Lectures on Stratification of Complex Analytic Sets

By

M.-H. Schwartz

Tata Institute of Fundamental Research, Bombay
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Foreword

The principle material of this course is taken from a paper of whitney [7]. In the first chapter we recall some classical theorem (sec [1] and [2]), explain the problem solved in [7] and give several examples. In chapter II we study stratifications of an analytic set having different properties. In chapter III we prove the theorem a) and b) of Whitney. The main lines of the proofs are taken form [7] but for the theorem b) our demonstrationis rather different (from the application of therorem a) a result on field of frames tangent to the strata of a stratification of an analytic set, along certain skeletons.

I have been lukcy enough to have the collaooration of Miss M.S. Rajwade and Dr. Raghavan Narasimhan who had ideas for many improvements and worte the present notes. I thank them very much for their help.

A-H. Sehwartz

Contents

Chapter 1

Preliminaries

In this course of lectures we shall deal with various decompositions of 1
$\mathbb{C}$-analytic set into manifold and then with some of their applications.

We shall first state some definitiens and a few theorem without proofs on holomorphic funtionns and $\mathbb{C}$-analytic sets which we shall use in what follows.

Definitions 1. *Let Ω be an open set in $\mathbb{C}^n$. A complex valued function f, defined on Ω is side to he holomorphic on Ω, if for every ζ in Ω, there exists $(\rho_i), \rho_i > 0, 1 \leq i \leq n, a_\alpha \in \mathbb{C}^n$, such that*

$$f(z) = f(\zeta) + \sum_{|\alpha| \geq 1} a_\alpha ((z - \zeta))^\alpha \text{ for } \mid z_i - \zeta_i \mid < \rho_i$$

$$\begin{aligned} \text{where} \quad \alpha &= (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \\ ((u))^\alpha &= u_1^{\alpha_1}, \ldots, u_n^{\alpha_n}, \\ \mid \alpha \mid &= \alpha_1 + \alpha_2 + \ldots + \alpha_n. \end{aligned}$$

We denote the set of holomorphic functions on $\Omega by \mathscr{O}\Omega$

Theorem 1 (Hartogs)**.** *A complex valued function f is holomorphic on Ω if and only if the partial derivatives $\dfrac{\partial f}{\partial z_i}$ exist at each point of Ω.*

If V is an analytic manifold and $a \in V$, we denote by $T(V, a)$ the tangent space to V at a.

Definition 2. Let V^n and W^m be two $\mathbb{C}$-analytic manifolds and $f : V^n \to$ 2
W^n an analytic map. For $a \in V^n$, let $(df)(a)$ denote the linear map $T(V^n, a) \to T(W^m, f(a))$, Then we define rank $(df)(a)$ as the dimension of the image of $T(V^n, a)$ by this map.

Remark. If $(x_1, \ldots, x_n)$ denote local coordinates in a neighbourhood U of a and if $f(x) = (f_1(x), \ldots, f_m(x))$ for $x \in U$, in local coordinates in a neighbourhood U of $f(a)$, then rank $(df)(a)$ = rank of the matrix $(\frac{\partial f_i}{\partial x_j}(a))_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ and it is independent of the coordinate neighbourhoods chosen. We now state the important.

Theorem 2 (Constant rank theorem)**.** *Let V^n and W^m be two $\mathbb{C}$ analytic manifolds and $f : V^n \to W^n$ be a holomorphic map. Let the rank $(df)(x) = r$, a constant, for $x \in$ an open set $\Omega \subset V^n$. Then for every $a \in \Omega$, there exist neighbourhoods U of a, V of $f(a)$, open cubes $Q_1 \subset \mathbb{C}^n, Q_2 \subset \mathbb{C}^m$ and biholomorphic maps $u : U \to Q_1, v : V \to Q_2$ such that if $g = v$ o f o u^{-1}, we have $g(x_1, \ldots, x_2, 0, \ldots, 0)$.*

Let Ω be an open set in $\mathbb{C}^n$ such that $0 \in \Omega$. Then we denote the inductive limit, $\varinjlim_{\Omega} \mathscr{O}^n_{\Omega}$, by $\mathscr{O}^n_o$ or by $\mathscr{O}^n$. It is clear that $\mathscr{O}^n$ is a ring and we call it the ring of germs of holomorphic functions at 0.

We shall assume the following properties of $\mathscr{O}^n$.

1. $\mathscr{O}^n$ is isomorphic to the ring of convergent power series in n variables with complex coefficients.

2. $\mathscr{O}^n$ is a local ring.

3 3. $\mathscr{O}^n$ is an integral domain.

4. $\mathscr{O}^n$ is a noetherian ring.

5. $\mathscr{O}^n$ is factorial.

Definition 3. [1] Given a $\mathbb{C}$ -analytic manifold M^n a subset V of M^n is defined to be a $\mathbb{C}$-analytic set if for every $a \in M^n$ there exists a

[1]We use the term analytic set for an analytic subvariety of an analytic manifold and the term analytic space for a space that is locally an analytic set.

neighbourhood U in M^n and a finite number of holomorphic functions $\left\{f_i\right\}$, $1 \leq i \leq m$ on U such that $U \cap V = \{z \in U \mid f_i(z) = 0, 1 \leq i \leq m\}$.

(4). A point a in V is said to be simple if there exists a neighbourhood U_a of a such that $U_a \cap V$ is an analytic submanifold of U_a.

Remark 1. An analytic set on a $\mathbb{C}$ -analytic manifold is closed.

Unless otherwise stated, in what follows, an analytic manifold and an analytic set will mean a $\mathbb{C}$ -analytic manifold and a $\mathbb{C}$ -analytic set respectively.

Notation. If $z = (z_1, \ldots, z_n)$ is a point in $\mathbb{C}^n$, z' will denote the point $(z_1, \ldots, z_{n-1})$ in $\mathbb{C}^{n-1}$. A poly-disc D^n in $\mathbb{C}^n$ will be $D_1 \times \cdots \times D_n$ where D_i are discs in $\mathbb{C}$. $\mathscr{M}^n$ will denote the maximal ideal (i.e. the ideal of germs vanishing at 0) of $\mathscr{O}^n$ and $\mathscr{O}^{n-1}$ will denote the ring of germs of holomorphic functions at $0'$ in $\mathbb{C}^{n-1}$. We identify $\mathscr{O}^{n-1}$ with a subring of $\mathscr{O}^n$.

Definitions 5. *A distiniuished polynomial in z_n of degree n is a polyno-* 4
mial $z_n^p + \sum_{k=1}^{p} a_k(z')z_n^{p-k}$, where $a_k(z')$ are holomorphic functions in z' on an open neighbourhood of $0'$ in $\mathbb{C}^{n-1}$ and $a_k(0') = 0$.

Theorem 3 (Weierstrass preparation theorem)**.** *Let $f \in \mathscr{M}^n$, $f \neq 0$. Then*

1. *There exists a basis $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ such that $f(0', z_n)$ does not vanish identically in any neighbourhood of $z_n = 0$ in $\mathbb{C}$.*

2. *With respect to any basis satisfying condition (1) above, there exists a unique distinguished polynomial P in $\mathscr{O}^{n-1}[z_n]$ such that $f = gP$ for some g in $\mathscr{O}^n$ and $g \notin \mathscr{M}^n$.*

3. *If $f(z) = \sum_{1}^{\infty} a_k(z)$ in a neighbourhood of 0, where $a_k(z)$ are homogeneous polynomials of degree k, and if p is the least integer such that $a_p(z) \not\equiv 0$, then p is the minimum degree of a distinguished polynomial P for which*

$$f = gP \text{ where } g \in \mathscr{O}^n \text{ and } g \notin \mathscr{M}^n.$$

Theorem 4 (The division theorem). *Let $f \in \mathscr{M}^n$ and suppose the basis for $\mathbb{C}^n$ so chosen that the condition (1) of Theorem 3 above is satisfied. Then if $f = uP$, where P is a distinguished polynomial of degree p in $\mathscr{O}^{n-1}_{z'}[z_n]$ and $u \in \mathscr{O}^n$, $u \notin \mathscr{M}^n$, then for any g in $\mathscr{O}^n$, there exists h in $\mathscr{O}^n$ and $r(z', z_n)$ in $\mathscr{O}^{n-1}_{z'}[z_n]$, with degree of $r < p$ such that*

$$g = hf + r$$

5 *and the h and r are unique.*

Definition 6. An analytic set V is said to be irreducible if, $V = V_1 \cup V_2$, where V_1 and V_2 are analytic sets, implies either $V = V_1$ or $V = V_2$.

Let V be an analytic set $\subset M, M$ being an analytic manifold. Then the germ of V at a point $a \in M$ is defined to be $\varinjlim_{\Omega} V \cap \Omega$ where Ω is a neighbourhood of a in M. The germ at a of an analytic set V is said to be irreducible if a has a fundamental system of neighbourhoods U such that $U \cap V$ is irreducible. The following Proposition is an easy consequence of the property 4 of $\mathscr{O}^n$.

Proposition 1. *The germ of an analytic set V at a can be written uniquely as $V_a = \bigcup\limits_{1 \leq i \leq r} V_{ia}$, where V_{ia} are irreducible germs of analytic sets $V_{ia} \not\subset \bigcup\limits_{j \neq i} V_{ja}$ for any i.*

Remarks 2. If I is the ideal in $\mathscr{O}_a$ of germs of holomorphic functions vanishing on V_a, I is prime if and only if V_a is irreducible.

3. Here we give an example of an irreducible analytic set V and a point $a \in V$ such that V_a is reducible. Let $V = \{z \in \mathbb{C}^2_{xy} \mid x^3 + y^3 - xy = 0\}$. Then since the set of simple points of V is connected, V is irreducible. Consider the points $x = x(t) = \dfrac{t}{1+t^3}$ and $y = y(t) = \dfrac{t^2}{1+t^3}$ for t sufficiently small. Then the points $(x(t), y(t))$ are in V. Further, $f(x,t) \equiv (1+t^3)x(t) - t = 0$ gives $f_t(0,0) = -1$ and hence by the implicit function theorem the equation $x = \dfrac{t}{1+t^2}$ can be solved for t,
6 for sufficiently small x, i.e. there exists $\varepsilon > 0$. such that for $|x| < \varepsilon, t =$

$t(x)$ is an analytic function of x. Thus the analytic set V_1 defined by $V_1 = \left\{z \in \mathbb{C}^2_{xy} \middle| |x| < \varepsilon, y = \dfrac{[t(x)]^2}{1+[t(x)]^3}\right\}$ is contained in V and similarly $V_2 = \left\{z \in \mathbb{C}^2_{xy} \middle| |x| < \varepsilon', y = \dfrac{[t'(x)]^2}{1-[t'(x)]^3}\right\}$ where $t'(x)$ is a solution of $(1-t^3)x + t = 0$, in V. Hence there is a neighbourhood U of 0 such that $U \cap V = (U \cap V_1) \cup (U \cap V_2)$ and thus V_o is reducible.

We now recall the important theorem of local representation of an analytic sec and some of its consequences that we shall need later.

Theorem 5. *Let I be a prime ideal in $\mathscr{O}^n, I \neq \{0\}, I \neq \mathscr{O}^n$. Then there exist*

(a) *a basis $(z_1, \ldots, z_k, \ldots, z_n)$ for $\mathbb{C}^n$, in integer $k \geq 0$ and a fundamental system of neighbourhoods $D^n = D^k \times D^{n-k}$ of $0, D^k \subset C^k_{z_1,\ldots,z_k} D^{n-k} \subset \mathbb{C}^{n-k}_{z_{k+1},\ldots,z_n}$' and if $\mathscr{O}^k$ denotes the ring of germs of holomorphic functions at $0''$ in $\mathbb{C}^k_{z_1,\ldots,z_k}$,*

(b) *there exist polynomials $P_{k+1}[x], Q_{k+j}[x], \tilde{Q}_{k+j}[x], 2 \leq j \leq n-k$ in $\mathscr{O}^k[x]$ with $\deg Q_{k+j}, \deg \tilde{Q}_{k+j} < \deg P_{k+j}$. such that I is generated by a finite number of holomorphic functions $f_1, \ldots, f_r$ on D^n and if S is the analytic set defined as the set of zeros of these functions in D^n, then following are satisfied*

1. *$\mathscr{O}^K \cap I = \{O\}$* **7**

2. *If $\eta : \mathscr{O}^n \to \mathscr{O}^n/I$ is the natural map, the quotient field of $(\mathscr{O}^n/I)$ is generated by $\eta(z_{k+1})$ over the quotient field of $\mathscr{O}^k$.*

3. *$P_{k+1}[x]$ is the minimal polynomial of $\eta(z_{k+1})$ over $\mathscr{O}^k$ and if $\delta =$ discriminant of P_{k+1} over $\mathscr{O}^k$, then $\delta z_{k+j} - Q_{k+j}[z_{k+1}]$ and*

$$\frac{\partial P_{k+1}}{\partial x}[z_{k+1}]z_{k+j} - \tilde{Q}_{k+j}[z_{k+1}]$$

are in I.

4. *For every* $z' \in D^k$ *with* $\delta(z') \neq 0$, *there exist precisely* p *points* (p = *degree of* $P_{k+1}[x]$)(z', z^i) *in* S,

$$z^i = \frac{Q_{k+j}[z^i_{k+1}]}{\delta(z')} = \frac{\tilde{Q}_{k+j}[z^i_{k+1}]}{P'_{k+1}[z_{k+1}]}$$

where $(z^i_{k+1})_{1 \leq i \leq p}$ *are the roots of* $P_{k+1}[x] = 0$.

5. *The points* $S' = \{z \in S | \delta(z') \neq O\}$ *are simple points of dimension* k *of* S *and* S' *is connected and dense in* S *and* $\pi : S' \cap D^n \to (D^k \cap \{z' | \delta(z') \neq 0\})$ *is a covering.*

6. *The projection* $\pi : V \cap D^n \to D^k$ *is proper and open.*

If 0 is in an anaytic set V and if V_0 is irreducible, let I = the ideal of germs at 0 of holomorphic functions vanishing on V_0. Then coordinate system $(z_1, \ldots, z_n)$ at 0 which satisfies the conditions (1)–(6) of the above theorem with respect to I, is said to be proper for V_0.

8 **Theorem 6** (H. Cartan)**.** *If* S *is an analytic set in an open set* $U \subset \mathbb{C}^n$, *for any* $a_0 \in U$, *there exist a neighbourhood* U_o *and a finite number of holomorphic functions* $f_1, \ldots, f_r$ *on* U_o *such that for any point* b *in* U_o, *the germs of* $f_1, \ldots, f_r$ *at* b *generate the ideal* I_b *associated to* S_b *over* $\mathscr{O}^n_b$.

Definitions 8. *If* a *is a simple point of* V, *let* U *be a neighbourhood of* a *such that* $U \cap V$ *is an analytic submanifold of* $\mathbb{C}^n$. *Then the dimension of* V *at* a, *denoted by* $\dim_a V$ *is defined to be the dimension of the submanifold* $U \cap V$.

9. For any point ζ in V, the dimension of V at ζ, denoted by $\dim_\zeta V$ is defined by $\varinjlim_{U_\zeta} \left(\operatorname*{Sup}_{\substack{z \text{ is a simple} \\ \text{point in} U_\zeta \cap V}} \dim_z V \right)$, where U_ζ is a neighnourhood of ζ.

Proposition 2. *If* V *is an irreducible analytic set,* V' *is another analytic set and* $V' \underset{\neq}{\subset} V$, *then* $\dim V' < \dim V$.

Proposition 3. *If V is an analytic set, $V \subset \Omega \subset \mathbb{C}^n$, and if $0 \in V$ and V_o is irreducible and if $\dim_o V = k$ and if $\{x_1, \ldots, x_n\}$ is a coordinate system in a neighbourhood U of 0 such that $\{x \in U | x_1 = \ldots = x_k = 0\} \cap V = \{0\}$ then by a linear change of coordinates, we can find a coordinate system $\{y_1, \ldots, y_n\} y_1 = x_1, \ldots, y_k = x_k$ such that $\{y_1, \ldots, y_n\}$ is a proper coordinate system for V_0, at 0.*

Remark 3. If an analytic set $V = \cup V_i$, where V_i are distinct irreducible analytic sets, the simple points of V are theorem simple points z of V_i, for each i, such that $z \notin V_j$ for $j \neq i$.

Theorem 7 (Hilbert's Nullstellensatz)**.** *For any ideal I in $\mathscr{O}^n$, there exists an integer $n = n(I)$ such that if $f \in \mathscr{O}^n$ and if f vanishes on the germ of the analytic set S_I defined by I, then $f^n \in I$.* 9

Proposition 4. *If V is an analytic set and $W \subset V$ is also an analytic set, $\overline{(V - W)}$ is an analytic set and* $\dim .\overline{(V - W)} \cap W < \dim \overline{(V - W)}$.

Proposition 5. *If V and W are analytic sets in an open set Ω in $\mathbb{C}^n$, then $V \cap W$ is an analytic set and* $\dim(V \cap W) \geq \dim V + \dim W - n$.

We now state the various types of decomositions of an analytic set that we shall consider.

(i) *Strict partitions into manifolds*

Definition 10. An analytic set V is said to be partitioned strictly in to manifolds if

$$V = \bigcup_i M_i, \text{ where}$$

1. M_i are analytic submanifolds of $\mathbb{C}^n$ with $M_i \cap M_j = \phi$for $i \neq j$ and
2. if $\dot{M}_i = \overline{M}_i - M_i, \dot{M}_i, \overline{M}_i$ are analytic sets.

(ii) *Canonical strict partitions into manifolds.*

We shall prove in Chapter II that any analytic set V can be canonically strictly partitioned into manifolds $V = \bigcup M_i$, where, if M_i is a manifold of maximum dimension p say, then it is a connected component of the set of simple points of V, if dimension p and

$$p = \max_{a \in V}(\dim_a V).$$

10 **(iii)** *Stratifications*

Definition 11. A strict partition into manifolds is a stratification if and only if

1. $\overline{M}_i - M_i = \dot{M}_i = \bigcup\limits_{j \epsilon J'_i} M_j$ for some subset J'_i of T or

2. $M_i \cap \overline{M}_j \neq \phi \Rightarrow M_i \subset \overline{M}_j$

Examples 1. Let V be the analytic set in $\mathbb{C}^3 xyt$ given by $V = \{z \in \mathbb{C}^3\text{xyt} \big| t^2 = x^2 = y^2\}$. Then V is a cone and if D is a generator of the cone, let $M_1 = D, M_2 = V - D$. Then $V = M_1 \cup M_2$ is a stratification. However, it is clear that this stratification is not uniquely determined as we may choose any generator for D and that either of two such stratifications is not finer than the other.

(iv) Whitney Stratifications

Given a stratification $V = \bigcup M_i$, let $M_i \subset \overline{M}_j$. Let z_0 be in M_i and consider a sequence of points $\{z_\nu\} \epsilon M_j, z_\nu \to z_0$. If $T(M_j, z)$ denotes the tangent space at z for any z in M_j, let $T(M_j, z_\nu) \to T$ (in a natural sense of Grassmann manifold that we shall describe later) as $z_\nu \to z_0$. The the pair (M_i, M_j) is said to be (a) *regular* at z_0, according to Whitney , if , for any such limit T, $T \supset T(M_i, z_0)$. This is clearly not the case in an arbitrary stratification. Consider

Example 2 (Whitney)**.** Let V be the analytic set in $\mathbb{C}^3_{xyt}$ given by $V = \{z \epsilon \mathbb{C}^3_{xyt} \big| y^2 = tx^2\}$ and the stratification $M_1 = \mathbb{C}_t$ and $M_2 = V - \mathbb{C}_t$.
11 Consider points z_ν on $\mathbb{C}_x$ such that $z_\nu \to 0$. It is clear that if T is the limit of $T(M_2, z_\nu)$, $T \not\supset T(M_1, 0)$. However if we have the stratification given by $M_0 = M_1 = \mathbb{C}_t - \{0\}$, $M_2 = V - \mathbb{C}_t$ then the pairs $(M_0, M_1), (M_1, M_2)$ are (a) regular at all points of $M_\circ$ and M_1 respectively.

But in a set other than a complex analytic set, it may not be possible to obtain a substratification which is (a) regular, i.e. for which all pairs (M_i, M_j) of strata, $M_i \subset \overline{M}_j$, are (a) regular.

Example 3. Let $f : \mathbb{R}^3_{xyt} \to \mathbb{R}$ be the continous function given by $f(x, y, t) = t\sin\left(\dfrac{x}{t}\sin\dfrac{1}{t}\right) - y$ if $t \neq 0$ and $f(x, y, 0) = -y$. We define the set V in $\mathbb{R}^3_{xyt}$ by $V = \{(x, y, t)\epsilon\mathbb{R}^3_{xyt} \big| f(x, y, t) = 0\}$. Consider the partition $M^1 = \mathbb{R}_x$ and $M^2 = V - \mathbb{R}_x$, of V.

