so long as no confusion is possible, we shall simply write (f,g) for $(f,g)_{\Gamma_0}$. We call (f,g) *the scalar product* of f and g. The following properties of the scalar product (f,g) are immediate from the definition:

1) (f,g) is linear in f.

2) $\overline{(f,g)} = (g,f)$.

3) $(f,f) \geq 0, (f,f) = 0 \Longrightarrow f = 0$.

4) $(f|A, g|A) = (f,g)$ for A in Γ .

148 Properties 1), 2) and 3) show that the scalar product (f,g) defines a positive-definite unitary metric on the space $[\Gamma_0, r, v]_0$.

For the explicit calculation of the scalar product of a cusp form and a Poincaré series, we need to prove

Lemma 5. *If $f(\tau)$ is a cusp form in $[\Gamma_0, r, v]$, then $|f(\tau)|y^{r/2}$ (with $\tau = x+iy$) is bounded in $\mathscr{G}$.*

Proof. If in $|f(\tau)|y^{r/2}$, we replace τ by $A<\tau>$ $(A \in \Gamma)$, then $|f(\tau)|y^{r/2}$ is transformed to $|(f|A)(\tau)|y^{r/2}$. In particular, when A belongs to Γ_0, $|(f|A)(\tau)| = |v(A)f(\tau)| = |f(\tau)|$ and therefore $|f(\tau)|y^{r/2}$ is left invariant by Γ_0. Thus, in order to complete the proof of the lemma, it is sufficient to prove that $|f(\tau)|y^{r/2}$ is bounded in a fundamental domain $\mathfrak{F}_0$ or Γ_0. Let $\rho_1, \rho_2, \ldots, \rho_\sigma$ be a complete system of inequivalent cusps of Γ_0 and let $\infty = A_j < \rho_j >$, $A_j \in \Gamma$ for $j = 1, 2, \ldots, \sigma$. Let $\mathfrak{p}_j (j = 1, 2, \ldots, \sigma)$ be cusp sectors at the cusps ρ_j of Γ_0. Then $\mathfrak{F}_0 = \bigcup_{j=1}^{\sigma} \mathfrak{p}_j$ is a fundamental domain for Γ_0 and for the proof of the lemma, it is sufficient to prove that $|f(\tau)|y^{r/2}$ is bounded in each $\mathfrak{p}_j$. It is obvious that $|f(\tau)|y^{r/2}$ is bounded in $\mathfrak{p}_j$ if and only if $|(f|A_j^{-1})(\tau)|y^{r/2}$ is bounded in $A_j < \mathfrak{p}_j >$. Since $(f|A{-}1_j)(\tau)$ has the Fourier expansion

$$\sum_{n+\kappa_j>0} c_n + \kappa_j e^{2\pi i(n+\kappa_j)\tau/N_j}.$$

it tends to zero exponentially as $y \to \infty$, uniformly in x. It follows that $|(f|A_j^{-1})(\tau)|y^{r/2}$ tends to zero as $y \to \infty$ and therefore is bounded in $A_j < \mathfrak{p}_j >$. Hence the lemma is proved. □

For the sake of brevity, we set $\gamma = \gamma(E, \Gamma_0)$ and **149**

$$e(\tau) = e^{2\pi i(n+\kappa)\tau/N}.$$

By the definition of Poincaré series, we have

$$G(\tau) := G_r(\tau, v, E, \Gamma_0, n+\kappa) = \sum_{\in\Gamma} \overline{v(L)}(e|L)(\tau).$$

For the scalar product (f, G) of a cusp form $f(\tau)$ in $[\Gamma_0, r, v]$ with $G(\tau)$, formal calculation yields

$$\begin{aligned}
(f,G) &= \frac{1}{\Im(\mathfrak{F}_0)} \iint\limits_{\mathfrak{F}_0} f(\tau) \sum_{L\in\gamma} v(L)\overline{(e|L)(\tau)} y^{r-2} dxdy \\
&= \frac{1}{\Im(\mathfrak{F}_0)} \sum_{L\in\gamma} \iint\limits_{\mathfrak{F}} (f|L)(\tau)\overline{(e|L)(\tau)} y^{r-2} dxdy. \\
&= \frac{1}{\Im(\mathfrak{F}_0)} \sum_{L\in\gamma} \iint\limits_{(\mathfrak{F}_0)_L} f(\tau)\overline{e(\tau)} y^{r-2} dxdy \\
&= \frac{2}{\Im(\mathfrak{F}_0)} \iint\limits_{\mathscr{L}} f(\tau)\overline{e(\tau)} y^{r-2} dxdy, \qquad (1)
\end{aligned}$$

where $\mathscr{L} = \bigcup\limits_{L\in\gamma} (\mathfrak{F}_0)_L$. The factor 2 appears on the right hand side of (1), because we can assume that both L and $-L$ belong to the set and their contributions to the sum are the same. The interchange of summation and integration would be justified in the above formal computation, if

we prove that the last integral converges absolutely. Let Z denote the subgroup of Γ_0 generated by U^N, where N is the least natural number so determined that U^N belongs to Γ_0. Let $\Gamma_0 = \bigcup_i ZL_i$, $L_i \in \Gamma_0$, be a coset decomposition of Γ_0 modulo Z. Then the set of all the matrices L_i is a
150 possible choice for the set γ and therefore $\mathscr{L}$ is a fundamental domain for Z. We decompose the set $\mathscr{L}$ into a countable number of sets $\mathscr{L}_k$ such that the images of $\mathscr{L}_k$ by elements Z belongs to the domain $\mathscr{G} = \{\tau | \tau = x+iy, 0 \leq x < N, y \geq 0\}$, which is a fundamental domain for Z. Since the integrand $f(\tau)\overline{e(\tau)}y^{r-2}dxdy$ is invariant under the transformation $\tau \to \tau + N$, we get immediately that

$$\frac{2}{\Im(\mathfrak{F}_0)} \int\int_{\mathscr{L}} f(\tau)\overline{e(\tau)}y^{r-2}dxdy = \frac{2}{\Im(\mathfrak{F}_0)} \int_0^\infty \int_0^N f(\tau)\overline{e(\tau)}y^{r-2}dxdy \quad (2)$$

Lemma 6 can be used to see that the integral on the right hand side of (2) converges absolutely for $r > 2$. Hence the formal computation for obtaining (1) is justified and we have indeed

$$(f, G) = \frac{2}{\Im(\mathfrak{F}_0)} \int_0^\infty \int_0^N f(\tau)e^{-2\pi i(n+\kappa)\overline{\tau}/N}y^{r-2}dxdy.$$

Using the Fourier expansion of f at the cusp ∞ given by

$$f(\tau) = \sum_{k+\kappa>0} c_{k+\kappa}(f)e^{2\pi i(k+\kappa)\tau/N}$$

with

$$c_{k+\kappa}(f) = \frac{1}{N}\int_0^N f(\tau)e^{2\pi i(k+\kappa)\tau/N}dx,$$

it follows immediately that $(f, G) = 0$ in case $n + \kappa = 0$ and otherwise,

$$(f, G) = \frac{2}{\Im(\mathfrak{F}_0)} \int_0^\infty \{\int_0^N f(\tau)e^{2\pi i(n+\kappa)\tau/N}dx\}e^{2\pi i(n+\kappa)(\tau-\overline{\tau})/N}y^{r-2}dy$$

$$= \frac{2N}{\Im(\mathfrak{F}_0)} c_{n+\kappa}(f) \int_0 e^{-4\pi(n+\kappa)y/N} y^{r-2} dy$$
$$= \frac{2N}{\Im(\mathfrak{F})} (4\pi(n+\kappa)/N)^{1-r} \Gamma(r-1) c_{n+\kappa}(f) \text{ for } n+\kappa > 0$$

The general case can be reduced to this particular case ($A = E$), if we note that 151

$$(f, G_r(\quad, v, A, \Gamma_0, n+\kappa)) =$$
$$= (f|A^{-1}, G_r(\quad, v, A, \Gamma_0, n+\kappa)|A^{-1})$$
$$= \frac{1}{\sigma(A, A^{-1})} (f|A^{-1}, G_r(\quad, v^{A^{-1}}, E, A, \Gamma_0 A^{-1}, n+\kappa)).$$

We are thus led to the *fundamental formula of the metrisation principle* as stated in

Theorem 22. *Let $f(\tau)$ be a cusp form belonging to $[\Gamma_0, r, v]$ for $r > 2$, and let*

$$(f|A^{-1})(\tau) = \sum_{n+\kappa>0} c_{n+\kappa}(f, A) e^{2\pi i(n+\kappa)\tau/N}$$

be the Fourier expansion of $f(\tau)$ at the parabolic cusp $A^{-1} < \infty >$ of Γ_0 with A in Γ. Then we have

$$(f, G_r(\quad, v, A, \Gamma_0, n+\kappa))$$
$$= \begin{cases} \frac{2N\Gamma(r-1)}{\sigma(A, A^{-1})\Im(\mathfrak{F}_0)} & \left(\frac{4\pi(n+\kappa)}{N}\right)^{1-r} c_{n+\kappa}(f, A), \text{ for } n+\kappa > 0 \\ 0 & \text{for } n+\kappa = 0 \end{cases}$$

We call two modular forms $f(\tau)$ and $g(\tau)$ orthogonal, if (f, g) exists and is equal to 0. The above theorem enables us to characterise the Poincaré series completely. In this connection, we prove

Theorem 23. *For $r > 2$, the Poincaré series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ is uniquely determined upto a constant factor, by the following properties:*

(i) $\underline{n+\kappa > 0}$. The series is a cusp form, which is orthogonal to all the forms $f(\tau)$ of the space $[\Gamma_0, r, v]$ for which the $(n+\kappa)$ – th Fourier coefficient $c_{n+\kappa}(f, A)$ at the cusp $A^{-1} < \infty >$ of Γ_0 vanishes. 152

(ii) $\underline{n+\kappa=0}$. *The series is orthogonal to all the forms of the space* $[\Gamma_0, r, v]_0$. *Moreover, the constant term in the Fourier series of* $G_r(\tau, v, A, \Gamma_0, 0)$ *at a parabolic cusp of* Γ_0 *is different from zero or equal to zero according as the cusp under consideration is equivalent to* $A^{-1} < \infty >$ *or not.*

Proof. (i) $\underline{n+\kappa} > 0$. The fact that the series $G_r(\tau, v, A, \Gamma_0, n+\kappa)$ has the property (i) is an immediate consequence of theorem 22. If $c_{n+\kappa}(G_r, A) = 0$ then applying theorem 22 to $f(\tau) = G_r(\tau, v, A, \Gamma_0, n+\kappa)$, this series vanishes identically and therefore $c_{n+\kappa}(f, A) = 0$ for every form $f(\tau)$ in $[\Gamma_0, r, v]$. Hence, if $g(\tau)$ is a cusp form which has the property (i) of theorem 23, there exists clearly a constant c such that

$$c_{n+\kappa}(g, A) = cc_{n+\kappa}(G_r, A) \Longrightarrow c_{n+\kappa}(g - cG_r, A) = 0.$$

This shows that the modular form $g - cG_r$ is orthogonal to g as well as to G_r and therefore $g - cG_r$ is orthogonal to itself. Hence $g - cG_r = 0$ i.e. $g = cG_r$.

(ii) $\underline{n+\kappa} = 0$. If $G_r(\tau, v, A, \Gamma_0, 0)$ has in its Fourier expansion at the cusp $B^{-1} < \infty >$ $(B \in \Gamma)$ a constant term different from zero, then

$$(G_r(\quad, v, A, \Gamma_0, 0)|B^{-1})(\tau) = \frac{1}{\sigma(A, B^{-1})} G_r(\tau, v^{B^{-1}}, AB^{-1}, B\Gamma_0 B^{-1}, 0)$$

does not vanish at the parabolic cusp ∞. But this is possible if and only if $AB^{-1}B\Gamma_0 B^{-1} = A\Gamma_0 B^{-1}$ contains a matrix M whose
153 second row is $(0, \pm 1)$ i.e. for some integral t, $A\Gamma_0 B^{-1}$ contains $\pm U^t$ or equivalently $\pm A^{-1} U^t = LB^{-1}$ for some L in Γ_0 and an integer t. The last condition means precisely that the cusps $A^{-1} < \infty >$ and $B^{-1} < \infty >$ are equivalent under Γ_0. It is now clear that the Eisenstein series $G_r = G_r(\tau, v, A, \Gamma_0, 0)$ satisfies the second assertion in (ii) of theorem 23; by theorem 22, G_r is orthogonal to $[\Gamma_0, r, v]_0$ i.e. G_r has the property (ii) of theorem 23. Let $g(\tau)$ be any modular form in $[\Gamma, r, v]$ which has the same property. We know that, for some constant c, $g - cG_r$ is a cusp form. But then $g - cG_r$ is orthogonal not only to G_r but also to g. Thus $g - cG_r$ is

also orthogonal to itself leading to $g = cG_r$, proving theorem 23 completely.

An other important consequence of theorem 22 is □

Theorem 24 (Completeness Theorem)**.** *The system of Eisenstein series for the group* Γ_0 *can be completed into a basis of the space* $[\Gamma_0, r, v], r > 2$*, by adding a finite number of Poincaré series* $G_r(\tau, v, A, \Gamma_0, n + \kappa)$ *for any (fixed and) given A.*

Proof. By theorem 17, it is sufficient to prove that the Poincaé series series $G_r(\tau, v, A, \Gamma_0, n + \kappa)$ for some fixed generate $[\Gamma_0, r, v]_0$. Let $\mathfrak{T}$ be the subspace of $[\Gamma_0, r, v]_0$ generated by the Poincaré series $G_r(\tau, v, A, \Gamma_0, n + \kappa)(n + \kappa > 0)$. Let t be the dimension of $\mathfrak{T}$ and $\{f_1, f_2, \ldots, f_t\}$ an orthonormal basis of $\mathfrak{T}$ i.e.

$$(f_i, f_k) = \delta_{ik} \text{ for } i, k = 1, 2, \ldots, t.$$

For an arbitrary element $f(\tau)$ of $[\Gamma_0, r, v]_0$, let

$$(f, f_i) = a_i \text{ for } i = 1, 2, \ldots, t.$$

It is obvious that the cusp form $f - \sum\limits_{i=1}^{t} a_i f_i$ is orthogonal to $f_1, f_2, \ldots, f_t$ **154**
and therefore to every cusp form $G_r(\tau, v, A, \Gamma_0, n + \kappa)$ $(n + \kappa > 0)$. But this implies, by theorem 22, that $f - \sum\limits_{i=1}^{t} a_i f_i = 0$ i.e. f belongs to $\mathfrak{T}$. Hence $\mathfrak{T} = [\Gamma_0, r, v]_0$, which proves theorem 25. □

By using the theory of the 'Weierstrass' points', Petersson has shown how to choose the values $n_1, n_2, \ldots, n_t$ so that the series $G_r(\tau, v, A, \Gamma_0, n_i + \kappa) i = 1, 2, \ldots, t$ form a basis of the space $[\Gamma_0, r, v]_0$.

In the case of the modular group Γ, theorem 21 enables us to make a very precise statement. We observed already that $f \to f\Delta^{-\kappa}$ defines a bijective linear mappain of $[\Gamma, r, v]$ onto $[\Gamma, r - 12\kappa, 1]$, showing that the dimension μ of $[\Gamma, r, v]$ is positive if and only if $r - 12\kappa \geq 0$ and $r - 12\kappa \neq 2$ (since $r - 12\kappa$ always is even). The assumption $r > 2$ guarantees the convergence of the Poincaré series. By theorem 21, we get

$$\frac{r}{12} = \mu + \kappa + \frac{a}{3} + \frac{b}{2} - 1 \text{ for } r - 12\kappa \geqq 0$$

and as we have proved already, this number coincides with the number of zeros of a modular form f in $[\Gamma, r, v]$ not vanishing identically.

Theorem 25. *If* $[\Gamma, r, v]$ *has positive dimension* μ *and* $r > 2$, *the series* $G_r(\tau, v, n+\kappa) := G_r(\tau, v, E, \Gamma, n+\kappa)$ *for* $n = 0, 1, 2, \ldots, \mu - 1$ *form a basis for* $[\Gamma, r, v]$.

Proof. Every form $f(\tau) \in [\Gamma, r, v]$ has a Fourier expansion of the type

$$f(\tau) = \sum_{n=0}^{\infty} c_{n+\kappa} e^{2\pi i(n+\kappa)\tau}.$$

The necessary and sufficient condition that $f(\tau)$ vanishes identically is
155 that $c_{n+\kappa} = 0$ for $n = 0, 1, \ldots, \mu - 1$. For, if the first μ coefficients $c_{n+\kappa}$ vanish, then $f(\tau)$ has unavoidable zeros of order $\kappa, \dfrac{a}{3}, \dfrac{b}{2}$ and an ordinary zero of order μ, which cannot happen unless $f(\tau) = 0$. Let

$$\varphi_m(\tau) = \sum_{n=0}^{\infty} c_{n+\kappa}^{(m)} e^{2\pi i(n+\kappa)\tau} (m = 0, 1, 2, \ldots, \mu - 1)$$

form a basis of $[\Gamma, r, v]$. Then the matrix

$$C = (c_{n+\kappa}^{(m)}) m, n = 0, 1, 2, \ldots, \mu - 1$$

is non-singular. If the matrix C were singular, then there exist complex numbers $x_0, x_1, \ldots, x_{\mu-1}$ not all zero such that

$$\sum_{m=0}^{\mu-1} x_m c_{n+\kappa}^{(m)} = 0, \quad n = 0, 1, 2, \ldots, \mu - 1.$$

This implies that

$$\sum_{m=0}^{\mu-1} x_m \varphi_m(\tau) = \sum_{n=0}^{\infty} \{\sum_{m=0}^{\mu-1} x_m c_{n+\kappa}^{(m)}\} e^{2\pi i(n+\kappa)\tau} = 0$$

showing that $\varphi_0(\tau)$, $\phi_1(\tau), \ldots, \varphi_{\mu-1}(\tau)$ are not linearly independent, which is a contradiction. Thus C is non-singular and therefore C^{-1} transforms $\{\varphi_0(\tau), \varphi_1(\tau), \ldots, \varphi_{\mu-1}(\tau)\}$ into a new basis for which the analogue of the matrix C is the unit matrix. Thus we can assume, without loss of generality, that already $\varphi_0(\tau), \varphi_1(\tau), \varphi_2(\tau), \ldots, \varphi_{\mu-1}(\tau)$ form such a basis i.e.

$$\varphi_m(\tau) = e^{2\pi i(m+\kappa)\tau} + \sum_{n=\mu}^{\infty} c_{n+\kappa}^{(m)} e^{2\pi i(m+\kappa)\tau} \quad (m = 0, 1, \ldots, \mu - 1).$$

We now prove the linear independence of $G_r(\tau, \nu, n + \kappa)(n = 0, 1, 2, \ldots, \mu - 1)$. Let

$$\sum_{n=0}^{\mu-1} x_n G_r(\tau, \nu, n + \kappa) = 0.$$

Then, by theorem 22, for $m + \kappa > 0$, $m < \mu$, it follows that **156**

$$0 = (\varphi_m(\tau), \sum_{n=0}^{\mu-1} x_n G_r(\tau, \nu, n + \kappa)) = \frac{6}{\pi}\Gamma(r - 1)(4\pi(n + \kappa))^{1-r} x_m.$$

As a result, for $\kappa > 0$, $x_m = 0$ for $m = 0, 1, \ldots, \mu - 1$, and if $\kappa = 0$, then $x_m = 0$ for $m = 1, 2, \ldots, \mu - 1$. But, in the latter case, $x_0 G_r(\tau, \nu, 0) = 0$ implies that $x_0 = 0$, since $G_r(\tau, \nu, 0) \not\equiv 0$. Hence $G_r(\tau, \nu, n + \kappa)$ $(n = 0, 1, 2, \ldots, \mu - 1)$ in any case are linearly independent and theorem 25 is proved.

We have already shown that the space $[\Gamma, 12, 1]$ is of dimension 1 and is generated by the modular form $\Delta(\tau)$, which has a Fourier expansion

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n\tau}$$

at the parabolic cusp ∞ of Γ, where $\tau(n)$ is Ramanujan's function. By using theorem 23, we shall now show that the Poincaré series $G_{12}(\tau, 1, n)$ vanishes identically if and only if $\tau(n) = 0$. If $\tau(n) = 0$, then, by theorem 23, the series $G_{12}(\tau, 1, n)$ is orthogonal to $\Delta(\tau)$ and therefore to all cusp form including itself, which implies that $G_{12}(\tau, 1, n) = 0$. Conversely, if $G_{12}(\tau, 1, n) = 0$, then obviously $\tau(n) = 0$. It has been proved in 1959 by D.H. Lehmer that $\tau(n) \neq 0$ for $n \leq 113740230287998$. □

4 The Fourier Coefficients of Integral Modular forms

Let Γ_0 be a subgroup of finite index in the modular group Γ and let v be
a multiplier system for the group Γ_0 and the weight $r > 0$. We shall find
157 some estimates for the Fourier coefficients $c_{n+\kappa}$ in the Fourier expansion
at the cusp ∞ of a modular form $f(\tau) \in [\Gamma_0, r, v]$, namely

$$f(\tau) = \sum_{n+\kappa \geq 0} c_{n+\kappa} e^{2\pi i(n+\kappa)\tau/N}, \tag{1}$$

where $N > 0$ is the least integer such that U^N belongs to Γ_0 and $v(U^N) = e^{2\pi i\kappa}$.

Let $s_\ell = A_\ell^{-1} < \infty > (\ell = 1, 2, \ldots, \sigma)$ with $A_\ell \in \Gamma$ be a complete system of pairwise inequivalent parabolic cusps of Γ_0. In particular, let $s_1 = \infty$ and $A_1 = E$. Let P_ℓ be the subgroup of Γ_0 consisting of those transformations of Γ_0 which leave s_ℓ fixed i.e.

$$P_\ell = \{L | L < s_\ell >= s_\ell, L \in \Gamma_0\}.$$

It is nothing but the group generated by $-E$ and $H_\ell = A_\ell^{-1} U^{N_\ell} A_\ell$ where N_ℓ is the least natural number such that $A_\ell^{-1} U^{N_\ell} A_\ell$ belongs to Γ_0. Let

$$\Gamma_0 = \bigcup_k P_\ell L_{k\ell}$$

be a coset decomposition of Γ_0 modulo P_ℓ. We set

$$M_{k\ell} = A_\ell L_{k\ell} = \begin{pmatrix} a_{k\ell} & b_{k\ell} \\ c_{k\ell} & d_{k\ell} \end{pmatrix}.$$

Since we can replace $L_{k\ell}$ by $-L_{k\ell}$, we can assume without loss of gen-
erality that either $c_{k\ell} > 0$ or $c_{k\ell} = 0$ and $d_{k\ell} = 1$. The mapping $M_{k\ell} \to$
$M_{k\ell}^{-1} < \infty >$ is one-to-one; because, if $M_{k\ell}^{-1} < \infty >= M_{pq}^{-1} < \infty >$, then
$s_\ell = A_\ell^{-1} < \infty >$ and $s_q = A_q^{-1} < \infty >$ are equivalent with respect to Γ_0,
which is possible only if $q = \ell$. It follows now that $L_{p\ell} L_{k\ell}^{-1} < s_\ell >= s_\ell$
158 i.e. $L_{p\ell} L_{k\ell}^{-1}$ belongs to P_ℓ and therefore $p = k$. Every rational number
s can be represented in the form $M_{k\ell}^{-1} < \infty >$. Since s is a parabolic

cusp of Γ_0, s is equivalent to $s_\ell = A_\ell^{-1} < \infty >$ for some suitable ℓ i.e. $s = L^{-1}A_\ell^{-1} < \infty >$ for L belonging to Γ_0. Let $L = K_\ell L_{k\ell}$, where K_ℓ belongs to P_ℓ. Then

$$s = L_{k\ell}^{-1}K_\ell^{-1}A_\ell^{-1} < \infty >= L_{k\ell}^{-1}A_\ell^{-1} < \infty >= M_{k\ell}^{-1} < \infty > .$$

We form the set

$$\mathfrak{F}^* = \bigcup_{n=-\infty}^{\infty} U^n < \mathfrak{K} >,$$

where $\mathfrak{F}$ is the fundamental domain of Γ given by

$$\mathfrak{F} = \{\tau||\tau| \geq 1, |2x| \leq 1\}.$$

It can be seen easily that

$$\mathscr{G} = \bigcup_{\ell-1}^{\sigma}\bigcup_{k} M_{k\ell}^{-1} < \mathfrak{F}^* >,$$

because $M_{k\ell}^{-1} < \mathfrak{F}^* >$ consists exactly of those images of $\mathfrak{F}$ under Γ, which have $M_{k\ell}^{-1} < \infty >$ as an improper vertex. Since $\mathfrak{F}^* \subset \{\tau|y \geq \frac{\sqrt{3}}{2}\}$, it follows that $M_{k\ell}^{-1} < \infty^* >$ is contained in the circle

$$(x + \frac{d_{k\ell}}{c_{k\ell}})^2 + (y - \frac{1}{\sqrt{3}c_{k\ell}^2}) \leq \frac{1}{3c_{k\ell}^4}, \tag{2}$$

when $c_{k\ell} > 0$. We shall now find out a set of necessary conditions, which have to be satisfied, if the domain $M_{k\ell}^{-1} < \mathfrak{F}^* >$ is to intersect a given line

$$\mathfrak{n} = \{\tau|0 \leq x \leq N, y = \eta\},$$

with $0 < \eta < \frac{\sqrt{3}}{2}$ in at least two points. If $c_{k\ell} = 0$, then $M_{k\ell}^{-1} <$ **159**
$\mathfrak{F}^* >= \mathfrak{F}^*$ and therefore $M_{k\ell}^{-1} < \mathfrak{F}^* >$ does not intersect $\mathfrak{n}$. Therefore $c_{k\ell}$ is necessarily greater than zero. Since $M_{k\ell}^{-1} < \mathfrak{F}^* >$ is contained in the circle defined in (1), the line $y = \eta$ must intersect this circle in two points, if it is to have at least two points in common with the set $M_{k\ell}^{-1} <$

$\mathfrak{F}^* >$. This means that the coordinates of the points of intersection given by

$$x = -\frac{d_{k\ell}}{c_{k\ell}} \pm \sqrt{\frac{2n}{\sqrt{3}c_{k\ell}^2} - \eta^2}, y = \eta$$

must satisfy the conditions

$$0 < c_{k\ell} < \sqrt{\frac{2}{\sqrt{3}\eta}}, \tag{3}$$

$$x_1 = x_1(M_{k\ell}) = -\frac{d_{k\ell}}{c_{k\ell}} - \sqrt{\frac{2n}{\sqrt{3}c_{k\ell}^2} - \eta^2} < N, \tag{4}$$

$$x_2 = x_2(M_{k\ell}) = -\frac{d_{k\ell}}{c_{k\ell}} + \sqrt{\frac{2\eta}{\sqrt{3}c_{k\ell}^2} - \eta^2} > 0. \tag{5}$$

If the condition (3) were not satisfied, then the line $y = \eta$ will intersect the circle (1) in at most one point. Moreover if $x_1 \geq N$ or $x_2 \leq 0$, then, from $x_1 < x_2$, the line $y = \eta$ and the circle will have again at most one point in common. From (4) and (5), we have

$$-\sqrt{\frac{2\eta}{\sqrt{3}c_{k\ell}^2} - \eta^2} < -\frac{d_{k\ell}}{c_{k\ell}} < N + \sqrt{\frac{2\eta}{\sqrt{3}c_{k\ell}^2} - \eta^2}$$

and therefore

$$-1 < -d_{k\ell}/c_{k\ell} < N + 1. \tag{6}$$

160 Moreover, it is immediate from (3) that the interval $x_1 \leq x \leq x_2$ is contained in the interval $|x + d_{k\ell}/c_{k\ell}| \leq hc_{k\ell}^{-1}\sqrt{\eta}$ with $h = \sqrt{\frac{2}{\sqrt{3}}}$. Consequently

$$\mathfrak{n}_{k\ell} = M_{k\ell}^{-1} < \mathfrak{F}^* > \cap \mathfrak{n} \subset \{\tau | y = \eta, |x + \frac{d_{k\ell}}{c_{k\ell}}| < hc_{k\ell}^{-1}\sqrt{\eta}\}.$$

It is obvious that $\mathfrak{n}_{k\ell}$ consists of at most a finite number of connected segments and

$$\mathfrak{n} = \bigcup_{k,\ell} \mathfrak{n}_{k\ell},$$

where $\mathfrak{n}_{k\ell}$ contains at least two points only if

$$0 < c_{k\ell} < \frac{h}{\sqrt{\eta}}, -1 < -\frac{d_{k\ell}}{c_{k\ell}} < N + 1.$$

Let

$$(f|A_\ell^{-1})(\tau) = \sum_{n+\kappa_\ell \geq 0} c_{n+\kappa_\ell}^{(\ell)} e^{2\pi i(n+\kappa_\ell)\tau/N_\ell}$$

be the Fourier expansion of $f(\tau)$ at the cusp $A_\ell^{-1} < \infty >$ of $\Gamma_0(\ell = 1, 2, \ldots, \sigma)$. If $\ell = 1$, this series is identical with the series (1). We set

$$\tau = M_{k\ell}^{-1} < \tau^* >= L_{k\ell}^{-1} A_\ell^{-1} < \tau^* >, L_{k\ell} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } A_\ell = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\begin{aligned} |f(\tau)| &= |f(L_{k\ell}^{-1} A_\ell^{-1} < \tau^* >)| = | - \gamma A_\ell^{-1} < \tau^* > +\alpha|^r |f(A_\ell^{-1}| < \tau^* >)| \\ &= |(-\gamma A_\ell^{-1} < \tau^* > +\alpha)^r |(-c\tau^* + a)^r (f|A_\ell^{-1})(\tau^*)| \\ &= |(-c_{k\ell}\tau^* + a_{k\ell})^r |(f|A_\ell^{-1})(\tau^*)|, \end{aligned}$$

which implies that 161

$$|f(\tau)| = |c_{k\ell}\tau + d_{k\ell}|^{-r} |(f|A_\ell^{-1}(\tau^*))|. \tag{7}$$

Since $|(f|A_\ell^{-1})(\tau^*)|$ is bounded for $\tau^* \in \mathfrak{F}^*$, we can assume after any necessary normalization that

$$|(f|A_\ell^{-1})(\tau^*)| \leq 1 \text{ for } \tau^* \in \mathfrak{F}^*, \ell = 1, 2, \ldots, \sigma, \tag{8}$$

