Lectures on
Geodesics Riemannian Geometry

By

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Introduction

The main topic of these notes is geodesics. Our aim is twofold. The first is to give a fairly complete treatment of the foundations of riemannian geometry through the tangent bundle and the geodesic flow on it, following the path sketched in [2] and [19]. We construct the canonical spray of a riemannian manifold (M, g) as the vector field G on $T(M)$ defined by the equation

$$i(G) \cdot d\alpha = -\frac{1}{2} dE,$$

where $i(G) \cdot d\alpha$ denotes the interior product by G of the exterior derivative $d\alpha$ of the canonical form α on (M, g) (see *(III.6)*) and E the energy (square of the norm) on $T(M)$. Then the canonical connection is introduced as the unique symmetric connection whose associated spray is G.

The second is to give global results for riemannian manifolds which are subject to geometric conditions of various types; these conditions involve essentially geodesics.

These global results are contained in Chapters IV, VII and VIII. Chapter IV contains first the description of the geodesics in a symmetric compact space of rank one (called here an S.C.-manifold) and the description of Zoll's surface (a riemannian manifold, homeomorphic to the two-dimensional sphere, non isometric to it and all of whose geodesics through every point are closed). Then we sketch results of Samelson and Bott to the effect that a riemannian manifold all of whose geodesics are closed has a cohomology ring close to that of an S.C.-manifold. In Chapter VII are contained the Hopf-Rinow theorem, the existence of a closed geodesic in a non-zero free homotopy class of a compact riemannian manifold, and the isometry between two simply connected and complete riemannian manifolds of the same constant sectional curvature.

In Chapter VIII one will find theorems of Myers and Synge, the Gauss-Bonnet formula and a result on complete riemannian manifolds of non-positive curvature. Then come the theorem of L.W. Green, which asserts that, on the two-dimensional real-projective space, a riemannian structure all of whose geodesics are closed has to be isometric to the standard one, and the theorem of E. Hopf: on the two-dimensional torus, a riemannian structure all of whose geodesics are without conjugate points has to be a flat one. As a counterpoint we have quoted the work of Busemann which shows that the theorems of Green and Hopf pertain to the realm of riemannian geometry, for they no longer hold good in G-spaces (see *(VIII.10)*). However, the result on complete manifolds with non-positive curvature is still valid in G-spaces.

We have included in Chapter VIII theorems of Loewner and Pu, which are "isoperimetric" inequalities on the two dimensional torus and the two-dimensional real projective space (equality implying isometry with the standard riemannian structures on these manifolds). These results do not involve geodesics explicitly, but have been included for their great geometric interest. One should also note that the results of Green, Hopf, Loewner and Pu are two-dimensional and so lead to interesting problems in higher dimensions.

The tools needed for these results are developed in various chapters: Jacobi fields, sectional curvature, the second variation formula play an important role; see also the formulas in *(VIII.4)* and *(VIII.8)*.

The reference for calculus is [36]; references for differential and riemannian geometry are [14]; [16], [17], [18], [19], [21], [33], [35], and lectures notes of I.M.Singer and a seminar held at Strasbourg University. We have used some of these references without detailed acknowledgement.

I am greatly indebted to N. Bourbaki, P. Cartier and J.L. Koszul for communication and permission to use ideas and results of theirs on connections. For most valuable help I am glad to thank here P. Cartier, J.L. Koszul, N. Kuiper and D. Lehmann.

Contents

Chapter 0

Preliminaries

1

1

In this chapter we formulate certain notions connected with the notion of a manifold in a form we need later on and fix some notations and conventions.

By means of a basis we identify a d-dimensional real vector space $\mathcal{V}$ with set $\mathbb{R}^d$ of all d-tuples of real numbers with the standard vector space structure and *denote* the element with 1 in the i^{th} place and zeros elsewhere *by* e_i. Then $\{e_i\}$ form a basis of $\mathbb{R}^d$, we call it the *canonical basis* of $\mathbb{R}^d$, and its *dual basis* $(u^1, \ldots, u^d)$ the canonical coordinate system in $\mathbb{R}^d$. (For $\mathbb{R}(= \mathbb{R}^1)$ we set $u^1 = t$). With this identification any real vector space becomes a manifold and this structure is independent of the basis chosen.

A *manifold* will always be a C^∞-manifold which is Hausdorff, para compact and of constant dimension d. Generally we denote it by M, a typical *chart of it* by (U, r) and *local coordinates* with respect to (U, r) by $x^i = u^i \circ r$. Let us note that because of para compactness partitions of unity for M exist.

We denote the *tangent space* of M at a by $T_a(M)$ and define the *tangent* bundle $T(M)$ of M to be

(0.1.1) $$\bigcup_{a\in M} \bigcup_{x\in T_a(M)} \{x\} = T(M)$$

and call elements of $T(M)$ *vectors* of M. Then we have the *natural* *projection map* p_M (or simply p) from $T(M)$ onto M which takes every vector of M at a onto a.

2 We *denote* the set of all *maps* (C^∞-maps, differentiable maps) of a manifold M into a manifold N *by* $D(M, N)$ and when $N = \mathbb{R}$ we write $F(M)$ for $D(M, \mathbb{R})$. Every f in $D(M, N)$ induces a map of $F(N)$ into $F(M)$ defined by

$$g \to g \circ f$$

and this in turn *a map* f^T of $T(M)$ into $T(N)$ defined by the equation

$$(f^T(x))(g) = x(g \circ f), \quad g \in F(N). \tag{0.1.2}$$

Then we have the following commutative diagram

$$\begin{array}{ccc} T(M) & \xrightarrow{f^T} & T(N) \\ \downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow{f} & N \end{array} \tag{0.1.3}$$

1.4

Let us note that if f^T restricted to $T_m(M)$ is one-one then we can choose a local coordinate system $(x^1, \ldots, x^e)$ for N at $n = f(m)$ such that $\{x^i \circ f\}$ $(i = 1, \ldots, d)$ form a local coordinate system at m for M. Also if f^T restricted to $T_m(M)$ is onto $T_n(N)$ then we can choose a local coordinate system $(x^1, \ldots, x^d)$ for M at m and a local coordinate system $(y^1, \ldots, y^e)$ for N at n such that $x^i = y^i \circ f$, $i = 1, \ldots, 0$.

1.5

We call a manifold N a *sub manifold* of M if

3 1) $N \subset M$,

2) the topology on N is induced by that on M,

3) to each point p of N there is a chart (U_p, r_p) in the atlas of M such that for some positive integer k

$$r_p(N \cap U_p) = \left\{(x_i) \in r_p(U_p) \,\middle|\, x_{k+1} = \ldots = x_d = 0\right\}.$$

1.6

Under these conditions a vector x of $T_m(M)$ will be in $i^T(T_m(N))$ (where $i : N \to M$ denotes the injection) if and only if $x(\varphi \circ i) = 0 \forall \varphi \in F(N)$ implies that $x = 0$.

In case the manifold M is an open sub manifold A of a finite dimensional real vector space V, we identify $T(A)$ with $A \times V$ as follows:

1.7

For $y \in V$, $f \in F(A)$, $a \in A$ set

$$\zeta_a^{-1}(y) \cdot (f) = \lim_{t \to 0} \left(\frac{f(a + ty) - f(a)}{t} \right).$$

We see that $\zeta_a^{-1}(y) \in T_a(M)$ and that the map ζ_a^{-1} is an isomorphism of V with $T_a(M)$. We denote its inverse by ζ_a and define a map from $T(A)$ onto V by setting

$$\zeta(x) = \zeta_{p_A(x)}(x). \tag{0.1.8}$$

Then the identification is given by the map

$$T(A) \ni x \to (p_A(x), \zeta(x)) \in A \times V. \tag{0.1.9}$$

Given (U, r) the maps $\zeta \circ r^T$ and $(p_{r(U)}, \zeta) \circ r^T$ are called the *principal part* and the *trivialising map* respectively.

(See Lang [19] : p. 49). **4**

The use of these maps instead of the explicit use of the local coordinates has the advantages of the latter without its tediousness. So, more explicitly, given (U, r) we write

$$x \underset{U}{=} (a, b)$$

$$a = (p_{r(U)} \circ r^T)(x) \quad \text{and} \quad b = (\zeta \circ r^T)(x).$$

Generally we take $(a^1, \dots, a^d)$, $(b^1, \dots, b^d)$ as the coordinate representations of a, b.

Now, we can consider the collection $(T(U), (p_{r(U)}, \zeta) \circ r^T)$ corresponding to the charts (U, r) of M as an atlas on $T(M)$ and thus define a structure of a manifold on $T(M)$. With this structure on $(T(M))$ we have $p_M \in D(T(M), M)$ and furthermore if N is a manifold and $f \in D(M, N)$ then $f^T \in D(T(M), T(N))$.

1.10

Since M is Hausdorff and hence there are functions with arbitrarily small support we see that the class of all sections of $T(M)$, i.e.,

$$\left\{ X \in D(M, T(M)) \,\middle|\, p \circ X = \mathrm{id}_M \right\}$$

can be identified with $\mathscr{C}(M)$, the set of all derivations of the $\mathbb{R}$-algebra $F(M)$ into itself. Elements of $\mathscr{C}(M)$ are *called vector fields* on M.

1.11

Now let us take (U, r). Suppose that a vector (a, b) is given in U. In (a, b) we vary the first coordinate a over U keeping b fixed. Thus, clearly, we
5 get a vector field over U. Now we take functions φ on M such that

(i) $0 \le \varphi \le 1$, and

(ii) φ is zero outside U and is 1 in a neighbourhood of a in U.

Then we set

$$X_{\ell c} = \begin{cases} \varphi(x) \cdot X(x) & \text{for } x \in U \\ 0_x & \text{for } x \notin U \end{cases}$$

This vector field so obtained is called a *locally constant vector field* at a.

If $f \in D(M,V)$ is a map of a manifold M into a finite dimensional real vector space, we define, besides f^T, *a map* df, called *the differential of* f, from $T(M)$ to V by setting

$$df = \zeta \circ f^T. \tag{0.1.12}$$

Further if M is a vector space V', then *denoting the Jacobian of* f by Df we have (with the Df of [36]):

1.13

$$Df \circ \zeta = df.$$

Let us write down explicitly the expression for df and Df. Let V be n-dimensional with a basis $\{\overline{e}_1, \ldots, \overline{e}_n\}$ and corresponding coordinate system $(\overline{x}^1, \ldots, \overline{x}^n)$ and, as usual, let (U, r) be a chart of M with local coordinate system $(x^1, \ldots, x^d)$. Then f on U is given by n C^∞-functions $\overline{f}^i = \overline{x}_i \circ f$ $(i = 1, \ldots, n)$ on U. The canonical local bases for $\mathscr{C}(U)$ and $\mathscr{C}(V)$ are

$$\left\{\left(\frac{\partial}{\partial x^i}\right)\right\} \quad \text{and} \quad \left\{\left(\frac{\partial}{\partial \overline{x}^i}\right)\right\} \quad \text{respectively.}$$

Then we have **6**

$$f^T\left(\frac{\partial}{\partial x^i}\right) = \sum_j \frac{\partial \overline{f}^j}{\partial x^i}\frac{\partial}{\partial \overline{x}^j}, i = 1, \ldots, d. \tag{0.1.14}$$

Now is customary let us write the contra variant (tangent) vector as a row vector. Now suppose that df takes a contra variant vector with coordinates $(x_1, \ldots, x_d)$ to one with $(\overline{x}_1, \ldots, \overline{x}_n)$, i.e.

$$(df)\left(\sum_i x_i \frac{\partial}{\partial x^i}\right) = \sum_i \overline{x}_i \frac{\partial f}{\partial \overline{x}^i}. \tag{0.1.15}$$

Then the coordinates are related by the matrix equation
(0.1.16)

$$\text{i.e.}\quad (x_1,\dots,x_n)=\begin{pmatrix} \dfrac{\partial \overline{f}^1}{\partial x^1} & \dfrac{\partial \overline{f}^2}{\partial x^1} & \cdots & \dfrac{\partial \overline{f}^n}{\partial x^1} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial \overline{f}^1}{\partial x^i} & \dfrac{\partial \overline{f}^2}{\partial x^i} & \cdots & \dfrac{\partial \overline{f}^1}{\partial x^i} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial \overline{f}^1}{\partial x^d} & \dfrac{\partial \overline{f}^2}{\partial x^d} & \cdots & \dfrac{\partial \overline{f}^n}{\partial x^d} \end{pmatrix}\cdot(\overline{x}_1,\dots,\overline{x}_n);\, X = J\cdot X$$

Hence the Jacobian is J.

2 Forms

We consider $\mathscr{C}(M)$ as an $F(M)$ module and denote its *dual* by $\mathscr{C}^*(M)$.
7 The elements of $\mathscr{C}^*(M)$ are called the *differential forms* on M.

2.1

Now we define the *cotangent bundle* $T^*(M)$ to be

$$T^*(M)=\bigcup_{a\in M}\bigcup_{x^*\in T_a(M)}\{x^*\},$$

and define a manifold structure on $T^*(M)$ in a way analogous to that on $T(M)$. For $\omega\in T^*(M)$, $X\in\mathscr{C}(M)$ by evaluation at a in M we see that $\omega(X)(a)$ depends only on $X(a)$ and that we can identify $\mathscr{C}^*(M)$ with

(0.2.2) $$\{\omega\in D(M,T^*(M))|p_M\circ\omega=\mathrm{id}_M\}=\mathscr{C}^*(M)$$

More generally, $\mathfrak{M}$ being any unitary module over a commutative ring A we write $L^s(\mathfrak{M})$ (resp. $E^s(\mathfrak{M})$) for the A-module of all multilinear (resp. alternating multilinear) forms of degree s on $\mathfrak{M}$. There is a map, called *multiplication*, from $L^s(\mathfrak{M})\times L^{s'}(\mathfrak{M})$ into $L^{s+s'}(\mathfrak{M})$ defined by $(\omega,\sigma)\to\omega\cdot\sigma$, where $(\omega\cdot\sigma)(X_1,\dots,X_s,X_{s+1}\dots,X_{s+s'})=$

$\omega(X_1, \ldots, X_s) \cdot \sigma(X_{s+1}, \ldots, X_{s+s'})$, and a map, called *exterior multiplication*, from $E^s(\mathfrak{M}) \times E^{s'}(\mathfrak{M})$ into $E^{s+s'}(\mathfrak{M})$, obtained by alternating the latter one.

Now we set

$$\begin{aligned} \mathscr{L}^s(M) &= L^s(\mathscr{C}(M)) \\ \textit{and} \quad \mathscr{E}^s(M) & E^s(\mathscr{C}(M)) \end{aligned}$$

considering $\mathscr{C}(M)$ as an $F(M)$-module, and

$$\begin{aligned} L^s(T(M)) &= \bigcup_{a \in M} \bigcup_{x \in L^s(T_a(M))} \{x\} \\ E^s(T(M)) &= \bigcup_{a \in M} \bigcup_{x \in E^s(T_a(M))} \{x\} \end{aligned}$$

considering the $T_a(M)$ as $\mathbb{R}$-modules. We can define a manifold struc- **8**
ture on $L^s(T(M))$ and also on $E^s(T(M))$ in a way analogous to that on $T^*(M)$. By evaluation at a of M we can show that for $\omega \in \mathscr{L}^s(M)$

$$\omega(X_1, \ldots, X_s)(a) \quad \text{depends only on} \quad (X_1(a), \ldots, X_s(a))$$

and hence we can identify $\mathscr{L}^s(M)$ and $\mathscr{E}^s(M)$ with

$$\begin{aligned} &\{\omega \in D(M, L^s(T(M)) \,|\, p_M \circ \omega = \mathrm{id}_M\} \\ \text{and} \quad &\{\omega \in D(M, E^s(T(M)) \,|\, p_M \circ \omega = \mathrm{id}_M\} \end{aligned} \tag{0.2.3}$$

respectively, and call them *s-forms on M* and *s-exterior forms on M* respectively. Now for $\omega \in \mathscr{L}^s(M)$ and $a \in M$, $\omega(a)$ makes sense and we often *denote it by* ω_a, and we have a similar convention for the elements of $\mathscr{E}^s(M)'s$.

2.4

Given a chart (U, r), with the notation

$$[i] = i_1 < \ldots < i_s, dx_{[i]} = dx_{i_1} \wedge \ldots \wedge dx_{i_s}$$

locally we can write any s-exterior form as

$$(0.2.5) \qquad \sum_{[i]} \omega_{[i]} dx_{[i]}, \quad \text{where} \quad \omega_{[i]} \in F(r(U)).$$

Every $f \in D(M, N)$ *induces a map* f^* of multilinear forms on N into those on M. We have, by definition

$$(0.2.6) \quad (f^*\omega)(x_1, \ldots, x_s) = \omega(f^T(x_1), \ldots, f^T(x_s)) \quad \text{for} \quad \omega \in \mathscr{L}^s(N)$$

and $x_1, \ldots, x_s \in T_a(M)$. Thus f takes $\mathscr{L}^s(N)$ into $\mathscr{L}^s(M)$ and further-
more it takes elements of $\mathscr{E}^s(N)$ into those of $\mathscr{E}^s(M)$ and hence can be
9 considered as a map of $\mathscr{E}^s(N)$ into $\mathscr{E}^s(M)$. f^* is a homomorphism of
R-modules having the following properties:

(0.2.7)
i) $f^*(\omega \cdot \sigma) = f^*(\omega) \cdot f^*(\sigma)$ for $\omega \in \mathscr{L}^s(N)$, $\sigma \in \mathscr{L}^{s'}(N)$,
ii) $f^*(\omega \wedge \sigma) = f^*(\omega) \wedge f^*(\sigma)$, for $\omega \in \mathscr{E}^s(N)$, $\sigma \in \mathscr{E}^{s'}(N)$
iii) for $g \in D(N, L)$, L a manifold, $(g \circ f)^* = f^* \circ g^*$.

We define *a map d, called exterior differentiation,* from $\mathscr{E}^s(M)$ into $\mathscr{E}^{s+1}(M)$ by setting

$$(0.2.8) \qquad (d\omega)(X_0, \ldots, X_s) = \sum_{i=0}^{s} (-1)^i X_i(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_s)) + \sum_{0 \le i < j \le s} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_s)$$

where $[X, Y] = X \circ Y - Y \circ X$ for X, $Y \in \mathscr{C}(M)$, $X_0, \ldots, X_s \in \mathscr{C}(M)$ and the element under $\wedge$ is to be deleted from the sequence of elements. This map has the following properties:

(0.2.9)
i) it is $\mathbb{R}$-linear
ii) $d \circ d = 0$
iii) $d(\omega \wedge \sigma) = (d\omega) \wedge \sigma + (-1)^{\text{degree of}\omega} \omega \wedge d\sigma$
iv) $d \circ f^* = f^* \circ d$ for $f \in D(M, N)$, N a manifold.

As an example we note

$$(0.2.10) \qquad (d\omega)(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

for $\omega \in \mathscr{C}^*(M)$; X, $Y \in \mathscr{C}(M)$.

For $X \in \mathscr{C}(M)$, $\omega \in \mathscr{L}^s(M)$ we define the *interior product* $i(x)\omega \in \mathscr{L}^{s-1}(M)$ by the equation

(0.2.11)
$$(i(X)\omega)(X_1, \ldots, X_{s-1}) = \omega(X, X_1, \ldots, X_{s-1}) \forall X_1, \ldots, X_{s-1} \in \mathscr{C}(M).$$

2.12

10

We call multilinear forms on $(\mathscr{C}(M))^s \times (\mathscr{C}^*(M))^t$ tensors of type (s, t) and denote the $F(M)$-module of such tensors by $\mathscr{L}_t^s(M)$.

3 Integration

3.1

a.1. Let V be a real vector space of finite dimension d. We call any nonzero element of $E^d(V)$ an *orientation on* V A basis $\{x_1, \ldots, x_d\}$, in that order, is *positive* with respect to the orientation S on V is

(0.3.2)
$$S(x_1, \ldots, x_d) > 0.$$

3.3

Let V be a vector space. V together with a symmetric positive definite bilinear form g on V, i.e. a bilinear form such that

$$g(x, y) = g(y, x), \quad \text{and} \quad g(x, x) > 0 \forall x \neq 0$$

is called a g-*euclidean* (or simply *euclidean* if there is no possible confusion) *space*. Sometimes we describe this situation as "V is provided with a *euclidean structure* g". Two vectors x and y are said to be *orthogonal relative* to g if $g(xy) = 0$. A basis $\{e_1, \ldots, e_d\}$ of V is called an *orthogonal basis relative* to g or simply an orthogonal basis of (V, g) if $e_1, \ldots, e_d$ are orthogonal i.e. $g(e_i, e_j) = 0$ if $i \neq j$. It is called an *orthonormal basis relative* to g if, further,

$$g(e_i, e_i) = 1.$$

3.4

Example. Let V be provided with a euclidean structure and be oriented
11 by S (i.e. S is an orientation on V). Then it admits a *canonical* orientation S_V defined by the equation

$$S_V(x_1, \dots, x_d) = 1$$

for every orthonormal basis $\{x_1, \dots, x_d\}$, positive relative to S.

3.5

2. *A volume t on V* is a map $V^d \to \mathbb{R}$ such that it is the absolute value of an orientation $S : t = |S|$, i.e.,

$$t(x_1, \dots, x_d) = |S(x_1, \dots, x_d)| \quad \text{for} \quad x_1, \dots, x_d \quad \text{in} \quad V, .$$

3.6

Example 3.2. Let V be g-euclidean. Then it admits a *canonical volume* t_V defined by $t_V(x_1, \dots, x_d) = (\det(g(x_i, x_j)))^{1/2}\ \forall x_1, \dots, x_d \in V$.

b. Let M be a manifold. We call an element $\sigma \in \mathscr{E}^d(M)$ an *orienting* or *volume form* if $\forall \mathrm{m} \in M : \sigma_m$ is an orientation of $T_m(M)$. In that case we also say that σ orients M or that M is oriented by σ.

3.7

Example. On $\mathbb{R}^d$ there is an orientation τ, called *the canonical orientation*, given by

$$\tau = du^1 \wedge \dots \wedge du^d. \tag{0.3.8}$$

We consider $\mathbb{R}^d$ always with this orientation.

Now there is the notion of a diffeomorphism between oriented manifolds *preserving orientation*. If M is oriented then every open submanifold of M is oriented in a natural way. Now let (U, r) be a chart of an oriented manifold M. Then on U there is an induced orientation and on $r(U) \subset \mathbb{R}^d$ there is an induced orientation. We say that (U, r) is *positive* if r preserves orientation between U and $r(U)$.

12 *If M is oriented*, using partition of unity, we can define a *notion of integration for elements* of $\mathscr{E}^d$, denoted by $\int_M \omega$ where $\omega \in \mathscr{E}^d(M)$, with some good properties some of which we proceed to state. The integral $\int_M \omega$ exists if ω has compact support. Further if M is an open sub manifold of an oriented manifold N and $\overline{M}$ is compact and ω is the restriction of an element of $\mathscr{E}^d(N)$ then also $\int_M \omega$ exists. Let M, N be oriented manifolds and f be a diffeomorphism of N with M preserving orientation, and let $\omega \in \mathscr{E}^d(M)$. Then if $\int_M \omega$ exists so does $\int_N f^*\omega$, and

$$\int\limits_N f^*\omega = \int\limits_M \omega. \tag{0.3.9}$$

3.10

Lemma. *Let $f \in D(N, M)$ where N, M are oriented and let $\omega \in \mathscr{E}^d(M)$ be an orienting form for M such that $\int_M \omega$ exists. Suppose that f is subjective, preserves orientation and that f_n^T is an isomorphism $\forall n \in N$. Then:*

$$\int\limits_N f^*\omega \geq \int\limits_M \omega.$$

Proof. Under the assumptions on f^T it follows that f is a local diffeomorphism. Now let us take an open covering $\{V_i\}$ of N such that f restricted to V_i is a diffeomorphism. Then $\{U_i = f(V_i)\}$ form a covering of M since f is onto. Let us take a partition of unity $\{\varphi_i\}$ on N subordinate to the covering $\{V_i\}$. Let us define $\overline{\varphi}_i$ on M by

$$\begin{aligned} \overline{\varphi}_i &= \varphi_i \circ f^{-1} \quad \text{on} \quad U_i \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

Then $\sum\limits_i \overline{\varphi}_i \geq 1$ since a point in M has at least one inverse. Now 13

$$\int\limits_N f^*\omega = \sum_i \int\limits_N \varphi_i \cdot f^*\omega = \sum_i \int\limits_M \overline{\varphi}_i \omega \quad \text{by (0.3.9)}$$

$$\geq \int_M \omega \quad \text{since} \quad \sum \overline{\varphi}_i \geq 1.$$

□

3.11

c. We call a domain D in M a *nice domain* if $\forall m \in b(D) = \overline{D} - D$ ($\overline{D}$ being the closure of D), there is an open neighbourhood U of m in M and a $\varphi \in F(M)$ such that

$$d\varphi_m \neq 0 \quad \text{and} \quad U \cap D = \varphi^{-1}(]-\infty, 0[).$$

3.12

Then it follows that $b(D)$ is a sub manifold of M of dimension $d-1$. The notion of orientation can be extended to manifolds with good boundary. Then one sees that $b(D)$ is oriented in a natural way if D is. Now let ω be a $d-1$ form on M and let us denote by i the injection of $b(D)$ into M.

Then Stokes theorem can be stated as follows:

3.13

If D is a nice domain of M such that $\overline{D}$ is compact then the Stokes' formula holds:

$$\int_D d\omega = \int_{b(D)} i^* \omega.$$

Let us note that if $b(D)$ is empty, then $\int_D d\omega = 0$.

d. On a manifold M a *positive odd d-form* is a map ω from $\mathscr{C}^d$ with values in the set of functions on M such that it is everywhere, locally, the absolute value of a local d-exterior form. A *volume element* θ on M
14 is a positive odd d-form such that, for every m, θ_m is a volume of $T_m(V)$. The volume element τ the absolute value of the canonical orientation on $\mathbb{R}^d$, is called the *canonical volume element* of $\mathbb{R}^d$.

3.14

On a manifold there is a notion of integration for positive odd d-forms and this allows us to define the integral of a positive odd d-form ω, which we denote by $\int_M \omega$.

3.15

If $f \in D(N, M)$ is a diffeomorphism, ω a positive odd d-form of M then we can, in a natural way, associate to ω a positive odd d-form $f^*\omega$ on N. Then if $\int_M \omega$ exists so does $\int_N f^*\omega$ and further we have

$$\int_M \omega = \int_N f^*\omega.$$

3.16

Remark. if M is a manifold oriented by ω then $|\omega|$ is a positive odd d-form and if $\int_M \omega$ exists then $\int_M |\omega|$ exists and

$$\int_M \omega = \int_M |\omega|$$

e. Let E be a differentiable fibre bundle over M

$$p : E \to M,$$

with E and M oriented, so that the fibres are oriented in a natural way. Let $\omega \in \mathscr{E}^d(M)$, $\varphi \in \mathscr{E}^f(E)$ (f being the dimension of a fibre) and $i_m : p^{-1}(m) \to E$ be the injection of the fibre $p^{-1}(m)$ into E. Using a partition of unity and Fubini's theorem we have

3.17 Integration along fibres

$$\int_E \varphi \wedge (p^*\omega) = \int_{m\in M} \Big(\int_{p^{-1}(m)} i_m^*(\varphi)\Big)\omega$$

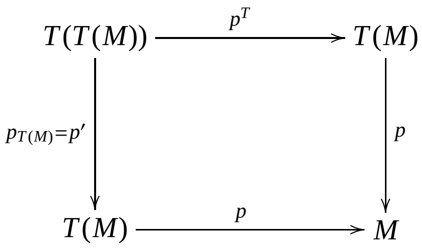
provided that M and the fibres are compact.

4

a) **Double tangent bundle.**

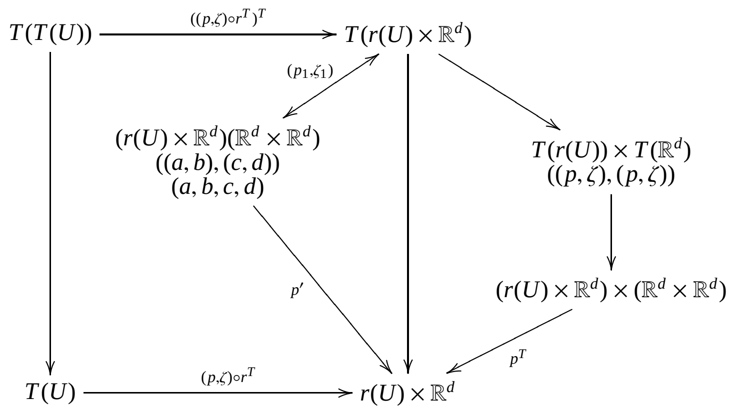
4.1

We have the following commutative diagram;



In the case of (U, r), a chart of M, the situation in detail is given by the following commutative diagram:

(0.4.2)



16 where the arrows with two heads denote isomorphisms and we identify corresponding spaces.

4.3

Generally we take $(a_1, \ldots, a_d)$, $(b_1, \ldots, b_d)$, $(c_1, \ldots, c_d)$, $(d_1, \ldots, d_d)$ as coordinate representatives of a, b, c, d if $z \underset{\cup}{=} (a, b, c, d)$.

4.4

Let Φ be a C^∞ function on $T(U)$. Then let us denote $\Phi \circ (r, \zeta \circ r^T)^{-1}$ on $T(r(U))$ by $\overline{\Phi}$ and the canonical coordinates in $T(r(U))$ by $(x^1, \ldots, x^d, y^1, \ldots, y^d)$. Then

$$z(d\Phi) = (a,b,c,d)(d\Phi) = = \sum_{i=1}^{d} \frac{\partial \overline{\Phi}}{\partial x^i}((a);(b))c_i + \sum_{i=1}^{d} \frac{\partial \overline{\Phi}}{\partial y^i}((a);(b))d_i. \tag{0.4.5}$$

As a particular case when $\varphi = d\Phi$, $\varphi \in C^\infty(U)$ we have

$$z(d\varphi) = (a,b,c,d)(d\overline{\varphi}) = \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2 \overline{\Phi}}{\partial x^j \partial x^i}(a) b_i c_j + \sum \frac{\partial \overline{\varphi}}{\partial x^i}(a) d_i.$$

Now let

$$T(T(U)) \ni z \underset{\cup}{=} (a,b,c,d).$$

Then we have

$$p'(z) \underset{\cup}{=} (a,b) \quad \text{and} \quad p^T(z) \underset{\cup}{=} (a,c). \tag{0.4.6}$$

4.7

Example. Let $\phi \in F(M)$ and let $\phi \circ r^{-1} = \phi$.

Then $\phi \in F(r(U))$ and we have

$$z(d\phi) = (a,b,c,d)(d\phi) = D\overset{o}{\phi}_a(d) + \overset{2}{D}\overset{o}{\phi}_a(b,c) \quad \text{by (0.4.5)} \tag{0.4.8}$$

where $D\overset{o}{\phi}$ is the Jacobian of $\overset{o}{\phi}$. Note that $\overset{2}{D}\overset{o}{\phi}$ is symmetric **17**
(see Dieudonne [36] : p. 174).

4.9

b) **Vertical vectors.** For x in $T(M)$ we define the space of *vertical vectors* V_x of $T(M)$ at x to be $(p_x^T)^{-1}(0)$. (From (0.4.6) it follows that $Y \underset{\cup}{=} (a, b, c, d)$ is vertical if and only if $c = 0$). From the injection

$$i_m : T_m(M) \to T(M) \quad \text{at} \quad m = p(x)$$

and the fact that $p(T_m(M)) = m$ we conclude that

$$(p^T \circ i_m^T)(T_x(T_m(M))) = 0$$

and hence that

$$i_m^T(T_{p(x)}(M)) \subset V_x.$$

But since p_x^T is surjective the kernel of p_x^T has dimension d and since the dimension of $i^T(T_{p(x)}(M))$, is d we have

(0.4.10) $$V_x = i_m^T(T_{p(x)}(M)).$$

For $y \in T(T(M))$ set $x = p_{T(M)}(y)$ and for y vertical set

(0.4.11) $$\xi(y) = (\zeta \circ (i_m^T)^{-1})(y).$$

(0.4.12) $$\begin{array}{ccc} T_x(T_{p(x)}(M)) & \xrightarrow{\zeta_x} & T_{p(x)}(M) \\ {\scriptstyle i_m^T}\downarrow & \nearrow & \\ V_x & & \end{array}$$

18 **4.13**

If $y \underset{\cup}{=} (a, b, c, d)$ then since y is vertical $c = 0$, and it follows that $\xi(y) \underset{\cup}{=} (a, d)$.

Note that ξ is an isomorphism between V_x and $T_m(M)$.

On $$T(M) \underset{M}{\times} T(M) = \{(x, y) \in T(M) \times T(M) \,|\, p(x) = p(y)\}$$

there is canonical manifold structure. Now we define a map

$$\xi^{-1} : T(M) \times T(M) \to T(T(M))$$

by the equation

$$\xi^{-1}(x, y) = \xi_x^{-1}(y) \quad \text{where} \quad \xi_x^{-1}(y) \tag{0.4.14}$$

is uniquely defined by the conditions

1) $\xi(\xi_x^{-1}(y)) = y$
2) $\xi_x^{-1}(y) \in V_x$.

4.15

If $x \underset{\cup}{=} (a, b)$, and $y \underset{\cup}{=} (a, c)$ then it follows that

$$\xi_x^{-1}(y) \underset{\cup}{=} (a, b, 0, c)$$

4.16

Lemma. *If $\omega \in F(T(M))$, restricted to $T_m(M)$ is linear for every m of M, and z is a vertical vector of $T(M)$, then*

$$z(\omega) = \omega(\xi(z)).$$

Proof. The proof follows from the definitions and that, for a linear map f in a vector space, one has $Df = f$. □

4.17

Lemma. *If $z \in T(T(M))$ and*

$$z(d\varphi) = 0 \quad \forall \varphi \in F(M)$$

then

$$z = 0$$

Proof. With the notation of 4.1 a) let $z \underset{\cup}{=} (a, b, c, d)$. Using 4.3 we get, for every linear function

$$z(d\phi) = D\phi_a(d) = \phi_a(d) = 0$$

and hence d is zero; and hence for every quadratic function

$$z(d\phi) = (D^2\phi_a)(b, c) = 0$$

and hence c is zero. □

4.18

Let h_θ *denote the map of* $T(M)$ *which takes every vector* x *into* θx*.* Then, with the usual notation relative to (U, r) we have

$$h_\theta(a, b) \underset{\cup}{=} (a, \theta b)$$

and

$$h_\theta^T(a, b, c, d) \underset{\cup}{=} (a, \theta \cdot b, c, \theta \cdot d).$$

c. **The canonical involution on $T(T(M))$.**

4.19

Theorem. *There is* an involution $z \to \bar{z}$ *of* $T(T(M))$ *with the following properties:*

1) $p^T(\bar{z}) = p'(z)$,

2) $p'(\bar{z}) = p^T(z)$

3) $\bar{z}(d\phi) = z(d\phi)\,\forall\ \in F(M)$

and is uniquely determined *by these conditions.*

Proof. In the case of (U, r) the map

$$z \underset{\cup}{=} (a, b, c, d) \to (a, c, b, d) \underset{\cup}{=} \bar{z}$$

20 has the properties 1, 2 and 3 thanks to formula (0.4.2) and the symmetry of $D^2\phi$. Conversely, if $z \underset{\cup}{=} (a, b, c, d)$ and $z' \underset{\cup}{=} (a', b', c', d')$,

$$p^T(z') = p'(z) \Leftrightarrow a = a' \quad \text{and} \quad b = c'$$
$$p'(z') = p^T(z) \Leftrightarrow a = a' \quad \text{and} \quad b' = c$$

and $z'(d\phi) = z(d\phi)\,\forall\phi \in F(M)$ together with $a = a'$, $b = c'$, $b' = c$ gives $d = d'$. Because of this uniqueness the problem becomes local which has a solution. The definition being intrinsic the involution is *canonical*. □

4.20

Remark. $\bar{z} = z \Leftrightarrow p'(z) = p^T(z)$.

d. **Another vector bundle structure on $T(T(M))$.**

4.21

Definition. *If $\theta \in \mathbb{R}$, $z, z' \in T(T(M))$ and $p^T(z) = p^T(z')$*

set $$z \oplus z' = \left(\overline{\bar{z} + \bar{z}'}\right)$$

and $$\theta \odot z = \overline{\theta \cdot \bar{z}}$$

Relative to (U, r) if $z \underset{\cup}{=} (a, b, c, d)$ and $z' \underset{\cup}{=} (a', b', c', d')$ the formula are

$$z \oplus z' \underset{\cup}{=} (a, b + b', c, d + d')$$

and

$$\theta \odot z \underset{\cup}{=} (a, \theta b, c, \theta d).$$

With this definition if f, g are curves (for a definition of a curve and related notions see §5) in $T(M)$ such that $p \circ f = p \circ g$ we have

$$(f + g)' = f' \oplus g' \quad \text{and} \quad (\theta \cdot f)' = \theta \odot f'.$$

This last definition of $\oplus$ and $\odot$ holds good for any vector bundle $E \xrightarrow{p} M$.

4.22

21 **Lemma.** *For $\phi \in F(M)$, $\theta \in \mathbb{R}$ and $z, z' \in T(T(M))$ such that $p^T(Z) = p^T(z')$, we have*

$$(z \oplus z')(d\phi) = z(d\phi) + z'(d\phi)$$

and

$$(\theta \odot z)(d\phi) = (\theta \cdot z)(d\phi).$$

Proof. The first part follows from the definition of $\oplus$ and property 3) of the involution $-$, and similarly the latter. $\square$

e. **The canonical forms μ and $d\mu$.**

We have the following commutative diagram:

(0.4.23)
$$\begin{array}{ccc} T(T^*(M)) & \xrightarrow{(p^*)^T} & T(M) \\ {\scriptstyle p_{T(M)}=p''}\downarrow & & \downarrow{\scriptstyle p} \\ T^*(M) & \xrightarrow[p^*]{} & M \end{array}$$

where p^* is the natural projection from $T^*(M)$ on M. For $z \in T(T^*(M))$ we denote $p''(z)(p^T(z))$ by $\mu(z)$. Then

(0.4.24) $$\mu \in \xi^1(T^*(M)) = \mathscr{C}^*(T^*(M)).$$

To describe locally the situation above we have a diagram similar to the one in (0.4.2). With a similar notation, if $z = (a, \beta, c, \delta)$ then $p''(z) = (a, \beta)$, $p^*(z) = (a, c)$ and hence

(0.4.25) $$\mu(z) = \beta(c) = \sum_i \beta_i c_i$$

22 To compute $d\mu$, let $z \underset{\cup}{=} (a, \beta, c, \delta)$ and $z' \underset{\cup}{=} (a, \beta', c, \delta')$, and let Z, Z' be local vector fields on T with constant principal parts such that

i) at (a, β) they are equal to z and z' respectively and

ii) $[Z, Z'] = 0$ in a neighbourhood of (a, β).

Then we have (0.2.10):

$$d\mu(Z, Z') = Z\mu(Z') - Z'\mu(Z)$$

around (a, β). Hence

$$d\mu(z, z') = \delta(c') - \delta'(c). \quad (0.4.26)$$

From this follows the

4.27

Lemma. *$d\mu$ is a non degenerate element of $\mathscr{E}^2(T^*(M))$.*

5 Curves

5.1

Definition. *A* curve *in M is an open interval I together with an $f \in D(I, M)$.*

We denote, generally, a curve by (I, f) and when no confusion is possible we omit I. Generally, whenever we consider a curve we assume that $0 \in I$. A curve can also be viewed as a point set E obtained as the image under f of an interval I. Hence we sometimes say "the curve E is parametrised by f", "the curve f is parametrised by $t \in I$" and by these we simply mean that the curve under consideration is (I, f).

5.2

23

When we take a closed interval $[a, b]$ and say f *is a curve from* $[a, b]$ to M we mean that there is an open interval $I \supset [a, b]$ such that $f \in D(I, M)$. We denote by P the element of $\mathscr{C}(\mathbb{R})$ such that $\zeta \circ P = 1$, so that $\left\{P = \frac{d}{\mathrm{dt}}\right\}$ is a basis of $\mathscr{C}(\mathbb{R})$, dual to the basis dt of $\mathscr{C}^*(\mathbb{R})$. Let us agree to denote the restriction of P to an interval by P itself.

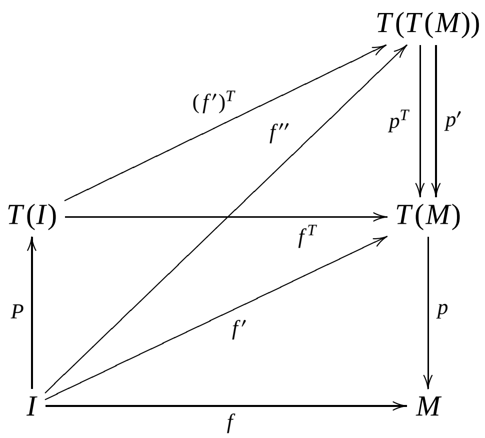
5.3

The curve $f^{-1}(t) = f(b - (t - a))$ is called the *inverse of the curve f* or the curve f described in the opposite way.

5.4

Definition. *If f is a curve in M we define the* speed *or the* tangent vector *f' to f by $f' = f^T \circ P$.*

(0.5.5)



5.7

Remarks. If N is a manifold and $g \in D(M, N)$ and f is a curve in M then $g \circ f$ is a curve in N and

$$(g \circ f)' = g^T \circ f'.$$

Sometimes this will be used as a geometric device for computation.

24

5.8

To compute $x(\phi)$ for $x \in T(M)$ and $\phi \in F(M)$ one can start with a curve f in M such that $f'(0) = x$ and observe

$$x(\phi) = f(0)(\phi) = (f^T \circ P)_0(\phi) = P(\phi \circ f)\big|_0 = \frac{d}{\mathrm{dt}}(\phi \circ f)\big|_0$$

b) We call the speed of the tangent vector f' of f *acceleration* f'' *of* f and thus

$$f'' = (f')' = (f')^T \circ P = (f^T)^T \circ P^T \circ P.$$

5.9

Remarks. We have $\overline{f''} = f''$, for

$$p^T \circ f'' = (p \circ f^T \circ P)^T \circ P = f^T \circ P = p' \circ f''$$

and (4.20) now gives the result.

5.10

We have $\dfrac{d^2(\phi \circ f)}{\mathrm{dt}^2} = f''(0)(d\phi)$. In fact

$$\begin{aligned} f''(0)(d\phi) &= ((f')^T \circ P)(0)(d\phi) = P(d\phi(f'))\Big|_0 \\ &= \frac{d}{\mathrm{dt}}(d\phi)(f^T \circ P)\Big|_0 = \frac{d}{\mathrm{dt}}\left(\frac{d}{\mathrm{dt}}(\phi \circ f)\right)\Big|_0 \\ &= \frac{d^2}{\mathrm{dt}^2}(\phi \circ f)\Big|_0 \end{aligned}$$

5.11 *Change of parameter.*

For $a, \theta \in \mathbb{R}$, *we define maps* τ_a and k_θ by setting

$$\tau_a(t) = a + t \quad \text{and} \quad k_\theta(t) = \theta \cdot t,$$

for any t in $\mathbb{R}$. It follows, directly, from the definition that

$$(f \circ \tau_k)' = f' \circ \tau_k, (f \circ k_\theta)' = \theta(f' \circ k_\theta)$$

and **25**

$$(f \circ k_\theta)'' = (h_\theta^T \circ f'' \circ k_\theta).$$

If ϕ is a diffeomorphism of an interval I' with I then $f \circ \phi$ is a curve. This new curve is called the curve obtained from the curve (I, f) by re-parametrisation by ϕ. This situation is sometimes described as "ϕ re-parametrises f."

6 Flows

In this article we fix, once for all, a manifold L and a vector field $X \in \mathscr{C}(L)$ of L.

a. **Integral curves.**

6.1

Definition. *An* integral *or* integral curve *of X is a curve $f \in D(I, L)$ such that*

$$f' = X \circ f.$$

From the fundamental existence theorem in the theory of differential equations we know that through any point of L there is an integral curve of X. Set

(0.6.2) $\quad \psi = \{(t, m) \in \mathbb{R} \times L | \exists I \supset [0, t]$ *and an* $f \in D(I, L)$

such that i) f is an integral of X, ii) $f(0) = m\}$. For $m \in L$ we define $t^+(m)$ by the equation

(0.6.3) $\quad t^+(m) = \sup \{t \in \mathbb{R} | (t, m) \in \psi\},$

and similarly $t^-(m)$. We can see that t^+ (resp. t^-) is lower (resp. upper) semi continuous on L. Since L is Hausdorff we can see that there exists a unique integral curve f_m of X defined over $]t^-(m), t^+(m)[$ with $f(0) = m$, and that it is maximal.

6.4

26

b. *Flow.* For $t \in]t^-(m), t^+(m)[$

6.5

set

$$\gamma(t, m) = f_m(t) = \gamma_t(m).$$

Then $\gamma \in D(\psi, L)$ and in open sets of L where γ_t, γ_s and γ_{t+s} are defined we have

$$\gamma_t \circ \gamma_s = \gamma_{t+s} \tag{0.6.6}$$

and, in particular, γ_t is a local diffeomorphism.

6.7

The family $\{\gamma_t\}$ is also called the local one parameter group of transformations generated by X. *We call γ the flow of X.*

6.8

Now let us suppose that a map f from a manifold M onto N is a diffeomorphism and X and Y are vector fields in M and N respectively such that

$$f^T \circ X = Y.$$

Then if γ_M and γ_N are flows of X and Y respectively we have

$$(\mathrm{id}_R, f) \circ \gamma_M = \gamma_N.$$

c. **Lie derivative.** Let $\omega \in \mathscr{L}^r(L)$, $m \in L$ and $t \in \mathbb{R}$ be sufficiently small. Then

$$(\gamma_t)^* \omega_{\gamma(t,m)} \in \mathscr{L}^r(T_m(L))$$

for every t and depends differentiably on t. So it makes sense to set

$$(\theta(X) \cdot \omega)(m) = \frac{d}{\mathrm{dt}}((\gamma_t)^* \omega_{\gamma(t,m)})|_{t=0}. \tag{0.6.9}$$

Note that **27**

$$\theta(X) \cdot \omega \in \mathscr{L}^r(L),$$

and that $\theta(X) \cdot \omega$ is called *the Lie derivative* of ω *with respect to* X. If $\omega \in \mathscr{E}^r(L)$, it is easy to see that $\theta_X \omega \in \mathscr{E}^r(L)$.

6.10

We can see that

$$\theta_X\omega(X_1,\ldots,X_r) = X(\omega(X_1,\ldots,X_r)) - \sum_{i=1}^{r}\omega(X_1,\ldots,[X,X_i],\ldots,X_r)$$

$\forall X_1,\ldots,X_r \in \mathscr{C}(L)$.

6.11

Further we recall that

$$\theta(X)\omega = 0 \Leftrightarrow \gamma_t^*(\omega) = \omega\,\forall t.$$

In this case we say that ω *is invariant by* X or *under the flow of* X. We also have

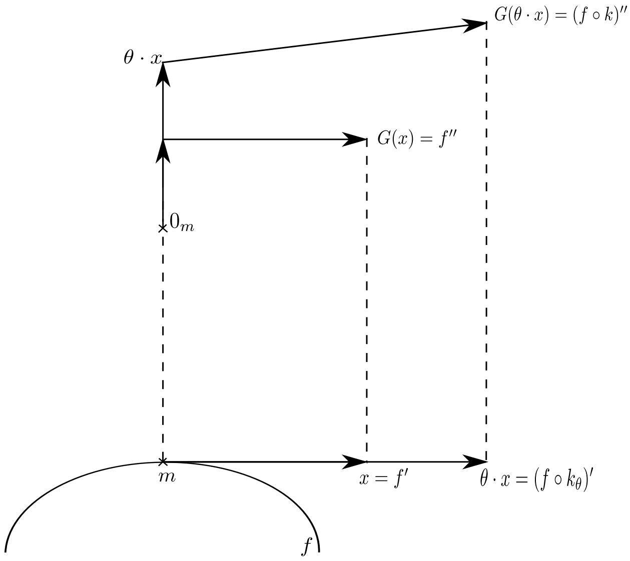
$$\theta(X) = i(X)\circ d + d\circ i(X). \tag{0.6.12}$$

Chapter 1

Sprays

Throughout this chapter M stands for a manifold and (U, r) for a typical **28**
chart of it.

1 Definition.



Our aim now is to define a "geodesic" so as to generalise a curve which is intuitively the shortest distance between two points on it, which are sufficiently close. Such a definition must have the following natural properties:

1) A geodesic curve is uniquely determined by its initial position and speed.

2) A change of parameter by a homothesy or a translation leaves
29 it a geodesic. So the family of geodesics determines a family of curves on $T(M)$ one passing through each point x of $T(M)$. Associating to each point x of $T(M)$ the tangent vector at x of the unique curve of the family passing through x we get a map (set theoretic) : $G : T(M) \to T(T(M))$.

We have by the very definition and from 5.9 together with 4.20

$$f'' = G \circ f' \tag{1}$$

and

$$p' \circ G = p^T \circ G = \mathrm{id}_{T(M)} \tag{2}$$

for any geodesic curve f. Further we have $f \circ k_\theta$ is a geodesic curve, and hence

$$(f \circ k_\theta)'' = G \circ (f \circ k_\theta)'. \tag{3}$$

But we have for $x = f'(0)$ and by 5.11:

$$(f \circ k_\theta)' = \theta f'(0) = \theta \cdot x$$

and

$$(f \circ k_\theta)''(0) = \theta \cdot (h^T_\theta \circ f''(0)) = \theta \cdot (h^T_\theta \circ G(x))$$

and so

$$G \circ h_\theta = \theta(h^T_\theta \circ G) \tag{4}$$

Thus from the concept of a geodesic we are led to a G with the properties (2) and (4). Keeping this in mind we reverse the process and
30 first define a spray and then a geodesic as its integral curve.

1.1

Definition. *A* spray *G on M is an element of $\mathscr{C}(T(M))$ such that*

1) $p^T \circ G = \mathrm{id}_{T(M)}$ *and*
2) $G \circ h_\theta = \theta(h^T_\theta \circ G)\, \forall \theta \in \mathbb{R}$

1.2

Note. From 2) it follows that

$$G(0_m) = 0\, \forall m \in M$$

Generally we denote a spray by G.

1.3

If N is an open sub manifold of M, G induces a spray on N and *we denote it by N_G.*

2 Geodesics

Now we define a geodesic curve.

2.1

Definition. *A* geodesic *of M relative to a spray G is a curve f in M such that f' is an integral of G, i.e. f is such that*

$$f'' = G \circ f'.$$

2.2

Remark. If $f : [0, 1] \to M$ is a geodesic, then the curve $g : [0, 1] \to M$ defined by $g(t) = f(1 - t)$ is a geodesic.

When no confusion is possible, we shall speak simply of a geodesic and omit the reference to the spray.

2.3

Proposition. *f is a geodesic if and only if there exists an integral g of G such that*

$$f = p \circ g.$$

31 *Proof.* If g is an integral of G then $g' = G \circ g$ and then if $f = p \circ g$, we have

$$f' = p^T \circ g = p^T \circ G \circ g = g \quad \text{since} \quad p^T \circ G = \mathrm{id}_{T(M)}.$$

Hence f is a geodesic. For the other part, if f is a geodesic we can take $g = f'$ for the integral of G. □

2.4

Remarks. From (0.6.2) one knows that given $x \in T(M)$ there is, locally, a geodesic f of M such that $f'(0) = x$, and that it is unique.

2.5

From the fact that $G(0_m) = 0$ we see that $\forall m \in M$ $f(\mathbb{R}) = m$ is a geodesic, called the *trivial geodesic* at m.

2.6

Proposition. *If $f \in D(I, M)$ is a geodesic so are*

$$f \circ \tau_a \in D(\tau_a^{-1}(I), M) \quad \textit{and}$$
$$f \circ k_\theta \in D(k_\theta^{-1}(I), M) \forall a \in \mathbb{R}, \forall \theta \in \mathbb{R}, \theta \neq 0.$$

This proposition is a direct consequence of the definitions.

2.7

Corollary. *For $x \in T(M)$, $\theta \in \mathbb{R}$ and $t \in \mathbb{R}$ we have*

$$\gamma(t, \theta x) = \theta \cdot \gamma(t\theta, x),$$

i.e. whenever one of these terms is defined so is the other and the equality holds.

Proof. Let f be the geodesic with $f'(0) = x$ and let $g = f \circ k_\theta$. Then g is the geodesic with $g'(0) = \theta \cdot x$ and we have by the definition of γ 32

$$f'(t) = \gamma(t, x)$$

and

$$g'(t) = \gamma(t, \theta x)$$

But then

$$\gamma(t, \theta x) = g'(t) = \theta \cdot f'(\theta \cdot t) = \theta \cdot \gamma(\theta \cdot t, x).$$

□

2.8

Corollary. *For $x \in T(M)$ and $\theta \in \mathbb{R}$ and $\neq 0$, we have*

$$\begin{aligned} & t^+(\theta \cdot x) = \theta^{-1}(t^+(x)) \\ \textit{and} \qquad & t^-(\theta \cdot x) = \theta^{-1}(t^-(x)) \end{aligned}$$

Proof. If f is the geodesic of M on $]t^-(x), t^+(x)[$ then

$$f \circ k_\theta \quad \text{on} \quad]\theta^{-1}t^-(x), \theta^{-1}t^+(x)[$$

is a geodesic of M with speed $\theta \cdot x$. Hence by the definitions of t^- and t^+ (see 6.4)

$$\begin{aligned} & t^-(\theta \cdot x) \leq \theta^{-1} \cdot t^-(x) \\ \text{and} \qquad & t^+(\theta \cdot x) \geq \theta^{-1} \cdot t^+(x). \end{aligned}$$

The other inequalities follow if we interchange the roles of f and $f \circ k_\theta$. □

3 Expressions for the spray in local coordinates

Let (U, r) be a chart and let $x \underset{\cup}{=} (a, b)$, $G(x) \underset{\cup}{=} (a, b, c, d)$ (see 4)

Then from the condition 1) for a spray and from (0.4.6) we have $b = c$. Setting $d = \psi(a, b)$ we have

$$\psi \in D((r(U) \times R^d), R^d), \quad \text{and}$$
$$G(\theta \cdot x) = (a, \theta b, \theta b, \psi(a, \theta b)),$$
$$h^T_\theta(G(x)) = (a, \theta b, b, \theta \cdot \psi(a, b)),$$

33 and

$$\theta h^T_\theta(G(x)) = (a, \theta b, \theta b, \theta^2 \cdot \psi(a, b)).$$

Now from the second condition for a spray we get

$$G(\theta \cdot x) = \theta(h^T_\theta(G(x)))$$

and hence

(1.3.1) $$\psi(a, \theta b) = \theta^2 \cdot \psi(a, b)\, \forall \theta \in \mathbb{R}$$

Using Euler's theorem on homogeneous functions we see that ψ is, for a fixed a, a quadratic form in b so that

(1.3.2) $$\psi(a, b) = (D^2\psi_a)(b \cdot b)$$

with $D^2\psi$ symmetric. Hence we have the following:

3.3

Proposition . *If G is a spray then there exists, locally, a unique $\delta \in D(r(U) \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ such that, for $a \in r(U)$, its restriction to $\{a\} \times \mathbb{R}^d \times \mathbb{R}^d$ is bilinear, symmetric and*

$$\psi(a, b) = \delta(a, b, b).$$

3.4

Setting $\Gamma^j_{ik}(a) = -\mathscr{U}^j(\delta(a, e_i, e_k))$ for $a \in r(U)$, we have

$$\Gamma^j_{ik} \in F(r(U)).$$

Let $f \in D(I, M)$ be a geodesic and let $a = r \circ (f|U)$, be its image in $r(U)$, then we have

$$f'(t) \underset{\cup}{=} (a(t), \frac{da(t)}{dt})$$
and
$$f''(t) \underset{\cup}{=} (a(t), \frac{da(t)}{dt}, \frac{da(t)}{dt}, \frac{d^2a(t)}{dt^2}).$$

So if we write

$$a(t) = \sum x^i(t)e_i$$

then we see that f is a geodesic if and only if

(1.3.5)
$$\frac{d^2x^j}{dt^2} + \sum_{i,k} \Gamma^j_{ik} \frac{dx^i}{dt} \cdot \frac{dx^k}{dt} = 0 \forall j.$$

34

4 The exponential map

Let us consider G and its flow. Setting

(1.4.1)
$$\Omega = (t^+)^{-1}(]1, +\infty[) \quad \text{(see 6a)}$$

we see, since t^+ is lower semi continuous, that Ω is open in $T(M)$. For every $m \in M$, 0_m the zero vector at m clearly belongs to Ω. Hence a neighbourhood of 0_m is in Ω.

4.2

If N is an open subset of M then by 1.3 there is a natural spray N_G on N and we can define $t^+_{N_G}$ and $t^-_{N_G}$ and Ω relative to N_G and N. We denote this open subset by N_Ω. We identify $T(N)$ with an open sub manifold of $T(M)$ and then, clearly $N_\Omega \subset \Omega \cap T(N)$. Now let x be in $\Omega \cap T(M)$. Then by (0.6.a) it follows that x belongs to N_Ω if the image of $]0, 1[$ under $p \circ \gamma(t, G(x))$ is in N.

4.3

Definition. *The map* $p \circ \gamma_1$ *from* Ω *into* M is called the exponential map (for the spray G) of $T(M)$ and is denoted *by exp.*

4.4

Note. $\exp \in D(\Omega, M)$ because p, γ are differentiable. *We denote the restriction of* exp to $T_m(M) \cap \Omega$ *by* $\exp_m$.

4.5

Lemma. *For x in Ω the map*

$$S : t \to \exp(tx)$$

defined for sufficiently small t is a geodesic of M and $S'(0) = x$. *In particular* $\exp(x)$ *is the point* $S(1)$ *of this geodesic.*

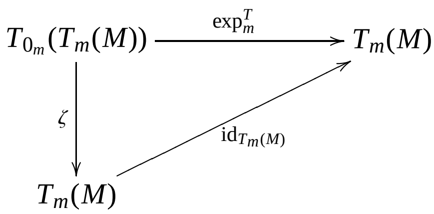
Proof. We have

$$\begin{aligned} S(t) = p \circ \gamma_1(tx) &= p \circ \gamma(1, tx) \\ &= p(t, \gamma(t, x)) \quad \text{by} \quad (2.7) \\ &= p(\gamma(t, x)) \end{aligned}$$

35 Hence S is a geodesic by (2.3). The fact that $S'(0) = x$ follows from the definition of γ. □

4.6

We have



Proof. For x in $T_m(M)$ and $\theta \in \mathbb{R}$ set $\mathscr{U}$ with:

$$\mathscr{U}(\theta) = \theta \cdot x$$

so that

$$\zeta_{0_m}(\mathscr{U}'(0)) = x.$$

Now we have

$$(\exp^T \circ \zeta_{0_m}^{-1})(x) = \exp^T(\mathscr{U}'(0)) = (\exp \circ \mathscr{U})'(0) = S'(0).$$

and by the preceding lemma we have

$$S'(0) = x.$$

□

4.7

Hence it follows by the inverse function theorem that exp is a diffeo-
morphism between suitable neighbourhoods of $0_m \in T_m(M)$ and m in **36**
M.

4.8

Corollary. *The map* $(p, \exp)$ *of* Ω *into* $M \times M$ *has maximal rank at* 0_m *for every* m *in* M.

Proof. Since the dimensions of Ω and $M \times M$ are the same it is enough to show that the map $(p, \exp)^T$ is injective. Now let $z \in T_{0_m}(T(M))$ be such that

$$(p, \exp .)^T(z) = 0.$$

Then since

(1) $$p^T(z) = 0 \text{ we have } z \in V_{0_m}.$$

By the previous proposition we have

(2) $$\exp^T(z) = \zeta(z) \text{ and hence } \zeta(z) = 0.$$

The corollary follows from (1) and (2). □

Chapter 2

Linear connections

1 Linear connection

37

1.1

Definition. *From 4.9 we see that, for x in $T(M)$, if H_x is transverse to V_x (i.e. a supplement to V_x in $T_x(T(M))$ then the restriction*

$$p^T|H_x : H_x \to T_p(x)(M)$$

is an isomorphism. So a natural question is whether it is possible to choose an interesting map $x \to H_x$ where x runs through M. For an open subset A of $\mathbb{R}^d$ we see that the choice

$$H_x = (\zeta_x^T)^{-1}(0) \tag{2.1.2}$$

gives us such a distribution of H_x. We look upon such a choice as a map

$$C : T(M) \underset{M}{\times} T(M) \ni (x, y) \to C(x, y) \in T(T(M)). \tag{2.1.3}$$

where x determines the fibre containing H_x and y, by means of the inverse of the isomorphism $(p^T|H_x) : H_x \to T_{p(y)}(M)$, the element $C(x, y)$ in H_x such that $p^T(C(x, y)) = y$.

Relative to (U, r), with the notation of 4 we have, by 1.1,

$$C((a, b), (a, b')) \underset{\cup}{=} (a, b, b', .). \tag{2.1.4}$$

Keeping this in mind we define a *connection*.

1.5

38 **Definition.** *A connection on $T(M)$ (or a linear connection on M) is a*

C_0) $C \in D(T(M) \underset{M}{\times} T(M), T(T(M)))$

such that

C_1) $(p, p^T) \circ C = \mathrm{id}\,|T(M) \underset{M}{\times} T(M)$

C_2) $(\theta \cdot x + \eta \cdot x', y) = \theta \odot C(x, y) \oplus \eta \odot C(x', y)$

C_3) $(x, \theta \cdot y + \eta \cdot y') = \theta \cdot C(x, y) + \eta \cdot C(x, y')$.

(Note that this definition would make sense for any vector bundle.)

Now let us consider *the image set* H_x of $\{x\} \times T_{p(x)}(M)$ under C : $H_x = C(x, T_{p(x)}(M))$.

By C_3) it follows that H_x is a vector subspace of $T_x(T(M))$; and from C_1) that

$$p^T(C(x, y)) = y$$

and hence that

i) C restricted to $\{x\} \times T_{p(x)}(M)$ is a monomorphism, and

ii) the only element common to V_x and H_x is zero.

Since the dimension of $T_x(T(M))$ is $2d$ and that of V_x is d it follows, now, that

(2.1.6) $$T_x(T(M)) = H_x + V_x \quad \text{(direct sum)).}$$

Now we give the following definitions.

1.7

Definition. *For $x \in T(M)$, H_x is called* the horizontal subspace *or* the horizontal component of $T_x(T(M))$ *relative to the connection C. Generally we omit the reference to C.*

1.8

39 **Definition.** *For $x \in T(M)$ the map*

1.9

$T_x(T(M)) \ni z \to z - C(x, p^T(z)) \in T_x(T(M))$ is the projection onto V_x parallel to H_x.

1.10

Definition. *For z in $T(T(M))$ the element*

1.11

$\mathscr{U}(z) = \xi(z - C(p'(z), p^T(z)))$ is called the *vertical component of z.*

1.12

Note. From the very definition we have

$$p \circ \mathscr{U} = p \circ p'.$$

1.13

Remark. By C_1) we have the following equations:
For $(x, y) \in T(M) \underset{M}{\times} T(M)$,

$$p^T(C(x,y)) = y = p'(C(y,x))$$
$$\text{and} \qquad p^T(C(y,x)) = x = p'(C(x,y)).$$

They show that the condition $C(x,y) = \overline{C(y,x)}$ is compatible with C_1), C_2), C_3).

In view of the above, the following definition makes sense:

1.14

Definition. *A connection C is called* symmetric *if*

1.15

C_4) $C(x,y) = \overline{C(y,x)}$ $\forall (x,y) \in T(M) \underset{M}{\times} T(M)$

Examples.

1.16

The *canonical connection* on an open subset $A \subset \mathbb{R}^d$ is defined by

$$H_x = (\zeta_x^T)^{-1}(0) \tag{2.1.17}$$

i.e.:

$$C((a,b),(a,c)) = (a,b,c,0) \quad \forall a \in A, \forall b,c \in \mathbb{R}^d.$$

40 From 4 one sees C_1), C_2), C_3) and C_4) are fulfilled so that this canonical connection is moreover symmetric.

1.18

If $f : M \to N$ is a diffeomorphism and C a connection on N, then $(x,y) \to ((f^{-1})^T)^T C(f^T(x), f^T(y))$ defines a connection on M.

Exercise. Use 1.16, 1.18 and partitions of unity to build up a connection on any manifold (a proof of that would follow also from 1.1.

1.19

Remark. If C is a connection on $T(M)$ then $G(x) = C(x,x)$ for every x in $T(M)$ is a spray, for

i) $p'(G(x)) = p'(C(x,x)) = x = p^T(C(x,x)) = p^T(G(x))$ and

ii) $(G \circ h_\theta)(x) = G(\theta \cdot x) = C(\theta x, \theta x) =$ by 4.21

$$= \theta \odot C(x, \theta, x) = h_\theta^T(\theta \cdot C(x, x)) =$$
$$= \theta \cdot h_\theta^T(C(x, x)) = (\theta \cdot (h_\theta^T \circ G))(x).$$

This spray is called *the spray associated to the* given connection.

1.20

Claim. *For an open subset $A \subset \mathbb{R}^d$ the geodesics relative to the spray associated to the canonical connection are open segments of straight lines.*

With the above notation, the definition shows that f is a geodesic if and only if

$$f'' - G \circ f' = 0,$$

i.e. if and only if $f'' - C(p'(f''), p^T(f'')) = 0$. By the definition of canonical connection C and being horizontal we have **41**

$$\zeta^T(C(p'(f''), p^T(f'')) = 0. \tag{2.1.21}$$

Hence, by (2.1.21), (4.20) and (5.9) f is a geodesic if and only if

$$\zeta^T(f'') = 0. \tag{2.1.22}$$

But

$$\zeta^T(f'') = \zeta^T \circ f'^T \circ P = (\zeta \circ f')^T \circ P = \frac{d}{\mathrm{dt}}(\zeta \circ f')$$

Hence f is a geodesic if and only if $\dfrac{d(\zeta \circ f)}{\mathrm{dt}} = 0$, i.e. if and only if $\zeta \circ f' = x$, a constant vector.