We have $t\dfrac{d}{dt}\left(\dfrac{1}{t}\sin\dfrac{1}{t}\right) = -\dfrac{1}{t}\left(\sin\dfrac{1}{t} + t\cos\dfrac{1}{t}\right)$. Let t_ν be a sequence of points such that $t_\nu \to 0$ and $\left[t\dfrac{d}{dt}\left(\dfrac{1}{t}\sin\dfrac{1}{t}\right)\right]_{t=t_\nu} = 0$. Then clearly $\tan\dfrac{1}{t_\nu} = -\dfrac{1}{t_\nu}$ and $\left|\sin\dfrac{1}{t_\nu}\right| = \dfrac{1}{\sqrt{1+t_\nu^2}} \to 1$. We suppose, without loss of generality, that $\sin\dfrac{1}{t_\nu} \to 1$. Let $x_{\nu,k} = \dfrac{2\pi k t_\nu}{\sin\dfrac{1}{t_\nu}}$ and $z_{\nu,k} = (x_{\nu,k}, 0, t_\nu)\epsilon V$.
We have, by direct computation,

$$\frac{\partial f}{\partial x}(z_{\nu,k}) = \sin\frac{1}{t_\nu}, \frac{\partial f}{\partial y}(z_{\nu,k}) = -1, \frac{\partial f}{\partial t}(z_{\nu,k}) = 0.$$

Let $z_\circ = (x_0, 0, 0)$ and $x_\nu = z_{\nu,k(\nu)}$ where $k(\nu)$ is the largest integer such that $k(\nu) \leq \dfrac{x_\circ}{2\pi t\nu}\sin\dfrac{1}{t\nu}$. Then clearly $z_\nu \in V$ and $z_\nu \to$ **12**
z_0. $T(\nu, z_{\nu,k(\nu)})$ has a limit, namely, the plane orthogonal to the vector $(1, -1, 0)$. This plane does not contain $\mathbb{R}_x$ and hence a condition corresponding to the condition (a) is not satisfied in this example.

However, it follows from the whitney's theorem that we shall prove, that such a situation cannot arise in an $\mathbb{C}$- analytic case.

If we assume that the stratification is (a) regular and if $M_i \subset \overline{M}_j$, M_i and M_j being two strata, let z_o be in M_i and $\{z_\nu\}$ be a sequence in $M_j, z_\nu \to z_0$. Let $\{\zeta_\nu\}$ be a sequence in $M_i, \zeta_\nu \to z_0$ and $\{\lambda_\nu\}$, a sequence of complex numbers such that $\lambda_\nu(z_\nu - \zeta_\nu)$, is convergent to (a vector) v say. Then the pair (M_i, M_j) is said to be *(b) regular* (according to Whitney) if every such $v\epsilon T = LimT(M_j, z_\nu)$.

Example 4 (Whitney)**.** Consider the analytic set

$$V = \{z \epsilon \mathbb{C}^3_{xyt} \mid t^2(x^2 - y^2) + x^3 - y^4 = 0\}.$$

Let

$$M_1 = \mathbb{C}_t, M_2 = V - \mathbb{C}_t, V = M_1 \cup M_2$$

be the stratification. Consider the points $z = \left(-\dfrac{1}{\nu^2}, 0, \dfrac{1}{\nu}\right)$ on M_2 and $\zeta_\nu = \left(0, 0, \dfrac{1}{\nu}\right)$ on M_1. Clearly $z_\nu, \zeta_\nu \to 0$ and $\nu^2(z_\nu - \zeta_\nu) \to (-1, 0, 0)$ and obviously the pair (M_1, M_2) is not (b) regular at 0 for the normal to the surface $f(x_1, y_1, t) = t^2(x^2 - y^2) + x^3 - y^4 = 0$ at $\left(-\dfrac{1}{\nu^2}, 0, \dfrac{1}{\nu}\right)$ is parallel to $\left(1, 0, -\dfrac{2}{\nu}\right)$ and this tends to $(1, 0, 0)$ at $\nu \to \infty$.

We give here a example of set V which is not $\mathbb{C}$ -analytic and a stratificaton $V = \cup M_i$, $M_1 \subset \overline{M_2}$ and the pair (M_1, M_2) is not (**b**) regular at any point of M_1.

13 **Example 5.** Let $V \subset \mathbb{R}^3_{xyt}$ be given as follows. Consider the parameters ρ, θ in $\mathbb{R}^2$ where $x = \rho\cos\theta, y = \rho\sin\theta$. Then $V = \{(x_1, y_1, t) \mid \rho = e^\theta, -\infty \leq \theta \leq 1\}$. Let $M_1 = \mathbb{R}_{xy} = \{(x, y, t) \mid x = y = 0\}$ and $M_2 = V - M_1$. Then for any point $(0, 0, t_0)$ on M_1, consider $z_\nu = \{(z, y, t) \mid \theta = -2\pi\nu, t = t_0\}$, i.e. $z_\nu = (e^{-2\pi\nu}, 0, t_0)$. Let $\zeta_\nu = (0, 0, t_0), \lambda_\nu = e^{2\pi\nu}$, i.e. $v = (1, 0, 0)$. Now the planes tangent to M_2 are all orthogonal to $(1, 0, 0)$ and the pair (M_1, M_2) is not (b) regular at any point $(0, 0, t_0)$ of M_1.

However, we shall prove that such a situation does not arise in $\mathbb{C}$-analytic sets. We shall prove the following

Theorem (Whitney) *Every stratification of a $\mathbb{C}$ -analytic set admits a substratification which is (a) and (b) regular.*

Chapter 2

Some theorems on stratification

Lemma 1. *If V is a $\mathbb{C}$-analytic set in an open set $U \subset \mathbb{C}^n$ and if $f_1, \ldots, f_r$ are holomorohic functions on U such that the germs of $f_1, \ldots, f_r$ at any point b in U generate the ideal I_b of V_b, then V is a submanifold of dimension p in a neighbourhood $U_0 \subset U$ of any point a_0 if and,only if rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z = n - p$ for z in $U' \cap V$ where U' is a certain neighbourhood of a_0.* 14

Proof. If $a_\circ$ is a simple point of dimension p, then there exists a neighbourhood U_0 of a_0 and holomorphic functions $g_1, \ldots, g_{n-p}$ on U_0 such that rank $\left(\dfrac{\partial g_i}{\partial z_j}\right)_z = n - p$ for any z in U_0 and the germs $(g_i)_z$ generate the ideal I_z for z in U_0. Clearly $g_i = \sum\limits_j \lambda_{ij} f_j$ and hence rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z \geq n - p$ for z in U_0. Since $(g_i)_z$ also generate I_z, we have conversely, rank $\left(\dfrac{\partial g_i}{\partial z_j}\right)_z \geq$ rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z \geq n - p$ i.e. rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z = n - p$ for z in U_0. Conversely if rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z = n - p$ for z in U', U' being a neighbourhood of a_0, we can find a subset $\{f_1, \ldots, f_{n-p}\}$ of $\{f_1, \ldots, f_r\}$ such that rank

$\left(\dfrac{\partial f_i}{\partial Z_j}\right)(a)_{\substack{i\leq n-p\\ j\leq n}} = n-p$. Consider $V' = \{z\epsilon U' \mid f_i(z) = 0, i \leq n-p\}$.

Let $U'' \subset U'$ be a neighbourhood of a_0 such that $U'' \cap V \subset V' \cap$
15 U'' $\dim V' = p$ and $V' \cap U''$ is a manifold. Hence by Proposition 2 of Chapter 1 if $V \cap U'' \underset{\neq}{\subset} V' \cap U''$, $\dim V = p' < p$ in U', and if b is a simple point of V in U'', rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)(b) = n - p'$ by the converse proved above. Thus rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)(b) = n-p' > n-p$ and we have a contradiction and this proves the lemma. $\square$

Lemma 2. *If V is an irreducible analytic set and if M is the set of its simple points and if* $\dim V = p$ *then* $\dot{V} = V - M$ *is an analytic set and* $\dim \dot{V} < p$.

Proof. For any $a_\circ \epsilon V$, there exists a neighbourhood U and holomorphic functions $f_1, \ldots, f_r$ on U (by Theorem 6, Chapter 1) such that the germs (f_{i_b}) generate the ideal I_b at any point b in U. By the above lemma a point b in U is a simple point if and only if rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_b = n-p$ i.e. a point z is in $U \cap \dot{V}$ if and only if determinants of all submatrices of order $\geq n-p$ of $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z$ are zero. Since $\dfrac{\partial f_i}{\partial z_j}$ and hence the determinants are holomorphic on U, the lemma is proved. $\square$

Lemma 3. *Let $V = UV_i$ be an analytic set when V_i are its irreducible components and let the maximum dimension of $V = p$. Then if $M = \{z\epsilon V \mid z$ is a simple point of dimension $p\}$ then the set $V_1 = V - M$ and the set $\dot{V}$ of singular points of V are analytic sets of dimension $< p$.*

16 *Proof.* It follows from the remark of Chapter 1 that

$$\dot{V} \bigcup_i \dot{V}_i \cup \left(\bigcup_{i\neq j} V_i \cap V_j\right).$$

By Lemmma 2 proved above, $\dot{V}_i$ is an analytic set and hence it follows that $\dot{V}$ is an analytic set of dimension $< p$. Also $V_1 = V - M =$

$\left(\bigcup\limits_{\dim V_i<p} V_i\right)\cup\left(\bigcup\limits_{i\neq j} V_i\cap V_j\right)\cup\left(\bigcup\limits_{\dim V_j=p}\dot{V}_j\right)$ is an analytic set of dimension $< p$. □

Proposition 1. *If V is any $\mathbb{C}$-analytic set there exists a strict partition $V = \bigcup M_i$ of V.*

Proof. Let M_1 be the set of simple points of maximum dimesion p say, of V and $M_1 = \cup M_i^p, M_i^p$ being the connected components of M_1. Then $\bar{M}_i^p$ is an analytic set and by the Lemma 2 above, $V - M_1$ is an analytic set of dim $< p$. Hence $\dot{M}_i^p = \bar{M}_i^p \cap (V - M_1)$ is an analytic set. Now consider $V_1 = V - M_1 \cdot \dim V_1 < p$. Let M_2 be the set of simple points of maximum dimension p_1, of V_1. Then if $M_2 = \bigcup M_i^{p_1} M_i^{-p_1}$ and $\dot{M}_i^{p_1}$ are analytic sets and $V - M_2$ is an analytic set of dimension $< p_1$ and so on. We finally get $V = \bigcup\limits_{i=0}^{P}(\bigcup\limits_{\nu} M_\nu^i)$ and this is clearly a strict partition into manifolds. This strict partition is a canonical one. □

Remark 1. If V is a $\mathbb{C}$ -analytic space then V has a strict partition into manifolds.

Proof. Let $V_k = \bigcup\limits_{\substack{V_\alpha\subset V\\ \text{and } \dim V_\alpha\le k}} V_\alpha$, V_α being irreducible analytic sets contained in V. Then V_k has a strict partition into manifolds and if $V_k = \underset{i}{U} M_{ki}$ is the partition, since $\{V_\alpha\}$ are locally finite, we define $M_i = \underset{k\to\infty}{\underrightarrow{\lim}} M_{ki}$ and it can be easily verified that $V = UM_i$ is a strict partition 17
in to manifolds. □

Examples. 1. Let $V \subset \mathbb{C}^3_{xyt}$ be given by

$$V = \left\{z\epsilon\mathbb{C}^3_{xyt}\middle| x(x^2 - y^2 - t) = 0, V(x^2 - y^2 - t) = 0\right\}.$$

Then clearly if $M_1 = \mathbb{C}_t$ and $M_2 = V - \mathbb{C}_t, V = M_1 \cup M_2$ is the canonical strict partition into manifolds. But this is not a stratification since $\dot{M}_2 = \{0\}$ is not a union of manifolds in the partition.

2. Let $V = V_1 \cup V_2 \cup V_3$, where

$$V = \left\{z\epsilon\mathbb{C}^5_{xytrs}\middle|x^2 - y^2 - t^2 = 0, r = s = 0\right\},$$
$$V_2 = \left\{z\epsilon\mathbb{C}^5_{xytrs}\middle|x = y = r = 0\right\}$$
and
$$V_3 = \left\{z\epsilon\mathbb{C}^5_{xytrs}\middle|t = y = s = 0\right\}.$$

Then if

$$M^0_1 = \{0\}, M^2_1 = V_1 - \{0\}, M^2_2 = V_2 - \{0\},$$
$$M^2_3 = V_3 - \{0\}, V = M^0_1 \cup \left(\bigcup_{i=1}^{3} M^2_i\right)$$

is the canonical strict partition into manifolds and clearly this is a stratification.

Notation. In what follows, a partition into manifolds of an analytic set V shall be written as $V = \cup M^k_\nu$ is a manifold of dimension k and $M^k = \cup M^k_\nu$ is the union of all manifolds of dimension k.

Lemma 4. *Let V be an analytic set and $V = \cup M^h_\nu$ be a strict partition into manifolds and $V = \cup S^k_\mu$, a stratification.Then the following are equivalent.*

18 *(1)* $S^h_\nu \cap M^k_\mu \neq \phi \Rightarrow S^h_\nu \subset \overline{M}^k_\mu$

(2) $S^h_\nu \cap M^k_\mu \neq \phi \Rightarrow S^h_\nu \subset M^k_\mu$ *(i.e.* $\{S^k_\mu\}$ *is a refinement of* $\{M^k_\nu\}$*).*

Proof. Obviously (2) $\Rightarrow$ (1). Conversely suppose that (1) holds. Let $z \in M^k_\mu \cap S^h_\nu$. Then there is a neighbourhood U of z such that $U \cap M^k_\mu = U \cap \overline{M}^k_\mu$. Since $S^h_\nu \subset \overline{M}^k_\mu$, it follows that $U \cap S^h_\nu \subset U \cap M^k_\mu$. If $A = \{z \in S^h_\nu | z \in M^k_\mu\}$ then this proves that A is open in S^h_ν. If $z_\nu \in A$ and $z_\nu \to z_o \in S^h_\nu$, let if possible, $z_0 \notin M^k_\mu$ and let $z_0 \in M^l_\lambda$, $(l, \lambda) \neq (k, \mu)$. Then by the same argument as above there is a neighbourhood U_0 of z_0 such that $U_0 \cap S^h_\nu \subset U_0 \cap M^l_\lambda$ but then there are $z_\nu \in M^l_\lambda \cap M^k_\mu$ and we have a contradiction. This proves that A is closed and hence that lemma is proved. □

Definition 1. For an analytic set V, a stratification $V = \cup S^k_\mu$ is defined to be a narrow stratification if for every open set U of V, the connected components of $S^k_\mu \cap U$ form a stratification of $V \cap U$.

Remark 2. An arbitrary stratification may not be a narrow stratification. For example, let $V = \mathbb{C}^2_{xy}$. Consider the stratification $M_1 = \{z \in \mathbb{C}^2_{xy} \big| x = 0\}$ and $M_2 = V - M_1$. Consider the point $a = (1, 1) \in M_2$. Let $f : \mathbb{C}^2_{xy} \to \mathbb{C}^n$ be a bolomorphic map.n sufficiently large, such that

(1) f is proper

(2) $f(z_1) \neq f(z_2)$ if $(z_1, z_o) \neq (0, a)$

(3) $f(a) = f(0)$ and 19

(4) rank $(df)(z) = 2$ for any $z \in \mathbb{C}^2_{xy}$.

Then $f(\mathbb{C}^2_{xy})$ is an analytic set and if $M_1 = f(M_1), N_2 = f(M_2 - \{a\})$ then $f(\mathbb{C}^2_{xy}) = N_1 \cup N_2$ is a stratification of $f(\mathbb{C}^2_{xy})$ but it is not a narrow stratification. For if U is a sufficiently small neighbourhood of $f(a)$, then $f^{-1}(U) = U_1 \cup U_2, U_1 \cap U_2 = \phi$, where U_1 and U_2 are neighbourhoods of 0 and a respectively and hence $N_2 \cap U$ has two connected components $f(U_1 \cap M_2)$ and $f(U_1 \cap M_2)$ and $U \cap f(V) = (N_1 \cap U) \cup f(U_1 \cap M_2) \cup f(U_2 \cap M_2)$ is not a stratification.

Proposition 2. *If V is a $\mathbb{C}$- analytic set and $V = \cup M^i_r$ is a strict partition into manifolds there exists a narrow stratification $V = \cup S^k_\mu$ which is a refinement of (M^i_r).*

Proof. We shall assume that there exist integers $n_0 > n_1 > . > n_k$ with the following properties

(1) for every $i \leq k$, there exists an analytic set $V_{i+1} \subset V$ with $V_0 = V$, dim $.V_{i+1} < n_i$ and dim $V_{i+1} = n_{i+1}$ if $i < k$ with

(2) $V - V_{i+1} = \bigcup\limits_{j=o}^{i} S^{n_j}_\nu$ where $(S^{n_j}_\nu)$ is a locally finite family of connected manifolds and $S^h_r \cap S^k_s = \phi$ if $(h, r) \neq (k, s).\overline{S}^h_r$ and $\dot{S}^h_r$ are analytic sets for $h \geq n_i$ and if for $h \geq k \geq n_i$ and U open, $S^h_{r,r'}$, $S^k_{s,s'}$ are

connected components of $S_r^h \cap U$ and $S_s^k \cap U$ respectively, then $S_{r,r'}^h \cap S_{s,s'}^k \neq \phi \Rightarrow S_{r,r'}^h \subset S_{s,s'}^k$,

(3) for $h \geq n_i, S_r^h \cap \overline{M}_s^k \neq \phi \Rightarrow S_r^h \subset \overline{M}_s^k$.

For $k = 0$, the above statement is trivial. Assuming that the result holds for $k = r - 1$ we shall prove it for $k = r$. □

20 Let $\dim V_r = n_r < n_{r-1}$ and let $M^n r = \cup M_\nu^{n_r}$ be the set of simple points of V_r of dimension n_r, where $M_\nu^{n_r}$ are its connected components. By Lemma 3 above, $\overline{M}_\nu^{n_r}$ and $V_n - M^{n_r}$ are analytic sets. We define $W_{r+1}^i, i = 1, 2$ as follows.

W_{r+1}^1, is the set of $z \in \bar{M}^{n_r}$ such that there is a neighourhood U of z and irreducible component $\sum$ of $U \cap \overline{S}_\nu^h$ with $h \geq n_r$ such that $0 \leq \dim_z \sum \cap \overline{M}_r^n < n_r$; W_{r+1}^2 is the set of $z \epsilon \overline{M}^{n_r}$ such that there is a neighbourhood U of z and an irreducible component $\sum_1$ of $U \cap \overline{M}_\mu^k, k$ and μ arbitrary, such that $0 \leq \dim_z U \cap \sum_1 \cap \overline{M}^{n_r} < n_r$.

Let $W_{r+1} = W'_{r+1} \cup W_{r+1}^2$. Since (M_μ^k) and (S_ν^h), $h > n_r$ are locally finite, W'_{r+1}, W_{r+1}^2 and hence W_{r+1} are anaytic sets. More over by Proposition 2 of Chapter 1, $\dim W_{r+1} < n_r$. Hence $S_\nu^{n_r} = M_\nu^{n_r} - W_{r+1}$ are connected manifolds and $\overline{S}_\nu^{n_r} = \overline{M}_\nu^{n_r}$ and $\dot{S}_\nu^{n_r} = (W_{r+1} \cap \bar{M}_\nu^{n_r}) \cup \dot{M}_\nu^{n_r}$ are analytic sets. Also if in an open set U we have $U \cap S_{\nu,\nu'}^{n_r} \cap \overline{S}_{\mu,\mu'}^k \neq \phi$ for $k \geq n_r$ then $\dim U \cap \overline{S}_{\nu,\nu'}^{n_r} \cap \overline{S}_{\mu,\mu'}^k = n_r$ by definition of W'_{r+1}. Hence $S_{\nu,\nu'}^{n_r} \subset \overline{S}_{\mu,\mu'}^k$ i.e. the property (2) is satisfied for $k = r$. Moreover if $S_\nu^{n_r} \cap \overline{M}_\mu^k \neq \phi$ for some k, it follows in the same way from the definition of W_{r+1}^2, that $\dim \overline{S}_\nu^{n_r} \cap \overline{M}_\mu^k = n_r$ and hence $S_\nu^{n_r} \subset \overline{M}_\mu^k$. Hence we prove by induction the existence of a sequence $n_0 > n_1 > \cdots > n_k$ satisfying the above three properties and hence there is a norrow stratification $V = \cup S_\mu^k$ which in fact satisfies the condition of Lemma 4 and hence it is a refinement of (M_μ^k).

Remark. (3) Let $\wedge_1, \ldots, \wedge_k$ be strict partitions of $\nu, \wedge_i$ given by $V =$
21 $\bigcup\limits_{h,\nu} M_\nu^{i,h}$ for each i. Then there exists a stratification $V = \cup S_\mu^k$ of V which is a refinement of $\wedge_i$ for each i. In the above proof we have

only to change W^2_{r+1} to $\bigcup\limits_{i=1}^{i} W'^{i}_{r+1}$, where W'^{i}_{r+1} is the set of points z in $\overline{M}^{n_r}$ for which there is a neighbourhood U_z and a connected $\sum$ component of some $U_z \cap \overline{M}^{i,h}_{\nu}$ such that $0 \leq \dim_z \sum \cap \overline{M}^{n_r} < n_r$.