The Fourier coefficient $c_{n+\kappa}$ of $f(\tau)$ is given by

$$c_{n+\kappa} = \frac{1}{N} \int_0^N f(\tau) e^{-2\pi i(n+\kappa)\tau/N} dx.$$

Choosing the path of integration along the line $y = \eta$, we obtain that

$$N|c_{n+\kappa}| \leq e^{2\pi(n+\kappa)\eta/N} \int_{\mathfrak{n}} |f(\tau)| dx$$

$$\leq e^{2\pi(n+\kappa)\eta/N} \sum_{k,\ell} \int\limits_{\mathfrak{n}_{k\ell}} |f(\tau)| dx. \tag{9}$$

But, by the definition of $\mathfrak{n}_{k\ell}$, we see that if τ belongs to $\mathfrak{n}_{k\ell}$, then $\tau^* = M_{k\ell} < \tau >$ belongs to $\mathfrak{F}^*$ and therefore, by (7) and (8),

$$|f(\tau)| \leq |c_{k\ell}\tau + d_{k\ell}|^{-r} \text{ for } \tau \in \mathfrak{n}_{k\ell}.$$

Consequently

$$|\int\limits_{\mathfrak{n}_{k\ell}} f(\tau) dx| \leq \int\limits_{|x+\frac{d_{k\ell}}{c_{k\ell}}| \leq hc_{k\ell}^{-1}\sqrt{n}} \{(c_{k\ell}x + d_{k\ell})^2 + c_{k\ell}^2 \eta^2\}^{-\frac{r}{2}} dx$$

$$\leq 2c_{k\ell}^{-r} \int\limits_0^{hc_{k\ell}^{-1}\sqrt{\eta}} (u^2 + \eta^2)^{-\frac{r}{2}} du$$

$$\leq 2c_{k\ell}^{-r} \{ \int^{\eta} \eta^{-r} du + \int_{\eta}^{hc_{k\ell}^{-1}\sqrt{\eta}} u^{-r} du \},$$

162 because $0 < \eta < hc_{k\ell}^{-1}\sqrt{\eta}$ by (3). It follows now that

$$\int\limits_{n_{k\ell}} |f(\tau)| dx = \begin{cases} 0(c_{k\ell}^{-r}\eta^{1-r}, c_{k\ell}^{-1}\eta^{\frac{1-r}{2}}) \text{ for } r > 0, r \neq 1 \\ 0(c_{k\ell}^{-1} \log(\frac{he}{c_{k\ell}\sqrt{\eta}})) \text{ for } r = 1 \end{cases} \tag{10}$$

Here and in the following, $0(\omega)$ means in general that $|\frac{0(\omega)}{\omega}|$ is bounded by a constant depending only on Γ_0 and r. Summing up the right hand side of (10) over all pairs of integers $(c_{k\ell}, d_{k\ell})$ (not necessarily coprime) which satisfy the inequalities (3) and (6), we get

$$\sum_{k,\ell} \int\limits_{\mathfrak{n}_{k\ell}} |f(\tau)| dx = \begin{cases} \eta^{1-r} o(\sum_{k,\ell} c_{k\ell}^{-r}) + \eta^{\frac{1-r}{2}} 0(\sum_{k,\ell} c_{k\ell}^{-1}), \text{ for } r > 0, r \neq 1 \\ 0(\sum_{k,\ell} c_{k\ell}^{-1}) \log(\frac{eh}{c_{k\ell}\sqrt{\eta}}), \text{ for } r = 1, \end{cases} \tag{11}$$

since the right hand side certainly includes the pairs (c_{kl}, d_{kl}) occurring on the left hand side. In what follows, all summations over k, ℓ are carried out in this sense. It can be proved easily that

$$\begin{aligned}\sum_{k,\ell} c_{k\ell}^{-\rho} &\leq (N+2) \sum_{c=1}^{[h\eta^{1/2}]} c^{1-\rho}\\ &= \begin{cases} o(1) & \text{for } \rho > 2,\\ o(\log \frac{1}{\eta}) & \text{for } \rho = 2,\\ o(\eta^{\rho/2-1}) & \text{for } 0 < \rho < 2.\end{cases}\end{aligned}$$

and

$$\sum_{k,\ell} c_{k\ell}^{-1} \log\{\frac{eh}{c_{k\ell}\sqrt{\eta}}\} \leq (N+2) \sum_{c=1}^{[h\eta^{-1/2}]} \log\{\frac{eh}{c\sqrt{\eta}}\} = o(\eta^{-1/2});$$

by Stirling's formula, namely **163**

$$n! = \alpha_n n^{n+\frac{1}{2}} e^{-n}, \lim_{n\to\infty} \alpha_n = \sqrt{2\pi},$$

with $h\eta^{-1/2} = n + \vartheta$, where $0 \leq \vartheta < l$ and n is an integer, we have indeed

$$\begin{aligned}\sum_{c=1}^{[h\eta^{-1/2}]} \log\{\frac{eh}{c\sqrt{\eta}}\} &= n + n\log(n+\vartheta) - \log\alpha_n - (n+\frac{1}{2})\log n + n\\ &= 2n + n\log(1+\frac{\vartheta}{n}) - \log\alpha_n - \frac{1}{2}\log n\\ &< 2n + \vartheta - \log\alpha_n - \frac{1}{2}\log n = o(n) = o(\eta^{-\frac{1}{2}}).\end{aligned}$$

Therefore, equation (11) gives

$$\sum_{k,\ell} \int_{\mathfrak{n}_{k\ell}} |(f(\tau)| dx = \begin{cases} o(\eta^{1-r}) & \text{for } r > 2,\\ o(\frac{1}{\eta}\log\frac{1}{\eta}) & \text{for } r = 2,\\ o(\eta^{-r/2}) & \text{for } 0 < r < 2.\end{cases} \tag{12}$$

With $n + \kappa = \dfrac{1}{\eta}$, (9) and (12) lead us then to

Theorem 26. *Let Γ_0 be a subgroup of finite index in the modular group and v a multiplier system for the group Γ_0 and weight $r > 0$. Let*

$$f(\tau) = \sum_{n+\kappa} c_{n+\kappa} e^{2\pi i(n+\kappa)\tau/N}$$

be the Fourier expansion of a modular form belonging to the space $[\Gamma_0, r, v]$. Then we have

$$c_{n+\kappa} = \begin{cases} o((n+\kappa)r - 1) & \text{for } r > 2, \\ o((n+\kappa)\log(n+\kappa)) & \text{for } r = 2, \text{ as } n+\kappa \to \infty, \\ o((n+\kappa)^{r/2}) & \text{for } 0 < r < 2, \end{cases}$$

164 For the Fourier coefficients of a cusp form $f(\tau)$ belonging to $[\Gamma_0, r, v]$ with $r \geq 2$, we have sharper estimates given by the following

Theorem 27. *Let*

$$f(\tau) = \sum_{n+\kappa>0} c_{n+\kappa} e^{2\pi i(n+\kappa)\tau/N}$$

be a cusp form in $[\Gamma_0, r, v]$, where Γ_0 is a subgroup of the modular group with finite index and v is a multiplier system for the group Γ_0 and weight r. Then

$$c_{n+\kappa} = o((n+\kappa)^{\frac{r}{2}}) \text{ as } n+\kappa \to \infty$$

for all $r \geq 2$.

Proof. Since $f(\tau)$ is a cusp form, by § 3, lemma 5, the function $|f(\tau)|y^{\frac{r}{2}}$ is bounded in $\mathscr{G}$ i.e.

$$|f(\tau)| \leq Cy^{-\frac{r}{2}} \text{ for some constant } C.$$

Therefore for the Fourier coefficient $c_{n+\kappa}$ we have

$$\begin{aligned} |c_{n+\kappa}| &= \frac{1}{N}\left|\int_0^N f(\tau) e^{-2\pi i(n+\kappa)\tau/N} dx\right| \\ &\leq Cy^{-\frac{r}{2}} e^{2\pi(n+\kappa)y/N} \end{aligned}$$

The estimate for $c_{\eta+\kappa}$ stated in theorem 27 follows immediately on taking $y = 1/(n+\kappa)$. □

We have seen in chapter 1 § 5, that the $n-th$ Fourier coefficient of the Eisenstein series $G_k(\tau)$, for any even integer $k \geq 4$, coincides with $d_{k-1}(n)$ but for a constant factor independent of n. But **165**

$$n^{k-1} < d_{k-1}(n) < \zeta(k-1)n^{k-1}, (k > 2)$$

since

$$d_{k-1}(n) = \sum_{d|n} (\frac{n}{d})^{k-1} < n^{k-1} \sum_{d=1}^{\infty} \frac{1}{d^{k-1}} = \zeta(k-1)n^{k-1}.$$

Therefore, for $k > 2$, the Fourier coefficients of the Eisenstein series increase more rapidly than the Fourier coefficients of cusp forms. Hence the estimates given in theorem 26 for $r > 2$ can not be sharpened, in general. However, sharper estimates for the Fourier coefficients of cusp forms belonging to the congruence subgroups of the modular group have been obtained by using some estimates of the so-called *Kloosterman sums* given by A. Weil. For this, we refer to a paper of K.B. Gundlach [1].

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Non-Analytic Modular Forms

1 The Invariant Differential Equations

The theory of non-analytic modular forms has a close connection with Siegel's researches in the theory of indefinite quadratic forms and is decisively influenced by this theory. The Eisenstein series 168

$$G(\tau,\overline{\tau});\alpha,\beta) = \sum_{(c,d)\neq(0,0)} (c\tau+d)^{-\alpha}(c\overline{\tau}+d)^{-\beta}, (\mathrm{Re}(\alpha+\beta)>2)$$

where $\alpha-\beta$ is an even integer and the sum runs over all pairs of integers $(c,d)\neq(0,0)$ is the prototype of a non-analytic modular form. In what follows, the functions $(c\tau+d)^{-\alpha}$ and $(c\overline{\tau}+d)^{-\beta}$ for real numbers $(c,d)\neq(0,0)$ and $\tau\in\mathscr{G}$ will be defined by

$$\begin{aligned}
(c\tau+d)^{-\alpha} &= e^{-\alpha\log(c\tau+d)}, (c\overline{\tau}+d)^{-\beta} = e^{-\beta\log(c\overline{\tau}+d)} \\
\text{with } \log(c\tau+d) &= \log|c\tau+d| + i\arg(c\tau+d), -\pi<\arg(c\tau+d)\leq\pi \\
\text{and } \log(c\overline{\tau}+d) &= \log|c\overline{\tau}+d| + i\arg(c\overline{\tau}+d), -\pi\leq\arg(c\overline{\tau}+d)<\pi;
\end{aligned}$$

here the branches of the logarithm are so chosen that always

$$\log(c\tau+d)+\log(c\overline{\tau}+d) \text{ is real.}$$

Let $S=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ be a real matrix with $|S|=ad-bc>0$ and let α,β be two complex numbers. Then for any $f(\tau,\overline{\tau})$, we define $f\underset{\alpha,\beta}{|}S$ by

$$(f\underset{\alpha,\beta}{|}S)(\tau,\overline{\tau}) = (c\tau+d)^{-\alpha}(c\overline{\tau}+d)^{-\beta} f(S<\tau>, S<\overline{\tau}>).$$

It is obvious that the series defined above satisfies the transformation
169 formula
$$(G(\quad , ; \alpha, \beta)|_{\alpha,\beta} S)(\tau, \overline{\tau}) = G(\tau, \overline{\tau}; \alpha, \beta)$$
for S belonging to the modular group. In this section, our aim is to find a differential equation, which has the given series $G(\tau, \overline{\tau}; \alpha, \beta)$ as a solution and which is left invariant by the transformations $f \to f \underset{\alpha,\beta}{|} S$.

Following a method of Selberg, we shall show here that this problem can be reduced in a natural way to the eigen-value problem for Laplace's differential operator in a certain 3-dimensional Riemannian space to be defined. Let $\mathscr{R}$ denote the direct product of the upper half-plane and the real t-axis i.e.
$$\mathscr{R} = \{(\tau, t) | \tau = x + iy, y > 0 \text{ and } t \text{ real}\}.$$
To any real matrix $R = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ with $|R| = 1$ and any real number a, we associate a transformation R_a of $\mathscr{R}$ defined by
$$R_a(\tau, t) = (R < \tau >, t + \arg(c\tau + d) + 2\pi a).$$
If R and S are two elements of the group Ω of all real matrices of determinant 1, and a, b are any two real numbers, then we define the composite of two transformations R_a and S_b by
$$(R_a \cdot S_b)(\tau, t) = R_a(S_b(\tau, t)).$$
We shall now show that the composite of two transformations R_a and S_b is again a transformation of the same type associated to RS. Let (m_1, m_2), (c, d) and (m_1^*, m_2^*) be the second rows of the matrices R, S and
170 RS respectively. Then, by definition, we have
$$\begin{aligned}(R_a \cdot S_b)(\tau, t) &= R_a(S < \tau >, t + \arg(c\tau + d) + 2\pi a)\\ &= (RS < \tau >, t + \arg(c\tau + d) + 2\pi a + \arg(m_1 S < \tau > + m_2) + 2\pi b)\\ &= (RS < \tau >, t + \arg(m_1^* \tau + m_2^*) + 2\pi(a + b + w(R, S)))\\ &= (RS)_{a+b+w(R,S)}(\tau, t),\end{aligned}$$
showing that
$$R_a \cdot S_b = (RS)_{a+b+w(R,S)}. \tag{1}$$

It is obvious from the definition of the mapping R_a that R_a is equal to the identity mapping on the space $\mathscr{R}$ when $R = E$ and $a = 0$ or $R = -E$ and $a = -\frac{1}{2}$. From (1), it follows that the mapping $(S^{-1})_{-b-w(S,S^{-1})}$ is the inverse $(S_b)^{-1}$ of the mapping S_b. Thus the set $\hat{\Omega}$ of all transformations R_a constitutes a group. If we substitute $R = -E$ and $a = -\frac{1}{2}$ in (1), we see that

$$S_b = (-S)_{b-\frac{1}{2}+w(-E,S)},$$

which shows that every element R_a of $\hat{\Omega}$ has two representations namely, one associated to the matrix R and the other to the matrix $-R$. Let Z denote the subgroup of Ω consisting of the two elements $\pm E$ and $\hat{Z}$ denote the subgroup of $\hat{\Omega}$ consisting of the elements $(\pm E)_a$ for every real number a. It can be verified that the kernel of the homomorphism

$$R_a \to RZ$$

from $\hat{\Omega}$ to Ω/Z is $\hat{Z}$ i.e. $\hat{\Omega}/Z$ and Ω/Z are isomorphic. Given a discrete subgroup Γ of Ω, we are interested in finding a *relation preserving rep-* **171**
resentation of Γ/Z in the group $\hat{\Omega}$ i.e. we want to find a special set of representatives for the cosets of a subgroup $\hat{\Gamma}$ of $\hat{\Omega}$ modulo $\hat{Z}$, where $\hat{\Gamma} = \{S_a | S \in \Gamma$, a arbitrary real$\}$, such that these representatives themselves form a group isomorphic with Γ/Z. We shall obtain one such representation, if we define a real-valued function $w(R)$ for every $R \in \Gamma$ satisfying the equations

$$R_{w(R)} = (-R)_{w(-R)} \tag{2}$$

$$R_{w(R)} S_{w(S)} = (RS)_{w(RS)} \tag{3}$$

for R and S belonging to Γ. For the existence of such a function $w(R)$, it is obviously necessary and sufficient that the following two equations are satisfied:

$$w(-R) = w(R) - \frac{1}{2} + w(-E, R) \tag{4}$$

$$w(RS) = w(R) + w(S) + w(R, S). \tag{5}$$

It can be shown that equations (4) and (5) are simultaneously solvable for a horocyclic group for which the system of generators and defining relations are well-known. We shall give here a solution for the modular group, which we denote by Γ. Since $V^3 = T^2 = -E$, we must have

$$(V_{w(V)})^3 = (T_{w(T)})^2 = (-E)_{w(-E)} = E_0,$$

where E_0 is the unit elements of $\hat{\Omega}$. This implies that

$$3w(V) + w(V, V) + w(V, V^2) = 2w(T) + w(T, T) = w(-E) = -\frac{1}{2}.$$

It now follows that

$$w(V) = \frac{1}{6}, \quad w(T) = \frac{1}{4}.$$

172 Let $\hat{\Gamma}_0$ be the subgroup of $\hat{\Omega}$ generated by $V_{\frac{1}{6}}$ and $T_{\frac{1}{4}}$. Then the correspondence $V_{\frac{1}{6}} \to VZ$ and $T_{\frac{1}{4}} \to TZ$ can be extended to a homomorphism of $\hat{\Gamma}_0$ onto Γ/Z. Therefore all the relations, which are satisfied by $v_{\frac{1}{6}}$ and $T_{\frac{1}{4}}$, will hold between VZ and TZ. But the converse is also true, because the defining relations $V^3Z = Z$ and $T^2Z = Z$ for the group Γ/Z are trivially satisfied by $V_{\frac{1}{6}}$ and $T_{\frac{1}{4}}$. Therefore the groups $\hat{\Gamma}_0$ and Γ/Z are isomorphic. This shows that if S_a and S_b are two elements of $\hat{\Gamma}_0$, then $S_a = S_b$ and therefore $a = b$. Thus we can write $a = w(S)$ and it is obvious that the function $w(S)$ satisfies the equations (4) and (5). From $T = UV$, it follows that

$$w(T) = w(U) + w(V), w(U) = \frac{1}{12}.$$

The group Ω/Z cannot have a relation-preserving representation in $\hat{\Omega}$; for, if it were true, then

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 \\ 0 & 1 \end{pmatrix} \text{ for real } \lambda > 0$$

would imply that

$$w\left(\begin{pmatrix}1 & \lambda^2\\ 0 & 1\end{pmatrix}\right) = w\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right) + w\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) + w\left(\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}\right)$$
$$= w\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) = \frac{1}{12},$$

whereas $w(U^2) = 2w(U) = \dfrac{1}{6}$, leading to a contradiction for $\lambda = \sqrt{2}$. 173

We define a positive definite metric form on $\mathscr{R}$ by

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + (dt - \frac{dx}{dy})^2. \tag{6}$$

This metric form on $\mathscr{R}$ is invariant under the transformation of $\hat{\Omega}$. In order to prove this, it is sufficient to prove that $dt - \dfrac{dx}{2y}$ is invariant under the transformations of $\hat{\Gamma}$, because we have already shown that $\dfrac{dx^2 + dy^2}{y^2}$ is invariant under Γ and therefore under $\hat{\Gamma}$. Let $S = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ and $(\tau^*, t^*) = S_a(\tau, t)$ for some real number a. Then

$$dt^* - dt = d(\arg\gamma\tau + \delta) = \frac{1}{2i}\{d\log(\gamma\tau + \delta) - d\log(\gamma\bar{\tau} + \delta)\}$$
$$= \frac{\gamma}{2i}\{\frac{d\tau}{\gamma\tau + \delta} - \frac{d\bar{\tau}}{\gamma\bar{\gamma} + \delta}\}$$

and

$$\frac{dx^*}{2y^*} - \frac{dx}{2y} = \frac{d\tau^* + d\bar{\tau}^*}{4y^*} - \frac{d\tau + d\bar{\tau}}{4y}$$
$$= \frac{1}{4y}\{(\frac{\gamma\bar{\tau} + \delta}{\gamma\tau + \delta} - 1)d\tau + (\frac{\gamma\tau + \delta}{\gamma\bar{\tau} + \delta} - 1)d\bar{\tau}\}$$
$$= \frac{\gamma}{2i}\{\frac{d\tau}{\gamma\tau + \delta} - \frac{d\bar{\tau}}{\gamma\bar{\tau} + \delta}\}$$

implying what we wanted to show. Hence $\mathscr{R}$ is a Riemannian space with the group $\hat{\Gamma}$ acting on it. The Laplacian Δ on the space $\mathscr{R}$ is given by

$$\Delta = y^2(\frac{\partial}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y\frac{\partial^2}{\partial x\partial t} + \frac{5}{4}\frac{\partial^2}{\partial t^2}. \tag{7}$$

We shall adopt the notation

$$(\varphi|S_a)(\tau, t) = \varphi(S_a(\tau, t)),$$

174 where $\varphi(\tau, t)$ is any function defined on $\mathscr{R}$ and S_a is an element of $\hat{\Gamma}$. We now formulate the indicated eigen-value problem. Let Γ_0 be a subgroup of finite index in Γ and v_1 an even abelian character of Γ_0. Moreover, let $\hat{\Gamma}_0$ denote the relation-preserving representation of Γ_0 in $\hat{\Omega}$. Then $\hat{\Gamma}_0$ consists of the transformations $S_{w(S)}(S \in \Gamma_0)$. We look for a real analytic function $\varphi(\tau, t)$ which satisfies the conditions:

1) $(\Delta + \lambda)\varphi(\tau, t) = 0$ for some real $\lambda \geq 0$,

2) $(\varphi|S_{w(S)})(\tau, t) = v_1(S)\varphi(\tau, t) \quad (S \in \Gamma_0)$,

3) $\lim\limits_{T\to\infty} \dfrac{1}{2T} \int\limits_{-T}^{T} \int\int\limits_{\mathfrak{F}_0} \varphi(\tau, t)\overline{\varphi(\tau, t)} y^{-2} dxdydt < \infty$,

where $\mathfrak{F}_0$ is a fundamental domain for Γ_0 in $\mathscr{G}$.

It can be seen that the functions

$$\varphi(\tau, t) = g(\tau)e^{-irt}$$

satisfy conditions 1), 2) and 3), where r is a given real number and $g(\tau)$ satisfies the conditions:

1') $\{y^2(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}) - iry\dfrac{\partial}{\partial x} + \lambda - \dfrac{5}{4}r^2\}g(\tau) = 0$,

2') $g(S < \tau >)e^{-ir \arg(c\tau+d)} = v_0(S)v_1(S)g(\tau)$, with $v_0(S) = e^{2\pi i w(S)}$ and $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0$,

3') $\int\int\limits_{\mathfrak{F}_0} g(\tau)\overline{g(\tau)} y^{-2} dxdy < \infty$.

175 By (4) and (5), we have

$$v_0(RS) = \sigma^{(r)}(R, S)v_0(R)v_0(S), v_0(-E) = e^{\pi i r}$$

showing that v_0 is a multiplier system for the group Γ_0 and weight r and therefore $v(S) = v_0(S)v_1(S)$ runs over all the multiplier systems for the group Γ_0 and weight r when v_1 runs over all the even abelian characters of Γ_0. As a matter of fact, $v_0(S)$ is the multiplier system for $\Delta^{r/12}(\tau)$ mentioned in theorem 19. Finally, we have

$$e^{-ir\arg(c\tau+d)} = e^{\frac{r}{2}\{\log(c\overline{\tau}+d)-\log(c\tau+d)\}} = (c\overline{\tau}+d)^{\frac{r}{2}}/(c\tau+d)^{\frac{r}{2}}.$$

Let $\alpha = \dfrac{q+r}{2}, \beta = \dfrac{q-r}{2}$ and q an arbitrary number. We introduce $f(\tau)$ by

$$g(\tau) = y^{\frac{q}{2}} f(\tau).$$

Then the conditions 1'), 2') and 3') lead to the following conditions for f:

1") $\{y^2(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}) - iry\dfrac{\partial}{\partial} + qy\dfrac{\partial}{\partial y} + \lambda - \dfrac{5}{4}r^2 + \dfrac{q}{2}(\dfrac{q}{2} - 1)\} f(\tau) = 0,$

2") $(f|S_{\alpha,\beta})(\tau) = v(S) f(\tau)$ for $S \in \Gamma_0$,

3") $\iint\limits_{\mathfrak{F}_0} f(\tau)\overline{f(\tau)} y^{\mathrm{Re}(\alpha+\beta)^{-2}} dxdy < \infty.$

Let us choose q in such a manner that the sum of the constant terms in the differential equation 1") vanishes. Then 1") reduces to

$$\Omega_{\alpha\beta} f(\tau) = 0 \tag{8}$$

with $\Omega_{\alpha\beta} = -y^2(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}) + i(\alpha-\beta)y\dfrac{\partial}{\partial x} - (\alpha+\beta)y\dfrac{\partial}{\partial y}$, where α and β **176**
are now given by

$$\alpha + \beta = 1 + \sqrt{5r^2 - 4\lambda + 1}, \alpha - \beta = r.$$

We shall show that the differential equation (8) has the invariant property mentioned above. Since we are interested in the functions $f(\tau)$ which are non-analytic in τ but analytic in both the independent variables $\tau = x + iy$ and $\overline{\tau} = x - iy$, we shall write $f(\tau, \overline{\tau})$ instead of $f(\tau)$ for the solutions of (8), as it seems to be a more suitable notation. Changing the variables from x, y to τ and $\overline{\tau}$ in $\Omega_{\alpha\beta}$ we obtain

$$\Omega_{\alpha\beta} = (\tau - \overline{\tau})^2 \frac{\partial^2}{\partial\tau\partial\overline{\tau}} - \beta(\tau - \overline{\tau})\frac{\partial}{\partial\tau} + \alpha(\tau - \overline{\tau})\frac{\partial}{\partial\overline{\tau}}, \tag{9}$$

with
$$\frac{\partial}{\partial\tau} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \frac{\partial}{\partial\overline{\tau}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{9a}$$

We look upon (9a) as a definition of $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial\overline{\tau}}$. The invariance property of the Laplacian Δ expressed by

$$\Delta((\varphi|S_a)(\tau,t)) = (\Delta\varphi|S_a)(\tau,t) \text{ for } S_a \in \hat{\Omega}$$

implies for $\Omega_{\alpha,\beta}$ the invariance property

$$\Omega_{\alpha,\beta}((f\underset{\alpha,\beta}{|}S)(\tau,\overline{\tau})) = (\Omega_{\alpha,\beta}\, f\underset{\alpha,\beta}{|}S)(\tau,\overline{\tau}), \text{ for } S \in \Omega. \tag{10}$$

This is the invariance property of the differential equation mentioned in the beginning. The two parameter domains defined by

$$1)\quad r,\lambda \text{ real and } \lambda \geq 0, \qquad 2)\quad \alpha,\beta \text{ both real} \tag{11}$$

will be of particular interest. In the theory of indefinite quadratic forms,
177 $(2\alpha, 2\beta)$ will occur as the signature of an indefinite quadratic form.

