Lectures On
Quadratic Jordan Algebras

By

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Preface

Until recently the structure theory of Jordan algebras dealt exclusively with finite dimensional algebras over fields of characteristic $\neq 2$. In 1966 the present author succeeded in developing a structure theory for Jordan algebras of characteristic $\neq 2$ which was an analogue of the Wedderburn-Artin structure theory of semi-simple associate rings with minimum condition on left or right ideals. In this the role of the left or right ideals of the associative thory was played by quadratic ideals (new called inner ideals) which were defined as subspaces Z of the Jordan algebra $\mathscr{J}$ such that $\mathscr{J} U_b \subseteq Z$ where $U_b = 2R_b^2 - mR_{b^2}, R_a$ the multiplication by b in $\mathscr{J}$. The operator U_b ($P(b)$ in the notation of Braun and Kocher), which in an associative algebra is $x \rightarrow bxb$, was introduced into abstract Jordan algebras by the author in 1955 and it has playedan increasingly important role in the theory and lits applications. It has been fairly clear for some time that an extension of the structure theory which was to encompass the characteristic two case would have to be "quadratic" in character, that is, would have to be based on the composition yU_x rather than the usual x, y (which is $\frac{1}{2}(xy + yx)$ in associative algebras). The first indication of this appeared already in 1947 in a paper of Kaplansky's which extended a result of Ancocheas's on Jordan homomorphisms (then called semi-homomorphisms) of associative algebras to the characteristic two case by redefining Jordan homomorphisms using the product xyx in place of $xy + yx$.

a completely satisfactory extension of the author's structure theory which include characteristic two or more precisely algebras over an arbitrary commutative ring has been given by McCrimmon in [5] and [6], McCrimmon's theory begins with a simple and beautiful axiomatization of the composition YU_x. In addition to the quadratic character of the mapping $x\o U_x$ of into its algebra of endomorphisms and the existence of unit 1 such that $U_1 = 1$ one has to assume only the so-called "fundamental formula" $U_xU_yU_x = U_{yU_x}$, one additional indentity, and the linearizations of these. Instead of assuming the linearizations it is equivalent and neater to assume that the two identities carry over on extension of the coefficient ring. If the coefficient ring Φ contains $\frac{1}{2}$ then the notion of a quadratic Jordan algebra is equivalent to the classical notion of a (linear) Jordan algebra there is a canonical way of passing from the operator U to the usual multiplication R and back. Based on these fundations one can carry over the fundamental notions (inverses, isotopy, powers) of the linear theory to the quadratic case and extend the Artin like structure theory to quadratic Jordan algebras. In particular, one obtains for the first time a satisfactory Jordan structure theory for finite dimensional algebras over a field of characteristic two.

In these lectures we shall detailded and self-contained exposition of McCrimmon's structure theiry including his recently developed theory of radicals and absolute zero divisors which constitute an important addition even to the classical linear theory. In our treatment we restrict attention to algebras with unit. This effects a sybstantial simplication. However, it should be noted that McCrimmon has also given an axiomatization for quadratic Jordan algebras withour unit and has developted the structure theory also for these. Perhaps the reader should be warned at the outsetthat he may find two (hopefully no more) parts of the exposition somewhat heavynamely, the derivation of the long list of identities in §1.3 and the proof of Osborn's thorem on algebras of capacity two. The first of these could have been avoided by proving a general theorem in identities due to Macdonald. However, time did not permit this. The simplification of the proof of Osborn's theorem remains an open problem. We shall see at the end of our exposition that this difficulty evaporates in the important special case of finite dimensional quadratic Jordan algebras over an algebraically closed field.

Nathan Jacobson

Contents

Chapter 0

Artinian Semi-simple Rings with Involution

In this chapter we shall determine the Artinian semi-simple rings with 1
involutions. The results are all well known. The final formulation in terms of matrix rings (in §2) will be particularly useful in the sequel. The results which we shall state without proof can be found in any standard text on associative ring theory.

1 Determination of the semi-simple artinian rings with involution.

Throughout these notes associative rings and algebras will be assumed to be *unital*, that is to contain a unit 1 such that $a1 = a = 1a$ for all a in the ring. We recall that such a ring is called *right artinian* if it satisfies one of the following equivalent conditions:

Minimum Condition: Any non-vacuous set of right ideals of the ring contains a minimal element.

Desending chain condition: There exist no properly descending infinite chain of right ideals $\mathfrak{I}_1 \supset \mathfrak{I}_2 \supset \mathfrak{I}_3 \cdots$

A right artinian ring is called *semi-simple* if it contains no non-zero nilpotent (two-sided) ideal. An ideal $\mathfrak{N}$ is nilpotent if there exists a positive integer N such that every product of Nz_i in $\mathfrak{N}$ is 0. Equivalently, if

$\mathscr{Z}\mathfrak{L}$ is defined to be the ideal generated by all be, $b \in \mathscr{Z}, c \in \mathfrak{L}$ and $\mathscr{Z}^m$ for $m = 1, 2, \ldots$ is defined by $\mathscr{Z}^1 = \mathscr{Z}, \mathscr{Z}^k = \mathscr{Z}^{k-1}\mathscr{Z}$ then $\mathfrak{N}^N = 0$.

2 We recall the fundamental Wedderburn-Artin structure theorems on semi-simple (right) artinian rings.

I. If $\mathfrak{a}$ is semi-simple artinian ($\neq 0$) then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_3$ where $\mathfrak{a}_i$ is an ideal which regarded as a ring is simple artinian. (A ring $\mathfrak{a}$ is simple if $\mathfrak{a} \neq 0$ and 0 and $\mathfrak{a}$ are the only ideals in $\mathfrak{a}$.) Conversely, if $\mathfrak{a}$ has the indicated structure then it is semi-simple artinian.

II. A ring $\mathfrak{a}$ is simple artinian if and only if $\mathfrak{a}$ is isomorphic to a complete ring Δ_n of $n \times n$ matrices over a division ring Δ. This is equivalent to isomorphism to the ring $\mathrm{End}_\Delta \mathcal{V}$ of linear transformations of an n dimensional (left) vector space $\mathcal{V}$ over a division ring Δ.

It is easily seen that the *simple components* $\mathfrak{a}_i$ in the first structure theorem are uniquely determined. In the second structure theorem, n and the isomorphism class of Δ are determined by $\mathfrak{a}$. This follows from the following basic isomorphism theorem.

III. Let $\mathcal{V}_i$, $i = 1, 2$, be a vector space over a division ring Δ_i and let ρ be an isomorphism of $\mathrm{End}_{\Delta_1}\mathcal{V}_1$ onto $\mathrm{End}_{\Delta_2}\mathcal{V}_2$. Then there exists a semi-linear isomorphism S of $\mathcal{V}_1$ onto $\mathcal{V}_2$ with associated isomorphism s of Δ_1 onto Δ_2 such that

$$A^\sigma = S^{-1}AS, \quad A \in End\mathcal{V}_1. \tag{1}$$

We now consider semi-simple artinian rings with involution. First, we give the basic definitions, which we formulate more generally
3 for rings which need not be associative. As in the associative case, we assume the rings are unital. Also homorphisms are assumed to map 1 into 1 and subrings contain 1.

Definition 1. A *ring with involution* is a pair $(\mathfrak{a}, J)$ where $\mathfrak{a}$ in a ring (with 1) and J is an involution (= anti-automorphism such that $J^2 = 1$) in $\mathfrak{a}$. A *homomorphism* σ of $(\mathfrak{a}, J)$ into a second ring with involution $(\mathscr{Z}, K)$ in a homomorphism of $\mathfrak{a}$ into $\mathscr{Z}$ (sending

1 into 1) such that $J\sigma = \sigma k$. A *subring* of $(\mathfrak{a}, J)$ is a subring $\mathscr{Z}$ (containing 1) of $\mathfrak{a}$ which is stable under J. An *ideal* of $(\mathfrak{a}, J)$ is an ideal of $\mathfrak{a}$ which is J-stable. (,) is *simple* if $\mathfrak{a} \neq 0$ and $\mathfrak{a}$ and 0 are the only ideals of $(\mathfrak{a}, J)$.

Let $(\mathfrak{a}, J)$ be simple and assume $\mathscr{Z}$ is an ideal $\neq 0$ in $\mathfrak{a}$. Then $\mathscr{Z} + \mathscr{Z}^J$ is an ideal in $(\mathfrak{a}, J)$. Hence $\mathscr{Z} + \mathscr{Z}^J = \mathfrak{a}$. Also $\mathscr{Z} \cap \mathscr{Z}^J$ is an ideal in $(\mathfrak{a}, J)$ so $\mathscr{Z} \cap \mathscr{Z} = 0$. Thus $\mathfrak{a} = \mathscr{Z} \oplus \mathscr{Z}$. If $\mathfrak{L}$ is an ideal in $\mathscr{Z}$ then $\mathfrak{L} + \mathfrak{L}^J$ is an ideal in $(\mathfrak{a}, J)$. It follows that $\mathscr{Z}$ is simple. This shows that if $(\mathfrak{a}, J)$ is simple then either $\mathfrak{a}$ is simple or $\mathfrak{a} = \mathscr{Z} \oplus \mathscr{Z}^J$ where $\mathscr{Z}$ is simple.

An associative ring with involution $(\mathfrak{a}, J)$ is *artinian semi-simple* if $\mathfrak{a}$ is artinian semi-simple. It follows from the first Wedderburn-Artin structure theorem that such a ring with involution is a direct sum of ideals which are artinian simple rings with involution and conversely. An artinian simple ring with involution is of one of the following types:$\mathfrak{a} = \mathscr{Z} \oplus \mathscr{Z}^J$ where $\mathscr{Z} \cong \Delta_n$, Δ a division ring or $\mathfrak{a} \cong \Delta_n$ (or $\cong$ End $\mathcal{V}$ where $\mathcal{V}$ is n dimensional vector space over a **4**
division ring Δ). We now consider the latter in greater detail.

Thus consider End $\mathcal{V}$ where $\mathcal{V}$ is an n-dimensional vector space over Δ. Assume End $\mathcal{V}$ has an involution J. Let $\mathcal{V}^*$ be the right vector space of linear functions on $\mathcal{V}$. We denote the elements of $\mathcal{V}^*$ as x^*, y^* etc. And write the value of x^* at y by $\langle y, x^*\rangle$. This gives a *bilinear pairing* of the left vector space $\mathcal{V}/\Delta$ with the right vector space $\mathcal{V}^*/\Delta$ in the sense that

$$\begin{gathered}\langle y_1 + y_2, x^*\rangle = \langle y_1, z^*\rangle + \langle y_2, x^*\rangle \\ \langle y_1 x_1^* + x_2^*\rangle = \langle y, x_1^*\rangle + \langle y, z_2^*\rangle \\ \langle \alpha y, x^*\rangle = \alpha\langle y, x^*\rangle, \langle y, x^*\alpha\rangle = \langle y,^* \rangle\alpha, \quad \alpha \in \Delta\end{gathered} \tag{2}$$

Also the pairing is *non-degenerate* in the sense that if $\langle y, x^*\rangle = 0$ for all $x^* \in \mathcal{V}^*$ then $y = 0$ and if $\langle y, x^*\rangle = 0$ for all $y \in \mathcal{V}$ then $x^* = 0$. Let Δ° be the opposite ring of $\Delta : \Delta^\circ$ is the same additive group as Δ and has the multiplication $\alpha \circ \beta = \alpha\beta$. Then if we put $\alpha x^* = x^*\alpha, \mathcal{V}^*$ becomes a(left) vector space over Δ° and the last equation in (2) becomes $\langle y, \alpha x^*\rangle = \langle y_1, x^*\rangle\alpha$.

Let $A \in \operatorname{End} \mathcal{V}$. Then we have a uniquely determined linear transformation A^* in $\operatorname{End} \mathcal{V}^*$ satisfying

$$\langle yA, x^* \rangle = \langle y_1, x^* A^* \rangle, \quad y \in \mathcal{V}, x^* \in \mathcal{V}^* \tag{3}$$

This is called the *transpose* of A. If we use the usual functional
5 notation $x^*(y)$ for $\langle y, x^* \rangle$ then $x^* A^*(y) = x^*$ is the resultant of A followed by $x^*(yA)$ or $x^* A^*$. It follows directly that

$$(A + B)^* = A^* + B^*, (AB)^* = B^* A^* \tag{4}$$

and $A \to A^*$ is bijective from $\operatorname{End} \mathcal{V}$ to $\operatorname{End} \mathcal{V}^*$. Thus $A \to A^*$ is an anti-isomorphism of $\operatorname{End} \mathcal{V}$ onto $\operatorname{End} \mathcal{V}^*$.

Now suppose $\operatorname{End} \mathcal{V}$ has an involution J. Since $A \to A^J$ and $A \to A^*$ are anti-isomorphisms, $A^J \to A^*$ is an isomorphism of $\operatorname{End} \mathcal{V}$ onto $\operatorname{End} \mathcal{V}^*$ (considered as left vector space over Δ°). Hence by the isomorphism theorem III we have a semi-linear isomorphism (S, s) (S with associated division ring isomorphism s) of $\mathcal{V}/\Delta$ onto $\mathcal{V}^*/\Delta^\circ$ such that

$$A^* = S^{-1} A^J S, \quad A \in \operatorname{End} \mathcal{V} \tag{5}$$

Now put

$$g(x, y) = \langle x, yS \rangle, \quad x, y \in \mathcal{V}. \tag{6}$$

Then g is additive in both factors, $g(\alpha x, y) = \alpha g(x, y)$ and

$$\begin{aligned} g(x, \alpha y) &= \langle x, (\alpha y)S \rangle = \langle x, \alpha^s (ys) \rangle \\ &= \langle x, (yS)\alpha^s \rangle = \langle x, yS \rangle \alpha^s \end{aligned}$$

The mapping $\alpha \to \alpha^s$ is an isomorphism of Δ onto Δ°; hence an anti-automorphism in Δ. The conditions just noted for g are that g is a sesquilinear form on $\mathcal{V}/\Delta$ relative to the anti-automorphism s
6 in Δ. Since the pairing $\langle\ ,\ \rangle$ is non-degenerate it follows that g is a non-degenerate form: $g(x, \mathcal{V}) = 0$ implies $x = 0$ and $g(\mathcal{V}, x) = 0$ implies $x = 0$. We have

$$g(x, yA^J) = g(x, ySA^*S^{-1}) = \langle x, ySA^* \rangle$$

$$= \langle xA, yS \rangle = g(xA, y).$$

Hence A^J is the uniquely determined *adjoint* of A relative to g in the sense that $g(x, yA^J) = g(xA, y), x, y \in \mathcal{V}$.

So far we have not used the involutorial character $J^2 = 1$ of J. We note first that the sesquilinear character of g implies that if $v \in V$ then $x \to g(v, x)s^{-1}$ is a linear function on V. Hence there exists a $v' \in V$ such that

$$g(v, x)^{s^{-1}} = g(x, v'), x \in V. \qquad (7)$$

Next we consider the linear mapping $x \to g(x, u)v$ in V where $u, v \in V$. Since

$$\begin{aligned} g(g(x, u)v, y) &= g(x, u)g(v, y) \\ &= g(x, g(v, y)^{s^{-1}} u), \end{aligned}$$

it is clear that if we put $A : x \to g(x, u)v$ then A^J is $y \to gA(v, y)^{s^{-1}} u = g(y, v')u$. Since $A^{J^2} = A$ this gives $g(x, u')v' = g(x, u)v$ for all x, u, v. This implies that $v' = \gamma v$, $\gamma \neq 0$, in Δ, (independent of v) so by (7), we have

$$g(x, y)^s = \delta g(y, x), \quad \delta \neq 0 \quad \text{in } \Delta. \qquad (8)$$

By (8) we have $g(x, y)^{s^2} = \delta g(x, y)\delta^s$. We can choose x, y so that 7
$g(x, y) = 1$. Then we get $\delta^s = \delta^{-1}$. If $\delta = -1$ we have $g(x, y)^s = -g(y, x)$. Then $g(x, y)^{s^2} = g(x, y)$ and $g(\alpha x, y)^{s^2} = \alpha^{s^2} g(x, y) = g(\alpha x, y) = \alpha g(x, y)$. Then $\alpha^{s^2} = \alpha$. So s is an involution in Δ and g is a semi-degenerate skew hermitian form relative to this involution. If $\delta \neq -1$ then we put $\rho = \delta + 1$ and $h(x, y) = g(x, y)\rho$. Then h is sesquilinear relative to the anti-automorphism $t : \alpha \to \rho^{-1}\alpha^s\rho$ and $h(x, y)^t = (g(x, y)\rho)^t = \rho^t g(x, y)^t = \rho^t\rho^{-1} g(x, y)^s\rho = \rho^t\rho^{-1}\delta g(y, x)\rho = \rho^t\rho^{-1}\delta h(y, x)$. Also $\rho^t\rho^{-1} = \rho^{-1}\rho^s$ so $\rho^t\rho^{-1}\delta = (1 + \delta)^{-1}(1 + \delta^s)\delta = 1$ since $\delta^s\delta = 1$. Hence $h(x, y)^t = h(y, x)$. Then $\alpha^{t^2} = \alpha$ so t is an involution in Δ and h is a non-degenerate hermitian form relative to this involution. Clearly $h(xA, y) = h(x, yA^J)$ so $A \to A^J$ is the adjoint mapping determined by h. We have now proved.

IV. Let V be a finite dimensional vector space over a division ring Δ and assume that End V has an involution J. Then Δ has an involution $j : \delta \to \overline{\delta}$ and there exists a non-degenerate hermitian or skew-hermitian form h on V such that J is the adjoint mapping determined by h.

The converse of IV is trivial. Given a non-degernate hermitian or skew-hermitian form h on V/Δ then the adjoint mapping relative to h is an involution in End V. We recall next how such forms are constructed. Let (Δ, j) be a division ring with involution and let V
8 be an n-dimensional vector space over Δ. Let $(v_1, v_2, \ldots, v_n)$ be a base for V/Δ and $H = (\eta_{ij}) \in \Delta_n$ be hermintian or skew hermitian, that is, $\overline{H}^t = \pm H$ where $\overline{H} = (\overline{\eta}_{ij})$ and the t denotes the transpose. If $x = \sum \xi_i v_i, y = \sum \eta_i v_i$ then we define

$$h(x, y) = \sum_{i,j=1}^{n} \xi_i \eta_{ij} \overline{\eta}_j. \tag{9}$$

Then direct verification shown that h is a hermitian or skew hermitian form accordings as H is hermitian or skew hermitian. Moreover, h is non-degernate if and only if H is invertible in Δ_n. Since it is clear that there exist hermitian invertible matrices for any involution j and any n (e.g the matrix 1) it follows that End V has an involution if and only if Δ has an involution. We remark that there exist Δ which have no involutions. For example, any finite dimensional central division algebra over the rationals Q of dimensionality > 4 has no involution. We remark also that if $\Delta = \Phi$ is field then $j = 1$ is an involution.

The construction we gave yields all hermitian and skew hermitian forms on V/Δ. Suppose h is a hermitian or skew hermitian form on V/Δ and, as before, $(v_1, v_2, \ldots, v_n)$ is a base for V/Δ. Then the matrix $H = (h(v_i, v_j))$ of h relative to the given base is hermitian or skew hermitian and if $= \sum \xi_i v_i, y = \sum \eta_i v_i$ then $h(x, y) = \sum \xi_i h(v_{i_1} v_j)\overline{\eta}_j$ as before.

Let h have the matrix $H = (\eta_{ij})$ relative to the base $(v_1, v_2, \ldots, v_n)$ and assume h is non-degenerate or, equivalently, H is invertible.

Let $A \in \text{End } V$ and write $v_i A = \sum \alpha_{ij} v_j$, so $(\alpha) = (\alpha_{ij})$ in the 9
matrix of A relative to this base. Let A^J be the adjoint of A relative to h and write $v_i A^J = \sum \beta_{ij} v_j$. It is clear that the defining conditions: $h(xA, y) = h(x, yA^J)$ are equivalent to the conditions $h(v_i A, v_j) = h(v_i, v_j A^J)$, $i, j = 1, \ldots, n$. Using the matrices (α) and (β) there n^2 conditions give the matrix condition $(\alpha)H = H(\overline{\beta})^t, H = (h(v_{ij} v_j))$, where $(\overline{\beta})^t$ is the transposed of the matrix $(\overline{\beta}) = (\overline{\beta}_{ij})$. For, $h(v_i A, v_j) = h(ikv_k, v_j) = h(\sum \alpha_{ik} v_k, v_j)$ and $h(v_j, v_j A^J) = h(v_i, \sum \beta_{jk} v_k) = \sum\limits_k h(v_i, v_k)\overline{\beta}_{jk}$ and $\sum\limits_k \alpha_{ik} h(v_k, v_j) = \sum\limits_k h(v_i, v_k)\overline{\beta}_{jk}$, $i, j = 1, \ldots, n$ are equivalent to the matrix condition we noted. Then the matrix of A^J is

$$(\beta) = H(\overline{\alpha})^t H^{-1}. \tag{10}$$

Now the mapping $K : (\alpha) \to H(\overline{\alpha})^t H^{-1}$ is an involution in Δ_n. Also $A \to (\alpha)$ is an isomorphism of End V onto Δ_n and since this maps $A^J \to (\alpha)^k$ it is an isomorphism of (End V, J) onto (Δ_n, K).

We note next that unless $j = 1$ then we may normalize h to be hermitian. Then suppose $j \neq 1$ and h is skew hermitian. Choose j so that $\overline{j} \neq j$ and put $\rho = \overline{j} - j \neq 0$. Then $h' = h\rho$ is sesquilinear relative to the involution $\alpha \to \beta^{-1}\overline{\alpha}\rho$ and $\rho^{-1}\overline{h'(x, y)}\rho = \rho^{-1}\overline{h(x, y)\rho}\rho = -(-h(y, x)\rho) = h'(y, x)$. Hence h' is hermitian. Clearly the adjoint mappings determined by h and h' are identical. If $j = 1$ then $\Delta = \Phi$ is commutative and again h is hermitian unless the characteristic is
$\neq 2$ and $h(x, y) = -h(y, x)$. Hence the two two cases we need 10
consider are 1) h is hermitian, 2) $\Delta = \Phi$ a field of characteristic $\neq 2$, $j = 1, h$, skew symmetric.

It is easily seen that in the first case unless $\Delta \neq \Phi$ a field of characteristic 2, $j = 1$ and $h(\ ,\ ) \equiv 0$ then there exists a base $(u_1, u_2, \ldots, u_n)$ such that the matrix $(h(u_i, u_j))$ is a diagonal matrix $\gamma = \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ where $\overline{\gamma_i} = \gamma_i \neq 0$. This is proved on pp. 152-157 and pp. 170-171 of Jacobson's *Lectures in Abstract Algebra*, Vol. II. The foregoing argument shows that (End V, J) is isomorphic to (Δ_n, K) where K is the involution $(\alpha) \to \gamma(\overline{\alpha})^t \gamma^{-1}$ in

Δ_n. If we take into account to the case omitted in 1) and 2) we see that it remains to assume that $\Delta = \Phi$ a field, $j = 1$ and $h(x, x) \equiv 0$. In this case $n = 2r$ is even and as is well known, there exists a base $(u_1, u_2, \ldots, u_n)$ such that the matrix $(h(u_i, u_j))$ is

$$s = \operatorname{diag}\{Q, Q, \ldots, Q\} \quad \text{where} \quad Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11}$$

Then (End V, J) is isomorphic to (Φ_n, K) where K is $(\alpha) \to S^{-1}(\alpha)^t S$. *2. Standard and canonical involutions in matrix rings.* Let

$(\mathscr{O}, j)$ be an associative ring with involution and let $\mathscr{O}_n$ be the ring of $n \times n$ matrices with entries in $\mathscr{O}$. As usual, we denote by e_{ij} the matrix whose (i, j) entry is 1 and other entries are 0 and we identify $\mathscr{O}$ with the set of scalar matrices $d = \operatorname{diag}\{d, \ldots, d\}$, $d \in \mathscr{O}$. Then $de_{ij} = e_{ij}d$ and every element of $\mathscr{O}_n$ can be written in one and only one way as $\sum d_{i_j} e_{i_j}$, $d_{i_j} \in \mathscr{O}$. Also we have the multiplication table

$$e_{ij} e_{kl} = \delta_{jk} e_{il} \tag{12}$$

11 and

$$\sum_1^n e_{ii} = 1. \tag{13}$$

Write $\overline{d} = d^j$ and consider the mapping $J_l : D = (d_{ij}) \to \overline{D}^t$. As is well-known and redily verified, J_1 is an involution in $\mathscr{O}_n$. We shall call this the *standard involution* (associated with j) in $\mathscr{O}_n$. More generally, let $C = \operatorname{diag}\{c_1, c_2, \ldots, c_n\}$ be a diagonal matrix with inverteble diagonal element $c_i = \overline{c}_i, C^{-1} = \operatorname{diag}\{c_1^{-1}, c_2^{-1}, \ldots, c_n^{-1}\}$. The the mapping

$$J_C : D \to C\overline{D}^t C^{-1} = CD^{J_1} C^{-1} \tag{14}$$

is an involution. We shall call such an involution a *canonical involution* (associated with j).

We shall now show that we have the following matrix form of the determination of simple artinian rings with involution given in §1.

V. Any artinian simple ring with involution is isomorphic to a matrix ring with canonical involution $(\mathscr{O}_n, J_c)$ where $\mathscr{O}$ is of one of the following types:

1. $\mathscr{O} = \Delta \oplus \Delta^\circ$, Δ a division ring, j the *exchange* involution $(a, b) \to (b, a)$.
2. $\mathscr{O} = \Delta$ a division ring, j an involution in Δ.
3. $\Delta = \Phi_2$ the ring of 2×2 matrices over a field Φ the involution $X \to Q^{-1}X^tQ$ where $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0. \end{pmatrix}$

The first possibility we noted for an artinian simple ring with involution is $\Delta_n \oplus \Delta_n^J$ are ideals Let $\mathscr{O} = \Delta \oplus \Delta^\circ$ and consider the matrix ring $\mathscr{O}_n$ with the standard involution J_l determined by the involution $j : (a, b) \to (b, a)$ in $\mathscr{O}$. Let $(a_{ij}), (b_{ij}) \in \Delta_n$ and consider the element $(a_{ij}) + (v_{ij})^J$ of $\Delta_n \oplus \Delta_n^J$. We map this into the element of $\mathscr{O}_n$ whose (i, j) entry is (a_{ij}, b_{ji}). Then direct verification shows that this mapping is an isomorphism of $(\Delta_n \oplus \Delta_n^J, J)$ onto $(\mathscr{O}_n, J_1)$. Thus these artinian simple rings with involution are matrix rings with standard involution with coefficient rings of the form 1 above. **12**

It remains to consider the simple artinian rings with involution $(\mathfrak{a}, J)$ such that $\mathfrak{a}$ is simple. The considerations of the last part of §1 show that such an $(\mathfrak{a}, J)$ is isomorphic either to a (Δ_n, J_c) where (Δ, J) is a division ring with involution and J_c is a corresponding canonical involution or to (Φ_{2r}, J_S) where J_S is the involution $A \to S^{-1}A^tS, S = \text{diag}\{Q, Q, \ldots, Q\}, Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The first possibility is the case 2 listed above. Suppose we have the second possibility. We consider the standard isomorphism of Φ_{2r} onto $(\Phi_2)_r$ which maps a $2r \times 2r$ matrix onto the corresponding $r \times r$ matrix of 2×2 blocks. It is easy to check that this is an isomorphism of (Φ_{2r}, J_S) onto $((\Phi_2)_r, J_l)$ where J_1 is the standard involution based on the involution $X \to Q^{-1}X^{t^l}Q$ in Φ_2. Thus we have the case 3.

13 We shall show later (Theorem of Herstein-Kleinfold-Osborn-McCrimmon, Chap.III) that the three possibilities for the coefficient ring $(\mathscr{O}, J)$ we noted have a uniform characterization as the simple rings with involution whose non-zero symmetric elements are invertible. It is easy to check that the rings with involution listed in $1, 2, 3$, have these properties. The converse of V is clear by re-tracing the steps.

We remark finally that the considerations of matrix rings with involutions can be generalized to the case in which the coefficient ring is not necessarily associative. If $(\mathscr{O}, J)$ is such a ring then the matrix ring $\mathscr{O}_n$ has the standard involution $D \to \overline{D}^t$. Also the notion of canonical involution J_c can be generalized. Here it is required that the entries c_i of the diagonal matrix C are the nucleous of $\mathscr{O}$, that is, there associate with all pairs of elelments a, b of $\mathscr{O}((a, b)c = a(bc), (ac)b = a(cb), (ca)b = c(ab))$.

Chapter 1

Basic Concepts

In this chapter we give the basic definitions and general results for 14
quadratic Jordan algebras. These algebras are Φ-modules for a commutative ringΦ equipped with a multiplicative composition which is linear in one of the variables and quadratic in the other. If the base ring Φ contains $\frac{1}{\mathscr{L}}$ then the notion of aquadratic Jordan algebra is equivalent to the usual notion of a (linear) Jordan algebra (see §4). The results of this chapter parallel those of Chapter I and a part of Chapter II of the author's book [4].

1 Special Jordan and quadratic Jordan algebras

It will be convenient from now on to deal with algebras over a (unital) commutative ring Φ. An associative algebra $\mathfrak{a}$ over Φ is a left (unital) Φ-module together with a product xy which is Φ-bilinear and associative. The results of chapter 0 carry over without change to algebras. We remark that rings are just algebras over $\Phi = \mathbb{Z}$ the ring of integers.

Let $(\mathfrak{a}, J)$ be an associative algebra with involution and let $\mathscr{H}(\mathfrak{a}, J)$ denote the subset of symmetric elements ($a^J = a$) of. It is clear that $\mathscr{H}(\mathfrak{a}, J)$ is a Φ-submodule. What other closure proportions does $\mathscr{H}(\mathfrak{a}, J)$ have? Clearly if $a \in \mathscr{H}(\mathfrak{a}, J)$ and $n = 1, 2, 3, \ldots$ then $a^n \in \mathscr{H} = \mathscr{H}(\mathfrak{a}, J)$. In particular, $a^2 \in \mathscr{H}$ and hence $ab + ba = (a + b)^2 - a^2 - b^2 \in \mathscr{H}$ if $a, b \in \mathscr{H}$. We note also that if, $a, b \in H$ then $aba \in \mathscr{H}$. We now observe that this last fact implies all the others since $\mathscr{H} isa \Phi$-module

and contains $1 \in \mathfrak{a}$. For, let $\mathscr{J}$ be any Φ-submodule of $\mathfrak{a}$ containing 1
15 and aba for every $a, b \in \mathscr{J}$. Then $\mathscr{J}$ contains $abc + cba = (a + c) - aba - cba, a, b, c$ in $\mathscr{J}$. Hence contains $ab + ba = ab1 + 1ba$. Also $\mathscr{J}$ contains $a^2 = a1a, a^3 = aaa$ and $a^n = aa^{n-2}a, n \geq 4$. In view of this it is natural to consider aba as the primary composition in $\mathscr{H}$ besides the module composition and the property that $\mathscr{H}$ contains 1.

There is one serious drawback in using the composition aba, namely, this is quadratic in a. It is considerably easier to deal with bilinear compositions. We now note that if Φ contains an element $\frac{1}{2}$ such that $\frac{1}{2} + \frac{1}{2} = 1$ (necessarily unique)then we can replace aba by the bilinear product $a \cdot b = \frac{1}{2}(ab + ba)$. More precisely, let $\mathscr{J}$ be a Φ-submodule of the associative algebra $\mathfrak{a}$ such that $1 \in \mathscr{J}$. Then $\mathscr{J}$ is closed under $a.b$ if and only if it closed under aba. We have see that if $\mathscr{J}$ is closed under aba then it is closed under $ab + ba$, hence, under $a \cdot b = \frac{1}{2}(ab + ba)$. Conversely, if $\mathscr{J}$ is closed under $a \cdot b$ then it is closed under aba since

$$2(a, b), a = \frac{1}{2}(ba^2 + a^2 b) + aba$$

so

$$2(b, a) \cdot a - b, a^2 = aba. \tag{1}$$

These observations lead us to define (tentatively) a *special quadratic Jordan algebra* $\mathscr{J}$ as a Φ-submodule of an associative algebra $\mathfrak{a}/\Phi$, Φ a commutative associative ring (with 1) containing 1 and aba for $a, b \in \mathscr{J}$. We call $\mathscr{J}$ a *special (linear) Jordan algebra* if Φ contains $\frac{1}{2}$. In this case the closure conditions are equivalent to : $1 \in \mathscr{J}$ and $a \cdot b = \frac{1}{2}(ab + ba) \in \mathscr{J}$ if $a, b \in \mathscr{J}$. We have seen that if $\mathfrak{a}, J)$ is an associative algebra with involution then $\mathscr{H}(\mathfrak{a}, J)$ the set of J-symmetric
16 elements is a special qudratic Jordan algebra. Of course, $\mathfrak{a}$ itself is a special quadratic Jordan algebra. We now give another important example as follows.

Let V be a vector space over a field Φ, Q a quadratic form on V and $C(V, Q)$ the corresponding Clifford algebra. Thus if $T(V)$ is the tensor algebra $\Phi \oplus V \oplus (V \otimes V) \oplus \ldots \oplus V^{(i)} \ldots, V^{(i)} = V \otimes V \otimes \ldots \otimes V$ (i times) with the usual multiplication then $C(V, Q) = T(V)/\overline{k}$ where $\overline{k}$ is the ideal in $T = T(V)$ generated by the elements $x \otimes x - Q(x), x \in V$. It is known that the mapping $\alpha + x \to \alpha + x + \bar{k}$ of $\Phi \oplus V$ into $C = C(V, Q)$ is injective. Hence we may identify $\Phi \oplus V$ with the corresponding subspace of C. Then C is generated by $\Phi \oplus V$ and we have the relation $x^2 = Q(x), x \in V$, in C. We claim that $\mathscr{J} \equiv \Phi + V$ is a special quadratic Jordan algebra in C. Let $a = \alpha + x, b = \beta + y, \alpha, \beta \in V$. Then $aba = (\alpha + x)(\beta + y)(\alpha + x) = \alpha^2\beta + 2\alpha\beta x + \alpha^2 y + \alpha(xy + yx) + \beta x^2 + xyx$. Now $x^2 = Q(x)$ gives $xy + yx = (x+y)^2 - x^2 y^2 = Q(x, y)$ where $A(x, y) = Q(x+y) - Q(x) - Q(y)$ is the symmetric bilinear form associated with Q. Hence $xyx = -yx^2 + Q(x, y)x$ and

$$aba = (\alpha^2\beta + \alpha Q(x, y) + \beta Q(x)) + (2\alpha\beta + Q(x, y))x + (\alpha^2 - Q(x))y \tag{2}$$

$\in \mathscr{J} = \Phi + V$. Since $\mathscr{J} \oplus 1$ and is a subspace of C/Φ it is clear that $\mathscr{J}$ is a special quadratic Jordan algebra.

2 Definition of Jordan and quadratic Jordan algebras

These notions arise in studying the properties of the compositions $a, b = \frac{1}{2}(ab + ba)$ and aba is an associative algebra, over Φ where in the first case $\Phi \ni \frac{1}{2}$. We note that $a \cdot b = b \cdot a$ and if $a^2 = a.a$ then $(a^2, b) \cdot a = \frac{1}{4}[(a^2 b + ba^2)a + a(a^2 b + ba^2)] = \frac{1}{4}(a^2 ba + aba^2 + ba^3 + a6^3 b), a^2 \cdot (b \cdot a) = \frac{1}{4}a^2(ab + ba) + (ab + ba)a^2 = \frac{1}{4}(a^3 b + ba^3 + a^2 ba + aba^2)$. These observations and the fac, which can be verified by experimentation, that other simple identities on $a \cdot b$ are consequences of $a \cdot b = b \cdot a$ and $(a^2 \cdot b) \cdot a = a^2 \cdot (b \cdot a)$ lead to the following 17

Definition 1′. An algebra $\mathscr{J}$ over Φ is called a *(unital linear) Jordan algebra* if 1)Φ contains $\frac{1}{2}$, 2)$\mathscr{J}$ contains an element 1 such that $a \cdot 1 = a = 1 \cdot a$, $a \in \mathscr{J}$, 3) the product $a \cdot b$, satisfies $a \cdot b = b \cdot a$, $(a^2 \cdot b) \cdot a = a^2 \cdot (b \cdot a)$ where $a^2 = a \cdot a$.

It is clear that if $\mathscr{J}$ is a special Jordan algebra then $\mathscr{J}$ is a Jordan algebra with $a \cdot b = \frac{1}{2}(ab + ba)$. If $\mathfrak{a}$ is an associative algebra over $\Phi \ni \frac{1}{2}$ then $\mathfrak{a}$ defines the Jordan algebra $\mathfrak{a}^+$ whose underlying Φ-module is $\mathfrak{a}$ and whose multiplication composition is $a \cdot b = \frac{1}{2}(ab + ba)$.

If $\mathscr{J}$ is Jordan we denote the mapping $x \to x \cdot a$ by R_a. This is a Φ-endomorphism of $\mathscr{J}$. We can formulate the Jordan conditions on $a \cdot b$ in terms of R_a and this will give our preferred definition of a Jordan algebra as follows:

Definition 1. A*(unital linear)Jordan algebra* over a commutative ring Φ (with 1)containing $\frac{1}{2}$ is a triple $(\mathscr{J}, R, 1)$ such that $\mathscr{J}$ is a (unital) left Φ-module, R is a mapping of $\mathscr{J}$ into End $\mathscr{J}$ (the associative Φ-algebra of endomorphisms of $\mathscr{J}$) such that

18 J1 $R : a \to R_a$ is a Φ-homomorphism.

J2 $R_1 = 1$

J3 $R_a R_{aR_a} = R_{aR_a} R_a$

J4 If L_a is defined by $xL_a = aR_x$ then $L_a = R_a$.

Definitions 1 and 1′ are equivalent: If $\mathscr{J}$ is Jordan in the sense of Defintion 1′ then we define R_a as $x \to x \cdot a$ and obtain $J_l - J_4$ of definition 1. Conversely, if $\mathscr{J}$ is Jordan in the sense of definition 1 then we define $a \cdot b = aR_b$. Then the conditions of Definition 1′ hold. Moreover, the passage from the bilinear composition $a \cdot b$ to the mapping R is the inverse of that from R to $a \cdot b$.

Let $\mathscr{J}$ be Jordan in the sense of the second definition and write $a \cdot b = aR_b = bR_a, a^2 = a \cdot a$. Then $[R_a R_{a^2}] = 0$ where $[AB] = AB - BA$ for the Φ-endomorphisms A, B. Let $(a) = [R_a, R_{a^2}]$ and consider the identity

$$0 = f(a+b+c) - f(a+b) - f(b+c) - f(a+c) + f(a) + f(b) + f(c).$$

This gives

$$\begin{aligned}[R_a R_{b\cdot c}] + [R_a R_{c\cdot b}] + [R_b R_{a\cdot c}] + [R_b R_{c\cdot a}] \\ + [R_c R_{a\cdot b}] + [R_c R_{b\cdot a}] = 0,\end{aligned}$$

as is readily checked. Since $a \cdot b = b \cdot a$ we get $2[R_a R_{b\cdot c}] + 2[R_b R_{a\cdot c}] +$
19 $2[R_c R_{a\cdot b}] = 0$. Since Φ contains $\frac{1}{2}$ we obtain

J5 $[R_a R_{b\cdot c}] + [R_b R_{a\cdot c}] + [R_c R_{a\cdot b}] = 0.$

Let $\underline{\rho}$ be a commutative associative algebra over Φ (= commutative associative ring extension of Φ). If m is a Φ-module we write $m_{\underline{\rho}} = \underline{\rho} \otimes_\Phi m$ regarded as (left unital)$\underline{\rho}$-module in the usual way. We have the Φ -homorphism $\nu : x \to 1 \otimes x$ of m into $m_{\underline{\rho}}$ as Φ-module. In the cases in which this is injective we shall identify x and $1 \otimes x$ and m and its image $1 \otimes m(= m^\nu)$. In any case $1 \otimes m$ generates $m_{\underline{\rho}}$ as $\underline{\rho}$-module. If n is a second Φ-module and η is a homorphism of m into n then there exists a unique homorphism $\eta_{\underline{\rho}}$ of $m_{\underline{\rho}}$ into $n_{\underline{\rho}}$ such that

$$\begin{array}{ccc} m & \xrightarrow{\eta} & n \\ \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\nu} \\ m_{\underline{\rho}} & \xrightarrow[\eta_{\underline{\rho}}]{} & n_{\underline{\rho}} \end{array} \tag{3}$$

is commutative. It follows that if γ End m and $\tilde{\gamma}$ denotes the resultant of ν: End $m \to$(End $m)_{\underline{\rho}}$ and the canonical mapping of (End $m)_{\underline{\rho}}$ into End $m_{\underline{\rho}}$. Then we have a unique homomorphism $\widetilde{\eta}$ of $m_{\underline{\rho}}$ into End $m_{\underline{\rho}}$ such that

$$\begin{array}{ccc} m & \xrightarrow{\eta} & \text{End } m \\ \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\widetilde{\nu}} \\ m_{\underline{\rho}} & \xrightarrow[\widetilde{\eta}]{} & \text{End } m_{\underline{\rho}} \end{array} \tag{4}$$

is commutative. 20

Now supppose $\Phi \ni \frac{1}{2}$ and $(\mathscr{J}, R, 1)$ is a Jordan algebra over Φ. Let $\widetilde{R}$ be homomorphism of $\mathscr{J}_{\underline{\rho}}$ into End $\mathscr{J}_{\underline{\rho}}$ determined as in (4) by R and put $\widetilde{1} = 1 \otimes 1$. If we use the definition of $\mathscr{J}$ $J5$, and the fact that $1 \otimes \mathscr{J}$ generates $\mathscr{J}_{\underline{\rho}}$ it is straight forward to check that $(\mathscr{J}_{\underline{\rho}}, \widetilde{R}, \widetilde{1})$ is a Jordan algebra.

We formulate next the notion of a (unital) quadratic Jordan algebra. This is arrived at by considering the properties of the product *aba* in an associative algebra or, equivalently, the mapping $U_a : x \to a \times a$. Note that $U_a \in$ End $\mathfrak{a}$ where $\mathfrak{a}$ the given associate algebra. Also $U : a \to U_a$ is quadratic is in the sense of the following

Definition 2. Let $\mathfrak{m}$ and $\mathfrak{n}$ be left (unital) Φ-modules, Φ-modules, Φ an arbitrary (unital) commutative associative ring. Then a mapping $Q : a \to Q(a)$ (or Q_a) of $\mathfrak{n}$ into $\mathfrak{m}$ is called *quadratic* if 1)$Q(\alpha a) =$
21 $\alpha^2 Q(a), \alpha \in \Phi, a \in \mathfrak{m}$, 2)$Q(a,b) \equiv Q(a+b) - Q(a) - Q(b)$ is Φ -bilinear from $\mathfrak{m}$ to $\mathfrak{n}$. The *kernel* of Q is the set of z such that $Q(z) = 0 = Q(a,z), a \in \mathfrak{m}$.

The associated Φ-bilinear mapping $Q(a,b)$ is symmetric: $Q(a,b) = Q(b,a)$. The kernel ker Q is a submodule. If Q and Q' are quadratic mappings of $\mathfrak{m}$ into $\mathfrak{n}$ then so is $Q + Q'$ and $\beta Q, \beta \in \Phi$. Hence the set of these mappings is a Φ-module. The resultant of a quadratic mapping and a Φ -homomorphism and of a Φ -homomorphism and a quadratic mapping is a quadratic mapping. If Q is a quadratic mapping of $\mathfrak{m}$ into $\mathfrak{n}$ and R is contained in ker Q then $Q(a+R) = Q(a)$ defines a quadratic mapping of $\overline{\mathfrak{m}} = \mathfrak{m}/R$ into $\mathfrak{n}$. If Q and Q' are quadratic mappings and $Q(a_i) = Q'(a_i), Q(a_i, a_j) = Q'(a_i, a_j)$ for all a_i, a_j on a set of generators $\{a_i\}$ then $Q = Q'$. In particular, if $Q(a_i) = 0$, $A(a_i, a_j) = 0$ then $Q = 0$. Let $\mathscr{F}$ be a free left module with base $\{x_i | i \in I\}$ and let $i \to b_i, \{i, j\} \to b_{ij}$ be mappings of the index set I and of the set I_2 of distinct unordered paris of elements $i_{1j}, i_{1j} I$ into n.

If $x \in \mathscr{F}$ and $x = \sum \xi_i x_i$ (finite sum) then we define $Q(x) = \sum \xi_i^2 b_i + \sum\limits_{i<j} \xi_i \xi_j b_{ij}$. Then it is easy to check that Q is a quadratic mapping of $\mathscr{F}$ into n. It $\mathscr{F}$ is free with base $\{x_i\}$ and $\underline{\rho}$ is a commutative associative algebra over Φ then $\mathscr{F}_{\underline{\rho}}$ is free with base $\{1 \otimes x_i\}$. It follows the remark just made that the following lemma holds for $m = \mathscr{F}$ free:

22 **Lemma .** *Let Q be a quadratic mapping of $\mathfrak{m}$ into $\mathfrak{n}$ where these are left modules over Φ and let $\underline{\rho}$ be an associative commutative ring extension of Φ. Then exists a unique quadratic mappings $Q_{\underline{\rho}}$ of $\mathfrak{m}_{\underline{\rho}}$ into $\mathfrak{n}_{\underline{\rho}}$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{Q} & \mathfrak{n} \\ \nu \downarrow & & \downarrow \nu \\ \mathfrak{m}_{\underline{\rho}} & \xrightarrow[Q_{\underline{\rho}}]{} & \mathfrak{n} \end{array} \tag{5}$$

Proof. Let $\mathscr{F} \xrightarrow{\eta} \mathfrak{m} \to 0$ be an exact sequence of modules where $\mathscr{F}$ is free and put $\mathfrak{K} = \ker \eta$ the kernel of η. Then we have the corresponding homomorphism $\eta_{\underline{\rho}}$ of $\mathscr{F}_{\underline{\rho}}$ onto $m_{\underline{\rho}}$ (as in (3)) and, as is well-known, $\ker \eta_{\underline{\rho}} = \underline{\rho}(1 \otimes \mathfrak{K})$ the $\underline{\rho}$ -submodule generated by $1 \otimes \mathfrak{K} = \{1 \otimes k | k \in \mathfrak{K}\}$. We have the isomorphism $\widetilde{x} + \underline{\rho}(1 \otimes \mathfrak{K}) \to \widetilde{x}^{\eta_{\underline{\rho}}}$ of $\mathscr{F}/\underline{\rho}(1 \otimes \overline{k})$ onto $m_{\underline{\rho}}$. We define Q^{η} of $\mathscr{F}$ to η by $\overline{Q}(x) = Q(x^{\eta}), x \in \mathscr{F}$. Since this is the resultant of η and Q it is aquadratic mapping. Also $\ker Q^{\eta} \supseteq \mathfrak{K}$. Since $\mathscr{F}_{\underline{\rho}}$ is $\underline{\rho}$ -free Q^{η} determines the quadratic mapping $Q^{\eta}_{\underline{\rho}}$ of $\mathscr{F}$ into η so that (5) is commutative for $m = \mathscr{F}$. We have $\ker Q^{\eta}_{\underline{\rho}} \supseteq \underline{\rho}(1 \otimes \mathfrak{K})$. Hence $\widetilde{x} + \underline{\rho}(1 \otimes \mathfrak{K}) \to Q^{\eta}_{\underline{\rho}}(\widetilde{x})$ is a quadratic mapping of $\mathscr{F}_{\underline{\rho}}/\underline{\rho}(1 \otimes \mathfrak{K})$ into $n_{\underline{\rho}}$. Using the isomorphism of $\mathscr{F}_{\underline{\rho}}/\underline{\rho}(1 \otimes \mathfrak{K})$ and $m_{\underline{\rho}}$ this can be transferred to the quadratic mapping $Q_{\underline{\rho}} : \overline{x}^{\eta_{\underline{\rho}}} \to Q_{\underline{\rho}}(\overline{x})$ of $m_{\underline{\rho}}$ into n. If $x \in \mathscr{F}$ then $1 \otimes x^{\eta} = (1 \otimes x)^{\eta_{\underline{\rho}}} \to Q^{\eta}_{\underline{\rho}}(1 \otimes x) = 1 \otimes Q^{\eta}(x) = 1 \otimes Q(x^{\eta})$. Hence $Q_{\underline{\rho}}$ satisfies the commutativity in (5). The uniqueness of $Q_{\underline{\rho}}$ is clear since 23
$1 \otimes \mathfrak{m}$ generates $\mathfrak{m}_{\underline{\rho}}$.

Let $\mathfrak{n} = \operatorname{End} \mathfrak{m}$. Then it is immediate from the lemma that if Q is a quadratic mapping of $\mathfrak{m}$ into End $\mathfrak{m}$ then there exists a unique quadratic mapping $\widetilde{Q}$ of $\mathfrak{m}_{\underline{\rho}}$ into End $\mathfrak{m}_{\underline{\rho}}$ such that commutativiy holds in:

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{Q} & \operatorname{End}\mathfrak{m} \\ \nu\downarrow & & \downarrow\widetilde{\nu} \\ \mathfrak{m}_{\underline{\rho}} & \xrightarrow[\widetilde{Q}]{} & \operatorname{End}\mathfrak{m}_{\underline{\rho}} \end{array} \tag{6}$$

where $\tilde{nu}$ is $A \to 1 \otimes A$ and $(\rho \times A) = \rho \otimes xA$.

We are now ready to define a quadratic Jordan algebra. We have two objectives in mind: first, to give simple axioms which will be adequate for studying the composition $axa = xU_a$ in associative algebras and second, to characterize the mapping $U_a \equiv 2R_a^2 - R_{a^2}$ in Jordan algebras. We recall that in an associative algebra $xU_a = x(2R_a^2 - R_{a^2})$. The following definiion is due to McCrimmon [5]. □

Definition 3. A *(unital) quadratic Jordan algebra* over a commutative associative ring Φ (with 1) is a triple $(\mathscr{J}, U, 1)$ where $\mathscr{J}$ is a (unital) left Φ -module, 1 a distinguished element of $\mathscr{J}$ and U is a mapping of $\mathscr{J}$ into End $\mathscr{J}$ such that

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QJ1 U is quadratic

QJ2 $U_1 = 1$

QJ3 $U_a U_b U_a = U_b U_a$

QJ4 If $U_{a,b} = U_{a+b} - U_a - U_b$ and $V_{a,b}$ is defined by $xV_{a,b} = aU_{x,b}$ then $U_b V_{a,b} = V_{b,a} U_b$.

QJ5 If $\underline{\rho}$ is any commuatative associative algebra over Φ and $\widetilde{U}$ is the quadratic mapping of $\mathscr{J}_{\underline{\rho}}$ into End $\mathscr{J}_{\underline{\rho}}$ as in (6), then $\widetilde{U}$ satisfies $QJ3$ and 5.

It is clear from QJ_5 that $(\mathscr{J}_{ub\rho}, \widetilde{U}, \widetilde{1}), \widetilde{1} = 1 \otimes 1$ is a quadratic Jordan algebra over $\underline{\rho}$. We remark also that QJ_4 states that $bU_{a,x}U_b = aU_{b,xU_b} \cdot$. Since the left side is symmetric in a and so is the right. Hence $aU_{b,xU_b} = xU_{b,aU_b} \cdot$. This gives the following addendum to QJ_4:

$$U_b V_{a,b} = V_{b,a} U_b = U_{aU_b,b} \cdot \qquad QJ4'$$

Let $\mathfrak{a}$ be an associative algebra over Φ and define U_a to be $x \to axa$. Then $U_a \in$ End $\mathfrak{a}$ and $QJ1$ and $QJ2$ are evidently satisfied since $U_{a,b}$ is the mapping $x \to axb + bxa$. We have $xU_a U_b U_a = a(b(axa)b)a$ and $xU_{bU_a} = xU_{aba} = (aba)x(aba)$ so $QJ3$ holds by the associative law. Now $xV_{a.b} = aU_{x,b} = xab + bax$. Hence $xU_b V_{a,b} = b(xba + abx)b =$
25 $bxbab + babxb$ and $V_{b,a}U_b = b(xba + abx)b = bxbab + babxb$. Thus $QJ4$ holds. Now $QJ5$ is clear since $\mathfrak{a}_{\underline{\rho}}$ is an associative algebra and the mapping $\widetilde{U}$ of $\mathfrak{a}_{\underline{\rho}}$ is $\widetilde{a} \to \widetilde{U}_a$ where $xU_{\widetilde{a}} = \tilde{a}\,\tilde{x}\,\tilde{a}$. We denote $(\mathfrak{a}, U, 1)$ by $\mathfrak{a}$.

If $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra, a *subalgebra* $\mathscr{B}$ of $\mathscr{J}$ is a Φ-submodule containing 1 and every $aU_b, a, b \in \mathscr{B}$, *a homomorphism* η of $\mathscr{J}$ into a second quadratic Jordan algebra is a module homomorphism such that $1^\eta = 1, (aU_b)^\eta = a^\eta U_b^\eta$. Monomorphism, isomorphism, automorphism are defined in the obvious way. A quadratic Jordan algebra $\mathscr{J}$ will be called *special* if there exists a monomorphism of $\mathscr{J}$ into algebra $\mathfrak{a}(q)$, $\mathfrak{a}$ associative. It is immediate that this is essentially the same definition we gave before.