(4) Proposition 2 can also be proved without using Lemma 4. We have only to change W^2_{r+1} to $W'^2_{r+1} = \Big\{ z \epsilon \overline{M}^{n_r}$ | There is a neighbourhood U of z and a component $\sum_1$ in U of some $\overline{M}^k_\mu$ such that $0 \leq \dim_z \overline{M}^{n_r} \cap \sum_1 < n_r$ or a comnonent $\sum'_1$ in U of some $\dot{M}^k_\mu$ such that $0 \leq \dim_z \sum'_1 \cap \overline{M}^{n_r} < n_r \Big\}$. Then it follows immediately that if $V = \bigcap S^k_\nu$ is the stratification obtained as in the proof of Proposition 2, $S^k_\nu \cap M^h_\mu \neq \phi \Rightarrow S^k_\nu \subset M^h_\mu$

Lemma 5 (Whitney)**.** *Let V be an analytic set of constant dimension $p, V \subset \mathbb{C}^n$ and let $a \epsilon V$. Then there exists a neighbourhood U of a and finite number of vector fields $v^1, \ldots, v^q$ defined on U such that*

(i) $v^k(z) = 0$, $1 \leq k \leq q$, *if z is a singular point of $V \cap U$ and*

(ii) $v^1(z), \ldots, v^q(z)$ *span the tangent space $T(V, z)$ if z is a simple point of $V \cap U$. We give here two proofs of this lemma.*

1st Proof. By Cartan's coherence theorem, there exists a neighbourhood U of a and a finite number of holomorphic functions $f_1, \ldots, f_q$ such that the germs of $f_1, \ldots, f_q$ at any point b in U, generate the ideal of germs of holomorphic functions at b, vanishing on V_b. It follows from Lemma 1 that z in V is a simple point of V if and only if rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z = n - p = r$. 22

In what follows $\lambda = (\lambda_1, \ldots, \lambda_n)$, $1 \leq \lambda_1 < \cdots < \lambda_r \leq n$ $\nu = (\nu_1, \ldots, \nu_r)$, $1 \leq \nu_1 < \nu_2 < \cdots < \nu_r \leq r$, and $\mu = (\mu_1, \ldots, \mu_{r+1})$, $1 \leq \mu_1 < \mu_2 < \cdots < \mu_{r+1} \leq n$. Also, we put $D_{\lambda\nu} = \det \left[\left(\dfrac{\partial f \lambda_j}{\partial z \nu_i}\right)\right]$ and $\mu^{(i)} = (\mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_{r+1})$ [a hat over a term means that the term is omitted].

We now define vectors $v^{\lambda\mu}$ as follows

$$\begin{aligned} v^{\lambda\mu} &= (v_k^{\lambda\mu})\epsilon\mathbb{C}^n \quad \text{where} \\ v_k^{\lambda\mu} &= 0 \quad \text{if} \quad k \notin \mu \\ \text{and} \qquad &= (-1)^{i-1} D_{\lambda\mu}(i) \quad \text{if} \quad k = \mu_i. \end{aligned}$$

Then obviously $v_{(z)}^{\lambda\mu} = 0$ for any z where rank $\left(\dfrac{\partial f_i}{\partial z_j}\right)_z < r$ i.e. for any z in the set of singular points of $V \cap U$.

We shall prove the condition (ii) for the vectors $(v^{\lambda\mu})$. Let z be a simple point of V. Then $v^{\lambda\mu} \in T(V, z)$ if and only if $\langle d f_j(z), v^{\lambda\mu}(z)\rangle = 0$ for $1 \leq j \leq q$.

But $\langle d f_j(z), v^{\lambda\mu}(z)\rangle$

$$\begin{aligned} &= \sum_{k=1}^{n} \left(\frac{\partial f_j}{\partial z_k}\right)(z) v_k^{\lambda\mu}(z) \\ &= \sum_{i=1}^{r+1} \frac{\partial f_j}{\partial z\mu_i}(z)(-1)^{i-1} D_{\lambda\mu}(i) \end{aligned}$$

and this is nothing but the determinant

$$\begin{bmatrix} \dfrac{\partial f_j}{\partial z\mu_1} & \dfrac{\partial f_j}{\partial z\mu_2} \cdots & \dfrac{\partial f_j}{\partial z\mu_{r+1}} \\ \dfrac{\partial f\lambda_1}{\partial z\mu_1} & \dfrac{\partial f\lambda_1}{\partial z\mu_2} \cdots & \dfrac{\partial f\lambda_1}{\partial z\mu_{r+1}} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial f\lambda_r}{\partial z\mu_1} & \dfrac{\partial f\lambda_r}{\partial z\mu_2} & \dfrac{\partial f\lambda_r}{\partial z\mu_{r+1}} \end{bmatrix}$$

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If $j \in \lambda$ this is clearly zero and if $j \notin \lambda$ this is the determinant of a submatrix of order $(r + 1)$ of the matrix $\left(\dfrac{\partial f_i}{\partial z_j}\right)$ and hence is zero. Thus $(v^{\lambda\mu})\epsilon T(V, z)$ for each pair (λ, μ). It now remains to prove that these vectors span $T(V, z)$. The dimension of $T(V, z) = \dim_z V = p$. Now there exists a pair (λ, ν) and a neighbourhood U' of z such that $D_{\lambda\nu}(\zeta) \neq 0$ for

$\zeta \epsilon U'$. Then define for each $\rho \notin \nu$, μ^ρ as the $(r+1)$-tuple which contains the integers $(\nu_1, \ldots, \nu_r, \rho)$. Then $v_\rho^{\lambda\mu^\rho} = \pm D_{\lambda\nu}$. Hence if $v^1, \ldots, v^p$ are the vectors defined by $v^{\lambda\mu^\rho}$, $\rho \notin \nu$, $v^1, \ldots v^p$ are linearly independent since their projections on $\mathbb{C}_{z\lambda'_1}, \ldots, z_{\lambda'_{n-r}} \lambda'_i \notin \nu$ are independent and hence span $T(V, z)$.

2nd Proof (R. Narasimhan). We use hero the properties of sheaves of germs of holomorphic functions on a manifold. Let V be an analytic set of constant dimension in an open set Ω in $\mathbb{C}^n$. Let $S \subset V$ be the set of singular points of V. Let F be the sheaf (on Ω) of germs of holomorphic mappings $g = (g_1, \ldots, g_n)$ into $\mathbb{C}^n$ such that a) $g(y) = 0$ for $y \in S$ and b) $\sum g_i(y) \dfrac{\partial}{\partial z_i} \in T(V, y)$ for $y \in V - S$. We have only to prove that F is coherant. By H. Cartan's coherence theorem, i.e. Theorem 6 of chapter 1, for any $a \in V$, there is a neighbourhood $W(\subset \Omega)$ of a and holomorphic functions $f_1, \ldots f_r$ on W such that the germs of $f_1, \ldots, f_r$ 24
at any point b in W generate the ideal I_b of V_b. Then for a holomorphic map $g : W \to \mathbb{C}^n$, $g_b \in F_b$ (the stalk of F at b)for every b if and only if (1) $\sum\limits_{i=1}^{n} g_i(z) \dfrac{\partial f_j}{\partial z_i}(z) = 0$, $j = 1, 2, \ldots, r$, if $z \epsilon V \cap W$ and (2)$g_i(z) = 0$, $i = 1, 2, \ldots, n$ if $z \epsilon S \cap W$.

Now, if $\varphi_1, \ldots, \varphi_k$ are holomorphic, then the sheaf of $(\alpha_1, \ldots, \alpha_k)$ such that $\sum \alpha_i \varphi_i = 0$ on an analytic set A is a quotient of the sheaf of relations between $(\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l)$ where the (ψ_j) generate the ideal sheaf of A and so is coherent. Since the intersection of finitely many coherent sheaves is again coherent, our lemma follows.

Proposition 3. *Let V be an analytic set, W, a $\mathbb{C}$-analytic manifold and $f : V \to W$ a holomorphic map. Then there exists a narrow stratification $V = \cup S_\nu^k$ of V such that if $f_{k,\nu} = f \mid S_\nu^k$, rank $df_{k,\nu}$ constant on S_ν^k.*

Such a stratification will be called a stratification consistent with f.

Proof. We assume that there exist integers $\dim V = n_0 > n_1 > \cdots > n_k$ such that for each $i \leq k$, there an analytic set $V_{i+1} \subset V$ with the following properties.

(1) $\dim V_{i+1} < n_i$ and $\dim V_{i+1} = n_{i+1}$ if $i < k$.

(2) $V - V_{i+1} = \cup S^h_\nu$, $h \geq n_i$, S^h_ν being connected manifolds of dimension h, $S^h_\nu \cap S^k_\mu = \phi$ if $(h,\nu) \neq (k,\mu)$ for which S^h_ν, $\dot{S}^h_\nu$ are analytic sets; furthermore,

$$S^h_\nu \cap \overline{S}^k_\mu \neq \phi \Rightarrow S^h_\nu \subset \overline{S}^k_\mu.$$

25 (3) if $f_{h,\nu} = f \mid S^h_\nu$, rank of $df_{h\nu}$ is constant an S^h_ν. The above statement is trivial for $k = 0$.

Assuming the existence of the above sequence for $k = r-1$, we shall prove it for $k = r$. Let $\dim V_r = n_r$ and M^{n_r} = the set of simple points of V_r of dimension n_r, and let $M^{n_r} - \cup M^{n_r}_\nu$, $M^{n_r}_\nu$ being the connected components of M^{n_r}. By Lemma 3 above, $\overline{M}^{n_r}_\nu$, $\dot{M}^{n_r}_\nu$ and $V_r - M^{n_r}$ are analytic sets. We defire $W^1_{r+1,\nu}$ and $W^2_{r+1,\nu}$ as follows. $W^1_{r+1,\nu} = \Big\{ z\epsilon\overline{M}^{n_r}_\nu \mid$ There exists a neighbourhood U of z and a connected component $\sum$ of S^k_μ with $k \geq n_r$ such that $0 \leq \dim_z U \cap \sum \cap \overline{M}^{n_r}_\nu < n_r \Big\}$.

Let $f^{n_r}_\nu = f \mid M^{n_r}_\nu$ and let the maximum rank of $df^{n_r}_\nu = k_r$ on $M^{n_r}_\nu$. Then $W^2_{r+1,\nu} = \Big\{ z\epsilon M^{n_r}_\nu \mid \operatorname{rank}(df^{n_r}_\nu)(z) < k_r \Big\}$.

Clearly by the same argument as in Proposition 2, $W^1_{r+1,\nu}$ is an analytic subset of $\overline{M^{n_r}_\nu}$ and $\dim W^1_{r+1,\nu} < n_r$. We shall now prove that $(W^2_{r+1,\nu} \cup M^{n_r}_\nu)$ and hence $\overline{W^2}_{r+1,\nu}$ is an analytic set of dimension $< n_r$. Since this problem is local we assume $V \subset \mathbb{C}^n$. By Lemma 5 above, for every $a\epsilon\overline{M^{n_r}_\nu}$ there exists a neighbourhood $U \subset \mathbb{C}^n$ and vector fields $v^1, \ldots, v^q$ on U such that $v^i(z) = 0$ for z in $\dot{M}^{n_r}_\nu U$ and (v^i) span $T(M^{n_r}_\nu, z)$, if z is a simple point of $\overline{M^{n_r}_\nu} \cap U$. Further, if z is a simple point of $\overline{M}^{n_r}_\nu$ we may choose U sufficiently small so that after a holomorphic change of coordinates, if $f = (f_i)$ and $df(z)$, the transformation $T(V,z) \to T(W, f(z))$. then if

$$\langle df(z), v^j(z)\rangle = w^j(z) \quad \text{are vectors in T(W, f(z)), then}$$

$$\text{rank} \quad (df^{n_r}_\nu)(z) = \text{dimension of the space spanned by } w^j(z).$$

26 Since $(w^j_i(z))$ are holomorphic functions on U, there exist holomor-

phic functions $h_1, \ldots, h_k$ on U such that

$$U \cap W^2_{r+1,\nu} = \left\{z \epsilon M^{n_r}_\nu \middle| h_1(z) = \cdots = h_k(z) = 0\right\}$$

where (h_i) are holomorphic on U. Let $U \cap \dot{M}^{n_r}_\nu = \Big\{z \epsilon U \mid g_1(z) = \ldots = g_l(z) = 0\Big\}$ where (g_i) are holomorphic on U. Then

$$U \cap (W^2_{r+1,\nu} \cup \dot{M}^{n_r}_\nu) = \left\{z \epsilon \overline{M}^{n_r}_\nu \cap \middle| h_i(z) g_j(z) = 0, i \leq k, i \leq l\right\}$$

and therefore $W^2_{r+1,\nu} \cup \dot{M}^{n_r}_\nu$ is analytic. Also $\dot{M}^{n_r}_\nu$ is an analytic set and hence $\overline{W^2_{r+1,\nu}} = clos(W^2_{r+1,\nu} \cup \dot{M}^{n_r}_\nu - \dot{M}^{n_r}_\nu)$ is an analytic set by Proposition 4 of Chapter 1. Let $W_{r+1,\nu} = W^1_{r+1,\nu} \cup \overline{W^2}_{r+1,\nu}$ and let $S^{n_r}_\nu = M^{n_r}_\nu - W_{r+1,\nu}$. Then $\overline{S^{n_r}_\nu} = \overline{M^{n_r}_\nu}$ and $\dot{S}^{n_r}_\nu = \dot{M}^{n_r}_\nu \cup W_{r+1,\nu}$ are analytic sets and by the definition of $W^1_{r+1,\nu}$, condition (2) of the induction hypothesis is satisfied. Further it follows from the definition of $W^2_{r+1,\nu}$ that rank $d f^{n_r}_\nu =$ constant on $S^{n_r}_\nu$ where $f^{n_r}_\nu = f \mid S^{n_r}_\nu$ and hence the proposition is proved by induction. □

Remark 6. If V is an analytic set, W an analytic manifold and $f : V \to W$ a holomorphic map and if $b \in W$, $Z = V \cap f^{-1}(b)$ is an analytic set. Moreover if $V = \cup M^i_\nu$ is a stratification of V. consistent with the restrictions of f, then the connected components $S^{\,i}_{\nu,j}$ of $Z \cap M^i_\nu$ form a strict partition of Z.

Proof. By the constant rank therorem in Chapter 1, $S^{\,i}_{\nu,j}$ is a manifold.

Let $Z \cap \overline{M}^i_\nu = \cup V_\alpha$, V_α being irreducible components of $Z \cap \overline{M}^i_\nu$.

Then if $V_\alpha \cap S^{\,i}_{\nu,j} \neq \phi$, then $V_\alpha \cap \dot{M}^i_\nu$ is proper analytic subset of V_α **27**
and $V_\alpha \cap \dot{M}^i_\nu = V_\alpha - V_\alpha \cap \dot{M}^i_\nu$ is connected and dense in V_α. Hence $\overline{S}^{\,i}_{\nu,j} = \bigcup\limits_{V_\alpha} \cap S^{\,i}_{\nu,j} \neq \phi V_\alpha$ is an analytic set and so is $\dot{S}^{\,i}_{\nu,j} = \overline{S}^{\,i}_{\nu,j} \cap (\bigcup\limits_{k \neq j} \overline{S}^{\,i}_{\nu,k} \cup \dot{M}^i_\nu)$. But the $S^{\,i}_{\nu,j}$ do not, in general, form a stratification of Z, as shown by the following: □

Example 3. Let V be the analytic set in $\mathbb{C}^3_{xyt}$ given by $V = \left\{z\epsilon\mathbb{C}^3_{xyt} \middle| y = x^2\right\}$. Consider the stratification $M^1 = \mathbb{C}_t$ and $M^2 = V - \mathbb{C}_t$. It is consistent with the restrictions of the holomorphic map $f : \mathbb{C}^3_{xyt} \to \mathbb{C}$ given by $f(z) = vt. f^{-1}(0) = Z = \mathbb{C}_t \cup (M^2 \cap t = 0)$ and this is clearly not a stratification.

Definition 2. If V is an analytic set in $\Omega \subset \mathbb{C}^n$, a function $f : V \to \mathbb{C}$ is said to be strongly holomorphic if for every $a\epsilon V$, there is a neighbourhood $U \subset \mathbb{C}^n$ and a holomorphic function F on U such that $F \mid V \cap U = f \mid V \cap U$.

Proposition 4. *Let V and W be analytic manifolds and $f : V \to W$ a strongly holomorphic map. Then there exists a stratification $V = \cup S^k_\mu$ of V such that rank df = constant on S^k_μ.*

Proof. Let $L_\circ = V$ and let max rank df on $V = r_\circ$. Let L_1 = the set of points of V such that rank $df < r_\circ$ on L_1. Then if $\dim V = n_\circ, L_1$ is an analytic set of dimension $n_1 < n_0$. Let max rank $df = r_1 < r_\circ$ on L_1 and so on. We get a finite sequence

$$V = L_0 \supset L_1 \supset \cdots \supset L_k, \dim L_k = 0.$$

max rank $df = r_i$ on $L_i, \dim L_i = n_i$,

$$n_0 > n_1 > \cdots > n_k = 0, r_0 > r_1 > \cdots > r_k.$$

28 Let $\wedge_i$ be a strict partition of V as follows.

$\overline{(V - L_i)}$ is an analytic set and L_i is an analytic set. Let $\overline{(V - L_i)} = \cup M^i_\nu$ and $L_i = \cup S^i_\nu$ be the respective stratifications of $\overline{(V - L_i)}$ and L_i. Then $V = (\cup M^i_\nu - L_i) \cup (\cup S^i_\nu)$ is a strict partition of V and we define $\wedge_i$ to be that strict partition. Then, by Remark 2 following Proposition 2 above, we have a stratification $V = \cup S^k_\mu$ which is a refinement of each $\wedge_i$. If $S^k_\mu \subset L_i$ and if $S^k_\mu \cap L_{i+1} \neq \phi$, then $S^k_\mu \subset L_{i+1}$ since the stratification is a refinement of $\wedge_{i+1}$. If $S^k_\mu \subset L_i$. and $S^k_\mu \cap L_{i+1} = \phi$. rank $df \leq r_i$ on S^k_μ and since $S^k_\mu \cap L_{i+1} = \phi$, rank $df = r_i$ on S^k_μ, i.e. $V = \cup S^k_\mu$ is a stratification with the required properties. $\square$

Remark 7. The above proposition can also be proved directly by changing $W^2_{r+1,\nu}$ in the proof of Proposition 3, to

$$W'^2_{r+1,\nu} = \left\{ z\epsilon \overline{M^{n_r}_\nu} \middle| \text{ rank } (df)(z) < \max \text{ rank } (df) \text{ on } \overline{M^{n_r}_\nu} \right\}$$

and then by defining

$$W_{r+1,\nu} = W'_{r+1,\nu} \cup W'^2_{r+1,\nu} \text{ and} S^{n_r}_r = M^{n_r}_\nu - W_{r+1,\nu}.$$

We expect to prove in the next chapter the following important theorem of Whitney.

Theorem. *Let V be an analytic set of dimension k and M manifold of dimension $m < k$ such that $M \subset V$ and $\overline{M}$ is an analytic set. Then there exist analytic sets W_a, W_b of dimensions $< m, W_a, W_b \subset \overline{M}$ such that if $z\epsilon M - W_a$. the pair (M, V) satisfies the condition (a) [stated in Chapter 1] of Whitney at az and if a $z \in M - W_b$, the pair (M, V) satisfies the* **29**
condition (b) of Whitney at z.

If we assume the above theorem it is easy to prove the Whitney's theorem stated in Chapter 1. We use the same reduction process as above.

Theorem. *For an analytic set V, there exists stratification $V = \cup S^k_\mu$ which is (a) and (b) regular.*

Proof. We prove by induction the existence of *a* someone of positive integers

$$n_0 > n_1 > \cdots > n_k$$

such that for each $i \leq k$, there exists an analytic set V_{i+1} in V such that $\dim V_{i+1} < n_i$ and $\dim V_{i+1} = n_{i+1}$ if $i < k$. which have. further following properties.

(1) $V - V_{i+1} = \cup S^k_\mu$, $k \geq n_i$ where S^k_μ are connected manifolds with $S^k_\mu \cap S^h_\nu = \emptyset$ if $(k\mu) \neq (h, \mu)$, $S^k_\mu \cap \overline{S}^h_\nu \neq \emptyset \Rightarrow S^k_\mu \subset S^{-h}_\nu$, and $\overline{S}^k_\mu$ and $\dot{S}^k_\mu$ are analytic sets.

(2) For $h > k \geq n_i$, if $S^h_\nu \subset \overline{S}^k_\mu$, the pair $(S^h_\nu, \overline{S}^k_\mu)$ is (a) and (b) regular.