We shall now define certain linear differential operators, which transform the series $G(\tau,\overline{\tau};\alpha,\beta)$ into one of the series $G(\tau,\overline{\tau};\alpha\pm 1,\beta\pm 1)$ and which are connected in a natural way to the operator $\Omega_{\alpha\beta}$. We set

$$K_\alpha = \alpha + (\tau-\overline{\tau})\frac{\partial}{\partial\tau} = \alpha + y\Big(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big). \tag{12}$$

$$\Lambda_\beta = -\beta + (\tau-\overline{\tau})\frac{\partial}{\partial\overline{\tau}} = -\beta + y\Big(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big). \tag{13}$$

It is an immediate consequence of the definition that

$$K_\alpha G(\tau,\overline{\tau};\alpha,\beta) = \alpha G(\tau,\overline{\tau};\alpha+1,\beta-1)$$
$$\Lambda_\beta G(\tau,\overline{\tau};\alpha,\beta) = -\beta G(\tau,\overline{\tau};\alpha-1,\beta+1),$$

where the differentiation is formally carried out term by term, which is quite justified under the assumption $\mathrm{Re}(\alpha+\beta) > 2$. It can now be deduced that $G(\tau,\overline{\tau};\alpha,\beta)$ satisfies the two differential equations:

$$\{\Lambda_{\beta-1}K_\alpha + \alpha(\beta-1)\}G(\tau,\overline{\tau};\alpha,\beta) = \{K_{\alpha-1}\Lambda_\beta + \beta(\alpha-1)\}G(\tau,\overline{\tau};\alpha,\beta) = 0.$$

But by the definition of $\Omega_{\alpha,\beta}$, K_α and Λ_β, we have

$$\Lambda_{\beta-1}K_\alpha + \alpha(\beta-1) = K_{\alpha-1}\Lambda_\beta + \beta(\alpha-1) = \Omega_{\alpha\beta}; \tag{14}$$

therefore the two differential equations are identical. With the help of (14), we obtain the following commutation relations:

$$\begin{aligned} \Omega_{\alpha\beta}\Lambda_{\beta-1} &= \Lambda_{\beta-1}\Omega_{\alpha+1,\beta-1}, \\ K_\alpha\Omega_{\alpha\beta} &= \Omega_{\alpha+1,\beta-1}K_\alpha \end{aligned} \tag{15}$$

Let $\{\alpha,\beta\}$ denote the space of functions $f(\tau,\overline{\tau})$, which are real analytic in $\mathscr{G}$ and are solutions of the differential equation $\Omega_{\alpha\beta}f = 0$. We define the operators K and Λ on the space $\{\alpha,\beta\}$ by **178**

$$Kf = K_\alpha f, \Lambda f = \Lambda_\beta f \text{ for any } f \in \{\alpha,\beta\}. \tag{16}$$

With the help of (15), we see immediately that

$$K\{\alpha,\beta\} \subset \{\alpha+1,\beta-1\}, \Lambda\{\alpha,\beta\} \subset \{\alpha-1,\beta+1\}.$$

This shows that, for integral $n \geq 0$, the $n-th$ iterate K^n (respectively Λ^n) of K (respectively Λ), is well-defined and is identical with the operator $K_{\alpha+n-1}\ldots K_{\alpha+1}K_\alpha$ (respectively $\Lambda_{\beta+n-1}\ldots\Lambda_{\beta+1}\Lambda_\beta$) on the space $\{\alpha,\beta\}$. For any real matrix S with positive determinant, we also define

$$f|S = \underset{\alpha,\beta}{f|S} \text{ for } f \in \{\alpha,\beta\}.$$

We shall now prove that the operators K and Λ commute with this operator corresponding to $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $|S| > 0$: namely,

$$(K|f)|S = K(f|S), \quad (\Lambda f)|S = \Lambda(f|S). \tag{17}$$

For the proof, we can assume without loss of generality that $|S| = 1$. Let us set $\tau_1 = S < \tau >$, $\overline{\tau}_1 = S < \overline{\tau} >$. Then indeed

$$\begin{aligned} (Kf)|S(\tau,\overline{\tau}) &= (c\tau+d)^{-\alpha-1}(c\overline{\tau}+d)^{-\beta+1}\{\alpha + (\tau_1-\overline{\tau}_1)\frac{\partial}{\partial\tau_1}\}f(\tau,\overline{\tau}_1) \\ &= (c\tau+d)^{-\alpha-1}(c\overline{\tau}+d)^{-\beta+1}\{\alpha + (\tau-\overline{\tau})\frac{c\tau+d}{c\overline{\tau}+d}\frac{\partial}{\partial\tau}\}f(\tau_1,\overline{\tau}_1) \end{aligned}$$

$$= \{\alpha + (\tau - \overline{\tau})\frac{\partial}{\partial\tau}\}(c\tau + d)^{-\alpha}(c\overline{\tau} + d)^{-\beta} f(\tau_1, \overline{\tau}_1)$$
$$= K(f|S)(\tau, \overline{\tau}).$$

179 The second statement follows in a similar way.

For any function $f(\tau, \overline{\tau})$ we define the operator $\mathbf{X}$ by

$$\mathbf{X} f(\tau, \overline{\tau}) = f(-\overline{\tau}, -\tau), \tag{18}$$

which is equivalent to the substitution $x \to -x$ leaving y fixed. It is an easy consequence of the definition that

$$\mathbf{X}\{\alpha, \beta\} = \{\beta, \alpha\}.$$

If $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to Ω and $S^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) S \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$, then, for any $f \in \{\alpha, \beta\}$, we have

$$(\mathbf{X} f)\underset{\beta,\alpha}{|} S^* = u(S)\mathbf{X}(f|S)_{\alpha,\beta}, \tag{19}$$

where $u(S)$ is the factor system given by

$$u(S) = u_{\alpha,\beta}(S) = \begin{cases} 1 & \text{for } c \neq 0 \\ e^{(\alpha-\beta)i\pi(1-\operatorname{sgn} d)} & \text{for } c = 0. \end{cases}$$

For, with $g = \mathbf{X} f$, we have $g(\tau, \overline{\tau}) = f(-\overline{\tau}, -\tau)$ and

$$((\mathbf{X})\underset{\beta,\alpha}{|} S^*)(\tau, \overline{\tau}) = (g\underset{\beta,\alpha}{|} S^*)(\tau, \overline{\tau}) = (-c\tau + d)^{-\beta}(-c\overline{\tau} + d)^{-\alpha} g(S^* < \tau >, S^* < \overline{\tau} >)$$
$$= u(S)(c(-\overline{\tau}) + d)^{-\alpha}(c(-\tau) + d)^{-\beta} f(S < -\overline{\tau} >, S < -\tau >)$$
$$= u(S)\mathbf{X}(f\underset{\alpha,\beta}{|} S)(\tau, \overline{\tau}).$$

The factor system $u(S)$ appears on the right hand side of 19, because the definitions of the powers of $(c\tau + d)$ and $(c\overline{\tau} + d)$ are different in the upper and lower half-planes. It can be verified easily that

$$\mathbf{X}^2 = 1, K\mathbf{X} = -\mathbf{X}\Lambda, \mathbf{X}K = -\Lambda\mathbf{X}.$$

180 We introduce here an operator Θ, which will be used in the theory of

Hecke operators in the next chapter and which acts on the space $\{\alpha,\beta\}$ under the assumptions

$$r = \alpha - \beta \text{ integral } \geq 0, \Gamma(\beta) \neq \infty.$$

It is defined by

$$\Theta = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\mathbf{X}\Lambda^r. \tag{20}$$

We shall show that

$$\Theta\{\alpha,\beta\} = \{\alpha,\beta\}, \quad \Theta^2 = 1.$$

By definition, we have

$$\Theta\{\alpha,\beta\} = \mathbf{X}\Lambda^r\{\alpha,\beta\} \subset \mathbf{X}\{\alpha, r, \beta + r\} = \mathbf{X}\{\beta,\alpha\} = \{\alpha,\beta\}.$$

Moreover $K\mathbf{X} = -\mathbf{X}\Lambda$ implies that the operator

$$\begin{aligned}\Theta^2 &= (\frac{\Gamma(\beta)}{\Gamma(\alpha)})^2\mathbf{X}\Lambda^r\mathbf{X}\Lambda^r = (-1)^r(\frac{\Gamma(\beta)}{\Gamma(\alpha)})^2K^r\mathbf{X}^2\Lambda^r \\ &= (-1)^r(\frac{\Gamma(\beta)}{\Gamma(\alpha)})^2K^r\Lambda^r = (-1)^r(\frac{\Gamma(\beta)}{\Gamma(\alpha)})^2K_{\alpha-1}K_{\alpha-2}\dots K_\beta\Lambda_{\alpha-1}\dots\Lambda_\beta\end{aligned}$$

on the space $\{\alpha,\beta\}$. But on the space $\{\beta+n-1, \alpha-1-n\}$, $K_{\beta+n}\Lambda_{\alpha-1-n} = -(\beta+n)(\alpha-1-n)$; therefore, it follows that

$$\Theta^2 = (\frac{\Gamma(\beta)}{\Gamma(\alpha)})^2\prod_{n=0}^{r-1}(\beta+n)(\alpha-1-n) = (\frac{\Gamma(\beta)}{\Gamma(\alpha)})^2\prod_{n=0}^{r-1}(\beta+n)^2 = 1.$$

Consequently $\{\alpha,\beta\} \subset \Theta\{\alpha,\beta\}$, which proves that

$$\Theta\{\alpha,\beta\} = \{\alpha,\beta\}.$$

2 Non-Analytic Forms

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For our later use, we determine all the periodic functions which are contained in the space $\{\alpha,\beta\}$ and increase at most as a power of y uniformly in x, when y tends to infinity. We define, for $y > 0$ and $\in = \pm 1$, the function $W(\in y;\alpha,\beta)$ by

$$W(\in y;\alpha,\beta) = y^{-\frac{1}{2}q} W_{\frac{1}{2}r\in,\frac{1}{2}(q-1)}(2y) \tag{1}$$

where $q = \alpha+\beta$, $r = \alpha-\beta$ and $W_{\ell,m}(y)$ is Whittaker's function which is a solution of Whittaker's differential equation

$$\{4y^2\frac{d^2}{dy^2} + 1 - 4m^2 + 4\ell y - y^2\}W(y) = 0 \tag{2}$$

We set, for $y > 0$,

$$u(y,q) = \frac{y^{1-q}-1}{1-q} = \begin{cases} \sum_{n=1}^{\infty} \frac{(\log y)^n}{n!}(1-q)^{n-1} & \text{for } q \neq 1 \\ \log y & \text{for } q = 1. \end{cases}$$

Lemma 6. *Let $g_\in(y)e^{i\in x} \in \{\alpha,\beta\}$ and let $g_\in(y) = o(y^k)$ for $y = \rightarrow \infty$ with a certain constant K if $\in \neq 0$. Then*

$$g_0(y) = a\ \ u(y,q) + b, g_\in(y) = aW(\in y,\alpha,\beta) \textit{ for } \in^2 = 1.$$

Proof. Since $g_\in(y)e^{i\in x}$ belongs to $\{\alpha,\beta\}$, it satisfies the differential equation $\Omega_{\alpha\beta} g_\in(y)e^{i\in x} = 0$, which shows that

$$\{y\frac{d^2}{dy^2} + q\frac{d}{dy} + \in r - \in^2 y\}g_\in(y) = 0. \tag{3}$$

If $\in = 0$, then 1 and the function $u(y,q)$ form a system of independent solutions and therefore $g_0(y) = au(y,q) + b$ for some constants a and

182 b. Let $\in^2 = 1$. Substituting $g_\in(y) = y^{-\frac{1}{2}q} W_0(\frac{1}{2}r \in, \frac{1}{2}(q-1), 2y)$ in (3) we see that $W(y) = W_0(\ell,m,y)$ is a solution of differential equation (2). But Whittaker's differential equation has two independent solutions, one of which tends to ∞ exponentially when $y \to \infty$ and therefore cannot occur in $g_\in(y)$; the other solution of (2) is $W_{\ell,m}(y)$ with the asymptotic behaviour given by

$$W_{\ell,m}(y) \approx e^{-\frac{1}{2}y}y^\ell\{1 + \sum_{n=1}^{\infty}\frac{1}{n!y^n}\prod_{q=1}^{n}[m^2 - (\ell + \frac{1}{2} - q)^2]\}. \tag{5}$$

Therefore we see by (1) that

$$g_{\in}(y) = aW(\in y; \alpha, \beta)$$

and the lemma is proved.

The behaviour of the functions mentioned in lemma 6 under the action of the operators K_α, Λ_β and $\mathbf{X}$ is given by □

Lemma 7. *1.* $\mathbf{X}1 = 1, \mathbf{X}u(y, q) = u(y, q)$,

$$\mathbf{X}W(\in y, \alpha, \beta)e^{i\in x} = W(-\in y, \beta, \alpha)e^{-i\in x} (\in^2 = 1).$$

2. $K_\alpha 1 = \alpha, K_\alpha u(y, q) = (1 - \beta)u(y, q) + 1$

$$K_\alpha W(\in y; \alpha, \beta)e^{i\in x} = \begin{cases} -W(\in y; \alpha + 1, \beta - 1)e^{i\in x} & \text{for } \in = 1 \\ -\alpha(\beta - 1)W(\in y; \alpha + 1, \beta - 1)e^{i\in x} & \text{for } \in = -1. \end{cases}$$

3. $\Lambda_\beta 1 = -\beta, \Lambda_\beta u(y, q) = (\alpha - 1)u(y, q) - 1.$

$$\Lambda_\beta W(\in y; \alpha, \beta)e^{i\in x} = \begin{cases} (\alpha - 1)\beta W(\in y; \alpha - 1, \beta + 1)e^{i\in x} & \text{for } \in = 1 \\ W(\in y; \alpha - 1, \beta + 1)e^{i\in x} & \text{for } \in = -1. \end{cases}$$

Proof. The assertion about the action of the operators $\mathbf{X}$, K_α and Λ_β **183**
on the functions 1 and $u(y, q)$ is trivial. Since $W(\in y; \alpha, \beta) = W(-\in y; \beta, \alpha)$, it follows that $\mathbf{X}(W(\in y; \alpha, \beta)e^{i\in x}) = W(-\in y; \beta, \alpha)e^{-i\in x}$. In order to prove the remaining statements in the lemma, we shall make use of certain identities between the solutions $W_0(\ell, m, y)$ of Whittaker's differential equation for different values of the parameters ℓ and m. It can be verified that

$$yW_0'(\ell, m, y) \pm (\ell - \frac{1}{2}y)W_0(\ell, m, y)$$

is again a solution of the type $W_0(\ell \pm 1, m, y)$. Let us assume that $W_0(\ell, m, y) = W_{\ell,m}(y)$. Then the asymptotic behaviour of this function shows that, but for a constant factor, $W_0(\ell \pm 1, m, y)$ is identical with $W_{\ell\pm1,m}(y)$. This constant factor can be determined by considering the asymptotic expansion (5) of $W_{\ell,m}(y)$. Thus we obtain the identities

$$yW_{\ell,m}'(y) - (\ell - \frac{1}{2}y)W_{\ell,m}(y) = -(m^2 - (\ell - \frac{1}{2})^2)W_{\ell-1,m}(y),$$

$$yW'_{\ell,m}(y) + (\ell - \frac{1}{2}y)W_{\ell,m}(y) = -W_{\ell+1,m}(y).$$

By the definition of the operator K_α, we have

$$\begin{aligned}
K_\alpha W(\in y;\alpha,\beta)e^{i\in x} &= \{\alpha + y(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y})\}y^{-\frac{1}{2}q}W_{\frac{1}{2}r\in,\frac{1}{2}(q-1)}(2y)^{ei\in x}\\
&= y^{-\frac{1}{2}q}e^{i\in x}\{\alpha - \frac{1}{2}q - \in y + y\frac{\partial}{\partial y}\}W_{\frac{1}{2}r\in,\frac{1}{2}(q-1)}(2y)\\
&= y^{-\frac{1}{2}q}e^{i\in x}(\alpha - \frac{1}{2}q - \in y)W_{\frac{1}{2}r\in,\frac{1}{2}(q-1)}(2y) + 2yW'_{\frac{1}{2}r\in,\frac{1}{2}(q-1)}(2y)\\
&= \begin{cases} -y^{-\frac{1}{2}q}W_{\frac{1}{2}r+1,\frac{1}{2}(q-1)}(2y)e^{ix} & \text{for } \in = 1\\ -\frac{q+r}{2}\frac{q-r-2}{2}y^{-\frac{1}{2}q}W_{-\frac{1}{2}r-1,\frac{1}{2}(q-1)}(2y)e^{-ix} & \text{for } \in = -1.\end{cases}\\
&= \begin{cases} -W(y;\alpha+1,\beta-1)e^{ix} & \text{for } \in = 1\\ -\alpha(\beta-1)W(-y;\alpha+1,\beta-1)e^{ix} & \text{for } \in = -1.\end{cases}
\end{aligned}$$

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The corresponding result for $\Lambda_\beta W(\in y;\alpha,\beta)e^{i\in x}$ can be proved in a similar way or could be derived from above with the help of the identity $\Lambda_\beta = -\mathbf{X}K_\beta\mathbf{X}$.

We shall now find the asymptotic behaviour of the function $W(\in y;\alpha,\beta)$ as $y \to 0$ and $y \to \infty$. □

Lemma 8. *For $y > 0$ and $\in = \pm 1$, the following asymptotic formulae hold:*

$$W(\in y;\alpha,\beta) \sim 2^{\frac{1}{2}r\in}y^{\frac{1}{2}(q-\in r)}e^{-y}(y \to \infty),$$
$$W(\in y;\alpha,\beta) \sim 2^{\frac{1}{2}(2-q)}\frac{\Gamma(q-1)}{\Gamma(\frac{q-\in r}{2})}y^{1-q}(y \to 0),$$

provided that in the latter case $\mathrm{Re}(q-1) > 0$, $\mathrm{Re}(q \pm \in r) > 0$.

Proof. The first formula follows from the asymptotic expansion (5) for $W_{\ell,m}(y)$ and the second from the integral representation

$$W_{\ell,m}(y) = \frac{y^{\frac{1}{2}-m}e^{-\frac{1}{2}y}}{\Gamma(m+\frac{1}{2}-\ell)}\int_0^\infty e^{-u}u^{m-\ell-\frac{1}{2}}(u+y)^{m+\ell-\frac{1}{2}}du,$$

for $\operatorname{Re}(m + \frac{1}{2} \pm \ell) > 0$ and $\operatorname{Re} m > 0$. It is now clear that

$$W_{\ell,m}(y) \sim \frac{\Gamma(2m)}{\Gamma(m + \frac{1}{2} - \ell)} y^{\frac{1}{2}-m} (y \to 0).$$

□

Let α and β be two arbitrary complex numbers with $r = \alpha - \beta$ real. **185**
Let Γ denote a horocyclic group and v a multiplier system for the group Γ and weight r. We assume that $-E$ belongs to Γ. By an *automorphic form of the type* $\{\Gamma, \alpha, \beta, v\}$, we mean a function $f(\tau, \overline{\tau})$ with the following properties:

1) $f(\tau, \overline{\tau})$ is real analytic in $\mathscr{G}$ and is a solution of the differential equation $\Omega_{\alpha\beta} f = 0$,

2) $(f|S)(\tau, \overline{\tau}) = v(S) f(\tau, \overline{\tau})$ for $S \in \Gamma$ and

3) If $A^{-1} < \infty >$ is a parabolic cusp of Γ, then with some constant $K > 0$, $(f|A^{-1})(\tau, \overline{\tau}) = o(y^K)$ for $y \to \infty$, uniformly in x.

We shall denote the space of these automorphic forms by $[\Gamma, \alpha, \beta, v]$. Let $N > 0$ be the least number so determined that $H = A^{-1}U^N A$ belongs to Γ and let

$$\sigma(A, H)v(H) = \sigma(H, A^{-1})v(H) = v^{A^{-1}}(U^N) = e^{2\pi i \kappa}, 0 \leq \kappa < 1.$$

Using the characterising properties 1), 2), 3) above for $f(\tau, \overline{\tau}) \in [\Gamma, \alpha, \beta, v]$, we shall show that, at the parabolic cusp $A^{-1} < \infty >$, there exists for f a Fourier expansion of the type

$$(f|A^{-1})(\tau, \overline{\tau}) = a_0 u(y, q) + b_0 + \sum_{n\kappa \neq 0} a_{n+\kappa} W(\frac{2\pi(n + \kappa)}{N} y; \alpha, \beta) e^{2\pi i (n+\kappa)x/N} \tag{6}$$

where a_0 and b_0 are equal to 0 in case $\kappa > 0$. Since the substitution **186**
$\tau \to \tau + N$ transforms $(f|A^{-1})(\tau, \overline{\tau})$ to $e^{2\pi i \kappa}(f|A^{-1})(\tau, \overline{\tau})$, we have, in any case, the Fourier expansion

$$(f|A^{-1})(\tau, \overline{\tau}) = \sum_{n=-\infty}^{\infty} \alpha_{n+\kappa}(y) e^{2\pi i (n+\kappa)x/N}$$

$$\text{with} \qquad \alpha_{n+\kappa}(y) = \frac{1}{N}\int_0^N (f|A^{-1})(\tau,\overline{\tau})e^{-2\pi i(n+\kappa)x/N}dx.$$

Because of property 3) above, we conclude that, for $y \to \infty$, $\alpha_{n+\kappa}(y)$ increases atmost as a power of y. But $\Omega_{\alpha|\beta}(f|A^{-1}) = (\Omega_{\alpha\beta}f)|A^{-1} = 0$, which implies that $\alpha_{n+\kappa}(y)e^{2\pi i(n+\kappa)x/N}$ satisfies the requirements of lemma 6. Therefore, the Fourier expansion of $(f|A^{-1})(\tau,\overline{\tau})$ may be seen to be of the type (6). Conversely, if $f(\tau,\overline{\tau})$ is such that for a real matrix A with $|A| = 1$, $(f|A^{-1})(\tau,\overline{\tau})$ has a series expansion as in (6), then lemma 8 shows that

$$(f|A^{-1})(\tau,\overline{\tau}) = o(y^K) \text{ for } y \to \infty,$$

with some positive constant K.

Applying K_α, Λ_β to (6) and making use of lemma 7 and (17) of § 1, we get

$$\begin{aligned}((K_\alpha f)|A^{-1})(\tau,\overline{\tau}) &= (1-\beta)a_0u(y,q) + a_0 + \alpha b_0 - \\ &- \sum_{n+\kappa>0} a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{N})y; \alpha+1, \beta-1)e^{2\pi i(n+\kappa)x/N} - \qquad (7)\\ &- \alpha(\beta-1)\sum_{n+\kappa<0} a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{N}y; \alpha+1,\beta-1)e^{2\pi i(n+\kappa)x/N},\end{aligned}$$

$$\begin{aligned}((\Lambda_\beta f)|A^{-1})(\tau,\overline{\tau}) &= (\alpha-1)a_0u(y,q) - a_0 - \beta b_0 + \\ &+ (\alpha-1)\beta\sum_{n+\kappa>0} a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{N}y; \alpha-1,\beta+1)e^{2\pi i(n+\kappa)x/N} + \qquad (8)\\ &+ \sum_{n+\kappa<0} a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{N}y; \alpha-1,\beta+1)e^{2\pi i(n+\kappa)x/N}.\end{aligned}$$

187 With the help of (7), (8) and § 2 of chapter 3, in which the transformation properties of a multiplier system have been described, the following relations can be established:

$$[\Gamma,\alpha,\beta,v]|S = [S^{-1}\Gamma S,\alpha,\beta,v^S] \text{ for } S \in \Omega, \qquad (9)$$

$$\mathbf{X}[\Gamma,\alpha,\beta,v] = [\Gamma^*,\beta,\alpha,v^*] \qquad (10)$$

with $v^*(S^*) = u(S)v(S)$, $S^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\Gamma^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$K_\alpha[\Gamma, \alpha, \beta, v] \subset [\Gamma, \alpha + 1, \beta - 1, v], \tag{11}$$

$$\Lambda_\beta[\Gamma, \alpha, \beta, v] \subset [\Gamma, \alpha - 1, \beta + 1, v]. \tag{12}$$

If, further $\alpha(\beta - 1) \neq 0$, then K_α maps the space $[\Gamma, \alpha, \beta, v]$ onto the space $[\Gamma, \alpha+1, \beta-1, v]$, because the operator $K_\alpha \dfrac{1}{\alpha(1-\beta)} \Lambda_{\beta-1}$ acts as the identity on the space $[\Gamma, \alpha+1, \beta-1, v]$ and therefore any $f \in [\Gamma, \alpha+1, \beta-1, v]$ is the image of $\dfrac{1}{\alpha(1-\beta)} \Lambda_{\beta-1} f \in [\Gamma, \alpha, \beta, v]$ under K_α. Similarly, it can be seen that Λ_β maps the space $[\Gamma, \alpha, \beta, v]$ onto the space $[\Gamma, \alpha - 1, \beta + 1, v]$, in case $\beta(\alpha - 1) \neq 0$. Thus from (11) and (12), we see that

$$K_\alpha[\Gamma, \alpha, \beta, v] = [\Gamma, \alpha + 1, \beta - 1, v], \text{ in case } \alpha(\beta - 1) \neq 0 \tag{11'}$$

$$\Lambda_\beta[\Gamma, \alpha, \beta, v] = [\Gamma, \alpha - 1, \beta + 1, v], \text{ in case } \beta(\alpha - 1) \neq 0 \tag{12'}$$

For $r = \alpha - \beta$ integral ≥ 0 and $\Gamma(\beta) \neq \infty$, we have

$$\Theta[\Gamma, \alpha, \beta, v] = [\Gamma^*, \alpha, \beta, v^*], \tag{13}$$

because of the relations **188**

$$\begin{aligned} \Theta[\Gamma, \alpha, \beta, v] &= \mathbf{X}\Lambda^r[\Gamma, \alpha, \beta, v] \subset \mathbf{X}[\Gamma, \alpha - r, \beta + r, v] \\ &= \mathbf{X}[\Gamma, \beta, \alpha, v] = [\Gamma^*, \alpha, \beta, v^*] \end{aligned}$$

and $\Theta^2 = 1$.

By complete induction on h, it follows from (8) that

$$\begin{aligned} &((\Lambda^h f)|A^{-1})(\tau, \overline{\tau}) = \\ &\frac{\Gamma(\alpha)}{\Gamma(\alpha - h)} a_0 u(y, q) + a_0 \sum_{\ell=0}^{h-1} (-1)^{h-1} \frac{\Gamma(\alpha)\Gamma(\beta + h)}{\Gamma(\alpha - 1)\Gamma(\beta + \ell + 1)} + (-1)^h \frac{\Gamma(\beta + h)}{\Gamma(\beta)} b_0 + \\ &+ \frac{\Gamma(\beta + h)\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha - h)} \sum_{n+\kappa>0} a_{n+\kappa} W(\frac{2\pi(n + \kappa)}{N} y; \alpha - h, \beta + h) e^{2\pi i(n+\kappa)x/N} \\ &+ \sum_{n+\kappa<0} a_{n+\kappa} W(\frac{2\pi(n + \kappa)}{N} y; \alpha - h, \beta + h) e^{2\pi i(n+\kappa)x/N}. \end{aligned}$$

where the Γ-functions $\dfrac{\Gamma(\beta+h)}{\Gamma(\beta)}$ and $\dfrac{\Gamma(\beta+h)}{\Gamma(\beta+\ell+1)}$ represent the value of the analytic functions $\dfrac{\Gamma(z+h)}{\Gamma(z)}$ and $\dfrac{\Gamma(z+h)}{\Gamma(z+\ell+1)}$ at the point $z=\beta$. Using lemma 7 and (19) of § 1 for $r=\alpha-\beta$ integral ≥ 0 and $\Gamma(\beta)\neq\infty$, we now see with $u(A^{-1})=u_{\alpha,\beta}(A^{-1})$ that

$$\begin{aligned}
u(A^{-1})(\Theta f|A^{*-1})(\tau,\overline{\tau}) &= u(A^{-1})\frac{\Gamma(\beta)}{\Gamma(\alpha)}((\mathbf{X}\Lambda^r f)|A^{*-1})(\tau,\overline{\tau}) = \\
&= \frac{\Gamma(\beta)}{\Gamma(\alpha)}(\mathbf{X}\{(\Lambda^r f)|A^{-1}\})(\tau,\overline{\tau}) \\
&= a_0 u(y,q) + \sum_{\ell=0}^{r-1}(-1)^{r-\ell}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha-\ell)\Gamma(\beta+\ell+l)}\cdot a_0 + (-1)^r b_0 + \\
&+ \frac{\Gamma(\alpha)}{\Gamma(\beta)}\sum_{n+\kappa>0} a_{n+\kappa}W(\frac{-2\pi i(n+\kappa)}{N}y;\alpha,\beta)e^{-2\pi(n+\kappa)x/N} + \\
&+ \frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{n+\kappa<0} a_{n+\kappa}W(\frac{-2\pi(n+\kappa)}{N}y;\alpha,\beta)e^{-2\pi i(n+\kappa)x/N}.
\end{aligned}$$

189 Let κ^* be so chosen that $\kappa\equiv-\kappa^*(\mod 1)$. Then $n+\kappa=-(n^*+\kappa^*)$ for some integer n^*. Therefore, if, finally, we replace n^* again by n, we obtain

$$\begin{aligned}
&u(A^{-1})(\Theta f|A^{*-1})(\tau,\overline{\tau}) \\
&= a_0 u(y,q) + \sum_{\ell=0}^{r-1}(-1)^{r-\ell}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\alpha-\ell)\Gamma(\beta+\ell+1)}a_0 + (-1)^r b_0 + \\
&+ \frac{\Gamma(\beta)}{\Gamma(\alpha}\sum_{n+\kappa^*>0} a_{-n-\kappa^*}W(\frac{2\pi(n+\kappa^*)}{N}y;\alpha,\beta)e^{2\pi i(n+\kappa^*)x/N} + \\
&+ \frac{\Gamma(\alpha)}{\Gamma(\beta)}\sum_{n+\kappa^*>0} a_{-n-\kappa^*}W(\frac{2\pi(n+\kappa^*)}{N}y;\alpha,\beta)e^{2\pi i(n+\kappa^*)x/N}.
\end{aligned}\tag{14}$$

Here $A^*=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)A\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)^{-1}$. Since $\left(\begin{smallmatrix}1&N\\0&1\end{smallmatrix}\right)^*=\left(\begin{smallmatrix}1&-N\\0&1\end{smallmatrix}\right)$, it can be proved, using just the properties of multiplier systems, that

$$v^{*^{A^{*-1}}}(U^N)v^{A^{-1}}(U^N)=1$$

corresponding to $\kappa+\kappa^* \equiv 0(\mod 1)$. If $r \equiv 0(\mod 2)$, then applying Θ on both sides of (14) and using the fact that $\Theta^2 = 1$, we see immediately that

$$\sum_{\ell=0}^{r-1}(-1)^{r-\ell}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha-\ell)\Gamma(\beta+\ell+1)} = 0 \quad (\text{for } r \equiv 0(\mod 2)), \tag{15}$$

which can also be proved directly.

We remark here that the space $[\Gamma, \alpha, v]$ of analytic modular forms of real weight α and multiplier system v is contained in the space $[\Gamma, \alpha, 0, v]$ and it is determined in this space by the condition $\dfrac{\partial f}{\partial \overline{\tau}} = 0$.

Following a method of Siegel, we shall now prove that, under some assumptions, the space $[\Gamma, \alpha, \beta, v]$ is a vector space of finite dimension over the complex number field.

Theorem 28. *Let Γ_0 be a subgroup of finite index in the modular group Γ. Let $r = \alpha - \beta$ be real and $p = \mathrm{Re}(\alpha+\beta) \geq 0$. Then the space $[\Gamma_0, \alpha, \beta, v]$ is of finite dimension over the complex number field.* **190**

Proof. We shall prove that theorem in a number of steps.

1) Since $r = \alpha - \beta$ is real, we have

$$|(c\tau+d)^\alpha(c\overline{\tau}+d)^\beta| = |c\tau+d|^p \text{ for real } (c,d) \neq (0,0).$$

For $L \in \Gamma_0$ and $A \in \Gamma$ we set

$$\tau^* = L < \tau >, \hat{\tau} = A < \tau > .$$

We shall denote by y, y^* and $\hat{y}$ the imaginary part of τ, τ^* and $\hat{\tau}$ respectively. Let f be a function in $[\Gamma_0, \alpha, \beta, v]$. Then obviously

$$y^{\frac{p}{2}}|f(\tau,\overline{\tau})| = y^{*\frac{p}{2}}|f(\tau^*,\overline{\tau^*})| = \hat{y}^{\frac{p}{2}}|(f|A^{-1})(\hat{\tau},\overline{\hat{\tau}})|.$$

Let $A_1^{-1} < \infty >, \ldots, A^{-1}_{\sigma<\infty>}$ with $A_\ell \in \Gamma$ for $1 \leq \ell \leq \sigma$ be a complete system of inequivalent parabolic cusps of Γ_0 and let $\mathscr{L}_1, \ldots, \mathscr{L}_\sigma$ be cusp sectors at these cusps. Then

$$\mathfrak{F}_0 = \bigcup_{n=1}^{\sigma} \mathscr{L}_n$$

is a fundamental domain for Γ_0. Let N_ℓ denote the width of the cusp sector $\mathscr{L}_\ell$ and let $N = \max\limits_{1\le\ell\le\sigma} N_\ell$. Net

$$(f|A_\ell^{-1})(\tau,\overline{\tau}) = a_0^{(\ell)}u(y,q) + b_0^{(\ell)} + \sum_{n+\kappa_\ell\neq 0} a_{n+\kappa_\ell}^{(\ell)} W(\frac{2\pi(n+\kappa_\ell)y}{N_\ell};\alpha,\beta)e^{2\pi i(n+\kappa_\ell)x/N_\ell}$$

be the Fourier expansion at the cusp $A_\ell^{-1} < \infty >$ for any f in
191 $[\Gamma_0,\alpha,\beta,v]$. Let us suppose that

$$a_0^{(\ell)} = b_0^{(\ell)} = a_{n+\kappa_\ell}^{(\ell)} = 0 \text{ for } |n+\kappa_\ell| \le m, \ell = 1,2,\ldots,\sigma.$$

Then we shall show that f vanishes identically for m chosen sufficiently large; this will establish the theorem. In the following, $C_1, C_2, \ldots$ will denote constants which depend only on Γ_0,α,β,v and not on f.