If $\mathfrak{a}$ is an associative algebra over $\Phi \ni \frac{1}{2}$ then we can form the Jordan algebra $\mathfrak{a}^+$ and the quadratic Jordan algebra $\mathfrak{a}(q)$. For $\mathfrak{a}^+$, $xR_a = \frac{1}{2}(a+a)$ and for $a^{(q)}$, $xU_a = axa$. The relation (1) : $2(x,a), a - xa^2 = axa$ shows that

$$U_a = 2R_a^2 - R_{a^2} \tag{7}$$

is the formula expressing U in terms of R. Conversely, we can express R in terms of U by noting that $U_{a,b} = U_{a+b} - U_a - U_b$ is $x \to axb + bxa$ so $V_a \equiv U_{a,1} = U_{1,a}$ is $x \to ax + xa$. Hencec

$$R_a = \frac{1}{2}V_a, V_a = U_{a,1} = U_{1,a}. \tag{8}$$

Now let $(\mathscr{J}, R, 1)$ be any Jordan algebra (over $\Phi \ni \frac{1}{2}$) and define U **26**
by (7). Then we claim that $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra. It is clear that $a \to U_a = 2R_a^2 - R_{a^2}$ is quadratic in a and $R_1 = 1$ gives $U_1 = 1$. Also, since $\mathscr{J}_{\underline{\rho}}$ is Jordan for any commutative associative algebra $\underline{\rho}$ over Φ, it is enough to prove that $QJ3$ and $QJ4$ hold. We need to recall some basic identities, namely,

J6 $R_aR_bR_c + R_cR_bR_a + R_{(a\cdot c)\cdot b} = R_{a\cdot b}R_c + R_{b\cdot c}R_a + R_{c\cdot a}R_c$

J7 $[R_c[R_aR_b]] = R_{c[R_aR_b]}$.

The first of these is obtained by writing $J5$ in element form: $(d\cdot a), (b\cdot c) - (d\cdot(b\cdot c))a$+etc., interchanging b and d and re-interpreting this as operation identity. This gives $R_aR_bR_c + R_cR_bR_a + R_{(a\cdot c)\cdot b} = R_aR_{b\cdot c} + R_bR_{a\cdot c} + R_cR_{a\cdot b}$. This and $J5$ give $J6$. To obtain $J7$ we interchange a and b in $J6$ and subtract the resulting relations frm $J6$. Special cases of $J5$ and $J6$ are

$J5'$ $[R_{a^2}R_b] + 2[R_{a,b}R_a] = 0$

$J6'$ $R_a^2R_b + R_bR_a^2 + R_{(a,b),b} = R_{a^2}R_b + 2R_{a,b}R_a$.

using (7) we obtain $U_{a,b} = 2(R_aR_b + R_bR_a) - 2R_{a,b}$. Then $xV_{a,b} = aU_{x,b}$ gives

$$V_{a,b} = 2(R_aR_b - R_bR_a + R_{a\cdot b}) \tag{9}$$

27 We shall now prove $QJ4$, which is equivalent to:

$$(2R_a^2 - R_{a^2})(R_{a\cdot b} + R_bR_a - R_aR_b) - (R_{a\cdot b} + R_aR_b - R_bR_a)(2R_a^2 - R_{a^2}) = 0$$

The left hand side of this after a little juggling becomes

$$\begin{aligned}
&2[R_a^2R_b + R_bR_a^2, R_a] + 2R_a[R_a + R_{a\cdot b}] + 2[R_aR_{a\cdot b}]R_a \\
&-[R_{a^2}R_b]R_a + R_a[R_bR_{a^2}] + [R_aR_{a^2} + R_{a^2}R_a, R_b] \\
&\quad + [R_{a\cdot ba^2}] \\
&= 2[R_a^2R_b + R_bR_a^2, R_a] + [R_aR_{a^2} + R_{a^2}R_aR_b] \\
&\quad + [R_{a\cdot b}R_{a^2}] \cdot (\text{by} J5') \\
&= 2[R_{a^2}R_b + 2R_{a\cdot b}R_a - R_{(a\cdot b)\cdot a'}R_a] + 2[R_aR_{a^2}, R_b] \\
&\quad + 2[R_{(a\cdot b)\cdot a}, R_a](J6' \text{ and } J5' \text{ with } b \to a\cdot b) \\
&= 4[R_{a\cdot b}R_a]R_a + 2[R_{a^2}R_b]R_a = 0(J5').
\end{aligned}$$

Hence $QJ4$ holds.

28 For the proof of $QJ3$ we begin with the following identity

J8. $[V_{a,b}V_{c,d}]V_{a,bV_{c,d}} - V_{aV_{d,c}}, b$

(cf. the author's book [2], (5) on p.325). To derive this we note that $J7$ shows that $[R_aR_b]$ is a derivation in $\mathscr{J}$. For any derivation D we have directly : $[V_{a,b}D] = V_{aD,b} + V_{a,bD}$. Also $[V_{a,b}R_c] = V_{a,bR_c} - V_{aR_c,b}$ follows directly from $J5$ and $J7$. Then $J9$, is a consequence of these two relations. We note next that the left hand side of $J8$ is skew in the pairs $(a, b), (c, d)$. Hence we have the consequence

$J9$ $V_{a,bV_{c,d}} - V_{aV_{d,c},b} = V_{cV_{b,a}}, d - V_{c,dV_{a,b}}$.

We now use the formula $xV_{a,b} = aU_{x,b}$ defining $V_{a,b}$ to write $J8$ and $J9$ in the following equivalent forms:

$J8'$ $U_{aU_{c,b},d} - U_{c,d}V_{a,b} = U_{b,d}V_{a,c} - V_{d,a}U - c, b$

$J9'$ $V_{c,d}V_{a,b} - V_{dU_{a,c},b} = V_{a,c}U_{b,d} - U_{a,d}V_{c,d}$.

Taking $d = aU_b, c = b$ in $J8'$ gives

$$2U_{aU_b} = U_{b,aU_b}V_{a,b} - V_{aU_b,a}U_b \tag{10}$$

Replacing $a \to b$, $c \to b$,$b \to a$, $d \to a$ in $J9'$ gives **29**

$$V_{aU_{b,a}} = V_{b,a}^2 - 2U_bU_b \tag{11}$$

If we substitute this in the last term of (10) we get $2U_{aU_b} = U_{b,a}U_b$ $V_{a,b} - V_{b,a}^2U_b + 2U_bU_aU_b$. Since $QJ4$ has the consequence $QJ4'$: $U_bV_{a,b} = V_{b,a}U_b = U_{aU_b}, b$ (as above) the foregoing reduces to $QJ3$[1]

3 Basic identities

In this section we shall derive a long but of identities which will be adequate for the subsequent considerations. No attempt has been made to reduce the set to a minimal one. On the contrary we have tried to list almost every identity which will occur in the sequel.

Let $(\mathscr{J}, U, 1)$ be a quadratic Jordan algebra over Φ. We write $aba = bU_a, abc = bU_{a,c}$ so $b \to aba$ is the Φ -endomorphism U_a for fixed a and $a \to aba$ is a quadratic mapping of $\mathscr{J}$ into itself for fixed b. We put $a^2 = 1U_a, a \circ b = (a+b)^2 - a^2 - b^2 = 1U_{a,b} = aV_{1,b}$ and $V_a = U_{a,1} = U_{1,a}$. We have $a \circ b = b \circ a, a \circ a = 2a^2, U_{a,a} = 2U_a, V_1 = V_{1,1} = 2$. Taking $b = 1$ in $QJ4$ gives $V_{a,1} = V_{1,a}$ so $1U_{a,} = aU_{1,x}$. Then $a \circ x = aV_x$. Since $a \circ x = x \circ a$ we have $xV_a = aV_x$: Also **30**
$xV_{1,a} = 1U_{a,x} = a \circ x = xV_a$ so $V_a = V_{1,a} = V_{a,1}$. We shall now apply a process of linearization to deduce consequences of $QJ3$ and $QJ4$. This method consists of applying $QJ3$ and $QJ4$ to $\mathscr{J}_{\underline{\rho}} = \Phi[\lambda]$ the polynomial algebra over Φ in the indeterminate λ. Since $\underline{\rho}$ is Φ -free

[1] The proof we have given of $QJ4$ was communicated to us by McCrommon, that of $QJ3$ by Meyberg. The first direct proof of QJ_3 was given by Macdonald. Subsequently he gave a general theorem on identities from which $QJ3$ and $QJ4$ are immediate consequences. See Macdonald [1] and the author's book [2] pp.40-48).

the canonical mapping of $\mathscr{J}$ into $\mathscr{J}_{\underline{\rho}}$ is injective so we may identify $\mathscr{J}$ with its image $1 \otimes \mathscr{J}$ in $\mathscr{J}_{\underline{\rho}}$ and regard $\widetilde{U}$ as the unique extension of U to a quadratic mapping of $\mathscr{J}_{\underline{\rho}}$ into End $\mathscr{J}_{\underline{\rho}}$. We write U for $\widetilde{U}$. The elements of $\mathscr{J}_{\underline{\rho}}$ can be written in one and only one way in the form $a_o + \lambda a_1 + \lambda^2 a_2 + \cdots + \lambda^n a_n, a_i \in \mathscr{J}$, and the endomorphism of $\mathscr{J}_{\underline{\rho}}$ can be written in one and only way as $A_o + \lambda A_1 + \cdots$. where A_i is the endomorphism in $\mathscr{J}_{\underline{\rho}}$ which extends the endomorphism A_i of $\mathscr{J}$. Now let $a, b, c \in \mathscr{J}$ and consider the identity $U_{a+\lambda c} U_b U_{a+\lambda c} = U_{bU_{a+\lambda c}}$ which holds in $\mathscr{J}_{\underline{\rho}}$ by $QJ5$. Comparing coefficients of λ and λ^2 we obtain

QJ6 $U_a U_b U_{a,c} + U_{a,c} U_b U_a = U_{bU_a} = U_{bU_a, bU_{a,c}}$

QJ7 $U_a U_b U_c + U_c U_b U_a + U_{a,c} U_b U_{a,c} = U_{bU_a, bU_c} + U_{b, U_{a,c}}$.

The same method applied to the variable a in $QJ6$ and b in $QJ4$ gives

$$U_a U_b U_{c,d} + U_{a,c} U_b U_{a,d} + U_{c,d} U_b U_a + U_{a,d} U_b U_{a,c}$$

QJ8 $= U_{bU_{a,c}, bU_{a,d}} + U_{bU_a, bU_{c,d}}$

QJ9 $V_{b,a} U_{b,c} + V_{c,a} U_b = U_{b,c} V_{a,b} + U_b V_{a,c}$

31 We remark that comparison of the coefficients of the other powers of λ in the foregoing identities yields identitites which we have already displayed. Also, if the method is applied to a variable in which the identity is quadratic, say $Q(a) = 0$ (e.g. $QJ3$ and the variable a) then we obtain in this way the bilinerizaation $Q(a, b) = Q(a + b) - Q(a) - Q(b) = 0$. We shall usually not display these bilinearizations.[2].

We shall now show that if $(\mathscr{J}, U, 1)$ satisifies $QJ1-4, 6-9$ then $QJ5$ holds, so $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra. Hence $QJ\,1-4, 6-9$ constitute an intrinsic set of conditions defining quadratic Jordan algebras. Let $\underline{\rho}$ be any commutative associative algebra over Φ and consider $\mathscr{J}_{\underline{\rho}}$ where it is assumed that $(\mathscr{J}, U, 1)$ satisfies $QJ1-4, 6-9$. If $a \in \mathscr{J}$ we put $a' = 1 \otimes a$. Then $QJ1 - 4, 6 - 9$ hold in $\mathscr{J}\underline{\rho}$ for all choices of

[2]An exception to this rule in $QJ8$ which is the bilinerization of $QJ7$ with respecet to c

the arguments in $\mathscr{J}' = 1 \otimes \mathscr{J}$. Also the bilinearizations of these conditions hold for all values of the argument in $\mathscr{J}'$. Since $QJ7 - QJ9$ are either Φ-linear (= Φ. endomorphisms) or Φ -quadratic in their arguments these hold for all choices of the arguments in $\mathscr{J}_{\underline{\rho}}$. Similarly, $QJ6$ holds for all $a \in \mathscr{J}'$ and all b, c in $\mathscr{J}_{\underline{\rho}}$. The validity of $QJ8$ in $\mathscr{J}_{\underline{\rho}}$ now implies that if $QJ6$ holds for the arguments $(a = a_1, b, c)$ and (a_2, b, c) in $\mathscr{J}_{\underline{\rho}}$ then it holds for $(a = a_1 + \rho a_2, b, c)$ for any $\rho \in \underline{\rho}$.

It follows from this that $QJ6$ holds in $\mathscr{J}_{\underline{\rho}}$. A similar argument using 32
$QJ9$ shows that $QJ4$ holds in $\mathscr{J}_{\underline{\rho}}$. Similarly, using $QJ6$ and $QJ7$ in $\mathscr{J}_{\underline{\rho}}$ one sees that if $QJ3$ holds for b in $\mathscr{J}_{\underline{\rho}}$ and $a = a_1, a = a_2$ in $\mathscr{J}_{\underline{\rho}}$ then it holds for b and $a = a_1 + \rho a_2$. It follows that $QJ3$ holds in $\mathscr{J}_{\underline{\rho}}$. We have therefore proved.

Theorem 1. *Let $\mathscr{J}$ be a left Φ -module, U a mapping of $\mathscr{J}$ into End $\mathscr{J}$ satisfying $QJ1 - QJ4, QJ6 - QJ9$. Then $QJ5$ holds so $\mathscr{J}$ is a quadratic Jordan algebra.*

The same argument implies the following result

Theorem 2. *Let $\mathscr{J}$ be a left Φ -module, U a quadratic mapping of $\mathscr{J}$ into End $\mathscr{J}$ such that $U_1 = 1$ and $QJ3, 4, 6-9$ and all their bilinearizations hold for all choices of the arguments in a set of generators of the Φ -module $\mathscr{J}$. Then $\mathscr{J}$ is a quadratic Jordan algebra.*

It is easy to prove by a Vandermonde determinant argument that if Φ is a field of cardinality $|\Phi| \geq 4$ then $QJ6-9$ follows from $QJ3, 4$ without the intervention of $QJ5$. Hence in this case $QJ1 - 4$ are a defining set of conditions for a quadratic Jordan algebra over Φ.

If we put $b = 1$ in $QJ3, 6$ and 7 we obtain respectively

$$U_a^2 = U_{a^2} \qquad \text{QJ 10}$$

$$U_a U_{a,c} + U_{a,c} U_a = U_{a^2, aoc} \qquad \text{QJ 11}$$

$$U_{a^2, c^2} + U_{aoc} = U_a U_c + U_c U_a + U_{a,c}^2 \qquad \text{QJ 12}$$

If we replace b by $b + 1$ in $QJ3, 6, 7$ and use the foregoing we obtain 33

$$U_a V_b U_a = U_{bU_a, a^2} \qquad \text{QJ13}$$

$$U_aV_bU_{a,c} + U_{a,c}V_bU_a = U_{bU_a,aoc} + U_{a^2,bU_{a,c}} \qquad \text{QJ14}$$

$$U_{a^2,bU_c} + U_{bU_a,c^2} + U_{bU_{a,c},aoc} \qquad \text{QJ15}$$
$$= U_aV_bU_c + U_cV_bU_a + U_{a,c}V_bU_{a,c}$$

Putting $c = 1$ in $QJ6$, $QJ7$ and $QJ9$ gives

$$U_aU_bV_a + V_aU_bU_a = U_{bU_a,boa} \qquad \text{QJ16}$$
$$U_{bU_a,b} + U_{boa} = U_aU_b + U_bU_a + V_aU_bV_a \qquad \text{(QJ17)}$$
$$U_bV_a + V_bV_{a,b} = V_{b,a}V_b + V_aU_b. \qquad \text{QJ18}$$

If we put $a = 1$ in $QJ6$ we get

$$U_bV_c + V_cU_b = U_{b,boc}. \qquad \text{QJ19}$$

Putting $c = 1$ in $QJ12$ and replacing b by $b+1$ in $QJ17$ give respectively:

$$2U_a = V_a^2 - V_{a^2} \qquad \text{QJ20}$$
$$V_{bU_a} + U_{a^2,b} + 2U_{a,aob} = U_aV_b = V_bU_a + V_aV_bV_a. \qquad \text{QJ21}$$

If we apply the two sides of $QJ3$ to 1 we obtain

$$a^2U_bU_a = (bU_a)^2 \qquad \text{QJ22}$$

34 which for $b = 1$ is

$$a^2U_a = (a^2)^2 \qquad \text{QJ23}$$

Next we put $a = 1$ in $QJ4'$ to obtain

$$V_bU_b = U_bV_b = U_{b,b^2} \qquad \text{QJ24}$$

Putting $b = 1$ in $QJ9$ gives

$$V_aV_c + V_{c,a} = V_cV_a + V_{a,c} \qquad \text{QJ25}$$

or

$$(x \circ a) \circ c + cU_{a,} = (x \circ c) \circ a + aU_c, \qquad \text{QJ 25'}$$

putting $x = c$ we obtain

$$\begin{aligned}(c \circ a) \circ c + cU_{a,c} &= (c \circ c) \circ a + aU_{c,c} \\ &= 2c^2 \circ a + 2aU_c\end{aligned}$$

which can be simplified by $QJ20$ to give

$$\{acc\} = c^2 \circ a. \qquad \text{QJ26}$$

It is useful to list also the bilinearization of this:

$$\{abc\} + \{bac\} = (a \circ b) \circ c \qquad \text{QJ 27}$$

which has the operator form **35**

$$V_{b,c} = V_a V_c - U_{b,c} \qquad \text{QJ 27}'$$

If we operate with the two sides of $QJ17$ on 1 we obtain

$$(a \circ b)^2 = a^2 U_b + b^2 U_a + 2aU_b \circ a - bU_a \circ b.$$

Also, if we replace a by $a + \lambda b$ in $QJ24$ and compare coefficients of λ we obtain

$$\begin{aligned}U_{a,b} V_a + U_b V_b &= U_{a,a \circ b} + U_{b,a^2} \\ V_a U_{a,b} &+ V_b U_a. \qquad \text{QJ28}\end{aligned}$$

Applying the first and last of these to b gives

$$\begin{aligned}bU_a \circ b &= -\{abb\} \circ a + \{ab \circ ab\} + 2b^2 U_a \\ &= -b^2 V_a^2 + 2b^2 U_a + \{aa \circ bb\} \qquad \text{QJ 26} \\ &= -b^2 \circ a^2 + \{aa \circ bb\} \qquad \text{QJ 20}\end{aligned}$$

which is symmetric in a and b. Hence we have

$$bU_a \circ b = aU_b \circ a \qquad \text{QJ29}$$

Using this and the foregoing formula for $(a \circ b)^2$ we obtain

$$(a \circ b)^2 = a^2 U_b + b^2 U_a + aU_b \circ a \qquad \text{QJ 30}$$

$$= a^2U_b + b^2U_a + bU_a \circ b$$

We wish to prove next 36

$$V_{aU_b,a} = V_{b,bU_a} \qquad \text{QJ 31}$$

In element form this is $\{caU_ba\} = \{cbbU_a\}$. Using $QJ27$ this is equivalent to $(c \circ aU_b) \circ a - \{aU_bca\} = (c \circ b) \circ bU_a - \{bcbU_a\}$ which is equivalent to

$$V_{aU_b}V_a - U_{aU_b,a} = V_bV_{bU_a} - U_{b,bU_a} \qquad \text{QJ 32}$$

If we interchange a and b in $QJ17$ and subtract we obtain $V_aU_bV_a - V_bU_aV_b = U_{bU_a,b} - U_{aU_b,a}$ which implies that $QJ32$ is equivalent to $V_aU_bV_a - V_bU_aV_b = V_bV_{bU_a} - V_{aU_b}V_a$. Hence it suffics to prove

$$(V_{aU_b} + V_aU_b)V_a = V_b(V_{bU_a} + U_aV_b) \qquad \text{QJ33}$$

We note next that bilinearization of $QJ29$ relative to b gives $bU_a \circ c + cU_a \circ b = aU_{b,c} \circ a = cV_{a,b} \circ a$. hence

$$V_{bU_a} + U_aV_b = V_{a,b}V_a \qquad \text{QJ 34}$$

Using this on the right hand side of $QJ33$ gives $V_bV_{a,b}V_a$ also, by $QJ34$, $V_{aU_b} = V_{b,a}V_b - U_bV_a$ so $V_{aU_b} + V_aU_b = V_{b,a}V_b - U_bV_a + V_aU_b = V_bV_{a,b}(QJ18)$. Hence the left hand side of $QJ33$ reduces to $V_b, V_{a,b}V_a$
37 also . This proves $QJ33$ and with it $QJ32$ and 31.

We shall now define the powers of a by $a^\circ = 1$, $a^1 = a, a^2 = 1U_a$, as before, and $a^n = a^{n-2}U_a, n\,2$. Then $QJ3$ implies that $U_{a^n} = U_a^n$. Also by induction on n we have

$$(a^m)^n = a^{m^n} \qquad \text{(QJ 35)}$$

We shall now prove

$$a^m \circ a^n = 2a^{m+n} \qquad \text{(QJ 36)}$$

by induction on $m + n$. This clear if $n + m \leqq 2$. Moreover, we may assume $m \leqq n$. We now note that $QJ36$ will follow if we can show that $U_aV_{a^n} = V_{a^n}U_a$. for then $a^m \circ a^n = b^mV_{a^n} = a^{m-2}U_aV_{a^n}$ (since $m \geqq 2$)$=$ $a^{n-2}V_{a^n}U_a = 2a^{n+n-2}U_a = 2a^{m+n}$. To prove the required

operation commutativity we shall show that V_{a^n} is in the subalgebra $\mathfrak{a}$ of End $\mathscr{J}$ generated by the commuting operations $U_a, V_a(QJ24)$. More genrally, we shall prove that U_{a^m,a^n} and $V_{a^m,a^n} \in \mathfrak{a}$. Since $V_{a^m,a^n} = V_{a^m}V_{a^n} - U_{a^m,a^n}(QJ27') = U_{a^m,1}U_{a^n,1} - U_{a^m,a^n}$ it suffices to show this for U_{a^m,a^n}. We use induction on $m+n$. The result is clear for $m+n \leqq 2$ by $QJ20$ and $U_{a,a} = 2U_a$ so we assume $m \geqq n, m \geqq 2$. If $n \geqq 2, U_{a^n,a^n} = U_aU_{a^{n-2},a^{n-2}}U_a$ by $QJ3$, so the result holds by induction in this case if $n = 1, U_{a^m,a} = U_{a^{m-2}U_{a,a}} = V_{a,a^{m-2}}U_a$ by $QJ4'$, so the result is valid in this case. Finally, if $n = 0, U_{a^m,1} = V_{a^m} = V_{a^{m-2}U_a} = V_{a,a^{m-2}}V_a - U_aV_{a^{m-2}}(QJ34)$. Hence the result holds in this case also. This completes **38**
the proof that $U_{a^m,a^n}, V_{a^m,a^n} \in \mathfrak{a}$ and consequently of $QJ36$.

We shall now prove a general theorem on operator identities involving the operators U_{a^m,a^n}, V_{a^m,a^n}

Theorem 3. *If $f(\lambda_1, \lambda_2, \ldots)$ is a polynomial in indeterminates $\lambda_1, \lambda_2, \ldots$ with coefficients in Φ such that $f(U_{a^{n_1}}, U_{a^{n_2}}, \ldots, U_{a^{n_n},a^{m_n}}, \ldots, V_{a^{n_1},a^{m_1}}, \ldots) = 0$ is an identity for all special quadratic Jordan algebras then this is an identity for all quadratic Jordan algebras.*

Proof. If X is an arbitrary non-vacuous set then there exists a free quadratic Jordan algebra $F(X)$ over Φ (freely) generated by X whose characteristic property is that $F(X)$ contains X and every mapping $X \to \mathscr{J}$ of X into a quadratic Jordan algebra $(\mathscr{J}, U, 1)$ has a unique extension to a homomorphism of $F(X)$ into $(\mathscr{J}, U, 1)$.[3] Let X contain more than one element one of which is denoted as x. It is clear from the universal property of $F(X)$ that of $f(U_{n1}, U_{n2}, \ldots, \ldots, \ldots) = 0$ holds in $F(X)$ then $f(U_{a^{n1}}, U_{a^{n2}}, \ldots) = 0$ holds in every quadratic Jordan algebra. Hence it suffices to prove $f(U_{n1}, U_{n2}, \ldots, \ldots) = 0$. Let Y be a set of the same cardinality as X and suppose $x \to y$ is a bijective mapping of X onto Y. Let $\Phi\{Y\}$ be the free associative algebra (with 1) gener- **39**
ated by Y and let $F_s(Y)$ be the subalgebra of $\phi\{y\}^{(q)}$ generated by Y. We have a homomorphism of $F(X)$ onto $F_s(Y)$ such that $x \to y$. If

[3]This is a special case of a general result proved in Cohn, *Universal Algebra*, pp.116-121 and p.170. A simple construction of free Jordan algebras and more generally of (linear) algebras defined by identities is given in Jacobson [2], pp.23-31. It is not difficult to modify this so that it applies to quadratic Jordan algebras.

$\mathscr{J}$ is a quadratic Jordan algebra then we denote the subalgebra of End $\mathscr{J}$ generated by the $U_a, a \in \mathscr{J}$, as Env $U(\mathscr{J})$. It is clear that Env $U(\mathscr{J})$ contains all $U_{a,b}$ and all $V_{a,b}$. Moreover it is easily seen that if $a \to a^\eta$ is a homomorphism of $(\mathscr{J}, U, 1)$ onto a second quadratic Jordan algebra $(\mathscr{J}', U', 1')$ then there exists a (unique) homomorphism of Env $U(\mathscr{J})$ onto Env $U(\mathscr{J}')$ such that $U_a \to U'_{a^\eta}, a \in \mathscr{J}$. Then also $U_{a,b} \to U'_{a^\eta,b^\eta}$ and $V_{a,b} \to V'_{a^\eta,b^\eta}$. In particular we have such a homoorphism of Env $U(F(X))$ onto Env $U(F_s(Y))$. Let $\mathscr{X}$ and $\mathscr{Y}$ respectively denote the subalgebra of Env $U(F(X))$ and Env $U(F_s(Y))$ generated by all $U_{x^n}, U_{x^n,x^m}, V_{x^n,x^m}$ and $U_{y^n}, U_{y^n,y^m}, V_{y^n,y^m}$. Then the restriction of our homomorphism of Env $U(F(X))$ onto Env $U(F(Y))$ to $\mathscr{X}$ is a homomorphism of $\mathscr{X}$ onto $\mathscr{Y}$ such that $U_{x^n} \to U_{y^n}, U_{x^n,x^m} \to U_{y^n,y^m}, V_{x^n,x^m} \to V_{y^n,y^m}$. Since $F(y)$ is special, $f(U_{y^{n1}}, U_{y^{n_2}}, \ldots, \ldots) = 0$ holds. It will follow that $f(U_{x^{n1}}, U_{x^{n2}}, \ldots) = 0$ holds in $F(X)$ if we can show that the homomorphism of $\mathscr{X}$ onto $\mathscr{Y}$ is an isomorphism. We have seen that $\mathscr{X}$ is generated by U_x and V_x and $\mathscr{Y}$ is generated by U_y and V_y. Since $U_x \to U_y$ and $V_x \to V_y$ the isomorphism will follow by showing that U_y and V_y are algebraically independent over Φ. Now in $\Phi\{Y\}^{(q)}$ we have $U_y = y_R y_L, V_y = y_R = y_L$ where a_R is $b \to ba$ and a_L is a $b \to ab$ and
40 y_L and y_R commute and are algebraically independent over Φ since if $z \in Y, z \neq y$, then $zk^k_R y^l_L = y^l z y^k$ and the elements $y^l z y^k, l, k = 0, 1, 2, \ldots$ are Φ -independent. Now $V_y = y_R + y_L$ and $U_y = y_R y_L$ are the "elementary symmetric" functions of y_R and y_L. The usual proof of the algebraic independence of the elementary symmetric function (e.g. Jacobson, *Lectures in Abstract Algebra*, p.108) carries over to show that U_y and V_y are algebraically independent operators in $\Phi\{Y\}^{(q)}$. It follows that they are algebraically independent operators also in $F_s(Y)$. This completes the proof of the theorem.

We now give two important instances of Theorem 3 which we shall need. Let $f(\lambda) \in \Phi[\lambda]$. an indeterminate and let $f(a)$ be defined in the obvious way if $a \in \mathscr{J}$ a quadratic Jordan algebra. Suppose $\mathscr{J}$ is a subalgebra of $\mathfrak{a}(q)$, $\mathfrak{a}$ associative. Then we claim that $U_{f(a)} U_{g(a)} = U_{(fg)(a)}$ holds in $\mathfrak{a}^{(q)}$, hence in $\mathscr{J}$. For, $xu_{(fg)(a)} = f(a)g(a)xf(a)g(a) = g(a)f(a)xf(a)g(a) = xU_{f(a)}U_{g(a)}$. It follows from Theorem 3 that

$$U_{f(a)} U_{g(a)} = U_{(fg)(a)} \qquad \text{QJ 37}$$

in any quadratic Jordan algebra. Another application of the theorem is the proof of

$$V_{a^n, a^n} = V_{a^{m+n}} \qquad \text{QJ 38}$$

This follows since in $\mathfrak{a}(q)$, $\mathfrak{a}$ associative, we have $xV_{a^m, a^n} = xa^m a^n + a^n a^m x = xa^{m+n} + a^{m+n}x = xV_{a^{m+n}}$. Similarly, one proves

$$V_{a^n} U_{a^n} = U_{a^n, a^{m+n}} \qquad \text{(QJ 39)}$$

and **41**

$$V_{a^n} = V_{a^{n-1}} V_a - V_{a^{n-2}} U_a, n \geq 2 \qquad \text{QJ 40}$$

The list of identities we have given will be adequate for the results which will be developed in this monograph. Other aspects of the theory require additional identities. Nearly all of these are consequences of the analogues for quadratic Jordan algebras of Macdomalli theorem. This result, which states that the extension of Theorem 3 to subalgebras with two generators is valid, has been proved by McCrimmon in [7]. □

4 Category isomorphism for $\Phi \ni \frac{1}{2}$. Characteristic two case.

We shall show first that if $\Phi \ni \frac{1}{2}$ then the two notions of Jordan algebra and quadratic Jordan algebra are equivalent. Let $CJ(CQJ)$ denote the category whose objects are Jordan algebras (quadratic Jordan algebras) over Φ with morphisms as homomorphisms. We have the following *Category Isomorphism Theorem*. Let $(\mathscr{J}, R, 1)$ be a Jordan algebra over a commutative ring Φ containing $\frac{1}{2}$. Define U by $U_a = 2R_a^2 - R_{a^2}$. Then $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra. Let $(\mathscr{J}, U, 1)$ be a quadratic Jordan algebra over Φ and define R by $R_a = \frac{1}{2}V_a, V_a = U_{a,1}$. Then $(\mathscr{J}, R, 1)$ is a Jordan algebra. The two constructions are inverses. Moreover, a mapping η of $\mathscr{J}$ is a homomorphism of $(\mathscr{J}, R, 1)$ if and only if it is a homomorphism of $(\mathscr{J}, U, 1)$. Hence $(\mathscr{J}, R, 1) \to (\mathscr{J}, U, 1)$, $\eta \to \eta$ is an isomorphism of the category CJ onto CQJ. **42**

Proof. Let $(\mathscr{J}, R, 1)$ be unital jordan over $\Phi \ni \frac{1}{2}$ and $U_a = 2R_a^2 - R_{a^2}$. Then we have shown in §2 that $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra.

We have $U_{a,b} = 2(R_aR_b + R_bR_a - R_{a,b})$ so $V_a = U_{a,1} = 2R_a$ and $\frac{1}{2}V_a = R_a$. Next let $(\mathscr{J}, U, 1)$ be a quadratic Jordan algebra over $\Phi \ni \frac{1}{2}$ and let $R_a = \frac{1}{2}V_a$. By $QJ24, 20, [V_a, V_{a^2}] = 0$ so $[R_aR_{a^2}] = 0$. Moreover, $a^2 = 1U_a = \frac{1}{2}(a \circ a) = aR_a$. Also $aV_b = bV_a$ gives $aR_b = bR_a$ and we have $R_1 = 1$ and $a \to R_a$ is a Φ -homomorphism of $\mathscr{J}$ into End $\mathscr{J}$. Hence $(\mathscr{J}, R, 1)$ is a linear Jordan algebra. By $QJ20$, $U_A = \frac{1}{2}V_a^2 - \frac{1}{2}V_{a^2} = 2R_a^2 - R_{a^2}$. This proves the assertions on the passage from $(\mathscr{J}, R, 1)$ to $(\mathscr{J}, U, 1)$ and back. The rest is clear.

We consider next the opposite extreme of the foregoing, namely, that in which $2\Phi = 0$ or, equivalently, $2(1) = 1 + 1 = 0$. Let $(\mathscr{J}, U, 1)$ be a quadratic Jordan algebra over Φ. We claim that $\mathscr{J}$ is a 2 -Lie algebra (= restricted Lie algebra of characteristic two) if we define $[ab] = a \circ b$ and $a^{[2]} = a^2$. We have $[aa] = a \circ a = 2a^2 = 0$ and $[[ab]c] + [[bc]a] + [[ca]b] = (a \circ b) \circ c = (b \circ c) \circ a + (c \circ a) \circ b = \{abc\} + \{bac\} + \{bca\} + \{cba\} + \{cab\} + \{acb\}(QJ27) = 2\{abc\} + 2\{bca\} + 2\{cab\} = 0$. Also $(a + b)^2 = a^2 + b^2 + [a, b]$ and $ba^2 = [[ba]a]$ since $V_{a^2} = V_a^2$ by $QJ20$. Hence the axioms for a 2-Lie algebra hold (Jacobson, Lie Algebras P. 6). This proves □

Theorem 4. *Let* $(\mathscr{J}, U, 1)$ *be a quadratic Jordan algebra over* Φ *such that* $2\Phi = 0$*. Then* $\mathscr{J}$ *is a* 2 *lie algebra relative to* $[ab] = a \circ b$ *and* $a^{[2]} = a^2$.

5 Inner and outer ideals. Difference algebras.

43 **Definition 4.** Let $(\mathscr{J}, U, 1)$ be a quadratic Jordan algebra. A subset $\mathscr{L}$ of $\mathscr{J}$ is called an *inner (outer) ideals* if $\mathscr{L}$ is a sub-module and $bab = aU_b(aba = bU_a) \in \mathscr{L}$ for all $a \in \mathscr{J}, b \in \mathscr{L} \mathscr{L}$ is an *ideal* if it is both an inner and an outer ideal.

The condition can be written symbolically as $\mathscr{J} U_{\mathscr{L}} \subseteq \mathscr{L}$ for an ideal, $\mathscr{L} U_{\mathscr{J}} \subseteq \mathscr{L}$ for an outer ideal. If $\mathscr{L}$ is an inner ideal and $a \in \mathscr{J}$ then $\mathscr{L}_{U_a}$ is an inner ideal since for $c = bU_a$, $\mathscr{J} U_c = \mathscr{J} U_{bU_a} = \mathscr{J} U_a U_b U_a \subseteq \mathscr{J} U_b U_a \subseteq \mathscr{L} U_a$. In particular, $\mathscr{J} U_a$ is an inner ideal called the *principal inner ideal determined by a*. This need not contain a. The inner ideal generated by a is $\Phi a + \mathscr{J} U_a$. For this contains a, is contained in every inner ideal containing a and is an inner ideal, since a typical element of $\Phi a + \mathscr{J} U_a$ is $\alpha a + bU_a, \alpha \in \Phi, b \in \mathscr{J}$, and $U_{\alpha a + bU_a} = \alpha^2 U_a + \alpha U_{a,bU_a} + U_a U_b U_a$. Since $U_{a,bU_a} = V_{a,b} U_a$ by $QJ4'$ we see that $\mathscr{J} U_{\alpha a + bu_a} \subseteq \mathscr{J} U_a$, so $\Phi a + \mathscr{J} U_a$ is an inner ideal. The outer ideal generated by a is the smallest submodule of $\mathscr{J}$ contianing a and stable under all $U_b, b \in \mathscr{J}$. The principal inner ideal detrermined by 1 is $\mathscr{J} U_1 = \mathscr{J}$. On the other hand, as we shall see, the outer ideal generated by 1 need not be $\mathscr{J}$. We shall call this the cone of $\mathscr{J}$.

If $\mathscr{L}$ is an outer ideal then $\{a_1 b a_2\} = bU_{a_1,a_2} = bU_{a_1+a_2} - bU_{a_1} - bU_{a_2} \in \mathscr{L}$ for $b \in \mathscr{L}, a_i \in \mathscr{J}$. In particular, $b \circ a = bu_{a,1} \in \mathscr{L}$, $b \in \mathscr{L}, a \in \mathscr{L}$. By $QJ27$ it follows that $\{ba_1 a_2\} \in \mathscr{L}, b \in \mathscr{L}, a_i \in \mathscr{J}$. If Φ contains $\frac{1}{2}$ then $\mathscr{L}$ is an outer ideal if and only if it is an ideal and if and only if $\mathscr{L}$ is an ideal in $(\mathscr{J}, R, 1)$ where $(\mathscr{J}, R, 1)$ is the Jordan **44**
algebra corresponding to $(\mathscr{J}, U, 1)$ in the usual way. For, if $\mathscr{L}$ is an outer ideal then $b \cdot a = \frac{1}{2} b \circ a \in \mathscr{L}, a \in \mathscr{J}, b \in \mathscr{L}$. On the other hand, if $\mathscr{L}$ is an ideal of $(\mathscr{J}, R, 1)$ then $bR_a = aR_b \in \mathscr{L}$ and this implies that bU_a and $aU_b \in \mathscr{L}$.

It is clear that the intersection of inner (outer) ideals is an inner (outer) ideal and the sum of outer ideals is an outer ideal. It is easily checked that the sum of an inner ideal and an ideal is an inner ideal.

Let $\mathscr{L}$ be an ideal in $\mathscr{J}, U, 1), b_i \in \mathscr{L}, a_i \in \mathscr{J}$. Then we have seen that $a_1 U_{a_2,b_2} = \{a_2 a_1 b_2\} \in \mathscr{L}$. Hence

$$\begin{aligned}(a_1 + b_1) U_{a_2+b_2} &= (a_1 + b_1)(U_{a_2} + U_{a_2,b_2} + U_{b_2})\\ &\equiv a_1 U_{a_2} (\mod \mathscr{L})\end{aligned}$$

It follows that if we define in $\overline{\mathscr{J}} = \mathscr{J}/\mathscr{L} = \{\overline{a} + \mathscr{L} | a \in \mathscr{J}\}, \overline{a_1} U_{\overline{a_2}} = \overline{a_1 U_{a_2}}$ then this is single valued. It is immediate that $(\overline{\mathscr{J}}, \overline{U}, \overline{1})$ is a quadratic Jordan algebra and we have the canonical homomorphism $a \to \overline{a}$ of $(\mathscr{J}, U, 1)$ onto $(\overline{\mathscr{J}, U} 1)$. Conversely, if η is a homomorphism of $(\mathscr{J}, U, 1)$ then $\mathscr{L} = \ker \eta$ is an ideal and we have the isomorphism $\overline{\eta} : \overline{a} \to a^\eta$ of $(\overline{\mathscr{J}} = \mathscr{J}/\mathscr{L}, \overline{U, 1})$ onto $(\mathscr{J}^\eta, U, 1)$. This fundamental theorem has its well-known consequences.

Examples. (1) Let $\mathscr{J} = \mathscr{H}(\mathbb{Z}_n)$ the quadratic Jordan algebra (over the ring of integers $\mathbb{Z}$) of $n \times n$ integral symmetric matrices. Let

$E = \operatorname{diag}\{1, \ldots, 1, 0, \ldots, 0\}$. Then E is an idempotent ($E^2 = E$) and the principal inner ideals $\mathscr{J} U_E = E \mathscr{H} E$ is the set of
45 matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

$A \in \mathbb{Z}_r$. Next let $B = (b_{ij}) \in \mathscr{H}(\mathbb{Z}_n)$ have even diagonal elements and let $A \in \mathscr{H}(\mathbb{Z}_n)$. Then (i, i) -entry of $C = ABA$ is

$$\begin{aligned} c_{ii} &= \sum_{j,k} a_{ij} b_{jk} a_{ki} = \sum_{j,k} a_{ij} b_{jk} a_{ik} \\ &= \sum_{j} a_{ij}^2 b_{jj} + 2 \sum_{j<k} a_{ij} b_{jk} a_{ik} \end{aligned}$$

Hence C has even diagonal elements. Thus the set $\mathscr{L}$ of integral symmetric matrices with even diagonal elements is an outer ideal in $\mathscr{H}(\mathbb{Z}_n)$. If $m \in \mathbb{Z}$ the set $m\mathscr{H}(\mathbb{Z}_n)$ of integral symmetric matrices whose entries are divisible by m is an ideal in $\mathscr{H}(\mathbb{Z}_n)$.

(2) Let $\underline{\rho} = \Phi(\lambda)$ the filed of rational expressions in an indeterminate λ over a field Φ of characteristic two. let $\mathscr{H}(\underline{\rho_n})$ be the set of $n \times n$ symmetric matrices with entries in $\underline{\rho}$. This is a quadratic Jordan algebra over Φ (with ABA as usual). Let $\mathscr{L}$ be the subset of matrices with diagonal entries in $\Phi(\lambda^2)$. Then $\mathscr{L}$ is an outer ideal containing 1. It is easy that this is the cone of $\mathscr{H}(\underline{\rho_n})$.

(3) Let Φ be a field of characteristic two, $\mathfrak{a} = \Phi[\lambda], \lambda$ an indeterminate. Consider $\mathfrak{a}^{(q)}$ and the subspace $\mathscr{L} = \Phi\lambda^2 + \sum_{i \geqq 4} \Phi\lambda^i$.

46 It is readily checked that $\mathscr{L}$ is an ideal in $\mathfrak{a}^{(q)}$. Let $\mathscr{J} = \mathfrak{a}^{(q)}/\mathscr{L}$ and put $\overline{\lambda} = \lambda + \mathscr{L}$. Then $\overline{\lambda} = \lambda + \mathscr{L}$. Then $\overline{\lambda}^2 = 0$ but $\overline{\lambda}^3 \neq 0$ in $\mathscr{J}$. If $\mathscr{J}$ is a special Jordan algebra, say, $\mathscr{J}$ is a subalgebra of $\mathfrak{L}^{(q)}, \mathfrak{L}$, associative, then the Jordan power X^n of $X \in \mathscr{J}$ coincides with the associative power X^n in $\mathfrak{L}$ since $X^n = X^{n-2} U_X = X X^{n-2} X$. Hence it is clear that in a special Jordan algebra $X^n = 0$ implies $X^{n+1} = 0$. It follows that $\mathscr{J} = \mathfrak{a}^{(q)}/\mathscr{L}$ is not special.

We have defined $\ker U = \{z | U_z = 0 = U_{z,a}, a \in \mathscr{J}\}$. This is an ideal since it is a submodule and if $a \in \mathscr{J}, z \in \ker U$, then $aU_z = 0$, and $U_{zU_a} = U_a U_z U_a = 0$. Also we can show that for $b \in \mathscr{J}$, $U_{zU_a,b} = 0$. To see this we note that $U_{z,a} = 0$, $a \in \mathscr{J}$ implies $V_z = U_{z,1} = 0$. Then $V_{z,a} = 0 = V_{a,z}$ by $QJ27'$. By $QJ9$, we have $\{b\{zba\}c\} + b\{zca\}b = \{\{bzc\}ab\} + \{(bzb)ac\}$ which gives (using a as operand): $V_{b,z}U_{b,c} + V_c, U_b = U_{b,b}V_{z,c} + U_{c,zU_b}$. This implies $U_{c,zU_b} = 0$ or $U_{b,zU_a} = 0$. Hence $zU_a \in \ker U$. The argument we have used show that every $\{\quad\}$ and $-U_-$ with one of the arguments $z \in \ker U$ is 0, with the exception of zU_a. In particular $2z = z \cdot 1 = 0$ which show that $\ker U = 0$ if $\mathscr{J}$ has no two torsion. We call $\mathscr{J}$ *nondegenerate* if $\ker U = 0$.

6 Special universal envelopes

A homomorphism of $(\mathscr{J}, U, 1)$ into $\mathfrak{a}(q)$ where $\mathfrak{a}$ is associative is called an *associative specialization of* $\mathscr{J}$ *into* $\mathfrak{a}$. A *special universal envelops* for $\mathscr{J}$ is a pair $(S(\mathscr{J}), \sigma_u)$ where $S(\mathscr{J})$ is an associative algebra and σ_u is an associative specilalization of into $S(\mathscr{J})$ such that if σ is an-
associative specialisation of $\mathscr{J}$ into an associative algebra $\mathfrak{a}$ then there 47
exists a unique homomorphism η of $s(\mathscr{J})$ into $\mathfrak{a}$ such that

$$\begin{array}{ccc} \mathscr{J} & \xrightarrow{\sigma_u} & S(\mathscr{J}) \\ \downarrow & \swarrow_{\eta} & \\ \mathfrak{a} & & \end{array} \tag{12}$$

is commutative. To construct an $(S(\mathscr{J}), \sigma_u)$ let $T(\mathscr{J})$ be the tensor algebra defined by the Φ -module $\mathscr{J}$: $T(\mathscr{J}) = \Phi \oplus (\mathscr{J} \oplus (\mathscr{J} \otimes \mathscr{J}) \oplus \ldots\ldots$ where all these tensor products are taken over Φ. Multiplication in $T(\mathscr{J})$ is defined by $(x_1 \otimes \ldots \otimes x_r)(x_{r+1} \otimes \ldots \otimes x_s) = x_1 \otimes \cdots \otimes x_s, x_i \in \mathscr{J}$, and the rule that the unit element Φ of 1_Φ is unit for $T(\mathscr{J})$. Then $T(\mathscr{J})$ is an associative algebra over Φ. Let k be the ideal in $T(\mathscr{J})$ generated by the elements $1 - 1_\Phi(1 \in \mathscr{J}), aba - a \otimes b \otimes a, a, b \in \mathscr{J}$. Put $S(\mathscr{J}) = T(\mathscr{J})/\mathfrak{K}$ and $a^{\sigma u} = a + \mathfrak{K}, a \in \mathscr{J}$. Then it is readily seen that $((\mathscr{J}), \sigma_u)$ is a special universal envelope for $(\mathscr{J}, U, 1)$.[4]. It is clear that we have an involution π' of $T(\mathscr{J})$ such that $(x_1 \otimes x_2 \ldots \otimes x_r)^{\pi'} = x_r \otimes x_{r-1} \otimes \ldots \otimes x_1, x_i \in \mathscr{J}$. Since $(aba - a \otimes b \otimes a)^{\pi'} = aba - a \otimes b \times a$ it is clear that $\mathfrak{K}^{\pi'} \subseteq \mathfrak{K}$. Hence π' induces an involution π in $S(\mathscr{J})/\mathfrak{K}$. We have $a^{\sigma_u \pi} = a^{\sigma_u}, a \in \mathscr{J}$, and since the a^{σ_u} generate $S(\mathscr{J})$ it is clear that π is the only involution satisfying $a^{\sigma_u \pi} = a^{\sigma u \pi}$. We shall call π the *main involution* of $S(\mathscr{J})$. If ξ is a homomorphism of $(\mathscr{J}, U, 1)$ into $(\mathscr{J}', U', 1')$ then we have a unique homomorphism ξ_u of $S(\mathscr{J})$ into $S(\mathscr{J}')$ such that

$$\begin{array}{ccc} \mathscr{J} & \xrightarrow{\xi} & \mathscr{J}' \\ \sigma_u \downarrow & & \downarrow \sigma_u \\ S(\mathscr{J}) & \xrightarrow[\xi]{} & S(\mathscr{J}') \end{array} \tag{13}$$

48 It is immediate that $(\mathscr{J}, U, 1)$ in special if and only if the mapping σ_u of $\mathscr{J}$ into $S(\mathscr{J})$ is injective. In this case it is convenient to identify $\mathscr{J}$ with its image in $S(\mathscr{J})$ and so regard $\mathscr{J}$ as a subset of $S(\mathscr{J}), \sigma_u$ as the injection mapping. Then $\mathscr{J}$ is a sub-algebra of the quadratic Jordan algebra $S(\mathscr{J})^{(q)}$ and the universal property of σ_u states that any homomorphism of $\mathscr{J}$ into an $\mathfrak{a}^{(q)}$, $\mathfrak{a}$ associative, has a unique extension to a homomorphism of $S(\mathscr{J})$ into $\mathfrak{a}$.

7 Quadratic Jordan algebras of quadratic forms with base points.

We consider a class of quadratic Jordan algebras $(\mathscr{J}, U, 1)$ over a field Φ satisfying the following conditions:

1. There exists a linear function T and a quadratic form Q on $\mathscr{J}$ (to Φ) such that

$$X^2 - T(X)X + Q(X) = 0 \tag{14}$$

[4]cf. Jacobson [2],pp. 65-72, for the corresponding discussion for Jordan algebras

[1]This condition is superfluous if Φ contains $\frac{1}{2}$ since in this case $X^K \cdot X = X^{k+1}$ so (14) implies (15).

$$X^3 - T(X)X^2 + Q(X)X = 0^1 \tag{15}$$

2. The same conditions hold for $\mathscr{J}_{\underline{\rho}}$ where $\underline{\rho}$ is any extension field of Φ and T and Q for $\mathscr{J}_{\underline{\rho}}$ are the extenstions of these functions 49
on $\mathscr{J}$ to a linear function and a quadratic form on $\mathscr{J}_{\underline{\rho}}$ respectively. (We assume $\mathscr{J}$ imbedded in $\mathscr{J}_{\underline{\rho}}$ and write, $U, 1$ for the U-operator and unit in $\mathscr{J}_{\underline{\rho}}$).

3. $\mathscr{J} \neq \Phi$

Taking $\underline{\rho} = \Phi(\lambda)$, an indeterminate and replacing x by $x + y$ in (14) we obtain

$$x \circ y = T(x)y + T(y)x - Q(x, y) \tag{16}$$

where $Q(x, y)$ is the symmeteric bilinear form associated with the quadratic form $Q(Q(x, y) = Q(x + y) - Q(x) - Q(y))$. Similarly, (15) and $X^3 = xU_x$ give

$$yU_x = T(x)x \circ y + T(y)x^2 - Q(x)y - Q(x, y)x - x^2 \circ y \tag{17}$$

Putting $y = 1$ in (16) gives $2x = T(x)1 + T(1)x - Q(x, 1)1$. If we take $x \notin \Phi 1$

$$T(1) = 2 \tag{18}$$

Then $x = 1$ in (14) gives

$$Q(1) = 1 \tag{19}$$

Also using the formulas for X^2 and $x \circ y$, (17) becomes

$$yU_x = Q(x)y + T(y)T(x)x - Q(x, y)x - T(y)Q(x)1 \tag{20}$$

We can write this in a somewhat more compact form by introducing $\overline{x} = T(x)1 - x$. Then (20) becomes 50

$$yU_x = Q(y, \overline{x})x - Q(x)\overline{y} \qquad 20'$$

Conversely, suppose we are given a quadratic form Q on a vector space $\mathscr{J}$ with a *base point* 1 such that $Q(1) = 1$. Define $T(x) = Q(x, 1), \overline{x} = T(x)1 - x$ and U_x by $(20')$ (or (20)). Then one can verify by direct calculation that $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra satisfying condition 1 and 2. We proceed to prove more that this by showing that $(\mathscr{J}, U, 1)$ is a special quadratic Jordan algebra satisfying 1 and 2. For this purpose we introduce the Clifford algebra of Q with base point 1 which is defined as follows. Let $T(\mathscr{J})$ be the tensor algebra over $\mathscr{J}$ and let $\mathscr{L}$ be the ideal in $T(\mathscr{J})$ generated by $1\Phi - 1$ and all $x \otimes x - T(x)x + Q(x)1$ where $1,\ x \in \mathscr{J}$ and $T(x) = Q(x, 1)$. Then we define *the clifford algebra* $C(\mathscr{J}, Q, 1)$ of *the quadratic form Q will base point* (such that $Q(1) = 1$) to be $T(\mathscr{J})/\mathscr{L}$. If $x \in \mathscr{J}$ we put $x^{\sigma_u} = x + \mathscr{L}$. Then we have $(x^{\sigma_u})^2 - T(x)x^{\sigma_u} + Q(x)1 = 0$. This implies that $x^{\sigma_u} y^{\sigma_y} + y^{\sigma_u} x^{\sigma_u} = T(x)y^{\sigma_u} + T(y)x^{\sigma_u} - Q(x, y)1 = 0$. Since $y^{\sigma_u} U_{x^{\sigma_u}} = x^{\sigma_u} y^{\sigma_u} x^{\sigma_u}$ we obtain

$$y^{\sigma_u} u_{x^{\sigma_u}} = Q(x)y^{\sigma_u} + T(y)T(x)x^{\sigma_u} - Q(x, y)x^{\sigma_u} - T(y)Q(x)^1. \qquad (21)$$

Also we have $1^{\sigma_u} = 1$. This and (21) show that $\mathscr{J}^{\sigma_u} = \{x^{\sigma_u} | x \in \mathscr{J}\}$ is a subalgebra of $C(\mathscr{J}, Q, 1)^{(q)}$. We shall that σ_u is injective. Then it
51 will follow from (20) and (21) that $(\mathscr{J}, U, 1)$ is a quadratic Jordan algebra and σ_u is an associative specialization of $\mathscr{J}$ in $C = C(\mathscr{J}, Q, 1)$. It is clear from the definition of $C(\mathscr{J}, Q, 1)$ that if $x \to x^{\sigma}$ is a linear mapping of $\mathscr{J}$ into an associative algebra $\mathfrak{a}$ suchthat $1^{\sigma} = 1$ and $(x^{\sigma})^2 - T(x)x + Q(x)1 = 0$ then there exists a unique homomorphism of $C(\mathscr{J}, Q, 1)$ into $\mathfrak{a}$ such that $x^{\sigma_u} \to x^{\sigma}, x \in \mathscr{J}$.