But then

$$f(t) = tx + y \quad \text{(for suitable values of } t\text{)}.$$

2 Connection in terms of the local coordinates

With the notation of 4 let $x \underset{\cup}{=} (a, b)$ and $y \underset{\cup}{=} (a, c)$. Then $(x, y) \in T(M) \underset{M}{\times} T(M)$. Now given a connection C on $T(M)$ we have, by (1.5).

2.1

$C(x, y) = (a, b, c, \delta(a, b, c))$ where δ depends on C, and then C_0) becomes

$$\delta \in D(r(U) \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d). \tag{2.2.2}$$

Now C_2) and C_3) together mean that the restriction of δ to $\{m\} \times \mathbb{R}^d \times \mathbb{R}^d$ for m in $r(U)$ is bilinear. Further C is symmetric if and only if δ restricted to $\{a\} \times \mathbb{R}^d \times \mathbb{R}^d$ is (see (4.19)). Now we prove a theorem which completes the remark (1.19).

2.3

42 **Theorem [2], th. 2, p. 173** *Given a spray G on M there exists a unique symmetric connection C on $T(M)$ such that*

$$C(x, x) = G(x).$$

Proof. In view of the first paragraph of this article, locally, the problem can be stated as follows:

Given a $\psi \in D(r(U) \times \mathbb{R}^d, \mathbb{R}^d)$ such that for a fixed m in $r(U)$, ψ is a quadratic form in the other variable to construct a $\delta \in D(r(U) \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ which, for fixed m in $r(U)$, is bilinear and symmetric in other variables and to prove its uniqueness. This has been done in (3.3). The uniqueness of δ shows that the problem is local. □

2.4

Remark. Let us write down C explicitly in terms of G. By (4.17) it is enough to express C in terms of G on $d\phi$ for $\phi \in F(M)$. We have

$$\begin{aligned} G(x+y)(d\phi) &= C(x+y, x+y)(d\phi) = C(x+y, x)(d\phi) + \\ &\qquad + C(x+y, y)(d\phi) = \\ &= (C(x, y) \oplus C(x, x))(d\phi) + (C(x, y) \oplus C(y, y))(d\phi) = \\ &= C(x, y)(d\phi) + C(y, x)(d\phi) + C(x, x)(d\phi) + C(y, y)(d\phi) = \\ &\qquad \text{by (4.19), (3)} \\ &= C(x, y)(d\phi) + C(x, y)(d\phi) + G(x)(d\phi) + G(y)(d\phi) = \end{aligned}$$

since C is symmetric
$$= 2C(x,y)(d\phi) + G(x)(d\phi) + G(y)(d\phi) \quad \text{by (4.22).}$$

Hence 43

(2.2.5) $$C(x,y)(d\phi) = \frac{1}{2}\left[G(x+y) - G(x) - G(y)\right] \quad (d\phi)$$

which shows, once again, the uniqueness (4.17).

3 Covariant derivation

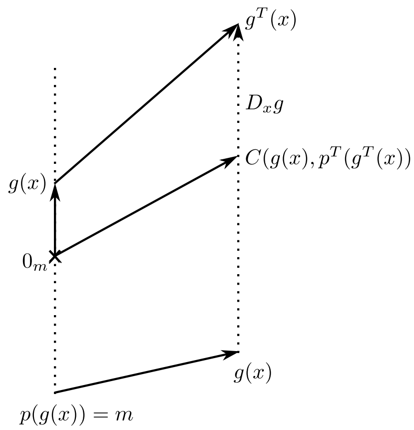
For the rest of this chapter we suppose given a manifold M with a connection C.

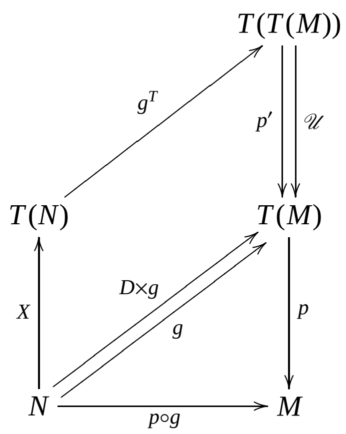
3.1

Definition. *Given a manifold N, an $X \in \mathscr{C}(N)$ and $g \in D(N, T(M))$, the map $N \to T(M)$ defined by*

(2.3.2) $$D_X g = \mathscr{U} \circ g^T \circ X$$

is called the covariant derivative of g with respect to X. *If $x \in T(N)$ we write $D_x g = \mathcal{U}(g^T(x))$.*





3.3

Remarks. Since X, g^T and $\mathcal{U}$ are differentiable we have

$$D_X g \in D(N, T(M)).$$

3.4

44 We have

$$
\begin{aligned}
p \circ D_X g &= p \circ \mathscr{U} \circ g^T \circ X = \\
&= p \circ p' \circ g^T \circ X \quad \text{by (1.12)} = \\
&= p \circ g
\end{aligned}
$$

and hence if g is a lift of f then $D_X g$ is again a lift of f.

3.5

Let $f \in D(I, M)$. Then f^T can be considered as a map from I to $T(M)$. Now we have: f is a geodesic for the spray G associated to the connection C if and only if

$$D_P f' = 0.$$

For we have

$$
\begin{aligned}
D_P f' &= \mathscr{U} \circ f'^T \circ T_P = \mathscr{U} \circ f'' = \xi(f'' - C(f', f')) = \\
&= \xi(f'' - G \circ f'), \quad \text{and} \quad \xi \text{ is an isomorphism}
\end{aligned}
$$

between the vertical vectors at a point and the tangent space containing that point.

Writing $f = p \circ g$ we have

$$
\begin{aligned}
\text{(2.3.6)} \qquad D_X g &= \xi(g^T \circ X - C(p' \circ g^T \circ X, p^T \circ g^T \circ X)) \\
&= \xi(g^T \circ X - C(g, f^T \circ X)).
\end{aligned}
$$

Now considering an ω in $\mathcal{E}^1(M)$ as a linear function on $T(M)$ and observing that $g^T \circ X - C(g, f^T \circ X)$ is vertical we obtain, by (4.16) **45**

$$
\begin{aligned}
\omega(D_X g) &= (g^T \circ X - C(g, f^T \circ X))(\omega) = \\
\text{(2.3.7)} \qquad &= X(\omega \circ g) - C(g, f^T \circ X)(\omega).
\end{aligned}
$$

In particular, for $\omega = d\phi$, where $\phi \in F(M)$,

$$\text{(2.3.8)} \qquad d\phi(D_X g) = X(d\phi \circ g) - (g, f^T \circ X)(d\phi).$$

Sometimes we write $D_X g(\phi)$ for $d\phi(D_X g)$ and $X(g(\phi))$ for $X(d\phi \circ g)$.

3.9

Example . For an open subset $A \subset \mathbb{R}^d$ with respect to the canonical connection we have

$$\nu = \zeta^T.$$

$$\text{(2.3.10)} \quad \text{For} \quad \begin{aligned} \nu(z) &= \nu(a, b, c, d) \\ &= \xi(z - C(p'(z), p^T(z))) \\ &= \xi((a, b, c, d) - (a, b, c, 0)) = \xi(a, b, o, d) = d. \end{aligned}$$

and $\zeta^T(a, b, c, d) = d$. Hence

$$D_X g = \nu \circ g^T \circ X = \zeta^T \circ g^T X = (\zeta \circ g)^T \circ X$$

where $g \in D(A, T(A))$.

3.11

In case $f = p \circ g$ is a constant map, i.e. in case $f(y) = m$ for every y in N, g can be considered as a map from N to $T_m(M)$, and in this case $\nu = \zeta$. Hence we have

$$D_X g = \nu \circ g^T X = \zeta \circ g^T \circ X = dg \circ X$$

46 and so $D_X g$ is nothing but the restriction of the differential map of g to the subset $\{X(y)\}$ of $T(N)$ In particular if g is a curve in $T_m(M)$, and $X = P$ on $\mathbb{R}$ we have

$$D_P g = \frac{dg}{\mathrm{dt}}$$

and hence $D_P g$ is the ordinary derivative of a vector valued function.

3.12

If $f \in D(N, M)$ and X, $Y \in \mathscr{C}(N)$ then $f^T \circ Y \in D(N, T(M))$ and by (2.3.8) we have

$$\text{(2.3.13)} \quad D_x(f^T \circ Y)(\phi) = X(Y(\phi \circ f)) - C(f^T \circ Y, f^T \circ X)(d\phi) \, \forall \phi \in F(M).$$

3.14

For $f \in D(N, M)$, $X \in \mathscr{C}(N)$, $Y \in \mathscr{C}(M)$ and $n \in N$ we have

$$D_{X(n)}(Y \circ f) = v(Y^T(f^T(X(n))) = D_{f^T(X(n))}Y.$$

In particular if $f = \mathrm{id}_M$ and $X, Y \in \mathscr{C}(M)$, then $D_X Y \in \mathscr{C}(M)$ and so D can be considered as a mapping $\mathscr{C}(M) \times \mathscr{C}(M) \to \mathscr{C}(M)$. For $f \in D(N, M)$, $\psi \in F(N)$, $X, Y \in \mathscr{C}(N)$ and $g, g' \in D(N, T(M))$ such that $p \circ g = p \circ g'$ we have

C.D. 1. $D_X g$ is $F(N)$ − linear in X,

C.D. 2. $D_X(g + g') = D_X g + D_X g'$

(2.3.15) C.D. 3. $D_X(\psi g) = X(\psi)g + \psi(D_X g)$

C.D. 4. $D_X(f^T \circ Y) - D_Y(f^T \circ X) = f^T \circ [X, Y]$

if C is symmetric.

Proof.

C.D. 1. follows directly from the definition. To prove the others, and for simplicity let us use the convention following (2.3.8). 47

C.D. 2. For $\phi \in F(M)$ we have, setting $f = p \circ g$,

$$\begin{aligned} D_X(g + g')(\phi) &= X(g + g')\phi - C(g + g', f^T \circ X)(d\phi) \quad \text{(by (2.3.8))} \\ &= X(g(\phi) + g'(\phi)) - (C(g, f^T \circ X) \oplus C(g', f^T \circ X))(d\phi) \\ &= X(g(\phi)) + X(g'(\phi)) - C(g, f^T \circ X))(d\phi) - C(g', f^T \circ X)(d\phi) \\ &\quad \text{by 4.22.} \end{aligned}$$

C.D. 3. $$\begin{aligned} D_X(\psi g)(\phi) &= X(\psi g)(\phi) - C(\psi \cdot g, f^T \circ x)(d\phi) = \\ &= X(\psi \cdot g(\phi)) - (\psi \odot C(g, f^T \circ X))(d\phi) = \\ &= X(\psi)g(\phi) + \psi X(g(\phi)) \\ &\quad -\psi \cdot C(g, f^T \circ X)(d\phi), \text{ by (4.22)} \end{aligned}$$

C.D. 4. $$\begin{aligned} &(D_X(f^T \circ Y) - D_Y(f^T \circ X))(\phi) = \\ &= X(Y(\phi \circ f)) - (f^T \circ Y, f^T \circ X)(d\phi) \end{aligned}$$

$$-Y(X(\phi \circ f)) - (f^T \circ X, f^T \circ Y)(d\phi) =$$

$= [X, Y](\phi \circ f)$ because C is symmetric, and
property 3) of the involution $-$, in (4.19)

$$= (f^T \circ [X, Y])(\phi) = (f^T \circ [X, Y])(\phi).$$

□

3.16

Lemma. *For* $f \in D(N, T(M))$, $X, Y \in \mathscr{C}(N)$ *the map*

$$(X, Y, f) \to ([D_X, D_Y] - D_{[X,Y]})(f)$$

if $F(N)$ *- trilinear.*

Proof. We have for $\psi \in F(N)$:

$$\begin{aligned}
&(([D_{\psi X}, D_Y] - D_{[\psi X,Y]})(f) \\
&= \psi D_X D_Y f - D_Y(\psi D_X f) - D_{[\psi X,Y]} f \quad \text{and by C.D. 1.,} \\
&= \psi D_X D_Y f - D_Y(\psi D_x f) - \psi D_{[X,Y]} f + Y(\psi) D_X f = \\
&= \psi D_X D_Y f - \psi D_Y D_X f - Y(\psi) \cdot D_X f \\
&\qquad - D_{[X,Y]} f + Y(\psi) \cdot D_X f \quad \text{by C.D. 3.,} \\
&\qquad = \psi([D_X, D_Y] - D_{[X,Y]}) f.
\end{aligned}$$

48 Therefore the mapping is $F(N)$-linear in X and since the mapping is antisymmetric in X, Y it is $F(N)$ - linear in Y. □

Applying C.D. 3. twice we obtain, by straightforward calculation

$$\begin{aligned}
&([D_X, D_Y] - D_{[X,Y]})(\psi \cdot f) \\
&= \psi([D_X, D_Y] - D_{[X,Y]} f).
\end{aligned}$$

4 The derivation law

4.1

In case $X, Y \in \mathscr{C}(M)$ and $\phi \in F(M)$, with $N = M$, C.D.'s can be written, respectively, as

D.L.1. $D_X Y$ is $F(M)$-linear in X,

D.L.2. $D_X(Y + Y') = D_X Y + D_X Y'$

D.L.3. $D_X(\phi Y) = X(\phi)Y + \phi \cdot D_X Y$

D.L.4. $D_X Y - D_Y X = [X, Y]$ if $\mathscr{C}$ is symmetric.

4.2

Thus from a connection we obtain a mapping from $\mathscr{C}(M) \times \mathscr{C}(M)$ into 49
$\mathscr{C}(M)$ satisfying D.L. 1, 2, 3. Any such map is called a *Derivation Law in M*. It is said to be *symmetric* (or without torsion) if D.L.4 holds for it.

Conversely, given a derivation law in M there is a connection which induces the given derivation law (see Koszul [18]: Th. 4. p. 94).

4.3

In local coordinates relative to (U, r), let $\{X_i\}$ denote the dual basis of $dx^i = d(\mathcal{U}i \circ r)$ and now let

4.4

$D_{X_i} X_k = \sum_j \lambda^j_{ik} X_j$. Then we have by 3.12

$$(D_{X_i} X_k)(x^j) = X_i(X_k(x^j)) - C(X_k, X_i)(dx^j).$$

But $X_k(x^j) = \delta^j_k$ a constant and hence $X_i(X_k(x^j))) = 0$.

Hence

$$\begin{aligned}(D_{X_i} X_k)(x^j) &= -C(X_k, X_i)(dx^j)\\ &= -\mathcal{U}^j(\delta(a, e_k, e_i)) = \Gamma^j_{ki} \quad \text{(see (3.4))}.\end{aligned}$$

4.5

Thus $\lambda^j_{ik} = \Gamma^j_{ki}$.

Given $\omega \in \mathscr{L}^r(M)$ and $X \in \mathscr{C}(M)$ we define a map

$$D\omega : \mathscr{C}^{r+1}(M) \to F(M) \quad \text{by setting}$$

4.6

$$(D\omega)(X, X_1, \ldots\ldots, X_r) = X(\omega(X_1, \ldots\ldots, X_r)) - \\ - \sum_{i=1}^{r} \omega(X_1, \ldots, X_{i-1}, D_X X_i, X_{i+1}, \ldots, X_r)$$

$\forall X_1, \ldots, X_r \in \mathscr{C}(M)$.

50 Then we have

i) $D\omega$ is $(r+1)$-additive

ii) $(D\omega)(\phi X, X_1, \ldots, X_r) =$

$$\phi \cdot X(\omega(X_1, \ldots, X_r)) - \sum_{i=1}^{r} \omega(X_1, \ldots, X_{i-1}, \phi \cdot D_X X_i, X_{i+1}, \ldots, X_r) =$$

by 4.1 D.L.1.

$$= \phi(D\omega)(X, X_1, \ldots, X_r)$$

and

iii) $D(\omega)(X, \phi X_1, \ldots, X_r) = X(\omega(\phi X_1, \ldots, X_r))$

$$- (D_X(\phi X_1), X_2, \ldots, X_r) - \sum_{i=2}^{r} \omega(\phi X_1, \ldots, D_X X_i, \ldots, X_r) =$$
$$= \phi X(\omega(X_1, \ldots, X_r)) + (X(\phi)\omega(X_1, X_2, \ldots, X_r)$$
$$- \omega(X(\phi), X_1, X_2, \ldots, X_r) - \omega(\phi D_X X_1, X_2, \ldots, X_r)$$
$$- \sum_{i=2}^{r} \phi\omega(X_1, \ldots, X_{i-1}, D_X X_i, X_{i+1}, \ldots, X_r) =$$
$$= \phi(D\omega)(X, X_1, \ldots, X_2)$$

and similarly for $i = 2, \ldots, r$.

4.7

Hence $D \in \mathscr{L}^{r+1}(M)$. However, in general, if $\omega \in \mathcal{E}^r(M)$, $D\omega$ may not belong to $\mathcal{E}^{r+1}(M)$). This form $D\omega$ is called the *covariant derivative* of ω with the respect to D or C (the derivation law or the connection). More generally:

Proposition 2.4.7 bis *Let* $X \in \mathscr{C}(N)$, $g_1, \dots, g_r \in D(N, T(M))$, $f \in D(N, M)$ *maps such that* $p \circ g_i = f \forall i = 1, \dots, r$; *and* $\omega \in \mathscr{E}^r(M)$. *Then* $(D\omega)(f^T \circ X, g_1, \dots, g_r) = X(\omega(g_1, \dots, g_r)) - \sum_{i=1}^{r} \omega(g_1, \dots, g_{i-1}, D_X g_i, g_{i+1}, \dots, g_r)$.

Proof. This follows from the local expression of $f_T \circ X$ given by (5.14) 51
and linearity. □

Now we set

$$D_X \omega = i(X) \circ D\omega \tag{2.4.8}$$

4.9

Proposition. *For* $\omega \in \mathscr{E}^1(M)$ *and* $(x, y) \in T(M) \underset{M}{=} T(M)$ *we have*

$$(D\omega)(x, y) = C(y, x)(\omega).$$

Proof. Let $X, Y \in \mathscr{C}(M)$ be such that

$$X(p(x)) = x \quad \text{and} \quad Y(p(y)) = y.$$

Then by (2.3.7) we have

$$\omega(D_X Y) = X(\omega \circ Y) - C(Y, X)(\omega)$$

and so

$$\begin{aligned} C(Y, X)(\omega) &= X(\omega(Y)) - \omega(D_X Y) = \\ &= (D\omega)(X, Y) \quad \text{by the definition of } D\omega. \end{aligned}$$

□

4.10

Lemma. *With the above notation*

$$(d\omega)(x, y) = (D\omega)(x, y) - (D\omega)(y, x)$$

provided that C *is symmetric.*

Proof. With the same notation, we have

$$\begin{aligned}
&(D\omega)(X, Y) - (D\omega)(Y, X) \\
&\quad = X(\omega(Y)) - \omega(D_X Y) - Y(\omega(X)) + \omega(D_Y(X)) \\
&\quad = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])
\end{aligned}$$

52 since $\mathscr{C}$ is symmetric $= (d\omega)(X, Y)$ (by the definition of $d\omega$: see (0.2.10)).

□

4.11

Definition. *For* $\phi \in F(M)$ $Dd\phi \in \mathscr{E}^2(M)$ *is called* the Hessian of ϕ *with respect to the connection* C.

4.12

Corollary. *If* C *is symmetric then*

$$Dd\phi \quad \textit{is symmetric} \quad \forall \phi \in F(M).$$

Proof. By the above proposition we have, with the same notation,

$$\begin{aligned}
(Dd\phi)(x, y) &= C(y, x)(d\phi) \\
&= \overline{C(y, x)}(d\phi) = C(x, y)(d\phi) \text{ (see 4.19 \& 1.15)} = \\
&= (Dd\phi)(y, x).
\end{aligned}$$

□

4.13

Proposition. *If f is a geodesic of M with respect to the spray associated to C, then for $\phi \in F(M)$ we have*

$$\frac{d^2(\phi \circ f)}{dt^2} = (Dd\phi)(f', f').$$

Proof. The fact that f is a geodesic implies that

$$f'' = C(f', f') \text{ and hence}$$
$$f''(d\phi) = C(f', f')(d\phi).$$
$$\text{But} \quad f''(d\phi) = \frac{d^2(\phi \circ f)}{dt^2} \quad \text{(see 5.10)}$$

and by the proposition above

$$C(f', f')(d\phi) = (Dd\phi)(f', f').$$

□

5 Curvature

5.1

Let $X, Y \in \mathscr{C}(M)$ and let us write

$$R(X, Y) = [D_X, D_Y] - D_{[X,Y]} \quad \text{(defect of brackets)}.$$

Then by (3.16) it follows that $R \in \mathscr{L}_1^3(M)$. This map R is called the curvature tensor of M for D. **53**

R has the following properties:

5.2

C.T. 1. $R(X, Y) = -R(Y, X)$,

C.T. 2. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ if C is symmetric.

The equality *C.T. 1.* follows from the definition of R, so let us consider *C.T. 2.* We have in succession, applying the definition of R and *D.L. 4.*, 4.1, six times.

$$
\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \\
&+ D_Y D_Z X - D_Z D_Y X - D_{[Y,Z]}X \\
&+ D_Z D_X Y - D_X D_Z Y - D_{[Z,X]}Y = \\
&= D_X(D_Y Z - D_Z Y) + D_Y(D_Z X - D_X Z) + D_Z(D_Y X - D_X Y) \\
&- D_{[Y,Z]}X - D_{[Z,X]}Y - D_{[X,Y]}Z \\
&= D_X[Y,Z] - D_{[Y,Z]}X + D_Y[Z,X] - D_{[Z,X]}Y + D_Z[Y,X] - D_{[Y,X]}Z = \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[XY]]
\end{aligned}
$$

which is zero by Jacobi identity in $\mathscr{C}(M)$.

Examples.

5.3

1. If the dimension of M is 1 then $R = 0$ by *C.T. 1.*

5.4

2. For an open subset A of $\mathbb{R}^d$ we have

$$R = 0$$

54 the connection being the canonical connection. To see this let x, y, z be vectors and X, Y, Z be vector fields such that $X(m) = x$, $Y(m) = y$, and $\zeta \circ Z = \{\zeta(Z)\}$. Then we have

$$D_X Z = \nu \circ Z^T \circ X = \zeta^T \circ Z^T \circ X = (\zeta \circ Z)^T \circ X.$$

But $\zeta \circ Z$ is constant and hence

$$(\zeta \circ Z)^T = 0.$$

This argument actually shows that

$$D_X D_Y Z = 0, D_Y D_X Z = 0, D_{[X,Y]}Z = 0.$$

5.5

Note. We can define the covariant derivative $D\Omega$ for any $\Omega \in \mathscr{L}^r_s(M)$ and then for R we can prove the Bianchi identity.

(2.5.6) $\quad D_X R(Y,Z) + D_Y R(Z,X) + D_Z R(X,Y) = 0 \quad$ (see [18]).

5.7

Lemma. *For* $\omega \in \mathscr{E}^1(M)$, X, Y, $Z \in \mathscr{C}(M)$:

$$([D_X, D_Y] - D_{[X,Y]})\omega = -R(X,Y)\cdot\omega$$

Proof. For any $Z \in \mathscr{C}(M)$ the left hand side, by definition 4.6, is equal to

$$\begin{aligned}
&(D_X D_Y \omega)(Z) - (D_Y D_X \omega)(Z) - (D_{[X,Y]}\omega)(Z)\\
&= X(D_Y\omega)(Z)) - (D_Y\omega)(D_X Z) - Y((D_X\omega))(Z))\\
&+ (D_X\omega)(D_Y Z) - [X,Y](\omega(Z)) + \omega(D_{[X,Y]}Z)\\
&= X(Y(\omega(Z))) - X(\omega(D_Y Z)) - Y(\omega(D_X Z))\\
&+ \omega(D_Y D_X Z) - Y(X(\omega(Z))) + Y(\omega(D_X Z))\\
&+ X(\omega(D_Y Z)) - \omega(D_X D_Y Z) - [X,Y](\omega(Z)) + \omega(D_{[X,Y]}Z)\\
&= \omega(R(Y,X)Z) = (R(Y,X)\cdot\omega)(Z).
\end{aligned}$$

□ **55**

5.8

Definition. *Given an* $X \in \mathscr{C}(M)$ *by* the horizontal lift *of* X, denoted by X^H, *we mean the mapping* $T(M)$ *to* $T(M)$ *defined by*

$$X^H(x) = C(x, X(p(x))),\ x \in T(M).$$

Since C is differentiable it follows that

$$X^H \in \mathscr{C}(T(M)).$$

5.9

Lemma. *For* $X \in \mathscr{C}(M)$ *and* $\omega \in \mathscr{E}^1(M)$

$$X^H(\omega) = D_X\omega$$

Proof. By (4.9) and the definition of X^H, we have

$$\begin{aligned} X^H(\omega)(x) &= (x, X(p(x)))\omega = (D\omega)(X(p(x)), x) = \\ &= (D_X\omega)(x)\ \forall x \in T(M). \end{aligned}$$

□

5.10

Proposition. *For* $X, Y \in \mathscr{C}(M)$ *and* $\omega \in \mathscr{E}^1(M)$

$$[X^H, Y^H] - [X, Y]^H \text{ is vertical}$$

$$\text{and} \qquad \omega(\xi([X^H, Y^H] - [X, Y]^H)) = -R(X, Y) \cdot \omega.$$

Proof. Since X^H and Y^H are projectable by p^T, i.e. images under p^T are again vector fields, we have

$$\begin{aligned} p^T \circ [X^H, Y^H] &= [p^T X^H, p^T Y^H] = [X, Y] \\ &= p^T \circ [X, Y]^H. \end{aligned}$$

□

56 Further we have (4.16):

$$\begin{aligned} \omega(\xi([X^H, Y^H] - [X, Y]^H)) &= ([X^H, Y^H] - [X, Y]^H) \cdot \omega \\ &= X^H(Y^H(\omega)) - Y^H(X^H(\omega)) - [X, Y]^H(\omega) = \\ &= X^H(D_Y\omega) - Y^H(D_X\omega) - D_{[X,Y]}\omega = \text{ by (5.9)} \\ &= D_X D_Y\omega - D_Y D_X\omega - D_{[X,Y]}\omega = \text{ by (5.9)} \\ &= -R(X, Y)\omega \text{ by (5.7).} \end{aligned}$$

5.11

Corollary. $\forall x \in T(M)$

$$\begin{aligned}&\xi([X^H, Y^H] - [X, Y]^H)_x\\&= -R(X(p(x)), Y(p(x)))x\end{aligned}$$

For the above proposition holds $\forall \omega \in \mathscr{E}^1(M)$ *and the elements of both sides of the equation are in* $T(M)$*.*

5.12 C.D. 5.

If N is a manifold, $X, Y \in \mathscr{C}(N)$ and $g \in D(N, T(M))$, then

$$([D_X, D_Y] - D_{[X,Y]})(g) = R(f^T \circ X, f^T \circ Y)(g)$$

where $f = p \circ g$.

Proof. We first prove two lemmas. □

5.13

Lemma. *If* $g \in D(N, T(M))$ *and* $f = p \circ g$ *then there exist, locally on* N*, functions* ψ_i *and, locally on* M*, vector fields* U_i *such that, locally*

$$g = \sum_i \psi_i (U_i \circ f).$$

57

Proof. Relative to a chart (U, r) of M let U_i be a basis for $\mathscr{C}(U)$. Then for $n \in f^{-1}(U)$ there exists $\psi_i(n)$ such that

$$g(n) = \sum_i \psi_i(n) U_i(f(n)).$$

The ψ_i are the required functions. □

5.14

Lemma. *For* $f \in D(N, M)$, $X, Y \in \mathscr{C}(N)$ *locally let*

$$f^T \circ X = \sum_i \psi_i \cdot (U_i \circ f) \quad \text{and} \quad f^T \circ Y = \sum_i \theta_i \cdot (U_i \circ f).$$

Then, locally

$$f^T \circ [X, Y] = \sum_i (X(\theta_i) - Y(\psi_i))(U_i \circ f) + \sum_{i,j} \theta_j \psi_i([U_i, U_j] \circ f).$$

Proof. $\forall \phi \in F(M)$:

$$\begin{aligned}
(f^T \circ [X, Y])(\phi) &= [X, Y](\phi \circ f) \\
&= X(Y(\phi \circ f)) - Y(X(\phi \circ f) = \\
&= X((f^T \circ Y)\phi) - Y((f^T \circ X)\phi) = \\
&= X(\sum_j \theta_j (U_j \circ f)\phi) - Y(\sum_i \psi_i (U_i \circ f)\phi) = \\
&= X(\sum_j \theta_j (U_j(\phi) \circ f)) - Y(\sum_i \psi_i (U_i(\phi) \circ f)) = \\
&= \sum_j X(\theta_j)(U_j \circ f)(\phi) + \sum_j \theta_j \circ X(U_j(\phi) \circ f) \\
&\quad - \sum_i Y(\psi_i)(U_i \circ f)(\phi) - \sum_i \psi_i X(U_i(\phi) \circ f)
\end{aligned}$$

58 But

$$\sum_j X(\theta_j)(U_j \circ f)(\phi) - \sum_j Y(\psi_j)(U_j \circ f)(\phi) = \sum_j (X(\theta_j) - Y(\psi_j))(U_j \circ f)(\phi)$$

and

$$\sum_j \theta_j X(U_j(\phi) \circ f) - \sum_i \psi_i Y(U_i(\phi) \circ f) =$$

$$= \sum_j \theta_j((f^T \circ X)U_j(\phi)) - \sum_i \psi_i((f^T \circ Y)U_i(\phi))$$
$$= \sum_j \theta_j(\sum_i \psi_i(U_i \circ f)U_j(\phi)) - \sum_i \psi_i(\sum_j \theta_j(U_j \circ f)U_i(\phi))$$
$$= \sum_{i,j} \psi_i \cdot \theta_j \cdot (U_i(U_j(\phi))) \circ f - \sum_{i,j} \psi_i\theta_j(U_j(U_i(\phi)) \circ f)$$
$$= \sum_{i,j} \psi_i \cdot \theta_j \cdot ([U_i, U_j] \circ f)(\phi).$$

Hence the result.

Now by (3.16) and (5.13) it is enough to prove the equation for $g = Z \circ f$ where $Z \in \mathscr{C}(M)$. Using the notations of (5.15) we see that

5.15

$$(D_X D_Y)(Z \circ f) - (D_Y D_X)(Z \circ f) - D_{[X,Y]}(Z \circ f)$$
$$= D_X(D_{f^T \circ Y} Z) - D_Y(D_{f^T \circ X} Z) - D_{f^T \circ [X,Y]} Z \text{ by (5.14)}$$
$$= (D_X D_{\sum_j \theta_j (U_j \circ f)} Z - D_Y D_{\sum_i \psi_i (U_i \circ f)} Z.$$
$$- (D_{(X(\theta_i) - Y(\psi_i))(U_i \circ f) + \sum_{ij} \theta_j \psi_i ([U_i, U_j] \circ f)} Z)$$
$$= D_X(\sum_j \theta_j D_{U_j \circ f} Z) - D_Y(\sum_i \psi_i D_{U_i \circ f} Z)$$
$$- (X(\theta_i) - Y(\psi_i)) D_{U_i \circ f} Z - \sum_{i,j} \psi_i \circ \theta_j \circ D_{[U_i, U_j] \circ f} Z$$

by 4.1 (D.L.1.)

$$= A - B - C - D \text{ say.}$$

The, by *D.L.2* and *D.L.3* **59**

$$A - B = \sum_j X(\theta_j)(D_{U_j \circ f} Z + \sum_j \theta_j D_X D_{U_j \circ f} Z$$
$$- \sum_i Y(\psi_i) D_{U_i \circ f} Z - \sum_i \psi_i D_Y D_{U_i \circ f} Z.$$

But the first and the third terms together equal C. Hence, by (3.14):

$$
\begin{aligned}
A - B - C - D &= \sum_j \theta_j D_{f^T \circ X}(D_{U_j} Z) - \sum_i \psi_i D_{f^T \circ Y}(D_{U_i} Z) \\
&\quad - \sum_{i,j} \psi_i \cdot \theta_j (D_{[U_i, U_j]} Z \circ f). \\
&= \sum_j \theta_j D_{\sum \psi_i (U_i \circ f)}(D_{U_j} Z) - \sum_i \psi_i D_{\sum_j \theta_j (U_j \circ f)}(D_{U_j} Z) \\
&\quad - \sum_{i,j} \psi_i \cdot \theta_j (D_{[U_i, U_j]} Z \circ f) \quad \text{by } (D.L.1) \\
&= \sum_{j,i} \theta_j \cdot \psi_i \cdot (D_{U_i} D_{U_j} Z \circ f) \\
&\quad - \sum_{i,j} \theta_j \cdot \psi_i \cdot (D_{U_j} D_{U_i} Z \circ f) \\
&\quad - \sum_{i,j} \psi_i \cdot \theta_j D_{[U_i, U_j]} Z \circ f \\
&= \sum_{i,j} \psi_i \cdot \theta_j \cdot \Big\{ (D_{U_i} D_{U_j} Z - D_{U_j} D_{U_i} Z - D_{[U_i, U_j]} Z) \Big\} \circ f \\
&= \sum_{i,j} \psi_i \cdot \theta_j \cdot \Big\{ R(U_i, U_j) Z \Big\} \circ f \\
&= \sum_{i,j} \psi_i \cdot \theta_j \cdot R(U_i \circ f, U_j \circ f)(Z \circ f) \\
&= R(\sum_i \psi_i (U_i \circ f), \sum_j \theta (U_j \circ f)(Z \circ f)) \\
&= R(f^T \circ X, f^T \circ Y)(Z \circ f) \ldots
\end{aligned}
$$

60 □

6 Convexity

Throughout this article let us denote by M a manifold with a spray G and by Ω the open set of $T(M)$ on which the function exp is defined.

6.1

Definition. *We say that M is* convex *if there exists $s \in D(M \times M, \Omega)$ such that*

$$(p, \exp) \circ s = \mathrm{id}_{M \times M} .$$

61

6.2

Since $M \times M$ as well as Ω is of dimension $2d$ the above equation gives that s^T on $T_{(m,n)}(M \times M)$ is one-one and hence onto $T_{s(m,n)}(\Omega)$. Hence s is a local diffeomrophism.

Supposing that M is convex, let m, $n \in M$. Then, for sufficiently small positive number ϵ, consider the map $f :] - \epsilon, 1 + \epsilon[\to M$ defined by the equation

$$f(t) = \exp .(t \cdot s(m, n)).$$

From (4.5) we conclude that f is a geodesic and, from the fact that $(p, \exp .) \circ s = \mathrm{id}_{M \times M}$ that,

i) $f(0) = \exp .(0_m)$, since $p(s(m, n)) = m$ and

ii) $f(1) = \exp((s(m, n)) = n$.

Hence f is a geodesic connecting m and n. In the case of an open subset A of $\mathbb{R}^d$, with the canonical connection, we know that the geodesics are segments of straight lines. Hence it follows that our convexity implies the usual one. Conversely, if we set

$$s(m, n) = \zeta_m^{-1}(n - m) \tag{2.6.3}$$

then, the convexity of A in the usual sense implies that $s(m, n) \in \Omega$ and hence it follows that convexity of A in the usual sense implies the convexity in our sense. This justifies our terminology.

6.4

Proposition. *If M is convex, then*

$$s \circ (p, \exp) = \mathrm{id}_{\Omega} .$$

6.5

62 **Remark.** From this it follows, if M is convex, that

$$(p, \exp) : \Omega \to M \times M$$

is a diffeomorphism and, in particular, that

$$\exp_m : \Omega \cap T_m(M) \to M$$

is a diffeomorphism for every $m \in M$, and in particular given $m, n \in M$ there is one and only one geodesic from m to n.

In view of 6.2, the proposition follows from the following lemma.

Lemma 6.5 (a)**.** *Let E and F be connected differentiable manifolds and $f : E \to F$ and $g : F \to E$ be differentiable maps such that g is a local diffeomorphism and $f \circ g = \mathrm{Id}_F$. Then $g \circ f = \mathrm{Id}_E$. (In other words, f is bijective.)*

Proof. Let $A = \{x | x \in E, g \circ f(x) = x\}$. It is sufficient to prove that $A = E$. Since the maps f and g are continuous, A is closed. Evidently $A \subset g(F)$. On the other hand, if $x = g(y)$, $y \in F$,

$$g \circ f(x) = g \circ (f \circ g)(y) = g(y) = x.$$

Hence $A = g(F)$; in particular A is non-empty. Finally, g being a local diffeomorphism, $A = g(F)$ is open in E. Hence A is a non-empty open and closed subset of the connected space E. It follows that $A = E$. □

As an immediate consequence of (6.4), we have

6.6

63 **Corollary.** *Let $f \in D(]-\epsilon, 1+\epsilon[, M)$ be a geodesic in M. Then, if M is convex,*

$$s(f(0), f(1)) = f'(0).$$

Proof. It is sufficient to check that $\exp f'(0) = f(1)$; (4.5) asserts precisely this. □

6.7

Definition. *We say that an open subset U of M is* convex *if U, as a sub manifold of M, with the induced spray, is convex.*

6.8

Theorem. *For every point m of a manifold M there is an open neighbourhood of m which is convex.*

Proof. First let us select a function $\phi \in F(M)$ such that

$$\phi(m) = 0, d\phi_m = 0 \text{ and } Dd\phi_m \text{ is positive definite,}$$

i.e. $(Dd\phi)(x, x) > 0$ for every non-zero x in $T_m(M)$. For example let us take any euclidean structure $\| \ \|$ on $T_m(M)$, and, locally, define ϕ to be $\| \ \|^2 \circ \exp^{-1}$. Then clearly

$$\phi(m) = 0 \quad \text{and} \quad d\phi_m = 0.$$

To prove that $(Dd\phi)(x, x) > 0$ for non-zero $x \in T_m(M)$, let $r > 0$ be such that

$$\{x \in T_m(M) | \ \|x\| < r\} \subset \Omega.$$

Set:

$$u : t \to tx(\text{on }] - r, r[) \text{ and } f_x = \exp \circ u.$$

Then $f_x'(0) = x$ and $\phi \circ f_x = \| \ \|^2 \circ u$ and hence, since f is a geodesic, by (4.13)

$$(Dd\phi)(x, x) = \frac{d^2(\phi \circ f_x)}{\mathrm{dt}^2}\Big|_{t=0} = 2 + \|x\| > 0.$$

Thanks to the continuity of $Dd\phi$, it is possible to find r' such that $0 <$ **64**
$r' < r$ and $Dd\phi$ is positive definite on $W = \exp(\{x \in T_m(M) \big| \|x\| < r'\})$. Since M is Hausdorff and $\overline{W}$ *compact in* $\exp(\{x \in T_m(M) \big| \ \|x\| < r\})$ then $\overline{W}$ is *compact in M*. Now let ${}^W G$ be the spray on W induced by G and W be the open set in $T(W)$ on which the corresponding $\exp .$ is defined. Now let us consider the map

$$(p, \exp .) : {}^W\Omega \to W \times W.$$

From (4.8) and inverse function theorem it follows that there is an $0 < r'' < r'$ and an open neighbourhood V of m in M such that for

$$V = \exp .(\{x \big| \, \|(x)\| < r''\}) \text{ the map}$$
$$(p, \exp .) : (p, \exp .)^{-1}(V \times V) \to V \times V \text{ is}$$

a diffeomorphism. Now let us set

$$s_1 = (p, \exp .)^{-1} : V \times V \to (p, \exp .)^{-1}(V \times V).$$

Since ϕ is continuous and is positive in V except at m and $\overline{V} - V$ is compact, being a closed subset of the compact set $\overline{W}$, it follows that

$$\epsilon = \inf_{x \in \overline{V} - V} \phi(x) > 0.$$

Then by the selection of V we have $\epsilon < r''$. Now let us set $U = V \cap \phi^{-1}(] - \infty, \epsilon_1[)$ where $0 < \epsilon_1 < \epsilon$. Note also that: $U = W \cap \phi^{-1}(] - \infty, \epsilon_1[) = \exp(\{x \in T_m(M) \big| \, \|x\| < \epsilon_1\})$. Then $m \in U$, U is open and we have

$$(p, \exp) \circ s_1 = \mathrm{id}_{U \times U}$$

65 since this equation holds in $V \times V$. Now if we show that $s_1(U \times U) \subset \Omega$ then it follows, by definition, that U is convex. Let $(x, y) \in U \times U$. Then 6.2 there is a geodesic f in V such that:

$$f_1(t_0) = x \quad \text{and} \quad f_1(t_1) = y.$$

But since f_1 is a geodesic we have (by our choice of W and by (4.13)):

$$\frac{d^2(\phi \circ f_1)}{\mathrm{dt}^2} = (Dd\phi)(f_1', f_1') > 0$$

and hence $\phi \circ f_1$ is a convex function. Further since $x, y \in U$ we have

$$\phi(x) < \epsilon_1, \phi(y) < \epsilon_1.$$

Hence by the convexity of $\phi \circ f_1$ it follows that

$$f_1 \big| [t_0, t_1] \subset W \cap \phi^{-1}(] - \infty, \epsilon_1[) = U.$$

Now the result follows from (4.2). □

7 Parallel transport

For the rest of this chapter let us suppose that a manifold M with a symmetric connection C is given.

7.1

Example. Let us consider $\mathbb{R}^d$ with its canonical connection. Let f be a curve in $\mathbb{R}^d$. If g is a lift of f into $T(\mathbb{R}^d)$ such that

$$D_P g = 0.$$

then, since (see (3.9)) $\nu = \zeta^T$. **66**

$$0 = D_P g = \nu \circ g^T \circ P = \zeta^T \circ g^T \circ P = (\zeta \circ g)^T \circ P,$$

and hence $\zeta \circ g$ is a constant, i.e. g is the field of vectors obtained by translating a constant vector along f. Keeping this in mind, we define a parallel lift.

7.2

Definition. *A lift g of a curve $f \in D(I, M)$ into $T(M)$ is said to be* a parallel lift of *f if*

$$D_P g = 0.$$

7.3

Example. A curve f is a geodesic, if and only if f' is a parallel lift of f (see 3.5).

Now we shall prove the existence and uniqueness of parallel lifts.

7.4

Proposition. *Let $f \in D(I, M)$. Then corresponding to every $t_0 \in I$ and $x_0 \in T_{f(t_0)}(M)$ there exists a unique parallel lift g of f such that*

$$g(t_0) = x_0.$$

Proof. Let $t \in I$ and (U, r) be a chart of M such that $f(t) \in U$. Then, by (5.13) relative to (U, r) the local existence of such a g is equivalent to the existence of functions ψ_i on $f^{-1}(U)$ satisfying the equations

$$\psi_i(t)(U_i \circ f)(t_0) = x_0,$$
$$D_P(\sum_i \psi_i(U_i \circ f)) = 0,$$

67 where $x \in T_{f(t)}(M)$ and $\{U_i\}$ is a basis for $\mathscr{C}(U)$. By (4.1) *C.D.2.* and *C.D.3.* this is the same as

$$\sum \psi_i(t)(U_i \circ f)(t_0) = x_0$$
$$\text{and} \quad \sum_i \frac{d\psi_i}{\mathrm{dt}}(U_i \circ f) + \sum_i \psi_i D_P(U_i \circ f) = 0.$$

Setting

$$D_P(U_i \circ f) = \sum_j a_{ji}(U_j \circ f)\,\forall i$$

and

$$x_0 = \sum_i b_i U_i(f(t_0))$$

we see that the above equations are equivalent to $\psi_i(t_0) = b_i$

$$\frac{d\psi_i}{\mathrm{dt}} + \sum_j a_{ij}\psi_j = 0.$$

But we know that this system being linear admits a unique solution in the whole domain of definition of the $\psi_i' s$. Hence locally the proposition is true. But since the definition of parallel lift is intrinsic and because we do have local solutions, we are done. ◻

7.5

Definition. *A path $C(t)$ in M consists of*

i) a family $f_i \in D(I_i, M)$, $i = 1, \ldots, k$ and *ii) a family* $[a_i, b_i] \subset I_i$ *of intervals such that*

$$f_i(b_i) = f_{i+1}(a_{i+1}) \ \textit{for} \ i = 1, \ldots, k-1.$$

$f_1(a_1)$ *is called the* origin *of the path C and* $f_k(b_k)$ *the* end.

Let $C = \{f_i \in D(I_i, M), [a_i, b_i] \subset I_i\}$ be a path in M. Then for every $x \in T_{f_1(a)}(M)$ we can consider the end point of the parallel lift g_1 of f_1 with $g_1(a_1) = x$; then the end point of the parallel lift g_2 of f_2 **68**
with $g_2(a_2) = g_1(b_1)$, and so on; and thus arrive at the point $g_k(b_k) \in T_{f_k}(b_k)(M)$.

7.6

The vector $g_k(b_k)$ is called *the parallel transport of x along C* and is denoted by

$$\tau(C)(x).$$

7.7

Proposition. *With the above notation* $\tau(C)$ *is an isomorphism between* $T_{f_1(a_1)}(M)$ *and* $T_{f_k(b_k)}(M)$.

Proof. By (4.1) *C.D.2.*-3, it follows that the mapping $\tau(C)$ is linear. Consider the path described in the opposite direction, i.e., the inverse path C^{-1} of C which consists of

$$\{f_i^{-1} \in D(I_i, M) | f_{k-i}^{-1}(t) = f_i(b_i - (t - a_i))\}.$$

This induces a linear map of $T_{f_k(b_k)}(M)$ into $T_{f_1(a_1)}(M)$. Further the parallel lift of C described in the opposite direction is the parallel lift of C^{-1} and hence

$$\tau(C^{-1}) \circ \tau(C) = \mathrm{id}_{T_{f_1(a_1)}}(M).$$

and $\tau(C) \circ \tau(C^{-1}) = \mathrm{id}_{T_{f_k(b_k)}}(M)$ by the same token. □

Given a path C we can cut off the path after a point $C(t)$ on C and then the new path gives, by the above proposition, an isomorphism of

$T_{f_1(a_1)}(M)$ onto $T_{C(t)}(M)$. *This isomorphism will be denoted by* $\tau(C,t)$, or τ_t. Now given a curve f and a lift g of f into $T(M)$ we can express $D_P g$ in terms of $\tau(C,t)$ and ordinary derivatives of functions. To see this let $f \in D(I, M)$, $t_0 \in I$ and let $\{x_1, \ldots, x_d\}$ be a fixed basis of $T_{f(t_0)}(M)$. Then, let us denote by $g_1, \ldots, g_d$ the parallel lifts of f into $T(M)$ with initial points $x_1, \ldots, x_d$ respectively. Then, by the previous proposition $\{g_1(t), \ldots, g_d(t)\}$ is a basis of $T_{C(t)}(M)$ and hence any lift g of f can be written as: 69

(2.7.8) $$g(t) = \sum \psi_i(t) g_i(t)$$

where $\psi_i \in F(I)$. By *C.D.2.*-3 we have

$$\begin{aligned} D_P g &= \sum_i \frac{d\psi_i}{\mathrm{dt}} \cdot g_i + \sum_i \psi_i (D_P g_i) \\ &= \sum_i \frac{d\psi_i}{\mathrm{dt}} \cdot g_i \end{aligned}$$

since g_i is a parallel lift and hence $D_P g_i = 0$. Now, *let us define* $\widehat{g}$ by the equation

(2.7.9) $$\widehat{g}(t) = \tau(C,t)^{-1}(g(t))$$

Then, we have the following proposition

7.10

Proposition. $D_P g = \tau(C,t)\left(\dfrac{d\widehat{g}(t)}{\mathrm{dt}}\right)$

Proof. $\tau(C(t))$ being linear we have

$$\widehat{g}(t) = \sum_i \psi_i(t)\widehat{g}_i(t) = \sum_i \psi_i(t) g_i(0) = \sum_i \psi_i(t) x_i$$

and hence

$$\frac{d\widehat{g}(t)}{\mathrm{dt}} = \sum_i \frac{d\psi_i}{\mathrm{dt}} \cdot x_i.$$

But $D_P g = \sum \dfrac{d\psi_i}{\mathrm{dt}} \cdot g_i(t)$. □

7.11

Remark. If C is a *loop*, i.e. *a path whose origin and end coincide*, in **70**
general,

$$\tau(C) \neq \mathrm{id}_{T_{f_1(a_1)}}(M).$$

For let $X, Y \in \mathscr{C}(M)$ be such that

$$[X, Y] = 0$$

and let C be a curvilinear parallelogram in M whose sides are integral curves of X and Y. Then the horizontal lift of C is made up of the integral curves of X^H and Y^H. But by (5.11)

$$\xi([X^H, Y^H]_x) = -R(X, Y)_x$$

and $-R(X, Y)_x$ is, in general, different from zero. Then, our contention follows from the geometric interpretation of the bracket of two vector fields.

8 Jacobi fields

8.1

Definition. A one-parameter family f of curves in a manifold M *is an* $f \in D(I \times J, M)$ *where* I, J *are open intervals.*

8.2

We denote the first coordinate of $I \times J \in \mathbb{R}^2$ by t and the second by α. We denote the canonical basis of $(I \times J)$ by P, Q (i.e. $P = \dfrac{\partial}{\partial t}$ and $Q = \dfrac{\partial}{\partial \alpha}$). The one parameter family f gives rise to a family $\{f_\alpha\}_{\alpha \in J}$ of curves defined by the equation

8.3

$f_\alpha(t) = f(t,\alpha)$, $\forall (t,\alpha) \in I \times J$, and to another family $\{C_t\}_{t\in I}$ of curves **71**
C_t, called *transversal curves*, which are defined by the equation

(2.8.4) $$C_t(\alpha) = f(t,\alpha), \quad (t,\alpha) \in I \times J.$$

Now setting

8.5

$\underline{\underline{P}} = f^T \circ P$ and $\underline{\underline{Q}} = f^T \circ Q$, we observe that $\underline{\underline{P}}$ appears as the family of tangent vectors to the curves $f'_\alpha s$ and $\underline{\underline{Q}}$ appears as the vectors of the variation of the $f'_\alpha s$ i.e. as the family of tangent vectors to the transversal curves $C'_t s$. Let us note that

(2.8.6) $$[P, Q] = 0.$$

8.7

Proposition. *With the above notation if C is symmetric and $\forall \alpha \in J$, f_α is a geodesic, then*

$$D_P D_P \underline{\underline{Q}} = R(\underline{\underline{P}}, \underline{\underline{Q}})\underline{\underline{P}}.$$

Proof. By 5.12 *C.D.5.*

$$D_Q D_P \underline{\underline{P}} = D_P D_Q \underline{\underline{P}} + R(\underline{\underline{Q}}, \underline{\underline{P}})\underline{\underline{P}} + D_{[Q,P]}\underline{\underline{P}}.$$

72 But since f_α are geodesic by 3.5 we have $D_P \underline{\underline{P}} = 0$ and by (2.8.6) we have $D_{[P,Q]}P = 0$.

Hence we have

(2.8.8) $$D_P D_Q \underline{\underline{P}} + R(Q, P)\underline{\underline{P}} = 0.$$

But since C is symmetric we have by (*C.D.4*) (see (2.3.15))

$$D_P \underline{\underline{Q}} - D_Q \underline{\underline{P}} = [P, Q] = 0$$

and hence

$$D_P D_P \underline{\underline{Q}} = D_P D_Q \underline{\underline{P}}$$

Now by (2.8.8) we have

$$D_P D_P \underline{\underline{Q}} = R(\underline{\underline{P}}, \underline{\underline{Q}})\underline{\underline{P}}.$$

□

Now we give the following definition

8.9

Definition. *A lift g of a geodesic f is called* a Jacobi field *f if*

$$D_P D_P g = R(f', g)f'.$$

8.10

Example. Let $f \in D(I, M)$ be a geodesic and let $\psi \in F(I)$.

Then $\psi f'$ is, by definition, a Jacobi field along f if and only if

$$D_P D_P(\psi f') = R(f', \psi f')f'.$$

But by *C.D.2.*-3 (2.3.15)

$$D_P D_P(\psi f') = \frac{d^2\psi}{\mathrm{dt}^2} \cdot f'$$

since f is a geodesic and hence $D_P f' = 0$ by 3.5. Further

$$\begin{aligned} R(f', \psi f')f' &= \psi R(f', f')f' \quad \text{since } R \in \mathscr{L}^1_3 \\ &= 0 \quad \text{by } \textit{C.T.1. } 5.2. \end{aligned}$$

Hence $\psi f'$ is a Jacobi field along f if and only if **73**

$$\frac{d^2\psi}{\mathrm{dt}^2} = 0$$

i.e. if and only if ψ is an affine function.

8.11

Let A be an open subset of $\mathbb{R}^d$ with its canonical connection. Let f be a geodesic. It is a segment of a straight line by (1.20). Further $R = 0$ for A by (2.5.4). Hence a lift g of f is a Jacobi field along f if and only if

$$D_P D_P g = 0.$$

But (by (3.9)):

$$D_P D_P g = D_P(\zeta \circ g)' = (\zeta \circ g)'' = \frac{d^2(\zeta \circ g)}{\mathrm{dt}^2}.$$

Hence g is a Jacobi field along f if and only if

(2.8.12) $$\zeta \circ g = tx + y, x, y \text{ in } \mathbb{R}.$$

Let us note that

(2.8.13) $$\widehat{g}(t) = tx + y.$$

8.14

Lemma. *Given a geodesic f of M and x, y in $T_{f(0)}(M)$ there exists at most one Jacobi field g along f such that*

$$g(0) = x \quad \textit{and} \quad (D_P g)(0) = y.$$

Proof. With the notation of (7), droping t in ζ_t for simplicity, we have

$$D_P g = \tau(\frac{d\widehat{g}}{\mathrm{dt}}),$$
$$D_P D_P g = \tau(\frac{d}{\mathrm{dt}})(\widehat{\tau}(\frac{d\widehat{g}}{\mathrm{dt}})) = \tau(\frac{d}{\mathrm{dt}}(\frac{d\widehat{g}}{\mathrm{dt}}))$$
$$= \tau(\frac{d^2\widehat{g}}{\mathrm{dt}^2}).$$

74 So g is a Jacobi field along f if and only if

(2.8.15) $$\frac{d^2\widehat{g}}{\mathrm{dt}^2} = \tau^{-1}(R(f', g)f').$$

Now for a given t let us define a transformation

$$\widehat{R}(t) = T_{f(0)}(M) \to T_{f(0)}(M)$$

by setting

$$\widehat{R}(t)(y) = \tau^{-1}(R(f', \tau(y))f'), \forall y \in T_{f(0)}(M). \tag{2.8.16}$$

Then g is a Jacobi field along f if and only if

$$\frac{d^2\widehat{g}}{\mathrm{dt}^2} = \widehat{R}(t)\widehat{g}. \tag{2.8.17}$$

Thus, if g is a Jacobi field along f then $\widehat{g}$ satisfies a second order homogeneous linear differential equation. Further, since

$$\tau_0 = \mathrm{id}_{T_{f(0)}(M)} \quad \text{and hence } \widehat{g}(0) = g(0)$$
$$\text{and } (D_P g)(0) = \tau_0(\frac{d\widehat{g}}{\mathrm{dt}}(0)) = \frac{d\widehat{g}}{\mathrm{dt}}(0)$$

the requirements $g(0) = x$ and $(D_P g)(0) = y$ become the initial conditions

$$\widehat{g}(0) = x \quad \text{and} \quad \frac{d\widehat{g}}{\mathrm{dt}}(0) = y$$

for the above differential equation. But under these circumstances it is known that the above differential equation can have at most one solution; the existence of the field would also follow from general theorems on differential equations, but in our case this will follow from (8.26). □

8.18

Remark. If g is a Jacobi field along f such that $g(0) = 0$ then **75**

$$\widehat{g}(t) = t\frac{d\widehat{g}}{\mathrm{dt}}(0) + \frac{t^3}{6}R(f'(0), \frac{d\widehat{g}}{\mathrm{dt}}(0))f'(0) + 0(t^3),$$

where $\frac{0(t^3)}{t^3} \to 0$ with t.

Proof. By (2.8.17)

$$\frac{d^2\widehat{g}}{dt^2}(0) = \widehat{R}(0)\widehat{g}(0) = 0, \quad \text{since } \widehat{g}(0) = g(0) = 0;$$

and

$$\frac{d^3\widehat{g}}{dt^3}(0) = \frac{d\widehat{R}}{dt}(0)\widehat{g}(0) + \widehat{R}(0)\cdot\frac{d\widehat{g}}{dt}(0)$$
$$= R(f'(0), \frac{d\widehat{g}}{dt}(0))f(0).$$

Now we have only to apply Taylor's formula with a remainder to $\widehat{g}$. □

Now let us see if, given a geodesic $f_0 \in D(I, M)$ there exists a Jacobi field passing through a given point x, above the geodesic with a speed whose vertical component y is given. For simplicity, let us suppose that all intervals occurring in this article contain zero.

Now let $S \in D(J, M)$ be any curve in M with $S'(0) = x$, and let $\widetilde{\mathscr{U}}$ (resp. $\widetilde{y}$) be the parallel lift of S such that

$$\widetilde{\mathscr{U}}(0) = f_0'(0) \quad (\text{resp. } \widetilde{y}(0) = y).$$

Now set

$$\widetilde{S}(\alpha) = \widetilde{\mathscr{U}}(\alpha) + \alpha\widetilde{y}(\alpha)\ \forall\alpha \in J \tag{2.8.19}$$

8.20

76 **Remark.** We have by (7.10)

$$D_Q\widetilde{S} = \tau_\alpha\left(\frac{d(f'(0) + \alpha\widetilde{y})}{d\alpha}\right) = \tau_\alpha(y)$$

and in particular

$$(D_Q\widetilde{S})(0) = y.$$

Now, since f_0 is a geodesic $tf_0'(0) \in \Omega$ for every t in I. Let $]a, b[= I$ and let

$$0 < \epsilon < \min\{|a|, b\},$$

and let

$$]a', b'[=]a + \epsilon, b - \epsilon[= I_\epsilon.$$

Then the compact set $[a', b']\widetilde{S}(0) = \{t\widetilde{S}(0)|t \in [a', b']\}$ is contained in Ω and Ω is an open set. Hence there is an open neighbourhood J_ϵ of $\alpha = 0$ such that

$$t\widetilde{S}(\alpha) \in \Omega, \forall t \in I_\epsilon \quad \text{and} \quad \alpha \in J_\epsilon. \tag{2.8.21}$$

Now set

$$f(t, \alpha) = \exp(t \cdot \widetilde{S}(\alpha)) \quad \text{for} \quad (t, \alpha) \in I_\epsilon \times J_\epsilon. \tag{2.8.22}$$

Then f is a one parameter family of geodesics and

$$f(t, 0) \quad \text{is} \quad f_0|I_\epsilon \quad \text{itself.}$$

Now we claim that

$$g(t) = \underline{\underline{Q}}(t, 0)\, \forall t \in I_\epsilon \tag{2.8.23}$$

is a Jacobi field along $f_0|I_\epsilon$ passing through x with a speed whose vertical component is y, i.e. that $g(0) = x$ and $(D_P g)(0) = y$. By (8.7) and the 77
definition of a Jacobi field it follows that g is a Jacobi field along f_0. To establish our claim we need only show that

$$g(0) = x \quad \text{and} \quad (D_P g)(0) = y.$$

We have

$$g(0) = \underline{\underline{Q}}(0, 0) = \frac{\partial C_0}{\partial \alpha}(0),$$

but $\quad C_0(\alpha) = \exp(0 \cdot \widetilde{S}(\alpha)) = S(\alpha)$ and hence

$$g(0) = (S'(\alpha))_{\alpha=0} = S'(0) = x. \tag{2.8.24}$$

Further we have, by the definition of g,

$$(D_p g)(0) = (D_P \underline{\underline{Q}})(0, 0).$$

Since the connection is symmetric, using (2.3.15) *C.D.4* and the fact that $[P, Q] = 0$ we have

$$(D_P \underline{\underline{Q}})(0,0) = D_Q \underline{\underline{P}}(0,0). \tag{2.8.25}$$

But we know that for every α

$$f_\alpha : t \to \exp(t\widetilde{S}(\alpha))$$

is a geodesic in M whose tangent vector at 0 is $\widetilde{S}(\alpha) = \underline{\underline{P}}(0, \alpha)$.

Hence

$$(D_Q \underline{\underline{P}})(0,0) = (D_Q \widetilde{S})(0) = y \quad \text{by the remark.}$$

Hence we have proved the following proposition.

8.26

Proposition . *Given a manifold M with a symmetric connection, a geodesic f_0 in M, a point m on f_0 and x and y in $T_m(M)$ there exists*
78 *a unique Jacobi field g along f_0 such that $g(0) = x$ and $(D_P g)(0) = y$. Furthermore, if we cut off the geodesic at both ends, then the Jacobi field can be realised as the vectors of the variations of a one parameter family f with f_α for $\alpha = 0$ coinciding with the corresponding restriction of f_0.*

Now let us follow the above notation and examine two special cases.

8.27

1. Let the Jacobi field g be such that $g(0) = x$ and $(D_P g)(0) = 0$. Then $y \equiv 0$ is a parallel lift of S with initial speed zero and so the vectors of variations of the one parameter family

$$\exp(t \cdot \widetilde{\mathscr{U}}(\alpha))$$

realise g. Further the initial tangent vectors of the $\{\exp t \cdot \widetilde{\mathscr{U}}(\alpha) : t \in I\}$ are got simply by parallel transport of $f_0'(0)$ along S.

8.28

2. Let the Jacobi field g be such that $g(0) = 0$. Then the trivial path $\{f(0)\}$ can be taken for S. Then a one parameter family can be given by $\exp(t\cdot(f_0'(0)+\alpha y))$. Hence all the curves f_α start from $f_0(0)$. Furthermore we have the following result.

8.29

Corollary. *Let $m \in M$, $0 \neq \mathscr{U} \in T_m(M) \cap \Omega$, and g be the Jacobi field along the geodesic*

$$f : t \to \exp(t \cdot \mathscr{U})$$

with $g(0) = 0$ and $(D_P g)(0) = y$. Then $\exp_m^T(\xi_u^{-1} y) = g(1)$.

Proof. Let us follow the notation of the previous proposition and take the one parameter family given by

$$\exp(t \cdot \widetilde{S}(\alpha)) \quad \text{where} \quad \widetilde{S}(\alpha) = f'(0) + \alpha \cdot y.$$

Let us consider the transversal curve **79**

$$C_1 = \exp(f'(0) + \alpha \cdot y).$$

We have

$$C_1(\alpha) = \exp .\widetilde{S}(\alpha),$$

and hence

$$g(1) = \underline{\underline{Q}}(1,0) = C_1'(0) = (\exp \circ \widetilde{S})'(0) = \exp^T(\widetilde{S}'(0))$$

But

$$\widetilde{S}'(0) = \xi_u^{-1} y.$$

□

8.30

Definition. *Given a* non trivial *geodesic* $f \in D(I, M)$ *and* t_1, t_2 *in* I *such that* $t_1 \neq t_2$*, the points* $f(t_1)$ *and* $f(t_2)$ *are said to be* conjugate points on f *(or simply conjugate points when there is no possible confusion over* f*) if there exists a Jacobi field* g *along* f *such that*

$$g(t_1) = 0, g(t_2) = 0 \quad \textit{and} \quad g \neq 0.$$

8.31

Remark . Let us note that the fact that $f(t_1)$ and $f(t_2)$ are conjugate points on a geodesic f *does not imply* that there is a one parameter family

$$f(t, \alpha)$$

of geodesics such that

$$f(t_1, \alpha) = f_0(t_1) \quad \text{and} \quad f(t_2, \alpha) = f_0(t_2) \forall \alpha.$$

All we can say, thanks to 8.28, is that there exists one satisfying the first condition, namely

$$f(t_1, \alpha) = f_0(t_1) \, \forall \alpha.$$

8.32

80 **Corollary .** *Given a non trivial geodesic* f*, and two points* $f(0)$ *and* $f(t)$ *on* f*, then they are non-conjugate on* f *if and only if* $\exp^T_{f(0)}$ *is of maximal rank at* $t \cdot f'(0) \in T_{f(0)}(M)$*.*

Proof. Let g be any Jacobi field along f. Then if we replace f by $f^0 = f \circ k_{t^{-1}}$, we get $f^{0\prime}(0) = t^{-1} f'(0)$ and $g^0 = g \circ k_{t^{-1}}$ is a Jacobi field along f^0, and $g^0(1) = g(t)$. It follows directly from definitions that $f(0)$ and $f(t)$ are conjugate points on f if and only if $f^0(0)$ and $f^0(1)$ are on f^0. So let us take t to be 1. (See (6.23)). □

Let us suppose that $f(0)$ and $f(1)$ are conjugate. Then there is a Jacobi field g along f such that

$$g(0) = 0, g(1) = 0 \quad \text{and} \quad g \neq 0.$$

Let the vertical component of the speed of g at zero be y; i.e. let $(D_P g)(0) = y$. Then $y \neq 0$ since otherwise $g \equiv 0$ by (8.14). Then by the previous corollary (8.29) we have:

$$\exp^T_{f(0)}(\xi^{-1}_{f'(0)}(y)) = g(1) = 0.$$

Hence $\exp^T_{f(0)}$ would not be of maximal rank at $f(0)$.

Now suppose that $\exp^T$ is not of maximal rank at $f'(0)$. Then there exists a z in $V_{f'(0)}$ such that $z \neq 0$ and $\exp^T z = 0$, and a vector y such that

$$\xi^{-1}_{p'(z)} y = z.$$

Also, by the corollary, for the Jacobi field g along f with

$$g(0) = 0 \quad \text{and} \quad (D_P g)(0) = y$$

we have

$$g(1) = \exp^T z = 0$$

and hence $f(0)$ and $f(1)$ are conjugate.

Chapter 3

Riemannian manifolds

1.1

Definition. *Let M be a manifold. An element $g \in \mathscr{L}^2(M)$ is said to* 81
define a Riemannian structure *(or* simply *r.s.) on M, if for every m in M, $g_m \in \mathscr{L}^2(T_m(M))$ defines a euclidean structure on $T_m(M)$ (see (3.3)). A manifold M with an r.s. is called a* Riemannian manifold *(or* simply *r.m.).* By (M, g) *we denote the r.m. M with the r.s.g.*

1.2

Example. 1. As a first example of an r.m. we endow $\mathbb{R}^d$ with the r.s., denoted by ϵ; defined by the equation

$$\epsilon(X, Y) = (\zeta \circ X) \cdot (\zeta \circ Y), \forall X, Y \in \mathscr{C}(\mathbb{R}^d)$$

where denotes the usual scalar product on $\mathbb{R}^d$.