2) We estimate $(f|A_\ell^{-1})(\tau,\overline{\tau})$ in $A_\ell < \mathscr{L}_\ell >$. Since the Fourier expansion of $(f|A_\ell^{-1})(\tau,\overline{\tau})$ converges in the whole of $\mathscr{G}$, we see that

$$|a_{n+\kappa_\ell}^{(\ell)} W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}\eta;\alpha,\beta)| \le C(\eta).$$

for all n,ℓ and $\eta > 0$. We normalise the modular form f in the beginning itself, so that $C(\eta) = 1$ for $\eta = \dfrac{1}{2\sqrt{3}}$. Moreover, with this η, we have $y \ge 3\eta$ for $\tau \in A_\ell < \mathscr{L}_\ell >$. Using the asymtotic formula for the function $W(\in y;\alpha,\beta)$ given in lemma 8, we obtain

$$\left|\frac{W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha,\beta)}{W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}\eta;\alpha,\beta)}\right| \sim (\frac{\eta}{y})^{\frac{1}{2}(p-\in r)} \quad e^{-2\pi|n+\kappa_\ell|(y-\eta)/N_\ell}$$

for $|n+\kappa_\ell| \to \infty$, where $\in = \operatorname{sgn}(n+\kappa_\ell)$. With the special choice of $\eta = \dfrac{1}{2\sqrt{3}}$ and for $\tau \in A_\ell < \mathscr{L}_\ell >$, it follows now that

$$\left|\frac{W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha,\beta)}{W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}\eta;\alpha,\beta)}\right| \le C_1 e^{-2\pi|n+\kappa_\ell|y/(3N_\ell)}$$

for $n + \kappa_\ell \neq 0$. This implies that

$$\begin{aligned}
|(f|A_\ell^{-1})(\tau,\overline{\tau})| &\leq \sum_{|n+\kappa_\ell|>m} \left| \frac{W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha,\beta)}{W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}\eta;\alpha,\beta)} \right| \\
&\leq C_1 \sum_{|n+\kappa_\ell|>m} e^{-2\pi|n+\kappa_\ell|y/(3N_\ell)} \\
&\leq C_2 e^{-2\pi my/(3N_\ell)}.
\end{aligned}$$

In particular, we see that the function 192

$$|(f|A_\ell^{-1})(\tau,\overline{\tau})|e^{-2\pi my/(3N_\ell)} \to 0 \text{ as } y \to \infty,$$

uniformly in x. Therefore it has in the domain $A_\ell < \mathscr{L}_\ell >$ a non-negative maximum M_ℓ which is attained at a finite point τ_ℓ of this domain. Thus

$$|(f|A_\ell^{-1}(\tau,\overline{\tau}))| \leq M_\ell e^{-2\pi my/(3N_\ell)} \text{ for } \tau \in A_\ell < \mathscr{L}_\ell >$$

and euqality holds when $\tau = \tau_\ell$. Let

$$M_\ell \leq M_h \text{ for } \ell = 1, 2, \ldots, \sigma.$$

3) We shall now estimate $(f|A_h^{-1})(\tau,\overline{\tau})$ in $\mathscr{G}$. For a given point $\tau \in \mathscr{G}$, we set $\tau' = A_h^{-1} < \tau >$, $\tau'' = L < \tau' >$, where L belongs to Γ_0 and is so chosen that τ'' belongs to $\mathfrak{F}_0$. Since $\mathfrak{F}_0 = \bigcup_{n=1}^{\sigma} \mathscr{L}_n$, the point τ'' belongs to at least one of the cusp sectors $\mathscr{L}_\ell$, $1 \leq \ell \leq \sigma$. Let $\tau'' \in \mathscr{L}_\ell$; then we set $\tau^* = A_\ell < \tau'' >$. We shall denote by y, y', y'' and y^* the imaginary parts of τ, τ', τ'' and τ^* respectively. By 1), we have

$$\begin{aligned}
|y^{\frac{p}{2}}(f|A_h^{-1})(\tau,\overline{\tau})| = y'^{\frac{p}{2}}|f(\tau',\overline{\tau'})| &= y''^{\frac{p}{2}}||f(\tau'',\overline{\tau''})| \\
&= y^{*\frac{p}{2}}|(f|A_\ell^{-1})(\tau^*,\overline{\tau^*})|.
\end{aligned}$$

Using the estimates of 2), we get 193

$$\begin{aligned}|(f|A_h^{-1})(\tau,\overline{\tau})| &\leq y^{-\frac{p}{2}} y^{*\frac{p}{2}} M_\ell e^{-2\pi m y^*/(3N_\ell)}\\ &\leq M_h(\frac{3pN_\ell}{4\pi emy})^{\frac{p}{2}} \leq M_h(\frac{3pN}{4\pi emy})^{\frac{p}{2}},\end{aligned}$$

because the function $y^{p/2}e^{-2\pi my/(3N_\ell)}$ has the maximum $(\frac{3pN_\ell}{4\pi em})^{p/2}$ at the point $y = \frac{3pN_\ell}{4\pi m}$. This estimate holds even when $p = 0$, in which case we assume that $p^p = 1$.

4) We proceed to find a bound for $|(f|A_h^{-1})(\tau_h, \overline{\tau}_h)|$ where τ_h is the point mentioned in 2) i.e. the point belonging to $A_h < \mathscr{L}_h >$ such that

$$M_h e^{-2\pi m y_h/(3N_h)} = |(f|A_h^{-1})(\tau_h, \overline{\tau}_h)| \text{ with } \tau_h = x_h + iy_h.$$

Let $\tau = x + \frac{i}{3}y_h$. Then

$$\begin{aligned}&a^{(h)}_{n+\kappa_h} W(\frac{2\pi(n+\kappa_h)}{N_h}\frac{y_h}{3};\alpha,\beta)\\ &\qquad = \frac{1}{N_h}\int_0^N (f|A_h^{-1})(\tau,\overline{\tau})e^{-2\pi i(n+\kappa_h)x/N_h}dx.\end{aligned}$$

With the help of the preceding step 3), we see now that

$$|a^{(h)}_{n+\kappa_h} W(\frac{2\pi(n+\kappa_h)}{N_h}\frac{y_h}{3};\alpha,\beta)| \leq M_h(\frac{9pN}{4\pi emy_h})^{p/2},$$

so that

$$\begin{aligned}M_h e^{-2\pi m y_h/(3N_h)} &= |(f|A_h^{-1})(\tau_h,\overline{\tau}_h)|\\ &\leq M_h(\frac{9pN}{4\pi emy_h})^{p/2}\sum_{|n+\kappa_h|>m}\frac{W(\frac{2\pi(n+\kappa_h)}{N_h}y_h;\alpha,\beta)}{W(\frac{2\pi(n+\kappa_h)}{3N_h}y_h;\alpha,\beta)}\\ &\leq C_3 M_h(\frac{9pN}{4\pi emy_h})^{p/2}\sum_{|n+\kappa_h|>m} e^{-\pi|n+\kappa_h|y_h/N_h}\end{aligned}$$

$$\leq C_4 M_h \left(\frac{9pN}{4\pi e m y_h}\right)^{p/2} e^{-\pi m y_h/N_h}$$

194 showing that

$$M_h e^{\pi m y_h/(3N_h)} \leq C_4 M_h \left(\frac{9pN}{4\pi e m y_h}\right)^{p/2}.$$

Thus

$$M_h e^{\pi m \eta/N} \leq C_5 M_h \left(\frac{3pN}{4\pi e m \eta}\right)^{p/2}$$

for $p \geq 2$. Now it is obvious that $M_h = 0$, if m is sufficiently large, which implies that $f = 0$. Hence the theorem is proved.

$\square$

3 Eisenstein Series

We associate to two functions $f(\tau, \overline{\tau}) \in \{\alpha, \beta\}$ and $g(\tau, \overline{\tau}) \in \{\alpha', \beta'\}$ a differential form $\omega(f, g)$ defined by

$$\omega(f, g) = y^{\gamma-1}\{f \Lambda_{\beta'} g d\overline{\tau} + g K_\alpha f d\tau\},$$

where α', β', γ are complex numbers which independently of α, β, will be so determined that $d\omega(f, g) = 0$. With the help of the differential equations satisfied by f and g, we see by simple calculation that

$$\begin{aligned} d\omega(f, g) &= \{\frac{\partial}{\partial \tau}(y^{\gamma-1} f \Lambda_\beta, g) - \frac{\partial}{\partial \overline{\tau}}(y^{\gamma-1} g K_\alpha f)\} d\tau \Lambda d\overline{\tau} \\ &= \left\{\frac{i}{2}(\gamma - 1)(\beta' - \alpha) y^{\gamma-2} f \cdot g + (\gamma - \alpha - \alpha') y^{\gamma-1} f \frac{\partial g}{\partial \tau} + \right. \\ &\quad + (\gamma - \beta - \beta') y^{\gamma-1} g \frac{\partial f}{\partial \tau}\} d\tau \Lambda d\overline{\tau}. \end{aligned}$$

This differential vanishes trivially, if any one of the following two conditions is satisfied: 195

$$\alpha' = \beta, \quad \beta' = \alpha, \quad \gamma = \alpha + \beta, \tag{1}$$

$$\alpha' = 1 - \alpha, \quad \beta' = 1 - \beta, \quad \gamma = 1. \tag{2}$$

We shall denote in the following by $\omega(f,g)|S(S \in \Omega)$ the differential form which is obtained by substituting $S < \tau >$ for τ in $\omega(f,g)$. We shall show that if the numbers α, β and α', β' satisfy any one of the conditions (1) and (2), then

$$\omega(f,g)|S = \omega(f|S, g|S).$$

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|S| = 1$. We set

$$\hat{\tau} = S < \tau > \text{ and } \hat{M} = M(S < \tau >, S < \overline{\tau} >),$$

where $M = M(\tau, \overline{\tau})$ denotes an arbitrary function or operator. By definition

$$\begin{aligned}
\omega(f|S, g|S) &= y^{\gamma-1}\{(f|S)\Lambda_\beta, (g|S)d\overline{\tau} + (g|S)K_\alpha(f|S)d\tau\} \\
&= y^{\gamma-1}\{(f|S)(\Lambda_\beta, g)|S\, d\overline{\tau} + (g|S)(K_\alpha f)|S\, d\tau\} \\
&= y^{\gamma-1}\Big\{(c\tau + d)^{-\alpha-\alpha'+1}(c\overline{\tau} + d)^{-\beta-\beta'-1}\hat{f}\hat{\Lambda}_\beta, \hat{g}d\overline{\tau} + \\
&\quad +(c\tau + d)^{-\alpha-\alpha'-1}(c\overline{\tau} + d)^{-\beta-\beta'+1}\hat{g}\hat{K}_\alpha\hat{f}d\tau\Big\} \\
&= \hat{y}^{\gamma-1}\{\hat{f}\hat{\Lambda}_\beta, \hat{g}d\overline{\hat{\tau}} + \hat{g}\hat{K}_\alpha\hat{f}d\hat{\tau}\} \\
&= \omega(f,g)|S.
\end{aligned}$$

We shall say that two spaces $[\Gamma, \alpha, \beta, v]$ and $[\Gamma, \alpha', \beta', v']$ are *adjoint of the first or the second kind* according as α, β and α', β' satisfy condition (1) or (2) above and $v \cdot v' = 1$. If f and g belong to the adjoint
196 spaces $[\Gamma, \alpha, \beta, v]$ and $[\Gamma, \alpha', \beta', v']$ respectively, then it follows from the transformation formula for $\omega(f,g)$ that

$$\omega(f,g)|S = \omega(f,g) \text{ for } S \in \Gamma.$$

We shall now examine what the existence of an invariant differential for two adjoint spaces of the first kind for a subgroup Γ_0 of finite index in the modular group means. Let $f \in [\Gamma_0, \alpha, \beta, v]$, $g \in [\Gamma_0, \alpha', \beta', v']$ and let $\alpha' = \beta, \beta' = \alpha$, $\gamma = q$ and $v \cdot v' = 1$. As before, let $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_\sigma$ be cusp sectors at the cusps $A_1^{-1} < \infty >, A_2^{-1} < \infty >, \ldots, A_\sigma^{-1}$ of Γ_0, which constitute a complete system of inequivalent parabolic cusps of Γ_0. Then

$$\mathfrak{F}_0 = \bigcup_{\ell=1}^{\sigma} \mathscr{L}_\ell$$

is a fundamental domain for Γ_0. Let $N_\ell, \ell = 1, 2, \ldots, \sigma$ be the width of the cusp sector $\mathscr{L}_\ell$. We remove the parabolic cusp from the domain $\mathscr{L}_\ell$ with the help of a circular arc c_ℓ, which is mapped by the transformation $\tau \to A_\ell < \tau >$ onto a segment s_ℓ of a line $y = y_\ell > 1$, and denote the remaining compact part by $\mathscr{L}_\ell^*$. Since $d\omega(f, g) = 0$, it follows that

$$\int_{\partial\mathscr{L}_\ell^*} \omega(f, g) = 0,$$

where $\partial\mathscr{L}_\ell^*$ is the boundary of $\mathscr{L}_\ell^*$ oriented in the positive direction. Consequently, we obtain

$$\sum_{\ell=1}^{\sigma} \int_{\partial\mathscr{L}_\ell^*} \omega(f, g) = 0$$

and here the sum of the integrals along those edges, which are equivalent but oriented in the opposite direction, vanishes, because $\omega(f, g)$ is invariant under the transformations of Γ_0. Using again the transformation formula for $\omega(f, g)$, we see that 197

$$\begin{aligned}\sum_{\ell=1}^{\sigma} \int_{\partial\mathscr{L}_\ell^*} &= \sum_{\ell=1}^{\sigma} \int_{c_\ell} \omega(f, g) = \sum_{\ell=1}^{\sigma} \int_{s_\ell} \omega(f, g)|A_\ell^{-1} \\ &= \sum_{\ell=1}^{\sigma} \int_{s_\ell} \omega(f|A_\ell^{-1}, g|A_\ell^{-1}) = 0 \qquad (3)\end{aligned}$$

Let

$$(f|A_\ell^{-1})(\tau, \overline{\tau}) = \varphi_\ell(y) + \sum_{n+\kappa_\ell \neq 0} a_{n+\kappa_\ell} W(\frac{2\pi(n + \kappa_\ell)}{N_\ell} y; \alpha, \beta) e^{2\pi i(n+\kappa_\ell)x/N_\ell}$$

and

$$(g|A_\ell^{-1})(\tau,\overline{\tau}) = \psi_\ell(y) + \sum_{n+\kappa'_\ell\neq 0} a_{n+\kappa'_\ell} W(\frac{2\pi(n+\kappa'_\ell)}{N_\ell}y;\alpha,\beta)e^{2\pi i(n+\kappa'_\ell)x/N_\ell}$$

be the Fourier expansions for $f(\tau,\overline{\tau})$ and $g(\tau,\overline{\tau})$ at the cups $A_\ell^{-1} < \infty >$ of Γ_0. Here

$$\varphi_\ell(y) = a'_\ell u(y,q) + a''_\ell \text{ and}$$
$$\psi_\ell(y) = b'_\ell u(y,q) + b''_\ell.$$

These functions vanish when v and consequently v' too, is ramified at the cusp $A_\ell^{-1} < \infty >$. Moreover we have $\kappa_\ell + \kappa'_\ell \equiv 0(\mod 1)$, because $v\cdot v' = 1$. By § 2, lemma 7, we get

$$\begin{aligned} K_\alpha(f|A_\ell^{-1}) &= (K_\alpha f)|A_\ell^{-1} \\ &= K_\alpha\varphi_\ell(y) - \sum_{n+\kappa_\ell>0} a_{n+\kappa_\ell} W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha+1,\beta-1)e^{2\pi i(n+\kappa_\ell)x/N_\ell} \\ &- \alpha(\beta-1)\sum_{n+\kappa_\ell<0} a_{n+\kappa_\ell} W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha+1,\beta-1)e^{2\pi i(n+\kappa_\ell)x/N_\ell} \end{aligned}$$

198 and

$$\begin{aligned} (\Lambda_{\beta'} g)|A_\ell^{-1} &= \Lambda_{\beta'}(g|A_\ell^{-1}) \\ &= \Lambda_{\beta'}\psi_\ell(y) + (\alpha'-1)\beta' \sum_{n+\kappa'_\ell>0} b_{n+\kappa'_\ell} W(\frac{2\pi(n+\kappa'_\ell)}{N_\ell}y; \\ &\qquad \alpha'-1,\beta'+1)e^{2\pi i(n+\kappa'_\ell x/N_\ell)}+ \\ &+ \sum_{n+\kappa'_\ell<0} b_{n+\kappa'_\ell} W(\frac{2\pi(n+\kappa'_\ell)}{N_\ell}y;\alpha'-1,\beta'+1)e^{2\pi i(n+\kappa'_\ell)x/N_\ell}. \end{aligned}$$

Since dy vanishes on S_ℓ, it is sufficient to calculate the value of $\omega(f|A_\ell^{-1}, g|A_\ell^{-1})$ modulo dy. By the definition of $\omega(f,g)$, we have

$$\omega(f|A_\ell^{-1}, g|A_\ell^{-1}) = y^{q-1}\{f|A_\ell^{-1}(\Lambda_\beta, g)|A_\ell^{-1}(K_\alpha f)A_\alpha^{-1}\}dx(\mod dy). \tag{4}$$

Let $X_\ell(y)dx$ denote the terms independent of x on the right hand side of equation (4). Then

$$
\begin{aligned}
& y^{1-q}X_\ell(y)\\
& = \varphi_\ell(y)\Lambda_{\beta'}\psi_\ell(y)K_\alpha\varphi_\ell(y)+\\
& + (\alpha'-1)\beta' \sum_{n+\kappa_\ell<0} a_{n+\kappa_\ell}b_{-n-\kappa_\ell}W(\frac{2\pi(n+\kappa)_\ell}{N_\ell}y;\alpha,\beta)W(-\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha'-1,\beta'+1)+\\
& + \sum_{n+\kappa_\ell>0} a_{n+\kappa_\ell}b_{-n-\kappa_\ell}W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha,\beta)W(-\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha'-1,\beta'+1)-\\
& - \sum_{n+\kappa_\ell>0} b_{-n-\kappa_\ell}W(-\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha',\beta')W(\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha+1,\beta-1)-\\
& - \alpha(\beta-1)\sum_{n+\kappa_\ell<0} b_{-n-\kappa_\ell}a_{n+\kappa_\ell}W(-\frac{2\pi(n+\kappa_\ell)}{N_\ell}y;\alpha',\beta')W(\frac{2\pi(n+\kappa_\ell)}{N_\ell};\alpha+1,\beta-1).
\end{aligned}
$$

Using the relations $W(-y;\alpha,\beta) = W(y;\beta,\alpha)$, $\alpha' = \beta$ and $\beta' = \alpha$, we obtain

$$y^{1-q}\chi_\ell(y) = \varphi_\ell(y)\Lambda_{\beta'}\psi_\ell(y) + \psi_\ell(y)K_\alpha\varphi_\ell(y).$$

Since the value of the integral of the terms containing x explicitly on **199**
the right hand side of (3) vanishes, in view of the integrand being then a periodic function of period N_ℓ, equal also to the length of s_ℓ, it follows that

$$\sum_{\ell=1}^{\sigma}\int_{s_\ell}\omega(f|A_\ell^{-1}, g|A_\ell^{-1}) = \sum_{\ell=1}^{\sigma}\int_{s_\ell}\chi_\ell(y)dx = 0 \tag{5}$$

But a simple calculation shows that

$$\chi_\ell(y) = b''_\ell a'_\ell - a''_\ell b'_\ell.$$

Therefore (5) implies that

$$\sum_{\ell=1}^{\sigma} N_\ell(b''_\ell a'_\ell - a''_\ell b'_\ell) = 0. \tag{6}$$

Let the notation be so chosen that the multiplier system v and therefore v' be unramified at the parabolic cusps $A_\ell^{-1} < \infty > (1 \le \ell \le \sigma_0)$. Then (6) leads to the bilinear relation

$$\sum_{\ell=1}^{\sigma_0} N_\ell(b''_\ell a'_\ell - a''_\ell b'_\ell) = 0$$

$$\text{i.e. } \mathcal{G}'\mathfrak{n}\mathfrak{b} = 0$$

where $\mathfrak{n}$ respectively $\mathfrak{b}$ denote the column vector with components $a'_1, a'_2, \ldots, a'_{\sigma_0}, a''_1, a''_2, \ldots, a''_{\sigma_0}$ respectively $b'_1, b'_2, \ldots, b'_{\sigma_0}, b''_1, b''_2, \ldots, b''_{\sigma_0}$ and

$$\mathfrak{n} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

is a $2\sigma_0$-rowed square matrix and D, the σ_0-rowed diagonal matrix with
200 $N_1, N_2, \ldots, N_{\sigma_0}$ on the diagonal. If there exist μ forms f with linearly independent vectors $\mathscr{G}$ and ν forms g with linearly independent vectors , then a consideration of the rank shows that

$$\mu + \nu \geq 2\sigma_0.$$

We formulate the results proved above in

Theorem 29. *Let Γ_0 be a subgroup of finite index in the modular group and let $A_\ell^{-1} < \infty >$ $(\ell = 1, 2, \ldots, \sigma_0)$ be a complete system of inequivalent parabolic cusps of Γ_0 at which a multiplier system v for the group Γ_0 and real weight $\alpha - \beta$ is unramified. Let $\mathscr{R}$ (respectively Γ) be the linear space of the vectors $\{\varphi_1(y), \ldots, \varphi_{\sigma_0}(y)\}$ (respectively $\{\psi_1(y), \ldots, \psi_{\sigma_0}(y)\}$), where $\varphi_\ell(y)$ (respectively $\psi_\ell(y)$) is the term independent of x in the Fourier expansion for $f \in [\Gamma_0, \alpha, \beta, v]$ (respectively $g \in [\Gamma_0, \beta, \alpha, v^1]$), at the parabolic cusp $A_\ell^{-1} < \infty >$, $\ell = 1, 2, \ldots, \sigma_0$. Then*

$$\textit{dimension } \mathscr{R} + \textit{ dimension } \gamma \leq 2\sigma_0.$$

Under the additional assumption $\mathrm{Re}(\alpha + \beta) > 2$, which enables us to give Eisentein series as explicit examples of modular forms, we shall prove that the spaces $\mathscr{R}$ and γ have dimension at least equal to σ_0, so that we have indeed the relation

$$\text{dimension } \mathscr{R} = \text{ dimension } \gamma = \sigma_0.$$

Let v be unramified at the cusp $A^{-1} < \infty >$. Then we define the Eisenstein series

$$G(\tau, \overline{\tau}; \alpha, \beta, v, A, \Gamma_0) = \sum_{M \in \gamma(A, \Gamma_0)} \{\sigma(A, L)v(L)(m_1\tau + m_2)^\alpha (m_1\overline{\tau} + m_2)^\beta\}^{-1}$$

201 where $M = AL = \begin{pmatrix} * & * \\ m_1 & m_2 \end{pmatrix}$ and $\gamma(A, \Gamma_0)$ has the same meaning as in chapter 3, § 2. As in the analytic case, the following transformation formulae can be proved:

$$\begin{aligned}
&(G(\quad ,; \alpha, \beta, v, A, \Gamma_0) \underset{\alpha,\beta}{|} S)(\tau, \overline{\tau}) \\
&\qquad = \frac{1}{\sigma(A, S)} G(\tau, \overline{\tau}; \alpha, \beta, v^S, AS, S^{-1}\Gamma_0 S) \text{ for } S \in \Gamma, \\
&(G(\quad ,; \alpha, \beta, v, A, \Gamma_0) \underset{\alpha,\beta}{|} L)(\tau, \overline{\tau}) \\
&\qquad = v(L) G(\tau, \overline{\tau}; \alpha, \beta, v, A, \Gamma_0) \text{ for } L \in \Gamma_0.
\end{aligned}$$

These transformation formulae show that the Eisenstein series is a modular form in $[\Gamma_0, \alpha, \beta, v]$. Moreover, it is not difficult to prove that the form $G(\tau, \overline{\tau}; \alpha, \beta, v, A, \Gamma_0)$ does not vanish at a cusp $B^{-1} < \infty >$ if and only if $B^{-1} < \infty >$ is equivalent to $A^{-1} < \infty >$ under Γ_0. Thus there exist as many linearly independent Eisentein series as the number of inequivalent parabolic cusps of Γ_0 at which v is unramified. Hence our assertion about the dimensions of $\mathscr{R}$ and γ is proved. We call a form $f \in [\Gamma_0, \alpha, \beta, v]$ a cusp form, when the functions $\varphi_1(y), \ldots, \varphi_{\sigma_0}(y)$ mentioned in theorem 29 vanish. Thus we have

Theorem 30. *Let Γ_0 be a subgroup of finite index in the modular group. If $\Gamma = \alpha - \beta$ is real and $\mathrm{Re}(\alpha + \beta) > 2$, then for every form f belonging to $[\Gamma_0, \alpha, \beta, v]$, there exists a linear combination $G(\tau, \overline{\tau})$ of Eisenstein series, so that $f(\tau, \overline{\tau}) - G(\tau, \overline{\tau})$ is a cusp form.*

Regarding the existence of cusp forms in the space $[\Gamma_0, \alpha, \beta, v]$, we prove the following

Theorem 31. *If, in addition to the assumptions of theorem 30, we assume that $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$ then the space $[\Gamma_0, \alpha, \beta, v]$ is generated by Eisenstein series and its dimension is equal to the number of inequivalent parabolic cusps of Γ_0 at which the multiplier system v is unramified.*

Proof. For the proof of the theorem, it is sufficient to prove that if $f \in$ 202
$[\Gamma_0, \alpha, \beta, v]$ is a cusp form, then $f = 0$. Following the proof of lemma 5

(chapter 3, § 3) it can be shown that $y^{p/2}|f(\tau,\overline{\tau})|$ is bounded in $\mathscr{G}$, where $p = \mathrm{Re}(\alpha+\beta)$. Let

$$f(\tau,\overline{\tau}) = \sum_{n+\kappa\neq 0} a_{n+\kappa} W(\frac{2\pi(n+\kappa)}{N} y;\alpha,\beta) e^{2\pi i(n+\kappa)x/N}$$

be the Fourier series of $f(\tau,\overline{\tau})$ at the parabolic cusp ∞ of Γ_0. Then

$$|a_{n+\kappa} y^{p/2} W(\frac{2\pi(n+\kappa)}{N} y;\alpha,\beta)| = |\frac{1}{N}\int_0^N y^{p/2} f(\tau,\overline{\tau}) e^{-2\pi i(n+\kappa)x/N} dx| \leq C$$

with a suitable positive constant C. But by § 2, lemma 8,

$$y^{p/2} W(\frac{2\pi(n+\kappa)}{N} y;\alpha,\beta)$$

is unbounded as $y \to 0$, because $p = \mathrm{Re}\, q > 2$; therefore, the above inequality can hold only if $a_{n+\kappa} = 0$ for $n+\kappa \neq 0$. Hence the theorem is proved.

In some special cases, using the method adopted in the proof of theorem 28, it can be proved that the cusp forms identically vanish even when the assumptions of theorem 31 are not satisfied. The following theorem is an example in this regard. □

Theorem 32. *Let v be an even abelian character of the theta group Γ_ϑ with $v^2 = 1$ and let $\alpha \geq 0$. Then the space $[\Gamma_\vartheta, \alpha, \alpha, v]$ contains no cusp form which does not vanish identically.*

First of all, we remark that there exist exactly 4 characters of Γ_ϑ of the type mentioned in theorem 32. Since $\Gamma[2]$ is a subgroup of index 2 and $\Gamma_\vartheta = \Gamma[2] \cup \Gamma[2]T$ is a coset decomposition of Γ_ϑ modulo $\Gamma[2]$, it
203 follows that $v_1(T) = -1$, $v_1(S) = 1$ for $S \in \Gamma[2]$ defines an even abelian character of Γ_ϑ and $v_1^2 = 1$. But we have proved already that Γ_ϑ is generated by T and U^2; therefore, v_1 is uniquely defined by $v_1(T) = -1$ and $v_1(U^2) = 1$. Let A denote the matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} -1 & 3 \\ -1 & 1 \end{smallmatrix}\right)$. Then it can be seen that the mapping $S \to ASA^{-1}$ is an automorphism of the group Γ_ϑ,

which maps T to U^2TU^{-2} and U^2 to TU^{-2}. This shows that $v_2(S) = v_1(ASA^{-1})$ is another even abelian character of Γ_ϑ such that $v_2^2 = 1; v_2$ is different from v_1 because $v_2(U^2) = -1$. A third character v_3 of Γ_ϑ of the same type as above is defined by v_1v_2 so that $v_3(T) = 1$ and $v_3(U^2) = -1$. Thus we have obtained four characters namely $1, v_1, v_2$ and $v_3 = v_1v_2$ of the desired type. Since the group Γ_ϑ is generated by T and U^2, these are all the even abelian characters v of Γ_ϑ, with $v^2 = 1$.

Proof of theorem 3.32. The case $\alpha > 1$ follows from theorem 31. If $\alpha = 0$, then any function $f \in [\Gamma_\vartheta, 0, 0, v]$ is a harmonic function. If $f \in [\Gamma_\vartheta, 0, 0, v]$ is a cusp form and does not vanish identically, then $f(\tau, \overline{\tau})$ attains its maximum at a finite point τ_0 of a fundamental domain of Γ_ϑ. But this contradicts the maximum modulus principle for harmonic functions unless f is constant in which case it vanishes identically; therefore the theorem is proved for $\alpha = 0$.

In what follows, we confine ourselves to the case $0 < \alpha \leq 1$. Let $\mathfrak{F}_0$ be the fundamental domain of Γ_ϑ given by

$$\mathfrak{F}_0 = \{\tau | \tau = x + iy, |x - 1| \leq 1, |\tau| \geq 1, |\tau - 2| \geq 1, y > 0\}.$$

The domain $\mathfrak{F}_0$ is decomposed by the circle $|\tau - 1| = \sqrt{2}$ into two parts, one of which, say $\mathscr{L}_1$, is unbounded and the other, $\mathscr{L}_2$ is bounded. Moreover, the above-defined elliptic transformation A maps $\mathscr{L}_2$ onto 204
$\mathscr{L}_1$. We set $A_1 = E$ and $A_2 = A$. Obviously $A_1 < \mathscr{L}_1 >= A_2 < \mathscr{L}_2 >= \mathscr{L}_1$, which implies that $y \geq 1$ for $\tau \in A_\ell < \mathscr{L}_\ell >$.

