Lectures on
An Introduction to Grothendieck's Theory of the Fundamental Group

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Preface

These lectures contain the material presented in a course given at the Tata Institute during the period December 1964 - February 1965. The purpose of these lectures was to give an introduction to Grothendieck's theory of the fundamental group in algebraic geometry with, as application, the study of the fundamental group of an algebraic curve over an algebraically closed field of arbitrary characteristic. All of the material (and much more) can be found in the "Séminaire de géométrie algébrique" of Grothendieck, 1960-1961 Exposé V, IX and X.

I thank Mr. S. Anantharaman for the careful preparation of the notes.

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Prerequisites

We assume that the reader is somewhat familiar with the notion and elementary properties of preschemes. To give a rough indication: Chap. I, § 1-6 and Chap. II, pages 1-14, 100-103 and 110-114 of the EGA ("Éléments de Géometrie Algébrique" of Grothendieck and Dieudon'e). We have even recalled some of these required elementary properties in Chap. I and II of the notes but this is done very concisely.

We need also all the fundamental theorems of EGA, Chap. III (first part); these theorems are stated in the text without proof. We do not require the reader to be familiar with them; on the contrary, we hope that the applications which have been made will give some insight into the meaning of, and stimulate the interest in, these theorems.

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1.1

Let A be a commutative ring with 1 and S a multiplicatively closed set in A, containing 1. We then form fractions $\frac{a}{s}$, $a \in A$, $s \in S$; two fractions $\frac{a_1}{s_1}$, $\frac{a_2}{s_2}$ are considered equal if there is an $s_3 \in S$ such that $s_3(a_1 s_2 - a_2 s_1) = 0$. When addition and multiplication are defined in the obvious way, these fractions form a ring, denoted by $S^{-1}A$ and called the ring of fractions of A with respect to the multiplicatively closed set S. There is a natural ring-homomorphism $A \to S^{-1}A$ given by $a \mapsto a/1$. This induces $a(1-1)$ correspondence between prime ideals of A not intersecting S and prime ideals of $S^{-1}A$, which is lattice-preserving. If $f \in A$ and S_f is the multiplicatively closed set $\{1, f, f^2, \ldots\}$, the ring of quotients $S_f^{-1}A$ is denoted by A_f. If p is a prime ideal of A and $S = A - p$, the ring of quotients $S^{-1}A$ is denoted by A_p; A_p is a local ring. 1

1.2

Let A be a commutative ring with 1 and X the set of prime ideals of A. For any $E \subset A$, we define $V(E)$ as the subset $\{\underline{p} : \underline{p}$ a prime ideal $\supset E\}$ of X. Then the following properties are easily verified:

(i) $\bigcap_\alpha V(E_\alpha) = V(\bigcup_\alpha E_\alpha)$ 2

(ii) $V(E_1) \cup V(E_2) = V(E_1 \cdot E_2)$

(iii) $V(1) = \emptyset$

(iv) $V(0) = X$.

Thus, the sets $V(E)$ satisfy the axioms for closed sets in a topology on X. The topology thus defined is called the *Zariski topology* on X; the topological space X is known as $\operatorname{Spec} A$.

Note. $\operatorname{Spec} A$ is a generalisation of the classical notion of an affine algebraic variety.

Suppose k is an algebraically closed field and let $k[X_1, \ldots, X_n] = k[X]$ be the polynomial ring in n variables over k. Let $\mathfrak{a}$ be an ideal of $k[X]$ and V be the set in k^n defined by $V = \{(\alpha_1, \ldots, \alpha_n) : f(\alpha_1, \ldots, \alpha_n) = 0 \forall f \in \mathfrak{a}\}$. Then V is said to be an affine algebraic variety and the Hilbert's zero theorem says that the elements of V are in $(1-1)$ correspondence with the maximal ideals of $k[X]/\mathfrak{a}$.

Remarks 1.3. (a) If $\mathfrak{a}(E)$ is the ideal generated by E in A, then we have: $V(E) = V(\mathfrak{a}(E))$

(b) For $f \in A$, define $X_f = X - V(f)$; then the X_f form a basis for the Zariski topology on X. In fact, $X - V(E) = \bigcup\limits_{f \in E} X_f$ by (i).

3 (c) X is not in general Hausdorff; however, it is T_0.

(d) $X_f \subset \bigcup\limits_{\alpha} X_{f_\alpha} \iff \exists$ an $n \in \mathbb{Z}^+$ such that f^n is in the ideal generated by the f_α's.

For any ideal $\mathfrak{a}$ of A, define $\sqrt{\mathfrak{a}} = \{a \in A : a^n \in \mathfrak{a}$ for some $n \in \mathbb{Z}^+\}$. $\sqrt{\mathfrak{a}}$ is an ideal of A and we assert that $\sqrt{\mathfrak{a}} = \cap\{\underline{p} : \underline{p}$ a prime ideal $\supset \mathfrak{a}\}$. To prove this, we assume, (as we may, by passing to $A/\mathfrak{a}$) that $\mathfrak{a} = (0)$. Clearly, $\sqrt{(0)} \subset \cap \underline{p}$. On the other hand, if $a \in A$ is such that $a^n \neq 0 \forall n \in \mathbb{Z}^+$, then $S_a^{-1}A = A_a$ is a non-zero ring and so contains a proper prime ideal; the lift p of this prime ideal in A is such that $a \notin p$.

It is thus seen that $V(\mathfrak{a}) = V(\sqrt{a})$ and $V(f) \supset V(\mathfrak{a}) \iff f \in \sqrt{\mathfrak{a}}$. Hence:

$$X_f \subset \bigcup_\alpha X_{f_\alpha} \iff V(f) \supset \bigcap_\alpha V(f_\alpha) = V(\bigcup_\alpha \{f_\alpha\})$$

and this proves (d).

(e) The open sets X_f are quasi-compact.

In view of (b), it is enough to consider coverings by $X_g's$; thus, if $X_f \subset \bigcup_\alpha X_{f_\alpha}$, then by (d), we have $f^n = \sum_{i=1}^r a_i f_{\alpha_i}$ (say); again by (d), we obtain $X_f = X_{f^n} \subset \bigcup_{i=1}^r X_{f_{\alpha_i}}$.

(f) There is a $(1-1)$ correspondence between closed sets of X and roots of ideals of A; in this correspondence, closed irreducible sets of X go to prime ideals of A and conversely. Every closed irreducible set of X is of the form $(\overline{x})$ for some $x \in X$; such an x is called a *generic point* of that set, and is uniquely determined. **4**

(g) If A is noetherian, Spec A is a noetherian space (i.e. satisfies the minimum condition for closed sets).

1.4 The Sheaf associated to Spec A

We shall define a presheaf of rings on Spec A. It is enough to define the presheaf on a basis for the topology on X, namely, on the $X_f's$; we set $\mathscr{F}(X_f) = A_f$. If $X_g \subset X_f$, $V(g) \supset V(f)$ and so $g^n = a_0 \cdot f$ for some $n \in \mathbb{Z}^+$ and $a_0 \in A$. The homomorphism $A_f \to A_g$ given by $\frac{a}{f}q \mapsto \dfrac{a \cdot a_0^q}{g^{qn}}$ is independent of the way g is expressed in terms of f and thus defines a natural map $\rho_g^f : \mathscr{F}(X_f) \to \mathscr{F}(X_g)$ for $X_g \subset X_f$. The transitivity conditions are readily verified and we have a presheaf of rings on X. This defines a sheaf $\widetilde{A} = \mathscr{O}_X$ of rings on X. It is easy to check that the stalk $\mathscr{O}_{\underline{p},X}$ of $\mathscr{O}_X$ at a point $\underline{p}$ of X is the local ring $A_{\underline{p}}$.

If M is an A-module and if we define the presheaf $X_f \mapsto M_f = M \otimes_A A_f$, we get a sheaf $\widetilde{M}$ of $\widetilde{A}$-modules (in short, an $\widetilde{A}$-Module $\widetilde{M}$), whose stalk at $\underline{p} \in X$ is $M \otimes_A A_{\underline{p}} = M_P$.

5 **Remark.** The presheaf $X_f \mapsto M_f$ is a sheaf, i.e. it satisfies the axioms (F_1) and (F_2) of Godement, Théorie des faisceaux, p. 109.

1.5

In this section, we briefly recall certain sheaf-theoretic notions.

1.5.1 Let $f : X \to Y$ be a continuous map of topological spaces. Suppose $\mathscr{F}$ is a sheaf of abelian groups on X; we define a presheaf of abelian groups on Y by $U \mapsto \Gamma(f^{-1}(U), \mathscr{F})$ for any open $U \subset Y$; for $V \subset U$, open in Y, the restriction maps of this presheaf will be the restriction homomorphisms $\Gamma(f^{-1}(U), \mathscr{F}) \to \Gamma(f^{-1}(V), \mathscr{F})$. This presheaf is already a sheaf. The sheaf defined by this presheaf is called the *direct image* $f_*(\mathscr{F})$ of $\mathscr{F}$ under f.

If U is any neighbourhood of $f(x)$ in Y, the natural homomorphism $\Gamma(f^{-1}(U), \mathscr{F}) \to \mathscr{F}_x$ given a homomorphism $\Gamma(U, f_*(\mathscr{F})) \to \mathscr{F}_x$; by passing to the inductive limit as "U shrinks down to $f(x)$", we obtain a natural homomorphism:

$$f_x : f_*(\mathscr{F}) \underset{f(x)}{\longrightarrow} \mathscr{F}_x.$$

If $\mathscr{O}_X$ is a sheaf of rings on X, $f_*(\mathscr{O}_X)$ has a natural structure of a sheaf of rings on Y. If $\mathscr{F}$ is an $\mathscr{O}_X$-Module, $f_*(\mathscr{F})$ has a natural structure of an $f_*(\mathscr{O}_X)$-Module.

The direct image $f_*(\mathscr{F})$ is a covariant functor on $\mathscr{F}$.

6 **1.5.2** Let $f : X \to Y$ be a continuous map of topological spaces and $\mathfrak{g}$ be a sheaf of abelian groups on Y. Then it can be shown that there is a unique sheaf $\mathscr{F}$ of abelian groups on X such that:

(a) there is a natural homomorphism of sheaves of abelian groups

$$\rho = \rho_{\mathfrak{g}} : \mathfrak{g} \to f_*(\mathscr{F})$$

and

(b) for any sheaf $\mathscr{H}$ of abelian groups on X, the homomorphism $\mathrm{Hom}_X(\mathscr{F}, \mathscr{H}) \to \mathrm{Hom}_Y(\mathfrak{g}, f_*(\mathscr{H}))$ given by $\rho \mapsto f_*(\varphi) \circ \rho_{\mathfrak{g}}$ is an isomorphism. The unique sheaf $\mathscr{F}$ of abelian groups on X with these properties is called the *inverse image* $f^{-1}(\mathfrak{g})$ of $\mathfrak{g}$ under f.

It can be shown that canonical homomorphism

$$(*) \qquad f_x \circ \rho_{f(x)} : \mathfrak{g}_{f(x)} \to f^{-1}(\mathfrak{g})_x$$

is an isomorphism, for every $x \in X$.

The inverse image $f^{-1}(\mathfrak{g})$ is a covariant functor on $\mathfrak{g}$ and the isomorphism $(*)$ shows that it is an exact functor.

If $\mathscr{O}_Y$ is a sheaf of rings on Y, $f^{-1}(\mathscr{O}_Y)$ has a natural structure of a sheaf of rings on X. If $\mathfrak{g}$ is an $\mathscr{O}_Y$-Module, $f^{-1}(\mathfrak{g})$ has a natural structure of an $f^{-1}(\mathscr{O}_Y)$-Module.

1.5.3 A *ringed space* is a pair $(X, \mathscr{O}_X)$ where X is a topological space 7
and $\mathscr{O}_X$ is a sheaf of rings on X, called the structure sheaf of $(X, \mathscr{O}_X)$. A morphism $\Phi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of ringed spaces is a pair (f, φ) such that

(i) $f : X \to Y$ is a continuous map of topological spaces, and

(ii) $\varphi : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$ is a morphism of sheaves of rings on Y.

Ringed spaces, with morphisms so defined, form a category. Observe that condition (ii) is equivalent to giving a morphism $f^{-1}(\mathscr{O}_Y) \to \mathscr{O}_X$ of sheaves of rings on X (see (1.5.2)).

If $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules, we denote by $\Phi_*(\mathscr{F})$ the sheaf $f_*(\mathscr{F})$, considered as an $\mathscr{O}_Y$-Module through φ. If $\mathfrak{g}$ is an $\mathscr{O}_Y$-Module, $f^{-1}(\mathfrak{g})$ is an $f^{-1}(\mathscr{O}_Y)$-Module and the morphism $f^{-1}(\mathscr{O}_Y) \to \mathscr{O}_X$, defined by φ, gives an $\mathscr{O}_X$-Module $f^{-1}(\mathfrak{g})_{f^{-1}} \otimes_{(\mathscr{O}_Y)} \mathscr{O}_X$; the stalks of this $\mathscr{O}_X$-Module are isomorphic to $\mathfrak{g}_{f(x)} \otimes_{\mathscr{O}_{f(x)}} \mathscr{O}_x$, under the identification $f^{-1}(\mathfrak{g})_x \simeq \mathfrak{g}_{f(x)}$. We denote this $\mathscr{O}_X$-Module by $\Phi^*(\mathfrak{g})$. In general, Φ^* is *not* an exact functor on $\mathfrak{g}$.

1.6 Affine Schemes

A ringed space of the form $(\operatorname{Spec} A, \widetilde{A})$, A a ring, defined in (1.4) is called an *affine scheme.*

8 **1.6.1** Let $\varphi : B \to A$ be a ring-homomorphism; φ defines a map

$$f = {}^a\varphi : X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$$
$$\underline{p} \mapsto \varphi^{-1}(\underline{p}).$$

Since ${}^a\varphi^{-1}(V(E)) = V(\varphi(E))$ for any $E \subset B$, ${}^a\varphi$ is a continuous map. Let $s \in B$; φ defines, in a natural way, a homomorphism

$$\varphi_s : B_s \to A_{\varphi(s)}.$$

In view of the remark at the end of (1.4), this gives us a homomorphism:

$$B_s = \Gamma(Y_s, \widetilde{B}) \to {}^A\varphi(s) = \Gamma(X_{\varphi(s)}, \widetilde{A}) = \Gamma(Y_s, f_*(\widetilde{A}))$$

and hence a homomorphism $\widetilde{\varphi} : \widetilde{B} \to f_*(\widetilde{A})$. If $x \in X$ the stalk map defined by $\widetilde{\varphi}$, namely

$$\widetilde{\varphi}_x : \mathscr{O}_{f(x)} \simeq B_{f(x)} \to \mathscr{O}_x \simeq A_x$$

is a local homomorphism (i.e. the image of the maximal ideal in $B_{f(x)}$ is contained in the maximal ideal of A_x).

9 **Definition 1.6.2.** A morphism $\Phi : (\operatorname{Spec} A, \widetilde{A}) \to (\operatorname{Spec} B, \widetilde{B})$ of two affine schemes, is a morphism of ringed spaces with the additional property that Φ is of the form $({}^a\varphi, \widetilde{\varphi})$ for a homomorphism $\varphi : B \to A$ of rings.

It can be shown that a morphism $\Phi = (f, \varphi)$ of ringed spaces is a morphism of affine schemes $(\operatorname{spec} A, \widetilde{A})(\operatorname{spec} B, \widetilde{B})$ if and only if the stalk-maps

$$\mathscr{O}_{f(x)} \to \mathscr{O}_x \quad \text{defined by} \quad \Phi(\text{rather, by } \varphi)$$

are local homomorphisms.

Remarks 1.6.4. (a) If M is an A-module, $\widetilde{M}$ is an exact covariant functor on M.

(b) For any A-modules M, N, $\mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is canonically isomorphic to $\mathrm{Hom}_A(M, N)$.

(c) If $(X, \mathscr{O}_X) = (\mathrm{spec}\, A, \widetilde{A})$, $(Y, \mathscr{O}_Y) = (\mathrm{spec}\, B, \widetilde{B})$ are affine, there is a natural bijection from the set $\mathrm{Hom}(X, Y)$ of morphisms of affine schemes $X \to Y$ onto the set $\mathrm{Hom}(B, A)$ of ring-homomorphisms $B \to A$.

(d) Let $(X, \mathscr{O}_X) = (\mathrm{Spec}\, A, \widetilde{A})$ be an affine scheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. Then one can show that $\mathscr{F}$ is quasi-coherent (i.e. for every $x \in X$, $\exists$ an open neighbourhood U of x and an exact sequence $(\mathscr{O}_X|U) \xrightarrow{(I)} (\mathscr{O}_X|U) \xrightarrow{(J)} \mathscr{F}|U \to 0) \Leftrightarrow \mathscr{F} \simeq \widetilde{M}$ for an A-module M. If we assume that A is *noetherian*, one sees that $\mathscr{F}$ is coherent $\Leftrightarrow$ $\mathscr{F}$ is $\simeq$ to $\widetilde{M}$ for a finite type A-module M.

(e) Let $X \xrightarrow{\Phi} Y$ be a morphism of affine schemes and $\mathscr{F}$ (resp. $\mathfrak{g}$) **10**
be a quasicoherent $\mathscr{O}_X$-Module (resp. $\mathscr{O}_Y$-Module). Then one can define a quasi-coherent $\mathscr{O}_Y$-Module (resp. $\mathscr{O}_X$-Module) denoted by $\Phi_*(\mathscr{F})$ (resp. $\Phi^*(\mathfrak{g})$) just as in (1.5.3); if $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$ and $\Phi = ({}^a\varphi, \widetilde{\varphi})$ for a $\varphi : B \to A$ and if $\mathscr{F} = \widetilde{M}$, M an A-module (resp. $\mathfrak{g} = \widetilde{N}$, N a B-module) then $\Phi_*(\mathscr{F})$ (resp. $\Phi^*(\mathfrak{g})$) is canonically identified with $\widetilde{[\varphi]^M}$, where $[\varphi]^M$ is the abelian group M considered as a B-module through φ (resp. $\widetilde{M \otimes_B A}$).

(For proofs see EGA Ch. I)

Chapter 2

Preschemes

Definition 2.1. A ringed space $(X, \mathscr{O}_X)$ is called a *prescheme* if every 11
point $x \in X$ has an open neighbourhood U such that $(U, \mathscr{O}_X|U)$ is an affine scheme.

An open set U such that $(U, \mathscr{O}_X|U)$ is an affine scheme is called an affine open set of X; such sets form a basis for the topology on X.

Definition 2.1.1. A morphism $\Phi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of preschemes is a morphism (f, φ) of ringed spaces such that for every $x \in X$, the stalk-map $\varphi_x : \mathscr{O}_{f(x)} \to \mathscr{O}_x$ defined by Φ is a local homomorphism.

Preschemes then form a category (Sch). In referring to a prescheme, we will often suppress the structure sheaf from notation and denote $(X, \mathscr{O}_X)$ simply by X.

2.1.2 Suppose $\mathscr{C}$ is any category and $S \in \mathrm{Ob}\,\mathscr{C}$. We consider the pairs (T, f) where $T \in \mathrm{Ob}\cdot\mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(T, S)$.

If (T_1, f_1), (T_2, f_2) are two such pairs, we define $\mathrm{Hom}((T_1, f_1), (T_2, f_2))$ to be the set of $\mathscr{C}$-morphisms $\varphi : T_1 \to T_2$, making the diagram

$$\begin{tikzcd} T_1 \arrow[rr, "\varphi"] \arrow[dr, "f_1"'] & & T_2 \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$

commutative.

12 This way we obtain a category, denoted by $\mathscr{C}|S$. In the special case $\mathscr{C}$ = (Sch), the category (Sch $/S$) = (Sch)$|S$ is called the category of S-preschemes; its morphisms are called S-morphisms. S itself is known as the base prescheme of the category.

Remark 2.1.3. Let $\operatorname{Spec} A$ be an affine scheme and Y any prescheme. Then $\operatorname{Hom}(A, \Gamma(Y, \mathscr{O}_Y))$ is naturally isomorphic to $\operatorname{Hom}(Y, \operatorname{Spec} A)$.

In fact, let (U_i) be an affine open covering of Y and $\varphi \in \operatorname{Hom}(A, \Gamma(Y, \mathscr{O}_Y))$. The composite maps

$$\varphi_i : A \xrightarrow{\varphi} \Gamma(Y, \mathscr{O}_Y) \xrightarrow{\text{restriction}} \Gamma(U_i, \mathscr{O}_Y)$$

give morphisms ${}^a\varphi_i : U_i \to \operatorname{Spec} A$, for every i, since the U_i are affine. It is easily checked that ${}^a\varphi_i = {}^a\varphi_j$ on $U_i \cap U_j$, $\forall\ i, j$. We then get a morphism ${}^a\varphi : Y \to \operatorname{Spec} A$; the map $\varphi \mapsto {}^a\varphi$ is a bijection from $\operatorname{Hom}(A, \Gamma(Y, \mathscr{O}_Y))$ onto $\operatorname{Hom}(Y, \operatorname{Spec} A)$ (cf. (1.6.4)(c)).

It follows that every prescheme X can be considered as a $\operatorname{Spec}\mathbb{Z}$-prescheme in a natural way:

$$(\text{Sch}) = (\text{Sch} / \operatorname{Spec}\mathbb{Z}) = (\text{Sch} /\mathbb{Z}).$$

Remark 2.1.4. Let $(X, \mathscr{O}_X)$ be a prescheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. Then
13 it follows from (1.6.4)(d) that $\mathscr{F}$ is quasi-coherent $\Leftrightarrow$ for every $x \in X$ and *any* affine open neighbourhood U of x, $\mathscr{F}|U \simeq \widetilde{M_U}$, for a $\Gamma(U, \mathscr{O}_X)$ - module M_U.

We may take this as our definition of a quasi-coherent $\mathscr{O}_X$-module.

2.2 Product of Preschemes

2.2.0 Suppose (X, f), (Y, g) are S-preschemes. We say that a triple (Z, p, q) is a product of X and Y over S if:

(i) Z is an S-prescheme

(ii) $p : Z \to X$, $q : Z \to Y$ are S-morphisms and

(iii) for any $T \in (\mathrm{Sch}/S)$, the natural map:

$$\begin{gathered}\mathrm{Hom}_S(T, Z) \to \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y) \\ f \mapsto (p.f, q.f)\end{gathered}$$

is a bijection.

The product of X and Y, being a solution to a universal problem, is obviously unique upto an isomorphism in the category. We denote the product (Z, p, q), if it exists by $X \underset{S}{\times} Y$ and call it the *fibre-product* of X and Y over S; p, q are called projection morphisms.

Theorem 2.2.1. *If X, $Y \in (\mathrm{Sch}/S)$, the fibre-product of X and Y over S always exists.*

We shall not prove the theorem here. However, we observe that if **14**
$X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$ and $S = \mathrm{Spec}\, C$ are all affine, then $\mathrm{Spec}(A \underset{C}{\otimes} B)$ is a solution for our problem. In the general case, local fibre-products are obtained from the affine case and are glued together in a suitable manner to yield a fibre product $X \underset{S}{\times} Y$.

(For details see EGA, Ch. I, Theorem (3.2.6)).

Remarks. (1) The underlying set of $X \underset{S}{\times} Y$ is *not* the fibre-product of the underlying sets of X and Y over that of S. However if $x \in X$, $y \in Y$ lie over the same $s \in S$, then there is a $z \in X \underset{S}{\times} Y$ lying over x and y. (For a proof see Lemma (2.3.1)).

(2) An open subset U of a prescheme X can be considered as a prescheme in a natural way. Suppose $S' \subset S$, $U \subset X$, $V \subset Y$ are open sets such that $f(U) \subset S'$, $g(V) \subset S'$; we may consider U, V as S'-preschemes. When this is done, the fibre-product $U \underset{S'}{\times} V$ is isomorphic to the open set $p^{-1}(U) \cap q^{-1}(V)$ in $Z = X \underset{S}{\times} Y$, considered as a prescheme.

This follows easily from the universal property of the fibre-product.

2.2.2 Change of base: Let X, S' be S-preschemes. Then the fibe-product $X \underset{S}{\times} S'$ can be considered as an S'-prescheme in a natural way:

$$\begin{tikzcd} X & X \underset{S}{\times} S' \arrow[l] \arrow[d] \\ S \arrow[from=u] & S' \arrow[l] \end{tikzcd}$$

15 When this is done, we say that $X \underset{S}{\times} S'$ is obtained from X by the base-change $S' \to S$ and denote it by $X_{(S')}$. Note that, in the affine case, this corresponds to the extension of scalars.

If X is any prescheme, by its *reduction* mod p, $p \in \mathbb{Z}^+$, (resp. mod p^2 and so on) we mean the base-change corresponding to $\mathbb{Z} \to \mathbb{Z}_{/(p)}$ (resp. $\mathbb{Z} \to \mathbb{Z}_{/(p^2)}$ and so on).

If $f : X \to X'$., $g : Y \to Y'$ are S-morphisms f and g define, in a natural way an S-morphism: $X \underset{S}{\times} Y \to X' \underset{S}{\times} Y'$, which we denote by $f \underset{S}{\times} g$ or by $(f, g)_S$. When $g = I_{S'} : S' \to S'$, we get a morphism: $f \underset{S}{\times} I_{S'} = f_{(S')} : X_{(S')} \to X'_{(S')}$.

2.3 Fibres

Let (X, f) be an S-prescheme and $s \in S$ be any point. Let $U \subset S$ be an affine open neighbourhood of s and $A = \Gamma(U, \mathscr{O}_S)$. If p_s is the prime ideal of A corresponding to s, $\mathscr{O}_{s,S}$ is identified with A_{p_s}. Denote by $k(s)$ the residue field of $\mathscr{O}_{s,S} = A_{p_s}$. The composite $A \to A_{p_s} \to k(s)$ defines a morphism $\operatorname{Spec} k(s) \to \operatorname{Spec} A = U \subset S$; i.e. to say, $\operatorname{Spec} k(s)$ is an S-prescheme in a natural way. Consider now the base-change $\operatorname{Spec} k(s) \to S$:

$$\begin{tikzcd} X \arrow[d, "f"] & X' = X \underset{S}{\times} \operatorname{Spec} k(s) \arrow[l, "p"'] \arrow[d, "q"] \\ S & \operatorname{Spec} k(s) \arrow[l] \end{tikzcd}$$

16 The first projection p clearly maps X' into the set $f^{-1}(s) \subset S$. We

claim that $p(X') = f^{-1}(s)$ and further that, when we provide $f^{-1}(s)$ with the topology induced from X, p is a homomorphism between X' and $f^{-1}(s)$.

To prove this, it suffices to show that for every open set U in a covering of X, p is a homeomorphism from $p^{-1}(U)$ onto $U \cap f^{-1}(s)$. In view of the remark (2) after Theorem (2.2.1) we may then assume that X, S are affine say $X = \operatorname{Spec} A$, $S = \operatorname{Spec} C$. That $p(X') = f^{-1}(s)$ will follow as a corollary to the following more general result.

Lemma 2.3.1. *Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ be affine schemes over $S = \operatorname{Spec} C$. Suppose that $x \in X$, $y \in Y$ lie over the same element $s \in S$. Then the set E of elements $z \in Z = X \underset{S}{\times} Y$ lying over x, y is isomorphic to* $\operatorname{Spec}(k(x) \underset{k(S)}{\otimes} k(y))$ *(as a set).*

Proof. One has a homomorphism $\psi : A \underset{C}{\otimes} B \to k(x) \underset{k(s)}{\otimes} k(y)$ which gives a morphism a_ψ : $\operatorname{Spec} k(x) \underset{k(s)}{\otimes} k(y) \to Z$; clearly the image of a_ψ is contained in the set $E = p^{-1}(x) \cap q^{-1}(y)$. That a_ψ is injective is seen by factoring ψ as follows: $A \underset{C}{\otimes} B \to A_x \underset{C_s}{\otimes} B_y \to k(x) \underset{k(s)}{\otimes} k(y)$. In order to see that a_ψ is surjective one remarks that for $z \in E$ the homomorphism $A \to A \underset{C}{\otimes} B \to k(z)$ factors through $k(x)$, similarly for B, therefore we have for $A \underset{C}{\otimes} B \to k(z)$ a factorisation $A \underset{C}{\otimes} B \to k(x) \underset{k(s)}{\otimes} k(y) \to k(z)$. Q.E.D. □

In the above lemma if we take $B = k(s)$ it follows that in the diagram 17

$$\begin{tikzcd}
 & X' = X \underset{S}{\times} \operatorname{Spec} k(s) = \operatorname{Spec}(A \underset{C}{\otimes} k(s)) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X = \operatorname{Spec} A \arrow[dr, "f"'] & & Y = \operatorname{Spec} B \arrow[dl, "g"] \\
 & s \in S = \operatorname{Spec} C &
\end{tikzcd}$$

the map $p : X' \to f^{-1}(s)$ is a bijection.

2.3.2 Returning to the assertion that X' is homeomorphic to the fibre $f^{-1}(s)$ (with the induced topology from X) we note that if $\varphi : C \to k(s)$ is the natural map, then $p : X' \to X$ is the morphism corresponding to $1 \otimes \varphi : A \to A \underset{C}{\otimes} k(s)$. To show that p carries the topology over, it is enough to show that any closed set of X', of the form $V(E')$, is also of the form $V((1 \otimes \varphi)E)$ for some $E \subset A$.

Now, any element of $A \underset{C}{\otimes} k(s)$ can be written in the form $\sum\limits_i a_i \otimes \left(\dfrac{\overline{c}_i}{\overline{t}}\right) =$
18 $\left(\sum\limits_i (a_i \otimes \overline{c}_i)\right) \cdot \left(1 \otimes \dfrac{1}{\overline{t}}\right)$ with $a_i \in A$, c_i, $t \in C$. Since $\left(1 \otimes \dfrac{1}{\overline{t}}\right)$ is a unit of $A \underset{C}{\otimes} k(s)$, we can take for $E \subset A$, the set of elements $\sum\limits_i a_i c_i$ where $\sum\limits_i a_i \otimes \left(\overline{c}_i / \overline{t}\right)$ is an element of E'. Q.E.D.

Note. The fibre $f^{-1}(s)$ can be given a prescheme structure through this homeomorphism $p : X' \to f^{-1}(s)$. If, in the above proof, we had taken $\mathscr{O}_s / \mathscr{M}_s^{n+1}$, instead of $k(s) = \mathscr{O}_s / \mathscr{M}_s$ we would still have obtained homeomorphisms $p_n : X \underset{S}{\times} \operatorname{Spec}\left(\mathscr{O}_s / \mathscr{M}_s^{n+1}\right) \to f^{-1}(s)$. The prescheme structure on $f^{-1}(s)$ defined by means of p_n, is known as the n^{th}-*infinitesimal neighbourhood of the fibre*.

2.4 Subschemes

2.4.0 Let X be a prescheme and $\mathscr{J}$ a quasi-coherent sheaf of ideals of $\mathscr{O}_X$. Then the support Y of the $\mathscr{O}_X$-Module $\mathscr{O}_X / \mathscr{J}$ is closed in X and $(Y, \mathscr{O}_{X/\mathscr{J}}|Y)$ has a natural structure of a prescheme. In fact, the question is purely local and we may assume $X = \operatorname{Spec} A$. Then $\mathscr{J}$ is defined by an ideal I of A and Y corresponds to $V(I)$ which is surely closed. The ringed space $(Y, \mathscr{O}_{X/\mathscr{J}}|Y)$ has then a natural structure of an affine scheme, namely, that of $\operatorname{Spec}(A/I)$.

Such a prescheme is called a *closed subscheme* of X. An *open subscheme* is, by definition, the prescheme induced by X on an open subset in a natural way. A *subscheme* of X is a closed subscheme of an open subscheme of X.

2.4.1 A subscheme may have the same base-space as X. For example, 19
one can show that there is a quasi-coherent sheaf $\mathscr{N}$ of ideals of $\mathscr{O}_X$ such that $\mathscr{N}_x$ = nil-radical of $\mathscr{O}_x$. $\mathscr{N}$ defines a closed subscheme Y of X, which we denote by X_{red} and which is *reduced*, in the sense that the stalks $\mathscr{O}_{y,Y}$ of Y have *no nilpotent elements*. X and Y have the same base space (A and A_{red} have the same prime ideals).

Consider a morphism $f : X \to Y$ of preschemes. Suppose $\varphi_X : X_{\text{red}} \to X$, $\varphi_Y : Y_{\text{red}} \to Y$ are the natural morphisms. Then $\exists$ a morphism $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_{\text{red}} \arrow[u, "\varphi_X"] \arrow[r, "f_{\text{red}}"'] & Y_{\text{red}} \arrow[u, "\varphi_Y"'] \end{tikzcd}$$

commutative. This corresponds to the fact that a homomorphism $\varphi : A \to B$ of rings defines a homomorphism $\varphi_{\text{red}} : A_{\text{red}} \to B_{\text{red}}$ such that

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ A_{\text{red}} \arrow[r, "\varphi_{\text{red}}"] & B_{\text{red}} \end{tikzcd}$$

2.4.2 A morphism $f : Z \to X$ is called an *immersion* if it admits a 20
factorization $Z \xrightarrow{f'} Y \xrightarrow{j} X$ where Y is a sub-scheme of X, $j : Y \to X$ is the canonical inclusion and $f' : Z \to Y$ is an isomorphism. The immersion f is said to be *closed* (resp. *open*) if Y is a closed subscheme (resp. open subscheme) of X.

Example . Let $X \xrightarrow{f} S$ be an S-prescheme. Then, there is a natural S-morphism $\Delta : X \to X \underset{S}{\times} X$ such that the diagram

$$\begin{tikzcd}
 & & X \arrow[dll, "I_X = \mathrm{id}_X"'] \arrow[dl, "\Delta"] \arrow[dd, "I_X"] \\
X \arrow[dd, "f"'] & X \underset{S}{\times} X \arrow[l, "p_1"] \arrow[dr, "p_2"'] & \\
 & & X \arrow[dll, "f"] \\
S & &
\end{tikzcd}$$

is commutative. Δ is called the diagonal of f. It is an immersion.

Definition 2.4.2.1. A morphism $f : X \to S$ is said to be *separated* (or X is said to be an S-scheme) if the diagonal $\Delta : X \to X \underset{S}{\times} X$ of f is a *closed* immersion.

A prescheme X is called a *scheme* if the natural map $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

Remark . Let Y be an affine scheme, X any prescheme and $(U_\alpha)_\alpha$ an
21 affine open c over for X. One can then show that a morphism $f : X \to Y$
is separated if and only if $\forall\ \alpha, \beta$,

(i) $U_\alpha \cap U_\beta$ is also affine

(ii) $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ is generated as a ring by the canonical images of $\Gamma(U_\alpha, \mathcal{O}_X)$ and $\Gamma(U_\beta, \mathcal{O}_X)$.

(For a proof see EGA, Ch. I, Proposition (5.5.6)).

2.4.3 Example of a prescheme which is not a scheme. Let $B = k[X]$, $C = k[Y]$ be polynomial rings over a field k. Then $\operatorname{Spec} B_X$ and $\operatorname{Spec} C_Y$ are affine open sets of $\operatorname{Spec} B$ and $\operatorname{Spec} C$ respectively; the isomorphism

$\dfrac{f(X)}{X^m} \mapsto \dfrac{f(Y)}{Y^m}$ of B_X onto C_Y defines an isomorphism $\operatorname{Spec} C_Y \xrightarrow{\sim} \operatorname{Spec} B_X$. By recollement of $\operatorname{Spec} B$ and $\operatorname{Spec} C$ through this isomorphism, one gets a prescheme S, which is *not* a scheme; in fact, condition (ii) of the proceding remark does *not* hold: for, $\Gamma(\operatorname{Spec} B, \mathscr{O}_S) \simeq B = k[X]$ and $\Gamma(\operatorname{Spec} C, \mathscr{O}_S) \simeq C = k[Y]$; the canonical maps from these into $\Gamma(\operatorname{Spec} B \cap \operatorname{Spec} C, \mathscr{O}_S) \simeq k[u, u^{-1}]$ are given by $X \mapsto u$, $Y \mapsto u$ and the image in each case is precisely $= k[u]$.

2.5 Some formal properties of morphisms

(i) every immersion is separated

(ii) $f : X \to Y$, $g : Y \to Z$ separated $\Rightarrow g \circ f : X \to Z$ separated.

(iii) $f : X \to Y$ a separated S-morphism $\Rightarrow f_{(S')} : X_{(S')} \to Y_{(S')}$ is separated for every base-change $S' \to S$.

(iv) $f : X \to Y$, $f' : X' \to Y'$ are separated S-morphisms $\Rightarrow f \underset{S}{\times} f' :$ **22**
$X \underset{S}{\times} X' \to Y \underset{S}{\times} Y'$ is separated.

(v) $g \circ f$ separated $\Rightarrow f$ is separated

(vi) f separated $\Leftrightarrow f_{\mathrm{red}}$ separated.