Assuming the existence of such a sequance for $K = r - 1$. let the dimension of $V_r = n_r < n_{r-1}$. Let $M^{nr} = \cup M_\nu^{n_r}$ be the set of simple points of V_r of dimension n_r, $M_\nu^{n_r}$ being its connected of components. Define the set $W^1_{r+1,\nu}$ as in proposition 3. Now if $M_\nu^{nr} \subset \bar{S}_\mu^k$, there are analytic sets $W_{a,\nu}^{k,\mu}$ and $W_{b,\nu}^{k,\mu}$ in $\overline{M_\nu^{n_r}}$ such that for any $z \in M_\nu^{n_r} - W_{a,\nu}^{k,\mu}$, $(M_\nu^{n_r}, \bar{S}_\mu^{-k})$
30 is (a) regular at z and for $z \in M_\nu^{n_r} - W_{b,\nu}^{k,\mu}(M_\nu^{n_r}, \bar{S}_\mu^k)$ is (b) regular at z. Let $W^2_{r+1,\nu} = \bigcup\limits_{M_\nu^{nr} \subset \bar{S}_\mu^k} (W_{a,\nu}^{k,\mu} \subset W_{b,\nu}^{k,\mu})$. Then $W^2_{r+1,\nu}$ is an analytic set of dimension $< n_r$. Let $W_{r+1,\nu} = W^1_{r+1,\nu} \cup W^2_{r+1,\nu}$ and $S_\nu^{\,n_r} = M_\nu^{n_r} - W_{r+1,\nu}$. Then clearly conditions (1) and (2) of the induction hypothesis are satisfide and the theorem is proved by induction. ◻

Proposition 3′. *In proposition 3 we can form a stratification which is also a whitney stratification.*

We have only to take, for $W_{r+1,\nu}$, $W_{r+1,\nu} \cup W^2_{r+1,\nu}$, where $W^2_{r+1,\nu}$ is as in the above proof.

Chapter 3

Whitney's Theorems

1 Tangent Cones

In what follows $G_{n,r}$ will denote the Grassmann manifold of $r - planes$ **31** through $\mathscr{O}$ in $\mathbb{C}^n$. We shall assume the classical result that $G_{n,r}$ is a compact $\mathbb{C}$-analytic manifold. $G_{n,1} = \mathbb{P}^{n-1}$ is the complex proiective space. If T is an r-plane in $\mathbb{C}^n$, $T^* = K(T)$ will denote the corresponding point in $G_{n,r}$ and for a vector $v \neq 0$ in $\mathbb{C}^n$, $K(v)$ will denote the corresponding point in $\mathbb{P}^{n-1}$. If $\alpha \in G_{n,r}, T(\alpha)$ will denote the r-plane in $\mathbb{C}^n$ such that $K \cdot T(\alpha) = \alpha$. If $r_1 < r_2$ and if $\alpha_1 \in G_{n,r_1}$, $\alpha_2 \in G_{n,r_2}$, $\alpha_1 \subset \alpha_2$ will mean that $T(\alpha_1) \subset T(\alpha_2)$.

Definition 1. Let V be an analytic set and a, a point in V; the tangent cone at a, denoted by $C(V, a)$ is difined to be $\{v \in \mathbb{C}^n|$. There is a sequence (b_ν) in V, $b_\nu \neq a$, and λ_ν in $\mathbb{C}$ such that $\operatorname*{Lim}_{\nu\to\infty} \lambda_\nu (b_\nu - a) = v\}$.

Remark 1. It follows trivially that if a is a simble point of V, $C(V, a) = T(V, a)$, i.e. the tangent space to V at a.

Definition 2. With the above notation, we define $C^*(V, a) = K[C(V, a) - 0] \subset \mathbb{P}^{n-1}$.

(3) If $v_1, \ldots, v_r$ are vectors in $\mathbb{C}^n$ we write dep. $(v_1, \ldots, v_r)$ when $v_1, \ldots, v_r$ are $\mathbb{C}$-linearly dependent.

(4) If $a \in \mathbb{C}^n$, we define $\tilde{\mathbb{C}}^n_a = \{(z,v) \mid z \in \mathbb{C}^n, v \in \mathbb{P}^{n-1}$ and if $K(\omega) = v$, dep. $(\omega, z-a)\}$. Clearly $\mathbb{C}^n \times \mathbb{P}^{n-1} \supset \tilde{\mathbb{C}}^n_a \supset \{a\} \times \mathbb{P}^{n-1}$ and $\Pi_1 : (\tilde{\mathbb{C}}^n_a - \{a\} \times \mathbb{P}^{n-1}) \to \mathbb{C}^n$ is injective. Also $\tilde{\mathbb{C}}^n_a = \mathrm{clos}(\tilde{\mathbb{C}}^n_a - \{a\} \times \mathbb{P}^{n-1})$.

32 **Remark 2.** $\widetilde{\mathbb{C}}^n_a$ is an analytic manifold of dimension n and $\{a\} \times \mathbb{P}^{n-1}$ is a submanifold of dimension $n-1$ of $\widetilde{\mathbb{C}}^n_a$.

Proof. If $(\omega_1, \ldots, \omega_n)$ denote homogeneous coordinates on $\mathbb{P}^{n-1}$, $\widetilde{\mathbb{C}}^n_a = \{(z,\omega) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid z_i\omega_j = \omega_i z_j\}$. Let $(z^0, \omega^0) \in \widetilde{\mathbb{C}}^n_a$. We may assume $\omega^0 = (\omega^0_1, \ldots, \omega^0_n)$ where $\omega^0_1 \neq 0$, Choose a neighbourhood U of (z^0, ω^0) such that if $(z,\omega) \in U$, then $\omega_1 \neq 0$. Then for any $(z_1, \omega) \in U \cap \widetilde{\mathbb{C}}^n_a$, we have

$$z_1 \frac{\omega_j}{\omega_1} = z_j,\ j \geq 2, \quad \text{i.e. } U \cap \widetilde{\mathbb{C}}^n_a = \{(z_1, \omega) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid z_j = \frac{\omega_j}{\omega_1} \cdot z_1,\ j \geq 2\},$$

and $(z_1, \frac{\omega_2}{\omega_1}, \ldots, \frac{\omega_n}{\omega_1})$ give the local coordinages in $U \cap \widetilde{\mathbb{C}}^n_a$ and this proves the remark. □

Definition 5. If V is an analytic set and $a \in V$, we define $V^{**}_a = \widetilde{\mathbb{C}}^n_a \cap (V \times \mathbb{P}^{n-1})$.

(6) V^*_a = closure of $[V^{**}_a - \{a\} \times \mathbb{P}^{n-1}]$ in $\mathbb{C}^n \times \mathbb{P}^{n-1}$.

Remark 3. Since $\{a\} \times \mathbb{P}^{n-1} \cap V^{**}_a$ is an analytic set, it follows from Proposition 4, Chapter 1 that V^*_a is an analytic set.

Remark 4. $\Pi_1 : (V^{**}_a - \{a\} \times \mathbb{P}^{n-1}) \to V$ is injective and $\dim_{(a,v)} V^*_a = \dim_a V$, where $(a,v) \in V^*_a$ This is obvious since

$$V^{**}_a - \{a\} \times \mathbb{P}^{n-1} = \{(z, K(z,a)) \mid z \neq a, z \in V\}.$$

Proposition 1. $V^*_a \cap \{a\} \times \mathbb{P}^{n-1} = a \times C^*(V,a)$.

Proof. Let $v \in C^*(V,a)$ and $v = K(\omega)$. Then there is a sequence $\{z_\nu\}$ in V, $z_\nu \neq a$ and $z_\nu \to a$ and a sequence $\{\lambda_\nu\}$ in $\mathbb{C}$ such that $\lambda_\nu(z_\nu - a) \to \omega$.
33 Consider the sequence $(z_\nu, K(z_\nu - a))$ in V^*_a. Obviously $(z_\nu, K(z_\nu - a)) \to (a,v)$. Conversels if $(z_\nu, v_\nu) \to (a,v).v_\nu = K(\omega_\nu)$, then dep $(z_\nu - a, \omega_\nu)$ and hence we have a sequence $\{\lambda_\nu\}$ in $\mathbb{C}$ such that $\lambda_\nu(z_\nu - a) \to \omega$, where $K(\omega) = v$. □

Proposition 2. *$C^*(V, a)$ is an analytic set and* $\dim_a V = \dim \Omega^*(V, a) + 1$.

Proof. Since the problem is local, we may assume $\dim .V = \dim_a V = p$. Then by Remark 4 above, $\dim V_a^* = p$ and $\dim C^*(V, a) \leq p - 1$, by Proposition 4 of Chapter 1. Also $\dim\{a\} \times \mathbb{P}^{n-1} = n - 1$ and hence it follows from Proposition 3 of Chapter 1 that $\dim C^*(V, a) \geq p + (n-1) - n = p - 1$, i.e. $\dim .C^*(V, a) = p - 1$ and this proves the Proposition.

In fact we shall use the following theorem and prove that $C^*(V, a)$ is an algebraic variety in $\mathbb{P}^{n-1}$. [See [2] for a proof of the following theorem.] □

Theorem (Remmert-Stein). *If $\Omega \subset \mathbb{C}^n$ is an open set and if $A \subset \Omega$ is an analytic set,* $\dim A \leq k - 1$ *and if $B \subset \Omega - A$ is an analytic set of constant dimension k, then $\overline{B}$ is an analytic set in Ω and* $\dim \overline{B} = k$.

Theorem (Chow). *Any analytic set in $\mathbb{P}^{n-1}$ is an algebraic set.*

Proof. Let $\Pi : \mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}$ be the natural map. Then if V is an analytic set in $\mathbb{P}^{n-1}$, $\dim V \geq 0$, then $W = \Pi^1(V)$ is an analytic set in $\mathbb{C}^n - \{0\}$ and $\dim W = \dim V + 1 > 0$. Hence by the theorem of Remmert and Stein stated above, $\overline{W}$ is analytic in $\mathbb{C}^n$. Obviously $0 \in \overline{W}$. Let U be a convex neighbourhood of 0 and $f^1, \ldots, f^k$ be homomorphic functions on U such that $U \cap \overline{W} = \{z \in U \mid f^i(z) = 0, 1 \leq i \leq k\}$. Let $f^i(z) = \sum_{r=1}^{\infty} P_r^i(z)$, where $P_r^i(z)$ is a homogeneous polynomial of degree 34
r. Since $z \in \overline{W} \Rightarrow \lambda z \in \overline{W}$, we have,

$$\begin{aligned}
U \cap \overline{W} &= \left\{z \in U \mid \lambda z \in U \cap \overline{W}, \mid \lambda \mid \leq 1\right\} \\
&= \left\{z \in U \mid \sum_{r=1}^{\infty} \lambda^r P_r^i(z) = 0, 1 \leq i \leq k, |\lambda| \leq 1\right\}. \\
&= \left\{z \in U \mid P_r^i(z) = 0, 1 \leq i \leq k, r \geq 1\right\}.
\end{aligned}$$

Now by Hilbert's basis theorem, there exist a finite number of polynomials, $P_1, \ldots, P_m$ among $\left\{P_r^i\right\}$, $1 \leq r < \infty$, $1 \leq i \leq k$, such that

$$\left\{z \in U \mid P_r^i(z) = 0, 1 \leq i \leq k, r \geq 1\right\}$$

$$= \left\{z \in U \mid P_j(z) = 0, 1 \leq j \leq m\right\}.$$

Thus $U \cap \overline{W}$ is the set of zeros of a finite number of homogeneous polynomials. Hence $\Pi(W) = V$ is an algebraic set in $\mathbb{P}^{n-1}$. □

Corollary. *In particular, $C^*(V, a)$ is an algebraic set in $\mathbb{P}^{n-1}$.*

Definition 7. If f is a holomorphic function in a neighbourhood of $a \in \mathbb{C}^n$, we have the series

$$f(a + z) = f^0 + f^1(z) + f^2(z) + \cdots$$

where $f^i(z)$ is a homogeneous polynomial of degree j in $z_1, \ldots, z_n$. if m is the smallest number such that $f^m(z) \equiv 0$, then f is said to have order m at a and for any such f, we define $f_a^*(z) = f^m(z)$ where m is the order of f at a.

Remark 5. In fact Whitney [6] has proved that if $a \in V$. V being an
35 analytic set in $\mathbb{C}^n$ and if I_a is the ideal of holomorphic germs vanishing on V_a, then there is a neighbourhood U of a such that

$$\left\{z \in U \mid f_a^*(f) = 0 \ \text{ for } \ f \in f_a\right\} = C(V, a) \cap U.$$

2 Wings

Definition 8. Let V be an analytic set, M, a manifold, $M \subset V$. Let $W \subset V$ be an analytic set with $\dim W < \dim V$ and U, an open set in M and l, a positive real number. Let $\widetilde{Z} = U \times [0, 1[Z = U \times]0, 1[$. Then we define a wing stretching from U into $V - W$ to be a set $B \subset V$ and a homeomorphism F of $\widetilde{Z}$ onto B for same $1 > 0$, where F satisfies the following conditions.

(1) For every λ, $0 \leq \lambda < l$, $F_\lambda(z) = F(z, \lambda)$ is a biholomorphic map from U onto $F_\lambda(U)$.

(2) F is differentiable in λ and $\dfrac{\partial F}{\partial \lambda}$ is continuous in z.

(3) If $z_1, \ldots, z_m$ are local coordinates in U, $z_j = x_j + iy_j$, where x_j and y_j are real, then the vectors $\dfrac{\partial F}{\partial x_1}, \dfrac{\partial F}{\partial y_1}, \ldots, \dfrac{\partial F}{\partial x_n}, \dfrac{\partial F}{\partial y_n}, \dfrac{\partial F}{\partial \lambda}$ in Z are linearly independent over $\mathbb{R}$.

(4) $F_0 \mid U$ = identity map and $F_\lambda(U) \subset V - W$ for $\lambda > 0$.

Remark 6. If $\widetilde{F} : U \times [0, 1[\to V$ defines a wing and if $(z_1, \ldots, z_m)$ are local coordinates in U, $\dfrac{\partial F}{\partial z_k}$ is continuous in $\widetilde{Z}$, for $0 \leq \lambda < l$.

Proof. We have only to check the continuity at points on $U \times \{0\}$. Since $\widetilde{F}$ is continuous on $U \times [0, 1[$, it is uniformly continuous on $U' \times [0, \delta]$, where $\bar{U}' \subset U$ and $0 < \delta < l$, i.e. $F_\lambda \to F_0$ uniformly on U' as $\lambda \to 0$. Hence by Weierstrass' theorem it follows that $\dfrac{\partial F_\lambda}{\partial z_k} \to \dfrac{\partial F_0}{\partial z_k}$ for $1 \leq k \leq n$. □

Remark 7. Let $z_i \in F_{\lambda_i}(U)$ and $z_i \to z \in F_0(U)$. Then $T(F_{\lambda_i}(U), z_i) \to$ **36**
$T(F_0(U), a)$.

Proof. It follows from conditions (1) and (4) in the definition of a wing, that

$$T(F_{\lambda_i}(U), z_i) = dF_{\lambda_j}[T(U, z_i')]$$

where $F_{\lambda_i}(z_i') = z_i$.

From the Remark 6 above, it follows that dF_λ is continuous on $[0, l[$ and hence follows the proof. □

Lemma 1. *Let V be an analytic set, $0 \in W \subset V$, W being an analytic subset of V, such that W_0, V_0 are irreducible and* $\dim_0 W = m < \dim_0 V = r$. *Then there exists a neighbourhood U of* 0 *and a basis* $(z_1, \ldots, z_n)$ *in U such that the basis is proper for V_0 as well W_0.*

Proof. Recalling Proposition 3 of Chapter 1, we have only to find a basis $(z_1, \ldots, z_m, \ldots, z_r, \ldots, z_n)$ in a neighbourhood U of 0 such that

$$(1) \qquad \begin{cases} \{z \in U \mid z_1 = 0, \ldots, z_m = 0\} \cap W = \{0\} \\ \text{and} \quad \{z \in U \mid z_1 = 0, \ldots, z_r = 0\} \cap V = \{0\}. \end{cases}$$

Let $a^0 \in V$, $b^0 \in W$, $a^0 \neq 0$, $b^0 \neq 0$ and a^0, b^0 simple points of V and W respectively. Choose a linear form $l_1(z)(= \sum_{j=1}^n \lambda_i z_j)$ such that $l_1(a^0) \neq 0$, $l_1(b^0) \neq 0$. Then by a holomorphic change of coordinates, we may suppose $l_1(z) = z_1$ and we have for some neighbourhood U_1 of
37 0, $W_1 = \{z \in U_1 \mid z_1 = 0\} \cap W$ is an analytic set of dimension $m - 1$ and $V_1 = \{z \in U_1 \mid z_1 = 0\} \cap V\}$ is an analytic set of dimension $n - 1$. Let $W_1 = \bigcup_\alpha W^1_\alpha$, $V_1 = \bigcup_\beta V^1_\beta$, W^1_α and V^1_β being irreducible components of W_1 and V_1 respectively. Choose a^1_α, b^1_β, simple points of W^1_α, V^1_β respectively and a linear form $l_2(z)$ such that $l_2(a^1_\alpha) \neq 0$, $l_2(b^1_\beta) \neq 0$ for all α and β and z_1 and $l_2(z)$ are linearly independent. By a change of coordinates let $l_2(z) = z_2$ and then there exists a neighbourhood $U_2 \subset U_1$ of 0 such that $W_2 = \{z \in U_2 \mid z_1 = z_2 = 0\} \cap W$ is an analytic set of dimension $m - 2$ and $\{z \in U_2 \mid z_1 = z_2 = 0\} \cap V$ is an analytic set of dimension $r - 2$. Proceeding this way, we finally have a basis $(z_1, \ldots, z_n)$ in a neighbourhood U of 0 such that conditions (1) are satisfied. $\square$

Remark 8. In the above lemma, if 0 is a simple point of W, then there exists a basis $(z_i, \ldots, z_n)$ in a neighbourhood U of 0 such that the basis is proper for V_0 and

$$U \cap W = \{z \in U \mid z_{m+1} = \ldots = z_n = 0\}.$$

Lemma 2. *Let $0 \in M \subset V$, where M is a manifold of dimension m, V is an analytic set with the germ V_0 irreducible and $\dim_0 V = r > m$. Let $W \subset V$ be an analytic set with $\dim W < r$. Then there exists a neighbourhood U of 0, an analytic set $V' \subset V$ in U such that*

(1) $U \cap M \subset U \cap V'$

(2) $\dim V' = m + 1$

(3) $\dim V' \cap W \leq m$.

38 *Proof.* By Remark 8, there is a neighbourhood U and coordinates $(z_1, \ldots, z_n)$ which are proper for V_0 and all irreducible components of W_0 and $M \cap U = \{z \in U \mid z_{m+1} = \ldots = z_n = 0\}$. Let Π_r denote the projection $(z_1, \ldots, z_n) \to (z_1, \ldots, z_r)$. Let $z_0 \in \Pi_r(U)$ be such that $z_0 \notin$

M, $z_0 \notin \Pi_r(W)$. (This is possible since $\dim W < r$, $m < r$). Let N be the $(m+1)$-plane spanned by M and the complex line defined by 0 and z_0. Then $\Pi_r^{-1}(N) \cap V$ is an analytic set. Also since $\Pi_r^{-1}(a) \cap V$ is finite for every a in $\Pi_r(U)$, $\dim \Pi_r^{-1}(N) \cap V \leq m+1$. Hence if $V' = \Pi_r^{-1}(N) \cap V$, $\dim V' = m+1$ and clearly V' satifies the conditions (1), (2) and (3) of the lemma. □

Proposition 3. *Let $a \in M \subset V$, M being a manifold of dimension m and V, an analytic set with V_a irreducible and* $\dim_a V = r > m$. *Let $W \subset V$ be an analytic set,* $\dim W < r$. *Then for any neighbourhood Ω of a in M, there is an open set $U \subset \Omega \subset M$ (U note necessarily a neighbourhood of a) and a wing stretching from U into $V - W$.*

Proof. For simplicity, we may assume a to be 0, By Remark 8 and Lemma 2 above, there is neighborhood U_1^n of 0 and coordinates $z_1, \ldots,$ z_n in U, which are proper for V_0 such that $M \cap U_1^n = \{z \in U_1^n \mid z_{m+1} = \ldots = z_n = 0\}$, and an analytic set V' in U_1^n such that

(1) $\dim V' = m+1$,

(2) $U_1^n \cap M \subset U_1^n \cap V' \subset U_1^n \cap V$ and

(3) $\dim V' \cap W \leq m$.

We shall prove that there is a wing stretching from an open set in Ω into $V' - W$.