Let $f(\tau, \overline{\tau})$ be a cusp form belonging to $[\Gamma_\vartheta, \alpha, \alpha, v]$. Since the width of the cusp sector $A_\ell < \mathscr{L}_\ell >$ is 2, we have a Fourier expansion of the type

$$(f|A^{-1})(\tau, \overline{\tau}) = \sum_{n+\kappa_\ell \neq 0} a^{(\ell)}_{n+\kappa_\ell} W(\pi(n + \kappa_\ell)y; \alpha, \alpha) e^{\pi i (n+\kappa_\ell)x} (\ell = 1, 2).$$

Here $\kappa_\ell = 0$ or $\dfrac{1}{2}(\ell = 1, 2)$, because $v^2 = 1$. Now $g = y^\alpha |f(\tau, \overline{\tau})|$ is invariant under the transformations of Γ_ϑ and for $\tau' = x' + iy' = A_2 < \tau >$, we obviously have

$$y^\alpha |f(\tau, \overline{\tau})| = y'^\alpha |(f|A_2^{-1})(\tau', \overline{\tau}')|.$$

It follows that g attains its maximum M at a point $\tau^* \neq \infty, 1$ of $\mathfrak{F}_0$. Let τ^* belong to $\mathscr{L}_\ell$. Then we set $\tau_0 = x_0 + iy_0 = A_\ell < \tau^* >$. It is obvious that $y_0 \geq 1$. Moreover

$$M = y^{\alpha_0} |(f|A_\ell^{-1})(\tau_0, \overline{\tau})|.$$

We shall now prove that $M = 0$; this will imply that $f = 0$ and prove the theorem when $0 < \alpha \leq 1$. From the integral representation

$$W(\pm y; \alpha, \alpha) = \frac{y^{-\alpha_e - y}}{\Gamma(\alpha)} \int\limits_0^\infty e^{-u} u^{\alpha-1} (1 + \frac{u}{2y})^{\alpha-1} du,$$

it is evident that

$$y^\alpha W(y; \alpha, \alpha) e^y$$

increases monotonically to 1 for $y \to \infty$, so that

$$(\frac{a}{y})^\alpha W(a; \alpha, \alpha) e^{a-y} \leq W(y; \alpha, \alpha) \leq y^{-\alpha} e^{-y} \text{ for } 0 < a \leq y. \tag{7}$$

205 Since $0 \leq 1 - \alpha < 1$, it follows that

$$\begin{aligned}
&(1 + \frac{u}{2y})^{1-\alpha} \leq 1 + (1-\alpha)\frac{u}{2y} \\
\Longrightarrow &(1 + \frac{u}{2y})^{\alpha-1} \geq (1 + (1-\alpha)\frac{u}{2y})^{-1} \geq 1 - (1-\alpha)\frac{u}{2y} \\
\Longrightarrow &\frac{1}{\Gamma(\alpha)} \int\limits_0^\infty e^{-u} u^{\alpha-1} (1 + \frac{u}{2y})^{\alpha-1} du \geq \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty e^{-u} u^{\alpha-1} \{1 - (1-\alpha)\frac{u}{2y}\} du \\
&= \frac{1}{\Gamma(\alpha)} \{\Gamma(\alpha) - \frac{(1-\alpha)}{2y} \Gamma(\alpha+1)\} = 1 - \frac{\alpha(1-\alpha)}{2y} \geq 1 - \frac{1}{8y}.
\end{aligned}$$

This implies that

$$y^\alpha W(y; \alpha, \alpha) e^y \geq 1 - 1/(8y) \text{ for } 0 < \alpha \leq 1 \text{ and } y > 0. \tag{8}$$

We shall now estimate $a^{(\ell)}_{n+\kappa_\ell}$ with the help of

$$a^{(\ell)}_{n+\kappa_\ell} W(\pi(n + \kappa_\ell)\rho y_0; \alpha, \alpha) = \frac{1}{2} \int\limits_0^2 (f|A_\ell^{-1})(\tau, \overline{\tau}) e^{-\pi i(n+\kappa_\ell)x} dx,$$

where $\tau = x + iy_0\rho$ with ρ an arbitrary constant belonging to the interval $0 < \rho < 1$. We obtain from above that

$$|a^{(\ell)}_{n+\kappa_\ell}| W(\pi(n+\kappa_\ell)\rho y_0; \alpha, \alpha) \leq M(\rho y_0)^{-\alpha} (n + \kappa_\ell \neq 0).$$

Using (7), we see that

$$\begin{aligned} M &= y_0^{\alpha}(f|A_\ell^{-1})(\tau_0 \overline{\tau}_0)| \\ &\leq \rho^{-\alpha} M \sum_{n+\kappa_\ell \neq 0} \frac{W(\pi|n+\kappa_\ell|y_0;\alpha,\alpha)}{W(\pi|n+\kappa_\ell|\rho y_0;\alpha,\alpha)} \\ &\leq \rho^{-\alpha} M \sum_{n+\kappa_\ell \neq 0} \frac{W(\pi|n+\kappa_\ell|y_0;\alpha,\alpha)}{W(\pi(1-\kappa_\ell)\rho;\alpha,\alpha)} \frac{(\pi|n+\kappa_\ell|\rho y_0)^{\alpha} e^{\pi|n+\kappa_\ell|\rho y_0}}{(\pi(1-\kappa_\ell)\rho)^{\alpha} e^{\pi(1-\kappa_\ell)\rho}} \end{aligned}$$

since $\kappa_\ell = 0$ or $\frac{1}{2}$ implies that $|n+\kappa_\ell| \geq 1-\kappa_\ell$. On using (8), this estimate gives the inequality 206

$$M \leq \frac{2M}{1 - \frac{1}{8\pi(1-\kappa_\ell)\rho}} \sum_{n+\kappa_\ell > 0} e^{-\pi(n+\kappa_\ell)(1-\rho)}.$$

Replacing $n + \kappa_\ell$ by $n + 1 - \kappa_\ell$ and summing up the right hand side of this inequality, we see that

$$M \leq \frac{2M}{1 - \frac{1}{8\pi(1-\kappa_\ell)\rho}} \frac{e^{-\pi(1-\kappa_\ell)(1-\rho)}}{1 - e^{-\pi(1-\rho)}} \tag{9}$$

It is now obvious that $M = 0$, if there exists a real number ρ with $0 < \rho < 1$ such that

$$\begin{aligned} & 2 < (1 - \frac{1}{8\pi(1-\kappa_\ell)\rho})(e^{\pi(1-\kappa_\ell)(1-\rho)} - e^{-\pi\kappa_\ell(1-\rho)}) \\ \text{i.e. } & 2 < (1 - \frac{1}{8\pi\rho})(e^{\pi(1-\rho)} - 1) \text{ when } \kappa_\ell = 0 \\ \text{and } & 1 < (1 - \frac{1}{4\pi\rho}) \sinh \frac{\pi}{2}(1-\rho) \text{ when } \kappa_\ell = \frac{1}{2}. \end{aligned}$$

For the first case, it is sufficient to take $\rho = \dfrac{1}{2}$, because

$$1 - \frac{1}{4\pi} > \frac{11}{12} \text{ and } e^{\frac{\pi}{2}} - 1 > 3.$$

For the second case, let the real number ρ be so determined that $\dfrac{\pi}{2}(1 - \rho) = 1.185$. Then $0 < \rho < 1$ and

$$\sinh\frac{\pi}{2}(1-\rho) = 1.482470\ldots > 1.4824$$
$$\frac{4\pi\rho}{4\pi\rho - 1} = \frac{4\pi - 9.48}{4\pi - 10.48} < 1.4794,$$

which together imply that the second inequality is possible. Hence the proof of theorem 32 is complete.

207 A direct consequence of theorem 31 is

Theorem 33. *Let Γ be the modular group and let α, β ge complex numbers such that $r = \alpha - \beta$ is real and* $\mathrm{Re}\,\alpha$, $\mathrm{Re}\beta$, $\mathrm{Re}(\alpha + \beta - 2) > 0$. *Then*

$$\textit{dimension } [\Gamma, \alpha, \beta, v] = \begin{cases} 1 & \textit{if } r \equiv 0(\mod 2), v = 1 \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By theorem 31, the dimension of $[\Gamma, \alpha, \beta, v]$ is equal to 1 if and only if the multiplier system v is unramified at ∞ i.e. $\kappa = 0$. Then it follows from (6) of chapter 3, § 1 that $r \equiv 0(\mod 2)$ and $v = 1$. Hence theorem 33 is proved. □

We shall now determine the Fourier coefficients of the Eisenstein series for the modular group Γ under the assumptions $r = \alpha - \beta \equiv 0(\mod 2)$ and $v = 1$. Instead of considering the series $G(\tau, \overline{\tau}; \alpha, \beta, 1, E, \Gamma)$, we consider the series

$$G(\tau, \overline{\tau}, \alpha, \beta) = \sum_{(m,n)\neq(0,0)} (m\tau + n)^{-\alpha}(m\overline{\tau} + n)^{-\beta},$$

because the Fourier coefficients turn out to be simple in this case. It is obvious that $G(\tau, \overline{\tau}; \alpha, \beta)$ defines a modular form in $[\Gamma, \alpha, \beta, 1]$ in case

$p = \mathrm{Re}(\alpha+\beta) > 2$. First of all, we find the Fourier series of the periodic function

$$f(\tau,\overline{\tau};\alpha,\beta) = \sum_{n=-\infty}^{\infty} (\tau+n)^{-\alpha}(\overline{\tau}+n)^{-\beta} (\rho > 2)$$

defined for $\tau \in \mathscr{G}$. Let

$$f(\tau,\overline{\tau};\alpha,\beta) = e^{\pi i(\beta-\alpha)/2} \sum_{n=-\infty}^{\infty} h_n(y;\alpha,\beta) e^{2\pi inx}$$

with **208**

$$\begin{aligned} h_n(y;\alpha,\beta) &= e^{\pi i(\alpha-\beta)/2} \int_{-\infty}^{\infty} \tau^{-\alpha}\overline{\tau}^{-\beta} e^{-2\pi inx} dx \\ &= \int_{-\infty}^{\infty} (-i\tau)^{-\alpha}(i\overline{\tau})^{-\beta} e^{-2\pi inx} dx \\ &= y^{1-\alpha-\beta} \int_{-\infty}^{\infty} (1-ix)^{-\alpha}(1+ix)^{-\beta} e^{-2\pi inyx} dx. \end{aligned}$$

Here the integrand is defined by

$$(1-ix)^{-\alpha} = e^{-\alpha \log(1-ix)}, (1+ix)^{-\beta} = e^{-\beta \log(1+ix)},$$

where the branch of the logarithm is chosen in such a way that $\log(1 \pm ix)$ is real for $x = 0$. In order to express the function

$$h(t;\alpha,\beta) = \int_{-\infty}^{\infty} (1-ix)^{-\alpha}(1+ix)^{-\beta} e^{-itx} dx \quad (\mathrm{Re}(\alpha+\beta) > 1)$$

in terms of Whittaker's function, we consider the gamma integrals

$$(1-ix)^{-\alpha}\Gamma(\alpha) = \int_{0}^{\infty} e^{-(1-ix)\xi} \xi^{\alpha-1} d\xi \quad (\mathrm{Re}\,\alpha > 0)$$

$$(1+ix)^{-\beta}\Gamma(\beta) = \int\limits_0^\infty e^{-(1+ix)\eta}\eta^{\beta-1}d\eta \quad (\mathrm{Re}\,\beta > 0)$$

which imply that

$$(1-ix)^{-\alpha}(1+ix)^{-\beta}\Gamma(\alpha)\Gamma(\beta) = \int\limits_0^\infty\int\limits_0^\infty e^{-(1-ix)\xi-(1+ix)\eta}\xi^{\alpha-1}\eta^{\beta-1}d\xi d\eta.$$

The substitution $\xi+\eta = u$, $\xi-\eta = t$ leads us to

$$\begin{aligned}&2^{q-1}\Gamma(\alpha)\Gamma(\beta)(1-ix)^{-\alpha}(1+ix)^{-\beta}\\ &\qquad = \int\limits_{-\infty}^{\infty} e^{itx}\left\{\int\limits_{u>|t|} e^{-u}(u+t)^{\alpha-1}(u-t)^{\alpha-1}(u-t)^{\beta-1}du\right\}dt,\end{aligned}$$

with $q = \alpha+\beta$. This shows that the function $(1-ix)^{-\alpha}(1+ix)^{-\beta}$ is the
209 Fourier transform of the u-integral. Let $\mathrm{Re}(\alpha+\beta) > 1$. Then the inverse of the above Fourier transform exists and we get

$$\frac{\Gamma(\alpha)\Gamma(\beta)2^{q-1}}{2\pi}\int\limits_{-\infty}^{\infty}(1-ix)^{-\alpha}(1+ix)^{-\beta}e^{-itx}dx = \int\limits_{u>|t|} e^{-u}(u+t)^{\alpha-1}(u-t)^{\beta-1}du.$$

Consequently, we obtain

$$h(t;\alpha,\beta) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}2^{1-q}\int\limits_{u>|t|} e^{-u}(u+t)^{\alpha-1}(u-t)^{\beta-1}du.$$

Making use of the integral representation of Whittaker's function given in the proof of lemma 8, we see that

$$h(t;\alpha,\beta) = \frac{2\pi}{\Gamma(\frac{q+\epsilon\Gamma}{2})}2^{-\frac{1}{2}q}|t|^{q-1}W(t;\alpha,\beta) \quad (\epsilon = \mathrm{sgn}\,t)$$

and in particular,

$$h(0;\alpha,\beta) = \frac{2\pi\Gamma(q-1)}{\Gamma(\alpha)\Gamma(\beta)}2^{1-q}.$$

By the definition of the function $h(t;\alpha,\beta)$, the Fourier coefficients of $f(\tau,\overline{\tau};\alpha,\beta)$ are given by

$$h_n(y;\alpha,\beta) = y^{1-q}h(2\pi ny;\alpha,\beta). \tag{10}$$

We shall now determine the Fourier expansion of the Eisenstein series $G(\tau,\overline{\tau};\alpha,\beta)$. Here we shall make use of the assumption $r = \alpha-\beta \equiv 0($ mod $2)$. Let us set $r = 2k$ (k integral). Then

$$\begin{aligned} G(\tau,\overline{\tau};\alpha,\beta) &= 2\sum_{n=1}^{\infty} n^{-\alpha-\beta} + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(m\tau+n)^{-\alpha}(m\overline{\tau}+n)^{-\beta} \\ &= 2\zeta(q) + 2\sum_{m=1}^{\infty} f(m\tau, m\overline{\tau};\alpha,\beta), \end{aligned}$$

because, for $m > 0$, $m\tau$ belongs to $\mathscr{G}$ with τ. Thus 210

$$G(\tau,\overline{\tau};\alpha,\beta) = 2\zeta(q) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} h_n(my;\alpha,\beta)e^{2\pi imnx}.$$

But from (10), it is obvious that

$$\begin{aligned} h_n(my;\alpha,\beta) &= (my)^{1-q}h(2\pi nmy;\alpha,\beta) \\ &= m^{1-q}y^{1-q}h(2\pi nmy;\alpha,\beta) \\ &= m^{1-q}h_{mn}(y;\alpha,\beta); \end{aligned}$$

therefore

$$G(\tau,\overline{\tau};\alpha,\beta) = 2\zeta(q) + 2(-1)^k\sum_{m=1}^{\infty} m^{1-q}\sum_{n=-\infty}^{\infty} h_{mn}(y;\alpha,\beta)e^{2\pi imnx}.$$

Collecting the terms for which $mn = \ell$, we get

$$\begin{aligned} G(\tau,\overline{\tau};\alpha,\beta) &= 2\zeta(q) + 2(-1)^k\sum_{\ell=-\infty}^{\infty}\{\sum_{\substack{d|\ell \\ d>0}} d^{1-q}\}h_\ell(y;\alpha,\beta)e^{2\pi i\ell x} \\ &= \varphi_k(y,q) + 2(-1)^k(\sqrt{2}\pi)^q \end{aligned}$$

$$\sum_{n\neq 0}\frac{d_{q-1}(n)}{\Gamma(\frac{q}{2}+\in k)}W(2\pi ny;\alpha,\beta)e^{2\pi inx}$$

with $\in = \operatorname{sgn} n, d_{q-1}(n) = \sum_{\substack{d|n\\ d>0}} d^{q-1}$ and

$$\begin{aligned}\varphi_k(y,q) &= 2\zeta(q) + 2(-1)^k\zeta(q-1)h_0(y;\alpha,\beta)\\ &= 2\zeta(q) + (-1)^k 2^{3-q}\pi\frac{\Gamma(q-1)\zeta(q-1)}{\Gamma(\frac{q}{2}+k)\Gamma(\frac{q}{2}-k)}\{(1-q)u(y,q)+1\}\\ &= 2\zeta(q) + (-1)^k 2^{3-q}\pi\frac{\Gamma(q-1)\zeta(q-1)}{\Gamma(\frac{q}{2}+k)\Gamma(\frac{q}{2}-k)}+\\ &\qquad (-1)^k 2^{3-q}\pi\frac{\Gamma(q-1)\zeta(q-1)}{\Gamma(\frac{q}{2}+k)\Gamma(\frac{q}{2}-k)}(1-q)u(y,q).\end{aligned}$$

211 The analytic continuation of the function $G(\tau,\overline{\tau};\alpha,\beta)-\varphi_k(y,q)$, which is so far defined for $\operatorname{Re} q > 2$, in the whole of the q-plane (for a fixed integer k) is obvious from the series as well as the estimate

$$W(\pm y;\alpha,\beta)\leq Ce^{-(1-\in)y} \text{ for } y\geq y_0>0, |\alpha|\leq m, |\beta|\leq m$$

with a positive constant $C = C(y_0,\in,m)$, where $\in, y_0$ and m are given positive numbers. In order to obtain this estimate for the function $W(y;\alpha,\beta)$, we consider the well-known integral representation for $W_{k,m}(y)$ (see Whittaker and Watson: A Course on Modern Analysis) of which the following two integrals are an immediate consequence:

$$W(y;\alpha,\beta) = \frac{2^{r/2}y^{-\beta}e^{-y}}{\Gamma(\beta)}\int_0^\infty u^{\beta-1}(1+\frac{u}{2y})^{\alpha-1}e^{-u}du \text{ for } \operatorname{Re}\beta>0,$$

$$W(y;\alpha,\beta) = \frac{-2^{r/2}\Gamma(1-\beta)e^{-y}y^{\beta}}{2\pi i}\int_{+\infty}^{(0+)}(-u)^{\beta-1}(1+\frac{u}{2y})^{\alpha-1}e^{-u}du$$
$$\text{for } \Gamma(1-\beta)\neq\infty.$$

In the second integral, the path of integration is a loop, which starts from ∞, circles round the point 0 in the positive direction so that the points 0 and $-2y$ are separated and then goes over again to ∞. The integrand is uniquely defined by the requirements $|\arg(-u)| \leq \pi$, in case u is not ≥ 0 and $|\arg(1+\frac{u}{2y})| < \pi$ in case $(1+\frac{u}{2y})$ is not ≤ 0. The estimate for the function $W(-y;\alpha,\beta)$ follows from that of $W(y;\alpha,\beta)$ because of the identity

$$W(-y;\alpha,\beta) = W(y;\alpha,\alpha).$$

For the analytic continuation of the function $\varphi_k(y,q)$ in the complex q-plane, we shall make use of the well-known facts that the functions $\xi(s) = s(l-s)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is an entire function of s, satisfies the func- **212**
tional equation $\xi(1-s) = \xi(s)$ and for $0 \leq \operatorname{Re} s \leq 1$ has the same zeros as the zeta function $\zeta(s)$ and does not vanish outside this strip. Further, we need the identity

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$$

for the Γ-function.

By the definition of the function $\varphi_k(y,q)$, we have

$$\varphi_k(y,q) = \frac{\pi^{q/2}}{(1-q)\Gamma(\frac{q}{2}+1)}\xi(1-q) + \frac{(-1)^k\pi^{q/2}\Gamma(\frac{q}{2}-1)}{(1-q)\Gamma(\frac{q}{2}+k)\Gamma(\frac{q}{2}-k)}\xi(q-1) + \\ + \frac{(-1)^k\pi^{q/2}\Gamma(\frac{q}{2}-1)}{\Gamma(\frac{q}{2}-k)\Gamma(\frac{q}{2}+k)}\xi(q-1)u(y,q).$$

But obviously

$$\frac{\Gamma(\frac{2-q}{2}+|k|)}{\Gamma(2-\frac{q}{2})} = (-1)^{k+1}\frac{\Gamma(\frac{q}{2}-1)}{\Gamma(\frac{q}{2}-|k|)};$$

therefore

$$\varphi_k(y,q) = \frac{\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2}+|k|)}\frac{1}{(1-q)}\left\{\frac{\Gamma(\frac{q}{2}+|k|)}{\Gamma(\frac{q}{2}+1)}\xi(1-q) - \frac{\Gamma(\frac{2-q}{2}+|k|)}{\Gamma(2-\frac{q}{2})}\xi(q-1)\right\} + \\ + \frac{(-1)^k\pi^{q/2}\Gamma(\frac{q}{2}-1)}{\Gamma(\frac{q}{2}+k)\Gamma(\frac{q}{2}-k)}\xi(q-1)u(y,q).$$

The expression in the brackets is an odd function of $q-1$ and therefore has at $q=1$ a zero of order at least equal to 1.

We introduce 213

$$G^*(\tau,\overline{\tau};\alpha,\beta) = \frac{q}{2}(1-\frac{q}{2})\pi^{-q/2}\Gamma(\frac{q}{2}+|k|)G(\tau,\overline{\tau};\alpha,\beta),$$

$$\begin{aligned}\varphi_k^*(y,q) &= \frac{q}{2}(1-\frac{q}{2})\pi^{-q/2}\Gamma(\frac{q}{2}+|k|)\varphi_k(y,q)\\ &= \frac{1}{1-q}\left\{(1-\frac{q}{2})\frac{\Gamma(\frac{q}{2}+|k|)}{\Gamma(\frac{q}{2})}\xi(1-q)\right.\\ &\qquad \left. -\frac{q}{2}\frac{\Gamma(1-\frac{q}{2}+|k|)}{\Gamma(1-\frac{q}{2})}\xi(q-1)\right\}+\\ &\quad +(-1)^{k+1}\frac{q}{2}\frac{\Gamma(\frac{q}{2})}{\Gamma(\frac{q}{2}-|k|)}\xi(q-1)u(y,q).\end{aligned}$$

It is obvious that $\varphi_k^*(y,q)$ as well as

$$\begin{aligned}&G^*(\tau,\overline{\tau};\alpha,\beta)-\varphi_k^*(y,q)\\ &= 2(-1)^k(2\pi)^{q/2}\frac{q}{2}(1-\frac{q}{2})\sum_{n\neq 0}\frac{\Gamma(\frac{q}{2}+|k|)}{\Gamma(\frac{q}{2}+\in k)}d_{q-1}(n)W(2\pi ny;\alpha,\beta)e^{2\pi inx}\end{aligned}$$

with $\in=\operatorname{sgn} n$, for the given integer $k=\frac{1}{2}(\alpha-\beta)$, is an entire function of q. The transformation formula of $G(\tau,\overline{\tau};\alpha,\beta)$ and therefore of the function $G^*(\tau,\overline{\tau};\alpha,\beta)$ was proved for $\operatorname{Re} q>2$ but remains valid by analytic continuation; thus

$$G^*(\tau\overline{\tau};\alpha,\beta)\in[\Gamma,\alpha,\beta,1].$$

With the help of

$$\zeta(0)=-\frac{1}{2},\zeta'(0)=-\frac{1}{2}\log 2\pi,\Gamma'(1)=-\gamma(\gamma=\text{Euler's constant})$$

we obtain, by simple calculation, that

$$\xi(0)=-1,\quad \xi'(0)=1+\frac{1}{2}(\gamma-\log 4\pi)$$

and finally, for some special values of k and q,

$$\varphi_k^*(y, q) = \begin{cases} -1 & \text{for } k = 0, q = 0 \\ -\frac{1}{y} & \text{for } k = 0, q = 2 \\ \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\{\sum_{h=1}^{k} \frac{1}{2h-1} + \frac{1}{2}(\gamma + \log \frac{y}{4\pi})\} & \text{for } k \geqq 0, q = 1. \end{cases}$$

Corresponding to $k = 0$, $q = 0, 2, 1$ we get the following Eisenstein series 214

$$G^*(\tau, \overline{\tau}; 0, 0) = -1, G^*(\tau, \overline{\tau}; 1, 1) = -\frac{1}{y},$$
$$G^*(\tau, \overline{\tau}; \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(\gamma + \log \frac{y}{4m}) + \sum_{n \neq 0} d_0(n) K_0(2\pi|n|y) e^{2\pi inx}$$

where

$$K_0(y) = \sqrt{\frac{\pi}{2}} W(\pm y; \frac{1}{2}, \frac{1}{2})$$

is the well-known Bessel function of pure imaginary argument. Since $\xi(q)$ vanishes only in the critical strip $0 < \operatorname{Re} q < 1$ of Riemann's zeta function $\zeta(q)$, it is easy to see that $\varphi_k^*(y, q)$ for given q and k (integral) does not represent a cusp form for any choice of α, β provided $k = \dfrac{1}{2}(\alpha - \beta)$ is integral.

Since $y^{\frac{1}{2}q} W(\in y; \alpha, \beta)$ with $y > 0, \in^2 = 1$, according to the definition of this function, depends only on $y, r \in$ and $(q-1)^2$, it is not hard to prove that G^* satisfies the following functional equations

$$G^*(\tau, \tau; \overline{\beta}, \alpha) = G^*(-\overline{\tau}, -\tau; \alpha, \beta),$$
$$G^*(\tau, \overline{\tau}; 1 - \alpha, 1 - \beta) = y^{q-1} G^*(\tau, \overline{\tau}; \beta, \alpha).$$

The following is an immediate consequence of the results proved above.

Theorem 34. *The linear space $[\Gamma, \alpha, \alpha, 1]$, where Γ is the modular group, has dimension 1 over the complex number field in case $\alpha \geq 0$ and in that case it is generated by $G^*(\tau, \overline{\tau}; \alpha, \alpha)$.* 215

Proof. By theorem 32, there exist no cusp forms in $[\Gamma, \alpha, \alpha, 1]$ which do not vanish identically. Thus by theorem 29, the space $[\Gamma, \alpha, \alpha, 1]$ is isomorphic with the space $\mathscr{R} = \gamma$ and therefore dimension $[\Gamma, \alpha, \alpha, 1] =$ dimension $\mathscr{R} \leq 1$. But, for $\alpha \geq 0$, there exists a form in $[\Gamma, \alpha, \alpha, 1]$, which does not vanish identically, namely, the Eisenstein series. Hence the theorem is proved. $\square$

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Dirichlet Series and Euler Products

1 Gamma Functions and Mellin Transforms

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In this section, we study some generalised gamma functions and Mellin transforms of some functions, which will be required in the sequel for the investigation of the connection between Dirichlet series and modular forms.