The above properties are *not all independent*. In fact, the following more general situation holds:

Let P be a property of morphisms of preschemes.

Consider the following propositions:

(i) every closed immersion has P

(ii) $f : X \to Y$ has P, $g : Y \to Z$ has $P \Rightarrow g \circ f$ has p

(iii) $f : X \to Y$ is an S-morphism having $P \Rightarrow f_{(S')} : X_{(S')} \to Y_{(S')}$ has P for any base-change $S' \to S$.

(iv) $f : X \to Y$ has P, $f' : X' \to Y'$ has $P \Rightarrow$ that $f \underset{S}{\times} f' : X \underset{S}{\times} X' \to Y \underset{S}{\times} Y'$ has P

(v) $g \circ f$ has P, g separated $\Rightarrow$ f has P.

(vi) f has $P \Rightarrow f_{\text{red}}$ has P.

If we suppose that (i) and (ii) hold then (iii) $\Leftrightarrow$ (iv). Also, (v), (vi) are consequences of (i), (ii) and (iii) (or (iv)).

Proof. Assume (ii) and (iii). The morphism $f \underset{S}{\times} f'$ admits a factorization:

$$\begin{tikzcd}
X \times X' \arrow[rr, "f \underset{S}{\times} f'"] \arrow[dr, "f \underset{S}{\times} I_{X'}"'] & & Y \underset{S}{\times} Y' \\
& Y \underset{S}{\times} X' \arrow[ur, "I_Y \underset{S}{\times} f'"'] &
\end{tikzcd}$$

23 By (iii), the morphisms $f \underset{S}{\times} T_{X'}$ and $I_Y \underset{S}{\times} f'$ have P and so by (ii) $f \underset{S}{\times} f'$ also has P. □

On the other hand, assume (i) and (iv). $I_{S'}$ being a closed immersion, has P by (i) and so $f_{(S')} = f \underset{S}{\times} I_{S'}$ has P by (iv).

Now assume (i), (ii) and (iii). If $g : Y \to Z$ is separated, $Y \xrightarrow{\Delta} Y \underset{Z}{\times} Y$ is a closed immersion and has P by (i); by making a base-change $X \xrightarrow{f} Y$ we get a morphism $X \simeq Y \underset{Y}{\times} X \xrightarrow{\Delta_Y \times I_X} Y \underset{Z}{\times} Y \underset{Y}{\times} X \simeq X \underset{Z}{\times} Y$ which, by (iii) has property p. The projection $p_2 : X \underset{Z}{\times} Y \to Y$ satisfies the diagram:

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "g \circ f"'] & X \underset{Z}{\times} Y \arrow[d, "p_2"] \\
Z & Y \arrow[l, "g"]
\end{tikzcd}$$

i.e. to say, p_2 is obtained from $g \circ f$ by the base-change $Y \to Z$ and so, by (iii) has P.

Finally, $f : X \to Y$ is the composite of $X \xrightarrow{\Delta_Y \times I_X} X \underset{Z}{\times} Y$ and $p_2 : X \underset{Z}{\times} Y \to Y$ and so by (ii) has P.

To prove (vi) from (i), (ii), (iii) use the diagram 24

$$\begin{tikzcd}
X_{\text{red}} \arrow[r, "f_{\text{red}}"] \arrow[d, "\varphi_X"'] & Y_{\text{red}} \arrow[d, "\varphi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and the facts that the canonical morphisms φ_X, φ_Y are closed immersions and so have P, that $\varphi_Y \circ f_{\text{red}} = f \circ g$ has P and that a closed immersion is separated, then use (v). Q.E.D.

We now remark that if we replace (i) of the above propositions by

(i)′ every immersion has P, then (i), (ii), (iii) imply (v)′ $g \circ f$ has P, g has $P \Rightarrow f$ has P.

2.6 Affine morphisms

Definition 2.6.1. A morphism $f : X \to S$ of preschemes is said to be *affine* (or *X affine over S*) if, for every affine open $U \subset S$, $f^{-1}(U)$ is affine in X.

It is enough to check that for an affine open cover (U_α) of S, the $f^{-1}(U_\alpha)$ are affine.

2.6.2 Suppose that $\mathscr{B}$ is a quasi-coherent $\mathscr{O}_S$-Algebra. Let (U_α) be an affine open cover of S; set $A_\alpha = \Gamma(U_\alpha, \mathscr{O}_S)$, $B_\alpha = \Gamma(U_\alpha, \mathscr{B})$ and $X_\alpha = \operatorname{Spec} B_\alpha$. The homomorphism $A_\alpha \to B_\alpha$ defines a morphism 25
$f_\alpha : X_\alpha \to U_\alpha$; the $X'_\alpha s$ then patch up together to give an S-prescheme $X \xrightarrow{f} S$; this prescheme X is affine over S, is such that $f_\alpha(\mathscr{O}_X) \simeq \mathscr{B}$, and is determined, by this property, uniquely upto an isomorphism. We denote it by $\operatorname{Spec} \mathscr{B}$. conversely, every affine S-prescheme is obtained

as Spec $\mathscr{B}$, for some quasi-coherent $\mathscr{O}_S$-Algebra $\mathscr{B}$. (For details, see EGA Ch. II, Proposition (1.4.3)).

Remarks. (a) Any affine morphism is separated. (Recall the remark at the end of (2.4.2).)

(b) If S is an affine scheme, a morphism $f : X \to S$ is affine $\Leftrightarrow$ X is an affine scheme.

(c) The formal properties (i) to (vi) of (2.5) hold, when P is the property of being affine.

(d) Suppose $X \xrightarrow{h} Y$ is an S-morphism. If f, g are the structural morphisms of X, Y resply, the homomorphism $\mathscr{O}_Y \to h_*(\mathscr{O}_X)$ defined by h, given an $\mathscr{O}_S$-morphism

$$\mathscr{A}(h) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X);$$

then we have a natural map: $\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_{\mathscr{O}_S}(g_*(\mathscr{O}_Y) \to f_*(\mathscr{O}_X))$ (the latter in the sense of $\mathscr{O}_S$-Algebras) defined by $h \mapsto \mathscr{A}(h)$. If Y is affine over S, it can be shown that this natural map is a *bijection*. (EGA Ch II, Proposition (1.2.7)). (Also, compare with remark (1.6.4) (c) for affine schemes, and with remark (2.1.3)).

2.7 The finiteness theorem

26 **Definition 2.7.0.** A morphism $f : X \to Y$ of preschemes is said to be of *finite type*, if, for every affine open set U of Y, $f^{-1}(U)$ can be written as $f^{-1}(U) = \bigcup_{\alpha=1}^{n} V_\alpha$, with each V_α affine open in X and each $\Gamma(V_\alpha, \mathscr{O}_X)$ a finite type $\Gamma(U, \mathscr{O}_Y)$-algebra.

It is again enough to check this for an affine open cover of Y. A_n affine morphism $f : X \to Y$ is of finite type $\Longleftrightarrow$ the quasi-coherent $\mathscr{O}_Y$-Algebra $f_\alpha(\mathscr{O}_X)$ is an $\mathscr{O}_Y$-Algebra of finite type. In particular, a morphism $f :$ Spec $B \to$ Spec A is of finite type $\Longleftrightarrow$ B is a finite type A-algebra (i.e. is finitely generated as A-algebra).

Definition 2.7.1. A morphism $f : X \to S$ is *universally closed* if, for every base-change $S' \to S$, the morphism $f_{(S')} : X \underset{S}{\times} S' \to S'$ is a closed map in the topological sense.

Definition 2.7.2. A morphism f is *proper* if

(i) f is separated

(ii) f is of finite type and

(iii) f is universally closed.

The formal properties of (2.5) hold then P is the property of being proper.

Definition 2.7.3. A prescheme Y is *locally noetherian*, if every $y \in Y$ has an affine open neighbourhood $\operatorname{Spec} B$, with B noetherian. It is said to be *noetherian*, if it can be written as $Y = \bigcup_{i=1}^{n} Y_i$ where the Y_i are affine **27**
open sets such that the $\Gamma(Y_i, \mathscr{O}_Y)$ are noetherian rings.

If $f : X \to Y$ is a morphism of finite type and Y is locally noetherian, then X is also locally noetherian.

2.7.4 Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces and f a morphism from X to Y. Let $\mathscr{F}$ be an $\mathscr{O}_X$-Module. We then define, for every $q \in \mathbb{Z}^+$, a presheaf of modules on Y by defining: $U \mapsto H^q(f^{-1}(U), \mathscr{F})$ for every open $U \subset Y$ (See EGA Ch 0, III § 12). The sheaf that this presheaf defines on Y is called the q^{th}-*direct image* of $\mathscr{F}$ and is denoted by $R^q f_*(\mathscr{F})$.

Theorem 2.7.5. *Let X, Y be preschemes, Y locally noetherian, and f a* proper *morphism from X to Y. Then, if $\mathscr{F}$ is any coherent $\mathscr{O}_X$-Module, the direct images $R^q f_*(\mathscr{F})$ are all coherent $\mathscr{O}_Y$-Modules.*

(For a proof see EGA Ch. III, Theorem (3.2.1)). This is the *theorem of finiteness* for proper morphisms.

Chapter 3

Étale Morphisms and Étale Coverings

Throughout this chapter, by a prescheme we will mean a locally noethe- **28**
rian prescheme and by a morphism, a morphism of finite type (unless it is clear from the context that the morphism is *not* of finite type, e.g., $S \leftarrow \operatorname{Spec} \mathscr{O}_{s,S}$, $\operatorname{Spec} A \leftarrow \operatorname{Spec} \widehat{A}$, A a noetherian local ring).

Definition 3.1.0. A morphism $f : X \to S$ is said to be *unramified* at a point $x \in X$ if (i) $\mathscr{M}_{f(x)}\mathscr{O}_x = \mathscr{M}_x$, (ii) $k(x)/k(f(x))$ is a finite separable extension.

Definition 3.1.1. A morphism $f : X \to S$ is said to be *flat* at a point $x \in X$ if the local homomorphism $\mathscr{O}_{f(x)} \to \mathscr{O}_x$ is flat (i.e., $\mathscr{O}_x$, considered as an $\mathscr{O}_{f(x)}$-module is flat; note that since the homomorphism $\mathscr{O}_{f(x)} \to \mathscr{O}_x$ is local, $\mathscr{O}_x$ will be *faithfully* $\mathscr{O}_{f(x)}$-flat).

Definition 3.1.2. A morphism $f : X \to S$ is said to be *étale* at a point $x \in X$ if it is both unramified and flat at x.

We say that $f : X \to S$ is unramified (resp. flat, étale) if it is unramified (resp. flat, étale) at every $x \in X$.

Remarks 3.1.3. (1) An unramified morphism $X \xrightarrow{f} S$ is étale at $x \in X \Leftrightarrow \widehat{\mathscr{O}}_{f(x)} \to \widehat{\mathscr{O}}_x$ is flat.

29 (2) A morphism $f : X \to S$ is unramified at

$$x \in X \iff f_{\operatorname{Spec} k(f(x))} : X \underset{S}{\times} \operatorname{Spec} k(f(x)) \to \operatorname{Spec} k(f(x))$$

is unramified at the corresponding point; in other words, *it is enough to look at the fibre for nonramification.*

(3) If $f : X \to S$ is unramified at $x \in X$ and $k(f(x)) \overset{\hookrightarrow}{=} k(x)$ then $\widehat{\mathcal{O}}_{f(x)} \to \widehat{\mathcal{O}}_x$ is surjective; if f is étale in addition, then $\widehat{\mathcal{O}}_{f(x)} \xrightarrow{\sim} \widehat{\mathcal{O}}_x$.

3.2 Examples and Comments

(1) A morphism $f : \operatorname{Spec} A \to \operatorname{Spec} k$ (k a field) is étale $\Leftrightarrow$ it is unramified $\Leftrightarrow A = \bigoplus_{i=1}^{r} K_i$, $K_{i/k}$ finite separable extensions.

(2) Let X and S be irreducible algebraic varieties, S normal. Then it can be shown that a dominant morphism $X \to S$ is étale $\Longrightarrow$ it is unramified.

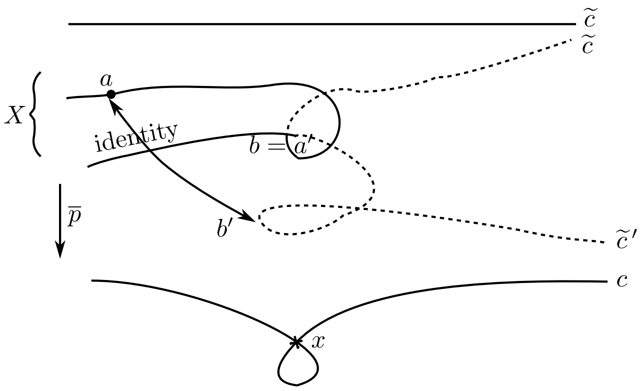
(3) If S is non-normal, an unramified dominant morphism $X \to S$ need not be étale; for instance, let c be an irreducible curve over an algebraically closed filed with an ordinary double point x, $\widetilde{c}$ its normalisation and $p : \widetilde{c} \to c$ the natural map.

(i) *p is unramified.* One has to prove this only at the points $a, b \in \widetilde{c}$ sitting over x. Now $\mathcal{M}_a$ is generated in $\mathcal{O}_a$ by a function with a simple zero at a, but such a function we
30 can find already in $\mathcal{O}_x$, for instance a function induced by a straight line through x not tangent to c at x.

(ii) *p is not étale*: otherwise, $\widehat{\mathcal{O}}_x \xrightarrow{\sim} \widehat{\mathcal{O}}_a$ but we know that $\widehat{\mathcal{O}}_x$ is *not* a domain, while $\widehat{\mathcal{O}}_a$ is.

(4) A connected étale variety over an irreducible algebraic variety *need not be irreducible*.

For instance, in the previous example we may take two copies $\widetilde{c}$ and $\widetilde{c}'$ of the normalisation of c and fuse them together in such a way that the points a, b on $\widetilde{c}$ are identified with the points b', a' on $\widetilde{c}'$. We then get a *connected* but reducible variety X and the morphism $\overline{p} : X \to c$ defined in the obvious manner is surely étale.



(5) Let X and S be irreducible algebraic varieties and suppose that S is normal and $f : X \to S$ a dominant morphism. Assume, in addition, that the function field $R(X)$ of X is a finite extension of degree n of the function field $R(S)$ of S. If $x \in X$ is such that the number of points in the fibre $f^{-1}(f(x))$ equals n, then it can be shown that f is étale in a neighbourhood of x. **31**

3.3

Our aim now is to give a necessary and sufficient condition for a morphism to be unramified.

3.3.0 Some algebraic preliminaries. Let $A \xrightarrow{\varphi} B$ be a homomorphism of rings defining an A-algebra structure on B. Then $B \underset{A}{\otimes} B$ is an A-algebra in a natural way and

$$p_1 : B \to B\underset{A}{\otimes}B \text{ given by } b \mapsto b \otimes 1$$
$$p_2 : B \to B\underset{A}{\otimes}B \text{ given by } b \mapsto 1 \otimes b$$

and $\mu : B\underset{A}{\otimes}B \to B$ given by $b_1 \otimes b_2 \mapsto b_1 b_2$ are all A-algebra homomorphisms. We may make $B\underset{A}{\otimes}B$ a B-algebra through p_1 i.e. by defining $b(b_1 \otimes b_2) = bb_1 \otimes b_2$ and the kernel I of μ is then a B-module. Since we have $\mu \circ p_1 = \mathrm{Id}$. and the equality $b_1 \otimes b_2 = (b_1 b_2 \otimes 1) - b_1(b_2 \otimes 1 - 1 \otimes b_2)$ is follows that $B\underset{A}{\otimes}B = (B\underset{A}{\otimes}1) \oplus I$, as a B-module.

32 We define the space $\Omega_{B/A}$ of A-differentials of B as the B-module I/I^2. We note that the B-module structure on $\Omega_{B/A} = I/I^2$ is natural, in the sense that it does not depend on whether we make $B\underset{A}{\otimes}B$, a B-algebra through p_1 or through p_2.

Some properties of $\Omega_{B/A}$

(1) The map $d : b \mapsto (1 \otimes b - b \otimes 1) \pmod{I^2}$ from B to $\Omega_{B/A}$ is A-linear. Also, since

$$\begin{aligned}(1 \otimes b_1 b_2 - b_1 b_2 \otimes 1) &= b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1)\\ &= +(1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1),\end{aligned}$$

we have: $d(b_1 b_2) = b_1 db_2 + b_2 db_1 \ \forall\, b_1, b_2 \in B$.

(2) Since I is generated as a B-module by elements of the form $(1 \otimes b_1 - b_1 \otimes 1)$, it follows that $\Omega_{B/A}$ is generated by the elements db.

(3) Let W be the B-submodule of $B\underset{A}{\otimes}B$, generated by elements of the form $(1 \otimes bb' - b \otimes b' - b' \otimes b)$; $\Omega_{B/A}$ is then isomorphic to $(B\underset{A}{\otimes}B)/W$.

In fact,

$$\Omega_{B/A} = I/I^2 \cong (B\underset{A}{\otimes}1 \oplus I)/(B\underset{A}{\otimes}1 \oplus I^2)$$

$$= (B\underset{A}{\otimes}B)/(B\underset{A}{\otimes}1 \oplus I^2)$$

33 and clearly $W \supset B\underset{A}{\otimes}1$; so $W = B \otimes 1 \oplus (W \cap I)$.

On the other hand, we have

$$(1 \otimes bb' - b \otimes b' - b' \otimes b) = -bb' \otimes 1 + (1 \otimes b - b \otimes 1)(1 \otimes b' - b' \otimes 1)$$

and hence: $W \subset B\underset{A}{\times}1 \oplus I^2$. Also clearly, $I^2 \subset W$. It follows that $W \cap I - I^2$, $W = B\underset{A}{\otimes}1 \oplus I^2$ and $\Omega_{B/A} \cong (B\underset{A}{\otimes}B)/W$.

(4) If V is any B-module and $D : B \to V$ any A-derivation of B in V, then there is a B-linear map $H_D : \Omega_{B/A} \to V$ such that

$$\begin{tikzcd}
B \arrow[rr, "D"] \arrow[dr, "d"'] \arrow[rr, phantom, shift right=6ex, "\circlearrowleft"] & & V \\
& \Omega_{B/A} \arrow[ur, "H_D"'] &
\end{tikzcd}$$

In fact, D defines an A-linear map $B\underset{A}{\otimes}B \to V$ given by $b_1 \otimes b_2 \mapsto b_1 Db_2$. This is certainly B-linear and is trivial on W. We then obtain a B-linear map $H_D : \Omega_{B/A} \to V$ such that $H_D(db) = Db\ \forall\ b \in B$. Also the correspondence $D \mapsto H_D$ is a B-isomorphism from the B-module of A-derivations $B \to V$, onto the B-module Hom_B $(\Omega_{B/A}, V)$. (This is the universal property of $\Omega_{B/A}$). In particular, the B-module of A-derivations of B is isomorphic to Hom_B
$(\Omega_{B/A}, B)$; and even more in particular, if $A = k$, $B = K$ are fields, 34
we see that the space of k-differentials $\Omega_{K/k}$ is the dual of the space of k-derivations of K and therefore is trivial if $K/_k$ is separably algebraic.

(5) Consider a base-change $A \to A'$ and the extension of scalars $B' = B\underset{A}{\otimes}A'$. From the universal property of the space of differentials, it

follows that

$$\Omega_{B'/A'} \simeq \Omega_{B/A} \underset{B}{\otimes} B' \Omega_{B/A} \underset{A}{\otimes} A'.$$

3.3.1 The sheaf of differentials of a morphism.

Let $f : X \to S$ be a morphism. Our aim now is to define a sheaf $\Omega_{X/S}$ on X. Assume, to start with, that $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$; then $\Omega_{B/A}$ is a B-module in a natural way and we define $\Omega_{X/S} = \widetilde{\Omega}_{B/A}$ on X. In the general case, consider the diagonal Δ of f in $X \underset{S}{\times} X$. Then Δ is a closed subscheme of an open subscheme U of $X \underset{S}{\times} X$ and $X \to \Delta$ is an isomorphism. Let I_X be the sheaf of ideals of $\mathscr{O}_U$ defining Δ. Then I_{X/I_X^2} is a sheaf of $\mathscr{O}_U$-modules whose support is contained in Δ. By means of the isomorphism $X \to \Delta$, we can lift this sheaf to a sheaf of abelian groups on X. Clearly the lift is independent of the choice of U. It remains to show that this sheaf is an $\mathscr{O}_X$-Module. To do this, we can assume again that $X \times S$ are affine and in this case, I_X is the sheaf $\widetilde{I}$ on $X \underset{S}{\times} X$ (where I is the ideal of $B \underset{A}{\otimes} B$ defined in 3.3.0) and therefore, our sheaf is $\widetilde{\Omega}_{B/A}$ considered above.

The $\mathscr{O}_X$-Module thus obtained will be called the *sheaf of differentials* of f (or of X over S) and will be denoted $\Omega_{X/S}$. The stalk $\Omega_{x,X/S}$ of this sheaf at a point $x \in X$ is canonically isomorphic to $\Omega_{\mathscr{O}_x/\mathscr{O}_{f(x)}}$. This is seen either from the universal property or by observing that the natural map $\Omega_{x,X/S} \to \Omega_{\mathscr{O}_x/\mathscr{O}_{f(x)}}$ given by

$$\sum_i \frac{b_i}{s_i} db_i' \mapsto \sum_i \frac{b_i}{s_i} d(b'_{i/1}) \quad (\text{with } X = \operatorname{Spec} B, b_i, b_i', s_i \in B, s_i \notin \underline{p}_x)$$

is an isomorphism.

Since, by assumptions, S is locally noetherian and f is of finite type it is cler from the definition that $\Omega_{X/S}$ is coherent.

Proposition 3.3.2. *For a morphism $f : X \to S$ and a point $x \in X$ the following are equivalent:*

(i) *f is unramified at x*

(ii) $\Omega_{x,X/S} = (0)$

(iii) $\Delta : X \to X\underset{S}{\times}X$ *is an open immersion in a neighbourhood of* x.

Proof. We may assume $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$. **36**

(i) $\Rightarrow$ (ii): Consider the base change:

$$\begin{tikzcd} x \in X \arrow[d] & X' = XS\underset{S}{\times}\operatorname{Spec} k(s) \arrow[l] \arrow[d] \\ s = f(x) \in S & \operatorname{Spec} k(s) = S' \arrow[l] \end{tikzcd}$$

If $x' \in X'$ is above x, then we have: $\Omega_{x,X/S} \underset{\mathscr{O}_x}{\otimes} \mathscr{O}_{x'} = \Omega_{x',X'/S'}$; this follows from property (5) of 3.3.0; furthermore,$\mathscr{O}_{x'} = \mathscr{O}_{x/\mathscr{M}_s\mathscr{O}_x} = k(x)$ since f is unramified at x. Therefore $\Omega_{x,X/S} \underset{\mathscr{O}_x}{\otimes} k(x) = \Omega_{x',X'/S'}$ and by Nakayama it suffices to show that the latter is (0). By the remark made above, $\Omega_{x',X'/S'} = \Omega_{k(x)/k(s)}$ and this is zero as $k(x)/k(s)$ is separably algebraic.

(ii) $\Rightarrow$ (iii)

Let z be the image of x in $X\underset{S}{\times}X$ under the diagonal $\Delta : X \to X\underset{S}{\times}X \cdot \Delta(X)$ is a closed subscheme of an open subscheme U of $X\underset{S}{\times}X$, and is defined, therefore, by a sheaf I of $\mathscr{O}_U$-ideals. If $\Omega_{x,X/S} = (0)$, then, we will have $I_z = I_z^2 = \ldots$; hence $I_z = (0)$ (Krull's intersection theorem). **37**
However, I is a sheaf of ideals of finite type and so I vanishes in a neighbourhood of z i.e. to say $\Delta : X \to X\underset{S}{\times}X$ is an open immersion in a neighbourhood of $x \in X$.

(iii) $\Rightarrow$ (i)

The question being local, we may assume that $\Delta : X \to X\underset{S}{\times}X$ is an open immersion everywhere. An open immersion remains an open immersion under base-change, and since we only have to look at the fibre over $f(x)$, for non-ramification at x, we may take $X = \operatorname{Spec} A$, $S = \operatorname{Spec} k$, k a field and A a k-algebra of finite type.

We are to show that $A = \bigoplus\limits_{\text{finite}} K_i$, where $K_{i/k}$ are finite separable field extensions; for this, we have to show that A is artinian and, if $\bar{k}$ is the algebraic closure of k, $A\underset{k}{\otimes}\overline{k}$ is radical-free. It is thus enough to show that $A\underset{k}{\otimes}\overline{k} = \bigoplus_{\text{finite}} \overline{k}$. By making the base-change $k \to \overline{k}$, we may assume k is algebraically closed.

Let $a \in X$ be any closed point of X. Since k is algebraically closed, $k(a) \simeq k$ and we have then

$$X \underset{i}{\overset{\sim}{\longleftarrow}} X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k(a) = X \times (a) \quad \text{(say)}.$$

$$X \times (a) \quad \text{is canonically imbedded in } X\underset{S}{\times}X$$

and let φ be the composite of the morphisms:

$$X \xrightarrow{i^{-1}} X \times (a) \hookrightarrow X\underset{S}{\times}X.$$

38 Then $\varphi^{-1}(\Delta) = (a)$ is open in X, since, by assumption, Δ is open in $X\underset{S}{\times}X$; i.e. any closed point of X is also open. But $X = \operatorname{Spec} A$ is quasi-compact and this implies that A has only finitely many maximal ideals. But A is a k-algebra of finite type and so the set of closed points of X is dense in X. It follows that A is artinian and we may write $A = \bigoplus_{i=1}^{n} A_i$ where the A_i are artinian local rings. We may then assume $A = A_i$; the open immersion $\Delta : X \to X\underset{S}{\times}X$ then gives an isomorphism $A\underset{k}{\otimes}A \to A$. This however means $A = k$. Q.E.D. □

3.3.3 Some properties of étale morphisms.

(1) An open immersion is étale.

(2) f, g étale $\Rightarrow g \circ f$ étale.

(3) f étale $\Rightarrow f_{(S')}$ étale for *any* base change $S' \to S$

(*not necessarily of finite type*), S', S locally noetherian.

This follows from condition (iii) of Proposition 3.3.2, the facts that an open immersion remains an open immersion and that a flat morphism remains flat under base change.

(4) f_1, f_2 étale $\Rightarrow f_1 \underset{S}{\times} f_2$ étale.

Follows from (2), (3) and the equality

$$f_1 \underset{S}{\times} f_2 = (f_1 \underset{S}{\times} I_{Y_2}) \circ (I_{X_1} \underset{S}{\times} f_2).$$

(5) $g \circ f$ étale, g unramified $\Rightarrow f$ étale. **39**

Let $f : X \to Y$ and $g : Y \to Z$. The question being local, we may assume from (iii) Proposition 3.3.2 that $\Delta = \Delta(Y) : Y \to Y \underset{Z}{\times} Y$ is an open immersion, hence étale. By the base-change $X \xrightarrow{f} Y$ we get $\Delta_{(X)} : X \to X \underset{Z}{\times} Y$ which again is étale by (3). The diagram:

$$\begin{tikzcd}
X \arrow[d, "g\circ f"'] & X \underset{Z}{\times} Y \arrow[l] \arrow[d, "p_2"] \\
Z & Y \arrow[l, "g"]
\end{tikzcd}$$

shows that the second projection $p_2 : X \underset{Z}{\times} Y \to Y$ is given by $(g \circ f)_{(Y)}$; so, again by (3), p_2 is also étale and from (2) we obtain that $f = p_2 \circ \Delta_{(X)} : X \xrightarrow{\Delta_{(X)}} X \underset{Z}{\times} Y \underset{p_2}{\longrightarrow} Y$ is étale.

(6) $g \circ f$ étale, f étale $\Rightarrow g$ étale.

The composite: $\mathscr{O}_{gf(x)} \Rightarrow \mathscr{O}_{f(x)} \to \mathscr{O}_x$ is flat and $\mathscr{O}_{f(x)} \to \mathscr{O}_x$ is faithfully flat, and thus $\mathscr{O}_{gf(x)} \to \mathscr{O}_{f(x)}$ is flat. It is clear that $k(f(x))/_{k(g\cdot f(x))}$ is a finite separable extension. Finally $\mathscr{M}_{f(x)}\mathscr{O}_x =$ **40**
$\mathscr{M}_x$ and $\mathscr{M}_{g\cdot f(x)}\mathscr{O}_x = \mathscr{M}_x = (\mathscr{M}_{g\cdot f(x)}\mathscr{O}_{f(x)})\mathscr{O}_x$; since $\mathscr{O}_{f(x)} \to \mathscr{O}_x$ is faithfully flat, it follows that $\mathscr{M}_{g\cdot f(x)}\mathscr{O}_{f(x)} = \mathscr{M}_{f(x)}$.

(7) An étale morphism is an open map.

In fact, we prove more generally:

Proposition 3.3.4. *A flat morphism is an open map.*

The proposition will follow as a consequence from the lemmas below. We first make the

Definition. A subset E of a noetherian topological space X is said to be *constructible* if E is a finite union of locally closed sets in X.

Lemma 3.3.5. *Let X be a noetherian topological space and E any set in X. Then, E is constructible $\Leftrightarrow$ for every irreducible closed set Y of X, $E \cap Y$ is either non-dense in Y or contains an open set of Y.*

Proof. $\Rightarrow$: Suppose $E = \bigcup_{i=1}^{n}(O_i \cap F_i)$, O_i open, F_i closed in X. Then $\overline{E \cap Y} \cap Y \subset \bigcup_{i=1}^{n}(F_i \cap Y)$ for any closed $Y \subset X$; if Y is irreducible with $E \cap Y$ dense in Y, this means $Y \subset F_i \cap Y$ for some i, i.e., $F_i \supset Y$; then $Y \cap E \supset (O_i \cap Y)$ open in Y.

41 $\Leftarrow$: We shall prove this by noetherian induction. Let $\mathscr{E}$ be the set of all closed sets F in X such that $E \cap F$ is not constructible. If $\mathscr{E} \neq \emptyset$ choose a minimal $F_0 \in \mathscr{E}$. By replacing X by F_0, we may assume that for every closed set F properly contained in X, $E \cap F$ is constructible. If X is reducible, say $X = X_1 \cup X_2$, X_1, X_2 both proper subsets of X and closed, then $E \cap X_1$, $E \cap X_2$ are both constructible and hence so is $E = (E \cap X_1) \cup (E \cap X_2)$. If X is irreducible, either

(i) $\overline{E} \neq X$ and so $E = E \cap \overline{E}$ is constructible or

(ii) $E \supset U \neq \emptyset$, U open, so that $E = U \cup (E \cap (X - U))$ is still constructible. This contradiction shows that $\mathscr{E} = \emptyset$. Q.E.D.

□

Lemma 3.3.6. *Let S be a noetherian prescheme and $f : X \to S$ a morphism. Then the image, under f, of any constructible set is constructible.*

Proof. Using the preceding lemma, the fact that S is noetherian and by passing to subschemes, irreducible components and so on, we are readily reduced to proving the following assertion: if X, S are both affine, reduced, irreducible and noetherian and if $f : X \to S$ is a morphism of finite type such that $\overline{f(X)} = S$, then $f(X)$ contains an open subset of S.

42 If $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$, our assumptions mean that A, B are noetherian integral domains and the ring-homomorphism $A \to B$ defining f is then easily checked to be an injection. The lemma will then follow from the following purely algebraic result. □

Lemma 3.3.7. *Let A be an integral domain and $B = A[x_1, \ldots, x_n]$ an integral A-algebra containing A. Then there exists a $g \in A$ such that for every prime ideal $\underline{p}$ of A with $g \notin \underline{p}$, there is a prime ideal $\mathscr{P}$ of B such that $\mathscr{P} \cap A = \underline{p}$.*

Proof. Choose a transcendence base $X_1, \ldots, X_k$ of B/A. Then the extension $B = A[x_1, \ldots, x_n]$ is algebraic over $A' = A[X_1, \ldots, X_k]$. By writing down the minimal polynomials of the x_i over A' and by dividing out these polynomials by a suitably chosen $g \in A$, $g \neq 0$, we can make the x_i integral over $A'\left[\frac{1}{g}\right]$. Consider now the tower of extensions.

$$\begin{tikzcd}
B\left[\frac{1}{g}\right] \arrow[d, dash] \\
A'\left[\frac{1}{g}\right] \arrow[d, dash] \\
A\left[\frac{1}{g}\right] \arrow[d, dash] \\
A
\end{tikzcd}$$

If $\underline{p}$ is a prime ideal of A such that $g \notin \underline{p}$ then there is a prime $\underline{p}'$ of $A\left[\frac{1}{g}\right]$ lying over $\underline{p}$. The prime ideal $\underline{p}' + (X_1, \ldots, X_k)$ of $A'\left[\frac{1}{g}\right]$ lies over $\underline{p}$. Now $\left[\frac{1}{g}\right]$ is a finite extension of $A'\left[\frac{1}{g}\right]$ and by Cohen-Seidenberg, $\exists$ a prime ideal $\mathscr{P}'$ of $B\left[\frac{1}{g}\right]$ lying over $\underline{p}$. The restriction $\mathscr{P}$ of $\mathscr{P}'$ to B then sits over $\underline{p}$. Q.E.D. □

Lemma 3.3.8. *Let $f : X \to S$ be a morphism (of finite type) and $T \subset X$* 43
be an open neighbourhood of a point $x \in X$. Assume that for every $s_1 \in S$ such that $f(x) \in \overline{(s_1)}$, there is a $t_1 \in T$ such that $f(t_1) = s_1$. Then $f(T)$ is a neighbourhood of $f(x)$.

Proof. Since the question is local, we may assume S noetherian and then by lemma 3.3.6 it follows that $f(T)$ is constructible. We may then write $f(T) = \bigcup\limits_{i=1}(O_i \cap F_i)$; and by choosing only those i, for which $f(x) \in O_i$, we may say that $f(T)$ is closed in a neighbourhood of $f(x)$. The hypotheses show that $f(T)$ is also dense in a neighbourhood of $f(x)$. It follows that $f(T)$ is itself a neighbourhood of $f(x)$. Q.E.D. □

Proof of Proposition 3.3.4.

In view of the lemma 3.3.8, it suffices to show that if $U \subset X$ is an open neighbourhood of $x \in X$, $f(U)$ contains all generisations of $f(x)$. The generisations of $f(x)$ are the points of Spec $\mathscr{O}_{f(x)}$ and those of x are points of Spec $\mathscr{O}_x$. But $\mathscr{O}_x$ is $\mathscr{O}_{f(x)}$-faithfully flat and then it is well-known for any prime ideal $\underline{p}$ of $\mathscr{O}_{f(x)}$, there is a prime-ideal $\mathscr{P}$ of $\mathscr{O}_x$ contracting to $\underline{p}$. U being open we have Spec $\mathscr{O}_x \subset U$. Hence Spec $\mathscr{O}_{f(x)} \subset f(U)$ (cf. Bourbaki, Alg. Comm. Ch. II, § 2, n°5, Cor. 4 to Proposition 11). Q.E.D.

3.4

44 A morphism $f : S' \to S$ is said to be an *effective epimorphism* if the sequence

$$S'' = S' \underset{S}{\times} S' \overset{p_1}{\underset{p_2}{\rightrightarrows}} S' \xrightarrow{f} S \quad \text{is exact}$$

i.e., if the sequence $\mathrm{Hom}(S, Y) \longrightarrow \mathrm{Hom}(S', Y) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \mathrm{Hom}(S'', Y)$ is exact, as a sequence of sets, $\forall Y$.

(We say that a sequence of sets $E_1 \xrightarrow{h_1} E_2 \overset{h_2}{\underset{h_2'}{\rightrightarrows}} E_3$ is exact if h_1 is an injection and $h_1(E_1) = \{x \in E_2 : h_2(x) = h_2'(x)\}$).

A morphism $f : S' \to S$ is *faithfully flat* if it is flat and surjective. Our aim now is to show that any such morphism is an effective epimorphism.

3.4.1 Some algebraic preliminaries; the Amitsur complex. A homomorphism of rings $f : A \to A'$ defines a sequence:

$$A \xrightarrow{f} A' \underset{p_2}{\overset{p_1}{\rightrightarrows}} A' \underset{A}{\otimes} A' = A'' \;\overset{p_{21}}{\underset{p_{31}}{\overset{p_{32}}{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}}}}\; A' \underset{A}{\otimes} A' \underset{A}{\otimes} A' = A''' \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}$$

where $p_1(a') = a' \otimes 1$; $p_2(a') = 1 \otimes a'$, $p_{21}(a' \otimes b') = a' \otimes b' \otimes 1$; $p_{31}(a' \otimes b') = a' \otimes 1 \otimes b'$; $p_{32}(a' \otimes b') = 1 \otimes a' \otimes b'$ and so on.