1.3

Let (M, g) be any r.m., and let U be an open subset of M. Then the restriction of g to U defines an r.s. on U, denoted sometimes by $g|U$. Whenever we consider an open subset of an r.m. as an r.m. it is the structure above we have in mind.

1.4

Remarks. Given an r.s. g on M, we have, for every m of M, a positive definite symmetric bilinear form g_m on $T_m(M)$ which depends differentiably on m, i.e. the map

$$g : m \to g_m$$

belongs to $D(M, L^2(T(M))$. The converse of this statement follows from (0.2.3).

The above examples make it possible to define an r.s. on every man-
82 ifold M. To see this, let (U_i, r_i) be a family of charts of M such that the U_i cover M and let $\{\varphi_i\}$ be a partition of unity (see (1)) subordinate to the covering $\{U_i\}$. By (1.3) on each U_i we have an r.s. namely $r_i^*(\epsilon_i)$ where $\epsilon_i = \epsilon | r_i(U_i)$ is the restriction of ϵ to $r_i(U_i)$. Now let us extend the form $\varphi_i r_i^*(\epsilon_i)$ to the whole of M by defining it to be zero outside U_i. Then

$$\varphi_i \cdot r_i^*(\epsilon_i) \in D(M, L^2(T(M)),$$

and set

$$h = \sum_i \varphi_i r_i^*(\epsilon_i).$$

Then we have for every X, Y in $\mathscr{C}(M)$, and m in M,

$$\begin{aligned} h(X, Y) &= \sum_i \varphi_i \cdot r_i^*(\epsilon_i)(X, Y) = \\ &= \sum_i \varphi_i r_i^*(\epsilon_i)(Y, X) \quad \text{since } \epsilon \text{ is symmetric} \\ &= h(Y, X), \end{aligned}$$

and

$$\begin{aligned} h(X, X)(m) &= \sum_i \varphi_i(m) r_i^*(\epsilon_i)(X, X) = \\ &= \sum_i \varphi_i(m) \epsilon(r_i^T(X), r_i^T(X))(m). \end{aligned}$$

Since $\sum \varphi_i(m) = 1$ there is an i_0 such that $\varphi_{i_0}(m) \neq 0$ and since $\varphi_i \geq 0$ and r_i are diffeomorphisms we have

$$\sum_i \varphi_i(m) \cdot \epsilon(r_i^T X(m), r_i^T X(m)) \geq \varphi_{i_0}(m) \cdot \epsilon(r_{i_0}^T X(m), r_{i_0}^T X(m)) > 0,$$

if $X(m) \neq 0$. Hence h is symmetric and positive definite and clearly **83**
bilinear. Hence every manifold admits an r.s.

Given two vectors x and y such that $p(x) = p(y)$ then $g(x, y)$ is defined and we call $g(x, y)$ the *scalar product* of x and y. Associated to this scalar product there is a *norm*, $\| \ \|$, on $T(M)$; namely

$$\|x\| = (g(x, x))^{\frac{1}{2}}. \tag{3.1.5}$$

1.6

We denote $g(x, x) = \|x\|^2$ by $E(x)$ and note that $E : T(M) \to \mathbb{R}$ is differentiable and $\| \ \|$ is continuous (E stands for *energy*).

1.7

On the tangent space $T_m(M)$ of M at m the topology induced by $T(M)$ and that induced by the norm $\| \ \|$ are the same.

1.8

Lemma. *$\forall x \in T(M)$ and $z \in V_x$ we have*

$$z(E) = 2g(x, \xi(z)).$$

Proof. Let

$$i : T_{p(x)}(M) \to T(M)$$

be the canonical injection. Then

$$z(E) = (i^T)^{-1}(z)(E \circ i)$$

where $E \circ i$ is the quadratic form on $T_{p(x)}(M)$ associated to $g_{p(x)}$. Then (see [36] and (0.4.11)):

$$z(E) = 2g(x, \zeta((i^T)^{-1})(z)) = 2g(x, \xi(z)).$$

□

1.9

84 **Definition.** *We say that an r.m.* $(M \cdot g)$ *is* isometric *to an r.m.* (N, h) *if there exists an* $f \in D(M, N)$ *such that*

i) f *is a diffeomorphism,*

ii) $f^* h = g$.

Then f *is called an* isometry *between* (M, g) *and* (N, h). *We say that* (M, g) *is* locally isometric *to* (N, h) *if,* $\forall m \in M$, *there exists an open neighbourhood* U *of* m *in* M *and a map* $f \in D(U, N)$ *such that:*

i) $f(U)$ *is open in* N,

ii) f *is an isometry between* $(U, g|U)$ *and* $(f(U), h|f(U))$.

1.10 Orthogonal vector fields.

Let us consider $(\mathbb{R}^d, \epsilon)$. Then the canonical basis $\dfrac{\partial}{\partial u^1}, \ldots, \dfrac{\partial}{\partial u^d}$ has the following two properties:

i) orthogonality: $\epsilon\left(\dfrac{\partial}{\partial u^i}, \dfrac{\partial}{\partial u^j}\right) = 0$ if $i \neq j$,

ii) commutativity: $\left[\dfrac{\partial}{\partial u^i}, \dfrac{\partial}{\partial u^j}\right] = 0$.

1.11

In the case of a manifold (M, g), locally, we can select an orthonormal basis for $\mathscr{C}(M)$. To see this let (U, r) be a chart of M and let $X_1, \ldots, X_d$ be a basis of $\mathscr{C}(U)$, say for example the one got by pulling back the canonical basis on $r(U)$. The define Y'_i and Y_i inductively by setting
85 $Y'_1 = Y_1 = \dfrac{X_1}{\|X_1\|}$, division by $\|X_1\|$ is possible since X_1 is a basis element and hence never zero,

$$Y'_i = X_i - \sum_{k=1}^{i-1} g(X_i, Y_k) Y_k$$

$$Y_i = \frac{Y'_i}{\|Y'_i\|}, \quad (Y'_i \text{ is never zero since } \{X_i\} \text{ is a basis}).$$

Then $Y_1, \ldots, Y_d$ is an orthonormal basis of $\mathscr{C}(U)$.

By pulling back the canonical basis on $\mathscr{C}(r(U))$ we get a commutative basis of $\mathscr{C}(U)$. But, in general, there is not any local orthonormal basis of $\mathscr{C}(M)$ which is also commutative. For, relative to (U, r) a chart of M, let $Y_1, \ldots, Y_d$ be an orthonormal basis of $\mathscr{C}(U)$ such that

$$[Y_i, Y_j] = 0.$$

Then by Frobenius's theorem there exists a local system of coordinates $y^1, \ldots, y^d$ such that

$$Y_i = \frac{\partial}{\partial y^i} \; i = 1, \ldots, d.$$

Then the map

$$f : m \to (y^1(m), \ldots, y^d(m))$$

gives a local isometry between U and $\mathbb{R}^d$. For clearly f is a diffeomorphism, and

$$\begin{aligned}(f^*\epsilon)(Y_i, Y_j) &= \epsilon(f^T(Y_i), f^T(Y_j)) \\ &= \epsilon\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \delta_{ij} = g(Y_i, Y_j),\end{aligned}$$

and hence $g = f^*\epsilon$. But in general r.m.'s are not locally isometric to $\mathbb{R}^d$ **86**
(see (5.2) and (5.4)).

2 Examples

2.1

Suppose that (M, g) is an r.m. and N any manifold such that there exists a map $f \in D(N, M)$:

$$f : N \to M$$

such that f_n^T is injective for every $n \in N$. Then $f^*g \in \mathscr{L}^2(N)$ and $(f^*g)_n$ is symmetric on $T_n(N)$. Moreover for $x \in T_n(N)$ if $(f^*g)(x, x) = 0$

then $0 = (f^*g)(x, x) = g(f_n^T(x),\ f_n^T(x))$ and since g is positive definite we have $f_n^T(x) = 0$. But f_n^T is injective and hence $x = 0$. Hence f^*g is positive definite. Therefore f^*g defines an r.s. on N. We apply this remark to the following cases.

A) Let N be an sub-manifold of (M, g). Then the injection

$$i : N \to M$$

has the properties stated above for f and hence we have an r.s., called the *induced r.s.,* on N. We denote it by $(g|N)$.

The study, initiated by Gauss, of surfaces S in $\mathbb{R}^3$, i.e. of two dimensional sub-manifolds S with the induced structure was the starting point for Riemann's original investigations on this subject. On the other hand, the theorem of Nash ([34]) states that if (M, g) is a connected r.m. then there exists an integer d' and a diffeomorphism f between M and a sub-manifold N of $\mathbb{R}^{d'}$ such that f is an isometry between (M, g) and $(N, \epsilon|N)$.

87 B) Let us consider the sphere $\mathbb{S}^d$:

$$\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} | x \cdot x = 1\} \subset (\mathbb{R}^{d+1}, \epsilon).$$

Then $\epsilon|\mathbb{S}^d$ is called the canonical r.s. on $\mathbb{S}^d$. Let us note that any element of the orthogonal group of $\mathbb{R}^{d+1}$ induces an isometry on $(\mathbb{S}^d, \epsilon|\mathbb{S}^d)$.

C) Suppose that (N, h) is an r.m., M is any manifold and $f \in D(M, N)$ is a *covering map* i.e. a map such that to each point n of N there is a neighbourhood U such that f restricted to each connected component of $f^{-1}(U)$ is a homeomorphism between that component and U. Then f^*h defines an r.s. on M and this situation is described by saying that f is a *Riemannian covering*. Note that, in this case, (M, f^*g) and (N, h) are locally isometric.

2.2

) Now let us suppose that (M, g) is an r.m. and that G is a discrete group of isometries of (M, g) without fixed points. If the quotient M/G is a manifold, then the canonical map p:

$$p : M \to M/G$$

is a covering map, and there is a unique r.s. on M/G, denoted by g/G, for which $p : M \to M/G$ is a Riemannian covering.

To see this, let $n \in M/G$ and let $m \in p^{-1}(n)$. Then since p is a covering map it is a local diffeomorphism, and we naturally define

$$h_n = (p^{-1})^* g_m$$

and h_n defines a euclidean structure on $T_m(M)$. Since G is a group of 88
isometries it follows that h_n is independent of the m chosen. Thus we have a map

$$h : n \to h_n$$

of N into $L^2(T(M/G))$. Furthermore since p is a local diffeomorphism, and since the differentiability is a local property it follows that h is differentiable. Hence there is an r.s. with the above properties.

D) Let us note two particular cases.

1. For (M, g) take $(\mathbb{S}^d, \epsilon|\mathbb{S}^d)$ and for G take the group generated by the antipodal map of $\mathbb{S}^d$, i.e. the one induced on $\mathbb{S}^d$ by the map $-\mathrm{id}_{\mathbb{R}^{d+1}}$ of $\mathbb{R}^{d+1}$. Then $\mathbb{S}^d/G$ is the real projective space $(P^d(\mathbb{R}))$, can).
2. For (M, g) take $(\mathbb{R}^d, \epsilon)$ and for G take a discrete subgroup of the group of all translations of $\mathbb{R}^d$ such that the rank of G is d. Then the *torus* $\mathbb{R}^d/G$ considered as an r.m. with the r.s. $\mathbb{R}^d/G$ is called a *flat torus* and denoted by $(\mathbb{R}^d/G, \epsilon/G)$.

3 Symmetric pairs

3.1

Let G be a Lie group and let $\lambda(s)$ and $\rho(s)$ denote the left and the right multiplications by $s \in G$. Further let $\underline{G}$ be the Lie algebra of G and

$\exp : \underline{G} \to G$ the associated exponential map. Let $X \in \underline{G}$; then the vector field defined by the one parameter group

$$\lambda(\exp(tX)), t \in \mathbb{R}$$

is the right invariant vector field whose value at e (the neutral element
89 of G) is X. To see this we have only to compute the speed of the curve $t \to (\exp(tX))g$, at $t = 0$.

But

$$(\exp(tX))g = \rho(g)(\exp(tX)),$$

so that

$$(\rho(g) \circ (\exp(tX)))'(0) = (\rho(g))^T (\exp(tX))'(0)$$

and by the definition of the exp map

$$(\exp(tX))'(0) = X.$$

Hence the result.

3.2

Definition. A symmetric pair (G, H, σ) *consists of*

1) *a connected Lie group G*

2) *a compact subgroup H of G and*

3) *an involutive automorphism σ of G such that*

$$\sum_0 \subset H \subset \sum$$

where $\sum$ is the subgroup of elements in G fixed by σ, and $\sum_0$ is the connected component of the identity, e of $\sum$.

When no confusion is possible, we shall speak of the symmetric pair (G, H) without any reference to σ.

Let M be the homogeneous space of left cosets of H in G, *p be the projection map from G to G/H. Set $m_0 = p(e)$, and let τ denote the left action of G on M*, i.e.

$$\tau(a)(xH) = ((ax).H), a \in G.$$

Let us denote the Lie algebra of H by $\underline{H}$ and identify it with the corresponding subalgebra of $\underline{G}$. Let us denote by $\underline{M}$ the set of elements X in $\underline{G}$ such that

(3.3.3) $$\sigma^T(X) = -X.$$

Then we have **90**

(3.3.4) $$\underline{G} = \underline{H} + \underline{M}$$

where the right hand side denotes the direct sum of *vector spaces*. We have further

(3.3.5) $$p^T|\underline{M} : \underline{M} \to T_{m_0}(M) \quad \text{is an isomorphism.}$$

3.6

Let $X \in \underline{M}$. Then for the two homomorphisms

$$t \to \sigma \circ \exp(tX)$$
$$t \to \exp(t(-X))$$

from $\mathbb{R}$ into G the differential maps are the same and hence

$$\sigma \circ \exp(tX) = \exp(t(-X)).$$

For every element a of G let us denote by $\operatorname{Ad}(a) = (\operatorname{Int}(a))^T_e$ the automorphism of the Lie algebra $\underline{G}$ corresponding to the automorphism

$$\operatorname{Int}(a) : u \to aua^{-1}$$

of G. Since $\operatorname{Int}(h)$ and σ commute $\forall h \in H$ we have

(3.3.7) $$(\operatorname{Ad} H)(\underline{M}) \subset \underline{M}.$$

3.8

Definition. *The map $\widehat{\sigma}$ from M to M is defined by the equation*

$$\widehat{\sigma} \circ p = p \circ \sigma \quad i.e. \quad \widehat{\sigma}(aH) = \sigma(a)H.$$

Now one sees that

(3.3.9) $$p^T \circ \operatorname{Ad} h = (\tau(h))^T \circ p^T, \forall h \in H,$$

(3.3.10) $$\widehat{\sigma}^T_{m_0} = -\operatorname{id}_{T_{m_0}(M)},$$

(3.3.11) $$\widehat{\sigma} \circ \tau(a) = \tau(\sigma(a)) \circ \widehat{\sigma}, \forall a \in G.$$

91

3.12

Proposition. *With the above notation, there exists an r.s. γ on M such that the transformations $\widehat{\sigma}$ and $\tau(a)$ for every a in G are isometries of (M, γ); moreover γ is unique (upto a positive constant) if* $\operatorname{Ad} H$ *acts irreducibly on $\underline{M}$.*

Proof. a) *Existence.* Since H is compact there is a euclidean structure $\widehat{\gamma}_e$ on $\underline{\mathrm{M}}$ which is invariant under the action of $\operatorname{Ad} H$. Now define

$$\gamma_{m_0} \in \mathscr{L}^2(T_{m_0}(M))$$

by the equation

(3.3.13) $$p^*(\gamma_{m_0})|\underline{\mathrm{M}} = \widehat{\gamma}_e.$$

Then by (3.3.9) we have

(3.3.14) $$(\tau(h))^*(\gamma_{m_0}) = \gamma_{m_0}, \forall h \in H.$$

Now, for $m \in M$, choose an $a \in G$ such that

$$\tau(a)(m_0) = m$$

and define γ_m by the equation

$$(\tau(a))\gamma_m = \gamma_{m_0}. \tag{3.3.15}$$

Then if a' is another element of G such that

$$\tau(a')(m_0) = m$$

we have $a' = ah$ for some $h \in H$, and (3.3.14) shows that γ_m is independent of the a chosen. Now we can see that the map

$$\gamma : m \to \gamma_m$$

defines an r.s. on M. By the very definition of γ it follows that 92
$\tau(a)$ is an isometry of (M, γ) for every a of G. Now let us consider $\widehat{\sigma}$. Since $\widehat{\sigma}$ is a diffeomorphism, to show that $\widehat{\sigma}$ is an isometry it is enough to show that γ is invariant under $\widehat{\sigma}^T$. To see this let $m \in M$. Then since M is a homogeneous space there exists an a in G such that

$$m = \tau(a) \cdot m_0.$$

Now let x and y be in $T_m(M)$, and then define x_0 and y_0 by the equations

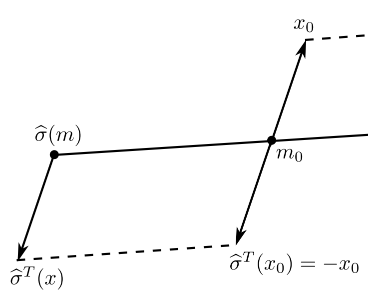
$$x = \tau(a)^T(x_0), \quad y = \tau(a)^T(y_0).$$

Then

$$\begin{aligned}
(\widehat{\sigma}^T)_m(x, y) &= \gamma_{\widehat{\sigma}^{-1}(m)}(\widehat{\sigma}^T(x), \widehat{\sigma}^T(y)) \\
&= \gamma_{\widehat{\sigma}^{-1}(m)}(\widehat{\sigma}^T \circ (\tau(a))^T(x_0), \widehat{\sigma}^T \circ (\tau(a))^T(y_0)) \\
&= \gamma_{\widehat{\sigma}^{-1}(m)}(\tau(\sigma(a))^T \circ \widehat{\sigma}^T(x_0), \tau(\sigma(a))^T \circ \widehat{\sigma}^T(y_0)) \\
&\qquad \text{by (3.3.11)} \\
&= \gamma_{\widehat{\sigma}^{-1}(m)}(\tau(\sigma(a))^T(-x_0), \tau(\sigma(a))^T(-y_0)) \\
&\qquad \text{by (3.3.10)} \\
&= \gamma_{m_0}(x_0, y_0) \text{ by the definition of } \gamma.
\end{aligned}$$

Hence the existence.

b) *Uniqueness.* Let γ_1 be any r.s. for which $\tau(a)$, for every a in G, is an isometry. Then, in particular $\tau(h)$ is an isometry of (M, γ_1) for every h of H and now by (3.3.9) it follows that $p^*(\gamma_1)|\underline{M}$ is a euclidean structure on $\underline{M}$ which is invariant under the action of $\operatorname{Ad} h$ for every h in H. Now the uniqueness is a consequence of the well known lemma of Schur. **93**



□

Remark. The transformation $\widehat{\sigma}$ defined above is called *the symmetry around* m_0. By (3.3.10) it follows that it is an involution having m_0 as isolated fixed point. Since G acts transitively on M, given any point m of M, there exists an a of G such that

$$\tau(a)(m_0) = m.$$

94 Then we see that the transformation

(3.3.16) $$\tau(a) \circ \widehat{\sigma} \circ \tau(a)^{-1}$$

is an involution having m as isolated fixed point. We call this involution the *symmetry around m* and denote it by

(3.3.17) $$\widehat{\sigma}_m.$$

Concerning these symmetries we have the following:

3.18

Proposition. *If $m, m' \in M$ are sufficiently near then $\exists\, n \in M$ such that*

$$\widehat{\sigma}_n(m) = m'.$$

Proof. Since G acts transitively on M we can suppose that $m = m_0$. Now since the map

$$p^T \circ (\exp|\underline{\mathrm{M}})^T : \underline{\mathrm{M}} \xrightarrow{(\exp|\underline{\mathrm{M}})^T} T_e(G) \xrightarrow{p^T} T_{m_0}(N),$$

is an isomorphism, the inverse function theorem shows, since m' is close enough to m_0; that there exists an $X \in \underline{\mathrm{M}}$ such that

$$m' = \exp(X).$$

Now $p(\exp(\frac{X}{2}))$ can be taken for n. □

3.19

The homogeneous Riemannian manifold (M, γ) is called the *symmetric space* associated to the symmetric pair (G, H).

3.20

Remark. 1. Let

$$f : t \to \exp(tX).$$

Then from the above proof it is clear that **95**

$$\widehat{\sigma}_{f(t_0)}(f(t)) = f(2t_0 - t).$$

3.21

Remark. 2. In fact, the condition that m and m' be sufficiently near is superfluous: (see 4.3).

4 The S.C.-manifolds

By K we denote one of the following fields: $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), $\mathbb{H}$ (the quaternions); the conjugation in K is $z \to \bar{z}$ and we set $k = \dim_{\mathbb{R}} K(k = 1, 2, 4)$. We, consider K^n (for an integer n) as a left vector space over K, denote by $e_1, \ldots, e_n$ its canonical basis and use freely the two identifications:

$$K^{n+1} = K^n \times K : (z_1, \ldots, z_n, z_{n+1}) = (z = (z, \ldots, z_n), z_{n+1}) = (z, z_{n+1}).$$

(3.4.1) $$K^n \subset K^{n+1} : (z_1, \ldots, z_n) = (z_1, \ldots, z_n, 0).$$

On K^n we consider the canonical hermitian structure $<, >$:

$$< (z_1, \ldots, z_n), (w_1, \ldots, w_n) >= \sum_i z_i \overline{w}_i.$$

By $\mathbb{U}(K^n)$ we denote the set of all K-endomorphisms of K^n which leave $<, >$ invariant; for $K = \mathbb{R}$, $\mathbb{C}$ we denote by $\mathbb{SU}(K^n)$ the subgroup of $\mathbb{U}(K^n)$ of elements of determinant equal to one. We also set:

(3.4.2) $$S0(n) = \mathbb{SU}(\mathbb{R}^n), 0(n) = \mathbb{U}(\mathbb{R}^n).$$

(3.4.2) $$SU(n) = \mathbb{SU}(\mathbb{C}^n), U(n) = \mathbb{U}(\mathbb{C}^n), n \geq 1$$

$$Sp(n) = (\mathbb{SU}(\mathbb{H}^n) =)\mathbb{U}(\mathbb{H}^n).$$

96 All these are compact Lie groups, and, with the exception of $0(n)$, all are connected. The component of the identity of $0(n)$ is $S0(n)$.

4.3

Proposition. $\mathbb{SU}(K^n)$ *acts transitively on the* K*-directions and also on the* $\mathbb{R}$*-directions of* K^n*; also* $S0(n)$ *acts transitively on the planes of* $\mathbb{R}^n$*; if* $K = \mathbb{C}$*, then* $n > 1$.

In particular $\mathbb{SU}(K^n)$ acts transitively on the sphere

(3.4.5) $$\mathbb{S}(K^n) = \{x \in K^n | < x, x >= 1\}; \mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1}).$$

On $K^{n+1} - 0$ the equivalence relation $R : z \sim z'$ if and only if $\exists\, \lambda \in K$ with $z = \lambda z'$ yields the *projective space*

$$K^{n+1} - \{0\} ar[d]_p$$

(3.4.6) $$P^n(K) = (K^{n+1} - \{0\})/R,$$

$$P^n(K)$$

on which $\mathbb{SU}(K^{n+1})$ acts transitively. So $P^n(K)$ is a homogeneous space; and is a connected compact manifold, of dimension k.n. We write $P^n(K) = M = G/H$ where $G = \mathbb{SU}(K^{n+1})$ and H is the subgroup of G leaving the point $m_0 = p(e_{n+1})$ fixed. We set

(3.4.7) $$H = \mathbb{S}(\mathbb{U}(K^n) \times \mathbb{U}(K)) = \mathbb{SU}(K^{n+1}) \cap (\mathbb{U}(K^n) \times \mathbb{U}(K))$$

where $\mathbb{U}(K^n) \times \mathbb{U}(K)$ is embedded in $\mathbb{U}(K^{n+1})$ by (3.4.1). For elements of H we use the notation $(f, \widehat{\lambda})$ where $f \in \mathbb{U}(K^n)$ and $\widehat{\lambda} \in \mathbb{U}(K)$ stands for the map $\mu \to \mu\lambda$ of K into itself which is associated to the element 97
$\lambda \in K$ (with $< \lambda, \lambda >= 1$), i.e.

(3.4.8) $$(f, \widehat{\lambda})(z, z_{n+1}) = (f(z), z_{n+1}\lambda).$$

From (3.4.7) and (3.4.7) we get the fibrations:

(3.4.10) $$\begin{array}{ccc} \mathbb{S}^n & \mathbb{S}^{2n+1} & \mathbb{S}^{4n+3} \\ \downarrow{\scriptstyle \mathbb{Z}_2} & \downarrow{\scriptstyle \mathbb{S}^1} & \downarrow{\scriptstyle \mathbb{S}^3} \\ P^n(\mathbb{R}) & P^n(\mathbb{C}) & P^n(\mathbb{H}) \end{array}$$

We show now that (G, H) is canonically a symmetric pair; define an endomorphism s of K^{n+1} and an automorphism σ of G by:

(3.4.11) $$\begin{aligned} & s(e_i) = e_i \ \text{ for } \ i = 1, \ldots, n \ \text{ and } \ s(e_{n+1}) = -e_{n+1} \\ & \sigma : g \to \sigma(g) = s \circ g \circ s. \end{aligned}$$

As is easily verified, the subgroup of elements fixed under σ is nothing but $H = \mathbb{S}(\mathbb{U}(K^n) \times \mathbb{U}(K))$.

We study now the complement $\underline{M}$ in $G = \underline{H} + \underline{M}$ as defined in (3), by introducing the map $r : K^n \to \underline{G}$ defined by:

$$r(z)(e_{n+1}) = z, r(z)(z) = -e_{n+1}, r(z)(u) = 0 \quad \forall u | < u, z > = 0 \tag{3.4.12}$$

when $< z, z >= 1$ and extension by linearity. Hence the one-parameter subgroup of G associated to $r(z) : t \to \exp(t \cdot r(z))$ is nothing but:

$$\exp(t \cdot r(z)) : \begin{cases} e_{n+1} \to (\cos t)e_{n+1} + (\sin t)z \\ z \to -(\sin t)e_{n+1} + (\cos t)z \\ u \to u \, \forall u \ \text{ with } \ < u, z >= 0. \end{cases} \tag{3.4.13}$$

Remark that $\dim(K^n) = k.n = \dim G/H = \dim \underline{M}$ and that $r(K^n) \subset \underline{M}$ for a direct computation yields:

$$(\exp(t.r(z)) = \exp(-t.r(z)).$$

98 Hence:

$$r(K^n) = \underline{M}, p^T \circ r : K^n \to T_{m_0}(M) \tag{3.4.14}$$

is an isomorphism. To find the action of $\mathrm{Ad}\, H$ on $\underline{M}$ we use (3.3.9) and (3.4.14). To $z \in K^n$ we associate the curve $l : t \to p(e_{n+1} + tz)$ in $P^n(K)$. For the image $\tau(h) \circ l$ under $h = \widehat{(f, \lambda)} \in H$, by (3.4.8) and definition of $P^n(K)$ we have:

$$t \to \tau(h)(p(e_{n+1} + tz)) = p(h(e_{n+1} + tz)) = p(\lambda \cdot e_{n+1} + t \cdot f(z)) = \\ = p(e_{n+1} + t \cdot \lambda^{-1} \cdot f(z))$$

hence by (5.7) and (3.3.9):

$$(\tau(h) \circ l)'(0) = (p^T \circ r)(\lambda^{-1} \cdot f(z)) = (\tau(h))^T (l'(0)) = \\ = ((\tau(h))^T \circ p^T \circ r)(z) = (p^T \circ \mathrm{Ad}(h))(r(z))$$

which yields:

$$\mathrm{Ad}(\widehat{(f, \lambda)}) = r \circ (\lambda^{-1} f) \circ r^{-1}. \tag{3.4.15}$$

4.16

Remarks. In the case $K = \mathbb{R}, \mathbb{C}$ the *homomorphism* $H = \mathbb{S}(\mathbb{U}(K^n) \times \mathbb{U}(K))) \to \mathbb{U}(K^n)$ defined by $\widehat{(f, \lambda)} \to \lambda^{-1} \cdot f$ in onto because the condition $\widehat{(f, \lambda)} \in \mathbb{SU}(K^{n+1})$ is equivalent to the condition $\lambda \cdot \det f = 1$, so that $\lambda^{-1} \cdot f = (\det f) \cdot f$ (and this is injective when $n > 1$.). This explains why one sometimes writes: $P^n(\mathbb{R}) = S0(n+1)/0(n)$; $P^n(\mathbb{C}) = SU(n+1)/U(n)$. One also writes:

$$P^n(\mathbb{H}) = \operatorname{Sp}(n+1)/\operatorname{Sp}(n) \times \operatorname{Sp}(1)$$

which is correct because $\mathbb{SU}(\mathbb{H}^n) = \mathbb{U}(\mathbb{H}^n)$. For the sphere $S0(n+1)/S0(n)$ the computation as above (except that there is no projection p) shows that $\operatorname{Ad} H$ is isomorphic to the action of $S0(n)$ in $\mathbb{R}^n$ (under **99**
(3.4.1)). From these remarks and from (4.3) we get:

4.17

Proposition. *For $P^n(K) = G/H$ the action of* $\operatorname{Ad} H$ *is transitive on the directions of $\underline{M}$; for $P^n(\mathbb{R})$ or $\mathbb{S}^n$ the action of* $\operatorname{Ad} H$ *is transitive on the planes of $\underline{M}$.*

In particular $\operatorname{Ad} H$ acts irreducibly on $\underline{M}$ so by (3.12) there exists (upto a positive constant) a canonical r.s. on G/H; in fact there exists a *canonical* r.s. on G/H, for we see from (3.4.14) and (3.4.15) that the canonical euclidean structure on K^n (deduced from its canonical hermitian form) gives a euclidean structure on $\underline{M}$ invariant by $\operatorname{Ad} H$. So we require by definition, that:

4.18

$r : K^n \to \underline{M}$ be a euclidean isomorphism. *We write* $(P^n(K), \text{can})$ *for this symmetric space* and leave the reader to check in the case of $\mathbb{S}^d = S0(d+1)/S0(d)$ that this procedure yields nothing but $(\mathbb{S}^d, \text{can})$ as defined in 2.1.

4.19

Remarks. we leave to the reader the proofs of the following facts:

A) the imbedding $K^n \to K^{n+1}$ yields an imbedding of $p^{n-1}(K)$ as a sub manifold of $P^n(K)$. Prove the canonical r.s. on the $P^n(K)$'s are *hereditary* in the sense that the induced r.s. on $P^{n-1}(K)$ by the canonical r.s. of $P^n(K)$ is its canonical r.s.;

B) the first fibration in (3.4.10) is a riemannian covering. The two last ones are excellent in the sense that for

$$F \quad \begin{array}{c} E \\ \big\downarrow \\ B \end{array} \quad P$$

100 first the r.s. induced on the fibres is the canonical one; second: if one writes at any $m \in E$ the orthogonal decomposition

$$T_m(E) = T_m(p^{-1}(p(m))) + N$$

then $p^T N : N \to T_{p(m)}(B)$ is a euclidean isomorphism.

To those symmetric spaces we add the symmetric space $P^2(\Gamma) = F_4/\operatorname{Spin}(9)$ *the Cayley projective plane* (see: [37]). We consider on it the canonical r.s. $(P^2(\Gamma), \text{can})$ such that all its geodesics are closed and of length π (see [14], p.356).

4.20

Definition. *Any of the* $(P^n(K), \text{can})$, $(\mathbb{S}^d, \text{can})$, $(P^2(\Gamma), \text{can})$ *is called as* S.C.-manifold.

We will use also a non compact symmetric space defined as follows. Let $S0_0(d, 1)$ be the identity component of the linear group of $\mathbb{R}^{d+1}$ which leaves invariant the quadratic form:

$$(x_1, \ldots, x_d, x_{d+1}) \to x_1^2 + \cdots + x_d^2 - x_{d+1}^2$$

For the same s and σ as in (3.4.2) we get the symmetric pair $(G, H) = (S0_0(d,1), S0(d))$. Again $S0(d)$ acts by Ad on $\underline{\mathrm{M}}$ as $S0(d)$ on $\mathbb{R}^d$; in particular:

4.21

Ad H acts transitively on the planes of $\underline{\mathrm{M}}$.

In particular by 3.12 we get (upto a positive constant) a canonical r.s. on $M = G/H$. In fact M is homeomorphic to $\mathbb{R}^d$; for,

$$A = \{z = (x_1, \ldots, x_d x_{d+1}) \in \mathbb{R}^{d+1} | x_{d+1} \geq 0, < z, z >= 1\} \text{ and}$$

define the injection **101**

$$i : A \to P^d(\mathbb{R}) \text{ by } i(x_1, \ldots, x_d, x_{d+1}) = p(x_1, \ldots, x_d, 1).$$

Claim . $S0_0(d,1)$ *acts transitively on* $i(A)$*; in fact, for any* z *with* $< z, z >= 1$*, we have in* $S0_0(d,1)$ *the one parameter subgroup*

$$t \to g(t) \begin{cases} e_{n+1} \to (ch\, t) \cdot e_{n+1} + (sh\, t) \cdot z \\ z \to (sh\, t) \cdot e_{n+1} + (ch\, t) \cdot z \\ u \to u \forall u \;\; with \;\; < u, z >= 0, \end{cases}$$

so that

$$\begin{aligned} g(t))(p(e_{n+1})) &= p(g(t)(e_{n+1}) = p(ch\, t) \cdot e_{n+1} + (sh\, t) \cdot z) = \\ &= p(e_{n+1} + (th\, t) \cdot z). \end{aligned}$$

Having an r.s. on $i(A)$*, we use a diffeomorphism between* A *and* $\mathbb{R}^d$ *to get a canonical r.s. on* $\mathbb{R}^d$ *which we denote by* $(\mathbb{R}^d, \mathrm{hyp})$*.*

4.22

$(\mathbb{R}^d, \mathrm{hyp})$ is called the *hyperbolic space of dimension d.*

5 Volumes

Let (M, g) be an r.m. Then for every $m \in M$, $T_m(M)$ has a euclidean structure by g i.e. the one given by the symmetric positive definite bilinear form g_m. Hence it has a canonical volume (see 3.6) t_m, and further if (M, g) is oriented then $T_m(M)$ admits a canonical volume form S_m induced by the oriented form σ', i.e. the orientation given by σ'_m.

5.1

102 **Proposition.** *With the above notation the map*

$$\theta : m \to t_m$$

is a volume element on M (called canonical volume element on M) and if M is oriented the map

$$\sigma : m \to S_m$$

is a volume form (called the canonical volume form).

Proof. Let us consider the second case first. We know that for $X_1, \ldots, X_d$ belonging to $\mathscr{C}(M)$, $\sigma(X_1, \ldots, X_d)$ is $F(M)$-multilinear. Hence to show that σ is a volume form we need only show that $\sigma(X_1, \ldots, X_d)$ is differentiable. To see this, since differentiability is a local property, let (U, r) be a chart of M, and let $Y_1, \ldots, Y_d$ be a basis of $\mathscr{C}(U)$, orthonormal relative to the restriction of g to U. We can suppose that Y_i in that order are positive relative to σ'. Then we have

$$\sigma(Y_1, \ldots, Y_d)(m) = 1, \forall m \in U,$$

and hence, since σ is $F(M)$-multilinear, it is differentiable on U. □

Now let us take up the first part. We know that t_m is a volume element on $T_m(M)$. So to show that θ is a volume element, it is enough to show that θ is, locally the modulus of a volume form. To see this let us take an orientation σ'_1 on U and then, clearly,

$$\theta = |\sigma_1|,$$

where σ_1 is related to σ'_1 as σ is to σ'.

5.2

103

Let us note that if $\{X_1, \ldots, X_d\}$ is any local basis of $\mathscr{C}(M)$ the formula (3.6) gives

$$\theta(X_1, \ldots, X_d) = (\det(g(X_i, X_j)))^{\frac{1}{2}}.$$

If (M, g) is a compact manifold then we know that

$$\int_M \theta$$

exists. We call this number the *volume of* (M, g) *and denote it by Vol* (M, g). Generally, if no confusion is possible, we omit the reference to g and say the volume of M and write $\mathrm{Vol}(M)$. If the dimension of the manifold is one, volume is called length and if it is two volume is called area. They are denoted by $lg(M)$ and $ar(M)$ respectively.

Let us note that in the oriented case, with the notation of (5.1),

$$\int_M \theta = \int_M \sigma$$

If N is a compact sub manifold of M then there is an induced r.s. on N (see (2.1)) A). Whenever we talk of volume of N without any reference to the r.s. on N this induced structure is the one that is considered. *And further, we set:* $\mathrm{Vol}(N, g|N) = \mathrm{Vol}(N)$.

5.3

In particular, suppose the dimension of N is equal to one; then if N admits the parametric representation $N = f(I)$ where $I = [0, 1]$ and f is a curve $f : I \to M$, we have:

$$lg(f(I)) = \int_{f(I)} \lambda = \int_I f^*\lambda = \int_0^1 (f^*\lambda)(P)$$

where λ is the length form of $(N, g|N)$. But $(f^*\lambda)(P) = \lambda(f^T \circ P) =$ 104

$\lambda(f') = \|f'\|$ so that

$$lg(N) = \int_0^1 \|f'\|\,\mathrm{dt} = lg(f(I)).$$

This leads to the following definition for paths (and a fortiori for curves):

Definition. *Let* $C = \{f_i \in D(I_i, M), [a_i, b_i] \subset I_i\}$ *be a path in an r.m.* (M, g). *Then* the length of C, denoted by $lg(C)$, *is by definition:*

$$lg(C) = \sum_i \int_{a_i}^{b_i} \|f_i'\|\,\mathrm{dt} = \sum_i lg(f_i[a_i, b_i]).$$

Now let us examine the volume element more closely.

5.4

Let (M, g) be an r.m. and let φ be a differentiable function on M which is everywhere positive. Then φg defined by the equation

$$\varphi \cdot g(X, Y) = \varphi(p_M(X)) \cdot g(X, Y), X, Y \in \mathscr{C}(M)$$

defines an r.s. on M. If we denote the volume element of (M, φ, g) by θ_φ and that of M by θ then, from (5.2) it follows that they are related by the equation

$$\theta_\varphi = \varphi^{d/2} \cdot \theta$$

5.5

B. Let (M, g) and (N, h) be two compact r.m.'s which are isometric. They by the definitions of isometry, volume and by (0.3.9) we have $\mathrm{Vol}(M, g) = \mathrm{Vol}(N, h)$.

105
5.6

C. Let (M, g) and (N, h) be two r.m.'s and let

$$p : (M, g) \to (N, h)$$

be a riemannian covering. Further let us suppose that N is compact and the covering has a finite number, k, of sheets. Then $\mathrm{Vol}(M, g)$ exists and further

$$\mathrm{Vol}(M, g) = k, \mathrm{Vol}(N, h).$$

To see this let us take an open covering $\{(U_i, r_i)\}$ by coordinate neighbourhoods of N, U_i taken so small as to make $p^{-1}(U_i)$ the union of k disjoint components $U_{i_1}, \ldots, U_{i_k}$ (this is possible because p is a k-sheeted covering map.) Then $\{U_{i_j}, r_{i_j} = r_i \circ p|U_{ij}\}$ is a complete atlas for (M, g). Now let us take a partition of unity $\{\varphi_i\}$ on N subordinate to the covering U_i. Then ψ_{i_j} defined by

$$\psi_{i_j} = \begin{cases} \varphi_i \circ p & \text{on } U_{i_j} \\ 0 & \text{outside } U_{i_j} \end{cases}$$

is a partition of unity on M subordinate to the covering $\{U_{i_j}\}$. Now since p is a riemannian covering map, denoting the volume element of (M, g) and (N, h) by θ_M and θ_N respectively, we have

$$\int\limits_{U_{i_j}} \psi_{i_j} \theta_M = \int\limits_{U_{i_j}} (\varphi_i \circ p)(p^* \theta_N) = \int\limits_{U_{i_j}} p^*(\varphi_i \theta)$$
$$= \int\limits_{U_i} \varphi_i \theta \text{ by (3.5.5).}$$

Hence 106

$$\mathrm{Vol}(M, g) = \sum_j \Big(\sum_i \int\limits_{U_{i_j}} \psi_{i_j} \theta_M\Big) = \sum_j \Big(\int\limits_N \theta\Big)$$
$$= k.\,\mathrm{Vol}(N, h).$$

Now let us note a particular case. If we take the real projective space $(P^d(\mathbb{R}), \text{can})$ for (N, h) and $(\mathbb{S}^d, \text{can})$ for (M, g) and the natural map of $\mathbb{S}^d$ onto $P^d(\mathbb{R})$ for p then we have $k = 2$. Hence

$$\text{Vol}(P^d(\mathbb{R}), \text{can}) = \frac{1}{2}\,\text{Vol}(\mathbb{S}^d, \text{can}).$$

5.7

D. Let us consider the case of a flat torus $\mathbb{R}^d/G$ (see (2.2) D.2). Let $(\tau_1, \ldots, \tau_d)$ be a set of generators for G. Then $\tau_1(0), \ldots, \tau_d(0)$ is a basis of $\mathbb{R}^d$. Let $(v_1, \ldots, v_d)$ be the corresponding coordinate system. Now let

$$W = \{x \in \mathbb{R}^d | 0 < v^i(x) < 1, \ \text{ for } \ i = 1, \ldots, d\}.$$

Then W is open in $\mathbb{R}^d$, $\overline{W}$ is compact and the restriction of

$$p : \mathbb{R}^d \to \mathbb{R}^d/G$$

to W is a diffeomorphism of W with $p(W)$. Also $p(\overline{W}) = \mathbb{R}^d/G$ and $\overline{W} - W$ is of measure zero. Hence

$$\begin{aligned}&\text{Vol}(R^d/G, \epsilon/G) = \text{Vol}(p(W), \epsilon/G) = \\ &= \text{Vol}(W, \epsilon) = |\det(\tau_1(0), \ldots, \tau_d(0))| \text{ by } ().\end{aligned} \qquad \text{(chap0:0.3.2)}$$

6 The canonical forms α and $d\alpha$

Let (M, g) be an r.m. Then there is a map

$$g^\sharp : \mathscr{C}(M) \to \mathscr{C}^*(M)$$

107 given by the equation

$$g^\sharp(X)(Y) = g(X, Y), \forall Y \in \mathscr{C}(M).$$

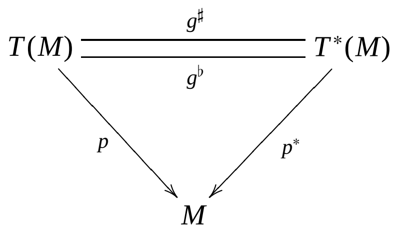
Clearly it is an $F(M)$-linear map and since g is non-degenerate it follows that $g^\sharp$ is an isomorphism of $F(M)$-modules. Let us note that

$$g^\sharp(X) = i(X)g.$$

6.1

We set $g^\flat = (g^\sharp)^{-1}$. Now evaluating $g^\sharp$ at each point of $T(M)$ we get a vector bundle isomorphism of $T(M)$ with $T^*(M)$. We denote this map also by the symbol $g^\sharp$ and it will be clear from the context which we mean by $g^\sharp$.

(3.6.2)



6.3

Remark. Let us note that $g^\sharp$ and $g^\flat$ give rise, in a natural, way, to isomorphisms of

$$\mathscr{L}^{r\pm i}_{s\mp i} \quad \text{with} \quad \mathscr{L}^r_s$$

between tensor spaces over (M, g) (these isomorphisms describe the operation usually called “raising” or “lowering” subscripts).

Now we utilise the maps $g^\flat$ and $g^\sharp$ to associate an element of $\mathscr{C}(M)$ **108**
with every φ and to pull μ from $T^*(M)$ to $T(M)$ where μ is defined in (0.4.24).

6.4

Definition. *For every $\varphi \in F(M)$ we set*

$$\operatorname{grad}\varphi = g^\flat(d\varphi).$$

If U is an open subset of $(\mathbb{R}^d, \epsilon)$ and $\varphi \in F(U)$ then we have

$$\operatorname{grad}\varphi = \sum_i \frac{\partial\varphi}{\partial u^i}\frac{\partial}{\partial u^i}.$$

6.5

Definition. $\alpha = (g^\sharp)^*(\mu)$.

If we have to consider the α's associated to several manifolds (M, g), $(N, h), \ldots$ at the same time, we write M_α, N_α for the corresponding α's.

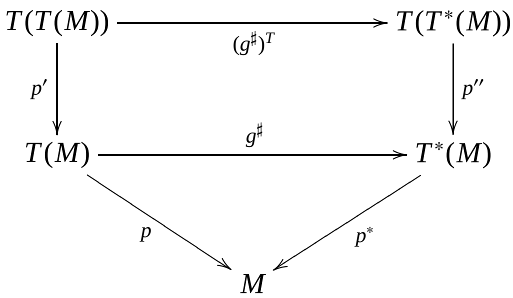
6.6

For every z in $T(T(M))$ we have

$$\alpha(z) = g(p'(z), p^T(z)).$$

Proof. For

$$\alpha(z) = (g^\sharp)^*(\mu(z)) = \mu(g^{\sharp T}(z)) =$$
$$= (p''(g^{\sharp T}(z)) = (p^{*T}(g^{\sharp T}(z)).$$



109 But $p^{*T} \circ g^{\sharp T} = (p^* \circ g^\sharp)^T = p_M^T$ and $p'' \circ g^{\sharp T} = g^\sharp \circ p_M$. Hence

$$(z) = (g^\sharp \circ p'_M(z))(p^T(z)) = g(p'(z), p^T(z)).$$

□

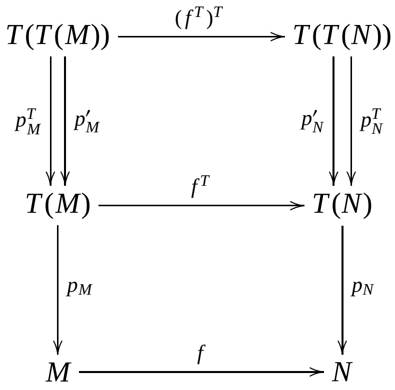
6.7

If $f : (M, g) \to (N, h)$ is an isometry, then

$$M_{\alpha = (f^T)^*({}^N(\alpha))} \quad \text{and} \quad d({}^M\alpha) = (f^T)^*(d({}^N\alpha)).$$

Proof. In view of the formula (2.4) IV. We need only prove the first equality. We have, by (6.6),

$$M_{\alpha(z)=g(p'_M(z),p^T_M(z))} \quad \text{and}$$
$$(f^T)^*({}^N\alpha)(z) = {}^N\alpha(f^T(z)) = h(p'_N(f^T)^T(z)), p^T_M(f^T)^T(z)$$



But we have

$$p'_N \circ (f^T)^T = f^T \circ p'_M \quad \text{and} \quad p^T_N \circ (f^T)^T = f^T \circ p^T_M.$$

Hence **110**

$$\begin{aligned} f^{T^*}({}^N\alpha) &= h(f^T(p'_M(z)), f^T(p^T_M(z)) = \\ &= (f^*h)(p'_M(z), p^T_M(z)) = \\ &= g(p'_M(z), p^T_M(z)) \text{ since } f \text{ is an isometry.} \end{aligned}$$

Hence the result. □

Since $g^\sharp$ is an isomorphism so is $(g^\sharp)^*$. Now by lemma (4.27) we get the following result.

6.8

Proposition. *$d\alpha$ is non-degenerate everywhere on $T(M)$.*

6.9

In particular it follows that $\overset{\alpha}{\wedge} d\alpha$ is a volume form for $T(M)$ (see also (2.5)).

6.10

Lemma. *For every $x \in T(M)$, $z \in V_x$ and $z' \in T_x(T(M))$ we have*

$$(d\alpha)(z, z') = g(\xi(z), p^T(z')).$$

Proof. Let us choose Z and Z' in $\mathscr{C}(T(M))$ such that

i) $p^T(Z) = 0$ and $Z(x) = z$ (for example let us take any $Y \in \mathscr{C}(M)$ such that $Y(p(x)) = \xi(z)$ and set $Z(x) = \xi_U^{-1}(Y(p(U))$ for every U in $T(M)$); and

ii) there exists an X in $\mathscr{C}(M)$ for which $p^T(Z') = X$; (for example let us take any Y in $\mathscr{C}(M)$ such that $Y(p(x) = p^T(z')$ and set $Z' = Y^T$ (see (4.19)).

Then

$$\begin{aligned} d\alpha(z, z') &= (d\alpha)(Z, Z')_x \\ &= Z(\alpha(Z'))_x - Z'(\alpha(Z))_x - \alpha([Z, Z'])_x. \end{aligned}$$

111 But by (6.6) we have

$$(Z) = g(p' \circ Z', p^T \circ Z) = g(p' \circ Z, 0) = 0;$$

$$\begin{aligned} \alpha([Z, Z]) &= g(p' \circ [Z, Z'], p^T \circ [Z, Z']) \\ &= g(p' \circ [Z, Z'], [p^T(Z), p^T(Z')]) \\ &= g(p \circ [Z, Z'], 0) = 0, \end{aligned}$$

and

$$\alpha(Z') = g(p' \circ Z', p^T \circ Z') = g(\mathrm{id})T(M), p^T \circ Z'),$$

$$\text{i.e.}\quad \alpha(Z')_y = g(y,(p^T \circ Z')(p(y))). \quad \text{Hence}$$
$$d\alpha(z,z') = Z(\alpha(Z'))_x = z(\alpha(Z')).$$

Now let us consider the map of $T(M)$ into $\mathbb{R}$ defined by

$$T(M) \to y \to \alpha(Z')_y = g(y, X(p(y)).$$

Clearly it is a linear function on $T(M)$. Hence, by (4.16) we have

$$\begin{aligned} z(\alpha(Z')) &= g(\xi(z), X(p(\xi(Z)) \\ &= g(\xi(z), p^T(z')). \end{aligned}$$

Note that this gives another proof of the non-degeneracy of $d\alpha$, already proved in (6.8). ◻

7 The unit bundle

7.1

Definition. *The subset* $U(M) = E^{-1}(1) = \{x \in T(M) \big| \; \|x\| = 1\}$ *of* $T(M)$ *is called the* unit bundle of M.

Let us set $W = E^{-1}(\,]-\infty, 1[\,) = \{x \in T(M) \big| \; \|x\| < 1\}$ *and observe that* $U(M)$ *is the boundary of* W *in* $T(M)$.

7.2

Lemma. *W is a nice domain.* **112**

Proof. We will be through if we show that $\forall x \in U(M) : (dE)_x \neq 0$. But by (1.8) we have

$$(dE_x)(\xi_x^{-1} x) = 2(g(x,x)) \neq 0.$$

Now by (3) we have the following result. ◻

7.3

Proposition. $U(M)$ *is a sub manifold of* $T(M)$ *of dimension* $2d - 1$.

7.4

Let us note that $U(M)$ is compact if M is. For, *the fibre* U_m at m is

$$U_m = \{x \in T_m(M) | g(x, x) = 1\}$$

and hence it is a sphere in the euclidean space $(T_m(M), g_m)$.

We denote the pullbacks of α and $d\alpha$ to the sub manifold $U(M)$ by α and $d\alpha$ themselves.

8 Expressions in local coordinates

Let (M, g) be an r.m. and let (U, r) be a chart of (M, g). Let $X_i = \left\{\dfrac{\partial}{\partial x^i}\right\}$ be the basis of $\mathscr{C}(U)$ dual to the basis $dx^i = d(u^i \circ r)$ of $\mathscr{C}^*(U)$. Then $\{(dx^i \cdot dx^j)\}$ (for this multiplication see (0.2.2)) is a basis of $\mathscr{L}^2(U)$. Hence there exists local functions $\{g_{ij}\}$ such that

8.1

$g = \sum\limits_{i,j} g_{ij} dy^i \cdot dx^j$, with $g_{ji} = g_{ij}$ since g is symmetric. Now let us take
113 any $X = \sum\limits_i p^i X_i$ and compute $g^\sharp(X)$. Setting $g^\sharp(X) = \sum p_i dx^i$, we have, by definition,

$$p_i = g^\sharp(X)(X_i) = g(\sum_j p^j X_j, X_i) = \sum_j g_{ij} p^j. \tag{3.8.2}$$

Since g is positive definite it follows that $\det(g_{ij})$ is never zero and hence p^i can be calculated in terms of p_j. So let

$$p^i = \sum_j g^{ij} p_j. \tag{3.8.3}$$

Then we have

$$\sum_1 g^{il} g_{lj} = \delta_{ij}.$$

8.4

Considering the set $\{x^1, \ldots, x^d, p_1, \ldots, p^d\}$ of coordinates on $T(U)$, and by (6.6).

$$\alpha|U = \sum_{i,j} g_{ij} p^i dx^j \tag{3.8.5}$$

and hence

$$d\alpha|U = \sum_{i,j,k} \frac{\partial g_{ij}}{\partial x^k} p^i dx^k \wedge dx^j + \sum_{i,j} g_{i,j} dp^i \wedge dx^j. \tag{3.8.6}$$

Chapter 4

Geodesics

1 The first variation

In this article, we study the following question; given two points m, n in an r.m. (M, g), look for a curve f with end points m, n which has minimal length among all curves in M with end points m, n. Then the first variation has to be zero for all one-parameter families of curves with fixed ends m, n. This will lead us to a *sufficient* condition involving the acceleration vector f'' of f and from there to a spray on (M, g) canonically, so to geodesics in (M, g). The necessity of this condition could be proved directly but we will not do it, for it will be a consequence of Chapter VII. We shall, in the course of these considerations, obtain a useful formula for a geodesic in (M, g), the first variation formula. **114**

Let us consider an r.m. $(M.g)$ and two points m, $n \in M$; let f be a one-parameter family of curves, for which we use the notation introduced in (8). We suppose that $f : (I =]-\epsilon, 1+\epsilon[\,) \times J \to M$ with $0 \in J$ and $\epsilon > 0$; moreover we suppose that $\|f_0'(t)\| - 1\ \forall t$.

For the lengths of the f_s's we define the *first variation* $l'(0)$ by:

(4.1.1) $$l'(0) = \frac{d}{ds}(lg(f_s|[0,1]))_{s=0}.$$

To compute $l'(0)$, if α is the canonical form on $T(M)$ (definition in (6.5), we set

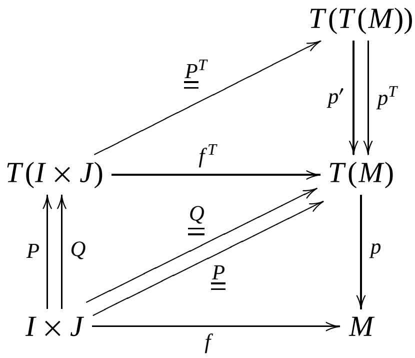
(4.1.2) $$\beta = \underline{\underline{P}}^*(\alpha).$$

115 We have:

$$E \circ \underline{\underline{P}} = \alpha(\underline{\underline{P}}^T \circ P) = \beta(P); \tag{4.1.3}$$

for, $\alpha(\underline{\underline{P}}^T \circ P) = g(p' \circ \underline{\underline{P}}^T \circ P, P^T \circ \underline{\underline{P}}^T \circ P)$. But $p' \circ \underline{\underline{P}} \circ P = \underline{\underline{P}}$ and $p^T \circ \underline{\underline{P}} \circ P = \underline{\underline{P}}$.

Since $\|f_0'(t)\| = 1$ we have, by the very definition of $Q = \dfrac{\partial}{\partial s}$, the following commutative diagram:



Further,

$$lg(f_s|[0,1]) = \int_0^1 \|f_s'(t)\|\,\mathrm{dt} = \int_0^1 (E \circ \underline{\underline{P}})^{\frac{1}{2}}(t,s)\,\mathrm{dt} =$$

$$= \int_0^1 \beta(P)^{\frac{1}{2}}(t,s)\,\mathrm{dt},$$

and

$$l'(0) = Q\Big(\int_0^1 (\beta(P))^{\frac{1}{2}}(t,s)\,\mathrm{dt}\Big)$$

$$= (1/2)\int_0^1 Q(\beta(P))(t,0)\cdot(\beta(P)(t,0))^{-1/2}\,\mathrm{dt} =$$

$$= (1/2)\int_0^1 Q(\beta(P))(t,0)\,\mathrm{dt}\,.$$

But **116**

$$d\beta(P,Q) = P\beta(Q)) - Q(\beta(P)) - \beta([P,Q]) \quad \text{and} \quad [P,Q] = 0$$

so we get

$$l'(0) = (1/2)\int_0^1 P(\beta(Q))(t,0)\,\mathrm{dt} - (1/2)\int_0^1 d\beta(P,Q)(t,0)\,\mathrm{dt}\,.$$

From the definition $P = \dfrac{\partial}{\partial t}$:

$$\int_0^1 P(\beta(Q))(t,0)\,\mathrm{dt} = [\beta(Q)]^{1,0}_{(0,0)}$$

but

$$\beta(Q) = \alpha(\underline{\underline{P}}^T \circ Q) = (\text{see (6.6)}) = g(p' \circ \underline{\underline{P}}^T \circ Q, p^T \circ \underline{\underline{P}} \circ Q) =$$
$$= g(\underline{\underline{P}}, \underline{\underline{Q}}) \text{ to the effect:}$$

(4.1.4) $$l'(0) = (1/2)[g(\underline{\underline{P}}, \underline{\underline{Q}})]^{(1,0)}_{(0,0)} - (1/2)\int_0^1 d\beta(P,Q)(t,0)\,\mathrm{dt}\,.$$

We can also write it as:

(4.1.5) $$l'(0) = (1/2)[g(\underline{\underline{P}}, \underline{\underline{Q}})]^{(1,0)}_{(0,0)} - (1/2)\int_0^1 d\alpha(f_0''(t), (\underline{\underline{P}}^T \circ Q)(t,0))\,\mathrm{dt}$$

for the acceleration $f_0''(t)$ of f_0 (see (5.8)).

Clearly:

1.6

If $f(0, s) = m$ and $f(l, s) = n \forall s \in J$, then

$$\left[g(\underline{\underline{P}}, \underline{\underline{Q}})\right]_{(0,0)}^{(1,0)} = 0.$$

But this does not imply that $f_0''(t)$ should be such that $i(f_0''(t))(d\alpha) = 0$ for $\underline{\underline{P}}^T \circ Q(t, 0)$ is not an arbitrary element in $T(T(M))$. For, adding (5.9), (6.10) and (1.8) we get:

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1.7

if z vertical then $(d\alpha)(f_0''(t), z) = -(1/2)z(E)$.

But $(\underline{\underline{P}}^T \circ Q)E = Q(\beta(P))$ so we have

$$\begin{aligned}
&(i(f_0''(t))(d\alpha) + (1/2)dE)(\underline{\underline{P}}^T \circ Q)(t, 0) \\
&= (P(\beta(Q)) - Q(\beta(P)) + (1/2)Q(\beta(P)))(t, 0) = \\
&= (P(\beta(Q)) - (1/2)Q(\beta(P))(t, 0)
\end{aligned}$$

and finally:

(4.1.8)

$$l'(0) = [g(P, Q)]_{(0,0)}^{(1,0)} - \int_0^1 ((i(f_0''(t)(d\alpha) + (1/2)dE))(\underline{\underline{P}}^T \circ Q)(t, 0)\,\mathrm{dt}$$

from which we get:

1.9 $i(f_0''(t)(d\alpha) + (1/2)dE = 0$ **is a sufficient condition for** f_0 **to be of critical length among all curves with ends** m, n**.**

And in particular

1.10

if the curve f_0 is such that $i(f_0'')(d\alpha) + (1/2)dE = 0$ then we have the *first variation formula*: $l'(0) = [g(\underline{\underline{P}}, Q)]_{(0,0)}^{(1,0)}$.

We are so lead to the following:

2 The canonical spray on an r.m. (M,g)

By Proposition (6.8) it follows that there exists one and only one $G \in \mathscr{C}(T(M))$ such that

(4.2.1) $$i(G)(d\alpha) = -\frac{1}{2}dE.$$

Regarding the G we prove the following proposition.

2.2

Proposition. *G is a spray on M.*

Proof. First let us prove that $p^T \circ G = \mathrm{id}_{T(M)}$. For any $x \in T(M)$ and $z \in V_x$ we have, on the one hand, by (1.8)

$$(d\alpha)(G(x),z) = -\frac{1}{2}z(E) = -g(x,\xi(z))$$

and, on the other by (6.10) **118**

$$(d\alpha)(G(x),z) = -g({}^T(G(x)),\xi(z))$$

and hence

$$g(p^T(G(x)) - x, \xi(z)) = 0.$$

Since ξ is an isomorphism between V_x and $T_x(M)$ and g is non - degenerate we have

$$p^T(G(x)) = x, \forall x \in T(M).$$

Now let us prove the homogeneity. □

First let us note that

$$\alpha \circ h^T_\theta = \theta \cdot \alpha,$$

In fact, $\forall z \in T(T(M))$ we have:

$$\begin{array}{ccc}
T(T(M)) & \xrightarrow{h_\theta^T} & T(T(M)) \\
{\scriptstyle p'}\downarrow\downarrow{\scriptstyle p^T} & & {\scriptstyle p'}\downarrow\downarrow{\scriptstyle p^T} \\
T(M) & \xrightarrow{h_\theta} & T(M) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
M & \xrightarrow[\mathrm{id}_M]{} & M
\end{array}$$

$$\begin{aligned}
\alpha(h_\theta^T(z)) &= g(p^T(h_\theta^T(z)), p' \circ h_\theta^T(z)) = \\
&= g((p \circ h_\theta)^T(z), h_\theta \circ p'(z)) = g(p^T(z), \theta \cdot p'(z)) = \\
&= \theta g(p^T(z), p'(z)) = \theta \cdot \alpha(z).
\end{aligned}$$

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Hence, we have, for every z, z' in $T(T(M))$,

$$d\alpha(h_\theta^T(z), h_\theta^T(z')) = \theta \cdot d\alpha(z, z').$$

Now for any $Z \in \mathscr{C}(T(M))$ we have

$$\begin{aligned}
d\alpha(\theta(h_\theta^T \circ G), h_\theta^T \circ Z) &= \theta \cdot d\alpha(h_\theta^T \circ G, h_\theta^T \circ Z)) \\
&= \theta^2 \cdot d\alpha(G, Z) = -\frac{1}{2}\theta^2 \cdot Z(E), \quad \text{and also}
\end{aligned}$$

$$\begin{aligned}
d\alpha(G \circ h_\theta, h_\theta^T \circ Z) &= -\frac{1}{2} h_\theta^T \cdot Z(E) \\
&= -\frac{1}{2} Z(E \circ h_\theta) = -\frac{1}{2} Z(\theta^2 \cdot E) = -\frac{1}{2}\theta^2 \cdot Z(E).
\end{aligned}$$

Again, as above, since $d\alpha$ is non-degenerate we have

$$G \circ h_\theta = \theta(h_\theta^T \circ G).$$

Now we give the following definition.

2.3

Definition. *The spray G of the above proposition is called* the canonical spray of the *r.m.* (M, g).

Whenever we talk of a spray on an r.m. we always mean the canonical spray. In view of this we can talk of *geodesics on an* r.m.

By the definition of isometry E is invariant under isometry and by (6.7), $d\alpha$ is also invariant under isometry and we have the following:

2.4

Proposition. *If* **120**

$$\lambda : (M, g) \to (N, h)$$

is an isometry, and ${}^{M}G$ (respectively ${}^{N}G$) is the canonical spray of (M, g) *(respectively on* (N, h)*) then*

$${}^{N}G = (\lambda^{T})^{T} \circ {}^{M}G \circ (\lambda^{-1})^{T}.$$

Proof. Let any $Z' \in \mathscr{C}(T(N))$; since λ is an isometry and hence in particular, a diffeomorphism, there exists a $Z \in \mathscr{C}(T(M))$ such that $Z' = (\lambda^{T})^{T} \circ Z \circ (\lambda^{-1})^{T}$. Set moreover:

$$G' = (\lambda^{T})^{T} \circ {}^{M}G \circ (\lambda^{-1})^{T}. \text{ Then, by ((6.7)), (0.2.6), (2.4),}$$

we have:

$$\begin{aligned}(i(G') \cdot d({}^{N}\alpha))(Z') &= d(((\alpha^{-1})^{T})^{*}(M\alpha))(G', Z') = \\ &= ((\lambda^{-1})^{T})^{*}(d({}^{M}\alpha))(G', Z') = \\ &= d({}^{M}\alpha)(((\lambda^{-1})^{T})^{T} \circ G' \circ \lambda^{T}, ((\lambda^{-1})^{T})^{T} \circ Z' \circ \lambda^{T}) \\ &= d({}^{M}\alpha)(G', Z') = (i({}^{M}G) \cdot d({}^{M}\alpha))(Z).\end{aligned}$$

□

Now, by the definition of ${}^{M}G$ and because λ is an isometry implies $M\ {}^{M}E = {}^{N}E \circ \lambda^{T}$, we have:

$$\begin{aligned}(i({}^{M}G) \cdot d({}^{M}\alpha))(Z) = -\frac{1}{2}d({}^{M}E)(Z) &= -\frac{1}{2}d({}^{N}E)((\lambda^{T})^{T} \circ Z \circ (\lambda^{-1})^{T}) = \\ &= -\frac{1}{2}d({}^{N}E)(Z').\end{aligned}$$

2.5

Corollary. *If $\lambda : (M,g) \to (N,h)$ is an isometry and f a geodesic in (M,g), then $\lambda \circ f$ is a geodesic in (N,h).*

121 *Proof.* Let (I, f) be an integral curve of ${}^M G$ in $T(M)$. Then $(I, \lambda^T \circ f)$ is an integral curve of ${}^N G = (\lambda^T)^T \circ {}^M G \circ (\lambda^{-1})^T$ in $T(N)$. □

For:

$$(\lambda^T \circ f)' = (\lambda^T \circ f)^T \circ P = (\lambda^T)^T \circ f' = (\lambda^T)^T \circ {}^M G \circ f =$$
$$= ({}^N G \circ \lambda^T) \circ f = {}^N G \circ (\lambda^T \circ f)'.$$

2.6

Corollary. *If $\lambda : (M,g) \to (N,h)$ is an isometry, then $\lambda^T({}^M\Omega) = {}^N\Omega$ and the following diagram is commutative:*

$$\begin{array}{ccc} M_\Omega & \xrightarrow{\lambda^T} & N_\Omega \\ {\scriptstyle M_{\exp}}\downarrow & & \downarrow{\scriptstyle N_{\exp}} \\ M & \xrightarrow{\lambda} & N \end{array}$$

Proof. Apply (4.5). □

We slightly generalize the above results in the:

2.7

Proposition. *Let (M,g) and (N,h) be two r.m.'s of the same dimension and a map $\lambda \in D(M,N)$ be such that $\lambda^* h = g$. Then λ is a local isometry; moreover, if (I, f) is a geodesic in (N,h) and $\widehat{f} : I \to M$ such that $\lambda \circ \widehat{f} = f$, then $\widehat{f}$ is a geodesic in (M,g).*

Proof. We note first the injectivity of $\lambda_m^T \forall m \in M$, which comes from $\lambda^* h = g$ and g positive definite; the equality of dimensions then implies λ_m^T is an isomorphism $\forall m \in M$; hence the inverse function theorem and $\lambda^* h = g$ imply the local isometry. □

Remark. 2.7 can be applied, in particular, to a riemannian *covering*. **122**

2.8

Example. Let us determine the canonical spray of $(\mathbb{R}^d, \epsilon)$.