We introduce the gamma functions

$$\Gamma(s;\alpha,\beta) = \int_0^\infty W(y;\alpha,\beta)y^{s-1}dy \tag{1}$$

as the Mellin transform of the function $W(y;\alpha,\beta)$, which satisfies the differential equation

$$yW''(y;\alpha,\beta) + qW'(y;\alpha,\beta) + (r-y)W(y;\alpha,\beta) = 0 \tag{2}$$

where $r = \alpha - \beta$ and $q = \alpha + \beta$. This differential equation has 0 as a 'place of determinacy' so that

$$W(y;\alpha,\beta) = o(y^{-K})(y \to 0)$$

for $K > K_0 = \max(0, \mathrm{Re}(q-1))$. Since $W(y;\alpha,\beta)$ tends to zero exponentially when $y \to \infty$, it follows that the function $\Gamma(s;\alpha,\beta)$, in any case, is regular in the half plane $\sigma = \operatorname{Re} s > K_0$. The integral representation of Whittaker's function mentioned in lemma 8 (chapter 4, § 2) gives

$$\begin{aligned} W(y;\alpha,\beta) &= y^{-\frac{1}{2}q} W_{\frac{1}{2}r,\frac{1}{2}(q-1)}(2y) \\ &= \frac{2^{1-\frac{1}{2}q}}{\Gamma(\beta)} \int\limits_1^\infty e^{-yu}(u-1)^{\beta-1}(u+1)^{\alpha+1} du \quad (\operatorname{Re}\beta > 0), \end{aligned} \tag{3}$$

217 which leads to

$$\Gamma(s;\alpha,\beta) = 2^{1-\frac{1}{2}q} \frac{\Gamma(s)}{\Gamma(\beta)} \int\limits_1^\infty (u-1)^{\beta-1}(u+1)^{\alpha-1} u^{-s} du \quad (\operatorname{Re}\beta > 0) \tag{4}$$

for $\sigma > \operatorname{Re}(q-1)$. Substituting $u = 1/(1-v)$ in (4), we get

$$\begin{aligned} \Gamma(s;\alpha,\beta) &= 2^{1-q/2} \frac{\Gamma(s)}{\Gamma(\beta)} \int\limits_0^1 v^{\beta-1}(1-v)^{s-q}(2-v)^{\alpha-1} dv \\ &= 2^{r/2} \frac{\Gamma(s)}{\Gamma(\beta)} \sum_{n=0}^\infty \binom{\alpha-1}{n} \frac{(-1)^n}{2^n} \int\limits_0^1 v^{\beta+n-1}(1-v)^{s-q} dv \\ &= 2^{r/2} \frac{\Gamma(s)}{\Gamma(\beta)} \sum_{n=0}^\infty \frac{(1-\alpha)_n}{n!2^n} \frac{\Gamma(\beta+n)\Gamma(s+1-q)}{\Gamma(s+n+1-\alpha)}, \end{aligned}$$

where, in general

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1), (a)_0 = 1.$$

We can thus, conclude finally that

$$\Gamma(s;\alpha,\beta) = 2^{r/2}\Gamma(s)\Gamma(s+1-q) \sum_{n=0}^\infty \frac{(1-\alpha)_n(\beta)_n}{n!2^n\Gamma(s+n+1-\alpha)} \tag{5}$$

This series converges for all s, α and β showing that $\Gamma(s;\alpha,\beta)$ is a meromorphic function, which has singularities at most at the poles of $\Gamma(s)$

$\Gamma(s+1-q)$. In particular, when $\alpha=\beta$, we see, from (4), that

$$\Gamma(s;\alpha,\alpha)=2^{1-\alpha}\frac{\Gamma(s)}{\Gamma(\alpha)}\int_1^\infty (u^2-1)^{\alpha-1}u^{-s}du.$$

On substituting $u^2=1/(1-v)$, we are led to

$$\begin{aligned}\Gamma(s;\alpha,\alpha)&=2^{-\alpha}\frac{\Gamma(s)}{\Gamma(\alpha)}\int_0^1 v^{\alpha-1}(1-v)^{\frac{s-1}{2}-\alpha}dv\\&=2^{-\alpha}\frac{\Gamma(s)\Gamma(\frac{s+1}{2}-\alpha)}{\Gamma(\frac{s+1}{2})}.\end{aligned}$$

Using Legendre's relation **218**

$$\Gamma(s)=\frac{1}{\sqrt{\pi}}2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}),$$

we obtain from above that

$$\Gamma(s;\alpha,\alpha)=\frac{2^{s-\alpha-1}}{\sqrt{\pi}}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}-\alpha). \tag{6}$$

If $\beta=0$, then it follows immediately from (5) that

$$\Gamma(s;\alpha,0)=2^{\alpha/2}\Gamma(s). \tag{7}$$

With the help of the differential equation (2), we shall show that the function $\Gamma(s;\alpha,\beta)$ satisfies the functional equation

$$\Gamma(s+2;\alpha,\beta)+(\beta-\alpha)\Gamma(s+1;\alpha,\beta)-s(s+1-\alpha-\beta)\Gamma(s;\alpha,\beta)=0. \tag{8}$$

It is obvious that

$$0=\int_0^\infty\{-y^2W''(y;\alpha-\beta)-qyW'(y;\alpha,\beta)+(y^2-ry)W(y;\alpha,\beta)\}y^{s-1}dy$$

$$= \int_0^\infty \{W(y;\alpha,\beta) - W''(y;\alpha,\beta)\}y^{s+1}dy - r\int_0^\infty W(y;\alpha,\beta)y^s dy - q$$
$$\int_0^\infty W'(y;\alpha,\beta)y^s dy$$
$$= \int_0^\infty \{W(y;\alpha,\beta) + \frac{r}{s+1}W'(y;\alpha,\beta) - \frac{s(s+1-q)}{s(s+1)}W''(y;\alpha,\beta)\}y^{s+1}dy. \tag{9}$$

But, by the definition of $\Gamma(s;\alpha,\beta)$, we have

$$\Gamma(s+2;\alpha,\beta) = \int_0^\infty W(y;\alpha,\beta)y^{s+1}dy,$$
$$\Gamma(s+1;\alpha,\beta) = \int_0^\infty W(y;\alpha,\beta)\ \ y^s dy = -\frac{1}{s+1}\int_0^\infty W'(y;\alpha,\beta)\ \ y^{s+1}dy,$$
$$\Gamma(s;\alpha,\beta) = \int_0^\infty W(y;\alpha,\beta)y^{s-1}dy = -\frac{1}{s}\int_0^\infty W'(y;\alpha,\beta)y^s dy$$
$$= \frac{1}{s(s+1)}\int_0^\infty W''(y;\alpha,\beta)y^{s+1}dy.$$

219 Therefore equation (8) is an immediate consequence of (9).

We now consider the determinant

$$D(s;\alpha,\beta) = \begin{vmatrix} \Gamma(s;\alpha,\beta) & \Gamma(s;\beta,\alpha) \\ -\Gamma(s+1;\alpha,\beta) & \Gamma(s+1;\beta,\alpha) \end{vmatrix} \tag{10}$$

which shall be of use later. With the help of (8), we see that $D(s;\alpha,\beta)$ satisfies the functional equation

$$D(s+1;\alpha,\beta) = \begin{vmatrix} \Gamma(s+1;\alpha,\beta) & \Gamma(s+1;\beta,\alpha) \\ -\Gamma(s+2;\alpha,\beta) & \Gamma(s+2;\beta,\alpha) \end{vmatrix}.$$

$$
\begin{aligned}
&= s(s+1-q)\begin{vmatrix}\Gamma(s+1;\alpha,\beta) & \Gamma(s+1;\beta,\alpha)\\ -\Gamma(s;\alpha,\beta) & \Gamma(s,\beta,\alpha)\end{vmatrix}\\
&= s(s+1-q)\ \ D(s;\alpha,\beta). \qquad (11)
\end{aligned}
$$

Let us set

$$H(s) = \frac{D(s;\alpha,\beta)}{\Gamma(s)\Gamma(s+1-q)}.$$

Then it can be seen easily from (11) that $H(s)$ is a periodic function of period 1. Since $H(s)$ is regular in the half plane $\sigma = \operatorname{Re} s > K_1$, where K_1 is a sufficiently large number, it follows that $H(s)$ is an entire function of s. We shall now show that $\lim_{s\to\infty} H(s) = 2$, which together with the periodicity of $H(s)$ will imply that $H(s) = 2$ for all s. In order to calculate the limit of $H(s)$ as $\operatorname{Re} s \to \infty$, we consider equation (5) which shows that

$$\Gamma(s;\alpha,\beta) = 2^{r/2}\frac{\Gamma(s)\Gamma(s+1-q)}{\Gamma(s+1-\alpha)}F(\beta,1-\alpha;s+1-\alpha;\frac{1}{2}), \qquad (12)$$

where $F(a,b;c;z)$ is the hypergeometric function defined by **220**

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n.$$

As a result, we have

$$H(s) = \frac{\Gamma(s)\Gamma(s+2-q)}{\Gamma(s+1-\alpha)\Gamma(s+1-\beta)}\begin{vmatrix}a_1(s) & a_2(s)\\ a_3(s) & a_4(s)\end{vmatrix}$$

with

$$
\begin{aligned}
a_1(s) &= F(\beta,1-\alpha;s+1-\alpha;\frac{1}{2}),\\
a_2(s) &= F(\alpha,1-\beta;s+1-\beta;\frac{1}{2}),\\
a_3(s) &= -\frac{s}{s+1-\alpha}F(\beta,1-\alpha;s+2-\alpha;\frac{1}{2}),\\
a_4(s) &= \frac{s}{s+1-\beta}F(\alpha,1-\beta;s+2-\beta;\frac{1}{2}).
\end{aligned}
$$

But is is well known that

$$\lim_{\sigma\to\infty}\frac{\Gamma(s+a)}{\Gamma(s)}e^{-a\log s}=\lim_{\sigma\to\infty}F(\alpha,\beta;s;\frac{1}{2})=1,$$

and therefrom it is clear that $\lim\limits_{\sigma\to\infty}H(s)=2$. Consequently, we have proved that

$$D(s;\alpha,\beta)=2\Gamma(s)\ \Gamma(s+1-\alpha-\beta). \tag{13}$$

By the general theory of Mellin transforms (see H. Mellin, Math. Ann. 68 (1910), 305-337), in order to invert relation (1) i.e. to show that

$$W(y;\alpha,\beta)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s;\alpha,\beta)y^{-s}ds \tag{14}$$

for $\sigma>\max(0,\mathrm{Re}(q-1))=K_0$, it suffices to know that $\Gamma(s;\alpha,\beta)$ is reg-
221 ular for $\sigma>K_0(s=\sigma+it)$ and satisfies the following growth condition:

$$\Gamma(s;\alpha,\beta)=o(e^{-(1-\in)\frac{\pi}{2}|t|})\text{ for }|t|\to\infty \tag{15}$$

uniformly in any given strip $\sigma_1\leq\sigma\leq\sigma_2$, with given $\in>0$. The regularity of the function $\Gamma(s;\alpha,\beta)$ in the half plane $\sigma>K_0$ has been already established. We shall therefore prove only the latter assertion. It is well known that

$$C_1\leq|\Gamma(s)|e^{\frac{\pi}{2}|t|}|t|^{\frac{1}{2}-\sigma}\leq C_2\text{ for }|t|\geq 1,\sigma_1\leq\sigma\leq\sigma_2$$

with certain positive constants $C_\ell=C_\ell(\sigma_1,\sigma_2)\quad(\ell=1,2)$. This inequality together with (12) leads to (15), because $\left|\dfrac{s+n-\alpha}{n}\right|\geq 1-\in$ and $|(s+1-\alpha)_n|\geq(1-\in)^n n!$ for $\sigma_1\leq\sigma\leq\sigma_2$, $|t|\geq t_0(\in,\alpha,\sigma_1,\sigma_2)$, for all $n\geq 1$ and for a given $\in>0$. Thus we have

$$\lim_{|t|\to\infty}F(\beta,1-\alpha,s+1-\alpha;\frac{1}{2})=1$$

uniformly in $\sigma_1\leq\sigma\leq\sigma_2$ and relation (14) is proved.

Finally, we shall calculate the Mellin transform $\xi(s)$ of an infinite series

$$F(y)=\sum_{n\neq 0}a_nW(\frac{2\pi n}{\lambda}y;\alpha,\beta)\ (\lambda>0) \tag{16}$$

with $a_n = o(|n|^{K_1})$ $(n \to \infty)$.

Let $K_0 = \max(0, \mathrm{Re}(q-1))$. We shall that $\xi(x)$ exists for $\sigma > \max(K_0, K_1 + 1)$; moreover, in this domain, we can calculate it by term-wise integration of the series $F(y)$. Without loss of generality, we can assume, in the very beginning, that

$$|a_n| < |n|^{K_1} \text{ for } n \neq 0.$$

Now it is not hard to prove that for a given $\in > 0$ and $K > K_0$, there exists a positive constant $C = C(K, \in)$, such that 222

$$|W(\pm y; \alpha, \beta)| < Cy^{-K} e^{-(1-\in)y} \text{ for } y > 0.$$

Having fixed σ, the number K is supposed in the sequel, to lie between σ and K_0. Consider

$$G(y) = 2C \sum_{n=1}^{\infty} n^{K_1} (\frac{2\pi ny}{\lambda})^{-K} e^{-2\pi(1-\in)ny/\lambda}$$

which is a series of positive terms and converges for $y > 0$. It is obvious from the above discussion that the series $G(y)$ majorises the series $F(y)$. Since the series

$$2C(\frac{\lambda}{2\pi})^{\sigma}(1-\in)^{K-\sigma}(\sigma - K)\sum_{n=1}^{\infty} n^{K_1-\sigma}$$

obtained by the term-wise integration of the series representing the function $G(y)y^{s-1}$ converges for $\sigma > \max(K_0, K_1 + 1)$, it follows that

$$\begin{aligned}
\xi(s) = \int_0^{\infty} F(y)y^{s-1}dy &= \sum_{n\neq 0} a_n \int_0^{\infty} W(\frac{2\pi n}{\lambda}y; \alpha, \beta)y^{s-1}dy \\
&= \sum_{n>0} a_n \int_0^{\infty} W(\frac{2\pi n}{\lambda}y; \alpha, \beta)y^{s-1}dy + \sum_{n<0} a_n \int_0^{\infty} W(\frac{2\pi |n|}{\lambda}y; \beta, \alpha)y^{s-1}dy \\
&= (\frac{\lambda}{2\pi})^s \Gamma(s; \alpha, \beta)\sum_{n>0}\frac{a_n}{n^s} + (\frac{\lambda}{2\pi})^s \Gamma(s; \beta, \alpha)\sum_{n<0}\frac{a_n}{|n|^s} \qquad (17)
\end{aligned}$$

under the assumption that Re $s > \max(K_0, K_1 + 1)$.

Now relation (17) can be inverted i.e.

$$F(y) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \xi(s) y^{-s} ds \text{ (for } \sigma > \max(K_0, K_1 + 1)) \tag{18}$$

and the right hand side of (18) can be calculated by term-wise integration of the series representing the function $\xi(s)y^{-s}$. Indeed, the function $\xi(s)$ satisfies the requirements for the inversion of relation (17), in view
223 of (15) and the fact that the Dirichlet series $\sum_{n>0} \dfrac{a_n}{n^s}$ and $\sum_{n<0} \dfrac{a_n}{|n|^s}$ can be majorised by $\sum_{n=1}^{\infty} n^{K_1 - \sigma} = \zeta(\sigma - K_1)$, which is independent of t.

2 Automorphic Forms and Dirichlet Series

The following lemma will be used often to prove the equality of functions of the space $\{\alpha, \beta\}$.

Lemma 9. *A function $g(x, y)$ belonging to the space $\{\alpha, \beta\}$ vanishes identically if and only if*

$$g(0, y) = \left(\frac{\partial q(x, y)}{\partial x} \right)_{x=0} = 0.$$

Proof. Since $g(x, y)$ satisfies the differential equation $\Omega_{\alpha,\beta} g = 0$, it can be written as a power series in x of the type

$$g(x, y) = \sum_{n=0}^{\infty} g_n(y) x^n,$$

so that the coefficients g_n satisfy the recursion formula

$$(n + 2)(n + 1) y g_{n+2} + (\beta - \alpha) i (n + 1) g_{n+1} + y g_n'' + (\alpha + \beta) g_n' = 0.$$

It is an obvious consequence that if $g_0 = g_1 = 0$, then $g_n = 0$ for all n and the lemma is proved.

In the following, we shall consider automorphic forms for the group $\Gamma < \lambda >$ generated by

$$u^{\lambda} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\lambda > 0).$$

We know from Hecke that the group $\Gamma < \lambda >$ is discontinuous if and only if λ satisfies any one of the two conditions:

1. $\lambda \geq 2$
2. $\lambda = 2\cos\pi/\ell$ $(\ell = 3, 4, 5, \ldots)$.

□

Hecke has also shown that the domain $\{\tau|\tau = x + iy, |2x| \leq \lambda, |\tau| \geq 1\}$ **224**
is a fundamental domain for the group $\Gamma < \lambda >$. It is evident that the subgroups of the modular group that occur among these groups are the modular group $\Gamma = \Gamma < 1 >$ itself and the theta group $\Gamma_{\vartheta} = \Gamma < 2 >$. Moreover, in case $\lambda > 2$, the group $\Gamma < \lambda >$ is no more a Grenzkreis group.

We shall now derive, for the multiplier system v of the group $\Gamma < 2\cos\pi/\ell >$ and weight r, certain relations which are analogues of the relations for the modular group given in chapter 3, § 1 and coincide with them in case $\ell = 3$. It is easy to see that the transformation

$$V = U^{-\lambda}T = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda = 2\cos\pi/\ell$$

leaves the point $-e^{-\pi i/\ell} = -\cos\pi/\ell + i\sin\pi/\ell$ fixed and therefore is an elliptic transformation. Since

$$V = \begin{pmatrix} 1 & e^{-\pi i/\ell} \\ 1 & e^{\pi i/\ell} \end{pmatrix}^{-1} \begin{pmatrix} e^{\pi i/\ell} & 0 \\ 0 & e^{-\pi i/\ell} \end{pmatrix} \begin{pmatrix} 1 & e^{-\pi i/\ell} \\ 1 & e^{\pi i/\ell} \end{pmatrix},$$

it follows that $V^{\ell} = -E$. If there exists an automorphic form for the group $\Gamma < 2\cos\pi/\ell >$, the multiplier system v and weight r, then it can be proved, as in chapter 3, § 1, that

$$v(V) = e^{(\pi ir/\ell)+2\pi ia/\ell} \ (0 \leq a < \ell), v(T) = e^{(\pi ir/2)+2\pi ib/2} \ (0 \leq b < 2).$$

Let $v(U^\lambda) = e^{2\pi i\kappa}, 0 \leq \kappa < 1$. Thus

$$v(T) = v(U^\lambda V) = v(U^\lambda)v(V)$$

implies that

$$\frac{\ell-2}{4\ell}r \equiv \kappa + \frac{a}{\ell} + \frac{b}{2} (\bmod 1).$$

225 We set $v(T)(-\tau)^r = \gamma(-i\tau)^r$ so that $v(T) = \gamma e^{\pi ir/2}$, showing that

$$e^{2\pi ib/2} = \gamma \text{ or } b = (1-\gamma)/2, (\gamma^2 = 1).$$

For the unramified case ($\kappa = 0$), we have

$$\frac{\ell-2}{4\ell}r = \frac{g-(1-\gamma)/2}{\ell} + \frac{1-\gamma}{4},$$

i.e.
$$r = \frac{4q}{\ell-2} + 1 - \gamma(\kappa = 0),$$

where g is an integer. If $r = 0$, then κ is a rational number. Let us write $\kappa = k/h$ with $h > 0$ and $(k, h) = 1$. Then it can be seen easily that h divides ℓ or 2ℓ according as 2 divides ℓ or does not divide ℓ.

An entire function $\varphi(s)$ is said to be of *finite genus*, if, in every strip $\sigma_1 \leq \sigma \leq \sigma_2$ $(s = \sigma + it)$,

$$\varphi(s) = o(e^{|t|^K}) \text{ for } |t| \to \infty$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, with some positive constant K.

Theorem 35. *Suppose we are given complex numbers α,β with real $r = \alpha - \beta$, real $\lambda > 0, \gamma = \pm 1$ and real κ with $0 \leq \kappa < 1$ such that the equalities $\gamma = 1$, $\kappa = 0$ and $\alpha = \beta = 0$ or 1 do not all hold at the same time. Let us consider*

I) functions $f(\tau, \overline{\tau})$ with the properties:

1) $f(\tau, \overline{\tau}) \in \{\alpha, \beta\}$,

2) $f(\tau, \overline{\tau}) = o(y^{K_1})$ for $y \to \infty$,
$f(\tau, \overline{\tau}) = o(y^{-K_2})$ for $y \to 0$,
uniformly in x with suitable positive constants K_1, K_2,

226 3) $f(\tau+\lambda, \bar{\tau}+\lambda) = e^{2\pi i\kappa} f(\tau, \bar{\tau})$

4) $f(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = \gamma(-i\tau)^{\alpha}(i\bar{\tau})^{\beta} f(\tau, \bar{\tau})$, *and*

II) *pairs of functions $\varphi = \varphi(s)$, $\psi = \psi(s)$ having the properties:*

1) *the functions φ and ψ are meromorphic and can be represented by Dirichlet series of the type*

$$\varphi(s) = \sum_{n+\kappa>0} \frac{a_{n+\kappa}}{(n+\kappa)^s}, \quad \psi(s) = \sum_{n+\kappa<0} \frac{a_{n+\kappa}}{|n+\kappa|^s}$$

in some half-plane,

2) *for ξ, η defined by*

$$\xi(s) = (\frac{\lambda}{2\pi})^s \{\Gamma(s;\alpha,\beta)\varphi(s) + \Gamma(s;\beta,\alpha)\psi(s)\},$$
$$\eta(s) + \lambda\frac{\alpha-\beta}{4\pi}\xi(s) = (\frac{\lambda}{2\pi})^{s+1}\{\Gamma(s+1;\alpha,\beta)\varphi(s) - \Gamma(s+1;\beta,\alpha)\psi(s)\}$$

and for $q = \alpha+\beta$, the functions

$$\xi(s) - \frac{a_0}{s(s+1-q)} - \frac{\gamma a_0}{(q-s)(1-s)} + \frac{b_0}{s} + \frac{\gamma b_0}{q-s} \tag{1}$$

and

$$\eta(s) + \lambda\frac{\alpha-\beta}{4\pi}\xi(s) - \lambda\frac{\alpha-\beta}{2\pi}\left\{\frac{\gamma b_0}{s-q} + \frac{\gamma a_0}{(s-q)(s-1)}\right\}, \tag{2}$$

for a suitable choice of the constants a_0 and b_0, are entire functions of finite genus and moreover, for $\kappa \neq 0$, a_0 and b_0 are both equal to zero.

3) *the functions φ and ψ satisfy the functional equations*

$$\xi(q-s) = \gamma\xi(s), \quad \eta(q-s) = -\gamma\eta(s).$$

Then the linear space of functions f mentioned in I) is mapped by means of the Mellin transformation onto the linear space of pairs of functions described in II) and this correspondence between the 227
two spaces is invertible.

Remark. As we shall see later, the function $f(\tau,\overline{\tau})$ has the Fourier expansion

$$f(\tau,\overline{\tau}) = a_0 u(y,q) + b_0 + \sum_{n+\kappa\neq 0} a_{n+\kappa} W(\frac{2\pi(n+\kappa)}{\lambda} y;\alpha,\beta) e^{2\pi i(n+\kappa)x/\lambda} \quad (3)$$

with a_0 and b_0 as determined by II), 2). Clearly, a one-one invertible correspondence $f \longleftrightarrow (\phi,\psi)$ may now be seen to exist only if the constants a_0 and b_0 are uniquely determined by the condition II), 2). It may be verified that this happens except when $\kappa = 0$, $\gamma = 1$ and $\alpha = \beta = 0$ or 1. We recall that both these possibilities have been excluded in the statement of theorem 35. It can also be checked that these are exactly the cases for which the identity

$$a_0 u(\frac{y}{\tau\overline{\tau}}, q) + b_0 = \gamma(-i\tau)^{\alpha}(i\overline{\tau})^{\beta}\{a_0 u(y,q) + b_0\}$$

has a non-trivial solution for a_0 and b_0.

Proof. A) We start with a function $f(\tau,\overline{\tau})$ with the properties mentioned in I) and prove the existence of a pair of functions φ and ψ with the properties in II). It is an immediate consequence of I), 1), 2), 3) that the function $f(\tau,\overline{\tau})$ has a Fourier expansion of the type

$$f(\tau,\overline{\tau}) = a_0 u(y,q) + b_0 + \sum_{n+\kappa\neq 0} a_{n+\kappa} W(\frac{2\pi(n+\kappa)}{\lambda} y;\alpha,\beta) e^{2\pi i(n+\kappa)x/\lambda},$$

where a_0 and b_0 both vanish when $\kappa \neq 0$. Since the function

$$g(x,y) = f(-\frac{1}{\tau}, -\frac{1}{\overline{\tau}})(-i\tau)^{-\alpha}(i\overline{\tau})^{-\beta} - \gamma^{f}(\tau,\overline{\tau})$$

satisfies the differential equation $\Omega_{\alpha\beta} g = 0$, it follows from
228 lemma 9 that the conditions

$$g(0,y) = [\frac{\partial}{\partial x} g(x,y)]_{x=0} = 0 \quad (5)$$

are equivalent with I), 4). It now follows, by simple calculation, that

$$F^{*}(\frac{1}{y}) y^{-q} = \gamma F^{*}(y),$$

$$H^*(\frac{1}{y})y^{-q} = -\gamma H^*(y) \tag{6}$$

with

$$\begin{aligned} F^*(y) &= f(iy, -iy) = a_0 u(y,q) + b_0 + F(y) \\ H^*(y) &= G(y) - \lambda\frac{\alpha-\beta}{4\pi}F^*(y) \\ H(y) &= G(y) - \lambda\frac{\alpha-\beta}{4\pi}F(y), \end{aligned} \tag{7}$$

where

$$\begin{aligned} F(y) &= \sum_{n+\kappa\neq 0} a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{\lambda}y;\alpha,\beta), \\ G(y) &= \sum_{n+\kappa\neq 0}(n+\kappa)a_{n+\kappa}yW(\frac{2\pi(n+\kappa)}{\lambda}y;\alpha,\beta). \end{aligned} \tag{8}$$

With the help of I), 2), the equation

$$a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{\lambda}y;\alpha,\beta) = \frac{1}{\lambda}\int\limits_0^{\lambda} f(\tau,\overline{\tau})e^{-2\pi i(n+\kappa)x/\lambda}dx$$

entails that

$$a_{n+\kappa}W(\frac{2\pi(n+\kappa)}{\lambda}y;\alpha,\beta) = o(y^{-K_2}) \text{ for } y\to 0.$$

Let us choose $y = \dfrac{c}{|n+\kappa|}$, where the constant c is so determined that $W(\pm\dfrac{2\pi}{\lambda}c;\alpha,\beta)\neq 0$. Then it results from above that

$$a_{n+\kappa} = o(|n+\kappa|^{K_2}) \quad (|n+\kappa|\to\infty). \tag{9}$$

This shows, as already proved in § 1, that the Mellin transforms **229**

$$\xi(s) = \int\limits_0^{\infty} F(y)y^{s-1}dy \tag{10}$$

$$\eta(s) = \int_0^\infty H(y)y^{s-1}dy = \int_0^\infty G(y)y^{s-1}dy - \lambda\frac{\alpha-\beta}{4\pi}\xi(s)$$

can be calculated by term-wise integration on making use of the series representation (8) for $F(y)$ and $G(y)$. Thus we obtain that $\xi(s)$ and $\eta(s)$ have the form II), 2), where

$$\varphi(s) = \sum_{n+\kappa>0} \frac{a_{n+\kappa}}{(n+\kappa)^s}, \quad \psi(s) = \sum_{n+\kappa<0} \frac{a_{n+\kappa}}{|n+\kappa|^s}$$

which converge in some half-plane, because of condition (9) satisfied by the coefficients $a_{n+\kappa}$. In order to get the functional equation for φ and ψ we proceed as follows. From (10), we have

$$\begin{aligned}\xi(s) &= \int_1^\infty F(y)y^{s-1}dy + \int_0^1 F(y)y^{s-1}dy \\ &= \int_1^\infty \{F(y)y^s + F(\frac{1}{y})y^{-s}\}\frac{dy}{y}\end{aligned}$$

and similarly

$$\eta(s) = \int_1^\infty \{H(y)y^s + H(\frac{1}{y})y^{-s}\}\frac{dy}{y}.$$

With the help of the transformation formulae

$$\begin{aligned}F(\frac{1}{y}) &= \gamma F(y)y^q + \gamma\{a_0u(y,q)+b_0\}y^q - a_0u(\frac{1}{y},q) - b_0, \\ H(\frac{1}{y}) &= -\gamma H(y)y^q + \lambda\frac{\alpha-\beta}{4\pi}\gamma\{a_0u(y,q)+b_0\}y^q + \lambda\frac{\alpha-\beta}{4\pi}\{a_0u(\frac{1}{y},q)+b_0\},\end{aligned}$$

which result from (6) and (7), we obtain by integrating the elementary terms that

$$\xi(s) = \int_1^\infty F(y)\{y^s + \gamma y^{q-s}\}\frac{dy}{y} - \frac{b_0}{s} - \frac{\gamma b_0}{q-s} + \frac{a_0}{s(s+1-q)} + \frac{a_0\gamma}{(q-s)(1-s)},$$

$$\eta s = \int_1^\infty H(y)\{y^s - \gamma y^{q-s}\}\frac{dy}{y} + \lambda\frac{\alpha-\beta}{4\pi}$$

$$\left\{\frac{b_0}{s} - \frac{\gamma b_0}{q-s} - \frac{a_0}{s(s+1-q)} + \frac{\gamma a_0}{(q-s)(l-s)}\right\}. \tag{11}$$

The functional equations for φ and ψ are now trivial consequences. Both the integrals are obviously entire functions of s, which are bounded in any strip $\sigma_1 \leq \sigma \leq \sigma_2$. Thus $\xi(s)$ and $\eta(s)$ are meromorphic functions of s, the singularities of which are explicitly given by (11). Finally, it remains to prove that the functions (1) and (2) are entire functions of finite genus. But this follows easily from (11) and the fact that **230**

$$\xi(s) = o(1), \quad \eta(s) = o(1) \text{ for } |t| \to \infty \tag{12}$$

uniformly in any strip $\sigma_1 \leq \sigma \leq \sigma_2$. Therefore the pair of functions φ and ψ defined above satisfies all the requirements of theorem 35, II).

B) We now start with a pair of functions $\varphi$and ψ with the properties mentioned in II) and prove the existence of a function f satisfying the properties mentioned in I). We define the function $f(\tau, \overline{\tau})$ by

$$f(\tau, \overline{\tau}) = a_0 u(y, q) + b_0 + \sum_{n+\kappa\neq 0} a_{n+\kappa} W(\frac{2\pi(n+\kappa)}{\lambda} y; \alpha, \beta) e^{2\pi i(n+\kappa)x/\lambda}$$

and prove that it has the desired properties. Since the Dirichlet series φ and ψ converge in some half-plane, it follows that the coefficients $a_{n+\kappa}$ satisfy the growth condition (9) with some positive constant K_2. Thus, for $\sigma > K_2 + 1$, the series converge absolutely and the considerations of §1 show that formula (10) can be inverted i.e.

$$F(y) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \xi(s) y^{-s} ds,$$

$$\sigma_0 > \max(K_2 + 1, \mathrm{Re}(\alpha + \beta - 1)). \tag{13}$$

$$H(y) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \eta(s)y^{-s}ds.$$

231 We choose the line of integration in such a way that the singularities of $\xi(s)$ and $\eta(s)$ lie in the half plane $\sigma < \sigma_0$. If we replace in the integrals (10) the functions $\xi(s)$ and $\eta(s)$ by the functions given in II), 2) and substitute the Dirichlet series for φ and ψ, then we obtain the functions F and G given in (8), by term-wise integration which is justified. Moving the line of integration in (13) from $\sigma = \sigma_0$ to $\sigma = \sigma_1 = \operatorname{Re}(\alpha + \beta) - \sigma_0$, we shall show that

$$F(y) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \xi(s)y^{-s}ds + \sum \operatorname{Res}\xi(s)y^{-s},$$

$$H(y) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \eta(s)y^{-s}ds + \sum \operatorname{Res}\eta(s)y^{-s}. \tag{14}$$

In order to prove (14), it is sufficient to show that for every $\in > 0$

$$\xi(s), \eta(s) = o(e^{-(1-\in)\frac{\pi}{2}|t|}) \text{ for } |t| \to \infty \tag{15}$$

uniformly in $\sigma_1 \leq \sigma \leq \sigma_0$. Such an estimate for $\xi(s)$ and $\eta(s)$ holds on the line $\sigma = \sigma_0$ because of (15) of § 1, and therefore holds also on the line $\sigma = \sigma_1$ in view of the functional equation II), 3). If t_0 is sufficiently large, then functions $\xi(s)e^{-\pi i(1-\in)s/2}$ and $\eta(s)e^{-\pi i(1-\in)s/2}$ are regular in the domain $t \geq t_0$, $\sigma_1 \leq \sigma \leq \sigma_0$ and are bounded on its boundary. But by II), 2) there exists a constant
232 K such that both the functions are $o(e^{t^K})$ in the interior of the above domain; therefore, by the principle of Phragmen-Lindelöf both the functions are bounded in the whole of the above domain. Consequently, (15) is proved for $t \to \infty$. In a similar way, it is proved for $t \to -\infty$. Replacing $\xi(s)$ and $\eta(s)$ by $\gamma\xi(q-s)$ and $-\gamma\eta(q-s)$ respectively in the integrals (14) and applying the substitution $s \to q - s$ we see that

$$F(y) = \gamma y^{-q} F(\frac{1}{y}) + \sum \operatorname{Res}\xi(s)y^{-s},$$

$$H(y) = -\gamma y^{-q} H(\frac{1}{y}) + \sum \operatorname{Res} \eta(s) y^{-s}. \tag{16}$$

The sum of the residues of $\xi(s)y^{-s}$ can be evaluated with the help of II), 2). As a matter of fact, it can be seen that

$$\sum \operatorname{Res} \xi(s) y^{-s} = -a_0 u(y,q) - b_0 + \gamma\{a_0 u(\frac{1}{y}, q) + b_0\} y^{-q}$$

$$\begin{aligned}\sum \operatorname{Res} \eta(s) y^{-s} &= \lambda \frac{\alpha-\beta}{2\pi} \gamma\{a_0 u(\frac{1}{y}, q) + b_0\} y^{-q} - \lambda \frac{\alpha-\beta}{4\pi} \sum \operatorname{Res} \xi(s) y^{-s} \\ &= \lambda \frac{\alpha-\beta}{4\pi} \{a_0 u(y,q) + b_0 + \gamma(a_0 u(\frac{1}{y}, q) + b_0) y^{-q}\}.\end{aligned}$$

Thus formulae (16) are identical with formulae (6) from which I), 4) follows by lemma 9. The assertions in I), 1), 2), 3) follow form the Fourier series of $f(\tau, \overline{\tau})$ and the estimate (9). Hence the theorem is proved.