We may then define homomorphisms of A-modules: **45**

$$\begin{aligned}
\partial_0 &= p_1 - p_2 \\
\partial_1 &= p_{21} - p_{31} + p_{32} \\
\partial_2 &= p_{321} - p_{421} + p_{431} - p_{432} \ldots
\end{aligned}$$

and so on. One then checks that $\partial_{i+1}\partial_i = 0 \forall i$; we thus get an augmented cochain complex:

$$A \underset{f}{\to} A' \xrightarrow{\partial_0} A'' \xrightarrow{\partial_1} A''' \xrightarrow{\partial_2} A'''' \xrightarrow{\partial_3} \ldots.$$

This complex is called the *Amitsur complex A'/A*.

Lemma 3.4.1.1. *If $A \xrightarrow{f} A'$ is faithfully flat, then*

(i) $A \xrightarrow{\sim} H^0(A'/A)$

(ii) $H^q(A'/A) = (0) \forall q > 0$.

Proof. Suppose B is any faithfully flat A-algebra.

Consider then the complex

$$B'/B \equiv B \underset{A}{\otimes} (A'/A) \equiv B \underset{1\otimes f}{\longrightarrow} B' = B \underset{A}{\otimes} A' \xrightarrow{1\otimes\partial_0} B'' = B' \underset{B}{\otimes} B' = B \underset{A}{\otimes} A'' \to$$

Since B is A-faithfully flat, it is enough to prove that $H^0(B'/B) \xleftarrow{\sim} B$ **46**
and $H^q(B'/B) = (0) \forall q > 0$. As a particular choice we may take $B = A'$. Then the homomorphism $A' \xrightarrow{f'=1\otimes f} A' \underset{A}{\otimes} A' = A''$ admits a section, i.e., $\exists \sigma : A'' \to A'$ such that $\sigma \circ f' = 1_{A'}$, namely, the homomorphism: $a' \otimes b' \mapsto a'b'$.

We may thus assume, without any loss of generality that the homomorphism $A \xrightarrow{f} A'$ admits a section $\sigma : A' \to A$ such that $\sigma \circ f = 1_A$:

$$\begin{tikzcd} A \arrow[r, "f"] & A' \arrow[r, "\partial_0"] \arrow[l, bend left, "\sigma"] & A'' \arrow[r, "\partial_1"] & A''' \ldots \arrow[r, "\partial_2"] & \cdot \end{tikzcd}$$

We construct now a homotopy operator in this complex, as follows:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & A' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\sigma"] & A' \underset{A}{\otimes} A' = A'' \arrow[r, shift left=2, "p_{21}"] \arrow[r, "p_{32}"] \arrow[r, shift right=2, "p_{31}"'] \arrow[d, "1\otimes\sigma"] & A' \underset{A}{\otimes} A' \underset{A}{\otimes} A' = A''' \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "1\otimes 1\otimes\sigma"] & {} \\ & A \arrow[r, "f"] & A' \underset{A}{\otimes} A \xrightarrow{\sim} A' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A' \underset{A}{\otimes} A' \underset{A}{\otimes} A \xrightarrow{\sim} A'' \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {} \end{tikzcd}$$

(i) $H^0(A'/A) = \ker \partial_0 \xleftarrow{\sim} A$.

47 Let $a' \in A'$ be such that $p_1(a') = p_2(a')$. Applying $1 \otimes \sigma$ we get $(1 \otimes \sigma)(a' \otimes 1) = (1 \otimes \sigma)(1 \otimes a')$ i.e. $a' \otimes 1 = 1 \otimes \sigma(a')$ in $A' \underset{A}{\otimes} A$. Under the canonical identification $A' \underset{A}{\otimes} A \xrightarrow{\sim} A'$, this means that

$$a' = f(\sigma(a')) \quad \text{i.e.} \quad a' \in f(A).$$

On the other hand, $f(A) \subset \ker \partial_0$ and f being faithfully flat, is injective. It follows that $\underline{\ker \partial_0 \xleftarrow{\sim} A}$.

(ii) By using the homotopy operator, one can show that $H^q(A'/A) = (0) \forall q > 0$; the proof is omitted. (We do not need it).

$\square$

3.4.2

Proposition 3.4.2.1. *A faithfully flat morphism is also an effective epimorphism.*

Proof. **Case (a). $S' = \operatorname{Spec} A'$, $S = \operatorname{Spec} A$ are affine.**

From local algebra, it follows that the ring homomorphism $A \xrightarrow{\varphi} A'$ defining $f : S' \to S$, is also faithfully flat. We have to show that, for any Y, the sequence

$$(*) \qquad \operatorname{Hom}(S, Y) \to \operatorname{Hom}(S', Y) \rightrightarrows \operatorname{Hom}(S' \underset{S}{\times} S', Y) \text{ is exact in } \mathcal{E}\mathrm{ns}\,.$$

(i) Suppose $Y = \operatorname{Spec} B$ is affine.

Then the sequence (*) is equivalent to a sequence: **48**

$$(*)' \qquad \operatorname{Hom}(B, Z) \to \operatorname{Hom}(B, A') \rightrightarrows \operatorname{Hom}(B, A' \underset{A}{\otimes} A').$$

The exactness of this sequence $(*)'$ now follows from assertion (i) of lemma 3.4.1.1.

(ii) Let Y be arbitrary.

Let $\varphi_1 : S \to Y$, $\varphi_2 : S \to Y$ be two morphisms such that $\varphi_1 \circ f = \varphi_2 \circ f$. Since f is a surjection, it is clear that $\varphi_1(s) = \varphi_2(s) \forall s \in S$.

Choose a point $s \in S$ and a point $s' \in S'$ with $f(s') = s$; let $y = \varphi_1(s) = \varphi_2(s) \in Y$. Choose an affine open neighbourhood $\operatorname{Spec} B$ of $y \in Y$ and an element $\theta \in A$ such that $s \in \operatorname{Spec} A_\theta$ and $\varphi_1(\operatorname{Spec} A_\theta) \subset \operatorname{Spec} B$, $\varphi_2(\operatorname{Spec} A_\theta) \subset \operatorname{Spec} B$.

Set $\theta^1 = \varphi(\theta) \in A'$. Then $s' \in \operatorname{Spec} A'_{\theta'}$, and $f(\operatorname{Spec} A'_{\theta'}) \subseteq \operatorname{Spec} A_\theta$; and by (i) it follows that φ_1 and φ_2, when restricted to $\operatorname{Spec} A_\theta$, define the same morphism of preschemes. Since $s \in S$ was arbitrary, this proves that $\operatorname{Hom}(S, Y) \to \operatorname{Hom}(S', Y)$ is injective.

Now, suppose that $\varphi' : S' \to Y$ is a morphism such that in **49**

$$\begin{tikzcd}
S'' = S' \underset{S}{\times} S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S^1 \arrow[r, "f"] \arrow[dr, "\varphi'"'] & S \\
& & Y.
\end{tikzcd}$$

We have $\varphi' \circ p_2 = \varphi' \circ p_1$. We want to find a morphism $\psi : S \to Y$ such that $\varphi' = \psi \circ f$.

In view of the injectivity established above, it is enough to define ψ locally. Let $s \in S$ and $s' \in S'$ with $f(s') = s$. Choose an affine open neighbourhood V of $\varphi'(s')$ in Y. Then the open neighbourhood $\varphi'^{-1}(V)$ of s' in S' is saturated under f : in fact, let $x_1 \in \varphi'^{-1}(V)$ and $x_2 \in S'$ such that $f(x_1) = f(x_2)$. Then, there is an $s'' \in S'' = S' \underset{S}{\times} S'$ such that $p_1(s'') = x_1$ and $p_2(s'') = x_2$; so

$$\varphi'(x_1) = \varphi' \circ p_1(s'') = \varphi' \circ p_2(s'') = \varphi'(x_2) \quad \text{and} \quad x_2 \in \varphi'^{-1}(V).$$

Now, f is flat and hence an open map (Prop. (3.3.4)) and so $f(\varphi'^{-1}(V))$ is an open neighbourhood of $s \in S$. Choose an element $\theta \in A$ such that $s \in \operatorname{Spec} A_\theta \subset f(\varphi'^{-1}(V))$; if $\theta^1 = \varphi(\theta) \in A'$ then $f(\operatorname{Spec} A'_{\theta'}) = \operatorname{Spec} A_\theta$, and $\operatorname{Spec} A'_{\theta'} = f^{-1}(\operatorname{Spec} A_\theta) \subset \varphi'^{-1}(V)$ since $\varphi'^{-1}(V)$ is saturated under f.

50 We then have a diagram

$$\begin{tikzcd}
& \operatorname{Spec}(A'_{\theta'} \underset{A_\theta}{\times} A'_{\theta'}) \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
& \operatorname{Spec} A'_{\theta'} \arrow[dl, "\varphi'"'] \arrow[d, "f"] \\
V & \operatorname{Spec} A_\theta = f(\operatorname{Spec} A'_{\theta'})
\end{tikzcd}$$

The problem of defining $\psi : \operatorname{Spec} A_\theta \to V$ is now a purely affine problem and we are back to (i).

Case (b). The general case.

Without loss of generality, we may assume S affine. Since f is a morphism of finite type, there is a finite affine open covering $(S'_\alpha)^n_{\alpha=1}$ of S'. Consider the disjoint union $S^* = \coprod^n_{\alpha=1} S'_\alpha$. S^* is affine and the morphism $S^* \to S$ defined in the obvious way is again faithfuly flat. Let Y be an arbitrary prescheme and F be the (contravariant) functor $X \mapsto \mathrm{Hom}(X, Y)$. We have a commutative diagram:

$$\begin{tikzcd} F(S) \arrow[r] & F(S') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(S'') \arrow[d] \\ F(S) \arrow[u, "identity"] \arrow[r] & F(S^*) \arrow[r, shift left] \arrow[r, shift right] & F(S^* \underset{S}{\times} S^*) \end{tikzcd}$$

The lower sequence is exact by case (a); the first vertical map is the identity and the second vertical map is clearly injective. Usual diagram - chasing shows that the upper sequence is also exact. Q.E.D. □ 51

3.5 Étale coverings

Definition. A morphism of preschemes, $f : X \to S$, is said to be *finite* if, for every affine open $U \subset S$, $f^{-1}(U)$ is also affine and the ring $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_S)$-*module* of finite type.

It is again enough to check the conditions for an affine open cover of S.

(1) If S is locally noetherian and $f : X \to S$ is a finite morphism, $f_*(\mathscr{O}_X)$ is a coherent $\mathscr{O}_S$-Module.

(2) A finite morphism remains finite under a base-change.

In particular, if $f : X \to S$ is finite and $s \in S$ any point, the morphism $f_{\mathrm{Spec}\, k(s)} : X \underset{S}{\times} k(s) \to k(s)$ is finite and this means that the fibre $f^{-1}(s)$ is finite, discrete.

(3) A finite morphism is proper.

In face, a finite morphism is affine and is hence separated; it remains finite under any base-change and so it is enough to show that a finite morphism is closed.

52 By obvious reductions, we may take X, S affine reduced and $\overline{f(X)} = S$. If $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$, the corresponding homomorphism $A \to B$ is an injection and B is a finite A-module. Cohen-Seidenberg then shows that $f(X) = S$.

(4) The following result holds:

Lemma 3.5.1 (Chevalley)**.** *Let S be (as always) locally noetherian and $f : X \to S$ a morphism of preschemes. Then the following conditions are equivalent:*

(a) *f is finite*

(b) *f is proper and affine*

(c) *f is proper and $f^{-1}(s)$ is* finite $\forall s \in S$.

(For a proof see EGA Ch.III (a) Proposition (4.4.2)).

Definition. A morphism $f : X \to S$ is said to be an *étale covering* if it is both étale and finite.

Let $X \xrightarrow{f} S$ be an étale covering. Then $f_*(\mathscr{O}_X)$ is a locally free $\mathscr{O}_S$-Algebra of finite rank. For any $s \in S$, the fibre $f_*(\mathscr{O}_X)\underset{S}{\otimes}k(s)$ is a finite direct sum $\sum_{i=1}^{n_s} K_i$ of finite separable field extensions K_i of $k(s)$. The rank of $f_*(\mathscr{O}_X)$ at $s \in S$ is then given by $\sum_{i=1}^{n_s}[K_i : k(s)]$ which equals the number of geometric points in the fibre $f^{-1}(s)$. This is constant in each
53 connected component of S; if S is connected, this constant rank is called the *degree* or the *rank* of the covering f. In this case, if this rank equals 1, then f is an isomorphism.

Note. For étale coverings we have properties similar to (2), (3) (with $S' \to S$ *not* necessarily of finite type) (4), (5) and (6) from 3.3.3; this follows immediately from 3.3.3 and properties of coverings.

A word of caution:

This concept of an "étale covering" (French: revêtement étale) should *not* be confused with the concept of a "covering in the étale topology" (French: famille couvrante). The latter concept is *not* treated in this course. We also note that an étale covering, as defined here, is not necessarily surjective if S is not connected.

Chapter 4

The Fundamental Group

Throughout this chapter, we shall denote by S a locally noetherian *con-* **54**
nected prescheme and by $\mathscr{C} = (\mathscr{E}t/S)$ the category of étale coverings of S. We note that the morphisms of $\mathscr{C}$ will all be étale coverings.

(See 3.3.3 and the note at the end of Ch. 3).

4.1 Properties of the category $\mathscr{C}$

($\mathscr{C}_0$) $\mathscr{C}$ has an initial object $\emptyset$ (the empty prescheme) and a final object S.

($\mathscr{C}_1$) Finite fibre-products exist in $\mathscr{C}$, i.e., if $X \to Z$ and $Y \to Z$ are morphisms in $\mathscr{C}$, then $X \underset{Z}{\times} Y$ exists in $\mathscr{C}$ (see (3.3.3))

($\mathscr{C}_2$) If $X, Y \in \mathscr{C}$, then the disjoint union $X \coprod Y \in \mathscr{C}$ (obvious).

($\mathscr{C}_3$) Any morphism $u : X \to Y$ in $\mathscr{C}$ admits a factorisation of the form

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "u_i"'] & & Y \\ & Y_1 \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where u_1 is an effective epimorphism, j is a monomorphism and $Y = Y_1 \coprod Y_2$, $Y_2 \in \mathscr{C}$.

55 In fact, u is an étale covering so u is both open and closed; if we write $u(X) = Y_1$ we have $Y = Y_1 \coprod Y_2$ and $u = u_1 : X \to Y_1$ is then an effective epimorphism

(see (3.4.2.1)).

Further, this factorisation of u into an epimorphism and a monomorphism is *essentially unique* in the sense that if

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "u_1'"'] & & Y \\ & Y_1 \arrow[ur, hook, "j'"'] & \end{tikzcd}$$

is another such factorisation, then there exists an isomorphism $\omega : Y_1 \to Y_1'$ such that $u_1' = \omega \circ u_1$ and $j = j' \circ \omega$.

(Because a factorisation into a product of an effective epimorphism and a monomorphism is unique).

($\mathscr{C}_4$) If $X \in \mathscr{C}$ and $\mathfrak{g}$ is a *finite* group of automorphisms of X acting, say, to the right on X, then the quotient $X/\mathfrak{g}$ of X by $\mathfrak{g}$ exists in $\mathscr{C}$ and the natural morphism $X \xrightarrow{\eta} X/\mathfrak{g}$ is an effective epimorphism.

The quotient, if it exists, is evidently unique upto a canonical isomorphism; also X is affine over S and therefore the existence of $X/\mathfrak{g}$ has only to be proved in the case X, S affine, say $X = \operatorname{Spec} A$, $S = \operatorname{Spec} B$ and $G =$ the group of B-automorphisms of A corresponding to $\mathfrak{g}$. Then $\operatorname{Spec} A^G$ (A^G is the ring of G-invariants of A) is the quotient we are looking for. (Compare with Serre, Groupes algébriques et corps de classes, p. 57). Our aim now is to show that $X/\mathfrak{g}$ is actually in $\mathscr{C}$. The ques-
56 tion is again local and we may assume $X = \operatorname{Spec} A$, $S = \operatorname{Spec} B$, with B noetherian, and $X/\mathfrak{g} = \operatorname{Spec} A^G$ as above. $X \to S$ is finite and so $X/\mathfrak{g} \to S$ is also finite. It remains to show that this morphism is étale. In order to do this, we first make some simplifications.

Suppose that $S' \to S$ is a flat affine base-change. We have a commutative diagram:

$$\begin{tikzcd}
X' = X \underset{S}{\times} S' \arrow[r] \arrow[d] & Y' = Y \underset{S}{\times} S' \arrow[r] \arrow[d] & \operatorname{Spec} B' = S' \arrow[d] \\
\operatorname{Spec} A = X \arrow[r] & X/\mathfrak{g} = \operatorname{Spec} A^G = Y \arrow[r] & \operatorname{Spec} B = S.
\end{tikzcd}$$

$\mathfrak{g}$ acts on X' in the obvious way, as a group of S'-automorphisms of X'. We assert that $Y' = Y \underset{S}{\times} S'$ is the quotient of X' with respect to this action. Indeed, we have an exact sequence of B-algebras: $0 \to A^G \to A \to \bigoplus_{\sigma \in G} A$, where $A \to \bigoplus_{\sigma \in G} A$ is the map given by $A \mapsto \sum_{\sigma \in G} (a - a^\sigma)$. Since B' is B-flat, we get an exact sequence: $0 \to A^G \underset{B}{\otimes} B' \to A \underset{B}{\otimes} B' \to \bigoplus_{\sigma \in G} (A \underset{B}{\otimes} B')$ and this proves that the subring of invariants of $A \underset{B}{\otimes} B'$ is $A^G \underset{B}{\otimes} B'$; hence our assertion.

Let $y \in Y = X/\mathfrak{g}$ and $s \in S$ be its image. Take for B' the local ring $\mathscr{O}_{s,S}$. Then there is a unique point $y' \in Y' = Y \underset{S}{\times} S'$ over y and one has $\mathscr{O}_{y',Y'} = \mathscr{O}_{y,Y}$; hence $Y \to S$ is étale at $Y \Leftrightarrow Y' \to S'$ is étale at y'. We **57**
may thus assume that $S = \operatorname{Spec} B$, B a noetherian, local ring. In view of the following lemma, we may assume B *complete*.

Lemma 4.1.1. *Let $X \xrightarrow{f} S$ be a morphism and $S' \xrightarrow{\varphi} S$ a faithfully flat base-change. Then f is étale $\Leftrightarrow$ $f_{(S')}$ is étale.*

Proof. $\Rightarrow$: is clear.

$\Leftarrow$: flatness of f is straightforward. To prove non-ramification one observes that in view of (5) (3.3.0), one has $\Omega_{X'/S'} = \varphi^*(\Omega_{X/S})$; but φ being faithfully flat, $\Omega_{X'/S'} = 0 \Leftrightarrow \Omega_{X/S} = 0$; one now applies proposition 3.3.2. Q.E.D. □

Let $x_1, \ldots, x_n$ be the points of X over s. By hypothesis each $k(x_i)/k(s)$ is a finite separable extension. We choose a sufficiently large finite galois extension K of $k(s)$ such that each $k(x_i)$ is imbedded in K. We now need the

Lemma 4.1.2. *Let B be a noetherian local ring with maximal ideal $\mathscr{M}$ and residue field k. Let K be an extension field of k. Then $\exists$ a noetherian local ring C and a local homomorphism $\varphi : B \to C$ such that (i) φ is B-flat and (ii) $C/\mathscr{M}C \cong K$. (see EGA. Ch. 0_{III}, Prop. (10.3.1)).*

58 In addition if $[K : k] < \infty$, we can choose C to be a finite B-algebra. (EGA. Ch. 0_{III}, Cor. (10.3.2)).

By making such a base-change $B \xrightarrow{\varphi} C$ we may assume that each $k(x_i)$ is trivial over $k(s)$. Under these assumptions we get $A = \bigoplus_{i=1}^{r} B$, a finite direct sum of copies of B. Under the action of $\mathfrak{g}$, the set $\{x_1, \ldots, x_n\}$ splits into disjoint subsets $\{x_1, \ldots, x_l\}$, $\{x_{l+1}, \ldots, x_m\}, \ldots$, on each of which $\mathfrak{g}$ acts transitively. The corresponding decomposition of A will then be given by $A = (\bigoplus_{i=1}^{l} B) \bigoplus (\bigoplus_{i=l+1}^{m} B) \bigoplus \ldots$. The action of G on each block, for instance, on a $(b_1, \ldots, b_l) \in \bigoplus_{i=1}^{l} B$, will then be just a permutation. The subring A^G will then be the direct sum $\Delta_1 \oplus \ldots \oplus \Delta_\alpha \oplus \ldots$ where Δ_1 is the diagonal of the block $\bigoplus_{i=1}^{l} B$ and so on. Each Δ is evidently isomorphic to B. Our assertion that $X/\mathfrak{g} \to S$ is étale is now clear.

Thus $X/\mathfrak{g} \in \mathscr{C}$; the natural morphism $X \xrightarrow{\eta} X/\mathfrak{g}$ is also then an étale covering. Therefore η will be an open map and thus if η is not surjective one could replace Y by the image of η; this is clearly impossible. Hence η is surjective and thus an effective epimorphism 3.4.2.1.

4.2

59 We shall now define a covariant functor F from $\mathscr{C}(\mathscr{E}t/S)$ to the category of finite sets. We shall fix once and for all a point $s \in S$ and an algebraically closed field $\Omega \supset k(s)$.

For any $X \in \mathscr{C}$, $F(X)$, by definition, will be the set of geometric points of X over $s \in S$, with values in Ω, i.e., is the set of all S-

morphisms $\operatorname{Spec}\Omega \to X$ for which the diagram is commutative.

$$\begin{tikzcd}
\operatorname{Spec}\Omega \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} k(s) \arrow[r] & S
\end{tikzcd}$$

We observe that if $x \in X$ sits above $s \in S$, then giving an S-morphism $\operatorname{Spec}\Omega \to X$ whose image is $x \in X$ is equivalent to giving a $k(s)$-monomorphism of $k(x)$ into Ω. Also note that for any $X \in \mathscr{C}$, $F(X)$ is a finite set whose cardinality equals the rank of X over S.

Properties of the functor *F*.

(F_0) $F(X) = \emptyset \Leftrightarrow X = \emptyset$.

(F_1) $F(S)$ = a set with one element;

$F(X \underset{Z}{\times} Y) = F(X) \underset{F(Z)}{\times} F(Y)$, $\forall X, Y, Z \in \mathscr{C}$.

(F_2) $F(X_1 \coprod X_2) = F(X_1) \coprod F(X_2)$.

(F_3) If $X \xrightarrow{u} Y$ is an effective epimorphism in $\mathscr{C}$, the map $F(u)$: 60
$F(X) \to F(Y)$ is *onto*.

In fact, if $y \in Y$ is a point above $s \in S$ and $y = u(x)$, $x \in X$, then any $k(s)$-monomorphism of $k(y)$ to Ω extends to a $k(s)$-monomorphism of $k(x)$ to Ω.

(F_4) Let $X \in \mathscr{C}$ and $\mathfrak{g}$ a finite group of S-automorphisms of X (acting to the right on X). Then $\mathfrak{g}$ acts in a natural way (again to the right) on $F(X)$ as expressed by

$$\operatorname{Spec}\Omega \to X \xrightarrow{\sigma \in \mathfrak{g}} X.$$

The natural map $\eta : X \to X/\mathfrak{g}$ (see ($\mathscr{C}_3$)) defines a surjection $F(\eta) : F(X) \to F(X/\mathfrak{g})$ (see (F_3)). In view of the commutativity of the diagram

$$\begin{tikzcd} \operatorname{Spec}\Omega \arrow[r] & X \arrow[rr, "\sigma\in\mathfrak{g}"] \arrow[dr, "\eta"'] & & X \arrow[dl, "\eta"] \\ & & X/\mathfrak{g} & \end{tikzcd}$$

it follows that $F(\eta)$ descends to a surjection $\widetilde{\eta} : F(X)/\mathfrak{g} \to F(X/\mathfrak{g})$. We claim that $\widetilde{\eta}$ *is actually a bijection.* This follows immediately from the

Lemma 4.2.1. (i) $\mathfrak{g}$ *acts transitively on the fibres of* η.

61 (ii) *Suppose* $y \in Y = X/\mathfrak{g}$ *and* $x \in \eta^{-1}(y)$. *Let* $\mathfrak{g}_d(x)$ *be the subgroup* $\{\sigma \in \mathfrak{g} : \sigma(x) = x\}$ *(called the* decomposition group of x*). Then we have:*

(a) $k(x)/k(y)$ *is a galois extension.*

(b) *the natural map* $\mathfrak{g}_d(x) \to$ *the galois group* $G(^{k(x)}/k(y))$ *is* onto.

Proof. (i) We may assume $X = \operatorname{Spec} A$, $Y = X/\mathfrak{g} = \operatorname{Spec} A^G$ as before (both noetherian); also one knows that A is finite on A^G. Let $\mathscr{P}$, $\mathscr{P}_1$ be prime ideals of A (i.e., points of X) such that $\mathscr{P}_1 \neq \sigma\mathscr{P}\ \forall\ \sigma \in G$, while $\mathscr{P} \cap A^G = \mathscr{P}_1 \cap A^G = \underline{p}$. We may assume $\mathscr{P}$ and $\mathscr{P}_1$ maximal (otherwise apply the flat base change $Y \leftarrow \operatorname{Spec} \mathscr{O}_{y,Y}$, where $y \in Y$ corresponds to $\underline{p}$). Then there is an $a \in \mathscr{P}_1$ such that $a \notin \sigma\mathscr{P}\forall\sigma$ (Chinese Remainder Theorem). Thus $b = \prod_\sigma \sigma(a) \notin \mathscr{P}$; but $b \in A^G$, so $b \in A^G \cap \mathscr{P}_1 = A^G \cap \mathscr{P}$-contradiction.

(ii) We have a diagram:

$$\begin{tikzcd} A \arrow[r] & A/\mathscr{P} \simeq k(x) \\ A^G \arrow[u] \arrow[r] & A^G/\underline{p} \simeq k(y), \arrow[u] \end{tikzcd}$$

62 and we know that $k(x)/k(y)$ is a finite, separable extension. Let $\theta \in A$ be such that $k(x) = k(y)(\overline{\theta})$. The polynomial $f = \prod_{\sigma}(T - \sigma\theta)$ is in $A^G[T]$, has θ as a root and splits completely in $A[T]$. The reduction $\overline{f}$ of $f \mod \underline{p}$ is in $k(y)[T]$, has $\overline{\theta}$ as a root and splits completely in $k(x)[T]$. It follows that $k(x)/k(y)$ is normal, hence galois.

Consider now the subgroup $G_d(\mathscr{P})$ of G corresponding to $\mathfrak{g}_d(x)$. We have $\mathscr{P} \neq \sigma^{-1}\mathscr{P}\ \forall\ \sigma \notin G_d(\mathscr{P})$ and by the Chinese Remainder Theorem we can choose a $\theta_1 \in A$ such that $\theta_1 \equiv \theta (\mathrm{mod} \mathscr{P})$ and $\theta_1 \equiv 0 (\mathrm{mod} \sigma^{-1}\mathscr{P})$, $\forall \sigma \notin G_d(\mathscr{P})$.

We have $k(x) = k(y)(\overline{\theta}_1)$. Consider now the polynomial $\overline{g} = \prod_{\sigma}(T - \overline{\sigma}\theta_1) \in k(y)[T]$. As $\overline{\theta}_1$ is a root of $\overline{g}$, for every $\varphi \in G(^{k(x)}/k(y))$, $\varphi(\overline{\theta}_1)$ is also a root of $\overline{g}$; hence $\varphi(\overline{\theta}_1) = \overline{\sigma}(\overline{\theta}_1)$ for some $\sigma \in G$. But $\varphi(\overline{\theta}_1) \neq 0$ and, by the choice of θ_1, $\overline{\sigma}(\overline{\theta}_1) = 0$ if $\sigma \notin G_d(\mathscr{P})$; hence $\varphi(\overline{\theta}_1) = \overline{\sigma}(\overline{\theta}_1)$ for some $\sigma \in G_d(\mathscr{P})$, i.e., $\varphi = \overline{\sigma}$ for some $\sigma \in G_d(\mathscr{P})$. Q.E.D.

□

(F_5) If $u : X \to Y$ is a morphism in $\mathscr{C}$ such that $F(u) : F(X) \to F(Y)$ is a bijection, then u is an isomorphism

From the fact that $F(u)$ is a bijection it follows (see the remark at 63
the end of Ch. 3) that the rank of $u : X \to Y$ is 1 at every $y \in Y$, hence (again by the same remark) u is an isomorphism.

A category which has the properties $(\mathscr{C}_0), \ldots, (\mathscr{C}_4)$ of 4.1 and from which there is given a functor F into finite sets with the above properties $(F_0), \ldots, (F_5)$ is called a *galois category*; the functor F itself is known as a *fundamental functor*.

4.3

Before we start our construction of the fundamental group of a galois category we motivate our procedure by two examples.

Example 1. Let S be a connected, locally arcwise connected, locally simply connected topological space and $\mathscr{C}$ the category of connected coverings of S; the morphisms of $\mathscr{C}$ are covering maps. Let $X \xrightarrow{p} S$ be such a covering. Fix a point $s \in S$; we define $F(X) = p^{-1}(s)$. Then $(X, p) \mapsto p^{-1}(s)$ is a covariant functor $F : \mathscr{C} \to \mathcal{E}\text{ns}$.

Each member of $\mathscr{C}$ determines (upto conjugacy) a sub-group of the fundamental group $\Pi_1(S, s)$; and to each subgroup H of $\Pi_1(S, s)$ there corresponds a member of $\mathscr{C}$ determining H. To the subgroup $\{e\}$ corresponds, what is known as, the universal covering $\widetilde{S}$ of S; and $\Pi_1(S, s)$ is isomorphic to the group of S-automorphisms of $\widetilde{S}$, i.e., to the group of covering transformations of $\widetilde{S}$ over S. Further we have the isomorphism (in $\mathcal{E}$ns):

$$\mathrm{Hom}_{\mathscr{C}}(\widetilde{S}, X) \xrightarrow{\sim} F(X), \quad \forall X \in \mathscr{C}.$$

64 The functor $F : \mathscr{C} \to \mathcal{E}\text{ns}$ is thus *representable* in the following sense.

Definition 4.3.1. A covariant functor $\mathscr{G}$ from a category $\mathscr{C}$ to $\mathcal{E}$ns is *representable* if $\exists$ an object $Y \in \mathscr{C}$ such that:

$$\mathrm{Hom}_{\mathscr{C}}(Y, X) \xrightarrow{\sim} \mathscr{G}(X) \ \forall\ X \in \mathscr{C}.$$

Example 2. Let k be a field and Ω a fixed algebraically closed field extension of k. Set $S = \operatorname{Spec} k$ and $\mathscr{C}$ = the category of connected étale coverings of S; any member of $\mathscr{C}$ is of the form $\operatorname{Spec} K$, where K/k is a finite separable field extension. For any $X \in \mathscr{C}$ we define $F(X)$ = the set of geometric points of X with values in Ω. Then, $F(X) \simeq \mathrm{Hom}_k(K, \Omega_s)$ is $X = \operatorname{Spec} K$, where Ω_s is the separable closure of k in Ω. If Ω_s is finite over k, we can further write $F(X) \simeq \mathrm{Hom}_{\mathscr{C}}(\operatorname{Spec} \Omega_s, X)$ and the functor $F : \mathscr{C} \to$ (finite sets), defined above, will be representable. However this is *not* the case in general; out we can find an indexed, filtered family $(N_i)_{i \in I}$ of finite galois extensions of k, (namely, the set of finite galois extensions contained in Ω_s) such that for any $X \in \mathscr{C}$, we can find an $i_0 = i_0(X)$ such that $F(X) \simeq \mathrm{Hom}_{\mathscr{C}}(\operatorname{Spec} N_i, X)$, $\forall i \geq i_0(X)$. In other words, we may write

$$F(X) \simeq \varinjlim_{i \in I} \mathrm{Hom}_{\mathscr{C}}(\operatorname{Spec} N_i, X), \quad \forall X \in \mathscr{C}$$

and the family $(\operatorname{Spec} N_i)_{i\in I}$ is in fact a projective family of objects in $\mathscr{C}$. **65**

Suppose now that $\mathscr{C}$ is any category and $\mathscr{G} : \mathscr{C} \to \mathscr{E}ns$ is a covariant functor. If $X \in \mathscr{C}$ and $\xi \in \mathscr{G}(X)$, we write, as a matter of notation, $\mathscr{G} \xrightarrow{\xi} X$. If $\mathscr{G} \xrightarrow{\xi} X$, and $\mathscr{G} \xrightarrow{\eta} Y$ and $X \xrightarrow{u} Y$ is a $\mathscr{C}$-morphism, we say that the diagram

$$\begin{tikzcd} \mathscr{G} \arrow[rr, "\xi"] \arrow[dr, "\eta"'] & & X \arrow[dl, "u"] \\ & Y & \end{tikzcd}$$

is commutative if $\mathscr{G}(u)(\xi) = \eta$.

If $\mathscr{G} \xrightarrow{\xi} X$, then for any $Z \in \mathscr{C}$, we have a natural map $\operatorname{Hom}_{\mathscr{C}}(X, Z) \to \mathscr{G}(Z)$ defined by $u \mapsto \mathscr{G}(u)(\xi)$.

Definition 4.3.2. We say that $\mathscr{G}$ is *pro-representable* if $\exists$ a projective system $(S_i, \varphi_{ij})_{i\in I}$ of objects of $\mathscr{C}$ and elements $\tau_i \in \mathscr{G}(S_i)$ (called the canonical elements of $\mathscr{G}(S_i)$) such that

(i) the diagrams

$$\begin{tikzcd} \mathscr{G} \arrow[rr, "\tau_i"] \arrow[dr, "\tau_j"'] & & S_i \\ & S_j \arrow[ur, "\varphi_{ij}(j\geq i)"'] & \end{tikzcd}$$

are commutative.

(ii) for any $Z \in \mathscr{C}$, the (natural) map **66**

$$\varinjlim_{i\in I} \operatorname{Hom}_{\mathscr{C}}(S_i, Z) \to F(Z)$$

is *bijective.*

In addition, if the φ_{ij} are *epimorphisms* of $\mathscr{C}$, we say that $\mathscr{G}$ is *strictly pro-representable.*

Thus, our functor F in Example 2 is pro-representable. Example 1 and 2 show that representable and pro-representable functors arise naturally in the consideration of the fundamental group.

4.4 Construction of the Fundamental group

4.4.1 Main theorem

(1) Let $\mathscr{C}$ be a galois category with a fundamental functor F. Then there exists a pro-finite group π (i.e., a group π which is a projective limit of finite discrete groups provided with the limit topology) such that F is an equivalence between $\mathscr{C}$ and the category $\mathscr{C}(\pi)$ of finite sets on which π acts continuously.

(2) If $\mathscr{C} \xrightarrow{F'} \mathscr{C}(\pi')$ is another such equivalence, then π' is continuously isomorphic to π and this isomorphism between π and π' is canonically determined upto an inner automorphism of π.

67 The profinite group π, whose existence is envisaged in assertion (1) above will be called the *fundamental group* of the galois category $\mathscr{C}$.

The theorem is a consequence of the following series of lemmas.

Definition 4.4.1.1. A category $\mathscr{C}$ is *artinian* if any "decreasing" sequence

$$T_1 \hookleftarrow_{j_1} T_2 \hookleftarrow_{j_2} T_3 \hookleftarrow_{j_3} \ldots$$

of monomorphisms in $\mathscr{C}$ is stationary, i.e., the j_r are isomorphisms for large r.

A (covariant) functor $F : \mathscr{C} \to \mathcal{E}\text{ns}$ is *left-exact* if it commutes with finite products i.e., if $F(X \times Y) = F(X) \times F(Y)$ and if, for every *exact* sequence $X \xrightarrow{u} Y \underset{u_2}{\overset{u_1}{\rightrightarrows}} Z$ in $\mathscr{C}$, the sequence

$$F(X) \xrightarrow{F(u)} F(Y) \underset{F(u_2)}{\overset{F(u_1)}{\rightrightarrows}} F(Z)$$

is exact as a sequence of sets.

If $Y \underset{u_2}{\overset{u_1}{\rightrightarrows}} Z$ are morphisms in $\mathscr{C}$, a *kernel* for u_1, u_2 in $\mathscr{C}$ is a pair (X, u) with $X \in \mathscr{C}$ and $u : X \to Y$ in $\mathscr{C}$ such that $X \xrightarrow{u} Y \underset{u_2}{\overset{u_1}{\rightrightarrows}} Z$

is exact in $\mathscr{C}$. Clearly a kernal is determined uniquely upto an isomorphism in $\mathscr{C}$.

