Lectures on
The Fourteenth Problem of Hilbert

By

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Chapter 0

In 1900, at the international Congress of Mathematicians in Paris, 1
Hilbert posed twenty-three problems. His complete address was published in Archiv.f. Math.U.Phys.(3),1,(1901) 44-63,213-237 (one can also find it in Hilbert's Gesammelte Werke).

The fourteenth problem may be formulated as follows: *The Fourteenth Problems. Let K be a field and* $x_1, \ldots, x_n$ *algebraically independents elements over* K. *Let L be a subfield of* $K(x_1, \ldots, x_n)$ *containing* K. *Is the ring* $K\,[x_1, \ldots, x_n] \bigcap L$ *finitely generated over* K*?*

The motivation for this problem is the following special case, connected with theory of invariants.

The Original fourteenth problem. Let K be a field and G a subgroup of the full linear group $GL(n, K)$. *Then G acts as a group of automorphisms of* $K[x_1, \ldots, x_n]$. *Let* I_G *be the ring of elements of* $K[x_1, \ldots, x_n]$. *invariant under* G. *Is* I_G *finitely generated over* K*?*

Contributions to the original fourteenth problem were made in particular cases. In fact it was proved that I_G is finitely generated in the following cases.

1. K is the complex number field and $G = SL(n, K)$ acting by means of its tensor representations. (D. Hilbert Math. Ann. 36(1890) 473 - 534).

2. K is the complex number field and G satisfies the following condition: there exists a conjugate G^* of G such that $A \in G^* \Rightarrow {}^t\overline{A} \in G^*$, where '$-$' and '$t$' indicate complex conjugation and transpose respectively (E. Fischer Crelle Journal 140(1911) 48 - 81). 2

3. K is an arbitrary field and G a finite group (E.Noether , for characteristic $K = 0$, Math. Ann. 77 (1916) 89 - 92, for characteristic $K \neq 0$ Göttinger Nachr. (1926 28-35).

4. K is the complex number field and G, a one parameter group (R. Weitzerböck Acta Math. 58 (1932) 231 - 293).

5. K is the complex number field and G, a connected semi-simple Lie group (H. Weyl Classical groups (1936) Princeton Univ. Press).

The next significant contribution was made by 0.Zariski in 1953. He generalized (Bull. Sci., Math.78(1954) 155-168) the fourteenth problem in the following way.

Problem of Zariski. Let K be a field and K $[a_1, \ldots, a_n]$ *an affine normal domain (i.e.a finitely generated integrally closed domain over K). Let L be a subfield of K* $(a_1, \ldots, a_n)$ *containing K. Is the ring K* $[a_1, \ldots, a_n] \bigcap L$ *finitely generated over K?*

He answered the question in the affirmative when trans. $\deg_K L \leq 2$. Later, in 1957, D. Rees (Illinois J. of Math. 2(1958) 145 - 149) gave a counter example to the problem of Zariski when trans. $\deg_K L = 3$.

Finally, in 1958, Nagata (Proceedings of the International Congress of Mathematicians, Edinburgh (1958) 459 - 462) gave a counter example to the original fourteenth problem itself. This counter example was in the case when trans. $\deg_K$L= 13. Then, in 1959, Nagata (Amer. J. Math. 91, 3(1959) 766 - 772) gave another counter example in the case when trans. $\deg_K L = 4$.

3 The groups occurring in these examples are commutative. So in view of Weyl's result we seek the answer to the original fourteenth problem in the case when G is a non- comutative, non-semi-simple Lie group. The examples mentioned above May be made to yield one with G non-commutative by considering what is essentially the direct product by a non-commutative group. More interesting is the case when G is a connected Lie group such that $[G, G] = G$. Even in this case the answer is in the negative as we shall see later in this course of lectures.

Chapter 1

A generalisation of the original fourteenth problem

1. We first generalise the original fourteenth problem in the follow- 4
ing way: *Generalised fourteenth problem. Let* K *be a field. Let* $R = K[a_1, \ldots, a_n]$ *be a finitely generated ring over* K *(*R *need not be an integral domain). Let* G *be a group of automorphism of* R *over* K*. Assume that for every* $f \in R$*,* $\sum_{g \in G} f^g K$ *is a finite dimensional vector space over* K*. Let* I_G *be the ring of invariants of* R *under* G*. Is* I_G *finitely generated over* K*?*

Remark 1. When $a_1, \ldots, a_n$ are algebraically independent and G is a subgroup of $GL(n, k)$ acting on R in the 'usual way', elements of R satisfy the finiteness condition we have imposed in the problem.

Remark 2. The omission of the assumption that $a_1, \ldots, a_n$ are algebraically independent is helpful. For instance let N be normal subgroup of G such that I_N is finitely generated. Then G acts on I_N, as N is normal in G. Hence G/N acts on I_N and $(I_N)_{G/N} = I_G$. So, for instance if we know that our generalized problem is true for (i) finite group (ii) connected semi-simple Lie groups, (iii) diagonalizable groups, then we can get immediately that: if G is an algebraic group whose radical N is diagonalizable then the answer to generalized problem is in the affirmative. Note that I_N need not be polynomial ring even if $a_1, \ldots, a_n$ are algebraically independent.

5 **Remark 3.** We do not assume R is an integral domain, as the constant field extension (of §3 below) of a finitely generated integral domain over a field K need not be an integral domain.

2. *Algebraic linear groups.* In this section by an affine variety we mean the set of rational points over K of a certain affine variety with points in an universal domain. We shall give Zariski topology to an algebraic variety. The group $GL(n,k)$, being the complement of the hypersurface defined by $\det(x_{ij}) = 0$ in K^{n^2}, is an affine variety (in fact it is isomorphic to the hypersurface $z\det(x_{ij}) = 1$ in K^{n^2+1}). The group operation of $GL(n,K)$ are regular i.e. $GL(n,K)$ is an algebraic group. In this course of lectures by an algebraic group we always mean closed subgroups of $GL(n,k)$. We define connectedness and irreducibility under Zariski topology. In the case of algebraic groups these two concepts coincide. If G_o is the connected component of an algebraic group G, then G_o is a normal subgroup of finite index in G. An algebraic group contains the largest connected normal solvable subgroup of G, which we call the *radical* of G. For details about algebraic groups we refer to seminaire C. Chevalley Vol.1 (1956-1958) and A. Borol Groups Linéaires Algébriques, Annals of Mathematics, 64 (1956). We remark that in the generalized problem we may assume that G is a subgroup of $GL(m,K)$ for some m. Because of the finiteness condition on the elements of $R = K[a_1, \ldots, a_n]$, $V = \sum\limits_{i=1}^{n} \sum\limits_{g\in G} a_i^g K$ is finite dimensional vector space. We choose a basis $b_1, \ldots, b_m$ for V. Then $R = K[b_1, \ldots, b_m]$
6 and G is a group of automorphisms of V. Therefore in the generalised problem we may assume $G \subseteq GL(n,k)$. From now on, whenever we say that a subgroup G of $GL(n,K)$ acts on $R = K[a_1, \ldots, a_n]$ we mean that it acts linearly on the vector space $\sum\limits_{i=1}^{n} Ka_i$. The following theorem shows that in the problem we may assume that G is an algebraic group

Theorem 1. *Let $R = K[a_1, \ldots, a_n]$ and let $G \subseteq G(n,k)$ act as a group of automorphisms of R over K. Let G^* be the closure of G in $GL(n,K)$.*

Then G^* acts as a group of automorphisms of R over K and $I_G = I_{G^*}$.

This theorem is a consequence of the following lemma (put $S = I_G$).

Lemma. *Let $S \subseteq R$ and let H be the set of elements of $GL(n,k)$ which induce automorphisms of R over K and leave every $s \in S$ invariant. Then H is algebraic.*

Proof. Let H_1 denote the set of elements of $GL(n,K)$ which induce automorphisms of R over K. Let $\mathcal{U}$ be the ideal of the ring of polynomials $K[x_1,\ldots,x_n]$ such that $K[a_1,\ldots,a_n] \approx K[x_1,\ldots,x_n]/\mathcal{U}$. Now $g \in H_1$ if and only if $\mathcal{U}^g = \mathcal{U}$. Let $f_1,\ldots,f_m$ generate $\mathcal{U}$. We may assume that $f_1,\ldots,f_m$ are linearly independent over K. Extend $f_1,\ldots,f_m$ to a linearly independent basis $f_1,\ldots,f_m, f_{m+1},\ldots,f_l$ of the vector space

$$V = \sum_{i=1}^{n} \sum_{g\in GL(n,k)} f_i^g K. \quad \text{Let} f_i^t = \sum_{j=1}^{l} \lambda_{ij} f_j, i = 1,\ldots,m,$$

where $t = (t_{rs}), t_{rs}$ are indeterminates. Then λ_{ij} are polynomials in t_{rs}. The condition $\mathcal{U}^g = \mathcal{U}$ is equivalent to □

$$\lambda_{ij}(g) = 0, i = 1,\ldots,m, j = m+1,\ldots,l.$$

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Hence H_1 is algebraic.

Let now $H = \left\{g \in H_1 \mid s^g = s,\, s \in S\right\}$.

It is enough to prove the lemma when S consists of a single element s. Let $s, s_1,\ldots,s_k$ be a linearly independent basis of the vector space $\sum_{h\in H_1} s^h K$. Let $y = (y_{rs})$ be a generic points or may component of H_1 over K. We have $s^y = \lambda_0 s + \sum_{i=1}^{k} \lambda_i s_i$, where λ_i are polynomials in y_{rs}. The condition $s^y = s$ is equivalent to $\lambda_\circ(s) = 1$, $\lambda_i(g) = 0$ for $i > 0$. Hence H is algebraic.

3. *Constant field extension.* We shall now study the behaviour of invariants under the constant field extensions. Let R' and R'' be two commutative rings containing a field K. Let G' (respectively G'') act on R' (respectively R'') as a group of automorphisms over K. Then the direct product $G' \times G''$ acts on $R \otimes_K R''$ as a group of automorphisms; in fact we have only to define $(a' \otimes a'')(g', g'') = a'^{g'} \otimes a''^{g''}$, $a' \in R'$, $a'' \in R''$, $(g', g'') \in G' \times G''$.

Lemma 1. *Let $I_{G' \times G''}$ denote the ring of elements of $R' \otimes_K R''$ invariant under $G' \times G''$ Let I'_G (respectively $I''_{G''}$) denote the ring of elements of R' (respectively R'') invariant under G' (respectively G''). Then $I_{G' \times G''} = I_{G'} \otimes_K I''_{G''}$.*

Proof. We have only to prove that $I_{G' \times G''} \subseteq I'_{G'} \bigotimes_K I''_{G''}$. Choose a linearly independent basis $(i'_{\lambda'})_{\lambda' \in \Lambda'}$ and $(i''_{\lambda''})_{\lambda'' \in \Lambda''}$ for the vector spaces $I'_{G'}$
8 and $I''_{g''}$ respectively over K. Extend these bases to a linearly independent bases of R' and R'' respectively, say

$$R' = \sum_{\mu' \in M'} i'_{\mu'} K, R'' = \sum_{\mu'' \in N''} i''_{\mu''} K \text{ with } \Lambda' \subseteq M' \text{ and } \Lambda'' \subseteq M''.$$

Let $f = \sum_{(\mu', \mu'') \in M' \times M''} k_{\mu'\mu''} \ (i_\mu \otimes i_{\mu''}) \in I_{G' \times G''}, k_{\mu'\mu''} \in K$.

For every $g \in G$, we have

$$f^{g'} = \sum_{(\mu', \mu'')} k_{\mu'\mu''} (i^{g'}_{\mu} \otimes i''_{\mu'}) = \sum_{\mu''} (\sum_{\mu'} k_{\mu'\mu''} i^{g'}_{\mu}) \otimes i''_{\mu''} = f.$$

Hence

$$\sum_{\mu'} k_{\mu'\mu''} i'^{g'}_{\mu'} = \sum_{\mu'} k_{\mu'\mu''} i'_{\mu'}, g' \in G', \mu'' \in M''.$$

Hence $k_{\mu'\mu''} = 0$, for $= \mu' \notin \Lambda'$. Similarly $k_{\mu'\mu''} = 0$, for $\mu'' \notin \Lambda''$. i.e. $f \in I'_{G'} \otimes_K L''_{G''}$ and the lemma is proved. □

Lemma 2. *With the above notation, $R' \otimes_K R''$ is finitely generated over K if and only if R' and R'' are finitely generated over K.*

Proof. It is clear that if R' and R'' are finitely generated over K, so is $R' \otimes_K R$. Now let $R' \otimes_K R$. be finitely generated over K, say $f_i = \sum_j \gamma'_{ij} \otimes \gamma''_{ij}$, $i = 1, \ldots, l$ are the generators. Then $R' = K\left[\gamma'_{ij}\right]$ and $R' = K\left[\gamma''_{ij}\right]$. □

Let $R = K[a_1, \ldots, a_n]$ and let G act on R as as group of automorphisms of R over K. Let K' be a filed containing K. Then G acts on $K' \otimes_K R$ which we denote by $K'[a_1, \ldots, a_n]$. Let $I'_{G'}$ be the invariant elements of $K' \otimes_K R$ under G. Then in Lemma 1 if we put $R' = K'$, $R'' = R, G' = \{1\}$, $G'' = G$, we get $I'_G = K' \otimes_K I_G$. As in Lemma 2 it **9**
follows that:

Proposition 1. *I'_G is finitely generated over K' if and only if I_G is finitely generated over K.*

The above proposition helps us to confine ourselves to a smaller field when K is 'too big'. Let G' be a subgroup of G such that G' is dense in the closure $\bar{G}$ of G. For instance when K is of characteristic zero or $\bar{G}$ is a torus group we can take G' to be finitely generated. When K is of characteristic $p \neq 0$ we may take G' to be countably generated. Let $K[x_1, \ldots, x_n]/\mathcal{U} \approx K[a_1, \ldots, a_n]$ with the $x_1, \ldots, x_n$ algebraically independent over K. Let K' be a subfield of K such that elements of G' are K'-rational and $\mathcal{U}$ is defined over K' For instance if we can choose G' finitely generated, K' can be chosen to be finitely generated over the prime field. If G' can be chosen countably generated, then K' can be choosen to be countably generated over the prime field. Now as the ideal $\mathcal{U}$ is defined over $K', K \otimes_{K'}$, $K'[a_1, \ldots, a_n] = K[a_1, \ldots, a_n]$. Hence by proposition 1, I_G is finitely generated over K is and only if $G'-$ invariants in K' $[a_1, \ldots, a_n]$ are finitely generated.

Of course, as we noted in §2, we can enlarge G' to an algebraic group.

3. Invariants of a finite group. We shall now consider the original 14th problems (in fact the generalised problem) when G is finite.

Theorem 2 (E. Noether)**.** *Let $R = K[a_1, \ldots, a_n]$ and G a finite group acting on R as a group of automorphisms of R over K. Then I_G is finitely* **10**
generated over K.

Proof. Let $G = \{1 = g_1, \ldots, g_h\}$. Let $a \in R$. Set

$$S_1 = \sum_{i=1}^{h} a^{g_i}, S_2 = \sum_{i<j} a^{g_i} a^{g_j}, \ldots, S_h = a^{g_1} \ldots a^{g_n}.$$

Then $S_i \in I_G$, $i = 1, \ldots, h$ and $a^h - S_1 a^{h-1} + \cdots + (-1)^h S_h = 0$. ◻

Hence R is integral over I_G. Now the theorem follows from the following Lemma:

Lemma. *Let $R = K[a_1, \ldots, a_n]$ and let S be subring of R containing K such that R is integral over S. Then S is finitely generated over K.*

Proof. There exist $G_{ij} \in S$, $1 \leq i \leq n$, $0 \leq j \leq m_i - 1$ such that

$$a_i^{m_1} + C_{im_i-1} a_i^{m_i-1} + \ldots + C_{io} = 0.$$

◻

Set $S' = K\left[C_{ij}\right]_{\substack{1 \leq i \leq n \\ 0 \leq j \leq n_i - 1}}$. Then R is finite S' module. As S' is noetherian, s is also finite S'-module. Hence S is finitely generated over K.

Corollary. *Let G_0 be a normal subgroup of G (G not necessarily finite) of finite index. Let G act on R as a group of automorphisms over K. If I_{G_0} is finitely generated over K, then, so is I_G.*

Proof. The group G/G_0 acts on I_{G_0} and $(I_{G_0})_{G/G_0} = I_G$ ◻

Remark. Suppose that R (of the above corollary) is an integral domain
11 Then the converse of the above Corollary is also true. Let $K_0 L$ be the quotient of I_{G_0} and I_G respectively. Then K_0 is a finite separable algebraic extension of L. Let I_G be finitely generated. As G/G_0 is finite, I_{G_0} is integral over I_G. Hence I_{G_0} is a finite I_G-module and therefore finitely generated over K.

Chapter 2

A generalization of the original fourteenth problem(contd.)

1. In this chapter we shall consider the generalized problems with certain assumption on the representation of G. In fact we shall prove the following: 12

Theorem 1. *Let $R = K[a_1, \ldots, a_n]$ and let $G \subseteq GL(n, K)$ act on R as a group of automorphisms of R over K. Suppose G satisfies the following condition: $(*)$ If λ is a representation of G by a finite dimensional $K(G)$-module $M \subset R$ and* $\lambda = \begin{pmatrix} 1 & *\ldots* \\ 0 & \\ \vdots & \lambda' \\ 0 & \end{pmatrix}$, *then λ is equivalent to* $\begin{pmatrix} 1, & 0\ldots0 \\ 0 & \\ : & \lambda' \\ 0 & \end{pmatrix}$. *Then I_G is finitely generated. We shall prove theorem in several steps.*

Lemma 1. *Let G satisfy the condition $(*)$ and let $f \in R$. Then there exists an $f^* \in I_G \bigcap (\sum\limits_g f^g K)$ such that $f - f^* \in \sum\limits_{g,g'} (f^g - f^{g'})K$.*

Proof. Let $M = \sum\limits_g f^g K$ and $N = \sum\limits_{g,g'} (f^g - f^{g'})K$. The vector spaces M and N are G-modules. Let the dimension of M be m. We shall prove

the lemma by induction on m. Suppose that for all $h \in R$ such that dimension of $\sum_g h^g K$ is $< m$, there exist $h^* \in I_G \bigcap (\sum_g h^g K)$ with $h - h^* \in \sum_{g,g'} (h^g - h^{g'})K$. The assertion is trivial when $m = 0$. Further if $f \in N$, we may take $f^* = 0$. Suppose $f' \notin N$. Then $M = Kf + N$ and f is G-invariant module N. Hence by the condition $(*)$ there exists a G-Module
13 N^* of dimension 1 in M and an $f' \in I_G$ with $M = Kf' + N^*$. If $f' \notin N$, then $M = Kf' + N$ and the lemma is proved. Suppose $f' \in N$. Set $f = \lambda f' + h$ with $\lambda \in K$, $h \in N^*$. Since $M_1 = \sum_g h^g K \subset N^*$, we have $\dim_K M_1 \leqslant \dim N^* = m - 1$. Hence by induction hypothesis there exists an $h^* \in I_G \bigcap (\sum_g h^g K)$ such that $h - h^* \in N_1 = \sum_{g,g_1} (h^g - h^{g1})K$. Since $N_1 \subseteq N$, the proof of Lemma 1 is complete with $f^* = h^*$. □

Remark. If R is a graded ring and $f \in R$ is homogeneous, the representations of G which occur in the above proof are all given by G-modules generated by homogeneous elements of the same degree.