□

In order to determine the singularities of the functions φ and ψ we solve the equations

$$\begin{aligned}\Gamma(s;\alpha,\beta)\ \varphi(s) + \Gamma(s;\beta,\alpha)\ \psi(s) &= (\frac{2\pi}{\lambda})^s \xi(s), \\ \Gamma(s+1;\alpha,\beta)\varphi(s) - \Gamma(s+1;\beta,\alpha)\psi(s) &= (\frac{2\pi}{\lambda})^{s+1} \{\eta(s) + \lambda \frac{\alpha-\beta}{4\pi} \xi(s)\},\end{aligned}$$

for φ and ψ. These equations are nothing but trivial modifications of those in theorem 35, II), 2). Using the value of the functional determinant $D(s;\alpha,\beta)$ obtained in § 1 we get **233**

$$\begin{aligned}2\Gamma(s)\Gamma(s+1-q)\varphi(s) &= (\frac{2\pi}{\lambda})^s \Gamma(s+1;\beta,\alpha)\xi(s) \\ &\quad + (\frac{2\pi}{\lambda})^{s+1} \Gamma(s;\beta,\alpha) \left\{\eta(s) + \lambda \frac{(\alpha-\beta)}{4\pi} \xi(s)\right\}, \\ 2\Gamma(s)\Gamma(s+1-q)\psi(s) &= (\frac{2\pi}{\lambda})^s \Gamma(s+1;\alpha,\beta)\xi(s) \\ &\quad - (\frac{2\pi}{\lambda})^{s+1} \Gamma(s;\alpha,\beta) \left\{\eta(s) + \lambda \frac{(\alpha-\beta)}{4\pi} \xi(s)\right\}.\end{aligned} \tag{17}$$

Let us set

$$w(s;\alpha,\beta) = \frac{\Gamma(s;\alpha,\beta)}{\Gamma(s)\Gamma(s+1-q)} \tag{18}$$

Then it is immediate from (5) of §1 that

$$w(s;\alpha,\beta) = 2^{\frac{1}{2}(\alpha-\beta)} \sum_{n=0}^{\infty} \frac{(\beta)_n(1-\alpha)_n}{2^n n!\Gamma(s+n+1-\alpha)}, \tag{19}$$

which implies that $w(s;\alpha,\beta)$ is an entire function of s. Moreover, the functional equation (8) of § 1 for $\Gamma(s;\alpha,\beta)$ shows that $w(s;\alpha,\beta)$ satisfies the functional equation

$$s(s+1-q)\ w(s+1;\alpha,\beta) + (\beta-\alpha)\ w(s;\alpha,\beta) - w(s-1;\alpha,\beta) = 0. \tag{20}$$

Expressing the function $\Gamma(s;\alpha,\beta)$ in (17) in terms of the function $w(s;\alpha,\beta)$, we obtain

$$\begin{aligned} 2\varphi(s) &= (\frac{2\pi}{\lambda})^s w(s+1;\beta,\alpha)s(s+1-q)\xi(s) + (\frac{2\pi}{\lambda})^{s+1} w(s;\beta,\alpha) \\ &\qquad \left\{\eta(s) + \frac{\lambda(\alpha-\beta)}{4\pi}\xi(s)\right\}, \\ 2\psi(s) &= (\frac{2\pi}{\lambda})^s w(s+1;\alpha,\beta)s(s+1-q)\xi(s) - (\frac{2\pi}{\lambda})^{s+1} w(s;\alpha,\beta) \\ &\qquad \left\{\eta(s) + \lambda\frac{\alpha-\beta}{4\pi}\xi(s)\right\}. \end{aligned} \tag{21}$$

In the following, $A_i(s)$ ($i = 1, 2\ldots$) will denote an entire function of s. From theorem 35, II), 2), we have the relations

$$\begin{aligned} s(s+1-q)\xi(s) &= s(s+1-q)\{\frac{\gamma b_0}{s-q} + \frac{\gamma a_0}{(s-q)(s-1)}\} + A_1(s), \\ \eta(s) + \lambda\frac{\alpha-\beta}{4\pi}\xi(s) &= \lambda\frac{(\alpha-\beta)}{2\pi}\{\frac{\gamma b_0}{2\pi} + \frac{\gamma a_0}{(s-q)(s-1)}\} + A_2(s), \end{aligned}$$

234 which, on using (20), give

$$2\varphi(s) = (\frac{2\pi}{\lambda})^s w(s-1;\beta,\alpha)\{\frac{\gamma b_0}{s-q} + \frac{\gamma a_0}{(s-q)(s-1)}\} + A_3(s),$$

$$2\psi(s) = (\frac{2\pi}{\lambda})^s w(s-1;\alpha,\beta)\{\frac{\gamma b_0}{s-q} + \frac{\gamma a_0}{(s-q)(s-1)}\} + A_4(s). \tag{22}$$

For computing the principal parts of the functions φ and ψ, we shall require the following special values of the function $w(s;\alpha,\beta)$ and its derivative:

$$w(0;\alpha,\beta) = \frac{2^{q/2}}{\Gamma(1-\alpha)}, w(q-1;\alpha,\beta) = \frac{2^{1-q/2}}{\Gamma(\beta)}(q=\alpha+\beta)$$
$$w'(0;1-\beta,\beta) = \frac{-\sqrt{2}}{\Gamma(\beta)}\{\frac{\Gamma'}{\Gamma}(\beta) + \log 2\}.$$

The first two values are immediate consequences of the series representation (19), when we take into consideration the fact that

$$(1-x)^{-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} x^n = \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)}{n!} x^n.$$

For the third, differentiating the series (19) term by term and substituting $s = 0$ and $\alpha = 1-\beta$ yield

$$\begin{aligned} w'(0;\alpha,\beta) &= -2^{\frac{1}{2}(\alpha-\beta)} \sum_{n=0}^{\infty} \frac{(\beta)_n(1-\alpha)_n}{2^n n!\Gamma(n+1-\alpha)} \frac{\Gamma'}{\Gamma}(n+1-\alpha) \\ &= -\frac{2^{\frac{1}{2}-\beta}}{(\Gamma(\beta))^2} \sum_{n=0}^{\infty} \frac{\Gamma'(n+\beta)}{2^n n!} = -\frac{2^{\frac{1}{2}-\beta}}{(\Gamma(\beta))^2}(\Gamma(\beta)2^{\beta})' \\ &= \frac{\sqrt{2}}{\Gamma(\beta)}\{\frac{\Gamma'}{\Gamma}(\beta) + \log 2\}. \end{aligned}$$

Consequently, we have 235

$$\varphi(s) = \begin{cases} \frac{\gamma\cdot 2^{q/2}}{q-1} & \frac{\pi}{\lambda}\{(\frac{\pi}{\lambda})^{q-1}\frac{a_0+(q-1)b_0}{\Gamma(\alpha)}\frac{1}{s-q} - \frac{a_0}{\Gamma(1-\beta)}\frac{1}{(s-1)}\} + A_5(s), \text{ for } q \neq 1 \\ \frac{\gamma\cdot 2^{1/2}}{\Gamma(\alpha)} & \frac{\pi}{\lambda}\{\frac{a_0}{(s-1)^2} + (b_0 + a_0\{\log\frac{\pi}{\lambda} - \frac{\Gamma'}{\Gamma}(\alpha)\})\frac{1}{s-1}\} + A_6(s), \text{ for } q = 1 \end{cases} \tag{23}$$

and

$$\psi(s) = \begin{cases} \frac{\gamma\cdot 2^{q/2}}{q-1}\frac{\pi}{\lambda}\{(\frac{\pi}{\lambda})^{q-1}\frac{a_0+(q-1)b_0}{\Gamma(\beta)}\frac{1}{s-q} - \frac{a_0}{\Gamma(1-\alpha)}\frac{1}{s-1}\} + A_7(s), \text{ for } q \neq 1 \\ \frac{\gamma\cdot 2^{1/2}}{\Gamma(\beta)}\frac{\pi}{\lambda}\{\frac{a_0}{(s-1)^2} + (b_0 + a_0\{\log\frac{\pi}{\lambda} - \frac{\Gamma'}{\Gamma}(\beta)\})\frac{1}{s-1}\} + A_8(s), \text{ for } q = 1. \end{cases} \tag{24}$$

In particular, we see that

$$(s-1)(s-q)\varphi(s) \text{ and } (s-1)(s-q)\psi(s)$$

are entire functions of s. Moreover, using the well-known asymptotic behaviour of the Γ-function, § 1, (15), and the fact that the functions $\xi(s)$ and $\eta(s)$ are of finite genus, it can be seen that $(s-1)(s-q)\varphi(s)$ and $(s-1)(s-q)\psi(s)$ are also entire functions of finite genus.

We remark here that conversely, from (23), (24) and the functional equations of φ and ψ, it cannot be concluded that the functions mentioned in (1) and (2) are entire functions.

From the various properties of $\Gamma(s;\alpha,\beta)$ derived in §1, it follows that the poles of the function $\xi(s)$ are contained in the sequences of numbers

$$1, 0, -1, -2, \ldots \text{ and } q, q-1, q-2, \ldots$$

But $\xi(q-s) = \gamma\xi(s)$; therefore the poles of $\xi(s)$ are also contained in the sequences of numbers

$$q-1, q, q+1, q+2, \ldots, \text{ and } 0, 1, 2, \ldots$$

236 In any case, the common points of these two sets of sequences of numbers are $0, 1, q-1$ and q. There will be some more common points, in case q is an integer and $q \leq -2$ or $q \geq 4$. Thus the poles of $\xi(s)$ are contained in

$$\begin{array}{ll} \{0, 1, 2, \ldots, q-1, q\}, & \text{for integral } q \geq 4, \\ \{q-1, q, \ldots, 0, 1, \}, & \text{for integral } q \leq -2, \text{ and} \\ \{0, 1, q-1, q\}, & \text{otherwise.} \end{array}$$

If $q = 4$, the regularity of $\xi(s)$ at $s = 2$ leads to a relation between some special values of φ and ψ. Since $\lim\limits_{s\to -n}(s+n)\Gamma(s) = \dfrac{(-1)^n}{n!}(n \geq 0)$, it follows that

$$\lim_{s\to 2}(s-2)\xi(s) = -\left(\frac{\lambda}{2\pi}\right)^2 \{w(2;\alpha,\beta)\varphi(2) + w(2;\beta,\alpha)\psi(2)\}.$$

But, for $q = 4$,

$$\begin{aligned}
w(2;\alpha,\beta) &= \frac{2^{\frac{1}{2}(\alpha-\beta)}}{\Gamma(\beta-1)}\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta-3)}{2^n n!(\beta-1)_n}\\
&= \frac{2^{\frac{1}{2}(\alpha-\beta)}}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\beta+n-1)(\beta-3)}{2^n n!}\\
&= \frac{2^{\frac{1}{2}(\alpha-\beta)}(\beta-1)}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\beta-3)}{2^n n!} + \frac{2^{\frac{1}{2}(\alpha-\beta)}}{2\Gamma(\beta)}\sum_{n=1}^{\infty}\frac{(\beta-2)_{n-1}}{2^{n-1}(n-1)!}.
\end{aligned}$$

Therefore, we get the condition

$$\frac{\beta-2}{\Gamma(\beta)}\varphi(2) + \frac{\alpha-2}{\Gamma(\alpha)}\psi(2) = o(\alpha+\beta=4)$$

Conversely, if this condition is satisfied, then $\xi(s)$ is regular at $s = 2$ provided $\alpha+\beta = 4$.

The case of analytic modular forms considered by Hecke appears as a particular case of our considerations when we assume that $\beta = 0$, **237**
$a_0 = 0$ and $\psi(s) = 0$. Under these assumptions we obtain, using (7) of § 1, that

$$\begin{aligned}
\xi(s) &= (\frac{\lambda}{2\pi})^s\Gamma(s;\alpha,0)\varphi(s) = 2^{\alpha/2}(\frac{\lambda}{2\pi})^s\Gamma(s)\varphi(s)\\
\eta(s) &= (\frac{\lambda}{2\pi})^{s+1}\{\Gamma(s+1;\alpha,0) - \frac{\alpha}{2}\Gamma(s;\alpha,0)\}\varphi(s)\\
&= 2^{\alpha/2}(\frac{\lambda}{2\pi})^{s+1}(s-\frac{\alpha}{2})\Gamma(s)\varphi(s).
\end{aligned}$$

The function $\xi(s)$ as given above is a constant multiple of the function $(\frac{\lambda}{2\pi})^s\Gamma(s)\varphi(s)$ considered by Hecke and both the functional equations for $\xi(s)$ and $\eta(s)$ lead to the same conclusion.

In the non-analytic case $\alpha = \beta$, by § 1, (6), we have

$$\begin{aligned}
\xi(s) &= (\frac{\lambda}{2\pi})^s\Gamma(s;\alpha,\alpha)\{\varphi(s)+\psi(s)\}\\
&= \frac{2^{-\alpha-1}}{\sqrt{\pi}}(\frac{\lambda}{\pi})^s\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}-\alpha)\{\varphi(s)+\psi(s)\},
\end{aligned}$$

$$\eta(s) = (\frac{\lambda}{2\pi})^{s+1}\Gamma(s+1;\alpha,\alpha)\{\varphi(s)-\psi(s)\}$$
$$= \frac{2^{-\alpha-1}}{\sqrt{\pi}}(\frac{\lambda}{\pi})^{s+1}\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2}+1-\alpha)\{\varphi(s)-\psi(s)\}.$$

For $\lambda \leq 2$, the linear space of functions $f = f(\tau,\overline{\tau})$ characterised in theorem 35, I) coincides with the space $[\Gamma < \lambda >, \alpha, \beta, v]$, where the multiplier system v is defined by

$$v(U^{\lambda}) = e^{2\pi i\kappa} \text{ and } v(T) = \gamma e^{\pi ir/2}.$$

We prove here this statement in the two special cases $\lambda = 1$ and 2. In these two cases, the assertion results easily from the following two lemmas.

238 **Lemma 10.** *Let Γ_0 be a subgroup of finite index in the modular group Γ and let $r = \alpha - \beta$ be real, where α and β are two complex numbers. If $f(\tau,\overline{\tau})$ belongs to the space $[\Gamma_0, \alpha, \beta, v]$, then*

$$f(\tau,\overline{\tau}) = o(y^{-K_2}) \textit{ for } y \to 0 \ (\textit{with } \tau = x+iy)$$

uniformly in x, with a positive constant K_2.

Proof. Let μ be the index of Γ_0 in Γ and

$$\Gamma = \bigcup_{n=1}^{\mu} \Gamma_0 S_n$$

be a coset decomposition of Γ modulo Γ_0. We may assume that $S_1 = E$. Consider

$$g(\tau,\overline{\tau}) = \sum_{n=1}^{\mu} |(f|S_n)(\tau,\overline{\tau})|,$$

which obviously does not depend upon the choice of the coset representatives S_n. Since, along with the set $\{S_n\}$, the set $\{S_n\ S\}$ for $S \in \Gamma$ is also a representative system for the left cosets of Γ modulo Γ_0 and since

$$|(c\tau+d)^{\alpha}(c\overline{\tau}+d)^{\beta}| = |c\tau+d|^{p} \quad (p = \mathrm{Re}(\alpha + beta)),$$

we obtain

$$(g|S)_{\frac{p}{2},\frac{p}{2}}(\tau,\overline{\tau}) = g(\tau,\overline{\tau}) \text{ for } S \in \Gamma,$$

showing that $y^{p/2}g(\tau,\overline{\tau})$ is invariant under the transformation of Γ. For a given point $\tau = x + iy$ with $y < \dfrac{\sqrt{3}}{2}$ we determine an equivalent point $\tau_0 = x_0 + iy_0 = S < \tau >$ with $y_0 \geq \dfrac{\sqrt{3}}{2}$, $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. It is obvious that $c \neq 0$ and 239

$$y_0 = \frac{1}{|c\tau + d|^2} \leq \frac{1}{c^2 y} \leq \frac{1}{y}.$$

Since $S_n < \infty >$ is a parabolic cusp of Γ_0, we have, by the definition of a non-analytic modular form,

$$(f|S_n)(\tau_0,\overline{\tau}_0) = o(y_0^{K_1}) \text{ for } y_0 \to \infty,$$

uniformly in x_0 with a positive constant K_1. Consequently,

$$g(\tau_0,\overline{\tau}_0) \leq C y_0^{K_1} \text{ for } y_0 \geq \frac{\sqrt{3}}{2},$$

with some suitable constant C. It follows from above that

$$\begin{aligned} |f(\tau,\overline{\tau})| \leq g(\tau,\overline{\tau}) = (\frac{y_0}{y})^{p/2} g(\tau_0\overline{\tau}_0) &\leq C y^{-p/2} y_0^{p/2+K_1} \\ &\leq C y^{-p-K_1} \text{ for } y \leq \frac{\sqrt{3}}{2}, \end{aligned}$$

because K_1 can be so chosen that $K_1 + \dfrac{p}{2} \geq 0$. Hence the lemma is proved. ◻

Lemma 11. *Let Γ_0 be a subgroup of finite index in the modular group Γ and let $f(\tau,\overline{\tau})$ be a continuous function in $\mathscr{G}$, which satisfies the transformation formula*

$$(f|S)_{\alpha,\beta}(\tau,\overline{\tau}) = f(\tau,\overline{\tau}) \text{ for every } S \in \Gamma_0,$$

where α and β are complex numbers with $\alpha - \beta$ real. Further, let

$$f(\tau,\overline{\tau}) = o(y^{K_1}) \text{ for } y \to \infty$$

$$f(\tau,\overline{\tau}) = o(y^{-K_2}) \textit{ for } y \to 0$$

uniformly in $x(\tau = x + iy)$ *with positive constants* K_1 *and* K_2. *Then*

$$(f\underset{\alpha,\beta}{|}A^{-1})(\tau,\overline{\tau}) = o(y^K) \textit{ for } y \to \infty,$$

240 *uniformly in* x *with a suitable constant* K *for every* A *belonging to* Γ.

Proof. Obviously, for the proof of the lemma, it is sufficient to confine ourselves to those elements $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of Γ for which $c \neq 0$. We set $\tau_1 = x_1 + iy_1 = A^{-1} < \tau >$ for $y \geq 1$. Then we have $y_1 = \dfrac{y}{|-c\tau + a|^2} \leq 1$ and

$$|f(\tau_1,\overline{\tau}_1)| \leq Cy_2^{-K_2} \text{ for } y_1 \leq 1.$$

with a certain constant C. Here K_2 can be so chosen that $K_2 \geq \dfrac{p}{2}$, where $p = \mathrm{Re}(\alpha + \beta)$. Let us consider the points τ with

$$|x - \frac{a}{c}| \leq m, \quad 1 \leq m \leq y,$$

m being a given constant. Then we have

$$\begin{aligned} |(f\underset{\alpha,\beta}{|}A^{-1})(\tau,\overline{\tau})| &= |f(\tau_1,\overline{\tau}_1)||-c\tau + a|^{-p} \\ &\leq Cy_1^{-K_2}|-c\tau + a|^{-p} \\ &= Cy^{-K_2}|-c + a|^{2K_2-p} \\ &= C|c|^{2K_2-p}y^{-K_2}\{y^2 + (x - \frac{a}{c})^2\}^{K_2-p/2} \\ &\leq C|\sqrt{2}c|^{2K_2-p}y^{K_2-p} \end{aligned}$$

But $(f\underset{\alpha,\beta}{|}A^{-1})(\tau,\overline{\tau})$ is periodic in x; therefore our assertion follows with $K = K_2 - p$.

In the sequel, we shall give some applications of theorem 35 to the spaces $[\Gamma < \lambda >, \alpha, \alpha, v]$ with $\lambda = 1$ or $2, \alpha > 0, \alpha \neq 1$ where the multiplier system v is determined by $v(U^\lambda) = e^{2\pi i\kappa}, v(T) = \gamma$, and satisfies
241 the condition $v^2 = 1$. We have proved already in (10) of chapter 4,

§ 2, that the operator $\mathbf{X} = \Theta(r = 0)$ maps the space $[\Gamma < \lambda >, \alpha, \alpha, v]$ onto the space $[(\Gamma < \lambda >)^*, \alpha, \alpha, v^*]$ with $(\Gamma < \lambda >)^*$ and v^* as defined in the above-mentioned chapter. But $v^2 = 1$ implies that $v^* = v$ and $(\Gamma < \lambda >)^* = \Gamma < \lambda >$ for $\lambda = 1$ or 2; therefore, the operator $\mathbf{X} = \Theta$ leaves the spaces $[\Gamma < \lambda >, \alpha, \alpha, v]$ invariant. Since every function f belonging to $[\Gamma < \lambda >, \alpha, \alpha, v]$ can be written as

$$f = \frac{1}{2}(f + \Theta f) + \frac{1}{2}(f - \Theta f),$$

if follows that the space $[\Gamma < \lambda >, \alpha, \alpha, v]$ can be represented as a direct sum

$$[\Gamma < \lambda >, \alpha, \alpha, v] = \mathscr{L}^{(1)}(\lambda, \alpha, v) + \mathscr{L}^{(-1)}(\lambda, \alpha, v)$$

so that

$$\Theta f = \in f \text{ for } f \in \mathscr{L}^{\in}(\lambda, \alpha, v), (\in = \pm 1).$$

Moreover, if $f \in \mathscr{L}^{\in}(\lambda, \alpha, v)$ has the Fourier expansion

$$f(\tau, \overline{\tau}) = a_0 u(y, q) + b_0 + \sum_{n+\kappa \neq 0} a_{n+\kappa} W(\frac{2\pi(n+\kappa)}{\lambda} y; \alpha, \alpha) e^{2\pi i(n+\kappa)x/\lambda},$$

we obtain, from (14) of chapter 4, § 2, the following relations for the coefficients:

$$a_0 = \in a_0, \quad b_0 = \in b_0, \quad a_{n+\kappa} = \in a_{-n-\kappa} \text{ for } n + \kappa \neq 0.$$

This shows that if f belongs to $\mathscr{L}^{(-1)}(\lambda, \alpha, v)$ then $a_0 = b_0 = 0$ i.e. f is a cusp form, in case $\lambda = 1$. Let φ and ψ be the Dirichlet series associated to the function $f \in \mathscr{L}^{(\in)}(\lambda, \alpha, v)$. Then $\varphi(s) = \psi(s)$ and

$$\xi(s) = \frac{2^{-\alpha}}{\sqrt{\pi}} (\frac{\lambda}{\pi})^s \Gamma(\frac{s}{2}) \Gamma(\frac{s+1}{2} - \alpha) \varphi(s), \eta(s) = 0 \text{ for } \in = 1,$$

$$\xi(s) = 0, \quad \eta(s) = \frac{2^{-\alpha}}{\sqrt{\pi}} (\frac{\lambda}{\pi})^{s+1} \Gamma(\frac{s+1}{2}) \Gamma(\frac{s}{2} + 1 - \alpha) \varphi(s) \text{ for } \in = -1.$$

We shall denote by $\vartheta^{\in}(\lambda, \alpha, \kappa\gamma)$ the linear space of meromorphic func- **242**
tions $\varphi(s)$, which with $\beta = \alpha$ and $\psi = \varphi$ satisfy the conditions of theorem 35, II), so that the linear mapping $f \to \varphi$ is an isomorphism between the spaces $\mathscr{L}^{(\in)}(\lambda, \alpha, v)$ and $\vartheta^{(\in)}(\lambda, \alpha, \kappa, \gamma)$. In the following theorem, we explicitly give a basis for the space $\vartheta^{(\in)}(\lambda, \alpha, \kappa, \gamma)$. □

Theorem 36. *Under the assumptions $\alpha > 0, \alpha \neq 1$ the space $\vartheta^{(\in)}(\lambda, \alpha, \kappa, \gamma)$ is generated by*

$$
\begin{aligned}
&\zeta(s)\zeta(s+1-2\alpha), \ \textit{in case } \lambda = 1, \kappa = 0, \gamma = 1, \in = 1,\\
&\left.\begin{aligned} &2^{-s}\zeta(s)\zeta(s+1-2\alpha)\\ &2^{-s}(2^s + 2^{2\alpha-s})\zeta(s)\zeta(s+1-2\alpha)\end{aligned}\right\}, \quad \textit{in case } \lambda = 2, \kappa = 0, \gamma = 1, \in = 1\\
&2^{-s}(2^s - 2^{2\alpha-s})\zeta(s)\zeta(s+1-2\alpha), \ \textit{in case } \lambda = 2, \kappa = 0, \gamma = -1, \in = 1,\\
&2^s L(s,\chi)\ L(s+1-2\alpha,\chi), \ \textit{in case } \lambda = 2, \kappa = \frac{1}{2}, \gamma = -1, \in = -1
\end{aligned}
$$

and 0, otherwise so long as $\lambda = 1$ or 2 and $\kappa = 0$ or $\frac{1}{2}$.

The functions $\zeta(s)$ and $L(s,\chi)$ are defined for Re $s > 1$ *by*

$$\zeta(s) = \int_{n=1}^{\infty} n^{-2} \textit{ and } L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

where x is the proper character modulo 4.

Proof. Using the well-known properties of the functions $\zeta(s)$ and $L(s,\chi)$, it can be shown without any difficulty that the given functions belong to the space $\vartheta^{\vartheta}(\lambda, \alpha, \kappa, \gamma)$. Thus to complete the proof of the theorem, it is sufficient to establish that the dimension of $\vartheta^{(\in)}(\lambda, \alpha, \kappa, \gamma)$ is not greater than the number of functions mentioned in theorem 36 in each individual case. Since

$$\text{dimension } \vartheta^{(\in)}(\lambda, \alpha, \kappa, \gamma) \leq \text{ dimension } [\Gamma < \lambda >, \alpha, \alpha, v],$$

243 it suffices to prove the following under the assumptions $\alpha > 0$ and $v^2 = 1$: namely,

$$\text{dimension } [\Gamma < \lambda >, \alpha, \alpha, v] \leq \begin{cases} 1, & \text{in case } \lambda = 1, v = 1,\\ 2, & \text{in case } \lambda = 2, v = 1\\ 1, & \text{in case } \lambda = 2, v = v_1 \text{ or } v_2\\ 0, & \text{otherwise,}\end{cases}$$

where v_1 and v_2 are even abelian characters of Γ_ϑ mentioned before the proof of theorem 32. Since under the given assumptions, $[\Gamma_\vartheta, \alpha, \alpha, v]$ does not contain any cusp form, which does not vanish identically, and since $\Gamma_\vartheta \subset \Gamma$, the dimension of the space $[\Gamma < \lambda >, \alpha, \alpha, v]$ for $\lambda = 1$ or 2 is at most equal, by theorem 29, to the number of inequivalent parabolic cusps of $\Gamma < \lambda >$ at which the multiplier system v is unramified. But we have already shown that in case $\lambda = 1$, the multiplier system v is unramified at ∞ only if $v = 1$ and in case $\lambda = 2$, the multiplier system v_1 respectively v_2, respectively $v_3 = v_1 v_2$ is ramified at 1 respectively ∞, respectively 1 as well as ∞. Therefore the above estimates hold for the dimension of $[\Gamma < \infty >, \alpha, \alpha, v]$. Hence the proof of the theorem is complete. □

Theorem 36 provides us three examples of functions which are uniquely fixed upto a constant factor by their functional equation and the fact that they can be represented by Dirichlet series in some half-plane. One of these functions, namely, the function defined by $2^s L(s, \chi) L(s + 1 - 2\alpha, \chi)$, is an entire function.

3 The Hecke Operations T_n

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In this section, we shall investigate the multiplicative properties of the Fourier coefficients of non-analytic modular forms in connection with the Euler product development of the corresponding Dirichlet series. For the sake of simplicity, we shall confine ourselves to the modular group Γ, though almost the same type of results as proved by Hecke and Petersson for the analytic case can be obtained for subgroups of the modular group, of arbitrary level.

For defining Hecke operators T_n, we consider the set O_n of all integral matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant n (n a natural number). Let $O_{n,g}$ denote the subset of O_n consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $(a, b, c, d) = g$. Then obviously

$$O_{n,g} = \left(\begin{smallmatrix} g & 0 \\ 0 & g \end{smallmatrix}\right) O_{ng^{-2},1} \text{ for } g^2 | n, O_n = \bigcup_{\substack{g^2|n \\ g>0}} O_{n,g}. \tag{1}$$

Since, with S, the set $\Gamma S\Gamma$ of matrices is also contained in O_n, it follows that O_n can be decomposed completely into left and right cosets modulo Γ. Moreover, any two left or right cosets are either identical or disjoint, because Γ is a group. For our later considerations, we shall need

Lemma 12. *1) The subset of O_n defined by*

$$\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | ad = n, \quad d > 0, b \bmod d\right\}$$

forms a complete system of representatives of left cosets of O_n modulo Γ.