Lemma 4.4.1.2. *Let $\mathscr{C}$ be a category in which finite products exists. Then finite fibre-products exist in $\mathscr{C}$ $\Leftrightarrow$ kernels exist in $\mathscr{C}$.* 68

Proof. $\Rightarrow$: Let $Y \underset{u_2}{\overset{u_1}{\rightrightarrows}} Z$ be morphisms in $\mathscr{C}$. We have a commutative diagram:

$$\begin{tikzcd}
 & & Z & \\
 & Y \arrow[ur, "u_1"] & & Y \arrow[ul, "u_2"'] \\
 & & Y \underset{Z}{\times} Y \arrow[ul, "p_1"] \arrow[ur, "p_2"'] \arrow[dr, "{(p_1,p_2)}"] & \\
(Y \underset{Z}{\times} Y) \underset{(Y\times Y)}{\times} Y \arrow[urr] \arrow[drr] & & & Y \times Y \\
 & & Y \arrow[ur, "\text{diagonal}"'] &
\end{tikzcd}$$

and it easily follows that $(Y \underset{Z}{\times} Y \underset{(Y\times Y)}{\times} Y$ is a solution for the kernal of u_1 and u_2.

$\Leftarrow$: Suppose $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ are morphisms in . If p and q are the canonical projections $X \times Y \xrightarrow{p} X$, $X \times Y \xrightarrow{q} Y$, we have an exact sequence:

$$\ker(fp, gq) \longrightarrow X \times Y \underset{gq}{\overset{fp}{\rightrightarrows}} Z.$$

It follows that $\ker(fp, gq)$ is a solution for the fibre-product $X \underset{Z}{\times} Y$. 69
Q.E.D. $\square$

In fact, we have shown that finite fibre products and kernels can be expressed in terms of each other. Hence, F commutes with finite fibre-products $\Leftrightarrow$ it is left-exact.

Corollary . *A fundamental functor is left-exact (see (F_1))*

Lemma 4.4.1.3. *A galois category is artinian.*

Proof. Let

$$T_1 \hookleftarrow_{j_1} T_2 \hookleftarrow_{j_2} \dots \hookleftarrow_{j_{r-1}} T_r \hookleftarrow_{j_r} T_{r+1} \dots$$

be a decreasing sequence of monomorphisms in $\mathscr{C}$. We have then:

$$\begin{aligned}
& T_{r+1} \xrightarrow{j_r} T_r \quad \text{is a monomorphism} \\
\Leftrightarrow\ & T_{r+1} \underset{\Delta}{\xrightarrow{\sim}} T_{r+1} \underset{T_r}{\times} T_{r+1} \\
\Rightarrow\ & F(T_{r+1}) \underset{\Delta}{\xrightarrow{\sim}} F(T_{r+1}) \underset{F(T_r)}{\times} F(T_{r+1}) \quad (\text{by } (F_1), (F_5)) \\
\Leftrightarrow\ & F(T_{r+1}) \xrightarrow{F(j_r)} F(T_r) \quad \text{is a monomorphism.}
\end{aligned}$$

Since the $F(T_r)$ are finite, this implies that the $F(j_r)$ are isomorphisms for large r; we are through by (F_5). Q.E.D. □

70 **Lemma 4.4.1.4.** *Let $\mathscr{C}$ be a galois category with a fundamental functor F. Then F is strictly pro-representable.*

Proof. With the notations of 4.3.2, consider the set $\mathscr{E}$ of pairs (X, ξ) with $F \xrightarrow{\xi} X$. We order $\mathscr{E}$ as follows:

$$(X, \xi) \geq (X', \xi') \Leftrightarrow \exists \quad \text{a commutative diagram:}$$

$$\begin{tikzcd}
F \arrow[rr, "\xi"] \arrow[dr, "\xi'"'] & & X \arrow[dl] \\
& X' &
\end{tikzcd}$$

We claim that $\mathscr{E}$ is *filtered* for this ordering; in fact, if (X, ξ), (X', ξ')

$\in \mathscr{E}$, in view of (F_1) we get a commutative diagram:

$$\begin{tikzcd}
 & & X \\
F \arrow[urr, "\xi"] \arrow[r, "(\xi,\xi')"] \arrow[drr, "\xi'"'] & X \times X' \arrow[ur, "p"] \arrow[dr, "p'"'] & \\
 & & X'
\end{tikzcd}$$

where p and p' are the natural projections.

We say that a pair $(X,\xi) \in \mathscr{E}$ is *minimal* in $\mathscr{E}$ if for any commutative **71**
diagram

$$\begin{tikzcd}
F \arrow[rr, "\xi"] \arrow[dr, "\eta"'] & & X \\
 & Y \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

with a monomorphism j, one necessarily has that j is an isomorphism.

(*) Every pair in $\mathscr{E}$ is dominated, in this ordering, by a minimal pair in $\mathscr{E}$.

Observe that $\mathscr{C}$ is artinian (Lemma 4.4.1.3).

(**) If $(X,\xi) \in \mathscr{E}$ is minimal and $(Y,\eta) \in \mathscr{E}$ then a $u \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ in a commutative diagram

$$\begin{tikzcd}
F \arrow[rr, "\xi"] \arrow[dr, "\eta"'] & & X \arrow[dl, "u"] \\
 & Y &
\end{tikzcd}$$

is uniquely determined.

In fact, if $u_1, u_2 \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ such that the diagrams 72

$$\begin{tikzcd} F \arrow[rr, "\eta"] \arrow[dr, "\xi"'] & & Y \\ & X \arrow[ur, "u_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[rr, "\eta"] \arrow[dr, "\xi"'] & & Y \\ & X \arrow[ur, "u_2"'] & \end{tikzcd}$$

are commutative then by $(\mathscr{C}_1)$ and Lemma 4.4.1.2 $\ker(u_1, u_2)$ exists; since F is left exact we get a commutative diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\xi"'] \arrow[dr, "\xi"] & Y \\ \ker(u_1, u_2) \arrow[r, hook, "j"'] & X \arrow[u, shift left, "u_1"] \arrow[u, shift right, "u_2"'] \end{tikzcd}$$

with a monomorphism j. As (X, ξ) is minimal j must be an isomorphism, i.e., $\underline{u_1 = u_2}$.

From (*), (**) it follows that the system I of minimal pairs of $\mathscr{E}$ is *directed.*

If $(X, \xi) \in I$, $(Y, \eta) \in \mathscr{E}$ and $u \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ appears in a commutative diagram

$$\begin{tikzcd} F \arrow[rr, "\xi"] \arrow[dr, "\eta"'] & & X \\ & Y \arrow[ur, "u"'] & \end{tikzcd}$$

73 then u must be an effective epimorphism.

In fact, be $(\mathscr{C}_3)$ we get a factorisation

$$\begin{tikzcd} Y \arrow[rr, "u"] \arrow[dr, "u_1"'] & & X_1 \coprod X_2 = X \\ & X_1 \arrow[ur, hook, "j"'] & \end{tikzcd}$$

with an effective epimorphism u_1 and a monomorphism j. By (F_2) and (F_3) we then obtain a commutative diagram:

$$\begin{tikzcd}
F \arrow[rr, "\xi"] \arrow[dr, "\xi"] \arrow[dd, "\eta"'] & & X = X_1 \coprod X_2 \\
& X_1 \arrow[ur, hook, "j"'] & \\
Y \arrow[ur, "u_1"'] & &
\end{tikzcd}$$

By minimality of (X, ξ), it follows that j is an isomorphism; thus u is an effective epimorphism. In particular:

*** The structure morphisms occurring in the projective family I are effective epimorphisms.

Consider now the natural map

$$\varinjlim_{i \in I} \mathrm{Hom}_{\mathscr{C}}(S_i, X) \to F(X), \quad X \in \mathscr{C}.$$

By (*) this is onto; by (**) it is injective. From (***) it thus follows 74
that F is strictly pro-representable. Q.E.D. □

Definition 4.4.1.5. Let $\mathscr{C}$ be a category with zero ($\emptyset$) in which disjoint unions exist in $\mathscr{C}$. An $X \in \mathscr{C}$ is *connected* in $\mathscr{C}$ $\Leftrightarrow$ $X \neq X_1 \coprod X_2$ in $\mathscr{C}$ with $X_1, X_2 \neq \emptyset$.

Note. In $(\mathscr{E}t/S)$, a prescheme is connected $\Leftrightarrow$ it is connected as a topological space.

With the notations of the preceding lemma, we have:

Lemma 4.4.1.6. (i) *$(X, \xi) \in \mathscr{E}$ is minimal $\Leftrightarrow$ X is connected in $\mathscr{C}$.*

(ii) *If X is connected in $\mathscr{C}$, then any $u \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ is an automorphism.*

(iii) *For any $X \in \mathscr{C}$, $\operatorname{Aut} X$ acts on $F(X)$ as follows:*
$F \xrightarrow{\xi} X \xrightarrow{\sigma \in \operatorname{Aut} X} X$. *It $X \in \mathscr{C}$ is connected, then for any $\xi \in F(X)$ the map $\operatorname{Aut} X \to F(X)$ defined by $u \mapsto F(u)(\xi) = u \circ \xi$, is an* injection.

Proof. (i) Suppose $X = X_1 \coprod X_2$ in $\mathscr{C}$, $X_1, X_2 \neq \emptyset$ and that $(X, \xi) \in \mathscr{E}$; then $\xi \in F(X) = F(X_1) \coprod F(X_2)$ say, $\xi \in F(X_1)$.

We then have a commutative diagram

$$\begin{tikzcd}
F \arrow[rr, "\xi"] \arrow[dr, "\xi"'] & & X \\
& X_1 \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

75 with a monomorphism j which is *not* an isomorphism. Thus (X, ξ) is *not* minimal.

On the other hand, let $X \in \mathscr{C}$ be connected and $(X, \xi) \in \mathscr{E}$. Suppose we have a commutative diagram:

$$\begin{tikzcd}
F \arrow[rr, "\xi"] \arrow[dr, "\eta"'] & & X \\
& Y \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

with a monomorphism j. By $(\mathscr{C}_3)$ we get a factorisation:

$$\begin{tikzcd}
Y \arrow[rr, "j"] \arrow[dr, "j_1"'] & & X = X_1 \coprod X_2 \\
& X_1 \arrow[ur, hook, "j_2"'] &
\end{tikzcd}$$

with an effective epimorphism j_1 and a monomorphism j_2. As j is a monomorphism, so is j_1 and thus j_1 is an isomorphism; since X is connected, $X_2 = \emptyset$ and one gets that $j = j_2 \circ j_1$ is an isomorphism.

(ii) As X is connected, it follows by $(\mathscr{C}_3)$ that u is an effective epimorphism; by (F_3), $F(u) : F(X) \to F(X)$ is onto and thus is a bijection ($F(X)$ finite). By (F_5) it follows that $u \in \operatorname{Aut} X$.

(iii) Let u_1, $u_2 \in \operatorname{Aut} X$ such that $F(u_1)(\xi) = F(u_2)(\xi)$, i.e., $\xi \in$ **76**
$\ker(F(u_1), F(u_2)) = F(\ker(u_1, u_2))$ (F is left-exact). We thus have a commutative diagram

$$\begin{tikzcd} F \arrow[rr, "\xi"] \arrow[dr, "\xi'"'] & & X \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] & X \\ & \ker(u_1, u_2) \arrow[ur, hook, "j"'] & & \end{tikzcd}$$

with a monomorphism j; as (X, ξ) is minimal by (i), j is an isomorphism, in other words, $\underline{u_1 = u_2}$. Q.E.D.

□

We briefly recall now the example 2 of 4.3. Assertion (iii) of the above lemma simply says in this case that if K/k is a finite separable extension field and if $\xi \in \operatorname{Hom}_k(K, \Omega)$, then the map $\operatorname{Aut} K \to \operatorname{Hom}_k(K, \Omega)$ given by $u \mapsto \xi \circ u$ is injective. We know that K/k is galois $\Leftrightarrow$ this map is also onto. Following this, we now make the

Definition 4.4.1.7. A connected-object $X \in \mathscr{C}$ is *galois* if for any $\xi \in F(X)$, the map $\operatorname{Aut} X \to F(X)$ defined by $u \mapsto u \circ \xi$ is a *bijection*. Note that this is equivalent to saying that the action of $\operatorname{Aut} X$ on $F(X)$ is *transitive*. Also observe that this definition is independent of F because the cardinality of $F(X)$ is the degree of the covering X over S; the action is already effective since X is connected (by (iii), Lemma 4.4.1.6). **77**

Lemma 4.4.1.8. *If $F \xrightarrow{\eta} Y$, then there is a galois object $X \in \mathscr{C}$, a $\xi \in F(X)$ and a $u \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$ such that the diagram*

$$\begin{tikzcd} F \arrow[rr, "\xi"] \arrow[dr, "\eta"'] & & X \arrow[dl, "u"] \\ & Y & \end{tikzcd}$$

is commutative. In other words, the system I_1 of galois pairs of $\mathscr{E}$ is cofinal *in $\mathscr{E}$.*

Proof. Let $(S_i)_{i\in I}$ be a projective system of minimal objects of $\mathscr{C}$ such that

$$F \xleftarrow{\sim} \varprojlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(S_i, *).$$

Let $\eta_1, \ldots, \eta_r$ be the elements of $F(Y)$. We can choose i large enough such that as u varies over $\mathrm{Hom}_{\mathscr{C}}(S_i, Y)$, the $u \circ \tau_i$ give all the $\eta' s$ (τ_i is the canonical element of $F(S_i)$). We then get:

$$F \xrightarrow{\tau_i} S_i \xrightarrow{\alpha} Y^r = \underbrace{Y \times \ldots \times Y}_{r \text{ times}} \xrightarrow{p_j} Y$$

where p_j is the j^{th} canonical projection $Y^r \to Y$; the elements $p_j \circ \alpha \circ \tau_i$,
78 $1 \le j \le r$, are precisely the elements $\eta_1, \ldots, \eta_r$ of $F(Y)$. By ($\mathscr{C}_3$) we get a factorisation:

$$\begin{tikzcd}
S_i \arrow[rr, "\alpha"] \arrow[dr, "\alpha_1"'] & & Y^r \\
& X \arrow[ur, hook, "\beta"'] &
\end{tikzcd}$$

with a monomorphism β and an effective epimorphism α_1. We claim that X *is galois.*

(i) ***X* is connected.**

Suppose $X = X_1 \coprod X_2$, X_1, X_2 in $\mathscr{C}$, $\neq \emptyset$; the element $\alpha_1 \circ \tau_i \in F(X_1)$, say. We can then choose a j large enough for us to get a commutative diagram:

$$\begin{tikzcd}
& S_i \arrow[rr, "\alpha_1"] & & X = X_1 \coprod X_2 \\
F \arrow[ur, "\tau_i"] \arrow[dr, "\tau_j"'] & & X_1 \arrow[ur, hook, "\alpha'"'] & \\
& S_j \arrow[uu, "\varphi_{ij}"] \arrow[ur, "\alpha'"'] & &
\end{tikzcd}$$

Hence $\beta' \circ \alpha' = \alpha_1 \circ \varphi_{ij}$ is an epimorphism, which is absurd.

(ii) Set $\xi = \alpha_1 \circ \tau_i \in F(X)$. We shall prove that the map $\operatorname{Aut} X \to F(X)$ 79
defined by $u \mapsto u \circ \xi$ is *onto*.

Let $\xi' \in F(X)$; we may assume i is so large that we get a commutative diagram:

$$\begin{tikzcd}
 & & X \\
F \arrow[urr, "\xi"] \arrow[r, "\tau_i"] \arrow[drr, "\xi'"'] & S_i \arrow[ur, "\alpha_1"'] \arrow[dr, "\alpha'_1"] & \\
 & & X
\end{tikzcd}$$

Our aim is to find a $\sigma \in \operatorname{Aut} X$ such that $\alpha'_1 = \sigma \circ \alpha_1$. Since X is connected, α'_1 is also an effective epimorphism. Since the manner in which a morphism in $\mathscr{C}$ is expressed as the composite of an effective epimorphism and a monomorphism is essentially unique, we will be through if we find a $\rho \in \operatorname{Aut} Y^r$ such that the diagram

$$\begin{tikzcd}
 & & X \arrow[r, hook, "\beta"] & Y^r \arrow[dd, "\rho"] \\
F \arrow[urr, "\xi"] \arrow[r, "\tau_i"] \arrow[drr, "\xi'"'] & S_i \arrow[ur, "\alpha_1"'] \arrow[dr, "\alpha'_1"] & & \\
 & & X \arrow[r, hook, "\beta"] & Y^r
\end{tikzcd}$$

is commutative. By assumption the elements $p_1 \circ \beta \circ \xi$, $1 \leq j \leq r$, 80 are all the distinct elements of $F(Y)$; so the morphisms $p_j \circ \beta \circ \alpha_1$ are all distinct. This means that the $p_j \circ \beta$ are all distinct; since α_1' is an effective epimorphism the $p_j \circ \beta \circ \alpha_1'$ are all distinct; and as X is connected and β is a monomorphism, it follows that the $p_j \circ \beta \circ \xi'$ are all distinct and therefore form the set $F(Y)$.

If we set $p_j \circ \beta \circ \xi = \eta_j$, $1 \leq j \leq r$, and $p_j \circ \beta \circ \xi' = \eta_{\rho(j)}$, $1 \leq j \leq r$, we get a permutation ρ of the set $\{1, 2, \ldots, r\}$; this permutation determines an automorphism of Y^r with the required property. Q.E.D. □

By this lemma we may clearly assume now that F is strictly pro-represented by a projective system (S_i) of galois objects of $\mathscr{C}$.

Let $\mathfrak{g}_i = \operatorname{Aut} S_i$ and θ_i be the bijection $\mathfrak{g}_i \to F(S_i)$ defined by $u \mapsto u \circ \mathscr{C}_i$ where $\mathscr{C}_i$ is the canonical element of $F(S_i)$. For $j \geq i$, we define $\psi_{ij} : \mathfrak{g}_j \to \mathfrak{g}_i$ as the composite

$$\mathfrak{g}_j \xrightarrow{\theta_j} F(S_j) \xrightarrow{F(\varphi_{ij})} F(S_i) \xrightarrow{\theta_i^{-1}} \mathfrak{g}_i.$$

For any $u \in \mathfrak{g}_j$, $\psi_{ij}(u)$ is the uniquely determined automorphism of
81 S_i which makes either one (and hence also the other) of the diagrams

$$\begin{tikzcd} & S_i \arrow[r, "\psi_{ij}(u)"] & S_i \\ F \arrow[ur, "\tau_i"] \arrow[dr, "\tau_j"'] & & \\ & S_j \arrow[r, "u"'] & S_j \arrow[uu, "\varphi_{ij}"'] \end{tikzcd} \qquad \begin{tikzcd} S_i \arrow[r, "\psi_{ij}(u)"] & S_i \\ S_j \arrow[u, "\varphi_{ij}"] \arrow[r, "u"'] & S_j \arrow[u, "\varphi_{ij}"'] \end{tikzcd}$$

commutative. It follows from this easily that the ψ_{ij} are group homomorphisms.

We thus obtain a projective system $\{\mathfrak{g}_i, \psi_{ij}\}_{i \in I}$ of finite groups with each ψ_{ij} surjective. Denote by $\{\pi_i, \psi_{ij}\}_{i \in I}$ the projective system of the opposite groups. The group $\pi = \varprojlim_{i \in I} \pi_i$ with the limit topology is profinite and we shall prove that it is the fundamental group of the galois category $\mathscr{C}$; we denote it by $\pi_1(S, s)$ when $\mathscr{C}$ and F are as in 4.1 and 4.2.

π_i acts on $\mathrm{Hom}_{\mathscr{C}}(S_i, X)$ to the left and hence π acts continuously on the set $\varinjlim_i \mathrm{Hom}_{\mathscr{C}}(S_i, X) \xrightarrow{\sim} F(X)$, to the left. Since $F(X)$ is finite, the action of π on $F(X)$ comes from the action of some π_i on $F(X)$.

4.4.1.9 We shall find it convenient now to introduce informally the notion of the procategory Pro $\mathscr{C}$ of $\mathscr{C}$. An object of Pro $\mathscr{C}$ = (called a pro-object of $\mathscr{C}$) will be a projective system $\widetilde{p} = (P_i)_{i\in I}$ in $\mathscr{C}$. If $\widetilde{P}$, $\widetilde{P'} = (P'_j)_{j\in J}$ are pro-objects of $\mathscr{C}$, we define $\mathrm{Hom}(\widetilde{P}, \widetilde{P'})$ as the double limit $\varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(P_i, P'_j)$. An object of $\mathscr{C}$ will be considered an object of Pro $\mathscr{C}$ in a natural way. 82

We may look at a pro-representable functor on $\mathscr{C}$, as a functor "represented" in a sense by a pro-object of $\mathscr{C}$. For instance, in the case of 4.4.1.4, we have:

$$\begin{aligned} F(X) &\xleftarrow{\sim} \varinjlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(S_i, X), \quad \forall X \in \mathscr{C} \\ &\simeq \mathrm{Hom}_{\mathrm{Pro}\,\mathscr{C}}(\widetilde{S}, X) \end{aligned}$$

where $\widetilde{S}$ is the pro-object $(S_i)_{i\in I}$ of $\mathscr{C}$.

Also for any $i \in I$, $\mathrm{Hom}_{\mathrm{Pro}\,\mathscr{C}}(\widetilde{S}, S_i) \simeq \mathrm{Hom}_{\mathscr{C}}(S_i, S_i) = \mathfrak{g}_i$ and we may then write:

$$\begin{aligned} \mathfrak{g} = \varprojlim_i \mathfrak{g}_i &= \varprojlim_i \mathrm{Hom}_{\mathscr{C}}(S_i, S_i) \\ &= \varprojlim_i \mathrm{Hom}_{\mathrm{Pro}\,\mathscr{C}}(\widetilde{S}, S_i) = \mathrm{Hom}_{\mathrm{Pro}\,\mathscr{C}}(\widetilde{S}, \widetilde{S}) \end{aligned}$$

and hence $= \mathrm{Aut}_{\mathrm{Pro}\,\mathscr{C}}\, \widetilde{S}$.

If we call $\widetilde{S}$ a pro-representative of F (in the case $\mathscr{C} = (\mathscr{E}t/S)$ we call it a *universal covering* of S) then π is the opposite of the group of automorphisms of $\widetilde{S}$. 83

Lemma 4.4.1.10. *Let $E \in \mathscr{C}(\pi)$; then $\exists$ an object $G(E) \in \mathscr{C}$, and a $\mathscr{C}(\pi)$-isomorphism $\gamma_E : E \to FG(E)$ such that the map* $\mathrm{Hom}_{\mathscr{C}}(G(E), X) \to \mathrm{Hom}_{\mathscr{C}(\pi)}(E, F(X))$ *given by $u \mapsto F(u)\circ\gamma_E$ is a bijection for all $X \in \mathscr{C}$. The assignment $E \mapsto G(E)$ can be extended to a functor $G : \mathscr{C}(\pi) \to \mathscr{C}$ such that F and G establish an equivalence of $\mathscr{C}$ and $\mathscr{C}(\pi)$.*

Proof. If $E = \coprod E_i$ is a decomposition of E into connected sets in $\mathscr{C}(\pi)$ and if $G(E_i)$ are defined we may define $G(E) = \coprod G(E_i)$. We may thus assume that π acts transitively on E. Fix an element $\mathscr{E} \in E$ and consider the surjection $\pi \to E$ defined by $\sigma \mapsto \sigma \cdot \mathscr{E}$. As E is finite, there is an i such that the diagram

$$\begin{tikzcd} \pi \arrow[rr] \arrow[dr] & & E \\ & \pi_i \arrow[ur] & \end{tikzcd}$$

($\pi \to \pi_i$ is the natural projection and $\pi_i \to E$ is the map $\sigma \mapsto \sigma \cdot \mathscr{E}$) is commutative. Let $H_i \subset \pi_i$ be the isotropy group of $\mathscr{E}$ in π_i. It is easily proved that the set π_i/H_i of left-cosets of $\pi_i \mathrm{mod} \cdot H_i$ is $\mathscr{C}(\pi)$-isomorphic
84 to E. We then define: $G(E) = G(\pi_i/H_i) = S_i/H_i^0$, the quotient of S_i by the opposite H_i^0 of H_i (remark: $H_i^0 \subset \mathfrak{g}_i \subset \operatorname{Aut} S_i$). By ($F_4$) we have: $F(G(E)) = F(S_i/H_i^0) \overset{\sim}{\leftarrow} F(S_i)/H_i^0 \simeq \pi_i/H_i \simeq E$, and hence a $\mathscr{C}(\pi)$-isomorphism $\gamma_E : E \to FG(E)$. If $j \geq i$ and $H_j \subset \pi_j$ is the isotropy group of $\mathscr{E}$ in π_j, then we have a $\mathscr{C}$-morphism $S_j/H_j^0 \to S_i/H_i^0$; since $F(S_j/H_j^0) \to F(S_i/H_i^0)$ is a $\mathscr{C}(\pi)$-isomorphism, it follows from (F_5) that $S_j/H_j^0 \overset{\sim}{\to} S_i/H_i^0$ and that $G(E)$ is independent of the choice of i (upto a $\mathscr{C}$-isomorphism).

Let $X \in \mathscr{C}$. Consider the map

$$\begin{aligned} \omega : \operatorname{Hom}_{\mathscr{C}}(G(E), X) &\to \operatorname{Hom}_{\mathscr{C}(\pi)}(E, F(X)) \\ u &\mapsto F(u) \circ \gamma_E \end{aligned}$$

(i) ω is an injection:

Let $u_1, u_2 \in \operatorname{Hom}_{\mathscr{C}}(G(E), X)$ be such that $F(u_1) \circ \gamma_E = F(u_2) \circ \gamma_E$. But $\gamma_E : E \to FG(E)$ is an isomorphism and so $\ker(F(u_1), F(u_2)) \underset{\cong}{\hookrightarrow} FG(E)$, i.e., $F(\ker(u_1, u_2)) \underset{\simeq}{\hookrightarrow} FG(E)$ (F is left-exact). It follows from (F_5) that $\ker(u_1, u_2) \underset{\simeq}{\hookrightarrow} G(E)$, in other words, $u_1 = u_2$.

85 **(ii) ω is a surjection.**

Let $E \xrightarrow{\alpha} F(X)$ be any $\mathscr{C}(\pi)$-morphism; put $\delta = \alpha(\mathscr{E}) \in F(X)$. The Pro $\mathscr{C}(\pi)$-morphism $\delta : \widetilde{S} \to X$ can be factored through some S_i:

$$\begin{tikzcd}
\widetilde{S} \arrow[rr, "\delta"] \arrow[dr, "\tau_i"'] & & X \\
& S_i \arrow[ur, "\delta_i"'] &
\end{tikzcd}$$

Let H'_i be the isotropy group of δ_i in π_i then $H'_i \supset H_i$ where H_i is as before (take i large enough). By the construction of $G(E) = S_i/H_i^0$, we have a morphism $\dfrac{S_i}{H_i^0} \to X$ making the diagram

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\tau_i"] & S_i \arrow[rr, "\delta_i"] \arrow[dr] & & X \\
& & S_i/H_i^0 \arrow[ur] &
\end{tikzcd}$$

commutative; one easily checks that this morphism goes to α under ω. It only remains to show that the assignment $E \mapsto G(E)$ can be extended to a functor. Let $E, E' \in \mathscr{C}(\pi)$ and $\theta \in \mathrm{Hom}_{\mathscr{C}(\pi)}$ (E, E'). To the composite $\gamma_{E'} \circ \theta : E \to FG(E')$ there corresponds **86**
a *unique* $\overline{\theta} \in \mathrm{Hom}_{\mathscr{C}}(G(E), G(E'))$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, "\theta"] \arrow[d, "\gamma_E"] & E' \arrow[d, "\gamma_{E'}"] \\
FG(E) \arrow[r, "F(\overline{\theta})"'] & FG(E')
\end{tikzcd}$$

is commutative. We set $G(\theta) = \overline{\theta}$. It follows easily that G is a covariant functor from $\mathscr{C}(\pi)$ to $\mathscr{C}$.

One now checks that there are functorial isomorphisms

$$\Phi : I_{\mathscr{C}} \to G \circ F$$

$$\text{and} \qquad \Psi : I_{\mathscr{C}(\pi)} \to F \circ G,$$

such that, for any $X \in \mathscr{C}$ and $E \in \mathscr{C}(\pi)$,

$$F(X) \xrightarrow{F(\Phi(X))} FGF(X) \xrightarrow{\Psi^{-1}(F(X))} F(X)$$
$$\text{and} \qquad G(E) \xrightarrow{G(\Psi(E))} GFG(E) \xrightarrow{\Phi^{-1}(G(E))} G(E)$$

and the identity maps.

This completes the proof of assertion (1) of Theorem (4.4.1). □

87 **Lemma 4.4.1.11.** *Let $\mathscr{C}$ be a galois category and F, F' be two fundamental functors $\mathscr{C} \to$ (finite sets). Suppose π, π' are the profinite groups defined by F, F' respectively as above; then π and π' are continuously isomorphic and this isomorphism is canonically determined upto an inner automorphism of π.*

Proof. We know that $F : \mathscr{C} \to \mathscr{C}(\pi)$ is an equivalence. Replacing F' by $F' \circ G$ (with G as before) we can assume that $\mathscr{C} = \mathscr{C}(\pi)$, that F is the trivial functor identifying an object of $\mathscr{C}(\pi)$ with its underlying set and π itself is the pro-object pro-representing this functor. Let $\widetilde{T} \in \operatorname{Pro}\mathscr{C}(\pi)$ pro-represent F'; first we show that $\pi \xrightarrow{\sim} \widetilde{T}$ in $\operatorname{Pro}\mathscr{C}$.

In order to do this, let $(T_j)_{j\in J}$ be a projective family of galois objects (with respect to F') of $\mathscr{C}$ such that $\widetilde{T} = \varprojlim_j T_j$; we denote the canonical maps $T_j \to T_i$ by q_{ij} and $\widetilde{T} \to T_j$ by q_j. Let $t_j \in T_j$ be a coherent system of points and consider the continuous maps $\alpha_j : \pi \in T_j$ determined by $\alpha_j(e) = t_j$; there exists a continuous $\alpha : \pi \to \widetilde{T}$ such that

$$\begin{tikzcd} \pi \arrow[rr, "\alpha"] \arrow[dr, "\alpha_j"'] & & \widetilde{T} \arrow[dl, "q_j"] \\ & T_j & \end{tikzcd}$$

is commutative. We have $\alpha(e) = \widetilde{t} = (t_j) \in \widetilde{T}$. The T_j are connected,
88 hence α_j transitive and as π is compact it follows that α is onto. Let $\mathfrak{H}$ be the isotropy group of $\widetilde{t}$, then we have $\pi/\mathfrak{H} \xrightarrow{\sim} \widetilde{T}$. ($\pi/\mathfrak{H}$ is the set

of left-cosets of $\pi \bmod . \mathfrak{H}$) and this is an isomorphism of topological spaces (it is a continuous map of compact Hausdorff spaces). However $F' = \mathrm{Hom}_{\mathrm{Pro}\,\mathscr{C}}(\widetilde{T}, *)$ and then clearly $F'(X) = X^{\mathfrak{H}}$ (the points of X invariant under $\mathfrak{H}$). We know that $F'(X) = \emptyset \Leftrightarrow X = \emptyset$; from this it follows that $\mathfrak{H} = (e)$ (take for X the π_i for larger and larger i). Hence $\alpha : \pi \xrightarrow{\sim} \widetilde{T}$ is an isomorphism of topological spaces. Our aim is to show that this is an isomorphism in $\mathrm{Pro}\,\mathscr{C}$.

For this it is enough to show the following: if $\beta : \widetilde{T} \to \pi$ is α^{-1} and if $p_i : \pi \to \pi_i$ are the canonical maps then every $\beta_i = p_i \circ \beta : \widetilde{T} \to \pi_i$ must factor through some T_j. In other words, we must find some morphism $T_j \to \pi_i$ making the diagram

$$\begin{tikzcd}
\widetilde{T} \arrow[r, "\beta"] \arrow[d, "q_j"'] \arrow[dr, "\beta_i"] & \pi \arrow[d, "p_i"] \\
T_j \arrow[r] & \pi_i
\end{tikzcd}$$

commutative. For this we must show that given i, $\exists j \in J$ such that 89
for every $t \in T_j$, $\exists$ an $s \in \pi_i$ such that $q_j^{-1}(t) \subset \beta_i^{-1}(s)$. Since the sets $\beta_i^{-1}(s)$ are open we can find for every point $x \in \beta_i^{-1}(s)$ an open neighbourhood U_x of the form $q_j^{-1}(t_x)$ for some $j_x \in J$ and $t_x \in T_{j_x}$, such that $U_x \subset \beta_i^{-1}(s)$. Since $\widetilde{T}$ is compact, $\exists$ a finite covering $U_{x_1}, \ldots, U_{x_N}$ of $\widetilde{T}$ of this type and $j > \max(j_{x_1}, \ldots, j_{x_N})$ satisfies our requirements.

This shows that $\pi \xrightarrow{\sim} \widetilde{T}$ in $\mathrm{Pro}\,\mathscr{C}$, and hence the map

$$\pi'^0 = \mathrm{Aut}_{\mathrm{Pro}\,\mathscr{C}}\, \widetilde{T} \to \mathrm{Aut}_{\mathrm{Pro}\,\mathscr{C}}\, \pi = \pi^0$$
$$\rho \mapsto \beta \circ \rho \circ \alpha$$

is a group isomorphism; it remains to be shown that this map is continuous and hence a homeomorphism. Take a fixed $\gamma^0 = \beta \circ \rho^0 \circ \alpha$ in π, some $i \in I$ and consider the commutative diagram

$$\begin{tikzcd}
\pi \arrow[r, "\gamma^0"] \arrow[d, "p_i"] & \pi \arrow[d, "p_i"] \\
\pi_i \arrow[r, "\gamma_i^0"] & \pi_i
\end{tikzcd}$$

90 Let U be the neighbourhood of γ^0 consisting of all $\gamma \in \pi$ such that $p_i \circ \gamma = \gamma_i^0 \circ p_i$. We want to find a neighbourhood V of ρ^0 such that $\beta \circ \rho \circ \alpha \in U \;\forall\, \rho \in V$. There is an index $j \in J$ and $l \in I$ and morphisms making all the following diagrams

$$\begin{tikzcd}
\pi \arrow[r, "\alpha"] \arrow[d, "p_1"] & \widetilde{T} \arrow[r, "\rho^0"] \arrow[d, "q_j"] & \widetilde{T} \arrow[r, "\beta"] \arrow[d, "q_j"] & \pi \arrow[dd, "p_i"] \\
\pi_l \arrow[r] \arrow[d, "p_l"] & T_j \arrow[r, "\rho_j^0"] & T_j \arrow[dr] & \\
\pi_i \arrow[rrr, "\gamma_i^0"] & & & \pi_i
\end{tikzcd}$$

commutative.

Consider all $\rho : \widetilde{T} \to \widetilde{T}$ such that $q_j \circ \rho = \rho_j^0 \circ q_j$; these ρ form a neighbourhood V of ρ^0 and $\beta \circ \rho \circ \alpha \in U \;\forall\, \rho \in V$. Hence we are through.

Finally we observe that the isomorphism $\pi \to \pi'$ is fixed as soon as $\alpha : \pi \to \widetilde{T}$ is fixed and this is in turn fixed by the choice of $\widetilde{t} = (t_j) \in \widetilde{T}$. By a different choice of $\widetilde{t}$ we obtain an isomorphism $\pi \to \pi'$ which differs from the first one by an inner automorphism of π (or π'). Q.E.D. $\square$

91 **Remark 4.4.1.12.** One can, in fact, show that $\mathrm{Pro}\,\mathscr{C}(\pi)$ is precisely the category of compact, totally disconnected (Hausdorff) topological spaces on which π acts continuously.

Chapter 5

Galois Categories and Morphisms of Profinite Groups

5.1

92

Suppose π and π' are profinite groups and $u : \pi' \to \pi$ a continuous homomorphism. Then u defines, in a natural way, a functor $H_u : \mathscr{C}(\pi) \to \mathscr{C}(\pi')$; and H_u being the identity functor on the underlying sets is fundamental.

On the other hand, let $\mathscr{C}, \mathscr{C}'$ be galois categories, π' a profinite group and $H : \mathscr{C} \to \mathscr{C}'$, $F' : \mathscr{C}' \to \mathscr{C}(\pi')$ be functors such that $F = F' \circ H$ is fundamental. Then we can choose a pro-object $\widetilde{S} = \{S_i, \varphi_{ij}\}_{i \in I}$ of $\mathscr{C}$ such that, $\forall X \in \mathscr{C}$, $F(X) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{Pro}\,\mathscr{C}}(\widetilde{S}, X)$. Moreover we may assume that the S_i are galois objects of $\mathscr{C}$, therefore the $F(S_i)$ are principal homogeneous spaces under the action of the π_i (on the right) (notations from Ch. 4) and if we identify $F(S_i)$, by means of the canonical element τ_i, with π_i then the maps $F(\varphi_{ij}) = \psi_{ij} : F(S_j) \to F(S_i)$ are group homomorphisms (see Ch. 4). However, in the present situation the $F(S_i)$ are not merely sets but are objects of $\mathscr{C}(\pi')$; as such, the group π' acts continuously upon the sets to the left, and this action commutes with the right-action of the π_i. This gives a continuous homomorphism

$u_i : \pi' \to \pi_i$ determined by the condition that for $\sigma' \in \pi'$ the $u_i(\sigma')$ is
93 the unique element of π_i such that $\sigma' \cdot \tau_i = \tau_i \cdot u_i(\sigma')$ (τ_i is the canonical element of $F(S_i)$); for $j \geq i$ we clearly have: $\psi_{ij} \circ u_j = u_i$; thus, we obtain a continuous homomorphism thus, we obtain a continuous homomorphism $u : \pi' \to \varprojlim_i \pi_i = \pi$ and u corresponds to H if we identify $\mathscr{C}$ with $\mathscr{C}(\pi)$.