Proposition 1. *Let R and G be as in Theorem 1 and let R' be a ring containing K. Let G act on R' as a group of automorphisms of R' over K. Assume that there is a surjective homomorphism φ of R onto R' such that $\varphi(a^g) = \varphi(a)^g$ for all $a \in R$. Let I'_G denote the G-invariant elements of R'. Then $\varphi(I_G) = I'_G$.*

Proof. We have only to show that $I'_G \subseteq \varphi(I_G)$. Let $f' \in I'_G$. Let $f \in R$ be such that $\varphi(f) = f'$. By Lemma 1 there exists an $f^* \in I_G$ such that $f - f^* \in N = \sum_{g,g'} (f^g - f^{g'})K$. But as f' is G-invariant, N is contained in the kernel of φ. Hence $\varphi(f^*) = f'$. Hence $\varphi(I_G) = I'_G$. □

We shall now prove Theorem 1 in the case when R is a graded ring i.e. $R = K[a_1, \ldots, a_n] \approx K[x_1, \ldots, x_n]/\mathcal{U}$ with $x_1, \ldots, x_n$ algebraically independent over K and $\mathcal{U}$, a homogeneous ideal of $K[x_1, \ldots x_n]$.

14 **Lemma 2.** *Let $S = \sum_{1=0}^{\infty} S_i$ be a graded ring. Assume that the ideal $\sum_{i>0} S_i$ have a finite basis. Them S is finitely generated over S_0.*

Proof. Let h_i, $i = 1, \ldots, r$ be a set of generators for I. We may assume that h_i are all homogeneous, say $h_i \in S_{j(i)}$, $j(i) > 0$. Then we assert that $S = S_0[h_1, \ldots, h_r]$. It is enough to prove that every homogeneous element f of S is in $S_0[h_1, \ldots, h_r]$. The proof is by induction on the degree i of f. For $i = 0$, there is nothing to prove. Assume $i > 0$. Suppose that for all $t < i$, $S_t \subseteq S_0[h_1, \ldots, h_r]$. We have $f = \sum_{k=1}^{r} g_k h_k$, $g_k \in S_{i-j}(i)$. By induction hypothesis $g_k \in S_0[h_1, \ldots, h_r]$, $1 \leq k \leq r$. Hence $f \in S_0[h_1, \ldots, h_r]$. □

Lemma 3. *Let R and G be as in Theorem 1. Let $f0_1, \ldots, f_r \in I_G$. Then* $(\sum_{i=1}^{r} f_i R) \bigcap I_G = \sum_{i=1}^{r} I_G f_i$.

Proof. The proof is by induction on r. For $r = 0$ there is nothing to prove. Assume $(\sum_{i=1}^{s} f_i R) \bigcap I_G = \sum_{i=1}^{s} f_i I_G$ for $s < r$. Let $f \in (\sum_{i=1}^{r} f_i R) \bigcap I_G$. Then $f = \sum_{i-1}^{i} h_i f_i$, $h_i \in R$. By Lemma 1 there exists an $h' \in N = \sum_{g,g'} (h_r^g - h_r^{g'})X$ such that $h_r + h' \in I_G$. As, for $g, g' \in G$, $\sum_{i=i}^{r} \sum (h_i^g - h_i^{g'}) f_i = 0$, there exist $h_i' \in R$, $1 \leq i \leq r-1$ with $\sum_{i=i}^{r-1} h_i' f_i + h' f_r = 0$. Hence $f - (h_r + h') f_r = \sum_{i=1}^{r-1} (h_i + h_i') f_i$.

But $f - (h_r + h') f_r \in I_G$ and therefore by induction hypothesis, there exist $h_i'' \in I_G$, $1 \leq i \leq r-1$ such that $\sum_{i=1}^{r-1} (h_i + h_i') f_i = \sum_{i=1}^{r-1} h_1'' f_i$.

The proof of Lemma 3 is complete. □ **15**

Remark. If R is graded and f_i are homogeneous, it is enough to assume the condition (*) for representations of G given by G-modules generated by homogeneous elements of R of the same degree.

Proposition 2. *Let $R = K[a_1, \ldots, a_n]$ be a graded ring. Assume that every representation λ of G given by a G-module generated by homoge-*

neous elements of R of the same degree and of the form $\lambda = \begin{pmatrix} 1 & * \cdots * \\ 0 & \\ \vdots & \lambda' \\ 0 & \end{pmatrix}$ *is equivalent to* $\begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & \\ \vdots & \lambda' \\ 0 & \end{pmatrix}$. *Then I_G is finitely generated over K.*

Proof. As R is graded, so is I_G. Let I be ideal of I_G of all elements of positive degree. □

As R is noetherian, the ideal R, I of R is finitely generated. Let $f_1, \ldots, f_r, \in, I$ generate R, I. We may assume the f_i to be homogeneous. By Lemma 3 and the remark following it we have $(\sum_{i=1}^{r}, R f_i) \bigcap I_G = \sum_{i=i}^{r} I_G f_i = I$. Hence by Lemma 2, I_G is finitely generated over K.

We shall now consider the non-graded case. Let R = K $[a_1, \ldots, a_n]$. Let be a transcendental element over R. Consider the homogeneous ring $R^* = K[a_1 t, \ldots, a_n t, t]$. Then G acts on R^* if we set $t^g = t$, for every $g \in G$. Further the G-module homomorphism $\varphi : R^* \to R$ defined by $\varphi(f(a_1 t, \ldots a_n t, t) = f(a_1 \ldots, a_n, 1)$ induces an isomorphism of a finite G-submodule of R^* generated by homogeneous elements of the same degree of R^* onto a finite G-submodule of R. Hence the condition
16 (*) is satisfied by all representation λ^* of G given by G-submodules of R^* generated by homogeneous elements of the same degree. Hence by proposition 2, L_G^* the ring of G-invariant elements of R^* is finitely generated over K. Further as every $f \in R$ is the image of a homogeneous in R^*, it follows from Proposition 1 and the remark after Lemma 1 that $\varphi(I_G^*) = I_G$. Hence I_G is finitely generated over K.

2. We now give examples where the condition (*) is satisfied. It is obvious that if every rational representation λ of G is completely reducible then it satisfies the condition (*). We shall later give some criteria of complete reducibility of rational representation of an algebraic group.

(1) Consider a torus group G (i.e. a connected algebraic linear group which is diagonalizable), acting on $R = K[a_1, \ldots, a_n]$. In this case if G is diagonalized each $K a_1^{i_1} \ldots a_n^{i_n}$ is a G-module. Thus R is the direct sum of simple G-modules. Hence every representation λ of G given by a G-sub-module of R is completely reducible. Hence L_G is finitely generated.

(2) Let K be a field of characteristic zero and let $G \subseteq GL(n, K)$ be semi-simple. Then it is well known that (see, Chevalley, Theorie des groupes de Lie, III) every rational representation of G is completely reducible. Hence in this case again I_G is finitely generated.

(3) Combining (1) and (2) we have : If G is an algebraic group whose radical is a torus group, then I_G is finitely generated.

We add here a Corollary to Theorem 1:

Corollary. *Assume that K is of characteristic zero and that an algebraic* 17
group $G \subseteq GL(n, K)$ acts on $R = K[a_1, \ldots, a_n]$. Let N be a normal subgroup of G which contains the unipotent part of the radical of G. If I_N = the set of N-invariants of R is finitely generated over K, then so is I_G

Proof. G acts on I_N and $(I_N)_G = I_G$. The closure of N being denoted by $\overline{N}$, the action of G on I_N is really an action of $G/\overline{N}$, whose radical is a torus group. Therefore by (3) above, I_G is finitely generated. □

As a further application of Theorem 1 we prove the following:

Theorem 2 (Generalization of Fischer's theorem)**.** *Let K denote the complex number field and let $G \subseteq GL(n, K)$ act on $R = K[a_1, \ldots, a_n]$ as a group of automorphisms of R over K. Assume that for every $A \in G$, ${}^t\overline{A}$ (complex conjugate of A) is in G. Then I_G is finitely generated over K.*

Proof. Let $\overline{G}$ be the closure of G. Then $\overline{G}$ also satisfies the hypothesis of the above theorem. Hence we may assume G is algebraic. Let H be the radical of G. Let H_u be the set of unipotent elements of H (we recall that $A \in GL(n, K)$ said to be unipotent if all the eigen values of A are

1). It is well known that H_u is an algebraic group. It is enough to prove that H_u consists of identity element only. For then H is a torus group (A connected linear algebraic group which has no unipotent elements other identity is a torus group if either the field is algebraically closed
18 or the group is solvable). It follows that I_G is finitely generated by (3) above. □

It remains to prove that H_u consists of identity element only. Let $A \subseteq H_u$. Now H is characteristic in G (i.e. admists all automorphisms of G). Considering the automorphism $g \to ({}^t\overline{g})^{-1}$, we see that $({}^t\overline{A})^{-1} \in H$ and hence $({}^t\overline{A})^{-1} \in H_u$ since $({}^t\overline{A})^{-1}$ is unipotent. The element ${}^t\overline{A}A$ is unipotent and hermitian. Hence ${}^t\overline{A}A = E$, the identity matrix. Hence A is unitary. But the only unipotent unitary matrix is the identity matrix. Thus $A = E$ and the theorem is proved.

3. Let $R = K[a_1, \ldots, a_n]$ be an integral domain and let $G \subseteq GL(n, K)$ act on R and satisfy the condition (*). Let V be the affine K-variety defined by R (the points of V lie say, in the algebraic closure $\overline{K}$ of K). Let L be the function field of R and let L_G be the field of G-invariant elements in L. The group G acts on V in a natural way. A subset F of V is said to be G-admissible if for every $P \in F$, $P^g \in F$, for every $g \in G$.

Theorem 3. *Let F be a G-admissible closed subset of V. Let $T = \left\{ f \middle| f \in L_G,\ f \text{ regular at every point of } F \right\}$. Then T is a ring of quotients of I_G.*

Proof. Let S denote the multiplicatively closed set of all $s \in I_G$ such that s does not vanish at any point of F. We shall show that $T = (I_G)_s$. We have only to show that $T = (I_G)_s$. Let $f \in T$. Consider the ideals $\mathcal{U} = \left\{ g \middle| g \in R, gf \in R \right\}$ and $\mathcal{U}(F) = \left\{ h \middle| h \in R, h(F) = 0 \right\}$. As the closed set defined by $\mathcal{U}$ and F are disjoint, we have, $\mathcal{U} + \mathcal{U}(F) = R$. Hence there exists a $g \in \mathcal{U}$, $g' \in \mathcal{U}(F)$ with $g + g' = 1$.

19 It is clear that the ideals $\mathcal{U}$ and $\mathcal{U}(F)$ are G-admissible. Hence $\sum_{\sigma,\sigma' \in G} (g^\sigma - g^{\sigma'})K \subseteq \mathcal{U}(F) \bigcap (\sum_\sigma g^\sigma K) \subset \sum_\sigma g^\sigma K \subseteqq \mathcal{U}$. By Lemma 1 of Theorem 1 there exists a $g^* \in I_G \bigcap \sum_\sigma g^\sigma K$ such that $g - g^* \in \sum_{\sigma,\sigma'} (g^\sigma -$

$g^{\sigma'})K$. Hence $g^*(P) = 1$ for every $P \in F$ and $g^* f \in R$. The theorem is proved. □

Consider the relation $\sim$ in V defined by $P \sim Q$ if $\{\overline{P^G}\} \bigcap \{\overline{Q^G}\} \neq \phi$ for $P, Q \in V$ (where P^G is the orbit of P, namely the set of all P^σ with $\sigma \in G$ and $\bar{\infty}$ denotes the closure in V).

Theorem 4. (1) $\sim$ *is an equivalence relation.* (2) *The quotient set $V/\sim$ i.e. the set of all equivalence classes by $\sim$ acquires the structure of an affine K-variety with coordinate ring I_G such that the natural mapping $V \to V/\sim$ is regular.*

Proof of (1): We shall in fact give the following characterization which proves that $\sim$ is an equivalence relation: $P \sim Q$ if and only if $f(P) = f(Q)$, for every $f \in I_G$. Let $P \sim Q$ and let $P' \in \{\overline{P^G}\} \cap \{\overline{Q^G}\}$. We have $f(P) = f(P^G) = f(P') = f(Q^G) = f(Q)$, for every $f \in I_G$. To prove the converse we remark that if F_1 and F_2 are two G-admissible disjoint closed sets, then as in the proof of Theorem 3 we can find an $f \in I_G$ such that $f(F_1) = 1$, $f(F_2) = 0$. The closed set $\{\overline{P^G}\}$, for $P \in V$ is G-admissible. If P, Q are not related by $\sim$ we can separate them by an $f \in I_G$. Hence (1) is proved.

Proof of (2): By Theorem 1, I_G is finitely generated over K. Let $f_1(a), \ldots, f_l(a)$ generate I_G. Let W be the K-affine variety defined by $I_G = K[f_1, \ldots, f_l]$. Consider the regular mapping $\varphi : V \to W$ defined **20**
by $\varphi(P) = \overline{P} = (f_1(P), \ldots f_l(P)) \in W \subseteq \overline{K}^l$. By (1), $\varphi(P) = \varphi(Q)$ if and only if $P \sim Q$. Hence $\overline{\varphi}^1(\overline{P}) = \left\{Q \middle| P \sim Q\right\} = \underline{P}$, say. Let $\overline{\mathcal{M}}$ be the maximal ideal corresponding to a point $\overline{P}$ of W. Then by Lemma 3 of Theorem 1 $\overline{\mathcal{M}R} \bigcap I_G = \overline{\mathcal{M}}$. Hence φ surjective and we identify the equivalence class $\underline{P}$ with the point $\overline{P}$ of W. By Theorem 3 the local ring at $\overline{P}$ of W is $(I_G)_{\overline{P}} = (\bigcap\limits_{x \in \underline{P}} \mho_x) \bigcap L_G$, where $\mho_y$ denotes the local ring at the point y of V.

Corollary 1. *Let $Q \in \{\overline{P^G}\}$. Then* $(I_G)_{\overline{P}} = \left(\bigcap\limits_{Q^* \in \{\overline{Q^G}\}} \mho_{Q^*}\right) \bigcap L_G$.

Proof. We have only to prove

$$\bigcap_{Q^*\in\{\overline{Q^G}\}\mho_{Q^*}} \subseteq (I_G)_{\overline{P}}. \quad \text{Let} \quad f \in \bigcap_{Q^*\in\{\overline{Q^G}\}} \mho_{Q^*}.$$

Then by Theorem 3, $f = h/g$, $h, g \in I_G$, $g(Q^*) \neq 0$, for every $Q^* \in Q^G$. But as $Q \in \{\overline{P^G}\}$, $g(Q) = g(P')$ for every $P' \in \underline{P}$. Hence $f \in (I_G)_{\overline{P}}$. □

Corollary 2. *Under the hypothesis of Corollary 1, if Q^G is closed, then* $\mho_Q \bigcap L_G = \left(\bigcap_{P^*\in\{\overline{P^G}\}} \mho_{p^*}\right) \bigcap L_G$.

Proof. For any $Q \in V$, $(\bigcap_{Q^*\in Q^G} \mho_{Q^*}) \bigcap L_G = \mho_Q \cap L_G$. As Q^G is closed, by Corollary 1, $\mho_Q \bigcap L_G = (I_G)_{\overline{P}} = (\bigcap_{P^*\in\{\overline{P^G}\}} \mho_{P^*}) \bigcap L_G$. □

As we noted before, our Theorem 1 implies the following

21 **Theorem 1′** *Let $R = K[a_l, \ldots, a_n]$ and let $G \subseteq GL(n, K)$ act on R as a group of automorphisms of R. over K. If every rational representation of G is completely reducible, then I_G is finitely generated.*

What we like to remark here is that when we want to prove this Theorem 1^*, we need not use the technique at the end of §1, and we can prove as follows: R is a homomorphic image of a polynomial ring $K[x_1, \ldots, x_n]$ on which G acts naturally. The polynomial ring is graded, hence the result on the graded case and Proposition 1 prove Theorem 1^*.

Chapter 3

The counter example

1. In this chapter we give some counter examples to the original 14th **22**
problem. For this we need some results on plane curves.

Lemma 1. *Let C be a curve of positive genus on the projective plane and let $P_1, \ldots, P_m, P_{m+1}, \ldots, P_n \in C (m \geq 1)$ and let $P_1, \ldots, P_m$ be independent generic points of C over $k(P_{m+1}, \ldots, P_n)$, where k is a field of definition for C. Then for any set of natural numbers $\alpha_i (1 \leq i \leq n)$ there does not exist any curve C' such that $C'.C = \sum\limits_{i=1}^{n} \alpha_i P_i$.*

Proof. Suppose there exists a C' with $C'.C = \sum\limits_{i=1}^{n} \alpha_i . P_i$, let C'' be a curve of the same degree as C' with $C''.C = \sum\limits_{j=1}^{l} \beta_j Q_j with Q_j$ algebraic over k. Then $C'.C$ is linearly equivalent to $C''.C$ (notation $C'.C \sim C''.C$) on C. Hence $\sum\limits_{i=1}^{m} \alpha_i P_i \sim \sum\limits_{j=1}^{l} \beta_j Q_j - \sum\limits_{t=m+1}^{n} \alpha_t P_t$. Let P be any point on C. Specializing the above linear equivalence under the specialization $(P_1, \ldots, P_m) \to (P, \ldots, P)$ over $\overline{k}(P_{m+1}, \ldots, P_n)$ (where $\overline{k}$ is the algebraic closure of k) we get $(\sum\limits_{i=1}^{m} \alpha_i) P_{\sim} \sum\limits_{j=1}^{l} \beta_j Q_j - \sum\limits_{t=m+1}^{n} \alpha_t P_t$. Thus for any two points P, Q on C, we have $(\sum \alpha_i) P \sim (\sum \beta_i) Q$. This contradicts the assumption that C has positive genus. □

Here after for a variety V, we denote by $K(V)$ the smallest field of definition for V containing k.