First of all from the definition of the scalar product of g we can identify α and μ, and $d\alpha$ and $d\mu$. Now by (0.4.26) we have, for every z, z' of $T(M)$,

$$d\alpha(z, z') = (\zeta^T(z)) \cdot (p^T(z') - (\zeta^T(z)) \cdot (p^T(z)),$$

and hence $\forall x \in T(M)$ we have

$$\begin{aligned} d\alpha(G(x), z) &= (\zeta^T(G(x))) \cdot (p^T(z)) - (\zeta^T(z)) \cdot (p^T(G(x))) = \\ &= (\zeta^T(G(x))) \cdot (p^T(z)) - \zeta^T(z) \cdot \zeta(x). \end{aligned}$$

On the other hand

$$E(x) = (\zeta(x)) \cdot (\zeta(x)),$$

and hence for $z \in T_x(T(M))$, by (0.4.5)

$$z(E) = 2(\zeta(x)) \cdot (\zeta^T(z)).$$

Hence

$$(\zeta^T \circ G(x)) \cdot p^T(z) = 0, \forall z \in T_x(T(M)).$$

But since g is non-degenerate and p^T is onto $T_x(M)$ we have

$$\zeta^T(G(x)) = 0. \tag{4.2.9}$$

Consequently, by ((2.1.22)), the geodesics are segments of straight lines. A more geometrical proof would be to use the symmetry around a line in $\mathbb{R}^d$ (which is an isometry) and apply the local uniqueness of geodesics and the lemma (2.6). We shall do this in a more general situation in §4.

3 First consequences of the definition

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3.1

Proposition. $G(E) = 0$.

Proof. In fact, by definition, we have

$$G(E) = (dE)(G) = -2d\alpha(G, G) = 0.$$

□

3.2

Corollary. *E is constant along a geodesic f in particular $\|f'\|$ is constant.*

Proof. In fact, we have, with usual notation,

$$\begin{aligned} P(E \circ f') &= (f'^T \circ P)(E) = f''(E) \\ &= (G \circ f')(E) = G(E) \circ f' = 0. \end{aligned}$$

This corollary shows that any geodesic is parametrised by either the arc length or by a constant times the arc length. Hence for a geodesic f we have

$$lg(f|[t_1, t_2]) = (t_2 - t_1)\|f'(t_1)\|. \tag{4.3.3}$$

□

Hence follows the

3.4

Corollary. *For every x in Ω, $\exp(x)$ is the end point of the geodesic f with $f'(0) = x$ and of length equal to $\|x\|$.*

3.5

Now, for every positive number r and every point m of (M, g) we set

$$\underline{B}(m, r) = \{x \in T_m(M) \big| \|x\| < r\}$$

and call it *the ball of radius r* in $T_m(M)$.

3.6

By (4.2) and (4.7) it follows that, to every point m of (M, g) there exists an $r(m) > 0$ such that

$$\underline{B}(m, r) \subset \Omega$$

and **124**

$$\exp_m |\underline{B}(m, r) \to \exp(\underline{B}(m, r))$$

is a diffeomorphism. We set:

$$B(m, r) = \exp(\underline{B}(m, r))$$

for such an image set and describe the situation when we have such a diffeomorphism *as* $\exp_m$ *is r-O.K.* Hence to every point m of (M, g) there is an $r(m) > 0$ such that $\exp_m$ is $r(m)$-O.K. In fact, something more is true. For, let us take, in the proof of (6.8), instead of an arbitrary euclidean structure on $T_m(M)$ the one that is induced by g. Then the following result is obtained:

3.7

For every m of (M, g) there exists an $r(m) > 0$ such that $\forall r' \in [0, r]$ one has:

$$B(m, r') \text{ is convex.}$$

3.8

Definition. *The flow of the vector field G on* $T(M)$ *is called* the geodesic flow of *the r.m.* (M, g).

3.9

Remark. Combining (5.7) and (3.1) we conclude that flow leaves the unit bundle $U(M)$ of M invariant. Furthermore we have the following result.

3.10

Proposition. *The geodesic flow leaves $d\alpha$ on $T(M)$ and α on $U(M)$ invariant.*

Proof. By (0.6.9) we have only to show that on $T(M)$

$$\theta(G)(d\alpha) = 0,$$

125 and on $U(M)$:

$$\theta(G) \cdot \alpha = 0.$$

Clearly, by 0.6.12, we have

$$\theta(G)(d\alpha) = (i(G)d + d \circ i(G))d\alpha = d(i(G)d\alpha) = d(-\frac{1}{2}dE) = 0.$$

Also

$$\theta(G)\alpha = d(i(G)\alpha) + i(G)d\alpha = d(\alpha(G)) - \frac{1}{2}dE.$$

By (6.6)

$$\alpha(G) = g(p^T \circ G, p' \circ G) = g(\mathrm{id}_{T(M)}, \mathrm{id}_{T(M)}) = E.$$

Hence

$$\theta(G) \cdot \alpha = \frac{1}{2}dE.$$

But $E \equiv 1$ on $U(M)$. Hence the result. □

4 Geodesics in a symmetric pair

A symmetric pair is an example of a manifold where one can write down all the geodesics explicitly. With the notations of (3) we prove the following proposition.

4.1

Proposition. *The geodesics of* (M, γ) *are the curves*

$$f : t \to (\tau(g) \circ p)(\exp tX)$$

where g runs through G and X through $\underline{M}$*.*

Proof. Since, for every element g of G, $\tau(g)$ is an isometry and G acts transitively on the manifold we need only consider the situation at m_0, i.e. for the curves $t \to p(\exp(t \cdot X))$. ◻

Let us fix $X \in \underline{M}$; and choose a positive r for which $\exp m_0$ is r-O.K. and moreover such that

(i) $B(m_0, r) = V$ is convex, and **126**

(ii) the only point in V fixed under $\widehat{\sigma}$ is m_0.

(This is possible: see the remark following the proof of (3.12)).

Let us note that by our construction

$$\widehat{\sigma}(V) = V$$

and denote the map

$$t \to p(\exp(t \cdot X))$$

by f. Let t_0 be such that

$$f|[-t_0, t_0] \subset V.$$

Then, since V is convex, for any t_1 such that $0 < t_1 < t_0$ there is a geodesic g from $f(-t_1)$ to $f(t_1)$. By reparametrising g if necessary, we can assume that

$$g(-t_1) = f(-t_1) \quad \text{and} \quad g(t_1) = f(t_1).$$

Further from (6.5) it follows that g is unique. Clearly by (2.5), (3.12), (3.20), $\widehat{\sigma} \circ g$ is a geodesic in V from $f(t_1)$ to $f(-t_1)$ and hence $\widehat{\sigma} \circ$

$g \circ k_{-1}$ (where $k_{-1} = -\,\mathrm{id}_{\mathbb{R}}$) is a geodesic from $f(-t_1)$ to $f(t_1)$. By the uniqueness of g we have

$$\widehat{\sigma} \circ g \circ k_{-1} = g.$$

In particular,

$$\widehat{\sigma}(g(0)) = g(0),$$

and since m_0 is the only point in V fixed by $\widehat{\sigma}$ we have

$$g(0) = m_0 = f(0).$$

127 That is to say: the unique geodesic from $f(-t_1)$ to $f(t_1)$ has to go through $f(0)$. Because the τ's are isometries and $\tau(\exp(\frac{t_1}{2} \cdot X))$ sends $f(-t_1)$ into $f(-\frac{t_1}{2})$ and $f(0)$ into $f(\frac{t_1}{2})$, we see, by the above argument (applied for $\frac{t_1}{2}$ instead of t_1) that g should also go through $f(\frac{t_1}{2})$. By the same token it follows that f and g coincide at all points $t_1 \frac{p}{q}$ where $\frac{p}{q}$ is a fraction in $[-1, 1]$ and q is a power of 2. Since such $\frac{p}{q}$ are dense in $[-1, 1]$ we see, using the continuity of f and g, that f and g coincide everywhere.

4.2

By (4.3) it follows that

$$p \circ \exp_{m_0} : \underline{\mathrm{M}} \to M$$

is onto. Hence given any point m of M there is an X in $\underline{\mathrm{M}}$ such that

$$m = p \circ \exp_{m_0} X$$

for (4.1) implies in particular: $\Omega \supset T_{m_0}(M)$.

4.3

Now clearly the symmetry around $p(\exp_{m_0}(\frac{X}{2}))$:

$$\sigma_{p(\exp_{m_0}(\frac{X}{2}))}$$

takes m_0 onto m and it follows that the assumption that m, m' be sufficiently close in (3.19) can be dropped.

5 Geodesics in S.C.-manifolds

5.1

Definition. *A curve $f \in D(\mathbb{R}, M)$ in an r.m. (M, g) is said to be a* closed geodesic *if*

(i) it is a geodesic,

(ii) it is periodic i.e. $\exists\, t_0 > 0 | f(t + t_0) = f(t) \forall t \in \mathbb{R}$.

5.2

Definition. *A closed geodesic is said to be* simply closed *if moreover there exists a period t_1 of f such that f is injective on* $[0, t_1[$. **128**

In this article, we consider the S.C.-manifolds (4.19) but the statements of (5.4) and (5.7) relative to $P^2(\Gamma)$ will not be proved; one can find some of them in [14] p.355.358 and the remaining ones in [38], Nos. 107 and 151.

5.4

Proposition. *In an S.C. manifold all geodesics are simply closed and of the same length. This length is 2π for ($\mathbb{S}^d$, can) and π for the others. Moreover, the geodesics for* $(P^d(\mathbb{R}), \mathrm{can})$ *are projective lines and for* $(\mathbb{S}^d, \mathrm{can})$ *they are great circles.*

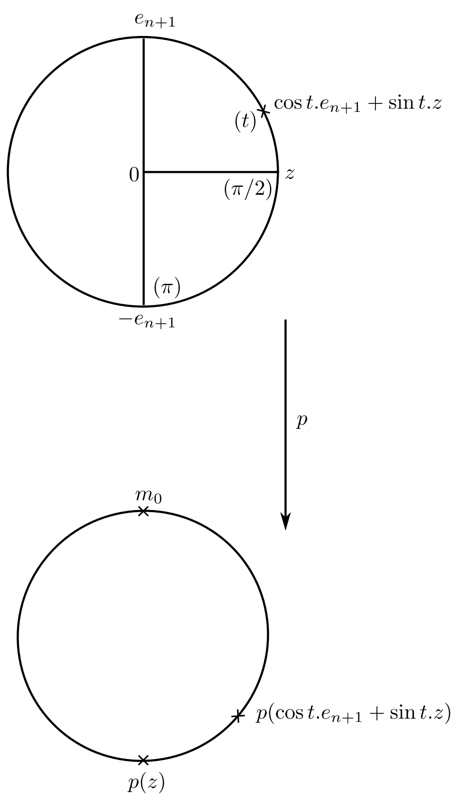
Proof. Because of the existence of a transitive family of isometries it is enough to examine the situation at one point, say m_0. By (4.1) (4.18) and (3.4.13) the geodesics through m_0 are $t \to \cos t.e_{d+1} + \sin t.z$ with $< z, z >= 1$ and $z \in \mathbb{R}^d$ for $(\mathbb{S}^d, \mathrm{can})$, and $t \to p(\cos t \cdot e_{n+1} + \sin t \cdot z)$ with $z \in K^n$, $< z, z >= 1$ for the $P^n(K)'s$ and in both cases they are parametrized by arc length. Hence the result for $(\mathbb{S}^d, \mathrm{can})$ is proved as also for $P^d(\mathbb{R})$. For a $P^n(K)$ if

$$p(\cos t.e_{n+1} + \sin t.z) = p(\cos t'.e_{n+1} + \sin t'.z)$$

then $\exists\, \lambda \in K$ with

$$\cos t.e_{n+1} + \sin t.z. = \lambda(\cos t'.e_{n+1} + \sin t'.z)$$

which implies that $\cos t = \lambda \cdot \cos t'$, $\sin t = \lambda \cdot \sin t'$ and hence $\lambda \in \mathbb{R}$
129 and $\lambda^2 = 1$; so $\lambda = \pm 1$ and all geodesics in $P^n(K)$ have π as the smallest period.



□

5.5

This result explains the terminology S.C.-manifolds (S.C. stands for “symmetric circled”).

5.6

Remark. In $(\mathbb{S}^d, \mathrm{can})$ two geodesics through $m \in \mathbb{S}^d$ meet again at the first time at the antipodal point $-m$ of $\mathbb{S}^d$. For the other S.C.-manifolds the situation will be described in the proposition below; in its statement in the case of $P^2(\Gamma)$ one should replace "z, z' are or not K-dependent" by the following: "z, z' belong or not to the same fibre of the canonical **130**
fibration $\mathbb{S}^{15} \to \mathbb{S}^8$".

5.7

Proposition. *Let (M, g) be an S.C.-manifold other than an $(\mathbb{S}^d, \mathrm{can})$; then two geodesics through m_0 with tangent vectors $p^T(r(z))$, $p^T(r(z'))$ at m_0 (for $z, z' \in K^n$, $< z, z >=< z', z' >= 1$ and $z' \neq \pm z$)*

i) never meet except at m_0 if z, z' are K-independent

ii) meet exactly at m_0 and at their mid-point at distance $\dfrac{\pi}{2}$ if z, z' are K-dependent.

Proof. By the argument in the proof of (5.4) our two geodesics meet at distances t, t' from m_0 if and only if $\exists\, \lambda \in K$ such that

$$\cos t . e_{n+1} + \sin t . z = \lambda (\cos t' \cdot e_{n+1} + \sin t' \cdot z')$$

hence

$$\cos t = \lambda \cdot \cos t' \quad \text{and} \quad \sin t . z = \lambda \cdot \sin t' \cdot z'.$$

We can suppose that $t, t' \in]0, \pi[$. Since $|\lambda| = 1$ either $\cos t = \cos t' = 0$ or $\lambda \in \mathbb{R}$. In the first case $t = t' = \dfrac{\pi}{2}$ and $z = \lambda \cdot z'$ so z, z' are K-dependent. In the second $\lambda \in \mathbb{R}$, $< z, z >=< z', z' >= 1$ and $\sin t \cdot z = \lambda \cdot \sin t' \cdot z$ would imply $z = \pm z'$, a contradiction. □

Remark. For $P^d(\mathbb{R})$ in fact $z \neq \pm z'$, implies z, z' are never $\mathbb{R}$-dependent so two distinct geodesics in $P^d(\mathbb{R})$ meet at most at one point as they should since they are projective lines.

5.12

Proposition. *For every point m of an S.C.-manifold (M, g) we have*

131 *1)* $\exp_m$ *is π-O.K. if (M, g) is an* $(\mathbb{S}^d, \mathrm{can})$

2) $\exp_m$ *is $\frac{\pi}{2}$-O.K. in the other cases.*

Proof. By (5.6) we see that for $M = \mathbb{S}^d$, $\exp_{m_0}$ is one-one on. $\underline{\mathrm{B}}(m_0, \pi)$ and if $M \neq \mathbb{S}^d$ by (5.7) that $\exp_{m_0}$ is one-one on $\underline{\mathrm{B}}(m_0, \frac{\pi}{2})$. Thus it suffices to show that $\exp_{m_0}$ is of maximal rank on $\underline{\mathrm{B}}(m_0, \pi)$ in the case of $\mathbb{S}^d$ and on $\underline{\mathrm{B}}(m_0, \frac{\pi}{2})$ in the other cases. We show, by example for the $P^n(K)$'s, that the expression we had for geodesics proves in particular that the map

$$h = \exp_{m_0} \circ p^T \circ r : K^n \to P^n(K)$$

is nothing but

(4.5.13) $\quad h : tz \to p(\cos t.e_{n+1} + \sin t.z) \quad$ with $\quad < z, z >= 1, t \in \mathbb{R}$.

We know that $h^T_{tz}(\zeta^{-1}_{tz} z) \neq 0$ by (6.32), and hence we study h^T_{tz} for z' with $< z', z' >= 1$ and orthogonal to z in the *euclidean* structure of K^n.

The curve $\alpha \to t(\cos\alpha \cdot z + \sin\alpha \cdot z')$ (whose speed for $s = 0$ is $\zeta^{-1}_{tz}(tz')$) has for image under h the curve

$$\alpha \to h(t(\cos\alpha \cdot z + \sin\alpha \cdot z') = p(\cos t \cdot e_{n+1} + \sin t(\cos\alpha \cdot z + \sin\alpha, z'))$$

whose speed for $z = 0$ is $p^T(\zeta^{-1}_{tz}(\sin t \cdot z'))$. Since $t \in]0, \frac{\pi}{2}[$ and $< z', z' >= 1$ one has $\sin t \cdot z'. \neq 0$. From $(p^T)^{-1}(0) \cap K^n = \{0\}$ one gets the proof. □

Remark. (5.12) would also follows from (7.1).

6 Results of Samelson and Bott

In the previous article we have seen that in an S.C.-manifold all geodesics are simply closed and of the same length. Now let us examine the converse. Given a connected riemannian manifold (M, g) such that for each point m of (M, g)

(i) each geodesic through m is simply closed; 132

(ii) the length of each geodesic through m is a number 1 independent of the geodesic; and

(iii) 1 is independent of m;

does it follow that (M, g) is isometric with an S.C.-manifold? (see article 8).

More generally we give the:

6.1

Definition. *Given an r.m.* (M, g) *and some* $m \in M$ *we say that* (M, g) *is a* C_m-manifold *if it is connected and if all geodesics through m are simply closed and of same length.*

Though there are no isometry relations between an arbitrary C_m-manifold and an S.C.-manifold we shall see that the cohomology ring of a C_m-manifold resembles that of an S.C.-manifold. Let us recall some definitions and results from algebraic topology.

Given a compact topological manifold M we denote its graded cohomology ring over a ring A by $H^*(M, A)$.

6.2

Definition. *The manifold M is said to be a* $TR(A)$-manifold *if* $H^*(M, A)$ *is a truncated polynomial ring.*

This means $H^*(M, A)$ is isomorphic to $A[X]/\mathfrak{A}$ where $\mathfrak{A}$ is an ideal generated by a positive power of X. The image under that isomorphism of X is a homogeneous element of $H^*(M, A)$; call its degree α. If $d = \dim M$, then define λ by $d = \alpha \cdot \lambda$. The following results are standard.

$\mathbb{S}^d$ is a $TR(\mathbb{Z})$ manifold with $\alpha = d$ and $\lambda = 1$

$P^n(\mathbb{C})$ is a $TR(\mathbb{Z})$ manifold with $\alpha = 2$ and $\lambda = n$

(4.6.3) $P^n(\mathbb{H})$ is a $TR(\mathbb{Z})$ manifold with $\alpha = 4$ and $\lambda = n$

$P^2(\Gamma)$ is a $TR(\mathbb{Z})$ manifold with $\alpha = 8$ and $\lambda = 2$

$P^d(\mathbb{R})$ is a $TR(\mathbb{Z}_2)$ manifold with $\alpha = 1$ and $\lambda = d$.

133 We shall prove a result due to Samelson to the effect that a C_m-manifold is a $TR(\mathbb{Z}_2)$-manifold.

6.4

Remark. In the definition of a C_m-manifold we have assumed that all the geodesics through m are of the same length. This property is not a consequence of the former namely that each geodesic is simply closed. To see this one can take either a suitable lens space or more precisely the following example due to I.M. Singer. A compact Lie group G is always a symmetric space, when associated to the symmetric pair $(G \times G, G)$ for the involutive automorphism $\sigma(g,h) = (h,g)$. For r.s. as in (3) the geodesics through e are the one-parameter subgroups (as follows from (4.1)). Take now $G = S0(3)$ and such an r.s. on it; if H is the subgroup generated by an element of order two then the quotient G/H is endowed with an r.s. by (3), which makes $G \to G/H$ a riemannian covering. Then the geodesics through $p(e)$ are all of same length (as in $S0(3)$) with one exception (the one which corresponds to that in $S0(3)$ which contains H) which is of length half that of the others.

134 However it is not known whether the assumption of equal length is, or is not, superfluous in the case of *simply connected* manifolds.

6.5

Proposition . *Let (M,g) be a C_m-manifold for which the (common) length of the geodesics through m equals 1. Then,*

1) $\Omega \supset T_m(M)$ *and* $M = \exp(\overline{\underline{B}(m, 1/2)})$ *in (particular M is compact)*

2) *if* $\dim M \geq 2$, *then:either M is simply connected, or the universal riemannian covering* $(\widetilde{M}, \widetilde{g}) \to (M,g)$ *is two-sheeted and* $(\widetilde{M}, \widetilde{g})$ *is a* $C_{\widetilde{m}}$*-manifold* $\forall \widetilde{m} \in p^{-1}(m)$*, with common length for the geodesics through* $\widetilde{m}$ *equal to 21.*

Proof. Because each geodesic is of length 1 we have

$$\underline{B}(m, \frac{1}{2}) \subset \Omega.$$

Since every geodesic, and hence $\exp_m$, is periodic with period 1 we have

$$T_m(M) \subset \Omega.$$

Since M is connected the Hopf-Rinow theorem (4.1) implies that $M = \exp(T_m(M))$. Now the periodicity of $\exp_m$ implies that

$$M = \exp_m(T_m(M)) = \exp_m(\overline{\underline{B}(m, 1/2)}).$$

Since $\overline{\underline{B}(m, 1/2)}$ is a closed ball and hence compact and $\exp_m$ is defferentiable and hence continuous, it follows that M is compact (for M is Hausdorff). This proves the first part. ◻

The second part of 2) follows from the remark after (2.7). To prove **135**
the first part of 2) we remark first that all the closed geodesics of period 1 in a C_m-manifold are homotopic as loops based at m (hereafter we use notions and notations of (6)) if $\dim M \geq 2$; in fact if γ_1, γ_2 are two such geodesics and $\lambda : [0, 1] \to U_m(M)$ a path connecting $\gamma_1'(0)$ to $\gamma_2'(0)$ in the unit tangent sphere $U_m(M)$ at m then the required homotopy is

$$(t, \alpha) \to \exp(t \cdot \lambda(\alpha))$$
$$\text{for} \quad \exp(0, \lambda(\alpha)) = \exp(1 \cdot \lambda(\alpha)) = m \forall \alpha \in [0, 1].$$

In particular the homotopy class in $\pi_1(M, m)$ of a closed geodesic γ of period 1 through m is independent of γ, say a. moreover γ^{-1} is again such a geodesic and belongs to a^{-1} hence $a = a^{-1}$ or $a^2 = 1$. Now we claim: any $b \in \pi_1(M, m) (b \neq 0)$ is a power of a (which yields the proposition). In fact let $b \in \pi_1(M, m)$, $\widetilde{m} \in p^{-1}(m)$ and $\widetilde{m}' \neq \widetilde{m}$. By (2.7) the covering (M, g) is complete hence by (4.10) there exists $\widetilde{\gamma} \in S(\widetilde{m}, \widetilde{m}')$. The projection $\gamma = p \circ \widetilde{\gamma}$ is a geodesic from m to m in (M, g) so by (5.2) has to be a multiple of a closed geodesic of period 1 through m; hence b is a power of a.

6.6

Theorem H. Samelson. *AC_m-manifold is a $TR(\mathbb{Z}_2)$ manifold.*

6.7

Lemma. *There exists a positive number r such that*

$$\underline{B}(m, \frac{1}{2}) \cap \exp_m^{-1}(B(m, r)) = \underline{B}(m, r).$$

136 *Proof.* Let r' be a positive number such that $\exp_m$ is r'-O.K. Then let us set

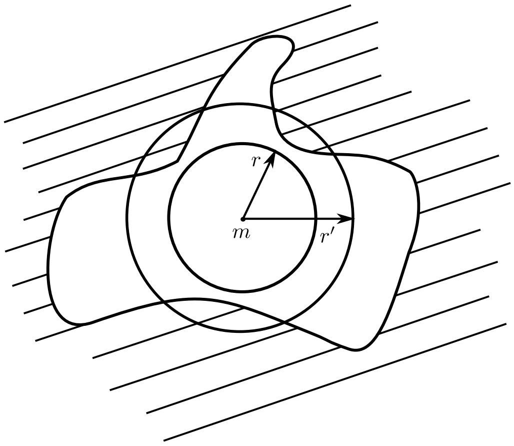
$$W = \exp_m \left\{ \overline{\underline{B}\left(m, \frac{1}{2}\right)} - \underline{B}(m, r') \right\}.$$

Note that $\underline{B}(m, \frac{1}{2}) - \underline{B}(m, r')$ is a closed subset of the compact ball $\underline{B}(m, \frac{1}{2})$ and hence compact. Therefore W is compact. Since the geodesics through m are *simply* closed it follows that $m \notin W$. Hence there is a positive number r such that

i) $0 < r < r'$ and

ii) $B(m, r) \cap W = \emptyset$.

Now let $n \in B(m, r)$ and let $n = \exp_m(x)$ where $||x|| < \frac{1}{2}$. Since $B(m, r) \cap W = \emptyset$ we see that $x \notin \underline{B}(m, \frac{1}{2}) - \underline{B}(m, r')$ and hence $x \in \underline{B}(m, r')$. But since $\exp_m$ is r'-O.K. and $r < r'$ we have $\exp_m$ is one-one on $\underline{B}(m, r)$ and since $n \in B(m, r)$ we have $x \in \underline{B}(m, r)$.



□

Proof of the proposition. By (5.4) we can multiply the r.s. on $(P^d(\mathbb{R})$, can) by a constant so as to make the common length of its geodesics equal to 1, the length of each closed geodesic of (M, g) through m. Let **137**
us fix a point n on $P^d(\mathbb{R})$ and a euclidean isomorphism

$$u : T_n(P^d(\mathbb{R})) \to T_m(M).$$

Now we *define a map* $f \in D(P^d(\mathbb{R}), M)$ by requiring the commutativity of the diagram:

$$\begin{array}{ccc} \overline{\underline{B}(n, \frac{1}{2})} & \xrightarrow{\;u\;} & T_m(M) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ P^d(\mathbb{R}) & \xrightarrow[\;f\;]{} & M \end{array}$$

By (5.12) $\exp_n$ is $\frac{1}{2}$-O.K. and hence the map

$$\exp_m \circ u \circ (\exp_n |\underline{B}(n, \frac{1}{2}))^{-1}$$

is well defined on $B(n, \frac{1}{2})$. On $B(n, \frac{1}{2})$ we define f to be this map. Now let $n' \in \exp(\underline{B}(n, \frac{1}{2}) - B(n, \frac{1}{2}))$. Then if $x \in -B(n, \frac{1}{2})$ and $\exp x = n'$, since the length of every geodesic is equal to 1 we have $\exp(-x) = n'$. By (5.7) no two geodesics through n meet outside n and hence it follows $\pm x$ are the only inverse images of n' in

$$\overline{\underline{B}(n, \frac{1}{2})} - B(n, \frac{1}{2}).$$

138 Since

$$\|u(x)\| = \|u(-x)\| = \frac{1}{2}$$

and since the common length of geodesics through m is 1, we conclude that

$$\exp(u(x)) = \exp(u(-x)).$$

Now, we extend f to $\underline{B}(n, \frac{1}{2})$ by setting

$$f(x) = \exp(u(x)).$$

By (4.5.13) f is a differentiable map. (We need only the fact that f is continuous). Now we claim that the topological degree mod 2 of f is one. First let us recall the process of getting the topological degree of a map.

First we can define the fundamental class mod 2 of the compact manifold M as follows.

We pick up a neighbourhood U of an arbitrary point m of M such that U is homeomorphic to an open ball. Then we consider the exact sequence

$$0 \to H_c^d(U, \mathbb{Z}_2) \xrightarrow{i} H^d(M, \mathbb{Z}_2) \to H^d(X - U, \mathbb{Z}_2) \to \cdots$$

([12]: p. 190. th. 4.10.1 and the following lines), in which $H_c^d(U, \mathbb{Z}_2)$ consists of exactly two elements 0 and γ_U. Then φ_M, the fundamental class mod 2 of M, is $i(\gamma_U)$. Then, , the degree mod 2 of a map

$$f : N \to M$$

from a compact manifold N into M is by definition such that

$$f^*(\varphi_M) = \delta \cdot \varphi_N.$$

In our case we define φ_M with the help of the ball $U = B(m, r)$ given by **139**
the lemma and for $N = P^d(\mathbb{R})$ and φ_N with $U' = B(n, r)$.

Let us note that the lemma together with the definition of f implies that

$$f^{-1}(U) = U'$$

and that $f|U'$ is a diffeomorphism. Then we have ([12]: 4.16., p. 199) the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_c^d(U) & \xrightarrow{\;i\;} & H_c^d(M) & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle f_c^*} & & \big\downarrow{\scriptstyle f^*} & & \\
0 & \longrightarrow & H_c^d(U') & \xrightarrow{\;i'\;} & H^d(P^d(\mathbb{R})) & \longrightarrow & \cdots
\end{array}$$

Since $f : U' \to U$ is a homeomorphism we have

$$f^*(\gamma_U) = \gamma_{U'}.$$

Since $\varphi_M = i(\gamma_U)$ and $\varphi_N = i'(\gamma_{U'})$ the commutativity of the diagram gives

$$\begin{aligned}(i' \circ f_c^*)(\gamma_U) = i'(f_c^*(\gamma_U)) = i'(\gamma_{U'}) = \varphi_N = (f^* \circ i)(\gamma_U) = \\ = f^*(\varphi_M) = \delta \cdot \varphi_M.\end{aligned}$$

Hence

$$\delta = 1.$$

Now we assert that

$$f^* : H^*(M, \mathbb{Z}_2) \to H(P^d(\mathbb{R}), \mathbb{Z}_2)$$

is injective. For Poincaré duality asserts that given any non-zero $e \in$ **140**
$H^*(M, \mathbb{Z}_2)$, $\exists\, e' \in H^*(M, \mathbb{Z}_2)$ such that

$$e \cup e' = \varphi_M.$$

Hence we have

$$f^*(e) \cup f^*(e') = f^*(e \cup e') = f^*(\varphi_M) = \varphi_N$$

and hence

$$f^*(e) \neq 0 \forall e \quad \text{i.e.} \quad f^* \quad \text{is injective.}$$

Hence $H^*(M, \mathbb{Z}_2)$ is isomorphic to a homogeneous sub-ring A of the truncated polynomial ring

$$H^*(P^d(\mathbb{R}), \mathbb{Z}_2) = \mathbb{Z}_2[X]/(X^{d+1}) \quad \text{(by (4.6.3)).}$$

Let $\overline{X}$ be the image of X in $H^*(P^d(\mathbb{R}), \mathbb{Z}_2)$ and $\overline{X}^\theta$ be the least positive power of $\overline{X}$ that is in A. Then $\exists\, e' \in H^*(M, \mathbb{Z}_2)$ such that $\overline{X}^\theta = f^*(e')$ and by Poincaré duality $\exists\, e'_1 \in H^*(M, \mathbb{Z}_2)$ such that $e' \cup e'_1 = \varphi_M$. Hence $f(e'_1) = \overline{X}^{d-\theta} \in A$. By the same token, for k with $d > k^\theta$ we have $\overline{X}^{k\theta} \in A$ and $\overline{X}^{d-k^\theta} \in A$. So the choice of θ implies $\exists\, k | d = k \cdot \theta$ and $H(M, \mathbb{Z}_2)$ is isomorphic to the truncated polynomial ring $\mathbb{Z}_2[X]/(X^{k+1})$.

6.9

Remark. In fact Samelson's result is sharper. But by using Morse theory Bott proved the following theorem:

6.10

Theorem [6]: p.375 *A simply connected C_m-manifold is a $TR(\mathbb{Z})$-manifold. The universal covering of an r.m. which is not simply connected is a homotopy sphere.*

141 We will not go into the proof of this theorem but be content with the remark that recent theorems in algebraic topology and (4.6.3) imply the following:

1) For a simply connected $TR(\mathbb{Z})$-manifold the cohomology ring $H^*(M, \mathbb{Z})$ is isomorphic to that of an S.C. manifold.

2) a simply connected C_m-manifold of dimension d odd and $d > 5$ is homeomorphic to $\mathbb{S}^d$.

3) a C_m-manifold, of dimension greater than or equal to 5 and which is not simply connected is homeomorphic to $P^d(\mathbb{R})$.

7 Expressions for G in local coordinates

Given (M, g) let us express G, locally, in terms of g. Let us follow the conventions of (4).

On $T(U)$ let us set

(4.7.1) $$G^{\bullet} = (g^{\sharp})^T \circ G \circ g^{\flat} \quad \text{and compute } G^{\bullet}.$$

Now for

$$\omega = (x^1, \ldots, x^d; p_1, \ldots, p_d) \in T^*(U)$$

let

(4.7.2) $$G^{\bullet}(\omega) = (x^1, \ldots, x^d; p_1, \ldots, p_d; v^1, \ldots, v^d; w_1, \ldots, w_d).$$

Let $\{Y_1, \ldots, Y_d; P^1, \ldots, P^d\}$ be the basis of $\mathscr{C}(T^*(U))$ dual to the basis $dx^1, \ldots, dx^d; dp_1, \ldots, dp_d$ of $\mathscr{C}^*(T^*(U))$.

Now set

(4.7.3) $$H = \frac{1}{2} E \circ g^{\flat}.$$

Then by the definition of G we have

$$i(G^{\bullet})(d\mu) = -dH.$$

Hence **142**

(4.7.4) $$d\mu(G^{\bullet}, Y_i) = -\frac{\partial H}{\partial x^i}$$

(4.7.5) $$d\mu(G, P^i) = -\frac{\partial H}{\partial p_i}$$

and hence by (0.4.26) we have

$$v^i = \frac{\partial H}{\partial p_i} \tag{4.7.6}$$

$$w_i = \frac{\partial H}{\partial x^i} \tag{4.7.7}$$

Now we compute G on U. By the definition of $g^\sharp$ we have

$$g^\sharp(x^1, \ldots, x^d; p^1, \ldots, p^d) = (x^1, \ldots, x^d; p_1, \ldots, p_d) \tag{4.7.8}$$

where

$$p_i = \sum_j g_{ij} p^j \quad \text{(see (8))}.$$

Now let

$$(g^\sharp)^T(x^1, \ldots, x^d; p^1, \ldots, p^d; v^1, \ldots, v^d; w^1, \ldots, w^d) = \tag{4.7.9}$$
$$= (x^1, \ldots, x^d; p_1, \ldots, p_d; v^1, \ldots, v^d; w_1, \ldots, w_d).$$

Then a straightforward calculation of the Jacobian $Dg^\sharp$ gives that

$$w_1 = \sum_{i,k} \frac{\partial g_{ij}}{\partial x^k} \cdot v^k \cdot p^j + \sum_j g_{ij} w^j. \tag{4.7.10}$$

Now let

$$G(x^1, \ldots, x^d; p^1, \ldots, p^d) = (x^1, \ldots, x^d; p^1, \ldots, p^d; p^1, \ldots, p^d; G^1, \ldots, G^d).$$

With the notation of ((1.3.2)) we have

$$G^i = -\sum_{j,k} \Gamma^i_{jk} p^j p^k. \tag{4.7.11}$$

143 Hence we have to compute Γ^i_{jk}. We have

$$G(x^1, \ldots, x^d; p_1, \ldots, p_d) = (x^1, \ldots, x^d; p_1, \ldots, p_d; p^1, \ldots, p^d; G_1, \ldots, G_d).$$

From (3.8.3) it follows that

$$H = \frac{1}{2} \sum_{j,k} g^{jk} p_j p_k \tag{4.7.12}$$

and hence

$$\frac{\partial H}{\partial x^i} = \frac{1}{2}\sum_{j,k}\frac{\partial g^{jk}}{\partial x^i}p_j p_k. \tag{4.7.13}$$

By (3.8.3) we have

$$\sum_l g^{il}g_{lj} = \delta_{ij}$$

and hence for every i

$$\sum_{j,k}\frac{\partial g^{jk}}{\partial x^i}p_j p_k = -\sum_{j,k}\frac{\partial g^{jk}}{\partial x^i}p^j p^k. \tag{4.7.14}$$

Hence by (4.7.7), (4.7.10) and (4.7.14) we have

$$\sum_{j,k}\frac{\partial g_{ij}}{\partial x^k}p^k p^j + \sum_j g_{ij}G^j = \frac{1}{2}\sum_{j,k}\frac{\partial g_{jk}}{\partial x^i}p^j p^k, \tag{4.7.15}$$

or

$$\sum_j g_{ij}G^i + \sum_{j,k}\frac{1}{2}\left(2\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^i}\right)p^j p^k = 0. \tag{4.7.16}$$

Now if we set **144**

$$\Gamma_{jik} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) \tag{4.7.17}$$

then we have

$$\sum_j g_{ij}G^j + \sum_{j,k}\Gamma_{jik}p^j p^k = 0. \tag{4.7.18}$$

Hence by the definition of g^{ij} we have

$$\Gamma^i_{jk} = \sum_l g^{il}\Gamma_{jlk} \tag{4.7.19}$$

where the Γ_{jlk} are given by (4.7.17).

Remark. (4.7.6) and (4.7.7) are nothing but Hamilton's equations.

7.1 Lagrange's equation

In local coordinates (4.2.1) also yields Lagrange's equations. On $T(U)$ we introduce the local vector fields

$$\left\{X_i = \frac{\partial}{\partial x^i}\right\}, \left\{P_i = \frac{\partial}{\partial p^i}\right\} \text{ and have :}$$
$$p^T([X_i, G]) = 0, \alpha([X_i, G]) = 0, p^T(X_i) = \xi(p_i)$$

So by (1.8)

$$\alpha(X_i) = \frac{1}{2}\frac{\partial E}{\partial p^i}.$$

And

$$d\alpha(G, X_i) = G(\alpha(X_i)) - X_i(\alpha(G)) - \alpha([G, X_i]) = \frac{1}{2}G\left(\frac{\partial E}{\partial p^i}\right) - X_i(E) =$$
$$= -\frac{1}{2}dE(X_i)$$

145 so:

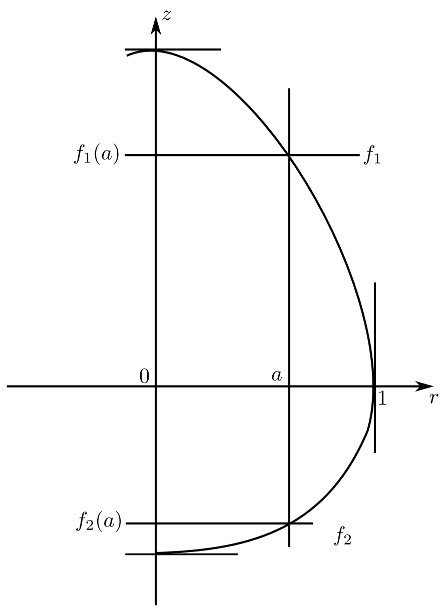
$$G\left(\frac{\partial E}{\partial p^i}\right) = \frac{\partial E}{\partial x^i}.$$

Working along a geodesic parametrized by t implies $G = \dfrac{d}{\mathrm{dt}}$ hence the Lagrange equations:

$$\frac{d}{\mathrm{dt}}\left(\frac{\partial E}{\partial p^i}\right) = \frac{\partial E}{\partial x^i}.$$

8 Zoll's surface

In this article we answer negatively the question which was posed at the beginning of article 6 by giving a counter example due to Zoll of a r.s. on $\mathbb{S}^2$ for which $\mathbb{S}^2$ is a C_m-manifold $\forall m \in \mathbb{S}^{-2}$ but not isometric to $(\mathbb{S}^2, \mathrm{can})$.



To construct the surface in the (x, y, z)-space we take in the (z, r)-plane two curves $r \to f_i(r)$ $(i = 1, 2)$ with $f_1,\ f_2 \in D([0, 1[, \mathbb{R}^+)$ and $f_i(1) = 0$, 146
$f_i'(0) = 0,\ f_i'' \leq 0$ and $f_i'(r) \to \underset{1}{\infty}(i = 1, 2)$.

We generate a surface of revolution M by setting:

$$M_1 = \left\{(x, y, z) | z = f_1(\sqrt{x^2 + y^2})\right\}, M_2 = \left\{(x, y, z) | z = -f_2(\sqrt{x^2 + y^2})\right\}$$

hence $M = M_1 \cup M_2$ is a C^1-manifold. We have two charts $(U_i, s_i)(i = 1, 2)$ on $U_i = M_i - (z^{-1}(0) \cup (x^{-1}(0) \cap y^{-1}(0)))$ defined as follows:

$$\text{if} \quad p(x, y, z) = (x, y) \ \text{ we set}$$
$$s_i(m) = (\|p(m)\|,\ \text{angle}\ (e_1, p(m))) \in \mathbb{R}^2 (i = 1, 2)$$

(polar coordinates). On U_i the local coordinates will be denoted by $x^1 = r = u^1 \circ s_i$, $x^2 = \varphi_s = u^2 \circ s_i$ $(i = 1, 2)$ so that

$$s_1^{-1}(u^1, u^2) = (u^1 \cdot \cos u^2, u^1 \cdot \sin u^2,\ f_i(u^1)). \tag{4.8.1}$$

We drop the i in s_i and in f_i. We have the vector fields

$$\{X_1, X_2\} \text{ dual basis of } \{dr, d\varphi\} \text{ on } \mathscr{C}(U_i),$$
$$\{Y_1, Y_2\} \text{ dual basis of } \{du^1, du^2\} \text{ on } \mathscr{C}(\mathscr{R}^2).$$

We endow M with the r.s. induced by $(\mathbb{R}^3, \text{can})$ as in (2.1). For $g = \sum_{i,j} g_{ij} dx^i dx^j$ we have by definition of ϵ (see (1.2)):

$$g_{ij} = g(X_i, X_j) = (\zeta \circ (s^{-1})^{T \cdot} \circ Y_i) \cdot (\zeta \circ (s^{-1})^T \circ Y_j) = (Ds^{-1} \circ Y_i) \cdot (Ds^{-1} \circ Y_j)$$

where Ds^{-1} is the $(\text{Jacobian})^{-1}$ of s, i.e. by (4.8.1):

$$Ds^{-1} = \begin{pmatrix} \cos u^2 & -u^1 \cdot \sin u^2 \\ \sin u^2 & u^1 \cdot \cos u^2 \\ f'(u^1) & 0 \end{pmatrix}$$

147 hence, setting $f' = \dfrac{df}{dr}$:

(4.8.2) $\quad g = (1 + f'^2) dr^2 + r^2 \cdot d\varphi^2, g_{12} = 1 + f'^2, g_{12} = 0, g_{22} = r^2$

hence by (3.8.3)

$$g^{11} = \frac{1}{1 + f'^2}, g^{12} = 0, g^{22} = \frac{1}{r^2};$$

thus by (4.7.17) and (4.7.19):

$$\Gamma_{122} = r, \Gamma_{121} = \Gamma_{222} = 0, \Gamma^2_{12} = \frac{1}{r}, \Gamma^2_{11} = \Gamma^2_{22} = 0.$$

Hence if

$$\psi : t \to (r(t), \varphi(t))$$

is a geodesic, then by (1.3.5) we have

$$\frac{d^2\varphi}{\mathrm{dt}^2} + \frac{2}{r} \cdot \frac{dr}{\mathrm{dt}} \cdot \frac{d\varphi}{\mathrm{dt}} = 0$$

i.e.

$$\frac{d}{\mathrm{dt}}\left(r^2 \frac{d\varphi}{\mathrm{dt}}\right) = 0.$$

Thus, along a given geodesic, there exists an a such that

$$r^2 \frac{d\varphi}{\mathrm{dt}} = a. \tag{4.8.3}$$

We suppose moreover that ψ is parametrized by arc length, so that by (4.3.3) and (4.8.2):

$$\left(1 + f'^2\right)\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\varphi}{ds}\right)^2 = 1$$

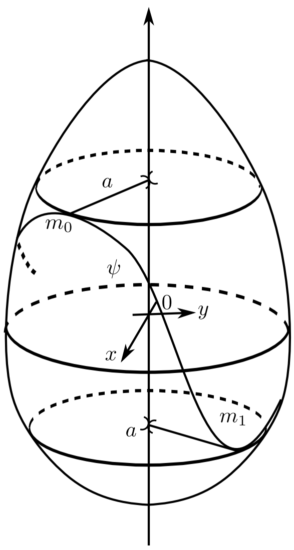
and hence 148

$$\left|\frac{d\varphi}{ds}\right| \le \frac{1}{r}.$$

Now by (4.8.3) we have

$$a \le \frac{1}{r(s)} \cdot r^2(s) = r(s),$$

and hence the geodesic never crosses the curve given by the section of M by the planes $z = f_i(a)$ $(i = 1, 2)$. Moreover:



(4.8.4) $$\frac{d\varphi}{dr} = \frac{a}{r}\left(\frac{1+f'^2}{r^2-a^2}\right)^{1/2}.$$

8.5

149 Let us note that (by continuity arguments, for our computations are not valid outside $U_1 \cup U_2$) that the geodesics corresponding to $a = 0$ are the meridians and that to $a = 1$ corresponds the equator. So to ensure that M is a C_m-manifold $\forall m \in M$ we have only to take care of the case $0 < a < 1$. Let then $m_0 = (x_0, y_0, f_1(a))$ and $m_1 = x_1, y_1, -f_2(a))$ be extreme points of ψ such that the part of ψ between them does not meet again the planes $z = f_1(a)$, $z = -f_2(a)$. Then - thanks to the symmetry of M-showing that ψ is simply closed amounts to showing that the variation $\Phi(a)$ of the angle φ from m_0 to m_1 is equal exactly to π. Since the integral

$$\int_a^1 \frac{a}{r}\left(\frac{1+f'^2}{r^2-a^2}\right)^{1/2} \cdot dr$$

exists we have by (4.8.4) applied for the part of ψ in U_1 and for the part in U_2:

(4.8.6) $$\Phi(a) = \int_a^1 \frac{a}{r} \cdot \frac{(1+f_1')^{1/2} + (1+f'^2_2)^{1/2}}{(r^2-a^2)^{1/2}} \cdot dr.$$

Then we have shown:

8.7

Theorem Darboux *The surface* $(M, \epsilon|M)$ *is a* C_m*-manifold* $\forall m \in M$ *if and only if* $\Phi(a) = \pi \forall a \in]0, 1[$, *with* $\Phi(a)$ *as in* (4.8.6).

8.8

Theorem Zoll: [30]). *There exists* $f_1,\ f_2 \in D([0,1[,\mathbb{R}^+)$ *such that:*

150 i) *M is a real-analytic sub manifold of* $\mathbb{R}^3$

ii) *M is a* C_m*-manifold* $\forall m \in M$

ii) *M is not isometric to* $(\mathbb{S}^2, \text{can})$.

Proof. A. To find $f_1,\ f_2$ we derive relations from $\Phi(a) = \pi \cdot a$. Set $g = (1+f'^2_1)^{1/2} + (1+f'^2_2)^{1/2}$, $\dfrac{1}{r} = x+1$, $\dfrac{1}{a^2} = \alpha+1$, $x = \alpha \cdot u$, $r \cdot g(r) = 2 \cdot \theta(x)$; then we must have

$$\int\limits_0^\alpha \frac{\theta(x)}{(\alpha - x)^{1/2}} \cdot dx = \pi\, \forall \alpha$$

and thus

$$\int\limits_0^1 \frac{\theta(\alpha \cdot u)}{(1-u)^{1/2}} \sqrt{\alpha} \cdot du = \pi\, \forall \alpha.$$

Differentiating the above equation with respect to α we have

(4.8.9) $$\theta(\alpha u) + 2\alpha \cdot u\theta'(\alpha u) = 0.$$

But the function

$$\theta(x) = \frac{k}{\sqrt{x}}$$

satisfies the above equation, and since it is known that

$$\int\limits_0^1 \frac{dx}{(x(1-x))^{1/2}} = \pi$$

it follows that the function $\dfrac{1}{\sqrt{x}}$ can be taken for $\theta(x)$. Hence we should try to find f_1 and f_2 such that:

$$(1+f'^2_1)^{1/2} + (1+f'^2) = \frac{2}{\sqrt{(1-r^2)}}.$$

(Note that the choice $f_1(r) = f_2(r) = (1 - r^2)^{1/2}$ makes the associated
151 surface a sphere). We set:

$$(1 + f'^2_1)^{1/2} = (1 - r^2)^{1/2} + \lambda(r)$$

and

$$(1 + f'^2_2)^{1/2} = (1 - r^2)^{1/2} - \lambda(r)$$

where in order that the equations have meaning we should have

$$\frac{1}{1 - r^2} \pm \lambda(r) \geq 1.$$

There exist such functions λ, for example the binomial expansion of $(1 - r^2)^{1/2}$ gives that the function

$$k \cdot r^2 \quad \text{for} \quad 0 < k < \frac{1}{2}$$

is such a function. Hence we set:

$$(1 + f'^2_1)^{1/2} = (1 - r^2)^{-1/2} + kr^2$$
$$(1 + f'^2_2)^{1/2} = (1 - r^2)^{-1/2} - kr^2,$$

and we know by (8.7) that M is a C_m-manifold $\forall m \in M$.

B. We check (i) now. Analyticity has to be checked only at $r = 1$ and we wish to express r as a function of z (instead of $z = f_1(r)$, $z = -f_2(r)$). We have (since $f'_1 < 0$, $f'_2 < 0$) and for $z > 0$:

$$\frac{dz}{dr} = f'_1 = \frac{-r}{\sqrt{(1 - r^2)}}(1 + 2k(1 - r^2)^{1/2} + k^2r^2(1 - r^2))$$

so if we set $s = (1 - r^2)^{1/2}$:

$$\frac{ds}{dz} = \frac{1}{1 + 2ks + k^2s^2(1 - s^2)} = b(s)$$

152 with $b(s)$ analytic in s. By Cauchy's theorem $\frac{ds}{dz} = b(s)$ has a unique analytic solution with $s(0) = 0$.

For $z < 0$ and $t = -(1 - r^2)^{1/2}$:

$$-\frac{dz}{dr} = f_2' = -\frac{r}{\sqrt{(1 - r^2)}}(1 - 2k\sqrt{(1 - r^2)} + k^2 r^2 (1 - r^2))$$

hence

$$\frac{\mathrm{dt}}{dz} = \frac{1}{1 + 2kt + k^2 t^2 (1 - t^2)} = b(t)$$

with the *same* b. So the unique analytic solution is the same for $z > 0$ and for $z < 0$.

C. We shall prove that $(M, \epsilon|M)$ is not isometric to $(\mathbb{S}^2, \mathrm{can})$. One can either compute the curvature of $(M, \epsilon|M)$ at a simple point or argue as follows: in $(\mathbb{S}^2, \mathrm{can})$ the meridians are the geodesics. The geodesics through two antipodal points m, m' have a closed geodesic as orthogonal trajectory, and this geodesic is the locus equidistant from m, m'. The same should hold in a surface isometric to $(\mathbb{S}^2, \mathrm{can})$. But for Zoll's surface and the two antipodal points $m = (0, 0, f_1(0))$, $m' = (0, 0, -f_2(0))$ the only orthogonal trajectory is the equator, which is not equidistant from m, m' since the lengths of the meridians from m to the equator and from the equator to m' are respectively:

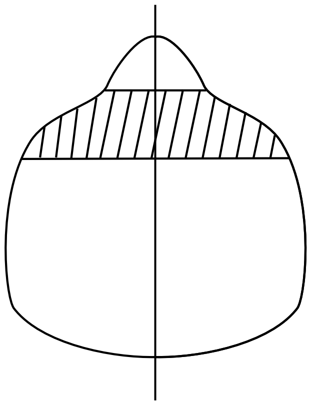
$$\int_0^1 (1 + f'^2_1)^{1/2} dr = \int_0^1 (1 - r^2)^{-1/2} dr + \int_0^1 kr^2 = \overline{2} + \frac{k}{3}$$

$$\int_0^1 (1 + f'^2_2)^{1/2} dr = \int_0^1 (1 - r^2)^{-1/2} dr - \int_0^1 kr^2 = \overline{2} - \frac{k}{3}.$$

□ **153**

8.10

Remarks. A. Check with the choice $\lambda(r) = \frac{3}{2} r^4$ that the meridians have inflexions



So there exist surfaces $(M, \epsilon|M)$ which are C_m-manifolds $\forall m \in M$ and which contain points with strictly *negative* curvature (Zoll: [30]).

B. We will see ((9)) the striking fact that if (M, g) is a C_m-manifold $\forall m \in M$ for M *homeomorphic* to $P^2(\mathbb{R})$ then (M, g) is *isometric* to $(P^2(\mathbb{R}), \text{can})$.

C. Note (4.8.3) is classical for motions with central acceleration (as it is the case for geodesics of a surface of revolution since the normal meets the z-axis). Note also the condition

$$\int_0^\alpha \frac{\theta(x)}{(\alpha - x)^{1/2}} \cdot dx = \pi \; \forall \alpha$$

154 is equivalent to the search of tautochronous motions on a vertical curve and yields the cycloidal pendulum.

Chapter 5

Canonical connection

1.1

Definition. *By (2.2) an r.m.* (M, g) *has a canonical spray* G. *BY (2.3)* **155**
there is a unique symmetric connection associated to G. *Whenever we consider* (M, g) *we consider it with the above connection and refer to* C *as the* connection *or* canonical connection on (M, g). *Hence given an r.m.* (M, g) *we can speak canonically of the concepts associated with a connection, namely, the derivation law, the curvature tensor, the parallel transport and Jacobi fields in* (M, g).

1.2

Example. Now let us examine the connection on $(\mathbb{R}^d, \epsilon)$. By (2.1.21) it follows that the spray G' associated to the canonical connection satisfies the equation

$$\zeta^T \circ G' = 0. \tag{5.1.3}$$

By (4.2.9) the canonical spray G on $(\mathbb{R}^d, \epsilon)$ satisfies the equation

$$\zeta^T \circ G = 0. \tag{5.1.4}$$

We have the direct sum $T_z(T(M)) = (\zeta^T)_z^{-1}(0) + (p^T)_z^{-1}(0)$ and since $p^T(G(z)) = p^T(G'(z)) = p'(z)$ we have $G = G'$. Hence by the uniqueness of the associated symmetric connection we see the canonical connection on $\mathbb{R}^d$ is the same as the connection on $(\mathbb{R}^d, \epsilon)$. Now we shall prove that

the connection is carried into the connection by isometries. We know by
156 (2.4) that the canonical spray is carried into the canonical spray and we have only to use ((2.2.5)).

1.5

Proposition. *If* (M, g) *and* (N, h) *are r.m.'s and*

$$f : (M, g) \to (N, h)$$

is an isometry, then

$$(f^T)^T({}^M C(x, y)) = {}^N C(f^T(x), f^T(y))\ \forall (x, y) \in T(M) \underset{M}{\times} T(M).$$

Proof. By (4.17) it is enough to show that both sides of the equality have the same effect on $d\varphi \in F(T(N))$ for every $\varphi \in F(N)$. We have

$$\begin{aligned}
&f^{T^T}({}^M C(x, y))(d\varphi) = {}^M C(x, y)(d\varphi \circ f^T) = {}^M C(x, y)(d(\varphi \circ f)) = \\
&\quad = \frac{1}{2}({}^M G(x + y)(d(\varphi \circ f)) - {}^M G(x)(d(\varphi \circ f)) - {}^M G(y)(d(\varphi \circ f))) = \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by } (2.2.5) \\
&\quad = \frac{1}{2}({}^M G(x + y)(d\varphi \circ f^T) - {}^M G(x)(d\varphi \circ f^T) - {}^M G(d\varphi \circ f^T)) = \\
&\quad = \frac{1}{2}(f^T)^T({}^M G(x + y))(d\varphi) - (f^T)^T({}^M G(x))(d\varphi) \\
&\qquad - (f^T)^T({}^M G(y))(d\varphi)) = \\
&\quad = \frac{1}{2}({}^N G(f^T(x + y))(d\varphi) - {}^N G(f^T(x))(d\varphi) - {}^N G(f^T(y))(d\varphi)) \\
&\quad = {}^N C(f^T(x), f^T(y))(d\varphi) \ \text{ by } ((2.2.5)) \text{ and since} \\
&\qquad f^T(x + y) = f^T(x) + f^T(y).
\end{aligned}$$

□

1.6

157 **Corollary.** *If* $f : (M, g) \to (N, h)$ *is an isometry, then*

i) ${}^{N}v \circ (f^T)^T = f^T \circ {}^{M}v$

ii) *if S is a manifold, $X \in \mathscr{C}(S)$ and $\varphi \in D(S, T(M))$, then* $N_{D_X}(f^T \circ \varphi) = f^T \circ D_X$

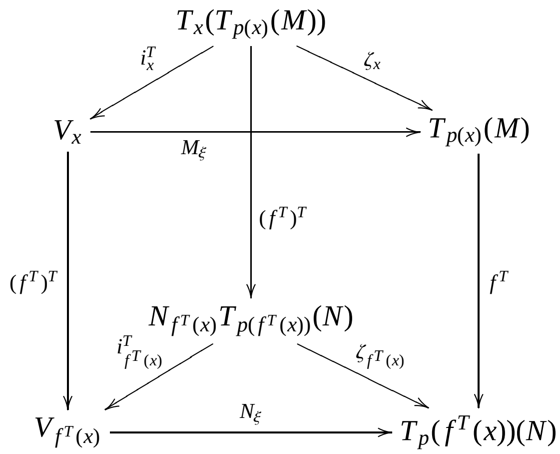
iii) *For every $X_1, Y_1 \in \mathscr{C}(M)$*

$$^{N}D_{f^T \circ X_1 \circ f^{-1}}(f^T \circ Y_1 \circ f^{-1}) = f^T \circ ({}^{M}D_{X_1}Y_1) \circ f^{-1}.$$

Proof. (i) For every Z in $\mathscr{C}(T(M))$ we have

$$^{M}v \circ Z = {}^{M}\xi(Z - C(p' \circ Z, p^T \circ Z)).$$

But from the definition of ξ it follows that the following diagram is commutative:



Hence 158

$$f^T \circ {}^{M}\zeta = {}^{N}\xi \circ (f^T)^T.$$

Hence

$$f^T \circ {}^{M}v \circ Z = {}^{N}\zeta \circ (f^T)^T \circ (Z - C(p' \circ Z, p^T \circ Z)) =$$
$$= {}^{N}\zeta((f^T)^T(Z) - (f^T)^T(C(p' \circ Z, p^T \circ Z))) =$$
$$= {}^{N}\zeta \circ ((f^T)^T \circ Z - C(f^T \circ p' \circ Z, f^T \circ p^T \circ Z)) \text{ by (1.5) } =$$
$$= {}^{N}\zeta((f^T)^T \circ Z - C(p' \circ (f^T)^T \circ Z, p^T \circ (f^T)^T \circ Z)) =$$
$$= {}^{N} v \circ (f^T)^T \circ Z.$$

(ii) We have

$$\begin{aligned} &{}^N D_X(f^T \circ \varphi) = {}^N v \circ (f^T \circ \varphi)^T \circ X \text{ by definition, } = \\ &= {}^N v \circ (f^T)^T \circ \varphi^T \circ X = \\ &= f^T \circ {}^M v \circ \varphi^T \circ X \text{ by (i)} = \\ &f^T \circ D_X \varphi \text{ by the definition of } D_X. \end{aligned}$$

(iii) We have only to apply (ii) and (3.14)

□

1.7 An application.

Let $f : (M, g) \to (N, h)$ be an isometry and let φ be a curve in (M, g) and ψ a parallel lift of φ. Then ${}^M D_P \psi = 0$. Hence by (ii) it follows that $f^T \circ \psi$ is a parallel lift of $f \circ \varphi$ in (N, h).

1.8

Example. Let (G, H) be a symmetric pair and (M, γ) be an associated
159 riemannian homogeneous manifold, as in (3). Let $X \in \underline{\mathrm{M}}$ and let

$$\varphi : t \to p(\exp(t.X))$$

be the associated geodesic and ψ a parallel lift of φ. Then we *claim :*

$$\psi(t) = (\tau(\exp(t.X)))^T (\psi(0))$$

i.e. the parallel transport is given by the tangent maps of the one parameter family of diffeomorphisms induced by X.

Proof. We know that for every t_0, $\tau(\exp(t_0 \cdot X))$ takes the image of φ into itself and is an isometry. Hence $(\tau(\exp(t_0 \cdot X)))^T$ takes ψ again into a parallel lift thanks to (1.7). But we do not know if it coincides with ψ. On the other hand we know that the map $\widehat{\sigma}_{\varphi(t_0)}$ is an isometry and hence preserves parallel lifts, maps the image of φ into itself and further $(\widehat{\sigma}_{\varphi(t_0)})^T$ is an isometry and acts as the negative of identity on $T_{(t_0)}(M)$. Hence $\psi(t)$ goes into $-\psi(-t)$. Therefore we try to represent $\tau(\exp(tX))$ as the composition of an even number of symmetries. □

Clearly $(\widehat{\sigma}_{\varphi(t_0)})^T \circ \psi$ is a parallel lift of $\widehat{\sigma}_{\varphi(t_0)} \circ \varphi$. But by (3.20) we have

$$(\widehat{\sigma}_{\varphi(t_0)} \circ f)(t) = f(2t_0 - t),$$

and hence with the notation of (5)

$$\widehat{\sigma}_{\varphi(t_0)} \circ f = f \circ \tau_{2t_0} \circ k_{-1}.$$

Now consider **160**

$$\psi_{t_0} = -\left(\widehat{\sigma}_{\varphi(t_0)}\right)^T \circ \psi \circ \tau_{2t_0} \circ k_{-1}.$$

Since $(\tau_{2t_0} \circ k_{-1})^2 = \mathrm{id}_{\mathbb{R}}$, it follows that ψ_{t_0} is a parallel lift of ψ and at t_0 is equal to

$$-\widehat{\sigma}_{\varphi(t_0)} \circ \psi(t_0) = \psi(t_0).$$

Hence by the uniqueness of parallel lifts (7.4) we have

$$\psi = \psi_{t_0}.$$

Now if we set $t_0 = 0$ we obtain

$$\psi = -\widehat{\sigma}^T \circ \psi \circ k_{-1}$$

and hence

$$\psi = -\widehat{\sigma}_{\varphi(t_0)} \circ (-\widehat{\sigma}^T \circ \psi \circ k_{-1}) \circ \tau_{2t_0} \circ k_{-1}.$$

But

$$\begin{aligned} \widehat{\sigma}_{\varphi(t_0)} \circ \widehat{\sigma} &= \tau(\exp(2t_0)X)) \text{ and hence} \\ \psi &= \tau(\exp 2t_0X) \circ \psi \circ k_{-1} \circ \tau_{2t_0} \circ k_{-1} \\ &= \tau(\exp 2t_0X) \circ \psi \circ \tau_{-2t_0} \text{ since} \\ &(\tau_{2t_0} \circ k_{-1})^2 = \mathrm{id}_{\mathbb{R}}\,. \end{aligned}$$

Hence in particular we have

$$\psi(2t_0) = (\exp(2t_0 \cdot X)))^T \psi(0).$$

2 Riemannian structures on $T(M)$ and $U(M)$

161

Let (M, g) be an r.m.

For every element x of $T(M)$ the tangent space at x is the direct sum of H_x and V_x:

$$T_x(T(M)) = H_x + V_x.$$

The map ξ defines an isomorphism between V_x and $T_{p(x)}(M)$ and so does the restriction of p^T to H_x between H_x and $T_{p(x)}(M)$:

$$\xi : V_x \to T_{p(x)}(M)$$
$$p^T|H_x : H_x \to T_{p(x)}(M).$$

By means of these isomorphisms and the euclidean structure on $T_{p(x)}(M)$ we define a euclidean structure $\overline{g}$ on $T_x(T(M))$. Precisely:

2.1

Definition. The canonical *r.s.* on $T(M)$, $\overline{g}$, *is defined by the equation*

$$\overline{g}(z, z') = g(v(z), v(z')) + g(p^T(z), p^T(z')) \; \forall z, z' \in T_x(T(M))$$
$$\forall x \in T(M).$$

2.2

Lemma. *With respect to this structure $\overline{g}$ for the function E on $T(M)$ we have (see 6.4):*

$$\operatorname{grad}(E) = \Xi \in \mathscr{C}(T(M))$$

where

$$\Xi(x) = 2\xi_x^{-1} x, \; x \in T(M).$$

162 *Proof.* Let Z be any vector field on $T(M)$. Then

$$Z(E) = (dE)(Z) = \overline{g}^{\sharp}(\operatorname{grad}(E))(Z) = \overline{g}(\operatorname{grad}(E), Z). \tag{5.2.3}$$

Let z be a horizontal vector. Then we have for $y = p^T(z)$:

$$z = C(x, p^T(z))$$

where C is the canonical connection on (M, g). But by ((2.2.5)) and (3.1) we have

$$z(E) = \frac{1}{2}(G(x+y)(E) - G(x)(E) - G(y)(E)).$$

Hence for every horizontal vector field Z we have

$$\overline{g}(\text{grad}(E), Z) = 0.$$

Hence $\text{grad}(E)$ is a vertical vector field. Let $x \in T(M)$ and let $z \in V_x$. Then, by (1.8) $z(E) = 2g(x, \xi(z))$, and by (2.1), since z and $\text{grad}\, E$ are vertical, we have:

$$\overline{g}(\text{grad}(E)_x, z) = g(\xi(\text{grad}(E)), \xi(z)).$$

Hence by (5.2.3) we have

$$g(2x - \xi(\text{grad}(E)_x), \xi(z)) = 0$$

for every vertical vector z. Since ξ is an isomorphism between V_x and $T_{p(x)}(M)$ and g is non-degenerate we have

$$2x - \xi(\text{grad}(E)_x) = 0.$$

Hence:

$$\text{grad}(E)_x = \xi_x^{-1}(2x) = 2\xi_x^{-1}(x).$$

Since $U(M)$ is a sub manifold of $T(M)$ there is an induced r.s. on $U(M)$. **163**

□

2.4

Definition. *The canonical* r.s. $\overline{\overline{g}}$ *on* $U(M)$ *is the structure* $\overline{g} \,|\, U(M) = \overline{\overline{g}}$.