We consider first the case in which $(Q, 1)$ is *pure* in the sense that $\mathscr{J} = \Phi 1 + V$ where V is a subspace such that $T(v) = 0, v \in V$. If the characteristic is $\neq 2$ then $T(1) = Q(1, 1) = 2 \neq 0$ and $\mathscr{J} = \Phi 1 \oplus (\Phi 1)^{\perp} (1\Phi 1)^{\perp}$ the orthogonal complement of $\Phi 1$ relative to $Q(x, y)$). Then $T(v) = Q(1, v) = 0$ for $v \in (\Phi 1)^{\perp}$ and $(Q, 1)$ is pure. If the characteristic is two then $T(1) = 0$ so $(Q, 1)$ is pure if and only if $T \equiv 0$. In this case V can be taken to be any subspace such that $\mathscr{J} = \Phi 1 \oplus V$. Now let $C(V, -Q)$ be the Clifford algebra of V relative to the restriction of $-Q$ to V. The canonical mapping of $\Phi 1 + V$ into $C(V, -Q)$ is injective so we can identify $\mathscr{J} = \Phi 1 \oplus V$ with the corresponding subset of $C(V, -Q)$. Let $x = \alpha 1 + v, \alpha \in \Phi, v \in V$. Then $T(x) = 2\alpha, Q(x) = \alpha^2 + Q(v)$ and in $C(V, -Q), x^2 = \alpha^2 1 + 2\alpha v + v^2 = (\alpha^2 - Q(v))1 + 2\alpha v$. Hence $x^2 - T(x) + Q(x) = 0$. It follows from the universal property of $C(\mathscr{J}, Q, q)$ that we have a homomorphism of $C(\mathscr{J}, Q, 1)$ into $C(V, -Q)$ such that $x^{\sigma_v} \to x$. Clearly this implies that σ_u is injective.

Suppose next that $(Q,1)$ isnot pure so the characteristic is two and $T \neq 0$. We can choose d so that $T(d) = 1$ and write $\mathscr{J} = \Phi d \oplus W$ **52**
where W is the hyperplane in $\mathscr{J}$ defined by $T(x) = 0$. Then $1 \in W$ and $W = \Phi 1 \oplus V, V$ a subspace. We again consider $C(V,Q)(Q = -Q)$ since char= 2) and we identify $W = \Phi 1 + V$ with the corresponding subset of $C(V,Q)$. Let D' be the derivation in the tensor algebra $T(V)$ such that $vD' = v + Q(v,d)1$. Since this maps $v \otimes v + Q(V)$ into 0 it maps the ideal $\mathfrak{K}$ defining $C(V,Q)$ into itself. Hence this induces a derivartion D in $C(V,Q)$ such that $vD = v + Q(v,d)1$. Now put $\mathscr{L} = C(V,Q)$ and let $\mathscr{L}[t,D]$ be the algebra of differential polynomials in an indeterminate t with coefficients in $\mathscr{L}$ such that

$$ct + tc = cD, \quad c \in \mathscr{L} \tag{22}$$

since the characteristic is two, D^2 is a derivation. Since $vD^2 = vD, v \in V$, and V generates $\mathscr{L}, D^2 = D$. Also $ct^2 + t^2c = cD^2$ so $c(t^2 + t) = (t^2 + t)c, c \in \mathscr{L}$. Since $t^2 + t$ commutes with t also, it is clear that this polynomial is in the center of $\mathscr{L}[t,D]$. Hence also $g(t) = t^2 + t + Q(d)1$ is in the center. Let $(g(t))$ be the ideal in $\mathscr{L}[t,D]$ generated by $g(t)$ and put $\mathscr{O} = \mathscr{L}[t,D]/(g(t))$. It is clear from the division algorithm (which is applicable to $g(t)$ since its leading coefficient is 1) that every element of $\mathscr{L}[t,D]$ is congrument modulo $(g(t))$ to an element of the form $c_o + c_1 t, c_i \in \mathscr{L}$. Also $c_o + c_1 t \equiv 0(\mod g(t))$ implies $c_o = c_1 = 0$. Hence we can identify $\mathscr{O}$ with the set of elements of the form $c_o + c_1 t, c_i \in \mathscr{L}$, and we have the realtions $vt + tv = v + Q(v,d), t^2 + t + Q(d) = 0$.
We have the injective linear mapping $x = \alpha 1 + v + \beta d \to y = \alpha 1 + v +$ **53**
$\beta t, \quad, \alpha\beta \in \Phi, v \in V$, of $\mathscr{J} = \Phi 1 + V + \Phi d$ into $\mathscr{O}$. Moreover, $T(x) = \beta$, $Q(x) = \alpha^2 + Q(v) + \beta^2 Q(d) + \beta Q(v,d) + \alpha\beta$ and

$$\begin{aligned} y^2 &= \alpha^2 + Q(v) + \beta^2 t^2 + \beta(vt + tv) \\ &= \alpha^2 + Q(v) + \beta^2(t + Q(d)1) + \beta(v + Q(v,d)) \\ &= T(x)y + Q(x)1 = T(x)y - Q(x)'. \end{aligned}$$

Hence by the universal property of $C(\mathscr{J}, Q, 1)$ we have a homomorphism of $C(\mathscr{J}, Q, 1)$ into $\mathscr{O}$ such that $(\alpha 1 + v + \beta d)^{\sigma_u} \to y = \alpha 1 + v + \beta t$. Clearly this implies that σ_u is injective.

We have now proved that $(\mathscr{J}, U, 1)$ is a special quadratic Joradan algebra and σ_u is an associative specialization of $\mathscr{J}$ into, $C(\mathscr{J}, Q, 1)$. We now take $y = 1$ in (20) to obtain $x^2 = Q(x)1 + 2T(x)x - T(x)x - 2Q(x)1$ (Since $T(x) = Q(x, 1), T(1) = Q(1, 1) = 2) = T(x)x - Q(x)$. Since $\mathscr{J}$ is special we have $x^3 - T(x)x^2 + Q(x)x = 0$ in $\mathscr{J}$. If $\underline{\rho}$ is an extension field of Φ then it is clear that the extension of U to a quadratic mapping of $\mathscr{J}_{\underline{\rho}}$ into End $\mathscr{J}_{\underline{\rho}}$ is given by (20) where Q and T are the extensions of Q and T to a quadratic form a linear function on $\mathscr{J}_{\underline{\rho}}$. It follows as in $\mathscr{J}$ that we have $x^2 - T(x)x + Q(x)1 = 0 = x^3 - T(x)x^2 + Q(x)x$ also $\mathscr{J}_{\underline{\rho}}$. Thus conditions 1 and 2 hold.

Now let σ be an associative specialization of $\mathscr{J}$ into $\mathfrak{a}$. Since σ is a homomorphism of $\mathscr{J}$ into $\mathfrak{a}^{(q)}$ we have $(x^k)^\sigma = (x^\sigma)^k, k = 0, 1, 2, \ldots$ Since $x^2 - T(x)x + Q(x)1 = 0$ in $\mathscr{J}$ we have $(x^\sigma)^2 - T(x)x^\sigma + Q(x)1 =$
54 0. By the universal property of $C(\mathscr{J}, Q, 1)$ we have a unique homomorphism of $C(\mathscr{J}, Q, 1)$ into $\mathfrak{a}$ such that $x^{\sigma_u} \to x^\sigma$. It follows that $(C(\mathscr{J}, Q, 1), \sigma_u)$ is a special universal envelope for $(\mathscr{J}, U, 1)$.

We shall call $(\mathscr{J}, U, 1)$ the *(quadratic Jordan) algebra of the form Q with base point* 1. If we to indicate Q and 1 then we use the notation Jord $(Q, 1)$ for this $(\mathscr{J}, U, 1)$.

8 The exceptional quadratic Jordan algebra

$\mathscr{H}(\mathscr{O}_3)$, $\mathscr{O}$ an Octonion algebra. A quadratic Jordan algebra which is not special will be called *exceptional*. We have already given one example of this sort, example (3) of §5. We shall now give the most important examples of exceptional quadratic Jordan algebra. These are based on Octonion algebras. We proceed to define these for an arbitrary basic field.

Let Φ be a field and let $\rho = \Phi[u]$ be the algebra over Φ with base $(1, u)$ over Φ where 1 is unit and

$$u^2 - y + \rho \quad (\rho = \rho 1), \tag{23}$$

$\rho \in \Phi$, $4\rho \neq -1$. This is a commutative associative algebra which has the involution

$$x = \alpha + \beta u \to \overline{x} = \alpha + \beta(1 - u), \alpha, \beta \in \Phi \tag{24}$$

Next we define a *quaternion algebra* over Φ which as Φ -module is a direct sum of two copies of $\Phi[u]$, so its elements are pairs $(a, b), a, b \in \Phi[u]$ with the usual vector space structure. We define a product in $O =$ **55**
$\Phi[u] \oplus \Phi[u]$ by

$$(a, b)(c, d) = (ac + \sigma\overline{d}b, da + b\overline{c}) \tag{25}$$

where $a, b, c, d \in \Phi[u]$ and σ is a fixed non-zero element of Φ. Then O is an associative algebra with $1 = (1, 0)$ and $\sigma/$ has the standard involution.

$$x(a, b) \to \overline{x} = (\overline{a}, -b). \tag{26}$$

Finally, let $O = \sigma/ \oplus \sigma/$ as vector space over Φ and define a product in O by (25) where σ replaced by $\tau \neq 0$ in Φ and the elements are now in O. The resulting algebra O is called an *Octonion algebra* over Φ. It has the standard involution (26). These algebras are not associative but are *alternative* in the sense that they satisfy the following weakening of the associative law called the *alternative laws:*

$$x^2y = x(xy), \quad yx^2 = (yx)x \tag{27}$$

In O we have $x + \overline{x} = t(x)$ where t is a linear funtion and $x\overline{x} = n(x) = \overline{x}x$ where $n(x)$ is quadratic form on O (values in Φ). t and n are called respectively the *trace* and *norm*.

We write $v = (0, 1)$ in $\mathscr{O}$. Then u and v generate $\mathscr{O}$ and we have the basic rules: $vu = \overline{u}v = (1 - u)v, u^2 = u + \rho, v^2 = \sigma$. Similarly, we put $w = (0, 1)$ in O and we have $wu = uw$, $wv = \overline{v}w = -vw$, $w^2 = \tau u, v, w$ generate O and every element of O can be written in one and only one **56**
was as $a + bw, a, b \in \mathscr{O}$. Suppose the base field Φ is algebraically closed. Then $\Phi[u]$ is a direct sum of two copies of Φ since the polynomial $\lambda^2 - \lambda - \rho$ is a product of distinct linear factors. Then $\Phi[u] = \Phi[e]$ where $e^2 = e, \overline{e} = 1 - e$. Thus in this case we may take $\rho = 0$. Also replacing v and w by mutliples of these elements we may suppose $v^2 = 1$, $w^2 = 1$. Then $(1, u, v, uv, w, uw, vw, (uv)w)$ is a base for O whose multiplication table has coefficients which are $0, \pm 1$. For arbitrary Φ we shall say that O is a *split* Octonion algebra if $\rho = 0$, $\sigma = 0 = \tau = 1$, or equivalently the base $(1, u, v, \ldots)$ has the multiplication table just indicated.

Now suppose Φ is of characteristic $\neq 2$, $\mathcal{O}$ an octonion algebra over Φ. Let $\mathcal{O}_3$ be the set of 3×3 matrices with entries in $\mathcal{O}$. This is an algebra over $\mathcal{O}$ with the usual vector space compositions and matrix multiplication. We have the standard involution in this algebra: $A \to \overline{A}^t$ where $\overline{A} = (\overline{a}_{ij})$ for $A = (a_{ij})$. Let $\mathscr{H}(\mathcal{O}_3)$ be the Φ -subspace of matrices satisfying $\overline{A}^t = A$. This is closed under the bilinear product $A \cdot B = \frac{1}{2}(AB + BA)$ and it is well-known that $(\mathscr{H}(\mathcal{O}_3), R, 1)$ is a Jordan algebra if $XR_A = X \cdot A$ (See Jacobson's book [2], p.21). We now consider the quadratic Jordan algebra $(\mathscr{H}(\mathcal{O}_3), U, 1)$ where $U_a = 2R_a^2 - R_{a^2}$ and we wish to analize the U operator in this algebra. For this we introduce the following notation.

$$\alpha[ii] = \alpha e_{ii}, \quad \alpha \in \Phi$$
$$a[ij] = ae_{ij} + \overline{a}e_{ij}, a \in \mathcal{O}, i \neq j \tag{29'}$$

57 Here the e_{ij} are the usual matrix units e_{ij} has 1 in the (i, j) -position 0's elsewhere. We have

$$a[ji] = \overline{a}[ij] \tag{27'}$$

and if $\mathscr{H}_{ii} = \{\alpha[ii], \alpha \in \Phi\}, \mathscr{H}_{ij} = \{a[ij] | i \neq j, a \in \mathcal{O}\}$, then

$$\mathscr{H} = (\mathcal{O}_3) = \mathscr{H}_{11} \oplus \mathscr{H}_{22} \oplus \mathscr{H}_{33} \oplus \mathscr{H}_{12} \oplus \mathscr{H}_{23} \oplus \mathscr{H}_{13} \tag{28}$$

The $\mathscr{H}_{ii}$ are one dimensional and the $\mathscr{H}_{ij}, i \neq j$, are eight dimensional so $\dim \mathscr{H} = 27$. Any $AU_B, A, B \in \mathscr{H}$ is a sum of elements xU_y where x, y are in the spaces $\mathscr{H}_{ij}$ and $xU_{y,z}$ where x, y, z are in the $\mathscr{H}_{ij}$ and y and z are not in the same subspace. It is easily checked that the non-zero xU_y, $xU_{y,z}$ of the type just indicated are the following:

(i) $\beta[ii]U_{\alpha[ii]} = \alpha\beta\alpha[ii]$

(ii) $\alpha[ii]U_{a[ij]} = \overline{a}\alpha a[jj]$

(iii) $b[ij]U_{a[ij]} = a\overline{b}a[ij]$

(It is easily seen that $(ax)a = a(xa)$ in any alternative algebra. Hence this is abbreviated to axa.)

(iv) $\{\alpha[ii]a[ij]b[ji]\} = (\alpha ab + \overline{\alpha(ab)})[ii]$

(v) $\{\alpha[ii]\beta[ii]a[ij]\} = \alpha\beta a[ij]$

(vi) $\{\alpha[ii]a[ij]\beta[jj]\} = \alpha\beta a[ij]$

(vii) $\{\alpha[ii]a[ij]b[jk]\} = \alpha ab[ik]$

(viii) $\{a[ij]\alpha[jj]b[jk]\} = \alpha ab[ik]$ **58**

(ix) $\{a[ij]b[ji]c[ik]\} = a(bc)[ik]$

(x) $\{a[ij]b[jk]c[ki]\} = (a(bc) + \overline{a(bc)})[ii]$.

Now let Φ_o be a subring of ϕ containing 1. Then $(\mathscr{H}, U, 1)$ can be regarded as a quadratic Jordan algebra over Φ_0. The foregoing formulas show that if O_o is a subalgebra of $(O/\Phi_o, j)$, that is, a subalgebra of O/Φ_o stable under j, then the subset $\mathscr{H}_o$ of $\mathscr{H}$ of matrices having entries in O_o is a subalgebra of $(\mathscr{H}, U, 1)$. This is clear since $\mathscr{H}_o$ is the set of sums of $\alpha[ii]$, $a[ij]$ where $\alpha, a \in O_o$. It is clear also that if $\mathfrak{K}$ is an ideal in (O_o, j) then the set $\mathscr{Z}$ of matrices with entries in $\mathfrak{K}$ is an ideal in $(\mathscr{H}_o, U, 1)$. Hence we have the quadratic Jordan algebra $(\mathscr{H}_o/\mathscr{Z}, \overline{U}, \overline{1})$.

If Φ has characteristic $\neq 2$ then $\overline{a} = a$ in O if and only if $a = \alpha \in \Phi$. This is not the case for characteristic two accordingly, in this case we let $\mathscr{H}(O_3)$ denote the set of 3×3 matrices with entries in O such that $\overline{A}^t = A$ and *the diagonal entries are in* Φ. (For Φ of characteristic $\neq 2$ the latter condition is implied by the former.) Then $\mathscr{H} = \mathscr{H}(O_3) = \sum\limits_{i\leq j=1}^{3} \mathscr{H}_{ij}$ where $\mathscr{H}_{ij}$ is as before. Then it is easily seen that there is a unique quadratic mapping of $\mathscr{H}$ into End $\mathscr{H}$ such that the formulas (i)-(x) hold and all other $xU_y, xU_{y,z}$ are 0 where x, y, z are in the subspaces $\mathscr{H}_{ij}, y$ and z not in the same subspace. For any characteristic we have

Theorem 5. $(\mathscr{H}, U, 1)$ *is a quadratic Jordan algebra.* **59**

Proof. The case in which the characteristic $\neq 2$ has been settled before, so we assume the characteristic is 2. Assume first that $\Phi = \mathbb{Z}_2$ the field of two elements and O is the split octonion algebra over Φ. Let O' be the split octonion algebra over the rationals Q, O_o the $\mathbb{Z}$ -subalgebra of $(O' j)$ of integral linear combinations of the base $(1, u, v, \ldots)$ Then we have the $\mathbb{Z}$ -quadratic Jordan algebra $(\mathscr{H}(O_{o3}), U, 1)/2\mathscr{H}(O_{0_3})$ which can be regarded as a $\mathbb{Z}_2$ -quadratic Jordan algebra. Moreover, it is clear that $(\mathscr{H}(O_3), U, 1)$ (over $\mathbb{Z}_2$) is isomorphic to this. Hence $\mathscr{H}(O_3, U, 1)$ is a quadratic Jordan algebra over $\mathbb{Z}_2$. Now let Φ be arbitrary of characteristic two, O an arbitrary octonion algebra. To prove $(\mathscr{H}(O_3), U, 1)$ is Jordan it is enough to show that the conditions $QJ3, 4, 6-9$ hold for the U -operator (Theorem 1). These hold if and only if they hold for $(\mathscr{H}(O_3)_\Omega, U, 1)$ where Ω is the algebraic closure of Φ. Also we may identify $\mathscr{H}(O_3)_\Omega$ with $\mathscr{H}(O_\Omega)_3$. Hence it suffices to assume Φ algebraically closed. Then O is split. Now it is clear from the definition of a split algebra that if O_o is the split octonion algebra over $\mathbb{Z}_2$ then $O = O_{o\Phi} = \Phi \otimes_{\mathbb{Z}2} O_0$ so $(\mathscr{H}(O_3), U1) = (\mathscr{H}(O_{o3})_\Phi, U, 1)$. Since we have just seen that the latter is a quadratic Jordan algebra it follows that $(\mathscr{H}(O_3), U, 1)$ is a quadratic Jordan algebra.

We have seen in Theorem 3 that a quadratic Jordan algebra over Φ with $2\Phi = 0$, is a 2-Lie algebra relative to $[a, b] = a \circ b = (a+b)^2 - a^2 - b^2$
60 and $a^{[2]} = a^2$. In particular, this holds for $(\mathscr{H}(O_3), U, 1)$ where O is an octonion algebra over a field of characteristic two. We now note that in this case $A^2 = 1U_A$ is the same as the square of the matrix $A \in \mathscr{H}$ as defined in O_3. To see this it is sufficient to show that $1U_{a[ij]} = (ae_{ij} + \overline{a} \cdot e_{ji})(ae_{ij} + \overline{a}e_{ji}) = n(a)(e_{ii} + e_{jj}) = n(a)[ii] + n(a)[jj], 1U_{\alpha[ii]} = (\alpha e_{ii})(\alpha e_{ii}) = \alpha^2[ii]$, $\{x1y\} = xy + yx$ if x, y are in different spaces $\mathscr{H}_{ij}$. By the defining formulas $1U_{a[ij]} = 1[ii]U_{a[ij]} + 1[ji]U_{a[ij]} = n(a)[jj] + n(a)[ii]$ (by (ii)), $1U_{\alpha[ii]} = 1[ii]U_{\alpha[ii]} = 1[ii]U_{\alpha[ii]} = \alpha^2[ii]$ (by (i)). By (v),$\{\alpha[ii]1c[ij]\} = \{\alpha[ii]1[ii]c[ij]\} = \alpha c[ij] = \alpha ce_{ij} + \alpha\overline{c}e_{ji}$. On the other hand, $\alpha e_{ii}(ce_{ii} + \overline{c}e_{ji}) + (ce_{ij} + \overline{c}e_{ji})(\alpha e_{ii}) = \alpha ce_{ij} + \alpha\overline{c}e_{ji}$. By $(Viii)$. $\{a[ij]1c[jk]\} = ac[ik]$. Also $a[ij]c[jk] + c[jk]a[ij] = (ae_{ij} + \overline{a}e_{ji})(ce_{jk} + \overline{c}e_{kj}) + (ce_{jk} + \overline{c}e_{kj})(ae_{ij} + \overline{a}e_{ji}) = ace_{ik} + \overline{ca}e_{ki} = ac[ik]$. The remaining $\{x1y\}$ and $xy + yx$ are 0. Hence we have proved our assertion and we have the following consequence of Theorems 4 and 5: □

Corollary . *Let O be an octonion algebra over a field of characteristic two, $\mathscr{H}(O_3)$ the set of 3×3 hermitian matrices in O_3 with diagonal entries in Φ. Then $\mathscr{H}(O_3)$ is a 2-Lie algebra relative to $AB = AB + BA$ and $A^{[2]} = A^2$.*

Theorem 5 has an important generalization in which the octonion

algebra is replaced by an alternative algebra with involution $(\mathcal{O}, j)$ *such that all norms* $d\overline{d} = dd^j, d \in \mathcal{O}$*, are in the nucleus. We recall that the nucleus of a non-associative algebra is the set of elements* α *such that* $[\alpha, x, y] = (\alpha x)y - \alpha(xy) = 0, [x, \alpha, y] = 0, [x, y, \alpha] = 0$ *for all* x, y *in* **61**
the algebra. In an alternative algebra the associator $[x, y, z] \equiv (xy)z - x(yz)$ *is an alternating function in the sense that* $[x, y, z]$ *is unchanged under even permutation of the arguments and changes sign under odd permutation. Hence* $\alpha \in N(\mathcal{O})$ *the nucleus of the alternative algebra* $\mathcal{O}$ *if and only if* $[\alpha, x, y] = 0, x, y \in \mathcal{O}$*. Now suppose* $(\mathcal{O}, j)$ *is an alternative algebra satisfying the condition that norms are in the nucleous* $N(\mathcal{O})$*. Let* N_o *be the* Φ *-submodule of* $N(\mathcal{O})$ *generated by the norms. If* $x, y \in \mathcal{O}$ *then* $(x + y)(\overline{x} + \overline{y}) - x\overline{x} - y\overline{y} = x\overline{y} + y\overline{x} \in N_o$*. In particular,* $t(x) = x + \overline{x} \in N_o$*. It follows that if* Φ *contains* $\frac{1}{2}$ *then* $\mathscr{H}(\mathcal{O}, j) \subseteq N(\mathcal{O})$ *so the condition in this case is that the symmetric elements of* $\mathcal{O}$ *are contained in the nucleus. Again suppose* $\mathcal{O}$ *arbitrary and* $(\mathcal{O}, j)$ *satisfies the norm condition. Then we have the following results (McCrimmon):*

1) $xN_o\overline{x} \subseteq N_o, \quad x \in \mathcal{O}$

2) $xN\overline{x} \subseteq N$

3) If $N' = N \cap \mathscr{H}(\mathcal{O}, j)$ *then* $xN'\overline{x} \subseteq N'$

Proof. 1. We shall use $(xa)\overline{x} = x(a\overline{x})$ which we write as $xa\overline{x}$. Also we shall need Moufang's identity: $(ax)(ya) = a(xy)a$ which holds in any alternative algebra. It is enough to prove $x(y\overline{y})\overline{x} \in N_o, x, y \in \mathcal{O}$. We have $x(y\overline{y}) = x(y(t(y) - y)) = x(yt(y)) - xy^2 = (xy)t(y) - (xy)t(y) - (xy)y = (xy)\overline{y}$. Hence $x(y\overline{y})\overline{x} = (x(y\overline{y}))(t(x) - x) = (x(y\overline{y}))t(x) - (x(y\overline{y}))x = ((xy)\overline{y})t(x) - (xy)(\overline{y}x)$ (by Moufang)= $(xy)(\overline{y}t(x) - \overline{y}x) = (xy)(\overline{y}\overline{x}) = (xy)(\overline{xy}) \in N_o$.

2. (2) We use $\alpha[x, y, z] = [\alpha x, y, z] = [x\alpha, y, z] = [x, y, z]\alpha$ for **62**
$x, y, z \in \mathcal{O}, \alpha \in N$, and $(xyx)z = x(y(xz))$ (see the author's book [2], pp. 18-19). We have to show that $[x\alpha\overline{x}, y, z] = 0$ if $\alpha \in N, x, y, z \in \mathcal{O}$. Since $x\overline{x} \in N$ this will follow by showing that $[x\alpha\overline{x}, y, z] = [x\overline{x}, y, z]\alpha$. For this we have the following calculation:

$$[x\alpha\overline{x}, y, z] = [x\alpha t(x), y, z] - [x\alpha x, y, z]$$

$$
\begin{aligned}
&= t(x)[x,y,z]\alpha + (x(\alpha(x(yz))) - (x(\alpha(xy)))z\\
&= t(x)[x,y,z]\alpha - (x\alpha)[x,y,z] + (x\alpha)((xy)z)\\
&- [x\alpha, xy, z] = (x\alpha)((xy)z)\\
&= g(x,y,z)\alpha
\end{aligned}
$$

where $g(x,y,z) = t(x)[x,y,z] - x[x,y,z] - [x,xy,z]$. Taking $\alpha = 1$ we have $g(x,y,z) = [x\overline{x}, y, z]$. Hence $[x\alpha\overline{x}, yz, z] = [x\overline{x}, y, z]\alpha = 0$.

3. is an immediate consequence of this.

Now consider the algebra $\mathcal{O}_3$ of 3×3 matrices with entries in $\mathcal{O}$. By $\mathscr{H}(\mathcal{O}_3)$ we shall now understand the set of hermitian matrices of $\mathcal{O}_3(\overline{A}^t = A)$*with* diagonal entries in $N' = N \cap \mathscr{H}(\quad, j)$. If $\mathcal{O}$ is associative then $\mathscr{H}(\mathcal{O}_3)$ is just the set of hermitian matrices. In any case the elements of $\mathscr{H}(\mathcal{O}_3)$ are sums of elements $\alpha[ii]$, $\alpha \in N'$, and $a[ij], a \in \mathcal{O}, i \neq j$. Since $N_o \subseteq N'$ and all traces are in N_o it is clear from (1) − (3) that the right hand sides of $(i) - (x)$ are contained in $\mathscr{H}(\mathcal{O}_3)$. Hence we can define a unique quadratic mapping of $\mathscr{H}(\mathcal{O}_3)$ into End $\mathscr{H}(\mathcal{O}_3)$ such that $(i) - (x)$ hold and the remaining $xU_y, xU_{y,z} = 0$ for x, y of the form
63 $\alpha[ii]$ or $a[ij]$. It has been proved by McCrimmon that $\mathscr{H}(\mathcal{O}_3, U, 1)$ is a quadratic Jordan algebra.

The algebras $\mathscr{H}(\mathcal{O}_3)$ with $\mathcal{O}$ not associative are exceptional. In fact, we have the following stronger result: □

Theorem 6. *If* $(\mathscr{H}(\mathcal{O}_3), U, 1)$ *is a homomorphic image of a special quadratic Jordan algebra then* $\mathcal{O}$ *is associative.*

Proof. The proof we sketch is due to Glennie and is given in detail on p.49 of the author's book [2]. One can show that the following identity holds in every $\mathfrak{a}^{(q)}$, $\mathfrak{a}$ associative:

$$
\begin{aligned}
&xzx \circ \{y(zy^2z)x\} - yzy \circ \{x(zy^2z)y\}\\
&\quad = x(z\{x(yzy)y\}z)x - y(z\{y(xzx)x\}z)y. \qquad (29)
\end{aligned}
$$

On the other hand, if one takes

$$x = 1[12],\ y = 1[23],\ z = a[21] + b[13] + c[32]$$

then one can see that the $(1,3)$ entry in the matrix on the left side of (29) is $a(bc)-(ab)c$ while the $(1,3)$ -entry on the right hand side is 0. Hence if (29) is to hold in $\mathscr{H}(O_3)$ then $a(bc)=(ab)c$ for all $a,b,c\in O$ so O is associative. Clearly this identity holds if $(\mathscr{H}(O_3),U,1)$ is a homomorphic image of a special quadratic Jordan algebra. □

9 Quadratic Jordan algebras defined by certian cubic forms.

In this section we assume the base ring Φ is an infinite field. We shall give another definition of the quadratic Jordan structure on $\mathscr{H}(O_3)$, O an octonion algebra over Φ. As before, $\mathscr{H}(O_3)$ denotes the set of $3\times$ 3 hermitian matrices with entries in O, diagonal entries in Φ. If $a=\sum_1^3\alpha_i[ii]+\sum_{i=1}^3 a_i[jk]$, where (i,j,k) is a cyclic permutation of $(1,2,3)$ and the notations are as in §8, then we define a "determinant" by **64**

$$N(a)=\det a=\alpha_1\alpha_2\alpha_3-\sum_1^3\alpha_i n(a_i)+t((a_1a_2)a_3) \tag{30}$$

Here $n(a)=a\overline{a}, t(a)=a+\overline{a}$ in O. It is known that $t((a_1a_2)a_3)=t(a_1(a_2a_3))$ so we write this as $t(a_1a_2a_3)$. Also it is known that $t(a_1a_2a_3)$ is unchanged under cyclic permutation of the arguments. If f is a rational mapping of $\mathscr{H}$ into a second finite dimensional space then we let $\Delta_a^b f$ denote the directional derivative of f at a in the direction b (see the author's book, pp, 215-221). In particular, if f is a polynomial function then we have $f(a+\lambda b)\equiv f(a)+(\Delta_a^b f)\lambda(mod\lambda^2)$ and $\Delta_a^b f$ is determined by this condition. Since N is polynomial mapping which is homogeneous of degree three we have

$$N(a+\lambda b)=N(a)+(\Delta_a^b N)\lambda+(\Delta_b^a N)\lambda^2+N(b)\lambda^3 \tag{31}$$

By (30) we have for $a=\sum\alpha_i[ii]+\sum a_i[jk], b=\sum\beta_i[ii]\sum b_i[jk]$ that

$$\Delta_a^b N=\sum_i\beta_i\alpha_j\alpha_k-\sum_i\beta_i n(a_i)-\sum_i\alpha_i t(\overline{a}_i,b_i)+\sum_i t(b_ia_j,a_i) \tag{32}$$

65 We define $T(a,b) = -\Delta_1^a \Delta^b logN = (\Delta_1^a N)(\Delta_1^b N) - \Delta_1^a(\Delta^b N)$ so T is a symmetric bilinear form in a and b. By (32), we have

$$T(a,b) = \left(\sum \alpha_i\right)\left(\sum \beta_i\right) - \sum_i \beta_i(\alpha_j + \alpha_k) - \sum t(\overline{a}_i, b_i)$$
$$= \sum \alpha_i\beta_i - \sum t(\overline{a}_i, b_i) \tag{33}$$

Since $t(a,b)$ is non-degenerate on O, $T(a,b)$ is non-degenerate on $\mathscr{H}$. If we define the "adjoint matrix", $a^\sharp$ by

$$a^\sharp \sum_i (\alpha_j\alpha_k - n(a_i)[ii] + \sum_i (\overline{a_j a_k} - \alpha_i a_i)[jk] \tag{34}$$

it is easy to check that

$$\Delta_a^b N = T(a^\sharp, b) \tag{35}$$

A straight forward verifiction using Moufang's identity shows that we have

$$a^{\sharp\sharp} = N(a)a. \tag{36}$$

It is clear from the definition of N that $N(1) = 1$. We now define $T(a) = T(a,1) = T(a,1^\sharp)$ since $1^\sharp = 1$ by (34). Then (33) gives $T(a) = \sum \alpha_i$. We define $a \times b = (a+b)^\sharp - a^\sharp - b^\sharp$. We have $T(a,b) = (\Delta_1^a N)(\Delta_1^b N) - \Delta_1^a(\Delta^b N) = T(a)T(b) - \Delta_1^a(\Delta^b N)$ (by (35)). Since N is cubic form (=homogeneous polynomial function of degree three we have $\Delta_i^a(\Delta^b N) = \Delta_x^c(\Delta^a(\Delta^b(^b N)))$ is independent of x and is symmetric in a,b,c. Hence $\Delta_1^a(\Delta^b N) = \Delta_b^a(\Delta^a N) = \Delta_b^a T(a^\sharp)$ (by (35))= $T(a \times b)$. Hence we have

$$T(a \times b) = T(a)T(b) - T(a,b) \tag{37}$$

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We define the *characteristic polynomial* $f_a(\lambda)N(\lambda 1 - a)$. By (31) and the definition of $T(a)$ we have

$$f_a(\lambda) = N(\lambda 1 - a) = \lambda^3 - T(a)\lambda^2 + S(a)\lambda - N(a) \tag{38}$$

where $S(a) = T(a^\sharp)$. Direct verification, using (34) and the foregoing definitions shows that

$$a^\sharp = a^2 - T(a)a + S(a)1 \tag{39}$$

where a^2 is the usual matrix square. Since $S(a) = T(a^\sharp)$ and T is linear and satisfies $T(1) = 3$ this gives

$$2T(a^\sharp) = T(a)^2 - T(a^2) \tag{40}$$

We now suppose that $\mathscr{H} = \mathscr{H}(O_3)$ is endowed with the quadratic Jordan structure given in §8. Then $a^2 = 1U_a$ is the usual square of a and $a^3 = aU_a$. We shall now establish the following formula for the U-operator in $\mathscr{H}$:

$$bU_a = T(a,b)a - a^\sharp \times b \tag{41}$$

We shall establish this using the foregoing formulas and the Hamilton-Cayley type theorem that

$$f_a(a) = a^3 - T(a)a^2 + S(a)a - N(a)1 = 0, \tag{42}$$

which we prove first. Suppose first that the characteristic is $\neq 2$. Then 67
$a^3 aU_a = \frac{1}{2}(aa^2 + a^2 a)$. Then one can verify (42) by direct calculation (see the author's book [2], p.232). Next assume char. $= 2$. Then we shall establish (42) by a reduction $\mod 2$ argument similar to that used in §8. We note first that we may assume the base field is algebraically closed. Then O is split and has a canonical base $(u_1, u_2, \ldots, u_8)$ with multiplication table in $\mathbb{Z}_2$ as in§8. We obtain a corresponding canonial base $(v_1, \ldots, v_{27})$ for O/Φ where $v_i = i[ii]$, $i = 1,2,3$ and v_j, for $j > 3$, has the form $u_k[12]$, $u_k[13]$ or $u_k[23]$, $k = 1,2,\ldots,8$. Now let $\xi_1, \xi_2, \ldots, \xi_{27}$ be indeterminates and consider the "generic" element $x = \sum\limits_1^{27} \xi_j v_j$ in $\mathscr{H}_{\underline{\rho}}, \underline{\rho} = \Phi(\xi), \xi = (\xi_1, \ldots, \xi_{27})$. By specialization it suffices to prove (42) for $a = x$. Let $\underline{\rho}_o = \mathbb{Z}_2(\xi), \mathscr{H}_o = \sum \underline{\rho}_o v_j$, so $\mathscr{H}_o$ is quadratic Jordan algebra over $\underline{\rho}_o$ and $x \in \mathscr{H}_o$. The functions S, T, N on $\mathscr{H}_o$ are the restriction of the corresponding ones on $\mathscr{H}_o$ are the restriction of the corresponding ones on $\mathscr{H}$. Hence it suffices to prove the result for x in $\mathscr{H}_o$. This follows by applying a $\mathbb{Z}$ -homomorphism of an algebra $\mathscr{H}' = \sum \underline{\rho}' v'_j$ where $\underline{\rho}' = \mathbb{Z}(\xi)$ and the v'_j are obtained from a canonical base of the split octonion algebra $\mathscr{O}'/Q$ as in §8.

We now begin with (42). A linearization of this relation by replacing a by $a + \lambda b$ and taking the coefficient of λ gives

$$bU_a = -a^2 \circ b + T(a)(a \circ b) + T(b)a^2 - T(a \times b)a - T(a^\sharp)b + T(a^\sharp, b)1$$

$$= -(a^\sharp + T(a)a - T(a)a - T(a^\sharp)1) \circ b + T(b)(a^\sharp +$$
$$T(a)a - T(a^\sharp)1) + T(a)a \times b - T(a^\sharp)b$$
$$- (T(a)T(b) - T(a,b))a + T(a^\sharp, b) \quad ((39),(37))$$
$$= -a^\sharp \circ b + T(a^\sharp)b + T(b)a^\sharp - (T(b)T(a^\sharp) - T(a^\sharp, b))$$
$$+ T(a,b)a$$
$$= T(a,b)a - a^\sharp \times b$$

68 using (37) and $a \times b = a \circ bT(a)b - T(b)a + T(a \times b)1$ which is the linearization of (39). Hence (41) holds.

We now assume we have a finite dimensional vector space $\mathscr{J}$ over an infinite field Φ equipped with a cubic form N, a point 1 satisfying $N(1) = 1$, such that:

(i) $T(a,b) = -\Delta_1^a \Delta^b \log N = (\Delta_1^a N)(\Delta_1^b N) - \Delta_1^a(\Delta^b N)$

is a non-degenerate symmetric bilinear form in a and b.

(ii) If $a^\sharp$ is defined by $T(a^\sharp, b) = \Delta_a^b N$ then $a^{\sharp\sharp} = N(a)a$.

We define

(iii) $bU_a = T(a,b)a - a^\sharp \times b$

where $a \times b = (a+b)^\sharp - a^\sharp - b^\sharp$. Then we have

Theorem 7. $(\mathscr{J}, U, 1)$ *is a quadratic Jordan algebra.*

Proof. We can linearize (ii) to obtain

$$a \times (a \times b) = N(a)b + T(a^\sharp, b)a \tag{43}$$
$$a^\sharp \times b^\sharp + (a \times b)^\sharp = T(a^\sharp, b)b + T(b^\sharp, a)a \tag{44}$$

we have $T(1,b) = (\Delta_1' N)(\Delta_1^b N) - \Delta_1'(\Delta^b N) = 3N(1)\Delta_1^b N - 2\Delta_1^b N$ (by Euler's theorem on homogeneous polynomial functions)$= \Delta_1^b N = T(1^\sharp, b)$.

69 Hence

$$1^\sharp = 1 \tag{45}$$

by the non-degeneracy of T. Since N is a cubic form $\Delta_x^c(\Delta^a\Delta^b N)$ is independent of x and symmetric in a, b, c. By (ii) we have $T(a \times c, b) = \Delta_c^a \Delta_N^b = \Delta_x^c(\Delta^a\Delta^b N)$. Hence $T(a \times c, b)$ is symmetric in a, b, c and in particular we have $T(a, b \times 1) = T(a \times b, 1) = \Delta_1^a \Delta^b N = (\Delta_1^a N)(\Delta_1^b N) - T(a, b) = T(a)T(b) - T(a, b)$ where $T(a) = T(a, 1^\sharp) = T(a, 1)$. This and the non-degeneracy of T imply

$$b \times 1 = T(b)1 - b \tag{46}$$

By (43) and the symmetry of $T(a \times c, b)$ we have $T(b, (c \times a^\sharp) \times a) = T(b \times a, c \times a^\sharp) = T(b \times a) \times a^\sharp, C = T(N(a)b, c) + T(T(a^\sharp, b)a, c) = T(b, N(a)c) + T(b, T(a, c)a^\sharp)$. Hence

$$(c \times a^\sharp) \times a = N(a)c + T(a, c)a^\sharp \tag{47}$$

Since $T(bU_a, c) = T(a, b)T(a, c) - T(a^\sharp \times b, c)$ is symmetric in b and c we have

$$T(bU_a, c) = T(b, cU_a) \tag{48}$$

Next we note that $T(bU_a)^\sharp, c) = T((T(a, b)a - a^\sharp \times b)^\sharp, c) = T(T(a, b)^2\, a^\sharp + (a^\sharp \times b)^\sharp - T(a, b)(a^\sharp \times b) \times a, c) = T9(a, b)^2 T(a^\sharp, c) - T(N)(a)a \times b^\sharp - N(a)T(a, b)b - T(a^\sharp, b^\sharp)a^\sharp, c - T(a, b)T(N(a)b + T(a, b)a^\sharp, c)$ (by (44), (47) and (ii))$= T(a^\sharp, b^\sharp)T(a^\sharp, c) - N(a)T(a \times b^\sharp, c) = T(b^\sharp U_{a\sharp}, c)$. Hence

$$(bU_a)^\sharp = b^\sharp U_{a\sharp} \tag{49}$$

70

Linearization of this relative to b gives

$$bU_a \times cU_a = (b \times c)U_{a^\sharp} \tag{50}$$

We can now provce $QJ3$. For this we consider $T(xU_{bU_a, y}) = T(T(bU_a, x)bU_a - (bU_a)^\sharp \times x, y)$. Since $T(bU_{a,x}) = T(b, xU_a)$ and $T((bU_a)^\sharp \times x, y) = T(b^\sharp U_{a^\sharp} \times x, y) = T(b^\sharp U_{a^\sharp}, x \times y) = T(b^\sharp, (x \times y)U_{a^\sharp}) = T(b^\sharp, xU_a \times yU_a)$ (by (50))$= T(b^\sharp \times xU_a, yU_a) = T((b^\sharp \times xU_a)U_a, y)$ the foregoing relation becomes $T(xU_{bU_a}, y) = T(T(b, xU_a)bU_a - (b^\sharp \times xU_a)U_a, y) = T(xU_aU_bU_a, y)$. Hence $QJ3$ holds. To prove $QJ4$ we note that the definition of U_a and $V_{a,b}$ give $xV_{a,b} = T(x,a)b + T(a,b)x - (x \times b) \times a$. Hence

$$xV_{a,b}U_a - xU_aV_{b,a} = T(x,a)bU_a - ((x \times b) \times a)U_a \\ - T(xU_a, b)a + (xU_a \times a) \times b$$

Using the symmetry of $T(x \times y, z)$ and (48) we obtain

$$T(T(x,a)bU_a - ((x \times b) \times a)U_a - T(xU_a, b)a + \\ (xU_a \times a) \times b, y) = T(b, T(x,a)yU_a - \\ T(b, ((yU_a \times a) \times x) - T(b, T(a,y)xU_a + T(b, y \times (xU_a \times a)).$$

It suffices to show this is 0 and this will follows by showing that
71 $T(x,a)yU_a - (yU_a \times a) \times x$ is symmetric in x and y. We have $T(x,a)yU_a - (yU_a \times a) \times x = T(x,a)T(y,a)a - T(x,a)a^\sharp \times y - T(a,y)(a \times a) \times x + ((a^\sharp \times y) \times x = T(x,a)T(y,a)a - T(x,a)a^\sharp \times y - 2T(a,y)a^\sharp \times x + N(a)(y \times x) + T(a,y)a^\sharp \times x = T(x,a)T(y,a)a - T(x,a)a^\sharp \times y - T(a,y)a^\sharp \times x + N(a)(y \times x)$. Since this is symmetric in x and y we have $QJ4$. Also we have $xU_1 = T(x)1 - 1 \times x = x$ by (46). To prove $QJ5$ we observe that if ρ is an extension field of Φ then $QJ3$ and $QJ4$ are valid for $\mathscr{J}_\rho$ since the hypothesis made on N carry over to the polynomial extension of N to $\mathscr{J}_\rho$. In particular, there hold if $\rho = \Phi(\lambda)$. Then the argument in §3 shows that $QJ6 - 9$ hold in $\mathscr{J}$. Hence $\mathscr{J}$ is a quadratic Jordan algebra by Theorem 1.

A cubic form N and element 1 with $N91) = 1$ satisfying (i)-(iii) will be called *admissible*. We shall now give another imporatant example of an admissible $(N, 1)$ which is due to Tits (see the author's book [2], pp.412-422). Let $\mathfrak{a}$ be a central simple associative algebra of degree three (so $\dim \mathfrak{a} = q$) and let n be the generic (= reduced) norm on $\mathfrak{a}$, t the generic trace. Let $\mathscr{J} = \mathfrak{a} \oplus \mathfrak{a} \oplus \mathfrak{a}$ a direct sum of three copies of $\mathfrak{a}$. We write the elements of $\mathscr{J}$ as triples $x = (a_0, a_1, a_2)$, $a_i \in \mathfrak{a}$. Let μ be a non-zero element of Φ and define

$$N(x) = n(a_o) + \mu n(a_1) + \mu^{-1} n(a_2) - t(a_o a_1 a_2) \tag{51}$$

If we put $1 = (1, 0, 0)$ we have $N(1) = 1$. It is not difficult to verify that $(N, 1)$ is admissible. Hence $\mathscr{J}$ with the U-operator defined by (iii) is a
72 quadratic Jordan algebra. It can be shown that these algebras are of the form $\mathscr{H}(O_3)$ and hence are exceptional also. ◻

10 Inverses

If a, b are elements of an associative algebra such that $aba = a, ab^2a = 1$ then a is invertiable by the second condition, so $ab = 1 = ba$ by the first. Thus a is invertible with inverse $b = a^{-1}$. Conversely, if a is invertible then $aa^{-1}a = a$ and $aa^{-2}a = 1$. This motivates the following:

Definition 5. An element a of a quadratic Jordan algebra $(\mathscr{J}, U, 1)$ is *invertible* if there exists ab in $\mathscr{J}$ such that $aba = a, ab^2a = 1$. Then b is called an *inverse* of a.

The foregoing remark shows that if $\mathscr{J} = \mathfrak{a}^{(q)}$, $\mathfrak{a}$ associative then a is invertible in $\mathscr{J}$ with inverse b if and only if $ab = 1 = ba$ in $\mathfrak{a}$. If σ is a homomorphism of $\mathscr{J}$ into $\mathscr{J}'$ and a is invertible with inverse b then a^σ is invertible in $\mathscr{J}'$ with inverse b^σ. In particular if $\mathscr{J}' = \mathfrak{a}^{(q)}$ then $a^\sigma b^\sigma = 1 = b^\sigma a^\sigma$.

We have the following

Theorem on Inverses 1. The following conditions are equivalent: (i) a is invertible, (ii) U_a is invertible in End $\mathscr{J}$, (iii) $1 \in \mathscr{J} U_a$(2) If a is invertible the inverse b is unique and $b = aU_a^{-1}$. Also $U_b = U_a^{-1}$. and if we put $b = a^{-1}$ then a^{-1} is invertible and $(a^{-1})^{-1} = a$. (3) We have $a \circ a^{-1} = 2, a^2 \circ a^{-1} = 2a, V = V_a U_a^{-1} = U_a^{-1} V_a$. (4) aba is invertibel if and only if a and b are invertible, in which case $(aba)^{-1} = a^{-1}b^{-1}a^{-1}$.

73

Proof. (1) If $b^2U_a = 1$ then $1 = U_1 = U_{b^2U_a} = U_a U_{b^2} U_a$ so U_a is invertible. Then (i)⇒.(ii). Evidently(ii) ⇒ (iii). Now assume (iii). Then there exists a c such that $1 = cU_a$. Then $1 = U_a U_c U_a$ so U_a is invertible. Then there exists ab such that $bU_a = a$. Hence $U_b U_b U_a = U_a$ and since U_a is invertible, $U_a U_b = 1 = U_b U_a$. Then $b^2 U_a = 1 U_b U_a = 1$ so a is invertible with b as inverse. Thus (iii) ⇒ (i).

(2) If a is invertible with b as inverse then $bU_a = a$ and since U_a^{-1} exists, $b = aU_a^{-1}$ is unique. Also $U_a U_b U_a = U_a$ gives $U_a U_b = 1 U_b U_a$ so $U_{a^{-1}} = U_a^{-1}$. Also U_b -invertible implies that b is invertible and its inverse is $bU_b^{-1} = bU_a = a$. This completes the

proof of (2).

(3) By $QJ24$ we have $U_aV_a = U_{a,a^2} = V_aU_a$. Hence $(a^{-1}\circ a)U_a = a^{-1}V_aU_a = a^{-1}U_aV_a = aV_a = 2a^2 = 2U_a$. Since U_a^{-1} exists this gives $a^{-1}\circ a = 2$. By $QJ20$ we have $a^{-1}\circ a^2 = a^{-1}V_{a^2} = a^{-1}(V_a^2 - 2U_a) = 4a - 2a = 2a$. Also, by $QJ13$ and 24, $U_aV_{a^{-1}}U_a = U_{a^{-1}U_{a,a^2}} = U_{a,a^2} = U_aV_a = V_aU_a$. Hence $V_{a^{-1}} = U_a^{-1}V_a = V_aU_a^{-1}$ (4). The first assertion is clear since $U_{aba} = U_aU_bU_a$. Also if a and b are invertible then $(aba)^{-1} = (aba)U_{aba}^{-1} = bU_aU_a^{-1}U_b^{-1}U_a^{-1} = b^{-1}U_a^{-1} = b^{-1}U_{a^{-1}} = a^{-1}b^{-1}a^{-1}$.

Remark by M.B. Rege. There are two other conditions for a to be invertible with b as inverse which can be added to those given in

(1): (iv) $aba = a$ and $ba^2b = 1$, (v) $aba = a$ and b is the only element
74 of $\mathscr{J}$ satisfying this condition. These are well-known for associative algebras. The associative case of (iv) applied to U_a and U_b given (iv) in the Jordan case. (v) is an immediate consequence.

If n is a positive integer then we define $a^{-n} = (a^{-1})^n$. Then it is easy to extend $QJ32, 33$ to all integral powers. It is easy to see also that for arbitrary integral $m, n, U_{a^m,a^n}, V_{a^m,a^n}$ are contained in the (commutative) subalgebra of End $\mathscr{J}$ generated by U_a, V_a and U_a^{-1}.

A quadratic Jordan algebra $\mathscr{J}$ is called a *division algebra* if $1 \neq 0$ in $\mathscr{J}$ and every non-zero element of $\mathscr{J}$ is invertible. If is an associative division algebra then $\mathfrak{a}^{(q)}$ is a quadratic Jordan division algebra. Also if $(\mathfrak{a}, J)$ is an associative division algebra with involution then $\mathscr{H}(\mathfrak{a}, J)$ is a quadratic Jordan division algebra since if $0 \neq h \in \mathscr{H}$ then $(h^{-1})^J = (h^J)^{-1} = h^{-1} \in \mathscr{H}$. If Q is a quadratic form with basic point 1 on a vector space $\mathscr{J}$ then we have seen that the quadratic Jordan algebra $\mathscr{J}$ =Jord $(Q, 1)$ is special and can be identified with a subalgebra of $C(\mathscr{J}, Q, 1)^{(q)}$ where $C(\mathscr{J}, Q, 1)$ is the Clifford algebra of Q with base point 1. In C we have the equation $x^2 - T(x)xQ(x) = 0, x \in \mathscr{J}$. Hence $x\overline{x} = Q(x)1 = \overline{x}x$ for $\overline{x} = T(x)1 - x$. This shows that x is invertible in C if and only if $Q(x) \neq 0$ in which case $x^{-1} = Q(x)^{-1}\overline{x}$. It follows that x is invertible in $\mathscr{J}$ if and only if $Q(x) \neq 0$. Hence $\mathscr{J}$ =Jord$(Q, 1)$ is a division algebra if and onky if Q is unisotropic in the sense that $Q(x) \neq 0$ if $x \neq 0$ in $\mathscr{J}$.

75 The existence of exceptional Jordan division algebras was first es-

tablished by Albert. Examples of these can be obtained by using Tits construction of algebras defined by cubic forms as in §9. In fact, it can be seen that if the algebra $\mathfrak{a}$ used in Tits construction is a division algebra and μ is not a generic norm in $\mathfrak{a}$ then the Tits' algebra defined by $\mathfrak{a}$ and μ is a division algebra.