23 **Lemma 2.** *Let $P_1, \ldots, P_t$ be independent generic points of the projective plane over a field k and d an integer such that a $t < \frac{d^2+3d}{2}$. Let L be the linear system of curves of degree d passing through $P_1, \ldots, P_t$. Let C be a generic member of L over $k(P_1, \ldots, P_t)$. Then $P_1, \ldots, P_t$ are independent generic points of C over $k(C)$.*

Proof. Set $u = \frac{d^2+3d}{2}$. Let $P_{t+1}, \ldots, P_u$ be independent generic points of C over $k(C)\,(P_1, \ldots, P_t)$. Then $P_1, \ldots, P_u$ are independent generic points of the projective plane over k as u is the dimension of the linear system L' of curves of degree d. Let $R_1, \ldots, R_t$ be independent generic points of C over $k\,(C, P_{t+1}, \ldots, P_u)$. Then we have a specialization $(P_1, \ldots, P_u) \to (R_1, \ldots, R_t, P_{t+1}, \ldots, P_u)$ over k because $P_1, \ldots, P_u$ are independent generic points of the projective plane. Let C specialize to C' under this specialization. If $C \neq C'$, then $R_1 + \cdots + R_t + P_{t+1} + \cdots + P_u \subset C.C'$. Since $R_1, \ldots, R_t, P_{t+1}, \ldots, P_u$ are independent generic points of C over $k(C)$ and the dimension of the trace L'_C on C of the linear system L' is $n-1$, we get a contradiction. Hence $C = C'$. Thus $(P_1, \ldots, P_t) \to (R_1, \ldots, R_t)$ is a specialisation over $k(C)$ and therefore $P_1, \ldots, P_t$ are independent generic points of C over $k(C)$. □

Proposition 1. *Let $P_1, \ldots, P_r$ be independent generic points of the projective plane over the prime field.*

(1) *Let C be a curve is degree d passing through the P_i with multiplicity m_i. Then $\frac{d}{\sum m_i} \frac{1}{\sqrt{r}}$, for $r = s^2, s \geq 4$.*

24 (2) *Furthermore if r' is a real number such that $r' > \frac{1}{\sqrt{r}}$, then there exists a curve C' of degree d' such that $r' > \frac{d'}{\sum m'_i} \frac{1}{r}$, where C' passes through P_i with with multiplicity m'_i.*

Proof. Let C specialize to $\sigma(C)$ under the specialization $(P_1, \ldots, P_r) \to (P_{\sigma(1)}, \ldots, P_{\sigma(r)})$ where σ runs through cyclic permutations of $1, \ldots, r$. Considering the curve $\sum_{\sigma} \sigma(C)$, we have only to prove (1) and (2) in the case when all the m_i are equal, say $m_i = m$. Thus we have to prove that $\frac{d}{m} > \sqrt{r} = s$. If there exists a curve C with $d/m \leq \sqrt{r}$, then by suitably raising the degree we can get actually the equality. Thus (1) is equivalent to:

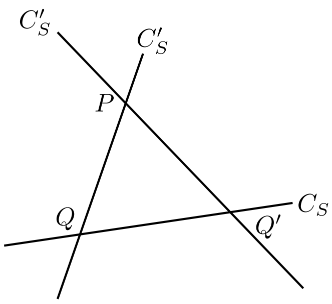
(1′) There does not exist a curve of degree sm passing through the P_i with multiplicity at least m.

Similarly (2) is equivalent to:

(2′) Given $r'' > \sqrt{r}$, there exist integers d, m and a curve C' of degree d passing through the P_i with multiplicity m such that $r'' > \frac{d}{m} > \sqrt{r}$.

□

Proof of (1′). Case (i) s even. Set $s' = \dfrac{s+2}{2}$. Let C_S, C_2, and $C'_{S'}$ be independent generic curves of degree s, s', s', respectively.



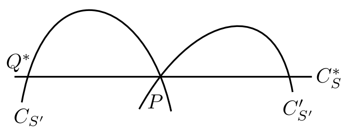
Let $C'_{S'}.C_{S'} = \sum_{i=1}^{s'^2} P_i, C_S.C_{S'} = \sum_{i=1}^{ss'} Q_i$ and $C_S.C'_S = \sum_{i=1}^{ss'} Q_i$. As the dimension of the linear system $L^{S'}$ of curves of degree s' is $\frac{s'^2+3s'}{2} > 2s$, among the P_i there exist $2s$ points say $P_1, \ldots, P_{2s}$ which are independent **25**
generic points of the projective plane over $k(C_s)$. Then by Lemma 2 $P_1, P_2, \ldots, P_{2s}$ are independent generic points of $C_{s'}$ over $k(C_S.C'_{S'})$, in fact over $k(C_S, C_{S'}, Q)$ as all the curves considered are independent generic curves.

Since $\frac{s^2+3s}{2} > 2s$, there exist curves of degree s passing through the points $P_1, \ldots, P_{2s}$. Let C^*_S be the generic curve passing through $P_1, \ldots, P_{2s}$.

Let

$$C_{S'}.C_S^* = \sum_{i=1}^{ss'-2s} Q_i^* + \sum_{i=1}^{ss'-2s} P_i,$$

$$C'_S.C_S^* = \sum_{i=1}^{ss'-2s} Q_i^{*'} + \sum_{i=1}^{2s} P_i.$$



By Lemma 2, $P_1, \ldots, P_{2s}$ are independent generic points of C_S^*. Further, we claim that the Q_i^* and $Q_i^{*'}$ are all distinct. For , consider the linear system $L = \left|C_S^*.C_{S'} - \sum P_i\right|$ on $C_{S'}$. $\sum Q_i^*$ is a generic member of L. Since degree of $L = ss' - 2s = 4.\frac{s'^2-3s'+2}{2} = 4g(C_{S'})$, where $g(C_{S'})$ denotes the genus of $C_{S'}, L$ has no base points. Hence each Q_i^* is a generic points of $C_{s'}$ over $k(C_{s'}, P_1, \ldots, P_{2s})$. In particular Q_i^* do not lie on $C'_{S'}$. Again the degree of the linear system $L - Q_i^* = ss' - 2s - 1 = 4g(C_{S'}) - 1 \geq 2g(C_{S'})$. Hence $L - Q_i^*$ has no base point and therefore $Q_j^*(j \neq i)$ are generic points of $C_{S'}$ over $k(C_{s'}, Q_i^*)$. Thus Q_i^* are all distinct. Similarly considering $C'_{s'}$, we see $Q_i^*, Q_i^{*'}$ are all distinct. We may renumber Q_i and Q'_i such that $Q_i, Q'_i, 1 \leq i \leq \frac{s'-2s}{2}$, are specialized to Q_i^*, $Q_i^{*'}$ under the specialization $(P_1, \ldots, P_{2s}, C_S) \to (P_1, \ldots, P_{2s}, C_S^*)$ over $k(C_S, C_S)$.

26 Assume that for some m there exists a curve of degree sm passing through $r = s^2$ independent generic points of the projective plane with multiplicity at least m. Then there exists a curve E of degree sm passing through the $P_i(1 \leq i \leq 2s)$, the $Q_i(1 \leq i \leq \frac{s^2-2s}{2})$ and the $Q'_i(1 \leq i \leq \frac{s^2-2s}{2})$ with multiplicity at least m. Then $C_{s'}$ is component of E. For otherwise, $E\ .C_{s'}$ is defined and contains $\sum_{i=1}^{2s} mP_i + \sum_{i=1}^{\frac{s^2-2s}{2}} mQ_i$. As the degree $E.C_{S'}$ is sms', we have $E.C_{s'} = \sum_{i=1}^{2s} mP_i + \sum_{i=1}^{\frac{s^2-2s}{2}} mQ_i$. Since the genus of $C_{s'}(= \frac{1}{2}s'(s' - 3) + 1)$ is positive we get a contradiction by Lemma 1. Hence $C_{s'}$ is a component of E. Let $E - C_{s'}$ specialize to E^* under the specialization $(P_1, \ldots, P_{2s}, C_s) \to (P_1, \ldots, P_{2s}, C_s^*)$.$E^*$ is a curve of degree $sm - s'$ passing through the $P_i(1 \leq i \leq 2s)$ and the

$Q_i^*(1 \le i \le \frac{s^2-2s}{2})$ with the the multiplicity at least (m-1) and through $Q_i'^*$ with the multiplicity at least m. We shall show the non-existence of such a curve by induction on m. We assert that C_s^* is component of E^*. For, if not, $E^*.C_s^*$ is defined and contains

$$\sum_{i=1}^{2s}(m-1)P_i + \sum_{i=1}^{\frac{s^2-2s}{2}}(m-1)Q_i^* + \sum_{i=1}^{\frac{s^2-2s}{2}} mQ_i^{*'}.$$

Inspecting the degrees we see that

$$E^*.C_s = \sum_{i=1}^{2s}(m-1)P_i + \sum_{i=1}^{\frac{s^2-2s}{2}}(m-1)Q_i' + \sum_{i=1}^{\frac{s^2-2s}{2}} mQ_i^{*'}.$$

Hence $((m-1)C_{s'} + mC_{s'}' - E^*).C_s^* = \sum_{i=1}^{2s} mP_i$. As the genus of $C_S^*(= \frac{1}{2}s(s-3)+1)$ is positive and since the P_i are independent generic points of C_s^* by lemma 2, we have a contradiction by Lemma 1.

Hence C_s^* is a component of E^*. If $m = 1$, degree of E^* is $s - s'$. **27**

Hence we get a contradiction. If $m > 1$, then $E^* - C_{S.}^*$ is of degree $s(m-1) - s'$ and passes through the P_i, Q_i^*, with multiplicity at least $m-2$, $m-2$ and $(m-1)$ respectively. Hence by induction hypothesis $E^* - C_{S.}^*$ does not exist and the lemma is proved, when r is even

Case (ii) s odd. Set $s' = \frac{s+1}{2}$. Let $C_S, C_{S'} C_{S'}'$ be independent generic curves. Let $P_1, \ldots, P_s$ be s points contained in $C_{S'}.C_{S'}'$. We take a generic curve C_S^* of degree s passing through $P_1, \ldots, P_s$ and proceed as in (i). We omit the details.

Remark. We remark that Proposition 1 is not true for $r \le 9$. The following are the example to that effect. (1) For $r = 1, 2$, a line passing through the P_i (2) $r = 3$, a line passing through two of the P_i (3) $r = 4, 5$, a conic through the P_i (4) $r = 6$, a conic through 5 of the P_i (5) $r = 7$, a cubic having a double point at one of the P_i and passing through all the P_i. (6) $r = 8$, a curve of degree 6 having a triple point at one of the P_i and double points at all the P_i (7) $r = 9$, a cubic passing through all the P_i.

Furthermore it is not known if $r \geq 9$, is sufficient to ensure the inequality of Proposition 1.

Proof of (2′). Let L^d be the linear system of curves of degree d. Let $f(x_0, x_1, x_2)$ be a homogeneous polynomial of degree d. The condition that the curve defined by f should pass through a point p with multiplicity m imposes $\frac{m(m+1)}{2}$ linear conditions (not necessarily independent).

28 Thus

$$\dim(L^d - \sum_{i=1}^{r} mP_i) \geq \frac{d(d+3)}{2} - r\frac{m(m+1)}{2} \geq 0$$

if

$$\frac{d(d+3)}{2} \geq r\frac{m(m+1)}{2}.$$

i.e. if $m \geq \dfrac{r - 3(\frac{d}{m})}{(\frac{d}{m})^2 - r}$. Now choose a rational number λ such that $r'' > \lambda > \sqrt{r}$. Writing $\lambda = \frac{d}{m}$ with sufficiently large m we get a curve C' of degree d passing through the P_i with multiplicity m such that $r'' > d/m > \sqrt{r}$ and (2′) is proved.

2. We now proceed to give the counter example where the transcendence degree of the sub-field L_G of invariants is four.

Let $a_{ij}, i = 1, 2, 3, j = 1, \dots, r$ be algebraically independent elements over the prime field Π . Let k be a field containing the a_{ij}. Consider the projective plans S over k. Set $P_i = (a_{1i}, a_{2i}, a_{3i}), P_i \in S, i = 1, 2, \dots, r$. Then the P_i are independent generic points of S over the prime field . Let $x_1, \dots, x_r, y_1, \dots, y_r$ be algebraically independent elements over k. Consider the subgroup G of $GL(2r, k)$ given by

$$G = \left\{ \sigma \in GL(2r, k) \middle| \sigma = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_r \end{pmatrix}, \right.$$

$$\left. B_i = \begin{pmatrix} c_i & c_i b_i \\ 0 & c_i \end{pmatrix}, \sum_{i=1}^{r} b_i a_{ji} = 0,\ j = 1, 2, 3,\ \prod_{i=1}^{r} c_i = 1 \right\}.$$

The group G acts on $k[x_1, y_1, \dots, x_r, y_r]$ with $\sigma(x_i) = c_i(x_i + b_i y_i)$, $\sigma(y_i) = c_i y_i, i = 1, \dots, r$.

Theorem 1. *The ring of invariants I_G is not finitely generated the k. Set* 29
$t = y_1, \ldots, y_r, u_i = t/y_i$*,* $v_i = x_i u_i$ *and* $w_j = \sum_{i=1}^{r} a_{ji} v_i$*. We need some lemmas.*

Lemma 1. $I_G = k[x_1, y_1, \ldots, x_r, y_r] \cap k(w_1, w_2, w_3, t)$.

Proof. It is sufficient to prove that the invariant sub-field $L_G = k(w_1, w_2, w_3, t)$. A straight forward verification shows that $w_1, w_2, w_3, t \in L_G$. Hence $k(w_1, w_2, w_3, t) \subseteq L_G$ as a_{ij} are independent over $\prod$, we have $k(x_1, \ldots, x_r, y_1, \ldots, y_r) = k(w_1, w_2, w_3, x_4, \ldots, x_r, y_1, \ldots, y_r)$. Now G operates on $k[w_1, w_2, w_3, x_4, \ldots, x_r, y_1, \ldots, y_r]$. As $w_i, t \in L_G$, to compute L_G it is enough to consider the action of G on $k(w_1, w_2, w_3, t)[x_4, \ldots, x_r, y_1, \ldots, y_{r-1}]$. Consider the subgroup H of G consisting of elements σ of G for which $c_i = 1, i = 1, \ldots, r, b_i = 0, i \geq 5$ and b_4 arbitrary. Since H is infinite, $k(w_1, w_2, w_3, t, x_4, \ldots, x_r, y_1, \ldots, y_{r-1})$ is a transcendental extensions of L_H the fixed field of H. Now, $L_H \supseteq k(w_1, w_2, w_3, t, x_5, \ldots, x_r, y_1, \ldots, y_{r-1})$. Therefore $L_H = k(w_1, w_2, w_3, t, x_5, \ldots, x_r, y_1, \ldots, y_{r-1})$. Next we consider the action of G on $k(w_1, w_2, w_3, t)[x_5, \ldots, x_r, y_1, \ldots, y_{r-1}]$ and consider the subgroup H_1 of G consisting of elements σ of G with $b_i = 0, i \geq 6, c_i = 1, i = 1, \ldots, r$. The fixed field L_{H_1} of H_1 is $k(w_1, w_2, w_3, t, x_6, \ldots, x_r, y_1, \ldots, y_{r-1})$. Proceeding in the same way we arrive at $k(w_1, w_2, w_3, t, y_1, \ldots, y_{r-1})$. Consider $k(w_1, w_2, w_3, t)[y_1, \ldots, y_{r-1}]$. $\sigma \in G$ acts on $k(w_1, w_2, w_3, t)[y_1, \ldots, y_{r-1}]$ with $\sigma(y_i) = c_i y_i, i = 1, \ldots, r-1$, where c_i are arbitrary non-zero elements of k. Hence $L_G = k(w_1, w_2, w_3, t)$. □

As w_1, w_2, w_3 are algebraically independent over k, we may regard
$H = k[w_1, w_2, w_3]$ as the homogeneous coordinate ring of the projective 30
plans S. Let $\mathscr{F}_i = (z_i, z_i')$ denote the prime ideal is H corresponding to the point P_i, where $z_i = a_{3i} w_1 - a_{1i} w_3, z_i' = a_{3i} w_2 - a_{2i} w_3$. Set $\mathcal{W}_n = \bigcap_{i=1}^{r} \mathscr{Y}_i^{(n)}$, for $n > 0$ and $\mathcal{W}_n = H$, for $n \leq 0$.

Lemma 2. $I_G = \left\{ \sum_n a_n t^{-n} \mid a_n \in \sqcap_n \right\}$.