Example. We shall apply the above to the particular case $\mathscr{C} = (\mathscr{E}t/S)$, $\mathscr{C}' = (\mathscr{E}t/S')$ where S, S' are, as usual locally noetherian, connected preschemes. Suppose $\varphi : S' \to S$ is a morphism of finite type and $s' \in S'$, $s = \varphi(s') \in S$. Let Ω be an algebraically closed field containing $k(s')$. We define functors $F : \mathscr{C} \to$ {Finite sets} and $F' : \mathscr{C}' \to$ {Finite sets} by defining:

$$F(X) = \mathrm{Hom}_S(\mathrm{Spec}\,\Omega, X), X \in \mathscr{C},$$
$$F'(X') = \mathrm{Hom}_{S'}(\mathrm{Spec}\,\Omega, X'), X' \in \mathscr{C}'.$$

Denote by π and π' the fundamental groups $\pi_1(S, s)$ and $\pi_1(S', s')$.

Corresponding to the morphism $\varphi : S' \to S$, we obtain a functor $\Phi : \mathscr{C} \to \mathscr{C}'$ given by $X \mapsto X \underset{S}{\times} S'$. We then have:

$$\begin{aligned} F'(X \underset{S}{\times} S') &= \mathrm{Hom}_{s'}(\mathrm{Spec}\,\Omega, X \underset{S}{\times} S') \\ &\simeq \mathrm{Hom}_s(\mathrm{Spec}\,\Omega, X) = F(X). \end{aligned}$$

94 Thus $F = F' \circ \Phi$; in view of the fact that F is fundamental and the equivalences $\mathscr{C} \overset{F}{\sim} \mathscr{C}(\pi)$, $\mathscr{C}' \overset{F}{\sim} \mathscr{C}(\pi')$, we obtain a continuous homomorphism $\pi' \to \pi$.

5.2

In this section, we shall correlate the properties of a homomorphism $u : \pi' \to \pi$ and those of the corresponding functor $H_u : \mathscr{C}(\pi) \to \mathscr{C}(\pi')$.

5.2.1 Suppose $u : \pi' \to \pi$ is onto; for any connected object X of $\mathscr{C}(\pi)$ (i.e., π acts transitively on X) any π-morphism $\pi \to X$ defined by, say,

$e \mapsto x$ is onto and therefore so is the map $\pi' \to X$ defined by $e' \mapsto x$; in other words, $H_u(X)$ is a connected object of $\mathscr{C}(\pi')$.

Conversely, suppose that for any connected $X \in \mathscr{C}(\pi)$, the object $H_u(X)$ is again connected in $\mathscr{C}(\pi')$. Write $\pi = \varprojlim_i \pi_i$ where the π_i are finite groups and the structure-homomorphisms $\pi_j \to \pi_i$, $j \geq i$, are all onto; this implies that the $\pi \to \pi_i$ are all onto and by our assumption then all the $\pi' \to \pi_i$ are onto. Since π and π' are both profinite it follows that $u : \pi' \to \pi$ is onto. Thus: *$u : \pi' \to \pi$ is onto $\Leftrightarrow$ for any connected $X \in \mathscr{C}(\pi)$, the object $H_u(X)$ is connected in $\mathscr{C}(\pi')$.*

5.2.2 A *pointed object* of $\mathscr{C}(\pi)$ is, by definition, a pair (X, x) with $X \in$ 95
$\mathscr{C}(\pi)$ and $x \in X$. By the definition of the topology on π, it is clear that giving a pointed, connected object of $\mathscr{C}(\pi)$ is equivalent to giving an open subgroup H of π; the object is π/H = set of left-cosets of $\pi \bmod H$ and the point is the class H. A final object of $\mathscr{C}(\pi)$ is a point e_c on which π acts trivially. We say that an $X \in \mathscr{C}(\pi)$ has a section if there is a $\mathscr{C}(\pi)$-morphism from a final object e_c to X; giving a section of X is equivalent to giving a point of X, invariant under the action of π. A pointed object (X, x) of $\mathscr{C}(\pi)$ admits a pointed section (i.e., e_c is mapped onto x) $\Leftrightarrow$ x is invariant under π.

Suppose $u : \pi' \to \pi$ is a homomorphism and H is an open subgroup of π such that $u(\pi') \subset H$. Let (X, x) be the pointed, connected object of $\mathscr{C}(\pi)$ determined by H. Then, in the action of π' on $H_u(X)$, x remains invariant, i.e., to say, the pointed object $(H_u(X), x)$ of $\mathscr{C}(\pi')$ admits a pointed section. The converse situation is clear. Thus:

For an open subgroup H of π, one has $u(\pi') \subset H \Leftrightarrow H_u(\pi/H)$ admits a pointed section in $\mathscr{C}(\pi')$.

5.2.3 We say that an $X \in \mathscr{C}(\pi)$ is *completely decomposed* if X is a finite disjoint sum of final objects of $\mathscr{C}(\pi)$, i.e., if the action of π on X is trivial.

Suppose $u : \pi' \to \pi$ is trivial, then for any $X \in \mathscr{C}(\pi)$, $H_u(X)$ is 96
completely decomposed in $\mathscr{C}(\pi')$. Conversely, assume that for any $X \in \mathscr{C}(\pi) H_u(X)$ is completely decomposed in $\mathscr{C}(\pi')$. Write $\pi = \varprojlim_i \pi_i$ as

usual; by assumption, each composite $\pi' \to \pi \to \pi_i$ is trivial. Hence $u : \pi' \to \pi$ is also trivial. Thus:

$u : \pi' \to \pi$ is trivial $\Leftrightarrow$ for any $X \in \mathscr{C}(\pi)$, $H_u(X)$ is completely decomposed in $\mathscr{C}(\pi')$.

5.2.4 Let H' be an open subgroup of π' and $X' \in \mathscr{C}(\pi')$ the connected, pointed object defined by H'. Assume that $\ker u \subset H'$; then $u(\pi')/u(H') \simeq \pi'/H'$ in $\mathscr{C}(\pi')$. This means that $u(H')$ is a subgroup of finite index of the pro-finite group $u(\pi')$ and hence is open in $u(\pi')$. Since π' is compact and π Hausdorff we can find an open subgroup H of π such that $H \cap u(\pi') \subset u(H')$.

Consider now the connected, pointed object $X = \pi/H$ of $\mathscr{C}(\pi)$. Denote by $H_u(X)_0$ the $\mathscr{C}(\pi')$-component of the pointed object $H_u(X)$ of $\mathscr{C}(\pi')$, containing the distinguished point of $H_u(X)$. There exists then an open subgroup H'_1 of π' such that $H_u(X)_0 \simeq \pi'/H'_1$ in $\mathscr{C}(\pi')$. We claim that $H'_1 \subset H'$; in fact, $u(H'_1) \subset H$ and so $u(H'_1) \subset H \cap u(\pi') \subset u(H')$, hence $H'_1 \subset u^{-1}(u(H'_1)) \subset u^{-1}(u(H')) = H'$, since, by assumption, H' is saturated under u.

97 Thus, there is a pointed $\mathscr{C}(\pi')$-morphism $H_u(X)_0 \simeq \pi'/H'_1 \to \pi'/H' \simeq X'$. If, on the other hand, we assume that $\exists$ a pointed, connected object X of $\mathscr{C}(\pi)$ such that we have a pointed $\mathscr{C}(\pi')$-morphism $H_u(X)_0 \simeq \pi'/H'_1 \to X' \simeq \pi'/H'$, then, we must have $H'_1 \subset H'$ and hence $\ker u \subset H'_1 \subset H'$. If u is surjective, then we can say that $H_u(X) \simeq X'$ (see 5.2.1). Also $\ker u \subset H'$ is a relation independent of the choice of the distinguished point in $X' \simeq \pi'/H'$. Thus:

$\ker u \subset H'$ $\Leftrightarrow$ $\exists$ *a connected object X of $\mathscr{C}(\pi)$ and a $\mathscr{C}(\pi')$-morphism of a connected $\mathscr{C}(\pi')$-component of $H_u(X)$ to $X' = \pi'/H'$. If u is onto then $X' = \pi'/H' \simeq H_u(X)$ for a connected object $X \in \mathscr{C}(\pi)$.*

In particular:

u is injective $\Leftrightarrow$ for every connected $X' \in \mathscr{C}(\pi')$, there is a connected $X \in \mathscr{C}(\pi)$ and a $\mathscr{C}(\pi')$-morphism from a $\mathscr{C}(\pi')$-component of $H_u(X)$ to X'.

5.2.5 Let $\pi' \xrightarrow{u'} \pi \xrightarrow{u''} \pi''$ be a sequence of morphisms of profinite groups. From 5.2.3 and 5.2.4 we obtain the following necessary and sufficient conditions for the sequence to be exact:

(a) $u'' \circ u'$ is trivial $\Leftrightarrow$ *for any* $X'' \in \mathscr{C}(\pi'')$, $H_{u'} \circ H_{u''}(X'')$ is *com-* 98
pletely decomposed in $\mathscr{C}(\pi')$.

(b) $\operatorname{Im} u' \supset \ker u'' \Leftrightarrow$ for any open subgroup H of π, with $H \supset \operatorname{Im} u'$, we also have $H \supset \ker u'' \Leftrightarrow$ *for any connected pointed object X of $\mathscr{C}(\pi)$ such that $H_{u'}(X)$ admits a pointed section in $\mathscr{C}(\pi')$, there is a connected object* $X'' \in \mathscr{C}(\pi'')$ and a $\mathscr{C}(\pi)$*-morphism of a* $\mathscr{C}(\pi)$*-component of $H_{u''}(X'')$ to X.*

5.2.6 Let $\mathscr{C}$ be a galois category with a fundamental functor F. Let $\widetilde{S} = (S_i)_{i \in I}$ be a pro-object of $\mathscr{C}$ with usual properties (in particular, S_i are galois) pro-representing F. Let π be the fundamental group of $\mathscr{C}$ determined by F. We know then that $\mathscr{C} \overset{F}{\sim} \mathscr{C}(\pi)$.

Let now $T \in \mathscr{C}$ be a connected object and $t \in F(T)$ be fixed for our considerations. We form the category $\mathscr{C}' = \mathscr{C}|T$; it is then readily checked that $\mathscr{C}'$ satisfies the axioms $(\mathscr{C}_0), \ldots, (\mathscr{C}_4)$ of Ch. IV. We have an exact functor $\mathscr{P} : \mathscr{C} \to \mathscr{C}'$ defined by $X \mapsto X \times T$. We now define a functor $F' : \mathscr{C}' \to \{\text{Finite sets}\}$ by setting, for any $X \in \mathscr{C}'$, $F'(X) =$ inverse image of t under the map $F(X) \to F(T)$. Again it is easily checked that $\mathscr{C}'$, equipped with F', is galois. A cofinal subsystem $\widetilde{S}'$ of $\widetilde{S}$ is defined by the condition: $S_i \in \widetilde{S}' \Leftrightarrow (S_i, \tau_i)$ dominates (T, t) in
the sense of Ch. 4. An $S_i \in \widetilde{S}'$ can be considered in the obvious way 99
as an object of $\mathscr{C}'$; it is then easily shown that they are galois in $\mathscr{C}'$ and the pro-object $\widetilde{S}'$ of $\mathscr{C}'$ pro-represents F' (To do the checking one may identify $\mathscr{C}$ and $\mathscr{C}(\pi)$).

Let H be the isotropy group of $t \in F(T)$ in π; let also N_i be the isotropy groups of $\tau_i \in F(S_i)$, $S_i \in \widetilde{S}'$. We have a diagram of the form

$$\begin{tikzcd}
\pi \arrow[rr] \arrow[dr] & & F(T) \ni t \\
& \tau_i \in F(S_i) \arrow[ur] &
\end{tikzcd}$$

which is commutative and thus $N_i \subset H$, $\forall i$. Since the $F(S_i)$ and $F(T)$ are all connected objects of $\mathscr{C}(\pi)$, we have $\frac{\pi}{N_i} \simeq F(S_i)$ and $\pi/H \simeq F(T)$.

The maps $F(S_i) \to F(T)$ are then the natural maps $\pi/N_i \to \pi/H$ and it follows that $F'(S_i) \simeq H/N_i$ in $\mathscr{C}(\pi)$. But as we have already remarked the S_i in $\widetilde{S'}$ form a system of galois objects with respect to F', pro-representing F' and one thus obtains:

$$\pi' \simeq \varprojlim_{S_i \in \widetilde{S'}} F'(S_i) = \varprojlim_{N_i \subset H} H/N_i \approx H.$$

Finally we remark that the composite functor $F' \circ \mathscr{P}$ is isomorphic with F and therefore fundamental; following the procedure of 5.1 we see that the corresponding continuous homomorphism $u : \pi' \to \pi$ is nothing but the canonical inclusion $H \hookrightarrow \pi$.

Chapter 6

Application of the Comparison Theorem an Exact Sequence for Fundamental Groups

6.1

100

As usual we make the convention that the preschemes considered are locally noetherian and the morphisms are of finite type (with the same remark as in the beginning of Ch. 3).

Definition 6.1.1. (a) A morphism $X \to \operatorname{Spec} k$, k a field, is said to be *separable* if, for any extension field K/k, the prescheme $X \underset{k}{\otimes} K$ is *reduced.*

(b) A morphism $X \xrightarrow{f} Y$ is separable if f is flat and for any $y \in Y$, $X \underset{Y}{\otimes} \operatorname{Spec} k(y)$ is separable over $k(y)$.

We shall now state a few results which we will need for our next main theorem. Proofs can be found in EGA.

Theorem 6.1.2. *Let $f : X \to Y$ be a proper morphism and $\mathscr{F}$ a coherent*

$\mathscr{O}_X$*-Module. If* $Y_1 \xrightarrow{q} Y$ *is a flat base-change*

$$\begin{tikzcd} X \arrow[d, "f"] & X_1 = X \times_Y Y_1 \arrow[l, "q_1"'] \arrow[d, "f_1"] \\ Y & Y_1 \arrow[l, "q"'] \end{tikzcd}$$

then we have the isomorphisms

$$R^n f_{1*}(q_1^*(\mathscr{F})) \xleftarrow{\sim} R^n f_*(\mathscr{F}) \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y_1}$$

for any $n \in \mathbb{Z}^+$*. (Prop. (1.4.15), EGA, Ch. III).*

101 **6.1.3** Suppose Y *is a noetherian prescheme and* $f : X \to Y$ *a proper morphism.* Let $Y' \hookrightarrow Y$ be a closed subscheme of Y, defined by a coherent Ideal $\mathscr{T}$ of $\mathscr{O}_Y$. The "inverse image" of Y' by f, namely the fibre-product $X \underset{Y}{\times} Y' = X'$ is then a closed subscheme of X, defined by the $\mathscr{O}_X$-Ideal $\mathscr{T} = f^*(\mathscr{T})\mathscr{O}_X$.

Let $\mathscr{F}$ be any *coherent* $\mathscr{O}_X$-Module; for $n \in \mathbb{Z}^+$, consider $\mathscr{F}_n = \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_X/\mathscr{T}^{n+1}$ (this is a coherent $\mathscr{O}_X$-Module, concentrated on the prescheme $X_n = (X', \mathscr{O}_X/\mathscr{T}^{n+1})$ and may also be considered as an $\mathscr{O}_{X_n}$-Module). Consider $R^q f_*(\mathscr{F}_n)$; this is a coherent $\mathscr{O}_Y$-Module (finiteness theorem), is concentrated on the prescheme $Y_n = (Y', \mathscr{O}_Y/\mathscr{T}^{n+1})$ and is in fact an $\mathscr{O}_{Y_n}$-Module. From the homomorphism $\mathscr{F} \to \mathscr{F}_n$ we obtain a homomorphism $R^q f_*(\mathscr{F}) \to R^q f_*(\mathscr{F}_n)$ and since the latter is an $\mathscr{O}_{Y_n}$-Module, we get a natural homomorphism:

$$(*) \qquad R^q f_*(\mathscr{F}) \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_Y/\mathscr{T}^{n+1} \to R^q f_*(\mathscr{F}_n).$$

As n varies, we get a projective system of homomorphisms. With the assumptions we have made about X, Y, f, the *comparison theorem* (EGA, Ch. III, Theorem (4.1.5)) states that in the limit, this gives an *isomorphism*:

$$(**) \qquad \varprojlim_n R^q f_*(\mathscr{F}) \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_Y/\mathscr{T}^{n+1} \xrightarrow{\sim} \varprojlim_n R^q f_*(\mathscr{F}_n).$$

102 (**Note:** Both sides of (**) are concentrated on Y' and they are also equal to $R^q\widehat{f}_*(\varprojlim_n \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_X/\mathscr{T}^{n+1})$ where $\widehat{f}$ is the morphism of the *ringed spaces* $\widehat{X} = (X', \varprojlim_n \mathscr{O}_X/\mathscr{T}^{n+1}) \to \widehat{Y} = (Y', \varprojlim_n \mathscr{O}_Y/\mathscr{T}^{n+1})$ obtained from the morphisms $f_n : X_n \to Y_n$, induced by f).

We shall now "specialise" the above comparison theorem to the case Y affine, say $Y = \operatorname{Spec} A$. Then Y' is defined by an ideal I of A. The first member of (**) corresponds to $\varprojlim_n H^q(X, \mathscr{F}) \otimes_A \frac{A}{I^{n+1}}$ which is precisely the completion of $H^q(X, \mathscr{F})$ under the I-adic topology, while the second member of (**) corresponds to $\varprojlim_n H^q(X, \mathscr{F}_n)$ and thus:

$$H^q(X, \mathscr{F})\widehat{\ } \xrightarrow{\sim} \varprojlim_n H^q(X, \mathscr{F}_n). \tag{***}$$

6.2 The Stein-factorisation

Let $X \xrightarrow{f} Y$ be a proper morphism. Then the coherent $\mathscr{O}_Y$-Algebra $f_*(\mathscr{O}_X)$ (finiteness theorem) defines a Y-prescheme $Y' \xrightarrow{q} T$, *finite* on Y. To the identity $\mathscr{O}_Y$-morphism $q_*(\mathscr{O}_{Y'}) = f_*(\mathscr{O}_X) \to f_*(\mathscr{O}_X)$ corresponds a Y-morphism $f' : X \to Y'$, i.e., we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f'"] \arrow[dr, "f"'] & & Y' = \operatorname{Spec} f_*(\mathscr{O}_X) \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

The morphism f' is again proper. This factorisation $f = q \circ f'$ is 103
known as the *Stein-factorisation* of f. For details the reader is referred to EGA Ch. III.

We shall now prove a theorem, which is of great importance to us.

Theorem 6.2.1. *Let $f : X \to Y$ be a separable, proper morphism. Let $X \xrightarrow{f'} Y' = \operatorname{Spec} f_*(\mathscr{O}_X) \xrightarrow{q} Y$ be the Stein-factorisation of f. Then $Y' \xrightarrow{q} Y$ is an étale covering.*

Proof. We have only to show that q is étale; this is a purely local problem and we may thus assume that Y is affine, say, $Y = \operatorname{Spec} A$.

We shall make a few simplifications to start with. Suppose $X_1 \to Y$ is a flat base-change; then from 6.1.2 one gets:

$$f_{(Y_1)_*}(\mathscr{O}_X \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y_1}) \simeq f_*(\mathscr{O}_X) \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y_1}$$

This means that in the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f'"] & & X_1 = X \underset{Y}{\times} Y_1 \arrow[ll] \arrow[d, "f'_{(Y_1)}"'] \\
\operatorname{Spec} f_*(\mathscr{O}_X) = Y' \arrow[d, "q"] & Y'_1 = Y' \underset{Y}{\times} Y_1 = \operatorname{Spec} f_{(Y_1)_*}(\mathscr{O}_X \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y_1}) \arrow[l] \arrow[d, "q_{(Y_1)}"] \\
Y & & Y_1 \arrow[ll]
\end{tikzcd}$$

the second vertical sequence is the Stein-factorisation of $f_{(Y_1)}$.

104 In view of this and the fact that it is enough to look at Spec $\mathscr{O}_{y,Y}$ for étaleness over $y \in Y$ we may make the base change $Y \leftarrow \operatorname{Spec} \mathscr{O}_{y,Y}$ and assume that A is a local ring. Again, in view of the same remark and Lemma 4.4.1 we may assume that A is *complete*.

Consider now the functor T on the category of finite type A-modules M, defined by $M \mapsto T(M) = \Gamma(X, \mathscr{O}_X \otimes_A M)$. We shall show that *the assumption "T is right-exact" implies the theorem.*

We remark that the *right-exactness of T is equivalent to the assumption that the natural map*

$$(+) \qquad \Gamma(X, \mathscr{O}_X) \underset{A}{\otimes} M \to \Gamma(X, \mathscr{O}_X \underset{A}{\otimes} M)$$

is an isomorphism. Denote by T' the right-exact functor

$$M \mapsto \Gamma(X, \mathscr{O}_X) \underset{A}{\otimes} M.$$

Suppose T is right exact. Then for any exact sequence $A^n \to A^m \to M \to 0$ we have a commutative diagram of exact sequences:

$$\begin{tikzcd}
T'(A^n) \arrow[r] \arrow[d, "\wr"] & T'(A^m) \arrow[r] \arrow[d, "\wr"] & T'(M) \arrow[r] \arrow[d] & 0 \\
T(A^n) \arrow[r] & T(A^m) \arrow[r] & T(M) \arrow[r] & 0
\end{tikzcd}$$

the first two vertical maps are isomorphisms; hence the third is also an isomorphism. Conversely, if (+) is an isomorphism, clearly T is right-exact. **105**

Now from the flatness of f and the assumed right exactness of T it follows that $\Gamma(X, \mathscr{O}_X \underset{A}{\otimes} M)$ is exact in M and thus [from the above remark] $\Gamma(X, \mathscr{O}_X)$ is A-flat. But $Y' = \operatorname{Spec} \Gamma(X, \mathscr{O}_X)$ and therefore $q : Y' \to Y$ is flat. It remains to show that q is unramified.

Let $y \in Y$ be the unique closed point of Y; denote by k the residue field $A/\mathscr{M} = k(y)$. Then $T(k) = \Gamma(X, \mathscr{O}_X \underset{A}{\otimes} k) = \Gamma(X_y, \mathscr{O}_{X_y})$ where X_y is the fibre $X \underset{Y}{\otimes} k(y)$. But as X is Y-proper, $T(k) = \Gamma(X, \mathscr{O}_X \underset{A}{\otimes} k)$ is an artinian k-algebra which, by the separability of X over Y, is radical-free for any base-change K/k, K an extension field of k. Hence $T(k) = \bigoplus_{i=1}^{r} K_i$ where the K_i are finite separable field extensions of k. Also, by our assumption, $T(k) = \Gamma(X, \mathscr{O}_X) \underset{A}{\otimes} k = \Gamma(X, \mathscr{O}_X)/\mathscr{M}\Gamma(X, \mathscr{O}_X)$. Let $y' \in Y'$ be a point above y. From the equality $\Gamma(X, \mathscr{O}_X)/\mathscr{M}\Gamma(X, \mathscr{O}_X) = \bigoplus_{i=1}^{r} K_i$, ($K_i/k$ are separable) it follows that the maximal ideal $\mathscr{M}$ of $\mathscr{O}_y$. generates the maximal ideal of $\mathscr{O}_{y'}$ and moreover that $\mathscr{O}_{y'}$ is unramified over $\mathscr{O}_y$. □

The proof of the theorem is thus complete modulo the assumption that T is right-exact. Before proceeding to prove this we may make some more simplifications. First we may assume that $T(k) = \Gamma(X_y, \mathscr{O}_{X_y}) = \bigoplus_{i=1}^{r} k$ (this can be done as in Lemma 4.1.2 by making a faithfully flat base-change which "kills" the extensions $K_{i/k}$). Then finally we may **106** assume $\Gamma(X_y, \mathscr{O}_{X_y}) = k$, i.e., X_y connected, because the connected components of X over y correspond bijectively with the connected components of X_y and it is enough to prove the theorem separately for each component over y.

Case (a). Assume A artinian.

Our aim is to show that for any finite type A-module M, $T'(M) \xrightarrow{\sim} T(M)$. We shall first show that $T'(M) \to T(M)$ is surjective.

For n large, one has $\mathscr{M}^n M = 0$ so that $T'(\mathscr{M}^n M) \to T(\mathscr{M}^n M)$ is onto. Assume now that $T'(\mathscr{M}^{i+1} M) \to T(\mathscr{M}^{i+1} M)$ is onto, and consider the exact sequence: $0 \to \mathscr{M}^{i+1} M \to \mathscr{M}^i M \to \frac{\mathscr{M}^i M}{\mathscr{M}^{i+1} M} \to 0$ of A-modules.

We get a commutative diagram

$$\begin{tikzcd}
T'(\mathscr{M}^{i+1}M) \arrow[r] \arrow[d] & T'(\mathscr{M}^{i}M) \arrow[r] \arrow[d] & T'\left(\frac{\mathscr{M}^i M}{\mathscr{M}^{i+1} M}\right) \arrow[r] \arrow[d] & 0 \\
T(\mathscr{M}^{i+1}M) \arrow[r] & T(\mathscr{M}^{i}M) \arrow[r] & T\left(\frac{\mathscr{M}^i M}{\mathscr{M}^{i+1} M}\right) &
\end{tikzcd}$$

107 of exact sequences (the second row is exact at the spot $T(\mathscr{M}^i M)$ because T is semi-exact, i.e., for $0 \to M' \to M \to M'' \to 0$ exact, the sequence $T(M') \to T(M) \to T(M'')$ is exact at the spot $T(M)$). The first vertical map is a surjection by assumption, the last is a surjection since $\dfrac{\mathscr{M}^i M}{\mathscr{M}^{i+1} M}$ is a finite direct sum of copies of k and by our assumption $\Gamma(X_y, \mathscr{O}_{X_y}) \simeq k$, we have $T'(k) \xrightarrow{\sim} T(k)$. It follows that $T'(M) \to T(M)$ is onto, by a downward induction. Finally consider an exact sequence $0 \to R \to A^p \to M \to 0$. We then obtain a commutative diagram

$$\begin{tikzcd}
T'(R) \arrow[r] \arrow[d, "\text{onto}"] & T'(A^p) \arrow[r] \arrow[d, "\wr"] & T'(M) \arrow[r] \arrow[d] & 0 \\
T(R) \arrow[r] & T(A^p) \arrow[r] & T(M) &
\end{tikzcd}$$

of exact sequences. If follows easily that $T'(M) \to T(M)$ is also injection. Thus, the theorem is completely proved in the case (a).

Case (b). The general case

Let A be a complete noetherian local ring. Denote by Y_1' the closed subscheme of Y defined by the maximal ideal $\mathscr{M}$. As in the comparison theorem we may define the functors

$$\begin{aligned} T_n(M_n) &= \Gamma(X_n, \mathscr{O}_X \underset{A}{\otimes} M \underset{A}{\otimes} A/\mathscr{M}^{n+1}) \\ &= \Gamma(X, \mathscr{O}_X \underset{A}{\otimes} M \underset{A}{\otimes} A/\mathscr{M}^{n+1}) \end{aligned}$$

where $M_n = M \underset{A}{\otimes} A/\mathscr{M}^{n+1}$. If M is an A-module of finite type, so **108**
is $\Gamma(X, \mathscr{O}_X \underset{A}{\otimes} M)$; as A is complete under the $\mathscr{M}$-adic topology, so is $\Gamma(X, \mathscr{O}_X \underset{A}{\otimes} M)$ and from 6.1.3 we obtain:

$$\Gamma(X, \mathscr{O}_X \underset{A}{\otimes} M) \xrightarrow{\sim} \varprojlim_n T_n(M_n).$$

To show that T is right-exact it is enough to show that for any exact sequence $M \xrightarrow{u} N \to 0$ of finite type A-modules, $T(M) \xrightarrow{T(u)} T(N) \to 0$ is again exact. But as each $A/\mathscr{M}^{n+1}$ is a complete artinian local ring, it follows from case (a) that $T_n(M_n) \xrightarrow{T_n(u)} T_n(N_n) \to 0$ is exact; also for each n, $\ker T_n(u)$ is a module of finite length over A. Thus, it is enough now to prove the

Lemma 6.2.2. *Let (K_n, φ_{nm}), (M_n, ψ_{nm}), (N_n, θ_{nm}), $n \in \mathbb{Z}^+$, be projective systems of abelian groups and $u = (u_n)$, $v = (v_n)$ be morphisms such that, for every $n \in \mathbb{Z}^+$ $0 \to K_n \xrightarrow{u_n} M_n \xrightarrow{v_n} N_n \to 0$ is exact. Assume, in addition, that for each n, $\exists m_0 = m_0(n)$ such that $\varphi_{nm}(K_m) = \varphi_{nm_0}(K_{m_0})$ $\forall$ $m \geq m_0(n) > n$ (this is the so-called Mittag-Leffler (ML) condition; it is certainly satisfied if the* $\ker u_n = K_n$ *are of finite length). Then the sequence of projective limits is also exact.*

Proof. The only difficult point is to show that $\varprojlim_n v_n$ is onto. By hy- **109**
pothesis, for each n, $\exists m_0(n) > n$ with $\varphi_{nm}(K_m) = \varphi_{nm_0}(K_{m_0})$ $\forall$ $m \geq m_0(n)$. By passing to a cofinal subsystem we may suppose, that, for any $n \in \mathbb{Z}^+$

$$\varphi_{nm}(K_m) = \varphi_{n,n+1}(K_{n+1}) \ \forall\ m \geq n+1.$$

Let now $(y_n) \in \varprojlim_n N_n$. Choose $x'_0 \in M_0$ with $v_0(x'_0) = y_0$. Assume inductively that we have chosen $(x_0, x_1, \ldots, x_{n-1}, x'_n)$ with (i) $\psi_{r,r+1}(x_{r+1}) = x_r$ for $0 \leq r \leq n-2$, and $\psi_{n-1,n}(x'_n) = x_{n-1}$. (ii) $v_r(x_r) = y_r$ for $0 \leq r \leq n-1$, and $v_n(x'_n) = y_n$.

Our aim now is to find $(x_0, x_1, \ldots, x_{n-1}, x_n, x'_{n+1})$ for which the above properties (i), (ii) hold when n is replaced by $n+1$. Choose $x''_{n+1} \in M_{n+1}$ such that $v_{n+1}(x''_{n+1}) = y_{n+1}$; then, $v_n(\psi_{n,n+1}(x''_{n+1}) - x'_n) = y_n - y_n = 0$ i.e. to say, $\psi_{n,n+1}(x''_{n+1}) - x'_n \in K_n$. By assumption, we can find $z_{n+1} \in K_{n+1}$ such that:

$$\begin{aligned}\psi_{n-1,n+1}(z_{n+1}) &= \varphi_{n-1,n+1}(z_{n+1})\\ &= \varphi_{n-1,n}(\psi_{n,n+1}(x''_{n+1}) - x'_n)\\ &= \psi_{n-1,n+1}(x''_{n+1}) - \psi_{n-1,n}(x'_n).\end{aligned}$$

We now set $x'_{n+1} = (x''_{n+1} - z_{n+1})$ and $x_n = \psi_{n,n+1}(x'_{n+1})$.

110 We then have:

$$\begin{aligned}\psi_{n-1,n}(x_n) &= \psi_{n-1,n+1}(x''_{n+1}) - \psi_{n-1,n+1}(z_{n+1})\\ &= \psi_{n-1,n}(x'_n) = x_{n-1}.\\ v_{n+1}(x'_{n+1}) &= v_{n+1}(x''_{n+1}) = y_{n+1}\end{aligned}$$

and finally $v_n(x_n) = v_n(\psi_{n,n+1}(x'_{n+1})) = y_n$. Q.E.D. □

6.3 The first homotopy exact sequence

6.3.1 Some properties of the Stein-factorisation of a proper morphism. Let Y be locally noetherian and $f : X \to Y$ be *proper*. Suppose $X \xrightarrow{f'} Y' \xrightarrow{q} Y$ is the Stein factorisation of f. By using the comparison theorem one can prove the

6.3.1.1 (*Zariski's connection theorem*).

The morphism $f' : X \to Y'$ is also proper. And for any $y' \in Y'$ the fibre $f'^{-1}(y')$ is *non-void and geometrically connected* (i.e., for any field $k' \supset k(y')$ the prescheme $X \underset{Y'}{\times} \operatorname{Spec} k'$ is connected).

(For a proof see EGA Ch. III § (4.3)).

One can then easily draw the following corollaries.

Corollary 6.3.1.2. *For any $y \in Y$, the connected components of the fibre $f^{-1}(y)$ are in a $(1-1)$ correspondence with the set of points of the fibre $q^{-1}(y)$ (which is finite and discrete).*

Corollary 6.3.1.3. *For any $y \in Y$, let $\overline{k(y)}$ be the algebraic closure of $k(y)$ and $\overline{X}_y$ be the prescheme $X \underset{Y}{\times} \overline{k(y)}$. The connected components of $\overline{X}_y$ (known as the geometric components of the fibre over y) are in $(1-1)$ correspondence with the geometric points of Y' over y.* **111**

6.3.2 Now, in addition, suppose that

(i) Y is connected.

(ii) f is separable

(iii) $f_*(\mathscr{O}_X) \overset{\sim}{\leftarrow} \mathscr{O}_Y$.

Assumption (iii) implies that $X \xrightarrow{f} Y$ is its own Stein-factorisation. From **??**, it follows that the fibres $f^{-1}(y)$ are (geometrically) connected. f is a closed map and therefore X is also connected. Similarly, for any $y \in Y$, $\overline{X}_y$ is also connected.

Fix $y \in Y$. We have a commutative diagram:

$$\begin{tikzcd} a \in X \arrow[d, "f"'] & \overline{X}_y \ni \overline{a} \arrow[l] \arrow[d, "\overline{f}"] \\ y \in Y & \overline{k(y)} \arrow[l] \end{tikzcd}$$

Let Ω be an algebraically closed field $\supset k(y)$ and let $\overline{a} \in \overline{X}_y$ be a geometric point over Ω; let $a \in X$ be the image of $\overline{a}$ in X. We have the fundamental groups $\pi_1(\overline{X}_y, \overline{a})$, $\pi_1(X, a)$, $\pi_1(Y, y)$ and continuous homo- **112**
morphisms: $\pi_1(\overline{X}_y, \overline{a}) \xrightarrow{\varphi} \pi_1(X, a)$, $\pi_1(X, a) \xrightarrow{\psi} \pi_1(Y, y)$ (See 5.1). We now have the following:

Theorem 6.3.2.1. *The sequence*

$$\pi_1(\overline{X}_y, \overline{a}) \xrightarrow{\varphi} \pi_1(X, a) \xrightarrow{\psi} \pi_1(Y, y) \to 0$$

is exact.

Proof. **(a) ψ is surjective.**

In view of 5.2.1 it is enough to show that, if Y'/Y is connected étale covering, then the étale covering $X' = X \underset{Y}{\times} Y'$ over X is connected.

$$\begin{tikzcd} X & X \underset{Y}{\times} Y' = X' \arrow[l] \arrow[d, "f' \text{ (proper)}"] \\ Y \arrow[from=u, "f"'] & Y' \text{ (connected)} \arrow[l] \end{tikzcd}$$

But $\mathscr{O}_Y = f_*(\mathscr{O}_X)$ and therefore, we have (from 6.1.2)

$$f'_*(\mathscr{O}_{X'}) = f'_*(\mathscr{O}_X \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y'}) = f_*(\mathscr{O}_{\mathscr{X}}) \underset{\mathscr{O}_Y}{\otimes} \mathscr{O}_{Y'} = \mathscr{O}_{Y'}.$$

It follows that the fibres of f' are connected, and hence $X' = X \underset{Y}{\times} Y'$ is also connected.

(b) $\psi \circ \varphi$ is trivial.

In view of 5.2.3 it is enough to show that if Y'/Y is any étale cov-
113 ering, the étale covering $(Y' \underset{Y}{\times} \overline{X}_y)/\overline{X}_y$ is completely decomposed. But $Y' \underset{Y}{\times} \overline{k(y)} = \coprod_{\text{finite}} \overline{k(y)}$ and hence: $Y' \underset{Y}{\times} \overline{X}_y \cong X \underset{=}{\times} (Y' \underset{\gamma}{\times} \overline{k(y)}) \coprod_{\text{finite}} \overline{X}_y$.