An element $a \in \mathscr{J}$ is called a *zero divisor* if U_a is not injective, equivalently, there exists $ab \neq 0$ in $\mathscr{J}$ such that $bU_a = 0$. Clearly, if a is invertible then it is not a zero divisor. An element z is called an *absolute zero divisor* if $U_z = 0$ and $\mathscr{J}$ is called *strongly non-degenerate* if $\mathscr{J}$ contains no absolute zero divisors $\neq 0$. This condition is stronger than the condition that $\mathscr{J}$ is non-degenerate which was defined by: ker $U = 0$, where ker $U = \{z | U_z = 0 = u_{z,a}, a \in \mathscr{J}\}$.

Let Φ be a field. An element $a \in (\mathscr{J}/\Phi, U, 1)$ is called *algebraic* if the subalgebra $\Phi[a]$ generated by a is finite dimensional. Clearly $\Phi[a]$ is the Φ-subspace spanned by the powers $a^m, m = 0, 1, 2, \ldots$, and we have the homomorphism of $\Phi[\lambda]^{(q)}$, λ an indeterminate, onto $\Phi[a]$ such that $\lambda \to a$. Let k_a be the kernal of this homomorphism. If the characteristic is $\neq 2$ then the ideals of $\Phi[\lambda]^{(q)}$ are the same as those of $\Phi[\lambda]^+$, which is the Jordan algebra associated with $\Phi[\lambda]^{(q)}$ by the category isomorphism. Since $ab = \frac{1}{2}(ab + ba) = a \cdot b$ in $\Phi[\lambda]$ we have $\Phi[\lambda]^+ = \Phi[\lambda]$ as algebras. Hence the ideals of $\Phi[\lambda]^{(q)}$ are ideals of the assoicative algebra $\Phi[\lambda]$ if char $\Phi \neq 2$. If char $\Phi = 2$, Example (3) of §5 shows that **76**
there exist ideals of $\Phi[\lambda]^{(q)}$ which are not ideals of $\Phi[\lambda]$. Let $\mathfrak{K}$ be an ideal $\neq 0$ in $\Phi[\lambda]^{(q)}$, $f(\lambda) \neq 0$ an element of $\mathfrak{K}$. Then $g(\lambda) f()^2 = g(\lambda) U_{f(\lambda)} \in \mathfrak{K}$. Hence $\mathfrak{K}$ contains the ideal $(f(\lambda)^2)$ of $\Phi[\lambda]$. The sum of all such ideals is an ideal $(m(\lambda))$ of $\Phi[\lambda]$. We may assume $m(\lambda)$ monic. In particular, if a is an algebraic element of $\mathscr{J}$ then $\mathfrak{K}_a$ contains a unique ideal $(m_a(\lambda))$ of $\Phi[\lambda]$ maximal in $\mathfrak{K}_a$ where $m_a(\lambda)$ is monic. We shall call $m_a(\lambda)$ the *minimim polynomial* of the algebraic element a. If $m_a(0) = 0$ so $m_a(\lambda) = \lambda h(\lambda)$ then $h(\lambda) \notin (m_a(\lambda))$ so these exists $ag(\lambda) \in \Phi[\lambda]$ such that $h(\lambda)g(\lambda) \in \mathfrak{K}_a$. Hence $h(a)g(a) \neq 0$ and $h(a)g(a)U_a = 0$. Thus $m_a(0) = 0$ implies that a is a zero divisor. On the other hand, suppose there exists a polynomial $f(\lambda)$ such that $f(a) = 0$ and $f(0) \neq 0$. Then we have a relation $1 = g(a)$ where $g(0) = 0$. Then $1 = g(a)^2$ and $g(\lambda)^2 = \lambda^2 h(\lambda)$. Then $1 = h(a)U_a$ and a is invertible by the Theorem

on Inverses. Hence an algebraic element a is either a zero divisior or is invertible according as $m_a(0) = 0$ or $m_a(0) \neq 0$. It is easily seen also that if a is algebraic then $\Phi[a]$ is a quadratic Jordan division algebra if and only if $\Re_a = (m_a(\lambda))$ where $m_a(\lambda)$ is irreducible. We have it to the reader to prove this.

If $\mathscr{J}$ is strongly non-degenerate then $\Re_a = (m_a(\lambda))$ for every algebraic element a. For, if $g(\lambda) \in \Re_a$ and $f(\lambda) \in \Phi[\lambda]$ then $U_{(fg)(a)}U_{g(a)} = 0(QJ34)$. Hence $(f,g)(a) = 0$ and $f(\lambda)g(\lambda) \in_a$. Thus $\Re_a$ is an ideal of $\Phi[\lambda]$ and $\Re_a = (m_a(\lambda))$ by definition of $(m_a(\lambda))$. ◻

11 Isotopes

77 This is an important notion in the Jordan theory which, like inverses, has an associative back ground.

Let $\mathfrak{a}$ be an associative algebra, c an invertible element $\mathfrak{a}$. Then we can define a new algebra $\mathfrak{a}(c)$ which is the same Φ-module as $\mathfrak{a}$ and which has the product $x_c y = xcy$. We have $(x_c y)_c = xcycz$ and $x_c(y_c z) = xcycz$ so $\mathfrak{a}^{(c)}$ is associative. Also $x_c c^{-1} = xcc^{-1} = x$ and $c_c^{-1} x = c^{-1}cx = x$ so c^{-1} is unit for $\mathfrak{a}(c)$. The mapping $c_R : x \to xc$ is an isomorphism of $\mathfrak{a}^{(c)}$ onto $\mathfrak{a}$ since $(x_c y)c_R = xcyc = (xc_R)(yc_R)$. An element u is invertible in $\mathfrak{a}$ if and only if, it is invertiable in $\mathfrak{a}^{(c)}$ since $uv = 1 = vu$ is equivalent to $u_c(c^{-1}vc^{-1}) = c^{-1} = (c^{-1}vc^{-1})_c u$. If d is invertible in $\mathfrak{a}$ (or $\mathfrak{a}^{(c)}$) then we can form the algebra $(\mathfrak{a}^{(c)})^{(d)}$. The product here is $x_{c,d} y = x_c d_c y = xcdcy$. Hence $(\mathfrak{a}^{(c)})^{(d)} = \mathfrak{a}^{(cdc)}$. In particular, if we taked $d = c^{-2}$ then we see that $(\mathfrak{a}^{(c)})^{c^{-2}} = \mathfrak{a}$. Finally, we consider the quadratic Jordan algebras $\mathfrak{a}^{(q)}$ and $\mathfrak{a}^{(c)(q)}$. The U-operator in the first is $U_a : x \to axa$ and in the second it is $U_a^{(c)} : x \to a_c x_c a = acxca$. Hence we have $U_a^{(c)} = U_c U_a$.

The considerations lead to the definition and basic properties of isotopy for quadratic Jordan algebras. Let $(\mathscr{J}, U, 1)$ be a quadratic Jordan algebra c an invertible element of $\mathscr{J}$. We define a mapping $U^{(c)}$ of $\mathscr{J}$ into End $\mathscr{J}$ by

$$U_a^{(c)} = U_c U_a \tag{52}$$

78 and we put $1^{(c)} = c^{-1}$. Evedently $U^{(c)}$ is a quadratic mapping we have

$U^{(c)}_{1^{(c)}} = U_c U_{c^{-1}} = 1$. Hence the axioms $QJ1$ and $QJ2$ for quadratic Jordan algebras hold. Also

$$\begin{aligned} U^{(c)}_{aU^{(c)}_b} &= U_c U_{aU_cU_b} = U_c U_b U_c U_a U_c U_b \\ &= U^{(b)}_b U^{(c)}_a U^{(c)}_b \end{aligned}$$

Hence $QJ3$ holds. Next we define $V^{(c)}_{a,b}$ by $xV^{(c)}_{a,b} = aU^{(c)}_{x,b}$ where $U^{(c)}_{a,b} = U^{(c)}_{a+b} - U^{(c)}_a - U^{(c)}_b = U_c U_{a+b} - U_c U_a = U_c U_b = U_c U_{a,b}$. Then $xV^{(c)}_{a,b} = aU_c U_{x,b} = xV_{aU_c,b}$. Thus $V^{(c)}_{a,b} = V_{aU_cb}$. Now we have

$$\begin{aligned} xV^{(c)}_{b,a} U^{(c)}_b &= bU^{(c)}_{x,a} U^{(c)}_b = bU_c U_{x,a} U_c U_b \\ &= bU_{xU_c,aU_c} U_b (\text{bilinearization of} QJ3) \\ &= xU_c U_{b,aU_c} U_b \\ &= xU_c U_b V_{aU_c,b} \quad (QJ4) \\ &= xU^{(c)}_b V^{(c)}_{a,b} \end{aligned}$$

Hence $QJ4$ holds for $(\mathscr{J}, U^{(c)}, 1^{(c)})$. It is clear also that these properties carry over to $\mathscr{J}_\rho$ for $\underline{\rho}$ any commutative associative algebra over Φ. Hence $(\mathscr{J}, U^{(c)}, 1^{(c)})$ is a quadratic Jordan algebra.

Definition 6. If c is an invertible element of $(\mathscr{J}, U, 1)$ then the quadratic Jordan algebra $\mathscr{J}^{(c)} = (\mathscr{J}, U^{(c)}, 1^{(c)})$ where $U^{(c)}_a = U_c U_a$, $1^{(c)} = c^{-1}$ is called the *c-isotope* of $(\mathscr{J}, U, 1)$. **79**

It is clear from the formula $U^{(c)}_a = U_c U_a$ and the fact that a is invertible in $\mathscr{J}$ if and only if U_a is invertible in End $\mathscr{J}$ that a is invertible in $\mathscr{J}$ if and only if it is invertible in the isotope $\mathscr{J}^{(c)}$. If d is another invertible element then we can form the d-isotope $(\mathscr{J}^{(c)})^{(d)}$ of $\mathscr{J}^{(c)}$. Its U operator is $U^{(c)(d)}$ where

$$\begin{aligned} U^{(c)(d)}_a &= U^{(c)}_d U^{(c)}_a = U_c U_d U_c U_a = U_{cdc} U_a \\ &= U^{(cdc)}_a \end{aligned}$$

Also we recall that cdc is invertible and $1^{(c)(d)} = (cdc)^{-1}$ since $d(U^{(c)}_d)^{-1} = d(U_c U_d)^{-1} = dU^{-1}_d U_{c^{-1}} = c^{-1} d^{-1} c^{-1} = (cdc)^{-1}$. Hence

$(\mathscr{J}^{(c)})^{(d)} = \mathscr{J}^{(cdc)}$. In this sense we have transitivity of the construction of isotopes. Also since $\mathscr{J}^{(1)} = \mathscr{J}$, $\mathscr{J}$ is its own isotope. Finally, we have $(\mathscr{J}^{(c)})^{(c^{-2})} = \mathscr{J}^{(cc^{-2}c)} = \mathscr{J}^{(1)} = \mathscr{J}$ so $\mathscr{J}$ is the c^{-2} -isotope of the c -isotope $(\mathscr{J}, c)$. In this sence the construction is symmetric.

Unlike the situation in the associative case isotopic quadratic Jordan algebras need not be isomorphic. An important instance of isotopy whihc gives examples of isotopic, non-isomorphic algebras is obtained
80 as follows. Let $(\mathfrak{a}, J)$ be an associative algebra with involution, h an invertible element of $\mathscr{H}(\mathfrak{a}, J)$. Then the mapping $K : x \to h^{-1}x^J h$ is also an involution in $\mathfrak{a}$. We claim that the quadratic Jordan algebra $\mathscr{H}(\mathfrak{a}, K)$ is isomorphic to the h-isotope of $\mathscr{H}(\mathfrak{a}, J)$. Let $x \in \mathscr{H}(\mathfrak{a}, J)$ then $xh_R = xh \in \mathscr{H}(\mathfrak{a}, K)$ since $(xh)^K = h^{-1}(xh)^J h = h^{-1}(hx)h = xh$. It follows that h_R is a Φ -isomorphism of $\mathscr{H}(\mathfrak{a}, J)$ onto $\mathscr{H}(\mathfrak{a}, K)$. Moreover, if $x, y \in \mathscr{H}(\mathfrak{a}, J)$ then $xU_y^{(h)}h_R = (xU_hU_y)h_R = yhxhyh = (xh_R)U_{yh_R}$. Hence h_R is an isomorphism of the quadratic Jordan algebra $\mathscr{H}(\mathfrak{a}, J)^{(h)}$ onto $\mathscr{H}(\mathfrak{a}, K)$.

It is easy to give examples such that $\mathscr{H}(\mathfrak{a}, J)$ and $\mathscr{H}(\mathfrak{a}, K)$ are not isomorphic. This gives examples of $\mathscr{H}(\mathfrak{a}, J)$ which is not isomorphic to the isotope to the isotope $\mathscr{H}(\mathfrak{a}, J)^{(h)}$ For example, let $\mathfrak{a} = \mathbb{R}_2$ the algebra of 2×2 matrices over the reals $\mathbb{R}$, J the standard involution in $\mathbb{R}_2$. If $a \in \mathscr{H}(\mathbb{R}_2)$ say $a = (a_{ij})$ then $\operatorname{tr} a^2 = \sum a_{ij}^2 \neq 0$. Hence a is not nilpotent. Let h =diag$\{1, -1\}$ and consider the involution $K : x \to h^{-1}xh$ in $\mathbb{R}_2$. Then $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathscr{H}(\mathbb{R}_2, K)$. Also $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}^2 = 0$. Hence $\mathscr{H}(\mathbb{R}_2, K)$ contains non-zero nilpotent elements. Thus $\mathscr{H}(\mathbb{R}_2)$ is not isomorphic to $\mathscr{H}(\mathbb{R}_2, K)$ and the latter is isomorphic to the isotope $\mathscr{H}(\mathbb{R}_2)^{(h)}$.

It is convenient (as in the foregoing discussion) to extend the notion of isotopy to apply to different algebras and to define isotopic mappings. Accordingly we give

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Definition 7. Let $(\mathscr{J}, U, 1)$, $(\mathscr{J}', U', 1)$ be quadratic Jordan algebras. A mapping η of $\mathscr{J}$ into $\mathscr{J}'$ is called an *isotopy* if η is an isomorphism of $(\mathscr{J}, U, 1)$ onto an isotope $\mathscr{J}'^{(c')}$ of $\mathscr{J}'$. If such a mapping exists then $(\mathscr{J}, U, 1)$ and $(\mathscr{J}', U', 1')$ are called *isotopic* (or *isotopes*)

Using $\eta = 1$ we see that the isotope $\mathscr{J}^{(c)}$ and $\mathscr{J}$ are isotopic in the sense of the present definition. Also it is clear that isomorphic algebras are isotopic. The definition gives $1^{\eta} = (c')^{-1}$ and $(xU_a)^{\eta} = x^{\eta}U'_{a^{\eta}}(c') = x^{\eta}U'_{c'}U'_{a^{\eta}}$ or $U_a\eta = \eta U'_{c'}U'_{a^{\eta}}$. Hence we have

$$U'_{a^{\eta}} = \eta^{*}U_a\eta \tag{53}$$

where $\eta^{*} = U'^{-1}_{c'}\eta^{-1}U'_{(c')^{-1}}\eta^{-1}$ is a module isomorphism of $\mathscr{J}'$ onto $\mathscr{J}$. Conversely, let η be a module isomorphism of $\mathscr{J}$ onto $\mathscr{J}'$ such that there exists a module isomorphism η^{*} of $\mathscr{J}'$ onto $\mathscr{J}$ satisfying (53). Then (53) implies that a is invertible in $\mathscr{J}$ if and only if a^{η} is invertible on $\mathscr{J}'$. Also $U'_1\eta = \eta^{*}\eta$ so $\eta^{*} = U'_{1^{\eta}}\eta^{-1} = U'^{-1}_{c'}\eta^{-1}$ where $c' = (1^{\eta})^{-1}$. Then $U^{\eta}_a = \eta U'_{c'}U'_{a^{\eta}} = \eta U'_{a^{(c')}}$ and η is an isomorphism of $(\mathscr{J}, U, 1)$ onto $(\mathscr{J}', U'^{(c')}, 1'^{(c')}, 1'^{(c')})$ so η is an isotopy of $\mathscr{J}$ onto $\mathscr{J}'$. Hence we have shown that a module isomorphism η of $\mathscr{J}$ onto $\mathscr{J}'$ is an isotopy if and only if there exists a module isomorphism η^{*} of $\mathscr{J}'$ onto $\mathscr{J}$ satisfying (53). It is cleat r that the isotopy η is an isomorphism of $\mathscr{J}$ onto $\mathscr{J}'$ if and only if $\eta^{*} = \eta^{-1}$ and $1^{\eta} = 1'$. The latter condition implies the former since we have $\eta^{*} = U'_1\eta^{-1}$. Hence an isotopy η is an isomorphism if and only if $1^{\eta} = 1'$. 82

If η is an isotopy of $\mathscr{J}$ onto $\mathscr{J}'$ and (53) holds then $U_{a'}\eta^{-1} = (\eta^{*})^{-1}U'_{a'}\eta - 1$ which shows that η^{-1} is an isotopy of $\mathscr{J}'$ onto $\mathscr{J}$. If $\mathscr{J}$ is an isotopy of $\mathscr{J}'$ onto $\mathscr{J}''$ and $U''_{a'\zeta} = \zeta U_{a'}\zeta, a' \in \mathscr{J}'$, then $U''_{a^{\eta\zeta}} = \zeta^{*}\eta^{*}U_a\eta\zeta$. Hence $\eta\zeta$ is an isotopy of $\mathscr{J}$ onto $\mathscr{J}''$ and $(\eta\zeta)^{*} = \zeta^{*}\eta^{*}$. It is clear from this that isotopy is an equivalence relation. Since η^{-1} is an isotopy it is an isomorphism of $\mathscr{J}'$ onto an isotope of $\mathscr{J}$. Hence η is also an isomorphism of an isotope of $\mathscr{J}$ onto $\mathscr{J}'$ (as well as of $\mathscr{J}$ on an isotope of $\mathscr{J}'$).

The set of isotopies of $\mathscr{J}$ onto $\mathscr{J}$ is a group of transformations of $\mathscr{J}$. Following Koecher, we call this the *structure group* of $\mathscr{J}$ and we denote it as Str $\mathscr{J}$. Clearly Str $\mathscr{J}$ contians the group of automorphisms Aut $\mathscr{J}$ as a subgroup. Moreover, Aut $\mathscr{J}$ is the subgroup of Str$\mathscr{J}$ of such that $1^{\eta} = 1$. If c is invertible then U_c is a module isomorphism of $\mathscr{J}$ onto $\mathscr{J}$ and $U_{aU_c} = U_cU_aU_c$. Hence (53) holds for $\eta = U_c, \eta^{*} = U_c$ so $U_c \in$ Str $\mathscr{J}$. It is clear from the foregoing discussion that U_c is an isomorphism of $(\mathscr{J}, U, 1)$ onto the $c^{-2} = (1U_c)^{-1}$ isotope

$(\mathscr{J}, U^{(c^{-2})}, c^2)$. In particular, if $c^2 = 1$ then U_c is an automorphism of $(\mathscr{J}, U, 1)$. The subgroup of Str $\mathscr{J}$ generated by the U_c, c invertible is called the *inner structure group*. We denote this as Instr. $\mathscr{J}$. If $\eta \in$ Str $\mathscr{J}$, $U_{a^\eta} = U_{1\eta}\eta^{-1}U_a\eta$ so $\eta^{-1}U_a\eta = (U_{1\eta})^{-1}U_{a^\eta}$ which implies that Instr $\mathscr{J}$ is a normal subgroup of Str $\mathscr{J}$. It follows also that Aut $\mathscr{J} \cap$ Instr
83 $\mathscr{J}$ is normal in Str $\mathscr{J}$. We call this the group of *inner automorphisms*.

We have seen that if η is an isotopy of $\mathscr{J}$ on $\mathscr{J}'$ then $\eta^* = (U_{1^\eta})^{-1}$ η^{-1}. since $(U_{1^\eta})^{-1} \in$ Str $\mathscr{J}'$ and η^{-1} is an isotopy of $\mathscr{J}'$ onto $\mathscr{J}$. We see that η^* is an isotopy of $\mathscr{J}'$ onto $\mathscr{J}$. In particular, if $\eta \in$Str $\mathscr{J}$ then $\eta^* \in$ Str $\mathscr{J}$. We have $(\eta\zeta)^* = \zeta^*\eta^*$ and $U_c^* = U_c$ for invertible c. Hence $\eta^* = (\eta^*)^{-1}U_{1^\eta}^{-1} = \eta U_{1^\eta}U_{1^n}^{-1} = \eta$ Thus $\eta \to \eta^*$ is an anti-automorphism of Str $\mathscr{J}$ such that $\eta^{**} = \eta$ and this stabilzes Instr $\mathscr{J}$.If η is an automorphism then $U_{a^\eta} = \eta^{-1}U_a\eta$ so $\eta^* = \eta^{-1}$.

Let c be invertible and consider the isotope $(||mathscrJ, c)$. Let $\eta \in$Str $\mathscr{J}$, so $U_{a^\eta} = \eta^* U_a^\eta$, $a \in \mathscr{J}$. Then $U_{a^\eta}^{(c)} = U_cU_{a^\eta} = U_c\eta^*U_a\eta = (U_c\eta^*U_c^{-1})U_cU_a\eta = (U_c\eta^*U_c^{-1})U_a^{(c)}\eta$. Hence $\eta \in$Str $\mathscr{J}^{(c)}$. By symmetry, Str $\mathscr{J}^{(c)}$ =Str$\mathscr{J}$. Similarly, one sees that Instr $\mathscr{J}^{(c)}$ =Instr $\mathscr{J}$.

If $\mathscr{Z}$ is an inner(outer) ideal in $\mathscr{J}$ then $\mathscr{Z}$ is an inner (outer) ideal of the isotope $\mathscr{J}^{(c)}$. For, if $\mathscr{Z}$ is inner and $b \in \mathscr{Z}$ then $\mathscr{J}U_b^{(c)} = \mathscr{J}U_cU_b = \mathscr{J}U_b \subseteq \mathscr{Z}$ and if $\mathscr{Z}$ is outer and $a \in \mathscr{J}$ then $\mathscr{Z}U_a^{(c)} = \mathscr{Z}U_cU_a \subseteq \mathscr{Z}$. Since an isotopy of $\mathscr{J}$ into $\mathscr{J}'$ maps an isotope of $\mathscr{J}$ onto $\mathscr{J}'$ it is clear that if $\mathscr{Z}$ is an inner (outer) ideal in $\mathscr{J}$ then $\mathscr{Z}^\eta$ is an inner (outer) ideal of $\mathscr{J}'$.

Chapter 2

Pierce Decompositions. Standard Quadratic Jordan Matrix Algebras

In this chapter we shall develop two of the main tools for the structure 84
theory: Peirce decomposition and the strong coordinatization Theorem. The corresponding discussion in the linear case is given in the author's book [2], Chapter III.

1 Idempotents. Pierce decompositions

An element e of $\mathscr{J} = (\mathscr{J}, U, 1)$ is *idempotent* if $e^2 = e$. Then $e^3 = eU_e = e^2U_e = (e^2)^2(QJ23) = e^2 = e$. Then $e^n = e^{n-2}U_e = e$ for $n \geqq 1$, by induction. Also $U_e^n = U_{e^n} = U_e$. The idempotents e and f are said to be *orthogonal* ($e \perp f$) if $e \circ f = fU_e = eU_f = 0$. If we apply $QJ12$ with $a = e$, $c = f$ to 1 we see that $e \circ f = 0$ and $fU_e = 0$ imply $eU_f = 0$. Hence e and f are orthogonal if $e \circ f = 0$ and either $eU_f = 0$ ar $fU_e = 0$. If e and f are orthogonal then $e + f$ is idempotent since $(e + f)^2 = e^2 + e \circ f + f^2 = e + f$. If e, f, g are orthogonal idempotents (that is, $e \perp f, g$ and $f \perp g$) then $(e + f) \circ g = (e + f)U_g = 0$ so $(e + f) \perp g$. It follows that if e, f, g, h are orthogonal idempotents then $e + f$ and $g + h$ are orthogonal idempotents. If e is idempotent then $f = 1 - e$

is idempotent since $f^2 = (1-e)^2 = 1 = -e \circ 1 + e = 1 - e$. Also $e \circ f = e \circ 1 - e \circ e = 2e - 2e = 0$ and $fU_\circ = (1-e)U_e = e - e = 0$ so e and f are orthogonal.

85 We recall that an endomorphism E of a module is called a projection if E is idempotent: $E^2 = E$, and the projections E and F are orthogonal if $EF = 0 = FE$. If E is a projection and X is an endomorphism satisfying the Jordan conditions: $EXE = EX + XE = 0$ then it clear that $EX = 0 = XE$. We now prove

Lemma 1. *If e and f are orthogonal idempotents in* $\mathscr{J}$ *then* U_e, U_f *and* $U_{e,f}$ *are orthogonal projections. If* e, f, g *are orthogonal idempotents then the projections* U_e, $U_{e,f}$ *and* $U_{f,g}$ *are orthogonal. If* e, f, g, h *are orthogonal idempotents then* $U_{e,f}$ *and* $U_{g,h}$ *are orthogonal.*

Proof. We have seen that $U_e^2 = U_{e^2}$, so $U_e^2 = U_e$ is a projection. We have $U_e U_{e,f} U_e = U_{e^2.eof}(QJ11) = 0$ and $U_e U_{e,f} U_e = U_{eU_{e,f}U_e}(QJ3) = 0$. Hence

$$U_e U_{e,f} = 0 = U_{e,f} U_e \tag{1}$$

Also $U_e U_f + U_f U_e = U_e^2 U_f + U_f U_{e^2} = -U_{e,f} U_e U_{e,f} + U_{eU_e, eU_f} + U_{eU_{e,f}}(QJ7) = 0$ by (1), $eU_f = 0$ and $eU_{e,f} = eef = e^2 \circ f = 0$. Since $U_e U_f U_e = U_{fU_e} = 0$ we have

$$U_e U_f = 0 = U_f U_e. \tag{2}$$

Now $e + f$ is idempotent so $U_{e+f} = U_c + U_{e,f} + U_f$ is idempotent. By (1) and (2) this gives $U_e + U_{e,f}^2 + U_f = U_e + U_{e,f} + U_f$. Hence

$$U_{e,f}^2 = U_{e,f} \tag{3}$$

86 Since $e \perp f, g, e \perp f + g$, so $U_e U_{f+g} = 0 = U_{f+g} U_e$. By (2) this gives

$$U_e U_{f,g} = 0 = U_{f,g} U_e. \tag{4}$$

we have $U_{e+f} U_{f,g} U_{e+f} = U_{fU_{e+f}}, gU_{e+f} = 0$ since g and $e + f$ are orthogonal. Since $U_{e+f} = U_e + U_f + U_{e,f}$ this, (1) and (4) gives $U_{e,f} U_{f,g} U_{e,f} = 0$. Also $U_{e,f} U_{f,g} + U_{f,g} U_{e,f} + U_f U_{e,g} + U_{e,g} U_f = U_{e \circ f fog} + U_{f^2, eog}$. (taking $b = 1, a = f, c = e, d = g$ in $QJ8$)= 0. Hence

$$U_{e,f} U_{f,g} = 0 = U_{f,g} U_{e,f}. \tag{5}$$

Finally, $e \perp g, h$ and $f \perp g, h$ so $e + f \perp g, h$. Hence, by (4), $U_{e,f} U_{g,h} = (U_{e+f} - U_e - U_f) U_{g,h} = 0$ and $U_{g,h} U_{e,f} = 0$.

A set of orthogonal idempotents $\{e_i | i = 1, 2, \ldots, n\}$ will be called *supplementery* if $\sum e_i = 1$. Then this gives

$$1 = U_1 = \sum_1^n U_{e_i} + \sum_{1<j} U_{e_i,e_j}. \tag{6}$$

The foregoing lemma shown that the $n(n+1)/2$ operators U_{e_i}, U_{e_i,e_j} with distinct subscripts are orthogonal projections. Since they are supplementary in End $\mathscr{J}$ in the sense that their sum is 1 we have

$$\mathscr{J} = \sum_{i \leq j} \bigoplus \mathscr{J}_{ij}, \mathscr{J}_{ii} = \mathscr{J} U_{e_i}, \mathscr{J}_{ij} = \mathscr{J} U_{e_i,e_j}, i < j \tag{7}$$

which we call the *pierce decomposition* of $\mathscr{J}$ *relative to the* e_i. We shall call $\mathscr{J}_{ij}$ the *pierce* (i, j)*-component* of $\mathscr{J}$ relative to the e_i. $\mathscr{J}_{ii} = U_{e_i}$ is **87**
an inner ideal called the *Pierce inner ideal determined by the idempotent* e_i.

We shall now derive a list of formulas for the products $a_{ij} U_{b_{kl}}$ where $a_{ij} \epsilon \mathscr{J}_{ij}, b_{kl} \epsilon \mathscr{J}_{kl}$. For this purpose we require □

Lemma 2. *If* $a_{ij} \epsilon \mathscr{J}_{ij}$ *then*

$$U_{e_i} U_{a_{ii}} U_{e_i} = U_{a_{ii}} \tag{8}$$

$$U_{a_{ij}} = U_{e_i} U_{a_{ij}} U_{e_j} + U_{e_j} U_{a_{ij}} U_{e_i} + U_{e_i e_j}, U_{a_{ij}} U_{e_i,e_j}, i \neq j \tag{9}$$

$$V_{a_{ii}} = U_{e_i} U_{a_{ii},e_i} U_{e_i} + \sum_{j \neq i} (U_{e_i} V_{aii} U_{e_j} + U_{e_j} V_{a_{ii}} U_{e_i} + U_{e_i,e_j} V_{a_{ii}} U_{e_i,e_j}) \tag{10}$$

$$U_{a_{ii},c_{ij}} = U_{e_i} U_{a_{ii},c_{ij}} U_{e_i,e_j} + U_{e_i,e_j} U_{a_{ii} c_{ij}} U_{e_i} i \neq j \tag{11}$$

Proof. The first is clear from $QJ3$ and $a_{ij} U_{e_i} = a_{ii}$. The second follows by taking $a = e_i, b = a_{ij}, c = e_j$ in $QJ7$. For (1) we have $V_{a_{ii}} = U_{1,a_{ii}} = U_{e_i,a_{ii}} + \sum_{j \neq i} U_{e_j a_{ii}}$. Then $U_{e_i,a_{ii}} = U_{e_i} U_{e_i,a_{ii}} U_{e_i}$ by $QJ3$ and $U_{e_j,a_{ii}} = U_{e_i} V_{a_{ii}} U_{e_j} + U_{e_j} V_{a_{ii}} U_{e_i} + U_{e_i,e_j} V_{a_{ii}} U_{e_i,e_j}$ follows by putting $a = e_i, b = a_{ii}, c = e_j$ in $QJ15$. Hence (10) holds. To obtain (11) we belinearrize $QJ6$ relative to b to obtain $U_a U_{b.d} U_{a,c} + U_{a,c} U_{b,d} U_a = U_{bU_a, du_{a,c}} + U_{dU_a, b} U_{a,c}$ and put $a = e_i, b = a_{ij}, c = e_j, d = c_{ij}$ in this.

88 To formulate the results on the products $a_{ij} U_{b_{kl}}$ of elements in Pierce components in a compact from we consider triples of unordered pairs of induced taken from $\{1, 2, \ldots, n\}$: (pq, rs, uv). In any pair pq we allow $p = q$ and we assume $pq = qp$. Also we identify $(pq, rs, uv) = (uv, rs, pq)$. We shall call such a triple *connected* if it can be written as

$$(pq, qr, rs)$$

It is easily seen that the only triples which are not connected are those of one of the following two forms:

$$\begin{aligned} &(pq, rs, -) \quad \text{with} \quad \{p, q\} \cap \{r, s\} = \\ &(pq, qr, qs) \quad \text{with} \quad r \neq p, q, s. \end{aligned}$$

We can now state the important □

Pierce decomposition theorm. Let $\{e_i | i = 1, 2, \ldots, n\}$ be a supplementary set of orthogonal idempotents, $\mathscr{J} = \sum \mathscr{J}_{ij}$ the corresponding Pierce decomposition of $\mathscr{J}$. Let $a_{pq} \epsilon \mathscr{J}_{pq}$ etc. Then for any connected triple (pq, qr, rs) we have

$$\{a_{pq} b_{qr} c_{rs}\} \epsilon \mathscr{J}_{ps} \qquad \text{PD 1}$$

$$b_{qr} U_{a_{pq}} = a_{pq} b_{qr} a_{pq} \epsilon \mathscr{J}_{PS} \quad \text{if} \quad pq = rs \qquad \text{PD 2}$$

If (pq, rs, uv) is not connected then

$$\begin{aligned} \{a_{pq} b_{rs} c_{uv}\} &= 0 \quad \text{and} \quad b_{rs} U_{a_{pq}} = a_{pq} b_{rs} a_{pq} \qquad \text{PD 3} \\ &= 0 \quad \text{for} \quad pq = uv \end{aligned}$$

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Also

$$\begin{aligned} &\{a_{pq} b_{qr} c_{rs}\} = (a_{pq} \circ b_{qr}) \circ c_{rs} \\ &\quad \text{if} (qr, pq, rs) \quad \text{is not connected} \qquad \text{PD 4} \\ &\{a_{pq} b_{qr} c_{rp}\} = ((a_{pq} \circ b_{qr}) \circ c_{rp}) U_{e_p} \quad \text{if} p \neq r. \end{aligned}$$

If $p \neq q$ then

$$a_{pq}V_{e_p} = a_{pq}, a_{pq}V_{a_{pp}}U_{b_{pp}} = a_{pq}V_{b_{pp}}V_{a_{pp}}V_{b_{pp}}. \qquad \text{PD 5}$$

(In other words, if $\overline{V}_{a_{pp}}$ denotes the restriction of $V_{a_{pp}}$ to $\mathscr{J}_{pq}$ then $a_{pp} \to \overline{V}_{a_{pp}}$ is a homomorphism of the quadratic Jordan algebra $(\mathscr{J}_{pp}, U, e_p)$ into $(\text{End } \mathscr{J}_{pq})^{(q)}$. Finally, we have

$$e_qU_{a_{pq}} = a_{pq}^2U_{e_p}, p \neq q. \qquad \text{PD 6}$$

Proof. We prove first $PD1-3$. The formulas in this set of $\{a_{pq}b_{rs}c_{uv}\}$ with $pq = ur$ are obtained by bilinearization of $b_{rs}U_{a_{pq}}$. Hence we may drop $\{a_{pq}b_{rs}c_{uv}\}$ for $pq = uv$. Then the only formula in $PD1-3$ involving just one index that we have to prove is $b_{ii}U_{a_{ii}}\epsilon\mathscr{J}$. This is clear from (8). Next we consider the formulas in PD $1-3$ which involve two distinct induces i, j. These are: 90

$$b_{ii}U_{a_{ij}}\epsilon\mathscr{J}_{jj} \qquad (12)$$
$$b_{ij}U_{a_{ij}}\epsilon\mathscr{J}_{ij} \qquad (13)$$
$$\{a_{ij}b_{ji}c_{ii}\}\epsilon\mathscr{J}_{ii} \qquad (14)$$
$$\{a_{ii}b_{ii}c_{ij}\}\epsilon\mathscr{J}_{ij} \qquad (15)$$
$$\{a_{ii}b_{ij}c_{jj}\}\epsilon\mathscr{J}_{ij} \qquad (16)$$
$$b_{ji}U_{a_{ii}} = 0, b_{ij}U_{a_{ii}} = 0, \{a_{ii}b_{jj}c_{ij}\} = 0, \{a_{ii}b_{jj}c_{jj}\} = 0 \qquad (17)$$

(12) and (13) follow from (9), and the first two equations in (17) follow from (8). (14) and (15) and the third part of (17) follow from (11). To prove (16) and the last part of (17) we note first that $\mathscr{J}_{ii}\circ\mathscr{J}_{ij} \subseteq \mathscr{J}_{ij}$ and $\mathscr{J}_{ii}\circ\mathscr{J}_{jj} = 0$ if $i \neq j$. The first of these is an immediate consequence of (10). Also (10) implies that $\mathscr{J}_{ii}\circ\mathscr{J}_{jj} \subseteq \mathscr{J}_{ii}$. By symmetry, $\mathscr{J}_{ii}\circ\mathscr{J}_{jj} \subseteq \mathscr{J}_{jj}$ and since $\mathscr{J}_{ii} \cap \mathscr{J}_{jj} = 0$ we have $\mathscr{J}_{ii} \circ \mathscr{J}_{jj} = 0$. By $QJ27$, we have $\{a_{ii}b_{ij}c_{jj}\} = -\{b_{ij}a_{ii}c_{jj}\} + (a_{ii} \circ b_{ij}) \circ c_{jj} = (a_{ii} \circ b_{ij}) \circ c_{jj}$ (third of (17)) $\epsilon\mathscr{J}_{ij}$. Also $\{a_{ii}b_{jj}c_{jj}\} = -\{b_{jj}a_{ii}c_{jj}\} + (a_{ii} \circ b_{jj}) \circ c_{jj} = 0$, by the first of (17) and $\mathscr{J}_{ii} \circ \mathscr{J}_{jj} = 0$. Next we consider $PD1-2$ for three distinct indices i, j, k. The formulas we have to establish are

$$\{a_{ij}b_{ij}c_{jk}\}\epsilon\mathscr{J}_{ik} \qquad (18)$$

$$\{a_{ij}b_{jj}c_{jk}\}\epsilon\mathscr{J}_{ik} \tag{19}$$
$$\{a_{ij}b_{ji}c_{ik}\}\epsilon\mathscr{J}_{ik} \tag{20}$$
$$\{a_{ij}b_{jk}c_{ki}\}\epsilon\mathscr{J}_{ii} \tag{21}$$

91 To prove these we make the following observation. Let S and T be non-vacaous disjoint subsets of the index set $\{1, 2, \ldots, n\}$ and put $e_s = \sum\limits_{i\epsilon S} e_i, e_T = \sum\limits_{j\epsilon T} e_j$. It follows easily, as before, that e_S and e_T are orthogonal idempotents. Also $\mathscr{J} U_{e_S} = \mathscr{J} U_{\sum e_i} \subseteq \sum\limits_{i\epsilon S} U_{e_i} + \sum\limits_{i,i''\epsilon S} \mathscr{J} U_{e_i} e_i$ and since the U_{e_i} and $U_{e_i,e_{i'}}$ are orthogonal projections with sum U_{e_S}, $\mathscr{J} U_{e_i} = \mathscr{J} U_{e_i} U_{e_S}$ and $\mathscr{J} U_{e_i,e'_i} = \mathscr{J} U_{e_i,e'_i} U_{e_S}$. Hence $\sum\limits_{i\epsilon S} \mathscr{J} U_{e_i} + \sum\limits_{i,i'\epsilon S} \mathscr{J} U_{e_i,e_i} \subseteq \mathscr{J} U_{e_S}$ and we have the equality $\mathscr{J} U_{e_S} = \sum\limits_{i\epsilon S} \mathscr{J} U_{e_i} + \sum\limits_{i,i'\epsilon S} \mathscr{J} U_{e_i,e'_i} = \sum\limits_{i,i'\epsilon S} \mathscr{J}_{ii'}$. Similary, we have $U_{e_S,e_T} = \sum\limits_{\substack{i\epsilon S\\ j\epsilon T}} \mathscr{J}_{ij}$. We now consider the supplementary set of orthogonal idempotents $\{e_j + e_k, e_l, l \neq j, k\}$. Since $a_{ii}\epsilon\mathscr{J} U_{e_i}, a_{ij}, b_{ij}, c_{ik}, c_{ki}\epsilon\mathscr{J} U_{e_i,e_s}, e_s = e_j + e_k$ and $b_{jj}, c_{jk}, b_{jk}\epsilon\mathscr{J} U_{e_S}$ (14)-(16) imply that the left hand sides of (18)-(21) are contained in $\mathscr{J}_{ij} + \mathscr{J}_{ik}, \mathscr{J}_{ij} + \mathscr{J}_{ik}, \mathscr{J}_{ij} + \mathscr{J}_{ik}, \mathscr{J}_{ii}$ respectively. Similarly, if we use the set of orthogonal idempotents $\{e_i + e_j, e_l, l \neq i, j\}$ we see that the left hand sides of (18)-(20) are contained in $\mathscr{J}_{ik} + \mathscr{J}_{ik}$. Since $(\mathscr{J}_{ij} + \mathscr{J}_{ik}) \cap (\mathscr{J}_{ik} + \mathscr{J}_{jk}) = \mathscr{J}_{ik}$ we obtain (18)-(20). Next
92 we consider the case of four distinct induces i, j, k, l. The only connected triple here is (ij, jk, kl). If we use the set of orthogonal idempotents $\{e_i + e_j + e_k, e_m, m \neq i, j, k\}$ as just indicated we obtain that $\{a_{ij}b_{jk}c_{kl}\}\epsilon\mathscr{J}_{il} + \mathscr{J}_{jl} + \mathscr{J}_{kl}$. Similarly, using $\{e_j + e_k + e_l, e_m, m \neq j, k, l\}$ we get that $\{a_{ij}b_{jk}c_{kl}\}\epsilon\mathscr{J}_{ij} + \mathscr{J}_{kj} + \mathscr{J}_{il}$. Taking the intersection of the right hand sides gives $\{a_{ij}b_{jk}c_{kl}\} \in \mathscr{J}_{il}$. Since a connected triple cannot have more than four distinct induces this concludes the proof of $PD1$ and $PD2$. We consider next the triples which are not connected. The first possibility is $(pq, rs, -)$ with $\{p, q\} \cap \{r, s\} = \phi$. Choose a subset S of the index set so that $p, q\epsilon S, r, s \notin S$ and put $e_S = \sum\limits_{i\epsilon S} e_i, e_T = \sum\limits_{j\notin S} e_j$. Then we can conclude $\{a_{pq}b_{rs}-\} = 0$ and $b_{rs}U_{a_{Pq}} = 0$ from (17) applied to the set of orthogonal idempotents $\{e_S, e_T\}$ since $a_{pq}\epsilon\mathscr{J} U_{e_S}$ and $b_{rs}\epsilon\mathscr{J} U_{e_T}$. Finally suppose we have (pq, qr, qs) where $r \neq p, q, s$. In

this case we obtain $\{a_{pq}b_{qr}c_{qr}\} = 0$ by applying the second part of (17) to the two orthogonal idempotents e_r and $e'_r = 1 - e_r$. This proves $PD3$. To prove $PD4$ we note that the hypothesis that (qr, pq, rs) is not connected and $PD3$ imply that $\{b_{qr}a_{pq}c_{rs}\} = 0$. The first part of $PD4$ follows from this and $QJ27$. For the second part of $PD4$ we note that $\{a_{pq}b_{qr}c_{rp}\}\epsilon\mathscr{J}_{pp}$ by $PD1$ so $\{a_{pq}b_{qr}c_{rp}\} = \{a_{pq}b_{qr}c_{rp}\}$. By $QJ27$, $\{a_{pq}b_{qr}c_{rp}\} = -\{b_{qr}a_{pq}c_{rp}\} + (a_{pq} \circ b_{qr}) \circ c_{rp}$. Since $\{b_{qr}a_{pq}c_{rp}\}\epsilon\mathscr{J}_{rr}$ and $r \neq p$, applying U_{e_p} to the two sides of the foregoing equations gives $PD4$. If $p \neq q$ we have $a_{pq}V_{e_p} = a_{pq}U_{1,e_p} = a_{pq}U_{e_q}, e_p + \sum_{l\neq q} a_{pq}Ue_l, e_p = a_{pq}U_{e_p}, e_q + \sum_{m\neq q,p} a_{pq}U_{e_m,e_p} + 2a_{pq}U_{e_p} = a_{pq}$ since a_{pq} since $a_{pq}\epsilon\mathscr{J}_{pq}$ and the U_{e_i}, U_{e_i,e_j} are orthogonal projections. This is the first part of $PD5$. The second part follows directly from $QJ21$ and **93**
$a_{pq}U_{a_{pp}} = 0 = a_{pq}U_{a_{pp},b_{pp}}$. To obtian $PD6$ we use $a^2_{pq}U_{e_p} = 1U_{a_{pq}}U_{e_p} = e_pU_{a_{pq}}U_{e_p} + e_qU_{a_{pq}}U_{e_p} + e_qU_{a_{pq}}U_{e_p}$ (by PD 3). Since $e_pU_{a_{pq}} \in \mathscr{J}_{pp}$ and $q_qU_{a_{pq}} \in \mathscr{J}_{pp}$ this reduced to $e_qU_{a_{pq}}$, which proves $PD6$.

The formaulas $PD4$ 4 imply some uesful associatively formulas for $\circ$. Suppose we have a connected triple (pq, qr, rs) such that (qr, pr, rs) and (pq, rs, qr) are note connected. Then we can apply PD also to $\{c_{ps}b_{qr}a_{pq}\}$ to obtain.

$$(a_{pq} \circ b_{qr}) \circ c_{rs} = a_{pq} \circ (b_{qr} \circ c_{rs}) \tag{22}$$

If $(qr.pr.rs)$ and (pq, rs, qr) are not connected. Special cases of this are

$$(a_{ii} \circ a_{ij}) \circ a_{jj} = a_{ii} \circ (a_{ij} \circ a_{jj}), i \neq j \tag{23}$$

$$(a_{ij} \circ a_{jj}) \circ a_{jk} = a_{ij} \circ (a_{jj} \circ a_{jk}), i, j, k \neq \tag{24}$$

$$(a_{ij} \circ a_{jk}) \circ a_{kl} = a_{ij} \circ (a_{jk} \circ a_{kl}, i, j, k, l \neq . \tag{25}$$

Similarly we have the following consequece of the second part of $PD4$:

$$((a_{pq} \circ b_{qr}) \circ c_{rp})U_{e_p} = (a_{pq} \circ (b_{qr\circ}c_{rp}))U_{e_p}, p \neq q, r \tag{26}$$

We note also that the PD theorem permits us to deduce the following formulas for the squaring composition and its bilinearization: **94**

$$a^2_{pq}\epsilon\mathscr{J}_{pp} + \mathscr{J}_{qq}, a_{pq} \circ a_{qr}\epsilon\mathscr{J}_{pr}\text{if}\{p, q\}\Phi\{qr\} \tag{27}$$

$$a_{pq} \circ a_{rs} = 0 \quad \text{if} \quad \{p, q\} \cap \{r, s\} = \emptyset.$$

We leave this to the reader to check. It is easily verified also that

$$\mathscr{J}_{ij}\{x | xV_{e_i} = x = xV_{e_j}\}, i \neq j \tag{28}$$

□

2 Standard quadratic Jordan matrix algebras

Let $(\mathscr{O}, j)$ be a (unital) non-associative algebra with involution, $\mathscr{O}_o$ a Φ-submodule of $\mathscr{H}(\mathscr{O}, j)$ containing all the *norms* $a\overline{a}(\overline{a} = a^j).a\epsilon\mathscr{O}$. Then $\mathscr{O}_o$ contains every $a\overline{b} + b\overline{a}, a, b\epsilon\mathscr{O}$ hence all the *tracis* $a + \overline{a}$. It follows that, if Φ contains $\frac{1}{2}$ then $\mathscr{O}_o = \mathscr{H}(\mathscr{O}, j)$.On the other hand, as the example of an octonian algebra with standard involution of a field of characteristic two shows, we may have $\mathscr{O}_o \subset \mathscr{H}(\mathscr{O}, j)$. We consider the algebra $\mathscr{O}_n$ of $n \times n$ matrices with entries in $\mathscr{O}$ and the standard involution $J_1 : A \to \overline{A}^t$ in $\mathscr{O}_n$. Let $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ be the set of matrices with entries in $\mathscr{O}$ satisfying $\overline{A}^t = A$ and having diagonal entries in $\mathscr{O}_o$. We use the notation we introduced in considering $\mathscr{H}(\mathscr{O}_3) \equiv \mathscr{H}(\mathscr{O}_3\Phi)$ and write

$$\begin{aligned} \alpha[ii] &= \alpha e_{ii}, \alpha\epsilon\mathscr{O}_o \\ a[ij] &= ae_{ij} + \overline{a}e_{ij}, a\epsilon\mathscr{O}, i \neq j. \end{aligned} \tag{29}$$

95 Then it is clear that $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ is the set of sums of the matrices $\alpha[ii]$ and $a[ii]$. Let $\mathscr{H}_{ii}(\mathscr{O}_o) = \{\alpha[ii] | \alpha\epsilon\mathscr{O}_o\}$, $\mathscr{H}_{ij} = \{a[ij] | a\epsilon\mathscr{O}\}$ for $i \neq j$. Then we have

$$\mathscr{H}(\mathscr{O}_n, o) = \sum_i \mathscr{H}_{ii}(\mathscr{O}_o) + \sum_{i<j} \mathscr{H}_{ij} \tag{30}$$

and the sum is direct. Let A^2 denote the usual square of the matrix $A\epsilon\mathscr{O}_n$ and put $A \circ B = AB + BA$. Then we have the following formulas:

$$\alpha[ii]^2 = \alpha^2[ii] \tag{M1}$$

$$a[ij]^2 = a\overline{a}[ii] + \overline{a}a[jj], i \neq j \tag{M2}$$

$$\alpha[ii] \circ a[ij] = \alpha a[ij], i \neq j \qquad \text{M3}$$

$$a[ij] \circ b[jk] = ab[ik], i, j, k \neq \qquad \text{M4}$$

$$\alpha[ii] \circ \beta[jj] = 0, \alpha[ii] \circ a[jk] = 0$$

$$a[ij] \circ b[kl] = 0, i, j, k, l \neq \qquad \text{M5}$$

It is clear that these formulas together with $a[ij] = \overline{a}[ji]$ determine A^2 for $A \epsilon \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ and they show that $A^2 \epsilon \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ if $A \epsilon \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$. Hence also $A \circ B \epsilon \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ if $A, B \epsilon \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$.