2.5

Proposition. *The volume elements $\overline{\theta}$ and $\overline{\overline{\theta}}$ of $T(M)$ and $U(M)$ are respectively given by the equations*

$$\overline{\theta} = |\overline{\sigma}|,\ \overline{\overline{\theta}} = |\overline{\overline{\sigma}}|$$

where

$$\overline{\sigma} = (-1)^{\frac{d(d-1)}{2}} \cdot \frac{1}{d!} \cdot \overset{d}{\wedge}(d\alpha)$$

and

$$\overline{\overline{\sigma}} = (-1)^{\frac{d(d-1)}{2}} \cdot \frac{1}{(d-1)!} \cdot \alpha \wedge (\overset{d-1}{\wedge}(d\alpha)).$$

2.6

Let us note, incidentally, that this implies that $T(M)$ and $U(M)$ are oriented canonically.

Proof. In view of (5.1) and (3.4) it is enough to show that $\overline{\theta}$ and $\overline{\overline{\theta}}$ take value one on some orthonormal basis of $T_x(T(M))$ and $T_x(U(M))$ for x in $T(M)$ and $U(M)$ respectively.

a) Let us consider $\overline{\theta}$. Let $x \in T(M)$ and let $\{x_1, \ldots, x_d\}$ be an orthonormal basis of $T_{p(x)}(M)$. Set $p_i = \xi_x^{-1} x_i$ and $z_i = C(x, x_i)$ ($i = 1, \ldots, d$). Then, by the definition of $\overline{g}$, it follows that $\{p_1, \ldots, p_d, z_1, \ldots, z_d\}$ is an orthonormal basis of $T_x(T(M))$. Further by (6.10) we have

$$(d\alpha)(p_i, p_j) = 0$$

$$(d\alpha)(p_i, z_j) = \delta_{ij}\ \forall i, j \qquad (5.2.7)$$

164 Hence by the definition of an exterior product through shuffles we have

$$(-1)^{\frac{d(d-1)}{2}} \overset{d}{\wedge}(d\alpha)(p_1, \ldots, p_d, z_1, \ldots, z_d) = \sum_{\sigma \in P_d} \prod_{i=1}^{d} (d\alpha)(p_{\sigma(i)}, z_{\sigma(i)})$$

where σ runs through all possible permutations of $\{1, \ldots, d\}$. Hence

$$(-1)^{\frac{d(d-1)}{2}} \overset{d}{\wedge}(d\alpha)(p_1, \ldots, p_d, z_1, \ldots, z_d) = d!.$$

b) Now let us consider $U(M)$. We want to look at the tangent vectors to $U(M)$ among those to $T(M)$. Let us consider the injection $i : U(M) \to T(M)$. If $x \in U(M)$ and $z \in T_x(T(M))$ is a vector in the image of $T_x(U(M))$ by i^T then there exists z_1 in $T_x(U(M))$ such that $i^T(z_1) = z$. But then since E is constant on $U(M)$ we have by (5.2.3) $0 = z_1(E \circ i) = i^T(z_1)(E) = z(E) = \overline{g}(\operatorname{grad}(E), z)$. Hence every such vector z is orthogonal to $\operatorname{grad}(E)$. But the subspace in $T_x(T(M))$ of vectors orthogonal to $\operatorname{grad}(E)_x$ is of dimension $2d-1$ which is equal to the dimension of $U(M)$. Since i_x^T is injective we have

$$\text{(5.2.8)} \qquad i_x^T(T_x(U(M)) = \{z | \overline{g}(\operatorname{grad}(E)_x, z) = 0\}.$$

□

2.9

Now let $x \in U(M)$ and let $\{x_1, \ldots, x_d\}$ be an orthonormal basis of $T_{p(x)}(M)$ with $x_1 = x$. Set, as above, $p_i = \xi_x^{-1} x_i$ and $z_i = C(x, x_i)$ for $i = 1, \ldots, d$. Then, since $\operatorname{grad}(E)_x = 2\xi_x^{-1}$ by (2.2), we see, by (5.2.7), that

$$\{p_2, \ldots, p_d, z_1, \ldots, z_d\}$$

is an orthonormal basis of $T_x(U(M))$ and the relations (5.2.7) are valid. **165**
Moreover (see 6.6): $\alpha(p_i) = 0\ \forall i$ and $\alpha(z_1) = g(x_1, x_1) = 1$. Hence

$$(-1)\frac{d(d-1)}{2}\alpha \wedge (\overset{d-1}{\wedge}(d\alpha))(p_2, \ldots, p_d, z_1, \ldots, z_d)$$
$$= \sum_{\sigma \in P_{d-1}} \alpha(z_1) \prod_{i=2}^{d-1} (d\alpha)(p_{\sigma(i)}, z_{\sigma(i)})$$

where σ runs through all permutations of $\{2, \dots, d\}$. Hence

$$\alpha \wedge (\overset{(d-1)}{\wedge} d\alpha)(p_2, \dots, p_d, z_1, \dots, z_d) = (d-1).$$

Hence the result.

2.10

Now let us compute $\mathrm{vol}(U(M), \overline{\overline{g}})$ when M is compact.

First let us suppose that M is oriented by a form $\sigma_1 \in \xi^d(M)$, and denote (see (5)) the canonical orienting form corresponding to σ_1 by σ. Let $\{p_1, \dots, p_d; z_1, \dots, z_d\}$ be as in the proof (a) of (2.5). Then we define $\tau \in \xi^d(T(M))$ by setting

$$\tau(p_1, \dots, p_d) = \pm,$$

according as $\{\xi p_1, \dots, \xi p_d\}$ is a positive basis of $T_m(M)$ or not, and zero for any other combination from $\{p_1, \dots, p_d; z_1, \dots, z_d\}$, and extending, multilinearly.

First let us note that the restriction of τ to $T_m(M)$ is the canonical
166 volume form of the euclidean space $(T_m(M), g_m)$. Then we have

(5.2.11) $$\overline{\sigma} = (-1)^{\frac{d(d-1)}{2}} \tau \wedge p_M^* \sigma$$

where $p_M : T(M) \to M$. For

$$\begin{aligned}(\tau \wedge p_M^* \sigma)&(p_1, \dots, p_d, z_1, \dots, z_d)\\ &= \tau(p_1, \dots, p_d) \cdot \sigma(p^T(z_1), \dots, p^T(z_d)) = \\ &= 1 \text{ by the definition of } \tau.\end{aligned}$$

Now let $\{p_1, \dots, p_d, z_1, \dots, z_d\}$ be as in (2.9). Then define ω by setting $\omega(p_2, \dots, p_d) = \pm 1$ according as $\{\xi p_1, \xi p_2, \dots, \xi p_d\}$ is positive or negative with respect to σ, and zero for any other combination from $\{p_1, \dots, p_d, z_1, \dots, z_d\}$ and extending linearly. We note that the restriction of ω to $U_m(M)$ is the canonical volume form on the sphere $(U_m(M), g_m | U_m(M))$ and check (as in the case of $\tau \wedge p_M^* \sigma$) that

(5.2.12) $$\overline{\overline{\sigma}} = \omega \wedge p_M^* \sigma.$$

Now we prove that

(5.2.13) $$\mathrm{Vol}(U(M), \overline{\overline{g}}) = \textcircled{d-1}. \mathrm{vol}(M, g),$$

where we use the notation:

(5.2.14) $$\textcircled{d} = \mathrm{Vol}(\mathbb{S}^d, \epsilon|\mathbb{S}^d).$$

Proof. First let us assume that M is oriented. Then $\omega \wedge p_M^* \sigma$ is a volume form on $U(M)$ and we have

$$\begin{aligned}\mathrm{Vol}(U(M), \overline{\overline{g}}) &= \int\limits_{U(M)} \omega \wedge p_M^* \sigma = \\ &= \int\limits_{x \in M} \left(\int\limits_{U_m(M)} \omega | U_m(M) \right) \sigma, \quad \text{by (3.17)}\end{aligned}$$

But 167

$$\int\limits_{U_m(M)} \omega | U_m(M) = \mathrm{Vol}(\mathbb{S}^{d-1}(T_m(M), g_m | \mathbb{S}^{d-1}(T_m(M))) = \text{(by (3.15))}$$

and because all euclidean structures on vector spaces of the same finite dimension are isometric, this

$$= \mathrm{Vol}(\mathbb{S}^{d-1}, \epsilon | \mathbb{S}^{d-1}) = \textcircled{d-1}.$$

Hence

$$\mathrm{Vol}(U(M), \overline{\overline{g}}) = \int\limits_{x \in M} \textcircled{d-1}\, \sigma = \textcircled{d-1} \cdot \mathrm{Vol}(M, g).$$

b) In the general case we take a partition of unity and proceed. Let $\{W_i, \varphi_i\}$ be a partition of unity on M, the $W_i' s$ being so small that they can be oriented. Then we have

(5.2.15) $$\sum (\varphi_i \circ p) = 1 \text{ on } U(M)$$

where $p : U(M) \to M$. Now let σ_i be an orienting form on W_i such that

$$\bar{\bar{\theta}}|U(W_i) = |\sigma_i|$$

where

$$\bar{\bar{\bar{\sigma}}}_i = \omega \wedge (p^* \sigma_i). \tag{5.2.16}$$

Then we have

$$\begin{aligned}
\int\limits_{U(W_i)} (\varphi_i \circ p)\bar{\bar{\theta}} &= \int\limits_{U(W_i)} (\varphi_i \circ p)\bar{\bar{\bar{\sigma}}} = \\
&= \int\limits_{U(W_i)} (\varphi_i \circ p)\omega \wedge p^* \sigma_i. \\
&= \int\limits_{m \in W_i} \left(\int\limits_{U_m(M)} \omega | U_m(M) \right) \varphi_i |\sigma_i| = \boxed{d-1} \int\limits_{W_i} \varphi_i |\sigma_i|. \qquad (5.2.17)
\end{aligned}$$

168 Now by (5.2.17) we have

$$\begin{aligned}
\mathrm{Vol}(U(M), \bar{\bar{g}}) = \int\limits_{U(M)} \bar{\bar{\theta}} &= \sum_i \int\limits_{U(W_i)} (\varphi_i \circ p)\bar{\bar{\theta}} = \\
&= \boxed{d-1} \sum_i \int\limits_{W_i} \varphi_i |\sigma_i| = \boxed{d-1} \mathrm{Vol}(M, g).
\end{aligned}$$

□

3 Dg = 0

Let us denote the canonical derivation law of the r.m. (M, g) *by* D. Since $g \in \mathscr{L}^2(M)$ and by 4.7, we can introduce the covariant derivative $\mathrm{Dg} \in \mathscr{L}^3(M)$. We shall prove that the 3-form Dg is zero.

3.1

First let us note that the form Dg is symmetric in the last two variables (because g is symmetric):

(5.3.2) $$\mathrm{Dg}(X, Y, Z) = \mathrm{Dg}(X, Z, Y)\ \forall X, Y, Z \in \mathscr{C}(M).$$

To start with we shall prove a lemma.

3.3

Lemma.

$$\mathrm{Dg}(y, x, x) = 2\,\mathrm{Dg}(x, x, y) \quad \forall (x, y) \in T(M) \underset{M}{\times} T(M).$$

Proof. With the notation of (8) let us take a one-parameter family f such that

$$\underline{\underline{P}}(0, 0) = x,\ \underline{\underline{Q}}(0, 0) = y,$$

and f_0 is a geodesic. □

Now let us follow the notations of (1). Then since f_0 is a geodesic by (4.2.1), and (1.7) we have;

(5.3.4) $$P(\beta(Q))(0, 0) = 1/2Q(\beta(P))(0, 0).$$

Further since the connection is symmetric and $[P, Q] = 0$ we have **169**

(5.3.5) $$D_P\underline{\underline{Q}} = D_Q\underline{\underline{P}}$$

and since f_0 is a geodesic

(5.3.6) $$(D_P\underline{\underline{P}})(t, 0) = 0.$$

Further, by the definition of D, we have, *at the point* $(t, 0)$:

$$\begin{aligned} P(\beta(Q)) &= P(g(\underline{\underline{P}}, \underline{\underline{Q}})) \\ &= \mathrm{Dg}(\underline{\underline{P}}, \underline{\underline{P}}, \underline{\underline{Q}}) + g(D_p\underline{\underline{P}}, \underline{\underline{Q}}) + g(\underline{\underline{P}}, D_p\underline{\underline{Q}}) \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Dg}(\underline{\underline{P}}, \underline{\underline{P}}, \underline{\underline{Q}}) + g(\underline{\underline{P}}, D_P\underline{\underline{Q}}) \quad \text{by (5.3.6)}\\
\text{and} \qquad Q(\beta(P)) &= Q(g(\underline{\underline{P}}, \underline{\underline{P}}))\\
&= \mathrm{Dg}(\underline{\underline{Q}}, \underline{\underline{P}}, \underline{\underline{P}}) + 2g(D_Q\underline{\underline{P}}, \underline{\underline{P}})\\
&= \mathrm{Dg}(\underline{\underline{Q}}, \underline{\underline{P}}, \underline{\underline{P}}) + 2g(D_P\underline{\underline{Q}}, \underline{\underline{P}}) \quad \text{by (5.3.5).}
\end{aligned}
$$

Hence by (5.3.4) we have, at the point $(t, 0)$

$$2\,\mathrm{Dg}(\underline{\underline{P}}, \underline{\underline{P}}, \underline{\underline{Q}}) = \mathrm{Dg}(\underline{\underline{Q}}, \underline{\underline{P}}, \underline{\underline{P}}),$$

and hence by the choice of the one parameter family

$$2\,\mathrm{Dg}(x, x, y) = \mathrm{Dg}(y, x, x).$$

Proof of $\mathrm{Dg} = 0$. Clearly, by the lemma

(5.3.7) $$\mathrm{Dg}(x, x, x) = 0 \quad \forall x \in T(M).$$

Further, we have, by the lemma,

$$
\begin{aligned}
&\mathrm{Dg}(y, x+y, x+y) = 2\,\mathrm{Dg}(x+y, x+y, y),\\
\text{i.e.} \qquad &\mathrm{Dg}(y, x, x) + \mathrm{Dg}(y, y, y) + 2\,\mathrm{Dg}(y, x, y)\\
&= 2\,\mathrm{Dg}(x, x, y) + 2\,\mathrm{Dg}(x, y, y) + 2\,\mathrm{Dg}(y, x, y) + 2\,\mathrm{Dg}(y, y, y).
\end{aligned}
$$

Again by the lemma and (5.3.7) we have

(5.3.8) $$\mathrm{Dg}(x, y, y) = 0.$$

170 Hence

$$
\begin{aligned}
&\mathrm{Dg}(x, y+z, y+z) = 0\\
\text{i.e.} \quad &\mathrm{Dg}(x, y, y) + \mathrm{Dg}(x, y, z) + \mathrm{Dg}(x, z, y) + \mathrm{Dg}(x, z, z) = 0.
\end{aligned}
$$

By (3.1) and (5.3.8) we have

$$2\,\mathrm{Dg}(x, y, z) = 0.$$

Hence

$$\mathrm{Dg} = 0.$$

Hence we have the following

3.9

Corollary.

(D.L.5) $X(g(Y,Z)) = g(D_x Y, Z) + g(Y, D_x Z)\ \forall X, Y, Z\ \ in\ \ \mathscr{C}(M).$

4 Consequences of Dg = 0

Now we shall complete the lemma (6.10) in the:

4.1

Proposition. *For every x of $T(M)$ we have*

$$(d\alpha)(z, z') = g(v(z)), p^T(z')) - g(v(z'), p^T(z))\ \forall z, z' \in T_x(T(M)).$$

Proof. We know that the result holds good if at least one of z and z' is a vertical vector (see (6.10)). In view of the bilinearity of both the sides and the direct sum decomposition

$$T_x(T(M)) = H_x + V_x$$

it is enough to prove the result $\forall z, z' \in H_x$. Now let $Z, Z' \in \mathscr{C}(M)$ be such that

(5.4.2) $Z(p(x)) = p^T(z)$ and $Z'(p(x)) = p^T(z')$.

Then we contend that

$$d\alpha(Z^H, Z'^H) = 0 \quad \text{(see (5.8))}.$$

We have by (0.2.10) **171**

(5.4.3) $d\alpha(Z^H, Z'^H) = Z^H(\alpha(Z'^H) - Z'^H(\alpha(Z^H) - \alpha([Z^H, Z'^H]).$

But for every $X \in \mathscr{C}(M)$ we have

$$\begin{aligned}\alpha(Z^H) \circ X &= g(p' \circ Z^H \circ X, p^T \circ Z^H \circ X)\\ &= g(X, Z)\ \text{ by (1.5) } C_1.,\end{aligned}$$

and hence

(5.4.4) $$\alpha(Z^H) = g^\sharp(Z).$$

Now by (5.4.4) and (5.4.3) we have

$$\begin{aligned} d\alpha(Z^H, Z'^H) &= Z^H(g^\sharp(Z')) - Z'^H(g^\sharp(Z)) - \alpha([Z^H, Z'^H]), \\ &= D_Z(g^\sharp(Z')) - D_{Z'}(g^\sharp(Z)) - \alpha([Z^H, Z'^H]) \text{ by (5.9).} \end{aligned}$$

Hence for every $X \in \mathscr{C}(M)$ we have

(5.4.5)
$$\begin{aligned} d\alpha(Z^H, Z'^H) \circ X &= D_Z(g^\sharp(Z'))(X) - D_{Z'}(g^\sharp(Z))(X) - \alpha([Z^H, Z'^H] \circ X) \\ &= Z(g(Z', X)) - g(Z', D_Z X) \\ &\quad - Z'(g(Z, X)) - g(Z, D_{Z'} X) - \alpha([Z^H, Z'^H] \circ X) \text{ by (2.4.8)} \\ &= g(D_Z Z', X) - g(D_Z, Z, X) - \alpha([Z^H, Z'^H] \circ X) \text{ by (3.9).} \end{aligned}$$

But

(5.4.6) $$\begin{aligned} \alpha([Z^H, Z'^H] \circ X) &= g(p' \circ [Z^H, Z'^H] \circ X, p^T \circ [Z^H, Z'^H] \circ X) \\ &= g(X, [Z, Z']) \quad \text{(see the proof of (5.10)).} \end{aligned}$$

Hence we have by (5.4.5) and (5.4.6)

$$\begin{aligned} (d\alpha)(Z^H, Z^H) \circ X &= g(D_Z Z' - D_{Z'} Z, X) - g(X, [Z, Z']) \\ &= g([Z, Z'], X) - g(X, [Z, Z']) \text{ by (4.1) D.L.4.} \\ &= 0 \text{ since } g \text{ is symmetric.} \end{aligned}$$

172 This being so for every X we have

$$(d\alpha)(Z^H, Z'^H) = 0.$$

Now we shall extend the result (3.9). Let N be any manifold and let h, $h' \in D(N, T(M))$ be such that $p \circ h = p \circ h'$. *Then we can define a function $g(h, h')$ on N by setting*

(5.4.7) $$g(h, h')(n) = g(h(n), h'(n)), \quad n \in N.$$

Since $h(n)$, $h'(n) \in T_{p(h(n))}(M)$, we have the following result. □

4.8

Corollary. *With the above notation we have*

(C.D.7) $$X(g(h,h')) = g(D_X h, h') + g(h, D_X h') \forall X \in \mathscr{C}(N).$$

Proof. This follows directly from (4) and Dg = 0. □

4.9

Corollary. *Parallel transport preserves g.*

Proof. Clearly it is enough to prove the result for a curve instead of for a path. Let $f \in D(I, M)$ be a curve and let h and h' be parallel lifts of f such that

$$h(0) = x \quad \text{and} \quad h'(0) = y.$$

Then we have

$$\frac{d}{\mathrm{dt}}(g(h(t), h'(t)) = P(g(h,h')) =$$
$$g(D_p h, h') + g(h, D_p h') \text{ by } (5.4.7)$$
$$= 0 \text{ since } D_p h = D_p h' = 0.$$

Hence $g(h,h')$ is constant on I. Hence **173**

$$g(h(t), h'(t)) = g(h(0), h'(0)) \; \forall t \in I.$$

□

5 Curvature

From the definition of R (see (5.1)) and from (1.6) iii) it follows, directly, that R is invariant under isometries, i.e.

5.1

Proposition. *If*

$$f : (M, g) \to (N, h)$$

is an isometry then $\forall X, Y, Z \in \mathscr{C}(M)$ *we have*

$$f^T \circ {}^M R(X,Y)Z \circ f^{-1} = {}^N R(f^T \circ X \circ f^{-1}, f^T \circ Y \circ f^{-1})(f^T \circ Z \circ f^{-1}).$$

5.2

Remark. Let us note that this result in particular implies that an r.m. in general, is not locally isometric to $\mathbb{R}^d$. For, for $\mathbb{R}^d$, $R = 0$ by (5.4).

Now let us consider the map

(5.5.3) $$(X, Y, Z, T) \to g(R(X, Y)Z, T); \mathscr{C}^4(M) \to \mathbb{R}$$

and prove

(5.5.4) C.T.3 $$g(R(X, Y)Z, T) = -g(R(X, Y)T, Z)$$

(5.5.5) C.T.4 $$g(R(X, Y)Z, T) = g(R(Z, T)X, Y)$$

Proof. C.T.3. We have by the definition of R

(5.5.6) $$g(R(X, Y)Z, Z) = g(D_X D_Y Z, Z) - g(D_Y D_X Z, Z) - g(D_{[X,Y]}Z, Z).$$

By by (3.9) D.L.5. we have

$$g(D_{[X,Y]}Z, Z) = \frac{1}{2}[X, Y](g(Z, Z))$$

$$g(D_X D_Y Z, Z) = X(g(D_Y Z, Z)) - g(D_Y Z, D_X Z)$$

and $$g(D_Y D_X Z, Z) = Y(g(D_X Z, Z)) - g(D_X Z, D_Y Z)$$

174 and hence by (5.5.6)

(5.5.7)
$$g(R(X, Y)Z, Z) = X(g(D_Y Z, Z)) - Y(g(D_X Z, Z)) - \frac{1}{2}[X, Y](g(Z, Z));$$

and again by (3.9)

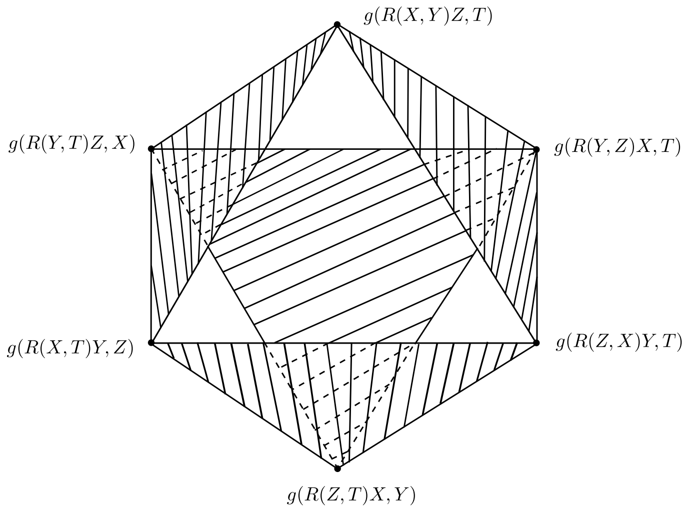
$$= X(\frac{1}{2}Yg(Z, Z)) - Y(\frac{1}{2}Xg(Z, Z)) - \frac{1}{2}[X, Y]g(Z, Z) = 0.$$

Now if we replace Z by $Z + T$ in (5.5.7) and carry out the cancellation we get C.T.3. □

Note. We sketch here the geometrical reason for which (C.T.3) holds: because parallel transport leaves g invariant, and thanks to (7.11), the endomorphism $R(X, Y)$ belongs to the tangent space of the orthogonal group (i.e. its Lie algebra); this fact is nothing but (C.T.3).

C.T.4

We use C.T.1., C.T.2. and C.T.3. and prove the result



In the octahedron, the sum of the terms at the three vertices of every shaded triangle is zero by (C.T.1), (C.T.2) and (C.T.3). Now adding the terms corresponding to the triangles above and subtracting the similar sum for the lower ones from it we get **175**

$$2(g(R(X,Y)Z,T)) - 2g(R(Z,T)X,Y)) = 0.$$

6 Jacobi fields in an r.m.

Suppose that $f \in D(I,M)$ is a geodesic in an r.m. (M,g) and that h is a Jacobi field along f. Then we have the following proposition.

6.1

Proposition. *If for some $t_0 \in I$,*

$$g(f'(t_0), h(t_0)) = g(f'(t_0), (D_P h)(t_0)) = 0$$

then

$$g(f', h) = 0 \quad on \quad I.$$

Proof. We have

$$(5.6.2) \qquad \begin{aligned} \frac{d}{\mathrm{dt}}(g(f', h)) &= P(g(f', h)) \\ &= g(D_P f', h) + g(f', D_P h) \text{ by (4.8) (C.D.7) =} \\ &= g(f', D_P h) \text{ by (7.3)} \end{aligned}$$

and again by (4.8) and (7.3).

$$(5.6.3) \qquad \begin{aligned} \frac{d^2}{\mathrm{dt}^2}(g(f', h)) = \frac{d}{\mathrm{dt}}(g(f', D_P h)) &= \frac{d}{\mathrm{dt}}(g(f', D_P h)) \\ &= g(f', R(f', h)f') \text{ by the definition of a} \\ &\qquad \text{of a Jacobi field (see (8.9))} \\ &= 0 \text{ by ((5.5.5) C.T.4) C.T.3.} \end{aligned}$$

Hence

$$g(f', D_P h) \quad \text{is constant on } I.$$

176 But, by hypothesis $g(f', D_P h)(0) = 0$, and hence $g(f', h))$ is constant on I. But $g(f', h)(0) = 0$. Hence $g(f, h) = 0$. □

6.4

Definition. *An r.m. of dimension two is called a* surface.

As an application of the above proposition we prove the existence of nice coordinate neighbourhoods for surfaces.

6.5

Application. *Given a surface* (M, g) *and a point* m *of* M *there exists a chart* (U, r) *such that* m *is in* U *and if we denote the local coordinates with respect to* (U, r) *by* x *and* y *then*

$$g|U = dx^2 + K(x, y)dy^2.$$

Proof. Let $\{x_0, y_0\}$ be an orthonormal base of $T_m(M)$ and let $S \in D(J'', M)$ be any curve in M such that

$$S'(0) = x_0 \quad \text{and} \quad \|S'(\alpha)\| = 1. \tag{5.6.6}$$

□

6.7

Then choose the lift $\widetilde{S}(\alpha)$ of S through y_0 such that at each point α, $S'(\alpha)$ and $\widetilde{S}(\alpha)$ form an orthonormal basis for $T_{S(\alpha)}(M)$ (see (1.2)). Now by taking a suitable sub interval J' of J'' and an interval I we can consider the one parameter family

$$\begin{aligned} f &: I \times J' \to M \\ f(t, \alpha) &= \exp(t \cdot \widetilde{S}(\alpha)). \end{aligned} \tag{5.6.8}$$

As in the proof of (8.26) we get

$$\begin{aligned} \underline{\underline{P}}(0,0) &= (f^T \circ P)(0,0) = x_0 \\ \underline{\underline{Q}}(0,0) &= (f^T \circ Q)(0,0) = y_0 \end{aligned} \tag{5.6.9}$$

and hence $f^T(0,0)$ is an isomorphism. Therefore by choosing I and J sufficiently small we see, by the inverse function theorem, that

$$f : I \times J \to f(I \times J) = U \tag{5.6.10}$$

is a diffeomorphism. Now set 177

$$f^{-1}|U = r \tag{5.6.11}$$

and

$$x = t \circ r, y = \alpha \circ r. \tag{5.6.12}$$

Then we have

$$g|U = g(\underline{\underline{P}}, \underline{\underline{P}})dx^2 + 2g(\underline{\underline{P}}, \underline{\underline{Q}})dx\, dy + g(\underline{\underline{Q}}, \underline{\underline{Q}})dy^2. \tag{5.6.13}$$

We have

$$g(\underline{\underline{P}}, \underline{\underline{P}})(0, \alpha) = \|\widetilde{S}(\alpha)\| = 1 \text{ by the definition of } \widetilde{S}(\alpha)$$

and

$$g(\underline{\underline{P}}, \underline{\underline{P}})(t, \alpha) = g(\underline{\underline{P}}, \underline{\underline{P}})(0, \alpha)$$

by (3.2) since for each α, $f_\alpha(t)$ is a geodesic, and hence

$$g(\underline{\underline{P}}, \underline{\underline{P}}) = 1. \tag{5.6.14}$$

Further from the proof of (8.26) it follows that for each α

$$t \to Q(t, \alpha) = h_\alpha(t) \tag{5.6.15}$$

is a Jacobi field along the geodesic $f_\alpha(t)$. Further

$$\begin{aligned} g(f'(0), h_\alpha(0)) &= g(\underline{\underline{P}}, \underline{\underline{Q}})(0, \alpha) \\ &= g(\widetilde{S}(\alpha), S'(\alpha)) = 0 \end{aligned} \tag{5.6.16}$$

by our construction; and also

$$\begin{aligned} g(f'_\alpha(0), D_P h_\alpha(0)) &= g(\underline{\underline{P}}, D_P \underline{\underline{Q}})(0, \alpha) \\ &= g(\underline{\underline{P}}, D_Q \underline{\underline{P}})(0, \alpha) \text{ by } 4.1 \text{ D.L.4.}, \\ &= g(\widetilde{S}, D_Q \widetilde{S})(0, \alpha) \\ &= \frac{1}{2} Q(g(\widetilde{S}, \widetilde{S})) \text{ by } (4.8) \\ &= 0 \text{ by the construction of } \widetilde{S}. \end{aligned} \tag{5.6.17}$$

178 Now (6.1) gives that

$$g(\underline{\underline{P}}, \underline{\underline{Q}})(t, \alpha) = 0 \tag{5.6.18}$$

Now the result follows if we set $K(x, y) = g(\underline{\underline{Q}}, \underline{\underline{Q}})$.

6.19

Proposition. *Let (M, g) be an r.m. let $f \in D(I, M)$ be a geodesic and let h be a Jacobi field along f. Then*

i) $\dfrac{d(E \circ h)}{\mathrm{dt}} = 2g(D_P h, h)$

ii *)* $\dfrac{d^2(E \circ h)}{\mathrm{dt}^2} = E \circ D_P h + g(R(f', h)f', h)$

iii) *if* $h(0) = 0$ *and* $(D_P h)(0) = y$ *with* $\|y\| = 1$ *then*

$$\|h(t)\| = t + \frac{t^3}{6} g(R(f'(0), y)f'(0), y) + 0(t^3),$$

where

$$\frac{0(t^3)}{t^3} \to 0 \quad \textit{with } t.$$

Proof. By (4.8) we have

$$\frac{d}{\mathrm{dt}}(E \circ h) = P(g(h, h)) = 2g(D_P h, h) \tag{5.6.20}$$

and again by (4.8), we have

$$\begin{aligned}
\frac{d^2}{\mathrm{dt}^2}(E \circ h) &= 2P(g(D_P h, h)) \\
&= 2g(D_P D_P h, h) + 2g(D_P h, D_P h) \\
&= 2g(R(f', h)f', h) + 2E \circ D_P h \quad \text{by the definition (8.9).}
\end{aligned}$$

Now with the notation of (2.7.9) by (8.18) we have **179**

$$\widehat{h}(t) = ty + \frac{t^3}{6}(R(f'(0), y)f'(0) + 0(t^3) \tag{5.6.21}$$

and hence

$$\|\widehat{h}(t)\|^2 = t^2 g(y, y) + \frac{t^4}{3} g(R(f'(0), y)f'(0), y) + 0(t^4)$$

$$= t^2(1 + \frac{t^2}{3} g(R(f'(0), y)f'(0), y) + 0(t^2)$$

Since $h(t)$ is the parallel transport of $\widehat{h}(t)$ along f we have by (4.9)

$$\|h(t)\| = \|\widehat{h}(t)\|$$

Now the rest is the expansion formula for

$$\sqrt{1+\delta}.$$

Sometimes it is convenient to re parametrise the arcs by their arc lengths and then deal with them. In the case of geodesics which are non trivial this re parametrisation is always possible. Let f be a geodesic. Then we know that $\|f'\|$ is constant, by (3.2), say θ. Then $\theta \neq 0$ and for $f \circ k_{-\theta}$ we have

$$\|(f \circ k_{-\theta})'\| = 1 \quad \text{(see (5.11))}.$$

□

6.22

Now for any vector x in $U(M)$ *let us denote the curve*

$$t \to \exp(tx)$$

by $\gamma_x(t)$.

By applying a change of parameter, from (8.29) we get the following result.

6.23

180 **Proposition.** *For every non zero* x *in* Ω *and* $y \in T_{p(x)}(M)$, *we have*

$$\exp_m^T(\zeta_x^{-1} y) = h(\|x\|)$$

where h *is the Jacobi field along* $\gamma_{x/\|x\|}$ *such that* $h(0) = 0$ *and* $(D_P h)(0) = y/\|x\|$.

Proof. By (8.29) we have

$$\exp_m^T(\zeta_x^{-1}y) = \mathring{h}(1)$$

where $\mathring{h}$ is the Jacobi field along

$$\mathring{f} : t \to \exp(t \cdot x)$$

such that

$$\mathring{h}(0) = 0 \quad \text{and} \quad (D_P\mathring{h})(0) = y.$$

But $\gamma\left(\dfrac{x}{x/\|x\|}\right) = \mathring{f} \circ k_{\|x\|} - 1$ and if we set $\mathring{f} \circ k_{\|x\|} - 1 = f$ and $\mathring{h} \circ k_{\|x\|} - 1 = h$ then h is a Jacobi field along f such that $h(0) = \mathring{h}(0) = 0$ and $h(\|x\|) = \mathring{h}(1) = \exp_m^T\left(\zeta_x^{-1}y\right)$.

But $D_P h = D_P(\mathring{h} \circ k_{\|x\|} - 1) = \|x\|^{-1} D_P \mathring{h}$, so that

$$D_P h(0) = \|x\|^{-1} D_P \mathring{h} = y/\|x\|.$$

□

6.24

Gauss Lemma. *For every x and y in $T_m(M)$ such that*

(5.6.25) $$x \neq 0, x \in \Omega \quad \textit{and} \quad g(x, y) = 0$$

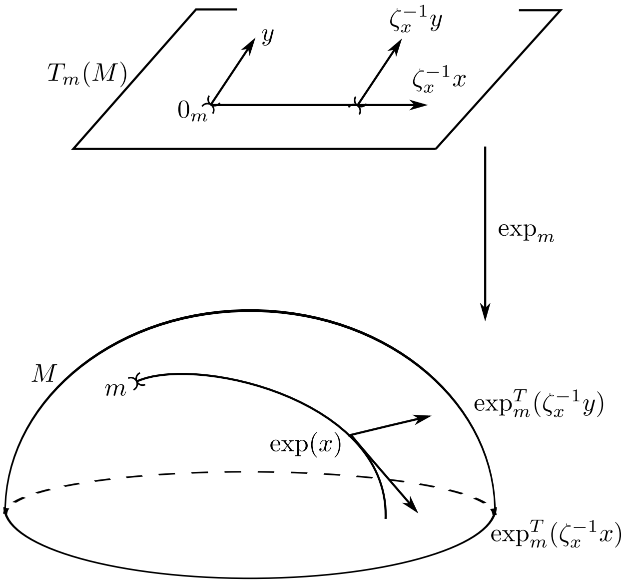
we have

$$g(\exp_m^T(\zeta_x^{-1}x), \exp_m^T(\zeta_x^{-1}y)) = 0.$$

Proof. Let h be the Jacobi field along $\gamma_{x/\|x\|}$ such that $h(0) = 0$ and **181**
$(D_P h)(0) = \dfrac{y}{\|x\|}$. Then we have $\gamma'_{x/\|x\|}(0) = \dfrac{x}{\|x\|}$ and hence

$$g(\gamma'_{x/\|x\|}, h)(0) = g\left(\frac{x}{\|x\|}, 0\right) = 0,$$

$$g(\gamma'_{x/\|x\|}, D_P h)(0) = g\left(\frac{x}{\|x\|}, \frac{y}{\|x\|}\right) = 0 \text{ by (5.6.25).}$$



Hence by (6.1) we have

(5.6.26) $$g(\gamma'_{x/\|x\|}, h) = 0,$$

and hence in particular

$$g(\gamma'_{x/\|x\|}, h)(x) = 0.$$

But by (6.23) we have

(5.6.27) $$h(\|x\|) = \exp_m^T(\zeta_x^{-1} y).$$

Further

$$\gamma_{\frac{x}{\|x\|}}(t) = \exp\left(t \cdot \frac{x}{\|x\|}\right) = (\exp_m \circ S)(t)$$

182 where

$$S(t) = t \cdot \frac{x}{\|x\|}.$$

Hence

$$\gamma'_{x/\|x\|} = \exp_m^T \circ S'.$$

But

(5.6.28) $$S'(t) = \zeta^{-1}_{\frac{tx}{\|x\|}} \left(\frac{x}{\|x\|}\right)$$

and hence

(5.6.29) $$\gamma'_{x/\|x\|} = \frac{1}{\|x\|} \exp_m^T(\zeta_x^{-1} x).$$

Now our result follows from (5.6.26), (5.6.27) and (5.6.29).

Now for an m in M and a non zero x in $T_m(M)$ set

(5.6.30) $$L_x = \left\{k\zeta_x^{-1} x | k \in \mathbb{R}\right\}$$

(5.6.31) $$N_x = \left\{\zeta_x^{-1} y | g(x, y) = 0\right\}.$$

Then we have the following corollary. □

6.32

Corollary. *With the above notation, we have*

i) $\|\exp_m^T(u)\| = \|u\|$, *if* $u \in L_x$

ii) $g(\exp_m^T(L_x), \exp_m^T(N_x)) = 0$.

Proof. The proof of the first part follows from the fact that L_x is one dimensional and by (5.6.29), (7.3) and (4.9). The second part follows from Gauss' lemma. □

Chapter 6

Sectional Curvature

Let (M,g) be an r.m. of dimension d greater than one, and let x and y **183**
be two linearly independent tangent vectors at a point m of M. *Let us denote the subspace generated by x and y by $P(x,y)$.* Let e_1 and e_2 be an orthonormal basis of $P(x,y)$. We recall that the norm on $\overset{2}{\wedge} T_m(M)$ is given by

$$\|x \wedge y\|^2 = \|x\|^2 \cdot \|y\|^2 - (g(x,y))^2.$$

Now we shall prove the following proposition.

0.33

Proposition. *For $(x,y) \in T(M) \underset{M}{\times} T(M)$, such that x and y are linearly independent* the quantity

$$A(x,y) = -\frac{g(R(x,y)x,y)}{\|x \wedge y\|^2}, \tag{6.1.2}$$

depends only on $P(x,y)$.

Proof. Let x' and y' be two vectors such that

$$P(x,y) = P(x',y').$$

Then we have numbers a, b, a', b' such that

$$x' = ax + by \quad \text{and} \quad y' = a'x + b'y.$$

Then

$$R(x', y') = R(ax + by, a'x + b'y) = (ab' - ba')R(x, y), \text{ by (5.2) (C.R.1).}$$

Hence

$$\begin{aligned} g(R(x', y')x', y') &= (ab' - ba')g(R(x, y)x', y') \\ &= (ab' - ba')g(R(x', y')x, y) \text{ by ((5.5.5) C.T.4)(C.T.4)} \\ &= (ab' - ba')^2 g(R(x, y)x, y) \end{aligned}$$

184 and further

$$x' \wedge y' = (ab' - a'b)(x \wedge y)$$

and hence

$$\|x' \wedge y'\|^2 = (ab' - ba')^2 \cdot \|x \wedge y\|^2.$$

□

0.3

For any two dimensional subspace P of the tangent space $T_m(M)$ *we denote the above quantity by*

$$A(P) = A(x, y)$$

and call the "function" A *the sectional curvature* of (M, g).

0.4

We set

$$\begin{aligned} \mathscr{P}(M) &= \{P \subset T_m(M) | m \in M \text{ and } \dim P = 2\} \\ A(M) &= \{A(P) | P \in \mathscr{P}(M)\}. \end{aligned}$$

0.5

Proposition. *Let*

$$\lambda : (M, g) \to (N, h)$$

be an isometry, then

$$^{M}A(P) = {}^{N}A(\lambda^{T}(P)) \forall P \in \mathscr{P}(M).$$

Proof. By (5.1) the curvature is invariant under isometry and by the definition of isometry the r.s. is invariant under it. Hence the result. ◻

2 Examples

If (M, g) is a surface then the tangent space at each point m of M is two dimensional and hence A assumes only one value on the set of two dimensional subspaces of $T_m(M)$. Hence A can be considered as a function on M itself. Now let us compute A and verify that it is nothing but the so-called Gaussian (or total) curvature of our surface (M, g). **185**

Let us take a coordinate system $\{x, y\}$ which has been constructed in (6.5) and follow the notations of that proof. We use propositions (8.7) and (7.10) and compute C. Let

(6.2.1) $$\omega : t \to \omega_\alpha(t)$$

be the parallel lift of f_α such that

(6.2.2) $$\omega_\alpha(0) = \underline{\underline{Q}}(0, \alpha).$$

Then by (4.9) we have

$$g(\omega_\alpha(t), \omega_\alpha(t)) = 1.$$

Since $f'_\alpha(t)$ is a parallel life of f_α by (4.9) we have

$$g(\omega_\alpha(t), f'_\alpha(t)) = g(\omega_\alpha(0), f'_\alpha(0)) = g(\underline{\underline{Q}}, \underline{\underline{P}})(0, \alpha) = 0.$$

Therefore

$$\underline{\underline{Q}}(t, \alpha) = \theta(t, \alpha)\omega_\alpha(t)$$

and hence

$$\widehat{\underline{\underline{Q}}}(t,\alpha) = \theta(t,\alpha)\omega_\alpha(0). \quad \text{(see (2.7.9))}. \tag{6.2.3}$$

186 But

$$g(\widehat{\underline{\underline{Q}}}(t,\alpha), \widehat{\underline{\underline{Q}}}(t,\alpha)) = g(\underline{\underline{Q}}(t,\alpha), \underline{\underline{Q}}(t,\alpha)) = k(t,\alpha)$$

and hence by (6.2.3)

$$\theta(t,\alpha) = (k(t,\alpha))^{1/2}. \tag{6.2.4}$$

Now by (8.7)

$$R(\underline{\underline{P}}, \underline{\underline{Q}})\underline{\underline{P}} = D_P D_P \underline{\underline{Q}} = \frac{\partial^2\theta(t,\alpha)}{\partial t^2} \cdot \omega_\alpha(t). \tag{6.2.5}$$

We have

$$g(R(\underline{\underline{P}}, \underline{\underline{Q}})\underline{\underline{P}}, \underline{\underline{Q}}) = g\left(\frac{\partial^2\theta(t,\alpha)}{\partial t^2}\omega_\alpha(t), \theta(t,\alpha)\omega_\alpha(t)\right) = \theta(t,\alpha)\frac{\partial^2\theta(t,\alpha)}{\partial t^2}$$

Further

$$\|\underline{\underline{P}} \wedge \underline{\underline{Q}}\|^2 = \theta(t,\alpha)^2.$$

Hence by the definition of A we have

$$A(t,\alpha) = -\frac{t}{\theta(t,\alpha)}\frac{\partial^2\theta(t,\ )}{\partial t^2}. \tag{6.2.6}$$

2.7

This is nothing but the value of the Gaussian curvature when computed in a local coordinate system of the type of (6.5) (see, for example [31]: formula $(3,2)$ on p.196).

2.8

The concept of sectional curvature is very powerful. Very simple restrictions on C can have deep implications. For example:

2.9

Proposition . *If* $A(M) \subset]-\infty, 0]$ *then for every* m *of* M, $\exp_m$ *is of maximal rank on* $T_m(M) \cap \Omega$.

Proof. To prove this we use (8.32). Let $x \in T_m(M) \cap \Omega$ and let $f : t \to$ **187**
$\exp(t \cdot x)$ be a geodesic, and let h be a Jacobi field along f such that

$$h(0) = 0 \quad \text{and} \quad h \not\equiv 0. \tag{6.2.10}$$

Then we have by (6.19) ii)

$$\frac{d^2(E \circ h)}{\mathrm{dt}^2} = 2E \circ D_P h + 2g(R(f', h)f', h) \geq \ldots \tag{6.2.11}$$
$$\ldots \geq 2g(R(f', h)f', h) = -A(f', h)\|f' \wedge h\|^2 \geq 0.$$

Hence the function $E \circ h$ is a non-negative convex function which is not identically zero and hence it can vanish at most at one point. Hence h cannot vanish outside 0 and we are through by (8.32). □

2.12

By 5.4 $R = 0$ for $(\mathbb{R}^d, \epsilon)$. Hence $A(\mathbb{R}^d) = \{0\}$.

2.13

Now let us consider the sectional curvature of the three symmetric pairs $(\mathbb{S}^d, \mathrm{can})$, $(P^d(\mathbb{R}), \mathrm{can})$, $(\mathbb{R}^d, \mathrm{hyp})$ (see (4)). We write any of them $M = G/H$; since G operates transitively and by isometries on M, (0.5) yields

$$A(M) = A(T_{m_0}(M)) \forall m_0 \in M. \tag{6.2.14}$$

By (4.17) and (4.21) we have in fact: $A(M)$ consists of a single real number, say k.

2.15

Definition. *An r.m.* (M, g) *for which* $\exists\, k \in \mathbb{R}$ *with* $A(M) = \{k\}$ *is called an r.m. of* constant sectional curvature.

Hence the three symmetric pairs are of constant sectional curvature and so is $(\mathbb{R}^d, \mathrm{can})$. The corresponding constant is 0 for $(\mathbb{R}^d, \mathrm{can})$, 1 for $(\mathbb{S}^d, \mathrm{can})$ or $(P^d(\mathbb{R}), \mathrm{can})$ by (6.3.12) and (4.19) $\underline{\mathrm{B}}$. We shall **********
188 move that the constant is < 0 for $(\dot{\mathbb{R}}^d, \mathrm{hyp})$. We do this by global arguments and use the results of Chapters VII and VIII. First $(\mathbb{R}^d, \mathrm{hyp})$ is *complete* in the sense of (4) because geodesics through m_0 are defined on the whole $\mathbb{R}^d$ (see 4.21 to 4.22). Now this remark together with the fact that the sectional curvature is constant and is equal to k and (3.7) enables us to exclude the case $k > 0$; and we are left to throw out the case $k = 0$. If $k = 0$ then $(\mathbb{R}^d, \mathrm{hyp})$ would be locally isometric to $(\mathbb{R}^d, \mathrm{can})$ by (6.4.35). By (1.8) the $\tau(\exp X)$ for $X \in \underline{\mathrm{M}}$ in the symmetric space $(\mathbb{R}^d, \mathrm{hyp})$ are products of symmetries around points of M; those symmetries are determined by the r.s. of (M, g) hence those symmetries are the same (locally) as in $(\mathbb{R}^d, \mathrm{can})$. In conclusion the $\tau(\exp X)$ are isomorphic to those in $(\mathbb{R}^d, \mathrm{can})$; but the latter are the translations and hence commute. That would imply $[\underline{\mathrm{M}}, \underline{\mathrm{M}}] = 0$; it is easy to check that for $(S0_0(d, 1), S0(d))$ this is *not* the case. By using (6.3.12) *we normalise k in* $(\mathbb{R}^d, \mathrm{hyp})$ so as to have $A(\mathbb{R}^d, \mathrm{hyp}) = \{-1\}$.

2.16

Proposition. *For any $k \in \mathbb{R}$ there exists a simply connected r.m.* (M, g) *such that $A(M, g) = \{k\}$. We may take*

$$
\begin{array}{ll}
(\mathbb{S}^d, k^{-1/2} \cdot \mathrm{can}) & \text{for } k > 0 \\
(\mathbb{R}^d, \mathrm{can}) & \text{for } k = 0 \\
(\mathbb{R}^d, (-k)^{-1/2} \cdot \mathrm{hyp}) & \text{for } k < 0.
\end{array}
$$

3 Geometric interpretation

189

3.1

Definition. *For $P \in \mathscr{P}(M)$ set*

$$\underline{S}(P, r) = \left\{ x \in P \,\middle|\, \|x\| = r \right\}$$

and when $\underline{S}(P, r) \subset \Omega$ *set:*

$$S(P, r) = \exp(\underline{S}(P, r)).$$

We call $S(P, r)$ the circle of radius r in P.

3.2

Note. Let us note that if P is a subspace of $T_m(M)$ and $\exp_m$ is r'-O.K. where $r < r'$, then the image of the circle $\underline{S}(P, r)$ is a compact submanifold of M of dimension one. So for r sufficiently small by 5.3 we can consider the length $(\lg(S(P, r)))$. To compute $\lg(S(P, r))$ let us take an orthonormal basis $\{x, y\}$ of P and the parametric representation

$$v : [0, 2\pi] \ni \alpha \to \text{r.e.}(\alpha) \in \underline{S}(P, r) \tag{6.3.3}$$

where

$$e(\alpha) = \cos\alpha \cdot x + \sin\alpha \cdot y.$$

Now let us set

$$S = \exp \circ v. \tag{6.3.4}$$

Then we have

$$\lg(S(P, r)) = \int_0^{2\pi} \|S'\| d\alpha. \tag{6.3.5}$$

We have

$$S' = \exp^T \circ v'$$

and by the definition of v

$$\zeta(v'(\alpha)) = \text{r.e.}\left(\alpha + \frac{\pi}{2}\right)$$

and hence

$$S' = \exp^T \left(\zeta^{-1}_{\text{r.e}(\alpha)} \left(\text{r.e.}\left(\alpha + \frac{\pi}{2}\right)\right)\right). \tag{6.3.6}$$

3.7

190

Now if we denote by h_α the Jacobi field along $\gamma_{e(\alpha)}$ such that

$$h_\alpha(0) = 0 \quad \text{and} \quad (D_P h_\alpha)(0) = e\left(\alpha + \frac{\pi}{2}\right),$$

then by (6.23) and by (6.3.6) we have

$$S'(\alpha) = h_\alpha(r).$$

Then by (6.19) iii) we have

$$\text{(6.3.8)}\quad \|S'(\alpha)\| = r + \frac{r^3}{6} g(R(e(\alpha), e(\alpha + \frac{\pi}{2}))e(\alpha), e(\alpha + \frac{\pi}{2})) + 0(r^3) = \\ = r - \frac{r^3}{6} A(P) + 0(r^3).$$

Hence by (6.3.5) we have

$$\text{(6.3.9)}\qquad \lg(S(P, r)) = 2\pi r - \frac{\pi r^3}{3} A(P) + 0(r^3).$$

Therefore we get

$$\text{(6.3.10)}\qquad A(P) = \lim_{r \to 0} \frac{3}{\pi r^3}(2\pi r - \lg(S(P, r))).$$

3.11

Application. *As an application of the above formula we note the following: Let* (M, g) *be an r.m. and let us denote the* A *of the manifold* (M, kg) *where* $k > d$ *by* A'. *Then we have by (5.4) and* (6.3.10).

$$\text{(6.3.12)}\qquad A'(P) = \frac{1}{k^2} A(P).$$

3.12

Example. Now let us calculate A for $(\mathbb{S}^d, \text{can})$. We have

$$\exp(S(\alpha)) = \cos r \cdot e_{d+1} + \sin\ \text{r.e}(\alpha)$$

by the proof of (5.12) and hence

$$\|S'\| = \sin r.$$

Therefore 191

$$\begin{aligned} \lg(S(P, r)) &= 2\pi \sin r \\ &= 2\pi\left(r - \frac{r^3}{6} + 0(r^3)\right) \end{aligned}$$

and hence $A(P) = 1 \forall P$. Instead of proceeding with circles we can proceed, in a similar manner, with discs $\underline{\text{B}}(m, r) \cap P$ (for some $P \in \mathscr{P}(M)$) and thus get

(6.3.15) $$A(P) = \lim_{r \to 0} \frac{12}{\pi r^4}(\pi r^2 - \text{ar}(\exp \underline{\text{B}}(m, r) \cap P)).$$

The proof is left to the reader as an exercise.

4 A criterion for local isometry

In this article we give a result of Elie Cartan which gives a criterion for local isometry.

To start with we shall prove a result which is purely algebraic and asserts, essentially, that the sectional curvature determines the curvature tensor.

4.1

Proposition. *Let V and V' be vector spaces with euclidean structures g and g', and let*

$$v : V \to V'$$

be a euclidean isomorphism. Let R and R' be bilinear maps

$$R : V \times V \to \operatorname{End}(V)$$
$$R' : V' \times V' \to \operatorname{End}(V')$$

such that properties C.T.1, C.T.2, C.T.3 (and hence C.T.4) hold for R and R'. Let A and A' be the maps defined by means of R and R' as in (6.1.2).
192 *Then*

i) *if an element x in V is such that for every y which is not linearly dependent on x we have*

$$A(x, y) = A'(v(x), v(y)) \tag{6.4.2}$$

then

$$R'(v(x), v(y))v(x) = v(R(x, y)x) \quad \forall y \in V.$$

and

ii) *if $A'(v(x), v(y)) = A(x, y)$ for every x and y in V which are linearly independent then*

$$R'(v(x), v(y))v(x) = v(R(x, y)z) \quad \forall x, y, z \in V.$$

Proof. i) Let y and z be elements in V such that the pairs $\{x, y\}$, $\{x, y\}$ and $\{x, y + z\}$ are linearly independent. Then by hypothesis we have

$$A(x, y + z) = A'(v(x), v(y + z)) - = A'(v(x), v(y) + v(z)). \tag{6.4.2}$$

Hence using trilinearity of R and C.T.4 we have by definition of A and A'

$$\frac{A(x, y) + A(x, z) - 2g(R(x, y)x, z)}{\|x \wedge (y + z)\|^2} = \dots \tag{6.4.3}$$
$$\dots = \frac{A'(v(x), v(y)) + A'(v(x), v(z)) - 2g(R(v(x), v(y))v(x), v(x)).}{\|v(x) \wedge (v(y) + v(z))\|^2}$$

By (6.1.2) and the fact that V is a euclidean isomorphism and (6.4.3) we have

(6.4.4) $$g(R(x,y)x,z) = g'(R'(v(x),v(y))v(x),v(z)).$$

Again since v is a euclidean isomorphism we have

$$g(R(x,y)x,z) = g(v^{-1}(R(v(x),v(y))v(x)),z)$$

and, since g is non-degenerate, varying z, we have 193

(6.4.5) $$v(R(x,y)x) = R'(v,(x),v(y))v(x).$$

The proof is trivial in case x and y are linearly dependent (C.T.1).

ii) Substituting $x+z$ for x in (6.4.5) and using (6.4.5) we have

$$v(R(x,y)z) + v(R(z,y)x) = R'(v(x),v(y))v(z) + R'(v(z),v(y))v(x).$$

i.e.

(6.4.6)
$$v(R(x,y)z) - R'(v(x),v(y))v(z) = -v(R(z,y)x) - R'(v(z),v(y))v(x).$$

This means that if we denote the map

$$(x,y,z) \to v(R(x,y)z) - R'(v(x),v(y))v(z)$$

by θ we have

(6.4.7) $$\theta(x,y,z) = -\theta(z,y,x).$$

But by C.T.1 we also have

(6.4.8) $$\theta(x,y,z) = -\theta(y,x,z).$$

Now by (6.4.7) and (6.4.8) we have

(6.4.9) $$\theta(x,y,z) = -\theta(z,y,x) = \theta(y,z,x)$$

i.e. a cyclic permutation of x, y, z does not alter θ. Hence we have

$$3\theta(x,y,z) = \theta(x,y,z) + \theta(y,z,x) + \theta(z,x,y).$$

But by using C.T.2. we have

$$\theta(x, y, z) + \theta(y, z, x) + \theta(z, x, y) = 0.$$

Hence

$$\theta(x, y, z) = 0.$$

□

194 **Example.** As an example let us note that if $A(M) = \{0\}$ then $R = 0$.

4.10

In what follows, if $x \in U(M)$, we denote the parallel transport along the geodesic

$$\gamma_x : t \to \exp tx$$

from $\gamma_x(0)$ to $\gamma_x(t)$ by $\tau(x, t)$.

Then the main theorem can be stated as follows:

4.11

Theorem. *Let (M, g) and (N, h) be two r.m.'s. Let m and n be points in M and N such that there exists*

i) *a euclidean isomorphism u between $(T_m(M), g_m)$ and $(T_n(N), g_n)$ and*

ii) *an $r > 0$ such that $\exp_m$ is r-O.K. and $\underline{B}(n, r) \subset {}^N\Omega$ and further that for every x, y in $U_m(M)$ which are linearly in dependent and every t in $]0, r[$ we have*
(6.4.12)
$$^N C(\tau(u(x), t)u(x), \tau(u(x), t)u(y)) = {}^M C(\tau(x, t)x, \tau(x, t)y).$$

Then there exists $\lambda \in D(B(m, r), B(n, r))$ such that

$$\lambda^*(h)|B(n, r) = g|(B(m, r)).$$

Proof. First let us prove a lemma. □

4.13

Lemma. *Under the assumptions of the above theorem, if $x \in U_m(M)$ and h and h_1 are Jacobi fields along γ_x and $\gamma_{u(x)}$ respectively such that*

$$h_1(0) = u(h(0)) \quad \textit{and} \quad (D_P h_1)(0) = u((D_P h)(0)) \tag{6.4.14}$$

then

$$h_1(t) = (\tau(u(x),t) \circ u \circ \tau(x,t)^{-1})h(t).$$

Proof. With the notations (2.7.9) and (2.8.16), by the definition of $\tau(x,t)$ **195**
we have

$$\frac{d^2\widehat{h}}{\mathrm{dt}^2} = {}^{\widehat{M}}R(t)\widehat{h}(t) = \tau(x,t)^{-1}({}^{M}R(\gamma_x'(t), \widehat{h}(t)\gamma_x'(t)). \tag{6.4.15}$$

Since $\gamma_x'(t)$ is a parallel lift of $\gamma_x(t)$ and $\tau(x,t)$ is the parallel transport along γ_x we have

$$\gamma_x'(t) = \tau(x,t)\gamma_x'(0) = \tau(x,t)x. \tag{6.4.16}$$

Hence, by (6.4.15) we have

$$\frac{d^2\widehat{h}}{\mathrm{dt}^2} = \tau(x,t)^{-1}({}^{M}R(\tau(x,t)x, \tau(x,t)\widehat{h}(t))(\tau(x,t)x)). \tag{6.4.17}$$

Let us note that the map

$$v_t = \tau(u(x),t) \circ u \circ \tau(x,t)^{-1} \tag{6.4.18}$$

From $T_{\exp tx}(M)$ to $T_{\exp t\cdot u(x)}(N)$ is a euclidean isomorphism since parallel transport preserves the euclidean structure (see (4.9)) and since u is a euclidean isomorphism.

Now applying the first part of (4.1) to v we have

$$\begin{aligned} &{}^{N}R(\tau(u(x),t)u(x), \tau(u(x),t)u(\widehat{h}(t)))(\tau(u(x),t))u(x)) = \dots \\ &\quad = v_t\left[{}^{M}R(\tau(x,t)x, \tau(x,t)\widehat{h}(t))(\tau(x,t)x))\right]. \end{aligned} \tag{6.4.19}$$

Hence since u is linear we have

$$\frac{d^2(u \circ \widehat{h})}{\mathrm{dt}^2} = u \circ \frac{d^2 h}{\mathrm{dt}^2} = \dots$$

(6.4.20)
$$\ldots = \tau(u(x),t)^{-1\,N}R(\tau(u(x),t)u(x),\tau(u(x),t)u(\widehat{h}(t))\tau(u(x),t)u(x).$$

196 Now let us note, as above (6.4.16), that

$$\gamma'_{u(x)}(t) = \tau(u(x),t)u(x)$$

and hence by (6.4.20)

$$\frac{d^2(u\circ\widehat{h})}{\mathrm{dt}^2} = {}^{\widehat{N}}R(t)((u\circ\widehat{h})(t)). \tag{6.4.21}$$

But by (2.8.4) we have

$$\frac{d^2\widehat{h_1}}{\mathrm{dt}^2} = {}^{\widehat{N}}R(t)\widehat{h}_1(t). \tag{6.4.22}$$

Hence $u\circ\widehat{h}$ and $\widehat{h}_1$ satisfy the same second order differential equation; and further the assumptions (6.4.14) are the same as

$$\widehat{h}_1(0) = (u\circ\widehat{h})(0) \text{ and } \frac{d\widehat{h}_t}{\mathrm{dt}}(0) = u\left(\frac{d\widehat{h}}{\mathrm{dt}}(0)\right) = \frac{d(u\circ\widehat{h})}{\mathrm{dt}}(0)$$

since u is linear and this means that $\widehat{h}_1$ and $u\circ\widehat{h}$ satisfy the same differential equation with the same initial conditions. Hence

$$\widehat{h}_1 = u\circ\widehat{h} \tag{6.4.23}$$

and hence

$$h_1(t) = v_t\circ h(t)\ \forall t. \tag{6.4.24}$$

□

Proof of the theorem. By (2.6) we have to define λ by:

$$\lambda = {}^N\exp_n \circ u\circ({}^M\exp_m|\underline{\mathrm{B}}(m,r))^{-1}; \tag{6.4.25}$$

we check that λ is indeed a local isometry. To show this we have only
197 to prove that the map

$$\lambda_a^T : T_a(M) \to T_{\lambda(a)}(N)$$

is a euclidean isomorphism $\forall a \in B(m, r)$. We first note, since u is linear, that

(6.4.26) $$u^T \circ \zeta_x^{-1} = \zeta_{u(x)}^{-1} \circ u, x \in T_m(M).$$

Now, for $a = m$, using (4.6) we have

$$\lambda_m^T = \zeta_{0_n} \circ u^T \circ \zeta_{0_m}^{-1} = \zeta_{0_n} \circ \zeta_{0_n}^{-1} \circ u = u$$

and hence we are through in case $a = m$. In the other case $\exists\, t \in]0, r[$ and $x \in U_m(M)$ such that

(6.4.27) $$a = {}^M \exp_m(t \cdot x).$$

Now let z be any element of $T_a(M)$. Since ${}^M \exp_m$ is r-O.K. and $a \in B(m, r)$ there exists an ω in $T_{t \cdot x}(T_m(M))$ such that

(6.4.28) $$z = {}^M \exp_m^T(\omega).$$

Now set

(6.4.29) $$y = \zeta(\omega).$$

By (6.23) we have

(6.4.30) $$z = h(\|t \cdot x\|) = h(t)$$

where h is the Jacobi field along γ_x such that

(6.4.31) $$h(0) = 0 \quad \text{and} \quad D_P h(0) = \frac{y}{\|t \cdot x\|} = \frac{y}{t}.$$

Further by the definition of λ and y we have

(6.4.32)
$$\lambda^T(z) = ({}^N \exp_n)^T \circ u^T \circ (\zeta_{tx}^{-1}(y)) = ({}^N \exp_n)^T \circ \zeta_{tx}^{-1}(u(y)) \quad \text{by (6.4.25)}$$

and hence, again by (6.23), we have

(6.4.33) $$\lambda^T(z) = h_1(\|t \cdot u(x)\|) = h_1(t)$$

(since u is a euclidean isomorphism) where h_1 is the Jacobi field along $\lambda_{u(x)}$ with

(6.4.34) $$h_1(0) = 0 \quad \text{and} \quad D_P h_1(0) = \frac{u(y)}{\|t \cdot u(x)\|} = \frac{u(y)}{t} = u\left(\frac{y}{t}\right).$$

Now considering v_t, h, h_1 and applying (because of (6.4.31) and (6.4.34)) the lemma we have

(6.4.35) $$h_1(t) = v_t \circ h(t) \ \forall t$$

where v_t is given by (6.4.18). But v_t is a euclidean isomorphism and hence the result follows by (6.4.33), (6.4.33) and (4.36).

4.35

Corollary. *Any two r.m.'s of constant sectional curvature k are locally isometric. Hence, by* (6.3.12) *any r.m. of constant sectional curvature k is locally isometric to*

$$(\mathbb{S}^d, k^{-1/2}, \text{can}) \ \ \textit{if} \ \ k > 0$$
$$(\mathbb{R}^d, \epsilon) \ \ \textit{if} \ \ k = 0$$
$$(\mathbb{R}^d, k^{-1/2}, \text{hyp}) \ \ \textit{if} \ \ k < 0.$$

4.36

Remarks. 1) For a global result on manifolds of constant sectional curvature see (7)

2) For a global result in general see [1]

3) In general the knowledge of sectional curvature (which is the same by (4.1) as that of curvature tensor) does not determine, even
199 locally, the r.s. In fact there exist r.m.'s (M, g) and (N, h) such that there exists a diffeomorphism λ such that

i) ${}^{M}A(P) = {}^{N}A(\lambda^{T}(P)) \forall P \in \mathscr{P}(M)$ and

ii) λ is not an isometry.

The theorem (4.11) says that if for all geodesics through a point, λ preserves the sectional curvature of planes corresponding under parallel transport along geodesics, then λ is a local isometry at this point.

5 Jacobi fields in symmetric pairs

In this article we assume that (M, g) is an r.m. associated to a symmetric pair (G, H) and follow the notation of (3) and of (4).