(c) Im $\varphi \supset$ ker ψ.

In view of 5.2.4 it is enough to prove that: Suppose $X' \xrightarrow{g} X$ is a connected étale covering of X and $\overline{X}'_y \xrightarrow{\overline{g}} \overline{X}_y$ admits a section σ (over $\overline{X}_y$). Then $\exists$ a connected étale covering Y'/Y such that $X' \xrightarrow{\sim} X \underset{Y}{\times} Y'$. We need, for proving this, the following

□

Lemma 6.3.2.2. *The composite $h : X' \xrightarrow{g} X \xrightarrow{f} Y$ is proper and separable.*

Proof. h is obviously proper and flat. We have only to show that the fibres of h are reduced, and remain reduced after any base-change $Y \leftarrow \operatorname{Spec} K$, K a field. For this, it is enough to prove that:

$$X'/X \text{ étale covering, } X \text{ reduced} \Rightarrow X' \text{ reduced.}$$

We may assume $X = \operatorname{Spec} A$, $X' = \operatorname{Spec} A'$; we have $A = A_{\text{red}}$ and we want to show that, $A' = A'_{\text{red}}$.

Let $(\mathscr{P}_i)_{i=1}^n$ be the minimal prime ideals of A. By assumption, the natural map $A \to \prod_{i=1}^n (A/\mathscr{P}_i)$ is an injection. So, $A' \to \prod_{i=1}^n (A'/\mathscr{P}_i A')$ **114**
is also injective and it is enough then to show that each $A'/\mathscr{P}_i A'$ is reduced. By making the base-change $A \to A/\mathscr{P}_i$, we may then assume that A is an integralldomain. Let $a \in X = \operatorname{Spec} A$ be the generic point of X; then $k(a) = K$ the field of fractions of A. The fibre over a is $= \operatorname{Spec}(A' \underset{A}{\otimes} K)$ and since this is non-ramified over $k(a) = K$, we have $A' \underset{A}{\otimes} K = \sum_{i=1}^r K_i$, $K_{i/K}$ finite separable field extensions. In particular, $A' \underset{A}{\otimes} K$ is reduced and hence for each $x' \in X'$, $\mathscr{O}_{x'} \underset{\mathscr{O}_{g(x')}}{\otimes} K$ is reduced and hence so is $\mathscr{O}_{x'} \subset \mathscr{O}_{x'} \underset{g(x')}{\otimes} K$. It follows that $A' = A'_{\text{red}}$.

Coming back to the proof of the assertion, let now $X' \xrightarrow{h'} Y' \to Y$ be the Stein-factorisation of $h : X' \to Y$. From the above lemma 6.3.2.2 and Theorem 6.2.1 it follows that $Y' \to Y$ is an étale covering. We have a commutative diagram

$$\begin{tikzcd}
& X' \arrow[dl] \arrow[d, "\alpha"] \arrow[dd, bend left, "h'"] \\
X \arrow[d, "f"'] & X'' = X \times_Y Y' \arrow[l, "p_1"'] \arrow[d, "f'"'] \\
Y & Y' = \operatorname{Spec} h_*(\mathscr{O}_{X'}) \arrow[l]
\end{tikzcd}$$

115 Our assertion will follow if we show that $\alpha : X' \to X'' = X \underset{Y}{\times} Y'$ is an isomorphism. We do this by showing that:

(i) α is an étale covering.

(ii) X'' is connected.

(iii) rank of α is 1 at *some* point of X''.

(see the remark at the end of Ch. 3).

(i) $Y' \to Y$ is an étale covering and so $X'' \to X$ is an étale covering. The composite $X' \to X'' \to X$ is the étale covering g and so α is an étale covering.

(ii) We know that h' is onto (6.3.1.1) and X' is connected (hypothesis). Hence Y' is connected and (ii) follows now from (a).

(iii) We make the base-change $Y \leftarrow \overline{k(y)}$ and obtain:

$$\begin{tikzcd}
 & & \overline{X'}_y \arrow[dll, "\overline{g}"'] \arrow[dl, "\overline{\alpha}"] \arrow[ddl, "\overline{h}'"] \\
\overline{X}_y \arrow[urr, bend left, "\sigma"] \arrow[d, "\overline{f}"] & \overline{X}''_y \arrow[l, "\overline{p}_1"] \arrow[d] & \\
\overline{k(y)} & \overline{Y}'_y \arrow[l] &
\end{tikzcd}$$

116 It is enough to show that rank $\overline{\alpha} = 1$ at *some* point of $\overline{X}''_y$. Since Y'/Y is an étale covering, $\overline{Y}'_y = \coprod_{i=1}^{n} \overline{k}_i$ each $\overline{k}_i = \overline{k(y)}$ and so $\overline{X}''_y = \overline{X}_y \underset{\overline{k(y)}}{\times} \overline{Y}'_y = \coprod_{i=1}^{n} \overline{X}_{y_i}$ each $\overline{X}_{y_i} = \overline{X}_y$. The section $\sigma : \overline{X}_y \to \overline{X}'_y$ is an étale covering and $\overline{X}_y$ is connected and hence $\sigma(\overline{X}_y)$ is a component Z of $\overline{X}'_y$; $\overline{\alpha}(Z)$ must then be some $\overline{X}_{y_i}$. Also the projection $\overline{p}_1$ is an isomorphism from $\overline{X}_{y_i}$

to $\overline{X}_y$; it follows that $Z \underset{(\sigma\circ\overline{p}_1)}{\overset{\overline{\alpha}}{\rightleftarrows}} \overline{X}_{y_i}$ are inverses of one another. Also we know that the number of components of $\overline{X}'_y$ is equal to the number of geometric points of Y' over y, i.e., is n. It follows that the number of connected components of $\overline{X}'_y$ and $\overline{X}''_y$ are the some. α is surjective because it is both open and closed and X'' is connected, therefore $\overline{\alpha}$ is surjective and étale and we get rank $\overline{\alpha} = 1$ at every pt. of $\overline{X}''_y$.Q.E.D. □

Remark 6.3.2.3. One may drop the assumptions $f_*(\mathscr{O}_X) = \mathscr{O}_Y$ and Y connected, in the above theorem; the assertion of the theorem will then be:

Denote by $\pi_0(\overline{X}_y, \overline{a})$, $\pi_0(X, a)$, $\pi_0(Y, y)$ the (pointed) sets of (connected) components of the preschemes $\overline{X}_y$, X, Y. Then, if f is proper **117**
and separable, we have the following exact sequence:

$$\pi_1(\overline{X}_y, \overline{a}) \to \pi_1(X, a) \to \pi_1(Y, y) \to \pi_0(\overline{X}_y, \overline{a}) \to \pi_0(X, a) \to \pi_0(Y, y) \to (1)$$

Proof. Assume to start with that X, Y are connected, dropping only the assumption $f_*(\mathscr{O}_X) \overset{\sim}{\leftarrow} \mathscr{O}_Y$. We have then the Stein-factorisation:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & Y \\ & Y' = \operatorname{Spec} f_*(\mathscr{O}_X). \arrow[ur, "q"'] & \end{tikzcd}$$

Applying the above theorem to f' we obtain an exact sequence: $\pi_1(\overline{X}_y, \overline{a}) \to \pi_1(X, a) \to \pi_1(Y', y') \to (e)$ where $y' \in Y'$ is the image of $a \in X$. We know then that $\pi_1(Y', y') \to \pi_1(Y, y)$ is an injection (5.2.6) and the quotient $\pi_1(Y, y)/\pi_1(Y', y')$ (set of left cosets mod $\pi_1(Y', y')$) is isomorphic to the set of geometric points of Y' over y. By corollary 6.3.1.3 this is isomorphic to $\pi_0(\overline{X}_y, \overline{a})$. We thus obtain the exact sequence:

$$\pi(\overline{X}_y, \overline{a}) \to \pi_1(X, a) \to \pi_1(Y, y) \to \pi_0(\overline{X}_y, \overline{a}) \to (1).$$

Now the assumptions about the connectivity of X and Y are dropped in turn to get the general exact sequence.

The procedure is obvious and the proof is omitted. □

Chapter 7

The Technique of Descents and Applications

7.1

Before stating the problem with which we shall be concerned in this section, in its most general form, we shall look at it in three particular cases of interest. **118**

Example 1. Let A, A' be rings and $\varphi : A \to A'$ be a ring-homomorphism. Let $\mathscr{C}_A$ (resp. $\mathscr{C}_{A'}$) be the category of A-(resp. A'-) modules. φ defines a covariant functor $\varphi^* : \mathscr{C}_A \to \mathscr{C}_{A'}$, viz., $\varphi^*(M) = M \underset{A}{\otimes} A' = M'$. Suppose $M, N \in \mathscr{C}_A$ and $u : M \to N$ is an A-linear map. Then $\varphi^*(u) = u \underset{A}{\otimes} 1_{A'} : M' \to N'$ is an A'-linear map.

Problem 1. Suppose $u' : M' = \varphi^*(M) \to \varphi^*(N) = N'$ is an A'-linear map. When can we say that $u' = \varphi^*(u)$ for an A-linear map $u : M \to N$?

Problem 2. Suppose $M' \in \mathscr{C}_{A'}$. When can we say that $M' = M \underset{A}{\otimes} A'$ for an $M \in \mathscr{C}_A$?

Example 2. Let S, S' be preschemes and φ be a morphism $\varphi : S' \to S$. Let $\mathscr{C}_S$ (resp. $\mathscr{C}_{S'}$) be the category of quasicoherent $\mathscr{O}_S$-(resp. $\mathscr{C}_{S'}$-) Modules. φ defines a (covariant) functor $\mathscr{C}_S \to \mathscr{C}_{S'}$ given by $\mathscr{F} \mapsto$

$\varphi^*(\mathscr{F})$. We again have:

Problem 1. If $u' : \mathscr{F}' = \varphi^*(\mathscr{F}) \to \mathfrak{g}' = \varphi^*(\mathfrak{g})$ is an $\mathscr{O}_{S'}$-morphism when can we say that $u' = \varphi^*(u)$ for an $\mathscr{O}_S$-morphism $u : \mathscr{F} \to \mathfrak{g}$?

119 **Problem 2.** If $\mathscr{F}'$ is any quasi-coherent $\mathscr{O}_{S'}$-Module, when can we say that $\mathscr{F}' = \varphi^*(\mathscr{F})$ for a quasi-coherent $\mathscr{O}_S$-Module?

Example 3. Let S, S' be preschemes and let $\mathscr{C}_S = (\mathrm{Sch}/S)$, $\mathscr{C}_{S'} = (\mathrm{Sch}/S')$. Suppose $\varphi : S' \to S$ is a morphism; φ defines a (covariant) functor $\varphi^* : \mathscr{C}_S \to \mathscr{C}_{S'}$ given by

$$X \mapsto \varphi^*(X) = X' = X \underset{S}{\times} S'.$$

We may again pose the two problems as in the previous examples.

It is clear that the two problems posed are of the same nature in the different examples and admit a generalisation in the following manner:

Let $\mathscr{C}$ be a category and suppose that for every $S \in \mathscr{C}$, we are given a category $\mathscr{C}_S$ such that for every morphism $\varphi : S' \to S$ in $\mathscr{C}$, we are given a covariant functor $\varphi^* : \mathscr{C}_S \to \mathscr{C}_{S'}$. Assume, in addition that

(1) If $S'' \underset{\psi}{\to} S' \underset{\varphi}{\to} S$ is a sequence in $\mathscr{C}$, there is a natural isomorphism: $\psi^* \circ \varphi^* \simeq (\varphi \circ \psi)^*$.

(2) If $S''' \underset{\varphi_3}{\longrightarrow} S'' \underset{\varphi_2}{\longrightarrow} S' \underset{\varphi_1}{\longrightarrow} S$ is a sequence in $\mathscr{C}$, the diagram, obtained from (1),

$$\begin{tikzcd}
& (\varphi_1\varphi_2\varphi_3)^* & \\
\varphi_3^* \circ (\varphi_1\varphi_2)^* \arrow[ur, "\sim"] & & (\varphi_2\varphi_3)^* \circ \varphi_1^* \arrow[ul, "\sim"'] \\
& \varphi_3^* \circ \varphi_2^* \circ \varphi_1^* \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

120 is *commutative.*

(3) $(\mathrm{Id})^* = \mathrm{Id}$.

(4) The isomorphism $\psi^* \circ \varphi^* \xrightarrow{\sim} (\varphi \circ \psi)^*$ of (1) has the property that if either ψ or φ is the identity, the isomorphism also becomes the identity.

Problem 1. Given $\xi, \eta \in \mathscr{C}_S$ and a morphism

$$u' : \xi' = \varphi^*(\xi) \to \varphi^*(\eta) = \eta' \quad \text{in} \quad \mathscr{C}_{S'}$$

when can we say that $u = \varphi^*(u)$ for a morphism $u : \xi \to \eta$ in $\mathscr{C}_S$?

Problem 2. Given $\xi' \in \mathscr{C}_{S'}$, when is ξ' of the form $\varphi^*(\xi)$ for a $\xi \in \mathscr{C}_S$?

We shall now obtain certain conditions necessary for the above two questions to have an answer in the affirmative.

For Problem 1. Suppose $\exists\, u : \xi \to \eta$ in $\mathscr{C}_S$ such that $u' = \varphi^*(u) : \xi' \to \eta'$. Assume that $S' \underset{S}{\times} S' = S''$ exists in $\mathscr{C}$; consider the sequence:

$$\begin{tikzcd} S & S' \arrow[l, "\varphi"'] & S'' \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_2"] \end{tikzcd}$$

Let $\varphi \circ p_1 = \varphi \circ p_2 = \psi$ and denote by ξ'', η'' the elements $\psi^*(\xi)$, $\psi^*(\eta)$ of $\mathscr{C}_{S''}$. Consider the morphism $p_1^*(u) : p_1^*(\xi') \to p_2^*(\xi')$; according to our assumption $p_i^* \circ \varphi^* \xrightarrow{\sim} \psi^*$. There is then a (dotted) morphism 121
making the diagram

$$\begin{tikzcd} p_1^*(\xi') \arrow[r, "p_1^*(u')"] \arrow[d, "\wr"] & p_1^*(\eta') \arrow[d, "\wr"] \\ \xi'' = \psi^*(\xi) \arrow[r, dashed] & \psi^*(\eta) = \eta'' \end{tikzcd}$$

commutative. In the following we also denote this morphism by $p_1^*(u')$, i.e., we identify $p_1^*(\xi')$ with ξ'' and $p_1^*(\eta')$ with η''. We introduce similarly $p_2^*(u')$. It is clear that with these notations we must have

$$\underline{p_1^*(u') = p_2^*(u')};$$

in fact α: both equal $\psi^*(u)$.

If this necessary condition is also sufficient (as they turn out to be in some cases) we say that $\varphi : S' \to S$ is a *morphism of descent.*

For Problem (2). Suppose $\exists\ \xi \in \mathscr{C}_S$, with $\varphi^*(\xi) = \xi'$. Then, with the above notations, we have:

$$p_1^*(\xi') \xrightarrow{\sim} \psi^*(\xi) \xleftarrow{\sim} p_2^*(\xi'),$$

122 i.e., an isomorphism $\alpha : p_1^*(\xi') \to p_2^*(\xi')$ making the diagram

$$\begin{tikzcd}
p_1^*(\xi') \arrow[rr, "\alpha"] \arrow[dr, "\sim"'] & & p_2^*(\xi') \arrow[dl, "\sim"] \\
& \psi^*(\xi) &
\end{tikzcd}$$

commutative. We shall now get conditions on α. Assume that $S''' = S' \underset{S}{\times} S' \underset{S}{\times} S'$ *exists in* $\mathscr{C}$; let q_i be the i^{th} projection $S''' \to S'$; also let $p_{ji} (j \geq i)$ be the morphism $(q_i, q_j)_S : S''' \to S''$. Consider the sequence:

$$\begin{tikzcd}
S & S' \arrow[l, "\varphi"'] & S'' \arrow[l, shift left, "p_1"'] \arrow[l, shift right, "p_2"] & S'''. \arrow[l, shift left=2, "p_{21}"'] \arrow[l, "p_{31}"] \arrow[l, shift right=2, "p_{32}"]
\end{tikzcd}$$

Let $\lambda = \varphi \circ p_1 \circ p_{21} = \varphi \circ p_1 \circ p_{31} = \cdots : S''' \to S$. We then have commutative diagrams of the type:

$$\begin{tikzcd}
& q_1^*(\xi') \arrow[dl, "\sim"'] & p_{21}^* p_1^*(\xi') \arrow[l, "\sim"'] \arrow[dd, "p_{21}^*(\alpha)"] \arrow[dr, "\sim"] & \\
\lambda^*(\xi) & & & p_{21}^* \psi^*(\xi) \arrow[lll, bend left=10, "\sim"] \\
& q_2^*(\xi) \arrow[ul, "\sim"] & p_{21}^* p_2^*(\xi') \arrow[l, "\sim"'] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

123 We also denote by $p_{21}^*(\alpha)$ the (dotted) morphism $q_1^*(\xi') \to q_2^*(\xi')$

which will make the diagram

$$\begin{tikzcd}
q_1^*(\xi') \arrow[d, dashed] & p_{21}^* p_1^*(\xi') \arrow[l, "\sim"'] \arrow[d, "p_{21}^*(\alpha)"] \\
q_2^*(\xi') & p_{21}^* p_2^*(\xi') \arrow[l, "\sim"']
\end{tikzcd}$$

commutative. Using this diagram and the "large" diagram above we see that

$$\begin{tikzcd}
& q_1^*(\xi') \arrow[dl, "\sim"'] \arrow[dd, "p_{21}^*(\alpha)"] \\
\lambda^*(\xi) & \\
& q_2^*(\xi) \arrow[ul, "\sim"]
\end{tikzcd}$$

is commutative. We make similar conventions and get similar commutative diagrams for $p_{32}^*(\alpha)$ and $p_{31}^*(\alpha)$. Then we must have

$$\underline{p_{32}^*(\alpha) p_{21}^*(\alpha) = p_{31}^*(\alpha)}$$

(the so-called "cocycle" condition). Finally, if $\Delta : S' \to S'' = S' \underset{S}{\times} S'$ is the diagonal, by the assumptions made at the beginning we must have **124**

$$\underline{\Delta^*(\alpha) = \text{identity}.}$$

If these necessary conditions are also sufficient and if φ is also a morphism of descent, we say that φ is a *morphism of effective descent.* An α of the above type is then called a *descent-datum* on ξ'.

Remark. Let $\mathscr{C}$ be the category of preschemes; if we agree that we take S and S' locally noetherian but if we *don't* assume $S' \to S$ of finite type then it may very well happen that $S'' = S' \underset{S}{\times} S'$ is *not* locally noetherian (e.g. $S = \operatorname{Spec} A$, $S' = \operatorname{Spec} \widehat{A}$, A a noetherian local ring). We shall deal with this difficulty in the sections 7.2.1.3, 7.2.1.4, 7.2.1.5.

Example (1).

Proposition 7.1.1. *A faithfully flat ring-homomorphism $\varphi : A \to A'$ is a morphism of effective descent.*

Proof. A (a) We shall first show that φ is a morphism of descent. Let M, N be A-modules and $u' : M\underset{A}{\otimes}A' \to N\underset{A}{\otimes}A'$ be an A'-linear map. Consider the commutative diagram

$$\begin{tikzcd}
M \arrow[r] & M\underset{A}{\otimes}A' = M' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "u'"] & M\underset{A}{\otimes}A'\underset{A}{\otimes}A' = M'' \arrow[d, shift right, "p_1^*(u')"'] \arrow[d, shift left, "p_2^*(u')"] \\
N \arrow[r] & N\underset{A}{\otimes}A' = N' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & N\underset{A}{\otimes}A'\underset{A}{\otimes}A' = N''
\end{tikzcd}$$

125 We have to show that if $p_1^*(u') = p_2^*(u')$ then $\exists$ an A-linear map $u : M \to N$ such that $u' = u\underset{A}{\otimes}I_{A'}$. This will follow, if we show that

(i) for any $M \in \mathscr{C}_A$, $M \to M\underset{A}{\otimes}A'$ is an injection,

(ii) for any $N \in \mathscr{C}_A$, $N \longrightarrow N' \underset{p_2}{\overset{p_1}{\rightrightarrows}} N''$ is exact, and

(iii) $u'(M) \subset \ker(N', p_1, p_2)$.

(i) We know that A' is a direct factor of $A'\underset{A}{\otimes}A'$ by means of the map $a' \mapsto a' \otimes 1$; thus, for any $M \in \mathscr{C}_A$, $M\underset{A}{\otimes}A' \to M\underset{A}{\otimes}A'\underset{A}{\otimes}A'$ makes $M\underset{A}{\otimes}A'$ a direct summand of $M\underset{A}{\otimes}A'\underset{A}{\otimes}A'$ and is in particular an injection. As A' is faithfully A-flat it follows that $M \to M\underset{A}{\otimes}A'$ is an injection.

(ii) We have to prove that the sequence

$$N \longrightarrow N' \underset{p_2}{\overset{p_1}{\rightrightarrows}} N''$$

is exact.

It suffices to prove this after tensoring the sequence with a ring B faithfully flat over A; note that we get a similar situation with the pair $B \to B' = B \underset{A}{\otimes} A'$ as with $A \to A'$. Take for B the ring A' itself. But then $A' \to A' \underset{A}{\otimes} A' = A''$ admits a section $\lambda : A'' \to A'$ given by $\lambda(a_1' \otimes a_2') = a_1' a_2'$. We may thus assume without loss of generality that $\varphi : A \to A'$ itself admits a section $\lambda : A' \to A$. **126**
Consider then the commutative diagram:

$$\begin{tikzcd}
N \arrow[r, "\psi=1\otimes\varphi"] & N\underset{A}{\otimes}A' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\mu=1\otimes\lambda"] & N\underset{A}{\otimes}A'\underset{A}{\otimes}A' \arrow[d, "\mu\otimes 1"] \\
& N \otimes A \xrightarrow{\sim} N \arrow[ul, <->, "\mathrm{Id}."] \arrow[r] & N\underset{A}{\otimes}A\underset{A}{\otimes}A' \xrightarrow{\sim} N\underset{A}{\otimes}A'
\end{tikzcd}$$

Let $x' = \sum(x_i \otimes a_i')$ be an element of $N' = N\underset{A}{\otimes}A'$ such that $p_1(x') = p_2(x')$; that is, $\sum(x_i \otimes a_i' \otimes 1) = \sum(x_i \otimes 1 \otimes a_i')$. On applying $\mu \otimes 1$ we obtain:

$$\sum(x_1 \otimes \lambda(a_i') \otimes 1) = \sum(x_i \otimes 1 \otimes a_i');$$

under the identification $N\underset{A}{\otimes}A\underset{A}{\otimes}A' \xrightarrow{\sim} N\underset{A}{\otimes}A'$, this means that

$$\sum(x_i \otimes a_i') = \sum(\lambda(a_i')x_i \otimes 1), \quad \text{i.e.,} \quad x' = \psi(\mu(x')) \in \psi(N).$$

(iii) By the commutativity of the diagram at the beginning of the proof, we have: $p_1^*(u') \circ p_1 = p_1 \circ u'$, $p_2^*(u') \circ p_2 = p_2 \circ u'$ and by assumption $p_1^*(u') = p_2^*(u')$ while for $m \in M$ one has $p_1(m) = p_2(m) = m \otimes 1 \otimes 1 \in M''$. Thus, for $m \in M$, $p_1 \circ u'(m) = p_2 \circ u'(m)$ and (iii) is proved.

(b) It remains to show that φ is effective. Suppose $N \in \mathscr{C}_{A'}$ and we have **127** a commutative diagram:

$$\begin{tikzcd}
 & N'\underset{A}{\otimes}A' = p_1^*(N') \arrow[dd, "\wr\alpha"] \\
N' \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & A'\underset{A}{\otimes}N' = p_2^*(N')
\end{tikzcd}$$

where α is an A'-isomorphism satisfying the "cocycle" condition. We want to find an $N \in \mathscr{C}_A$ such that $N\underset{A}{\otimes}A' \underset{\rho}{\xrightarrow{\sim}} N'$ and such that

$$\begin{tikzcd}
N\underset{A}{\otimes}A'\underset{A}{\otimes}A' \arrow[r, "p_1^*(\rho)"] \arrow[d, "\wr"', "\text{(natural)}"] & N'\underset{A}{\otimes}A' \arrow[d, "\wr\alpha"] \\
A'\underset{A}{\otimes}N\underset{A}{\otimes}A' \arrow[r, "p_2^*(\rho)"] & A'\underset{A}{\otimes}N'
\end{tikzcd}$$

commutes.

Set $N = \ker(\alpha \circ p_1 - p_2) \in \mathscr{C}_A$ (this choice of N is motivated by (ii) of (a)). We always have an A'-*linear* map $N\underset{A}{\otimes}A' \xrightarrow{\rho} N'$. To show that ρ is an A'-isomorphism, it is enough to show that ρ is an A-isomorphism; and for this, we may assume, as in (a), that there is a section $\lambda : A' \to A$ for $\varphi : A \to A'$.

128 For the sake of clarity and ease, we now go back to the general case and prove: □

Lemma 7.1.2. *If $\varphi : S' \to S$ admits a section $\sigma : S \to S'$, then φ is a morphism of effective descent.*

Proof. Let $\xi' \in \mathscr{C}_{S'}$ and $\alpha : p_1^*(\xi') \to p_2^*(\xi')$ be an isomorphism such that $p_{32}^*(\alpha)p_{21}^*(\alpha) = p_{31}^*(\alpha)$ (notations as before). Our aim is to find an

$\eta \in \mathscr{C}_S$ and an isomorphism $\eta' = \varphi^*(\eta) \xrightarrow{\rho} \xi'$ such that the diagram

$$\begin{tikzcd}
& p_1^*(\eta') \arrow[r, "\sim", "p_1^*(\rho)"'] \arrow[dl, "\sim"'] & p_1^*(\xi') \arrow[dd, "\wr\,\alpha"] \\
\eta'' & & \\
& p_2^*(\eta') \arrow[r, "\sim", "p_2^*(\rho)"'] \arrow[ul, "\sim"] & p_2^*(\xi')
\end{tikzcd}$$

is commutative.

We have the diagram:

$$\begin{tikzcd}
S & S' \arrow[l, "\varphi"'] & S' \underset{S}{\times} S' \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_2"] \\
& S \arrow[ul, "\mathrm{Id.}"] \arrow[u, "\sigma"'] & S \underset{S}{\times} S' \xrightarrow{\sim} S' \arrow[l, "\mathscr{P}"'] \arrow[u, "\sigma \underset{S}{\times} 1"']
\end{tikzcd}$$

with $p_1 \circ (\sigma \times 1) = \sigma \circ \varphi$ and $p_2 \circ (\sigma \times 1) =$ identity. Hence, if $\eta = \sigma^*(\xi')$ **129**
we have $\eta' = \varphi^* \sigma^*(\xi') = (p_1 \circ (\sigma \times 1))^*(\xi') = (\sigma \times 1)^* p_1^*(\xi')$ and $(*****)^*$ $(\sigma \times 1)^* p_2^*(\xi') = \xi'$. We then get a $\theta : \eta' \xrightarrow{\sim} \xi'$, namely, $\theta = (\sigma \times 1)^*(\alpha)$. We obtain thus a $\beta : p_1^*(\eta) \to p_2^*(\eta')$ which makes the diagram

$$\begin{tikzcd}
p_1^*(\xi') \arrow[r, "\sim", "\alpha"'] & p_2^*(\xi') \\
p_1^*(\eta') \arrow[u, "\wr", "p_1^*(\theta)"'] \arrow[r, "\sim", "\beta"'] & p_2^*(\eta') \arrow[u, "p_2^*(\theta)", "\wr"']
\end{tikzcd}$$

commutative. However, we do *not*, in general, have a commutative diagram

$$\begin{tikzcd}
p_1^*(\eta') \arrow[rr, "\sim", "\beta"'] & & p_2^*(\eta') \\
& \eta'' = (p_1 \circ \varphi)^*(\eta) \arrow[ul, "\sim", "\text{(natural)}"'] \arrow[ur, "\sim"', "\text{(natural)}"] &
\end{tikzcd}$$

130 Therfore, we want to modify $\theta : \eta' \to \xi'$ to a $\rho : \eta' \xrightarrow{\sim} \xi'$ by means of an automorphism $\gamma : \eta' \to \eta'$ (i.e., $\rho = \theta \circ \gamma$) such that the diagram

$$\begin{tikzcd}
p_1^*(\eta') \arrow[rr, "\sim", "\beta"'] & & p_2^*(\eta') \\
p_1^*(\eta') \arrow[u, "p_1^*(\gamma)"] & & p_2^*(\eta') \arrow[u, "p_2^*(\gamma)"'] \\
& \eta'' \arrow[ul, "\sim"] \arrow[ur, "\sim"'] \arrow[u, phantom, "\text{(natural)}"] &
\end{tikzcd}$$

is commutative. This ρ will satisfy our requirements. We first observe that $\beta : p_1^*(\eta') \to p_2^*(\eta')$ also satisfies the cocycle condition. We may consider β as an element of $\mathrm{Aut}(\eta'')$. To find a γ in $\mathrm{Aut}\,\eta'$ such as above, i.e., such that $\beta = p_2^*(\gamma) \circ p_1^*(\gamma)^{-1}$ we have only to prove the following: $\square$

Lemma 7.1.3. *Let $\eta \in \mathscr{C}_S$ and consider the "complex"*

$$(\mathrm{Aut}) \equiv \mathrm{Aut}\,\eta' \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \mathrm{Aut}\,\eta'' \overset{p_{31}^*,\ p_{21}^*}{\underset{p_{32}^*}{\rightrightarrows\!\!\to}} \mathrm{Aut}\,\eta'''$$

(notations as before). If there is a section $\sigma : S \to S'$ then

$$\underline{H^1(\mathrm{Aut}) = (e)}.$$

131 *Proof.* By definition, the 1-cocycles are elements $\beta \in \operatorname{Aut}\eta''$ for which the cocycle condition is satisfied and the 1-coboundaries are those $\beta \in \operatorname{Aut}\eta''$ which are of the form $\beta = p_2^*(\gamma) \circ p_1^*(\gamma)^{-1}$, for a $\gamma \in \operatorname{Aut}\eta'$. Consider now the corresponding complex in $\mathscr{C}$; the section $\sigma : S \to S'$ defines a homotopy operator for this complex in the following manner:

$$
\begin{tikzcd}
S & S' \arrow[l, "\varphi"'] & S'' \arrow[l, shift left, "p_1"'] \arrow[l, shift right, "p_2"] & S''' :::::: \arrow[l, shift left=2, "p_{21}"'] \arrow[l, "p_{31}"] \arrow[l, shift right=2, "p_{32}"] \\
& S \arrow[ul, "\mathrm{Id.}"] \arrow[u, "\sigma"'] & S \underset{S}{\times} S' \xrightarrow{\sim} S' \arrow[l, "\varphi"'] \arrow[u, "\sigma\times 1"'] & S \underset{S}{\times} S' \underset{S}{\times} S' \xrightarrow{\sim} S'' \arrow[l, shift left, "p_1"'] \arrow[l, shift right, "p_2"] \arrow[u, "\sigma\times 1\times 1"']
\end{tikzcd}
$$

with $p_1 \circ (\sigma \times 1) = \sigma \circ \varphi$, $p_2 \circ (\sigma \times 1) = \mathrm{Id.}$,

$$(\sigma \times 1) \circ p_1 = p_{21} \circ (\sigma \times 1 \times 1), \quad (\sigma \times 1) \circ p_2 = p_{31} \circ (\sigma \times 1 \times 1),$$

and

$$p_{32} \circ (\sigma \times 1 \times 1) = \mathrm{Id}\,.$$

If β is a 1-cocycle, set $\gamma = (\sigma \times 1)^*(\beta)$. We then have:

$$
\begin{aligned}
p_2^*(\gamma) \circ p_1^*(\gamma)^{-1} &= p_2^*(\sigma \times 1)^*(\beta) \circ (p_1^*(\sigma \times 1)^*(\beta))^{-1} \\
&= (((\sigma \times 1)_{p_2})^*(\beta)) \circ (((\sigma \times 1)_{p_1})^*(\beta))^{-1} \\
&= (\sigma \times 1 \times 1)^* p_{31}^*(\beta) \circ ((\sigma \times 1 \times 1)^* p_{21}^*(\beta))^{-1} \\
&= (\sigma \times 1 \times 1)^* (p_{31}^*(\beta) \cdot p_{21}^*(\beta)^{-1}) \\
&= (\sigma \times 1 \times 1)^* p_{32}^*(\beta) = \beta.
\end{aligned}
$$

Q.E.D. □

Example 2. $\mathscr{C} = (\mathrm{Sch})$, $S, S' \in (\mathrm{Sch})$; $\mathscr{C}_S$ (resp. $\mathscr{C}_{S'}$) is the category of 132
quasi-coherent $\mathscr{O}_S$-(resp. $\mathscr{O}_{S'}$-)-Modules.

Proposition 7.1.4. *If $\varphi : S' \to S$ is faithfully flat, quasi compact then φ is a morphism of effective descent.*

(A morphism f is quasi-compact if $f^{-1}(U)$ is quasi-compact for every quasi-compact: U).

(*Note:* We do *not* assume here that φ is a morphism of finite type-our hypothesis is much weaker).

Proof. **Case (a)** *S, S' both affine.* The proposition in this case follows from Proposition 7.1.1.

Case (b) *S affine.* There is a finite affine open cover $(S'_i)_{i\in I}$ of S' (φ quasi-compact). Set $\widetilde{S} = \coprod_{i\in I} S'_i$. Then $\widetilde{S}$ is affine and the morphism $\widetilde{S} \xrightarrow{\psi} S$ (composite of the natural map $\theta : \widetilde{S} \to S'$ and φ) is also faithfully flat. Now consider the diagram:

$$
\begin{tikzcd}
 & \widetilde{S} \arrow[dl, "\psi"'] \arrow[dd, "\theta"', "\text{(natural)}"] & \widetilde{S}\times_S\widetilde{S} \arrow[l, shift left, "q_1"'] \arrow[l, shift right, "q_2"] \arrow[dd] & \widetilde{S}\times_S\widetilde{S}\times_S\widetilde{S} \arrow[l, shift left=2, "q_{31}"'] \arrow[l, "q_{32}"] \arrow[l, shift right=2, "q_{21}"] \\
S & & & \\
 & S' \arrow[ul, "\varphi"] & S'\times_S S' \arrow[l, shift left, "p_1"'] \arrow[l, shift right, "p_2"] & S'\times_S S'\times_S S' \arrow[l, shift left=2, "p_{31}"'] \arrow[l, "p_{32}"] \arrow[l, shift right=2, "p_{21}"]
\end{tikzcd}
$$

133 If $\xi, \eta \in \mathscr{C}_S$ and $u' : \xi \to \eta'$ is a $\mathscr{C}_{S'}$-morphism then u' defines a $\mathscr{C}_{\widetilde{S}}$-morphism $\widetilde{u} : \psi^*(\xi) \to \psi^*(\eta)$. And from the equality $p_1^*(u') = p_2^*(u')$ and the commutativity of the above diagram follows $q_1^*(\widetilde{u}) = q_2^*(\widetilde{u})$. By case (a), $\exists$ a $u : \xi \to \eta$, a $\mathscr{C}_S$-morphism, such that $\widetilde{u} = \psi^*(u)$; it is immediate that $u' = \varphi^*(u)$; in fact this holds in every open set S'_i.

If $\xi' \in \mathscr{C}_{S'}$ and α is a descent-datum for ξ', $\widetilde{\alpha}$ defined in the obvious way is a descent-datum for $\theta^*(\xi') = \widetilde{\xi}$ and again by case (a), $\exists \xi \in \mathscr{C}_S$ such that $\widetilde{\xi} \xleftarrow{\sim} \psi^*(\xi)$. It is easy to see that $\xi' \xleftarrow{\sim} \varphi^*(\xi)$.