Let V_A in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ denote the endomorphism $X \to X \circ A$, and suppose from now on that $n \geq 3$. Suppose we have the following identity in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$:

$$[V_A, V_{B \circ C}] + [V_B V_{A \circ C}] + [V_C V_{A \circ B}] = 0 \qquad (31)$$

If we write $[A, B, C]_o$ for the associator $(A \circ B) \circ C - A \circ (B \circ C)$ then **96**
(31) is the same as

$$[A, D, B \circ C]_o + [B, D, A \circ C]_o + [C, DA \circ B]_o = 0 \qquad (31')$$

Assume first $n \geq 4$ and take $A = a[ij]$, $B = b[jk]$, $C = c[kl]$, $D = 1[ll]$ where $i, j, k, l \neq$. This gives $[a, b, c][il] = 0$ where $[a, b, c] = (ab)c - a(bc)$ the associator in $\mathscr{O}$. Hence $[a, b, c] = 0$, $a, b, c \epsilon \mathscr{O}$, so $\mathscr{O}$ must be associative if $n \geq 4$ and (31) holds. Next let $n = 3$. Let $\alpha \epsilon \mathscr{O}_o$ and take $A = \alpha[ii]$, $B = b[jk]$, $C = l[kk]$, $D = d[ij]$ in (25') where $i, j, k \neq$. This gives $[\alpha, d, b][ik] = 0$ so

$$[\alpha, a, b] = 0, \quad \alpha \epsilon \mathscr{O}_o, a, b \epsilon \mathscr{O} \qquad (32)$$

Assume next that in addition to (25) we have the identity

$$[V_A V_{A \circ B}] + [V_B V_{A^2}] = 0 \qquad (33)$$

or

$$[A, D.A \circ B]_0 + [B, D, A^2]_o = 0 \qquad (33')$$

Taking $A = a[ij], D = 1[kk], B = b[jk], i, j, k \neq$, we obtain $[\overline{a}, a, b]$ $[jk] = 0$ so $[\overline{a}, a, b] = 0, a, b \epsilon \mathscr{O}$. Since $\mathscr{O}_o$ contians all the traces $a + \overline{a}$ we have by (33), $[a + \overline{a}, a, b] = 0$ and, by the result just proved,

$[a, a, b] = 0$. Applying the involution we obtain $[b, a, a] = 0$. Thus $\mathscr{O}$ must be alternative. Then (33) implies that $\mathscr{O}_o \subseteq N(\mathscr{O})$ the nucleus of $\mathscr{O}$. In particular, we see that we have the conditions on $\mathscr{O}$ we noted in **97**
§1.8, namely, $\mathscr{O}$ is alternative and all norms $x\overline{x} \epsilon N(\mathscr{O})$. We saw also that if these conditions hold then the Φ -module N_o spanned by the norms and $N' = N(\mathscr{O}) \cap \mathscr{H}(\mathscr{O}, j)$ have the property that $xN_o\overline{x} \subseteq N_o$, $xN'\overline{x} \equiv N_o, x\epsilon\mathscr{O}$. We note also that the two conditions (31) and (33) are consequences of $[V_A V_{A^2}] = 0$ and the hypothesis that this carries over $\mathscr{O}_{\underline{\rho}}$ where $\underline{\rho}$ is commutative associative algebra over the base ring Φ. Our results give the following

Lemma . *Let $(\mathscr{O}, j)$ be an algebra with involution over $\Phi, \mathscr{O}_o$ a Φ-submodule of $\mathscr{O}$ containing all $x\overline{x}, x\epsilon\mathscr{O}$. Let $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ be the Φ-module of matrices $A\epsilon\mathscr{O}_n$ such that $\overline{A}^t = A$ and the diagonal elements of A are in $\mathscr{O}_o$. If $A\epsilon\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ let A^2 be usual square of $A, A \circ B = AB + BA$. Assume $n \geqq 3$.Then the identities (31) and (33) in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ imply that $\mathscr{O}$ is associative if $n \geqq 4$ and $\mathscr{O}$ is alternative and $\mathscr{O}_o \subseteq N(\mathscr{O})$ if $n = 3$.*

We can now prove the following

Theorem 1. *Let $(\mathscr{O}, j)$ be an algebra with involution, $\mathscr{O}_o$ a Φ -sub module of $\mathscr{H}(\mathscr{O}.j)$ containing all the norms $x\overline{x}, x\epsilon\mathscr{O}$. Let $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ be the set of $n \times n$ matrices with entries in $\mathscr{O}$ such that $\overline{A}^t = A$ and the diagonal elements are in $\mathscr{O}_o$. Assume $n \geq 3$. Then there exists at most one quadratic Jordan structure on $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ satisfying the following conditions: $1U_A = A^2$ the usual matrix square, the elements $e_i = 1[ii] = e_{ii}, i = 1, 2, \ldots, n$ are a supplementary set a orthogonal*
98 *idempotents in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$, the submodule $\mathscr{H}_{ij} = \{a[ij] | a\epsilon\mathscr{O}\}$, $i \neq j$ is the pierce (i, j) -module and $\mathscr{H}_{ii}(\mathscr{O}_o) = \{\alpha[i, i] | \alpha\epsilon\mathscr{O}_o\}$ is the pierce (i, i)-module relative to the set $\{e_i\}$. Necessary condition for the existence of such a structure are: $\mathscr{O}$ associative if $n > 3$, $\mathscr{O}$ alternative with $\mathscr{O}_o \subseteq N\mathscr{O}$ if $n = 3$ and $x\mathscr{O}_o\overline{x} \subseteq \mathscr{O}$, $x\epsilon\mathscr{O}$.*

Proof. Suppose we have a quadratic mapping U so that $(\mathscr{J}, U, 1)$ is quadratic Jordan and the given conditions hold. Then we shall establish the following formulas for the U operator, in which $i, j, k, l \neq, \alpha, \beta\epsilon\mathscr{O}_o$, $a, b, \epsilon\mathscr{O}$:

QM 1 $\beta[ii]U_{\alpha[ii]} = (\alpha(\beta)\alpha[ii]$

QM 2 $\alpha[ii]U_{[ij]} = \overline{a}(\alpha a)[jj]$

QM 3 $b[ij]U_{a[ij]} = a(\overline{b}a)[ij]$

QM 4 $\{\alpha[ii]a[ij]b[ji]\} = ((\alpha a)b + \overline{(\alpha a)b})[ii]$

QM 5 $\{\alpha[ii]\beta[ii]a[ii]\} = \alpha(\beta a)[ij]$

QM 6 $\{\alpha[ii]a[ij]\beta[jj]\} = \alpha(a\beta)[ij]$

QM 7 $\{\alpha[ii]a[ij]b[jk]\} = \alpha(ab)[ik]$

QM 8 $\{a[ij]\alpha[jj]b[jk]\} = a(\alpha b)[ik]$

QM 9 $\{a[ij]b[ji]c[ik]\} = a(bc)[ik]$

QM 10 $\{a[ij]b[jk]c[ki]\} = (a(bc) + \overline{a(bc)})[ii]$

QM 11 $\{a[ij]b[jk]c[kl] = a(bc)[il]$.

□

The formulas $QM4 - QM11$ are immediate consequences of $PD4$ 99
and $M1-M4$. To prove $QM1$ we note that $\beta[ii]U_{\alpha[ii]}\epsilon\mathscr{H}(\mathscr{O}_o)$ so this has the form $\gamma[ii], \gamma\epsilon\mathscr{O}_o$. Then $\gamma[ij] = \beta[ii]U_{\alpha[ii]}o1[ij] = 1[ij]V_{\beta[ii]U_{\alpha[ii]}} = 1[ij]V_{\alpha[ii]}V_{\beta[ii]}(PD6) = (\alpha\beta)\alpha[ij](M3)$ Hence $QM1$ holds. For $QM2$ we recall the identity

$$U_bV_a = V_{b,a}V_b + V_aV_b - V_bV_{a,b} \qquad \text{(QJ 18)}$$

Let $k \neq i, j$. Then $\alpha[ii]U_{a[ij]} \circ 1[jk] = \alpha[ii]U_{a[ij]}V_{1[jk]} = \alpha[ii]$ $V_{a[ij],1[jk]}, V_{a[ij]} + \alpha[ii]V_{1[jk]}U_{a[ij]} - \alpha[ii]V_{a[ij]}V_{1[jk],a[ij]} = \overline{a}(\alpha a)[jk]$ by $PD4$ and $M1 - 4$. Since $\alpha[ii]U_{a[ij]}\epsilon\mathscr{J}_{jj}$ this proves $QM2$. To prove $QM3$ we again use $QJ18$ to write $b[ij]U_{a[ij]} \circ 1[jk] = b[ij]U_{a[ij]}V_{1[jk]} = b[ij]V_{a[ij],1[jk]}V_{a[ij]}$ since the other two terms given by $QJ18$ are 0 be $PD3$. Also the first term is $\{1[jk]b[ij]a[ij]\} \circ a[ij] = ((1[jk] \circ b[ij]) \circ a[ij]) \circ a[ij] = a(\overline{b}a)[ik]$. Hence $QM3$ holds. The PD theorem and the argument just used shows that U is unique. Since the identity $(V_AV_{A^2}) =$

0 holds in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ and in extensions obtained by extending the ring Φ, the Lemma implies that is associative if $n > 3$ and alternative with $\mathscr{O}_0 \subseteq N(\mathscr{O})$ if $n = 3$. It is clear from $QM2$ that $x\mathscr{O}_o\overline{x} \subseteq \mathscr{O}_o$. This completes the proof.

The conditions for $n > 3$ given in this theorem are clearly sufficient since in this case $\mathscr{O}_n$ is associative and $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ is a subalgebra of $\mathscr{O}_n^{(q)}$. Moreover, it is easy to see that the square $1U_A$ in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ co-
100 incides with the usual A^2 (cf. the proof of the Corollary to Theorem 1.5 (§1.8)) and the conditions on the $e_i 1[ii]$ hold. The unique quadratic Jordan structure given in this case is that defined by $BU_A = ABA$. We now consider the case $n = 3$. Suppose $\mathscr{O}$ is an alternative algebra such that all norms $x\overline{x} \epsilon N(\mathscr{O})$. Let N_o be the submodule generated by all norms, $N' = \mathscr{H}(\mathscr{O}, j) \cap N(\mathscr{O})$. Then we have shown in §1.8 that $xN_o\overline{x} \subseteq N_o$ and $xN'\overline{x} \subseteq N'$. Hence we can take these as choices for the submodule $\mathscr{O}_o$. It is clear also that any $\mathscr{O}_o$ satisfying the conditions of the theorem satisfies $N' \supseteq \mathscr{O}_o \supseteq N_o$. It can be verified by a rather lengthy fairly direct calculation that $\mathscr{H}(\mathscr{O}_3, N')$ with the usual 1 and the U operator defined by $QM1-11$ is a quadratic Jordan algebra. We omit the proof of this (due to McMrimmon). In the associative case $N' = \mathscr{H}(\mathscr{O}, j) \cap N(\mathscr{O}) = \mathscr{H}(\mathscr{O}, j)$ and $\mathscr{H}(\mathscr{O}_n, N') = \mathscr{H}(\mathscr{O}_n)$ the complete set of hermitian matrices with entries in $\mathscr{O}$. Accordingly, we shall now define a *standard quadratic Jordan matrix algebra* to be any algebra of the form $\mathscr{H}(\mathscr{O}_n), n = 1, 2, 3, \ldots,$ to be any algebra of the form $\mathscr{H}(\mathscr{O}_n), n = 1, 2, 3, \ldots$, where $(\mathscr{O}, j)$ is an associative algebra with involution *or* an algebra $\mathscr{H}(\mathscr{O}_3, N')$ where $(\mathscr{O}, j)$ is alternative with involution such that all norms $x\overline{x}\epsilon N(\mathscr{O})$ and $N' = \mathscr{H}(\mathscr{O}, j) \cap N(\mathscr{O})$. For the sake of uniformity we abbreviate $\mathscr{H}(\mathscr{O}_3, N') = \mathscr{H}(\mathscr{O}_3)$. It is easily seen that if $n \geqq 3$ and N_o is the submodule of $\mathscr{H}(\mathscr{O}, j)$ spanned by the norms then $\mathscr{H}(\mathscr{O}_n, N_o)$ is the come of $\mathscr{H}(\mathscr{O}_n)$.

101 **Theorem 2.** *Let $\mathscr{H}(\mathscr{O}_n)$, $n \geqq 3$, be a standard quadratic Jordan matrix algebra. A subset $\mathscr{J}$ of $\mathscr{H}(\mathscr{O}_n)$ is an outer ideal containing 1 if and only if $\mathscr{J} = \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ where $\mathscr{O}_o$ is a Φ-submodule of $N' = \mathscr{H}(\mathscr{O}, j) \cap N(\mathscr{O})$ such that $1\epsilon\mathscr{O}_o$ and $x\mathscr{O}_o\overline{x} \subseteq \mathscr{O}_o$, $x\epsilon\mathscr{O}$. $\mathscr{J} = \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ is simple if and only if $(\mathscr{O}, j)$ is simple.*

Proof. Let $\mathscr{O}_o$ be a submodule of N' containing 1 and every $x\alpha\overline{x}$,

$x\epsilon\mathscr{O}$, $\alpha\epsilon\mathscr{O}_o$. Then $\mathscr{O}_o$ contains all the norms $x\bar{x}$ and all the traces $x+\bar{x}$. It is therefore clear from $QM1-11$ (especially $QM1,2,4,10$) that $\mathscr{H}(\mathscr{O}_n,\mathscr{O}_o)$ is an outer ideal. Since $1\epsilon\mathscr{O}_o$, $\mathscr{H}(\mathscr{O}_n,\mathscr{O}_o)$ contains $1=\sum_1^n 1[ii]$. Conversely, let $\mathscr{J}$ be an outer ideal of $\mathscr{H}(\mathscr{O}_n)$ containing 1. Then $\mathscr{J}$ contains $e_i=1U_{e_i}, e_i=1[ii]$ and every $a[ij]=\{e_ie_ia[ij]\}$, $a\epsilon\mathscr{O}$, $i\neq j$. Also, if $\beta[ii]\epsilon\mathscr{J}$ then $\beta[jj]=\beta[ii]U_{1[ij]}\epsilon\mathscr{J}$ for $j\neq i$. If $b\epsilon\mathscr{J}$ and $b=\sum_{i\leqq j} b_{ij}$, $b_{ij}\epsilon\mathscr{H}_{ij}$, then $b_{ii}=bU_{e_i}\epsilon\mathscr{J}$. It is clear from these results that $\mathscr{J}=\mathscr{H}(\mathscr{O}_n,\mathscr{O}_o)$ where $\mathscr{O}_o$ is a submodule of N' containing 1. Since $\alpha[ii]U_{a[ij]}=\bar{a}\alpha a[jj]\epsilon\mathscr{J}$ if $\alpha\epsilon\mathscr{O}_o$, $a\epsilon\mathscr{O}, i\neq j$, it is clear that $a\mathscr{O}_o\bar{a}\subseteq\mathscr{O}_o$, $a\epsilon\mathscr{O}$. Let $\mathscr{Z}$ be an ideal in $(\mathscr{O},j)$ and let k be the subset of $\mathscr{J}=\mathscr{H}(\mathscr{O}_n,\mathscr{O}_o)$ of matrices all of whose entries are in $\mathscr{Z}$. Then inspection of $QM1-QM11$ shows that k is an ideal of $\mathscr{J}$. Hence simplicity of $\mathscr{J}$ implies simplicity of $(\mathscr{O},j)$. Conversely, suppose $(\mathscr{O},j)$ is simple and let k be an ideal $\neq 0$ in $\mathscr{J}$. If $b\epsilon k$ and $b=\sum_{i\leqq j} b_{ij}, b_{ij}\epsilon\mathscr{H}_{ij}$, then operating on b with U_{e_i} or U_{e_i,e_j} shows that ev- **102**
ery $b_{ij}\epsilon k$. Let $\mathscr{Z}=\{b\epsilon\mathscr{O}|b[12]\epsilon k\}$. We now use the formulas $M1-M5$ for the squaring operator, which are consequences of $QM1-QM11$. Since k is an ideal it follows from $M4$ that $\mathscr{Z}=\{b|b[ij]\epsilon k, i\neq j\}$ and $\mathscr{Z}$ is an ideal of $(\mathscr{O},j)$. Also, by $M3$, if $\beta[ii]\epsilon k$ then $\beta\epsilon\mathscr{Z}$. It is clear from these results that $\mathscr{Z}\neq 0$ so $\mathscr{Z}=\mathscr{O}$, hence $a[ij]\epsilon k$ for all $a\epsilon\mathscr{O}$, $i\neq j$. Now let $\alpha\epsilon\mathscr{O}_o$. Then, by $QM2$, $\alpha[jj]=\alpha[ii]U_{1[ij]}\epsilon k$ if $i\neq j$. Hence $k=\mathscr{J}$ and $\mathscr{J}$ is simple.

It is clear from the first part of Theorem 2 that $\mathscr{H}(\mathscr{O}_n.N_o)$, N_o the submodule generated by the norms is the core of $\mathscr{H}(\mathscr{O}_n)$ (=outer ideal generated by:1). □

3 Connectedness and strong connectedness of orthogonal idem-potents

In this section we shall give some lemmas on orthogonal idem-potents which will be used to prove the Strong Coordinatization Theorem (in the next section) and will play a role in the structure theory of chapter III.

Definition 1. If e_1 and e_2 are orhtogonal idempotents in $\mathscr{J}$ then e_1 and e_2 are *connected* (strongly *connected*) in $\mathscr{J}$ if the Pierce submodule $\mathscr{J}_{12} = \mathscr{J}U_{e_1,e_2}$ contains an element u_{12} which is invertible in $\mathscr{J}U_e, e = e_1 + e_2$ (satisfies $u_{12}^2 = e_1 + e_2$). Then we say also that e_1 and e_2 are *connected*(*strongly connected*) by U_{12}.

Note that $u^2 = 1$ implies $U_u^2 = 1$ so u is invertible. Thus strong connectedness implies connectedness. Moreover, if e_1 and e_2 are strongly
103 connected by u_{12} then u_{12} is its own inverse in $\mathscr{J}U_e$. For, $u_{12}U_{u_{12}} = u_{12}U_{e_1,e_2}U_{u_{12}} = e_1V_{u_{12},e_2}U_{u_{12}} = e_1U_{u_{12}}V_{e_2,U_{12}}(QJ4) = u_{12}^2U_{e_2}V_{e_2,u_{12}}$ $(PD6) = e_2V_{e_2,u_{12}} = \{e_2e_2u_{12}\} = e_2^2 \circ u_{12} = u_{12}$.

If U_{12} connects e_1 and e_2 then $\mathscr{J}_{11}U_{u_{12}} \subseteq \mathscr{J}_{22}$, $\mathscr{J}_{22}U_{u_{12}} \subseteq \mathscr{J}_{11}$, $\mathscr{J}_{12}U_{u_{12}} \subseteq \mathscr{J}_{12}$ by $PD2$. Hence if U' is the inverse of the restriction of $U_{u_{12}}$ to U_e then $\mathscr{J}_{12}U' = \mathscr{J}_{12}$. Hence the inverse $u_{21} = u_{12}U'$ of u_{12} in $\mathscr{J}U_e$ is contained in $\mathscr{J}_{12}$. If $f = 1 - e$ then f and e are orthogonal idempotents and since $e_i \epsilon \mathscr{J}U_e$, $\{e_1, e_2, f\}$ is a supplementary set of orthogonal idempotents in $\mathscr{J}$. It is clear also from the PD formulas that $\mathscr{J}U_e + \mathscr{J}U_f$ is a subalgebra of $\mathscr{J}$, that U_e and U_1 are ideals in this subalgebra and $\mathscr{J}U_e + \mathscr{J}U_f = \mathscr{J}U_e \oplus \mathscr{J}U_f$. If follows that if $x\epsilon \mathscr{J}U_e$, $y \epsilon \mathscr{J}U_f$ then $x + y$ is invertible in $\mathscr{J}U_e + \mathscr{J}U_f$ if and only if x is invertible in $\mathscr{J}U_e$ and y is invertible in $\mathscr{J}U_f$. In this case, the definition of invertibility shows that $x + y$ is invertible in $\mathscr{J}$. Also if x' and y' respectively are the inverse of x and y in $\mathscr{J}U_e$ and $\mathscr{J}U_f$ then $x' + y'$ is the invese of $x + y$ in $\mathscr{J}U_e + \mathscr{J}U_f$ and so in $\mathscr{J}$. In particular, if $u_12 \epsilon \mathscr{J}_{12} = \mathscr{J}U_{e_1,e_2}$ is invertible in $\mathscr{J}U_e$ with inverse $u_{21}\epsilon \mathscr{J}_{12}$ then $c_{12} = f + u_{12}$ is invertible in $\mathscr{J}$ with inverse $c_{21} = f + u_{21}$. We have the Pierce decomposition $\mathscr{J} = \mathscr{J}U_e \oplus \mathscr{J}U_{e,f} \oplus \mathscr{J}U_f$ relative to $\{e, f\}$ and we have the following usefull formulas for the action of $U_{c_{12}} = U_{u_{12}} + u_f + u_{f,u_{12}}$ on these submodules:

$$xU_{c12} = xU_{u12}, x\epsilon \mathscr{J}U_e(PD3)$$

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$$\begin{aligned} yU_{C_{12}} &= y \circ u_{12}, y\epsilon \mathscr{J}U_{e,f}(PD3,4,5) \\ zU_{c_{12}} &= z, z\epsilon \mathscr{J}U_f(PD3) \end{aligned} \tag{34}$$

If $U_{12}^2 = e$ so $u_{21} = u_{12}$ then $c_{12}^2 = 1$ and $U_{c_{12}}$ is an automorphism of $\mathscr{J}$ such that $U_{c_{12}^2} = 1$ (see §1.11).

Lemma 1. *Let e_1, e_2, e_3 be pairwise orthogonal idempotents in $\mathscr{J}$ such that e_1 and e_2 are connected (strongly connected) by u_{12} and e_2 and e_3 are connected (strongly connected) by u_{23}. Then e_1 and e_3 are connected (strongly connected) by $u_{13} = u_{12} \circ u_{23}$ In the strongly connected case $c_{12} = u_{12} + 1 - e_1 - e_2, c_{23} = u_{23} + 1 - e_2 - e_3$, $c_{13} = u_{13} + 1 - e_1 - e_3$ satisfy $c_{ij}^2 = 1$, $U_{c_{ij}}$ is an automorphism such that $U_{c_{ij}}^2 = 1$ and we have*

$$c_{13} = c_{12}U_{c_{23}} = c_{23}U_{c_{12}} \qquad (35)$$

$$U_{c_{13}} = U_{c_{23}}U_{c_{12}}U_{c_{23}} = U_{c_{12}}U_{c_{23}}U_{c_{12}} \qquad \text{so}$$

Proof. Put $e_4 = 1 - e_1 - e_2 - e_3$ so $\{e_i | i = 1, 2, 3, 4\}$ is a supplementary set of orthogonal idempotents. Let $\mathscr{J} = \sum \mathscr{J}_{ij}$ be the corresponding Pierce decomposition. Then $c_{12} = u_{12} + e_3 + e_4, c_{23} = u_{23} + e_1 + e_4$. Since u_{12} is invertible in $\mathscr{J}_{11} + \mathscr{J}_{12} + \mathscr{J}_{22}$ with inverse u_{21}, c_{12} is invertible in $\mathscr{J}$ with inverse $c_{21} = u_{21} + e_3 + e_4$. Similarly, c_{23} is invertible with inverse $c_{32} = u_{32} + e_1 + e_4$. By the Theorem on inverse $c_{13} = c_{23}U_{c_{12}}$ is 105
invertible with inverse $c_{31} = c_{32}U_{c_{21}}$. We have

$$\begin{aligned} c_{13} = c_{23}U_{c_{12}} &= u_{23}u_{c_{12}} + e_1U_{c_{12}} + e_4U_{c_{12}} \\ &= u_{13} + u_{12}^2 + e_1U_{u_{12}} + e_4 \quad \text{(by(34))} \\ &= u_{13} + u_{12}^2U_{e_2} + e_4(PD6). \end{aligned}$$

Similarly, $c_{31} = u_{31} + e_1U_{u_{21}} + e_4$. Since $u_{12}^2U_{e_2}$ and $u_{21}^2U_{e_2} \epsilon \mathscr{J}_{22}$ it follows that u_{13} is invertible in $\mathscr{J}_{11} + \mathscr{J}_{13} + \mathscr{J}_{31}$ with inverse u_{31}. Hence e_1 and e_3 are connected by $u_{13} = u_{12} \circ u_{23}$. If $u_{12}^2 = e_1 + e_2$ and $u_{23}^2 = e_2 + e_3$ then $c_{12}^2 = 1$ and $c_{23}^2 = 1$. Then $U_{c_{12}}, U_{c_{23}}$ are automorphisms with square 1. Also $c_{13} = c_{23}U_{c_{12}}$ satisfies $c_{13}^2 = 1$ so $U_{c_{13}}$ is an automorphism and $U_{c_{13}}^2 = 1$. The formula for c_{13} now becomes $c_{13} = u_{23} \circ u_{12} + e_2 + e_4$. A similar calculation gives $c_{12}U_{c_{23}} = u_{12} \circ u_{23} + c_2 + c_4$. Hence $c_{12}U_{c_{23}} = c_{23}U_{c_{12}}$ so (35) and its consequence $U_{c_{23}}U_{c_{12}}U_{c_{23}} = U_{c_{12}}U_{c_{23}}U_{c_{12}}$ hold.

The following lemma is of technical importance since it permits the reduction of considerations on connected idempotents to strongly connected idempotents. □

Lemma 2. *Let $\{e_i i = 1, \ldots, n\}$ be a supplementary set of orthogonal idempotents in $\mathscr{J}$, $\mathscr{J} = \sum \mathscr{J}_{ij}$ the corresponding Pierce decomposition. Assume e_1 and e_j, $j > 1$, are connected by u_{1j} with inverse u_{ji} in $\mathscr{J}_{11} + \mathscr{J}_{1j} + \mathscr{J}_{jj}$. Then*

$$u_j = u_{ij}^2 U_{e_j} \quad and \quad v_j = u_{j1}^2 U_{e_j}, j > 1 \tag{36}$$

106 *are inverses in $\mathscr{J}_{jj}$ and if we put $u_1 = e_1 = v_1$ then $u = \sum_1^n u_i$, $v = \sum_1^n v_i$ are inverses in $\mathscr{J}$. The set $\{u_i\}$ is a supplementry set of orthogonal idempotents in the v-isotope $\widetilde{\mathscr{J}} = \mathscr{J}^{(v)}$ and u_j is strongly connected by u_{1j} to u_1 in $\mathscr{J}$. The Pierce submodule $\widetilde{\mathscr{J}}_{ij}$ of $\widetilde{\mathscr{J}}$ relative to the u_i coincides with $\mathscr{J}_{ij}$. Moreover, $\mathscr{J}_{11} = \widetilde{\mathscr{J}}_{11}$ are algebras and $\mathscr{J}_{jj}$ and $\widetilde{\mathscr{J}}_{jj}$, $j > 1$, are isotpic. Also, if $j > 1$, $x_{11} \epsilon \mathscr{J}_{11}$, $x_{1j} \epsilon \mathscr{J}_{1j}, x_{1j} V_{x_{11}} = x_{1j} \widetilde{V}_{x_{11}}$ where $\widetilde{V}$ is the V-operator in $\widetilde{\mathscr{J}}$.*

Proof. It is clear that $u_{1j}^2, u_{j1}^2 \epsilon \mathscr{J}_{11} + \mathscr{J}_{jj}$ and these are inverses in $\mathscr{J}_{11} + \mathscr{J}_{jj}$. It follows that $u_j = u_{1j}^2 U_{ej}$ and $v_1 = u_{jl}^2 U_{e_j}$ are inverses in $\mathscr{J}_{jj}$ and $u = \sum_1^n u_i$, $v = \sum_1^n v_i (u_1 = e_1 = v_1)$ are inverses in $\mathscr{J}$. Now consider the isotope $\widetilde{\mathscr{J}} = \mathscr{J}^{(v)}$ with unit element u. We have $U_{u_i}^{(v)} = U_v U_{ui} = \sum_j U_{v_i} U_{u_i} + \sum_{j<k} U_{v_j, v_k} U_{u_i}$. It is clear from the PD. Theorem ($PD1-3$) that $\mathscr{J} U_{v_j} \subseteq \mathscr{J}_{jj}$, $\mathscr{J} U_{v_j, v_k} \subseteq \mathscr{J}_{jk}$ and $\mathscr{J}_{jk}$ and $\mathscr{J}_{pq} U_{u_i} = 0$ unless $p = q = i$. Hence $U_{u_i^{(v)}} = U_{v_i} U_{u_i}$ and $\mathscr{J} U_{v_i} U_{u_i} = \mathscr{J}_{ii} U_{v_i} U_{u_i}$. Also since the restrictions of U_{v_i} and U_{u_i} to $\mathscr{J}_{ii}$ are inverses we have

$$U_{u_i}^{(v)} = U_{e_i}. \tag{37}$$

Similarly, replacing e_i by $e_i + e_j, u_i$ by $u_i + u_j, v_i$ by $v_i + v_j$, $i \neq 1$. We obtain $U_{u_i + u_j}^{(v)} = u_{e_i + e_j}$. This and (37) imply

$$U_{u_i, u_j}^{(v)} = U_{e_i, e_j}, i \neq j \tag{38}$$

107 Now $uU_{ui}^{(v)} \left(\sum_1^n u_k \right) U_{e_i} = u_i, uU_{u_i, u_j}^{(v)} = \sum_k u_k U_{e_i, e_j} = 0$ and $u_i U_{U_j}^{(v)} = u_i U_{e_j} = 0$ if $i \neq j$. These shows that the u_i are orthogonal idempotents

in the isotope $\widetilde{\mathscr{J}} = \mathscr{J}^{(v)}$. Since their sum is u they are supplementary. Then (37) and (38) show that $\widetilde{\mathscr{J}_{ii}} = \mathscr{J}_{ii}$, $\widetilde{\mathscr{J}_{ij}} = \mathscr{J}_{ij}$ for the corresponding Pierce submodules. Since $u_{1j} \epsilon \mathscr{J}_{ij}, u_j \epsilon \mathscr{J}_{1j}$. More over, $uU^{(v)}_{u_{1j}} = uU_v U_{u_{lj}} = vU_{u_{lj}} = (e_1 + v_j)U_{u_{1j}} = e_1 U_{u_{1j}} + v_1 U_{u_{1j}} = u^2_{1j} U_{e_j} + u^2_{j1} U_{e_j} U_{u_{1j}}$ (PD 6 and ((36)))$= u_j + e_1 U_{u_{jl}} U_{u_{lj}} = u_j + e_1$. Hence u_{1j} strongly connects e_1 and u_i in $\widetilde{\mathscr{J}}$. Let $x_i, y_i \epsilon \mathscr{J}_{ii} = \widetilde{\mathscr{J}_{ii}}$, $x \epsilon \mathscr{J}_{lj} = \widetilde{\mathscr{J}_{lj}}$, $j > 1$. Then $x_i U^{(v)}_{y_i} = x_i U_v U_{y_i} = x_i U_{v_i} U_{y_i}$. If $i = 1$, $v_1 = e_1$ so $x_1 U^{(v)}_{y_1} = x_i U_{y1}$. Thus $\mathscr{J}_{ii}$ and $\widetilde{\mathscr{J}_{ii}}$ are isotopic $\mathscr{J}_{11}$ and $\widetilde{\mathscr{J}_{11}}$ are identical as algebras. Finally, $x\widetilde{V}_{x_1} = uU^{(v)}_{x,x_1} = uU_v U_{x,x_1} = vU_{x,x_1} = \{xvx_1\} = \{xe_1x_1\} = \{x \sum\limits_1^n e_i x_1\} = xV_{x_1}$. This completes the proof. ◻

Lemma . *Let $\{e_i| = i = 1, 2, \ldots, n\}$ be a supplementary set os orthogonal idempotents in $\mathscr{J}$ such that e_1 is strongly connected to e_j, $j > 1$, by u_{1j}. Put $c_{1j} = u_{1j} + 1 - e_1 - e_j$ as above and $U_{(1j)} = U_{c_{1j}}$.Then there exists a unique isomorphism $\pi \to U_\pi$ of the symmetric group S_n into* Aut $\mathscr{J}$ *such that $(1j) \to U_{(ij)}$. Moreover, $e_i U_\pi = e_{i\pi}$ and if $i\pi = i\pi'$, $j\pi = j\pi'$ then $U_\pi = U_{\pi'}$ on $\mathscr{J}_{ij}$ ($i = j$ allowed).*

Proof. We have seen that $U_{(1j)}$ is an automorphism of period two. Now it is known that the symmetric group S_n is generated by the transpositions $(1j)$ and that the defining realations for there is $(1j)^2 = 1, ((1j)$ $(1k)^3 = 1$, $((1j)(1k)(1j)1l))^2 = 1, j, k, l \neq$. By lemma 1, we have $U_{(1j)}U_{(1k)}U_{(1j)} = U_{(1k)}U_{(1j)}U_{(1k)}$. Hence $(U_{(1j)}U_{(1k)})^3 = U_{(1j)}U_{(1k)}$ 108
$U_{(1j)}U_{(1k)}U_{(1j)}U_{(1k)} = U_{(1j)}U_{(1k)}U_{(lj)}U_{(1j)}U_{(1k)}U_{(1j)} = 1$. Also, if j, k, $l \neq$, then $U_{(1j)}U_{(1k)}U_{(1j)}$ $U_{(1l)}U_{(1j)}$ $U_{(1j)}U_{(1k)}U_{(1j)}U_{(1l)} = U_{c_{1_l}U_{c_{1_j}}U_{c_{1_j}}}$ $U_{c_{1_1}} = U_{c_{1_1}} U_{c_{jk}} U_{c_{1l}}$ where $c_{jk} = c_{lk} U_{c_{1j}}$. The form of c_{jk} derived in Lemma 1 and (34) imply that $c_{1l} U_{c_{jk}} = c_{1l}$. Hence $U_{c_{1l}} U_{c_{jk}} U_{c_{1l}} = 1$. These relations imply that we have a unique monpmorphism of S_n into Aut $\mathscr{J}$ such that $(1j) \to U_{(1j)} = U_{c_{1j}}$. By (34) $e_1 U_{(1j)} = e_1 U_{c_{1j}} = e_1 U_{U_{1j}} = u^2_{1j} U_{e_j} (PD6) = e_j$. Hence $e_i U_\pi = e_{i\pi}$ for $\pi \epsilon S_n$. Now suppose π and π' satisfy $e_i \pi = e_i \pi'$, $e_j \pi' = e_j \pi'$. Put $\pi'' = \pi' \pi^{-1}$. Then $e_i \pi'' = e_i, e_j \pi'' = e_j$ so π'' is a product of transpositions which fix i and j. If (kl) is such a transposition then $U_{(kl)} = U_{(1l)} U_{(1k)} U_{(1l)}$. By (34) this acts as identity on $\mathscr{J}_{ij}$. Hence π'' is 1 on $\mathscr{J}_{ij}$ and $\pi = \pi''$ on $\mathscr{J}_{ij}$. ◻

4 Strong coordinatization theorem

We shall now obtain an important characterization of the quadratic Jordan algebras $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o), n \geqq 3$, or equivalently, in view of Theorem 2, of the outer ideals containing 1 in standard quadratic Jordan algebras $\mathscr{H}(\mathscr{O}_n), n \geqq 3$. We shall call a triple $(\mathscr{O}, j, \mathscr{O}_o)$ a *coordinate algebra*, if $(\mathscr{O}, j)$ is an alternative algebra with involution and $\mathscr{O}_o$ is a Φ -submodule of $N'(\mathscr{O}) = \mathscr{H}(\mathscr{O}, j) \cap N(\mathscr{O})$ suchthat $1 \epsilon \mathscr{O}_o$ and $x\mathscr{O}_o\overline{x} \subseteq \mathscr{O}_o, x \epsilon \mathscr{O}$. We call $(\mathscr{O}, j, \mathscr{O}_o)$ *associative* if $\mathscr{O}$ is associative. We shall show that $\mathscr{J} = \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o), n \geqq 3$, is characterised by the following two conditions: (1) $\mathscr{J}$ contains $n \geqq 3$ supplementary strongly connected orthog-
109 onal idempotents, (2) $\mathscr{J}$ is non-degenerate in the sense that $\ker U = 0$. Consider an $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o), n \geqq 3$. Let $e_i = 1[ii]$, $u_{1j} = 1[1j]$, $j > 1$ (notation as in §2). Then the e_i are orthogonal idempotents and $\sum\limits_1^n e_i = 1$. Also u_{1j} is in the Pierce (i, j) component of $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ relative to the e_i and $u_{1j}^2 = e_1 + e_j$ so e_1 and e_j are strongly connected by u_{1j}. Hence (1) holds. To prove (2) we recall that $\ker U$ is an ideal (§1.5) so if $z = \sum\limits_{1 \leqq j} z_{ij}[ij] \epsilon \ker U$ then $z_{ii}[ii] = zU_{e_i} \epsilon \ker U$ and $z_{ij}[ij] = zU_{e_i, e_j}, i \neq j$, $\ker U$. We recall also that all products involving an element of $\ker U$ are 0 except those of the form $zU_a, z \in \ker U$. Hence $z_{ij}[ij] = 0$ and $z_{ii}[ij] = z_{ii}[ii] \circ 1[ij] = 0$. Then $z_{ii} = 0$ and $z = 0$. Thus $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ is non-degenerate.

We shall now prove that the conditions (1) and (2) are sufficient for a quadratic Jordan algebra to be isomorphic to an algebra $\mathscr{H}(\mathscr{O}_n . \mathscr{O}_o)$, $n \geqq 3$. We have the following

Strong coordinatization Theorm. Let $\mathscr{J}$ be a quadratic Jordan algebra satisfying:(1) $\mathscr{J}$ is non-degenerate, (2) $\mathscr{J}$ contains $n \geqq 3$ supplementary strongly connected orthogonal idempotents. Then there exists a coordinate algebra $(\mathscr{O}, j\mathscr{O}_o)$ which is associative if $n \geqq 4$ such that $\mathscr{J}$ is isomorphic to $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$. More precisely, let $\{e_i | i = 1, \ldots, n\}$ be a supplementary set of strongly connected orthogonal idempotents and let e_1 be strongly connected to $e_j, j > 1$, by u_{1j}. Then there an isomorphism η of $\mathscr{J}$ onto $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ such that $e_i^\eta = 1[ii], i = 1, \ldots, n,$, $u_{1j}^\eta = 1[1j], j = 2, \ldots, n$.

Proof. Put $c_{ij} = u_{ij} + 1 - e_1 - e_j$. By lemma 3 of §3 we have an isomorphism $\pi \to U_\pi$ of S_n into Aut $\mathscr{J}$ such that $U_{(ij)} = U_{c_{1j}}, e_i U_\pi = e_{i\pi}$. Then for the Pierce module $\mathscr{J}_{pq}$ (relative to the e_i) we have $\mathscr{J}_{pq}U_\pi = \mathscr{J}_{pq}U_{p\pi,q\pi}$. Also if $\pi, \pi' \epsilon S_n$ satisfy $p\pi = p\pi', q\pi = q\pi'$ then U_π and $U_{\pi'}$ have the same restrictions to $\mathscr{J}_{pq}$. This implies that if $p\pi = p$ and $q\pi = q$ then U_π is the identity on $\mathscr{J}_{pq}$. By Lemma 1 of §3, $c_{jk} = c_{1j}U_{c_{1k}}$, for $i, j, k \neq$ satisfies $c_{jk} = c_{kj} = c_{1k}U_{c_{1j}} = u_{jk} + 1 - e_j - e_k$ where $u_{jk} = u_{1j} \circ u_{1k}$ also $u_{jk}^2 = e_j + e_k, U_{(jk)} = U_{(1k)}U_{(1j)}U_{(1k)} = U_{c_{jk}}$. By (34), if $i, j, k \neq$ then $x_{ij}U_{(jk)} = x_{ij} \circ u_{jk}$. Hence $(x_{ij} \circ u_{jk}) \circ u_{jk} = x_{ij}U_{(jk)}^2 = x_{ij}$. In particular, if $1, j, k \neq$ then $u_{jk} \circ u_{1k} = (u_{1j} \circ u_{lk}) \circ u_{lk} = u_{1j}$. We note also that 110

$$U_\pi^{-1} U_{e_i} U_\pi = U_{e_i\pi} \tag{39}$$

since for $\pi = (1j)$ we have $U_{(1j)}U_{e_i}U_{(1j)} = U_{c_{1j}}U_{e_i}U_{e_{1j}} = U_{e_i}U_{c_{1j}} = U_{e_i}U_{1j} = U_{e_{i(1j)}}$ and $U_{\pi\pi'}^{-1}U_{e_i}U_{\pi\pi'} = U_{\pi'}^{-1}U_\pi^{-1}U_{e_i}U_\pi U_{\pi'}$. Let $\mathscr{O} = \mathscr{J}_{12}$ and define for x, y

$$xy = xU_{(23)} \circ yU_{(13)} \tag{40}$$

Since $xU_{(23)}\epsilon\mathscr{J}_{13}$ and $yU_{(13)}\epsilon\mathscr{J}_{23}, xy\epsilon\mathscr{O} = \mathscr{J}_{12}$. Also the product is Φ -bilinear. Define for $x\epsilon\mathscr{O}$.

$$j : x \to \overline{x} = xU_{(12)} \tag{41}$$

Since $U_{(12)}$ maps $\mathscr{J}_{12}$ into itself and $U_{(12)}^2 = 1$, j is a Φ-isomorphism of $\mathscr{J}_{12}$ such that $j^2 = 1$. Also if $x, y\epsilon\mathscr{O}$, 111

$$\begin{aligned}
\overline{xy} &= (xU_{(23)} \circ yU_{(13)})U_{(12)}\\
&= xU_{(23)}U_{(12)} \circ yU_{(13)}U_{(12)}\\
&= yU_{(12)}U_{(23)} \circ xU_{(12)}U_{(13)}\\
&= \overline{y}\overline{x}\\
xu_{12} &= xU_{(23)} \circ 12^U(13)\\
&= xU_{(23)} \circ U_{23}\\
&= xU_{23}^2\\
&= x\\
\overline{u}_{12} &= u_{12}^3 = u_{12} \quad \text{(see§3)}
\end{aligned}$$

These relations imply that u_{12} acts as the unit element of $\mathscr{O}$ relative to its product and j is an involution in $\mathscr{O}$.

We now define n^2 "coordinate mappings" $\eta_{pq}, p, q = 1, 2, \ldots, n$ of $\mathscr{J}$ into $\mathscr{O}$ as follows:

$$\eta_{ij} = U_{e_i,e_j} U_\pi \quad \text{if} \quad i \neq j, i\pi = 1, j\pi = 2 \tag{42}$$

$$\eta_{ii} = U_{e_i} U_\pi V_{c_{12}} \quad \text{if} \quad i\pi = 1 \tag{43}$$

112 It is clear from the preliminary remark that this is independent of the choice of π. Also $\eta_{pq} = 1$ on all the Pierce submodules except $\mathscr{J}_{pq}$. If $i \neq j$, U_π is a Φ isomorphism of $\mathscr{J}_{ij}$ onto $\mathscr{O} = \mathscr{J}_{12}$. Hence η_{ij} is a Φ-isomorphism of $\mathscr{J}_{ij}$ onto $\mathscr{O}$. Since $\mathscr{J}_{ii} U_{e_i} U_\pi = \mathscr{J}_{11}$ it is clear also that η_{ii} is a Φ homomorphism of $\mathscr{J}_{ii}$ onto $\mathscr{O}_o \equiv \mathscr{J}_{11}^{\eta_{11}} = \mathscr{J}_{11} V_{c_{12}}$. We now prove for $x \epsilon \mathscr{J}$

$$\overline{x^{\eta_{pq}}} = x^{\eta_{pq}}. \tag{44}$$

If $i \neq j$, we have $\overline{x^{\eta_{ij}}} = x U_{e_i,e_j} U_\pi U_{(12)} = x U_{e_j,e_i} U_{\pi'}$, where $i\pi' = 2$, $j\pi' = 1$. Hence $\overline{x^{\eta_{ij}}} = x^{\eta_{ji}}$. To prove $\overline{x^{\eta_{ii}}} = x^{\eta_{ii}}$ we require

$$V_{c_{12}} U_{c_{12}} = V_{c_{12}} = U_{c_{12}} V_{c_{12}} \tag{45}$$

We have $V_{c_{12}} U_{c_{12}} U_{c_{12}^2, c_{12}} = U_{C_{12}} V_{c_{12}}$ by $QJ24$. Since $c_{12}^2 = 1$ and $U_{1,c_{12}} = V_{c_{12}}$ we have (45). Now $\overline{x^{\eta_{ii}}} = x U_{e_i} U V_{c_{12}} = x U_{e_i} U V_{c_{12}} = X^{\eta_{ii}}$.

Define the mapping η of $\mathscr{J}$ onto $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ by

$$x = \sum_{p \leqq q} x^{\eta_{pq}} [pq]. \tag{46}$$

This is a Φ-homomorphism of $\mathscr{J}$ onto $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ since, if $x \epsilon \mathscr{J}_{pq}$, $x = x^{\eta_{pq}}[pq]$. It is clear that $\mathscr{J}_{pq}^{\eta} = \mathscr{H}_{pq}$ where these are defined as usual. We have $e_i^\eta = e_i U_{e_i} U_\pi V_{c_{12}}[ii] = e_1 V_{c_{12}}[ii] = e_1 V_{u_{12}}[ii] = U_{12}[ii] = 1[ii](U_{12} = 1 \text{ in } \mathscr{O})$. Hence $1^\eta = 1$ also. Also $u_{12}^\eta = n_{12}[12]$
113 and if $1, 2, j \neq$ then $u_{ij}^\eta = u_{1j} U_{(2j)}[ij] = (U_{1j} = (u_{1j} \circ u_{2j})[1j] = u_{12}[1j] = 1[1j]$ (since $u_{1j} \circ u_{2j} = u_{12}$ was shown in the first paragraph).

We now consider $\mathscr{J}$ relative to the squaring operation ($a^2 = 1U_a$) and we shall prove for $i, j, k \neq, x_{ij} \epsilon \mathscr{J}_{ij}$ etc:

$$(x_{ij} \circ y_{jk})^\eta = x_{ij}^\eta \circ y_{jk}^\eta \tag{47}$$

$$(x_{ii} \circ y_{ij})^\eta = x_{ii}^\eta \circ y_{ij}^\eta \tag{48}$$

$$(x_{ii}^2)^\eta = (x_{ii}^\eta)^2 \tag{49}$$

$$(x_{ij}^2)^\eta = (x_{ij}^\eta)^2 \tag{50}$$

For (47) let π be a permutation such that $i\pi = 1$, $j\pi = 3$, $k\pi = 2$ and put $x_{ij}U_\pi = x\epsilon\mathscr{J}_{13}, y_{jk}U_\pi = y\epsilon\mathscr{J}_{23}$. Then

$$\begin{aligned} x_{ij}^\eta \circ y_{jk}^\eta &= xU_{(23)}[ij] \circ yU_{(13)}[jk] \\ &(xU_{(23)})(yU_{(13)})[ik] \\ &= (xU_{(23)}^2 \circ yU_{(13)}^2)[ik] \\ &= (x \circ y)[ik] \\ &= (x_{ij} \circ y_{jk})U_\pi[ik] \\ &= (x_{ij} \circ y_{jk})^\eta. \end{aligned}$$

For (48) let π be a permutation such that $i\pi = 1$, $j\pi = 2$. Put $x_{ii}U_\pi =$ **114**
$x\epsilon\mathscr{J}_{11}, y_{ij}U_\pi = y\epsilon\mathscr{J}_{12}$. Then

$$\begin{aligned} x_{ii} \circ y_{ij} &= xV_{c_{12}}[ii] \circ y[ij] \\ &= ((xV_{c_{12}})y)[ij] \\ &= (xV_{c_{12}}U_{(23)} \circ yU_{(13)})[ij]. \end{aligned}$$

Since $x\epsilon\mathscr{J}_{11}, xU_{(23)} = x$ (see first paragraph) and $xV_{c_{12}} = x \circ u_{12}$. Hence $xV_{c_{12}}U_{(23)} = (x \circ u_{12})U_{(23)} = x \circ u_{12}U_{(23)} = x \circ (u_{12} \circ u_{23}) = x \circ u_{13} = xV_{c_{13}} = xV_{c_{13}}U_{c_{13}}$(cf. (45))= $(x \circ u_{13})U_{c_{13}}$. Then $xV_{c_{12}}U_{23} \circ yU_{(13)} = ((x \circ u_{13}) \circ y)U_{(13)} = ((x \circ y) \circ u_{13})U_{13} = x \circ y = (x_{ii} \circ y_{ii})U_\pi$. Hence $x_{ii} \circ y_{iij} = (x_{ii} \circ y_{ij})U_\pi[ij] = (x_{ii} \circ y_{ij})^\eta$.

For (49) choose π so that $i\pi = 1$ and put $x = x_{ii}U_\pi\epsilon\mathscr{J}_{11}$. Then

$$\begin{aligned} (x_{ij})^2 &= (xV_{c_{12}}[ii])^2 = (xV_{c_{12}})^2[ii] \\ &= (xV_{c_{12}}U_{(23)} \circ V_{c_{12}}U_{(13)}[ii] \\ &= (xV_{c_{13}} \circ xV_{c_{12}})U_{(13)}[ii] \quad \text{(proof of(48))} \\ &= ((x \circ u_{13}) \circ (x \circ u_{12}))U_{(13)}[ii] \\ &= ((x \circ (x \circ U_{12})) \circ u_{13})U_{(13)}[ii] \end{aligned}$$

$$
\begin{aligned}
&= (x^2 \circ u_{12})U^2_{(13)}[ii] \\
&= (x^2 \circ u_{12})[ii] \\
&= x^2_{ii}U_\pi V_{c_{12}}[ii] \\
&= (x^2_{ii})^\eta .
\end{aligned}
$$

115 For (50) let π satisfy $i\pi = 1,\ j\pi = 2$ and put $x = x_{ij}U_\pi \epsilon \mathscr{J}_{12}$. Then

$$
\begin{aligned}
(x_{ij})^2 &= x_{ij}U_\pi[ij]^2 = (x[ij])^2 \\
&= x\overline{x}[ii] + \overline{x}x[jj].
\end{aligned}
$$

Now $x\overline{x} = xU_{(23)} \circ xU_{(12)}U_{(13)} = xU_{(23)} \circ xU_{(13)}U_{(23)} = (x \circ xU_{(13)})U_{(23)} = (x \circ (x \circ u_{13}))U_{(23)} = (x^2 \circ u_{13})U_{(23)} = (x^2 U_{e_1} \circ u_{13})U_{(23)} = x^2 U_{e_1} \circ u_{12} = x^2 U_{e_1} V_{c_{12}} = x^2_{ij}U_\pi U_{e_1} V_{c_{12}}$. Similary, $\overline{x}x = x^2_{ij}U_\pi U_{e_2} V_{c_{12}}$. Hence

$$
(x_{ij})^2 = (x^2_{ij}U_\pi U_{e_1} U_{e_{12}})[ii] + (x^2_{ij}UU_{e_2}V_{c_{12}})[jj].
$$

On the other hand, $x^2_{ij} = x^2_{ij}U_{e_i} + x^2_{ij}U_{e_j}$ so $(x^2_{ij})^\eta = (x^2_{ij}U_{e_i}U_\pi V_{c_{12}})[ii] + (x^2_{ij}U_{e_j}U_\pi U_{(12)}V_{c_{12}}V_{c_{12}})[jj]$ and $x^2_1 U_{e_i}U_\pi V_{c_{12}} = x^2_{ij}\ U_\pi\ U_{e_1}V_{c_{12}}$, $x^2_{ij}U_{e_j}U_\pi\ U_{(12)}V_{c_{12}} = x^2_{ij}U_\pi U_{e_2}U_{c_{12}}V_{c_{12}} = x^2_{ij}U_\pi U_{e_2}V_{c_{12}}$. Hence (50) holds.

It is clear from these formulas that $(x^2)^\eta = (x^\eta)^2$ for $x\epsilon\mathscr{J}$ and $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ is closed under squaring. We now introduce a 0-operator in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ by $QM1 - 11$ and the formulas giving 0. Clearly $AU_B\epsilon\mathscr{O}_n$ but it is not immediately clear that $AU_g\epsilon\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$. We claim that
116 $xU^\eta) = x^\eta U_{y^\eta}, x, y, \epsilon\mathscr{J}$. It is sufficient to prove this and $\{xyz\}^\eta = \{x^\eta y^\eta z^\eta\}$ for x, y, z in Pierce components. We note first that since η maps $\mathscr{J}_{pq}$ into $\mathscr{H}_{pq}$ we have $(xU_{e_i})^\eta = x^\eta U_{1[ii]}$, $(xU_{e_i,e_j})^\eta = x^\eta U_{1[ii],1[jj]}$. From this it follows that we can carry over the proof of theorem 1. For the relation corresponding to $QM4$ we have $\{x_{ii}yz\}^\eta = (((x_{ii} \circ y_{ij}) \circ z_{ij})U_{e_i})^\eta = ((x_{ii} \circ z_{ij}) \circ z_{ij})U_{1[ii]}$ and $\{x^\eta_{ii}y^\eta_{ij}z^\eta_{ij}\} = ((x_{ii} \circ y_{ij}) \circ z_{ij})U_{1[ii]}$ by the formulas in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$. Hence $\{x_{ii}y_{ij}z_{ij}\}^\eta = \{x^\eta_{ii}y^\eta_{ij}z^\eta_{ij}\}$ holds. The formulas corresponding to $QM5 - 11$ are obtained in a similar manner. For $QM2$ we note that $u^\eta_{jk} = 1[jk]$, $j \neq k$ and $V_{1[jk]}$ is injective on $\mathscr{H}_{ij}$ if $i, j, k \neq$. Hence it suffices to prove $(x_{ii}U_{y_{ij}})^\eta \circ u^\eta_{jk} = x^\eta_{ii}U^\eta_{y_{ij}} \circ u^\eta_{jk}$. Now $(x_{ii}U_{y_{ij}})^\eta \circ u^\eta_{jk} = (x_{ii}U_{y_{ij}}V_{u_{jk}})^\eta = (x_{ii}V_{y_{ij},u_{jk}}V_{y_{ij}})^\eta$ as in the proof of $QM2$

in Theorem 1) = $(\{x_{ii}y_{ij}u_{jk}\} \circ y_{ij})^\eta = \{x^\eta_{ii}y^\eta_{ij}u^\eta_{jk}\} \circ y^\eta_{ij} = x^\eta_{ii}U^\eta_{y_{ij}} \circ u^\eta_{jk}$ by the formulas in $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$. A similar argument applies to $QM1, 3$. Hence $(xU_y)^\eta = x^\eta U_{y^\eta}$ which with $\mathscr{J}^\eta = \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ implies that $AU_B \epsilon \mathscr{H}(\mathscr{O}_n.\mathscr{O}_o)$ for $A, B \epsilon \mathscr{H}(\mathscr{O}_n.\mathscr{O}_o)$. It follows that $(\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o), U, 1)$ is Jordan and η is a homomorphism. It now follows from Theorem 1 that $\mathscr{O}$ is associative if $n \geqq 4$ and alternative with $\mathscr{O}_o \subseteq N(\mathscr{O})$ if $n = 3$. Also $x\mathscr{O}_o\overline{x} \subseteq \mathscr{O}_o$ if $x \epsilon \mathscr{O}$.