Let f be a geodesic in M. Then by (1.8) the parallel transport $\tau(f', t)$ along f is given by the tangent maps of isometries. Hence by (1.6) we have the:

5.1

Proposition. *Let m be a point of* (M, g). *Then*

i) $\tau(x,t)(R(x,y)x) = R(\tau(x,t)x, \tau(x,t)y)\tau(x,t)x \ \forall x \in U_m(M), \forall y \in T_m(M)$ *and*

ii) $A(x,y) = A(\tau(x,t)x, \tau(x,t)y)$ *if x, y are linearly independent.*

5.2

Remark. The converse of (5.1) ii) is true: an r.m. where parallel transport preserves sectional curvature is locally isometric to a symmetric space: see Chapter IV of [14]. The relation (5.1) i) gives a complete in- **200**
formation about Jacobi fields of M. For now the formula (6.4.15) gives

(6.5.3) $$\frac{d^2\widehat{h}}{\mathrm{dt}^2} = R(x, \widehat{h}(t))x.$$

Now let us introduce the endomorphism $\overline{R}(x)$ *of* $T_m(M)$ *by setting:*

(6.5.4) $$\overline{R}(x)y = R(x,y)x \quad \forall y \in T_m(M).$$

Then by ((5.5.5) C.T.4) we have

$$g(\overline{R}(x)y,z) = g(R(x,y)x,z) = g(R(x,z)x,y) = g(\overline{R}(x)z,y)$$

and hence $\overline{R}(x)$ is symmetric. Now by the reduction process of quadratic forms to diagonal form we see that there exist real numbers $\lambda_1,\ldots,\lambda_d$ and an orthonormal basis $\{x_1,\ldots,x_d\}$ of $T_m(M)$ such that

(6.5.5) $$\overline{R}(x)x_i = \lambda_i x_i, i = 1,\ldots,d.$$

But since $\overline{R}(x)x = 0$, and x is in $U_m(M)$ we can assume that $x_1 = x$ and $\lambda_1 = 0$. For any lift h of f into $T(M)$ there correspond functions $\psi_1,\ldots,\psi_d$ such that

$$\widehat{h}(t) = \sum_i \psi_i(t)x_i$$

and by (7.10) we have

$$\frac{d^2\widehat{h}(t)}{dt^2} = \sum_i \frac{d^2\psi_i(t)}{dt^2}\cdot x_i.$$

Further if h is a Jacobi field then by (6.5.3) we have

(6.5.6) $$\frac{d^2\psi_i}{dt^2} = \lambda_i\psi_i.$$

201 In particular we define for $i = 1,\ldots,d$ a Jacobi field h_i by

$$h_i(0) = 0 \quad \text{and} \quad (D_P h_i)(0) = x_i.$$

Then we have

(6.5.7) $$h_i(t) = \begin{cases} (-\lambda_i)^{-1/2}\cdot \sin(\sqrt{-\lambda_i}\cdot t)\cdot x_i & \text{if } \lambda_i < 0 \\ t\cdot x_i & \text{if } \lambda_i = 0 \\ (\lambda_i)^{-1/2}\cdot \mathrm{sh}(\sqrt{\lambda_i}\cdot t)\cdot x_i & \text{if } \lambda_i > 0. \end{cases}$$

Let us note that by ((5.5.5) C.T.4)

(6.5.8) $$\lambda_i = g(\overline{R}(x)x_i,x_i) = g(R(x,x_i)x,x_i) = -A(x,x_i).$$

6 Sectional curvature of S.C.-manifolds

In this article we assume that (M, g) stands for an S.C.-manifold and follow the notations of (4). We already know (by (2)) that $A(M) = \{1\}$ for $(\mathbb{S}^d, \text{can})$ and $(P^d(\mathbb{R}), \text{can})$. In the following proposition we think of the remark (5.6) for the case of $P^2(\Gamma)$: then "y is orthogonal to $K \cdot x$" stands for "y is orthogonal to the fibre through x". Note that the proof below, for the case of $P^2(\Gamma)$, uses only (5.7).

6.1

Proposition. *Let (M, g) be an S.C.-manifold, which is $(P^n(K), \text{can})$ with $n > 1$, $K \neq \mathbb{R}$ or $P^2(\Gamma)$.*

Then

$$A(M) = \{1, 4\};$$

more precisely, if x, y are $\mathbb{R}$-independent: **202**

i) $A(x, y) = 4$ *if x, y are K-dependent*

ii) $A(x, y) = 1$ *if y is orthogonal to $K \cdot x$.*

Proof. ***** (6.6.2). By (5.4) all geodesics through m_0 are closed and are of length π. By (8.26) every Jacobi field along a geodesic γ_x can be realised as the variations of vectors along a family of geodesic curves. Hence it follows that any Jacobi field along a geodesic through m_0 which vanishes at m_0 for $t = 0$ vanishes again for $t = \pi$. Now let $x \in U_{m_0}(M)$ and the corresponding objects

$$\{x = x_1, x_2, \ldots, x_d\}, \{\lambda_1 = 0, \lambda_2, \ldots, \lambda_d\}, \{h_1, h_2, \ldots, h_d\}$$

be as in (5). Hence $h_i(\pi) = 0$ implies, necessarily by (6.5.7) that $\lambda_i < 0$ and we have

$$h_i(t) = (-\lambda_i)^{-1/2} \cdot \sin(\sqrt{-\lambda_i} \cdot t) \cdot x_i. \tag{6.6.3}$$

Again $h_i(\pi) = 0$ implies $\lambda_i \in \{-1, -4, -9, \ldots\}$. Now by (8.32) and (5.12) we know $h_i(t)$ cannot vanish on $]0, \frac{\pi}{2}[$, hence $\lambda_i = -1$ or -4.

This implies already that $A(M) \subset \{1, 4\}$ for the eigen values of $\overline{R}(x)$ on the orthogonal complement of x are 1 or 4 (by (6.5.8)).

(i) Now let x_i be K-dependent on x and let h be the Jacobi field along γ_x given by the variations of vectors along the one parameter family of geodesics

$$\gamma_{(\cos\alpha)\cdot x+(\sin\alpha)\cdot x_i}(t).$$

203 Then, since each $(\cos\alpha)\cdot x + (\sin\alpha)\cdot x_i$ is K-dependent on x by (5.7) (ii), it follows that all of them meet at $t = \dfrac{\pi}{2}$ and hence

$$h_i\left(\frac{\pi}{2}\right) = 0,$$

hence $\lambda_i = -4$.

(iii) Suppose conversely that $\lambda_i = -4$ and set

$$e(\alpha) = (\cos\alpha)\cdot x + (\sin\alpha)\cdot x_i \quad \text{and} \quad S(\alpha) = \exp\left(\frac{\pi}{2}\cdot e(\alpha)\right).$$

Then (see (6.3.6)) we have

$$S'(\alpha) = \exp^T_{m_0}\left(\zeta^{-1}_{\frac{\pi}{2}\cdot e(\alpha)}\left(\frac{\pi}{2}\cdot e\left(\alpha+\frac{\pi}{2}\right)\right)\right).$$

But by (6.23) we have

$$S'(\alpha) = h\left(\frac{\pi}{2}\right) \tag{6.6.6}$$

where h is the Jacobi field along $\gamma_{e(\alpha)}$ satisfying the equations

$$h(0) = 0 \quad \text{and} \quad D_P h(0) = e\left(\alpha + \frac{\pi}{2}\right).$$

Since the subspaces generated by the pairs $\{e(\alpha), e(\frac{\pi}{2}+\alpha)\}$ and $\{x, x_i\}$ are the same we have

$$A\left(e(\alpha), e\left(\frac{\pi}{2}+\alpha\right)\right) = A(x, x_i) = 4.$$

Since 4 is the *maximum value* of the quadratic form $A(e(\alpha), *)$, it follows that $e(\alpha + \frac{\pi}{2})$ is an eigen vector of $\overline{R}(e(\alpha))$. Hence in view of (6.6.3) we have

$$h\left(\frac{\pi}{2}\right) = 0$$

and hence by (6.6.6) we have

$$S'(\alpha) = 0 \ \forall \alpha.$$

Hence $S(\alpha) = S(0) = \exp(\frac{\pi}{2} \cdot x)$ for every α, and hence in partic- **204**
ular

$$\exp\left(\frac{\pi}{2} \cdot x_i\right) = S(1) = S(0) = \exp\left(\frac{\pi}{2} \cdot x\right).$$

Then by (5.7) we see that $\{x, x_i\}$ are K-dependent. We conclude that $\lambda_i = -4$ if and only if x_i is K-dependent on x. Since $\{x_i\}$ is an orthogonal basis, we see that $\lambda_i = -1$ if and only if x_i is orthogonal to $K \cdot x$. Hence $\overline{R}(x)$ is -4 times the identity on the orthogonal complement of x in Kx and -1 times the identity on the orthogonal complement of Kx(in T_{m_0}). Thus, if $K \neq \mathbb{R}$, $\exists\, i$ such that $\lambda_i = -4$; if $n > 1$, then $\exists\, i$ such that $\lambda_i = -1$.

□

6.8

Remark. Let (M, g) be any S.C.-manifold and $x \in U(M)$. Then (upto a permutation of the i's) we have for the associated set $\{\lambda_i\}(i \geq 2)$

$P^d(\mathbb{R}), \mathbb{S}^d(\mathbb{R})$: $\lambda_i = -1 \forall i$
$P^n(\mathbb{C})$: $\lambda_2 - 4, \lambda_i = -1 \quad i \geq 3$
$P^n(\mathbb{H})$: $\lambda_2 = \lambda_3 = -4, \lambda_i = -1 \quad i \geq 5$
$P^2(\Gamma)$: $\lambda_i = -4 : i = 2, \ldots, 8, \lambda_i = -1 : i = 9, \ldots, 16.$

7 Volumes of S.C. manifolds

Now we give an example of calculation. We show how the exponential map and Jacobi fields can be used to estimate the volume of an r.m. In sharper recent results this has been systematically done to get bounds for Vol(M, g) in terms of those of $A(M)$ (see [33], Ch. 11)). Here we shall be content with the treatment of a case where the computation can be explicitly carried out and, at the same time, is simple enough to exhibit **205** clearly the method involved. The same method works in more general situations but we do not have space here to handle them.

We compute the volume of an S.C. manifold. Since we know (see (5.6)) that

$$\mathrm{Vol}(\mathbb{S}^d, \mathrm{can}) = 2\,\mathrm{Vol}(P^d(\mathbb{R}), \mathrm{can}),$$

we may assume that (M, g) is different from $\mathbb{S}^d$. First let us note that the fact that any point m can be joined to m_0 by a geodesic (see proof of (5.4)) together with the fact that each geodesic is of length π implies that

$$\exp_{m_0}(\overline{\underline{B}(m_0, \pi/2)}) = M.$$

But $\exp_{m_0}$ is a C^∞-map and the set $\overline{\underline{B}(m_0, \pi/2)} - \underline{B}(m_0, \pi/2)$ has measure zero. Hence we have

$$\begin{aligned}\mathrm{Vol}(M, g) &= \mathrm{Vol}(\exp_{m_0}(\overline{\underline{B}(m_0, \pi/2)})) \\ &= \mathrm{Vol}(\exp_{m_0}(\underline{B}(m_0, \pi/2)) \\ &= \mathrm{Vol}(B(m_0, \pi/2)).\end{aligned}$$

By (5.12) $\exp_{m_0}$ is $\pi/2$-O.K., and if we fix an orientation on $T_{m_0}(M)$ and pass to the canonical volume form σ on $B(m_0, \pi/2)$, (see (5))we have

$$\mathrm{Vol}(B(m_0, \pi/2)) = \int_{B(m_0,\pi/2)} \sigma = \int_{\underline{B}(m_0,\frac{\pi}{2})} (\exp_{m_0})^*\sigma.$$

206 To compute the latter we look upon $\underline{B}(m, \pi/2) - \{0\}$ as diffeomorphic to the product manifold

$$W =]0, \pi/2[\times U_{m_0}(M),$$

under the map

$$n : W \ni (t, x) \to t \cdot x \in \underline{B}(m_0, \pi/2) - \{0\}.$$

Here n clearly is a diffeomorphism and since the point m_0 has zero measure we have

$$\mathrm{vol}(M, g) = \int_{\underline{B}(m_0,\pi/2)} (\exp_{m_0})^*\sigma$$

$$= \int_W (\exp_{m_0} \circ n)^* \sigma.$$

Now let us introduce two projection maps on W defined by the equations

$$p(t, x) = t \quad \text{and} \quad q(t, x) = x.$$

Denoting the canonical volume form on $U_{m_0}(M)$ by θ we get a volume form

$$\omega = p^*(\mathrm{dt}) \wedge q^*(\theta)$$

on W. But then there exists a function φ such that

$$(\exp_{m_0} \circ n)^* \sigma = \varphi\omega.$$

and we will be through if we know φ.

7.1

Lemma. *With the above notation*

$$\varphi(t, x) = \left(\frac{\sin 2t}{2}\right)^{\alpha} \quad (\sin t)^{d-\alpha-1}$$

for every x in $\mathbb{S}^{d-1}$ and t in $]0, \pi/2[$, where

$$\alpha = 0 \ \ for \ \ P^d(\mathbb{R}); \alpha = 1 \ \ for \ \ P^d(\mathbb{C});$$
$$\alpha = 3 \ \ for \ \ P^d(\mathbb{H}); \alpha = 7 \ \ for \ \ P^2(\Gamma).$$

207

Proof. Let us follow the notation of article 6. The tangent space $T_{(t,x)}(W)$ is in a natural way isomorphic to $T_t(\,]0, \pi/2[\,) \times T_x(U_{m_0}(M))$. But (see 2.9) the vectors $\{\zeta_x^{-1} x_2, \ldots, \zeta_x^{-1} x_d\}$ form a basis for $T_x(U_{m_0}(M))$ and hence

$$\{(P, 0), (0, \zeta_x^{-1} x_2), \ldots, (0, \zeta_x^{-1} x_d)\}$$

is a basis for $T_{(t,x)}(M)$ such that

$$\omega((P, 0), (0, \zeta_x^{-1} x_2), \ldots, (0, \zeta_x^{-1} x_d)) = 1.$$

Hence we have

$$\varphi(t,x) = ((\exp_{m_0} \circ n)^*)((P,0),(0,\zeta_x^{-1}x_2),\dots,(0,\zeta_x^{-1}x_d)) =$$
$$= (\exp^T_{m_0}(n^T(P,0),\exp^T_{m_0}(n^T(0,\zeta_x^{-1}x_2),\dots,)$$
$$\dots,\exp^T_{m_0}(n^T(0,\zeta_x^{-1}x_d))).$$

Since the map

$$n : (t,x) \to t\cdot x$$

is bilinear, we have

$$n^T(0,\zeta_x^{-1}x_i) = t\cdot\zeta_{t\cdot x}^{-1}x_i \quad \text{and} \quad n^T(P,0) = \zeta_{tx}^{-1}x.$$

Further by (6.32) the images

$$\exp^T_{m_0}(\zeta_{tx}^{-1}x_1),\exp^T_{m_0}(\zeta_{tx}^{-1}x_2),\dots$$

are mutually orthogonal and

$$\|\exp^T_m(\zeta_{tx}^{-1}x)\| = \|\zeta_{tx}^{-1}x\| = 1.$$

208 Hence we have

$$\varphi(t,x) = \prod_{i>2}\|\exp^T_{m_0}(\zeta_{tx}^{-1}t\cdot x_i)\|.$$

But by (6.23) we have

$$\exp^T_{m_0}(\zeta_{t\cdot x}^{-1}(t\cdot x_i)) = h_i(\|tx\|) = h_i(t)$$

where h_i is the Jacobi field along γ_x such that

$$h_i(0) = 0 \quad \text{and} \quad (D_P h_i)(0) = \frac{t\cdot x_i}{\|t\cdot x_i\|} = x_i.$$

But then by (6.5.7) we have

$$h_i(t) = (-\lambda_i)^{-1/2}\cdot\sin(\sqrt{-\lambda_i}\cdot t)\cdot x_i$$

and hence

(6.7.2) $$\varphi(t,x) = \prod_{i>2}(-\lambda_i)^{-1/2}\cdot\sin(\sqrt{-\lambda_i}\cdot t)$$

and we are through by (6.8). □

Now let us note that since all euclidean structures of the same dimension are isomorphic we have

$$\mathrm{Vol}(U_m(M), g_m) = \mathrm{Vol}(\mathbb{S}^{d-1}, \mathrm{can})\ \forall m \in M.$$

We set

$$\mathrm{Vol}(\mathbb{S}^d, \mathrm{can}) = \textcircled{d}\,. \tag{6.7.3}$$

Hence by (3.17):

$$\int\limits_W \varphi(t,x)\cdot p^*(\mathrm{dt}) \wedge q^*(\theta) = \int\limits_{U_{m_0}(M)} \left(\int\limits_0^{\pi/2} \varphi(t,x)\cdot \mathrm{dt}\right)\cdot \theta =$$

$$= \left(\int\limits_{U_{m_0}(M)} \theta\right)\cdot \int\limits_0^{\pi/2} \varphi(t,x)\cdot \mathrm{dt}$$

since φ is independent of x. **209**

Hence

$$\mathrm{Vol}(M,g) = \textcircled{d-1}\cdot \int\limits_0^{\pi/2} \left(\frac{\sin 2t}{2}\right)^{\alpha} \cdot \sin^{d-\alpha-1} t\cdot \mathrm{dt}\,. \tag{6.7.4}$$

First (6.7.4) gives for the $\textcircled{d}$'s the recurrence formula:

$$\textcircled{d} = 2\cdot \mathrm{Vol}(P^d(\mathbb{R}), \mathrm{can}) = 2\textcircled{d-1}\int\limits_0^{\pi/2} \sin^{d-1} t\cdot \mathrm{dt}\,.$$

The value of $\int_0^{\pi/2} \sin^{d-1} t\cdot \mathrm{dt}$ is well known. Using this, we get

$$\textcircled{2n} = 2\cdot \frac{(2\pi)^n}{(2n-1)\cdot(2n-3)\ldots 3.1}, \textcircled{2n-1} = 2\cdot \frac{\pi^n}{(n-1)!} \tag{6.7.5}$$

$$\mathrm{Vol}(P^{2n}(\mathbb{R}), \mathrm{can}) = \frac{(2\pi)^n}{(2n-1)\cdot(2n-3)\ldots 3.1}, \mathrm{Vol}(P^{2n-1}(\mathbb{R}), \mathrm{can}) = \frac{\pi^n}{(n-1)!}$$

Note, in particular, that

$$\textcircled{2} = \mathrm{Vol}(\mathbb{S}^2, \mathrm{can}) = 4\pi, \mathrm{Vol}(P^2(\mathbb{R}), \mathrm{can}) = 2\pi. \tag{6.7.6}$$

For $P^n(\mathbb{C})$ we compute

$$\int_0^{\pi/2} \frac{\sin 2t}{2} \cdot \sin^{2n-2} t \cdot \mathrm{dt} = \int_0^{\pi/2} \sin^{2n-1} t \cdot \cos t \cdot \mathrm{dt} = \frac{1}{2n};$$

with (6.7.5) this gives

$$\mathrm{Vol}(P^n(\mathbb{C}), \mathrm{can}) = \frac{\pi^n}{n!} \tag{6.7.7}$$

210 By the same kind of straightforward computations,

$$\mathrm{Vol}(P^n(\mathbb{H}), \mathrm{can}) = \frac{\pi^{2n}}{(2n+1)!}, \tag{6.7.8}$$

$$\mathrm{Vol}(P^2(\Gamma), \mathrm{can}) = \frac{\pi^8}{11.10.9.8.7.6.5.4.}. \tag{6.7.9}$$

7.10

Remark. Using (3.4.10) and (4.19), and the formula (3.17) one would get

$$\mathrm{Vol}(P^n(\mathbb{C}), \mathrm{can}) = \textcircled{2n+1}/\textcircled{1},$$
$$\mathrm{Vol}(P^n(\mathbb{H}), \mathrm{can}) = \textcircled{4n+3}/\textcircled{3},$$

which agrees with (6.7.5), (6.7.6), (6.7.7).

Note also that

$$P^2(\Gamma) = \textcircled{23} / \textcircled{7}.$$

But here it is impossible to get that formula through a fibration because one knows from algebraic topology (non existence of elements of Hopf invariant one) that there does not exist a fibration

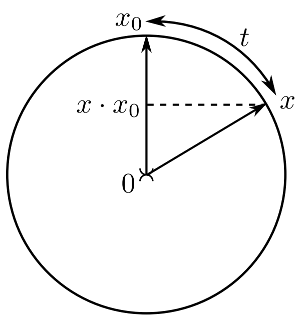
$$\begin{array}{c} \mathbb{S}^{23} \\ \big\downarrow {\scriptstyle \mathbb{S}^7} \\ P^2(\Gamma) \end{array}$$

Next, we give a formula which will be used later on:

(6.7.11)
$$I = \int_{x \in \mathbb{S}^d} |x \cdot x_0| \cdot \sigma = \frac{2}{d} \boxed{d-1},$$

where σ is the canonical volume form of $(\mathbb{S}^d, \mathrm{can})$ and x_0 a fixed point in $\mathbb{S}^d$.

Proof. If x is at the distance t from x_0 *on* $\mathbb{S}^d$ then $x \cdot x_0 = \cos t$. We **211**
make the computations on $P^d(\mathbb{R})$ instead of $\mathbb{S}^d$ using the remark that $x \to |x \cdot x_0|$ is invariant under the antipodal map of $\mathbb{S}^d$.



Thus $I = 2\int_{p^d(\mathbb{R})} p(|x{\cdot}x_0|){\cdot}\overline{\sigma}$ where $\overline{\sigma}$ denotes the volume element on $p^d(\mathbb{R})$. We follow the computation of $\mathrm{Vol}(P^d(\mathbb{R}), \mathrm{can})$ as above, taking $p(x_0) = m_0$. This gives

$$I = 2 \int_{U_{m_0}(M)} \left(\int_0^{\pi/2} \varphi(t,x) \cdot |\cos t| \cdot \mathrm{dt} \right) = \ldots$$
$$= 2\boxed{d-1} \int_0^{\pi/2} \sin^{d-1} t \cdot \cos t \cdot \mathrm{dt} = \frac{2}{d} \boxed{d-1}.$$

□

8 Ricci and Scalar curvature

In this article we assume that the dimension d of the manifold (M, g) is greater than one.

For every point m of M and positive number r we get

$$\begin{aligned} &\underline{S}(m,r) = \{x \in T_m(M) \,\big|\, \|x\| = r\} \\ \text{and when}\quad &\underline{S}(m,r) \subset \Omega \quad \text{we set} \\ &S(m,r) = \exp(\underline{S}(m,r)). \end{aligned}$$

If $\exp_m$ is r-O.K. then for every r' in $]0, r[$ $S(m, r')$ is a sub manifold of
212 M of dimension $d-1$. Now we calculate the volume

$$\sigma(m, r') = \mathrm{Vol}(S(m, r'), g|S(m, r')).$$

First let us relate $U_m(M)$ and $\underline{S}(m, r)$ by defining the map

$$\overline{r}' : U_m(M) \ni x \to r' \cdot x \in \underline{S}(m, r')$$

so that if $\exp_m$ is r-O.K. then taking some orientation on $T_m(M)$ we get by means of $\exp^{-1}_{m_0} |B(m, r)$ and $\overline{r}'$ a volume form σ on $S(m, r')$. Then we have

$$\sigma(m, r') = \int\limits_{S(m,r')} \sigma = \int\limits_{U_m(M)} (\exp_m \circ \overline{r}')^* \sigma.$$

But if we denote the canonical volume form on $U_m(M)$ by θ then there exists a φ in $F(U(M))$ such that

$$(\exp_m \circ \overline{r}')^* \sigma = \varphi \cdot \theta, \tag{6.8.1}$$

and we will be through if we can compute φ. Now let $\{x = x_1, x_2, \ldots, x_d\}$ be an orthonormal basis of $T_m(M)$. Then by the definition of θ we have

$$\theta(\zeta_x^{-1} x_2, \zeta_x^{-1} x_3, \ldots) = 1.$$

Now let us note that

$$\overline{r}'^T (\zeta_x^{-1} x_i) = \zeta_{r'x}^{-1} (r' x_i) \quad \text{for} \quad i = 2, \ldots, d$$

and that Gauss' Lemma (see (6.24)) gives that the vectors

$$(\exp_m \circ \overline{r}')^T(\zeta_x^{-1} x_i) = \exp_m^T(\zeta_{r'x}^{-1} r' x_i), i \geq 2$$

213 are all orthogonal to the vector $\exp_m^T(\zeta_{r'x}^{-1} x)$ and hence are tangent to $S(m, r')$. Hence we have

$$\begin{aligned}\varphi(x) &= ((\exp_m \circ \overline{r}')^* \sigma)(\zeta_x^{-1} x_2, \ldots, \zeta_x^{-1} x_d)\\ &= \sigma(\exp_m^T(\zeta_{r'x}^{-1} r x_2), \ldots, \exp_m^T(\zeta_{r'x}^{-1} r x_d).\end{aligned}$$

But by (6.23)

$$\exp_m^T(\zeta_{r'x}^{-1} r' x_i) = h_i(r')$$

where h_i is the Jacobi field along γ_x satisfying the conditions

$$h_i(0) = 0 \quad \text{and} \quad (D_P h_i)(0) = \frac{r' x_i}{\|r' x_i\|} = x_i.$$

Hence

$$\begin{aligned}\varphi(x) &= \sigma(h_2(r'), \ldots, h_d(r')) = \|h_2(r') \wedge \ldots \wedge h_d(r')\| = \ldots \text{ (by (4.9))}\\ &= \|\tau(x, r')^{-1} h_2(r') \wedge \ldots \wedge \tau(x, r')^{-1} h_d(r')\| = \ldots\end{aligned}$$

(6.8.2)

$$= \|\widehat{h}_2(r') \wedge \ldots \wedge \widehat{h}_d(r')\|.$$

But by (8.18) we have

$$h_i(r') = r' x_i + \frac{r'^3}{6} R(x, x_i)x + 0(r'^3).$$

Hence we have (for $\overline{R}$ as introduced in (5)):

$$\varphi(x) = \|(r')^{d-1}(x_2 \wedge \ldots \wedge x_d) + \frac{(r')^{d+1}}{6}$$

(6.8.3)

$$\sum_{i=2}^{d} x_2 \wedge \ldots \wedge x_{i-1} \wedge \overline{R}(x) x_i \wedge x_{i+1} \ldots \wedge x_d + 0(r'^{d+1})\|.$$

But 214

$$\sum_{i=2}^{d} x_2 \wedge \ldots \wedge x_{i-1} \wedge \overline{R}(x)x_i \wedge x_{i+1} \wedge \ldots \wedge x_d = \ldots$$

$$= \sum_{i=2}^{d} g(\overline{R}(x)x_i, x_i)(x_2 \wedge x_3 \wedge \ldots \wedge x_d)$$

$$= (\text{Trace } (\overline{R}(x))) \cdot (x_2 \wedge \ldots \wedge x_d).$$

8.4

Definition. *We set for $x \in T(M)$:*

$$\text{Ric}(x) = -\,\text{Trace}(\overline{R}(x))$$

and call it the Ricci curvature of (M, g) at x.

Then we have

(6.8.5) $$\varphi(x) = (r')^{d-1}\left(1 - \frac{(r')^2}{6}\,\text{Ric}(x) + 0((r')^2)\right)$$

8.6

Note. If $x \in U(M)$, we have, since x and x_i are orthonormal,

$$g(\overline{R}(x)x_i, x_i) = g(R(x, x_i)x, x_i)$$
$$= -A(x, x_i).$$

Hence $\{x = x_1, x_2, \ldots, x_d\}$ are orthonormal and we have

$$\text{Ric}(x) = \sum_{i=2}^{d} A(x, x_i).$$

8.7

Remarks. i) Ric is a quadratic form on $\mathscr{C}(M)$. To see this let us define for $X, Y \in \mathscr{C}(M)$ a map $\overline{\overline{R}}(X, Y) : \mathscr{C}(M) \to \overline{\overline{C}}(M)$ by the equation

$$\overline{\overline{R}}(X, Y)(Z) = R(X, Z)Y.$$

215 Then the trace of the map $\overline{\overline{R}}(X, Y)$ is a symmetric bilinear map on $\mathscr{C}(M) \times \mathscr{C}(M)$: for, if $\{x_1, \ldots, x_d\}$ is an orthonormal base of $T_m(M)$, then we have

$$\text{Trace}(\overline{\overline{R}}(x, y)) = \sum_i g(R(x, x_i)y, x_i) =$$
$$\sum_i g(R(y, x_i)x, x_i \text{ by C.T.4 } = \text{Trace}(\overline{\overline{R}}(y, x)).$$

Hence we have

$$\text{Ric}(X) = \text{Trace}(R(X)) = \text{Trace}(\overline{\overline{R}}(X, X)).$$

ii) Since Ric can be considered as a quadratic form on $T_m(M)$ the trace of Ric gives a function on M.

8.8

Definition. *We set*

$$\Gamma = \text{Trace}(\text{Ric})$$

and call it the scalar curvature *of M.*

Then if $\{x_i\}$ is an orthonormal basis of $T_m(M)$ we have

$$\Gamma(m) = \sum_i \text{Ric}(x_i) = \sum_{i \neq j} A(x_i, x_j). \tag{6.8.9}$$

In particular if the dimension is 2 *we have* $\Gamma = 2A$ *where A, the sectional curvature, is considered as a function on the manifold.*

Now let us take up the calculation of $\sigma(m, r)$. By (6.8.1) we have

$$\sigma(m, r) = \int\limits_{U_m(M)} \varphi(x)\theta =$$

$$= \int\limits_{U_m(M)} (r')^{d-1}\left(1 - \frac{r'^2}{6}\text{Ric}(x) + 0(r')^2\right) = \tag{6.8.10}$$

$$= (r')^{d-1} \cdot \boxed{d-1} - \frac{(r')^{d+1}}{6} \int\limits_{U_m(M)} \mathrm{Ric}(x) \cdot \theta + 0((r')^{d+1}).$$

Since Ric is a quadratic form on $T_m(M)$, there exist constants $\lambda_1, \dots, \lambda_d$ **216**
and a system of orthonormal vectors $x_1, \dots, x_d$ such that, if $x = x^1 x_1 + \cdots + x^d x_d$, then

$$\mathrm{Ric}(x) = \sum \lambda_i (x^i)^2.$$

Then

$$\int\limits_{U_m(M)} \mathrm{Ric}(x)\theta = \sum_i \lambda_i \int\limits_{U_m(M)} (x^i)^2 \theta.$$

But since by symmetry

$$\int\limits_{U_m(M)} (x^i)^2 \theta = \int\limits_{U_m(M)} (x^j)^2 \theta \quad \forall i, j$$

we have

$$d \cdot \int\limits_{U_m(M)} (x^1)^2 \theta = \int\limits_{U_m(M)} ((x^1)^2 + (x^2)^2 + \cdots + (x^d)^2)\theta = \dots$$

$$= \int\limits_{U_m(M)} \theta \text{ since } \|x\| = 1.$$

Hence we have

(6.8.11) $$\int\limits_{U_m(M)} \mathrm{Ric}\,\theta = \sum \lambda_i \frac{\boxed{d-1}}{d} = \frac{\boxed{d-1}}{d} \cdot \mathrm{Trace}(\mathrm{Ric}) = \dots$$

$$= \Gamma(m) \cdot \frac{\boxed{d-1}}{d}.$$

Hence by (6.8.10) we have

(6.8.12) $$\sigma(m, r') = \boxed{d-1} (r')^{d-1} \left(1 - \frac{(r')^2}{6} \Gamma(m) + 0(r')^2\right).$$

Now by (6.8.12) we have

(6.8.13) $$\Gamma(m) = \lim_{r \to 0} \frac{6d}{r^{d+1}} (r^{d-1} \cdot \boxed{d-1} - \sigma(m, r)).$$

8.14

Remark. When the dimension of the manifold is two the Gaussian curvature has deep implications on the topology of the manifold (see Gauss-Bonnet formula (6.1)). But for dimension greater than two the situation is quite different. For example on $\mathbb{S}^3$ we can define a homogeneous r.s. for which Γ is constant and is of given sign. Further let M be a compact manifold which is oriented by a form ω. Then there exists an r.s. g on M such that, if we denote the canonical orientation of (M, g) by ω_1, then **217**

$$\int_M \Gamma\omega_1$$

has a preassigned sign, (see [3]). However, if Γ is everywhere positive then there are certain topological restrictions on M (see [20]: theorem 2, p.9).

Chapter 7

The metric structure

1.1

218

In this chapter we assume that all the manifolds we consider are *connected.*

1. Identity of balls

1.2

Definition. *For every pair of points m and n in* (M, g) *we set*

$$\mathscr{P}(m, n) = \{C | C \text{ is a path from } m \text{ to } n\}.$$

Since we have assumed that M is connected it follows that M is arc wise connected and hence that

$$\mathscr{P}(m, n) \neq \emptyset \ \forall m, n \in M.$$

Now set

$$d(m, n) = \inf_{C \in \mathscr{P}(m,n)} \lg(C). \tag{7.1.3}$$

1.4

Then, since the length of a path is non-negative we have

$$d(m, n) \geq 0;$$

since every path from m to n gives rise to a path from n to m with the same length we have

$$d(m, n) = d(n, m);$$

and since a path from m to p and another from p to n give rise to one from m to n we have

$$d(m, n) \leq d(m, p) + d(p, n).$$

Hence d has all the properties of a metric structure except perhaps the one which asserts that $d(m, n) = 0$ only if $m = n$. Now we shall proceed
219 to show that this property is also valid; in fact, we shall prove much more, namely, that locally geodesics realise the distance d.

1.5

Now throughout this article, let us fix a point m and a positive number r such tht $\exp_m$ is r-O.K., and set:

$$\lambda = (\exp_m |B(m, r))^{-1}.$$

1.6

Lemma. *Let n and n' be points in $B(m, r)$ and let the map*

$$C : [0, t_0] \to M$$

be a path such that

i) *the image of C does not contain m,*

ii) *the image of C lies wholly in $B(m, r)$.*

Then we have

$$\lg(C) \geq \left| \|\lambda(n')\| - \|\lambda(n)\| \right|,$$

and equality holds if and only if

i) *the three points 0_m, $\lambda(n)$ and $\lambda(n')$ are in a straight line and*

ii) $\lambda \circ C$ *lies in the segment* $[\lambda(n), \lambda(n')]$ *and is injective.*

Proof. Since C is made up of finite number of curves and for any three real numbers a, b and c we have

$$|a - c| \leq |a - b| + |b - c|$$

we need only prove that results for curves. So let us assume that C is a curve:

$$C \in D([0, t_0], M),$$

and set 220

$$b = \lambda \circ C. \tag{7.1.8}$$

Since by i) 0_m is not in the image of C the map $\dfrac{b}{||b||}$ makes sense and we write x for it:

$$x = \frac{b}{||b||} \tag{7.1.9}$$

Then we have

$$b' = ||b||' \cdot \zeta_x^{-1} x + ||b|| x'. \tag{7.1.10}$$

By the definition of x we have $||x|| = 1$ and hence we have

$$g(\zeta_x^{-1} x, x') = 0$$

and hence by Gauss' lemma (see (6.24)) we have

$$g(\exp_m^T(||b|| \cdot \zeta_x^{-1} x), \exp_m^T(||b|| \cdot x')) = 0. \tag{7.1.12}$$

Now by (7.1.8) and (7.1.10) we have

$$\begin{aligned} ||C'||^2 &= || \exp_m^T \circ b' ||^2 = \\ &= || \exp_m^T(||b||' \cdot \zeta_x^{-1} x)||^2 + 2g(\exp_m^T(||b||' \zeta_x^{-1} x), \exp_m^T(||b|| \cdot x')) + \\ &\quad + || \exp^T(||b|| \cdot x')||^2 = \end{aligned} \tag{7.1.13}$$

$$= \|\exp_m^T(\|b\|' \cdot \zeta_x^{-1} x)\|^2 + \|\exp_m^T(\|b\| x')\|^2, \quad \text{by (7.1.12).}$$

But by (6.32) i) we have

$$\|\exp_m^T(\zeta_x^{-1} x)\| = \|x\| = 1$$

and hence

$$\|C'\|^2 = (\|b\|')^2 + \|\exp_m^T(\|b\| x')\|^2, \geq (\|b\|')^2 \tag{7.1.14}$$

221 equality holding if and only if $\|\exp_m^T x'\| = 0$. □

1.15

Since $\exp_m$ is r-O.K., this is equivalent to saying that

$$\|C'\| \geq \|b\|',$$

and that equality holds if and only if $x' = 0$.

Now we have

$$\lg(C) = \int_0^{t_0} \|C'\| \,\mathrm{dt} \geq \int_0^{t_0} \|b\|' \,\mathrm{dt} \geq \left| \|\lambda(n')\| - \|\lambda(n)\| \right|. \tag{7.1.16}$$

1.17

Further the last inequality becomes equality if and only if $\|b\|'$ is of the same sign.

1.18

Hence by (1.15), (7.1.16) and (1.17) we have

$$\lg(C) \geq \left| \|\lambda(n')\| - \|\lambda(n)\| \right|$$

where equality holds if and only if $x' = 0$ and $\|b\|'$ is of the same sign.

This means that equality holds if and only if x is constant, and b is injective. This means that b should lie on the line joining 0_m and $\lambda(n)$ and b is injective.

1.19

Note. In the case $\lambda \circ C$ lies on a line through 0_m and is injective we say, for simplicity, that C is *a monotonic image by* $\exp_m$ *of the segment* $[(\lambda \circ C)(0), (\lambda \circ C)(t_0)]$.

1.20

Lemma. *If* $0 < r' < r$, *then* 222

$$\exp(\overline{\underline{B}(m, r')}) = \overline{B(m, r')}$$

and the latter is a compact set.

Proof. Since $\overline{\underline{B}(m, r')}$ is compact, exp continuous and M Hausdorff, $\exp(\overline{\underline{B}(m, r')})$ is compact. In particular $\exp(\overline{\underline{B}(m, r')})$ is closed.

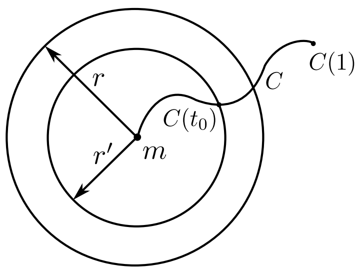
But since $\exp(\underline{B}(m, r')) \subset \exp(\overline{\underline{B}(m, r')})$ it follows that

$$\overline{\exp(\underline{B}(m, r'))} \subset \exp(\overline{\underline{B}(m, r')}).$$

In our case the other inclusion follows from the fact that exp is a diffeomorphism on $\underline{B}(m, r)$. □

1.21

Lemma. *Let*



$$C : [0, 1] \to M$$

be a path in M such that

$$C(0) = m \quad \textit{and} \quad C[(0,1]) \not\subset B(m,r).$$

Then $\forall r' \in]0, r[,\ \exists\, t_0 \in]0, 1[$ *such that*

$$\|\lambda(C(t_0))\| = r' \quad \textit{and} \quad C([0,t_0[) \subset B(m,r').$$

Proof. Let $J = \{s \in]0, 1[| C(s) \notin B(m,r')\}$. By (1.20) and the continuity of C, J is closed so that $t_0 = \inf J > 0$. We claim that t_0 has the property required above. By continuity: $C(t_0) = \lim_{t \underset{<}{\longrightarrow} t_0} C(t)$; and $C(t) \subset B(m,r') \forall t < t_0$ since, J being closed, $t_0 \in J$.

Hence by (1.20): $C(t_0) \in \overline{B(m,r')} = \exp(\overline{B(m,r')})$, i.e. $\|\lambda(C(t_0))\| \leq$
223 r'. But $t_0 \in J$ implies that $\|\lambda(C(t_0))\| \geq r'$. Further, $C([0,t_0[) \subset B(m,r')$ by definition of t_0. □

1.22

Lemma. *Let $n \in B(m,r)$ and let*

$$C : [0, t_0] \to M$$

be a path, such that $C(0) = n$ and $C(t_0) = m$ and whose image lies in $B(m,r)$. Then

i) $\lg(C) \geq \|\lambda(n)\|$ *and*

ii) *in* i) *the equality holds if and only if C is a monotonic image by* $\exp_m$ *of the segment* $[\lambda(n), 0_m]$.

Proof. Let $t_1 = \inf_{t \in [0,t_0]} t | C(t) = m$.

Then for every sufficiently small positive number ϵ we have

$$\lg(C) \geq \lg(C|[0,t_1]) \geq \lg(C|[0,t_1-\epsilon]). \tag{7.1.23}$$

Since $m \notin C([0,t_1-\epsilon])$ by (1.6) we have

$$\lg(C|[0,t_1-\epsilon]) \geq \Big| \|\lambda(n)\| - \|(\lambda \circ C)(t_1-\epsilon)\| \Big|. \tag{7.1.24}$$

Since $\lg(C|[0,t])$ is a continuous function of t, taking limit as $\epsilon \to 0$ we have

$$\begin{aligned} \lg(C|[0,t_1]) &\geq \lim \Big| \|\lambda(n)\| - \|(\lambda \circ C)(t_1 - \epsilon)\| \Big| = \\ &= \Big| \|\lambda(n)\| - \|\lambda(m)\| \Big| = \|\lambda(n)\|. \end{aligned} \tag{7.1.25}$$

□

1.26

To prove ii) let us note that if equality holds above then

(i) $\lg(C|[t_1,t_0]) = 0$ and hence $C([t_1,t_0]) = \{m\}$ and

(ii) in the following sequence of inequalities equality should hold everywhere:

$$\begin{aligned} &\lg(C|[0,t_1]) = \lg(C|[0,t_1-\epsilon]) + \lg(C|[t_1-\epsilon,t_1]) \\ &\geq \Big| \|\lambda(n)\| - \|(\lambda \circ C)(t_1-\epsilon)\| \Big| + \|(\lambda \circ C)(t_1-\epsilon)\| \text{ (by i) } \geq \|\lambda(n)\|. \end{aligned}$$

Hence it follows that the above inequalities are equalities. Now 224
from (1.6) it follows from the first equality that $\forall\epsilon$:

$$(\lambda \circ C)(t_1-\epsilon) \in [0_m, \lambda(n)] \quad \text{and that} \quad C|[0,t_1-\epsilon]$$

is a monotone image by $\exp_m$ of the segment $[\lambda(n), (\lambda \circ C)(t_1-\epsilon)]$.

1.28

Lemma. *Let n be a point in M and let*

$$C : [0,t_0] \to M$$

be a path from m to n such that

$$C([0,t_0]) \not\subset B(m,r).$$

Then

$$\lg(C) \geq r.$$

Proof. Let $0 < r' < r$. Then by (1.21) $\exists t_1$ such that

$$\|\lambda(C(t_1))\| = r' \quad \text{and} \quad C([0, t_1[) \subset B(m, r').$$

Hence, by (1.22) we have

$$\lg(C) \geq \lg(C|[0, t_1]) \geq \|\lambda(C(t_1))\| = r'.$$

This being true for every $r' < r$ we are through. □

1.29

Theorem. *Let $n \in B(m, r)$. Then*

i) *$d(m, n) = \|\lambda(n)\|$, and the geodesic*

$$\gamma_{\lambda(n)/\|\lambda(n)\|}\Big|_{[0,\lambda(n)]}$$

realises the distance $d(m, n)$, and

ii) *it is the only one, i.e. if $C \in \mathscr{P}(m, n)$ is such that* $\lg(C) = d(m, n)$, *then C is a monotone image by* $\exp_m$ *of the segment* $[0_m, \lambda(n)]$.

225 *Proof.* The geodesic defined above has length $\|\lambda(n)\|$: see (4.3.3). Hence $d(m, n) \leq \|\lambda(n)\|$.

But if $C \in \mathscr{P}(m, n)$ then by (1.22) i) and (1.28) we have

$$\lg(C) \geq \|\lambda(n)\|.$$

Hence $d(m, n) \geq \|\lambda(n)\|$ and the first part follows.

To prove ii) let $C \in \mathscr{P}(m, n)$ and let $\lg(C) = \|\lambda(n)\|$. Then by (1.28) the image of C lies in $B(m, r)$ and the part ii) states the same thing as (1.22) ii). □

1.30

Notation. We set

$$D(n, s) = \{n' \in M | d(n, n') < s\}.$$

1.31

Corollary. *If* $\exp_n$ *is* s*-O.K. then*

$$B(n, s) = D(n, s).$$

Proof. We have only to prove that $D(n, s) \subset B(n, s)$; so let $n' \in D(n, s)$. Then $s' = d(n, n') < s$ and hence by the definition of d there exists a $C \in \mathscr{P}(n, n')$ such that $s' < \lg(C) < s$. Then by (1.28) the image of C has to lie in $B(n, s)$. Hence

$$n' \in B(n, s).$$

□

2 The metric structure

2.1

Proposition. *d is a metric structure.*

Proof. By (1.4) we have only to prove that $d(m, n) = 0$ implies $m = n$. Now let $d(m, n) = 0$. Then there exists an $r > 0$ such that $\exp_m$ is r-O.K. Hence by (1.31) it follows that $n \in B(m, r)$. Then by (1.29) the result follows. □

2.2

226

Whenever we consider the manifold (M, g) as a metric space, *it is this metric we deal with.* Hence we speak of *distance* without any mention of the metric structure involved. In particular, we may speak of bounded sets, of completeness..., in a r.m. (M, g).

2.3

Remark. Throughout we have assumed that M is Hausdorff. This is a necessary condition for a topology to come from a metric; Hausdorffness was used only in proofs of (1.20) and (1.21).

2.4

Proposition. *For a riemannian manifold (M, g), the underlying topology of M considered as a differentiable manifold and that induced by the metric structure are the same.*

Proof. Let M be a point of M and let $\exp_m$ be r-O.K. Then by (1.31) we have

$$D(m, r') = B(m, r') \quad \text{if} \quad 0 < r' < r.$$

But every neighbourhood of m in the topology induced by the metric d contains $D(m, r')$ for sufficiently small r', and every neighbourhood of m in the original topology contains $B(m, r')$ for sufficiently small r' because λ is a diffeomorphism. Hence the neighbourhood system of m in either topology is the same as that in the other. Hence the result. □

The following is a direct consequence of the above proposition.

2.5

Corollary. *The function $d : M \times M \to \mathbb{R}$ is continuous.*

2.6

227 **Corollary.** *We have*

$$\overline{D(m, r)} = \{n \in M | d(m, n) \leq r\} \forall r > 0.$$

Proof. In view of (1.31) and (2.5) we have only to prove that

$$\{n \in M | d(m, n) = r\} \subset \overline{D(m, r)}.$$

Suppose n be such that $d(m, n) = r$; $\forall \epsilon > 0$ (small enough) $\exists C \in \mathscr{P}(m, n)$ such that $\lg(C) \leq r + \epsilon$; by continuity we can find n_ϵ on the image of C with $d(m, n_\epsilon) = r - \epsilon$ and so: $n_\epsilon \in D(m, r)$; moreover:

$$d(n, n_\epsilon) \leq \lg(C|n \text{ to } n_\epsilon) = \lg(C) - \lg(C|m \text{ to } n_\epsilon) \leq r + \epsilon - d(m, n_\epsilon) \leq 2\epsilon$$

in particular $n_\epsilon \xrightarrow{\epsilon \to 0} n$, hence $n \in \overline{D(m, r)}$. □

2.7

Example. Let m be a point of $(\mathbb{S}^d, \text{can})$ and $\sigma(m)$ its antipodal point. Then since $\sigma(m) \notin B(m, \pi)$ and $\exp_m$ is π-O.K. (see (5)) we have $d(m, \sigma(m)) \geq \pi$.

But since there exists a great circular arc (geodesic) of length π joining m and $\sigma(m)$ we have

$$d(m, \sigma(m)) \leq \pi$$

Hence

$$d(m, \sigma(m)) = \pi.$$

3 Nice balls

If $\exp_m$ is r-O.K. then we know that $D(m, r) = B(m, r)$ and further that for any point n in $B(m, r)$ the distance $d(m, n)$ is realised by a geodesic and that such a realisation is essentially unique. Further (see (6.8) and **228** (3.7) we know that for sufficiently small r, $B(m, r)$ is convex and hence any two points n and n' of it can be joined by a geodesic $f_{n,n'}$ which is completely contained in $B(m, r)$ and that such a curve is unique. So a natural question is whether $f_{n,n'}$ realises the distance between n and n' and if so if it is the unique path with that property. Before trying to answer this question we give a name to the balls for which this (and more) is true.

3.1

Definition. *A ball $B(m, r)$ is called a* nice ball *if*

i) *$B(m, r)$ is convex*

ii) *$\exp_n$ is r-O.K. $\forall n \in B(m, r)$ and*

iii) *$\forall n$, $n' \in B(m, r)$ the geodesic $f_{n,n'}$ realises $d(n, n')$ uniquely, i.e., that $d(n, n') = \lg(f_{n,n'})$ and upto a re parametrisation by a function $f_{n,n'}$ is the only path that realises $d(n, n')$.*

Now we shall prove that there are nice balls with assigned centres.

3.2

Lemma. *Suppose that $x \in U_m(M)$ and that $\exists\, t_0 > 0$ such that*

i) $[0, t_0[\cdot x \subset \Omega$ *and*

ii) *there exists a sequence $\{t_n\}$ of numbers in $[0, t_0[$ and a point p in M such that $t_n \to t_0$ and $\gamma_x(t_n) \to p$ as $n - \infty$.*

Then $t_0 x \in \Omega$.

229 *Proof.* a) First let us prove that

$$\gamma_x(t) \to P \quad \text{as} \quad t \to t_0.$$

Since d is a metric it is enough to prove that $d(\gamma_x(t), p) \xrightarrow{t \to t_0} 0$. We have

$$d(\gamma_x(t), p) \leq d(\gamma_x(t), \gamma_x(t_n)) + d(\gamma_x(t_n), p).$$

Since d is continuous by (2.5) and since $\gamma_x(t_n)$ tends to p as n tends to infinity by hypothesis the second term on the right hand side tends to zero as n tends to infinity. Further, by (4.3.3),

$$\lg(\gamma_x|[t_n, t]) \leq |t - t_n|$$

since $x \in U_m(N)$. Hence

$$\lg(\gamma_x|[t_n, t]) \leq |t - t_0| + |t_0 - t_n|,$$

and hence

$$d(\gamma_x(t), p) \leq d(\gamma_x(t_n), p) + |t_n - t_0| + |t - t_0|.$$

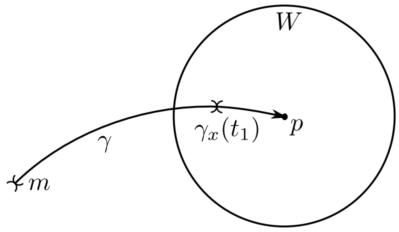
So for t in $[t_0 - \epsilon, t_0]$ we have

$$d(\gamma_x(t), p) \leq d(\gamma_x(t_n), p) + |t_n - t_0| + \epsilon,$$

and hence the result.

b) Let W be a convex ball with centre at p, and let

$$s : W \times W \to {}^{W}\Omega$$



be the associated map (see (6)). Since $\gamma_x(t) \xrightarrow{t \to t_0} t_0$ then $\exists\, t_1 \in$ 230
$[0, t_0[$ such that

$$\gamma_x([t_1, t_0) \subset W.$$

Set $u = \gamma'_x(t_1)$; by (6.6) we have

$$s(\gamma_x(t_1), \gamma_x(t)) = (t - t_1) \cdot u \quad \text{for} \quad t_1 < t < t_0.$$

Now s being continuous we have

$$s(\gamma_x(t_1), p) = \lim_{t \to t_0} s(\gamma_x(t_1), \gamma_x(t)) = \lim_{t \to t_0} (t - t_1) \cdot u = (t_0 - t_1) \cdot u.$$

Hence by the definition of s it follows that $(t_0 - t_1) \cdot u \in {}^{W}\Omega$. Since ${}^{W}\Omega$ is an open set there exists a positive number δ such that

$$[t_1, t_0 + \delta] \cdot u \subset {}^{W}\Omega$$

and since ${}^{W}\Omega \subset \Omega$ we have

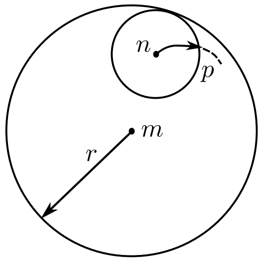
$$[t_1, t_0 + \delta] \cdot u \subset \Omega.$$

But this means simply that the geodesics γ_x can be extended to the point $\exp(t_0 + \delta) \cdot x$ and hence $t_0 \cdot x \in \Omega$. ◻

Lemma 7.3.2 bis *Let m be a point of (M, g) such that* $\exp_m$ *is r-O.K. Then*

$$\Omega \supset \underline{B}(n, r - d(m, n)) \cdot nB(m, r).$$

Proof. Let $n \in B(m, r)$, $x \in U_n(M)$.



We have to show that (see (6.4))

$$s = t_G^+(x) \geq r - d(m, n).$$

We suppose the contrary and deduce a contradiction.

231 Let

$$s < r - d(m, n).$$

Then setting $r' = d(m, n) + s$ we have $r' < r$.

Further $\forall t \in [0, s]$, we have

$$\begin{aligned} d(m, \exp(tx)) &\leq d(m, n) + d(n, \exp(tx)) \\ &\leq d(m, n) + s = r' < r. \end{aligned}$$

Now we take a sequence $\{t_n\}$ of numbers in $[0, s[$ tending to s as $n \to \infty$. Then the sequence of points $\{\exp t_n x\}$ lies in $B(m, r')$ and since $\overline{\underline{B}(m, r')}$ is compact (see (1.20)) and $\overline{\underline{B}(m, r')} = \exp(\overline{\underline{B}(m, r')})$, a sub sequence of $\{\exp t_n x\}$ converges to a point $p \in B(m, r)$. Hence the above lemma gives that

$$s \cdot x \in \Omega$$

and this contradicts the definition of $s = t_G^+(x)$. $\square$

3.3

Theorem. *For every m of M there exists an $r > 0$ such that $B(m, r)$ is a nice ball.*

Proof. Let r_1 be a positive number such that $\exp_m$ is r_1-O.K. and $B(m, r_1)$ is convex.

We claim that we have only to set $r = \frac{r_1}{3}$.

i) By (3.7) $B(m, r)$ is convex.

ii) Let n be in $B(m, r)$. Then, by (1.31) we get $d(m, n) < r$. Hence by (28):

$$\underline{B}\left(n, r_1 - \frac{r_1}{3}\right) = \underline{B}(n, 2r) \subset \Omega.$$

Further by (6.5) $\exp_n$ is $2r$-O.K. and in particular r-O.K.

iii) Let n, $n' \in B(m, r)$. Then $d(n, n') \leq d(n, m) + d(m, n') < 2r$ so we are through because of (1.29).

□

3.4

Corollary. *Let K be a compact subset of M. Then there exists $\delta > 0$ such that* $\exp_m$ *is δ − O.K.$\forall m \in K$.* 232

Proof. By the compactness of K we can cover it with a finite number of nice balls $B(m_i, r_i)$. Then we are through if we set $\delta = \inf_i r_i$. □

3.5

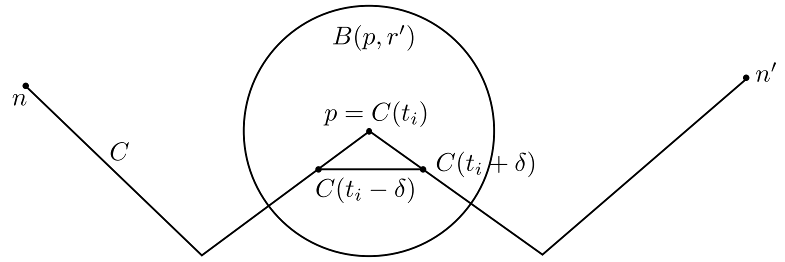
Corollary. *Let n, $n' \in M$ and $C \in \mathscr{P}(n, n')$ be such that*

$$\lg(C) = d(n, n').$$

Then (upto injective re parametrisation) C is a geodesic.

Proof. Let $C = \{[a_i, b_i] \subset I_i, f_i \in D(I_i, M)\}$.

Since being a geodesic is a local property, it is enough to show that for each point $n = f_i(t_i)$ there is a neighbourhood of t_i in which C is a geodesic. Now let $B(n, r')$ be a nice ball.



Then since C is continuous there exists a $\delta > 0$ such that

$$C([t_i - \delta, t_i + \delta]) \subset B(n, r').$$

Now let f be the geodesic in $B(n, r')$ from $C(t_i - \delta)$ to $C(t_i + \delta)$.

Then by the definition of a nice ball, if C restricted to $(t_i - \delta, t_i + \delta)$ were not a geodesic, we would have $\lg(C) > \lg(f)$, and hence replacing that part of C by f we would get

$$d(n, n') < \lg(C|[a_1, t_i - \delta]) + \lg(C|[t_i + \delta, b_k]) + \lg f < \lg(C),$$

a contradiction. □

233 As an application of the above corollary we prove the following result called *"the corner condition"* or *"the strict triangle inequality"*.

3.6

Application. *Let f be a geodesic from n to n' and g be a geodesic from n' to n'' both being parametrised by the arc length. Then if $f'(n') \neq g'(n')$ then $d(n, n'') < d(n, n') + d(n', n'')$.*

Proof. For if $d(n, n'') = d(n, n') + d(n', n'')$ the above corollary gives that the path from n to n'' given by f from n to n' together with g from n' to n'' is a geodesic. Then since both are parametrised by arc length we should have $f'(n') = g'(n')$. □

3.7

We insert here a result we will need later on (see (10)). Let $B(m,r)$ be a nice ball, n, $n' \in B(m,r)$. We associate to n, n' and $t \in [0,1]$ the point (n,n',t) defined as follows: (n,n',t) is on *the* geodesic from n to n' in $B(m,r)$ and such that $d(n,(n,n',t)) = t \cdot d(n,n')$. In particular $(n,n',\frac{1}{2})$ can be called the *mid-point* of n, n'.

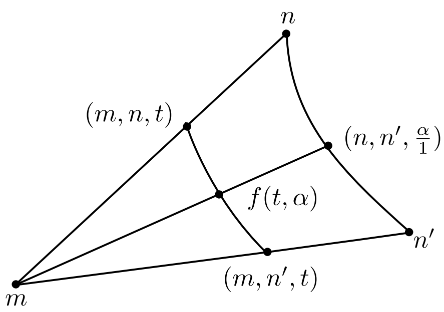
3.8

Proposition. *Let* (M,g) *be an r.m. such that* $A(M) \subset]-\infty,0]$ *and let* $B(m,r)$ *be a nice ball. Then*

$$d((m,n,t),(m,n',t)) \leq t \cdot d(n,n')\ \forall t \in [0,1], \forall n,n' \in B(m,r).$$

Proof. Set $1 = d(n,n')$ and define a one parameter family of curves $f : [0,1] \times [0,1] \to M$ by:

$$f(t,\alpha) = \left(m, \left(n, n', \frac{\alpha}{1}\right), t\right).$$



Then with the notations of (8): each f_α is a geodesic, so $t \to \underline{\underline{Q}}(t,\alpha)$ is a Jacobi field along f_α. By $A(M) \subset]-\infty,0]$ and (6.19) we have $\frac{1}{2}\frac{d^2}{\mathrm{dt}^2}(E \circ$ 234
$\underline{\underline{Q}}) \geq E \circ D_P\underline{\underline{Q}}$. We normalize $\underline{\underline{Q}}$ in H, so that $\underline{\underline{Q}} = \varphi \cdot H$ with $E \circ H = 1$. Then: $g(H, D_P H) = 0$ so that $E \circ D_P\underline{\underline{Q}} = \left(\frac{d\varphi}{\mathrm{dt}}\right)^2 + E \circ D_P H \geq \left(\frac{d\varphi}{\mathrm{dt}}\right)^2$ as

$E \circ \underline{\underline{Q}} = \varphi^2$, we get $\dfrac{d^2\varphi}{\mathrm{dt}^2} > 0$. So φ is a positive function, vanishing at $t = 0$, in a particular $\varphi(t) \leq t \cdot \varphi(1)$, which reads $\|\underline{\underline{Q}}(t,\alpha)\| \leq t\|\underline{\underline{Q}}(1,\alpha)\|$. But $\underline{\underline{Q}}(t,\alpha)$ is the speed of the transverse curve c_t which connects (m, n, t) and (m, n', t) so: $d((m, n, t), (m, n', t)) \leq \lg(c_t) = \int_0^1 \|\underline{\underline{Q}}(t,\alpha)\| d\alpha$

$$\leq \int_0^1 t\|\underline{\underline{Q}}(1,\alpha)\| d\alpha = t \cdot \lg(c_1) = t \cdot d(n, n').$$

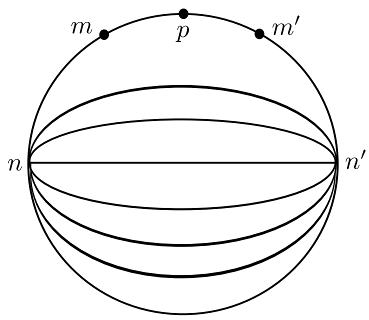
□

4 Hopf-Rinow theorem

4.1

Definition. *We set:* $\mathscr{T}(n, n') = \{C \in \mathscr{P}(n, n') | \lg(C) = d(n, n')\}$. *Then each path is a geodesic by (3.5) and we will always normalize it requiring it to be parametrized by its arc length. We put only such paths in* $\mathscr{T}(n, n')$.

Let us note that, in general, $\mathscr{T}(n, n')$ is neither empty nor consists of a single element, as can be seen from the manifold $(\mathbb{S}^d - p$, can) obtained from the sphere $\mathbb{S}^d$ by deleting one point p: $\mathscr{T}(m, m') = \emptyset$ and $\mathscr{T}(n, n')$ has continuous elements.



4.2

235

Suppose that $\mathscr{T}(n, n')$ is non-empty. Then there exists an

$$f : [0,1] \to M$$

such that f is a geodesic parametrised by the arc length, $f(0) = n$ and $f(1) = n'$. Let x be the initial speed of f. Then the curve

$$t \to \gamma_x(t) = \exp(tx)$$

is a geodesic through n with the initial speed x. Hence it follows that f is the restriction of γ_x to $[0,1]$ and hence that

$$n' = \exp(1\ x)$$

$$\text{i.e.} \qquad n' \in \exp(\overline{\underline{B}(n,1)}) \quad \text{and} \quad 1 \cdot x \in \Omega,$$

since f is parametrised by the arc length and hence x is a unit vector.

4.3

Theorem 4.1 (Hopf-Rinow)**.** *The following four properties are equivalent.*

i) *Every closed bounded set in M is compact.*

ii) *M is complete.*

iii) $\Omega = T(M)$.

iv) *There exists a point m in M such that $T_m(M) \subset \Omega$.*

Proof. We shall prove that

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)}.$$

Since every Cauchy sequence is bounded the first implication is clear. □

To prove that $\Omega = T(M)$ it is enough to show that $\forall x \in U(M)$ one has $t^+(x) = +\infty$.

Suppose $t^+(x) < \infty$; then there exists a Cauchy sequence

$$\{t_n\} \subset [0, t^+(x)[$$

236 such that

$$t_n \to t^+(x).$$
$$n \to \infty$$

Then we have: $d(\exp(t_i \cdot x), \exp(t_j \cdot x)) \leq |t_i - t_j| \cdot ||x|| = |t_i - t_j|$ and hence $\exp t_i x$ is a Cauchy sequence. So if M is complete this sequence has a limit point and an application of (3.2) gives the second implication

The third implication is clear.

So we are left with proving the fourth implication. It will

be a consequence of the following:

4.4

Proposition. *If for some point m in M*

$$T_m(M) \subset \Omega$$

then

$$\mathscr{T}(m, n) \neq \emptyset \quad \forall n \in M$$

(In fact if K is closed and bounded, $\exists\, r | K \subset \overline{D(m, r)}$ *but* $\overline{D(m, r)} \subset \exp(\overline{B}(m, r))$ *which is compact. So K is closed in a compact, hence compact)*

Proof of the proposition. Set:

$$D_t = \overline{D(m, t)}, F_t = \{n \in D_t | \mathscr{T}(m, n) \neq \emptyset\}, I = \{t \geq 0 | D_t = F_t\}.$$

We remark that F_t is closed: this follows easily from the fact that $\Omega \supset T_m(M)$. Moreover $\exists\, r | I \supset [0, r]$, for (1.29) implies that $I \supset [0, r]$ as soon as $\exp_m$ is r-O.K. We are going to prove that I is both closed and open.

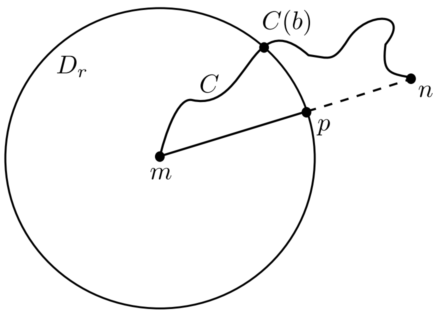
4.5

Lemma. *Let $r \in I$, $n \notin D_r$; then $\exists\, p \in D_r$ such that $d(m,p) = r$ and $d(m,n) = d(m,p) + d(p,n)$.* **237**

Proof of the lemma. $D_r = F_r$ being compact, $\exists\, p \in D_r$ such that

$$d(p,n) = \inf\{d(n,q) | q \in D_r\}.$$

We claim: p is the right one. For $\forall \epsilon > 0$, $\exists\, C \in \mathscr{P}(m,n)$ such that $\lg(C) < d(m,n) + \epsilon$. Because $t \to d(m, C(t))$ is continuous $\exists\, b$ such that $d(C(b), m) = r$.



But:

$$r + d(p,n) \leq r + d(C(b),n) \leq$$
$$\leq \lg(C|m \text{ to } C(b)) + \lg(C|C(b) \text{ to } n) = \lg(C) \leq$$
$$\leq \epsilon + d(m,n)$$
$$\text{so}: r + d(p,n) \leq \epsilon + d(m,n).$$

Letting $\epsilon \to 0$, we obtain

$$r + d(p,n) \leq d(m,n) \leq d(m,p) + d(p,n) \leq r + d(p,n)$$

hence

$$d(m,p) + d(p,n) = d(m,n).$$

4.6

I is closed. Let $\{r_i\} \subset I$ be such that $\lim\limits_{i\to\infty} r_i = r$, and let $n \in D_r$. If 238
$d(m,n) < r$, then $\exists\, i | d(m,n) < r_i$ so $\mathscr{T}(m,n) \neq \emptyset$. If $d(m,n) = r$ and if $n \notin D_{r_i} \forall i$ then by (4.5) $\exists\, p_i \in D_{r_i}$ with $d(m,n) = r = r_i + d(p_i,n)$. When $i \to \infty : d(p_i,n) \to 0$, so $p_i \to n$. But $\forall i : p_i \in F_{r_i} \subset F_r$ and F_r is closed; so $n \in F_r$. In any case then $n \in F_r$ i.e. $D_r = F_r$.