Assuming Lemma 2 we shall first prove Theorem 1 and later prove Lemma 2. Suppose I_G is finitely generated, say $I_G = k[f_1, \ldots, f_m]$. We *may assume* $f_j = h_j t^{-j}, h_j \in \mathcal{W}_j, h_j$ *homogeneous,* $j = 1, \dots, m$. *Set* $r_j = \dfrac{\text{degree } h_j}{j}$, $j = 1, \dots, m$ *and* $r^* = \min_{i \leq j \leq m} r_j$. *For any monomial* $f_1^{i_1} \dots f_m^{i_m} = h_1^{i_1} \dots h_m^{i_m} t^{-i_1 - 2i_2 - \cdots - mi_m}$, *we have* $\dfrac{\text{degree } (h_1^{i_1} \dots h_m^{i_m})}{i_1 + 2i_2 + \cdots + mi_m} \geq r^*$. *Hence for any homogeneous* $a_n \in \mathcal{W}_h$, *we have* $\dfrac{\text{degree } a_n}{n} \geq r^*$. *By proposition 1,* $r^* > s = \sqrt{r}$, *for* $r \geq 4$. *Again by the same proposition there exists an* $a_n \in \mathcal{W}_n$, *for some n, such that degree* $a_{n/n} < r^*$. *This is a contradiction and therefore* I_G *is not finitely generated.*

It now remains to prove Lemma 2. We first prove

$$I_G \subseteq \left\{ \sum_n a_n t^{-n} \mid a_n \in H \right\}. \tag{$*$}$$

As a_{ij} are algebraically independent over k we have

$$k[v_1, \dots, v_r] = k[w_1, w_2, w_3, v_4, \dots, v_r].$$

Hence

$$k\left[x_1, \dots, x_r, y_1..y_r, \frac{1}{y_1}, \dots, \frac{1}{y_r}\right] = k\left[w_1, w_2, w_3, x_4, \dots, x_r, y_1, \dots y_r, \frac{1}{y_1}, \dots, \frac{1}{y_r}\right].$$

Now

$$k\left[w_1, w_2, w_3, x_4, \dots, x_r, y_1, \dots y_r, \frac{1}{y_1}, \dots, \frac{1}{y_r}\right] \bigcap k(w_1, w_2, w_3, y_1, \dots, y_r) = k\left[w_1, w_2, w_3, y_1, \dots, y_r, \frac{1}{y_1}, \dots, \frac{1}{y_r}\right].$$

Now by lemma 1,

$$I_G \subseteq k\left[w_1, w_2, w_3, y_1, \dots, y_r, \frac{1}{y_1}, \dots, \frac{1}{y_r}\right] \bigcap k(w_1, w_2, w_3, t)$$

$$= k\left[w_1, w_2, w_3, t, \frac{1}{t}, y_2, \ldots, y_r, \frac{1}{y_2}, \ldots, \frac{1}{y_r}\right] \bigcap k(w_1, w_2, w_3, t)$$
$$= k\left[w_1, w_2, w_3, t, \frac{1}{t}\right].$$

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Hence $I_G \subseteq \left\{\sum_n a_n t^{-n} \middle| a_n \in H\right\}$. Let $\mho_i$ be the valuation ring $k[y_1, \ldots, x_r, y_1, \ldots, y_r]_{(y_i)}, 1 \leq i \leq r$ and let V_i be the corresponding normalized valuation. We prove

$$\mathscr{Y}_i^m = \mathscr{Y}_i^{(m)} = \left\{f \in H \mid V_i(f) \geq m\right\}, \qquad **$$

for every m. Since $z_i = \sum_{j \neq i}(a_{3i}a_{1j} - a_{1i}a_{3j})x_j u_j, V_i(z_i) = 1$. Similarly $V_i(z'_i) = 1$. It is easy to check that $w_3, z_i/t$, and z'_i/t modulo the maximal ideal $\mathscr{M}_i$ of $\mho_i$ are algebraically independent over k. Now $k[w_1, w_2, w_3] = k[z_i, z'_i, w_3]$. If $h \notin \mathscr{Y}_i$, then $h \equiv h_1 \pmod{\mathscr{M}_i}$, with $0 \neq h_1 \in k[w_3]$. Hence $V_i(h) = 0$. Therefore $\mathscr{Y}_i^{(m)} \subseteq \left\{h \in H \mid V_i(h) \geq m\right\}$. To prove $(**)$ we need only to prove that, for $f \in H$, $V_i(f) \geq m$ implies $f \in \mathscr{Y}_i^m$. This in turn will be proved if we prove that $f \in \mathscr{Y}_i^m$, $f \notin \mathscr{Y}_i^{m+1}$ implies $V_i(f) = m$. If $f \in \mathscr{Y}_i^m$, $f \notin \mathscr{Y}_i^{m-1}$ then f can be written as $f = \sum_{j=}^{m} h_j z_i^j z_i'^{m-j}$, with $h_j \in k[w_3]$, $h_l \neq 0 \mod \mathscr{M}$, for some l. Hence $f/z_i'^m = \sum_{j=0}^{m} h_j (\frac{z_i}{z_i})^j$. As z_i/z'_i and w_3 modulo m_i are algebraically independent over k, $f/z_i'^m \not\equiv 0 \pmod{\mathscr{M}_i}$. Hence $V_i(f) = V(z_i'^m) = m$ and $(**)$ is proved. Next, we prove

$$V_i(\sum_n a_n t^{-n}) = \min_n V_i(a_r t^{-n}),$$

for $a_n \in H$. Let $d = \min_n V_i(a_n t^{-r})$. For every n, $V_i(a_n t^{-n}) \geq d$ i.e. 32
$V_i(a_n) \geq n + d$. Hence by $(*, *)$, $a_n \in \mathscr{Y}_i^{n+d}$. Let $a_n = \sum_{j=0}^{n+d} h_{nj} z_i^j z_i'^{n+d-j}$. Then $\sum_n \frac{a_n}{t^{n+d}} = \sum_n \sum_j h_{nj} (\frac{z_i}{t})^j (\frac{z'_i}{t})^{n+d-j}$. If $V_i(a_m t^{-m}) = d$ for some m, then

$0 \neq h_{ml} \mod \mathscr{M}_i \in k[w_3]$ for some l. Since z_i/t and z'_i/t modulo $\mathscr{M}_i$ are algebraically independent over $k(w_3 \mod \mathscr{M}_i)$, $\sum\limits_{n}^{m} \frac{a_n}{t^{n+d}} \not\equiv 0 \pmod{\mathscr{M}_i}$. Hence $V_i(\sum\limits_n a_n t^{-n}) = d$.

Now suppose $f \in I_G$. Then by $(*) f$ can be written as $f = \sum\limits_n a_n t^{-n}$, $a_n \in H$. As $I_G \subseteq \mho_i, 1 \leq i \leq r, V_i(f) \geq 0$. Hence by $(***)$, $V_i(a_n t^{-n}) \geq 0$, that is $V_i(a_n) \geq n$ and $a_n \in \mathscr{Y}_i^n$. Hence $I_G \subseteq \left\{ \sum a_n t^{-n} \mid a_n \in \mho_n \right\}$. On the other hand if $a_n \in \mathscr{W}_n$, then $V_i(a_n) \geq n, i = 1, \ldots, r$. Hence a_n is divisible by t^n. Thus $I_G \supseteq \left\{ \sum a_n t^{-n} \mid a_n \in \mathscr{W}_n \right\}$ and the proof of Lemma 2 is complete.

3. In the counter example we have given to the original 14th problem in section 2, the group G is commutative. In this section we give an example, where the group G is such that $[G, G] = G$.

If we make use of the structure of the ring I_H of invariants of the subgroup H of G given in the last section such that all the c_i are 1 then we can give a direct construction of the required example. But since we did not give the explicit structure of I_H, we do it in an indirect way.

Let a_{ij}, $i = 1, 2, 3$, $j = 1, \ldots, r (r = 16)$ be algebraically independent real numbers over the field of relational numbers. Set

33 $V = \left\{ (b_1, \ldots, b_r) \mid b_i \text{ real}, \sum\limits_j a_{ij} b_j = 0 (i = 1, 2, 3, s = \text{an even} \right.$ number ≥ 4,

$$G_o = \left\{ \begin{pmatrix} c_1 B_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & c_r B_r \end{pmatrix} B_i = \begin{pmatrix} 1 & \ldots\ldots & b_{il} \\ \ddots & 0 & \\ & \ddots & b_{is} \\ 0 & & 1 \end{pmatrix}, \begin{array}{c} (b_{li}, \ldots, b_{ri}) \in V, \\ c_i \text{ real}, \prod_{c_i} = 1 \end{array} \right\}$$

Let K be the real number field. G_o acts on the polynomial ring

$$R_0 = K[x_{11}, x_{12}, \ldots, x_{1s}, y_1, x_{21}, x_{22}, \ldots, x_{2s}, y_2, \ldots, x_{r1}, x_{s2}, \ldots, x_{rs}, y_r].$$

Let I_0 be the ring of G_o - invariants in R_o. Then setting $t = \prod y_j$

(i) $w_{ij} = \sum\limits_\alpha a_{i\alpha} x_{\alpha j} t / y_\alpha \in I_o$

(ii) $I_o = R_o \bigcap K(w_{11}, \ldots, w_{1s},\ \ w_{21}, \ldots, w_{2s},\ \ w_{31}, \ldots, w_{3s}, t)$.

Since $I_o/I_o \bigcap (\sum\limits_{j \geq 2} x_{ij}R)$ is isomorphic to previous example I_G, we see that I_o is not finitely generated. We see also easily that every element of I_o can be expressed in the form $\sum\limits_i a_i t^{-i}$ (finite sum) such that $a_i \in \mathscr{Y} = K[w_{11}, \ldots, w_{3s}]$, $a_i t^{-i} \in I_o$.

For each $T \in SL(s, K)$, let

$$T_r = \begin{pmatrix} T' & & 0 \\ & \ddots & \\ 0 & & T' \end{pmatrix}, T' = \begin{pmatrix} T & 0 \\ & \\ 0 & 1 \end{pmatrix}$$

$$(T' \in SL(s+1, K)), T_r \in SL(r(s+1), K)$$

and $\quad G_1 = \left\{ T_r \mid T \in SO(s, K) \right\}$.

Then one can easily see that I_o is G-admissible and the module $M_i = \sum\limits_j w_{ij}K$ is also G_1-admissible and T is the transformation on N_i given by T_r. Therefore for a general linear transformation $T \in SL(s, K)$ we have $I_{o/I_o \cap (\sum\limits_{j \geq 2} x_{ij}R)}T$ is isomorphic to the previous example I_G.

Therefore we see that the degree $a_i > 4i$ (for $i > 0$) and that for any rational number α greater than $\frac{1}{4}$. there is a homogeneous form a_l in w_{ij} **34**
such that $\frac{\deg\ a_l}{r_l} = \alpha$ and $a_l t^{-l} \in I_o$.

Now consider a copy of R_o say R'_o. The isomorphism of R_o onto R'_o, we denote by "/". Let G_2 be the subgroup in $SL(2r(s+1), K)$ generated by

$$G^*_o = \left\{ \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \mid B_i \in G_o \right\} \text{ (i.e. } G^*_o = G_o \times G_o)$$

$$G^*_1 = \left\{ \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \mid T \in G_1 \right\}.$$

Then the ring of G^*_o- invariants is $I_o \otimes I'_o$. Let M_{id} be the module of homogeneous forms a_i in $\mathscr{Y}$ of degree d such that $a_i t^{-1}$ is in I_o. Then each M_{id} which is not zero defines a representation of G_1. Hence we can choose a linearly independent basis of M_{id} say $f_{idl}, \ldots, f_{idn}(i,d)$ such that the representation is orthogonal. Then $f^*_{id} = \sum_j f_{idj} f'_{idj}$ is G^*_1-invariant and $f^*_{id} t^{-i}$ is G_2-invariant. Therefore the same reasoning as in the previous example I_G can be applied to the ring I_2 of G_2-invariant and we see that I_2 is not finitely generated.

Let C be the complex number field and let $\bar{G_2}$ be the closure of G_2 in $SL(2r(s+1), C)$. We want to show that $\bar{G_2}$ contains a connected normal algebraic subgroup $\bar{G}$ such that

(i) $\bar{G_2}/\bar{G}$ is a torus group and

(ii) $[\bar{G}, \bar{G}] = \bar{G}$

If the existence of $\bar{G}$ is shown then $\bar{G}$ is the given example.

Let G_{oo} be the subgroup of G_0 defined by $c_1 = \ldots = c_r = 1$. Then we can consider the same construction as done for G^*_0, G_2 and $\bar{G}_{20}$.

We claim that $\bar{G}_{20}$ is the required $\bar{G}$.

35 *Proof.* Since $\bar{G}_{20}$ is normal in G_2 we see that $\bar{G}_{20}$ is normal in $\bar{G}_2$. Obviously $\bar{G_2}/_{\bar{G}_{20}}$ is a torus group. It is also obvious that $\bar{G}_{20}$ in connected. Therefore we have only to show that $[\bar{G}_{20}, \bar{G}_{20}] \supseteq \bar{G}_{20}$. Since $[\bar{G}_{20}, \bar{G}_{20}]$ is a closed subgroup we have only to show that both G^*_1 and G^*_{oo} are in $[\bar{G}_{20}, \bar{G}_{20}]$. Since G^*_1 is semi-simple and since $G^*_1 \subseteq \bar{G}_{20}$ we see that $G^*_1 \subseteq [\bar{G}_{20}, \bar{G}_{20}]$. Consider the set $\{B^*_1\} = \left\{\begin{pmatrix} B & O \\ O & E \end{pmatrix}\right\}$ (E =identity in $SL(x(s+1), k)B \in G_\infty$). For a fixed $T^* \in G^*_1, B^*_1 \rightsquigarrow T^{*-1} B^{*-1}_1 T^* B^*_1$ gives a homomorphism B^*_1 and B^*_1 (because B^*_1 is isomorphic to a vector space over K). Since s is even, there is a $T^* \in G^*_1$ such that $T^{*-1} B^{*-1}_1 T^* B^*_1 = \begin{pmatrix} E & O \\ O & E \end{pmatrix}$ implies $B^*_1 = \begin{pmatrix} E & O \\ O & E \end{pmatrix}$. Therefore we see that image is dense in B^*_1. Therefore $[\bar{G}_{20}, \bar{G}_{20}]$ contains the set $\{B^*_1\}$. Similarly $[\bar{G}_{20}, \bar{G}_{20}]$ contains all the $\begin{pmatrix} E & O \\ O & B \end{pmatrix}$, $(B \in G_{oo})$. Therefore $[\bar{G}_{20}, \bar{G}_{20}]$ contain G^*_{oo} and the proof is complete. □

Chapter 4

Theorem of Weitzenböck

Theorem 1 (Weitzenböck)**.** *Let K be the complex number field and G a complex one parameter Lie subgroup of $GL(n, K)$ acting on the ring of polynomials $K[x_1, ..., x_n]$. Then the ring of invariants I_G of $K[x_1, ..., x_n]$ is finitely generated over K. (see also C.S. Seshadri; On a theorem of Weitzenböck in invariant Theory, J. Math. Kyoto Univ. 1-3 (1962), 403-409).* **36**

Let $G = \{e^{tA} | t \in K\}$ be a one parameter Lie group, A being a constant matrix. Let $A = N + S$, where$N$ and S are nilpotent and semi-simple matrices respectively with $NS = SN$. Let $G_1 = \{e^{tN} | t \in K\}$ and $G_2 = \{e^{tS} | t \in K\}$. If $N \neq O$, then the mapping $t \rightarrow e^{tN}$ of the additive group K onto G_1 is an algebraic isomorphism, since N is nilpotent. Since every element of G_2 is semi-simple, the closure $\bar{G_2}$ of G is a torus group. Further since the elements of G_2 and $\bar{G_2}$ commute, the mapping $(g_1, g_2) \rightarrow g_1 g_2$ of $G_1 \times \bar{G_2}$ onto is an algebraic homomorphism. Hence $G_1\bar{G_2}$ is a closed subgroup of $GL(n, k)$. Hence the closure $\bar{G}$ of G is contained in $G_1\bar{G_2}$. Since $G_1 \subseteq G$ and $G_2 \subseteq G$, it follows that $\bar{G} = G_1\bar{G_2}$. Thus $\bar{G}$ is a torus group or a direct product of *a* torus group and the additive group K. Hence Theorem 1 is equivalent to the following, by virtue of Chapter 2, Section 2, Corollary to Theorem 1.

Theorem 2. *Let G be a unipotent algebraic group of dimension one*

37 *of $GL(n,k)$ acting on $K[x_1,\dots,x_n]$, where K is a field of characteristic zero. Then I_G is finitely generated over K.*

Our proof is mostly due to C. S. Seshadri.

We first prove two lemmas. Unless otherwise stated K denotes a field of characteristic zero.

Lemma 1. *Let V be an affine variety and G a connected algebraic group acting on V. Let W be a subvariety of V and H a subgroup of G. Let W be stable under H and suppose $\left\{x^h \mid h \in H\right\} = \left\{x^g \mid g \in G\right\} \cap W$, for every $x \in W$. Let f be a H-invariant regular function on W. Then there exists a G-invariant rational function f^* on $\overline{W^G}$ such that f^* is integral over the local ring $\mho_x$ in $\bar{W}_G$ for every $x \in W^G$; f^* takes unique value at x, for $x \in W^G$, and such that f^* induces f on W. (Lemma of Seshadri).*

Proof. Assume first $\bar{W}^G$ is normal. The function f defines a regular function F on $W \times G$ defined by $F(x,g) = f(x)$, for $x \in W, g \in G$.

Let Q be the regular mapping of $W \times G$ into $\bar{W}^G$ defined by $\varphi(x,y) = x^g$. If $x^g = x'^{g'}$, for $x, x' \in W, g, g' \in G$ then $x^{gg'^{-1}} = x' = x^h$ for some $h \in H$, by hypothesis. Thus $f(x') = f(x^h) = f(x)$. Thus F is constant on each fibre $\varphi^{-1}(x^g)$, for $x \in W, g \in G$. Let A be the coordinate ring of $\bar{W}^G$. Let U be the affine variety defined by the affine ring A[F]. The generic fibre of the projection of U onto W^G is reduced to a single point. As the ground field K is of characteristic zero, U and W^G are birationally equivalent. Thus F induces a rational function f^* on $\bar{W}^G$. By definition of F, f^* is G-invariant. Further f^* assumes the unique finite value $f(x)$ at x^g, for $x \in W, g \in G$. As $\bar{W}^G$ is normal f^* is regular value at x^g and the lemma is proved in the case $\bar{W}^G$ is normal. □

38 Suppose $\bar{W}^G$ is not normal. Without loss of generality we may assume $\bar{W}^G = V$. Let $\tilde{V}$ be the derived normal model of V. Let $\tilde{W}_1, \dots, \tilde{W}_s$ correspond to W, in $\tilde{V}$. How G operates on $\tilde{V}$. As W^G is dense in V, so

are $\tilde{W}_i^G$ in $\tilde{V}$. The functions f induces H-invariant regular functions f_i on $\tilde{W}_i 1 \leq i \leq s$. By what we have just proved, we get G-invariant rational function f_i^*, $1 \leq i \leq s$ of $\tilde{V}$ which is regular on $\tilde{W}_i^G$, and induces f_i. Now f_i^*, and f_j^* take the same value on $\tilde{W}_i^G \cap \tilde{W}_j^G$. As $\tilde{W}_i^G$ contains a non-empty open-set of $\tilde{V}$, $f_1^* = \ldots = f_s^* = f^*$. Thus f^* is regular on $\tilde{W}_i^G$, $1 \leq i \leq s$. Hence f^* is integral over the local ring of x^g in W^G, for $x \in W, g \in G$. Lemma 1 is completely proved.

Remark 1. If the ground field is of characteristic $p \neq 0$, then Lemma 1 is true under the following modification of the rationality condition for $f^* : f^*$ is in a purely inseparable extension of the function field of $\tilde{W}^G$.

Remark 2. It is interesting to note that W^G is open. For, each fibre $\varphi^{-1}(x^g)$ is irreducible and of dimension equal to dimension of H, because of the condition on the orbits we have imposed. Thus φ has no fundamental points and W^G is open.