We assume that V_0' is irreducible and that the basis $(z_1, \ldots, z_n)$ is proper for V_0' and satisfies the condition of Remark 8. Then $U_1^n \cap M$ is the analytic set given by $\{z \in U_1^n \mid z_{m-1} = \ldots = z_n = 0\}$. Let I be 39
the ideal of germs at 0 of holomorphic functions vanishing on V_0' and let $\eta : \theta^n \to \theta^n / I$ be the natural projection. Then with the notation of Theorem 5 of Chapter 1, there exists a distinguished polynomial $P_{m+2}[x]$ in $\theta^{m+1}[x]$ such that P_{m+2} is the minimal polynomial of $\eta(z_{m+2})$ over θ^{m+1}, $\eta(z_{m+2})$ generating the quotient field of θ^n/I over the quotient field of θ^{m+1}. Let δ be the discriminant of P_{m+2}. Let C in $\Pi_{m-1}(U_1^n)$ be the analytic set given by

$$C = \left\{ z \in \Pi_{m+1}(U_1^n) \mid \delta(z) = 0 \quad \text{or} \quad z \in \Pi_{m+1}(V' \cap W \cap U_1^n) \right\}.$$

Then dimension of $C = m$ and if $D = (\overline{C - M}) \cap M$, by Proposition 4 of Chapter 1, $\dim D < m$. Hence given an open set $\Omega < M$, there is an open set $U_1^m \subset \Omega$ such that $U_1^m \cap D = \phi$, i.e. $(U_1^m \times \{0\}) \cap (\overline{C - M}) = \phi$. Hence there is an open set U_1^1 in $\mathbb{C}$, $0 \in U_1^1$, such that $(U_1^m \times U_1^1) \cap (\overline{C - M}) = \phi$. This implies that (i) if $(z_1, \ldots, z_{m+1}) \in U_1^m \times U_1^1$ and $z^{m+1} \neq O$, then $\delta(z_1, \ldots, z_{m+1}) \neq 0$ and $(z_1^0, \ldots, z_{m+1}) \notin (\overline{C - M})$.

Let $\widetilde{z}^0 = (z_1^0, \ldots, z_m^0) \in U_1^m$. Let $z_0 = (z_1^0, \ldots, z_m^0, O, \ldots, O) \in \mathbb{C}^n$. By Proposition 1 of Chapter 1, $V'_{z_0} = \bigcup_{i=1}^k V_{z_0}^i$ where $V_{z_0}^i$ are irreducible germs of analytic sets and $V_{z_0}^i \not\subset \bigcup_{j \neq i} V_{z_0}^j$ for any i. We assume that $V_{z_0}^i$ are germs of analytic sets, defined by analytic sets V^i in a neighbourhood U_2^n of z_0, $\Pi_{m=1}(\Pi^n) \subset \Pi_1^m \times \Pi_1^1$ and that $\Pi_{m+1}(V' \cap U_2^n) = \Pi_{m+1}(\Pi_2^n)$. Now z_0 is an isolated point of $U_2^n \cap V^1 \cap \Pi_{m+1}^{-1}(\widetilde{z}^0, O)$. Hence there is an open set Π_1^{n-m-1} in $\mathbb{C}^{n-m-1}$, $0 \in U_1^{n-m-1}$, such that $(U_1^m \times U_1^1 \times \supset$
40 $\Pi_1^{n-m-1}) \cap V^1 \cap \Pi_{m+1}^{-1}(\widetilde{z}^0, 0)$ is empty and hence there is an open set $U_2^m \times \Pi_2^1 = U_1^m \times U_1$ such that

(i) $(\widetilde{z}^0, 0) \in U_2^m \times U_2^1$ and $\Pi_{m+1}^{-1}(z) \cap (U_2^m \times U_2^1 \times \partial U_1^{n-m-1}) \cap V^1 = \phi$ if $z \in U_2^m \times U_2^1$,

(ii) Π_{m+1}; $(U_2^m \times U_2^1 \times U_1^{n-m-1}) \cap V^1 \to U_2^m \times U_2^1$ is surjective. It follows that $\Pi_{m+1} : (U_2^m \times U_2^1 \times U_1^{n-m-1}) \cap V^1 \to U_2^m \times U_2^1$ is proper and surjective.

Let

$$X = (U_2^m \times U_2^1 \times U_1^{n-m-1}) \cap V^1 \quad \text{and}$$

$$U_2^m \times U_2^1 = U^{m+1} = \left\{ z \in \mathbb{C}^{m+1} \middle| \left| z_i - z_i^0 \right| < \rho_i, 1 \leq i \leq m, \left| z_{m+1} \right| < \rho_{m+1} \right\}.$$

Since

$$z \in (U^{m+1} - M) \Rightarrow \delta(z) \neq O, \Pi_{m+1} : \left[X - \Pi_{m+1}^{-1} \left(M \cap U^{m+1} \right) \right] \to (U^{m+1} - M)$$

is a covering of p sheets say. Moreover, since $V_{z^0}^1$ is irreducible, we may assume that $\left[X - \Pi_{m+1}^{-1} \left(M \cap U^{m+1} \right) \right]$ is connected.

Let

$$Y_0 = \left\{ z \in \mathbb{C}^{m+1} \middle| \left| z_i - z_i^0 \right| < \rho_i, 1 \leq i \leq m, 0 < \left| z_{m+1} \right| < \rho_{m+1}^{1/p} \right\}$$

and $\quad Y = \left\{ z \in \mathbb{C}^{m+1} \middle| \left| z_i - z_i^0 \right| < \rho_i, 1 \leq i \leq m, \left| z_{m+1} \right| < \rho_{m+1}^{1/p} \right\},$

and consider the covering $(\Pi' \big| Y_0) : Y_0 \to (U^{m+1} - M)$, where $\Pi' : Y \to U^{m+1}$ is given by

$$\Pi'(z_1, \ldots, z_{m+1}) = (z_1, \ldots, z_m, z_{m+1}^p).$$

Then there is a map $f_0 : Y_0 \to X - \Pi_{m+1}^{-1}(U^{m+1} \cap M)$ such that $\Pi' = \Pi_{m+1} \circ f_0$ on Y_0. By Riemann's extension theorem, $f_0 : Y_0 \to U^{m+1} \times U_1^{n-m-1}$ can be extended to a holomorphic function on Y, the extention being denoted by f, and since X is closed in $U^{m+1} \times U_1^{n-m-1}$ and Y_0 is dense in Y, it follows that $f(Y) \subset X$ and $\Pi' = \Pi_{m+1} \circ f$ on Y. Also, since Π' and $\Pi_{m+1}|X$ are proper, f is proper and $f(Y) = X$. this **41**
implies that

$$X \cap \Pi_{m+1}^{-1}(z_1, \ldots, z_m, 0) = (z_1, \ldots, z_m, 0, \ldots, 0) = f(z_1, \ldots, z_m, 0) \text{ in } X.$$

Now consider $U_2^m \times [0, \delta_{m+1})$ and let $g : U_2^m \times [0, \delta_{m+1}) \to Y$ be given by $g(z_1, \ldots, z_m, \lambda) = (z_1, \ldots, z_m, \lambda^{1/p})$ where $\lambda^{1/p}$ is the positive p^{th} root of λ_0 for $\lambda > 0$.

Let $\widetilde{Z} = U_2^m \times (0, \delta_{m+1})$ and $Z = U_2^m \times (0, \delta_{m+1})$ and $\widetilde{F} \cdot \widetilde{Z} \to V'$ be defined by $\widetilde{F} = f \circ g$. Then we claim that $\widetilde{F}$ defines the wing with the required properties. It is obvious that $\widetilde{F}$ is a homeomorphism and that $\widetilde{F}(z_1, \ldots, z_m, 0) = (z_1, \ldots, z_m, 0, \ldots, 0)$. Also for every $\lambda \geq 0 \widetilde{F}_\lambda : U_2^m \to F_\lambda(U_2^m)$ is biholomorphic. In fact, $\widetilde{F}$ is analytic in λ on Z and hence $\dfrac{\partial F}{\partial \lambda}$ is continuous on Z. Also $\Pi_{m+1}(F(z, \lambda)) = (z, \lambda)$, hence condition 3 in the definition of wing is trivially verified. Also because of (1), for $\lambda > 0$, $F_\lambda(U_2^m) \subset V' - W$. $\square$

Remark 9. If the open neighbourhood $\Omega \subset M$ of a contains a simple point of $\overline{(V' - W)}$, the proposition is trivial.

Remark 10. In fact the wing that we obtained in Proposition 3 stretches into $\left\{ z \in V' \mid z \text{ is a simple point of } V' \text{ and } z \notin W \right\}$ i.e. $F_\lambda(U_2^m) \subset \{z \in V' \mid z \text{ is a simple point of } V' \text{ and } z \notin W\}$ for $\lambda > 0$.

3 The singular set S_a

Let Ω be an open set in $\mathbb{C}^n$ and $V \subset \Omega$ be an irreducible analytic set of dimension r. Let $W \subset V$ be an irreducible analytic set and $\dim W = m < r$. We shall prove that there is an analytic set $S_a \subsetneqq W$ such that for every simple point z of W with $z \notin S_a$, the pair (W, V) is (a) regular at z.

42 In what follows G will denote the Grassmann manifold $G_{n,r}$, G' will denote the Grassmann manifold $G_{n,m}$ and $\overset{\circ}{V}$, $\overset{\circ}{W}$ will denote the sets of simple points of V and W respectively and $\dot{V}$, $\dot{W}$, the sets of singular points of V and W respectively. If $\alpha \in G$, $T(\alpha)$ will be the r-plane corresponding to α. Consider $C^*(\overset{\circ}{V}) = \{(z, \alpha) \mid z \in \overset{\circ}{V}, T(\alpha) = T(V, z)\}$. Clearly $C^*(\overset{\circ}{V}) \subset \overset{\circ}{V} \times G$ is an analytic set. Let $C^*(V) =$ closure of $C^*(\overset{\circ}{V})$ in $\Omega \times G$. For $z \in V$, we define $C^*(V, z)$ as follows.

$$z \times C^*(V, z) = C^*(V) \cap \{z\} \times G.$$

Proposition 4. *$C^*(V)$ is an analytic set in $\Omega \times G$ and $C^*(V, z)$ is an analytic subset of G.*

Proof. Let $z \in V$. By Lemma 5 of Chapter 2 there exists a neighbourhood $U \subset \mathbb{C}^n$ of z and holomorphic vector fields $v^1, \ldots, v^q$ on U such that $v^i(z) = 0$, $1 \leq i \leq q$, for $z \in \dot{V} \cap U$ and $\{v^i(z)\}$, $1 \leq i \leq q$ generate $T(V, z)$ if $z \in \overset{\circ}{V} \cap U$.

Now for any $\alpha \in G$, the r-plane $T(\alpha)$ defines upto a complex non-zero factor, an r-vector $\widehat{\alpha}$ in the exterior algebra of $\mathbb{C}^n$. Moreover there exists a neighbourhood U' of α such that the co-ordinates of $\widehat{\alpha}$ are holomorphic on U'. For any vector v, if $\widehat{\alpha} = (\alpha_1, \ldots, \alpha_r)$, we define $v \wedge \widehat{\alpha} = v \wedge \alpha_1 \wedge \ldots \wedge \alpha_r$ and the equation $v \wedge \widehat{\alpha} = 0$ is independent of the choice of holomorphic coordinates of $\widehat{\alpha}$. Hence if we define

$$C_U^{**}(V) = \left\{(z, \alpha) \mid z \in V \cap U, \alpha \in G, v^i(z) \wedge \widehat{\alpha} = 0, 1 \leq i \leq q\right\}$$

43 where $\widehat{\alpha}$ has the above meaning, $C_U^{**}(V)$ is an analytic set in $U \times G$. Further, $v^i(z) \wedge \alpha = 0$, $1 \leq i \leq q$, if and only if all the vectors $v^i(z) \in T(\alpha)$. Since $\{v^i(z)\}$, $1 \leq i \leq q$ span $T(V, z)$ if $z \in \overset{\circ}{V}$ and $\dim T(V, z) =$

$\dim T(\alpha) = r$, we have, for $z \in \overset{\circ}{V}$, $(z, \alpha) \in C^{**}_U(V)$ if and only if $T(\alpha) = T(V, z)$. It follows from Proposition 4 of Chapter 1 that $U \cap C^*(V) =$ closure of $[C^{**}_U(V) - \dot{V} \times G]$ in $U \times G$ is an analytic set and further,

$$\begin{aligned} &\dim C^*(V) = \dim V = r \\ \text{and}\quad &\dim C^*(V) \cap (\dot{V} \times G) \leq r - 1. \end{aligned}$$

It follows that $C^*(V, z)$ for any $z \in V$ is an analytic set in G. □

Lemma 3. *If z is a simple point of W, then the following are equivalent*

(1) $\alpha \in C^*(V, z) \Rightarrow T(\alpha) \supset T(\overset{\circ}{W}, z)$

(2) (W, V) *is (a) regular at z.*

Proof. The proof is trivial. We assume that condition (1) holds. If $q_i \in \overset{\circ}{V}$, $q_i \to z$ and if $T(\overset{\circ}{V}, q_i) \to T$, then clearly $T^* \in C^*(V, z)$ where T^* is the element in G, corresponding to T. It follows from (1) that $T \supset T(\overset{\circ}{W}, z)$, i.e. (W, V) is (a) regular at z. Conversely, if we assume that (W, V) is (a) regular at z and if $\alpha \in C^*(V, z)$, then there is a sequence $\{q_i\}$ in $\overset{\circ}{V}$, $q_i \to z$ and $T^*(\overset{\circ}{V}, q_i) \to \alpha$. Then $T(\alpha) \supset T(\overset{\circ}{W}, z)$ and the condition (1) is satisfied.

Consider the set C^* in $\Omega \times G \times G'$, given by

$$C^* = \Big\{(z, \alpha, \alpha') \mid z \in W, \alpha \in C^*(V, z), \alpha' \in T^*(W, z)\Big\}.$$

Then if

$$\begin{aligned} A &= \Big\{(z, \alpha, \alpha') \mid z \in W, \alpha \in C^*(V, z), \alpha' \in G'\Big\} \\ \text{and}\quad B &= \Big\{(z, \alpha, \alpha') \mid z \in W, \alpha \in G, \alpha' \in C^*(W, z)\Big\}, \end{aligned}$$

it follows from Proposition 1 above, that A and B and hence $C^* = A \cap B$ 44
are analytic sets. Let

$$R^* = \Big\{(z, \alpha, \alpha') \mid z \in W, \alpha \in C^*(V, z), \alpha' \in C^*(W, z), T(\alpha') \subset T(\alpha)\Big\}.$$

Then R^* is an analytic set and it follows from Proposition 4 of Chapter 1 that S_a^* = closure of $(C^* - R^*)$ in $\Omega \times G \times G'$ is an analytic set. Let $\Pi_1 : \Omega \times G \times G' \to \Omega$ be the projection $\Pi_1(z, \alpha, \alpha') = z$, and let $\Pi_1(S_a^*) = S_a$. We shall prove in the following two propositions that S_a is an analytic set with

(1) $\dim S_a < \dim W$ and

(2) if $z \in (\mathring{W} - S_a)$, (W, V) is (a) regular at z.

$\square$

Proposition 5. *With the above definition of S_a, if $z \in (\mathring{W} - S_a)$, (W, V) is (a) regular at z.*

Proof. If $z \in (\mathring{W} - S_a)$ and if $\alpha \in C^*(V, z)$, then $(z.\alpha, \alpha') \in C^*$ where $T(\alpha') = T(\mathring{W}, z)$. Since $z \notin S_a$, $(z, \alpha, \alpha') \notin S_a^*$ and hence $(z, \alpha, \alpha') \in R^*$, i.e. $T(\alpha') = T(\mathring{W}, z) \subset T(\alpha)$. The Proposition now follows from Lemma 1 above. $\square$

To prove the next proposition, we shall use the following

Theorem (Remmert). *If V is an analytic space and $f \cdot V \to \Omega' \subset \mathbb{C}^m$ is a holomorphic, proper map, then*

(1) *$f(V)$ is an analytic set in Ω'*

(2) $\dim f(V) = \max\limits_{\substack{z \text{ simple} \\ \text{point of } V}}$ *(rank (df) (z)).*

45 **Proposition 6.** *S_a is an analytic set and* $\dim S_a < \dim W = m$.

Proof. Since G and G' are compact, $\Pi_1 : \Omega \times G \times G' \to \Omega$ is proper and hence $S_a = \Pi_1(S_a^*)$ is an analytic set by (1) of the theorem above. Also if $\dim S_a = m$, it follows from (2) of the same theorem that there exists a simple point z_0^{**} of S_a^* such that rank $(d\Pi_1)(z_0^{**}) = m$ and hence by the constant rank theorem stated in Chapter 1, there is neighbourhood U^{**} of z_0^{**} such that $U^{**} = U_0 \times U \times U'$ and $\Pi_1(U^{**} \cap S_a^*)$ is a submanifold of dimension m, i.e. if $z_0^{**} = (z_0, \alpha_0, \alpha'_0)$, z_0 is a simple point of S_a, of dimension m. Since z_0 is a simple point of W, we may assume that $\Pi_2 : U^{**} \cap C^*$ is an isomorphism onto $\Pi_2(U^* \cap C^*) = \widehat{C}$ where $\Pi_2 : \Omega \times G \times G' \to \Omega \times G$ is the projection $\Pi_2(z, \alpha, \alpha') = (z, \alpha)$. Let $\Pi_2(U^{**} \cap S_a^*) = \widetilde{S}_a$.

(1) Since $z_0^{**} \notin R^*$, there is a vector $v_0 \in T(\overset{\circ}{W}, z_0)$ such that $v_0 \notin T(\alpha_0)$. Let for simplicity $z_0 = 0$.

With a suitable change of coordinates we can assume that $S_a \cap U_0 = \{(z_1, \ldots, z_n) \mid z_{m+1} = \ldots = z_n = O\}$ and $V_0 = \dfrac{\partial}{\partial z_1}$. Consider the analytic set $L^* = \{(z, \alpha) \mid \alpha = \alpha_0, z_2 = \ldots = z_n = 0\}$ in $U_0 \times U$. This is of dimension 1 and rank $(d\Pi_1)(z_0^*) = 1$. It follows from (1) and from the constant rank theorem that there is a neighbourhood $U_2^* = U_1 \times U_2$ of z_0^*, U_1 and U_2 being neighbourhoods of 0 and α_0 respectively such that

(2) $(\Pi_1 \mid (U_2^* \cap L^*) : L^* \to U_1 \cap L$ is an analytic isomorphism and

(3) if $z^* \in L^* \cap U_2^*$ and $z^* = (z, \alpha)$, we have

$$T(L, z) \not\subset T(\alpha).$$

Now $U_2^* \cap L^* \subset C^*(V)$ and **46**

$$\dim(\dot{V} \times G) \cap C^*(V) < \dim C^*(V) = r.$$

Hence by Proposition 3 of §2, there is an open set $U_3^* \subset U_2^* \cap L^*$ and a wing B^* defined by $F^* : U_3^* \times [O, \delta) \to C^*(V)$ such that $F_\lambda^*(t) \in (C^*(V) - \dot{V} \times G)$ for $\lambda > O$. Let $\Pi_1(U_3^*) = U_3$. Define $F : U_3 \times (0, \delta) \to V$ by $F = \Pi_1 \circ F \circ \Pi_1^{-1}$ (since by (2), $\Pi_1 : U_2^* \cap L^*$ is an analytic isomorphism). Since, for $\lambda > 0$, $F_\lambda^*(U_3^*) \subset (C^*(V) - \dot{V} \times G)$, $F_\lambda : U_3 \to F_\lambda(U_3)$ is an analytic isomorphism for $\lambda > 0$ and it is easy to varify that $B = F(U_3 \times [0, \delta))$ is a wing which is homeomorphice with B^*. Set $B_\lambda = F_\lambda U_3$. Choose a sequence q_i in B_{λ_i} such that $q_i \to p$ in $U_3 = B_0$. Then by remark 7 of §2, $T(B_{\lambda_i}, q_i) \to T(B_0, p) = T(L, p)$. Let

$q_i = F(t_i, \lambda_i)$ and let $q_i^* = (q_i, \alpha_i) = F^*(t_i, \lambda_i)$. Then $q_i^* \to (p, \alpha) \in U_3^*$. Now $T(L, p) = \operatorname{Lim} T(B_{\lambda_i}, q_i)$ and $T(B_{\lambda_i}, q_i) \subset T(\alpha_i)$, since $q_i \in \overset{\circ}{V}$; hence $T(L, p) \subset T(\alpha)$, $(p, \alpha) \in U_3^*$. But this contradicts the condition (3) above and hence it follows that

$$\dim S_a < m.$$

From Proposition 2 and Proposition 3 follows the □

Theorem (a)(Whitney). *If V is an irreducible analytic set in an open set $\Omega \subset \mathbb{C}^n$ and if $W \underset{\neq}{\subset} V$ is an irreducible analytic subset, then there exists an analytic set $S_a \underset{\neq}{\subset} W$ such that for any $z \in \overset{\circ}{W} - S_a$, (W, V) is (a) regular at z.*

4 Theorem (b)

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Lemma 4. *Let $z_0 \in W \subset V, W, V$ being analytic sets such that W_{z_0} and V_{z_0} are irreducible and $\dim_{z_0} W = m < \dim_{z_0} V = r$. Then there exists a neighbourhood U of z_0 and an analytic set X of dimension 1 in U such that $z_0 \in X$ and*

$$U \cap (X - \{z_0\}) \subset U \cap (V - W).$$

Proof. Let for the sake of simplicity $z_0 = 0$. We have only to recall the proof of Lemma 1, §2. We have linear forms $l_1, \ldots, l_m$ and a neighbourhood U' of 0 such that $\{z \in U' \mid l_i(z) = 0, 1 \leq i \leq m\} \cap W$ is an analytic set of dimension 0 and $V' = \{z \in U' \mid l_i(z) = 0, 1 \leq i \leq m\} \cap V$ is an analytic set of dimension $r - m$. Let X be a one dimensional analytic subset of V', $0 \in X$. Then clearly there exists a neighbourhood U of 0 such that $U \subset U'$ and $U \cap [X - \{z_0\}) \subset V - W$.