4.7

I is open. Suppose $I = [0, r_0]$ and get a contradiction as follows. From the compactness of $F_{r_0} = D_{r_0}$ we can by (3.4) find $r > 0$ such that $\exp_p$ is r-O.K. $\forall p \in D_{r_0}$.

Claim: $F_{r'} = D_{r'} \forall r' | r_0 < r' < r_0 + r$. In fact if $n \in M$ with $r_0 < d(m,n) < r' < r_0 + r$ then $n \notin D_{r_0}$. By (4.5) $\exists\, p$ with $d(m,p) = r_0$ and $d(m,p) + d(p,n) = d(m,n)$ hence $d(p,n) < r_0 + r - r = r$ so by (1.29) $\exists\, g \in \mathscr{T}(p,n)$ and $\exists\, f \in \mathscr{T}(m,n)$ and $\lg(f \cup g) = d(m,n)$ so $f \cup g \in \mathscr{T}(m,n)$. Let us note the results proved in the course of the proof of the Hopf-Rinow theorem as corollaries.

4.8

Corollary. *If (M,g) satisfies any one of the four equivalent conditions of (4.3) then*

$$\mathscr{T}(n,n') \neq \cdot\emptyset \quad \forall \quad n,n' \in M.$$

4.9

Corollary. *If (M,g) satisfies any one of the four equivalent conditions of* (6.4.3) *then for any point n of M and every $s > 0$ we have*

$$D(n,s) = \exp(\underline{B}, (n,s)).$$

Corollary. *If (M,g) is complete and n, n' and n'' are points of M such that $d(n,n'') = d(n,n') + d(n',n'')$ then $\exists\, f \in \mathscr{T}(n,n')$ such that the image set of f contains n'.*

239 For one can take $f_1 \in \mathscr{T}(n, n')$ and $f_2 \in \mathscr{T}(n', n'')$ and joining them get the element $f_1 \cup f_2$ of $\mathscr{T}(n, n'')$ by (3.5).

4.10

Remarks. 1) The converse of (4.8) is false. For example in a euclidean ball for any two points n and n' we have $\mathscr{T}(n, n') \neq \emptyset$. But it is not complete.

2) The completeness, in general, depends on g. For example the manifold $\mathbb{S}^1 \times \mathbb{R}$ with the product r.s. is homeomorphic to the manifold $\mathbb{R}^2 - \{0\}$ with the r.s. induced from $\mathbb{R}^2$. But the former is complete and the later is not. But if M is compact then for any r.s. $g(M, g)$ is complete by (2.4) and hence complete.

3) The Hopf-Rinow theorem and corollary (4.7) are the starting points for obtaining global results in riemannian geometry. We shall give typical examples in the next articles and in Chapter VIII.

4.12

Symmetric pairs give r.m.'s. $M = G/K$ which are always complete: in fact use (4.1) to check (iv) in (4.3) for $m = m_0 = p(e)$.

5 A covering criterion

5.1

Proposition. *Let two r.m.'s (M, g) and (N, h) of the same dimension and a map $p \in D(M, N)$ be such that*

i) *p is onto,*

ii) *$g = p^*h$,*

iii) *(M, g) is complete.*

Then p is a covering map.

Proof. By the definition of a covering map we have to show that given any point n of N there exists a neighbourhood V of n such that each connected component of $p^{-1}(V)$ is homeomorphic to V through p. We
240 shall show that if $\exp_n$ is s-O.K., then $B(n, s)$ is a neighbourhood with the above property.

a) Let $m \in p^{-1}(n)$; then from ii) it follows that p_m^T is injective, and since the manifolds are of the same dimension that p_m^T is bijective. Hence by (ii) it follows that p_m^T is a euclidean isomorphism between $T_m(M)$ and $T_n(N)$. Let us set

$$\lambda = (\exp_n | B(n, s))^{-1}.$$

Then since (M, g) is complete and hence $T(M) = {}^M\Omega$ we can define the map

$$q = {}^M\exp \circ (p_m^T)^{-1} \circ \lambda : B(n, s) \to M.$$

Let us note, since p_m^T is a euclidean isomorphism, that the image of q is $B(m, s)$. Further since $g = p^* h$ by (2.6) we have

$$p \circ {}^M\exp = {}^N\exp \circ p_m^T$$

and hence

$$\begin{aligned}(p_m^T)^{-1} \circ \lambda \circ p \circ {}^M\exp &= (p_m^T)^{-1} \circ \lambda \circ {}^N\exp \circ p_m^T \\ &= (p_m^T)^{-1} \circ \mathrm{id}_{\underline{B}(n,s)} \circ p_m^T = \mathrm{id}_{\underline{B}(m,s)}\end{aligned}$$

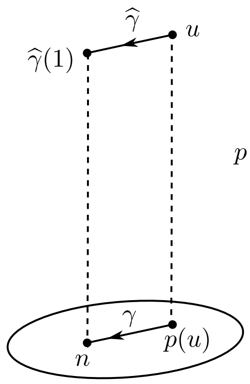
and hence $(p_m^T)^{-1} \circ \lambda \circ p$ is the inverse of ${}^M\exp$ on $\underline{B}(m, s)$. In particular ${}^M\exp$ is s-O.K., and p and q are diffeomorphisms which are inverses of each other.

b) Now suppose that m' is any point in $p^{-1}(n)$. Let $u \in B(m, s) \cap B(m', s)$. Since by a) $\exp_m$ is s-O.K, by (1.29) there is a unique geodesic γ_1 parametrised by the arc length from m to u realising
241 the distance between m and u, and a similar one γ_2 from m' to

u. Then since p is a local isometry by ii) it follows that $p \circ \gamma_1$ is a geodesic (parametrised by the arc length) realising the distance between n and $p(u)$, and $p \circ \gamma_2$ is also one such. Hence $p \circ \gamma_1 = p \circ \gamma_2$. Therefore γ_1 and γ_2 are lifts of the same curve having a common point u. Since p is local diffeomorphism $\gamma_1 = \gamma_2$ and hence $m = \gamma_1(0) = \gamma_2(0) = m'$. So we have $B(m, s) \cap B(m, s') = \emptyset$ as soon as $m \neq m'$.

c) We now prove that

$$p^{-1}(B(n, s)) = \bigcup_{m \in p^{-1}(n)} B(m, s).$$



Let $u \in p^{-1}(B(n, s))$ and let

$$\gamma : [0, 1] \to N$$

be an element of $\mathscr{T}(p(U), n)$ and $\widehat{\gamma}$ be the geodesic in (M, g) such that

$$\widehat{\gamma}'(0) = (p_u^T)^{-1}(\gamma'(0)).$$

Since (M, g) is complete the geodesic $\widehat{\gamma}$ is defined for every value of the parameter. Since $p \circ \widehat{\gamma}$ has the initial speed $\gamma'(0)$ it follows that

$$p \circ \widehat{\gamma} = \gamma$$

and hence

$$(p \circ \widehat{\gamma})(1) = \gamma(1) = n$$

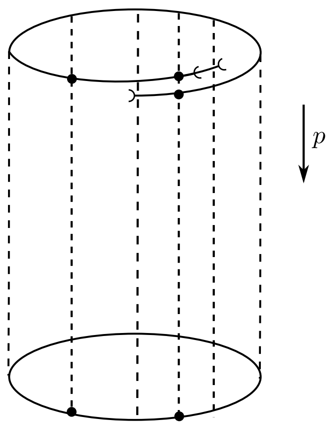
and hence $\widehat{\gamma}(1)$ is in $p^{-1}(n)$. Hence u is in $\bigcup_{m \in p^{-1}(n)} B(m, s)$.

□

5.2

Remarks. 1) We have assumed that M and N are of the same dimension.
242 Actually this can be deduced from ii) Since we have seen that p_m^T is injective for every m of M it follows that given any point m there exists a neighbourhood U of m which is mapped by a C^∞-map into $p(U)$ by means of p. Hence the dimension of N is greater than or equal to that of M. If it is greater, then U will be mapped into a set of measure zero in N, and since M is paracompact and connected it is the union of countable number of the neighbourhoods of the type U; it follows that $p(M)$ is of measure zero. But by i) $p(M) = N$ and hence this is a contradiction.

2) Completeness of (M, g) is essential as this picture shows:



5.3

Proposition. *Let (M, g) be a complete r.m, and let m be a point of M such that* $\exp_m$ *is of maximal rank everywhere. Then the map* $\exp_m$ *is a* *covering map.*

Proof. First let us note that $\exp_m$ is onto by (4.8). By (2.1) $\widehat{g} = (\exp_m)^* g$ is a r.s. on the manifold $T_m(M)$. Hence if we prove that $(T_m(M), g)$ is complete then by (5.1) we shall be through. Let $x \in T_m(M)$, $x \neq 0$ and 243
$v : \mathbb{R} \to T_m(M)$ be defined by $v(t) = t.x$. Then $\exp_m \circ v$ is a geodesic in (M, g) (because (M, g) is complete). Hence by (2.7), v is a geodesic in $(T_m(M), \widehat{g})$. This being so $\forall \times \in T_m(M)$ we have fulfilled condition (iv) of (4.3) for $0_m \in T_m(M)$. □

5.4

Remark. If we suppose, in addition to the maximality of rank for $\exp_m$ and completeness of (M, g), that M is simply connected then the above result gives that $\exp_m$ is actually a diffeomorphism.

5.5

Corollary . *If (M, g) is complete and simply connected and further $A(M) \subset]-\infty, 0]$ then M is diffeomorphic to $\mathbb{R}^d$.*

Proof. By (2.9) it follows that $\exp_m$ is of maximal rank everywhere for every m. Now we have only to apply the above remark. □

5.6

Remark. 1) The completeness assumption on (M, g) cannot be removed as the example $((\mathbb{R}^3 - \{0\}), \text{can})$ shows.

2) There is a converse to the above corollary see (7.2).

6 Closed geodesics

In this article we prove that if (M, g) is a compact r.m, then in every non-zero free homotopy class there exists a closed geodesic (see (5)) the length of which is the least among the lengths of all paths in that class. Given a non-zero homotopy class, (M, g) being compact we construct a curve in that class which is of the smallest length and then show that it is a closed geodesic. Before proceeding further we recall a few definitions and results.

244 **6.1**

1) For any points m_1 and m_2 of M let us *denote* the class of all continuous maps σ of $I = [0, 1]$ into M such that

$$\sigma(0) = m_1, \sigma(1) = m_2$$

by L_{m_1,m_2}. If m_1, m_2 and m_3 are points of M and, σ and ζ are elements of L_{m_1,m_2} and L_{m_2,m_3} respectively then there is an element $\zeta \circ \sigma \in L_{m_1,m_3}$ which corresponds to the pair (σ, ζ) in a natural way:

$$(\zeta \circ \sigma)(t) = \begin{cases} \sigma(2t) \text{ if } \circ \leq t \leq \frac{1}{2} \\ \zeta(2t-1) \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

If m_1 and m_2 are points of M then there is a homotopy relation in the class L_{m_1,m_2} : σ and ζ in L_{m_1,m_2} are *homotopic* if there exists a map

$$f : I \times I \to M$$

such that

$$f(t, 0) = \sigma(t),\ f(t, 1) = \zeta(t)\ \forall t \in I$$

and $f(0, \alpha) = m_1$, $f(1, \alpha) = m_2$ $\forall \alpha \in I$. *We denote this by* $\sigma \sim \zeta$.

The particular case when m_1 and m_2 coincide is of special importance for us. In this case, we set $L_{m_1} = L_{m_1,m_1}$ and call elements of L_{m_1} *loops* based at m_1; in L_m there is a law of composition $\circ$ and the relation $\sim$. This relation is an equivalence relation in L_m and the law of composition passes down to the quotient set which is denoted by $\pi_1(M, m)$. Further this law of composition in the quotient set defines the structure
245 of a group on this quotient set. $\pi_1(M, m)$ is called the *fundamental group* of M at m.

6.2

Definition. *Let m_1, $m_2 \in M$, $\sigma \in L_{m_1}$, $\zeta \in L_{m_2}$. Then they are* free homotopic *if there exists a map*

$$f : I \times I \to M$$

such that

$$f(t,0) = \sigma(t),\ f(t,1) = \zeta(t),\ f(0,\alpha) = f(1,\alpha)\ \forall t \in I, \forall \alpha \in I.$$

If this is so then the map F:

$$I \times I \ni (t,\alpha) \to F(t,\alpha) = \begin{cases} f(0, 3\alpha \cdot t) \text{ if } 0 \le t \le \frac{1}{3} \\ f(3t-1,\alpha) \text{ if } \frac{1}{3} \le t \le \frac{2}{3} \\ f(0, 3(1-t)\cdot\alpha) \text{ if } \frac{2}{3} \le t \le 1 \end{cases}$$

is a homotopy between σ and $f(0,\alpha)^{-1} \circ \zeta \circ f(0,\alpha)$, where $f(0,\alpha) \in L_{m_1,m_2}$. The quotient set is denoted by $\overline{\overline{F}}(M)$*; it does not carry a natural group structure. But it has a zero element, the class of loops reduced to a point. Conversely if γ is a path from m_1 to m_2 and Γ is a homotopy between σ and $\gamma^{-1} \circ \zeta \circ \gamma$ then the map Φ:*

$$I \times I \ni (t,\alpha) \to \Phi(t,\alpha) = \begin{cases} \gamma(\alpha - 3t \cdot \alpha) \text{ if } 0 \le t \le \frac{1}{3} \\ \Gamma(3t-1,\alpha) \text{ if } \frac{1}{3} \le t \le \frac{2}{3} \\ \gamma((3t-2)\cdot\alpha) \text{ if } \frac{2}{3} \le t \le 1 \end{cases}$$

is a free homotopy between σ and ζ. Hence, we obtain (6.3). An element σ of L_{m_1} is free homotopic to an element ζ of L_{m_2} if and only if there exists a path γ from m_1 to m_2 such that the loops σ and $\gamma^{-1} \circ \zeta \circ \gamma$ are homotopic. Thus there is a map

$$f : \bigcup_{m \in M} \pi_1(M,m) \to F(M)$$

from $\bigcup_{m \in M} \pi_1(M,m)$ into the free homotopy group of M. Further from **246**

6.3

if follows that the image $f(u)$ of an element u in $\pi_1(M,m)$ is not zero in $\overline{F}(M)$ if u is not. Further from (6.3) we conclude that for every a in $\overline{F}(M)$, if we write a_m for $\pi_1(M,m) \cap f^{-1}(a)$, then

(7.6.4) $$v a_m v^{-1} = a_m \ \forall v \in \pi_1(M,m).$$

6.5

3. Let

$$p : \widetilde{M} \to M$$

be the universal covering of M. Then let σ be an element of $\pi_1(M,m)$, $\widetilde{m} \in p^{-1}(m)$ and $\widetilde{m'}$ be the end point of the lift $\widehat{\sigma}$ of σ through $\widetilde{m}$. Then $\widetilde{m'}$ depends only on the class of σ in $\pi_1(M,m)$, not on σ itself. This is related to the *deck transformations* of the covering $p : \widetilde{M} \to M$ as follows: a deck transformation is a map $f : \widetilde{M} \to \widetilde{M}$ such that $p \circ f = p$; they operate transitively on a given fibre $p^{-1}(m)$; and there exists a map from $\pi_1(M,m)$ into the set of all deck transformations, denoted by $u \to \widehat{u}$ such that, given $u \in \pi_1(M,m)$, then $\widehat{u}(m)$ is the common end point $\widetilde{m'}$ of lifts through $\widetilde{m}$ of all loops $\sigma \in u$.

We endow $\widetilde{M}$ with the r.s. $\widetilde{g} = p^* g$ (see (2.1) C) so that $(\widetilde{M},\widetilde{g}) \to (M,g)$ is a riemannian covering; and denote by $\widetilde{d}$ the metric structure of $(\widetilde{M},\widetilde{g})$. Note $\widetilde{g} = p^* g$ and $p \circ f = p$ imply $f^* \widetilde{g} = \widetilde{g}$, i.e. deck transformations are isometries.

We fix from now on an $a \in \overline{F}(M)$ such that $a \neq 0$ and define a function φ on M by setting: $\varphi(m) = \inf\{\widetilde{d}(\widetilde{m},\widehat{u}(\widetilde{m})), \widetilde{m} \in p^{-1}(m), u \in a\}$
247 the geometrical meaning of which is: $\varphi(m)$ is the infimum of lengths of all loops in L_m whose free homotopy class is in a; we are going to show that φ is continuous on M and that its minimum on the compact set M is realized by a closed geodesic.

We remark that, for a lift $\widetilde{\sigma}$ in $(\widetilde{M},\widetilde{g})$ of a curve σ in (M,g), we have: $\lg(\widetilde{\sigma}) = \lg(\sigma)$.

6.6

Lemma. *Let m be a point of M. Then*

$$\varphi(m) = \inf\{\widetilde{d}(\widetilde{m}, \widehat{u}(\widetilde{m})) | u \in a_m\} \forall \widetilde{m} \in p^{-1}(m).$$

Proof. Let $\widetilde{m'} \in p^{-1}(m)$; since deck transformations act transitively and each deck transformation is induced by an element of $\pi_1(M, m)$, then $\exists\, v \in \pi_1(M, m)$ such that $\widetilde{m'} = \widehat{v}(\widetilde{m})$.

Then we have

$$\widetilde{d}(\widetilde{m'}, \widehat{u}(\widetilde{m'})) = \widetilde{d}(\widehat{v}(\widetilde{m}), \widehat{u}(\widehat{v}(\widetilde{m}))) = \widetilde{d}(\widetilde{m}, (\widehat{v^{-1} \circ u} \circ v)(\widetilde{m}))$$

since each deck transformation is an isometry. But, by (6.3), $v^{-1} \circ u \circ v \in a_m$ for $u \in a_m$. Hence the result. □

6.7

Lemma. *If $p : (N, h) \to (M, g)$ is an r. covering and (M, g) is complete then (N, h) is complete.*

Proof. We check iii) of (4.3); let $y \in U(N)$ and $x = p^T(y)$; the geodesic $t \to {}^M\exp(tx)$ is defined for all values of t since (M, g) is complete; its lift in (N, h) is by (2.7) nothing but the geodesic $t \to {}^N\exp(ty)$, which is therefore defined for all values of t. □

6.8

Lemma. *If (M, g) is complete, then $\forall m \in M$, $\forall \widetilde{m} \in p^{-1}(m)$, $\exists\, u \in a_m$ such that $\varphi(m) = \widetilde{d}(\widetilde{m}, \widehat{h}(\widetilde{m}))$.*

Proof. By (6.6) **248**

$$\varphi(m) = \inf\{\widetilde{d}(\widetilde{m}, \widehat{d}(\widetilde{m})) | u \in a_m\}.$$

Looking for an infimum we can work inside a given ball $\overline{D(m, r)}$ for a suitable r, which is compact by (5.3); then $p^{-1}(m) \cap \overline{D(m, r)}$ is a discrete set in the compact set $\overline{D(m, r)}$, so is finite; a fortiori the points $\widehat{u}(m)$ in $\overline{D(m, r)}$ are finite in number. □

6.9

Lemma. *φ is continuous on M.*

Proof. Let m be a point of M and let r be a positive real number such that $B(m,r)$ is a nice ball. Let $n \in B(m,r)$, $\widetilde{m} \in p^{-1}(m)$. By the proof of (5.1) $\exists \widetilde{n}$ with $B(\widetilde{m},r) \cap p^{-1}(n) = \{\widetilde{n}\}$, and further $\widetilde{d}(\widetilde{m},\widetilde{n}) = d(m,n)$. But $\forall u \in \pi_1(M,m)$:

$$\widetilde{d}(\widetilde{m},\widehat{u}(\widetilde{m})) \leq \widetilde{d}(\widetilde{m},\widetilde{n}) + \widetilde{d}(\widetilde{n},\widehat{u}(\widetilde{n})) + \widetilde{d}(\widehat{u}(\widetilde{m}),\widehat{u}(\widetilde{n}))$$

and since $\widehat{u}$ is an isometry we have

$$\widetilde{d}(\widehat{u}(\widetilde{n}),\widehat{u}(\widetilde{m})) = \widetilde{d}(\widetilde{n},\widetilde{m}) = d(n,m).$$

Hence we have

$$\widetilde{d}(\widetilde{m},\widehat{u}(\widetilde{m})) \leq \widetilde{d}(\widetilde{n},\widehat{u}(\widetilde{n})) + 2d(m,n)$$

Hence by the definition of φ we have

$$\varphi(m) \leq \widetilde{d}(\widetilde{n},\widehat{u}(\widetilde{n})) + 2d(m,n)$$

and hence taking the infimum on the right as u varies through a_m,

$$\varphi(m) \leq \varphi(n) + 2d(m,n).$$

Since $B(m,r)$ is a nice ball we can interchange the roles of m and n and get

$$\varphi(n) \leq \varphi(m) + 2d(m,n).$$

Hence

$$|\varphi(m) - \varphi(n)| \leq 2d(m,n) < 2r.$$

249 This holds for any r small enough, so we are through. □

6.10

Theorem . *Let* (M, g) *be a compact r.m., and let* a *be a non zero free homotopy class on* M*. Then* $\exists\, \gamma \in a$ *such that:*

i) γ *is a closed geodesic,*

ii) $\lg(\xi) \geq \lg(\gamma)\ \forall \zeta \in a$.

Proof. Since M is compact and φ is continuous $\exists\, m \in M$ such that

$$\varphi(m) \leq \varphi(n)\ \forall n \in M.$$

Let $\widetilde{m} \in p^{-1}(m)$; then by (6.8) $\exists\, u \in a_m$ such that

$$\varphi(m) = \widetilde{d}(\widetilde{m}, \widehat{u}(\widetilde{m})).$$

By (6.7) and by (4.8), $\mathscr{T}(\widetilde{m}, \widehat{h}(\widetilde{m})) \neq \emptyset$. Let

$$\widetilde{\gamma} : [0, 1] \to \widetilde{M}, \widetilde{\gamma} \in \mathscr{T}(\widetilde{m}, \widehat{u}(\widetilde{m}));$$

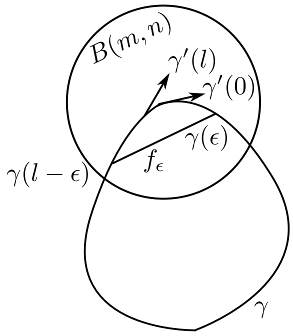
and set $\gamma = p \circ \widetilde{\gamma}$. We check ii) first: we have $\lg(\gamma) = \varphi(m)$ by construction. Then let $\tau \in a$ with $\tau \in u$; and let $\widetilde{n} \in p^{-1}(n)$ and $\widetilde{\tau}$ be the lift of τ through $\widetilde{n}$. Then by (6.6)

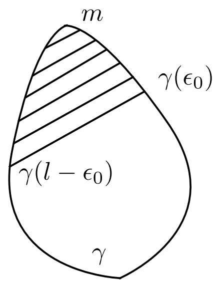
$$\lg(\tau) = \lg(\widetilde{\tau}) \geq \widetilde{d}(\widetilde{n}, \widehat{u}(\widetilde{n})) \geq \varphi(n) \geq \varphi(m) = \lg(\gamma).$$

We prove i) now: we have to prove that γ has no corner. Suppose, by contradiction, that γ did have a corner at m; choose a nice ball $B(m, r)$. Suppose, to simplify the notation, that $m = \gamma(0) = \gamma(1)$; denote by f_ϵ the unique geodesic in $B(m, r)$ from $\gamma(\epsilon)$ to $\gamma(1 - \epsilon)$ (see (3.1)); because $\gamma'(0) \neq \gamma'(1)$, we have, by (3.6),

$$\lg(\gamma|[0, \epsilon]) + \lg(\gamma|[1 - \epsilon, 1]) < \lg(f_\epsilon).$$

The family of loops $\{f_\epsilon \cup (\gamma|[\epsilon, 1-\epsilon])\}$ for $\epsilon \in [0, \epsilon_0]$, with $\epsilon_0 < r$, is a free homotopy, so $f_{\epsilon_0} \cup (\gamma|[\epsilon_0, 1 - \epsilon_0]) \in a$ and $\lg(f_{\epsilon_0} \cup (\gamma|[\epsilon_0, 1 - \epsilon_0]) < \lg(\gamma)$, **250**
contradicting ii),





□

6.11

Remarks. 1) Let us note that the compactness condition on M is, in general, necessary. For if we take the curve

$$\{(x,y) \in \mathbb{R}^2 | y = e^{-x}\}$$

and take the surface $\sum$ of revolution it generates in $\mathbb{R}^3$ by rotation around the x-axis the theorem does not hold for $\sum$. For the cross sections of the surface by the planes orthogonal to the x-axis are all in the same non zero free-homotopy class but the infinimum of their lengths *iz* zero.

2) From (6.10) one sees that if (M, g) is any non-simply-connected compact *m.*, it has at least one closed geodesic. This remains true when this (M, g) is simply connected compact but the proof is far more difficult, for it uses Morse theory: the oldest proof is in [11]

7 Manifolds with constant sectional curvature

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In (4) we have seen that every r.m. of constant sectional curvature is upto a scalar, locally isometric to one of the three standard r.m. of constant sectional curvature:

$$(\mathbb{S}^d, \text{can}), (\mathbb{R}^d, \epsilon), (\mathbb{R}^d, \text{Hyp}).$$

These standard r.m.s are complete. Use (4.12) and the fact that the geodesics in $(\mathbb{R}^d, \epsilon)$ are well known (see (4.2.9)). Further they are simply connected. Now we prove that every simply connected r.m. with constant sectional curvature is essentially one of the manifolds listed above.

7.1

Proposition . *Let (M, g) be an r.m. of constant sectional curvature k which is complete and simply connected. Then (M, g) is isometric to*

$$(\mathbb{R}^d, (-k)^{-\frac{1}{2}} \cdot \text{Hyp})), (\mathbb{R}^d, \epsilon), (\mathbb{S}^d, k^{-\frac{1}{2}} \cdot \text{can})$$

$(k < 0, k = 0, k > 0)$.

Proof. Let us denote each of the simply connected r.m.s of constant sectional curvature by (N, h).

a) Suppose first that $k \leq 0$. Then by (2.9) exp has maximal rank everywhere, and hence, by (5.4), we conclude that for any point n of N, $\exp_n$ is ∞-O.K. Let m be any point in M. Then, since (M, g) is complete, $\exp_m$ is defined on the whole of $T_m(M)$. Now if u is any euclidean isomorphism from $T_n(N)$ to $T_m(M)$, then by the proof of (4.11) (see (6.4.25)) it follows that the map

$$\lambda = \exp_m \circ u \circ \exp_n^{-1}$$

is a local isometry between N and M. Then by (5.1) it follows 252
that λ is a covering map. But since (M, g) is simply connected λ is one-one and hence λ is an isomorphism.

b) Now suppose that $k > 0$. Let n be a point of N and let $n' \neq n$ and different from the antipodal point of n. Then by (5.12) and (4.18) it follows that $\exp_n$ and $\exp_{n'}$ are $(\pi/\sqrt{k})$-O.K. Now let m be any point of (M, g) and let $u : T_n(N) \to T_m(M)$ be any euclidean isomorphism. Then since (M, g) is complete $\exp_m$ is defined on $T_m(M)$ and hence by the proof of Cartan's theorem (4.11) the map

$$\lambda = {}^M\exp_m \circ u \circ ({}^N\exp_n | B(n, \pi/\sqrt{k}))^{-1}$$

is a local isometry; in particular $\lambda^T_{n'}$ is an euclidean isomorphism, so again

$$\lambda' = {}^M\exp_{\lambda(n')} \circ \lambda^T_{n'} \circ ({}^N\exp_{n'} |B(n', \pi/\sqrt{k}))^{-1}$$

is a local isometry.

□

We claim that λ, λ' coincide on their common domain of definition, i.e. $B(n, \pi/\sqrt{k}) \cap B(n', \pi/\sqrt{k})$. For we have $\lambda^T_{n'} = (\lambda')^T_{n'}$ hence the assertion follows from the diagram (2.6).

Now since $B(n, \pi/\sqrt{k})$ together with $B(n', \pi/\sqrt{k})$ covers N and λ and λ' coincide on the intersection $B(n, \pi/\sqrt{k}) \cap B(n', \pi/\sqrt{k})$ we have a map λ'' from N to M. Since λ and λ' are local isometries it follows that λ'' is. Now if we can show that $\lambda''(N) = M$ then by (5.1) it follows that
253 λ'' is a covering map, but since M is simply connected we get that λ'' is an isometry. Since N is compact $\lambda''(N)$ is compact (M is Hausdorff) and hence $\lambda''(N)$ is closed in M. But since λ and λ' are local isometries and hence in particular open maps we have

$$\lambda''(N) = \lambda(B(n, \pi/\sqrt{k})) \cup \lambda'(B(n', \pi/\sqrt{k}))$$

is open in M. But M is connected and hence:

$$\lambda''(N) = M.$$

Chapter 8

Some formulas and applications

In this chapter we assume that all the manifolds we consider are of dimension greater than or equal to two. **254**

1 The second variation formula

To investigate whether a geodesic γ with $\gamma(0) = m$, $\gamma(1) = n$ belongs to $\mathscr{T}(m, n)$ or not, we are led to proceed as follows (where we use the notations of (8) and (1).

1.1

Let f be a one parameter family

$$f :]-\epsilon, 1+\epsilon[\times J \to M(\epsilon > 0)$$

with fixed end points, i.e.

$$f(0, \alpha) = \gamma(0) \quad \text{and} \quad f(1, \alpha) = \gamma(1)\ \forall \alpha \in J,$$

and such that $f_0 = \gamma$. Then the first variation formula (see (4.1.2), (2.1)) gives that

$$l'_f(0) = \frac{\partial}{\partial \alpha}(\lg(f_\alpha|[0, 1]) = 0.$$

Then if γ is in $\mathscr{T}(\gamma(0), \gamma(1))$, we have

$$\lg(f_\alpha|[0,1]) \geq \lg(\gamma|[0,1]) = \lg(f_0|[0,1]);$$

therefore the so-called *second variation* $l''_f(0)$ verifies:

$$l''_f(0) = \frac{\partial^2}{\partial\alpha^2}(\lg(f_\alpha|[0,1])) \geq 0.$$

Later on we will use the fact that $l''_f(0) < 0$ implies that $\gamma \notin \mathscr{T}(m, n)$ (see (1.12). So we wish to compute $l''_f(0)$.

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1.2

Now let us assume that f is such that

i) f_0 is a geodesic parametrised by the arc length,

ii) $g(\underline{\underline{P}}, \underline{\underline{Q}})(t, 0) = 0$ and then compute $l''_f(0)$ (we do not require that the end points be fixed).

We have

$$\begin{aligned} l''_f(0) &= \frac{\partial^2}{\partial\alpha^2}(\lg(f_\alpha|[0,1])) \\ &= \ldots\ldots = Q(Q\int_0^1 g(\underline{\underline{P}}, \underline{\underline{P}})^{1/2}\,\mathrm{dt})) \\ &= \int_0^1 Q(Q(g(\underline{\underline{P}}, \underline{\underline{P}})^{1/2}))\,\mathrm{dt}; \end{aligned} \tag{8.1.3}$$

$$\begin{aligned} Q(g(\underline{\underline{P}}, \underline{\underline{P}})^{1/2}) &= 1/2\, Q(g(\underline{\underline{P}}, \underline{\underline{P}}))g(\underline{\underline{P}}, \underline{\underline{P}})^{-\frac{1}{2}} \\ &= g(D_Q\underline{\underline{P}}, \underline{\underline{P}})g(\underline{\underline{P}}, \underline{\underline{P}})^{-\frac{1}{2}} \quad \text{by (4.8) (C.D.7).} \end{aligned} \tag{8.1.4}$$

Hence, using the fact that

$$Q(Q(g(\underline{\underline{P}}, \underline{\underline{P}})^{1/2})) = Q(g(D_Q\underline{\underline{P}}, \underline{\underline{P}}))g(\underline{\underline{P}}, \underline{\underline{P}})^{1/2},$$

we obtain

(8.1.5) $$-(1/2)g(D_Q\underline{\underline{P}},\underline{\underline{P}})\cdot Q(g(\underline{\underline{P}},\underline{\underline{P}}))\cdot g(\underline{\underline{P}},\underline{\underline{P}})^{-3/2} = \ldots$$
$$\ldots = g(D_QD_Q\underline{\underline{P}},\underline{\underline{P}}) + g(D_Q\underline{\underline{P}},D_Q\underline{\underline{P}})g(\underline{\underline{P}},\underline{\underline{P}})^{\frac{1}{2}} - g(D_Q\underline{\underline{P}},\underline{\underline{P}})^2 g(\underline{\underline{P}},\underline{\underline{P}})^{-3/2}.$$

Now we use (1.2). We have

$$\begin{aligned} g(D_Q\underline{\underline{P}},\underline{\underline{P}}) &= g(D_P\underline{\underline{Q}},\underline{\underline{P}}) \text{ since } [\underline{\underline{P}},\underline{\underline{Q}}]=0 \text{ and } D \text{ is symmetric}\\ &= P(g(\underline{\underline{Q}},\underline{\underline{P}})) - g(\underline{\underline{Q}},D_P,\underline{\underline{P}}) \text{ by (4.8).}\end{aligned}$$

Since f_0 is a geodesic $(D_P\underline{\underline{P}})(t,0)=0$ and hence by (1.2) ii) we have

(8.1.6) $$g(D_Q\underline{\underline{P}},\underline{\underline{P}})(t,0)=0.$$

Again since D is symmetric and $[\underline{\underline{P}},\underline{\underline{Q}}]=0$ we have **256**

$$\begin{aligned} g(D_QD_Q\underline{\underline{P}},\underline{\underline{P}}) = g(D_QD_P\underline{\underline{Q}},\underline{\underline{P}}) &= g(D_PD_Q\underline{\underline{Q}},\underline{\underline{P}}) + g(R(\underline{\underline{Q}},\underline{\underline{P}})\underline{\underline{Q}},\underline{\underline{P}})\\ &\qquad\text{by (5.12) (C.D.5)}\\ &= P(g(D_Q\underline{\underline{Q}},\underline{\underline{P}})) - g(D_Q\underline{\underline{Q}},D_P\underline{\underline{P}}) + g(R(\underline{\underline{P}},\underline{\underline{Q}})\underline{\underline{P}},\underline{\underline{Q}}).\end{aligned}$$

Now since f_0 is a geodesic we have $(D_P\underline{\underline{P}})(t,0)=0$ and hence we have

(8.1.7) $$g(D_QD_Q\underline{\underline{P}},\underline{\underline{P}})(t,0) = P(g(D_Q\underline{\underline{Q}},\underline{\underline{P}}))(t,0)+g(R(\underline{\underline{P}},\underline{\underline{Q}})\underline{\underline{P}},\underline{\underline{Q}})(t,0).$$

Combining (8.1.5), (8.1.6) and (8.1.7) with the fact that f_0 is parametrised by the arc length and hence $g(\underline{\underline{P}},\underline{\underline{P}})(t,0)=1$ we have the following proposition.

Proposition 1.8. *If f is a one parameter family of curves satisfying the conditions (1.2)* (i) *and* (ii) *then*

$$l''_f(0) = \Big[g(D_Q\underline{\underline{Q}},\underline{\underline{P}})\Big]_{(0,0)}^{(1,0)} + \int_0^1 (\|D_P\underline{\underline{Q}}\|^2 + g(R(\underline{\underline{P}},\underline{\underline{Q}})\underline{\underline{P}},\underline{\underline{Q}}))(t,0)\,dt$$

Noticing the fact that the integral in (1.8) depends only on the curve f_0 and on the lift $Q(t,0)$ of that curve we give the following definition.

Definition 1.9. *Let*

$$\gamma : [0,1] \to M$$

be a curve and let h be a lift of γ *into* $T(M)$.

Then we define

$$l_h''(0) = \int_0^1 (\|D_P h\|^2 + g(R(\gamma', h)\gamma', h))(t)\,\mathrm{dt}\,.$$

257 *In general it is easier to construct a lift h along a curve* γ *and compute* $l_h''(0)$ *then to construct a family f of curves and compute* $l_f''(0)$. *A relation between the two processes is given by the:*

Proposition 1.10. *Let*

$$\gamma : [0,1] \subset I \to (M, g)$$

be a geodesic parametrised by the arc length and let h be a lift of γ *into* $T(M)$ *such that*

$$g(\gamma', h) = 0.$$

Then, there exists an interval J containing zero and a one parameter family

$$f :]-\epsilon, 1+\epsilon[\times J \to M (\epsilon > 0)$$

of curves such that

a) $f_0 = \gamma$ *and (1.2)* i) ii) *hold for f*

b) $l_f''(0) = l_h''(0)$.

Proof. We wish to set:

$$f(t, \alpha) = \exp(\alpha \cdot h(t)).$$

But the right hand side makes sense only if $\alpha \cdot h(t) \in \Omega$. So we adjust J so that this happens. Let $0 < \epsilon' < \epsilon$; the map

$$[-\epsilon', 1+\epsilon'] \times J \ni (t, \alpha) \to \alpha \cdot h(t) \in T(M)$$

is continuous. Under this map the compact set $[-\epsilon', 1+\epsilon'] \times \{0\}$ goes into the open set Ω since $\Omega \ni 0_P \forall p \in M$. Hence there exists $r > 0$ such that under the above map $[-\epsilon', 1+\epsilon'] \times [-r, r]$ goes into Ω. Hence we can set

$$f(t,\alpha) = \exp(\alpha \cdot h(t)) \forall (t,\alpha) \epsilon [-\epsilon', 1+\epsilon'] \times [-r, r].$$

Then **258**

$$f_0(t) = \exp(0 \cdot h(t)) = \exp(0_{p(h(t))}) = \exp(0_{\gamma(t)}) = \gamma(t)$$

and hence f_0 is a geodesic parametrised by the arc length; and since $\underline{\underline{Q}}(t,0) = h(t)$ we have

$$g(\underline{\underline{P}}, \underline{\underline{Q}})(t,0) = g(\gamma', h)(t) = 0.$$

□

1.11

Hence f satisfies a) i). For ii) we see that $h(t) = \underline{\underline{Q}}(0,t)$ (as in (8)), and hence:

$$l''_f(0) = \Big[g(D_Q \underline{\underline{Q}}, \underline{\underline{P}})\Big]_{(0,0)}^{(1,0)} + \int\limits_0^1 (\|D_P \underline{\underline{Q}}\|^2 + g(R(\underline{\underline{P}}\underline{\underline{Q}})\underline{\underline{P}}, \underline{\underline{Q}}))(t,0)\,\mathrm{dt}\,.$$

But since the map

$$\alpha \to \exp(\alpha \cdot h(t))$$

is a geodesic we have

$$(D_Q \underline{\underline{Q}}) = 0.$$

Hence we are through by (1.11) and the definition of $l''_h(0)$.

Corollary 1.12. *Let m and n be points of M and let $\gamma \in \mathscr{P}(m,n)$ be a geodesic in (M,g); suppose that there exists a lift $h(t)$ of γ such that*

1) $g(h, \gamma') = 0$,

2) $h(0) = h(1) = 0$,

3) $l''_h(0) < 0$.

Then $\gamma \notin \mathscr{T}(m, n)$.

Proof. Let us take the one parameter family defined in the above proposition. Then we have

$$\frac{\partial}{\partial\alpha}(\lg(f_\alpha|[0,1])) = 0$$

259 and

$$\frac{\partial^2}{\partial\alpha^2}(\lg(f_\alpha|[0,1])) = l''_r(0) = l''_h(0) < 0.$$

Hence using Taylor's formula with a remainder we see that there exist α's such that

$$g(f_\alpha|[0,1]) < g(f_0|[0,1]). \tag{8.1.13}$$

But we have

$$\begin{aligned} & f(0,\alpha) = \exp(\alpha \cdot h(0)) = \exp(\alpha \cdot 0_{\gamma(0)}) = \gamma(0) = m \\ \text{and} \quad & f(1,\alpha) = \exp(\alpha \cdot h(1)) = \exp(\alpha \cdot 0_{\gamma(1)}) = \gamma(1) = n \end{aligned}$$

and hence by (8.1.13) there exist curves from m to n with length less than that of γ. Hence γ is not in $\mathscr{T}(m, n)$. □

2 Second variation versus Jacobi fields

2.1

Notation. For $x \in U(M)$ let us set (see (8.30)):

$$j(x) = \inf\{t > 0 | \gamma_x(0) \text{ and } \gamma_x(t) \text{ are conjugate on } \gamma_x\}.$$

If $\exp_m$ is r-O.K. then by (8.32) it follows that

$$j(x) \geq r \ \forall x \in U_m(M).$$

Hence $j(x)$ is a strictly positive function on $U(M)$.

2.2

Proposition. *Let* $x \in U(M)$, $l \in [0, j(x)]$ *and let*

$$h : [0,1] \to T(M)$$

be a lift of γ_x *such that* $h(0) = h(1) = 0$. *Then* $l_h''(0) \geq 0$ *and equality holds if and only if h is a Jacobi field.*

Proof. Let $\{x = x_1, x_2, \ldots, x_d\}$ be a basis of $T_{m=p(x)}(M)$ and let h_i be the Jacobi field along γ_x such that $h_i(0) = 0$ and $(D_P h_i)(0) = x_i$ (see (8.26)). *Then* $\forall t \in [0,1] : \{h_1(t), \ldots, h_d(t)\}$ *form a basis of* $T_{\gamma_x(t)}(M)$: for if for some $t_0 > 0$ we have:

$$\sum_i a_i h_i(t_0) = 0.$$

then the Jacobi field $k : t \to \sum_i a_i h(t)$ vanishes at $t = 0$ and at $t = t_0$ so 260
by (8.30): $k = 0$.

Hence

$$D_P k = \sum_i a_i D_P h_i = 0;$$

in particular

$$(D_P k)(0) = \sum_i a_i x_i = 0.$$

Since $\{x_1, \ldots, x_d\}$ is a basis of $T_m(M)$ we have

$$a_1 = \ldots = a_d = 0. \qquad \text{Q.E.D.}$$

Hence corresponding to every lift h of γ_x into $T(M)$ we have functions $\{f_1, \ldots, f_d\}$ on $]0, 1[$ such that

$$h(t) = \sum_i f_i(t) h_i(t). \tag{8.2.3}$$

Then we have

$$D_P h = \sum_i f_i' h_i + \sum_i f_i (D_P h_i)$$

and setting

$$(8.2.4) \qquad u = \sum_i f_i' h_i \quad \text{and} \quad v = \sum_i f_i (D_P h_i)$$

we have

$$D_P h = u + v.$$

Hence

$$(8.2.5) \qquad \|D_P h\|^2 = \|u\|^2 + 2g(u, v) + \|v\|^2$$

and

$$(8.2.6) \quad g(R(\gamma_x', h)\gamma_x', h) = g(R(\gamma_x', \sum f_i h_i)\gamma_x', h) =$$

$$= \sum_i f_i g(R(\gamma_x', h_i)\gamma_x', h) = \sum_i f_i g(D_P D_P h_i, h),$$

(since h_i are Jacobi fields).

□

261 **2.6**

$$= g(\sum_i f_i D_P D_P h_i, h) = g(D_P v - \sum_i f_i' D_P h_i, h)$$

$$= g(D_P v, h) - \sum_i f_i' g(D_P h_i, h).$$

$$= P(g(v, h)) - g(v, D_P h) - \sum_{i,j} f_i' f_j g(D_P h_i, h_j =$$

by (4.8) (C.D.7)

$$= P(g(v, h)) - g(v, u + v) - \sum_{i,j} f_i' f_j g(D_P h_i, h_j).$$

But if we set

$$\xi = g(D_P h_i, h_j) - g(D_P h_j, h_i) \quad \text{(a classical Sturmain argument)}$$

then

$$\begin{aligned}
\xi' = P(\xi) &= g(D_P D_P h_i, h_j) + g(D_P h_i, D_P h_j)\\
&\qquad - g(D_P D_P h_j, h_i) - g(D_P h_j, D_P h_i)\\
&= g(D_P D_P h_i, h_j) - g(D_P D_P h_j, h_i) =\\
&= g(R(\gamma'_x, h_i)\gamma'_x, h_j) - g(R(\gamma'_x, h_j)\gamma'_x, h_i) = 0\\
&\qquad\qquad\qquad\qquad \text{by ((5.5.5) C.T.4).}
\end{aligned}$$

Hence ξ is constant, but $\xi(0) = 0$ for $h_i(0) = 0 \forall i$. So $\xi = 0$. Hence we have

$$\begin{aligned}
\sum_{i,j} f'_i f_j g(D_P h_i, h_j) &= \sum_{i,j} f'_i f_j g(D_P h_j, h_i)\\
&= g(\sum_j f_j D_P h_j, \sum_i f'_i h_i) = g(u, v).
\end{aligned}$$

Hence by (8.2.5) and (8.2.6) we have

$$\|D_P h\|^2 + g(R(\gamma'_x, h)\gamma'_x, h) = P(g(v, h)) + \|u\|^2.$$

Hence 262

$$l''_h(0) = [g(v, h)]_0^1 + \int_0^1 \|u\|^2 \, dt,$$

and since $h(0) = h(1) = 0$ we have

(8.2.7) $$l''_h(0) = \int_0^1 \|u\|^2 \, dt\,.$$

Hence $l''_h(0) \geq 0$ and equality holds if and only if u is identically zero. But since $\{h_1(t), \ldots, h_d(t)\}$ from a basis for every t, u is identically zero if and only if each f'_i is zero, i.e. if and only if each f_i is a constant.

Hence $h = \sum_i f_i h_i$ is a Jacobi field.

2.9

Proposition. *Let $x \in U(M)$ and $l > j(x)$; then*

$$\gamma_x|[0,1] \in \mathscr{T}(\gamma_x(0), \gamma_x(1)).$$

Proof. First we prove two lemmas. □

2.10

Lemma. *Let h be a Jacobi field along γ_x. Then*

$$l_h''(0) = \Big[g(D_p h, h)\Big]_0^1 .$$

Proof. We have by definition

$$\begin{aligned} l_h''(0) &= \int_0^1 g(D_p h, D_P h) + g(R(\gamma_x', h)\gamma_x', h)\,\mathrm{dt} = \\ &= \int_0^1 g(D_P h, D_P h) + g(D_P D_P h, h)\,\mathrm{dt} \quad \text{(since } h \text{ is a Jacobi field)} \\ &= \int_0^1 P(g(D_P h, h))\,\mathrm{dt} \quad \text{by (4.8) (C.D.7)} = [g(D_P h, h)]_0^1 . \end{aligned}$$

□

2.11

Lemma. *Let h be a Jacobi field along γ_x such that*

$$h(0) = 0 \quad \textit{and} \quad (D_P h)(0) \neq 0.$$

Then given any positive number η we can choose a positive number ϵ
263 *less than η and a Jacobi field k along $\gamma_x|[-\epsilon, \epsilon]$ such that*

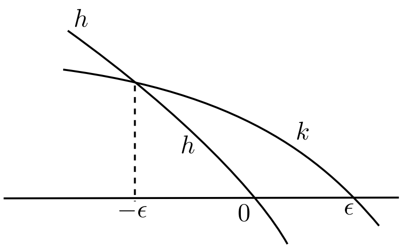
i) $k(-\epsilon) = h(-\epsilon)$

ii) $k(\epsilon) = 0$

iii) $g(D_P h, h)(-\epsilon) < g(D_P k, k)(-\epsilon)$.

Proof. Let r be a positive number such that $B(m, r)$ is convex. Then $\forall t \in]0, r[$ there exists a Jacobi field k_t along $\gamma|[-t, t]$ such that

(8.2.12) $$k_t(-t) = h(t) \quad \text{and} \quad k_t(t) = 0.$$



In fact we set f_α = the geodesic from $\exp(\alpha \cdot h(-t))$ to $\gamma(t)$ parametrised by the arc length, then the family $\{f_\alpha\}$ of the curves f_α so obtained is a one-parameter family of geodesics and the $k(s) = \underline{\underline{Q}}(s, 0)$ satisfy our requirements. Now let us set

$$u = (D_P h)(0) \quad \text{and} \quad \widehat{k_t}(t') = a_t + t' b_t + 0(t').$$

Then

$$\widehat{h}(t') = t' \cdot u + 0(t'), \widehat{h}'(t') = u + 0(1)$$

and

$$\widehat{k_t'}(t') = b_t + 0(1).$$

Then by (8.2.12) we have

$$a_t - t b_t + 0(t) = \widehat{k_t}(-t) = \widehat{h}(-t) = -tu + 0(t)$$

and

$$\widehat{k_t}(t) = a_t + t b_t + 0(t) = 0.$$

Hence **264**

$$b_t = \frac{u}{2} + 0(t).$$

□

2.13

Hence

$g(D_P h, h)(-t) = g(\widehat{h'}(-t), \widehat{h}(-t)) = g(u+0(1), -tu+0(t)) = -t\|u\|^2+0(t),$

and

$$g(D_P k_t, k_t)(-t) = g(\widehat{k_t'}(-t), \widehat{k_t}(-t))$$
$$= g\left(\frac{u}{2} + 0(1), \widehat{h}(-t)\right) = g\left(\frac{u}{2} + 0(1), -tu + 0(t)\right) = \frac{t}{2}\|u\|^2 + 0(t).$$

Since u is non zero by our hypothesis we have $\|u\| > 0$ and hence by (2.13) we can choose t so small that

$$g(D_P h, h)(-t') < g(D_P k, k)(-t')$$

for $0 < t' < t$.

Now let us take up the proof of (2.6). Since $1 > j(x)$ there exists a $t_0 \in]0, 1[$ and a Jacobi field h along γ_x such that

$$h \neq 0, h(0) = h(1) = 0.$$

Set $\eta = -t_0$. Then applying the lemma (2.11) for the Jacobi field h around the point t_0 we get $\epsilon \in]0, \eta[$ and a Jacobi field k_ϵ along $\gamma_x|[t_0 - \epsilon, t_0 + \epsilon]$. Now let f be the lift of γ_x such that:

$$h_1 = f|[0, t_0 - \epsilon] = h|[0, t_0 - \epsilon]$$
$$h_2 = f|[t_0 - \epsilon, t_0 + \epsilon] = k_\epsilon$$
$$h_3 = f|[t_0 + \epsilon, 1] = 0.$$

265 Then we have

$$l''_f(0) = l''_{h_1}(0) + l''_{h_2}(0) + l''_{h_3}(0).$$

But by (2.10) we have

$l''_{h_1}(0) = [g(D_P h_1, h_1)]_0^{1-\epsilon} = g(D_P h_1, h_1)(1-\epsilon)$ since $h_1(0) = h(0) = 0;$

and

$$l''_{h_2}(0) = [g(D_P h_2, h_2)]_{1-\epsilon}^{1+\epsilon} = -g(D_P h_2, h_2)(1-\epsilon) \text{ since } h_2(1+\epsilon) = 0$$

and

$$l''_{h_3}(0) = 0 \text{ since } h_3 = 0.$$

Hence $l''_f(0) < 0$ so we are done with (1.12).

2.14

Remark . The two propositions (2.2) and (2.6) are essential parts of Morse's index theorem.

3 The theorems of Synge and Myers

First let us recall that Ricci curvature (Ric) is a function from $T(M)$ to $\mathbb{R}$.

3.1

Proposition. *Let (M, g) be an r.m. of dimension d such that*

$$\mathrm{Ric}(U(M)) \subset [k, \infty[$$

for some $k > 0$. Let γ be a geodesic in $\mathscr{P}(m, n)$ where m and n are points of M. Now if

$$\lg(\gamma) > \pi \left(\frac{d-1}{k}\right)^{1/2}$$

then $\gamma \notin \mathscr{T}(m, n)$. **266**

Proof. Let us suppose that γ is parametrised by the arc length and then take an orthonormal basis $\{x_1 = \gamma'(0), x_2, \ldots, x_d\}$ of $T_{\gamma(0)}(M)$. Let h_i be the parallel lift along γ with $h_i(0) = x_i$, and set

$$s_i(t) = \sin\left(\frac{\pi}{1} t\right) \cdot h_i(t).$$

Then since h_i is a parallel lift we have $D_P h_i = 0$ and hence by (4.1) D.L.3 we have

$$(D_P s_i)(t) = \frac{\pi}{1} \cos\left(\frac{\pi t}{1}\right) \cdot h_i(t),$$

and since parallel transport preserves g we have

(8.3.2) $$||(D_P s_i)(t)||^2 = \frac{\pi^2}{1^2} \cos^2\left(\frac{\pi t}{1}\right).$$

Again since parallel transport preserves g by the definition of A (see (6.1.2)) and since $\gamma'(0)$ and $h_i(0)$ are orthogonal we have

$$g(R(\gamma', h)\gamma', h) = -A(\gamma', s_i)\|s_i\|^2 = -A(\gamma', h_i)\sin^2\frac{\pi t}{1}.$$

Hence by the definition (1.9) we have

$$l''_{s_i}(0) = \int_0^1 \frac{\pi^2}{1^2}\cos^2\left(\frac{\pi}{1}t\right) - \sin^2\left(\frac{\pi t}{1}\right)A(\gamma', h_i)\,\mathrm{dt}\,. \tag{8.3.3}$$

By (8.6) we have

$$\sum_{i=2}^{d} A(\gamma', h_i) = \mathrm{Ric}(\gamma') \geq k$$

and hence

(8.3.4)

$$\sum_{i=2}^{d} l''_{s_i}(0) \leq \int_0 \frac{\pi^2}{l^2}\cdot(d-1)\cos^2\left(\frac{\pi t}{1}\right) - k\sin^2\left(\frac{\pi t}{1}\right)\mathrm{dt} = \frac{1}{2}\left(\frac{\pi^2}{1^2}(d-1) - k\right).$$

267 Hence if $l > \pi\left(\frac{d-1}{k}\right)^{\frac{1}{2}}$ then $\sum_{i=2}^{d} l''_{s_i}(0) < 0$ and hence

$$\exists\, i | l''_{s_i}(0) < 0;$$

hence, by (1.12), $\gamma \notin \mathscr{T}(m, n)$. □

3.5

Definition. *The real number $d(M, g)$ defined as*

$$d(M, g) = \sup\{d(m, n) | m, n \in M\}$$

is called the diameter of the *r.m.* (M, g).

3.6

Remarks. 1) If $d(M, g) < \infty$ and the manifold (M, g) is complete then by Hopf-Rinow theorem it follows that M is compact.

2) Let us note that (8.6) implies that if

$$A(M) \subset [k, \infty[\text{ then } \operatorname{Ric}(U(M)) \subset [(d-1)k, \infty[$$

3.7

Corollary Myers *Suppose that* (M, g) *is a complete r.m. such that there exists a* $k > 0$ *satisfying*

$$\operatorname{Ric}(U(M)) \subset [k, \infty[.$$

Then

$$d(M, g) \leq \pi \left(\frac{d-1}{k}\right)^{\frac{1}{2}}.$$

In particular M *is compact and the fundamental group of* M *is finite.*

Proof. Suppose that there exist m and n in M such that

$$d(m, n) > \pi \left(\frac{d-1}{k}\right)^{\frac{1}{2}}.$$

Then by (8.4.8) there exists a geodesic γ joining m and n such that

$$\lg(\gamma) > \pi \left(\frac{d-1}{k}\right)^{\frac{1}{2}}.$$

But this contradicts (3.1); hence $d(M, g) \leq \pi \left(\frac{d-1}{k}\right)^{\frac{1}{2}}$. **268**

By (3.6) it follows that M is compact. Now let

$$p : (\widetilde{M}, \widetilde{g}) \to (M, g)$$

be the universal riemannian covering of (M, g). Then since p is a local isometry we obtain

$$\operatorname{Ric}(U(\widetilde{M})) = \operatorname{Ric}(U(M)) \subset [k, \infty[.$$

Hence the first part of the corollary gives that $\widetilde{M}$ is compact. Hence the covering $p : \widetilde{M} \to M$ has finite number of sheets. □

3.8

Remarks. 1) This result was first proved by Bonnet for two dimensional manifolds.

2) In the above inequality (3.7) the equality occurs for

$$\left(\mathbb{S}^d, \left(\frac{d-1}{k}\right)^{\frac{1}{2}} \cdot \mathrm{can}\right) \subset \mathbb{R}^{d+1}.$$

The proof here would not yield easily the result that equality in (3.7) is attained only if (M, g) is isometric to $(\mathbb{S}^d, (\frac{d-1}{k})^{\frac{1}{2}} \cdot \mathrm{can})$. However the validity of this result can be deduced easily from [33]: Corollary 4, p.256-257.

3.9

Theorem. *(Synge) Let (M, g) be an r.m. such that*

i) *M is compact*

ii) *M is orientable*

iii) *the dimension of M is even*

iv) $A(M) \subset]0, \infty[$.

Then M is simply connected.

269 *Proof.* By contradiction: suppose that M is not simply connected; then (see (6.3)) there exists in M at least one non-zero free homotopy class. Let γ denote the closed geodesic which fulfills the conditions i) and ii) of theorem (6.10). Let $l = \lg(\gamma)$ and $\tau(t)$ denote the parallel transport along γ from 0 to t. Since $\gamma(0) = \gamma(1)$ then $\tau(1)$ is an endomorphism of $T_{\gamma(0)}(M)$; by (4.9) we see that $\tau(1)$ belongs to the orthogonal group of $T_{\gamma 0}(M)$. In fact $\tau(1)$ is a rotation. For: suppose that σ is the orienting form and that $\{\gamma'(0) = x_1, \ldots, x_d\}$ is a positive orthonormal base of $T_{\gamma(0)}(M)$. Then the function

$$\sigma(\tau(t)(x_1), \ldots\ldots, \tau(t)(x_d))$$

never vanishes since σ is never zero and $\tau(t)$ is an isomorphism; hence

$$\sigma(\tau(1)(x_1), \ldots, \tau(1)(x_d)) = 1.$$

Set

$$H = \left\{ y \in T_{\gamma(0)}(M) | g(\gamma'(0), y) = 0 \right\};$$

then since parallel transport preserves g and $\tau(1)(\gamma'(0)) = \gamma'(1) = \gamma'(0)$ by (3.5) we get $\tau(1)(H) = H$. But H is odd dimensional and hence $\exists\, y \in H, y \neq 0$, with $\tau(1)(y) = y$ (because $\tau(1)$ is a rotation).

Let h be the parallel lift of γ such that $h(0) = y$. since $\tau(1)(y) = y$ we have

$$h(0) = h(1) = y.$$

Now let f be the parameter family of curves associated to h in the man- 270
ner of (1.10). Then by the definition of f we see that f_α is a closed curve $\forall \alpha$ and (see (1.9)):

$$l''_f(0) = l''_h(0) = -\int_0^1 A(\gamma', h) \|h\|^2 \,\mathrm{dt} < 0$$

since $h(t)$ is non zero and $A(M) \subset]0, \infty[$. Hence $\exists\, \alpha_0$ such that

$$\lg(f_{\alpha_0}) < \lg(\gamma).$$

Moreover the restriction $f|[0,1] \times [0, \alpha_0]$ defines a free homotopy between γ, and f_{α_0}. This contradicts our choice of γ. □

3.12

Remark. The manifold $(P^3(\mathbb{R}), \mathrm{can})$ shows that this result cannot be improved.

4 A formula

We prove an integral formula (4.7) which will be used in articles 7 and 9 (see [13]: theorem 5.1).

In this article we assume that the r.m. (M,g) *is complete*. In particular (4.3) *the geodesic flow* is defined on the whole $\mathbb{R}\times M$. We denote the flow by G_t. We recall the fact that Ric is the Ricci curvature of (M,g) (see (8)).

4.1

Definition. We say that an r.m. (M,g) *is* $NCB(k)$ *if* k *is a positive number and (see (2.1))*

$$j(U(M)) \subset [k,\infty[.$$

271 *Now for any positive number* k *and for any* $x \in U(M)$ *we set:*

$$(8.4.2) \quad f_k(x) = \int_{-k/2}^{k/2} \left((d-1)\frac{\pi^2}{k^2}\sin^2\left(\frac{\pi t}{k}\right) - \cos^2\left(\frac{\pi t}{k}\right)\cdot \mathrm{Ric}(G_t(x))\right)\mathrm{dt}\,.$$

Note that by the definition of a flow and because γ_x' *is an integral curve of* G *we have:*

$$G_t(x) = \gamma_x'(t).$$

4.3

Proposition. *If* (M,g) *is* $NCB(k)$ *then*

$$f_k(x) \geq 0 \ \forall x \in U(M),$$

equality occurring for a given x if and only if $\forall t \in \left[-\frac{k}{2}, \frac{k}{2}\right]$ and $\forall y$ which is linearly independent of $G_t(x)$ we have

$$A(G_t(x), y) = \pi^2/k^2.$$

Proof. As in the proof of (3.1) for a given x let $\{x = x_1, x_2, \ldots, x_d\}$ be an orthonormal basis of $T_{p(x)}(M)$, let h_i be the parallel lift of γ_x through x_i and let s_i be the lift defined by the equation

$$s_i(t) = \cos\left(\frac{\pi t}{k}\right) \cdot h_i(t) \quad \text{for} \quad -\frac{k}{2} \le t \le \frac{k}{2}. \tag{8.4.4}$$

Then (1.9):

$$l''_{s_i}(0) = \int_{-k/2}^{k/2} \left(\frac{\pi^2}{k^2} \sin^2\left(\frac{\pi t}{k}\right) - A(G_t(x), h_i(t)) \cos^2\left(\frac{\pi t}{k}\right)\right) \mathrm{dt}\,. \tag{8.4.5}$$

Hence by (8.6) and (8.4.2) we have

$$f_k(x) = \sum_{i=2}^{d} l''_{s_i}(0). \tag{8.4.6}$$

Now let us note that, by the definition, s_i vanishes at $\pm\frac{k}{2}$ and the fact **272**
that (M, g) is an $NCB(k)$ implies that $j(\gamma'(-\frac{k}{2})) \ge k$. Hence by (2.1) and (2.2) we have

$$l''_{s_i}(0) \ge 0 \ \forall i = 2, \ldots, d$$

and equality occurs if and only if s_i is a Jacobi field. Now suppose that $f_k(x) = 0$. Then s_i is a Jacobi field $\forall i$. Hence

$$D_P D_P s_i = R(G_t(x), s_i)(G_t(x)).$$

But

$$D_P D_P S_i = -\frac{\pi^2}{k^2} \cos\left(\frac{\pi t}{k}\right) \cdot h_i(t)$$

and

$$R(G_t(x), s_i)G_t(x) = \cos\left(\frac{\pi t}{k}\right) \cdot R(G_t(x), h_i(t))G_t(x).$$

Further $\cos(\frac{\pi t}{k})$ does not vanish in $]-\frac{k}{2}, \frac{k}{2}[$ and hence we have

$$R(G_t(x), h_i(t))G_t(x) = -\frac{\pi^2}{k^2} \cdot h_i(t) \forall i.$$

and since $\{h_1(t), h_2(t), \ldots, h_d(t)\}$ form a basis of $T_{\gamma_x(t)}(M)$ (see the proof of (2.2)) we have (see (6.8.2))

$$\overline{R}(G_t(x))y = -\frac{\pi^2}{k^2} y$$

whenever y is orthogonal to $G_t(x)$. In particular if $\{G_t(x), y\}$ are orthonormal we have

$$A(G_t(x), y) = -g(R(G_t(x), y)G_t(x), y) = \frac{\pi^2}{k^2} g(y, y) = \frac{\pi^2}{k^2}.$$

Before stating the next proposition let us recall that Γ denotes the scalar curvature of (M, g) (see (8)) □

4.7

273 **Proposition.** *Let (M, g) be an oriented compact manifold. Then if (M, g) is $NCB(k)$ we have*

$$\mathrm{Vol}(M, g) \geq \frac{k^2/\pi^2}{d(d-1)} \cdot \int_M \Gamma \cdot \sigma$$

(where σ is the canonical volume form of (M, g)) and the equality occurs if and only if $A(M) = \{\pi^2/k^2\}$.

Further, if $k = \infty$, $\int \Gamma \cdot \sigma = 0$, then $A(M, g) = \{0\}$.