Corollary. *If furthermore G is semi-simple and codimension of* $(W^G)^C$ *in* $\bar{W}^G$ *is at least 2, then the ring* I_H *of H-invariants in the coordinate ring of W is finitely generated.*

Proof. As codim $(W^G)^C \geq 2$, any function f on $\bar{W}^G$ which is integral over the local rings of points of W^G in $\bar{W}^G$ is integral over the coordinate ring A of $\bar{W}^G$. Let B be the ring got by adjoining all rational functions **39**
f^* on $\bar{W}^G$, as in Lemma 1. Since B is contained in the derived normal ring of A, B is an affine ring and G acts on B. As G is semi-simple, the ring of invariants I_G of B is finitely generated. By Lemma 1, the ring I_H of H-invariants in the coordinate ring of W is the homomorphic image of I_G and hence is finitely generated. □

Lemma 2. *Let K be of characteristic zero and let* ρ *be a rational repre-*

sentation of the additive group $H = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in K \right\}$ *in* $GL(n, k)$*. Then there exists a rational representation* ρ^* *of* $SL(2, k)$ *in* $GL(n, \lambda)$ *such that* $\rho^* = \rho$ *on* H*.*

Proof. We shall denote the element $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ of H by λ. Choose λ such that $\rho(\lambda) \neq 1 (\lambda \neq 0$ is enough). Let the Jordon normal form of $\rho(\lambda)$ be

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}, \quad \text{where} \quad A_i = \begin{pmatrix} 110 & . & 0 \\ 011 & & ..0 \\ .. & .. & \\ & & 1 \\ 0.. & . & 1 \end{pmatrix}.$$

Let A_i have n_i rows. Let ρ_i^* be the representation of $SL(2, K)$ given by homogeneous forms of degree $n_i - 1$ in two variables. It is easy to check that the Jordan normal form of $\rho_i^*(\lambda)$ is A_i. Hence we may assume that $\rho_i^*(\lambda) = A_i$. We take $\rho^* = \begin{pmatrix} \rho_i^* & 0 \\ 0 & \rho_r^* \end{pmatrix}$. □

Remark 2. Lemma 2 is not true in the case where K is of characteristic $p \neq 0$, because the group H has non-faithful rational representations which are not trivial.

40 **Proof of theorem 2.** Let G be given by a rational representation ρ of $H = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in K \right\}$ in GL(n,K). By Lemma 2 we extend this representation to *a* rational representation ρ^* of SL(2, K). Let

$$G' = \left\{ \begin{pmatrix} \rho^*(g) & 0 \\ 0 & g \end{pmatrix} \mid g \in SL(2, k) \right\}$$

$$\text{and} \qquad H' = \left\{ \begin{pmatrix} \rho(\lambda) & 0 \\ 0 & \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \end{pmatrix} \mid \lambda \in K \right\}$$

The group G' acts on $K = [x_1, \ldots, x_n, x_{n+1}, x_{n+2}]$. Let $V = K^n \times K^2$ and W the subvariety defined by the equations $x_{n+1} - 1 = 0 = x_{n+2}$. The group G' acts on V and W is stable under H'. If for a point

$$Q = (a_1, \dots, a_n, 1, 0) \in W, \quad Q^{g'} \in W,$$

where
$$g' = \begin{pmatrix} \rho^*(g) & 0 \\ 0 & g \end{pmatrix} \in G', g \in SL(2, K).$$

Then $(1, 0)^g = (1, 0)$. Hence $g \in H$. That is $g' \in H'$. Hence the orbit condition of Lemma 1 is satisfied. As $W^{G'}$ contains the complement of the hyperplane $x_{n+1} = 0, W^{G'}$ is dense in V. Further $(W^{G'})^C$ is contained in the variety defined by the equations $x_{n+1} = 0 = x_{n+2}$. The conditions of the corollary to Lemma 1 are satisfied. Hence the ring of H'-invariants $I_{H'}$ in the coordinate ring of W is finitely generated. That is I_G is finitely generated.

Chapter 5

Zarisk's Theorem and Rees' counter example

1. $\mathfrak{G}$**-transform**. Let R be an integral domain with quotient field L. Let **41**
$\mathfrak{G}$ be an ideal of R. The set

$$S(\mathfrak{G}; R) = \left\{ f \mid f \in L, f\mathfrak{G}^n \subseteq R \text{ for some } n \right\}$$

is defined to be the $\mathfrak{G}$- transform of R.

Remark 1. $S(\mathfrak{G}; R)$ is an integral domain containing R.

Remark 2. It is clear that if $\mathfrak{G}$ and $\mathfrak{b}$ are two ideals with finite basis such that $\sqrt{\mathfrak{G}} = \sqrt{\mathfrak{G}}$, then $S(\mathfrak{G}; R) = s(\mathfrak{b}; R)$ (we recall that for an ideal $\mathfrak{G}$ in R, $\sqrt{\mathfrak{G}} = \left\{ x \mid x \in R, x^n \in \mathfrak{G}, \text{for some } n \right\}$).

Remark 3. If R is a noetherian normal ring and height $\mathfrak{G} \geq 2$, then $S(\mathfrak{G}, R) = R$.

Proof. Let $f \in S(\mathfrak{G}, R)$. The $f\mathfrak{G}^n \subseteq R$, for some n. As $\mathfrak{G}$ is not contained in any prime ideal of height 1, $f \in R_{\mathscr{Y}}$, for every prime ideal $\mathscr{Y}$ of height 1. Hence $f \in R$, R being normal. □

Corollary . *Let R be as in Remark 3 and $\mathfrak{G} = \mathfrak{G}_1 \cap \mathfrak{G}_2$ with height $\mathfrak{G}_2 \geq 2$. Then $S(\mathfrak{G}, R) = s(\mathfrak{G}_1, R)$.*

Remark 4. If R is an affine ring and height $\mathfrak{G} \geq 2$, then $S(\mathfrak{G}; R)$ is integral over R.

Proof. By passing to the derived normal ring we may assume that R is normal any apply Remark 3.

For any integer $n \geq 0$, set $\mathfrak{G}^{-n} = \left\{ f \mid f \in L, f\mathfrak{G}^n \subseteq R \right\}$, Then $S(\mathfrak{G}; R) = \bigcup_n \mathfrak{G}^{-n}$. We shall abbreviate $S(\mathfrak{G}; R)$ to S when there is no confusion. □

42 **Remark 5.** If R is noetherian, $\mathfrak{G}^{-n}$ is a finite R-module.

Proof. $\mathfrak{G}^{-n} \subseteq R\frac{1}{a}, a \neq 0, a \in \mathfrak{G}^{-n}$. □

Proposition 1. *Let $\mathfrak{G}$ be an ideal in R with a finite basis $a_1, \ldots, a_n$, ($a_i \neq 0, i = 1, \ldots, n$). Let $t_1, \ldots, t_{n-1}$ be algebraically independent elements over R. Set $t_n = (1 - \sum_{i=1}^{n-1} a_i t_i) \mid a_n$; this is an element of $R[t_1, \ldots, t_{n-1}, \frac{1}{a_n}]$. Then*

(i) *$S = R[t] \cap L$, where t stands for $(t_1, \ldots, t_n)$.*

(ii) *Further if R is normal, then $S = R^* \cap L$, where R^* is the derived normal ring $R[t]$.*

Proof. (i) Let $c = f(t) \in R[t] \cap L$. Choose r greater than the degree of f. Then since $\sum_{i=1}^{n} a_i t_i = 1$, we have

$$a_i^r c \in R[t_1, \ldots, t_{i-1}, \hat{t}_i, t_{i+1}, \ldots, t_n],$$

where $\hat{}$ on t_i indicates that t_i has been omitted. As $t_i, \ldots, t_{i-1}, \hat{t}_i, t_{i+1}, \ldots, t_n$ are algebraically independent over $R, a_i^r c \in R, i = 1, \ldots, n$. Hence$\mathfrak{G}^{nr} c \subseteq R$, *i.e.*$c \in S$. Thus $R[t] \cap L \subseteq S$. On the other hand let $c \in S$. Then $c\mathfrak{G}^r \subseteq R$, for some r. Let $m_1, \ldots, m_l$ be all the monomials in the a_i of degree r. Raising the equation $\sum_{i=1}^{n} a_i t_i = 1$ to the rth power, we have $\sum_{i=1}^{l} m_i f_i = 1$, where $f_i \in R[t]$.

Since $cm_i \in R, i = 1, \ldots, t$, we have $c = \sum_{i=1}^{l} (cm_i) f_i \in R[t]$. Hence $S \subseteq R[t] \cap L$ and proof of (i) is complete.

(ii) To prove that $S = R^* \cap L$, we have only to show the inclusion $S \supseteq R^* \cap L$. Let $c \in R^* \cap L$. Then c is integral over $R[t]$ and we have a monic equation $c^s + f_1(t)c^{s-1} + \ldots + f_s(t) = 0$, $f_i(t) \in R[t], i = 1, \ldots, s$.

Let $r \geq \max_{1 \leq i \leq s}$ (degree of f_i). Then $a_i^r c$ is integral over $R[t_1, \ldots, t_{i-1},$ **43**
$\hat{t}_i, t_{i+1}, \ldots, t_n]$. As R is normal and $t_1, \ldots, t_{i-1}, \hat{t}_i, t_{i+1}, \ldots, t_n$ are algebraically independent, $R[t_1, \ldots, t_{i-1}, \hat{t}_i, t_{i+1}, \ldots, t_n]$ is normal. Hence $a_i^r c \in R$. Therefore $c\mathfrak{G}^{nr} \subseteq R$, i.e., $c \in S$. Hence $R^* \cap L \subseteq S$ and (ii is proved).

□

Let R be an affine ring over a ground field K. Let L' be a field with $K \subseteq L' \subseteq L$, where L is the quotient field of R. Then, is $R \cap L'$ an affine ring? Let us call this *Generalized Zariski's Problem.* We recall that this is just the restatement of Zarisk's Problem (see Chapter 0) without the hyper thesis of normality on R. We shall later in this chapter give a counter example to the above problem (with trans $deg_k L' = 2$).

Let R be an affine ring. By proposition 1 it follows that :

(i) If there exists an ideal $\mathfrak{G}$ in R such that $S(\mathfrak{G}; R)$ is not finitely generated, then $S(\mathfrak{G}; R)$ is a counter example to the Generalized Zariski's Problem.

(ii) If further R is normal and if there exists an ideal $\mathfrak{G}$ in R such that $S(\mathfrak{G}; R)$ is not finitely generated, then $S(\mathfrak{G}; R)$ is a counter example to Zariski's Problem.

2. Krull Rings and $\mathfrak{G}$ - transforms

We shall now proceed to prove the converse of proposition 1 (see proposition 4) in the case when R is normal. For that we need some generalities on Krull rings. We say that an integral domain R is *Krull ring* if there exists a set I of discrete valuations of the quotient field L **44** of R, such that (i) $R = \bigcap_{v \in I} R_v$, where R_v denotes the valuation ring of v. (ii) For $a \in R$, $a \neq 0$, $v(a) = 0$, for all but a finite number of $v \in I$.

Proposition 2. *An integral domain R is a Krull ring if and only if the following two conditions are satisfied:*

(i) *For every prime ideal $\mathscr{Y}$ of height 1, $R_{\mathscr{Y}}$ is a discrete valuation ring.*

(ii) *Every principal ideal of R is the intersection of a finite number of primary ideals of height one.*

For the proof of Proposition 2, we refer to: M. Nagata "Local Rings" Interscience Publishers, Now York, 1962).

Remark 1. It easily follows that $R = \bigcap R_{\mathscr{Y}}$, where $\mathscr{Y}$ runs through all prime ideals of height 1.

Remark 2. Let $R = \bigcap_{v \in I} R_v$ be a Krull ring defined by a family of discrete valuations $\{R_v\}_{v \in I}$ of the quotient field of R. Then for a prime ideal $\mathscr{Y}$ of R of height 1, we have $R_{\mathscr{Y}} = R_v$, for some $v \in I$. (For prof see M. Nagata "Local rings", Interscience Publishers, New York, 1962).

Proposition 3. *If R is a Krull ring then the $\mathfrak{G}$-transform S of R is also a Krull ring.*

Proof. Let J be the set of those prime ideals $\mathscr{Y}$ of height 1 which do not contain $\mathfrak{G}$. We shall prove that $S = \bigcap_{\mathscr{Y} \in J} R_{\mathscr{Y}}$. Let $x \in S$. Then $x\mathfrak{G}^n \subseteq R$, for some n. As $\mathfrak{G}^n \not\subset \mathscr{Y}$, for $\mathscr{Y} \in J$, $X \in \bigcap_{\mathscr{Y} \in J} R_{\mathscr{Y}}$. ◻

45 Conversely let $y \in \bigcap_{\mathscr{Y} \in J} R_{\mathscr{Y}}$. Now as R is a Krull ring, $\mathfrak{G}$ is contained in a finite number of prime ideals of height 1. Therefore there exists an $n \geq 0$ such that $y\mathfrak{G}^n \subseteq R_{\mathfrak{g}}$ where $\mathfrak{g}$ is any prime ideal of height 1 containing $\mathfrak{G}$. Hence $y\mathfrak{G}^n \subseteq \bigcap_{\mathscr{Y}} R_{\mathscr{Y}}$, $\mathscr{Y}$ running through all prime ideals of height 1. By the remark after Proposition 2, $y\mathfrak{G}^n \subseteq R$. Hence $y \in S$ and the proposition is proved.

Proposition 4. *Let R be an affine normal ring over a ground field K. Let L' be a field such that $K \subseteq L' \subseteq L$, where L is the quotient field of R. Set $R' = R \cap L'$. Then there exists an affine normal ring $\mathscr{O}$ and an ideal $\mathfrak{G}$ of $\mathscr{O}$ such that $R' = S(\mathfrak{G}; \mathscr{O})$.*

We shall prove the assertion in several steps.

(i) R' is a Krull ring.

Proof. We have $R = \bigcap\limits_{\mathscr{Y}} R_{\mathscr{Y}}$, where y runs through all prime ideals of height 1. Now $R' = \bigcap\limits_{\mathscr{Y}}(R_{\mathscr{Y}} \bigcap L')$. Hence R' is a Krull ring. ◻

(ii) For every prime ideal $\mathfrak{g}$ of height 1 in R' there exists a prime ideal $\mathfrak{g}$ of height 1 in R lying above $\mathfrak{g}$ *i.e.*$\mathfrak{g} \bigcap R' = \mathfrak{g}'$.

Proof. We may assume that L' is the quotient field of R'. We have $R' = \bigcap\limits_{\mathscr{Y}}(R_{\mathscr{Y}} \bigcap L')$, as in (i) By Remark 2 following Proposition 2, we have $R'_{\mathfrak{g}'} = R_{\mathfrak{g}} \bigcap L'$ for some prime ideal $\mathfrak{g}$ of height 1 of R. This proves (ii). ◻

(iii) There exists a normal affine ring $\mathscr{O}' \subseteq R'$ such that for every prime ideal ρ' of height 1 of R', we have height $(\mathfrak{g}' \bigcap \mathscr{O}') = 1$.

Proof. We may assume that L' is the quotient field of R'. Take a normal affine ring $R'' \subseteq R'$ such that L' is the quotient field of R''. Let Q' be the set of prime ideals $\mathfrak{g}$ of height 1 in R' with height $(\mathfrak{g}' \bigcap R'')\rangle 1$. Let T be set of prime ideals $\mathscr{Y}$ of height 1 of R such that heights $(\mathscr{Y} \bigcap R'') > 1$. Let V, V'' be the affine varieties defined by R and R'' respectively. For a $\mathscr{Y} \in T, \mathscr{Y} \cap R''$ defines isolated fundamental subvariety of V'' with respect to V under the morphism $f : V \to V''$ defined by the inclusion $R'' \subseteq R$. Hence T is finite. Therefore by (ii), Q' is finite. Let R_1 be an affine normal ring such that $R'' \subseteq R_1 \subseteq R'$. Let Q'_1 be the set of prime ideals $\mathfrak{g}'_1$ of height 1 of R' such that height $(\mathfrak{g}'_1 \bigcap R_1) > 1$. Then $Q'_1 \subseteq Q'$. We next prove that for a $\mathfrak{g}' \in Q'$, there exists 46

an affine normal ring R_1 with $R'' \subseteq R_1 \subseteq R'$ such that $\mathfrak{g}' \notin Q'_1$, where Q'_1 is as above. We claim that $R'_{\mathfrak{g}'}$ is a discrete valuation ring such that $\underset{K}{trans.deg} R'_{\mathfrak{g}'/\mathfrak{g}'R'\mathfrak{g}'} = \underset{K}{trans.deg}.\ L' - 1$. Let $\mathfrak{g}$ be a prime ideal of height 1 of R lying above $\mathfrak{g}'$ (see (*ii*)). Let $x_1, \dots x_r \in R$ be such that (i) $x_1, \dots, x_r$ are algebraically independent over $R'(ii) x_1, \dots, x_r \bmod \mathfrak{g}$ from a transcendence base for $R/\mathfrak{g}$ over $R'/\mathfrak{g}'$. Such a choice is possible since $R_{\mathfrak{g}}$ and $R'_{\mathfrak{g}'}$ are valuation rings. Since R is affine and $\mathcal{G}$ is of height 1, we have $\underset{K}{\text{trans.deg.}} R/\mathcal{G}$ $= \underset{K}{\text{trans.deg.}} L - 1$. Also $\underset{K}{\text{trans.deg.}} R/\mathcal{G} = \underset{K}{\text{trans.deg.}} R'/\mathcal{G}' + r$. Hence $\underset{K}{\text{trans.deg.}} R'/\mathcal{G}' = \underset{K}{\text{trans.deg.}} L - 1 - r \geq \underset{K}{\text{trans.deg.}} L' - 1$. Hence $\underset{K}{\text{trans.deg.}} R'/\mathcal{G}' = \underset{K}{\text{trans.deg}} L' - 1$. Since R'' is affine and height $(\mathcal{G} \bigcap R'') > 1$, we have trans. ${}^{R''}/_{(R'' \bigcap \mathcal{G}) \leq} \underset{K}{\text{trans.deg.}} L' - 2$ Let $y_1, \dots, y_l \in R'$ be such that $y_1, \dots, y_l \mod \mathcal{G}'$ from a transcendence base of $R'/\mathcal{G}'$ over ${}^{R''}/_{\mathcal{G} \bigcap R''}$. Let R''' be the derived normal ring of $R''[y_1, \dots, y_l]$. Then height $(\mathcal{G}' \bigcap R'') = 1$. Since
47 Q' is finite, in a finite number of steps we arrive at a normal affine ring $\mathcal{O}' \subseteq R'$, with the same quotient field as R' such that for every prime ideal $\mathfrak{g}'$ of height 1 of R' we have height $(\mathfrak{g}' \bigcap \mathcal{O}') \leq 1$. But $\mathcal{O}'$ and R' have the same quotient field. Therefore height $(\mathcal{O}' \bigcap \mathfrak{g}') = 1$. This proves (iii). □

(iv) Let $\mathcal{O}$ be as in (iii). Let P be the set of prime ideals $\mathscr{G}$ of height 1 in $\mathscr{G}'$ for which there does not exist any prime ideal of height 1 in R, lying over $\mathcal{G}$. Let V, V' be the affine varieties defined by R and $\mathscr{G}'$ and let $f : V \to V'$ be the morphism induced by the inclusion $\mathscr{G}' \subseteq R$. There are only a finite number of subvarieties of codimension 1 in V' to which there do not correspond any subvariety of codimension 1 in V. Hence the set P is finite. To prove Proposition 4, we take $\mathcal{O} = \mathcal{O}', \mathfrak{G} = \bigcap_{\mathscr{Y} \in P} \mathscr{Y}$. Let F be the set of prime ideals of height 1 which do not contain $\mathfrak{G}$. Then $S(\mathfrak{G}; \mathcal{O}) = \bigcap_{\mathfrak{g} \in F} \mathcal{O}_{\mathfrak{g}}$ (See Proposition 3). We have, $R' = \bigcap_{\mathfrak{g}'} R'_{\mathfrak{g}'}$, where $\mathfrak{g}'$ runs through prime ideals of R' of height 1. Now it follows easily by our construction of $\mathscr{O}$ that $R' = S(\mathfrak{G}; \mathscr{O})$.