It remains to prove that η is an isomorphism. Let $\mathfrak{K} = \ker^\eta$. Since $\mathfrak{K}$ = is an ideal, $\mathfrak{K} = \sum \mathfrak{K}_{ij} \mathfrak{K} = \mathfrak{K} \cap \mathscr{J}_{ij}$ also since η_{ij} is injective if $i \neq j$ we have $\mathfrak{K}_{ij} = 0$, if $i \neq j$. Hence $\mathfrak{K} = \sum \mathfrak{K}_{ii}$ and it suffices to show $\mathfrak{K}_{ii} = 0, i = 1, \ldots, n$. Let $z \epsilon \mathfrak{K}_{ii}$ and consider the products $\{xyz\}, x, y$ in **117**
Pierce modules $\mathscr{J}_{pq}$. Such a product is 0 since $\mathfrak{K}_{pq} = \mathfrak{K} \cap \mathscr{J}_{pq} = 0$ Unless $x, y \mathscr{J}_{ij}, i \neq j$ or $x, y, \epsilon \mathscr{J}_{ii}$. In the first case we note that $x \circ z = 0$ since $\mathfrak{K}_{ij} = 0$ so $\{xyz\} = 0$ by $PD4$. In the second case we write $x = wU_{u_{ij}}$ (u_{ij} as above), $w \epsilon \mathscr{J}_{jj}$ where $j \neq i$. This can be done since $\mathscr{J}_{jj}U_{u_{ij}} = \mathscr{J}_{jj}U_{(ij)} = \mathscr{J}_{ii}$. Then $\{xyz\} = wU_{u_{ij}}V_{y,z} = 0$ by $QJ9$, the PD relations and $\mathfrak{K}_{ij} = 0$. Our result implies that $\{xyz\} = 0$ for all $x, y \epsilon \mathscr{J}$ so $U_{x,z} = 0$ for all x. We show that $U_z = 0$. For this it suffices to show that $xU_z = 0$ if $x \epsilon \mathscr{J}_{ii}$ or $wU_{u_{ij}}u_z = 0$ for $w \epsilon \mathscr{J}_{jj}, i \neq j$. This follows from $QJ17$ and the PD relations. We have now shown that $\mathfrak{K}_{ii} \subseteq \ker U$. Hence $\mathfrak{K}_{ii} = 0$ and the proof is complete. ◻

Remarks . The hypothesis that $\mathscr{J}$ is non-degenerate is used only at the last stage of the proof. If this is dropped the argument shows that $\mathfrak{K} = \ker \eta \subseteq \ker U$. The converse inequality holds since $(\ker U)^\eta$ is contained in the radical of U in $\mathscr{J}^\eta = \mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$. Since $\mathscr{H}(\mathscr{O}_n, \mathscr{O}_o)$ is non-degenerate we have $(radU)^\eta = 0$ so $\ker U \equiv \mathfrak{K}$ and $\mathfrak{K} = \ker U$ in any case. This gives a characterization of the algebras satisfying the first hypothesis of the S.C.T. IF $\mathscr{J}$ has no two torsion $\ker u = 0$ so in this case we can drop the second hypothesis and obtain the conclusion of S.C.T.

The S.C.T can be strenghtened to give a coordinatization Theorem in which the second hypothesis is replaced by the weaker one that the e_i are connected. In this case we get an isomorphism onto a "canonical matrix algebra" (cf. Jacobson [3], p.137).

Chapter 3
Structure Theory

In this chapter we shall develop the structure theory of quadratic Jordan algebras which is analogous to and is intimately connected with the structure with the structure theory of semi-simple Artinion associative rings. We consider first the theory of the radical of a quadratic Jordan algebra which was given recently by McCrimmon in [5]. McCrimmon's definition of the radical is analogous to that of the Jacobson radical in the associative case and is an important new notion even for Jordan algebras. We shall determine the structure of the quadratic Jordan algebras which are semi-simple (that is have 0 radical) and satisfy the minimum condition for principal inner ideals. Such an algebra is a direct sum of simple ones satisfying the same minimum condition. The simple quadratic Jordan algebras satisfying the minimum condition are either division algebras, outer ideals containing 1 in algebras $\mathscr{H}(\mathfrak{a}, J)$ where $(\mathfrak{a}, J)$ is simple Artinian with involution, outer ideals containing 1 in quadratic Jordan algebras of quadratic forms with base points (§1.7), or certain isotopes $\mathscr{H}(\mathscr{O}_3, J_c)$ of algebras $\mathscr{H}(\mathscr{O}_3)$, $\mathscr{O}$ an actonion algebra over a field (§1.8, 1.9). The only algebras in this list which are of capacity $\geqq 4$ (definition in §6) are outer ideals cintaining 1 in $\mathscr{H}(\mathfrak{a}, J), (\mathfrak{a}, J)$ simple Artinian with involution. In this sense the general case in the classification of simple quadratic Jordan algebras satisfying the minimum condition is constituted by the algebras defined by the $(\mathfrak{a}, J)$ we determined in Chapter0.

118 **119**

These results reduce in the case of (linear) Jordan algebras to those given in Chapter IV of the author's book [2].

1 The radical of a quadratic Jordan algebra

As in the associative ring theory the notation of the radical will be based on the Quasi-invertiblity which we define as follows:

Definition 1. An element z of a quadratic Jordan algebra $\mathscr{J}$ is called *quasi-invertible* if $1 - z$ is invertible. If the inverse of $1 - z$ is denoted as $1 - w$ then w is called *the quasi-inverse* of z.

The condition that z be quasi-invertible with quasi-inverse w are

$$(1 - w)U_{1-z} = 1 - z, (1 - w)^2 U_{1-z} = 1. \tag{1}$$

Since $U_{1-z} = 1+U_z-V_z$ the conditions are $1+z^2-2z-w-wU_z+w\circ z = 1 - z, 1 + z^2 - 2z - 2w - 2wU_z + 2w \circ z + w^2 + w^2 U_z - w^2 \circ z = 1$. These reduce to

$$w + z - z^2 - w \circ z + wU_z = 0 \tag{2}$$

$$2w - z^2 + 2wU_z - w^2 U_z + 2z - 2w \circ z - w^2 + w^2 \circ z = 0. \tag{3}$$

The quasi-inverse of z is

$$w = (z^2 - z)U^{-1}_{1-z} \tag{4}$$

since $(1-z)^{-1} = (1-z)U^{-1}_{1-z} = (1-z)^2 U^{-1}_{1-z} - (z^2-z)U^{-1}_{1-z} = 1-(z^2-z)U^{-1}_{1-z}$.

120 Also $(1 - z) \circ (1 - w) = 2$ which gives

$$z \circ w = 2(z + w) \tag{5}$$

Then $1 - z = (1 - w)U_{1-z} = 1 - w + (1 - w)U_z - (1 - w) \circ z = 1 - w + (1 - w)U_z - 2z + 2z + 2w = 1 + w + (1 - w)U_z$. Thus

$$w + z + (1 - w)U_z = 0 \tag{6}$$

An immediate consequence of this is

Lemma 1. *If Z is an inner ideal and $z \epsilon Z$ is quasi-invertible then the quasi-inverse w of z is in Z.*

We prove next

Lemma 2. *If z^n is quasi-invertible then z is quasi-invertible.*

Proof. We have $1 - \lambda^n = (1 - \lambda)(1 + \lambda + \ldots + \lambda^{n-1})$. Hence $U_{1-z^n} = U_{1-z}U_y = U_y U_{1-z}$ where $y = 1 + z + \ldots + z^{n-1}$ $(QJ37)$. Since z^n is quasi-invertible, U_{1-z^n} is invertible. Hence $1 - z$ is invertible and z is quasi-invertible. □

Remarks. The argument shows also that $-(z + z^2 + \ldots + z^{n-1})$ is quasi-invertibel. The converse of lemma 2 is false since -1 is quasi-invertible in any (linear) Jordan algebra. On the other hand, $1 = (-1)^2$ is not quasi-invertible.

An immediate corollary of Lemma 2 is: Any nilpontent element is quasi-invertible. By a *nilpotent* z we mean an element such that $z^n = 0$ for some n. Then $z^n = 0$ for all $m > 2n$. Since no idempotent $\neq 1$ is 121
invertibel and c idempotent implies $1 - e$ idempotent it is clear that no idempotent $\neq 0$ is quasi-invertible.

Definition 2. An ideal (inner ideal, outer ideal) Z is called*quasi - invertible* if very $z \epsilon Z$ is quasi-invertible Z is called *nil* if every $z \epsilon Z$ is nilpotent.

The foregoing result shows that if Z is nil then Z is Quasi-invertible.

Lemma 3. *If Z is a quasi-invertible ideal and $u \epsilon \mathscr{J}$ is invertible then $u - z$ is invertible for every z.*

Proof. $u - z$ is invertible if and only if $(u - z)^2 U_u^{-1}$ is invertible. We have $(u - z)^2 U_u^{-1} = (u^2 - u \circ z + z^2)U_u^{-1} = 1 - w$ where $w = (u \circ z - z^2)U_u^{-1} \epsilon Z$. Then w is quasi-invertible and $u - z$ is invertible. □

Lemma 4. *If Z and $\mathfrak{L}$ are quasi-invertible ideals then $Z + \mathfrak{c}$ is a quasi-invertible ideal.*

Proof. Let $x \epsilon Z, y \epsilon \mathfrak{c}$. Then $1 - (x + y) = (1 - x) - y$ is invcertible by lemma 3 since $1 - x$ is invertible and $y \epsilon \mathfrak{c}$. Hence $x + y$ is quasi-invertible. Thus every element of $Z + \mathfrak{c}$ is quasi-invertible. □

Lemma 5. *If z is quasi-invertible in $\mathscr{J}$ and η is a homomorphism of $\mathscr{J}$ into $\mathscr{J}'$ then z^η is quasi-invertible in $\mathscr{J}'$. If Z is a quasi-invertible ideal and $\bar{z} = z + Z$ is quasi invertible in $\overline{\mathscr{J}} = \mathscr{J}/Z$ then z is quasi invertible in $\mathscr{J}$.*

Proof. The first statement is clear since invertible elements are mapped into invertible elements by a homomorphism. To prove the second result consider $1 - z$ where $\bar{z}$ where $\bar{z}$ is quasi-invertible in $\overline{\mathscr{J}} = \mathscr{J}/Z$. We
122 have a $w \epsilon \mathscr{J}$ such that $(1 - w)^2 U_{1-z} = 1 - y, y \epsilon Z$. Since Z is quasi-invertible, $1 - y$ is invertible. Hence $1 - z$ is invertible (Theorem on inverses). Thus z is quasi-invertible.

We are now in position to prove our first main result. □

Theorem 1. *There exists a unique maximal quasi-invertible ideal $\mathfrak{R}$ in $\mathscr{J}$. $\mathfrak{R}$ contains every quasi-invertible ideal and $\overline{\mathscr{J}} = \mathscr{J}/\mathfrak{R}$ contains no quasi-invertible ideal $\neq 0$.*

Proof. Let $\{Z_\alpha\}$ be the collection of quasi-invertible ideals of $\mathscr{J}$ and put $\mathfrak{R} = \cup Z_\alpha$. If $x, y \epsilon \mathfrak{R}$, $x \epsilon Z_\alpha, y \epsilon Z_\beta$ for some α, β. By lemma 4, $Z_\alpha + Z_\beta = Z_\gamma$. Hence $x + y \epsilon \mathfrak{R}$ and $x + y$ is quasi-invertible. It follows that $\mathfrak{R}$ is a quasi-invertible ideal. Clearly $\mathfrak{R}$ contains every quasi-invertible ideal so $\mathfrak{R}$ is the unique maximal quasi-invertible ideal. Now let $\overline{Z}$ be a quasi-invertible ideal of $\overline{\mathscr{J}} = \mathscr{J}/\mathfrak{R}$. Then $\overline{Z} = Z/\mathfrak{R}$ where Z is an ideal of $\mathscr{J}$ containing $\mathfrak{R}$. Let $z \epsilon Z$. Then $\bar{z} = z + \mathfrak{R}$ is quasi invertible in $\overline{\mathscr{J}}$. Hence by Lemma 5, z is quasi-invertible in $\mathscr{J}$. Hence Z is a quasi-invertible ideal of $\mathscr{J}$. Then $Z \subseteq \mathfrak{R}, Z = \mathfrak{R}$ and $Z = Z/\mathfrak{R} = 0$.

The ideal $\mathfrak{R}$ is called the (*Jacobson*)*radical* of $\mathscr{J}$ and will be denoted also as rad $\mathscr{J} \cdot \mathscr{J}$ is called *semi-simple* if rad $\mathscr{J} = 0$. The second statement of Theorem 1 is $\overline{\mathscr{J}} = \mathscr{J}/$ rad $\mathscr{J}$ is semi-simple. Since nil ideals are quasi-invertible rad $\mathscr{J}$ contains every nil ideal. □

2 Properties of the radical

We show first that rad $\mathscr{J}$ is independent of the base ring Φ. This is a consequence of

123 **Theorem 2.** *Let $\mathscr{J}$ be a quadratic Jordan algebra over Φ, $\Re = \operatorname{rad}\mathscr{J}$ and γ the radical of $\mathscr{J}$ regarded as a quadratic Jordan algebra over $\mathbb{Z}$. Then $\Re = \gamma$.*

Proof. It is clear that $\Re \subseteq \gamma$. The reverse inequality will follow if we can show that $\Phi\gamma$ the Φ-submodule generated by γ is a quasi-invertible ideal. The elements of $\Phi\gamma$ have the form $\sum \alpha_i z_i$, $\alpha_i \epsilon \Phi$, $z_i \epsilon \gamma$. If $a \epsilon \mathscr{J}$ then $(\sum \alpha_i z_i)U_a = \sum \alpha_i (z_i U_a) \epsilon \Phi\gamma$ since $z_i U_a \epsilon \gamma$. Also $aU_{\sum \alpha_i z_i} = \sum_i \alpha_i^2 a U_{z_i} + \sum_{i<j} \alpha_i \alpha_j (aU_{z_i,z_j})$ since $aU_{z_i}, aU_{z_i,z_j} \epsilon \gamma$ for any $a \epsilon \mathscr{J}$. Next we show that $z = \sum \alpha_i z_i$ is quasi-invertible. The result just proved show that $z^3 = zU_z \epsilon \gamma$. Hence this is quasi-invertible and z is quasi-invertible by lemma 2. Hence $\Phi\gamma$ is a quasi-invertible ideal, which completes the proof. ◻

Definition 3. An element $a \epsilon \mathscr{J}$ is called *regular* if $a \epsilon \mathscr{J} U_a \mathscr{J}$ is called *regular* if every $a \epsilon \mathscr{J}$ is regular.

If Φ is a field and a is an algebraic element of $\mathscr{J}$ then $\Phi[a]$ is finite dimensional (§1.10). Then $\Phi[a]\Phi[a]U_a \supseteq \Phi[a]U_{a^2}(U_{a^2} = U_a^2) \supseteq \Phi[a]U_{a^3} \ldots$ Hence we have an n such that $\Phi[a]U_{a^n} = \Phi[a]U_{a^{n+1}} = \ldots$ Then $a^{2n} \epsilon \Phi[a]U_{a^n} = \Phi[a]U_{a^{2n}}$. Hence a^{2n} is regular. Thus if a is algebraic (Φ a field) then there exists a power of a which is regular.

Theorem 3. *(1)* rad $\mathscr{J}$ *contains no non-zero regular elements. (2) If Φ is a field and $z \epsilon \operatorname{rad}\mathscr{J}$ is algebraic then z is nilpotent.*

Proof. (i) Let $z \epsilon \operatorname{rad}\mathscr{J}$ be regular, so $z = xU_z$ for some $x \epsilon \mathscr{J}$. Suppose 124
first that x is invertible. Since $zU_x \epsilon \operatorname{rad}\mathscr{J}$, $x - zU_x$ is invertible by lemma 3. Hence $U_{x-zU_x} = U_x - U_{x,zU_x} + U_x U_z U_x$ is invertible in End $\mathscr{J}$. Now

$$
\begin{aligned}
zU_{x-zU_x} &= zU_x - zU_{x,zU_x} + zU_x U_z U_x \\
&= zU_x - zV_{x,z}U_x + zU_x U_z U_x \quad (QJ4') \\
&= zU_x - xU_{z,z}U_x + zU_x U_z U_x \\
&= zU_x - 2zU_x + xU_z U_x
\end{aligned}
$$

Since $xU_2 = z$ this is 0 and since U_{x-zU_x} is invertible, $z = 0$. Now let x be arbitrary. Since $z = xU_z = xU_{xU_z} = xU_z U_x U_z$. We may replace x by $xU_z U_x$ and thus assume $x \in \operatorname{rad}\mathscr{J}$. Now $z^2(xU_z)^2 =$

$z^2U_xU_z(QJ22) = yU_z$ where $y = z^2U_x\epsilon$ rad $\mathscr{J}$. Then $u = 1 + x - y$ is invertible by lemma 3. Also $uU_z = z^2 + z - z^2 = z$. Hence $z = 0$ since u is invertible by the first case. (2) If $z\epsilon$ rad $\mathscr{J}$ is algebraic then we have seen that z^{2n} is regular for some n. Since $z^{2n}\epsilon$ rad $\mathscr{J}$, (1) implied that $z^{2n} = 0$. Hence z is nilpotent. □

Theorem 4. *If u is an invertible element of $\mathscr{J}$ then* rad $\mathscr{J}^{(u)}$ = rad $\mathscr{J}$.

Proof. Since rad $\mathscr{J}$ is an ideal in $\mathscr{J}$ it is ideal in the isotope $\mathscr{J}^{(u)}$. If $z \in$ rad $\mathscr{J}$ then $u^{-1} - z$ is invertible by lemma 3. Then $u^{-1} - z$ is invertible in $\mathscr{J}^{(u)}$. Since u^{-1} is the unit of $\mathscr{J}^{(u)}$ this states that z is quasi-invertible in $\mathscr{J}^{(u)}$. Thus rad $\mathscr{J}$ is a quasi-invertible ideal of
125 $\mathscr{J}^{(u)}$. Hence rad $\mathscr{J} \subseteq$ rad $\mathscr{J}^{(u)}$. By symmetry rad $\mathscr{J}$ = rad $\mathscr{J}^{(u)}$. □

3 Absolute zero divisors.

We recall that z is an absolute zero divisor (§1.10) if $U_z = 0$. We shall now show that such a z generates a nil ideal of $\mathscr{J}$. For the proof we shall need some information on the ideal (inner ideal, outer ideal) generated by a non-vacuous subset S of $\mathscr{J}$. Clearly the outer ideal generated by S is the smallest submodule containing S which is stable under all $U_x, x\epsilon\mathscr{J}$. This is the set of Φ-linear combinations of the elements of the form $sU_{x_1}U_{x_2}\cdots U_{x_k} s \in S, x_i \in \mathscr{J}$. We have seen also that the linear ideal generated by a single element s is $\Phi s + \mathscr{J}U_s$ (§1.5). Beyond this we have no information on the inner ideal generated by a subset. We now prove

Lemma 6. *If Z is an inner ideal then the outer ideal $\mathfrak{c}$ generated by Z is an ideal.*

Proof. The elements of $\mathfrak{c}$ are sums of elments of the form $bU_{x1}U_{x2}\ldots U_{xk}$, $b\epsilon Z, x_i\epsilon\mathscr{J}$. We have to show that $\mathfrak{c}$ is an inner ideal. For this it suffices to prove that if $a, x_i\epsilon\mathscr{J}$, $b\epsilon Z$ then $aU_{bU_{x1}}\ldots U_{x_k}\epsilon\mathfrak{c}$ and for $c, d\epsilon\mathfrak{c}, aU_{c,d}\epsilon\mathfrak{c}$. Since $aU_{c,d} = \{cad\}$ the second is clear since $c\epsilon\mathfrak{c}$ and $\mathfrak{c}$ is an outer ideal (§1.5). For the first we use

$$aU_{bU_{x_1}\ldots U_{x_k}} = aU_x\ldots U_{x_1}U_bU_{x_1}\ldots U_x.$$

Since Z is an inner ideal $aU_x \cdots U_{x_1} U_b \epsilon Z$. Hence $aU_{bU_{x_1} \cdots U_x} \epsilon \mathfrak{c}$.

Let z be an absolute zero divisor. Then the inner ideal generated by z is Φz. Hence the last result shows that the ideal generated by z is the set of Φ linear combinations of elments of the form $zU_{x_1} U_{x_2} \cdots U_{x_k}, x_i \epsilon \mathscr{J}$. 126
Since the ideal generated by a set of elements is the sum of the ideals generated by the individual elements we see that the ideal $\mathfrak{M} = zer \mathscr{J}$ generated by all the absolute zero divisors is the set of Φ -linear combinations of elements $zU_{x_1} U_{x_2} \cdots U_x, z$ an absolute zero divisor, $x_i \epsilon \mathscr{J}$. Since for $\alpha \epsilon \Phi, \alpha z U_{x_1} U_{x_2} \cdots U_{x_k}$ is an absolute zero divisor we have the following □

Lemma 7. *The ideal $\mathfrak{M} = zer \mathscr{J}$ generated by the absolute zero divisors is the set of sums $z_1 + z_2 + \cdots + z_k$ where the z_i are absolute zero divisors*

We prove next

Lemma 8. *If z is an absolute zero divisor and y is nilpotent then $x = y+z$ is nilpotent.*

Proof. We show first that for $n = 0, 1, 2, \ldots$

$$x^n = y^n + zM_{n-1} \tag{7}$$

where

$$M_{-1} = 0, M_o = 1, M_{n+1} = V_{y^{n+1}} + M_{n-1} U_y, n \geqq 0. \tag{8}$$

We note that by (8) we have $M_1 = V_y, M_2 = V_{y^2} + U_y$ and in general

$$M_{2k} = V_{y^{2k}} + V_{y^{2k-2}} U_y + V_{y^{2k+1}} U_{y^2} + \cdots + U_{y^n}, \geq 1 \tag{9}$$
$$M_{2k+1} = V_{y^{2k+1}} + V_{y^{2k-1}} U_y + \cdots + V_y U_{y^k}, k \geqq 1$$

Now (7) is clear if $n = 0, 1$. Assume it for n. Then $x^{n+2} = x^n U_x =$ 127
$x^n(U_y + U_{y,z})$ (since $U_z = 0$)$= y^n U_y + zM_{n-1} U_y + y^n U_{y,z} + zM_{n-1} U_{y,z}$. Now $y^n U_y = y^{n+2}$ and $y^n U_{y,z} = \{zy^n y\} = zV_{y^n, y} = zV_{y^{n+1}}$ $(QJ38)$. Hence

$$x^{n+2} = y^{n+2} + z(M_{n-1} U_y + V_{y^{n+1}}) + zM_{n-1} U_{y,z}$$

Thus we shall have (7) by induction if we can show that $zM_{n-1} U_{y,z} = 0$. By (9), this will follow if we can show that $zV_{y^i} U_{y^j} U_{y,z} = 0$ for

$i > 0$, $j \geqq 0$ and $zU_{y^j}U_{y,z} = 0$ for $j \geqq 0$. For the second of these we use $zU_{y^j}U_{y,z} = yzU_{y^j}z = yV_{zU_{y^j,z}} = yV_{y^j,y^jU_z}$ $(QJ31) = 0$ since $U_z = 0$. For the first we use the bilinearization of $QJ31$ relative to a: $V_{bU_{a,c,b}} = V_{aU_b,c} + V_{cU_b,a}$ in $zV_{y^i}U_{y^j}U_{y,z} = yzV_{y^i}U_{y^j}z = yzU_{y^j,y^{i+j}}z$ $(QJ39) = yV_{zU_{y^j,y^i,j,z}} = y(V_{y^jU_z,y^{i+j}} + V_{y^{i+j}U_z,y^i}) = 0$. This proves (7). Since y is nilpotent it is clear from (9) that $y^n = 0$ and $M_n = 0$ for sufficiently large n. Hence $x^n = 0$ by (7).

Repeated application of Lemma 8 shows that if the z_i are absolute zero sivisors then $z_1 + z_2 + \cdots + z_k$ is nilpotent. Hence, by lemma 7, we have □

Theorem 5. *The ideal* ker $\mathscr{J}$ *generated by the absolute zero divisors is a nil ideal.*

It is clear from this that ker $\mathscr{J} \subseteq$ rad $\mathscr{J}$. It clear also that a simple quadratic Jordan algebra contains no absolute zero divisors$\neq 0$.

4 Minimal inner ideals

128 An inner ideal Z in $\mathscr{J}$ is called *minimal (maximal)* if $Z \neq 0 (Z \neq \mathscr{J})$ there exists no inner ideal $\mathfrak{c}$ in $\mathscr{J}$ such that $Z \supset \mathfrak{c} \supset 0 (Z \subset \mathfrak{c} \subset \mathscr{J})$. If Z is a minimal inner ideal then $\mathscr{J}U_b = 0$ or $\mathscr{J}U_b = Z$ for every $b \epsilon Z$ since $\mathscr{J}u_b$ is an inner ideal contained in Z. Similarly, either $ZU_b = 0$ or $ZU_b = Z$, $b \epsilon Z$. We shall now prove a key result on minimal inner ideals which will serve as the starting point of the structure theory. As a preliminary to the proof we note the following

Lemma . *Let $a, b \epsilon \mathscr{J}$ satisfy $aU_b = b$. Then $E = U_aU_b$ and $F = U_bU_a$ are idempotent elements of End $\mathscr{J}$ and if $d = bU_a$ then b and d are* related *in the sense that $dU_b = b$ and $bU_d = d$.*

Proof. Since $aU_b = b, U_bU_aU_b = U_b$. Then $U_aU_bU_aU_b = U_aU_b$ and $U_bU_aU_bU_a = U_bU_a$ so E and F are idempotents. If $d = bU_a$ then $dU_a = bU_aU_b = aU_aU_aU_b = aU_b = aU_b = b$ and $bU_d = aU_bU_aU_bU_a = aU_bU_a = bU_a = d$. We shall now prove the following □

Theorem on Minimal Inner Ideals. Any minimal inner ideal Z of $\mathscr{J}$ is of one of the following types: $IZ = \Phi z$ where z is a non-zero absolute zero divisor, II $Z = \mathscr{J}U_b$ for every $b \neq 0$ in Z but $ZU_b = 0$ and $b^2 = 0$ for every $b \in Z$, III Z is a Pierce inner ideal $\mathscr{J}U_e, e^2 = e$, such that (Z, U, e) is a division algebra. Moreover, if $\mathscr{J}$ contains no idempotent $\neq 0, 1$ and contains a minimal inner ideal Z of type II then $2\mathscr{J} = 0$ and for every $b \neq 0$ in Z there exists an element $d\epsilon\mathscr{J}$ such that

(i) $dU_b = b, bU_d = d, b^2 = 0 = d^2, b \circ d = 1, \mathscr{O} = \mathscr{J}U_d$ is a minimal inner ideal of type II. **129**

(ii) $c = b + d$ satisfies $c^2 = 1$, $c^{-1} = c$ and in the isotope $\mathscr{J}^{(c)}$, b and d are supplementary strongly connected orthogonal idempotents such that the Pierce inner ideals $\mathscr{J}U_b^{(c)}Z$, $\mathscr{J}U_d^{(c)} = \mathscr{O}$ are minimal of type III.

Proof. Suppose first that Z contains an absolute zero divisor $z \neq 0$. Then Φz is a non-zero inner ideal contained in Z so $Z = \Phi z$, by the minimality of Z. From now on we assume that Z contains no absolute zero divisor $\neq 0$. Then $Z = \mathscr{J}U_b$ for every $b \neq 0$ in Z. Also $ZU_b = 0$ or $ZU_b = Z$. Suppose Z contains $ab \neq 0$ such that $ZU_b = 0$ and let $y\epsilon Z$. Then there exists an $a\epsilon\mathscr{J}$ such that $aU_b = y$. Then $ZU_y = ZU_bU_aU_b = 0$. Thus either $ZU_b = 0$ for every $b\epsilon Z$ or $ZU_b = Z$ for every $b \neq 0$ in Z. In the first case $\mathscr{J}U_bi = \mathscr{J}U_b^2 = \mathscr{J}U_bU_b \subset ZU_b = 0, b\epsilon Z$ and since $b^2 = 1U_b\epsilon Z$ and Z contains no absolute zero divisors $\neq 0$ we have $b^2 = 0$, $b\epsilon Z$. Thus Z is of type II. Now assume $ZU_b = Z$ for every $b \neq 0$ in Z. Let b be such an element and let $a\epsilon Z$ satisfy $aU_b = b$. By the lemma, b and $d = bU_a$ are related. Also $d\epsilon Z$ and $E = U_bU_d$ and $F = U_dU_b$ are idempotent operators. We have $\mathscr{J}E = \mathscr{J}U_bU_d = Z$ and $\mathscr{J}F = Z$ so the restrictions $\overline{E}$ and $\overline{F}$ of E and F to Z are Z the identiy on Z. Put $e = d^2U_b\epsilon Z$, $f = b^2U_d\epsilon Z$. Then $e \neq 0$ since $\mathscr{J}U_e = \mathscr{J}U_bU_d^2U_b = Z$ and similarly $f \neq 0$. Also $e^2 = (d^2U_b)^2 = b^2U_d^2U_b = b^2U_d\overline{F} = b^2U_d = f$. Similarly, $f^2 = e$. Then $e^2 = (f^2)^2 = f^2U_f(QJ23) = f^2U_dU_b^2U_d = f^2FE = f^2 = e$. Then e is a non-zero idempotent in $Z, Z = \mathscr{J}U_e$ and this is a division algebra since $ZU_b = Z$ for every non-zero b hence Z is type III. **130**

Now suppose $\mathscr{J}$ contains no idempotents $\neq 0, 1$ and contains the minimal inner ideal Z of type II. Let $b \neq 0$ in Z and let $a \epsilon \mathscr{J}$ satisfy $aU_b = b$. We claim that $c = a^2U_b = 0$. Otherwise c is a non-zero element of Z and there exist $b_o, c_o \epsilon \mathscr{J}$ suchthat $b_oU_c = b$, $c_oU_c = c$. Put $e = c_oU_bU_a$. Then $e^2 = (c_oU_bU_a)^2 = a^2U_{c_OU_b}U_a = a^2U_bU_{c_o}U_bU_a = cU_{c_o}U_bU_a = cU_{c_o}U_cU_{b_o}U_cU_a = cU_{b_o}U_cU_a$ (since $cU_{c_o}U_c = c$ by the Lemma) $= c_oU_cU_{b_o}U_cU_a = c_oU_{b_oU_c}U_a = c_oU_bU_a = e$. Since $eU_aU_b = c_oU_bU_a^2U_b = c_oU_bU_{a^2}U_b = c_oU_{a^2U_b} = c_oU_c = c \neq 0$. More over, if $e = 1$ then $0 = b^2U_a = 1U_bU_a = eU_bU_a = c_oU_bU_aU_bU_a = c_oU_bU_a$ (Lemma)$= e$. Hence e is an idempotent $\neq 0, 1$ contrary to hypothesis. Thus we have shown that $a^2U_b = 0$ for every $a \epsilon \mathscr{J}$ such that $aU_b = b$. Put $d = bU_a$. Then $bU_d = d$, $dU_b = b$ and $d^2U_b = 0$. Then $d^2 = (bU_d)^2 = d^2U_bU_d = 0$. By $QJ30$, $(b \circ d)^2 = b^2U_d + d^2U_b + d^2U_b + bU_d \circ b = b \circ d$. By $QJ17$, $bU_{b \circ d} = -bU_{d,b} + bU_dU_b + bU_bU_d + bV_aU_bV_d = -b^2 \circ d + b + 0 + dV_bU_bV_d = b$ (since $dV_bU_b = dU_bV_b = b \circ b = 0$). Hence $b \circ d$ is an idempotent $\neq 0$ and so $b \circ d = 1$. We have now established all the relations on b, d in (i). Now put $c = b + d$. Then $c^2 = b^2 + b \circ d + d^2 = 1$. Hence U_c is an automorphism such that $U_c^2 = 1$ and $bU_c = bU_b + bU_{b,d} + bU_d = d$. Thus U_c maps the minimal inner ideal $Z = \mathscr{J}U_b$ onto the minimal inner ideal $\mathscr{O} = \mathscr{J}U_d$. Next we consider $(d+1)U_b = dU_b + b^2 = b$. As before, this implies $(d+1)^2U_b = 0$ so $0 = 2dU_b = 2b$. Then $4Z = 4\mathscr{J}U_b = \mathscr{J}U_{2b} = 0$. Since $2Z$ is
131 an inner ideal contained in Z, which is minimal, this implies $2Z = 0$. Applying the automorphism U_c shows that $2\mathscr{O} = 2ZU_c = 0$. Also $2b = 0$ implies $2\mathscr{J}U_{b,d} = U_{2b,d} = 0$. We now have $2\mathscr{J} = 2\mathscr{J}U_c = 2\mathscr{J}U_b + 2\mathscr{J}U_{b,d} + 2\mathscr{J}U_d = 0$.

Next we consider the isotope $\mathscr{J}^{(c)}$. We have $c^{-1} = cU_c = (b + d)U_{b+d} = bU_b + bU_{b,d} + bU_d + dU_bdU_{b,d} + dU_d = b + d = c$. Hence c is the unit of $\mathscr{J}^{(c)}$. Since $cU_b^{(c)} = cU_cU_b = cU_b = (b+d)U_b = b$, b is idempotent in $\mathscr{J}^{(c)}$. Hence $d = c - b$ is an idempotent orthogonal to b in $\mathscr{J}^{(c)}$. We have $cU_1^{(c)} = cU_cU_1 = c$ and $1 = b \circ d = 1U_{b,d} \epsilon \mathscr{J}U_{b,d}^{(c)}$. Hence b and d are strongly connected by 1 in $\mathscr{J}^{(c)}$. Finally, $\mathscr{J}U_b^{(c)} = \mathscr{J}U_cU_b = \mathscr{J}U_b = Z$ and $\mathscr{J}U_d^{(c)} = \mathscr{O}$, so Z and $\mathscr{O}$ are the Pierce inner ideals determined by the idempotents b and d in $\mathscr{J}^{(c)}$. Since Z and $\mathscr{O}$ are minimal inner ideals of $\mathscr{J}$ they are minimal inner ideals of

$\mathscr{J}^{(c)}$. Clearly they are of type III in $\overline{\mathscr{J}}$.

All the possibilities indicated in the theorem can occur. To see this consider $\Phi_2^{(q)}$ the special quadratic Jordan algebra of 2×2 matrices over a field Φ. Then Φe_{11} is a minimal inner ideal of type III and Φe_{12} is a minimal iner ideal of type II in $\mathscr{J} = \Phi_2^{(q)}$. Next let $\mathscr{J} = \Phi 1 + \Phi_{e_{12}}$. This is a subalgebra of $\Phi_2^{(q)}$ and e_{12} is an absolute zero divisor in $\mathscr{J}$. Hence Φe_{12} is a minimal inner ideal of type I. Finaly, assume Φ has characteristic two. Then $\mathscr{J} = \Phi 1 + \Phi e_{12} + \Phi e_{21}$ is a subalgebra of $\Phi_2^{(q)}$ since $x = \alpha 1 + \beta e_{12} + \gamma e_{21}, \alpha, \beta, \gamma \epsilon \Phi$, then $x^2 = (\alpha^2 + \beta\gamma)1$ so x^2 and $xy + yx \epsilon \Phi 1$ for $x, y \epsilon \mathscr{J}$. Then $xyx + yx^2 \epsilon \Phi x$ and $xyx \epsilon \Phi x + \Phi y$. The formula for x^2 shows that $\mathscr{J}$ contains no idempotents $\neq 0, 1$. Also **132**
$Z = \Phi e_{12}, \mathscr{O} = \Phi e_{21}$ are minimal inner ideals of type II in $\mathscr{J}$ and $b = e_{12}, d = e_{21}$ satisfy (i). □

5 Axioms for the structure theory.

We shall determine the structure of the quadratic Jordan algebras which satisfy the following two conditions: (1) strong non-degeneracy (=non-existance of absolute zero divisors $\neq 0$), (2) the descending chain condition (D C C) for principal inner ideals.The latter is equivalent to the minimum condition for principal inner ideals. We have called a quadratic Jordan algebra $\mathscr{J}$ regular if for every $a \epsilon \mathscr{J}$ there exists $x \epsilon \mathscr{J}$ such that $xU_a = a$ (§1.10, Definition 3 of §2). Clearly this implies strong non-degeneracy.

Lemma 1. *If $\mathscr{J}$ is strongly non-degenerate and satisfies the DCC for principal inner ideals then every non-zero inner ideal Z of $\mathscr{J}$ contains a minimal inner ideal of $\mathscr{J}$.*

Proof. If $b \neq 0$ is in Z then $\mathscr{J} U_b$ is a principal inner ideal contained in Z and $\mathscr{J} U_b \neq 0$ by the strong non-degeneracy. By the minimum condition (= D C C) for principal inner ideals contained in contains a minimal element $\mathfrak{R}$. We claim that $\mathfrak{R}$ is a minimal inner ideal of $\mathscr{J}$. Otherwise, we have an inner ideal Z such that $\mathfrak{R} \supset Z \supset 0$. The argument used for Z shows that Z contains a non-zero principal inner ideal $\mathscr{J} U_c$ and $\mathfrak{R} \supset \mathscr{J} U_c$ contrary to the choice of $\mathfrak{R}$.

As a first application of this result we note that the foregoing conditions (1) and (2) are equivalent to : (1) and (2′) $\mathscr{J}$ is semi-simple. For, we have ◻

133 **Theorem 6.** *If a quadratic Jordan algebra $\mathscr{J}$ satisfies the DCC for principal inner ideals then $\mathscr{J}$ is semi-simple if and only if $\mathscr{J}$ is strongly non-degenerate.*

Proof. If z is an absolute zero divisor then the ideal generated by z is nil and so is contained in rad $\mathscr{J}$. Hence is $\mathscr{J}$ is semi-simple, so rad $\mathscr{J}$ = 0, then $\mathscr{J}$ contains no absolute zero divisors $\neq 0$. Conversely, suppose *rad* $\mathscr{J} \neq 0$. Then we claim that $\mathscr{J}$ contains non-zero absolute xero divisors. Otherwise, we can apply lemma 1 to conclude that rad $\mathscr{J}$ contains a minimal inner ideal $\mathfrak{K}$ of $\mathscr{J}$. $\mathfrak{K}$ is not of type III since rad $\mathscr{J}$ contains no non-zero idempotents. Also $\mathfrak{K}$ is not of type II since in this case the Theorem on Minimal Inner Ideals shows that every element of $\mathfrak{K}$ is regular. Hence $\mathfrak{K}$ is os type I and $\mathscr{J}$ contains an absolute zero divisor $\neq 0$, contrary to hypothesis.

It is immediate that if $\mathscr{J}$ satisfies the DCC for principal inner ideals, or is strongly non-degenerate, or is regular then the same condition holds for every isotope $\mathscr{J}^{(c)}$. The same is true of the quadratic Jordan algebra $(\mathscr{J}U_e, U, e)$ if e is an idempotent in $\mathscr{J}$. This follows from ◻

Lemma 2. *Let e be an idempotent in $\mathscr{J}$. Then any inner (principal inner) ideal of $(\mathscr{J}U_e, U, e)$ is an inner(principal inner) ideal of $\mathscr{J}$ and any absolute zero divisor of $\mathscr{J}U_e$ is an absolute zero divisor of $\mathscr{J}$. Moreover, if $\mathscr{J}$ is regular then $\mathscr{J}U_e$ is regular.*

Proof. If Z is an inner ideal of $\mathscr{J}U_e$ and $b \in Z$ then $b = bU_e$. Hence $\mathscr{J}U_b = \mathscr{J}U_{bU_e} = (\mathscr{J}U_e)U_bU_e \subseteq ZU_e = Z$. Hence Z is an inner ideal of $\mathscr{J}$. If Z is principal in $\mathscr{J}U_e, Z = \mathscr{J}U_eU_b, b\epsilon\mathscr{J}U_e$. Then
134 $b = bU_e$ so $Z = \mathscr{J}U_eU_eU_bU_e = \mathscr{J}U_eU_bU_e = \mathscr{J}U_b$ is a principal inner ideal of $\mathscr{J}$. Let $z\epsilon\mathscr{J}U_e$ be an absolute zero-divisor in $\mathscr{J}U_e$. Then $z = zU_e$. Hence if $x\epsilon\mathscr{J}$ then $xU_z = (xU_e)U_zU_e = 0$. Thus z is an absolute zero divisor in $\mathscr{J}$. FInally, suppose $\mathscr{J}$ is regular and let $a = aU_e\epsilon\mathscr{J}U_e$. Then $a = xU_a$ for some $x\epsilon\mathscr{J}$. Hence $a = xU_a =$

$xU_eU_aU_e = (\gamma U_e)U_eU_aU_e = x(U_e)U_a$. Since $xU_e \epsilon \mathscr{J} U_e$ this shows that $\mathscr{J} U_e$ is regular. ◻

We shall now give the principal examples of quadratic Jordan algebras which are strongly non-degenerate and satisfy the DCC for principal inner ideals.

Examples. 1. If $\mathfrak{a}$ is asemi-simple right Artinian algebra then it is well-known that $\mathfrak{a}$ is left Artinian and every right (left) ideal of $\mathfrak{a}$ has a complementary right (left) ideal. Let $a \epsilon \mathfrak{a}$. Then $\mathfrak{a}$ a is a left ideal so there exists a left ideal $\mathfrak{I}$ such that $\mathfrak{a} = \mathfrak{a}a \oplus \mathfrak{I}$. Then $1 = e + e'$ where $e \epsilon \mathfrak{a}a, e' \epsilon \mathfrak{I}$. It follows that $\mathfrak{a}a = \mathfrak{a}e$ and $e^2 = e$. Then $ae = a$ and $e = xa$. Hence $axa = a$ so $a \epsilon \mathfrak{a}^{(q)} U_a$. Thus $\mathfrak{a}^{(q)}$ is regular and consequently strongly non-degenerate. We note that $a\mathfrak{a}a = a\mathfrak{a} \cap \mathfrak{a}a$. Clearly $a\mathfrak{a}a \subseteq a\mathfrak{a} \cap a\mathfrak{a}_a$. On the other hand, if $x = au = va$ then, by regularity, $x = xyx$, $y \epsilon \mathfrak{a}$ so $x = auyva \epsilon a\mathfrak{a}a$. Hence $a\mathfrak{a} \cap \mathfrak{a}a \subseteq a\mathfrak{a}a$. If $a\mathfrak{a}a \supseteq b\mathfrak{a}b$ then $b = bzb = awa \epsilon a\mathfrak{a}a$. Then $a\mathfrak{a} \supseteq b\mathfrak{a}$ and $\mathfrak{a}a \supseteq \mathfrak{a}b$. Since $\mathfrak{a}$ satisfies the descending chain condition on both left and right ideals it is now clear that $\mathfrak{a}^{(q)}$ satisfies the descending chain condition for principal inner ideals.

2. Let $(\mathfrak{a}, J)$ be an associative algebra with involution. Suppose $\mathfrak{a}^{(q)}$ **135**
is regular. Then $\mathscr{H}(\mathfrak{a}, J)$ is regular. For if $h \epsilon \mathscr{H}(\mathfrak{a}, J)$ there exists an $a \epsilon \mathfrak{a}$ such that $hah = a$. Then $ha^J h = h$ so $h = ha(ha^J h) = h(aha^J)h$ and $aha^J \epsilon \mathscr{H}$. Hence $\mathscr{H}$ is regular. We note next that if $\mathfrak{a}^{(q)}$ is regular and satisifies the descending chain condition for principle inner ideals then $\mathscr{H}$ satisfies these conditions. We have seen that $\mathscr{H}$ is regular. Now suppose $\mathscr{H} U_{b_1} \supseteq \mathscr{H} U_{b_2} \supseteq \mathscr{H} U_{b_3} \ldots$ where $b_i \epsilon \mathscr{H}$. Then $b_{i+1} \in \mathscr{H} U_{b_{i+1}}$ by regularity so $b_{i+1} \in \mathscr{H} U_{b_i}$ and so $b_{i+1} = b_i h_i b_i, h_i \in \mathscr{H}$. Then $\mathfrak{a}U_{b_i}U_{b_i}U_{b_i} \subseteq \mathfrak{a}b_i$. Hence $\mathfrak{a}U_{b_1} \supseteq \mathfrak{a}U_{b_2} \supseteq \ldots$ is a descending chain of principal inner ideals of $\mathfrak{a}(q)$. Hence we have an m such that $\mathfrak{a}U_{b_m} = \mathfrak{a}U_{b_{m+1}} = \ldots$ By regularity, $b_i \epsilon \mathfrak{a} U_{b_i}$ so if $n \geqq m$, $b_n = b_{n+1}a_{n+1}b_{n+1}, a_{n+1} \epsilon \mathfrak{a}$. Then $b_n = b_{n+1}a^j_{n+1}b_{n+1}$. Also, by regularity of $\mathscr{H}$, $b_n = b_n k_n b_n$, $k_n \epsilon \mathscr{H}$. Then $b_n = (b_{n+1}a_{n+1}b_{n+1})k_n(b_{n+1}a^j_{n+1}b_{n+1}) = b_{n+1}l_{n+1}$

b_{n+1} where $l_{n+1} = a_{n+1}b_{n+1}k_nb_{n+1}a^j_{n+1} \in \mathscr{H}$. Hence $b_n \epsilon \mathscr{H}U_{b_{n+1}}$ and $\mathscr{H}U_{b_n} \subseteq \mathscr{H}U_{b_{n+1}}$. Thus $\mathscr{H}U_{b_m} = \mathscr{H}U_{b_{m+1}} = \ldots$ and $\mathscr{H}$ satisfies the DCC for principal inner ideals.

It is clear from (1) and the foregoing results that if $(\mathfrak{a}, J)$ is semi-simple Artinian with involution then $\mathscr{H}(\mathfrak{a}, J)$ is regular and satisfies the DCCfor principal inner ideals.

3. If $\mathscr{J}$ is a quadratic Jordan algebra over Γ and Φ is a subring Γ then $\mathscr{J}/\Phi$ and $\mathscr{J}/\Gamma$ have the same principal inner ideals. Hence $\mathscr{J}/\Phi$ has DCC on these if and only if this holds for $\mathscr{J}/\Gamma$. It
136 is clear also that $\mathscr{J}/\Phi$ is regular if and only if $\mathscr{J}/\Gamma$ is, and is strongly non-degenerate if and only if $\mathscr{J}/\Gamma$ is. Now let Γ be a field and let $\mathscr{J} =$ Jord $(Q, 1), Q$ a quadratic form on $\mathscr{J}/\Gamma$ with vase point 1 (cf. §1.7). We have the formulas $yU_x = Q(x, \overline{y})x - Q(x)\overline{y}, \overline{x} = T(x)1 - x, T(x) = Q(x, 1)$ in $\mathscr{J}$. If $Q(x) \neq 0$ then x is invertible and $\mathscr{J}U_x = \mathscr{J}$(§1.7). If $Q(x) = 0$ the $Q(x) = 0$ the formula for U_x shows that $\mathscr{J}U_X \subseteq \Gamma x$. This implies that $\mathscr{J}/\Gamma$, hence $\mathscr{J}/\Phi$, satisfies the DCC for principal inner ideals. If $x \epsilon \mathscr{J}$ satisfies $Q(x) = 0$, $Q(x, y) = 0$, $y \epsilon \mathscr{J}$, then $U_x = 0$. On the other hand, suppose Q is non-degenerate. Then for $x \epsilon \mathscr{J}$ either $Q(x) \neq 0$ there exists a y such that $Q(x, y) \neq 0$. In either case $x \epsilon \mathscr{J}U_x$. Hence $\mathscr{J} =$ Jord $(Q, 1)$ has non-zero absolute zero divisors or is regular according as Q is degenerate or not.

   We remark that the formula for U shows that if $\mathfrak{K}$ is a subspace such that $Q(k) = 0, k \epsilon \mathfrak{K}$, then $\mathfrak{K}$ is an inner ideal. This can be used to construct examples of algebras which are regular with DCC on principal inner ideal but not all inner ideals.

4. Let O be an octonion algebra over a field Γ, Φ a subring of Γ. We consider $\mathscr{H}(O_3)$ as quadratic Jordan algebra over Φ(cf. §§1.8, 1.9).since $\mathscr{H}(O_3)$ is a finite dimensional vector space over Γ, $\mathscr{H}/\Gamma$ satisfies the DCC for principal inner ideals. Hence $\mathscr{H}/\Phi$ satisfies this condition. We proceed to show that $\mathscr{H}$ is strong non-degenerate. Let $e_i = 1[ii]$, $f_i = 1 - e_i$ (notations as in §1.7). Then $\mathscr{H}U_{f_3} = \mathscr{H}_{11} \oplus \mathscr{H}_{12} \oplus \mathscr{H}_{22} = \{\alpha[11] + \beta[22] + a[12] | \alpha, \beta \epsilon \Phi, a \epsilon O\}$. If $x\alpha[11] + \beta[22] + a[12]$ the Hamilton-Cayley theorem in $\mathscr{H}(O_3)$

shows that $x^3 - T(x)x^2 + S(x)x = 0$ (since $N(x) = 0$, see §1.9). Here $T(x) = \alpha+\beta$ and $S(x) = T(x^\sharp) = \alpha\beta - n(a)$ by direct calcula- **137** tion. Also, direct calculation using the usual matrix square shows that $x^2 - T(x)x + S(x)f_3 = 0$. Hence $(\mathscr{H}U_{f_3}, U, f_3)$ =Jord (S, f_3) (see §1.7). Since the symmetric form $n(a, b)$ of the norm form $n(a)$ of O is non-degenerate the same is trur of the symmetric bilinear form of $S(x) = \alpha\beta - n(a)$. Hence $S(x)$ is non-degenerate so $\mathscr{H}U_{f_3}$ = Jord (S, f_3) is strongly non-degenerate. By symmetry, $\mathscr{H}U_{f_i}$ is strongly non-degenerate for $i = 1, 2$ also. Now let $\epsilon\mathscr{H}$ be an absolute zero divisor in $\mathscr{H}$. Then zU_{f_i} is an absolute zero divisor in $\mathscr{H}U_{f_i}$ so $zU_{f_i} = 0$, $i = 1, 2, 3$. Clearly this implies $z = 0$ so $\mathscr{H}$ is strongly non-degenerate.

5. It is not difficult to shows by an argument similar to that used in 2 that if $\mathscr{J}$ is regular then any ideal Z and $\mathscr{J}$ contianing 1 is regular and if $\mathscr{J}$ is regular and satisfies the minimum condion then the same is true of Z. We leave the proofs to the reader.

6 Capacity

An Idempotent $e\epsilon\mathscr{J}$ is called *primitive* If $e \neq 0$ and e is the only non-zero idempotent of $\mathscr{J}U_e$. If e is not primitive and e' is an idempotent $\neq 0$, e in $\mathscr{J}U_e$ then $e = e' + e''$ where e' and e'' are orthogonal idempotents $\neq 0$. Conversely if $e = e' + e''$ where e' and e'' are orthogonal idempotents then $e', e''\epsilon\mathscr{J}U_e$ (cf. §2.1) so e is not primitive. Hence e is primitive if and only if it is impossible to write $e = e' + e''$ where e' and e'' are non-zero orthogonal idempotents. An idempotent e is called *completely primitive* if $(\mathscr{J}U_e, U, e)$ is a division algebra. Since a division algebra contains no idempotents $\neq 0, 1$ it is clear that if e is completely primitive then e is primitive.

Lemma 1. *If $\mathscr{J} \neq 0$ satisfies the DCC for Pierce inner ideals then* **138** *$\mathscr{J}$ contains a (finite) supplementary set of orthogonal primitive idempotents.*

Proof. Consider the set of non-zero Pierce inner ideals of $\mathscr{J}$. By the DCC on these there exists a minimal element $\mathscr{J}U_{e_1}$ in the set. Clearly

e_1 is primitive. If $e_1 = 1$ we are done. Otherwise, put $f_1 = 1 - e_1$ so $f_1 \neq 0$ and consider $\mathscr{J} U_{f_1}$. Since f_1 is an idempotent the hypothesis carries over to $\mathscr{J} U_{f_1}$. Hence $\mathscr{J} U_{f_1}$ contains a primitive idempotent e_2 and this is orthogonal to e_1. If $1 = e_1 + e_2$ we are done. Otherwise, put $f_2 = 1 - e_1 - e_2$ and apply the argument to obtain a primitive idempotent e_3 in $\mathscr{J} U_{f_2}$. Also $\mathscr{J} U_{f_1} \supseteq \mathscr{J} U_{f_2}$ since $f_1 = e_2 + f_2, e_2 \neq 0$. Now e_3 is orthogonal to e_1 and e_2 so if $1 = e_1 + e_2 + e_3$ we are done. Otherwise, we repeat the argument with $f_3 = 1 - e_1 - e_2 - e_3$. Then $\mathscr{J} U_{f_2} \supset U_{f_2} \supset U_{f_3} \supset \ldots$ Since the DCC holds for Pierce inner ideals this process terminates with a supplementary set of orthogonal primitive idempotents. □

Definition 4. A quadratic Jordan algebra $\mathscr{J}$ is said to *have a capacity* if it contains a supplementary set of orthogonal completely primitive idempotents. Then the minimum number of elements in such a set is called *the capacity* of $\mathscr{J}$.

Theorem 7. *If $\mathscr{J}$ is strongly non-degenerate and satisfies the DCC for principal inner ideals then $\mathscr{J}$ has an isotope $\mathscr{J}^{(c)}$ which has a capacity. If $\mathscr{J}$ has no two torsion then $\mathscr{J}$ itself has a capacity.*

139 *Proof.* Let $\{e_i\}$ be a supplementary set of orthogonal primitive idempotents in $\mathscr{J}$ (Lemma 1). Suppose for some $i, \{e_i\}$ is not completely primitive. Since the hypothesis carry over to $\mathscr{J} U_{e_i}, U_{e_i}$ contains a minimal inner ideal Z (Lemmas 1, 2 of §5). Since $\mathscr{J} U_{e_i}$ is not a division algebra $Z \subset \mathscr{J} U_{e_i}$ Now Z is not of type I by the strong non-degeneracy and it is not of type III since $Z \subset \mathscr{J} U_{e_i}$ and e_i is primitive. Hence Z is of tyoe II. Also since $\mathscr{J} U_{e_i}$ contains no idempotent $\neq 0$, e_i the Theorem on Minimal Inner Ideals implies that $2\mathscr{J} U_{e_i} = 0$ and $\mathscr{J} U_{e_i}$ contains an element c_i such that $c_i^2 = e_i$, $c_i^{-1} = c_i$ (in $\mathscr{J} U_{e_i}$) and in the isotope $(\mathscr{J} U_{e_i})^{(c_i)}$, $c_i = b_i + d_i$ where b_i, d_i are orthogonal idempotents such that the corresponding pierce inner ideals are minimal of type III. Let $c_1 = e_j$ if e_j is completely primitive; otherwise let c_j be as just indicated. Put $c = \sum c_j$. Then c is invertible and it is clear that $\mathscr{J}^{(c)}$ has a capacity.