In what follows, V is an irreducible analytic set of dimension r in Ω, Ω an open set in $\mathbb{C}^n$, W is an analytic subset of V. For any anlytic set A, $\overset{\circ}{A}$ is the set of simple points of A and $\dot{A}$ is the set of singular points of A. G will denote the Grassmann manifold of r planes in $\mathbb{C}^n$ and $\mathbb{P}^{n-1} = R$ will denote the complex projective space. Let $0 \in W \subset V$,

$\dim_0 W = m < r$. By Lemma 5 of Chapter 2, there is a neighbourhood U of 0 and holomorphic vector fields $v^1, \ldots, v^k$ in U such that $v^i(z) = 0$, $1 \leq i \leq k$, if $z \in U$ is a singular point of V and $(v^i(z))$ span $T(v, z)$ is z is a simple point of V. We now define an analytic set $C_0 \subset W \times V \times \mathbb{P} \times G$ as follows.

$$C_0 = \Big\{(\zeta, z, v^*, \alpha) \mid \zeta \in W, z \in V, v^* \in \mathbb{P}, \alpha \in G. \text{ if } K(v) = v^*, \\ \text{dep. } (z - \zeta, v) \text{ and } v^i(z) \wedge \alpha = 0, 1 \leq i \leq k\Big\}.$$

(For notation, see §1) **48**

Clearly C_0 is an analytic set. Let C^{**} = closure of $\Big[C_0 - (W \times \dot{V} \cup W \times W) \times \mathbb{P} \times G\Big]$ in $W \times V \times \mathbb{P} \times G$. By Proposition 4 of Chapter 1, C^{**} is an analytic set and $(\zeta, z, v^*, \alpha) \in C^{**}$ if and only if there are sequences $z_\nu \in \overset{\circ}{V}, z_\nu \notin W, \zeta_\nu \in W, \lambda_\nu \in \mathbb{C}$ such that $z_\nu \to z$, $\zeta_\nu \to \zeta$, $\lambda_\nu(\zeta_\nu - z_\nu) \to v$ where $K(V) = V^*$ and $T(V, \zeta_\nu) \to T(\alpha)$.

Let Δ be the diagonal in the set $W \times W$ and let $\tilde{C}^* = C^{**} \cap \Delta \times \mathbb{P} \times G$. If $\Pi_2 : W \times V \times \mathbb{P} \times G \to V \times \mathbb{P} \times G$ is the projection $\Pi_2(\zeta, z, v^*, \alpha) = (z, v^*, \alpha)$, let $C^* = \Pi_2 \tilde{C}^*$ Clearly C^* is an analytic set in $\Omega \times \mathbb{P} \times G$. Now, let $0 \in W \subset V$ and W_0 and V_0 be irreducible such that 0 is a simple point of W, $\dim_0 W = m < \dim_0 V = r$. Then we remark that we can choose a neighbourhood U of 0 and a basis $(z_1, \ldots, z_n)$ in U such that $W \cap U \Big\{z \in U \mid z_{m+1} = \ldots = z_n = 0\Big\}$ and moreover, if $\Pi_m : \mathbb{C}^n \to \mathbb{C}^m$ is the projection $\Pi_m(z_1, \ldots, z_n) = (z_1, \ldots, z_m)$ then $\Pi_m^{-1}(z) \cap V \not\subset \dot{V}$ for $z \in \Pi_m(U)$. We have only to choose a basis $(z_1, \ldots, z_n)$ such that $W \cap U = \Big\{z \in U \mid z_{m+1} = \ldots = z_n = 0\Big\}$ and $\Pi_m^{-1}(\cap) \cap V \not\subset \dot{V}$. (Since the set of simple points is open in V, by shrinking U if necessary, we then have, $\Pi_m^{-1}(z) \cap V \not\subset \dot{V}$ for $z \in \Pi_m(U)$). Such a choice of basis is possible since the set of simple points is dense in V. With respect to such a basis if $z^0 = (z_1^0, \ldots, z_m^0, 0, \ldots, 0) \in U \cap W = M$, M_{z^0} will denote **49**
the transverse plane at z^0, i.e $\{z \in U \mid z_i = z_i^0, 1 \leq i \leq m\}$. Let $\mathbb{P}'$ denote the projective space of $\mathbb{C}^{n-m} = \Big\{(z_{m+1}, \ldots, z_n) \mid (z_1, \ldots, z_n) \in \mathbb{C}^n\Big\}$. We

then define $\sigma_0 \subset (V \cap U) \times \mathbb{P}' \times G$ as follows

$$\sigma_0 = \left\{(z, K(z - \Pi_m z), \alpha) \mid z \in \overset{\circ}{V} \cap U, z \notin W, T(\alpha) = T(V, z)\right\}.$$

Let σ = closure of σ_0 in $U \times \mathbb{P}' \times G$.

In the proof of the Theorem (b) we shall use the sets and notations introduced above. □

Theorem Whitney. *If V is an analytic set, W its analytic subset, V, W being irreducible and* $\dim W = m < r = \dim V$, *then there exists an analytic subset S_b of W such that* $\dim S_b < \dim W$ *and if $z \in \overset{\circ}{W}$, $z \notin S_b$, then the pair (W, V) is (b) regular at z.*

Proof. Consider the analytic set C^* as defined above. Let $R^* \subset V \times \mathbb{P} \times G$ be the analytic set defined by $R^* = \left\{(z, v^*, \alpha) \mid z \in V, v^* \in \mathbb{P}, \alpha \in G, V^* \subset \alpha\right\}$. Here $v^* \subset \alpha$ means that if $v \in \mathbb{C}^n$ such that $K(v) = v^*$, $v \subset T(\alpha)$. Then by proposition 4 of Chapter 1, S_b^* = closure of $C^* - R^*$ in $V \times \mathbb{P} \times G$ is an analytic set. Let $\Pi : V \times \mathbb{P} \times G \to V$ be the projection $\Pi(z, v^*, \alpha) = z$. Then Π is proper and hence by Remmert's proper mapping theorem stated in §3, $\Pi(S_b^*) = S_b$ is an analytic set, $S_b \subset W$. We claim that if $z \in \overset{\circ}{W}$ and $z \notin S_b$ then (W, V) is (b) regular at z. This is obvious for if (W, V) is not (b) regular at z, there are sequences $\zeta_\nu \in W$, $z_\nu \in \overset{\circ}{V} - W$, $\lambda_\nu \in \mathbb{C}$ such that $\zeta_\nu \to z$, $z_\nu \to z$, $\lambda_\nu(z_\nu - \zeta_\nu) \to v$, $T(V, z_\nu) \to T$ and $v \notin T$. But then $z^* = (z, K(v), K(T)) \in C^* - R^*$ and $z = \Pi z^* \in S_b$, a contradiction. □

50 We now proceed to prove that $\dim S_b < m$. If possible let $\dim S_b = m$. Since $\Pi : S_b^* \to S_b$ is proper, by Remmert's theorem stated in §3, there exists a simple point z_0^* of S_b^* (in particular $z_0^* \in C^* - R^*$) and a neighbourhood U_0^* of z_0^* such that $\Pi(U_0^* \cap S_b^*)$ is a manifold of dimension m and $\Pi(U_0^* \cap S_b^*) = U_0 \cap W = U_0 \cap S_b$, $\Pi z_0^* = z_0$ being a simple point of S_b and $\Pi(U_0^*) = U_0$.

By Theorem (a) of Whitney, there exists an analytic set $S_a \subset W$ such that $\dim S_a < m$ and if $z \in (\overset{\circ}{W} - S_a)$, then (W, V) is (a) regular at z. Hence we may assume that for U_0^* obtained above, $U_0 \cap S_a = \emptyset$.

We assume, without loss of generality, that $\dot{V} \subset W$ and that V_{z_0} is irreducible. By the remark made above we put $z_0 = 0$ and obtain a neighbourhood $U \subset U_0$ of 0 and a basis $(z_1, \ldots, z_n)$ such that $U \cap W = M = \{z \in U \mid z_{m+1} = \ldots = z_n = 0\}$ and for any $z \in \Pi_m(v)$, if N_z is the transverse plane, $N_z \cap V \not\subset V$. We not construct σ_0 and σ as above. Consider the holomorphic map $\psi : V \times \mathbb{P}' \times G \to W \times V \times \mathbb{P} \times G$ given by $\psi(z, v^*, \alpha) = (\Pi_m z, z, v^*, \alpha)$. Now, $(z, v^*, \alpha) \in \sigma \Rightarrow (\Pi_m z, z, v^*, \alpha) \in C^*$. Hence $\psi^{-1}(C^*) = \sigma$ and σ is an analytic set. Moreover, the set $\{(z, K(z - \Pi_m z), T(V, z)) \mid z$ is a simple point of V in $U\}$ is a connected set of simple points of σ and is dense in σ. Hence σ is irreducible.

We now prove that $\Pi(\sigma - R^*) \supset M \cap U$. If $z \in M$, $z = \Pi(z^*)$, $z^* = (z, v^*, \alpha) \in C^* - R^*$, then there are sequences $\zeta_\nu \in W$, $z_\nu \in (\overset{\circ}{-}W)$, $\lambda_\nu \in \mathbb{C}$ such that z_ν, $\zeta_\nu \to z$, $\lambda_\nu(z_\nu - \zeta_\nu) \to v$ and $K(v) = v^*$ and $T(V, z_\nu) \to T(\alpha)$. Consider $z_\nu - \zeta_\nu = z_\nu - \Pi_m z_\nu + \Pi_m z_\nu - \zeta_n$. Since **51**
$|\Pi_m z_\nu - \zeta_\nu| \leq |z_\nu - \zeta_\nu|$, there exists a subsequence $\{\lambda_{\nu_k}\}$ of $\{\lambda_\nu\}$ such that $\lambda_{\nu_k}(\Pi_m z_{\nu_k} - \zeta_{\nu_k})$ converges to v'' say. (v'' may be the zero vector). Clearly $v'' \in T(M, z)$ and since by our assumption (W, V) is (a) regular at any point in M, $v'' \in T(\alpha)$. Hence if $\lambda_{\nu_k}(z_{\nu_k} - \Pi_m z_{\nu_k}) \to v'$, $v' \in T(\alpha)$, i.e. $v' \neq 0$ and $(z, K(v'), \alpha) \in \sigma - R^*$. Hence we have proved that $\Pi(\sigma - R^*) \supset M \cap U$.

Let σ' = closure of $(\sigma \cap \Pi^{-1}(M) - R^*)$ in $V \times \mathbb{P}' \times G$. Then σ' is an analytic set and $\Pi\sigma' = M$. Again, using Remmert's proper mapping theorem, there exists a simple point z_1^* of σ' (in particular $z_1^* \notin R^*$) and a neighbourhood U_1' of z_1^* in the set of simple points of $\sigma'(U_1' \cap R^* = \emptyset)$ such that if $\Pi_1 = \Pi|\sigma'$, $\operatorname{rank}(d\Pi_1)(z_1^*) = m$ for $z^* \in U_1'$ and $\Pi_1(U_1') = M_1$, M_1 being an open set in M. Hence, using the constant rank theorem and assuming U_1' to be sufficiently small, we obtain an analytic set $M' \subset U_1'$, $z_1^* \in M'$ such that $(\Pi_1|M') : M' \to M_1$ is an analytic isomorphism. Consider $W_1 = \Pi^{-1}(W)$. Then $\dim W_1 < r = \dim \sigma$. Hence by Lemma 2, there exists a neighbourhood U_1^* of z_1^* in $\mathbb{C}^n \times \mathbb{P}' \times G$, $U_1^* \cap R^* = \emptyset$, and an irreducible analytic set $\sum^* \subset U_1^*$, $\sum^* \subset \sigma$ such that $\dim \sum^* = m + 1$, $\sum^* \supset M'$ and $\dim \sum^* \cap W_1 = m$. Now consider a point z_2^* in M' such tht z_2^* does not lie on other components of $\sum^* \cap W_1$. If $\Pi_2 = \Pi| \sum^*$, there exists a neighbourhood U_2^* of z_2^* such that z^* is an isolated point of $\Pi_2^{-1}(\Pi_2 z^*)$ if $z^* \in U_2^*$. In fact $\Pi_2^{-1}(\Pi_2 z^*) = z^*$. Hence $\Pi_2(\sum^* \cap U_2^*) = \sum$

52 is an irreducible analytic set in $\Pi_2 U_2^* = U_2$ if U_2^* is sufficiently small. Also $\sum \subset V$ and $\dim \sum = m+1$. Now $N_{z_2} \cap \sum \cap W = \{z_2\}$ and if $B = N_{z_2} \cap \sum$, $\dim B \geq 1$ by Proposition 5 of Chapter 1.

If $A = N_{z_2} \cap \sum \cap W$, then, if $A^* = \Pi_2^{-1}(A)$, $B^* = \Pi_2^{-1}(B)$ then $\dim A^* = 0$ and $\dim B^* \geq 1$. In fact $A^* = \{z_2^*\}$. [If $\dim_{z_2^*} B^* < 1$, we may choose U_2^* sufficiently small and then $\dim B \cap \Pi(U_2^*) < 1$, which is a contradiction. Hence $\dim_{z_2^*} B^* \geq 1$.] Hence there exists a point z_3^* in $A^* \subset B^*$, such that $\dim_{z_3^*} B^* \geq 1$ and since we assumed above that $U_1^* \cap R^* = \emptyset$, $z_2^* \in \sigma - R^*$. We may assume tht $B^*_{z_2^*}$ is irreducible and then, by Lemma 4 above, there exists a neighbourhood U_3^* of z_2^* and a one-dimensionla analytic set X^* in U_3^*, $z_2^* \in X^*$, such that $X^* - z_2^* \subset B^* - A^*$. Then if $\Pi X^* = X$, X is an analytic set in ΠU_3^* such that $\dim X = 1$ and $X - z_2 \subset (\overset{\circ}{V} - W)$. Let $z_2^* = (z_2, v^*, \alpha)$, then $v^* \notin \alpha$ since $z_2^* \notin R^*$. Let ζ_ν^* be a sequence in $B^* - A^*$, ζ_ν^* simple points of B^*, $\zeta_\nu^* \to z_3^*$, $\zeta_\nu^* = (z_\nu, v_\nu^*, \alpha_\nu)$. Then $z_\nu \to z_2$, $K(z_\nu - \Pi_m z_\nu) = K(z_\nu - z_2) = v_\nu$ and $T(V, z_\nu) = T(\alpha_\nu)$ where $v_\nu^* \to v^*$ and $T(V, z_\nu) \to T(\alpha)$. Hence $v^* \in C(X, z_2)$. Also $\dim C^*(X, z_2) = 1$ (see §1 for notation). Hence $v^* = C^*(X, z_2)$. Also since $\dim X = 1$, and z_ν are simple points of X, if $KT(X, z_\nu) \to T^*$, $T^* = C^*(X, z_2)$. Hence $v^* \in \operatorname*{Lim}_{\nu\to\infty} K(T(X, z_\nu)) \subset \operatorname*{Lim}_{\nu\to\infty} K(T(V, z_\nu)) = \alpha$. Hence we have a contradiction and this proves that the assumption that $\dim S_b = m$ is false.

Chapter 4

Whitney Stratifications and pseudofibre bundles

1 Pseudo fibre spaces

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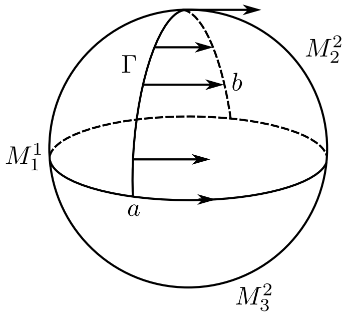
The situation we with to consider is suggested by the following example.

Let V be a $\mathbb{C}$-analytic manifold with a narrow stratification $\{M_i\}$ satisfying conditions (a) and (b) of Whitney. Let $\mathscr{V}$ be the tangent bundle of V and $\mathscr{M}_i$ the tangent bundle of M_i. $\mathscr{M}_i$ can be identified naturally with a subset of $\mathscr{V}$, and let $\mathscr{V}' = \cup \mathscr{M}_1$. $\mathscr{V}'$ consists of vectors at points $x \in V$ which are tangent to the M_i containing x. $\mathscr{V}'$ has a topology induced from that of $\mathscr{V}$ (under which it is not necessarily locally compact). $\mathscr{V}'$, with its natural projection on V is called the (complex) tangent pseudo fibre bundle (or pseudo bundle) of the stratification.

Real tangent pseudo bundles are similarly defined.

Remark that the homotopy lifting theorem is not in general valid, as shown by the following example.

Let $V = S^2$ be the two dimensional sphere, and $V = M_1^1 \cup M_2^2 \cup M_3^2$, where M_1^1 is a great circle and M_2^2, M_3^3 the two (open) hemispheres of $V - M_1^1$. Let Γ be half of a great circle orthogonal to M_1^1 as shown. Then Γ is homotopic semicircle of M_1^1. This homotopy cannot be lifted to $\mathscr{V}'$ such that in initial curve is lifted to the field of tangent vectors to M_2^2 54
orthogonal to Γ (as shown in the figure).



We now proceed to the general definition of a pseudo fibre space (or pseudo fibration).

Let V be a manifold of class C^1 and dimension N. Let $\{M_i\}$ be a narrow stratification of V into connected C^1 locally closed submanifolds M_i such that each $\overline{M}_i$ is a union of strata and suppose that $\{M_i\}$ is a locally finite family.

Let $\mathscr{K}$ be a triangulation of V consistent with the above stratification, i.e. each open simplex K_i is contained in M_j for some j, and suppose that the open simplices are C^1 submanifolds of V. We suppose that the following fineness condition is satisfied.

$(*)$ *If* $K_j \subset M_i$ *then* $\overline{K}_j \cap \dot{M}_i$ *is a single closed simplex* $\overline{K}_h$ *contained in the boundary* $\dot{K}_j$ *of* K_j (*unless* $\overline{K}_j \cap \dot{M}_i = \emptyset$).

This condition can always be ensured by passing to a sufficiently fine barycentric subdivison of $\mathscr{K}$.

Finally we suppose given a piecewise differentiable cell decomposition $\mathscr{D} = (D_i)$ of V (into open cells D_i; the decomposition is not consistent with (M_i)) which is dual to $\mathscr{K}$. This dual cell decomposition is obtained as follows. If $\mathscr{K}$ is a simplicial complex whose support V is a combinatorial manifold, let $\mathscr{K}_1$ be the barycentric subdivision of $\mathscr{K}$. Let p_i be the barycentre of $K_i \in \mathscr{K}$. The q-simplices of $\mathscr{K}_1$ have for vertices the sets $(p_{i_q}, \ldots, P_{i_q})$ with $K_{i_j} \supset \overline{K}_{i_{j+1}} (j = 0, \ldots, q-1)$; we
55 denote this q-simplex by $(p_{i_0}, \ldots, p_{i_q})$ and call p_{i_0} the first, and p_{i_q} the last vertex. We have $K_{i_q} = \bigcup (p_{i_0}, \ldots, p_{i_q})$, the union being over those simplices with p_{i_q} as last vertex. For any i, let

$$D_i = \bigcup (p_i, p_{i_1}, \ldots, p_{i_q})$$

the union being taken over all the simplices $(p_i, \dots, p_{i_q})$ of $\mathscr{K}_1$ for which p_i (the barycentre of K_i) is the first vertex. Then, if V is a combinatorial manifold of dimension N, each D_i is a cell and if K_i has dimension k, D_i is of dimension $N - k$. Further D_i is a cell decomposition of V and has the following two properties:

(i) Each K_i of dimension k meets exactly one D_j of dimension $N - k$ (viz. the D_i described above).

(ii) If $K_i \cap D_j \neq \emptyset$, then $\dim(K_i \cap D_j) = \dim K_i + \dim D_j - N$. The definitions that we now give will depend, *a priori*, on $\mathscr{K}$ and $\mathscr{D}$. Note also that we could give the definitions below when V is a manifold with boundary and $\mathscr{K}$ is a cell decomposition.