Let R be an integral domain and let $\mathfrak{G}$ be an ideal of R. We say that the $\mathfrak{G}$-transform of R is *finite* if $S(\mathfrak{G}; R) = R[\mathfrak{G}^{-n}]$ for some $n \geq 0$.

Theorem 1. *Let R be a normal affine ring and $\mathfrak{G}$ be an ideal of R. Then the $\mathfrak{G}$-transform S of R is finite if and only if the $\mathfrak{G}$ P- transform of P is finite for any $P = R_{\mathscr{Y}}$, where $\mathscr{Y}$, is a prime ideal of R.*

Proof. Clearly $\mathfrak{G}^{-n}P = (\mathfrak{G}P)^{-n}$, for every n. Hence if $\mathfrak{G}'$ transform of R finite, then so is $\mathfrak{G}P$- transform of P, for every P. Conversely assume S is not finite. We then define an increasing sequence of normal rings by induction as follows: 48

Set $R_0 = R$. Having defined $R_j, 0 \leq j \leq i$, we define R_{i+1} as the derived normal ring of $R_i\left[(\mathfrak{G}(R_i))^{-1}\right]$. Then we have $S \supset R_i \supset R_{i-1}\left[(\mathfrak{G}_{i-1})^{-1}\right]$, where such R_i is an affine normal ring. By the definition of $\mathfrak{G}$-transform, $S = \underset{i}{\cup} R_i$. Further $\mathfrak{G}R_i$- transform of R_i is also S. We claim that height $\mathfrak{G}R_i = 1$ for every i. For if height $\mathfrak{G}R_i > 1$, then the $\mathfrak{G}R_i-$transform of R_i is integral over R_i and therefore $S = R_i$. This contradicts the assumption that S is not finite. Let $\mathfrak{G}_i$ be the intersection of those prime ideals of height 1 of R_i which contains $\mathfrak{G}R_i$. Then we have $\mathfrak{G}_i \subseteq \mathfrak{G}_{i+1}$. For let $\mathscr{R}$ be a prime ideals of height 1 in R_{i+1} containing $\mathfrak{G}R_{i+1}$. If height $(y \cap R_i = 1)$, then by definition of "$\mathfrak{G}_i$, we have $\mathfrak{G}_i \subseteq \mathscr{Y} \cap R_i$". ◻

Otherwise let height $\mathscr{R} \cap R_i \rangle 1$. Since the $\mathfrak{G}$-transform of R_i is S (see remark after the definition of $\mathfrak{G}$-transform), we have $(\mathfrak{G}R_i)^{-1}\mathfrak{G}_i^m \subseteq R_i$, for some m. Now if $\mathfrak{G}_i \not\subset \mathscr{R} \cap R_i$, then $(\mathfrak{G}R_i)^{-1} \subseteq (R_i)_{\mathscr{R}\cap R_i}$. Hence $R\left[(\mathfrak{G}R_i)^{-1}\right] \subseteq (R_i)_{\mathscr{R}\cap R_i}$. Therefore, $R_{i+1} \subseteq (R_i)_{\mathscr{R}\cap R_i}$, since R_i is normal. Hence $(R_{i+1})_{\mathscr{R}} = (R_i)_{\mathscr{R}\cap R_i}$. This contradicts the assumption that height $\mathscr{R} \cap R_i > 1$. Hence $\mathfrak{G}_i \subseteq \mathscr{R} \cap R_i$. Therefore $\mathfrak{G}_i \subseteq \mathfrak{G}_{i+1}$, for $i \geq 0$. Set $\mathfrak{G}^* = \underset{i}{\cup} \mathfrak{G}_i$. Then $\mathfrak{G}^*$ is a proper ideals of $S = \underset{i}{\cup} R_i$. Let $\mathscr{Y}^*$ be a prime 49
ideal containing $\mathfrak{G}^*$, got $\mathscr{Y} = \mathscr{Y}^* R_i 1$ and $P\mathfrak{G}R_{\mathscr{Y}}$. We now consider the

$\mathfrak{G}P$-transfer of P. We can $(R_i)_i = (R_{\mathscr{Y}})_i$ Where $T = R - \mathscr{Y}$ and $(R_{\mathscr{Y}})_i$ are defined by n infection as follows:

We set $(R_{\mathscr{Y}})_o = R_{\mathscr{Y}}$ and having defined $(R_{\mathscr{Y}})_o, \ldots, (R\mathscr{Y})_i$ we define $(R_{\mathscr{Y}})_{i+1}$ as the derived normal ring of $(R_{\mathscr{Y}})_i\left[(\mathfrak{G}(R_{\mathscr{Y}})_i)^{-1}\right]$. Since height $\mathfrak{G}(R_i)_T = 1$ by our construction, $(R_i)_T$ cannot be the $\mathfrak{G}P$-transform of P. Hence $\mathfrak{G}P$-transform is not finite.

3. Geometric meaning of the $\mathfrak{G}$-transform.

Let V be a variety. We call an affine variety V', an *associated affine variety* of V if i) $V' \supseteq V$. ii) The set of divisors of V' coincide with that $V i.e.$ the set of local rings of rank 1 of V and V' are the same.

Theorem 2. *Let F be a proper closed set of an affine variety V and let $\mathfrak{G}$ be an ideal which defines F in the affine ring R of V. Then $V - F$ has an associated affine variety if and only if the $\mathfrak{G}$-transform S of R is finite; in this case S defines an associated affine variety and S contains and is integral over the affine ring of any associates affine variety of $V - F$.*

Lemma 1. *Let R be an integral domain and let $\mathfrak{G}$ be an ideal of R. Set $R' = R\left[\mathfrak{G}^{-n}\right]$ or $S(\mathfrak{G}; R)$. Then the correspondence $\mathscr{Y}' \rightsquigarrow \mathscr{Y}' \cap R$ establishes a $1-1$ corresponding between the set of prime ideals of R' not containing $\mathfrak{G}$ and the set of prime ideals of R not containing $\mathfrak{G}$. Further, for a prime ideals $\mathscr{Y}'$ of R' with $\mathscr{Y}' \not\supset \mathfrak{G}$, we have $R'_{\mathscr{Y}'} = R_{\mathscr{Y}' \cap R}$.*

50 *Proof.* Let $\mathscr{Y}$ be a prime ideals of R which does not contains $\mathfrak{G}$. Let $a \in \mathscr{U}, a \notin \mathscr{Y}$. Then $R' \subseteq R\left[\frac{1}{a}\right]$. Since $\mathscr{Y}R\left[\frac{1}{a}\right]$ is a prime ideal, so is $\mathscr{Y}' = \mathscr{Y}R\left[\frac{1}{a}\right] \bigcap R'$. Further $\mathscr{Y}' \bigcap R = \mathscr{Y}$ and $R'_{\mathscr{Y}'} = R_{\mathscr{Y}}$.

Conversely if $\mathscr{Y}'$ is a prime ideal of R' which does not contain $\mathfrak{G}$, then $\mathscr{Y} = \mathscr{Y}' \bigcap R$ does not contain $\mathfrak{G}$ and $R_{\mathscr{Y}} = R'_{\mathscr{Y}'}$. □

Lemma 2. *Let $\mathfrak{G}$ be an ideal of a noetherian domain R. Let S be the $\mathfrak{G}$-transform of R and R' a subring of S containing R. Then the $\mathfrak{G}R'$-transfer of R' is S.*

Proof. Let S' be the $\mathfrak{G}R'$-transform of R'. We have only to prove that $S' \subseteq S$. Let $f \in S'$. Then there exists an n such that $f\mathfrak{G}^n \subseteq R'$. Let $a_1, \ldots, a_l \in \mathfrak{G}^n$ generate $\mathfrak{G}^n$. There exists an m such that $fa_i\mathfrak{G}^m \subseteq R$. Hence $f\mathfrak{G}^{n+m} \subseteq R$ and the lemma is proved. □

Corollary. *With the above notation, if $a \in S$, then $aS : \mathfrak{G}S = aS$ and consequently $\mathfrak{G}S$ is not of height 1.*

Proof. Let $f \in S$, $f\mathfrak{G} \subseteq S$ a i.e. $\frac{f}{a}\mathfrak{G} \subseteq S$. $\frac{f}{a} \in S$.

We now prove Theorem 2. Suppose S is finite. Then S is an affine ring and defines an associated affine variety by lemma 1 and Corollary. to Lemma 2. Conversely assume that V' is an associated affine variety of $V - F$. Let R' be the affine ring of V'. Let $x' \in R'$. Set $\mathfrak{G}_{x'} = \left\{y \mid y \in R, yx' \in R\right\}$. Since $x' \in R_{\mathscr{Y}}$, for $\mathscr{Y} \not\supset \mathfrak{G}$ (by the hypothesis) we have $\mathfrak{G}_{x'} \not\subset \mathscr{Y}$, for $\mathscr{Y} \not\supset \mathfrak{G}$. Hence $\mathfrak{G}_{x'}$ contains a power of $\mathfrak{G}$. Therefore $x' \in S$ i.e., $R' \subseteq S$. Since the divisors of V' and $V - F$ are the same, height $(\mathfrak{G}R') \geq 2$. Hence S is integral over R'. Hence S is an affine ring and therefore finite.

Let V be an affine variety defined by an affine ring R and let F be a closed set defined by an ideal $\mathfrak{G}$. □ **51**

Theorem 3′ *The variety $V - F$ is affine if and only if $1 \in \mathfrak{G}S$, where S is the $\mathfrak{G}$-transform of R. In this case F is pure of codimension 1 and S is the affine ring of $V - F$.*

Proof. Suppose $V-F$ is affine. Then $V-F$ is an associated affine variety of $V - f$. Let R' be the coordinated ring of $V - F$. Then $R' \subseteq S$ (by Theorem 2). Now $1 \in \mathfrak{G}R'$. Hence $1 \in \mathfrak{G}S$. Conversely suppose that $1 \in \mathfrak{G}S$. Then $1 \in \mathfrak{G}\mathfrak{G}^{-n}$ for some n. Set $R' = R\left[\mathfrak{G}^{-n}\right]$. Since $\mathfrak{G}R' \ni 1$, Lemma 1 of Theorem 2, proves that the affine variety defined by R' is $V - F$. $R' = S$ because $\mathfrak{G}R' \ni 1$ (by virtue of Lemma 2). □

It now remains to prove that F is pure of codimension 1 if $V - F$ is affine. Suppose the contrary. Let F_1 be an irreducible component of F with codimension $F_1 > 1$. Let $f \in R$, with f not vanishing on F_1 and vanishing on all tho other irreducible components. Then considering $R\left[\frac{1}{f}\right]$ we may suppose that $F = F_1$. But this would mean that S is integral over R. Hence $1 \notin \mathfrak{G}S$. Contradiction.

Theorem 3. *$V - F$ is an affine variety if and only if $\mathfrak{G}(\mathfrak{G}(P)^{-n(P)} \ni 1$ for every local ring P of F (the integer $n(P)$ depending on P).*

Proof. If $V - F$ is affine, then by Theorem 3, $1 \in \mathfrak{G}S$ i.e $1 \in \mathfrak{G}\mathfrak{G}^{-n}$ for some n. Hence $1 \in \mathfrak{G}(\mathfrak{G}P)^{-n}$ for every P of F. Conversely assume that $1 \in \mathfrak{G}(\mathfrak{G}P)^{-n(P)}$ for every P of F. Then by lemma 2 of Theorem 2, the $\mathfrak{G}P$-transform of P is finite for every P of F. Hence by Theorem 2, S is
52 finite. Then $1 \in \mathfrak{G}S$ and $V - F$ is an affine variety. □

Corollary 1. *If V is an affine variety and F a divisorial closed subset of F such that some multiple of F is locally principle, then $V - F$ is an affine variety.*

Corollary 2. *If V is a non-singular affine variety and F a divisorial closed subset of V, then $V - F$ is an affine variety.*

Corollary 3. *If V is an affine curve and F a closed subset of V, then $V - F$ is again affine.*

Proof. It is sufficient to prove the Corollary when F consists of a single point P. If P is normal then by theorem 3′, $V - F$ is affine. If P is not normal we consider the derived normal ring P' of P. Let C be the conductor of P' with respect to P. Then $\mathfrak{G}^n \subseteq C$ for some n, $\mathfrak{G}$ being an ideal which defines F. By considering $R\left[\mathfrak{G}^{-n}\right]$ we are reduced to the case when F consists of normal points. This proves the Corollary. □

4. Zariski's theorem and some related results.

Theorem 4 (Zariski's)**.** *Let R be an affine ring over a ground filed K and let Ω be the quotient field of R. Let L be a subfield of Ω containing K.*

(1) *If* trans.deg $._K L = 1$ *, then $R \cap L$ is an affine ring*

(2) *If R is normal and* $\underset{K}{\text{trans.deg}} L = 2$ *, then $R \bigcap L$ is an affine ring.*

Proof of (1) is a consequence of Proposition 4, Theorem 2 and Corollary 3 to theorem 3′.

Proof of (2) By virtue of Proposition 4, (2) is contained in the following theorem:

Theorem 4′ *Let R be an affine normal ring of dimension 2 over a ground field K. Then for any ideals $\mathfrak{G}$ if R the $\mathfrak{G}$-transform $S(\mathfrak{G}, R)$ is finite.* **53**

We require the following lemma

Lemma. *Let S be a Krull ring, $\mathfrak{g}'$ a prime ideal of height 1. Let $\mathfrak{G}$ be an ideal of S such that $\mathfrak{G}R_{\mathfrak{g}} = \mathfrak{g}R_{\mathfrak{g}}$. Then $\mathfrak{G} : \mathfrak{g} \not\subset \mathfrak{g}$.*

Proof. Since R is a Krull ring $R_{\mathfrak{g}}$ is a discrete valuation ring.Therefore there is an $a \in \mathfrak{G}$ such that $aR_{\mathfrak{g}} = \mathfrak{g}R_{\mathfrak{g}}$. Since R is a Krull ring, $aR = \mathfrak{g} \cap \mathfrak{g}_1 \cap \ldots \cap \mathfrak{g}_n, \mathfrak{g}_i$ being primary ideals of height 1 different from $\mathfrak{g}$. Then $\mathfrak{g} \not\supset \mathfrak{g}_1 \cap \ldots \cap \mathfrak{g}_n = aR : \mathfrak{g} \subseteq \mathfrak{G} : \mathfrak{g}$. □

Proof of Theorem 4′**.** We may assume that the ideal $\mathfrak{G}$ is pure of height 1. Choose an element $b \in \mathfrak{G}$ such that $\mathscr{V}_{\mathscr{Y}_i}(b) = n_i, 1 \leq i \leq r$ where $\mathscr{U}_{\mathscr{Y}_i}$ is the normal valuation corresponding to $\mathscr{Y}_i$. Then $\mathscr{U} \cap \mathscr{U}^1 = bR$ where $\mathscr{U}'$ is an ideal pure of height 1 such that $\mathscr{U}$ and $\mathscr{U}'$ do not have common prime divisors of height 1. We have $\mathfrak{G}S = bS$ (cf. Cor to Lemma 2, p.50). Choose an element a $a \in \mathfrak{G}$ such that a is not contained in any of the prime divisors of $\mathfrak{G}$ and $\mathscr{V}_{\mathscr{Y}_i}(a) > \mathscr{V}_{\mathscr{Y}_i}(b) = n_i, 1 \leq i \leq r$. Let x be a transcendental element over R. Extend the ground field K to $K(x)$. We remark that the $\mathfrak{G}K(x)[R]$-transform of $R' = K(x)[R]$ is finite if and only if the $\mathfrak{G}$-transfer of R is finite. Now $\mathfrak{G}R' = \mathscr{K}_1^{(n_1)} \bigcap \ldots \bigcap \mathscr{K}_r^{(n_r)}$, where $\mathscr{K}$ is the centre on R' of the valuation $\mathscr{V}_{\mathscr{K}_i}$ on $\Omega(x)$ define by $\mathscr{V}_{\mathscr{K}_i}(\sum a_1 x^i) = \min\limits_j \mathscr{V}_{\mathscr{Y}_i}(a_j)$. The element a and b do not have a common prime divisor in S. The choice of the element a and the following any lemma show that we may assume that bS is a prime ideal. **54**

Lemma. *Let S be a Krull ring and let $a, b \in c$ with a, b not having a common prime divisor. Then $ax - b$ prime in $S[x]$.*

Now assume that the $\mathfrak{G}$-transform S of R is not finite. Then as in the proof of Theorem 1 (of this chapter), there exist normal local rings $(P_i, \mathscr{W})$, $0 \leq i < \infty$ such that $P_0 = R_{\mathscr{Y}}$ for suitable prime ideal $\mathscr{Y} \supset \mathfrak{G}$ and $(P_i, \mathscr{W}_i)$ dominates $(P_{i-1}, \mathscr{W}_{i-1})$. Furthermore $S^* = \bigcup_i P_i = S_M$, where M is a maximal ideals of S. Set $\mathscr{W}^* = \cup \mathscr{W}_i$. Consider $\mathscr{W} = \mathscr{W}^* \cap R$. Since $bS \cap R = \mathfrak{G}'$, the canonical mapping $\varphi : R/\mathscr{V}' \to S/bS$ is an injection. For $s \in S$, we have $s\mathfrak{G}^m \subseteq R$ for some m. Since $\mathfrak{G}^m \not\subset \mathfrak{G}'$ is of dimension 1, by the theorem of Krull-Akizuki (see M. Nagata "Local rings", Theorem 33.2, p.115) S/bS is noetherian and S/M is of finite length over $R/\mathscr{W}$. In particular $\mathscr{W}^*$ is finitely generated. Hence exists all such that $\mathscr{W}_1 S^* = \mathscr{W}^*$ and that $S^*/\mathscr{W}^*$ is of finite length over $P_1/\mathscr{W}_1$.

We now proceed to prove that S^* is noetherian. Since height $(\mathscr{W}^*) = 2$, by virtue of a theorem of Cohen we need only prove that every prime ideal of height 1 of S^* is finitely generated. (see M. Nagata, "Local rings" Theorem 3.4, p.8).