Proof. Let $\overline{\overline{\sigma}} = \omega \wedge (p^*\sigma)$ (see (6.2.5)) be the volume form of $U(M)$ so that for every m of M, $\omega | U_m(M)$ is the volume form of $U_m(M)$. Now set

$$\varphi_k(x, t) = (d-1)\frac{\pi^2}{k^2} \sin^2\left(\frac{\pi t}{k}\right) - \cos^2\left(\frac{\pi t}{k}\right) \cdot \mathrm{Ric}(G_t(x)) \ \forall x \in U(M), t \in \mathbb{R}$$

and define I by the equation

$$I = \int\limits_{U(M)} f_k(x)\cdot \overline{\overline{\sigma}} = \int\limits_{U(M)} \left\{ \int\limits_{-\frac{k}{2}}^{\frac{k}{2}} \varphi_k(x,t)\cdot \mathrm{dt} \right\} \overline{\overline{\sigma}}.$$

Since M is compact we may interchange the order of integration and obtain

$$I = \int\limits_{-k/2}^{k/2} \left(\int\limits_{U(M)} \varphi_k(x,t)\overline{\overline{\sigma}} \right) \mathrm{dt} = A - B, \tag{8.4.8}$$

where

$$A = \int\limits_{-k/2}^{k/2} \left(\int\limits_{U(M)} (d-1)\frac{\pi^2}{k^2}\sin^2\left(\frac{\pi t}{k}\right)\overline{\overline{\sigma}} \right) \mathrm{dt} \tag{8.4.9}$$

and

$$B = \int\limits_{-k/2}^{k/2} \left(\int\limits_{U(M)} \cos^2\left(\frac{\pi t}{k}\right)\mathrm{Ric}(G_t(x))\overline{\overline{\sigma}} \right) \mathrm{dt}\,. \tag{8.4.10}$$

But 274

$$A = (d-1)\frac{2}{k^2}\cdot \int\limits_{-k/2}^{k/2} \sin^2\frac{\pi t}{k} \left(\int\limits_{U(M)} \overline{\overline{\sigma}} \right) \mathrm{dt}\,. \tag{8.4.11}$$

$$= (d-1)\frac{\pi^2}{k^2}\boxed{d-1}\,\mathrm{Vol}(M,g) \int\limits_{-k/2}^{k/2} \sin^2\frac{\pi t}{k}\cdot \mathrm{dt} =$$

$$= (d-1)\frac{2}{k^2}\boxed{d-1}\,\mathrm{Vol}(M,g)\cdot \frac{k}{2} \text{ by (5.2.13).}$$

By (3.10) and (2.5) we have:

$$G_t(\overline{\overline{\sigma}}) = \overline{\overline{\sigma}}.$$

Hence by (0.3.9) we have

$$\int_{U(M)} \mathrm{Ric}(G_t)\cdot\overline{\overline{\sigma}} = \int_{U(M)} (\mathrm{Ric}\circ G_t)G_t(\overline{\overline{\sigma}}) = \int_{U(M)} G_t(\mathrm{Ric}\circ\overline{\overline{\sigma}}) = (\text{by } (0.3.9))$$

$$= \int_{U(M)} \mathrm{Ric}\circ\overline{\overline{\sigma}} = \int_{U(M)} \mathrm{Ric}\circ(\omega\wedge p^*\sigma) =$$

$$= \int_{m\in M}\left(\int_{U_m(M)} \mathrm{Ric}\,|U_m(M)\right)\cdot\omega))\sigma \text{ by (3.17).}$$

But by (6.8.11) we get

$$\int_{U(M)} \mathrm{Ric}\,\overline{\overline{\sigma}} = \int_M \frac{d-1}{d}\Gamma(m)\cdot\sigma = \frac{d-1}{d}\int_M \Gamma\cdot\sigma$$

and hence

$$B = \int_{-k/2}^{k/2} \cos^2\left(\frac{\pi^t}{k}\right)\frac{d-1}{d}\int_M \Gamma\cdot\sigma\,\mathrm{dt} = \frac{d-1}{d}\frac{k}{2}\int_M \Gamma\cdot\sigma. \tag{8.4.12}$$

But

$$\int_{-k/2}^{k/2} \sin^2\frac{\pi t}{k}\,\mathrm{dt} = \int_{-k/2}^{k/2} \cos^2\frac{\pi t}{k}\,\mathrm{dt} = \frac{k}{2}.$$

275 Hence we have

$$I = \frac{(d-1)k}{2}\left[\frac{(d-1)\cdot\pi^2}{k^2}\mathrm{Vol}(M,g) - \frac{1}{d}\int_M \Gamma\cdot\sigma\right].$$

But by (4.3) $I \geq 0$ and since $k\cdot(d-1) > 0$ we have

$$\mathrm{Vol}(M,g) \geq \frac{k^2\pi^{-2}}{(d-1)d}\int_M \Gamma\cdot\sigma \tag{8.4.13}$$

where the equality occurs if and only if $I = 0$ i.e. if and only if $f_k(x) = 0$ for every x.

But by (4.3), this happens if and only if whenever

$$x \in U(M), t \in]-\frac{k}{2}, \frac{k}{2}[\text{ and } y \text{ is linearly independent of } G_t(x),$$

we have

$$A(G_t(x), y) = \frac{\pi^2}{k^2}.$$

This is clearly equivalent to the fact that

$$A(M) = \left\{\pi^2/k^2\right\}.$$

Now suppose that (M, g) is $NCB(\infty)$ and $\int_M \Gamma \cdot \sigma = 0$. First by (4.3) letting k tend to infinity we have $f_\infty(x) \geq 0 \ \forall x \in U(M)$ and then by (8.4.8), (8.4.10) and (8.4.12) we have

$$I = \int_{U(M)} f_\infty(x) \cdot \sigma = -\frac{k(d-1)}{2d} \int_M \Gamma \cdot \sigma = 0.$$

Hence

(8.4.14) $$f_\infty(x) = 0 \ \forall x \in U(M).$$

Now by (8.2.7) we have

$$0 \leq l''_{s_i(k)} = \int_{-k/2}^{k/2} \frac{\pi^2}{k^2} \sin^2\left(\frac{\pi t}{k}\right) - A(G_t(x), h_i(t)) \cos^2\left(\frac{\pi t}{k}\right) \mathrm{dt}$$

and because of (8.4.6) and (8.4.14) **276**

$$\sum_{i=2}^{d} l''_{s_i(k)}(0) \to 0 \text{ as } k \to \infty.$$

Since each of the terms is non-negative we have

$$\lim_{k\to\infty}\int_{-k/2}^{k/2}\left(\frac{\pi^2}{k^2}\sin^2\left(\frac{\pi t}{k}\right)-A(G_t(x),h_i(t)\cos^2\left(\frac{\pi t}{k}\right)\right)\mathrm{dt}=0.$$

But by (8.2.7) the integrand is non-negative, and hence

$$\cos^2\left(\frac{\pi t}{k}\right)A(G_t(x),h_i(t))\leq\frac{\pi^2}{k^2}\sin^2\frac{\pi t}{k}.$$

Fixing t and letting k to infinity we have

$$A(G_t(x),h_i(t)\leq 0.$$

Hence the sequence $f_n(t)$ of functions defined by the equation

$$f_n(t)=\begin{cases}-A(G_t(x),h_i(t))\cos^2\left(\frac{\pi t}{n}\right) \text{ in }]-\frac{n}{2},\frac{n}{2}[\\ 0 \text{ outside}\end{cases}$$

is a non decreasing sequence of functions. Hence

$$0=\lim_{n\to\infty}\int f_n(t)\,\mathrm{dt}=\int\lim_{n\to\infty}f_n(t)\,\mathrm{dt}=\int A(G_t(x),h_i(t)\,\mathrm{dt}\,.$$

Since $A(G_t(x),h_i(t))$ is of the same sign we have

$$A(G_t(x),h_i(t))=0\ \forall t.$$

Putting $t=0$ we get

$$A(x,x_i)=0$$

whenever x and x_i are orthogonal. But the sectional curvature is a function of the two dimensional subspaces and hence the result. □

5 Index of a vector field

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In the case of a compact oriented r.m. (M,g), we have seen that $\int_M \Gamma\cdot\sigma$ plays a role. In the two dimensional cases we have $\Gamma=2A$, and the

Gauss-Bonnet formula (see next article) says that

$$\int_M A \cdot \sigma = 2\pi \cdot \chi(M),$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M. In the next section we give that part of the proof which involves riemannian geometry. Now we collect some facts from algebraic topology.

1. Let us take $\mathbb{R}^d$ with its canonical r.s. ϵ and denote by $\widehat{\omega} \in \mathscr{E}^d(U(\mathbb{R}^d))$ the form called ω in (5.2.11). Let $m \in \mathbb{R}^d$ and B an open set of $\mathbb{R}^d$ containing m and X a vector field on $\mathbb{R}^d$ which does not vanish in B except at m. We define $\overline{X}$ on $B - \{m\}$ by the equation

$$\overline{X} = X/\|X\|.$$

Then

$$\overline{X} \in D(B - \{m\}, U(\mathbb{R}^d)).$$

Let $r > 0$ be sufficiently small; we consider the number

$$\frac{1}{\boxed{d-1}} \int_{S(m,r)} (X)^*(\widehat{\omega}) \tag{8.5.1}$$

(for $\boxed{d-1}$ see (6.7.3) and for $S(m, r)$ see (8). It is known that the above number is an integer depending only on X and m but not on r. This number is called *the index of the vector field X at m and is denoted by $i(X, m)$.* In fact this index can be viewed as the degree (in the sense of **278**
algebraic topology; see the proof of (6.6) or [35]: theorem 4.2 p. 127) of the map:

$$\zeta \circ \overline{X} \circ \zeta_m^{-1} : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}. \tag{8.5.1.bis}$$

2. Let M be an oriented manifold and let X be an element of $\mathscr{C}(M)$ having an isolated zero at the point m of M. Now let (A, U) be a positive chart around m, such that X does not vanish on $A - \{m\}$. Then it is known that for a sufficiently small positive number r

$$i(U^T \circ X \circ U^{-1}, U(m)) = \frac{1}{\boxed{d-1}} \int_{S(U(m),r)} (U^T \circ X \circ U^{-1})^*(\widehat{\omega}) \tag{8.5.2}$$

depends only on X and m but not on (A, U).

This number is called *the index of X at m and is denoted by* $i(X, m)$. If we define $\widehat{\omega}_0$ by:

$$\widehat{\omega}_0 = \widehat{\omega}|U_0(\mathbb{R}^d)$$

then

(8.5.3) $$i(X, m) = \frac{1}{(d-1)} \int\limits_{S(U(m),r)} (U^T \circ X \circ U^{-1})^*(\zeta^*(\widehat{\omega}_0)).$$

3. It is known that on a compact oriented r.m. (M, g) there exist vector fields which vanish only at a finite number of points, and that for any such vector field X, if we denote the set of points where it vanishes by m_s, we have

(8.5.4) $$\sum_x i(X, m_s) = \chi(M)$$

279 where $\chi(M)$ is the Euler-Poincaré characteristic of M. A proof of this fact is usually got either by using the Lefschetz fixed point theorem or by using Morse theory.

6 Gauss-Bonnet formula

In this section we assume that (M, g) is a compact oriented surface, that σ denotes its canonical volume form and that A is its sectional curvature. We follow the notation of the preceding article.

6.1

Theorem. *(Gauss-Bonnet formula)*

$$\int\limits_M A \cdot \sigma = 2\pi \cdot \chi(M).$$

6.2

Lemma. *For the form* $\omega \in \mathscr{E}^1(U(M))$ *(defined in* (5.2.11)*) we have*

$$d\omega = -p^*(A \cdot \sigma).$$

Proof of the lemma. For every element x of $U(M)$ let $\underline{x}$ be the unique element of $T_{p(x)}(M)$ such that $\{x, \underline{x}\}$ forms a positive orthonormal basis of $T_{p(x)}(M)$.

By the definition of ω.

$$\omega(z) = 0 \ \forall z \in H_x, \omega(\zeta_x^{-1}\underline{x}) = 1. \tag{8.6.3}$$

First let us note that a basis of $T_x(U(M))$ is given by

$$\{a_1 = \zeta_x^{-1}\underline{x}, a_2 = C(x, x), a_3 = C(x, \underline{x})\}$$

and then that to prove the lemma we have only to check the equality on pairs of these vectors. To compute $d\omega$ let us extend a_1, a_2, a_3 to vector fields on $U(M)$.

6.4

280

We take arbitrary vector fields X, $\underline{X}$ on M such that

$$X(p(x)) = x, \underline{X}(p(x)) = \underline{x} \quad \text{and} \quad [X, \underline{X}] = 0$$

and set

$$Z(y) = \xi_y^{-1} y \ \forall y \in U(M).$$

Then clearly Z, X^H, X^H are extensions of a_1, a_2 and a_3 respectively. By construction we have

$$\left[X^H, Z\right] = \left[\underline{X}^H, Z\right] = 0,$$

for p^T and v vanish on both since they can be taken inside the brackets. Hence by (0.2.10) we have

$$d\omega(X^H, Z) = 0 \quad \text{and} \quad d\omega(\underline{X}^H, Z) = 0, \tag{8.6.5}$$

and further

$$p^*(A\cdot\sigma)(X^H, Z) = A\cdot\sigma(p^T\circ X, p^T\circ Z) = 0$$

since Z is vertical, and similarly

$$p^*(A\cdot\sigma)(\underline{X}^H, Z) = 0.$$

Now on the one hand we have

$$\begin{aligned} d\omega(a_2, a_3) = d\omega(X^H, \underline{X}^H)_x &= -\omega(R(x, \underline{x})x) \text{ by (5.10)} \\ &= g(R(x, \underline{x})x, \underline{x}) = -A(x, \underline{x}) \end{aligned}$$

since x and $\underline{x}$ are orthonormal.

On the other hand

$$\begin{aligned} p^*(A\cdot\sigma)(a_2, a_3) &= A(p(x))\sigma(p^T(a_2), p^T(a_3)) \\ &= A(p(x))\sigma(x, \underline{x}) = A(p(x)) \end{aligned}$$

since $\{x, \underline{x}\}$ is a positive orthonormal basis of $T_{p(x)}(M)$. Hence the lemma.

281 **Proof of the theorem.**

a) Let X be a vector field on M which vanishes only on the finite set $\{m_s\}$. Let $M = M - \{m_s\}$ and on M set $\overline{X} = \dfrac{X}{\|X\|}$. We choose a positive number ϵ such that

   i) $\exp_{m_s}$ is ϵ-O.K. $\forall s$

   ii) $B(m_s, \epsilon) \cap B(m_t, \epsilon) = \phi$ if $s \neq t$.

   In particular, m_s is the only zero of X in $B(m_s, \epsilon)$, hence

$$M \cap B(m_s, \epsilon) = B(m_s, \epsilon) - \{m_s\} \forall s.$$

b) For every r in $]0, \epsilon[$, the set $M_r = M - \bigcup\limits_s B(m_s, r)$ is a nice domain of M, for $\exp_{m_s}$ is ϵ-O.K. and $\underline{B}(m_s, r)$ is a nice domain in $T_{m_s}(M)$. The boundary of M_r is the union $\bigcup\limits_s S(m, r)$. Note that the set $S(m_s, r)$ as the boundary of $B(m_s, r)$ has the orientation opposite to that on it as a part of the boundary of $B(m_s, \epsilon) - \overline{B(m_s, r)}$.

c) By (6.2) noting that $p \circ \overline{X} = \mathrm{id}_M$ we have

$$(\overline{X})^* d\omega = -\overline{X}(p^*(C \cdot \sigma)) = -A \cdot \sigma;$$

and further by (2.4) iv,

$$(\overline{X})^*(d\omega) = d((\overline{X})^* \omega).$$

Hence

$$\int_{\dot{M}_r} A \cdot \sigma = -\int_{\dot{M}_r} d((\overline{X})^* \omega) = -\int_{b(\dot{M}_r)} i^*(\overline{X})^* \omega$$

by Stokes formula ((3.13)) and hence

(8.6.6) $$\int_{\dot{M}_r} A \cdot \sigma = \sum_s \int_{S(m_s, r)} i^*((\overline{X})^* \omega).$$

Let $\lambda_s = (\exp_{m_s} | B(m_s, \epsilon))^{-1}$. Then λ_s is a diffeomorphism of **282**
$\underline{S}(m_s, r)$ onto $S(m_s, r)$. Hence by (0.3.9) we have

(8.6.7) $$\int_{S(m_s,r)} i^*((\overline{X})^* \omega) = \int_{\underline{S}(m_s,r)} \overline{(\lambda^T \circ X \circ \lambda^{-1})}(((\lambda^T)^{-1})^*(\omega).$$

By (8.5.3) we have (since ① $= 2\pi$)

(8.6.8) $$i(X, m_s) = \frac{1}{2\pi} \int_{S(m_s,r)} \left(\lambda^T \circ X \circ \lambda^{-1}\right)^* (\lambda^*(\widehat{\omega}_0)),$$

d) Now let us look at $((\lambda^T)^{-1^*})(\omega)$ and $\zeta^*(\widehat{\omega}_0)$.

We have $\lambda_i^{-1} = \exp_{m_i}$ and hence

$$((\lambda^T)^{-1})^*(\omega) = (\exp^T)^*(\omega);$$

from (4.6) it follows that

$$\left[(\exp^T)^* \omega\right]_{0_{m_i}} = [\zeta^*(\omega)]_{0_{m_i}} = \zeta^*(\widehat{\omega}_0).$$

Hence

$$((\lambda^T)^{-1})^*(\omega) - \zeta^*_{m_s}(\widehat{\omega}_0)_n \to 0 \quad \text{as} \quad n \to m_s.$$

Now $\forall m \in M$ and $\forall z \in T_{0_m}(T_m(M))$ we have (4.6),

$$\|\lambda^{-1}(z)\| = \|\exp^T(z)\| = \|\zeta^{-1}_{0_m}(z)\| = \|z\|$$

and hence

$$\overline{(\zeta^T \circ X \circ \lambda^{-1})_n} - (\lambda^T \circ X \circ \lambda^{-1})_n \to 0 \quad \text{as} \quad n \to m_s.$$

Hence by (8.6.7) and (8.6.8) it follows that

$$\frac{1}{2\pi} \int\limits_{S(m_s,r)} ((\overline{X}^*)(\omega) - i(X, m_s))\omega \to 0 \quad \text{as} \quad r \to 0.$$

283 Hence we have

$$\lim_{r\to 0} \int\limits_{\dot{M}_r} A \cdot \sigma = 2\pi \left(\sum_s i(X, m_s) \right) = 2\pi \cdot \chi(M) \quad \text{by (8.5.4).}$$

But since $\{m_s\}$ has measure zero we have

$$\lim_{r\to 0} \int\limits_{\dot{M}_r} A \cdot \sigma = \int\limits_M A \cdot \sigma.$$

6.9

Corollary. *If M is homeomorphic to $\mathbb{S}^2$ then*

$$\int\limits_M A \cdot \sigma = 4\pi,$$

and if M is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ then

$$\int\limits_M A \cdot \sigma = 0.$$

6.10

Remarks. a) In the case of dimensions greater than three we have a formula:

$$\chi(M) = \int_M \varphi \cdot \sigma$$

where φ is an element of $F(M)$ depending on the curvature tensor R of (M, g). The relation between φ and R is complicated. The proof of the above formula proceeds analogously; one proves that there exists a form ψ such that

$$d\psi = p^*(\varphi \cdot \sigma)$$

on $U(M)$; here ψ is not the form ω of (5.2.11) but one far more complicated, which involves R: for details see [9]: pp. 38-41.

b) The formula $\int_M A \cdot \sigma = 2\pi(\sum_s i(X, m_s))$ proves that the sum $\sum_s i(X, m_s)$ does not depend on the vector field X on M (put any r.s. on M) so we know that this sum is an invariant. Then (8.5.4) gives us its value.

c) The formula $\int_M A \cdot \sigma = 2\pi(\sum_s i(X, m_s))$ proves that $\int_M A \cdot \sigma$ does **284**
not depend on the r.s. chosen on M but only on the differentiable manifold M.

d) In order to be self-contained, we prove (6.9) without appeal to (8.5.4); we will, in fact, use (6.1) in the next articles only for the sphere and for the torus. When M is homeomorphic to the torus we use the fact there exists on M a nowhere zero vector field. Then there is a map $\overline{X} : M \to U(M)$ and, by Stokes formula,

$$\int_M A \cdot \sigma = -\int_M d((\overline{X})^* \omega) = 0.$$

When M is homeomorphic to the sphere, we compute $\sum_s i(X, m_s)$ for a particularly simple vector field on $\mathbb{S}^2$ and apply the remark c). Let p denote the projection $(x, y, z) \to (x, y)$ in $\mathbb{R}^3$ and X the vector field on $\mathbb{S}^2$ such that

$$\forall m \in \mathbb{S}^2 : p^T(X(m)) = \zeta^{-1}_{p(m)}(\underline{p(m)})$$

where $x \to \underline{x}$ denotes rotation by $\frac{\pi}{2}$ in $\mathbb{R}^2$. Because the restrictions of p to the northern and southern hemispheres are charts, the formula (8.5.2) (8.5.1.bis) give the index of X at the north and the south poles P, P':

$$i(X, P) = i(X, P') = 1.$$

In fact the maps $\mathbb{S}^1 \to \mathbb{S}^1$ so obtained are the rotations by $\frac{\pi}{2}$ or by $-\frac{\pi}{2}$, and the degree of either is 1.

7 E. Hops's theorem

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Using (2.6) we prove the:

7.1

Lemma. *If (M, g) is a complete r.m., and if $m \in M$ and $r > 0$ are such that $\exp_m |\underline{B}(m, r)$ is injective, then $\exp_m$ is r-O.K.*

Proof. If the result were false, $\exists\, x \in U_m(M)$ with $1 = j(x) < r$. Let ϵ be so that $1 < 1 + \epsilon < r$ and let $n = \gamma_x(1 + \epsilon)$. Then, by (2.6), the restriction $\gamma_x|[0, 1 + \epsilon] \notin \mathscr{T}(m, n)$. Hence $1 + \epsilon > d(m, n)$. By (8.4.8) we see that $\exists\, y$ such that $||y|| = d(m, n)$ and $\exp(y) = n$. Hence $\exp(y) = \exp((1 + \epsilon)x)$ and $y \neq (1 + \epsilon)x$, which is the required contradiction. □

7.2

Proposition. *Let m be a point of a complete r.m. (M, g).*
Then the following conditions are equivalent:

1) $\exp_m : T_m(M) \to M$ *is injective.*

2) *M is simply connected and* $\exp_m^T$ *is everywhere of maximal rank.*

Proof. The fact that (2) implies (1) is (5.3). Now by (7.1) from (1) it follows that $\exp_m$ is of maximal rank, and hence $\exp_m$ is a diffeomorphism between $T_m(M)$ and M. Hence (1) implies (2).

Now suppose that (M, g) is complete and that for every point m of M, $\exp_m$ is injective on T_m. Then it follows that $\forall m,\ n \in M$ the set $\mathscr{T}(m, n)$ consists of exactly one element. Then the geodesics have the set theoretic properties of straight lines in euclidean or hyperbolic geometry. It is of interest to see whether there exist compact r.m.s (M, g) whose universal riemannian covering $(\widetilde{M}, \widetilde{g})$ have geodesics sharing that property. In the case of surfaces any surface with $A(M) \subset]-\infty, 0]$ will **286**
do. But if we require that M be homeomorphic to a torus we have the following theorem due to E. Hopf ([15]). □

7.3

Theorem. *If* (M, g) *is an r.m. such that*

1) *M is* $NCB(\infty)$

2) *M is homeomorphic to* $\mathbb{S}^1 \times \mathbb{S}^1$

then $A(M, g) = \{0\}$.

Proof. By the inequality in the first part of (4.7), letting k tend to infinity, we get

$$\int_M A \cdot \sigma = 0,$$

and then by the last part of (4.7) and (6.9) we get:

$$A(M) = \{0\}.$$

□

7.4

Remarks. 1) For the dimensions greater than two the question is open.

2) For further considerations see article 10.

8 Another formula

In this article, we prove an inequality of integral geometry, which is slightly weaker than a formula of Santalo: see the remark at the end of the article.

Let us consider (M, g) a complete, oriented r.m. of dimension d and $A \subset M$ a sub manifold of M of dimension $d-1$. For points $n \in A$ we identify $T_n(A)$ with its image in $T_n(M)$ and define

$$U(A) = \bigcup_{n \in A} (U(M) \cap T_n(A))$$
$$U(M)|A = \bigcup_{n \in A} T_n(M).$$

287 Clearly the orthogonal of $T_n(A)$ is one-dimensional and we assume from now on that we can choose unit vectors in this one dimensional subspace such that they can be extended into a *vector field N of M*; then, (see (2.9)) we have (8)

$$U(A) = (U(M)|A) \cap (g^\sharp(N)^{-1}(0))$$

8.1

Lemma. *The subset*

$$W = \{U(M)|A\} - U(A)$$

of $U(M)$ is a sub manifold of dimension $2d-2$.

Proof. Suppose that locally A is given by $\varphi^{-1}(0)$ where $\varphi \in F(M)$. Then $U(M)|A$ is given by $(\varphi \circ p)^{-1}(0)$ (for $p : U(M) \to M$). Further $U(M)$

is a $(2d - 1)$ dimensional. Hence it follows that $U(M)|A$ is $(2d - 2)$ dimensional. But $U(A)$ is closed in $U(M)|A$ since, by (1.10), $U(A) = (g^\sharp(X))^{-1}(0)$. Hence the given set is an open sub manifold of $U(M)|A$. Hence the result.

In this section we compute the volume of the set

$$E_1 = \{G_t(W)|0 < t < 1\}, \tag{8.8.2}$$

where G_t is the geodesic flow.

To do this we introduce the map

$$f : \mathbb{R} \times W \to U(M)$$

defined by the equation

$$f(t, x) = G_t(x) \tag{8.8.3}$$

($(G_t(x)$ makes sense, since M is complete and hence G_t is defined for all t.).

We define an orientation on A in a natural way by means of N: a basis $\{x_2, \ldots, x_d\}$ of $T_n(A)$ is said to be a positive basis if $\{N(n), x_2, \ldots, x_d\}$ is a positive basis for $T_n(M)$ relative to the orientation on M. Let α be **288**
the volume form of this oriented r.m. $(A, g|A)$. We use the symbol ω exclusively for the restriction of ω (see (5.2.12)) to $U(M)|A$. Then the canonical volume form $\overline{\overline{\alpha}}$ of $(U(M)|A))$ is given by the equation,

$$\overline{\overline{\alpha}} = \omega \wedge (p^*(\alpha)). \tag{8.8.4}$$

On the other hand a volume form β on $\mathbb{R} \times W$ is given by

$$\beta = p_1^*(\mathrm{dt}) \wedge p_2^*(\overline{\overline{\alpha}}) \tag{8.8.5}$$

where p_1 and p_2 denote the projections from $\mathbb{R} \times W$ to $\mathbb{R}$ and W respectively.

Hence $\exists \varphi \in F(\mathbb{R} \times W)$ such that

$$f^*(\overline{\overline{\sigma}}) = \varphi \cdot \beta \tag{8.8.6}$$

where $\overline{\overline{\sigma}}$ is the canonical volume form of $(U(M), \overline{\overline{g}})$. Clearly the computation of $\mathrm{Vol}(W, g)$ amounts to that of φ. □

8.7

Lemma. $\forall x \in W$, $t \in \mathbb{R}$ *we have*

$$\varphi(t, x) = \varphi(0, x).$$

Proof. This merely asserts the invariance of $\overline{\overline{\sigma}}$ under the geodesic flow. To be more precise let τ_t denote the transformation of $\mathbb{R} \times W$ into itself taking (s, x) into $(s + t, x)$. Then since $G_{t+s} = G_t \circ G_s$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R} \times W & \xrightarrow{f} & V(M) \\ {\scriptstyle \tau_t}\downarrow & & \downarrow{\scriptstyle G_t} \\ \mathbb{R} \times W & \xrightarrow[f]{} & V(M) \end{array}$$

Hence we have

$$\begin{aligned} \varphi \cdot \beta &= f^*(\overline{\overline{\sigma}}) = f^*(G_t^*(\overline{\overline{\sigma}})) = (G_t \circ f)^*(\overline{\overline{\sigma}}) \text{ [by (3.10) and (5.2.7)]} \\ &= (f \circ \tau_t)^*(\overline{\overline{\sigma}}) = \tau_t^*(f^*(\overline{\overline{\sigma}})) = \tau_t^*(\varphi \circ \beta) = \tau_t^*(\varphi)\tau_t^*(\beta). \end{aligned}$$

289 But by construction β is invariant under τ_t and hence we have

$$\varphi \cdot \beta = \tau_t^*(\varphi) \cdot \beta.$$

Since β is never zero we have

$$\varphi(0, x) = \varphi(t, x).$$

□

8.8

Lemma. $\forall x \in W : \varphi(0, x) = g(x, N(p(x)))$.

Proof. Let $p(x) = n$ and let $\{y_1, \dots, y_d\}$ and $\{x_1, \dots, x_d\}$ be orthonormal bases of $T_n(M)$ with $y_1 = N(n)$ and $x_1 = x$. Then if we set

$$u_i = C(x, y_i) \quad \text{and} \quad v_i = \zeta_x^{-1} x_i \quad \text{for} \quad i = 1, \dots, d,$$

we see, by definition of the r.s. on $T(M)$, that

$$\{u_2, u_3, \dots, u_d; v_2, \dots, v_d\}$$

is an orthonormal basis of $T_x(W)$. Hence by the definition of β we have

$$\beta((P, 0), (0, u_2), (0, u_d); (0, v_2), \dots, (0, v_d)) = 1.$$

Hence

$$\varphi(0, x) = f^*(\overline{\overline{\sigma}})(P, 0); (0, u_2), \dots, (0, u_d); (0, v_2), \dots, (0, v_d)) = \dots$$
$$\dots = (f^T(P, 0), f^T(0, u_2), \dots, f^T(0, v_d)).$$

Since $G_0 = \mathrm{id}_{U(M)}$ we have

$$f^T_{(0,x)}(0, z) = z \forall z \in T_x(W);$$

and $f^T_{(0,x)}(P, 0)$ is the initial speed of the curve $t \to G_t(x)$. But this is $C(x, x)$ by the definition of G_t (see (0.33)) and hence we have

$$\varphi(0, x) = \overline{\overline{\sigma}}(C(x, x), u_2, \dots, u_d, v_2, \dots, v_d).$$

Now let us write **290**

$$x = g(x, N(n)) \cdot N(n) + k \quad \text{with} \quad g(k, N(n)) = 0.$$

Then

$$C(x, x) = g(x, N(n)) \cdot C(x, N(n)) + C(x, k) = g(x, N(x)) \cdot u_1 + C(x, k).$$

□

Since k is orthogonal to $N(n)$ it is a linear combination of $y_2, \dots, y_d$ and hence $C(x, k)$ is that of $u_2, \dots, u_d$. Hence we have

$$\overline{\overline{\sigma}}(C(x, k), u_2, \dots, u_d, v_2, \dots, v_d) = 0.$$

Hence we have

$$\varphi(0, x) = g(x, N(n)) \cdot \overline{\overline{\sigma}}(u_2, \dots, u_d, v_2, \dots, v_d) = g(x, N(n)).$$

8.9

Remarks. This lemma together with (8.7) in particular shows that f is of maximal rank. Hence E is open in $U(M)$.

Now we come to our computation.

8.10

Proposition. *If* $\mathrm{Vol}(A, g|A)$ *is finite, then* $\mathrm{Vol}(E_1, \overline{\overline{g}})$ *is finite and*

$$\mathrm{Vol}(E_1, \overline{\overline{g}}) \frac{21 \cdot \boxed{d-2}}{d-1} \cdot \mathrm{Vol}(A, g|A).$$

Proof. We know (8.9) that f^T is everywhere of maximal rank and that f is onto. Hence we have by (3.10)

$$\mathrm{Vol}(E_1, \overline{\overline{g}}) \leq \int_{[0,1]\times W} |f^*(\overline{\overline{\sigma}})|$$

But the definition of β and φ and (3.17) gives

$$\int_{[0,1[\times W} |f^*(\overline{\overline{\sigma}})| = \int_{]0,1[\times W} |\varphi|(p_1^*(\mathrm{dt}) \wedge p_2^*(\overline{\overline{\alpha}})) = \int_0^1 \int_{\{t\}\times W} |\varphi| p_2^*(\overline{\overline{\alpha}})\,\mathrm{dt}$$

291 But neither φ nor $p_2^*(\overline{\overline{\alpha}})$ depends on t by (8.7), and hence we have

$$\int_{\{t\}\times W} |\varphi| p_2^*(\overline{\overline{\alpha}}) = \int_{\{0\}\times W} |\varphi| p_2^*(\overline{\overline{\alpha}}) = \int_W |\varphi| \overline{\overline{\alpha}}$$

where we have identified $\{0\} \times W$ with W (since $G_0 = \mathrm{id}_{U(M)}$). □

Hence we have

$$\mathrm{Vol}(E_1, \overline{\overline{g}}) \leq 1 \cdot \int_W |\varphi| \overline{\overline{\alpha}}.$$

Now since $U(M)|A$ is of measure zero in $U(M)$, we have

$$\int_W |\varphi|\overline{\overline{\alpha}} = \int_{U(M)|A} |\varphi|\overline{\overline{\alpha}}.$$

Using the fact that $\overline{\overline{\alpha}} = \omega \wedge p^*(\alpha)$ together with (3.17), (8.8), we get

$$\int_{U(M)} |\varphi|\overline{\overline{\alpha}} = \int_{n\in A} \left(\int_{U_n(M)} |g(x, N(n))|(\omega|U_n(M)) \right) \alpha.$$

But by (6.7.11)

$$\int_{U_n(M)} |g(x, N(n))|(\omega|U_n(M)) = \frac{2}{d-1} \cdot \enclose{circle}{d-2} \;\forall n;$$

hence

$$\int_{U(M)} |\varphi|\overline{\overline{\alpha}} = \frac{2}{d-1} \cdot \enclose{circle}{d-2} \int_A \alpha = \frac{2 \cdot \enclose{circle}{d-2}}{d-1} \cdot \mathrm{Vol}(A, g|A).$$

8.11

Remark. If we assume that A, in addition to satisfying the conditions of the above sections, is *convex* (i.e. everywhere locally A is given by $\varphi^{-1}(0)$ with φ having a non-negative Hessian form $D\, d\varphi$ (see (4.11)) and if we set

$$W^+ = \{x \in U_n(M) | g(x, N(p(x))) > 0\},$$

then, one can show that, for sufficiently small l, actually f is a diffeo- **292**
morphism between $]0, l[\times W^+$ and $f(]0, l[\times W^+)$, so that one has the equality

$$\mathrm{Vol}(f(]0, l[\times W^+)) = \frac{1 \cdot \enclose{circle}{d-2}}{d-1} \cdot \mathrm{Vol}(A, g|A)$$

(see [27]: formula (21) p. 488).

9 L.W. Green's theorem

In (8) we have built up an r.m. (M, g) which is C_m-manifold $\forall m \in M$, which is homeomorphism to $\mathbb{S}^2$ but is not isometric to $(\mathbb{S}^2, k.\text{can})$ for any real number k. It is somewhat striking that if we replace "homeomorphic to $\mathbb{S}^2$" by "homeomorphic to $P^2(\mathbb{R})$", then there exists a positive real number k such that (M, g) is isometric to $(P^2(\mathbb{R}), k.\text{can})$. This theorem (9.5) is due to L.W. Green: [13]

For the rest of this section let us assume that (N, h) is a C_m-manifold $\forall m \in N$ and that N is homeomorphic to $P^2(\mathbb{R})$.

Under the above assumption it follows that the common length of the geodesics through a point n is independent of n. For if n' is any other point, then by the Hopf-Rinow theorem and (4.8) it follows that there exists a geodesic γ through n and n', and all geodesics through n' have the same length as γ. Hence, by multiplying, if necessary, the r.s. on (N, h) by a positive constant we can assume that *the length of each geodesic is* π. Now let (M, g) be the universal riemannian covering of (N, h). Then by (6.5) it follows that

293 i) M is homeomorphic to $\mathbb{S}^2$

ii) (M, g) is a C_m-manifold for every m of M, and every geodesic starting from m and of length π ends at $\sigma(m)$, where σ is the non trivial deck transformation of (M, g).

9.1

Lemma. *$\forall m, m' \in M$,*

$$d(m, m') \leq \pi,$$

and equality occurs if and only if $m' = \sigma(m)$.

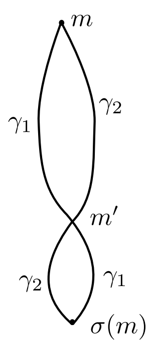
Proof. $\leq$ follows from ii). Now $d(m, \sigma(m)) < \pi$ is absurd for take $\gamma \in \mathscr{T}(m, \sigma(m))$. Then $p \circ \gamma$ would be a geodesic from $p(m)$ to $p(m)$ and of length $< \pi$ contradicting the definition of a C_m-manifold. □

9.2

Lemma. *For every m of M* $\exp_m$ *is π-O.K.*

Proof. Because of (7.1) we need only show that $\exp_m |B(m,\pi)$ is injective. Now let γ_1 and γ_2 be two geodesics meeting at a point m' different from both m and $\sigma(m)$. Then we can suppose that the length of the curve γ_1 between m and $\sigma(m)$ is less than or equal to that of γ_2. Then since γ_1 and γ_2 are distinct geodesics we have $\gamma_1'(m') \neq \gamma_2'(m')$. Hence by (3.8) we have

$$\begin{aligned} d(m,\sigma(m)) &< \lg \gamma_1| \text{ from } m \text{ to } m' + \lg \gamma_2| \text{ from } m' \text{ to } \sigma(m) \\ &\leq \lg \gamma_2| \text{ from } m \text{ to } \sigma(m) = \pi. \end{aligned}$$



But this contradicts (9.1). □ **294**

9.3

Corollary. *One has: $ar(M,g) \geq 4\pi$ and equality occurs if and only if $A(M,g) = \{1\}$.*

Proof. By (9.2), (M,g) is $NCB(\pi)$. Then the results follows from (4.7) and (6.9). □

9.4

Lemma . *Any closed geodesic γ_1 in (M, g) meets any geodesic γ of length π.*

Proof. Let us suppose that they do not meet and that γ is a geodesic from m to $\sigma(m)$. Then since M is homeomorphic to the sphere and γ_1 is a simple closed curve (by Jordan curve theorem) it divides the rest of M into two connected components. Since γ is connected it is in one connected component. Let m' be a point in the other. Then let γ_2 be a geodesic curve from m to $\sigma(m)$ passing through m'. Then m' and $\sigma(m)$ are in two different connected components of $M - \gamma_1$ and γ_2 from m' to $\sigma(m)$ joins them. Hence γ_2 from m' to $\sigma(m)$ meets γ_1 and similarly γ_2 from m to m' does. Hence γ_2 meets γ_1 in at least two points m_1 and m_2. Then we have a part of the geodesic curve γ_2, of length less than π, joining m_1 and m_2; and a part of the geodesic curve γ_1 with the same property. But this contradicts the fact that $\exp_{m_1} | B(m_1, \pi)$ is injective. Hence the result. □

9.5

Theorem. *(L.W. Green) Let (N, h) be an r.m. such that*

i) *N is homeomorphic to $P^2(\mathbb{R})$,*

ii) *(N, h) is a C_n-manifold $\forall n \in N$ (the common length being π). Then (N, h) is isometric to $(P^2(\mathbb{R}), \text{can}\,.)$.*

295 *Proof.* (a) Let A be a closed geodesic of length 2π chosen once for all. It is a sub manifold of (M, g). Using the notation of (8) and (8.10) we have

$$\text{Vol}(E_\pi, \overline{\overline{g}}) \leq 8\pi^2. \qquad (8.9.6)$$

(b) Now we claim that

$$E_\pi = U(M) - U(M)|A.$$

Since, any two geodesics through a point m meet only at $\sigma(m)$ by the definition of E_π it follows that

$$E_\pi \subset U(M) - U(M)|A.$$

Now let $x \in U(M) - U(M)|A$, and let γ be the curve:

$$t \to \exp t.x, \quad t \in [-\pi, 0].$$

But (9.4) it follows that $\exists\, t_0 \in [-\pi, 0]$ such that $\exp(t_0 x) \in A$ and hence, if we set $y = \gamma'(t_0)$ we have $y \in U(M)|A$. Clearly since A is a geodesic we have

$$G_t(U(A)) = U(A)$$

and hence $y \notin U(A)$. Further $t_0 \neq -\pi$. Hence $y \in W$ and

$$G_{-t_0}(y) = x \in E_\pi.$$

(c) Now since $U(M)|A$ is of measure zero in $U(M)$ we have by (5.2.13)

$$\mathrm{Vol}(E_\pi, \bar{\bar{g}}) = \mathrm{Vol}(U(M), \bar{\bar{g}}) = 2\pi \cdot \mathrm{ar}(M, g).$$

But this together with (8.9.6) gives

$$\mathrm{ar}(M, g) \leq 4\pi.$$

But then by (9.3) we have first

$$\mathrm{ar}(M, g) = 4\pi,$$

$$\text{and then} \qquad A(M, g) = \{1\}.$$

Now by (7.1) it follows that (M, g) is isometric to $(\mathbb{S}^2, \mathrm{can})$ and, **296**
since the deck transformation σ is such that

$$d(m, \sigma(m)) = \pi,$$

σ is nothing but the antipodal map of $\mathbb{S}^2$ hence (N, h) is isometric to $(P^2(\mathbb{R}), \mathrm{can})$.

□

9.7

Remarks. 1) The theorem was proved by Green under a slightly different assumption, namely that (M, g) is a complete two dimensional r.m. such that $j(x) = \pi$ for every x in $U(M)$. In this form, the theorem was conjectured by Blaschke a long time ago.

2) In the proof of (9.5) one can replace the use of (8.10) by that of (12.9).

3) The question is open for the $P^d(\mathbb{R})$'s, $d \geq 3$.

10 Concerning G-spaces

In order to bring out the results that are special to riemannian geometry and do not belong to distance geometry in general, H.Busemann was led to introduce metric spaces having the following properties.

i) Every closed bounded set is compact.

ii) Any two points can be joined by a geodesic.

iii) Geodesics can be prolonged locally.

iv) These prolongations are unique.

The results that follow are in [7]; the precise definition of the spaces is as follows.

10.1

297

A G- space M is a metric space M whose distance map d verifies the following conditions:

i) every closed bounded set is compact;

ii) $\forall m, n \in M$, $\exists\, p$ such that

$$p \neq m, p \neq n \text{ and } d(m,n) = d(m,p) + d(p,n);$$

iii) $\forall a \in M \exists\, r_a > 0$ such that $\forall m, n \in M$ with

$$d(a, m) < r_a, d(a, n) < r_a, \exists\, p \text{ with}$$
$$p \neq m, p \neq n \text{ and } d(m, p) = d(m, n) + d(n, p);$$

iv) if $m, n \in M$ and p_1, p_2 are such that

$$d(m, p_i) = d(m, n) + d(n, p_i)(i = 1, 2), d(n, p_1) = d(n, p_2)$$

then $p_1 = p_2$.

Under the above conditions i) and ii) one can define a set analogous to $\mathscr{T}(m, n)$ and the notion of geodesics. Then iii) gives local prolongability and iv) uniqueness of such prolongations.

Whether a G-space is necessarily a topological manifold is an open question in dimensions greater than two.

Now any complete r.m. is indeed a G-space; for i) comes from (4.3); ii) follows from (4.8), iii) comes from the existence of nice balls and iv) from (3.6).

The essential result of Busemann is that the theorem (5.5) pertains to G-spaces theory, whereas the theorems (7.3) and (9.5) pertain to riemannian geometry.

Concerning (5.5) the first thing to do is to find a metric definition equivalent to that of non positive curvature for a r.m.; such a definition is suggested by (3.8); in a G-space M and inside a ball $D(a, r_a) = \{m \in$ **298**
$M | d(a, m) < r_a\}$ one proves the uniqueness of a geodesic between two points, in particular of a midpoint $(n, n', \frac{1}{2})$ $\forall n, n' \in D(a, r_a)$, defined by

$$d(n, (n, n', \frac{1}{2})) + d((n, n', \frac{1}{2}), n') = d(n, n').$$

Then

10.2

Definition. *A G-space is said to have non positive curvature if* $\forall a \in M \exists \circ < s_a < r_a$ *such that*

$$d((m, n, \frac{1}{2}), (m, n', \frac{1}{2})) \leq \frac{1}{2} d(n, n')\ \forall n, n' \in D(a, s_a).$$

Then Busemann proved the following theorems [7] (38.2) and (39.1):

10.3

Theorem. *A simply connected G-space M with non positive curvature is such that one can take $r_a = \infty$ in (10.1) iii) $\forall a \in M$. In particular any two geodesics in M meet in at most one point and M is homeomorphic to some $\mathbb{R}^d$.*

On the other hand the theorems of L.W. Green and E. Hopf are not true for G-spaces, as shown by the following

10.4

Theorem. *([7], theorem (33.3)) There exists on $\mathbb{S}' \times \mathbb{S}$ a G-space structure whose universal covering is not isomorphic to the canonical G-space $\mathbb{R}^2$, such that any two geodesics meet at most at one point.*

Theorem (33.3) of [7] gives far more: it characterizes completely the systems of curves in $\mathbb{R}^2$ which are the set of geodesics of the universal covering of a G-space structure on $\mathbb{S}^1 \times \mathbb{S}^1$; then it is easy to pick out such a system which is not arguesian, although the canonical G-space structure on $\mathbb{R}^2$ is arguesian.

10.5

299 **Theorem.** *(Skornyakov: [28], see also [8]) Let $\sum$ be any system of closed Jordan curves in $P^2(\mathbb{R})$ such that any two distinct points in $P^2(\mathbb{R})$ lie exactly on one curve of $\sum$. Then there exists a G-space structure on $P^2(\mathbb{R})$ whose set of geodesics is $\sum$.*

11 Conformal representation

If (M, g) is an oriented surface we can define a map

$$J : T(M) \to T(M)$$

by extending linearly the map

$$x \to \underline{x}$$

appearing in the beginning of the proof of (6.2). It follows directly from the definition that the map J has the following properties:

(i) $J^2 = -\operatorname{id}_{T(M)}$

(ii) $g \circ J = g$

i.e. $g(J \circ X, J \circ Y) = g(x, Y)$, $\forall X, Y \in \mathscr{C}(M)$

(iii) If $f : M \to M$ is an isometry, then

$$f^T \circ J = J \circ f^T \text{ if } f \text{ preserves orientation,}$$
$$f^T \circ J = -J \circ f^T \text{ if } f \text{ reverses orientation.}$$

(iv) If h is any other r.s. on M such that $h \circ J = h$ then there exists a φ in $F(M)$ such that $h = \varphi \cdot g$. Now suppose that on the tangent bundle $T(M)$ of an even dimensional manifold M a map J

$$J : T(M) \to T(M)$$

satisfying the equation **300**

$$J^2 = -\operatorname{id}_{T(M)}$$

is given. Then in order that there exists on M a complex analytic structure for which J coincides with the multiplication by i it is necessary and sufficient that the map defined by

$$[X, Y] - J \circ [J \circ X, J \circ Y] + J \circ [J \circ X, Y] + [J \circ X, J \circ Y] = 0$$
$$\forall X, Y \in \mathscr{C}(M)$$

be the zero map: see [23].

In the case of our (M, g), because of the bilinearity of the expression above, we can assume that $J \circ X = Y$ in the expression above; then it is zero. Hence it follows that every two dimensional oriented r.m. admits a complex analytic structure the associated J of which satisfies ii).

This construction, applied to $(\mathbb{S}^2, \text{can})$ leads to the Riemann sphere.

11.1

Remark. In out two dimensional case for the existence, we do not actually need the deep theorem of [23]. What we need is "the existence of isothermal coordinates" which is easier: see [10]. *Now for the rest of this article, let* $\widetilde{h}_0$ denote the canonical r.s. on $\mathbb{S}^2$, and h_0 the canonical r.s. on a flat torus (see (2.2)) or that on $P^2(\mathbb{R})$. It is clear that a flat torus admits a canonical complex structure, coming down from that of $\mathbb{R}^2$, and we denote the associated multiplication by $\sqrt{-1}$ on its tangent bundle by J_0; we denote also by J_0 the multiplication by $\sqrt{-1}$ on the tangent bundle of the Riemann sphere $(\mathbb{S}^2, \widetilde{h}_0)$. Then the conformal
301 representation theorem for Riemann surface gives, in particular the

11.2

Theorem. *Let M be a one dimensional complex manifold. Then*

i) *if M is homeomorphic to* $\mathbb{S}^1 \times \mathbb{S}^1$ *there exists a flat torus* (N, h_0) *and a diffeomorphism*

$$\lambda : N \to M$$

such that

$$\lambda^T \circ J_0 = J \circ \lambda^T$$

ii) *if M is homeomorphic* $\mathbb{S}^2$ *there exists a diffeomorphism* $\lambda : \mathbb{S}^2 \to M$ *such that*

$$\lambda^T \circ J_0 = J \circ \lambda^T$$

(the condition $\lambda^T \circ J_0 = J \circ \lambda^T$ *simply means indeed that* λ *is holomorphic).*

Now suppose that we take the complex analytic structure associated to the r.s. g on (M, g). *Then on the flat torus* (N, h_0) *or on* $(\mathbb{S}^2, \text{can})$

$$\lambda^*(g) \circ J_0 = g \circ \lambda^T \circ J_0 = g \circ J \circ \lambda^T = g \circ \lambda^T = \lambda^*(g)$$

which together with (iv) gives the

11.3

Proposition. *Let (N, g) be an r.m. Then*

(1) *if M is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ then there exists a flat torus (N, h_0) and a diffeomorphism $\lambda : N \to M$ and a $\varphi \in F(N)$ such that $\lambda^*(g) = \varphi \cdot h_0$.*

(2) *if M is homeomorphic to $\mathbb{S}^2$, then there exists a diffeomorphism $\lambda : \mathbb{S}^2 \to M$ and a $\varphi \in F(\mathbb{S}^2)$ such that $\lambda^*(g) = \varphi \cdot \widetilde{h}_0$.*

11.4

Corollary. *Let (M, g) be an r.m. that is homeomorphic to $P^2(\mathbb{R})$. Then* **302**
there exists a diffeomorphism

$$\lambda : P^2(\mathbb{R}) \to M \ \text{ and a } \ \varphi \in F(P^2(\mathbb{R})) \ \text{ such that}$$

$\lambda^*(g) = \varphi \cdot h_0$.

Proof. Let $(\widetilde{M}, \widetilde{g})$ be the universal riemannian covering of (M, g)

$$\begin{array}{c} (\widetilde{M}, \widetilde{g}) \\ \Big\downarrow p \\ (M, g) \end{array} \qquad (p^*(g) = g)$$

Then $\widetilde{M}$ is homeomorphic to $\mathbb{S}^2$ and for the J associated to g and by (8.2.3), there exists a diffeomorphism $\mu : \mathbb{S}^2 \to M$ such that

$$\mu^T \circ J_0 = J \circ \mu^T$$

Let us denote the non trivial deck transformation of $\widetilde{M}$ by $\widetilde{\sigma}$ and set

$$\widehat{\sigma} = \mu^{-1} \circ \widetilde{\sigma} \circ \mu$$

The deck transformation $\widetilde{\sigma}$ is an isometry (see in (6.5)) reversing the orientation and hence by (iii) we have

$$\widetilde{\sigma}^T \circ J = -J \circ \widetilde{\sigma}^T$$

which gives

$$\widehat{\sigma}^T \circ J_0 = -J \circ \widehat{\sigma}^T$$

Hence $\widehat{\sigma}$ is an automorphism of $\mathbb{S}^2$ which reverses the sign of J, i.e. an antiholomorphic map, and further $\widehat{\sigma}$ is an involution without fixed points (since $\widetilde{\sigma}$ is). All antiholomorphic maps of the Riemann sphere being of the type

$$z \to \frac{a\bar{z} + b}{c\bar{z} + d}$$

303 one deduces that there exists a holomorphic map $\theta : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\widehat{\sigma} = \theta \circ \sigma \circ \theta^{-1}$ where $\sigma = -\operatorname{id}|\mathbb{R}^3|\mathbb{S}^2$ is the antipodal map on $\mathbb{S}^2$. Now if we set $\widetilde{\lambda} = \mu \circ \theta$ then $\widetilde{\lambda}$ is a diffeomorphism between $\mathbb{S}^2$ and M and we have

$$\widetilde{\lambda}^{-1} \circ \widetilde{\sigma} \circ \widetilde{\lambda} = \sigma$$

and hence we have a map λ defined by the following commutative diagram

$$\begin{array}{ccc} (\widetilde{M}, \widetilde{g}) & \xleftarrow{\ \widetilde{\lambda}\ } & (\mathbb{S}^2, \widetilde{h}_0) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (M, g) & \xleftarrow{\ \lambda\ } & (P^2(\mathbb{R}), h_0) \end{array}$$

Since $\widetilde{\lambda}$ is holomorphic by (iv) there exists a ψ in $F(\mathbb{S}^2)$ such that

$$(\widetilde{\lambda})^*(\widetilde{g}) = \psi \widetilde{h}_0.$$

We have

$$(\widetilde{\lambda})^*(\widetilde{g}) = \widetilde{\lambda}^*(p^*(g)) = p^*(\lambda^*)(g)$$

and hence

$$p^*((\lambda^*(g)) = \psi p^*(h_0)$$

which implies, since p^T is of maximal rank, that there exists a φ in $F(P^2(\mathbb{R}))$ such that

$$\psi = \varphi \cdot p \quad \text{and} \quad \lambda^*(g) = \varphi \cdot h_0$$

Hence the result. □

12 The theorems of Loewner and Pu

In this article we consider a given manifold M of dimension two and various r.s. on it.

We set

$$a(g) = \mathrm{ar}(M, g)$$

and in case $\overline{F}(M)$ *is non trivial* (see (6)) **304**

$$c(g) = \inf\{\mathrm{lg}(\tau)) | \tau \in \overline{F}(M) - \{0\}\}$$

i.e. $c(g)$ is the infimum of the lengths of all closed curves in (M, g) which are not homotopic to zero. The theorem of Loewner gives a lower bound for $\dfrac{a(g)}{(c(g))^2}$ for any r.s. on $\mathbb{S}^1 \times \mathbb{S}^1$ and that of P_u in the case of $P^2(\mathbb{R})$. Before giving them we compute $a(g)/(c(g))^2$ in the standard cases.

A) Let us take $P^2(\mathbb{R})$ and denote its canonical r.s. by h_0. By (6.7.6) $a(h_0) = 2\pi$. To compute $c(h_0)$ we look at the universal riemannian covering:

$$(\mathbb{S}^2, \mathrm{can}) \quad (P^2(\mathbb{R}), h_0).$$

For every τ in $\overline{F}(P^2(\mathbb{R})) - \{0\}$ the lift $\overline{\tau}$ of τ through m ends in $\sigma(m)(\sigma = -\mathrm{id}_{\mathbb{R}^3} |\mathbb{S}^2)$ so that $\mathrm{lg}(\widetilde{\tau}) \geq \pi$ and equality occurs if $\widetilde{\tau}$ is a geodesic. Hence we have

(8.12.1)
$$\frac{a(h_0)}{c^2(h_0))^2} = \frac{2}{\pi}.$$

B) Now let us take a flat torus $(T, h_0) = (\mathbb{R}^2/G, \epsilon/G)$.

Let us denote the metric on $(\mathbb{R}^d, \epsilon)$ by ρ. Now let $\{s, t\}$ be a system of generators of G such that

$$\rho(0, s(0)) = \inf\{\rho(0, k(0)) | k \neq 0, k \in G\}.$$

We set $a = s(0)$ and $b = t(0)$. To compute $c(h_0)$ we look at the universal riemannian covering:

$$(\mathbb{R}^2, \epsilon) \to (T, h_0).$$

By our selection of s it follows that the lift of any closed curve τ is such that

$$\lg(\tau) = \lg(\widetilde{\tau}) \geq \rho(0, a),$$

305 and equality is attained. Further we have

$$\rho(0, a) \leq \rho(0, b), \quad \text{and} \quad \rho(0, a) \leq \rho(a, b)$$

because $s - t \in G$, we have

$$c(h_0) \leq \frac{1}{3} \quad \text{(perimeter of } 0ab).$$

On the other hand we have by (5.7)

$$a(h_0) = \det(\overline{0a}, \overline{0b}) = 2 \quad \text{(area of } 0ab).$$

But the well known isoperimetric inequality for triangles gives that

$$\text{(8.12.2)} \qquad \text{(area of } 0ab) \leq \frac{\sqrt{3}}{36} \quad \text{(perimeter of } 0ab)^2$$

where equality occurs if and only if $0ab$ is equilateral.

12.3

Definition . *A flat torus in $\mathbb{R}^2$ is said to be equilateral if there exists a similitude of $\mathbb{R}^2$ transforming it into the torus got by the group G generated by*

$$s : 0 \to s = (0, 1) \quad \textit{and} \quad t : 0 \to t = \left(\frac{1}{2}, \frac{3}{2}\right).$$

Then by (8.12.2) *we have*

12.4

Lemma. *For a flat torus*

$$a(h_0)/c^2(h_0) \geq \frac{\sqrt{3}}{2}$$

where equality occurs if and only if the torus is equilateral.

C) The underlying idea of the proofs of the theorems of Loewner and Pu is as follows: first using 11 we get an isometry between (M, g) and an r.s. of the type $\varphi \cdot h_0$ on the standard manifold. Then we use the transitive group of isometries these standard manifolds possess to average the function φ in such a way $a(h)$ decreases and $c(h)$ increases. First we prove two lemmas, relative to the 306
following situation: (N, ϵ) is a compact surface and $\mathscr{C}$ a compact Lie group operating transitively, by isometries, on (N, ϵ); denote by τ any volume form on $\mathscr{C}$ such that

$$\operatorname{Vol}(\mathscr{C}) = \int_{\mathscr{C}} \tau = 1.$$

We suppose given on N an everywhere strictly positive function and we consider the average $\widehat{\varphi}$, by $\mathscr{C}$, of its square root, i.e. we set

$$\widehat{\varphi} = \left(\int_{t \in \mathscr{C}} (\varphi \circ t)^{\frac{1}{2}} \cdot \tau \right)^2 .$$

In this context we have the following two lemmas.

12.5

Lemma. *$a(\widehat{\varphi} \cdot \epsilon) \leq a(\varphi \cdot \epsilon)$ and the equality holds if and only if the function φ is constant on N.*

Proof. Let $\sigma_{\varphi\epsilon}$ be the volume element of the r.m. $(N, \varphi \cdot \epsilon)$; by (5.4), $\sigma_{\varphi\epsilon} = \varphi\sigma_\epsilon$ so that

$$a(\widehat{\varphi} \cdot \epsilon) = \int_N \sigma_{\widehat{\varphi}\epsilon} = \int_N \widehat{\varphi}\sigma_\epsilon = \int_N \int_{\mathscr{C}} (\varphi \circ t)^{\frac{1}{2}} \cdot \tau)^2 \sigma_\epsilon;$$

by Schwarz' inequality

$$\left(\int_{\mathscr{C}} (\varphi \circ t)^{\frac{1}{2}} \cdot \tau\right)^2 \leq \int_{\mathscr{C}} (\varphi \circ t) \cdot \tau \int_{\mathscr{C}} \tau = \int_{\mathscr{C}} (\varphi \circ t) \cdot \tau$$

and

$$a(\widehat{\varphi} \cdot \epsilon) \leq \int_{N \times \mathscr{C}} (\varphi \circ t)(\tau \times \sigma_\epsilon) = \int_{t \in \mathscr{C}} \int_N (\varphi \circ t) \cdot \sigma_\epsilon) \cdot \tau.$$

But the fact that t is an isometry implies that $t^*\sigma_\epsilon = \sigma_\epsilon$ and hence

$$\int_N (\varphi \circ t) \cdot \sigma_\epsilon = \int_N t^*\varphi \cdot t^*\sigma_\epsilon = \int_N t^*(\varphi \cdot \sigma_\epsilon) = \int_N \varphi\sigma_\epsilon \quad \text{(by (3.15))}$$
$$= a(\varphi \cdot \epsilon);$$

307 Consequently

$$a(\widehat{\varphi} \cdot \epsilon) \leq \int_{t \in \mathscr{C}} a(\varphi \cdot \epsilon) \cdot \tau = a(\varphi \cdot \epsilon) \cdot \mathrm{Vol}(\mathscr{C}) = a(\varphi \cdot \epsilon).$$

Equality occurs if and only if it occurs in Schwarz' inequality, ie. if and only if the function $t \to \varphi \circ t$ on $\mathscr{C}$ is constant, which implies that φ itself is constant because $\mathscr{C}$ acts transitively on N. □

12.6

Lemma. $c(\widehat{\varphi} \cdot \epsilon) \geq c(\varphi \cdot \epsilon)$.

Proof. Let γ be any curve in N; we suppose it given in the form $\gamma : [0, 1] \to N$. Then

$$\lg(\gamma, \widehat{\varphi} \cdot \epsilon) = \int_0^1 ((\widehat{\varphi} \cdot \epsilon)(\gamma', \gamma'))^{\frac{1}{2}} \cdot \mathrm{dt} = \int_0^1 \widehat{\varphi}^{\frac{1}{2}} (\epsilon(\gamma' \cdot \gamma')^{\frac{1}{2}} \, \mathrm{dt} =$$

$$= \int_0^1 \left(\int_{t \in \mathscr{C}} (\varphi \circ t)^{\frac{1}{2}} \cdot \tau\right) (\epsilon(\gamma', \gamma'))^{\frac{1}{2}} \cdot \mathrm{dt} =$$

$$= \int\limits_{[0,1]\times\mathscr{C}} (\varphi \circ t)^{\frac{1}{2}} (\epsilon(\gamma', \gamma'))^{\frac{1}{2}} \cdot \tau \times \mathrm{dt} =$$

$$= \int\limits_{\mathscr{C}} \left(\int\limits_0^1 (((\varphi \circ t) \cdot \epsilon)(\gamma', \gamma'))^{\frac{1}{2}} \cdot \mathrm{dt} \right) \tau$$

But since t is an isometry

$$((\varphi \circ t)\epsilon)(\gamma', \gamma') = (t^*\varphi \cdot t^*\epsilon)(\gamma', \gamma') = t^*(\varphi \cdot \epsilon)(\gamma', \gamma') = \dots$$
$$\dots = (\varphi \cdot \epsilon)(t^T \circ \gamma', t^T \circ \gamma') = (\varphi \cdot \epsilon)(t \circ \gamma)', (t \circ \gamma)').$$

Now

$$\int\limits_0^1 ((\varphi \cdot \epsilon)(t \circ \gamma)', (t \circ \gamma))^{\frac{1}{2}} \cdot \mathrm{dt} = \lg(t \circ \gamma, \varphi \cdot \epsilon)$$

so we get

$$\lg(\gamma, \widehat{\varphi} \cdot \epsilon) = \int\limits_{t \in \mathscr{C}} \lg(t \circ \gamma, \varphi \cdot \epsilon) \cdot \tau.$$

Suppose now that τ is closed and non homotopic to zero in N; then the same holds for the curves $t \circ \gamma$, because the t's are diffeomorphisms; hence by the definition of $c(\varphi \cdot \epsilon)$, **308**

$$\lg(\gamma, \widehat{\varphi} \cdot \epsilon) \geq \int\limits_{\mathscr{C}} c(\varphi \cdot \epsilon) \cdot \tau = c(\varphi \cdot \epsilon) \cdot \mathrm{Vol}(\mathscr{C}) = c(\varphi \cdot \epsilon).$$

Now this holds for any such γ, so that, by the definition of $c(\widehat{\varphi} \cdot \epsilon)$, we get

$$c(\widehat{\varphi} \cdot \epsilon) \geq c(\varphi \cdot \epsilon).$$

□

12.7

Proposition. *With the hypothesis of the lemmas, we have*

$$\frac{a(\varphi \cdot \epsilon)}{(c(\varphi \cdot \epsilon))^2} \geq \frac{a(\epsilon)}{(c(\epsilon))^2}$$

and equality holds if and only if φ is constant.

Proof. We have only to remark that the transitivity of $\mathscr{C}$ implies the average $\widehat{\varphi}$ is constant on N; let k be that constant value, then $a(\widehat{\varphi}\cdot\epsilon) = k^2 \cdot a(\epsilon)$ and $c(\widehat{\varphi}\cdot\epsilon) = k\cdot c(\epsilon)$; hence the two lemmas yield

$$\frac{a(\varphi\cdot\epsilon)}{(c(\varphi\cdot\epsilon))^2} \geq \frac{k^2\cdot a(\epsilon)}{k^2\cdot(c(\epsilon))^2} = \frac{a(\epsilon)}{(c(\epsilon))^2}$$

□

12.8

Corollary. *(Loewner, unpublished) Let* (M, g) *be such that* M *is homeomorphic to* $\mathbb{S}^1 \times \mathbb{S}^1$. *Then*

$$\frac{a(g)}{(c(g))^2} \geq \frac{\sqrt{3}}{2}$$

where equality holds if and only if (M, g) *is isometric to an equilateral flat torus.*

Proof. By (8.2.3) there exists a flat torus (N, h), a diffeomorphism f from N to M and $\varphi \in F(N)$ such that

$$f^*(g) = \varphi\cdot h.$$

309 So that (M, g) and (N, φ, h) are isometric under f and in particular

$$a(g) = a(\varphi\cdot h), \quad c(g) = c(\varphi\cdot h).$$

Now the corollary follows from (12.7) combined with (12.4). □

12.9

Corollary. *(Pu: [25]) Let* (M, g) *be such that* M *is homeomorphic to* $P^2(\mathbb{R})$. *Then*

$$\frac{a(g)}{(c(g))^2} \geq \frac{2}{\pi}$$

where equality holds if and only if there is $k > 0$ *such that* (M, g) *is isometric to* $(P^2(\mathbb{R}), k.\,\mathrm{can})$.

Proof. This time we get the transitive group $\mathscr{C}$ of isometries of $(P^2(\mathbb{R})\cdot$ can) by writing $P^2(\mathbb{R})$ as the homogeneous space

$$P^2(\mathbb{R}) = S\,0(3)/0(2).$$

□

12.10

Remarks. A. The result (12.8) has been generalized by Blatter in [5], to compact orientable manifolds of genus $\gamma \geq 2$; precisely: for any integer $\gamma \geq 2$ there exists a real number n_γ such that

$$\frac{a(g)}{(c(g))^2} \geq n_\gamma$$

for any surface (M, g) where M is compact, orientable, of genus γ. The proof is definitely deeper than the one above. *But here the equality* is never attained. We sketch here the proof (based on an idea of N. Kuiper): suppose (M, g) is such that the equality is attained and *call m-curve* in (M, g) a curve of length equal to $c(g)$ (they are closed geodesics); then first: given any point $m \in M$ and any neighbourhood V of m there exists an m-curve which **310**
meets V (in fact, if not, change slightly the r.s. inside V in order to have a smaller $a(g)$; this change would not affect $c(g)$ and so it is impossible). By continuity one shows now that $\forall m \in M$ there exists a one-parameter family f of m-curves such that $f(0, 0) = m$. Looking at the corresponding Jacobi fields, such a Jacobi field either never vanishes or vanishes at least at two points (thanks to the fact that M is oriented); but it cannot vanish at two points for (2.6) would imply that the geodesic is not an m-curve. Finally following such a family by continuity one would get a vector field on M, which is never zero, contradicting the fact that the genus of M is greater than one.

B. The theorems of Loewner and Pu are answers to particular cases of the following general question: given a compact differentiable manifold M and some homology or homotopy class or set of classes δ, and an r.s. g on M, set

$$a(g) = \mathrm{Vol}(M, g) \quad \text{and} \quad c(g) = \inf\{\mathrm{Vol}(N, g|N) | N \in \delta\}.$$

Then: does one have an inequality $a(g) \geq k \cdot (c(g))^n$, valid for any g; if so, what are the cases of equality? (the integer n is defined so that the quotient $\dfrac{a(g)}{(c(g))^n}$ be homogeneous of degrees zero).

The good candidates for generalizing the Pu theorem are the S.C.-manifolds (excepting the spheres), when one takes for δ the class of a
311 projective subspace. Except in the case of $P^2(\mathbb{R})$ the question is completely open. However, the answer cannot be so simple; in fact, for the complex projective space either k has not the value of the canonical r.s. or the equality can be attained for an r.s. non isometric to the canonical one. The proof of this fact consists simply in deforming the canonical kählerian r.s. by adding to it $\sqrt{-1} \cdot d'd'' f$ (where f is a real function); Stokes formula (3.13) and an inequality of Wirtinger show easily that both $a(g)$ and $c(g)$ are unchanged.

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