We begin with a lemma.

Lemma 1. *Given $K_i \subset M^k$, the strata which meet the closed simplex $\overline{K}_i$ can be arranged so that they give rise to a sequence*

$$\overline{M}_1 \subsetneqq \dots \subsetneqq \overline{M}_h = \overline{M}^k, \dim M_{j+1} > \dim M_j. \tag{1}$$

Further $\overline{K}_i \cap \overline{M}_j$ form a strictly increasing sequence of simplices

$$\overline{K}^1 \subsetneqq \dots \subsetneqq \overline{K}^h = \overline{K}_i. \tag{2}$$

Proof. Because of the fineness condition, it is sufficint to prove (1). Let 56
M^q, $M^{q'}$, $q \neq q'$ be distinct strata of dimensions q, q' respectively (with $q \geq q'$) meeting $\overline{K}_i$. If $q = k$, then $M^q = M^k \subset K_i$; hence $M^{q'} \cap \overline{M}^q \neq \emptyset$, and hence $\overline{M}^{q'} \subset \overline{M}^q$; since M^q, $M^{q'}$ are distinct, we have $q' < q$.

If $q \leq k - 1$, then $M^q \subset \dot{M}^k$, $M^{q'} \subset \dot{M}^k$. By our fineness condition, there is a simplex $K \subset M^{k'}$, $k' < k$ such that $\dot{M}^k \cap \overline{K}_i = \overline{K}$. Thus, we may replace k by $k' < k$. Proceeding thus, we reach a K' lying in a stratum M^l of dimension $l = q$ and the previous argument applies.

In the whole of this chapter, we suppose that $\mathscr{K}$, $\mathscr{D}$ are given satisfying the hypotheses made above.

The local coordinates (or charts) of our pseudo-fibration will be defined on subsets of the following type.

Let $\bar{K}$ be a closed simplex of $\mathscr{K}$ and K^0 one of its vertices. Let

$$L = \bar{K} \cap \overset{\circ}{\mathrm{St}}(K^0) \tag{3}$$

where $\overset{\circ}{\mathrm{St}}(K^0)$ is the open star of K^0 in $\mathscr{K}$. If $K^0 \in M^p$, then, by Lemma 1,

$$L = \bigcup_{q=p}^{k} L \cap M^q \quad \text{(where} \quad K \subset M^k), \tag{4}$$

M^q being, as usual, a stratum of dimension q.

57 We suppose given, for each dimension k such that there is an $M^k \neq \emptyset$, $0 \leq k \leq N$, a fibre type F_k, i.e. a locally compact topological space F_k. We suppose that if $h \leq k$, we are given a family $\mathscr{M}_{k,h}$ of continuous injections $\mu_{kh} : F_h \to F_k$; we suppose that this family of injections is non-empty if F_h, F_k are. We suppose that for $h \leq k \leq l$, and $\mu_{kh} \in \mathscr{M}_{kh}$, $\mu_{lk} \in \mathscr{M}_{lk}$, we have $\mu_{lk} \circ \mu_{kh} \in \mathscr{M}_{lh}$.

We now construct the models for our pseudo-fibrations on sets of the type L (in (3) above).

Let

$$L = \bar{K} \cap \mathrm{St}^0(K^0) = \bigcup_{q=p}^{k} L \cap M^q.$$

We find $\mu_q \in \mathscr{M}_{kq}$ (when $L \cap M^q \neq \emptyset$), such that $\mu_k = \mathrm{id}_{F_k}$ and if $\alpha_q = \mu_q(F_q)$, then $\alpha_{q'} \subset \alpha_q$ if $q' \leq q$ (so that $\alpha_q \subset \alpha_k = F_k$); further, we suppose that if $q' \leq q$, there is $\mu_{qq'} \in \mathscr{M}_{qq'}$ such that $\mu_{q'} = \mu_q \circ \mu_{qq'}$. Let

$$\mathscr{L} = \bigcup_{q=p}^{k} (L \cap M^q) \times \alpha_q : \tag{5}$$

then

$$L \times \alpha_q \subset \mathscr{L} \subset L \times F_k$$

and we put on $\mathscr{L}$ the topology induced from that of $L \times F_k$. □

Remark. It would be possible to work with sets $L' = \bar{L} \cap \mathrm{St}(K^0)$, where $\mathrm{St}(K^0)$ is the closure of $\mathrm{St}^0(K^0)$, instead of the sets L above in view of our fineness condition.

58 **Definition 1.** A pseudo-fibre space, or a pseudo-fibration ξ on a C^1 manifold V with the data of a stratification, a triangulation and a dual cell decomposition as above is a hausdorff space ξ and a projection $\bar{\omega} : \xi \to V$ (not necessarily surjective) such that for each set L as in (3), there is a homeomorphism g of $\mathscr{L}$ onto $\xi(L) = \overline{\omega}^{-1}(L)$. The pair $(g, \mathscr{L})$ is called a chart of ξ.

Lemma 2. *$\bar{\omega}$ is an open map.*

We omit the proof (see [3])

Definition 2. A pseudo-fibration is called a pseudo vector bundle (or pseudo-bundle) if each F_k is a finite dimensional vactor space over $\mathbb{R}$ or $\mathbb{C}$, and $\mathscr{M}_{kh}$ consists of all linear injections of F_h into F_k.

Let ξ be a pseudo-bundle such that $F_K = \mathbb{R}^k$ (resp. $F_{2k} = \mathbb{C}^k$, $M^{2k+1} = \emptyset$). Let $W_{r,k}$ be the set of all r-frames in F_k, i.e. the set of all ordered r-tuples of vectors linearly independent over $\mathbb{R}$ (resp. $\mathbb{C}$). of course, if $k < r$ (resp. $k < 2r$) then $W_{r,k} = \emptyset$.

Let $\mathscr{M}_{r,k,h}$ be the set of injections of $W_{r,h}$ into $W_{r,k}$ induced by linear injections of F_h in F_k. Then we may construct a pseudo-fibration with $W_{r,k}$ as fibre type (and $\mathscr{M}_{r,k,h}$ as given injections) for which charts are obtained as follows.

Let $(\mathscr{L}, g)$ be a chart of ξ. Let $\alpha_{r,q}$ be the space of r-frames in α_q, and let

$$\mathscr{L}_r = \bigcup_q (L \cap M^q) \times \alpha_{r,q}.$$

Let ξ_r be the union $\bigcup_{x \in V} \xi_r(x)$, $\xi_r(x)$ being the space of r-frames in 59
$\xi(x) = \bar{\omega}^{-1}(x)$. Clearly, the map

$$g : \mathscr{L} \to \xi(L)$$

induces a bijection

$$g_r : \mathscr{L}_r \to \xi_r(L) = \bigcup_{x \in L} \xi_r(x).$$

It is clear that there is a unique topology on ξ_r making ξ_r into a pseudo-fibration for which the $(\mathscr{L}_r, g_r)$ are charts.

ξ_r is called the associated pseudo-fibration of r-frames in ξ.

2 Obstructions in pseudo-fibrations

Let ξ be a pseudo-fibration with fibre type F_k. Let ν_k be the smallest integer $\nu \geq 0$ such that $\pi_\nu(F_k) \neq 0$. We will make the following hypothesis: $\rho = k - \nu_k$ is a *positive integer independent of* k. (Here of course k runs over those integers with $M^k \neq \emptyset$ for some stratum M^k).

In the example given above, we have $F_k = W_{r,k}$; here $\rho = r - 1$ in the case of real bundles, $\rho = 2r - 1$ in the case of complex bundles.

The problem we consider is that of the existence and homotopy of continuous sections of ξ (i.e. continuous maps $s : U \to \xi$, $U \subset V$, such that $\bar{\omega} \circ s = \mathrm{id}_U$.)

60 **Proposition 1.** *The obstruction dimension to skeleton-wise extension of a section over* $\mathscr{D}$ *is* $N - \rho + 1$; i.e. *if* $\mathscr{D}^q$ *is the q-skeleton of* $\mathscr{D}$, *then any section s of* ξ *over* $\mathscr{D}^{N-\rho-1}$ *can be extended to* $\mathscr{D}^{N-\rho}$.

Proof. We begin by remarking that if $N \geq \rho$, every vertex D^0 of $\mathscr{D}^0$ lies in a K^N, and since $N \geq \rho$, $F_N \geq \emptyset$, so that a section s of ξ over $\mathscr{D}^0$ exists if $N - \rho \geq 0$. Let $m \leq N - \rho$, and suppose that the section s is constructed on the $(m-1)$-skeleton $\mathscr{D}^{m-1}$. To extend s to $\mathscr{D}^m$, we choose any m-cell D^m of $\mathscr{D}^m$. Let $T^n \in \mathscr{K} \cap \bar{D}^m$. We proceed by induction on n. We have $T^n = D^{N-h+n} \cap K^h$, $K^h \subset M^k$; here $N - h + n \leq m \leq N - \rho$ so that $n \leq h - \rho \leq k - \rho = \nu_k$. By our induction hypothesis (on n) $s|\dot{T}^n$ is already constructed; further, if $n = 0$, s can be extended to T^n. Suppose therefore that $n \geq 1$.

Choose now an L such that $\bar{T}^n \subset L \subset \bar{K}^h \cap \mathrm{St}(K^0)$. [Such an L exists: there is a unique K^{N-m} such that $D^m \cap K^{N-m} = T^0$ is a vertex. Then, by Lemma 1, $K^{N-m} \subset \bar{K}^h$, and we choose for K^0 a vertex of $\bar{K}^h \cap K^{N-m}$.] Consider the chart $g : \mathscr{L} \to \xi(L)$. Then $g^{-1}s$ defines a section of $\mathscr{L}$ on $\dot{T}^n$, i.e. a map $s' : T^n \to \alpha_h \subset \alpha_k$ (since $K^h \subset M^k$); since $\nu_k \geq n$, this can be extended to a continuous map $s' : T^n \to \alpha_k$, and so gives rise to a section $g(s') = s : T^n \to \xi(L)$. Proceeding thus, we obtain an extension of the section s to $\mathscr{D}^m$. Since this can be done for $m \leq N - \rho$, the proposition is proved.

Before we proceed to the next proposition, we make a few remarks.

Let $I = [0, 1]$, and let $\widehat{V} = V \times I$, $\widehat{M}_i = M_i \times I$. Let $\widehat{\mathscr{K}}$ be the *cell-*
61 *decomposition* of $\widehat{V}$ (whose (closed) cells are the sets $\bar{K}_i \times \{0\}$, $\bar{K}_i \times \{1\}$,

$\overline{K}_i \times I$. Let $\widehat{\xi} = \xi \times I$ and $\widehat{\overline{\omega}} = \overline{\omega} \times \mathrm{id}_I$. We define the structure of pseudo-fibration (on the manifold V with boundary, and corresponding to the cell-decomposition $\mathscr{K}$; cf. remark on page 45) on $\widehat{\xi}$ as follows.

If $L = \overline{K} \cap \mathrm{St}^0(K^0) \subset V$ is a set defining a chart of ξ, let $\widehat{L} = L \times I$, and $\widehat{g}$ the bijection of $\widehat{\mathscr{L}} = \mathscr{L} \times I$ onto $\widehat{\xi}(\widehat{L}) = \widehat{\overline{\omega}}^{-1}(\widehat{L})$ given by $\widehat{g} = g \times \mathrm{id}_I$. The fibre type of $\widehat{\xi}$ and the injections between the fibres are the same as for ξ.

Let $s_{N-\rho}$, $s'_{N-\rho}$ be two sections of ξ over $\mathscr{D}^{N-\rho}$, the $N-\rho$ skeleton of $\mathscr{D}$. We identify them with sections on $\mathscr{D}^{N-\rho} \times \{0\}$, $\mathscr{D}^{N-\rho} \times \{1\}$ respectively, of $\widehat{\xi}$. $\square$

Proposition 2. *Two sections $s_{N-\rho}$, $s'_{N-\rho}$ of ξ on $\mathscr{D}^{N-\rho}$ are homotopic on $\mathscr{D}^{N-\rho-1}$; in fact a given homotopy on $\mathscr{D}^{N-\rho-2}$ can be extended to $\mathscr{D}^{N-\rho-1}$.*

Proof. We do not consider the case when F_N is not connected, for we would then have $\rho > N$. If F_N is connected, any two sections of ξ over $\mathscr{D}^0$ are homotopic.

Let $m \leq N - \rho - 1$. By induction on m, suppose given a homotopy between s, s' on $\mathscr{D}^{m-1}$. Let D^m be an m-cell of $\mathscr{D}^m$. Then, with the notations as above,

$$T^n = D^{N-h+m} \cap K^h, K^h \subset M^h; N - h + n \leq m \leq N - \rho - 1$$

so that

$$\nu_k \geq n + 1 \geq 1.$$

This implies that F_k is connected, so that (if $n = 0$) any two sections 62
on T^0 are homotopic. Suppose (by induction on n) that, for $n \geq 1$, the homotopy between s, s' on $\mathscr{D}^{m-1}$ is extended to all the $T^\lambda \subset \overline{D}^m$ for which $\lambda < n$. Then s on $T^n \times \{0\}$, s' on $T^n \times \{1\}$ and the given homotopy on $\dot{T}^n \times I$ define a section of $\widehat{s}$ on the whole boundary of $T^n \times I$ and we have only to show that this section can be extended to $\overline{T}^n \times I$.

To prove this, we choose L with $\overline{T}^n \subset L \subset \overline{M}^k$ and a chart $\widehat{g} : \widehat{\mathscr{L}} \to \widehat{\xi}(\widehat{L})$ as above. Clearly $\widehat{g}^{-1}\widehat{s}$ is a section of $\widehat{\mathscr{L}}$ on the boundary of $T^n \times I$, hence gives rise to a map of the boundary of $T^n \times I$ into F_k. Since $k \geq h \geq n + \rho + 1$, so that $\nu_k \geq n + 1$, this can be extended to a map of $\bar{T}^n \times I$ into F_k, and so gives rise to a section of $\widehat{\mathscr{L}}$ on $\bar{T}^n \times I$, and its image by $\widehat{g}$ is a section of $\widehat{\xi}$ on $\bar{T}^n \times I$ extending $\widehat{s}$. This proves that the given homotopy on $\mathscr{D}^{m-1}$ can be extended to $\mathscr{D}^m$ if $m \leq N - \rho - 1$. The proposition follows. □

Proposition 3. *Suppose $F_p \neq \emptyset$ and $K^0 \in M^p$. Then ξ has a section on the open star U of K^0 in $\mathscr{K}$. Moreover, if F_p is arcwise connected, any two sections over U are homotopic.*

Proof. To construct a section s on U, we proceed by induction on the dimension h of simplices $K^h \subset U$. Clearly, since $F_p \neq \emptyset$, $s|K^0$ exists. Suppose $s|K^l$ given for all $l < h$, and $K^h \subset U \cap M^k$, $L = U \cap \bar{K}^h$. Consider a chart $g : \mathscr{L} \to \xi(L)$. We are given a section of ξ on $\dot{L} \cap U$, hence a section of $\mathscr{L}$ on $\dot{L} \cap U$, a fortiori a map of $\dot{L} \cap U$ into F_k. Since
63 $\dot{L} \cap U$ is a hemisphere on the boundary of L, this can be extended to a map $L \to F_k$. Since the interior of L, which is K^h, has the property that $K^h \times F_k \subset \mathscr{L}$, this gives us a section of $\mathscr{L}$ on L extending the given section on $L \cap U$, and the image by g gives us a section of ξ on L extending the given section on $\dot{L} \cap U$.

Suppose now that F_p is connected. We use the notation before Proposition 2. Given sections s_0, s_1 of $\widehat{\xi}$ on $U \times \{0\}$ and $U \times \{1\}$, we have to extend it to a section $\widehat{s}$ of $U \times I$. Let $K^h \times I \subset U \times I$ and suppose $\widehat{s}$ given on $K^l \times I$ for all $l < h$. Let $K^h \subset U \cap M^k$, $L = U \cap \bar{K}^h$ and let $\widehat{g} : \widehat{\mathscr{L}} \to \widehat{\xi}(\widehat{L})$ be a chart. As before, this leads to a map of $L \times \{0\} \cup L \times \{1\} \cup (\dot{L} \cap U) \times I$ into F_k, and if $h \geq 1$, this union is not the whole of the boundary of $L \times I$, and the map therefore extends to $L \times I$, which, as before gives us a section of $\widehat{\xi}$ on $L \times I$ extending $s_0|L \times \{0\}$ and $s_1|L \times \{1\}$ (and the section defining the homotopy on $L \times I$). If $h = 1$, since F_p is arcwise connected, the problem is trivial. Our proposition follows by induction on h. □

3 Local structure of pseudo vector bundles

Let $\{M_i\}$ be a stratification of the complex manifold V. Let ξ be a topological space $\bar{\omega} : \xi \to V$ a continuous map such that for $x \in M^k$, $\bar{\omega}^{-1}(x)$ is homeomorphic to $F_k = \mathbb{C}^k$. We look for conditions that $\bar{\omega} : \xi \to V$ be a pseudo-fibration. We shall apply these considerations to the tangent fibration of a Whitney stratification in §4.

In this section, if $K_i \subset M^p$, we write ${}_pK_i$ or simply ${}_pK$ for K_i. Let **64**
${}_pU = \mathrm{St}^0({}_pK)$ and $\widetilde{U}_p = \overline{{}_pU} \cap \bigcup_{q \geq p} M^q$.

The conditions that we impose on our space ξ are the following.

Φ_1. To each ${}_pK \subset M^p$, we can associate a non-empty family $\Phi(p)$ of mappings

$$\varphi_p : \widetilde{U}_p \times F_p \to \xi(\widetilde{U}_p) = \bar{\omega}^{-1}(\widetilde{U}_p)$$

such that φ_p is continuous, and $\varphi_p \mid \{x\} \times F_p$ is a $\mathbb{C}$-linear injection into $\xi(x)$.

Φ_2. Suppose that ${}_pK \subset \overline{{}_mK}, {}_mK \subset M^m$, $p \leq m$. Then, clearly, $\overline{{}_pU} \supset \overline{{}_mU}$, and $\widetilde{U}_p \supset \widetilde{U}_m$.

Let μ be a linear injection of F_p in F_m. Then, given a φ_p as in Φ_1, there is a $\varphi_m \in \Phi(m)$

$$\varphi_m : \widetilde{U}_m \times F_m \to \xi(\widetilde{U}_m)$$

such that

$$\varphi_p \mid \widetilde{U}_m \times F_p = \varphi_m(\mathrm{id}_{\widetilde{U}_m} \times \mu),$$

i.e. $\varphi_p(x, \zeta) = \varphi_m(x, \mu(\zeta))$ for $x \in \widetilde{U}_m$, $\zeta \in F_p$.

Proposition 4. *If ξ is a topological space with a map $\bar{\omega} : \xi \to V$ for which a family of maps $\{\varphi\} = \{\Phi(p)\}_{p \geq 0}$ satisfying Φ_1 and Φ_2 exist, then ξ carries a natural structure of pseudo-vector bundle.*

Proof. It is clearly sufficient to construct charts for $\bar{\omega} : \xi \to V$. **65**

Let $L = \bar{K}^m \cap \mathrm{St}^0(K^0)$, $K^0 \in M^p$, $K^m \subset M^k$.

Let $\alpha_p \subset \ldots \subset \alpha_k = F_k$ be a family of subspaces of $F_k = \mathbb{C}^k$ such that $\alpha_q \approx F_q$, and let

$$\mathscr{L} = \bigcup_{q=p}^{k} (L \cap M^q) \times \alpha_q \subset L \times F_k.$$

Let π_F, π_L be the projections of $\mathscr{L}$ into F_k, L respectively. Let $e_1, \ldots, e_k$ be a k-frame in F_k such that $e_1, \ldots, e_q \in F_q$ (and so span F_q). By our fineness condition, there exists, for each q, a $q^K = K^h$ such that

$$K^h \subset L \cap M^q \subset \bar{K}^h, \quad \text{i.e.} \quad q^K \subset L \cap M^q \subset \overline{{}_qK}.$$

Thus

$$\pi_F^{-1}(\alpha_q) = \bigcup_{h \geq q}(L \cap M^h) \times \alpha_q = (L \cap \bigcup_{h \geq q} M^h) \times \alpha_q = (L \cap \widetilde{U}_q) \times \alpha_q.$$

We will construct an isomorphism $g : \mathscr{L} \to \xi(L)$ inductively by constructing maps

$$g_q : \pi_F^{-1}(\alpha_q) \to \xi(L \cap \widetilde{U}_q).$$

When $q = p$, $\pi_F^{-1}(\alpha_q) = (L \cap \widetilde{U}_p) \times \alpha_p$; now, by Φ_1, there is a map

$$g_p : \pi_F^{-1}(\alpha_p) \to \xi(L \cap \widetilde{U}_p),$$

which is the restriction to $L \cap \widetilde{U}_p$ of a map $\widetilde{U}_p \times \alpha_p \to \xi(\widetilde{U}_p)$. g_p is an injection on each fibre $\{x\} \times \alpha_p$. Now suppose that

$$g_q : \pi_F^{-1}(\alpha_q) \to \xi(L \cap \widetilde{U}_q)$$

66 is determined as the restriction to $L \cap \widetilde{U}_q$ of a map $\varphi_q : \widetilde{U}_q \times \alpha_q \to \xi(\widetilde{U}_q)$. Let h be the smallest integer $> q$ occurring among the $\alpha_q, \ldots, \alpha_k$. By Φ_2, there is a map $\varphi_h : \widetilde{U}_h \times \alpha_h \to \xi(\widetilde{U}_h)$ such that $\varphi_h | \widetilde{U}_h \times \alpha_q = \varphi_q \mid \widetilde{U}_h \times \alpha_q$; we may take $g_h = \varphi_h \mid L \cap \widetilde{U}_h$. This gives us finally a map g of $\mathscr{L} = \cup \pi_F^{-1}(\alpha_q)$ into $\xi(L)$ which is injective on the fibres. Since the fibres of $\mathscr{L}$ and $\xi(L)$ at any point have the same dimension, g is an isomorphism. $\square$

Remark. If ξ is a complex pseudo vector bundle as above, two mapping $\varphi_p, \varphi'_p \in \Phi(p)$ [cf. Φ_1] are isotonic, i.e. there is a continuous family $\varphi_p(t)$, $0 \leq t \leq 1$ of maps in $\Phi(p)$ on ${}_pU$ with $\varphi_p(0) = \varphi_p$, $\varphi_p(1) = \varphi'_p$.