Let $\mathfrak{g}^*$ be a prime ideal of height 1. Set $\mathfrak{g} = \mathfrak{g}^* \cap R$. Then $S^*\mathfrak{g}^* = R_{\mathfrak{g}}$. Hence by the lemma proved we have $\mathfrak{g}'' = S^* : \mathfrak{g} \not\subset \mathfrak{g}^*$. But $\mathfrak{g}S^* \subseteq$
55 $\mathfrak{g}''$. We claim that height $\mathfrak{g}'' \geq 2$. For if $\mathfrak{g}'' \subseteq \mathcal{R}$, a prime ideal of height 1 of S^*, then 1. $\mathfrak{g} = S^*_{\mathcal{K}} = S^*_{\mathfrak{g}^*}$. Hence $\mathscr{K} = \mathfrak{g}^*$ a contradiction to the fact that $\mathfrak{g}''4''$ |.Hence either $\mathfrak{g}'' = S^*$ or $\mathfrak{g}''$ *is* $\mathscr{N}\mathscr{W}$-primary. Hence $\mathscr{N}\mathscr{W}^{*t} \subseteq \mathfrak{g}''$ for some t. Since $S/\mathscr{N}\mathscr{W}^{*r}$ is artinian and $\mathscr{N}\mathscr{W}^{*r}$ is finitely generated we conclude that $\mathfrak{g}''$ is finitely generated. Now $\mathfrak{g}''/_{\mathfrak{g}\mathfrak{g}''}$ is a finitely generated over $S^*/\mathfrak{g}^*$. But in $S^*/\mathfrak{g}^*$ the only prime ideals are $\boxplus^*/\mathfrak{g}^*$ and (0) and $S^*/\mathfrak{g}^*$ is Noetherian. Hence $\mathfrak{g}^* \cap \mathfrak{g}''/\mathfrak{g}^*\mathfrak{g}''$ is finitely generated. Hence $\mathfrak{g}'' \cap \mathfrak{g}^* \diagup \mathfrak{g}S^*$, being residue class module of $\mathfrak{g}'' \cap \mathfrak{g}^* \diagup \mathfrak{g}^*\mathfrak{g}''$, is finitely generated. Hence $\mathfrak{g}$" $\cap\, \mathfrak{g}^*$ is finitely generated. Since $S^*/\mathfrak{g}''$ and $S^*/\mathfrak{g}^*$ are noetherian, we have $S^* \diagup \mathfrak{g}'' \cap \mathfrak{g}^*$ is noetherian. Hence $\mathfrak{g}^*/\mathfrak{g}'' \cap \mathfrak{g}^*$ is finitely generated. Hence $\mathfrak{g}^*$ is finitely generated. Hence S^* is noetherian.

The ring, P_1 being a geometric normal local ring, is analytically normal. Further S^* and P_1 have the some quotient field, $\mathscr{N}\mathscr{W}_1 S^* = \mathscr{N}\mathscr{W}^*$ and $S^*/_{\mathscr{N}\mathscr{W}^*}$ is of finite length over $R/_{\mathscr{N}\mathscr{W}^*}$ and therefore over $P_1/\mathscr{N}\mathscr{W}_1 P_1$. Hence by Zariski's main Theorem (see M. Megata, "Local ring" Theorem 37.4 , P.137) $P_1 = S^*$. This is contradiction to the construction of the P_i.

Theorem 5. *Let V be a normal affine variety of dimension 2 and lot F be a divisorial closed subset of V. Then $V - F$ is affine.*

Proof. Let $\mathscr{U}$ be the ideal defining F. By Theorem 4 the $\mathscr{U}$-transfer S of R is finite. Let V' be the affine variety defined by S. Since height $\mathscr{U} \leq 2$ and V is of dimension 2, $V-F$ is isomorphic to an open subset V'' of V' such that $V' - V''$ consists of atmost a a finite number of points. 56
Let $\times' \in V' - V''$. Consider the morphism $V' \to V$ induced by the inclusion $R \subseteq S$. Since $f_{-1} f(\times')$ is discrete, by Zariski's Main Theorem f is bio-holomorphic at x'. Let W be an irreducible component of F passing through $f(x')$. Since x' and $f(X')$ are bio-holomorphic there is subvariety of codimension 1 of V' passing through x' and laying over W. This contradicts the fact that V' is the associated affine variety of $V - F$. Hence $V - V''$ is empty and $V - F$ is affine. □

We now proceed to give an example to show that Theorem 4(2) is false if we do not assume that R is normal. Take $R = K[X, Y, Z]/(f)$. Where

(1) $f(X, Y, Z) = Y(Z + YT) + X(u_1 YZ + U_2 Z^2)$

(2) f is irreducible

(3) $T, U_1, \in U_2 \in K[X, Y]$.

Let the image X, Y, Z, T, U_1, U_2 be denoted by x, y, z, t, u_1 *and* u_2 respectively so that $R = K[x, y, z] = 0$. Set $\mathscr{Y} = (x, y)$. Since $R/_{\mathscr{Y}} = K[X, Y, Z]/_{(f,X,Y)} \approx K[Z]$, the ideal $\mathscr{Y}$ is prime. Similarly $\mathfrak{g} = (y, z)$ is prime. We shall show that the $\mathscr{Y}$-transfer of R is not finite. We first prove that

(*) $\mathscr{Y}^{-1} = R + z_1 R, z_1 = (z + yt)/X.$

Proof. We have $(x) = \mathscr{Y} \cap (x, 2 + yt)$. For let $\lambda x + \mu y = ax + \beta(z + yt)$. Then $\mu y^2 \in (x)$ and therefore $\mu y^2 t \in (x)$ i.e. $\mu yz \in (x)$. But z is not a zero divisor module (x). Hence | $(y \in (x)$. Therefore $(x) = \mathscr{Y} \cap (x, z + yt)$. Lot now $g \in \mathscr{Y}^{-1}$. Then $g = \frac{\gamma}{x} = \frac{\gamma}{y}$ $i.u.\gamma \in (x) : (y)$. Since $(x) = \mathscr{Y} \bigcap (x, Z + yt)$, we have $(x) : (y) = (x, Z + yt) : (y)$. 57
But $\ell \in K[x, y]$ and therefore y is not a zero divisor module $(x, z +$

$yt)$. Hence $(x, Z + yt) : (y) = (x, Z + yt)$. Hence $\gamma \in R + z_1 F$. This proves (*). Set $R_1 = R[\mathscr{Y}^{-1}] = R[z_1] = K[x, y, z_1] \quad K[X, Y, Z_1]/_{\mathscr{U}}$ (say) where $Z_1 = (Z + YT)/_X$. We have $f_1(X, YZ) = X f_1(X, Y, Z_1)$ where $f_1(X, YZ) = X f_1(X, Y, Z_1) = Y(Z_1 + Y t_1) + X(U_1' Y Z_1 + U_2' Z_1^2), T_1 = U_2 T^2 - U_1 T, U_!' = U_1 - 2U_2 T, U_2' = U_2' X$. Now $f(x, y, z) = X f_1(x, y, z_1) = 0$. Hence $f_1(x, y, z_1) = 0$ *i.e.* $f_1(x, y, z_1) = 0 \in \mathscr{U}$. We claim next that the element $f_1(x, y, z_1) = 0$ is prime in $K[X, Y, Z_1]$. Suppose $g_2(x, y, z_1)$. Then one of the g_1 say g_2 is a unit in $K\left[X, Y, Z, \frac{1}{X}\right]$. Thus $f_1(x, y, z_1) = X^r g_1(x, y, z_1) = 0$ for some $r \geq 0$. But $f_1(x, Y, Z_1)$ is not divisible by in $K[X, Y, Z_1]$. hence $f_1(X, Y, Z_1)$ is irreducible. Hence it follow that $R_1 = R[\mathscr{Y}^{-1}] = K[X, Y, Z_1]/_{(f_1)}$. Further f_1 satisfied the same condition as f. Proceeding in the same way we see that the $\mathscr{Y}$-transform S of R is obtained by the successive adjunction of elements $Z_1, Z_2, Z_3, \ldots$, where $Z_{n+1} \frac{z_n + y t_n}{X}, tn \in K[x, y]$. This shows that S is not finite. □

5. Rees' counter example.

Let K be a field of an arbitrary characteristic and lot C be a non-singular plane cubic curve defined over K. For a natural number n and a fixed point Q of $C, T_n = \left\{P \mid np \text{ or } nQ\right\}$ is a finite act, because C is of positive genus. We choose here Q to be a point of inflexion (it is well known that a non-singular plan cubic has 9 points of inflexion). Then
58 $P \in T_{3d}$ if and only if there is a plane curve C_d of degree d such that $C_d.C = 3dP$. Thus we are that the set of points P and C, such that $C_d.C$ is a multiple of P on suitable curve C_d of positive degree, is a countable set. Therefore there is a point P of C such that no np $(0. > 0)$ is linearly equivalent to any $C.C_d$ (on C). (Note that the system of $C_d.C$ is complete linear this follows from the arithmetic normality of C.) We fix such a point P also and we enlarge K, if necessary, so that P and Q are rational over K.

Let $H = K[x, y, z]$ be the homogeneous coordinate ring of $C, (x, y, z)$ being a generic point of C over K. Let $\mathscr{Y}$ be the prime ideals of H which defines P. Let t be a transcendental element over H and consider the ring S generated by all of at^{-n} with $a \in \mathscr{Y}^{(n)}$ (n runs through all natural numbers) over $H[t]$.

We want to show that:

(1) S is not finitely generated over K.

(2) There is a normal affine ring R such that $S = R \cap L$,where L is the field of quotient of S.

This S is the Roes' counter example to Zariski problem in the case of transcendence degree 3.

Proof of (1). Consider $\deg \mathscr{Y}^{(n)}$ (=minimum if degree of elements of $\mathscr{Y}^{(n)}$). if a homogeneous element h of degree d is in $\mathscr{Y}^{(n)}$, then h defines $C_d.C$ with a suitable plane curve C_d of degree d. Since $h \in \mathscr{Y}^{(n)}, C_d.C$ contains nP. By the choice of $P, C_d.C \neq nP$, whence $3d = degC_d.C >$ n and $d > n/3$. Let σ be one of 1, 2, 3 and such that $m'\sigma$ in divisible by 59
3. Set $d = (n + \sigma)/3$. Since C is an abelian variety, there is a point R of C such that $n(P - Q) + (R - Q) \sim 0$. Then

$$nP + R + (\sigma - 1) \sim 3U \sim C_d.C,$$

where C_d is a curve of degree d. Since C is a non-singular plane curve, the system of all $C_d.C$ (with fixed d) is a complete linear system, whence there is a C_d such that $nP + R + (\sigma - 1)Q = C_d.C$. Let h be the homogeneous form of degree d in H defined by C_d. Then $h\ \mathscr{Y}(n)$. Thus we see that

$$(*) \qquad \deg \mathscr{Y}^{(n)} = (n + \sigma)/3, 1 \leq \sigma \leq 3.$$

This $(*)$ being shown, we see that S is not finitely generated over K by the same way an in the construction of the fourteenth problem in Chapter III.

Proof of (2). We first show that

$$(**) \qquad S = H[t, t^{-1}] \cap V,$$

where V is the valuation ring obtained as follows:

Let p be a prime element of the valuation ring $H_{\mathscr{Y}}$. Then t/p is transcendental over $H_{\mathscr{Y}}$, whence we have a valuation ring $H_{\mathscr{Y}}(t/p)(= H_{\mathscr{Y}}[t/p]_{pH_{\mathscr{Y}}[t/p]}).H_{\mathscr{Y}}(t/p)$ is independent of the particular choice of p and this valuation ring is denoted by V.

It is obvious that $S \leq H[t, t^{-1}] \cap V$. Let f be an arbitrary element of $H[t, t^{-1}] \cap V$. Then f is of the form $\sum a_i t^i$ (finite sum) with $a_i \epsilon H$ 60
(i may be negative). Let v be a valuation defined by V. Then by the construction of $V, v(\sum a_i t^i) = \min V(a_i t^i) \geq 0$. Therefore $a_i \epsilon \mathscr{Y}^{(-i)}$ if $i < U$, which implies that $f \epsilon S$. Thus $(**)$ is proved.

This being settled, it remains only to prove the following lemma by virtue of Proposition 1 (§1).

Lemma. *Let R be a normal affine ring of a function field L over a field K and let $V_1, \ldots, V_n$ be divisional valuation rings of L (i.e., V_i are discrete valuation rings of L over K such that trans.* $\deg_k L - 1 =$ *trans.* deg. *of the residue class field of V_i). Then there is a normal affine ring $\mathcal{V}$ with an ideal $\mathfrak{G}$ such that $R \cap V_1 \cap \cdots \cap V_n = S(\mathfrak{G}; \mathcal{V})$.*

The proof is substantially the same as that of Proposition 4 (§2) and we omit the detail.

Chapter 6

Complete reducibility of rational representation of a matrix group

This chapter is mostly a representation of M. Nagata: Complete re- 61
ducibility of rational representation of a matric group, J. Math. Kyoto Univ. 1-1 (1961), 87-99.

It is well known in the classical case that every rational representation of a semi-simple algebraic linear group is completely reducible. But the same argument becomes false in the case where the universal domain is of characteristic $p \neq 0$. For instance, when K is a universal domain of characteristic 2, the simple group $SL(2, K)$ has the following rational representation ρ which is not completely reducible:

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac & bd \\ 0 & a^2 & b^2 \\ 0 & c^2 & d^2 \end{pmatrix}.$$

(This ρ is not completely reducible because ac and bd are not linear polynomials in a^2, b^2, c^2, d^2.) Therefore it is an interesting question to ask conditions for an algebraic linear group G so that every rational representation of G is completely reducible.

Now, our answer of the above question, can be stated as follows:

(1) When $p \neq 0$: Every rational representation of G is completely reducible if and only if there is a normal subgroup $G_\circ$ of finite index such that $(i) G_\circ$ is a subgroup of a torus group (i.G., diagonalizable) and (ii) the index of $G_\circ$ in G is prime to p. If G is
62 connected, then the above condition is equivalent to the condition that the representation of G by homogeneous forms of degree p is completely reducible. On the other hand, if G is an algebraic group (which may not be connected), then the complete reducibility of all rational representations of G is equivalent to the condition that every element of G is semi-simple (i.e., diagonalizable).

(2) When $p = 0$: Each of the following two conditions is equivalent to the complete reducibility of all rational representations of G.

(I) The closure of G has a faithful rational representation which is completely reducible.

(II) The radical of the closure of G is a torus group.

We shall prove also the following interesting theorem concerning the complete reducibility of rational representations of a connected algebraic linear group:

If G is a connected algebraic linear group, then every rational representation of G is completely reducible if (and only if) the following is true:

If $\rho' = \begin{pmatrix} 1 & \tau \\ 0 & \rho \end{pmatrix}$ is a rational representation of G, then ρ' is equivalent to the representation $\begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$.

1. Preliminaries on connected algebraic linear groups.

63 Throughout this chapter, K denotes a universal domain of an arbitrary characteristic, unless the contrary is explicitly stated. Let G be a connected algebraic linear group contained in $GL(n, K)$. A Borel subgroup B of G is defined to be a maximal connected solvable subgroup of G. Then as was proved by Borel, the following is true:

Lemma 1. *The homogeneous variety G/B is a projective variety. On the other hand, every element of G is in some conjugate of B.*

Now we have:

Lemma 2. *If $u \in G$ is unipotent, G being a connected algebraic linear group, then there is a closed connected unipotent subgroup of G which contains u.*

Proof. By the last half of Lemma 1, we see that u is in a Borel subgroup B of G. Since B is solvable, the set U of all unipotent elements of B is a closed connected subgroup, which proves the assertion. □

On the other hand, the following was proved by Borel:

Lemma 3. *If a connected algebraic linear group G consists merely of semi-simple elements, then G is commutative, hence is a torus group.*

Next we shall concern with an algebraic group which is not connected:

Lemma 4. *Let G be an algebraic linear group and let $G_\circ$ be the connected component of the identity of G. Then each coset $G_\circ g (g \in G)$ contains an element of finite order.*

Proof. Let A be the smallest algebraic group containing g and let $A_\circ$ be the connected component of the identity of A. Since A is commutative and since $A_\circ$ is infinitely divisible (in the additive formulation), $A_\circ g$ 64
contains an element of finite order, which proves the assertion. □

2. Preliminaries on group representations.

Let G be an abstract group, let $G_\circ$ be a normal subgroup of G and let K be a field of characteristic p which may be zero, throughout this section, except for in Lemmas 8 and 9.

The following lemma is well known :

Lemma 5. *If a finite K-module M is a simple $K-G$-module, then M is the direct sum of a finite number of $K-G_\circ$–modules which are simple.*

Corollary . *If a representation ρ of G in $GL(n, K)$ is completely reducible, then the restriction of ρ on $G_\circ$ is completely reducible.*

The converse of the above Corollary is not true in general if $p \neq 0$, but we have:

Lemma 6. *Let ρ be a representation of G in $GL(n, K)$. If the restriction $\rho_\circ$ of ρ on $G_\circ$ is completely reducible and if the index $t = [G : G_\circ]$ is finite and not divisible by p, then ρ itself is completely reducible.*

Proof. If ρ is not completely reducible, then ρ contains a representation of the form $\begin{pmatrix} \rho_1 & \tau \\ 0 & \rho_2 \end{pmatrix}$ which is not completely reducible and such that ρ_1, ρ_2 are irreducible. Hence we may resume that $\rho = \begin{pmatrix} \rho_1 & \tau \\ 0 & \rho_2 \end{pmatrix}$ and that ρ_1, ρ_2 are irreducible. Let the representation module of ρ be M^*.M^* contains the representation module M of ρ_2 and M^*/M is the representation
65 module of ρ_1. Since ρ_0 is completely reducible, we see that M is a direct summand of M^* are a $G_\circ - K-$ module. Hence $M^* = M \oplus N_1 \oplus \cdots \oplus N_r$, where N_i are simple $G_\circ - K-$ modules. For each N_i we fix a linearly independent basis $a, \ldots, a_i s$ over K; we note here that the number s is independent of i because M^*/M is a simple $G - K$-module (remember the well know proof of Lemma 5). For each $(r, s)-$ matrix $b = (b_{ij})$ over the module M, we define $N(b) = \sum_{ij}(a_{ij} + b_{ij})K$. Thus we have a one-one correspondence between all of b and all of submodules N such that $M^* = M \oplus N$ as a K-module. We may assume, on the other hand, that ρ_1 is given by the linearly independent basis $a_{11}, \ldots, a_{rs}$ modulo M of M^*/M. Each $g \in G$ defines a linear transformation $f(g)$ on the module of (r, s)-matrices over M^* as follows: If $(x_{11}, \ldots, x_{rs})\rho_1(g) = (y_{11}, \ldots, x_{rs})$, then $(x_{ij}).f(g) = (x_{ij})$. We define also an (r, s) - matrix $c(g)$ over M by the relation $N(c(g)) = N(0)^g$. If b and b' are such that $N(b)^g = N(b')$, then we have $(a_{ij} + b_{ij})^g = (a_{ij} + b'_{ij}).f(g)$. Since $(a_{ij})^g = (a_{ij} + c(g)).f(g)$, we see that $b' = c(g) + b^g.f(g)^{-1}$. Thus:

$$N(b)^g = N(c(g) + b^g.f(g)^{-1}). \tag{1}$$

If we apply this formula to the case where $b = c(h)$ with $h \in G$, then we have

$$c(hg) = c(g) + c(h)^g.f(g)^{-1}. \tag{2}$$

Now, let $g_1 \ldots, g_t$ be such that $G = \sum G_\circ g_i$ and we set $d = t^{-1}$ $(\sum c(g_i))$. We want to show that $N(d)^g = N(d)$ for every $g \in G$. Indeed, $c(g) + d^g . f(g)^{-1} = c(g) + t^{-1}(\sum c(g_i)^g . f(g)^{-1}) = c(g) + t^{-1}(\sum(c(g_i g) - c(g)) = t^{-1}(\sum c(g_i g)) = d$. Therefore $M^* = M \oplus N(d)$ is a representation **66**
module of G, which completes the proof. ◻

Corollary. *If $p = 0$, then the coverer, of the Corollary to Lemma 5 is true, provided that the index $[G : G_\circ]$ is finite.*

Lemma 7. *Let H be a subgroup of finite index of G. If a representation ρ of H in $GL(n, K)$ is not completely reducible, then the representation ρ^* of G included by ρ is not completely reducible.*

Proof. Let M be the representation module of ρ. M contains an $K - H$-module N which is not a direct summand of M. Let M^* be the representation module of ρ^*. Then M^* is of the form $M \oplus \sum M_{g_i}$ where g_i are such that $G = H + \sum H_{g_i}(g_i \notin H)$. It is obvious that $\sum M_{g_i}$ is H-admissible. M^* contains $N^* = N \oplus \sum N_{g_i}$. If N^* is a direct summand of M^* as a $G - K$-module , then we have $M \oplus \sum M_{g_i} = N \oplus \sum N_{g_i} \oplus N'$ as an $H - K -$ Module . Then we see that $M = N \oplus (M \cap (\sum N_{g_i} + N'))$ as an $H - K-$ module, which is a contradiction. Hence N^* is not a direct summand of M^* and ρ^* is not completely reducible. ◻

Corollary. *If a finite group G^* has order which is divisible by p, then G^* has a representation which is not completely reducible.*

Proof. G^* has an element a whose order is p. Then the sub-group $\{a^i\}$ is represented by $\left\{ \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \right\}$, and we see the assertion by Lemma 7. ◻

Next we observe relationship between rational representations of a matric group G and those of the closure of G.