It is clear from the definition that $\mathscr{J}$ has capacity 1 is and only if $\mathscr{J}$ is a division algebra. We consider next the algebras of capacity two and we shall prove the following usefull lemma for these □

Lemma 2. *Let $\mathscr{J}$ have capacity two, so $1 = e_1 + e_2$ where the e_i are orthogonal completely primitive idempotents, $\mathscr{J} = \mathscr{J}_{11} \oplus \mathscr{J}_{12} \oplus \mathscr{J}_{22}$ the corresponding Pierce decomposition. If $x \epsilon \mathscr{J}_{12}$ either $x^2 = 0$ or x invertible. The set of absolute zero divisors of $\mathscr{J}$ is the set of $x \epsilon \mathscr{J}_{12}$ such that $x^2 = 0$ and $x \circ y = 0$, $y \epsilon \mathscr{J}_{12}$ and this set is an ideal. Either e_1 and e_2 are connected or every element of $\mathscr{J}_{12}$ is an absolute zero* **140**
divisor. $\mathscr{J}$ is simple if and only if $\mathscr{J}_{12} \neq 0$ and $\mathscr{J}$ is strongly non-degenerate. If $\mathscr{J}$ is simple e_1 and e_2 are connected and every outer ideal containing 1 in $\mathscr{J}$ is simple of capacity two.

Proof. If $x \epsilon \mathscr{J}_{12}, x^2 = x_1 + x_2, x_i \epsilon \mathscr{J}_{ii}$. Since $\mathscr{J}_{ii}$ is a division algebra, either $x_i = 0$ or x_i is invertible in $\mathscr{J}_{ii}$. Clearly if $x \neq 0$ and $x_2 \neq 0$ then x^2 and hence x is invertible. Suppose $x_1 = 0$ so $x^2 = x_2 \epsilon \mathscr{J}_{22}$. Then since $e_1 \circ x = x$, and $V_x V_{x^2} = V_{x^2} V_x$ we have $x_2 \circ x = x^2 \circ (e_1 \circ x) = (x^2 \circ e_1) \circ x = (x_2 \circ e_1) \circ x = 0$. By PD 5, if $a_2 \epsilon \mathscr{J}_{22}$ the mapping $a_2 \to \overline{V}_{a_2}$ the restriction of V_{a_2} to $\mathscr{J}_{12}$ is a homomorphism of $(\mathscr{J}_{22}, U, e_2)$ into (End $\mathscr{J}_{12})^{(q)}$. Since $\mathscr{J}_{22}$ is a division algebra this is a monomorphism and the image is a division subalgebra of (End $\mathscr{J}_{12})^{(q)}$. We recall also that invertibility in (End $\mathscr{J}_{12})^{(q)}$ is equivalent to invertibility in End $\mathscr{J}_{12}$. Since we had $xV_{x_2} = 0$ it now follows that either $x_2 = 0$ or $x = 0$. In either case $x^2 = x_2 = 0$. Thus $x_1 = 0$ implies $x^2 = 0$ and, by symmetry, $x_2 = 0$ implies $x^2 = 0$. It is now clear that either $x^2 = 0$ or x is invertible.

Let $x \epsilon \mathscr{J}_{12}$ satisfy $x^2 = 0$, $x \circ y = 0$ for all $y \epsilon \mathscr{J}_{12}$. Let $a \epsilon \mathscr{J}_{11}$. Then $aU_x \epsilon \mathscr{J}_{22}$ and $(aU_x)^2 = x^2 U_a U_x = 0$. Since $\mathscr{J}_{22}$ is a division algebra this implies that $aU_x = 0$. Similarly $bU_x = 0$ if $b \epsilon \mathscr{J}_{22}$. By $QJ17$, $U_x = U_{x \circ e_1} = U_x U_{e_1} + U_{e_1} U_x + V_s x U_{e_1} V_x - U_{e_1} V_x - U_{e_1 U_x, e_1} = U_x U_{e_1} + U_{e_1} U_x + V_x U_{e_1} V_x$ since $e_1 U_x = 0$ by the PD theorem. If $y \epsilon \mathscr{J}_{12}$ we have $yU_{e_1} = 0 = y_{12} V_x$. Hence $yU_x = yU_x U_{e_1} \epsilon \mathscr{J}_{11}$. By symmetry $yU_x \epsilon \mathscr{J}_{22}$
so $yU_x = 0$ Thus $U_x = 0$ and x is an absolute zero divisor. Conversely **141**
suppose x is an absolute zero divisor. Then xU_{e_i} is an absolute zero divisor in the division algebra $\mathscr{J}_{ii}$ so $xU_{e_i} = 0$. Then $x = xU_{e_1, e_2} \epsilon \mathscr{J}_{12}$. ALso $x^2 = 1U_x = 0$ and if $y \epsilon \mathscr{J}_{12}$ then $y \circ x \epsilon \mathscr{J}_{11} + \mathscr{J}_{22}$ and $(y \circ x)^2 = y^2 U_x + x^2 U_y + yU_x \circ y (QJ30) = 0$. As before, this implies that $y \circ x = 0$. Hence the set of absolute zero divisors coincides with the set of $x \epsilon \mathscr{J}_{12}$ such that $x^2 = 0$ and $x \circ y = 0, y \epsilon \mathscr{J}_{12}$. To see that this set is an ideal it is enough to prove that it is closed under addition. This is immediate.

Suppose e_1 and e_2 are not connected. Then $x^2 = 0$ for all $x \epsilon \mathscr{J}_{12}$. Then $x \circ y = (x+y)^2 - x^2 - y^2 = 0$. for all $x, y \epsilon \mathscr{J}_{12}$. Then the preceding result shows that every $x \epsilon \mathscr{J}_{12}$ is an absolute zero divisor.

Now suppose $\mathscr{J}_{12} \neq 0$ and $\mathscr{J}$ is strongly non-degenerate. Let Z be an ideal $\neq 0$ in $\mathscr{J}$. We have $Z = ZU_{e_1} \oplus ZU_{e_2} \oplus ZU_{e_1,e_2}$ and $Z_{ii} \equiv ZU_{e_i} = Z \cap \mathscr{J}_{ii}, Z_{12} \equiv ZU_{e_1,e_2} = Z \cap \mathscr{J}_{12}$. Since $\mathscr{J}_{ii}$ is a division algebra and Z_{ii} is an ideal in $\mathscr{J}_{ii}$ either $Z_{ii} = 0$ or $Z_{ii} = \mathscr{J}_{ii}$. Since $Z \neq 0$, $Z_{22} \neq 0$. If $Z_{11} \neq 0$ so $Z_{11} = \mathscr{J}_{11}$ then $Z \supseteq e_1 \circ \mathscr{J}_{12} = \mathscr{J}_{12}$. Similarly, if $Z_{22} \neq 0$ then $Z \supseteq \mathscr{J}_{12}$. Next suppose $Z_{12} \neq 0$ and let $x \neq 0$ in Z_{12}. Since x is not an absolute zero divisor either $x^2 \neq 0$ or there exists $ay \epsilon \mathscr{J}_{12}$ such that $x \circ y \neq 0$. In either case, since x^2 and $x \circ y \epsilon \mathscr{J}_{11} + \mathscr{J}_{22}$ we obtain either $Z_{11} \neq 0$ or $Z_{22} \neq 0$. Then, as before, $Z_{12} = \mathscr{J}_{12}$. Thus $Z \supseteq \mathscr{J}_{12}$. Since $\mathscr{J}_{12} \neq 0$ and $\mathscr{J}$ is strongly non-degenerate $\mathscr{J}_{12}$ contains as invertible element. Then Z contains an invertible element and so $Z = \mathscr{J}$. Hence $\mathscr{J}$ is simple. If $\mathscr{J}_{12} = 0$ then $\mathscr{J} = \mathscr{J}_{11} \oplus \mathscr{J}_{22}$
142 and the $\mathscr{J}_{ii}$ are ideals. Hence in this case $\mathscr{J}$ is not simple. Also if $\mathscr{J}$ contains absolute zero divisors$\neq 0$ then the set of these is an ideals and $\mathscr{J}$ is not simple. Hence simplicity of $\mathscr{J}$ implies $\mathscr{J}_{12} \neq 0$ and $\mathscr{J}$ is strongly non-degenerate.

If $\mathscr{J}$ is simple $\mathscr{J}_{12}$ contains an invertible element. Then e_1 and e_2 are connected. If Z is an outer ideal containing 1 then contains the e_i and $\mathscr{J}_{12} = \mathscr{J}_{12} \circ e_i$ (of. the proof of Theorem 2.2) Clearly, this and the previous results imply that Z is simple of capacity two. □

7 First structure theorem

The results of the last section have put us into position to prove rather quickly the

First structure Theorem. Let $\mathscr{J}$ be a strongly non-degenerate quadratic Jordan algebra satisfying the DCC for principal inner ideals (equivalently, by Theorem 6 $\mathscr{J}$ is semi-simple with DCC for principal inner ideals). Then $\mathscr{J}$ is a direct sum of ideals which are simple quadratic Jordan algebras satisfying the DCC on principal inner ideals. Conversely, if $\mathscr{J} = \mathscr{J}_1 \oplus \ldots \oplus \mathscr{J}_s$ where the $\mathscr{J}_i$ are ideals which are simple quadratic Jordan algebras with DCC on principal inner ideals then $\mathscr{J}$ is strongly non-degenerate with DCC on prinicipal inner ideals.

Proof. By Theorem 7, $\mathscr{J}$ has an isotope $\tilde{\mathscr{J}} = \mathscr{J}^{(c)}$ whose unit $c^{-1} = c$ is a sum of completely primitive orthogonal idempotenets e_i. Let $\mathscr{J} \sum \tilde{\mathscr{J}}_{ij}$ be the corresponding Pierce decompostion. It is clear that $\tilde{\mathscr{J}}$ and hence every Pierce inner ideal of $\mathscr{J}$ is strongly non-degenerate. If $c = e_1$ so $\tilde{\mathscr{J}} = \mathscr{J}_{11}$ then $\mathscr{J}$ is a division algebra and the result is clear. Hence assume the number of e_i is > 1. Let $i \neq j$ and consider **143**
the Pierce inner ideal $\mathscr{J} U_{e_i+e_j} = \mathscr{J}_{ii} + \tilde{\mathscr{J}}_{ii} + \tilde{\mathscr{J}}_{ij} + \tilde{\mathscr{J}}_{jj}$. By lemma 2 of the proceding section, either $\tilde{\mathscr{J}}_{ij} = 0$ or e_i and e_j are connected and $\tilde{\mathscr{J}}_{ii} + \tilde{\mathscr{J}}_{ij} + \tilde{\mathscr{J}}_{jj}$ is simple. Since connectedness of orthogonal idempotents is a transitive relation (§2.3) we may decompose the set of indices i into non over-lapping subsets $I_1, I_2, \ldots, I_s$ such that if $i, j \epsilon I_k$, $i \neq j$, the e_i and e_j are connected but if $i \epsilon I_k$ and $j \epsilon I_l$, $k \neq l$, then e_i and e_j are not connected so $\mathscr{J}_{ij} = 0$. Put $1_k = \sum\limits_{i \epsilon I_k} e_i$, $\tilde{\mathscr{J}}_k = \tilde{\mathscr{J}} U^{(1)}_{1_k}$. $\tilde{\mathscr{J}}_k = \tilde{\mathscr{J}}_1 \oplus \ldots \oplus \tilde{\mathscr{J}}_s$ and the Pierce relations show that $\tilde{\mathscr{J}}_k$ is an ideal. We claim that $\mathscr{J}_k$ is simple. We may suppose $I_k = \{1, 2, \ldots, m\}, m > 1$. Then $\tilde{\mathscr{J}}_k = \sum\limits_{i \geqq j=1}^{m} \tilde{\mathscr{J}}_{ij}$ and e_i and e_j are connected if $i \neq j \epsilon \{1, \ldots, m\}$. Also $\tilde{\mathscr{J}}_{ii} + \tilde{\mathscr{J}}_{ij} + \tilde{\mathscr{J}}_{jj}$ is simple. Let Z be a non-zero ideal in $\tilde{\mathscr{J}}_k$. Then, as before, $Z = \sum Z_{ij}$ where $Z_{ij} = \tilde{\mathscr{J}}_{ij} \cap Z$. Since $\tilde{\mathscr{J}}_{ii} + \tilde{\mathscr{J}}_{ij} + \tilde{\mathscr{J}}_{jj}$ is simple either Z contains this or $Z_{ii} = Z_{ij} = Z_{jj} = 0$. Clearly $Z \neq 0$ implies that for some $i \neq j$ we have Z_{ii}, Z_{jj} or $Z_{ij} \neq 0$. Then $Z \supseteq \tilde{\mathscr{J}}_{ii}, \tilde{\mathscr{J}}_{jj}$ and consequently $Z \supseteq \tilde{\mathscr{J}}_{ll}, \tilde{\mathscr{J}}_{lr}$ for all $l, r \epsilon \{1, \ldots, m\}$. Then $Z = \tilde{\mathscr{J}}_k$ and $\tilde{\mathscr{J}}_k$ is simple. Since $\mathscr{J}$ is an isotope of $\tilde{\mathscr{J}}$ we have $\mathscr{J} = \mathscr{J}_1 \oplus \ldots \oplus \mathscr{J}_s$ where $\mathscr{J}_i = \tilde{\mathscr{J}}_i$ as module, id an ideal of $\mathscr{J}$ which is a simple algebra (since any ideal of $\mathscr{J}_i$ is an ideal of $\mathscr{J}$ because of the direct decomposition) Since $\mathscr{J}_i$ is a Pierce inner ideals of $\mathscr{J}$ it satisfies the DCC for principal inner ideals.

Conversely, suppose $\mathscr{J} = \mathscr{J}_1 \oplus \mathscr{J}_2 \oplus \ldots \oplus \mathscr{J}_s$ where $\mathscr{J}$ is an ideal and is a simple quadratic Jordan algebra with unit 1_i, satisfying the DCC **144**
in principal inner ideals. Since the absolute zero divisors generate a nil ideal $\mathscr{J}_i$ is strongly non-degenerate. If z is an absolute zero divisor in $\mathscr{J}$ in $\mathscr{J}$ then $z = \sum z_i, z_i = z U_{1_i}$, and z_i is an absolute zero divisor of $\mathscr{J}_i$. Hence $z_i = 0$, $i = 1, 2, \ldots, s$ and $z = 0$. Thus $\mathscr{J}$ is strongly non-

degenerate. Let $a\epsilon\mathscr{J}$ and write $a = \sum a_i, a_i = aU_{1_i}$, then it is immediate that $\mathscr{J}U_a = \sum \mathscr{J}U_{a_i} = \sum \mathscr{J}_i U_{a_i}$. Also if $b = \sum b_i, b_i = bU_{1_i}$ then $\mathscr{J}U_a \supseteq \mathscr{J}U_b$ if and only if $\mathscr{J}U_{a_i} \supseteq \mathscr{J}U_{b_i}$, $i = 1, 2$. It follows from this that the minimum condition for principal inner ideals carries over from the $\mathscr{J}_i$ to $\mathscr{J}$.

It is easy to show that if Z is an ideal of $\mathscr{J}$ then $Z = \mathscr{J}_{i_1} + \mathscr{J}_{i_2} + \ldots + \mathscr{J}_{i_k}$ for some subset $\{i_1, \ldots, i_k\}$ of the index set. Clearly this implies that the decompostion $\mathscr{J} = \mathscr{J}_1 \oplus \mathscr{J}_2 \oplus \ldots \oplus \mathscr{J}_s$ into simple ideals is unique. We shall call the $\mathscr{J}_i$ *the simple components* of $\mathscr{J}$. □

8 A theorem on alternative algebras with involution.

Our next task is to determine the simple quadratic Jordan algebras which satisfy the DCC for principal inner ideals. By passing to an isotope we may assume $\mathscr{J}$ has a capacity. If the capacity is 1, $\mathscr{J}$ is a division algebra. We shall have nothing further to say about this. The case of capacity two will be treated in the next section by a rather lengthy direct analysis. The determination for capacity 3 will be based on the Strong coordinatization Theorem supplemented by information by information on the coordinate algebra. Both for this and for the study of the capacity two case we shall need to determine the coordinate algebra $(\mathscr{O}, j, \mathscr{O}_o)$
145 (§2.4 for the definition) such that $(\mathscr{O}, j)$ is simple and the non-zero elements of $\mathscr{O}_o$ are invertible. We now consider this problem.

We define an *absolute zero divisor* in an altenative algebra $\mathscr{O}$ to be an element z such that $zaz = 0$ for all $a\epsilon\mathscr{O}$.

Definition 5. An alternative algebra with involution $(\mathscr{O}, j)$ is called a *composition algebra* if 1 $\mathscr{O}$ has no absolute zero divisors $\neq 0$ and 2) for any $x\epsilon\mathscr{O}$, $Q(x) = x\bar{x}\epsilon\Phi 1$.

A complete determination of these algebras over a field is given in the following.

Theorem 8. *Let* $(\mathscr{O}, j)$ *be a composition algebra over a field* Φ. *Then* $(\mathscr{O}, j)$ *is of one of the following types: I a purely is separable field* P/Φ

of exponent one and charateristic two, $j = 1$. II $(\mathscr{O}, j) = (\Phi, 1)$. III $(\mathscr{O}, j)$ a quadratic algebra with standard involtion (§1.8).IV. $(\mathscr{O}, j) = (\mathfrak{a}, j)$, $\mathfrak{a}$ a quaternion, j the standard involution. V. $(\mathscr{O}, j)$ an octonion algebra with standard involution.

Proof. We have $x\overline{x} = Q(x)\epsilon\Phi$, from which it is immediate that Q is a quadratic form on $\mathscr{O}/\Phi$ whose associated bilinear form satisfies $Q(x, y)1 = x\overline{y} + y\overline{x}$. Hence

$$T(x) \equiv x + \overline{x} = Q(x, 1)1\epsilon\Phi 1. \tag{10}$$

Also $[x, \overline{x}, y] = [x; x + \overline{x}, y] - [x, x, y] = 0$ for all $y\epsilon\mathscr{O}$ and $Q(\overline{x})x = xQ(\overline{x}) = x(\overline{x}x) = (x\overline{x})x = Q(x)x$. Hence

$$Q(x) = Q(\overline{x})Q(x, y) = Q(\overline{x}, \overline{y}) \tag{11}$$

Next we note that $Q(xz, y) - Q(x, y\overline{z}) = (xz)\overline{y} + y(\overline{z}\,\overline{x}) - x(z\overline{y}) - (y\overline{z})\overline{x} =$ **146**
$[x, z, \overline{y}] - [y, \overline{z}, \overline{x}] = -[x, y, z] - [y, z, x] = [x, y, z] - [x, y, z] = 0$ (by $x + \overline{x}\epsilon N(\mathscr{O})$ and the alternating character of $[x, y, z]$. Hence $Q(xz, y) = Q(x, y\overline{z})$ and $Q(xz, y) = Q(\overline{zx}, \overline{y}) = Q(\overline{z}, \overline{y}x) = Q(z, \overline{x}y)$). Thus

$$Q(xz, y) = Q(x, y\overline{z}) = Q(z, \overline{x}y). \tag{12}$$

We have $x(\overline{x}y) = Q(x)y = (yx)\overline{x}$ so by bilinearization we have

$$x(\overline{z}y) + z(\overline{x}y) = Q(x, z)y = (yz)\overline{x} + (yx)\overline{z} \tag{1}$$

We suppose first that $Q(x, y)$ is generate which means that we have a non-zero z such that $Q(x, z) = 0$ for all x. Then $x\overline{z} + z\overline{x} = 0$ and $z + \overline{z} = 0$ so $xz = z\overline{x}$. Also $z^2 = -z\overline{z} = -Q(z)1$. Hence $zxz = -Q(z)\overline{x}, x\epsilon\mathscr{O}$. If $Q(z) = 0, z$ is an absolute zero divisor contrary to hypothesis. Hence $Q(z) \neq 0$ and $\overline{x} = \alpha zxz$, $\alpha = -Q(z)^{-1}, x\epsilon\mathscr{O}$. Then $xy = \overline{\overline{\overline{yx}}} = \alpha z(\overline{y}\,\overline{x})z = \alpha(z\overline{y}(\overline{x}z) = \alpha(yz)(zx)$. In particular, $xz = \alpha z^2(zx) = zx$ and consequently $\overline{x} = \alpha zxz = \alpha z^2 x = x$. Thus $j = 1$ and consequently $\overline{xy} = \overline{y}\,\overline{x}$ given $xy = yx$ so $\mathscr{O}$ is commutative. Also $z + \overline{z} = 0$ gives $2z = 0$ and $x = \alpha zxz$ gives $2x = 0$. Hence $2\mathscr{O} = 0$. This implies that $\mathscr{O}$ has no 3 torsion. Then

$$3[x, y, z] = [x, y, z] + [y, z, x] + [y, z, x] + [z, x, y]$$

$$= (xy)z - x(yz) + (yz)x - y(zx) + (zx)y - z(xy)$$
$$= [xy, z] + [yz, x] + [zx, y] = 0$$

147 and commutativity imply that $\mathscr{O}$ is associative. Since $x^2 = x\overline{x} = Q(x)1$ and $xyx = Q(x)y$ it is clear that $Q(x) \neq 0$ if $x \neq 0$ so $\mathscr{O}$ is a purely inseparable extension field of exponent one over Φ.

Now assume $Q(x, y)$ is non-degenerate. If $\mathscr{O} = \Phi 1$ we have type II. Hence assume $\mathscr{O} \supset \Phi 1$. If $x \epsilon \mathscr{O}$ we have $x^2 - T(x)x + x\overline{x} = x^2 - (x + \overline{x})x + x\overline{x} = 0$ so $x^2 - Q(x, 1)x + Q(x)1 = 0$. If Φ has characteristic two then $Q(1) = 1$ and $Q(1, 1) = 2 = 0$. Hence we can choose a $u \epsilon \mathscr{O}$ such that $Q(1, u) = 1$. Then $u^2 u + \rho 1$ and $4\rho + 1 = 1 \neq 0$ so $\Phi[u]$ is a quadratic algebra. If Φ has characteristic $\neq 2$ then $Q(1, 1) \neq 0$ and we can choose $q \epsilon \mathscr{O}$ such that $Q(1, q) = 0$ and $Q(q) = \beta \neq 0$. Put $u = q + \frac{1}{2}1$. Then $T(u) = 1$ and $Q(u) = \frac{1}{4} + \beta$, $u^2 = u + \rho_1, \rho = -\beta - \frac{1}{4}$. Since $4\rho + 1 = -4\beta \neq 0$, $\Phi[u]$ is a quadartic algebra. Hence in both cases we obtian a quadratic subalgebra $\Phi[q]$ which is a subalgebra of $(\mathscr{O}, j)$ since $\overline{u} = 1 - u$. Thus the induced involution is the standard one in $\Phi[q]$. It is clear also that $\Phi[u]$ is non-isotropic as a subspace relative to $Q(x, y)$. Now let Z be any finite deminsional non-isotropic subalgebra of $(\mathscr{O}, j)$ and assume $Z \subset \mathscr{O}$. As is well-known, $\mathscr{O} = Z \oplus Z^{\perp}$ and $Z^{\perp} \neq 0$ is non-isotropic. Hence there exists $a\, v \epsilon Z^{\perp}$ such that $Q(v) = -\sigma \neq 0$. Since $1 \epsilon Z, Q(1, v) = 0$ so $\overline{v} = -v$ and $v^2 = \sigma 1$. If $a \epsilon Z$ then $Q(a, v) = a\overline{v} + v\overline{a} = -av + v\overline{a} = 0$ so

$$av = v\overline{a}, a \epsilon \mathscr{Z} \tag{14}$$

If $a, b \epsilon Z, Q(av, b) = Q(v, \overline{a}b)$(by (12)) $= 0$. Hence $\mathscr{Z}v = \{xv | x \epsilon \mathscr{Z}\}$
148 $\subseteq \mathscr{Z}^{\perp}$ and $\mathfrak{L} = \mathscr{Z} + \mathscr{Z}v = \mathscr{Z} \oplus \mathscr{Z}v$ has dimensionality$= 2 dim \mathscr{Z}$. Also $Q(av, bv) = Q((av)\overline{v}, b) = Q(a(v\overline{v}), b) = -\sigma Q(a, b)$. It follows that $x \to xv$ is a linear isomorphism of $\mathscr{Z}$ onto $\mathscr{Z}v$ and $\mathscr{Z}v$ and $\mathfrak{L}$ are non-isotropic. By (13) with $z = v$ we obtain

$$a(bv) = (ba)v, (av)b = (a\overline{b})v, a, b, \epsilon \mathscr{Z}. \tag{15}$$

Also $(av)(bv) = v(\overline{a}b)v$ (Moufang identity)$= (\overline{b}a)v^2$. Hence

$$(av)(bv) = \sigma \overline{b}a, a, b \epsilon \mathscr{Z}. \tag{16}$$

We have $\overline{a+bv} = \overline{a} - v\overline{b} = \overline{a} - bv$. We apply these considerations to the quadratic subalgebra $\Phi[u]$. If $\mathscr{O} = \Phi[u]$ we have case III. Otherwise, we take $\mathscr{Z} = \Phi[u]$ and obtian the quaternion algebra $\mathfrak{a} = \Phi[u] + \Phi[u]v$. If $\mathscr{O} = \mathfrak{a}$ we have caseIV. Otherwise, we take $\mathscr{Z} = \mathfrak{a}$ and repeat the argument. Then $\mathfrak{L} = \mathcal{O}$ is an octonion algebra. We now claim that $\mathscr{O} = \mathcal{O}$ so we have case V. Otherwise, we can apply the construction to $\mathscr{Z} = \mathcal{O}$ and obtain $\mathfrak{L} = \mathscr{Z} + \mathscr{Z}v$ such that (15) and (16) hold. Put $x = a + bv, y = dv, a, b, d \epsilon \mathcal{O}$. Then we have $[\overline{x}, x, y] = 0$ and $\overline{x}(xy) = (\overline{a} - bv)(\sigma\overline{d}b + (da)v)$. Since $(\overline{x}y)y$ is a multiple of $y = dv$ this implies that $\overline{a}(\overline{d}b) = (\overline{a}\overline{d})b$. Since this holds for all $a, b, d \epsilon \mathcal{O}$ we see that $\mathcal{O}$ we see that $\mathcal{O}$ must be associative. Since it is readily verified that it is not we have a contradiction. This completes the proof that only $I - V$ can occur.

It is readily seen that the algebras with involution $I - V$ are composition algebras. We prove next □ 149

Theorem 9 (Herstein-Kleinfeld-Osborn-McCrimmon)**.** *Let* $(\mathscr{O}, j, \mathscr{O}_o)$ *be a coordinate algebra (over any* Φ*) such that* $(\mathscr{O}, j)$ *is simple and every non-zero element of* $\mathscr{O}_o$ *is invertible in* $\mathscr{O}_o$*. Then we have one of the following alternatives:*

- *I.* $\mathscr{O} = \Delta \oplus \Delta^{\circ}, \Delta$ *an associative division algebra j the exchange involution,* $\mathscr{O}_o = \mathscr{H}(\mathscr{O}, j)$*.*
- *II. an associative division algebra with involution.*
- *III. a split quaternion algebra* Γ_2 *over its center* Γ *which is a field over* Φ*, standard involution,* $\mathscr{O}_o = \Gamma$*.*
- *IV. an algebra of octonions over its center* Γ *which is a field over* Φ *standard involution,* $\mathscr{O}_o = \Gamma$*.*

Proof. We recall that the hypothesis that $(\mathscr{O}, j\mathscr{O}_o)$ is a coordinate algebra means that $(\mathscr{O}, j)$ is an alternative algebra with involution, $\mathscr{O}_o$ is a Φ submodule of $\mathscr{O}$ contianed in $\mathscr{H}(\mathscr{O}, j) \cap N((O))$ and containing 1 and every $xd\overline{x}, d\epsilon\mathscr{O}_o, x\epsilon\mathscr{O}$. Hence $\mathscr{O}_o$ contains all the norms $x\overline{x}$ and all the traces $x + \overline{x}$. Then $[x, \overline{x}, y] = 0, x, y\epsilon\mathscr{O}$. We recall also the following realtion in any alternativfe algebra

$$n[x,y,z] = [nx,y,z] = [xn,y,z] = [x,y,z]n \tag{17}$$

for $n\epsilon N(\mathscr{O}), x, y, z\epsilon\mathscr{O}$ (see the author's book pp. 18-19).

Suppose first that $\mathscr{O}$ is not simple. Then $\mathscr{O} = \Delta \oplus \overline{\Delta}(\overline{\Delta} = \Delta^j)$ where Δ is an ideal. The elements of $\mathscr{H}(\mathscr{O}, j)$ are the elements $a + \overline{a}, a\epsilon\Delta$. Hence $\mathscr{H}(\mathscr{O}, j) = \mathscr{O}_o$. Since these are in $N(\mathscr{O})$ every $a\epsilon N(\mathscr{O})$ so Δ and
150 $\overline{\Delta} \subseteq N(\mathscr{O})$. Then $\mathscr{O} = N(\mathscr{O})$ is associative. Also if $a \neq 0$ is in Δ then $a + \overline{a}$ is invertible which implies a is invertible. Hence Δ is a division algebra and we have case I.

From now on we assume $\mathscr{O}$ simple. Then its center $C(\mathscr{O})$(defined as the subset of $N(\mathscr{O})$ of elements which commute with every $x\epsilon\mathscr{O}$) is a field over Φ (see the author's book p.207). It follows that $\Gamma = \mathscr{H}(\mathscr{O}, j) \cap C(\mathscr{O})$ is a field over Φ. We can regard $\mathscr{O}$ as an algebra over Γ when we wish to do so. We note first that the following conditions on $a\epsilon\mathscr{O}$ are equvalent: (i)$a\overline{a} \neq 0$, (ii) as has a right inverse (iii) $\mathfrak{a}a \neq 0$, (iv) a has a left inverse, Asssume (i). Then $a\overline{a}$ is invertible in $N(\mathscr{O})$ so we have ab such that $(a\overline{a})b = 1$. Then $a(\overline{a}b) = 1$ and a has a right inverse. Hecne $(i) \Rightarrow (ii)$. Next assume $\overline{a}a = 0$. Then $0 =(\overline{a}, a)b = \overline{a}(ab)$ and $ab \neq 1$.Hence $(ii) \rightarrow (iii)$. By symmetry, $(iii) \rightarrow (iv) \rightarrow$ and $(iv) \rightarrow (i)$. Let $z\epsilon\mathfrak{z}, a\epsilon\mathscr{O}$. Then $(az)(\overline{az}) = (az)(\overline{z}\,\overline{a}) = (az)(\overline{z}(a + \overline{a})) - (az)(\overline{z}a) = ((az)\overline{z})(a + \overline{a}) - a(z\overline{z})a = 0$. Hence $az\epsilon\mathfrak{z}$. Also $\overline{\mathfrak{z}} = \mathfrak{z}$ so $za\epsilon\mathfrak{z}$. Moreover, $\mathfrak{z}$ is closed under multiplication by elements of Φ and $1 \notin \mathfrak{z}$. Hence if $\mathfrak{z}$ is closed under addition it is an ideal $\neq \mathscr{O}$, of $(\mathscr{O}, j)$ and so $\mathfrak{z} = 0$. Then every non-zero element of $\mathscr{O}$ has a left and a right inverse in $\mathscr{O}$.

Suppose $\mathfrak{z} = 0$. If $\mathscr{O}$ is associative $(\mathscr{O}, j)$ is an associative algebra with involution and we have case II. Next assume $\mathscr{O} = N(\mathscr{O})$. We claim that in this case $N(\mathscr{O}) = C(\mathscr{O})$. By (17), if $n\epsilon N(\mathscr{O}), x\epsilon\mathscr{O}$, $[nx] = nx - xn\epsilon N(\mathscr{O})$ and n commutes with all associators Direct verification shows that if $x, y\epsilon\mathscr{O}, n\epsilon N(\mathscr{O})$ then $[xy, n] = [xn]y + x[yn]$ where
151 $[ab] = ab - ba$ and $x[x,y,z] = [x^2, y, z] - [x, xy, z]$. The last implies that $0 = [x[x,y,z], n] = [xn][x,,y,z]$. Hence we have

$$[xn][x,y,z] = 0, n\epsilon N(\mathscr{O}), x, y, z, \epsilon\mathscr{O} \tag{18}$$

Bilinearization of this gives

$$[xn][w,y,z] + [wn][x,y,z] = 0 \tag{19}$$

Suppose $x \notin N(\mathscr{O})$. Then we can choose y, z such that $[x,y,z] \neq 0$ so this has a right inverse. Since $[x,n]\epsilon N(\mathscr{O})$ this and (18) imply $[xn] = 0$. If $x\epsilon N(\quad)$, (19) gives $[x,n][w,y,z] = 0$. Since $N(\mathscr{O}) \neq \mathscr{O}$ we can choose $[w,y,z] \neq 0$ and again conclude $[xn] = 0$. Hence $[xn] = 0$ for all x and $N(\mathscr{O} = C(\mathscr{O})$ if $\mathfrak{z} = 0$ and $\mathscr{O}$ is not associative. In this case $x\overline{x}\epsilon C(\mathscr{O}) \cap \mathscr{H}(\mathscr{O}, j) = \Gamma$. Also we have no absolute zero divisors since $\mathscr{O}$ is a division algebra. Treating $(\mathscr{O}, j)$ as a algebra over Γ we have a composition algebra. Since $\mathscr{O}$ is not associative we have the octonion case and we shall have case IV if we can show that $\mathscr{O}_o = \Gamma$. Since $N(\mathscr{O}) \subseteq \Gamma$ for an actonion algebra over a field, $\mathscr{O}_o \subseteq \Gamma$. To prove the opposite inequality it is enough to show that every element of Γ is a trace. Now O contains a quadratic algebra $\Phi[u]$ in whixh $u^2 u + \rho_1$ and $\overline{u} = 1 - u$. Thus $u + \overline{u} = 1$ and if $\gamma\epsilon\Gamma$ then $\gamma = \gamma u + \overline{\gamma u}$ is a trace.

It remains to consider the situation in which $\mathfrak{z}$ is not closed under addition. Then we have $z_1, z_2 \epsilon \mathfrak{z}$ such that $z_1 + z_2 = u$ is invertible. Hence $e_1 + e_2 = 1$, $e_i = z_i u^{-1} \epsilon \mathfrak{z}$ and $\overline{e_1} + \overline{e_2} = 1$. Also $\overline{e_i} e_i = 0$ and $\overline{e_1} = \overline{e_1}(e_1 + e_2) = \overline{e_1} e_2 = (\overline{e_1} + \overline{e_2}) e_2 = e_2$, Then $e_2 e_1 = 0, e_1 + e_2 = 1$ 152
so the e_i are orthongonal idempotents and $\overline{e_1} = e_2, \overline{e_2} = e_1$. Let $\mathscr{O} = \mathscr{O}_{11} \oplus \mathscr{O}_{12} \oplus \mathscr{O}_{21} \oplus \mathscr{O}_{22}$ be the corresponding Pierce decomposition (see shafer [1] pp. 35-37 and the author's [2' pp. (165-166). Since $\mathscr{O}$ is simple $\mathscr{O}_{12} + \mathscr{O}_{21} \neq 0$ and Since $\mathscr{O}_{12}\mathscr{O}_{21} + \mathscr{O}_{12} + \mathscr{O}_{21} + \mathscr{O}_{21}\mathscr{O}_{12}$ is an ideal, $\mathscr{O}_{12}\mathscr{O}_{21} = \mathscr{O}_{11}$, $\mathscr{O}_{21}\mathscr{O}_{12} = \mathscr{O}_{22}$. Also since $\overline{e_1} = e_2$, $\overline{e_2} = e_1$ we have $\overline{\mathscr{O}}_{11} = \mathscr{O}_{22}$, $\overline{\mathscr{O}}_{22} = \mathscr{O}_{11}$, $\overline{\mathscr{O}}_{12} = \mathscr{O}_{12}$, $\overline{\mathscr{O}}_{21} = \mathscr{O}_{21}$. Let $x = x_{11} + x_{12} + x_{21} + x_{22}$ where $x_{ij} \epsilon \mathscr{O}_{ij}$. Then the Pierce relations give

$$x_{11}\overline{x}_{11} = x_{22}\overline{x}_{22} = x_{11}\overline{x}_{21} = x_{22}\overline{x}_{12} = x_{12}\overline{x}_{22} = x_{21}\overline{x}_{11} = 0.$$

Also $x_{12} = e_1 x e_2 \epsilon \mathfrak{z}$ since $e_i \epsilon \mathfrak{z}$ so $x_{12}\overline{x}_{12} = 0$. Similarly, $x_{21}\overline{x}_{21} = 0$. If $y\epsilon\mathscr{O}_{12}$, $y + \overline{y}\epsilon\mathscr{O}_{12} \cup \mathscr{O}_o \subseteq \mathfrak{z} \cap \mathscr{O}_o = 0$. Similarly, if $y\epsilon\mathscr{O}_{21}, y + \overline{y} = 0$. Hence $x_{11}\overline{x}_{12} + x_{12}\overline{x}_{11} = 0 = x_{22}\overline{x}_{21} + x_{21}\overline{x}_{22}$. Combining we see that

$$x\overline{x} = x_{11}\overline{x}_{22} + x_{22}\overline{x}_{11} + x_{12}\overline{x}_{21} + x_{21}\overline{x}_{12} = y + \overline{y}$$

where $y = x_{11}\overline{x}_{22} + x_{12}\overline{x}_{21}\epsilon\mathscr{O}_{11}$. We show next that if $y\epsilon\mathscr{O}_{11}$ then $y = \overline{y}\epsilon\Gamma$. Since $y + \overline{y}\epsilon\mathscr{O}_o \subseteq N(\mathscr{O})$ it is enough to show that $y + \overline{y}$ commutes with every $x\epsilon\mathscr{O}$. We have seen that if $x_{ij}\epsilon\mathscr{O}_{ij}$ then $x_{ij} = \overline{-x}_{ij}$ if $i \neq j$ so $x_{ii}x_{ij} = \overline{-x}_{ij}\overline{x}_{ii} = x_{ij}\overline{x}_{ii}$. Hence if $y\epsilon\mathscr{O}_{11}$ then $(y + \overline{y})x_{ij} = yx_{ij} + \overline{y}x_{ij} = x_{ij}\overline{y} + x_{ij}y = x_{ij}(y + \overline{y})$. Also

$$\begin{aligned}(y + \overline{y})(x_{12}x_{21}) &= ((y + \overline{y})x_{12})x_{21} = (x_{12}(y + \overline{y}))x_{21}\\ &= x_{12}((y + \overline{y})x_{21}) = x_{12}(x_{12}(x_{21}(y + \overline{y}))\\ &= x_{12}x_{21}(y + \overline{y}).\end{aligned}$$

153 Similarly, $[y+\overline{y}, x_{21}x_{12}] = 0$ so $y+\overline{y}$ commutes with every x and $y+\overline{y}\epsilon\Gamma$. Thus we have $x\overline{x}\epsilon\Gamma$. We claim that if $Q(x, y) = x\overline{y} + y\overline{x}\epsilon\Gamma$ then this is non-degenerate. The formulas (12) show that the set of z such that $Q(z, x) = 0$ for all $x\epsilon\mathscr{O}$ is an ideal of $(\mathscr{O}, j)$. Hence if this is not 0 it contains 1. But $Q(1, e_1) = e_1 + \overline{e}_1 = 1$. Hence $Q(x, y)$ is non-degenerate. Then the proof of Theorem 7 shows that we have one of cases II-V of that theorem, one sees easily that the only possibilities allowed here are $(\mathscr{O}, j)$ is split quaternion or split octoion over Γ. As before, we have $\mathscr{O}_o = \Gamma$ in the octonion case and we are in case IV. In the split quaternion case, $\mathscr{O} = \Gamma_2$, the argument used before shows that $\mathscr{O}_o \supseteq \Gamma$. If the characteristic is $\neq 2\Gamma = \mathscr{H}(\mathscr{O}, j)$, hence, $\mathscr{O}_o = \Gamma$. If the characteristic is two then it is easily seen that we have a base of matrix units e_{ij} such that $\overline{e}_{11} = e_{22}, \overline{e}_{22} = c_{11}, \overline{e}_{12} = e_{12}, \overline{e}_{21} = e_{21}$. If $a\epsilon\mathscr{O}_o$, $a\epsilon\mathscr{H}(\mathscr{O}, j)$ so $a = \alpha 1 + \beta e_{12} + \gamma e_{21}, \alpha, \beta, \gamma\epsilon\Gamma$. Since $e_{12}a\overline{e}_{12} = \gamma e_{21}\epsilon\mathscr{O}_0$ and the non-zero elements of $\mathscr{O}_o$ are invertible, $\gamma = 0$. Similarly $\beta = 0$ so again $\mathscr{O}_o = \Gamma$. Thus we have case III. ◻

9 Simple quadratic Jordan algebras of capacity two

Let $\mathscr{J}$ be of capacity two, so $1 = e_1 + e_2$ where the e_i are completely primitive orthogonal idempotents, $\mathscr{J} = \mathscr{J}_{11} \oplus \mathscr{J}_{12} \oplus \mathscr{J}_{22}$ the corresponding Pierce decomposition. Then $\mathscr{J}_{ii}$ is a division algebra. Put $\mathfrak{m} = \mathscr{J}_{12}$. If $x_i\epsilon\mathscr{J}_{ii}$ then $\nu_i : x_i \to \overline{V}_{x_i}$ the restricition of V_{x_i} to $\mathfrak{m}$ is a homomorphism of $\mathscr{J}_{ii}$ into $(\text{End } \mathfrak{m})^{(q)}$. Since $\mathscr{J}_{ii}$ is a division algebra,

ν_i is a monomorphism. Hence $\mathscr{J}_{ii}$ is special so this can be indentified 154
with a subalgebra of $S(\mathscr{J}_{ii})^{(q)}$ where $S(\mathscr{J}_{ii})$ is the special universal envelope of $\mathscr{J}_{ii}$ (see §1.6). If π is the main involution of $S(\mathscr{J}_{ii})$ then $\mathscr{J}_{ii} \subseteq \mathscr{H}(S(\mathscr{J}_{ii}), \pi)$ and $s(\mathscr{J}_{ii})$ is generated by $\mathscr{J}_{ii}$. The homomorphism ν_i has a unique extension to a homomorphism of $S(\mathscr{J}_{ii})$. The latter permits us to consider $\mathfrak{m}$ as a right $s(\mathscr{J}_{ii})$ module in the natural way. If $m\epsilon\mathfrak{m}$ and $x_i\epsilon\mathscr{J}_{ii}$ then the definitions give $mx_i = mV_{x_i} = m \circ x_i$ and if $x_i, y_i, \ldots, z_i\epsilon\mathscr{J}_{ii}$ then

$$m(x_i y_i \ldots z_i) = (\ldots((m \circ x_i) \circ y_i) \circ \ldots \circ z_i) \tag{20}$$

Also by the associativity conseqences of the PD theorem $(m \circ x_1) \circ x_2 = (m \circ x_2) \circ x_1$ from which follows

$$(ma_1)a_2 = (ma_2)a_1, a_i\epsilon S(\mathscr{J}_{ii}). \tag{21}$$

We recall that if $m\epsilon\mathfrak{m}$, either $m^2 = 0$ or m is invertible (Lemma 2 of §6). Suppose $m^2 = 0$. Then $x_i U_m = 0$ for $x_i\epsilon\mathscr{J}_{Iii}$ (proof of Lemma 2 §6). Then $(x_i \circ m)^2 = x_i^2 U_m + m^2 U_{x_i} + x_i U_m \circ x_i (QJ30) = 0$. This implies that if m is invertible and $x \neq 0$ then mx_i is invertible. Otherwise $(mx_2)^2 = 0$ and $(mx_i y_i)^2 = (mx_i \circ y)^2 = 0$ for all $y_i\epsilon\mathscr{J}_{ii}$. If we choose y_i to be the inverse of x_i and $\mathscr{J}_{ii}$ we obtain the contradiction $m^2 = 0$. If m is invertible and, as in Lemma 2 of §2.3. we put $u = c_1 + m^2 U_{e_2}, \cdot v = e_1 + m^{-2} U_{e_2}$ then we have seen that in the isotope $\tilde{\mathscr{J}} = \mathscr{J}^{(v)}, u_1 = e_1$ and $u_2 = m^2 U_{e_2}$ are supplementary orthogonal idempotents which are strongly connected by m. The Pierce submodule $\mathscr{J}_{ij}$ relative to the u_i coincides with $\mathscr{J}_{ij}$. Moreover, $\mathscr{J}_{11} = \tilde{\mathscr{J}}_{11}$ as quadratic Jordan 155
algebras, and for $m\epsilon\mathfrak{m} = \mathscr{J}_{12} = \overline{\mathfrak{m}} = \tilde{\mathscr{J}}_{12}$ and $x_1\epsilon\mathscr{J}_{11}$ we have $mV_{x_1} = m\overline{V}_{x_1}$. Hence the $S(\mathscr{J}_{11})$ module structure on $\mathfrak{m}$ is unchanged in passing from $\mathscr{J}$ to $\tilde{\mathscr{J}}$. Also $\mathscr{J}_{22}$ and $\tilde{\mathscr{J}}_{22}$ are isotopic so $\tilde{\mathscr{J}}_{22}$ is a division algebra and u_1 and u_2 are completely primitive in $\tilde{\mathscr{J}}$. Clearly, $\mathscr{J}$ is of capactiy two. Since the isotope $\tilde{\mathscr{J}}$ is determined by the choice of the invertible element m it will be convenient to denote this as $\mathscr{J}_m$.

We shall now assume $\mathscr{J}$ simple and we shall prove the following structure theorem which is due to Osborn [11] in the linear case and to McCrommon in the quadratic case.

Theorem 10. *Let $\mathscr{J}$ be a simple quadratic Jordan algebra of capacity two. Then either $\mathscr{J}$ is isomorphic to an outer ideal $\ni 1$ of a quadratic Joradan algebra of a non-degenerate quadratic form on a vector space over a field P/Φ or $\mathscr{J}$ is isomorphic to an outer ideal $\ni 1$ of an algebra $\mathscr{H}(\mathscr{O}_2, J_H)$ where $(\mathscr{O}, J)$ is either an associative division algebra with involution or $\mathscr{O} = \Delta \oplus \Delta^j, \Delta$ an associative division algebra and J_H is the involution $X \to H^{-1}X^{-1}H$, $H \epsilon \mathscr{H}(\mathscr{O}_2)$.*

We have seen in §1.11 that $\mathscr{H}(\mathscr{O}_2, J_H)$ is isomorphic to the H-isotope of $\mathscr{H}(\mathscr{O}_2)$. Now consider Jord $(Q, 1)$ the quadratic Jordan algebra of a quadartic form Q with base point 1 on a vector space over a field P. Let u be an invertible elements so $Q(u) \neq 0$. Then $Q' = Q(u)Q$ is a quadratic form which has the base point $u^{-1} = Q(u)^{-1}\overline{u}$ since $Q(u)Q(u^{-1}) = Q(Q(u)^{-1}\overline{u}) = Q(Q(u, 1)1 - u) = Q(u)^{-1}Q(u) = 1$. Now
156 *consider Jord $(Q(u)Q, u^{-1})$. Put $x' = Q(u)Q(x, u^{-1})u^{-1} - x$ and let U' denote the U-operator in this algebra. A straight forward calculation shows that $xU'_a = Q(a, \overline{xU_u})a - Q(a)\overline{xU_u} = xU_vU_a$. It follows that Jord $(Q(u)Q, u^{-1})$ is identical with the u-isotope of Jord $(Q, 1)$. These remarks show that to prove Theorem 10 it suffices to show that there exists an isotope of $\mathscr{J}$ which is isomorphic to an outer ideal containing 1 in a Jord $(Q, 1)$ with non-degenerate Q(over a field) or to an outer ideal containing 1 in an $H(\mathscr{O}_2)$. By passing to an isotope we may assume at the start that $1 = e_1 + e_2$ where the e_i are orthogonal completely primitive and are strongly connected by an element $u \epsilon \mathscr{J}_{12}$. The proof will be divided into a series of lemmas. An important point in the argument will be that except for trivial cases $\mathfrak{m} = \mathscr{J}_{12}$ is spanned by invertible elements. This fact is contained in*

Lemma 1. *Suppose $\mathscr{J}_{11} \neq \{0, \pm e_1\}$ (that is, $\mathscr{J}_{11} \neq \mathbb{Z}_2$ or $\mathbb{Z}_3$). Let $m, n \epsilon \gamma \mathfrak{m}$, m invertible. Then there exist $x_1 \neq 0, y_1 \neq 0$ in $\mathscr{J}_{11}$ such that $x_1 \circ m + y_1 \circ n$ is invertible. Any element of $\mathfrak{m}$ is a sum of invertible elements.*

Proof. Since $x \epsilon \mathfrak{m}$ is either invertible or $x^2 = 0$, if the result is false, then $x_1 \circ m + y_1 \circ n)^2 = 0$ for all $x_1 \neq 0, y_1 \neq 0$ in $\mathscr{J}_{11}$. By $QJ30$ and the PD theorem the component in $\mathscr{J}_{22}$ of this element is

$$x_1^2 U_m + y_1^2 U_n \{x_1 \circ m, e_1, y_1 \circ m\} = 0 \tag{22}$$

Take $(x_1, y_1) = (x_1, e_1), (e_1, y_1), (e_1, e_1), (x_1, y_1)$ and add the first two equations thus obtained to the negative of the last two. This gives

$$\{z_1 \circ m e_1 w_1 \circ n\}, z_1 = x_1 - e_1, w_1 = y_1 - e_1 \tag{23}$$

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Using this and (22) we obtain

$$z_1^2 U_m + w_1^2 U_n = 0 \quad \text{if} \quad z_1, w_1 \neq 0, -e_1 \quad \text{in} \quad \mathscr{J}_{11}. \tag{24}$$

In particular, $w_1^2 U_m + w_1^2 U_n = 0$, so, by (24) $(z_1^2 - w_1^2)U_m = 0$ and since m is invertible, $z_1^2 = w_1^2$ is $z_1, w_1 \neq 0, -e_1$. Let $z_1 \neq 0, \pm e_1$ so $z_1 - e_1 \neq 0, -e_1$ and so $z_1^2 = (z_1 - e_1)^2 = z_1^2 - 2z_1 + e_1$. Hence $2z_1 = e_1$. Also since $-z_1 \neq 0, \pm e_1$ we have also $-2z_1 = e_1$. Then $4z_1 = 0$ and since $\mathscr{J}_{11}$ is a quadratic Jordan division algebra, $2z_1 = 0$. This gives $e_1 = 0$ contrary to $\mathscr{J}_{11} \neq 0$. Hence the first statement holds. For the second we note that $\mathfrak{m}$ contains an invertible element m and if n is any element of $\mathfrak{m}$ then there exist $x_1, y_1 \neq 0$ in $\mathscr{J}_{11}$ such that $p = x_1 \circ m + y_1 \circ n$ is invertible. Then if z_1 is the inverse of y_1 in $\mathscr{J}_{11}, n = z_1 \circ p - z_1 \circ (x_1 \circ m)$ and $z_1 \circ p$ and $-z_1 \circ (x_1 \circ m)$ are invertible elements of $\mathfrak{m}$.