2) There exists a common system of representatives for left and right
245 *cosets of O_n modulo Γ.*

Proof. 1) For every $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in O_n$, there exists a matrix $L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ such that $LS = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, because the equation $\gamma a + \delta c = 0$ has a solution for γ and δ with $(\gamma, \delta) = 1$ and then with suitable α and β we can construct a matrix $L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belonging to Γ. Since $-E$ belongs to Γ, it follows that every left coset of O_n modulo Γ contains a matrix of the type $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $d > 0$. If

$$L\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ 0 & d^* \end{pmatrix} \text{ with } L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma,$$

then $\gamma = 0$ and from $d > 0$, $d^* > 0$, we have $\alpha = \delta = 1$ implying that $a = a^*$, $d = d^*$ and $b^* = b + \beta d$ i.e. $b^* \equiv b (\bmod d)$. Hence the assertion 1) of the lemma is proved.

2) Since any matrix $S \in O_{n,1}$ has 1 and n as its elementary divisors, we have

$$O_{n,1} = \Gamma S_n \Gamma \text{ with } S_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$

Let $\{S_n L_i\}$ $(i = 1, 2, \ldots, \rho(n))$ be a system of representatives of left cosets of $O_{n,1}$ modulo Γ, with $L_i \in \Gamma$. Then the matrices $A_i = L_i' S_n L_i$ $(i = 1, 2, \ldots, \rho(n))$ (L_i', the transpose of L_i) in any case form a system of representatives for the left cosets of $O_{n,1}$ modulo Γ and therefore

$$O_{n,1} = \bigcup_{i=1}^{\rho(n)} \Gamma A_i.$$

By transposition, we get

$$O_{n,1} = \bigcup_{i=1}^{\rho(n)} A_i \Gamma.$$

Thus there exists a common representative system for the left and right cosets of $O_{n,1}$ modulo Γ. Our assertion 2) now follows from the decomposition (1) above. **246**

Let α, β be complex numbers such that $r = \alpha - \beta$ is an integer. Let V_n denote a system of representatives of left cosets of O_n modulo Γ. We define the linear operator T_n on the space $[\Gamma, \alpha, \beta, 1]$

$$f|T_n = n^{q-1} \sum_{S \in V_n} f|S \, (q = \alpha + \beta). \tag{2}$$

The definition of the operator T_n is independent of the choice of V_n, because, if we replace S belonging to V_n by LS for any L in Γ, then

$$f|(LS) = (f|L)|S = f|S \text{ for } f \text{ in } [\Gamma, \alpha, \beta, 1].$$

$\square$

Theorem 37. *The linear space* $[\Gamma, \alpha, \beta, 1]$ *is mapped into itself by* $f \to f|T_n$.

Proof. Let $f = f(\tau, \overline{\tau})$ be an element of $[\Gamma, \alpha, \beta, 1]$. Then by (10) of chapter 4, § 1, $f|T_n$ belongs to $\{\alpha, \beta\}$ and for $L \in \Gamma$, we have

$$(f|T_n)|L = n^{q-1} \sum_{S \in V_n} f|(SL) = f|T_n,$$

because, along with S, the matrix SL also runs over a system of representatives of left cosets of O_n modulo Γ. Thus, in order to complete the proof of the theorem, it remains to show that, at the parabolic cusp ∞ of Γ,

$$(f|T_n)(\tau,\overline{\tau}) = o(y^K) \text{ for } y \to \infty$$

247 with some positive constant K. For this, we shall compute explicitly the Fourier expansion of $f|T_n$. In the following, we shall take, for V_n, the special set of representatives given in lemma 12. Let

$$f(\tau,\overline{\tau}) = a(0)u(y,q) + b(0) + \sum_{k\neq 0} a(k)W(2\pi y;\alpha,\beta)e^{2\pi i\kappa x} \tag{3}$$

be the Fourier expansion of $f(\tau,\overline{\tau})$ at the parabolic cusp ∞. Then

$$\begin{aligned}(f|T_n)(\tau,\overline{\tau}) &= n^{q-1}\sum_{\substack{d|n\\ d>0}} d^{-q}\sum_{b \bmod d}\{a(0)u(\frac{ay}{d},q)+b(0)\}+\\ &\quad + n^{q-1}\sum_{\substack{d|n\\ d>0}} d^{-q}\sum_{b \bmod d}\sum_{k\neq 0} a(k)W(\frac{2\pi ak}{d}y;\alpha,\beta)e^{2\pi ik\frac{ax+b}{d}}.\\ &= \sum_{\substack{d|n\\ d>0}}(\frac{n}{d})^{q-1}\{a(0)u(\frac{ny}{d^2},q)+b(0)\}+\\ &\quad + \sum_{\substack{k\neq 0\\ d|n\\ d>0}}(\frac{n}{d})^{q-1}\sum_{k\neq 0} a(kd)W(\frac{2\pi nk}{d};\alpha,\beta)e^{\frac{2\pi ink}{d}}x\end{aligned}$$

Let us set $m = \dfrac{nk}{d} = ak$. Then $kd = \dfrac{m}{a}\cdot\dfrac{n}{a}$, where a runs over all positive divisors of (m,n). Writing d in place of a, we obtain from above, by a brief calculation, that

$$(f|T_n)(\tau,\overline{\tau}) = a^*(0)u(y,q) + b^*(0) + \sum_{k\neq 0} a^*(k)W(2\pi ky;\alpha,\beta)e^{2\pi ikx} \tag{4}$$

with

$$a^*(0) = d_{q-1}(n)a_0, \quad b^*(0) = d_{q-1}(n)b(0)$$

$$a^*(m) = \sum_{\substack{d|(m,n)\\ d>0}} d^{q-1} a\left(\frac{mn}{d^2}\right). \tag{5}$$

Consequently, theorem 37 is proved. □ **248**

Theorem 38. *The operators $T_n = T(n)$ $(n = 1, 2, \ldots)$ commute with each other and satisfy the composition rule*

$$T(m)T(n) = \sum_{\substack{d|(m,n)\\ d>0}} T(mn/d^2)d^{q-1}.$$

Proof. (i) Let $(m,n) = 1$. For $f \in [\Gamma, \alpha, \beta, 1]$, we have, by definition,

$$\begin{aligned} f|T(m)T(n) &= (f|T(m))|T(n) \\ &= (nm)^{q-1} \sum_{\substack{a'd'=m\\ b' \bmod d'}} \sum_{\substack{ad=n\\ b \bmod d}} f|\begin{pmatrix} a' & b' \\ 0 & d'\end{pmatrix}\begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \\ &= (nm)^{q-1} \sum_{\substack{a'd'=m\\ b' \bmod d'}} \sum_{\substack{ad=n\\ b \bmod d}} f|\begin{pmatrix} a'a & a'b+b'd \\ 0 & dd'\end{pmatrix}. \end{aligned}$$

But, for $(m,n) = 1$, the product $d'd$ runs over the positive divisors of mn when d' (respectively d) runs over positive divisors of m (respectively n) and $a'b + bd'$ runs through all the residue classes modulo dd' when b' (respectively b) does so modulo d' (respectively d); therefore, the matrix $\left(\begin{smallmatrix} a'a & a'b+b'd \\ 0 & dd'\end{smallmatrix}\right)$ runs over a system V_{mn} and we have

$$T(m)T(n) = T(mn).$$

(ii) Let $m = p$, $n = p^r$, $r \geq 1$ and p a prime number. We shall show that

$$T(p^r)T(p) = T(p^{r+1}) + p^{q-1}T(p^{r-1}).$$

Let f be any element of the space $[\Gamma, \alpha, \beta, 1]$. Then from

$$f|T(p) = p^{q-1}\{f|\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} + \sum_{\ell \bmod p} f|\begin{pmatrix} 1 & \ell \\ 0 & p\end{pmatrix}\}$$

and 249

$$f|T(p^r) = p^{r(q-1)} \sum_{\substack{0\le t\le r\\ b_t \bmod p^t}} f|\begin{pmatrix} p^{r-t} & b_t\\ 0 & p^t\end{pmatrix},$$

it follows that

$$f|T(p^r)T(p) = p^{(r+1)(q-1)} \sum_{\substack{0\le t\le r\\ b_t \bmod p^t}} f|\begin{pmatrix} p^{r+1-t} & b_t\\ 0 & p^t\end{pmatrix} +$$
$$+ p^{(r+1)(q-1)} \sum_{\substack{0\le t\le r\\ b_t \bmod p^t}} \sum_{\ell \bmod p} f|\begin{pmatrix} p^{r-t} & p^{r-t}\ell + b_t p\\ 0 & p^{t+1}\end{pmatrix}.$$

It is obvious that the first sum along with the term $t = r$ from the second double sum gives the term $f|T(p^{r+1})$. Thus, on simplifying the second sum by taking out the factor p, we get

$$\begin{aligned} f|T(p^r)T(p) &= f|T(p^{r+1}) + p^{(r+1)(q-1)}p^{-q}\\ &\quad \sum_{\substack{0\le t< r\\ b_t \bmod p^t}} \sum_{\ell \bmod p} f|\begin{pmatrix} p^{r-l-t} & p^{r-1-t}\ell + b_t\\ 0 & p^t\end{pmatrix}\\ &= f|T(p^{r+1}) + p^{(r+1)(q-1)}p^{q-1}p^{-q+1}\\ &\quad \sum_{\substack{0\le t< r\\ b_t \bmod p^t}} f|\begin{pmatrix} p^{r-1-t} & b_t\\ 0 & p^t\end{pmatrix}\\ &= f|T(p^{r+1}) + p^{q-1} f|T(p^{r-1}). \end{aligned}$$

This shows that $T(p^r)$ is a polynomial in $T(p)$ with complex numbers as coefficients and therefore the operators $T(p^r)(r = 0, 1, 2, \ldots)$ commute with each other. By (i), it follows trivially that the operators $T(n)$ commute with each other.

250 (iii) In order to prove the second assertion of the theorem, it is sufficient, because of (i), to prove that

$$T(p^r)T(p^s) = \sum_{0\le u\le \min(r,s)} p^{u(q-1)}T(p^{r+s-2u}).$$

Without loss of generality, we can assume that $r \leq s$. The assertion is clearly true for $r = 0, 1$. Let us assume that $1 \leq r < s$ and that the assertion is proved for r and $r - 1$ instead of r. Then we shall establish it for $r + 1$ instead of r. By (ii) and the induction hypothesis, we have

$$\begin{aligned} T(p)T(p^r)T(p^s) &= T(p^{r+1})T(p^s) + p^{q-1}T(p^{r-1})T(p^s) \\ &= \sum_{0\leq u\leq r} p^{u(q-1)}T(p)T(p^{r+s-2u}), \end{aligned}$$

which implies that

$$\begin{aligned} T(p^{r+1})T(p^s) &= \sum_{0\leq u\leq r} p^{u(q-1)}T(p^{r+s+1-2u}) \\ &+ \sum_{0\leq u\leq r} p^{(u+1)(q-1)}T(p^{r+s-1-2u}) - \\ &- p^{q-1}T(p^{r-1})T(p^s) \\ &= \sum_{0\leq u\leq r} p^{u(q-1)}T(p^{r+s+1-2u}) + p^{(r+1)(q+1)}T(p^{s-1-r}) \\ &= \sum_{0\leq u\leq r+1} p^{u(q-1)}T(p^{r+1+s-2u}). \end{aligned}$$

Hence the assertion is proved for all r, s and the proof of theorem 38 is complete.

□

Theorem 39. *Under the assumptions that $r = \alpha - \beta$ is an even non-neqative integer and $\Gamma(\beta) \neq \infty$, we have, for all natural numbers n and for all $f \in [\Gamma, \alpha, \beta, 1]$.*

$$(\Theta f)|T(n) = \Theta(f|T(n)),$$

where Θ is the operator defined in chapter IV, §1, (20). **251**

Proof. By (17) and (19) of chapter 4, § 1, we have

$$(\Theta f)|T(n) = n^{q-1} \sum_{S\in V_n} (\Theta f)|S$$

$$= n^{q-1}\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{S\in V_n}\mathbf{X}(\Lambda^r f|S^*)$$
$$= n^{q-1}\frac{\Gamma(\beta)}{\Gamma(\beta)}\sum_{S\in V_n}\mathbf{X}\Lambda^r(f|S^*)$$
$$= \Theta n^{q-1}\sum_{S\in V_n} f|S^*,$$

where $S^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}$. But S^*, along with S, runs over a representative system of left cosets of O_n modulo Γ; therefore it follows that

$$(\Theta f)|T(n) = \Theta(f|T(n)).$$

In the following, unless otherwise stated, $\alpha-\beta$ will be an even non-negative integer and $\Gamma(\beta)\neq\infty$. Under these assumptions, the operator Θ is well-defined on the space $[\Gamma,\alpha,\beta,1]$ and maps it onto itself. Since $\Theta^2 = 1$, the space $[\Gamma,\alpha,\beta,1]$ can be expressed as a direct sum of the subspaces $\mathscr{L}^{(1)}_{\alpha,\beta}$ and $\mathscr{L}^{(-1)}_{\alpha\beta}$ defined by

$$\mathscr{L}^{(\in)}_{\alpha\beta} = \{f|\Theta f = \in f, f\in[\Gamma,\alpha,\beta,1]\}(\in^2 = 1).$$

Let $f\in[\Gamma,\alpha,\beta,1]$ have a Fourier expansion of the type (3). Then, by
252 (14) of chapter 4, § 2, we have

$$\begin{aligned}\Theta f(\tau,\overline{\tau}) = a(0)u(y,q) + b(0) +& \\ +\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{k>0} a_{(-k)}W(2\pi ky;\alpha,\beta)e^{2\pi ikx}+& \\ +\frac{\Gamma(\alpha)}{\Gamma(\beta)}\sum_{k<0} a(-k)W(2\pi ky;\alpha,\beta)e^{2\pi ikx}.&\end{aligned}\tag{6}$$

If we further assume that f belongs to $\mathscr{L}^{(\in)}_{\alpha\beta}$, then $\Theta f = \in f$. Therefore, comparing the coefficients of the Fourier expansion (6) and that of f, we obtain

$$\begin{aligned}&\in a(0) = a(0), \in b(0) = b(0),\\ &\in a(k) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}a(-k)\quad (k>0)\end{aligned}\tag{7}$$

This shows that the Eisenstein series $G^*(\tau, \overline{\tau}; \alpha, \beta)$ introduced in the previous chapter belongs to $\mathscr{L}^{(1)}_{\alpha\beta}$. Since the term $\varphi^*_k(y)$ $(r = 2k)$ independent of x in the Fourier expansion of $G^*(\tau, \overline{\tau}; \alpha, \beta)$ under the assumption of theorem 39 never vanishes, it follows, by theorem 29, that with the help of $G^*(\tau, \overline{\tau}; \alpha, \beta)$, any modular form of the space $\mathscr{L}^{(1)}_{\alpha\beta}$ can be reduced to a cusp form. Moreover, the space $\mathscr{L}^{(-1)}_{\alpha\beta}$ consists of cusp forms only. It is an immediate consequence of theorem 39 that the operators $T(n)$ leave the spaces $\mathscr{L}^{(\in)}_{\alpha\beta}$ invariant.

In addition to the spaces $\mathscr{L}^{(\in)}_{\alpha\beta}$, we shall be interested also in the space $\mathscr{L}_\alpha$ consisting of analytic modular forms belonging to the space $[\Gamma, \alpha, 0, 1]$, where α is an even integer ≥ 4. With the help of the Eisentein series $G(\tau, \overline{\tau}; \alpha, 0)$, which belongs to the space $\mathscr{L}_\alpha$, every form of $\mathscr{L}_\alpha$ can be reduced to a cusp form. Moreover, the operators $T(n)$ leave the space **253**
$\mathscr{L}_\alpha$ invariant.

It follows immediately from the preceding results that the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \psi(s) = \sum_{n=1}^{\infty} \frac{a(-n)}{n^s}$$

associated to the forms of $\mathscr{L}^{(\in)}_{\alpha\beta}$ differ from each other only by a constant factor so that

$$f \to \varphi(s)$$

is a one-one invertible correspondence between modular forms and Dirichlet series except when $\alpha = \beta = 0$ or 1. □

Theorem 40. *The Eisenstein series $G^*(\tau, \overline{\tau}; \alpha, \beta)$ is an eigen-function of all operators $T(n)$ and*

$$(G^*(\quad , ; \alpha, \beta) | T(n))(\tau, \overline{\tau}) = d_{q-1}(n) G^*(\tau, \overline{\tau}; \alpha, \beta) \quad (n \geq 1).$$

Proof. Since we have seen already in the previous chapter that the m-th Fourier coefficient of the series $G^*(\tau, \overline{\tau}; \alpha, \beta)$ is, upto a constant factor, dependent only on $\operatorname{sgn} m$, equal to $d_{q-1}(m)$, it is obvious from (5) that the assertion of the theorem is equivalent with

$$d_{q-1}(n) d_{q-1}(m) = \sum_{\substack{d|(m,n) \\ d>0}} d^{q-1} d_{q-1}(mn/d^2) (m, n \geq 1). \tag{8}$$

The proof of theorem 37 shows that the term independent of x in the Fourier expansion of $(G^*(\ ,\ ;\alpha,\beta)|T(n))(tau,\overline{\tau})$ is obtained by multiplying $\varphi^*_k(y)$ with $d_{q-1}(n)$. Therefore our assertion (8) follows immediately from theorem 38.

254 The theorem above shows that in order to prove the decomposability of the linear spaces $\mathscr{L}^{(\in)}_{\alpha\beta}(\in^2= 1)$ and $\mathscr{L}_\alpha(\beta = 0)$ as direct sums of subspaces of dimension 1, which are invariant under the operators $T(n)$, it is sufficient to confine ourselves to the linear spaces of cusp forms which are invariant by the operators $T(n)$ and are contained in any one of the three above-mentioned spaces. We shall denote one such space by γ and prove with the help of Petersson's Metrisation Principle that it can be decomposed into subspaces of dimension 1 which are invariant under the operators $T(n)$.

Let Γ_0 be a subgroup of finite index in the modular group and let $\mathfrak{F}_0$ be a fundamental domain for Γ_0 consisting of a finite number of hyperbolic triangles. Let f and g be two modular forms belonging to the space $[\Gamma_0,\alpha,\beta,1]$ such that at least one of them is a cusp form. Then we define as in chapter 3, § 3, the scalar product of f and g by

$$(f,g) = \frac{1}{\Im(\mathfrak{F}_0)}\iint\limits_{\mathfrak{F}_0} f\overline{g}y^{p-2}dxdy \quad (p = \operatorname{Re} q = \operatorname{Re}(\alpha+\beta)),$$

where $\Im(\mathfrak{F}_0)$ denotes the hyperbolic area of $\mathfrak{F}_0$. In the same way as for the analytic modular forms, it can be proved that the scalar product is independent of the choice of a fundamental domain and does not depend upon the group Γ_0 in the sense described before. □

Theorem 41. *The operators $T(n)$ acting on the space $[\Gamma,\alpha,\beta,1]$ are Hermitian operators i.e.*

$$(f|T(n),g) = (f,g|T(n)),$$

255 *for any two cusp forms f and g belonging to the space $[\Gamma,\alpha,\beta,1]$.*

Proof. For $S\in\mathscr{O}_n$, we have

$$\Gamma[n]\subset S\Gamma S^{-1}. \tag{9}$$

Since the principal congruence subgroups are normal subgroups of Γ and $S = L_1 \begin{pmatrix} g & 0 \\ 0 & gd \end{pmatrix} L_2$ with $L_i \in \Gamma (i = 1, 2)$ and $g^2 d = n$, it suffices to prove (9) for $S = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, where d is a divisor of n. If $L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a matrix belonging to $\Gamma[n]$, then the matrix

$$S^{-1} L S = \begin{pmatrix} \alpha & \beta d \\ \gamma d^{-1} & \delta \end{pmatrix}$$

obviously belongs to Γ. Thus assertion (9) is proved. Since the matrix nS^{-1} along with S is in $\mathcal{O}_n$, relation (9) remains true when S is replaced by S^{-1}.

Let f and g be two modular forms belonging to the space $[\Gamma, \alpha, \beta, 1]$. Then it is an immediate consequence of (9) that, for $S \in \mathcal{O}_n$, the forms $f|S$ and g are modular forms for the groups $S^{-1}\Gamma[n]S$ and $\Gamma[n]$. Moreover, it is obvious that, when it exists, the scalar product of $f|S$ and g defined for either of the two groups is the same. If $\mathfrak{F}_n$ is a fundamental domain for $\Gamma[n]$, then $S^{-1} < \mathfrak{F}_n >$ is a fundamental domain for the group $S^{-1}\Gamma[n]S$. Let us set $S = \sqrt{n} S^* (|S^*| = 1)$ and $p = \operatorname{Re} q$. Then

$$\begin{aligned} (f|S, g) &= \frac{1}{\Im(\mathfrak{F}_n)} \iint\limits_{S^{-1} < \mathfrak{F}_n >} f|S \cdot \overline{g} y^{p-2} dx dy \\ &= \frac{n^{-q/2}}{\Im(\mathfrak{F}_n)} \iint\limits_{S^{-1} < \mathfrak{F}_n >} f|S^* \cdot \overline{g} y^{p-2} dx dy. \end{aligned}$$

Since the substitution $\tau \to S^{*-1} < \tau >$ with $S^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms the **256**
function $(f|S^*)(\tau, \overline{\tau})\overline{g(\tau, \overline{\tau})} y^p$ into the function

$$\begin{aligned} &\frac{f(\tau, \overline{\tau})\overline{g(S^{*-1} < \tau >, S^{*-1} < \overline{\tau} >)} y^p}{(cS^{*-1} < \tau > + d)^\alpha (cS^{*-1} < \overline{\tau} > + d)^\beta |(-c\tau + a)^\alpha (-c\overline{\tau} + a)^\beta|^2} \\ &= f(\tau, \overline{\tau})\overline{(g|S^{*-1})(\tau, \overline{\tau})} y^p \end{aligned}$$

and leaves the measure $\dfrac{dx dy}{y^2}$ invariant, it follows that

$$(f|S, g) = \frac{n^{-q/2}}{\Im(\mathfrak{F}_n)} \iint\limits_{\mathfrak{F}_n} \overline{f \cdot g|S^{*-1}} y^{p-2} dx dy$$

$$= \frac{1}{\mathfrak{I}(\mathfrak{F}_n)} \iint\limits_{\mathfrak{F}_n} \overline{f \cdot g|nS^{-1}} y^{p-2} dxdy$$
$$= (f, g|nS^{-1}).$$

Let V_n denote a common representative system of left and right cosets of O_n modulo Γ; such a representative system exists, by lemma 12. By the definition of the operator $T(n)$, we have

$$(f|T(n), g) = n^{q-1} \sum_{S \in V_n} (f|S, g) = n^{q-1} \sum_{S \in V_n} (f, g|nS^{-1})$$
$$= (f, g|T(n)),$$

because it can be seen easily that whenever S runs through a representative system of right cosets of O_n modulo Γ, then nS^{-1} runs through the left cosets of O_n. Hence the theorem is proved. □

Lemma 13. *Let $\mathfrak{m}$ be a set of pairwise commuting Hermitian matrices. Then there exists a unitary matrix U such that the matrix $\overline{U}'HU =$*
257 *$U^{-1}HU$ for every H in $\mathfrak{m}$ is a diagonal matrix.*

Proof. It suffices to prove the assertion for a finite set $H_1, H_2, \ldots, H_r$ of Hermitian matrices, because there exist in $\mathfrak{m}$ only finitely many linearly independent matrices over the field of real numbers. We shall prove the assertion by induction on r. It is well-known that the assertion is true for $r = 1$. Let us assume $r > 1$ and the assertion to be true for any set of $r - 1$ mutually commuting Hermitian matrices. Since $\overline{U}'HU$ is Hermitian along with H and since the unitary matrices form a group, we can assume, without loss of generality, that $H_1, H_2, \ldots, H_{r-1}$ are already diagonal matrices. We can also assume that the matrix $H = \sum_{k=1}^{r-1} x_k H_k$ with x_k as real variables has the form

$$H = \begin{pmatrix} \ell_1 E^{(k_1)} & & & & 0 \\ & \ell_2 E^{(k_2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \ell_t E^{(k_t)} \end{pmatrix},$$

where $\ell_1, \ell_2, \ldots, \ell_t$ are pairwise distinct linear forms in the variables x_k and in general $E^{(k)}$ denotes the k-rowed unit matrix. From $HH_r = H_r H$, it follows that H_r must be of the form

$$H_r = \begin{pmatrix} A_1^{(k_1)} & & & & 0 \\ & A_2^{(k_2)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & A_t^{(k_t)} \end{pmatrix}$$

where each $A^{(k_i)}$ is a Hermitian matrix. We now find unitary matrices U_i so that $\bar{U}_i' A_i^{(k_i)} U_i (i = 1, 2, \ldots t)$ are diagonal matrices. Let us set **258**

$$U = \begin{pmatrix} U_1 & & & & & 0 \\ & U_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & U_t \end{pmatrix}.$$

Then it is obvious that U transforms the matrices $H_1, H_2, \ldots, H_r$ into diagonal matrices. $\square$

Theorem 42. *A linear space γ of cusp forms contained in any one of the spaces $\mathscr{L}_{\alpha\beta}^{(\epsilon)}(\epsilon^2 = 1)$ and $\mathscr{L}_\alpha(\beta = 0)$ and invariant under the operators $T(n)$ has an orthonormal basis $g_1, g_2, \ldots, g_t$ (t = dimension γ) so that*

$$g_i | T_n = \rho_i(n) g_i$$

for all $n \geq 1$ and $i = 1, 2, \ldots, t$.

Proof. Let $\{f_1, f_2, \ldots, f_t\}$ be an orthonormal basis for γ i.e.

$$(f_i, f_k) = \delta_{ik} \quad (\delta_{ik}, \text{ Kronecker's symbol}).$$

Let

$$f_i | T_n = \sum_{k=1}^{t} \lambda_{ik} f_k.$$

Then, by theorem 41, we have

$$\begin{aligned}\lambda_{ik}(n) &= (f_i|T_n, f_k) = \overline{(f_k, f_i|T_n)}\\ &= \overline{(f_k|T_n, f_i)} = \overline{\lambda_{ki}(n)}\end{aligned}$$

showing that the matrix $\Lambda(n) = (\lambda_{ik}(n))$ is Hermitian. But, by theorem 38, the operators $T(n)$ commute with each other; therefore, the matrices $\Lambda(n)$, which define a representation of the operators $T(n)$, com-
259 mute pairwise. Thus, by lemma 13, there exists a unitary matrix U such that

$$U^{-1}\Lambda(n)U = \begin{pmatrix} \rho_1(n) & & & & & 0 \\ & \rho_2^{(n)} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \rho_t(n) \end{pmatrix}$$

If

$$\begin{pmatrix} f_1 \\ \vdots \\ f_t \end{pmatrix} = U \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix},$$

then $g_1, g_2, \ldots, g_t$ constitute a basis of γ having the properties mentioned in theorem 42.

Denoting the Fourier coefficients of g_i by $a_i(m)$ with $m \geq 1$, we claim that for every $n \geq 1$, $\rho_i(n)$ is equal to $a_i(n)$ upto a constant factor independent of n. In fact, from (5), we have

$$\rho_i(n)a_i(m) = \sum_{\substack{d|(m,n)\\ d>0}} a_i(mn/d^2)d^{q-1}(m, n \geq 1)$$

and on taking $m = 1$,

$$\rho_i(n)a_i(1) = a_i(n)n = 1, 2, \ldots$$

Consequently $a_i(1) \neq 0$, for $i = 1, 2, \ldots, t$. Moreover, it follows immediately that

$$\rho_i(n)\rho_i(m) = \sum_{\substack{d|(m,n)\\ d>0}} \rho_i(mn/d^2)d^{q-1} \tag{10}$$

If we drop the condition of orthonormality for the basis $\{g_1, g_2, \ldots, g_t\}$ **260**
of γ and assume only the orthogonality with $a_i(1) = 1$, then the Dirichlet series

$$\varphi_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}$$

associated with $g_i(s)$ $(i = 1, 2, \ldots, t)$ has an Euler product development of the type

$$\varphi_i(s) = \prod_p \{1 - a_i(p)p^{-s} + p^{q-1-2s}\}^{-1},$$

where p runs over all primes. For, it can be seen, by using (10), that

$$\sum_{k=0}^{\infty} \frac{a_i(p^k)}{p^{ks}} \{1 - a_i(p)p^{-s} + p^{q-1-2s}\} = 1$$

and

$$\prod_p \{\sum_{k=0}^{\infty} \frac{a_i(p^k)}{p^{ks}}\} = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s}.$$

Thus it follows, from theorem 40, that the Dirichlet series corresponding to the normalised Eisenstein series (i.e. with $a_i(1) = 1$) has an Euler product development i.e.

$$\sum_{n=1}^{\infty} \frac{d_{q-1}(n)}{n^s} = \prod_p \{1 - d_{q-1}(p)p^{-s} + p^{q-1-2s}\}^{-1} = \zeta(s)\zeta(s + 1 - q).$$

Consequently, we have proved the following □

Theorem 43. *For every linear space* $\mathscr{L}_{\alpha\beta}^{(\epsilon)}(\epsilon^2 = 1)$ *and* $\mathscr{L}_\alpha(\beta = 0)$ *there exists a basis* $\{h_1, \ldots, h_t\}$ *so that the Dirichlet series corresponding to* h_i *can be represented as an Euler product.*

Finally, we remark that the functions mentioned in theorem 36 have **261**
an Euler product development which can be obtained using the well-known product representation

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1} v$$

for the functions $\zeta(s)$ and $L(s, \chi)$. In case $\lambda = 2$, these Euler products coincide with the Euler products of the Dirichlet series corresponding to certain modular forms of level 4 as for as the contributions of the odd primes are concerned, because the space $[\Gamma_\vartheta, \alpha, \alpha, v](v^2 = 1)$ is contained in the space $[\Gamma[4], \alpha, \alpha, 1]$.

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