Lemma 8. *Let G be a matric group and let G^* be the closure of G. Let* **67**
ρ^* be a rational representation of G^* and let ρ be the restriction of ρ^* on G. Then ρ is irreducible if and only if ρ^* is irreducible. ρ is completely reducible if and only if ρ^* is completely reducible.

Proof. $\rho(G)$ is dense in $\rho^*(G^*)$ and we see the assertions easily. ◻

Lemma 9. *Let N be a normal subgroup of a matric group G and let ρ be an irreducible rational representation of G into $GL(n, K), K$ being an universal domain. If N consists only of unipotent elements, then N is contained in the kernel of the irreducible representation ρ.*

Proof. Since the set of all unipotent matrices in $GL(n, K)$ is closed, and since the image of unipotent element under a rational representation is again unipotent, the closure N^* of $\rho(N)$ consists only of unipotent elements. Therefore N^* is nilpotent, hence is solvable. Therefore we may assume that every element (a_{ij}) of $\rho(N)$ is such that $a_{ij} = 0$ if $i > j$, whence $a_{ii} = 1$ for every i. On the other hand, the Corollary to Lemma 5 says that the restriction of ρ on N is completely reducible, whence $\rho(N)$ must consist only of the identity, which completes the proof. □

3. The main result in the case where G is connected and $p \neq 0$.

Theorem 1. *Let K be a universal domain of characteristic $p \neq 0$ and let G be a connected matric group contained in $GL(n, K)$. Then the following three conditions are equivalent to each other:*

(I) *Every rational representation of G is completely reducible.*

(II) *G is contained in a term group, i.e., there is an element a of $GL(n, K)$ such that $a^{-1}Ga$ is a subgroup of the diagonal group.*

68 (III) *The representation of G by homogeneous forms of degree p is completely reducible.*

Proof. It is obvious by virtue of Lemma 8 that such of the above conditions for G is equivalent to that for the closure of G. Therefore we may assume that G is a connected algebraic linear group. It is well known that (II) implies (I) and it is obvious that (I) implies (III). Thus we have only to show that (III) implies (II). Assume that (III) is true and that (II) is not true and we shall lead to a contradiction. Lemma 3 shows that G contains an element g which is not semi-simple. Then the unipotent part g_u of g is different from the identity and is contained in G (cf. Borel's paper "Groupes leneaires algebriques, Ann. of Math 64, No.1 (1956) 20-82), hence G contains a connected closed unipotent subgroup $U \neq 1$

by Lemma 2. The representation module F_p of the representation $_p$ of G by homogeneous forms of degree p is nothing but the module of homogeneous forms of degree p in n variables $X_1, \ldots, X_n$ on which element g of G operates by the rule $h(X_1, \ldots, X_n)^g = h((X_1, \ldots, X_n)g)$. F_p contains $M = \sum X_i^p K$, which is also a representation module of G. Hence (III) implies that M is a direct summand of F_p. Thus $F_p = N \oplus M$. For each monomial $n_{i_1 \ldots i_n} = X_1^{i_1} \ldots X_n^{i_n}$ with i_j such that $i_j < p$ and $\sum i_j = p$, there is a uniquely determined element $m_{i_1 \ldots i_n}$ of M such that $f_{i_1 \ldots i_n} = n_{i_1 \ldots i_n} + m_{i_1 \ldots i_n}$ form linearly independent basis for N. We note that N and M are representation modules of U. Hence we have only to show that: □

The decomposition $F_p = N \oplus M$ as a representation module of the connected closed unipotent group Ulead us to a contradiction.

Let $u = (u_{ij})$ be a generic point of U over the universal domain K. 69
We may replace U with conjugate of U. Hence we may assume first that $u_{ij} = 0$ if $i > j$, where $u_{ij} = 1$ for i. set $K^* = K(\left\{u_{ij}^p\right\})$, and we choose $(K, 1)$ that $u_{k1} \notin 1^*, u_{ijj \in k^*, u_{ij} \in K^*}$ if $i > K$ and such that $u_{kj} \in K^*$ if $j > 1$. for each $A^{-1}UA$ (A being a triangular unipotent matrix), we can associate such $a(k, 1)$ and we may assume that the pair $(k, 1)$ for U is lexicographically smallest among those $(k, 1)$ for $A^{-1}UA$. assume for a moment that is a linear relation $\sum_i \alpha_i u_{ki} \in K^*$ with $\alpha_1 \in k$ and $\alpha_1 \neq 0$. We may assume that $\alpha_1 = 1$ and that $\alpha_i = 0$ if $u_{ki} \in k^*$. Hence, in particular, $\alpha_1 = \ldots = \alpha_k = \alpha_{1+1} = \ldots \alpha_n = 0$. Consider the unit matrix 1 and the matrix $c^1 = (c_{ij}^1)$ such that (i) $c_{ij}^1 = 0$ if $j \neq 1$, (ii) $c'_{i1} = \alpha_i$ if $i \neq 1$ and (iii) $c_{11}^1 = 0$. Set $c = 1 + c'$. Then obviously $c^{-1} = 1 - c^1$. Since $c^{-1}u \equiv u$ modulo k^* , We see easily that such a $(k, 1)$ defined for $c^{-1}Uc$ has the same K and a smaller 1 than our $(k, 1)$, which is a contradiction. Therefore:

$$\text{(1)} \qquad \text{If } \alpha_i \in K \text{ and if } \alpha_1 \neq 0, \text{ then } \sum i\alpha_i u_{ki} \notin K^*.$$

Now, let $a = (a_{ij})$ be an arbitrary element of U. Then ua is also a generic point of U over K. Since $u_{kj}(j > l)$ is in K^*, the (k, j). component of ua must be in K^*. This shows by virtue of (1) above that $a_{lj} = 0$ if $j > 1$. since a is arbitrary, we see that $u_{lj} = 0$ for every $j \neq l$.Thus X_1

is U-invariant. Now we consider the elements $f_{i_1} \ldots i_n (i_j < p, \sum i_j = p)$. We denote by g_j the element $f_{i_l} \ldots i_n$ such that $i_j = 1, i_1 = p^{-1}$ for each $j = k, k+1, \ldots, 1-1, 1+1, \ldots, n$. Since we have

$$(2) \qquad (X_k X_l^{p-l})^u = \sum u_{kj} Z_j X_l^{p-l},$$

70 we see that

$$(3) \qquad g_k^u = \sum_{j \neq l} u_{kj} g_j.$$

Consider the coefficient of Z_1^p in g_k^u; let it be d. (3) shows that d is a linear combination of $u_k j (j \neq l)$ with coefficients in K. On the other hand, (2) shows that $d - u_{kl}$ must be in K^*. Thus we have a contradiction to (1) above, which completes the proof of Theorem 1.

4. The main result in the case where $p \neq 0$.

Theorem 2. *Let K be a universal domain of characteristic $p \neq 0$ and let G be a matric group contained in $GL(n, k)$. Then the following conditions are equivalent to each other:*

(I) Every rational representation of G is completely reducible.

(II) There is a normal subgroup G_o of finite index such that (i) G_o is a subgroup of a torus group and (ii) the index of G_o in G is not divisible by p.

(III) The connected component G_o of the identity of G is a subgroup of a torus group and $[G : G_o]$ is not divisible by p.

If G is an algebraic linear group, then the above conditions are equivalent to the following condition:

(IV) Every element of G is semi- simple.

Proof. It is obvious that (III) implies (II) and that (II) implies (I) by virtue of Lemma 6. Therefore, by Lemma 8, we have only to prove the
71 equivalence of (I), (III), (IV) in the case where G is an algebraic linear group. Thus we assume that G is algebraic let G_o be the connected component of the identity of G. Assume first that (IV) is true. Then G_o

consists merely of semi simple elements, hence G_o is a torus group by Lemma 3. If a semi-simple a has a finite order, then the order is prime to p. Therefore Lemma 4 implies that$[G : G_o]$ is not divisible by p. Thus (IV) implies (III). As we have remarked above, (III) implies (I). Assume now that (IV) is not true. Then, as we have seen in the proof of Theorem 1, there is a unipotent element u of G which is different from the identity, If $u \in G_o$, then G_o has a rational representation which is not completely reducible, hence G itself has such one by Lemma 7 or by the corollary to Lemma 8. If $u \notin G_o$, then the finite group G/G_0 has a representation which is not completely reducible, which is a rational representation of G. Thus we see that (I) is not true. Therefore (I) implies (IV), which completes the proof of Theorem 2. □

5. The main result in the case where $p = 0$.

Theorem 3. *Let K be a universal domain of characteristic* $p = 0$ *and let G be a matric group contained in* $GL(n, k)$*. Then the following conditions are equivalent to each other:*

(I) Every rational representation of G is completely reducible.

(II) The closure of G has a faithful rational representation which is completely reducible.

(III) The radical of the closure of G is a torus group.

Proof. It is obvious that (I) implies (II) by virtue of Lemma 8. Lemma 9 shows that (II) implies (III). In order to show that (III) implies (I), we shall prove the following lemma: □

Lemma 10. *Let G be a connected algebraic linear group and let R be* 72
the radical of G. If R is a torus group, then there is a closed connected normal subgroup S such that (i) $G = RS$ *and (ii)* $R \cap S$ *is a finite group. Furthermore, R is contained in the center of G (hence R is the connected component of the identity of the center of G).*

Proof. For the fact that R is contained in the center of G, see Borel's paper. Let S be the subgroup generated by all unipotent elements of

G. Then S is obviously a normal subgroup. Each unipotent element is in a closed connected unipotent subgroup of G, hence S is generated by closed connected subgroups, and therefore S is a closed connected subgroup of G. Now, we may assume that R is a diagonal group and that each $g \in G$ is given by

$$g = \begin{pmatrix} \rho_1(g) & \tau_{12}(g)\cdots & \tau_{1r}(g) \\ 0 & \rho_2(g)\cdots & \tau_{2r}(g) \\ & \cdots\cdots\cdots\cdots & \\ 0 & \cdots\cdots & \rho_r(g) \end{pmatrix}$$

with irreducible representations $\rho_1, \ldots, \rho_r$. If $u \in G$ is unipotent, then $\rho_i(u)$ is unipotent, whence the determinant of $\rho_i(u)$ is 1. Therefore we see that if $s \in S$, then the determinant of $\rho_i(s)$ is 1. On the other hand, since R is in the center of $G, \rho_i(R)$ is in the center of $\rho_i(G)$, hence by the famous lemma of Schur every element of $\rho_i(R)$ is of the form $k.\rho_i(1)$ with $k \in K$. Therefore we see that $R \cap S$ is a finite group. Since S a is closed normal subgroup. RS is a closed normal subgroup. Since G/R is
73 semi- simple, we see that G/RS is semi-simple, unless $G = RS$. If $G \neq RS$, then G/RS contains a non-trivial unipotent element, whence there must be a unipotent element of G outside of RS, which is a contradiction to our construction of S. Therefore $G = RS$, which completes the proof.

Now we proceed with the proof of Theorem 3. By the Corollary to Lemma 6, we may assume that G is connected. Lemma 8 allows us to assume that G is an algebraic linear group. Let R be the radical of G and let S be the normal subgroup given in Lemma 10. Since $R \cap S$ is a finite group and since $G = RS$, we see that S is semi-simple, whence every rational representation of S is completely reducible. Let ρ be an arbitrary rational representation of G. We may assume that $\rho(R)$ is a diagonal group, whence the completes reducibility of the restriction of ρ on S implies the complete reducibility of ρ, which completes the proof. □

6. Another result.

Let G be a connected algebraic linear group with universal domain K, throughout this section.

Theorem 4. *Every rational representation of G is completely reducible if (and only if) the following is true:*

If $\rho'' = \begin{pmatrix} 1 & 0 \\ 0 & \rho' \end{pmatrix}$ *is a rational representation of G, then* ρ'' *is equivalent to the representation* $\begin{pmatrix} 1 & 0 \\ 0 & \rho' \end{pmatrix}$.

Proof. Let$\rho = \begin{pmatrix} \rho_1 & \tau \\ 0 & \rho_2 \end{pmatrix}$ be a rational representation of G. We have only to show that ρ is equivalent to the representation $\begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$. □

Since quad $\rho(ab) = \rho(a)\rho(b)$, we have 74

(1)
$$\tau(ab) = \rho_1(a)\tau(b) + \tau(a)\rho_2(b) \text{ for any } a, b - G.$$

Let x be a generic point of G over K consider $f(x) = T(x)\,\rho_2(x)^{-1}$. $f(a)$ is then well defined for any $a \in G$. The relation (1) implies that $f(ab) = \rho_1(a)\tau(1)\rho_2(b)^{-1}\rho_2(a)^{-1} + \tau(a)\rho_2(a)^{-1} = \rho_1(a)f(b)\rho_2(a)^{-1} + f(a)$ for any $a, b \in G$, whence

(2)
$$f(xa) = \rho_1(x)f(a)f_a(x)^{-1} + f(x) \text{ for any } a \in G.$$

Let m, n be such that T is an(m,n) matrix and consider the module L of all (m, n)-matrices over $K(x)$. Each element C of G defines an K-linear map ϕ_g or. L as follows:

$$\phi_g(w_{ij}(x)) = (w_{ij}(xg)).$$

Thus L becomes $K - G$ - module. Let M be the set of all $\rho_1(x)c\rho_2$ $(x)^{-1}$ with (m, n)-matrices c over K. Then M is a finite K-module contained in L. Since $\rho_1(xa)c\rho_2(xa)^{-1} = \rho_1(x)(\rho_1(a)c\rho_2(a)^{-1}, x\rho_2$

$(x)^{-1}(a \in G)$, M is G-admissible. Set $N = f(x)K + M$. Then the relation (2) shows that N is also a finite $K-G$ -module. We consider a representation ρ^* of G by the module N. The relation (2) shows that $f(x)$ is G-invariant modulo M, hence either $f(x) \in M$ or ρ^* is equivalent to a representation of the form $\begin{pmatrix} 1 & \lambda \\ 0 & \rho' \end{pmatrix}$. The former case implies that $f(x) + \rho_1(x)c\rho(x)^{-1} = 0$ with some (m, n)-matrix c over K. By our assumption, the latter case implies that there is an element $\rho_1(x)c\rho_2(x)^{-1}$ of M such that $f(x) + \rho_1(x)_c\rho_2(x)^{-1}$ is G-
75 invariant. Hence, in any case, there is an (m, n)-matrix c over K such that $f(x)+\rho_1(x)c\rho_2(x)^{-1}$ is G-invariant. Set $\tau^* = \tau - c\rho_2 + \rho_1 c$. Then, transforming ρ by the matrix $\begin{pmatrix} \rho_1(1) & c \\ 0 & \rho_2(1) \end{pmatrix}$, we see that ρ is equivalent to the representation $\begin{pmatrix} \rho_1 & \tau^* \\ 0 & \rho_2 \end{pmatrix}$. Set $f^*(x) = \tau^*(x)\rho_2(x)^{-1}$. Then $f^*(x) = f(x) - c + \rho_1(x)c\rho_2(x)^{-1}$, which is G-invariant by our choice of c. Therefore $f^*(xa) = f^*(x)$ for any $a \in G$, whence $f^*(x) = f^*(xx^{-1}) = 0$. This shows that $\tau^* = O$, which completes the proof of Theorem 4.

We note by the way that the matrix $f(x)$ has an interesting property as follows:

Proposition. *Assume that* $\rho = \begin{pmatrix} \rho_1 & \tau \\ 0 & \rho_2 \end{pmatrix}$ *is a rational representation of* G. *Set* $H = \{h | h \in G, \tau(h) = 0\}$. *Then the homogeneous variety* $G/H = \{gH\}$ *is a quasi-affine variety, on which the coordinates of a point* gH *are given by* $f(g)$.

Proof. Since $\tau(1) = 1$, the formula (1) in the above proof shows that $\tau(a^{-1}) = -\rho_1(a)^{-1}\tau(a)\rho_2(a)^{-1}$, hence $\tau(a^{-1}b) = \rho_1(a)^{-1}\left[\tau(b)\rho_2(b)^{-1} - \tau(a)\rho_2(a)^{-1}\right]\rho_2(b)$. There fore $f(a) = f(b)$ if and only if $aH = bH$, which prove the assertion. □

Remark. Note that the above preposition only proves that G/H is a quasi-affine and not affine as stated in M. Nagata: Complete reducibility of rational representations, J. Math. hyeto Univ., 1-1 (1961), 87-99.