If ${}_pK = K^0 \in M^p$ is a vertex, we remark, using a chart, that φ_p, φ'_p correspond to sections of the associated bundle ξ_p of p-frames over ${}_pU$, and two such sections are homotopic by Proposition 3.

In the general case, one proceeds by induction, as in the proof of Proposition 3.

4 Pseudo-fibration corresponding to a Whitney stratification

We prove in this section the following

Theorem. *Let $\{M_i\}$ be a Whitney stratification of a complex manifold V and $\mathscr{V}'$ the space of tangent vectors to the strata* (see beginning of §1). *Then $\mathscr{V}'$ carries a natural structure of pseudo-vector bundle.*

Proof. By the triangulation theorem for analytic sets, there is a triangulation $\mathscr{K}$ of V compatible with $\{M_i\}$. We suppose (by suitable subdivision) that $\mathscr{K}$ satisfies the fineness condition of §1 (Condition (*) on p. 45), and that the open star of any simplex is contained in a coordinate neighbourhood of V. **67**

Let $K^l \subset M^p$ (p is the complex dimension of M^p). Then, by the finenss condition, there is a vertex K^0 of $\bar{K}^l$, $K^0 \in M^p$. Then $\widetilde{U}_l \subset \widetilde{U}_0$, and we shall construct the maps $\varphi \in \Phi(p)$ on $\widetilde{U}_0 \times \mathbb{C}^p$, i.e. $\varphi : \widetilde{U}_0 \times \mathbb{C}^p \to \mathscr{V}'(\widetilde{U}_0)$. This means, of course, that there is a continuous field of p-frames in $\widetilde{U}_0$ compatible with the stratification; further, since K^0 is a vertex, $\widetilde{U}_0$ is the star of K^0.

We will use the following lemma; its proof is to be found in [5]. $\square$

Lemma 2. *Let X be a Euclidean complex, Y a subcomplex. Then Y has a fundamental system of tubular neighbourhoods Θ in X such that the segments $]y, \dot{x}]$, $y \in Y$, $\dot{x} \in \Theta$, form a partition of $\Theta - Y$.*

We consider a Euclidean complex homeomorphic to $\mathscr{K}$; we use the same notation in this euclidean complex as in V. Let $U = U_0$ be as above; let $\dot{U}$ be its boundary and M_j a stratum of dimension $> p$ with $\dot{M}_j \cap U \neq \emptyset$. Consider the subcomplex $X = \bar{M}_j \cap \dot{U}$ of $\dot{U}$, and the subcomplex $Y = \dot{M}_j \cap \dot{U}$ of X. Let Θ be a tubular neighbourhood as in Lemma 2, and $t \in [0, 1]$ the parameter for the directed segment $[y, x]$ (parametrized linearly). For $u \in [0, 1]$, let λ_u be the homothesy having $y_0 = K^0$ as centre, the dilatation being u. Let T be the complex **68**
generated by the $\lambda_u(\Theta)$, $0 \leq u \leq 1$, and let $y_u = \lambda_u(y)$ (and similr notation for other points). clearly, the segments $]y_u, x_u]$ form a partition of $T - \dot{M}_j \cap \dot{U}$. T is called a "conical neighbourhood" of $\dot{M}_j \cap \dot{U}$ in

$\bar{M}_j \cap \dot{U}$. T is called a "conical neighbourhood" of $\dot{M}_j \cap \dot{U}$ in $\bar{M}_j \cap \dot{U}$. We also speak of conical neighbourhoods in $\mathscr{K}$ on the original manifold V. Remark it is not a real neighbourhood since it is not a neighbourhood at K^0.

We now proceed to the construction of a field of p-frames in $\widetilde{U}$. We may suppose that V is an open set in $\mathbb{C}^N$ because of our hypothesis that the star of any simplex is contained in a co-ordinate neighbourhood.

If $K^0 \in M^p$ with $p = 0$, the statement is trivial; let then $p \geq 1$. Let $y_0 = K^0 \in M^p = M$. Let $e_i(y_0)$, $1 \leq i \leq p$ be a basis of $T(M, y_0)$. We shall extend the p-frame $Z = \{e_i(y_0)\}$ to the complexes $M_j^q \cap \widetilde{U}$ by induction on $q = \dim M_j^q$. Suppose this to be done for all M_j^q of dimension $q < m$, and let N be a stratum of dimension m such that $\bar{N} \cap \bar{U} \neq \emptyset$. We suppose furthermore that all the vectors already constructed on $M_j^q \cap \widetilde{U}$ tend to zero at a point of $\bar{U} - \widetilde{U}$ which is subcomplex of $\dot{U}$ by the definition of $\widetilde{U}$. For a point x in the closure of the complement of the conical neighbourhood T of $\dot{N} \cap \bar{U}$ in $N \cap \bar{U}$, let $e_i(x)$ be the orthogonal projection (with respect to the metric induced on U from $\mathbb{C}^N$) on the tangent space $T(N, x)$ of the translate of $e_i(y_0)$ to x. If $x \in T$, $x \notin \dot{N}$, then x is on a unique segment $[y, x_1]$ corresponding to parameter
69 value $t \in [0, 1]$. Let $\xi(x)$ [resp. $\eta(x)$] be the projection on $T(N, x)$ of the translate of $e_i(x)$ [resp. $e_i(y)$]. (Note that, by induction, the e_i are defined on $\dot{N} \cap \widetilde{U}$). We have already supposed that these can be extended to $\dot{N} \cap \bar{U}$ and vanish on $\bar{U} - \widetilde{U}$). We set

$$e_i(x) = t\xi(x) + (1 - t)\eta(x).$$

The field e_i is continuous on $N \cap \bar{U}$ (it may have zeros). We prove that it is continuous on $\bar{N} \cap \bar{U}$. Let $y \in \dot{N} \cap \bar{U}$, and let $y \in M_j$. *In fact, by Whitney's condition (a),* if x is near y, then the orthogonal projection $v = \eta(x)$ of the translate of $e_i(y) \neq 0$ to x on $T(N, x)$ is near v. In fact, if this were not true, we could find a sequence of points $x_i \in N$ tending to y such that (the Grassmannian being compact) $\kappa T(x_i, N)$ converges to a limit κT such that T is transverse to $T(y, M_j)$; this is impossible since our stratification is, by assumption, a Whitney stratification. Since, as x tends to y, the parameter value tends to zero, $e_i(x) = \eta(x) + t(\xi(x) - \eta(x))$ is near the translate of $e_i(y)$ to x.

It is clear from the above construction that we may find continuous fields $e_1, \ldots, e_p$ in $\overline{N} \cap \overline{U}$ extending the fields on $\dot{N} \cap \overline{U}$ (which form a p-frame on $\dot{N} \cap \widetilde{U}$). There is a neighbourhood W of y_0 such that the $e_i(x)$ form a p-frame at x for $x \in W$. By induction, the $e_i(x)$ form a p-frame at points of $\dot{N} \cap \widetilde{U}$. Consequently, there is a neighbourhood T' of all the points of $\dot{N} \cap \widetilde{U}$ in $\overline{N} \cap \overline{U}$ on which the e_i remain a p-frame (T' may not be a neighbourhood in $\overline{N} \cap \overline{U}$). It is immediate that we may take $T' \subset T$ and suppose that $\dot{N} \cap \widetilde{U}$ is a retract of T'. **70**

In $T'' = T' \cup W$, the e_i form a p-frame. Let $H = N \cap \widetilde{U} - T'' \subset N$. Moreover, if T' and W are suitably chosen, then each $K^l \cap H$ is a cell. (This is obvious for the euclidean complexes, and the general case can be reduced to this by a homeomorphism.)

We now extend the p-frame from T'' to H by doing this stepwise on the cells $K^l \cap H$. Let $\mathscr{T}_p$ be the (locally trivial) fibre space of p-frames tangent to N. We suppose, by induction on l, that the frame is extended to the complex $\mathscr{K}^{l-1} \cap H$ and consider a cell $K^l \cap H$.

First suppose that $\widetilde{U} = \overline{U}$ (i.e. that $\dot{U} \cap M^p = \emptyset$). In this case, T'' being suitably chosen, $K^l \cap H$ is a hemisphere on the boundary $\dot{K}^l$. Hence, the following lemma implies that any section of $\mathscr{T}_p$ on $\dot{K}^l \cap H$ can be extended to $K^l \cap H$, and our result would follow.

Lemma. *Let Δ be a convex polyhedron in $\mathbb{R}^l$ and $\mathscr{T}$ a locally trivial fibre space on Δ. Let S be an open linear simplex contained in $\dot{\Delta}$. Then any section of $\mathscr{T}$ on $\dot{\Delta} - S$ can be extended to Δ.*

Proof. Since the pair $(\Delta, \dot{\Delta} - S)$ is homeomorphic to the pair $(I^l, I^{l-1} \times \{0\})$ [I being the unit interval on $\mathbb{R}$], we replace Δ by I^l and S by $\dot{I}^l - I^{l-1} \times \{0\}$. If s is a section of $\mathscr{T}$ on $I^{l-1} \times \{0\}$, by the homotopy lifting theorem (lift the trivial homotopy of $I^{l-1} \times \{0\}$), there is a map $F : I^l (= I^{l-1} \times I) \to \mathscr{T}$ with $p \cdot F(x,t) = (x,t)$ and $F(x,0) = s(x)$ [here $p : \mathscr{T} \to I^l$ **71**
is the projection]. Clearly F is a section of $\mathscr{T}$ on I^l extending s. □

In the case when $\widetilde{U} \neq \overline{U}$, let $Y = \dot{K}^l \cap (\overline{U} - \widetilde{U})$. Then $Y \subset \dot{U}$. We consider a sequence $\{Y_\nu\}$ of neighbourhoods of Y in K^l such that ($Y_\nu - Y$ is a subcomplex of a suitable subdivision of $K^l - Y$ and) $\overline{Y}_{\nu+1} \subset Y_\nu$, $\bigcap_{\nu=1}^{\infty} Y_\nu = Y$.

The argument used above shows that the given p-frame can be extended to $K^l \cap H - Y_\nu$. For a suitable choice of the $\{Y_\nu\}$, we can apply the lemma above to extend any section of $\mathscr{T}_p$ on $K^l \cap H - Y_\nu$ to $K^l \cap H - Y_{\nu+1}$. Thus, in both cases, the given p-frame can be extended to $K^l \cap H$.

This proves that we can construct continuous p-frames on $\widetilde{U}$, and Φ_1 is proved.

To prove Φ_2, we have only to prove that if ${}_pK \subset {}_m\overline{K}$, ${}_mK \subset M^m$, and if $\{e_i\}_{1 \le i \le p}$ is a continuous p-frame on $\widetilde{U}_p$, these can be extended to an m-frame on $\widetilde{U}_m$.

The proof is exactly similar to that given above: we choose a vertex y'_0 of ${}_m\overline{K}$ in M^m, and vectors $e_{p+1}(y'_0), \dots, e_m(y')$ at y'_0 linearly independent of $e_1(y'_0), \dots, e_p(y'_0)$ and apply the above reasoning; one has to consider neighbourhoods where the fields constructed on $\overline{N} \cap \widetilde{U}$, $\{U = \mathrm{St}^0(y'_0)\}$ are independent of the $e_i (1 \le i \le p)$ and replace the fibre space $\mathscr{T}_p$ by the space $\mathscr{T}_{p,m}(e_1, \dots, e_p)$ of $m - p$ vectors which are independent of the $e_1, \dots, e_p$.

72 Fields of frames tangent to a Whitney stratification

From the results of §1 and the above theorem, it follows that the r-frames of the fibres of $\mathscr{V}'$ form a pseudofibre space $\mathscr{V}'_r$. The fibre type of $\mathscr{V}'_r$ over a stratum M^k (of complex dimension k) is empty for $k \le r$ and, for $k \ge r$, is the manifold of r-frames in $\mathbb{C}^k$, which has the homotopy type of the Stiefel manifold $U(k)/U(k-r)$.

Hence the first non-zero homotopy group of the fibre $F_{k,r}$ over M^k is $\pi_{2k-2r+1}(F_{k,r})$; hence $\rho = 2r - 1$. If N is the complex dimension of V, we deduce from the results of §2 the following

Proposition. *With the notation of §2, the obstruction dimension to skeletonwise extension, over $\mathscr{D}$, of a continuous field of r-frames "tangent to the strata" is $2p = 2(N - r + 1)$. Two such fields, defined on $\mathscr{D}^{2p-1}$, are homotopic on $\mathscr{D}^{2p-2}$. Further, if $y_0 \in M^p$, $p \ge r$, then, on the open star of y_0 in $\mathscr{K}$, there exists a continuous field of r-frames tangent to the strata.*

Whitney has posed the following question: Can one find locally, families of (real) analytic or semi-analytic fields of vectors which are linearly independent and consistent with the stratification?

Our proposition shows that *continuous* fields of this kind exist.

Whitney has also shown that, in general, holomorphic fields with this property do not exist (see [6]).

Obstruction classes. 73

Let us consider first the simple example when the stratification of V consists of a (closed) submanifold M of complex dimension k and $M' = V - M$. Let μ be the homomorphism of $H^i(M, \mathbb{Z})$ into $H^{i+N-k}(V, \mathbb{Z})$ which is the Thom-Gysin isomorphism followed by the canonical map $H^*(V, V - M; \mathbb{Z}) \to H^*(V, \mathbb{Z})$. Then one can define classes $c_q(M) \in H^{2q}(M, \mathbb{Z})$ which coincide with the Chern classes of the tangent bundle of M when M is a closed submanifold of V, and define

$$\hat{c}_p(M) = \mu(c_q(M)) \in H^{2q}(V, \mathbb{Z}), N - p = k - q (= r - 1),$$
$$\hat{c}_p(M') = c_p(V) - \hat{c}_p(M) \in H^{2p}(V, \mathbb{Z}).$$

[One has $c_p(M') = \hat{c}_p(M')$.] It can be shown that there is a section of $\mathscr{V}_r'$ over $\mathscr{D}^{2p}(N - p = r - 1)$ if and only if

$$c_q(M) = 0, c_p(M') = 0.$$

The definition of these classes $c_p(M)$ can be generalized to any stratification; they depend, in general, one the dual complex $\mathscr{D}$ (see [5]). However, the definition of the classes $\hat{c}_p(M)$ can be generalized in such a way as to be independent of $\mathscr{D}$ [5]. If M_i is a stratum of dimension k, we have

$$\hat{c}_p(M_i) \in H^{2P}(V, \mathbb{Z}), \hat{c}_p(M_i) = \sum_p \hat{c}_p(M_i) = \sum_{p=N-k}^{N} \hat{c}_p(M_i).$$

These classes have the property that

$$\sum_i \hat{c}_p(M) = c_p(V), \sum_i \hat{c}(M_i) = c(V).$$

74

Relationship with a stratification consistent with a mapping.

Let f be a holomorphic mapping of a complex manifold V into another W (of the same dimension N). We have seen in Chapter 2 (Proposition 3 and Remark 6) that there exist Whitney stratifications of V and W consistent with f (the restriction of f to any stratum M of V has constant rank). It can be shown that the local topological degree of f [which is the limit, when U shrinks to a point x, of the maximum number of points in a fibre $g^{-1}g(y)$ of the restriction g of f to a neighbourhood U of x] is constant on M_i; we denote this constant by $m(M_i)$. If we denote by $c_p(M_i, \mathbb{Q})$, $\hat{c}_p(M_i, \mathbb{Q}), \ldots$ the images of $c_p(M_i)$, $\hat{c}_p(M_i)$ under the natural map $H^{**}(V, \mathbb{Z}) \to H^{**}(V, \mathbb{Q})(H^{**} = \sum_{p\geq 0} H^{2p})$ then one can prove the following result:

$$f^*c(W, \mathbb{Q}) = \sum_i m(M_i)\hat{c}(M_i, \mathbb{Q}).$$

This result is far from trivial even when the stratification of V contains only two strata as in the example above.

We end these notes with the following proposition concerning the existence of holomorphic fields of vectors tangent to the strata, which may however admit zeros (unlike in the theorem above with $r = 1$).

Proposition (R. Narasimhan). *If $\mathscr{V}'$ is, as in the above theorem, the pseudovector bundle defined by a stratification of V, the sheaf of germs of holomorphic sections of $\mathscr{V}'$ is coherent.*

75 More generally we have

Proposition. *Let V be a complex manifold, $\{M_i\}$ a locally finite family of locally closed analytic submanifolds such that $\overline{M}_i$ is analytic for each i. Let $\mathscr{F}$ be the sheaf of germs of holomorphic vector fields ξ_x such that $\xi_x \in \mathscr{F}_x$ if and only if $\xi_x(y) \in T(M_i, y)$ for all y near x and all i such that $y \in M_i$. Then $\mathscr{F}$ is coherent. [$\mathscr{F}_x$ = germs $\mathscr{T}_x(V)$ of all vector fields on V is $x \notin \cup\overline{M}_i$.]*

Proof. Let $\mathscr{F}^i$ be the sheaf of germs of holomorphic vector fields ξ_x such that $\xi_x \in \mathscr{F}^i_x$ if and only if $\xi_x(y) \in T(M_i, y)$ for all y near x such that $y \in M_i$. Then $\mathscr{F} = \cap\mathscr{F}^i$ and every point of V has a neighbourhood U such that $\mathscr{T}_x(V) = \mathscr{F}^i_x$ for $x \in U$ and all but finitely many i. Hence the intersection $\mathscr{F} = \cap \mathscr{F}^i$ is locally finite, and it suffices to prove the proposition when the family $\{M_i\}$ contains only one element, say M; further, the theorem being local, we may suppose that V is an open set in $\mathbb{C}^n$, so that the sheaf $\mathscr{T}(V)$ of germs of holomorphic vector fields can be identified naturally with $\mathscr{O}^n$. Moreover (by choosing V small enough), we may suppose that there are holomorphic functions $g_1, \ldots, g_q$ in V which generate the ideal of holomorphic functions vanishing on $\bar{M}$ at any point of V. Then, clearly, an element $\xi = (a_1, \ldots, a_n) \in \mathscr{O}^n_x$ belongs to $\mathscr{F}_x$ if and only if, in a neighbourhood Ω of x,

$$\sum a_i \frac{\partial g_k}{\partial z_i} = 0 \quad \text{on} \quad \Omega \cap M, \quad \text{for each} \quad k,$$

hence, if and only if $\sum a_i \frac{\partial g_k}{\partial z_i} = 0$ on $\Omega \cap \bar{M}$. For each $k = 1, \ldots, q$, let 76
$\mathscr{G}_k$ denote the subsheaf of $\mathscr{O}^n$ consisting of germs $(a_1, \ldots, a_n)$ such that

$$\sum_{i=1}^{n} a_i \frac{\partial g_k}{\partial z_i} = 0 \quad \text{on} \quad \bar{M}.$$

Then, clearly, $\mathscr{F} = \bigcap_{k=1}^{q} \mathscr{G}_k$. Further, if

$$\mathscr{R}_k = \mathscr{R}_\lambda\left(\frac{\partial g_k}{\partial z_1}, \ldots, \frac{\partial g_k}{\partial z_n}, g_1, \ldots, g_q\right)$$

is the sheaf of relations between the functions in parantheses, $\mathscr{R}_k$ is coherent and $\mathscr{G}_k$ is a quotient of $\mathscr{R}_k$. Hence $\mathscr{G}_k$ is of finite type. Since further $\mathscr{G}_k$ is a subsheaf of $\mathscr{O}^n$, it is coherent and hence so is $\mathscr{F}$. $\square$

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