Case (c). *S, S' arbitrary.* Let (T_j) be an affine open cover of S and $T'_j = \varphi^{-1}(T_j)$. Then the T'_j form an open cover of S'. Let $T = \coprod_j T_j$ and $\theta : T \to S$ the natural map. If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{O}_S$-Modules denote by $\Phi(S)$ the group $\mathrm{Hom}_S(\mathscr{F}, \mathscr{G})$, by $\Phi(S')$ the group

$\mathrm{Hom}_{S'}(\mathscr{F}', \mathscr{G}')$ and so on. If we set $T' = \coprod_j T'_j$ and $\theta : T' \to S'$ is the natural map, we have a commutative diagram:

$$\begin{tikzcd}
\Phi(S) \arrow[r, "\varphi^*"] \arrow[d] & \Phi(S') \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] \arrow[d] & \Phi(S'') \\
\Phi(T) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \Phi(T') \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] \arrow[d, shift left] \arrow[d, shift right] & \Phi(T'') \\
\Phi(T \times_S T) \arrow[r] & \Phi(T' \times_{S'} T') &
\end{tikzcd}$$

By case (b), the morphisms $T'_j \to T_j$ are morphisms of effective descent and therefore clearly so is $T' = \coprod_j T'_j \to \coprod_j T_j = T$. It follows that the second row is exact while $\Phi(T \times_S T) \to \Phi(T' \times_{S'} T')$ is an injection. As it is clear that $\theta : T \to S$ is a morphism of (effective) descent, the first column is also exact. Usual diagram-chasing shows that the first row is also exact, in other words, that φ is a morphism of descent. We show similarly that φ is also effective. Q.E.D. $\square$ **134**

Example 3. We shall not discuss the problems (1) and (2) in their general form, in this case. However, if we restrict ourselves to the case of *preschemes affine over S, S'* then *a faithfully flat, quasi-compact morphism $S' \to S$ is a morphism of effective descent.* In fact, such preschemes are defined by quasi-coherent $\mathscr{O}_S$-and $\mathscr{O}_{S'}$-Algebras and we are essentially back to example (2). [It is not difficult to see that u (resp. $\mathscr{F}$) in example (2), problem (1) (resp. problem (2)) is an $\mathscr{O}_S$-Algebra homomorphism (resp. an $\mathscr{O}_S$-Algebra) provided we start with $\mathscr{O}_S$-Algebras and homomorphisms of $\mathscr{O}_S$-Algebras instead of $\mathscr{O}_S$-Modules (look at the proof of Proposition 7.1.1)].

7.2

Let S be a locally noetherian prescheme and X, Y be étale coverings of S. Let $S_0 \hookrightarrow S$ be a closed subscheme of S defined by a Nil-Ideal $\mathscr{F}$

of $\mathcal{O}_S$ ($\mathcal{F}$ is a Nil-Ideal of $\mathcal{O}_S \Leftrightarrow \mathcal{F}_s \subset$ nil-radical of $\mathcal{O}_{sS}$, $\forall s \in S$;
135 this is equivalent to saying that S_0 and S have the same base space or $(S_0)_{\mathrm{red}} = S_{\mathrm{red}}$). We then have an obvious functor $(\mathcal{E}t/S) \xrightarrow{\Phi} (\mathcal{E}t/S_0)$ given by $X \mapsto X \underset{S}{\times} S_0 = X_0$. We assert that Φ defines an equivalence of categories in the sense of the following

7.2.1 Main Theorem

(a) $\mathrm{Hom}_S(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{S_0}(X_0, Y_0)$

(b) If $X_0 \in (\mathcal{E}t/S_0)$, then $\exists\, X \in (\mathcal{E}t/S)$

and an isomorphism $X \underset{S}{\times} S_0 \xrightarrow{\sim} X_0$.

The theorem will follow from the series of lemmas below.

Lemma 7.2.1.1. *Let S be locally noetherian and $f : X \to S$ a separated morphism of finite type. Then f is an open immersion $\Leftrightarrow$ f is étale and universally injective.*

(*Note:* Universally injective - radiciel = injective + radiciel residue field extensions).

Proof. $\Rightarrow$ clear.

$\Leftarrow$: f is an open map and thus by passing to an open sub-scheme of S, we may assume that $f(X) = S$. Clearly f is then a homeomorphism onto. Since étale remains étale under base-change, $f_{(S')}$ is still a homeomorphism onto, for any base-change $S' \to S$; it follows that f is universally closed and thus proper. By Chevalley's lemma we deduce that f is finite. Suppose then that $X = \mathrm{Spec}\,\mathcal{A}$, where $\mathcal{A}$ is a coherent
136 $\mathcal{O}_S$-Algebra. As f is étale and universally injective, it follows that $\mathcal{A}$ is locally free of rank 1 at every point of S, hence $X \xrightarrow{\sim} S$. Q.E.D. □

Lemma 7.2.1.2. *Let $f : X \to S$ be a separated étale morphism of finite type (S locally noetherian). Suppoe $Y \to S$ is a morphism of finite type and $Y_0 = V(\mathcal{F})$ is a closed subscheme of Y defined by a Nil-Ideal $\mathcal{F}$ of*

$\mathscr{O}_Y$. *Let* $\alpha_0 : Y_0 \to X$ *be an* S*-morphism. Then* $\exists$ *a* unique S*-morphism* $\alpha : Y \to X$ *making the diagram*

$$\begin{tikzcd} X \arrow[d, "f"'] & Y_0 \arrow[l, "\alpha_0"'] \arrow[d, hook] \\ S & Y \arrow[l] \arrow[ul, "\alpha"'] \end{tikzcd}$$

commutative.

Proof. By making the base-change $Y \to S$ and considering the morphism $Y_0 \to X \underset{S}{\times} Y$ we are reduced to proving the lemma in the case $S = Y$. That is, given a Y-morphism $Y_0 \xrightarrow{\alpha_0} X$, we want to extend α_0 to a section α of $X \to Y$.

Now suppose $\sigma : Y \to X$ is a section of $X \xrightarrow{f} Y$; $f \cdot \sigma =$ identity is étale and f is étale, hence σ is étale. Also $f \cdot \sigma$ is a closed immersion and f is separated and thus σ is a closed immersion. Y being locally noetherian its connected components are open and we may then assume Y connected. $\sigma(Y)$ will then be a connected component of X, isomorphic to Y under σ; in view of lemma 7.2.1.1 then, the sections of f are in $(1-1)$ correspondence with the components X_i of X such that $f|X_i$ *is* **137**
surjective and universally injective on X_i to Y.

Now Y_0, Y have the same base-space and hence so have $X_0 = X \underset{Y}{\times} Y_0$ and X; also, the morphism $f_{(Y_0)} : X_0 \to Y_0$ obtained from f by the base-change $Y_0 \to Y$ is topologically the same map as f; hence, f is universally injective and surjective on X_i to $Y \Leftrightarrow f_{(Y_0)}$ is so on $(X_i)_0$ to Y_0, i.e., the set of sections of $X \xrightarrow{f} Y$ is the same as the set of sections of $X_0 \xrightarrow{f_{(Y_0)}} Y_0$ and the $(1-1)$ correspondence is given in the obvious way. The lemma now follows from the fact that $Y_0 \xrightarrow{\alpha_0} X$ can be considered as a section for $X_0 \xrightarrow{f_{(Y_0)}} Y_0$. Q.E.D. □

This lemma proves part (a) of Theorem 7.2.1. To prove part (b) we need a generalisation of the lemma.

Warning. In the following sections 7.2.1.3–7.2.1.5 until the proof of (b) of Theorem 7.2.1, we drop the assumptions made in 6.1 that the preschemes are locally noetherian and the morphisms are of finite type. We begin by making the

Definition 7.2.1.3. A morphism $f : X \to Y$ of preschemes is of *finite presentation* if

(i) f is quasi-compact

(ii) $\Delta : X \to X \underset{Y}{\times} X$ is quasi-compact,

(iii) for every $x \in X$, $\exists$ a nbd. U_x of $x \in X$ and a nbd. $V_{f(x)}$ of $f(x)$ in
138 Y such that $f(U_x) \subset V_{f(x)}$ and $\Gamma(U_x, \mathscr{O}_X)$ is a $\Gamma(V_{f(x)}, \mathscr{O}_Y)$-algebra of finite presentation; i.e., $\Gamma(U_x, \mathscr{O}_X) \cong \Gamma(V_{f(x)}, \mathscr{O}_Y)[T_1, \dots, T_s]/\mathfrak{a}$ where $\mathfrak{a}$ is a finitely generated ideal of a polynomial algebra $\Gamma(V_{f(x)}, \mathscr{O}_Y)[T_1, \dots, T_s]$.

An example of a morphism of finite presentation is a finite type separated morphism $X \xrightarrow{f} Y$ with Y locally noetherian.

Lemma 7.2.1.4. *Let U be a* noetherian *prescheme and $(\mathscr{A}_\alpha)_{\alpha \in I}$ an inductive family of quasi-coherent $\mathscr{O}_U$-Algebras. Let V_α be the U-affine prescheme* $\operatorname{Spec} \mathscr{A}_\alpha$ *defined by $\mathscr{A}_\alpha$, $\forall_{\alpha \in I}$, and $V = \operatorname{Spec} \mathscr{A}$ where $\mathscr{A} = \varinjlim_\alpha \mathscr{A}_\alpha$.*

(a) *Suppose we have a diagram:*

$$\begin{tikzcd}
 & X_\alpha \arrow[dr] & & Y_\alpha \arrow[dl] & X_\beta \arrow[dr] & & X_\beta \arrow[dl] & X \arrow[dr] & & Y \arrow[dl] \\
U & & V_\alpha \arrow[ll] & & & V_\beta \arrow[lll] & & & V' \arrow[lll] &
\end{tikzcd}$$

where X_α, Y_α are finitely presented preschemes over $V_\alpha \forall\ \alpha \in I$ such that $\forall \beta \geq \alpha$, $X_\beta = X_\alpha \underset{V_\alpha}{\times} V_\beta$

$$Y_\beta = Y_\alpha \underset{V_\alpha}{\times} V_\beta; \quad \textit{let} \quad X = X_\alpha \underset{V_\alpha}{\times} V, \quad Y = Y_\alpha \underset{V_\alpha}{\times} V$$

Then, $\varprojlim_{\alpha} \mathrm{Hom}_{V_\alpha}(X_\alpha, Y_\alpha) \xrightarrow{\sim} \mathrm{Hom}_V(X, Y)$.

(b) *Suppose is a finitely presented prescheme over V. Then, for all large $\alpha \in I$, $\exists\ X_\alpha/V_\alpha$, finitely presented, such that (i) $\forall \beta \geq \alpha$, $X_\beta \simeq X_\alpha \underset{V_\alpha}{\times} V_\beta$ and (ii) $X \simeq X_\alpha \underset{V_\alpha}{\times} V$.* **139**

(c) *With assumptions as in (a), if the Y'_αs and Y are locally noetherian, X/Y étale covering $\Rightarrow$ X_α/Y_α étale covering already, for some $\alpha \in I$.*

Proof. (a) We shall be content with merely observing that it is clear how to prove this in case everything is affine.

(b) Since U is noetherian, in view of (a) we may assume that $U = \mathrm{Spec}\, C'$, $V_\alpha = \mathrm{Spec}\, A_\alpha$, $V = \mathrm{Spec}\, A$ where $A = \varprojlim_{\alpha} A_\alpha$.

Case (1). *Assume $X = \mathrm{Spec}\, B$.*

X/V finitely presented $\Rightarrow$ B/A finitely presented; let

$$B \cong A[T_1, \ldots, T_s]/\mathfrak{a}, \mathfrak{a}$$

finitely generated, say, by $P_1, \ldots, P_n \in A[T_1, \ldots, T_s]$. Choose α so large that the coefficients of the P_i come from A_α. Consider the ideal $\mathfrak{a}_\alpha = (P_1, \ldots, P_n)$ of $A_\alpha[T_1, \ldots, T_s]$ and set $B_\alpha = A_\alpha[T_1, \ldots, T_s]/\mathfrak{a}_\alpha$ and $X_\alpha = \mathrm{Spec}\, B_\alpha$; for $\beta \geq \alpha$, set $X_\beta = X_\alpha \underset{V_\alpha}{\times} V_\beta$.

Case (2). *X arbitrary.*

Let $(X_i)_{i=1}^r$ be a finite affine open cover of X. In view of case (1) and (a), it is enough now to show that each $X_i \cap X_j$ is quasi-compact. But the underlying space of $X_i \cap X_j$ is that of $X \underset{(X \underset{V}{\times} X)}{\times} (X_i \underset{V}{\times} X_j)$ as is seen from the commutative diagram: **140**

$$\begin{tikzcd}
X \underset{(X \underset{V}{\times} X)}{\times} (X_i \underset{V}{\times} X_j) \arrow[r] \arrow[d] & X_i \underset{V}{\times} X_j \arrow[d, "\text{(natural)}"] \\
X \arrow[r, "\Delta"] & X \underset{V}{\times} X
\end{tikzcd}$$

As Δ is a quasi-compact morphism the "top"-morphism is also quasi-compact; since $X_i \underset{V}{\times} X_j$ is quasi-compact, our result follows.

(c) Here again we give only some indications and leave the details to the reader. To start with we may assume everything affine. Say $U = \operatorname{Spec} C$, $V_\alpha = \operatorname{Spec} A_\alpha$, $V = \operatorname{Spec} A$, $X_\alpha = \operatorname{Spec} B_1^\alpha$, $Y_\alpha = \operatorname{Spec} B_2^\alpha$, $X = \operatorname{Spec} B_1$, $Y = \operatorname{Spec} B_2$. We also note that we can assume $V_\alpha = Y_\alpha$, and $V = Y$. One checks that $\Omega_{B_{1/B_2}} \xleftarrow{\sim} \varprojlim_\alpha \Omega_{B_1^\alpha/B_2^\alpha}$, say. Now if X/Y is unramified then $\Omega_{B_1/B_2} = 0$. But since $\Omega_{B_1^{\alpha_0}/B_2^{\alpha_0}}$ is a finite $B_1^{\alpha_0}$-module one has $\Omega_{B_1^\alpha/B_2^\alpha} = 0$ for large α. Also, if X/Y is finite and flat then B_1 is a locally free B_2-module of finite rank but then the same is true for B_1^α with respect to B_2^α for large α. Assertion (c) follows. Q.E.D. $\square$

141 **Lemma 7.2.1.5.** *Suppose we have a commutative diagram:*

$$\begin{tikzcd}
X & X_{(T)} = X \underset{S}{\times} T \arrow[l] \arrow[d] & Y_0 = V(\mathscr{F}) \arrow[l, "\alpha_0"'] \arrow[d, hook] \\
S \arrow[from=u, "f"] & T \arrow[l] & Y \arrow[l]
\end{tikzcd}$$

with: S noetherian, f étale and separated, $T \to S$ affine, $Y \to T$ finitely presented and $Y_0 = V(\widetilde{\mathscr{F}})$ where $\mathscr{F}$ is a Nil-Ideal of $\mathscr{O}_Y$. Then $\exists$ a unique *$\alpha : Y \to X_{(T)}$ keeping the diagram still commutative.*

Proof. Since $T \to S$ is affine, $T = \operatorname{Spec} \mathscr{B}$, where $\mathscr{B}$ is a quasi-coherent $\mathscr{O}_S$-Algebra. Write $\mathscr{B} = \varprojlim_\lambda \mathscr{B}_\lambda$ where the $\mathscr{B}_\lambda'$s are $\mathscr{O}_S$-sub-Algebras, of finite type, of $\mathscr{B}$. Set $T_\lambda = \operatorname{Spec} \mathscr{B}_\lambda$. By (a) and (b) of Lemma 7.2.1.4

we have, for large λ, a situation of the following type:

$$\begin{tikzcd}
 & X\underset{S}{\times}T_\lambda \arrow[dl] \arrow[dd] & & Y_{0,\lambda} \arrow[ll, "\alpha_{0,\lambda}"'] \arrow[dd] & \\
S\,X \arrow[dd] & & X\underset{S}{\times}T \arrow[ll] \arrow[dd] & & Y_0 = V(\mathscr{F}) = Y_{0,\lambda}\underset{Y_\lambda}{\times}Y \arrow[ul] \arrow[dd, hook] \\
 & \operatorname{Spec}\mathscr{B}_\lambda = T_\lambda \arrow[dl] & & Y_\lambda \arrow[ll] \arrow[uull, dashed, "\alpha_\lambda ?"] & \\
S\,S & & \operatorname{Spec}\mathscr{B} = T \arrow[ll] \arrow[ul] & & Y = Y_\lambda\underset{T_\lambda}{\times}T \arrow[ll] \arrow[ul]
\end{tikzcd}$$

For large λ, $\exists$ inductive families (Y_λ), $(Y_{0,\lambda})$ and morphisms $\alpha_{0,\pi}$: **142**
$Y_{0,\lambda} \to X\underset{S}{\times}T_\lambda$ such that $\varinjlim_\lambda \alpha_{0,\lambda} = \alpha_0$ (see diagram). Also it is not very difficult to see (using arguments similar to those in Lemma 7.2.1.4) that the $Y_{0,\lambda}$ are subschemes defined by Nil-Ideals $\mathscr{F}_{0,\lambda}$ of $\mathscr{O}_{Y_\lambda}$. Now T_λ is of finite type over S and is therefore noetherian; also, since $Y \to T$ is of finite presentation so is $Y_\lambda \to T_\lambda$ and is in particular of finite type. Hence Lemma 7.2.1.2 applies and we get a (dooted) morphism $\alpha_\lambda : Y_\lambda \to X\underset{S}{\times}T_\lambda$ keeping the diagram commutative. Passing to the limit with respect to λ we get an $\alpha : T \to X\underset{S}{\times}T$ making the diagram commutative. The uniqueness of α follows from Lemma 7.2.1.2 and (a) of Lemma 7.2.1.4. Q.E.D. □

7.2.1.6 Proof of (b) of Theorem 7.2.1

Case 1. Assume $S = \operatorname{Spec} A$, A a noetherian, complete local ring; S_0 is then given by $\operatorname{Spec}(A/J) = \operatorname{Spec} A_0$, J a nil-ideal of A. Let $X_0 \xrightarrow{f_0} S_0$ be the given étale covering; assume that if $s_0 \in S_0$ is the unique closed point, the residual extensions at the points of $f_0^{-1}(s_0)$ are all trivial.

Then X_0 is given by $\operatorname{Spec} B_0$ where B_0 is a finite direct-product of copies of A_0, say $B_0 = \bigoplus_{i=1}^r A_0$; $X = \operatorname{Spec} B$, with $B = \bigoplus_{i=1}^r A$ is then a solution for our problem.

Case 2. From among the assumptions in case 1, drop completeness of A **143**

and the triviality of residual extensions along the fibre $f_0^{-1}(s_0)$.

$$
\begin{tikzcd}
S = \operatorname{Spec} A & & \operatorname{Spec} A' = S' \arrow[ll, "\text{faithfully flat}"', "\text{quasi-compact}"] \\
& X_0 \arrow[d, "f_0"] & X'_0 = X_0 \underset{S_0}{\times} S'_0 \arrow[l] \arrow[d, "f'_0"] \\
& \operatorname{Spec} A_0 = S_0 \arrow[uul, hook] & S'_0 = \operatorname{Spec}\left(\frac{A'}{JA'}\right) \arrow[l]
\end{tikzcd}
$$

In this case we can choose a complete, noetherian, local overring A' of A such that the following situation holds: *and such that* if $s'_0 \in S'_0$ is the unique closed point, then the residual extensions along the fibre $f'^{-1}_0(s'_0)$ *are all trivial.* By case 1, $\exists$ an étale covering X'/S' such that $X'_0 \xleftarrow{\sim} X' \underset{S'}{\times} S'_0$.

144 We have then the following situation:

$$
\begin{tikzcd}
 & X' \arrow[d] & p_1^*(X') \arrow[dr] & & p_2^*(X') \arrow[dl] & \\
S & S' \arrow[l] & & S'' = S' \underset{S}{\times} S' \arrow[ll, shift right, "p_2"'] \arrow[ll, shift left, "p_1"] & & S''' = S' \underset{S}{\times} S' \underset{S}{\times} S' \arrow[ll, shift left=2, "p_1"] \arrow[ll, "p_{31}"'] \arrow[ll, shift right=3, "p_2"'] \\
 & X_0 \arrow[d, "f_0"] & X_0' \arrow[l] \arrow[d, "f_0'"] & p_1^*(X_0') \arrow[dr] & & p_2^*(X_0') \arrow[dl] & \\
 & S_0 \arrow[uul, hook] & S_0' \arrow[l] & & S_0'' = S_0' \underset{S_0}{\times} S_0' \arrow[ll, shift right, "p_2"'] \arrow[ll, shift left, "p_1"] & & S_0''' = S_0''' \underset{S_0}{\times} S_0' \underset{S_0}{\times} S_0'' \arrow[ll, shift left=2, "p_{21}"] \arrow[ll, "p_{31}"'] \arrow[ll, shift right=3, "p_{32}"']
\end{tikzcd}
$$

Since X'_0/S'_0 "comes from below", it clearly has a descent-datum $\alpha_0 : p_1^*(X'_0) \xrightarrow{\sim} p_2^*(X'_0)$ satisfying the cocycle condition. We want to "lift" this isomorphism α_0 to an $\alpha : p_1^*(X') \to p_2^*(X')$. For this one may be tempted to use part (a) of Theorem 7.2.1 which we already have proved. But observe that we are now in a type of situation we anticipated while making the remark preceding Proposition 7.1.1 – we *cannot* make sure that S'', S''_0 are locally noetherian and hence *cannot* apply part (a). *It is here that Lemma 7.2.1.5 comes to our rescue.* By using this lemma
145 in the obvious way we get an isomorphism $\alpha : p_1^*(X') \xrightarrow{\sim} p_2^*(X')$; the assertion of uniqueness in this lemma proves that the α we obtained satisfies the cocycle condition. As $S' \to S$ is faithfully flat and quasi-compact, it is a morphism of effective descent for étale coverings (an easy corollary to Proposition 7.1.4) and thus $\exists$ an étale covering X/S such that $X' \xleftarrow{\sim} X \underset{S}{\times} S'$. Therefore

$$X'_0 \xleftarrow{\sim} X' \underset{S'}{\times} S'_0 \simeq X \underset{S}{\times} S' \underset{S'}{\times} S'_0 \simeq (X \underset{S}{\times} S_0) \underset{S_0}{\times} S'_0.$$

Also, by construction, the descent-datum for X'/S' goes down to that for X'_0/S'_0. It follows that $X \underset{S}{\times} S_0 \xrightarrow{\sim} X_0$.

Case 3. S *arbitrary.*

By part (a), which we have already proved, it is enough to prove the existence of an étale covering X/S (with the required property) *locally.*

Let $s \in S$ and $U = \operatorname{Spec} A$ be an affine open neighbourhood of s. We may assume $S = \operatorname{Spec} A$, A noetherian. The local ring A_s is given by $A_s = \varinjlim_{\substack{f \in A \\ f(s) \neq 0}} A_f$; we then have $S \leftarrow \operatorname{Spec} A_f \leftarrow \operatorname{Spec} A_s$ and the given étale covering X_0 over $S_0 = \operatorname{Spec}(A/J)$, J a nil-ideal, defines an étale covering $X_{s,0}$ over $\operatorname{Spec}(A_s/JA_s) = \operatorname{Spec}(A/J)_s$ and then, by case 2, an étale covering X_s over $\operatorname{Spec} A_s$. By (b) and (c) of lemma 7.2.1.4, $\exists f \in A$,
146 $f(s) \neq 0$ and an étale covering $X_f/\operatorname{Spec} A_f$ such that $X_s \xleftarrow{\sim} X_f \underset{A_f}{\otimes} A_s$. The covering X_f is a solution for our problem in the neighbourhood $\operatorname{Spec} A_f$ of s. Q.E.D.

Remark. The Theorem 7.2.1 shows that if $S_0 \hookrightarrow S$ is such that $(S_0)_{\text{red}} = S_{\text{red}}$ then the natural functor $(\mathscr{E}t/S) \xrightarrow{\Phi} (\mathscr{E}t/S_0)$ is an equivalence; in particular, it proves that $\pi_1(S_{\text{red}}, s) \xrightarrow{\sim} \pi_1(S, s)$.

Proposition 7.2.2. *Let $S' \xrightarrow{f} S$ be a faithfully flat, quasi-compact, radiciel morphism. Then the natural functor $(\mathscr{E}t/S) \xrightarrow{\Phi} (\mathscr{E}t/S')$ is an equivalence.*

Proof. Consider the diagonal $S' \xrightarrow{\Delta} S'' = S' \underset{S}{\times} S'$.

Since f is radiciel $\Delta(S') = S''$ (Cor. (3.5.10) – EGA, Ch.I), i.e. S' is a closed subscheme of S'', having the same base-space.

(a) Given $X, Y \in (\mathscr{E}t/S)$, set $X' = X \underset{S}{\times} S'$, $Y' = Y \underset{S}{\times} S'$.

To prove $\mathrm{Hom}_S(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{S'}(X', Y')$.

Since $f : S' \to S$ is faithfully flat, quasi-compact it is a morphism of descent for étale coverings. We have thus only to show that if $u' \in \mathrm{Hom}_{S'}(X', Y')$, and if $S' \overset{p_1}{\underset{p_2}{\leftleftarrows}} S'' = S' \underset{S}{\times} S'$ are the canonical projections, then, considered as morphisms from $X'' = X' \underset{S'}{\times} S''$ to $Y'' = Y' \underset{S'}{\times} S''$, $p_1^*(u)$ and $p_2^*(u)$ are equal. We have the diagram: 147

$$\begin{tikzcd}
S & & S' \arrow[ll] & & & S'' = S' \underset{S}{\times} S' \arrow[lll, shift left] \arrow[lll, shift right] & & & S' \arrow[lll, "\Delta"'] & \\
& X' \arrow[ur] \arrow[rr, "u'"] & & Y' \arrow[ul] & X'' \arrow[ur] \arrow[rr, shift left, "p_1^*(u')"] \arrow[rr, shift right, "p_2^*(u')"'] & & Y'' \arrow[ul] & X' \arrow[ur] \arrow[rr, "u'"] & & Y' \arrow[ul]
\end{tikzcd}$$

In view of our remarks about Δ, and Theorem 7.2.1, it is enough to show that $\Delta^*(p_1^*(u')) = \Delta^*(p_2^*(u'))$. But each of $\Delta^*(p_1^*(u'))$, $\Delta^*(p_2^*(u'))$ is equal to u' considered as morphisms form X' to Y'.

(b) Given $X' \in (\mathscr{E}t/S')$, to show that $\exists\ X \in (\mathscr{E}t/S)$ such that $X' \xleftarrow{\sim} X \underset{S}{\times} S'$.

For this, again it is enough to find a descent-datum $\alpha : p_1^*(X') \xrightarrow{\sim} p_2^*(X')$. We have the diagram:

$$\begin{tikzcd}
S & S' \arrow[l] & & S'' \arrow[ll, shift left, "p_2"] \arrow[ll, shift right, "p_1"'] & & S''' \arrow[ll, shift right=2, "p_{31}"'] \arrow[ll, "p_{32}" description] \arrow[ll, shift left=2, "p_{21}"] & \\
& & p_1^*(X') \arrow[ur] & & p_2^*(X') \arrow[ul] & & S' \arrow[ulll, "\substack{\Delta_1 \\ \text{diagonal}}"] \arrow[ul, "\substack{\Delta_2 \\ \text{diagonal}}"'] \\
X' \arrow[uu] & & & & & & X' = \Delta_1^* p_1^*(X') = \Delta_1^* p_2^*(X') \arrow[u]
\end{tikzcd}$$

148 in view of our remarks about $\Delta_1 : S' \to S''$ and Theorem 7.2.1, the identity morphism $i : X' = \Delta_1^* p_1^*(X') \to \Delta_1^* p_2^*(X') = X'$, lifts to an isomorphism $\alpha : p_1^*(X') \to p_2^*(X')$. The other diagonal morphism $\Delta_2 : S' \to S''' = S' \underset{S}{\times} S' \underset{S}{\times} S'$ also imbeds S' as a closed subscheme of S''' having the same base-space. Then one checks easily that α satisfies the cocycle condition again by using Theorem 7.2.1. Q.E.D. $\square$

Proposition 7.2.2 simply says that if $S' \to S$ is faithfully flat, quasi-compact and radiciel, $s' \in S'$ and $s \in S$ its image, then $\pi_1(S', s') \xrightarrow{\sim} \pi_1(S, s)$.

7.3

Let k be an algebraically closed field and X, Y be connected k-preschemes. Suppose X is k-proper and Y locally noetherian. Let $a \in X$, $b \in Y$ be geometric points with values in an algebraically closed field extension K of k. Consider a geometric point $c = (a, b) \in X \underset{k}{\times} Y$ over a and b. We claim first that $X \underset{k}{\times} Y$ is connected. Since Y is connected and

$X \underset{k}{\times} Y \to Y$ is proper, it is enough to show that the fibres of $X \underset{k}{\times} Y \to Y$ are connected and for this one has only to show that for any field $k' \supset k$

(*) $X \underset{k}{\times} k'$ is connected.

The question is purely topological and we may assume $X = X_{\text{red}}$. Looking at the Stein-factorisation $X \to \operatorname{Spec}\Gamma(X, \mathscr{O}_X) \to k$ it follows **149**
(by making use of 6.3.1.1 and the finiteness theorem) that $\Gamma(X, \mathscr{O}_X) = k$, since X is connected and k algebraically closed. On the other hand, if $X' = X \underset{k}{\times} k'$, one has $\Gamma(X', \mathscr{O}_{X'}) = \Gamma(X, \mathscr{O}_X) \underset{k}{\otimes} k'$ (flat base-change) hence $= k'$; again looking at the Stein-factorisation we see that X' is connected 6.3.1.1.

We can then form $\pi_1(X \underset{k}{\times} Y, c)$ and form the product of the natural maps $\pi_1(X \underset{k}{\times} Y, c) \to \pi_1(X, a)$, and $\pi_1(X \underset{k}{\times} Y, c) \to \pi_1(Y, b)$. We then have the

Proposition 7.3.1. *With the assumptions made above* $\pi_1(X \underset{k}{\times} Y, c) \xrightarrow{\sim} \pi_1(X, a) \times \pi_1(Y, b)$.

Proof. We may assume $X = X_{\text{red}}$.

Case (1). *Assume $K = k$.*

The morphism $X \underset{k}{\times} Y \to Y$ is proper and separable and the fibre over $b \in Y$ is $\simeq$ to $X \underset{k}{\times} k = X$. We then get the exact sequence:

$$\pi_1(X, a) \to \pi_1(X \underset{k}{\times} Y, c) \to \pi_1(Y, b) \to e \quad \text{(Theorem 6.3.2.1)}.$$

But the fibre over b, namely, $X \underset{k}{\times} k = X$ is imbedded in $X \underset{k}{\times} Y$ and the composite $X = X \underset{k}{\times} k \hookrightarrow X \underset{k}{\times} Y \xrightarrow{p_1} X$ is identity. This means that $\exists$ continuous sections for the homomorphism $\pi_1(X, a) \to \pi_1(X \underset{k}{\times} Y, c)$. Thus, we get:

(i) $e \to \pi_1(X, a) \to \pi_1(X \underset{k}{\times} Y, c) \to \pi_1(Y, b) \to e$

is exact,

(ii) and the sequence splits (also *topologically*). 150

Case (2). *K arbitrary.*

The reasoning as in case (1) gives an isomorphism

$$\pi_1(X \underset{k}{\times} Y \underset{k}{\times} K, c') \xrightarrow{\sim} \pi_1(X \underset{k}{\times} K, a') \times \pi_1(Y \underset{k}{\times} K, b')$$

where c', a', b' are points of $(X \underset{k}{\times} Y) \underset{k}{\times} K$, $X \underset{k}{\times} K$, $Y \underset{k}{\times} K$ respectively, above c, a, b. The theorem then is a consequence of the following: $\square$

Proposition 7.3.2. *Let k be an algebraically closed field and $X \to k$ a proper connected k-scheme. Let k' be an algebraically closed field extension of k and let $a' \in X \underset{k}{\times} k'$ be any geometric point. If $a \in X$ be the image of a', then*

$$\pi_1(X \underset{k}{\times} k, a') \xrightarrow{\sim} \pi_1(X, a).$$

Proof. That $X \underset{k}{\times} k'$ is connected follows from assertion (*) of 7.3; the same assertion also proves that if Z is a connected étale covering of X then $Z' = Z \underset{k}{\times} k'$ is a connected étale covering of $X' = X \underset{k}{\times} k'$. In other words, $\pi_1(X \underset{k}{\times} k', a') \to \pi_1(X, a)$ is *surjective* (cf. 5.2.1). We shall prove that it is injective by showing that every connected étale covering Z' of $X' = X \underset{k}{\times} k'$ is of the form $Z \underset{k}{\times} k'$ for some $Z \in (\mathscr{E}t/X)$.

By Lemma 7.2.1.4 $\exists$ a k-algebra A of *finite type*, $A \subset k'$, and $Z_A \in$
151 $(\mathscr{E}t/(X \underset{k}{\times} A))$ such that $Z' \xleftarrow{\sim} Z_A \underset{A}{\times} k$. If $Y = \operatorname{Spec} A$, $\exists\, y \in Y$ such that $k(y) = k$ [since A is of finite type over (the algebraically closed) k]. One can apply case (1) of Theorem 7.3.1 to the k-rational point $(a, y) \in X \underset{k}{\times} Y$ to obtain

$$\pi_1(X \underset{k}{\times} Y, (a, y)) \xrightarrow{\sim} \pi_1(X, a) \times \pi_1(Y, y).$$

If Z_A is defined by an open subgroup H of $\pi_1 = \pi_1(X \underset{k}{\times} Y, (a, y))$ this means that $\exists$ open (normal) subgroups $G \subset \pi_1(X)$ and $G' \subset \pi_1(Y)$ (defining galois coverings $\widetilde{X}/X$, $\widetilde{Y}/Y$ respectively) such that $H \supset G \times G'$ (i.e. such that Z_A is obtained as a quotient of the galois covering $\widetilde{X} \underset{k}{\times} \widetilde{Y}$

of $X \times Y$). Let $\overline{Z}_A$ be the lift of the covering $Z_{A/(X\times_k Y)}$ to $X\times_k\widetilde{Y}$; we than have the commutative diagram:

$$\begin{tikzcd}
& \widetilde{X}\times_k\widetilde{Y} \arrow[dl] \arrow[d] \arrow[dr] & \\
X\times_k\widetilde{Y} \arrow[dr] & \overline{Z}_A \arrow[l] \arrow[r] & Z_A \arrow[dl] \\
& X\times_k Y &
\end{tikzcd}$$

We claim that $\overline{Z}_A$ is connected. 152

In fact, let $y \in Y = \operatorname{Spec} A$ be the generic point of Y; then $k(y) =$ field of fractions of $A \subset k$. For any $\widetilde{y} \in \widetilde{Y}$ lying above $y \in Y$ we thus have $k(\widetilde{y}) \subset k'$ since k' is algebraically closed. If we then apply the base-change $X\times_k\widetilde{Y}\times_{\widetilde{Y}}k' \to X\times_k\widetilde{Y}$ to the étale covering $\overline{Z}_A \to X\times_k\widetilde{Y}$, we obtain the étale covering

$$Z_A\times_Y k' = Z' \to X'$$

which is connected. It follows that $\overline{Z}_A$ is connected. Thus the morphism $\widetilde{X}\times_k\widetilde{Y} \to \overline{Z}_A$ is surjective and $\overline{Z}_A$ is sandwiched between $\widetilde{X}\times_k\widetilde{Y}$ and $X\times_k\widetilde{Y}$; this implies that $\overline{Z}_A/(X\times_k\widetilde{Y})$ is defined by an open subgroup of $\pi_1(X\times_k\widetilde{Y}) = \pi_1(X) \times G'$ which contains $\pi_1(\widetilde{X}\times_k\widetilde{Y}) = G \times G'$; i.e. $\overline{Z}_A = \widetilde{X}_1\times_k\widetilde{Y}$ for an $\widetilde{X}_1 \in (\mathscr{E}t/X)$. We now obtain

$$\begin{aligned}
(\overline{Z}_A)_{\widetilde{y}} &= \text{the fibre of } \overline{Z}_A \text{ over } \widetilde{y} \\
&= \widetilde{X}_1\times_k\widetilde{Y}\times_{\widetilde{Y}}k(\widetilde{y}) = \widetilde{X}_1\times_k k(\widetilde{y}) \\
\text{and} \quad Z' &= Z_A\times_Y k' = (Z_A)_y \times_{k(y)} k' \\
&= (\overline{Z}_A)_{\widetilde{y}} \times_{k(\widetilde{y})} k = \widetilde{X}_1\times_k k'
\end{aligned}$$

Q.E.D. □

Chapter 8

An Application of the Existence Theorem

8.1 The second homotopy exact sequence

Let A be a *complete noetherian local ring* and $S = \operatorname{Spec} A$; let $s_0 \in S$ be 153
the closed point of S. Let X be a *proper S-scheme* and set $\overline{X}_0 = X \underset{S}{\times} \overline{k(s_0)}$.

Let $\overline{a}_0 \in \overline{X}_0$ be a geometric point of $\overline{X}_0$ (over some fixed algebraically closed field $\Omega \supset k(s_0)$) and $a_0 \in X$ be its image in X. Assume that $\overline{X}_0$ is *connected*.

$$\begin{tikzcd}
a_0 \in X \arrow[d, "\text{proper}"', "f"] & \overline{X}_0 \ni \overline{a}_0 \arrow[l] \arrow[d] \\
s_0 \in \operatorname{Spec} A & \overline{k(s_0)} \arrow[l]
\end{tikzcd}$$

Theorem 8.1.1. *The sequence*

$$e \to \pi_1(\overline{X}_0, \overline{a}_0) \to \pi_1(X, a_0) \to \pi_1(S, s_0) \to e$$

is exact*; and we have the isomorphism*

$$\pi_1(S, s_0) \cong G(\overline{k(s_0)}/k(s_0)).$$

(Note: Compare with Theorem 6.3.2.1).

The proof of the theorem is a consequence of the following results. 154

Proposition 8.1.2. *In the above Theorem 8.1.1 assume A is an artinian local ring. With the same notation, the same assertions hold.*

Proof. Since the introduction of nilpotent elements does not affect the fundamental groups (Theorem 7.2.1). We may assume that $\underline{A = k(s_0)}$. In this case, the assertion $\pi_1(S, s_0) \cong G(\overline{k(s_0)}/k(s_0))$ is clear. Next, if characteristic $k(s_0) = p$ and if $k' = (k(s_0))^{p^{-\infty}}$, then the morphism $\operatorname{Spec} k(s_0) \leftarrow \operatorname{Spec} k'$ is faithfully flat, quasi-compact and radiciel and hence by Proposition 7.2.2, we may replace $\underline{k(s_0)}$ by $(k(s_0))^{p^{-\infty}}$, i.e., we may assume $k(s_0)$ is *perfect*. In this case, $\overline{k(s_0)}$ is the inductive limit $\varinjlim_{i\in I} k_i$ of finite galois extensions k_i of $k(s_0)$; set $X_i = X \underset{k(s_0)}{\times} k_i$ and $a_i =$ image of $\overline{a}_0$ in X_i.

$$\begin{tikzcd}
a_0 \in X \arrow[d] & X_i \ni a_i \arrow[l] \arrow[d] & \overline{X}_0 \ni \overline{a}_0 \arrow[l] \arrow[d] \\
k(s_0) & k_i \arrow[l] & \overline{k(s_0)} \arrow[l]
\end{tikzcd}$$

155 By Lemma 7.2.1.4 an étale covering of $\overline{X}_0$ is determined by an étale covering of some X_i and the latter is uniquely determined modulo passage to X_j, $j \geq i$.

One thus gets the isomorphism:

$$\pi_1(\overline{X}_0, \underline{\overline{a}}_0) \xrightarrow{\sim} \varinjlim_i \pi_1(X_i, a_i)$$

[The injectivity follows from the fact that for any open subgroup H of $\pi = \pi_1(\overline{X}_0, \overline{a}_0)$ there exists, by Lemma 7.2.1.4 an index i and an open subgroup $H^{(i)}$ of $\pi^{(i)} = \pi_1(X_i, a_i)$ such that $\pi/H \simeq \pi^{(i)}/H^{(i)}$. The surjectivity follows because otherwise there would exist a set $E \in \mathscr{C}(\pi^{(i)})$ with two points a and b in $\dot{E}$ such that a and b are in the same connected component of E with respect to the action of all $\pi^{(j)}(j \geq i)$ but a and b lie in different components with respect to the action of π; again by

7.2.1.4 this is impossible because a connected component of E in $\mathscr{C}(\pi)$ can be realised in some $\mathscr{C}(\pi^{(j)})$].

On the other hand, we assert that each X_i/X is galois and $G(k_i/k(s_0)) \xrightarrow{\sim} \mathrm{Aut}(X_i/X)$.

In fact, suppose we have a situation of the following type:

$$\begin{tikzcd} a_0 \in X \arrow[d] & X' = X \underset{k}{\times} k' \ni a' \arrow[l, "\varphi"'] \arrow[d] \\ k & k' \arrow[l] \end{tikzcd}$$

with X universally connected over k. Then X'/X is a connected étale **156**
covering. Also, we have: $\deg(X'/X) = \operatorname{rank}\varphi =$ number of geometric points in the fibre $\varphi^{-1}(a_0) = \deg(k'/k) =$ number of automorphisms of $k'/k \leq$ number of automorphisms of $X'/X \leq$ number of geometric points in the fibre $\varphi^{-1}(a_0)$ (because X is *connected* - see the proof of Lemma 4.4.1.6). Hence $\mathrm{Aut}(k'/k) \xrightarrow{\sim} \mathrm{Aut}(X'/X)$ and X'/X is galois. Now for every $i \in I$ we have an exact sequence:

$$(e) \to \pi_1(X_i, a_i) \to \pi_1(X, a_0) \to \mathrm{Aut}(X_i/X) \to (e)$$

(see 5.2.6). Since each π_1 is pro-finite, by passing to the projective limit we obtain an *exact* sequence:

$$(e) \to \pi_1(\overline{X}_0, \overline{a}_0) \to \pi_1(X, a_0) \to G(\overline{k(s_0)}/k(s_0)) = \pi_1(S, s_0) \to (e)$$

□

Proposition 8.1.3. *Let A be a complete, noetherian local ring and $S = \operatorname{Spec} A$. Let X be a* proper *S-scheme such that if $s_0 \in S$ is the closed point of S the fibre $X_0 = X \underset{S}{\times} k(s_0)$ is universally connected. Let a'_0 be a geometric point of X_0 with values in some algebraically closed $\Omega \supset k(s_0)$. If $a_0 \in X$ is the image of a'_0 then canonically $\pi_1(X, a_0) \xleftarrow{\sim} \pi_1(X_0, a'_0)$.*

Proof. This will come from the fact that the natural functor $(\mathscr{E}t/X) \xrightarrow{\Phi} (\mathscr{E}t/X_0)$ is an equivalence.

(a) If $Z, Z' \in (\mathscr{E}t/X)$ then

$$\mathrm{Hom}_X(Z, Z') \xrightarrow{\sim} \mathrm{Hom}_{X_0}(Z \underset{X}{\times} X_0, Z' \underset{X}{\times} X_0).$$

157 In fact, let $\mathscr{A}(Z)$, $\mathscr{A}(Z')$ be the coherent, locally free $\mathscr{O}_X$-Algebras defining Z, Z' over X. If $\mathscr{M}$ is the maximal ideal of A, set $A_n = A/\mathscr{M}^{n+1}$, $X_n = X \underset{A}{\times} A/\mathscr{M}^{n+1}$, $Z_n = Z \underset{X}{\times} X_n$ and so on, $n \in \mathbb{Z}^+$. We then have:

$$\mathrm{Hom}_X(Z, Z') \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_X-\mathrm{Alg.}}(\mathscr{A}(Z'), \mathscr{A}(Z)).$$

Now for the $\mathscr{O}_X$-Modules, we have:

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{A}(Z'), \mathscr{A}(Z)) = \Gamma(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}(Z'), \mathscr{A}(Z)))\\
&\xrightarrow{\sim} \varprojlim_n \Gamma(X_n, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}(Z'), \mathscr{A}(Z)) \underset{A}{\otimes} A/\mathscr{M}^{n+1})
\end{aligned}$$

by the Comparison theorem, (where $\mathscr{H}om_{\mathscr{M}_X}$ is the sheaf of germs of $\mathscr{O}_X$-homomorphisms); since the $\mathscr{A}(Z')$, $\mathscr{A}(Z)$ are coherent and locally free, we have:

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{A}(Z'), \mathscr{A}(Z)) \underset{A}{\otimes} A/\mathscr{M}^{n+1} = \mathscr{H}om_{\mathscr{O}_{X_n}}(\mathscr{A}(Z'_n), \mathscr{A}(Z_n));$$

therefore,

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{A}(Z'), \mathscr{A}(Z)) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{\mathscr{O}_{X_n}}(\mathscr{A}(Z'_n), \mathscr{A}(Z_n)).$$

However, this holds also for homomorphisms of $\mathscr{O}_X$-Algebras, because the condition for an $\mathscr{O}_X$-Module homomorphism to be an $\mathscr{O}_X$-Algebra homomorphism can be expressed by means of commutativity in diagrams of $\mathscr{O}_X$-Modules. Therefore, $\mathrm{Hom}_X(Z, Z') \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{X_n}(Z_n, Z'_n)$. But by Theorem 7.2.1, the natural map $\mathrm{Hom}_{X_n}(Z_n, Z'_n) \to \mathrm{Hom}_{X_0}(Z_0, Z'_0)$
158 is an isomorphism; therefore one obtains:

$$\varprojlim_n \mathrm{Hom}_{X_n}(Z_n, Z'_n) \xrightarrow{\sim} \mathrm{Hom}_{X_0}(Z_0, Z'_0).$$

Q.E.D. □

(b) If Z_0 is an étale covering over X_0, then there exists an étale covering Z/X such that $Z_0 \xleftarrow{\sim} Z \underset{X}{\times} X_0$.

For proving this we need another powerfull theorem from the EGA.

8.1.4 The Existence Theorem for proper morphisms Let A be a noetherian ring and I and ideal of A such that A is complete for the I-adic topology. Let $Y = \operatorname{Spec} A$ and $f : X \to Y$ be a *proper* morphism. Set $A_n = A/I^{n+1}$, $n \in \mathbb{Z}^+$, and $Y_n = \operatorname{Spec} A_n$, $X_n = X \underset{Y}{\times} Y_n$. Suppose, for every n, $\mathscr{F}_n$ is a coherent $\mathscr{O}_{X_n}$-Module such that $\mathscr{F}_{n-1} \simeq \mathscr{F}_n \underset{\mathscr{O}_{X_n}}{\otimes} \mathscr{O}_{X_{n-1}}$. Then $\exists$ a coherent $\mathscr{O}_X$-Module $\mathscr{F}$ such that, for each n, $\mathscr{F}_n \xleftarrow{\sim} \mathscr{F} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_{X_n}$.

(For a proof see EGA, Ch. III, (5.1.4).)

Coming back to the proof of (b), the étale covering $Z_0 \to X_0$ is defined by a coherent locally free $\mathscr{O}_{X_0}$-Algebra $\mathscr{B}_0$. By Theorem 7.2.1, for each n, we get a coherent locally free $\mathscr{O}_{X_n}$-Algebra such that $\mathscr{B}_{n-1} \xleftarrow{\sim} \mathscr{B}_n \underset{\mathscr{O}_{X_n}}{\otimes} \mathscr{O}_{X_{n-1}}$. By the existence theorem, $\exists$ a coherent $\mathscr{O}_X$-Algebra $\mathscr{B}$ such that $\mathscr{B}_n \xleftarrow{\sim} \mathscr{B} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_{X_n}$. Set $Z = \operatorname{Spec} \mathscr{B}$. We claim that Z/X is the 159
étale covering we are looking for. It is clear that $Z \underset{X}{\times} X_0 \xrightarrow{\sim} Z_0$. It remains to show that Z/X is étale.

We first observe that since A is a local ring and f is closed any (open) neighbourhood of the fibre $f^{-1}(s_0)$ is the whole of X. Also, if Z/X is étale over the points of the fibre $X_0 = f^{-1}(s_0)$, then Z is étale over X at points of an open neighbourhood of X_0; for, if $x_0 \in X_0$ then $\mathscr{B}_{x_0}$ is free over $\mathscr{O}_{x_0, X}$ and hence $\mathscr{B}$ is a free $\mathscr{O}_X$-Module in a neighbourhood of x_0; and similarly for non-ramification (use Ch. 3, Proposition 3.3.2). Therefore it is enough to prove that $Z \to X$ is étale at points of Z lying over the fibre $X_0 = f^{-1}(s_0)$.

Let then $x_0 \in X_0$; choose an affine open neighbourhood U of x_0 in X such that $\mathscr{B}_0$ is $\mathscr{O}_{X_0}$-free over $U_0 = U \cap X_0$. Set $\Gamma(U, \mathscr{O}_X) = C$ and $J = \mathscr{M}C$, the ideal generated by $\mathscr{M}$; then, for $n \in \mathbb{Z}^+$, $C_n = C/J^{n+1} \xrightarrow{\sim} \Gamma(U, \mathscr{O}_{X_n})$. Similarly, if $M = \Gamma(U, \mathscr{B})$ then

$$M_n = M/J^{n+1}M \xrightarrow{\sim} \Gamma(U, \mathscr{B}_n).$$

We know that $M_0 = \Gamma(U, \mathscr{B}_0) \xrightarrow{\sim} \Gamma(U_0, \mathscr{B}_0)$ is free as a $C_0 = \Gamma(U_0, \mathscr{O}_{X_0})$-module. Choose a basis for M_0 over C_0. Lift this basis to M and let L be the free C-module on these elements and $\varphi : L \to M$ be the natural C-linear map; then we have an exact sequence of C-modules:

$$0 \to \ker\varphi \to L \to M \to \operatorname{coker}\varphi \to 0.$$

160 For $n \in \mathbb{Z}^+$, set $L_n = L/J^{n+1}L$ and let φ_n be the C_n-linear map $L_n \xrightarrow{\varphi_n} M_n$ defined by φ. We claim that each φ_n is an isomorphism.

That φ_0 is an isomorphism is clear from the construction. Assume that $\varphi_0, \ldots, \varphi_{n-1}$ are all isomorphisms and consider the exact sequence

$$0 \to J^n/J^{n+1} \to C_n \to C_{n-1} \to 0$$

of C_n-modules. Since L_n, M_n are C_n-flat (recall that $\mathscr{B}_n$ is $\mathscr{O}_{X_n}$-flat) we get the following commutative diagram of exact sequences of C_n-modules:

$$\begin{tikzcd}
0 \arrow[r] & J^n/J^{n+1} \underset{C_n}{\otimes} M_n \arrow[r] & M_n \arrow[r] & M_{n-1} \arrow[r] & 0 \\
0 \arrow[r] & J^n/J^{n+1} \underset{C_n}{\otimes} L_n \arrow[r] \arrow[u] & L_n \arrow[r] \arrow[u, "\varphi_n"] & L_{n-1} \arrow[r] \arrow[u, "\varphi_{n-1}"] & 0.
\end{tikzcd}$$

But

$$J^n/J^{n+1} \simeq \frac{\mathscr{M}^n}{\mathscr{M}^{n+1}} \underset{A}{\otimes} C \cong \bigoplus_{i=1}^{r} \Gamma(U, \mathscr{O}_{X_0}),$$

$$J^n/J^{n+1} \underset{C_n}{\otimes} M_n \simeq \bigoplus_{i=1}^{r} \Gamma(U, \mathscr{B}_0) = \bigoplus_{i=1}^{r} M_0,$$

$$\text{and} \quad J^n/J^{n+1} \underset{C_n}{\otimes} L_n \simeq \bigoplus_{i=1}^{r} L_0.$$

161 It follows that the first vertical map is an isomorphism; so is φ_{n-1} by inductive assumption. Hence φ_n is an isomorphism.

If $\widehat{C}$, $\widehat{L}$, $\widehat{M}$ are the J-adic completions of C, L, M and if $\widehat{\varphi}$ is the $\widehat{C}$-linear map defined by φ, it follows that $\widehat{\varphi}$ is an isomorphism. Thus, one obtains: $(\operatorname{coker}\varphi)\widehat{\ } = 0$ and $\ker\varphi \subset \ker(L \to \widehat{L})$. It follows then that $\mathscr{B}$ is locally free at $x_0 \in X_0$, i.e., $Z \to X$ is flat at points over X_0. It is also unramified at points above X_0 since $Z_0 \to X_0$ is unramified and for non-ramification it suffices to look at the fibre.

The proof of Proposition 8.1.3 is complete.

Proof of Theorem 8.1.1. The isomorphism $\pi_1(S, s_0) \xleftarrow{\sim} G(\overline{k(s_0)}/k(s_0))$ follows from Proposition 8.1.3 when $X = S$.

Consider now the diagram

$$\begin{tikzcd}
a_0 \in X \arrow[d] & X_0 \ni a_0' \arrow[l] \arrow[d] & \overline{X}_0 \ni \overline{a}_0 \arrow[l] \arrow[d] \\
s_0 \in S & k(s_0) \arrow[l] & \overline{k(s_0)} \arrow[l]
\end{tikzcd}$$

By Proposition 8.1.2 we get the exact sequence:

$$(e) \to \pi_1(\overline{X}_0, \overline{a}_0) \to \pi_1(X_0, a_0') \to G(\overline{k(s_0)}/k(s_0)) \to (e).$$

One now uses the isomorphisms $\pi_1(X_0, a_0') \xrightarrow{\sim} \pi_1(X, a_0)$ (Proposition 8.1.3) and $G(\overline{k(s_0)}/k(s_0)) \xrightarrow{\sim} \pi_1(S, s_0)$ to get he required exact sequence. Q.E.D.

Chapter 9

The Homomorphism of Specialisation of the Fundamental Group

9.1

With the notations and assumptions of Chapter 8, let s_1 be an arbitrary 162
point of $S = \operatorname{Spec} A$ and $\overline{X}_1 = X \underset{S}{\times} \overline{k(s_1)}$. If $\overline{a}_1$ is a geometric point of $\overline{X}_1$ and a_1 its image in X and if $\overline{X}_1$ is connected we get a sequence

$$\pi_1(\overline{X}_1, \overline{a}_1) \to \pi_1(X, a_1) \to \pi_1(S, s_1) \to (e)$$

such that the composites are trivial. We have continuous isomorphisms $\beta : \pi_1(X, a_1) \to \pi_1(X, a_0)$ and $\alpha : \pi_1(S, s_1) \to \pi_1(S, s_0)$. Consider now the diagram:

$$\begin{tikzcd}
 & \pi_1(\overline{X}_1, \overline{a}_1) \arrow[r] & \pi_1(X, a_1) \arrow[r] \arrow[d, "\beta"] & \pi_1(S, s_1) \arrow[r] \arrow[d, "\alpha"] & (e) \\
(e) \arrow[r] & \pi_1(\overline{X}_0, \overline{a}_0) \arrow[r] & \pi_1(X, a_0) \arrow[r] & \pi_1(S, s_0) \arrow[r] & (e).
\end{tikzcd}$$

This is *not* necessarily commutative; but it is *commutative upto an inner automorphism of* $\pi_1(S, s_0)$. (It is readily seen in the example of section 5.1 that, with the notation therein, the continuous homomorphism there, is determined upto an inner automorphism of $\pi_1(S, s)$ if we take for s a point different from $\varphi(s')$-see Ch. 4.)

Now the lower row is *exact*. This implies that there is a continuous
163 homomorphism $\pi_1(\overline{X}_1, \overline{a}_1) \to \pi_1(\overline{X}_0, \overline{a}_0)$ which is clearly determined upto an inner automorphism of $\pi_1(X)$. This is called the *homomorphism of specialisation of the fundamental group.*

9.2

With the same assumptions as above further suppose X/S separable. Since we have assumed $\overline{X}_0$ connected, the condition $f_*(\mathscr{O}_X) = \mathscr{O}_S$ is automatically satisfied and then the upper sequence in the diagram of 9.1 is also exact. In this case the homomorphism of specialisation is *surjective.*

9.3

Let Y be locally noetherian and $X \to Y$ be a separable, proper morphism with fibres universally connected. Suppose $y_0, y_1 \in Y$ with $y_0 \in (\overline{y}_1)$. Let $\overline{a}_0$, $\overline{a}_1$ be geometric points of $\overline{X}_0 = X \underset{Y}{\times} \overline{k(y_0)}$ and $\overline{X}_1 = X \underset{Y}{\times} \overline{k(y_1)}$ respectively. Then there is a (natural) homomorphism of specialisation $\pi_1(\overline{X}_1, \overline{a}_1) \to \pi_1(\overline{X}_0, \overline{a}_0)$ which is *surjective.*

In fact, in view of Proposition 7.3.2 we may make the base-change $Y \leftarrow \operatorname{Spec} \widehat{\mathscr{O}}_{y_0, Y}$ and apply the above considerations to points above y_0 and y_1.

This is the so-called *semi-continuity of the fundamental group.*

Appendix to Chapter IX

The Fundamental Group of an Algebraic Curve Over an Algebraically Closed Field

Let k be an algebraically closed field and X a proper, nonsingular, connected algebraic curve over k (i.e., a k-scheme of dimension 1). **164**

Case 1. *Charac.* $k = 0$.

We compute $\pi_1(X)$ in this case by assuming results of transcendental geometry.

In view of Lemma 7.2.1.4 and Proposition 7.3.2 one can assume that $k = \mathbb{C}$, the field of complex numbers. Then one has an analytic structure on X and on every $X' \in (\mathscr{E}t/X)$ (see GAGA - J.P. Serre); one can then show that the natural functor

$$(\mathscr{E}t/X) \to \left\{ \begin{array}{l} \text{Finite topological coverings of the} \\ \text{analytic space } X^h \text{ defined by } X \end{array} \right\}$$

is an equivalence [cf. GAGA and Espaces fibrés algébriques - Śeminair Chevally 1958, (Anneaux de Chow) - See especially Prop. 19, Cor. to Prop. 20 and Cor. to Theorem 3].

One thus obtains:

$\pi_1(X, a) \simeq \widehat{\pi_1^{\text{top}}}(X^h, a)$, the completion of the topological fundamental group $\pi_1^{\text{top}}(X^h, a)$ with respect to subgroups of finite index. This group $\pi_1^{\text{top}}(X^h, a)$ is "known" if X is a nonsingular, proper, connected curve of genus g. It is a group with $2g$ generators u_i, $v_i (i = 1, \ldots, g)$ subject to one relation, namely

$$(u_1 v_1 u_1^{-1} v_1^{-1})(u_2 v_2 u_2^{-1} v_2^{-1}) \ldots (u_g v_g u_g^{-1} v_g^{-1}) = 1.$$

Before going to the case charac. $k = p \neq 0$, we shall briefly recall some evaluation-theoretic results. (Ref. S. Lang-Algebraic Numbers or J.P. Serre-Corps Locaux). **165**

Let V be a discrete valuation ring with maximal ideal $\mathscr{M}$ and field of fractions K. Let K'/K be a finite galois extension and V' the integral closure of V in K'. Choose any maximal ideal $\mathscr{M}'$ of V' and consider the decomposition group of $\mathscr{M}'$, i.e., the subgroup $\mathfrak{g}_d(\mathscr{M}')$ of $G(K'/K)$ of elements leaving $\mathscr{M}'$ stable. Then there is a natural homomorphism $\mathfrak{g}_d(\mathscr{M}') \to G(k(\mathscr{M}')/k(\mathscr{M}))$; the kernel $\mathfrak{g}_i(\mathscr{M}')$ of this homomorphism is called the *inertial group* of $\mathscr{M}'$. The maximal ideals of V' are transformed into each other by the action of the galois group $G(K'/K)$ and the various inertial groups are conjugates to each other (Compare with Lemma 4.2.1). For the following definition it is irrelevant which of the $\mathscr{M}'$ we take and therefore we shall write $\mathfrak{g}_i$, or sometimes $\mathfrak{g}_i(K'/K)$ instead of $\mathfrak{g}_i(\mathscr{M}')$. We say that K' is *tamely ramified* over V if the order of $\mathfrak{g}_i$ is prime to the characteristic of $k(\mathscr{M})$ and that K' is *unramified* over V if $\mathfrak{g}_i$ is trivial.

One has then the following:

(1) If K'/V is tamely ramified then the inertial group $\mathfrak{g}_i(K'/K)$ is *cyclic.*

(2) If $K'' \supset K' \supset K$ is a tower of finite galois extensions and if V' is a localisation of the integral closure of V in K' and if K'/V, K''/V' are tamely ramified then K''/V is tamely ramified.

166 In addition, one has an exact sequence:

$$(e) \to \mathfrak{g}_i(K''/K') \to \mathfrak{g}_i(K''/K) \to \mathfrak{g}_i(K'/K) \to (e)$$

(cf. Corps Locaux, Ch. I, Propostion 22).

(3) Let $\tau \in \mathscr{M}$ be a uniformising parameter and $n \in \mathbb{Z}^+$ be prime to charac. $k(\mathscr{M})$. If K contains the n^{th} roots of unity, then $K' = \dfrac{K[X]}{(X^n - \tau)}$ is a finite galois extension, tamely ramified, with galois group $= \mathfrak{g}_i(K'/K) \simeq \mathbb{Z}/n\mathbb{Z}$.

Lemma (Abhyankar). *Let L, K' be finite galois extensions of K, tamely ramified over V with order $\mathfrak{g}_i(L/K)$ $(= n)$ dividing order $\mathfrak{g}_i(K'/K)(= m)$. Let L' be the composite extension of L and K'. Then L' is unramified over the localisations of the integral closure V' of V in K'.*

Proof. One checks that $\exists$ a monomorphism $\mathfrak{g}_i(L'/K) \to \mathfrak{g}_i(K'/K) \times \mathfrak{g}_i(L/K)$ such that the projections $\mathfrak{g}_i(L'/K) \to \mathfrak{g}_i(K'/K)$ and $\mathfrak{g}_i(L'/K) \to \mathfrak{g}_i(L/K)$ are onto (see 2)). Since the orders of $\mathfrak{g}_i(L/K)$ and $\mathfrak{g}_i(K'/K)$ are prime to p = charac. $k(\mathscr{M})$, so is the order of $\mathfrak{g}_i(L'/K)$, i.e., L'/V is tamely ramified; hence $\mathfrak{g}_i(L'/K)$ is cyclic (1)). As $n|m$, it follows that each element of $\mathfrak{g}_i(L'/K)$ has order dividing m. But $\mathfrak{g}_i(L'/K) \to \mathfrak{g}_i(K'/K)$ is onto. Thus, $\mathfrak{g}_i(L'/K) \xrightarrow{\sim} \mathfrak{g}_i(K'/K)$. From (2) the kernel of this map is $\mathfrak{g}_i(L'/K')$ which is therefore trivial. Q.E.D. □

Case 2. *Charac.* $k = p \neq 0$. **167**

Definition. We say that a Y-prescheme X is *smooth* at a point $x \in X$ (or $X \to Y$ is *simple* at x) if $\exists$ an open neighborhood U of x such that the natural morphism $U \to Y$ admits a factorisation of the form

$$U \xrightarrow{\text{étale}} \operatorname{Spec} \mathscr{O}_Y [T_1, \ldots, T_n] \to Y$$

(the T_i are indeterminates).

Note. Smoothness is stable under base-change.

Definition. For a Y-prescheme X, the *sheaf of derivations* of X over Y is the dual of the $\mathscr{O}$-Module $\Omega_{X\dagger Y}$; it is denoted by $\mathfrak{g}_{X\dagger Y}$.

We shall assume the following

Theorem 1. *(SGA, 1960, III, Theorem (7.3)).*

Let A be a complete noetherian local ring with residue field k. If X_0/k is a projective, smooth scheme such that

$$\begin{aligned} & H^2(X_0, \mathfrak{g}_{X_0|k}) = (0) \\ \text{and} \quad & H^2(X_0, \mathscr{O}_{X_0}) = (0); \end{aligned}$$

then $\exists$ a projective, smooth A-scheme X such that $X \underset{A}{\otimes} k \simeq X_0$.

Let k be an algebraically closed field of charac. $p \neq 0$ and X_0 a **168**
nonsingular (= smooth in this case) connected curve, proper (hence, as

is well-known projective) over k. Consider the ring $A = W(k)$ of Witt-vectors; this is a complete, discrete valuation ring with residue field k; if K is the field of fractions of A, then charac. $K = 0$, and K is complete for the valuation defined by A. The conditions of Theorem 1 are satisfied by X_0. Therefore $\exists$ a projective smooth A-scheme X such that $X_0 \overset{\sim}{\leftarrow} X \underset{A}{\otimes} k$. The X_0 is universally connected; but then it follows that the generic fibre is also universally connected (Use Stein-factorisation; Note that Spec A has only two points). Furthermore, as X is smooth over A, the local rings of X are regular (one has to show that if $X \to Y$ is étale and Y is regular then X is regular). Finally we mention the important fact that by Proposition 8.1.3, we have: $\pi_1(X) \overset{\sim}{\leftarrow} \pi_1(X_0)$.

We may thus replace X_0 by X. Let a_0 be the closed point of Spec A and a_1 the generic point; set $X_K = X \underset{A}{\otimes} K$, $\overline{X}_K = X \underset{A}{\otimes} \overline{K}$, where as usual $\overline{K}$ is the algebraic closure of K. If $\overline{a}_0 \in X_k = X \underset{A}{\otimes} k$ is a geometric point over a_0 and $\overline{a}_1 \in \overline{X}_K$, a geometric point over a_1 then we have the homomorphism of specialisation

$$\pi_1(\overline{X}_K, \overline{a}_1) \to \pi_1(X_k, \overline{a}_0) \simeq \pi_1(X)$$

which is surjective (9.2).

169 From case 1 one already "knows" about $\pi_1(\overline{X}_K, \overline{a}_1)$ (charac. $\overline{K} = 0$). Thus it only remains to study the kernel of the above epimorphism. This amounts by 5.2.4 to studying the following question: Given a connected étale covering $\overline{Z}$ of $\overline{X}_K$ which is *galois*, when does there exist a $Z \in (\mathscr{E}t/X)$ such that $Z \overset{\sim}{\leftarrow} Z \underset{A}{\otimes} \overline{K}$. [Remember: a connected étale covering is galois if the degree of the covering equals the number of automorphisms. Also note: we have integral schemes because, for instance, our schemes are connected and regular. Therefore we may consider the function fields $R(\overline{Z})$ and $R(\overline{X}_K)$ of $\overline{Z}$ and $\overline{X}_K$. Clearly if $\overline{Z}$ is galois over $\overline{X}_K$ then $R(\overline{Z})$ is a galois extension of $R(\overline{X}_K)$.].

By Lemma 7.2.1.4, $\exists$ a finite subextension K_1 of K and a $Z_{K_1} \in (\mathscr{E}t/X_{K_1}$
$= X \underset{A}{\otimes} K_1)$ such that $\overline{Z} \overset{\sim}{\leftarrow} Z_{K_1} \underset{K_1}{\otimes} \overline{K}$. Then the above question takes the form: Given a connected $\overline{Z} \in (\mathscr{E}t/\overline{X}_K)$, which is galois when does

there exist a $Z \in (\mathscr{E}t/X)$ and a finite extension K_2 of K_1 such that $Z_{K_2} = Z_{K_1} \underset{K_1}{\otimes} K_2 \overset{\sim}{\longleftarrow} Z \underset{A}{\otimes} K_2$?

Now, for any finite extension K' of K, let A' be the integral closure of A in K'; then by a corollary to Hensel's lemma it follows that A' is again a discrete valuation ring whose residue field is again k (recall that k is algebraically closed). Thus $\pi(X) \simeq \pi_1(X_k) \simeq \pi_1(X_{A'})$ where $X_{A'} = X \underset{A}{\otimes} A'$, again by Proposition 8.1.3. Thus the question becomes: Given $Z_{K_1} \in (\mathscr{E}t/X_{K_1})$, universally connected and galois, when does 170
there exist a finite extension K_2 of K_1 such that $Z_{K_2} = Z_{K_1} \underset{K_1}{\otimes} K_2$ comes from a Z_{A_2} in $(\mathscr{E}t_{/X_{A_2}} = X \underset{A}{\otimes} A_2)$ (where A_2 is the integral closure of A in K_2)?

Consider any finite extension K' of K_1 and let A' be the integral closure of A in K'. We are given a situation of the form:

$Z_{K'}$

$X_{K'}$ $\quad + \; p'$- generic point of the fibre over the closed point.

generic point

$\sim$ Spec $K' - \underbrace{0 \quad \cdot}_{\text{Spec } A'}$-closed point (residue field k).

The Z_{K_1} (resp. $Z_{K'} = Z_{K_1} \underset{K_1}{\otimes} K'$) are connected and hence, as we have already remarked, integral. Let $R(Z_{K_1})$, $R(Z_{K'})$, $R(X_{K_1})$ and $R(X_{K'})$ be the function fields of Z_{K_1}, $Z_{K'}$, X_{K_1} and $X_{K'}$. Then $R(Z_{K_1})$ is a finite galois extension of $R(X_{K_1})$ and

$$R(Z_{K'}) = R(Z_{K_1}) \underset{K_1}{\otimes} K' = R(Z_{K_1}) \underset{R(X_{K_1})}{\otimes} R(X_{K'}).$$

Our aim now is to choose, if possible, the field $K' \supset K_1$ such that $R(Z_{K'})$ is *unramified* over $X_{A'}$. By this we mean the following: consider the normalisation Z' of $X' = X_{A'}$ in $R(Z_{K'})$; we want Z' to be unramified 171
over X'.

(**Note:** Since $Z_{K'}$ is regular, so certainly normal, we have that $Z' \underset{A'}{\otimes} K' \overset{\sim}{\longrightarrow} Z_{K'}$, because the process of normalisation is unique. – See EGA, II, § (6.3) and also S. Lang, Introduction to Algebraic Geometry, Ch. V).

We can also express this as follows: we want Z' to be unramified over the whole of X' and not merely on the open subscheme $X_{K'} = X' \underset{A'}{\otimes} K'$.

Consider the integral closure A_1 of A in K_1. Let p be the generic point of the fibre in $X_{A_1} = X \underset{A}{\otimes} A_1$ over the closed point (as a space, this is nothing but X_0). Then p is a point of codimension 1 in X_{A_1}; furthermore X_{A_1}/A_1 is smooth and therefore the local ring $\mathscr{O}_1$ of p (as a point in X_{A_1}) is regular. Therefore $\mathscr{O}_1$ is a discrete valuation ring in $R(X_{A_1}) = R(X_{K_1})$. Similarly define $\mathscr{O}'$ in $R(X_{K'}) = R(X_{A'})$ for any $K' \supset K_1$. It is easily checked that $\mathscr{O}'$ is the integral closure of $\mathscr{O}_1$ in $R(X_{K'})$. Now, any open set in $X_{A'}$, containing $X_{K'}$ and the generic point p' of the fibre over the closed point of Spec A' is such that its complement is of codimension ≥ 2 in X'_A. We have the following theorem depending on the so-called *purity of the branch locus*:

Theorem 2. *(SGA, 1960-1961 expose X, Cor (3.3))*

Let P be a locally noetherian regular prescheme and U the comple-
172 ment of a closed set of codimension ≥ 2 in P. Then the natural functor $(\mathscr{E}t/P) \xrightarrow{\Phi} (\mathscr{E}t/U)$ is an equivalence. (For a proof see SGA, X, 1962).

[For the special case of the theorem which we need, there is a direct proof in SGA, X, 1961, p. 16. However even for that proof one needs the following result (which we have *not* proved in these lectures). An unramified covering of a normal prescheme is étale. (SGA, I, 1960-1961, Theorem (9.5)).]

Thus, it is enough to prove that $\exists K' \supset K_1$ such that $R(Z')$ is unramified over X' at the point p' because Z' is clearly flat over $X' = X_{A'}$ at the point p (the local ring $\mathscr{O}'$ of p' in X' is a discrete valuation ring).

If τ is a uniformising parameter of A_1, it is also a parameter of $\mathscr{O}_1$. Let then $n \in \mathbb{Z}^+$ be such that $(n, p) = 1$ and set $K' = K_1[T]/(T^n - \tau)$. Then K'/K_1 is a finite galois extension and $R(X_{K'}) \simeq R(X_{K_1})[T]/(T^n - e)$. Hence $R(X_{K'})$ is tamely ramified over $\mathscr{O}_1$, and has an inertial group of order n.

Assume now that the degree of the galois covering $\overline{Z}$ over $\overline{X}_K$ is prime to $p =$ charac.k. Then one may take n equal to this degree. By Abhyankar's lemma one then has that $R(Z')$ is unramified over X' above the point p'. Thus one has "proved" the

Proposition 3. *The kernel of the (surjective) homomorphism of specialisation $\pi_1(\overline{X}_K, \overline{a}_1) \to \pi_1(X_0, \overline{a}_0)$ is contained in the inter-section of the kernels of continuous homomorphisms from $\pi_1(\overline{X}_K)$ to finite groups of order prime to p.*

Therefore if, for a profinite group $\mathscr{G}$, we denote by $\mathscr{G}(p)$ the profinite group $\varprojlim \mathscr{G}_i$ where the limit is taken over all quotients $\mathscr{G}_i$ of $\mathscr{G}$ which are finite of order prime to p, then we have: 173

$$\pi_1^{(p)}(\overline{X}_K, \overline{a}_1) \xrightarrow{\sim} \pi_1^{(p)}(X_0, \overline{a}_0).$$

(One can also say: $\mathscr{G}^{(p)}$ is the quotient of $\mathscr{G}$ by the closed normal subgroup generated by the p-Sylow subgroups of $\mathscr{G}$). Since we "know", by topological methods, the group $\pi_1(\overline{X}_K, \overline{a}_1)$ we obtain:

Theorem 4. *If X is a nonsingular, connected, proper curve of genus g over an algebraically closed field k of charac. $p \neq 0$, then $\pi_1^{(p)}(X) \xrightarrow{\sim} \mathscr{G}^{(p)}$ where $\mathscr{G}$ is the completion with respect to subgroups of finite index of a group with $2g$ generators u, u_i, $v_i (i = 1, 2, \ldots, g)$ with one relation:*

$$(u_1 v_1 u^{-1} v_1^{-1})(u_2 v_2 u_2^{-1} v_2^{-1}) \ldots (u_g v_g u_g^{-1} v_g^{-1}) = 1.$$

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