We are assuming that e_1 and e_2 are strongly corrected by $u \epsilon \mathfrak{m}$. Then $\eta = U_u$ is an automorphism of period two in $\mathscr{J}, \eta$ maps $\mathfrak{m}$ onto itself and exchange $\mathscr{J}_{11}$ and $\mathscr{J}_{22}$. Hence η defines an isomorphism of $\mathscr{J}_{11}$ onto $\mathscr{J}_{22}$. This extends uniquely to an isomorphism η of $(S(\mathscr{J}_{11}), \pi)$ onto $(S(\mathscr{J}_{22}), \pi)$. We have $u^3 = u$(§2.3). Hence $u^\eta = uU_u = u^3 = u$. We shall now derive a number of results in which u plays a distinguished roles. These will be applied later to any invertible $m \epsilon \mathfrak{m}$ by passing to the isotope $\mathscr{J}_m (= \mathscr{J}^{(v)}$ as above). Let $x \epsilon \mathscr{J}$. Then $(x \circ n)^\eta = x^\eta \circ u^\eta \circ u$. Also $x^\eta \circ u = xU_u V_u = xU_{u,u^2} = xU_{u,1} = xV_u = x \circ u$. Thus we have

$$(x \circ u)^\eta = x^\eta \circ u = x \circ u, x \epsilon \mathscr{J} \tag{25}$$

We prove next □ **158**

Lemma 2. *Let $m \epsilon \mathfrak{m}$. Then $m + m^\eta = ux_1, x_1 = (u \circ m)U_{e_1}$. If $m + m^\eta = 0$ then $u \circ m = 0$.*

Proof. Let $m, n\epsilon\mathfrak{m}$. Then $mU_n = mU_{e_1,e_2}U_n = e_jV_{m,e_i}U_n = -e_jV_{n,e_i}U_{m,n} + e_jU_nV_{e_i}, m + e_1U_{m,n}V_{e_i}, n$ (by $QJ9$) $= -\{nnm\} + \{n^2U_{e_i}e_im\} + \{(m\circ n)U_{e_i}e_in\}$ (PD 6 and its bilinearization). Hence $mU_n = -n^2 \circ m + n^2U_{e_i} \circ m + (m\circ n)U_{e_i} \circ m$ (replacing e_i by $e_i + e_j$). Since $1 = U_{e_i} + U_{e_j} + U_{e_i,e_j}$ this gives

$$mU_n = -n^2U_{e_j} \circ m + (m \circ n)U_{e_i} \circ m, n, \in \mathfrak{m} \tag{26}$$

Taking $n = u$ we get $m^\eta = mU_n = -n\mu_n = -m+ux_1, x_1 = (m\circ u)U_{e_1}$. This is the first statment of the lemma. If $m + m^\eta = 0$ we have $x_1 = (u\circ m)U_{e_1} = 0$. Applying η gives $(u\circ m^\eta)U_{e_2} = 0 = (u\circ m)U_{e_2}$, by (25). Since $u \circ m\epsilon\mathscr{J}_{11} + \mathscr{J}_{22}$ the realtions $(u \circ m)U_{e_i} = 0$, $i = 1, 2$, imply $u \circ m = 0$. □

Lemma 3. *If $m\epsilon\mathfrak{m}$ satisfies $mx_1 = mx_1$ for all $x_1\epsilon\mathscr{J}_{11}$ then $ma^\pi = ma^\eta, a\epsilon S(\mathscr{J}_{11})$.*

Proof. Since $\mathscr{J}_{11}$ generates $S(\mathscr{J}_{11})$ it suffices to prove the conclusion for $a = x_1, x_2 \ldots x_k$, $x_i \in \mathscr{J}_{11}$. We use induction on k. Assume $m(x_1, x_2 \ldots x_k)^\pi = m(x_1 \ldots x_k)^\eta$. Then $m(x_1 \ldots x_{k+1})^\pi = mx^\eta_{k+1}(x_1 \ldots x_k)^\pi = m(x_1 \ldots x_k)^\pi x^n_{k+1}$ (by (21))$=$ $m(x_1 \ldots x_k)^\eta x^\eta_{n+1} = m$
159 $(x_1 \ldots x_{k+1})^\eta$ which proves the inductive step. We have $ux_1 = ux^\eta_1, x_1\epsilon \mathscr{J}_{11}$, by (25). Hence Lemma 3 and (25) give

$$ua^\pi = ua^\eta = (ua)^\eta, a\epsilon S(\mathscr{J}_{11}). \tag{27}$$

Now suppose $ua = 0$ for an $a\epsilon S(\mathscr{J}_{11})$. Then for $b\epsilon S(\mathscr{J}_{11})$, $uba = ub^{\eta\pi}a = ub^{\pi\eta}a = uab^{\pi\eta}$ (by (21))$=$ 0. Hence we have □

Lemma 4. *If $ua = 0$ for $a\epsilon S(\mathscr{J}_{11})$ then $uba = 0$ for all $b\epsilon S(\mathscr{J}_{11})$. We prove next*

Lemma 5. *Let $n\epsilon\mathfrak{n} = uS(\mathscr{J}_{11})$, $a\epsilon S(\mathscr{J}_1)$. Then $n(a + a^\pi) = nx_1, x_1 = (u \circ ua)U_{e_1}$. Also if $y_1\epsilon\mathscr{J}_{11}$ then $n(a^\pi y, a) = nz_1$ where $z_1 = y^\eta_1 U_{ua}\mathscr{J}_{11}$.*

Proof. We have $ua^\pi + ua = (ua)^\eta + (ua)$ (by (5))$=$ $ux_1, x_1 = (u \circ ua)U_{e_1}$ (by Lemma 1). Hence $u(a^\pi + a - x_1) = 0$ so, by Lemma 4, $n(a^\pi + a - x_1) = 0$ for all $n\epsilon uS(\mathscr{J}_{11})$. This proves the first statment. To prove the second it suffices to show that $n(a^{\pi}y_1a) = n(y_1^{\eta}U_{ua})$ and $n(a^{\pi}y_1b + b^{\pi}y_1a) = n(y^{\eta}U_{ua,ub})$ for $a = t_1 \ldots z_1, y_1, t_1, \ldots, z_1 \epsilon \mathscr{J}_{11}$, $b\epsilon S(\mathscr{J}_{11}$, $b\epsilon(\mathscr{J}_{11})$. For $m\epsilon\mathfrak{m}$, we have $y_1^{\eta}U_{mt_1} = y_1^{\eta}U_{m\circ t_1} = y_1^{\eta}U_mU_{t_1}$ by $QJ17$ and the PD theorem. Iteration of this gives $y_1U_{ua} = y_1^{\eta}U_{ut_1} \ldots z_1 = y_1^{\eta}U_uu_{t_1} \ldots U_{z_1} = y_1U_{t_1} \ldots U_{z_1} = z_1 \ldots t_1y_1t_1 \ldots z_1$ (in $A(\mathscr{J}_{11})) = a^{\pi}y_1a$. Next we use the first statement of the lemma to obtain

$$n(a^{\pi}uy_1b + b^{\pi}y_1a) = n((u \circ ua^{\pi}y_1b)U_{e_1}, n\epsilon\mathfrak{n} \tag{28}$$

If $m, n\epsilon^{\mathfrak{m}}, y_1\epsilon\mathscr{J}_{11}$ then $\{my_1n\} = ((m \circ y_1) \circ n)U_{e_2}$ (PD theorem)= $((m \circ y_1) \circ (n \circ e_1)U_{e_2} = \{m \circ y_1e_1n\}$. Since $\{my_1n\}$ is symmetric in m and n we have $\{m \circ y_1e_1, n\} = \{me_1n \circ y_1\}$. If we take $a = t_1 \ldots z_1$, **160**
$t_1, \ldots, z_1\epsilon\mathscr{J}_{11}$ then we can iterate this to obtain

$$\{my_1ae_1n\} = \{m, y_1, na^{\pi}\} = \{my_1e_1na\} \tag{29}$$

Since $\mathscr{J}_{11}$ generates $S(\mathscr{J}_{11})$ this holds for all $\epsilon S(\mathscr{J}_{11})$. In particular, we have

$$\{u.e_1ub^{\pi}y_1a\} = \{ua^{\pi}y_1ub^{\pi}\} \tag{30}$$

Now $(u \circ ua^{\pi}y_1b)U_{e_1} = \{ue_1ua^{\pi}y_1b\}U_{e_1} + \{ue_2u_a^{\pi}y_1b\}U_{e_1} = \{ue_2ua^{\pi}y_1b\}U_{e_1}$ (PD theorem) $= \{ue_2ua^{\pi}y_1b\} = \{ue_1ub^{\pi}y_1a\}^{\eta}$ (by (27))= $\{ua^{\pi}y_1ub\}^{\eta}$ (by (30))= $\{uay_1ub\}$ (by (27)). Going back to (28) we obtain $n(a^{\pi}y_1b + b^{\pi}y_1a) = n\{uay_1^{\eta}ub\} = ny_1^{\eta}U_{ua,ub}$ as required. This completes the proof. We obtain next an important corollary of Lemma 5 namley ◻

Lemma 6. *Let* $\mathfrak{n}$ *be as in Lemma 5 and let ua be invertible,* $a\epsilon S(\mathscr{J}_{11})$. *Then there exists* $ab\epsilon S({}_{11})$. *Such that* $nab = n = nba, n$.

Proof. The hypothesis implies that U_{ua} is a invertible. Hence $z_1 = e_2U_{ua} \neq 0$ in $\mathscr{J}_{11}$. Then z_1^{-1} exits in $S(\mathscr{J}_{11})$. Applying the second part of Lemma 5 to $y_1 = e_1$ shows that $na^{\pi}a = nz_1$ holds for all $n\epsilon\mathfrak{n}$. Hence replacing n by nz_1^{-1} gives $nba = n$ for $b = z_1^{-1}a$. Also $(ua)^{\eta} = ua^{\pi}$ is invertible so $w_1 = e_2U_{ua}\pi$ invertible in $S(\mathscr{J}_{11})$ and $naa^{\pi} = nw_1, n\epsilon\mathfrak{n}$. **161**
Multiplying by w_1^{-1} on the right gives $nac = n$, $c = a^{\pi}w_1^{-1}$. It now follows that $nb = na$ and $nab = n = nba$, $n\epsilon\mathfrak{n}$. ◻

Lemma 7. *If* $x_2 \epsilon \mathscr{J}_{22}$ *and* $m\epsilon\mathfrak{m}$ *is invertible then* $mx_2 = mx_1m_1$, $x_1 = x_2U_m^{-1}, m_1 = m^2U_{e_2}$.

Proof. We have $x_2 = x_1U_m$, where $x_1 = x_2U_m^{-1}$. Then $mx_2 = x_2 \circ m = x_1U_mV_m = x_1U_{m,m^2} = \{mx_1m^2\} = \{mxx_1m^2U_{e_1}\} = \{mx_1m_1\} = (m \circ x_1) \circ m_1 = mx_1m_1$. We prove next □

Lemma 8. *If* $\mathfrak{m}_o = \{m\epsilon\mathfrak{m} | m^\eta = -m\}$, $\mathfrak{m}^* = \{m\epsilon\mathfrak{m} | mx_1 = mx_1^\eta, x_1\epsilon\mathscr{J}_{11}\}$ *then* $\mathfrak{m}_o \subseteq \mathfrak{m}^* \cup \mathfrak{n}, \mathfrak{n} = u \in s(\mathscr{J}_{11})$.

Proof. Let $m\epsilon\mathfrak{m}_o$ and assume $m\epsilon\mathfrak{m}^*$. Then we have an $x_1\epsilon\mathscr{J}_{11}$ such that $n = mx_1 - mx_1^\eta \neq 0$ and we have to show that $m\epsilon\mathfrak{n}$. Since $m^\eta = -m, n = mx_1 + (mx_1)^\eta = uy_1, y_1 = (u \circ mx_1)U_{e_1}$, by Lemma 2. Now m is invertible since otherwise, $m^2 = 0$ and hence $xU_m = 0$ and $x^2U_m = 0$ for $x = x_1 - x_1^\eta \epsilon \mathscr{J}_{11} + \mathscr{J}_{22}$. Then $n^2 = (m \circ x)^2 = 0$ by QJ 30. However, $n = uy_1$ and Since $y_1\epsilon\mathscr{J}_{11}$ is $\neq 0$ u is invertible, n is invertible. This contradiction proves m invertible. We have $n = mx_1 - mx_1^\eta = mx_1 - m(x_1^\eta U_m^{-1})(m^2U_{e_1})$ (by Lemma 7)= $a = x_1 - (x_1^\eta U_m^{-1})(m^2U_{e_1}\epsilon(\mathscr{J}_{11})$. Thus

$$n = uy_1 = ma, y_1 \neq 0 \quad \text{in} \quad \mathscr{J}_{11}, a\epsilon su(\mathscr{J}_{11}) \tag{31}$$

We now apply lemma 6 to m(replacing u) in the isotope $\mathscr{J}_m$. Since $ma = n$ is invertible in $\mathscr{J}$, hence in $\mathscr{J}_m$, and since the $S(\mathscr{J}_{11})$ module
162 structure on $\mathfrak{m}$ is unchanged in passing from $\mathscr{J}$ to $\mathscr{J}_m$ it follows from Lemma 6 that there exists $ab \in S(\mathscr{J}_{11})$ such that $mab = m$. Then $m = nb = uy_1b\epsilon\mathfrak{n}$ as requried.

As before, Let ν_1 be the monomorphism $x_1 \to \overline{V}_{x_1}$ of $\mathscr{J}_{11}$ into (End $\mathfrak{m}^{(q)}$. Also let ν_1 denote the (unique) extension of this to a homomorphism of $s(\mathscr{J}_{11})$ into End $\mathfrak{m}$ and let $\mathcal{E}_1 = S(\mathscr{J}_{11})^{\nu_1}$. Then $S(\mathscr{J}_{11})^{\nu_1}$ is the algebra of endomorphisms generated by the $\overline{V}_{x_1, x_1}\epsilon\mathscr{J}_{11}$.

We shall now prove the following important result on $\mathcal{E}_1$. □

Lemma 9. *The involution* π *in* $S(\mathscr{J}_{11})$ *induces an involution* π *in* $\mathcal{E}_1$. *If we identity* $\mathscr{J}_{11}$ *with its image* $\mathscr{J}_{11}^{\nu_1}$ *in* $\mathcal{E}_1$ *then* $\mathscr{J}_{11} \subseteq \mathscr{H}(\mathcal{E}_1\pi)$, $\mathscr{J}_{11}$ *contians* 1 *and every* $a^\pi x_1 a$, $x_1\epsilon\mathscr{J}_{11}, a\epsilon\mathcal{E}_1$. *Also* $(\mathcal{E}_1, \pi)$ *is simple and the nonzero elements of* $\mathscr{J}_{11}(\subseteq \mathcal{E}_1)$ *are invertible.*

Proof. If $\mathscr{J}_{11} = \{0, \pm e_1\}$ then $\mathcal{E}_1 = \mathbb{Z}_2$ or $\mathbb{Z}_3$ and the result is clear. From now on we assume $\mathscr{J}_{11} \neq \{0, \pm e_1\}$ so lemma 1 is applicable. In particular, $\mathfrak{m}$ is spanned by invertible elements. To show that π induces an involution in $\mathcal{E}_1$ we have to show that $(\ker \nu_1)^\pi \subseteq \ker \nu_1$ and for this it suffices to show that if $k \epsilon \ker \nu_1$ and $m \epsilon \mathfrak{m}$ is invertible then $mk^\pi = 0$. Let $\mathscr{J}_m$ be the isotope of $\mathscr{J}$ defined by m as before. By (27) applied to m (in place of u) we have $mk^\pi = (mk)^\eta = 0$. Hence the first statement is proved. It is clear that $\mathscr{J}_{11} \subseteq \mathscr{H}(\mathcal{E}_1, \pi)$ and $1 \epsilon \mathscr{J}_{11}$. Let $x_1 \epsilon \mathscr{J}_{11}, a \epsilon S(\mathscr{J}_{11})$, m an invertible element of $\mathfrak{m}$. Then, by Lemma 5, there exists an element $y_m \epsilon \mathscr{J}_{11}$ such that $xa^\pi x_1 a = xy_m$ for all x in $mS(\mathscr{J}_{11})$. Let n be a second invertible element of and let $y_n \epsilon \mathscr{J}_{11}$ satisfy $xa^\pi x_1 a = xy_n, x \epsilon nS(\mathscr{J}_{11})$. As in Lemma let $u_1 \neq 0$, $v_1 \neq 0$ be elements of $\mathscr{J}_{11}$ such that $p = mu_1 + nv_1$ is invertible and let $y_p \epsilon \mathscr{J}_{11}$ satisfy $xa^\pi x_1 a = xy_p, x \epsilon pS(\mathscr{J}_{11})$. Suppose $y_m \neq y_n$. Then $d_1 = y_m - y_n \neq 0$
is invertible in $\mathscr{J}_{11}$ with inverse d_1^{-1}. Then $nv_1 d_1^{-1}(a^\pi x_1 a - y_n) = 0$ so 163
$pd_1^{-1}(a^\pi x_1 a - y_n) = mu_1 d_1^{-1}(a^\pi x_1 a - y_n) = mu_1 d_1^{-1}(y_m - y_n) = mu_1$. Hence $m \epsilon pS(\mathscr{J}_{11})$ and, similarly, $n \epsilon pS(\mathscr{J}_{11})$. Then $ma^\pi x_1 a = my_p$ and $na^\pi y_1 a = ny_p$. This implies $y_p = y_m = y_n$ contradicting $y_m \neq y_n$. Hence there exists an element $y_1 \epsilon \mathscr{J}_{11}$ such that $ma^\pi x_1 a = my_1$ for all invertible $m \epsilon \mathfrak{m}$. Since $\mathfrak{m}$ is spanned by invertible elements this gives the second statement of the Lemma. We note next that the non-zero elements of $\mathscr{J}_{11}$ are invertible in $\mathcal{E}_1$ since $\mathscr{J}_{11}$ is a division subalgebra of $\mathcal{E}_1^{(q)}$.

It remains to show that $(\mathcal{E}_1, \pi)$ is simple. Let $a \epsilon \mathcal{E}_1$ then $a^\pi a$, $aa^\pi \epsilon \mathscr{J}_{11}$ and the non-zero element of $\mathscr{J}_{11}$ are invertible; the proof of the theorem of Herstein-Kleinfeld. Osbon-McCrimmon shows that either $aa^\pi = 0 = a^\pi a$ or a is invertible. Let u be an element strongly connecting e_1 and e_2, as before. By Lemma 5, we have $ua^\pi a = ue_1^\eta U_{ua} = ue_2 U_{ua} = u((ua)^2 U_{e_1})$. Hence $a^\pi a = 0$ implies $(ua)^2 U_{e_1} = 0$. Since $(ua)^2 \epsilon \mathscr{J}_{11} + \mathscr{J}_{22}$ this implies $(ua)^2$ not invertible. Then ua is not invertible and $(ua)^2 = 0$. Thus we see that if $a \epsilon \mathcal{E}_1$ then either $a^\pi a$ and aa^π are invertible on $(ua)^2 = 0$.

Now let Z be a proper ideal of $(\mathcal{E}_1, \pi)$ and let $z \epsilon Z$. Then $z + z^\pi, z^\pi z \epsilon Z \cap \mathscr{J}_{11}$. Since Z contains no invertible elements, we have $z + z^\pi = 0 = zz^\pi$. Hence $z^2 = 0$ and $(uz)^2 = 0$. By Lemma 2, if $m \epsilon \mathfrak{m}$, $m + m^\eta = ux_1, x_1 =$

$(u\circ m)U_{e_1}$. Then $f(m) = x_1 = (u\circ m)U_{e_1}$ defines a Φ-homomorphism of $\mathfrak{m}$ into $\mathscr{J}_{11}$. Since η is an automorphism of $\mathscr{J}$ mapping $\mathfrak{m}$ onto $\mathfrak{m}$ and
164 $\mathscr{J}_{11}$ onto $\mathscr{J}_{22}$ it is clear that η defines an isomorphism of $\mathcal{E}_1$ onto the subalgebra $\mathcal{E}_2$ of End $\mathfrak{m}$ generated by $\mathscr{J}_{22}^{\nu_2}$ extending the isomorphism of $\mathscr{J}_{11}$ onto $\mathscr{J}_{22}$. Moreover, we have $(ma)^\eta = m^\eta a^\eta$, $m\epsilon\mathfrak{m}, a\epsilon\mathcal{E}_1$. Iteration of the result of Lemma 7 shows that if m is an invertible element of $\mathfrak{m}$ then for any $a\epsilon\mathcal{E}_1$ there exists $ab\epsilon\mathcal{E}_1$ such that $ma^\eta = mb$. Then $uf(ma) = ma+(ma)^\eta = ma+m^\eta a^\eta = m(a-a^\eta)+(m+m^\eta)+(m+m^\eta)a^\eta = m(a-b)+uf(m)a^\eta = m(a-b)+ua^\eta f(m) = m(a-b)+ua^\pi f(m)$ (by (27)). Hence

$$m(a-b) = u(f(ma) - a^\pi f(m)) \tag{32}$$

In particular, taking $a = z\epsilon\mathscr{Z}$ we obtain w so that $mz^\eta = mw$ and $m(z-w) = ur$, $r = f(mz)+zf(m)\epsilon\mathcal{E}_1$. Since $wz\epsilon\mathscr{Z}$ we have $w^\pi z+z^\pi w = 0$. Also $mw^2 = mz^\eta w = mwz^\eta = m(z^\eta)^2 = 0$ since $z^2 = 0$. Hence w is not invertible and consequently $w^\pi w = 0$. Then $(z-w)^\pi(z-w) = z^\pi z - z^\pi w - w^\pi z + w^\pi w = 0$. This relation and the second part of Lemma 5 applied to the isotope $\mathscr{J}_n$ imply that $m(z-w)$ is not invertible in this isotope. Hence $m(z-w)$ is not invertible in $\mathscr{J}$ and consequently $(ur)^2 = (m(z-w))^2 = 0$. Then a reversal of the argument shows that $r^\pi r = 0$. Then r is not invertible and since $r = f(mz)+zf(m)$ and z is in $\mathscr{Z}$ which is a nil ideal, $f(mz)$ is not invertible. Since $f(mz)\epsilon\mathscr{J}_{11}$ it follows that $f(mz) = 0$. Hence we have $mz + (mz)^\eta = 0$. By the second part of Lemma 2, this implies $mz\circ u = 0$. Then $0 = (mz\circ u)U_{e_2} = \{ue_1mz\}$ (by linearization of $x^2U_{e_2} = e_1U_x, x\epsilon\mathfrak{m}$)= $\{uz^\pi e_1 m\}$ (by (29))= $-\{uze_1m\} = -(uz\circ m)U_{e_2}$. If we replace m by m^η in this and apply η we obtian $(uz\circ m)U_{e_1} = 0$. Hence we have proved that $uz\circ m = 0$ for all invertible m. It follows that
165 this holds for all $m\epsilon\mathfrak{m}$ and since $(uz)^2 = 0$, Lemma 2 of §6, shows that this is an absolute zero divisor. Since $\mathscr{J}$ is simple we have $uz = 0$. On passing to the isotpe $_m$ we can replace u by any invertible $m\epsilon\mathfrak{m}$. Then $mz = 0$ for all invertible $m\epsilon\mathfrak{m}$ so $z = 0$. Hence $Z = 0$ and $(\mathcal{E}_1, \pi)$ is simple.

Lemma 9 shows that $(\mathcal{E}_1, \pi, \mathscr{J}_{11})$ is an associative coordinate algebra satisfying the hypotheses of the Herstein-Kleinfeld-Osborn-McCrimmon theorem. Also in the present case $\mathscr{J}_{11}$ generates $\mathcal{E}_1$. This excludes case III given in that theorem so we have only the possibilities

I and II given in the theorem. It is convenient to separate the cases in which $\mathcal{E}_1$ is a division algebra into the subcases: π non-trivial and $\pi = 1$ in which case $\mathcal{E}_1$ is field.

Accordingly, the list of possibilities for $(\mathcal{E}_1, \pi, \mathscr{J}_{11})$ we

I $\mathcal{E}_1 = \Delta \oplus \Delta^\pi, \Delta$ an assoiative division algebra, $\mathscr{H}(\mathcal{E}_1, \pi) = \mathscr{J}_{11}$.

II $\mathcal{E}_1$ an associative division algebra, $\pi \neq 1$

III $\mathcal{E}_1$ a field, $\pi = 1$.

□

Lemma 10. *If $\mathcal{E}_1$ is of thye I or II then $\mathfrak{m} = u\mathcal{E}_1 (= uS(\mathscr{J}_{11}))$ and if $\mathcal{E}_1$ is of type III then $\mathfrak{m} = \mathfrak{m}^* = \{m \epsilon \mathfrak{m} | mX_1 = mX_1^\eta, X_1 \epsilon \mathscr{J}_{11}\}$ (as in Lemma 7).*

Proof. We show first that in types I and II any $m \epsilon \mathfrak{m}$ such that $m^\eta = -m$ as contained in $u\mathcal{E}_1$. By Lemma 7, it is enough to show this for for m with $m^\eta = -m$ and $mx_1^\eta = mx_1, x_{1_{11}}$. Then, by Lemma 3, $ma^\pi = ma^\eta = ma$, $a \epsilon S(\mathscr{J}_{11})$. Now in types I and II there exists an invertible element **166**
a in $\mathcal{E}_1$ such that $a = +b - b^\pi$. In the case I we choose b invertible in Δ then b^π is invertible in Δ^π and $a = b - b^\pi$, is invertible in $\mathcal{E}_i = \Delta \oplus \Delta^\pi$. in case II we choose an element b is the division algebra $\mathscr{E}_1$ such that $b^\pi \neq b$. This can be done since $\pi \neq 1$. Then $a = b - b^\pi \neq 0$ is invertible. Now let m be as indicated $(m^\eta = -m, ma^\pi = ma^\eta, a \epsilon S(\mathscr{J}_{11}))$. Then $ma = mb - mb^\pi = mb - mb^\eta = mb + m^\eta b^\eta = ma + (ma)^\eta = ux_1, x_1 \epsilon \mathscr{J}_{11}$. Then $m = ux_1 a^{-1} \in uS(\mathscr{J}_{11})$.

Suppose we have type I and let $m \epsilon \mathfrak{m}$. Then $m + m^\eta = ux_1, x_1 \epsilon \mathscr{J}_{11}$. Since the type is $I, x_1 = a + a^\pi, a \epsilon S(\mathscr{J}_{11})$. Then $m + m^\eta = ua + ua^\pi = ua + (ua)^\eta$ (by (27)). Hence $(m - ua)^\eta = m^\eta - (ua)^\eta = ua - m$. Then $m - ua \epsilon uS(\mathscr{J}_{11})$ and $m \epsilon US(\mathscr{J}_{11}) = u\mathcal{E}_1$.

Suppose we have type II. Then $\mathscr{J}_{11} \neq \{0, \pm e_1\}$ so $\mathfrak{m}$ is spanned by invertible elements. Hence it suffices to show that if $m \epsilon \mathfrak{m}$ is invertible then $m \epsilon u\mathcal{E}_1$. By Lemma 2, $m + m^\eta = uf(m)$ where $f(m) \epsilon \mathscr{J}_{11}$. By Lemma 6, if $a \epsilon \mathcal{E}_1$ there exists $a\, b \epsilon \mathcal{E}_1$ such that $ma^\eta = mb$. By (31), $m(a - b) = u(f(ma) - a^\pi f(m))$ where $f(m), f(ma) \epsilon \mathscr{J}_{11}$. If $a - b$ is invertible for some a this implies $m\epsilon u\mathcal{E}_1$. Otherwise, since $\mathcal{E}_1 u$ a division algebra, $a = b$ for all a. Then $f(ma) = a^{\pi} f(m)$ and applying II, $f(ma)^{\pi} = f(ma) = f(m)a$. Hence $a^{\pi} f(m) = f(m)a$. In particular, $x_1 f(m) = f(m)x_1, x_1 \epsilon \mathscr{J}_{11}$ and since $\mathscr{J}_{11}$ generates $\mathcal{E}_{1,f(m)}$ is in the center of $\mathcal{E}_1$. Then $(a^{\pi} - 1) f(m) = 0$. Since we can choose a so that $a^{\pi} - a$ is inverible, $f(m) = 0$. Then $m + m^{\pi} = 0$. and $m\epsilon u\mathcal{E}_1$ by the result proved before.

167 Now suppose we have type III. The case $\mathscr{J}_{11} = \{0, \pm e_1\}$ is trivial so we may assume $\mathfrak{m}$ is spanned by invertible elements. It suffices to show that if $m\epsilon\mathfrak{m}$ is invertible then $mx_1 = mx_1^{\eta}, x_1\epsilon\mathscr{J}_{11}$. As in the last case, $m + m^{\eta} = uf(m)$ and if $a\epsilon\mathcal{E}_1$ then there exists $b\epsilon\mathcal{E}_1$ such that $m(a - b) = u(f(ma) - af(m))$ (since $\pi = 1$). If $a - b \neq 0$, $m = uc$, $c\epsilon\mathcal{E}_1$ and $mx_1 = ucx_1 = ux_1c$ (by commutativity of $\mathcal{E}_1$)$= ux_1^{\eta}c = ucx_1^{\eta} = mx_1^{\eta}$ (by (27) and (21)). Hence the result holds in this case. It remains to conisder the case in which $a = b$ for all a. Then $f(ma) = af(m) = f(m)a, a\epsilon\mathcal{E}_1$. Then $mx_1 + (mx_1)^{\eta} = uf(mx_1) = uf(m)x_1 = (m + m^{\eta})x_1$. Then $(mx_1)^{\eta} = m^{\eta}x_1$ and $mx_1 = mx_1^{\eta}$, $x_1\epsilon\mathscr{J}_{11}$ as required. We can now complete the □

Proof of Theorem 10. Suppose first $(\mathcal{E}_1, \pi\mathscr{J}_{11})$ is of type I or II. Since $\mathfrak{m} = u\mathcal{E}_1$ and $\mathscr{J}_{11}^{\eta} = \mathscr{J}_{22}$ any element of $\mathscr{J}$ can be written in the form $x_1 + y_1^{\eta} + ua, x_1, y_1\epsilon\mathscr{J}_{11}, a\epsilon\mathcal{E}_1$. Also a is unique since $ua = 0$ implies $\mathfrak{m}_a = (u\mathcal{E}_1)a = 0$ (Lemma 4). Hence the mapaping

$$\zeta : x_1 + y_1^{\eta} + ua \to x_1[11] + y_1[22] + a[21] \tag{33}$$

is a module isomorphism of $\mathscr{J}$ onto $\mathscr{H}((\mathcal{E}_1)_2, \mathscr{J}_{11})$. It is clear that the mapping $\eta' : X \to 1[12]X1[12] = X_{U_1}[12]$ is an automorphism in $\mathscr{H}((\mathcal{E}_1)_2, \mathscr{J}_{11})$ and by inspection we have $(X^{\eta})^{\zeta} = (x^{\zeta})^{\eta'}$. We shall now show that ζ is an algebra isomorphism. Because of the properties of the Pierce decomposition, the relation between η and η' and the quadratic Jordan matrix algebra properties $QN1 - QM6$ this will follow if we can
168 establish the following formulas:

(i) $(x_1 U_{y_1})^{\zeta} = Y_1 x_1 y_1[11]$

(ii) $(x_1 U_{ua})^{\zeta} = (a^{\pi} x_1 a)[11]$

(iii) $((ua)U_{ub})^{\zeta} = ba^{\pi}b[21]$

(iv) $\{x_1 ua^{\pi} ub\}^{\zeta} = (X_1 ab + 1(X_1 ab)^{\pi})[11]$

(v) $\{x_1 y_1 ua\}^{\zeta} = (ay_1 x_1)[21]$

(vi) $\{y_1^{\eta} ua x_1\}^{\xi} = y_1 a x_1 [21]$.

Since $x_1 U_{y_1} = y_1 x_1 y_1$ in $\mathcal{E}_1$, (i) is clear. For (ii) we use Lemmas 5 and 10 to obtain $a^{\pi} x_1 a = x_1^{\eta} U_{ua}$. For (iii) we have

$$\begin{aligned}(ua)U_{ab} &= -(ub)^2 U_{e_1} \circ ua + (ua\circ)ubU_{e_2} \circ ub \quad (by(26))\\ &= -e_1 U_{ub} \circ ua + e_1 U_{ua,ub} \circ ub \quad (PD)\\ &= -e_1^{\eta} U_{ub} \circ ua + ((e_1^{\eta} U_{ua^{\pi},ub^{\pi}}) \circ ub^{\pi})^{\eta}\\ &= -uab^{\pi} b + (u(b^{\pi} ab^{\pi} + b^{\pi} ba^{\pi}))^{\eta}\\ &= -uab^{\pi} b + uba^{\pi} b + uab^{\pi} b\\ &= uba^{\pi} b.\end{aligned}$$

This implies (iii). For (iv), we use $\{x_1 ua^{\pi} ub\} = ((x_1 \circ ua^{\pi}) \circ ub))U_{e_1} = (ua^{\pi} x_1 \circ ub)U_{e_1} = e_1^{\eta} U_{ua^{\pi} x_1, ub} = x_1 ab + b^{\pi} a^{\pi} x_1$. This gives (iv). For (v) we have $\{x_1 y_1 ua\} = (ua \circ y_1) \circ x_1 = uay_1 x_1$. (vi) follows from $\{y_1^{\eta} ua x_1\} = (ua \circ y_1^{\eta}) \circ x_1 = uy_1^{\eta} a x_1 = uy_1 a x_1$ This completes the proof of the first part. Now suppose we have type III. Then $mx_1 = mx_1^{\eta}, m\epsilon\mathfrak{m}, x_1 \epsilon \mathscr{J}_{11}$. **169**
Also $\mathcal{E}_1$ is a field and $\pi = 1$. Hence, by Lemma 3, $ma = ma^{\eta}$, $a\epsilon\mathcal{E}_1$. We consider the mapping $Q : m \to -m^2 U_{e_1}$ of $\mathfrak{m}$ into $\mathcal{E}_1$. We claim that Q is quadratic mapping of $\mathfrak{m}$ as $\mathcal{E}_1$ module into $\mathcal{E}_1$. If $m\epsilon\mathfrak{m}, x_1 \epsilon \mathscr{J}_{11}$ then $(mx_1)^2 U_{e_1} = (m \circ x_1)^2 U_{e_1} = m^2 U_{x_1} U_{e_1} = m^2 U_{e_1} U_{x_1} = x_1 (m^2 U_{e_1}) x_1^2$ since $\mathcal{E}_1$ is commutative. It follows that for $a = x_1 \ldots z_1$, $x, \ldots z_1 \epsilon \mathscr{J}_{11}$, we have $(ma)^2 U_{e_1} = (m^2 U_{e_1}) a^2$. We show next that if $m, n\epsilon\mathfrak{m}$ and $y_1 \epsilon \mathscr{J}_{11}$ then $(mx_1 \circ n)U_{e_1} = ((mon)U_{e_1} x_1$. Since this is clear for $n = 0$ and both sides are in $\mathcal{E}_1$ it suffices to show that $(mx_1 \circ n)U_{e_1} \circ n = (((m \circ n)U_{e_1})x_1) \circ n$. Since $\mathcal{E}_1$ is commutative $(((m \circ n)U_{e_1})x_1) \circ n = (n \circ x_1) \circ ((m \circ n)U_{e_1}) = \{nx_1 (m \circ n)U_{e_1}\}$ (PD theorem) $= \{nx_1 m \circ n\} = x_1 U_{n, m\circ n} = x_1 (U_n V_m + V_m U_n)$ $(QJ19)$ $= m(x_1 U_n)^{\eta} + (mx_1)U_n$. By $QJ33$, we have $(mx_1)U_n = (mx_1)v_{e_1} U_n V_{e_1} = (mx_1)V_n U_{e_1} V_n - (mx_1)V_{e_1 U_n} =$

$(mx_1)V_nU_{e_1}V_n - (mx_1)V_{n^2U_{e_2}}$ (PD 6)= $((mx_1 \circ n)U_{e_1}) \circ n - (mx_1)V_{n^2U_{e_2}}$. This and the following relation give

$$(((m \circ n)U_{e_1})x_1) \circ n - (mx_1 \circ n)U_{e_1}) \circ n \tag{34}$$
$$= m(x_1U - n)^{\eta} - (mx_1)V_{n^2U_{e_2}}$$

The left hand side is a multiple (in $\mathcal{E}_1$) of n and the right hand side is a multiple of m. Hence if m and n are $\mathcal{E}_1$ independent then we obtian $(((m \circ n)U_{e_1})x_1) \circ n = (((mx_1)U_{e_1})x_1) \circ n$ which gives the required relation. Now suppose $m = na$, $a\epsilon\mathcal{E}_1$. Then again (34) will yield the result
170 provided we can prove that $n(x_1U_n)^{\eta} = (nx_1)V_{n^2U_{e_2}}$. This follows since $n(x_1U_n)^{\eta} = n(x_1U_n) = x_1U_nV_n = x_1U_{n,n^2} = \{nx_1n^2\} = \{nx_1n^2U_{e_2}\} = (n \circ x_1) \circ n^2U_{e_2} = (nx_1)V_{n^2U_{e_2}}$.

We have now proved that for $Q(m) = -m^2U_{e_1}$ we have $Q(ma) = a^2Q(m)$ for all $a = x_1 \ldots z_1, x_1, \ldots, z_1\epsilon\mathscr{J}_{11}$ and $Q(mx_1, n) = x_1Q(m, n)$. The latter implies that $Q(ma, n) = aQ(m, n)\, m, n\epsilon\mathfrak{m}$, $a\epsilon\mathcal{E}_1$. This and the first result imply that $Q(m) = a^2Q(m)$ for all $a\epsilon\mathcal{E}_1$. Then Q is a quadratic mapping. We note next that $m^2U_{e_2} = (m^2U_{e_1})^{\eta}$. Since $m(m^2U_{e_1}) = m(m^2Ue_1)^{\eta}$ it suffices to show that $m(m^2U_{e_1}) = m(m^2U_{e_2})$. We have $m(m^2U_{e_1}) = m^2U_{e_1} \circ m = \{m^2U_{e_1}e_1m\} = \{m^2e_1m\} = e_1U_{m,m^2} = e_1U_mV_m = m^2U_{e_2}V_m = m \circ m^2U_{e_2} = mm^2U_{e_2} = mm^2U_{e_2}$. Thus $m^2 = m^2U_{e_1} + m^2U_{e_2} = m^2U_{e_1} + (m^2U_{e_1})^{\eta}$. Since the elements $m\epsilon\mathfrak{m}$ such that $m^2 = 0$ and $m \circ n = 0$, $n\epsilon\mathfrak{n}$, are absolute zero divisors it now follows that Q is non-degenerate.

We now introduce $\mathfrak{R} = f_1\epsilon_1 \oplus f_2\epsilon_1 \oplus \mathfrak{m}_a$ direct sum of $\mathfrak{m}$ and two one dimensional (right) vector spaces over $\mathcal{E}_1$ and extend Q to $\mathfrak{R}$ by defining $Q(f_1a + f_2b + m) = ab + Q(m), a, b \in \mathcal{E}_1$. Then Q is a non-degenerate quadratic form on $\mathfrak{R}$ and $Q(f) = 1$ for $f + f_1 + f_2$. Hence we can form Jord (Q, f). It is immediate that $\mathscr{J}' \equiv f_1\mathscr{J}_{11} + f_2\mathscr{J}_{11} + \mathfrak{m}$ is an outer ideal containing f in $\mathfrak{R}$ = Jord (Q, f). We now define the mapping ζ of $= \mathscr{J}$ onto $\mathfrak{R}$ by $x_1 + y_1^{\eta} + m \to f_1x_1 + f_1y_1 + m$. It is easy to check that this is a monomorphism.

10 Second structure theorem.

Let $\mathscr{J}$ be a simple quadratic Jordan algebra satisfying the minimum condition (for principal inner ideals). Then $\mathscr{J}$ contains no absolute zero divisors $\neq 0$ since these generate a nil ideal (Theorem 5). By Theorem 7. $\mathscr{J}$ has an isotope $\tilde{\mathscr{J}}$ which has a capacity. If the capacity is one then $\tilde{\mathscr{J}}$, hence $\mathscr{J}$, is a division algebra. If the capacity is two then the structure of $\overline{\mathscr{J}}$ is given by Theorem 10. This implies that $\mathscr{J}$ itself has the form given in Theorem 10. Now assume the capacity of $\tilde{\mathscr{J}}$ is $n \geqq 3$ and let $\tilde{1} = \sum\limits_{1}^{n} f_i$ be decomposition of the unit $\tilde{1}$ of $\tilde{\mathscr{J}}$ into orthogonal completely primitive idempotents. Since $\tilde{\mathscr{J}}$ is simple the proof of the First structure Theorem shows that every f_j, $j > 1$, is connected to f_1. Since we can replace $\tilde{\mathscr{J}}$ by an isotope, by Lemma 2 of §2.3, we may assume the connectedness is strong. Then we can apply the strong coordinatization Theorem to conclude that $\tilde{\mathscr{J}}$ is isomorphic to an algebra $\mathscr{H}(O_n, O_o)$ with the coordinate algebra (O, j, O_o). By Theorem 2.2, $\mathscr{H}(O_n, O_o)$ is an outer ideal in $\mathscr{H}(O_n)$ and the simplicity of $\mathscr{H}(O_n, O_o)$ implies that (O, j) is simple. The Pierce inner ideal determined by the idempotent 1[11] in $\mathscr{H} = \mathscr{H}(O_n, O_o)$ is the set of elements $\alpha[11], \alpha \epsilon O_0 \subseteq N(O)$. Since this Pierce inner ideal is a division algebra it follows that every non-zero element of O_o is invertible in $N(O)$. Hence (O, j, O_o) satisfies the hypothesis of the Herstein-Kleinfeld-Osborn-McCrimmon theorem. Hence (O, j, O_o) has one of the types $I - V$ given in the $H - K - O - M$ theorem. If the type is I-IV then the consideration of chapter 0 show that O_n with its standard involution J_1 is a simple Aritinian algebra with involution. Since $\mathscr{H}(O_n, O_o)$ is an outer ideal containing 1 in $\mathscr{H}(O_n)$ it follows that $\mathscr{H}$ is isomorphic to an outer ideal containing 1 in an $\mathscr{H}(\mathfrak{a}, J), (\mathfrak{a}, J)$ simple Artinian with involution. Then $\mathscr{J}$ also has this form. The remaining type of coordinate algebra allowed in the $H-K-O-M$ theorem is an octoion algebra with standard involution over a field Γ with $\Gamma = O$. In this case we must have $n = 3$. Thus if we take into account the previous results we see that $\mathscr{J}$ is of one of the following types. 1) a division algebra, 2) an outer ideal containing 1 in a Jord $(Q, 1)$ where $(Q, 1)$ is a non-degenerate quadratic form with base point on a vector space, 3)an outer ideal con-

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taining 1 in an $\mathscr{H}(\mathfrak{a}.J)$ where $(\mathfrak{a}, J)$ is simple Artinian with involution, 4)an isotope of an algebra $\mathscr{H}(O_3)$ where O is an octonian algebra over a field Γ/Φ with standard involution.

We now consider the last possibility in greater detail. Let $c \sum \gamma_i[ii]$, $\gamma_i \neq 0$ in Γ. Since c is an invertible element of $N(O_3)$ it determines an involution $J_c : X \to c^{-1}X^t c$ in O_3. Let $\mathscr{H}(O_3, J_x)$ denote the set of matrices in O_3 which are symmetric under J_c and have diagonal elements in Γ. If $\alpha\epsilon\Gamma$ we put $\alpha\{ii\} = \alpha[ii] = \alpha e_{ii}$ and if $a\epsilon O$ we put $a\{ij\} = ae_{ij} + \gamma_j^{-1}\gamma_i\overline{a}e_{ji}$, $i \neq j$. Then $\alpha[ii], a\{ij\}\epsilon\mathscr{H}(O_3, J_c)$ and every element of $\mathscr{H}(O_3, J_c)$ is a sum of $\alpha[ii]$ and $a\{ij\}$. If $X\epsilon\mathscr{H}(O_3)$ then $Xc\epsilon\mathscr{H}(O_3 J_c)$ since $c^{-1}(\overline{Xc})^t c = Xc$. In view of the situation for algebras $\mathscr{H}(\mathfrak{a}, J)$, $\mathfrak{a}$ associative, it is natural to introduce a quadratic Jordan structure in $\mathscr{H}(O_3, J_c)$ so that bijective mapping $X \to Xc$ of $\mathscr{H}(O_3)^{(c)}$ onto $\mathscr{H}(O_3, J_c)$ becomes an isomorphism of quadratic Jordan algebras. We shall call $\mathscr{H}(O_3, J_c)$ endowed with this structure a cononical quadratic Jordan matrix algrbra. It is easy to check that
173 the elements $e_i = 1\{ii\}$ are orthogonal idempotents in $\mathscr{H}(O_3, J_c)$ and $\sum e_i$ is the unit of $\mathscr{H}(O_3, J_c)$. The pierce spaces relative to these are $\mathscr{H}(O_3, J_c)_{ii} = \{\alpha\{ii\}|\epsilon\Gamma, \mathscr{H}(O_3, J_c)_{ij} = \{a\{ij\}|a\epsilon O\}, i \neq j$. It is easy to verify that the formulas for the U-operator for elements in these submodules are identical with (i)-(x) of §1.8 with the exceptions that (ii) and (iii) become

$$\alpha\{ii\}U_{a\{ij\}} = \gamma_j^{-1}\gamma_i\overline{a}\gamma a\{ij\} \qquad (ii)'$$

$$b\{ij\}U_{a\{ij\}} = \gamma_j^{-1}\gamma_i a\overline{b}a\{ij\} \qquad (iii)'$$

It is clear from these formulas that if $\rho \neq 0$ in Γ then $\mathscr{H}(O_3, J_{\rho c}) = \mathscr{H}(O_3, J_c)$. We note next that if $a_i\epsilon O$, $n(u_i) \neq 0$, $\delta_i = n(u_i)\gamma_i$ and d =diag$\{\gamma_1, \gamma_2, \gamma_3\}$ then there exists an isomorphism of $\mathscr{H}(O_3, J_c)$ onto $\mathscr{H}(O_3, J_d)$ fixing the e_{ii}. First, one can verify directly that if $u\epsilon O$, $n(u) \neq 0$, then the Γ -linear mapping of $\mathscr{H}(O_3, J_c)$ onto $\mathscr{H}(O_3, J_d), d =$ diag$\{\gamma_1, n(u)\gamma_2, n(u)\gamma_3\}$ such that $e_{ii} \to e_{ii}$, $a\{12\} \to au\{12\}'$, $a\{23\} \to u^{-1}a\overline{u}\{23\}'$, $a\{13\} \to a\overline{u}\{13\}'$ where $a\{ij\}' = ae_{ij} + \delta_j^{-1}\delta_i\overline{a}e_{ji}, \delta_1 = \gamma_1, \delta_i = n(u)\gamma_i, i = 2, 3$, is an isomorphism. (Because of the Pierce relations it is sufficent to verify $QM(ii)$], $(iii)'$ and $M4$, §2.2). Similarly, one can define isomorphism of $\mathscr{H}(O_3, J_c)$ onto $\mathscr{H}(O_3, J_d), d$ =diag$\{n(u)$

$\gamma_1, \gamma_2, n(u)\gamma_3\}$ or diag $\{n(u), \gamma_1 n(u)\gamma_2, \gamma_3\}$. Combining these and taking account the fact that $\mathscr{H}(O_3, J_{\rho_c}) = \mathscr{H}(O_3, J_c)$ we obtain an isomorphism of $\mathscr{H}(O_3, J_c)$ onto $\mathscr{H}(O_{3,d})$ fixing the e_{ii}, d as above. We shall now prove the following

Lemma . *Let $\mathscr{J}$ be a quadratic Jordan algebra which is an isotope of $\mathscr{H}(O'_3)$, O' octonian over a field Γ. Then $\mathscr{J}$ is isomorphic to a canonica Jordan matrix algebra $\mathscr{H}(O_3, J_c)$ where O is an octonian algebra.* **174**

Proof. It is easily seen that if (O', j) is an octonion algebra with standard involution then (O', j) is simple. Hence, by Theorem 2.2, $\mathscr{H}(O'_3)$ and every isotope $\mathscr{J}$ of $\mathscr{H}(O'_3)$ is simple. Let $e_1, e_2, \ldots, e_k$ be a supplementary set of primitive orthogonal idempotents in $\mathscr{J}$ (Lemma 1 of §6). We show first that $k = 3$ and the e_i are completely primitive. Since $\mathscr{J}$ is not a division algebra 1 is not completely primitive. Hence if 1 is primitive then the Theorem on Minimal Inner Ideals shows that an isotope of $\mathscr{J}$ has capacity two. Since this is simple it follows. from Theorem 10 that this algebra is special. Since an isotope of a special quadratic Jordan algebra is special this implies that $\mathscr{H}(O'_3)$ is special. Since this is not the case (§1.8), 1 is not primitive, so $k > 1$. If $k > 3$ or $k = 3$ and one the e_0 is not completely primitive then the Minimal Inner Ideal Theorem implies that $\mathscr{J}$ has an isotope containing $l > 3$ supplementary orthogonal completely primitive idempotents. Then the foregoing results show that this isotope, hence $\mathscr{H}(O'_3)$ is special. Since this is ruled out we see tjhat $k = 2$ or 3 and if $k = 3$ then the e_i are completely primitive. It remains to exclude the possibility $k = 2$. In this case the arguments just used show that we may assume e_1 completely primituve, e_2 not. By the *MII* theorem and Lemma 2 of §2.3 we have an isotope $\tilde{\mathscr{J}} = \mathscr{J}^{(v)}$ where $v = e_1 + v_2, v_2 \epsilon \mathscr{J} U_{e_2}$ such that the unit of $\tilde{\mathscr{J}}$ is $e_1 + u_2$, $u_2 \epsilon \mathscr{J} U_{e_2}$ and u_2 is a sum of two completley primitive strongly connected orthogonal **175**
idempotents in $\tilde{\mathscr{J}}$. Then $\mathscr{J}\overline{U}_{u_2} = \mathscr{J} U_{e_1+v_2} U_{u_2} = \mathscr{J} U_{e_2}$ and $\tilde{\mathscr{J}}\overline{U_{u_2}}$ is an isotope of $\mathscr{J} U_{e_2}$. Since $\tilde{\mathscr{J}}$ is exceptional the foregoing results show that we can identify $\tilde{\mathscr{J}}$ with an algebra $\mathscr{H}(O''_3)$ where O'' is an octonion algebra. Moreover, we can identify e_1 with 1[11]. Then, as we saw in §5, $\tilde{\mathscr{J}} \tilde{U}_{u_2}$ is the quadratic Jordan algebra of a quadratic form S with base point such that the associated symmetric bilinear form is

non-degenerate. Since $\mathscr{J} U_{e_2}$ is an isotope of $\tilde{\mathscr{J}} \tilde{U}_{u_2}$ it is the quadratic Jordan algebra of a quadratic form Q with base point such that $Q(x, y)$ is non-degenerate. Moreover, $\mathscr{J} U_{e_2}$ is not a division algebra since e_2 is not completely primitive. Hence we can choose $x \neq 0$ in $\mathscr{J} U_{e_2}$ such that $Q(x) = 0$. Then $x^2 = T(x)x$, $T(x) = Q(x, 1)$, and if $T(x) \neq 0$, $e = T(x)^{-1} x$ is an idempotent $\neq 0$, e_2, contrary to the primitivity of e_2. Hence $T(x) = 0$. Since Q is non-degenerate there exists $a y$ in $\mathscr{J} U_{e_2}$ such that $Q(x, y) = 1$ and we may assume also that $Q(y) = 0$. Then, as for x, we have $T(y) = 0$. Since $Q(a, b)$ is non-degenerate there exists a w such that $T(w) = Q(w, 1) = 1$. Then $z = w - Q(x, w)y - Q(y, w)x$ sarisfies $Q(x, z) = 0 = Q(y, z)$, $T(z) = 1$. Put $e = z + x - Q(z)y$. Then $T(e) = 1$ and $Q(e) = 0$. Hence e is an idempotent $\neq 0$, e_2. This contradiction proves our assertion on the idempotents.

Now let e_1, e_2, e_3 he supplementary completely primitive orthogonal idempotents in $\mathscr{J}$. These are connected so we have an isotope $\mathscr{J} = \mathscr{J}^{(v)}$ where $v = e_1 + v_2 + v_3$, $v_i \epsilon \mathscr{J} U_{e_i}$ and the unit u is a sum of three strongly connected primitive orthogonal idempotents.

176 As before, we can identify $\tilde{\mathscr{J}}$ with an $\mathscr{H}(O_3)$. O an octonian algebra over a field, e_1 with 1[11]. Then $\mathscr{J}$ is the isotope of $\mathscr{H}(O_3)$ determined by an element of the form $e_1 + e_2 + e_3, e_i \epsilon \mathscr{H}(O_3) U_1[ii]$. Then $e_i = \gamma_i[ii]$. Then $\mathscr{J}$ is isomorphic to $\mathscr{H}(O_3, J_c)$.

The foregoing lemma and previous results prove the direct part of the □

Second structure Theorem. Let $\mathscr{J}$ be a simple quadratic Jordan algebra satisfying DCC for principal inner ideals. Then $\mathscr{J}$ is of one of the follwing types: 1) a quadratic Jordan division algebra, 2) an outer ideal containing 1 in a quadratic Jordan algebra of a non-degenerate quadratic form with base point over a field Γ/Φ, 3) an outer ideal containing in $\mathscr{H}(\mathfrak{a}, J)$ where $(\mathfrak{a}, J)$ is simple associative Artinian O with involution, 4) a canonical Jordan matrix algebra $\mathscr{H}(O_3, J_c)$ where is an octonion algebra over a field Γ/Φ and c =diag$\{1, \gamma_2, \gamma_3\}, \gamma_i \neq 0$ in Γ. Conversely, any algebra of one of the types 1)-4) satisfies the DCC for principal inner ideals and all of these are simple with the exception of certian algebras of type 2) which are direct sums of two division algebras isomorphic to outer ideals of $\Omega^{(q)}$.

We consider the exceptional case indicated in the foregoing statement. Let $\mathscr{J}$ =Jord$(Q, 1)$ where Q is a non-degenerate quadratic form on $\mathscr{J}/\Gamma$ with base point 1. We have seen in §5 that $\mathscr{J}$ satisfies the DCC for principal inner ideals and $\mathscr{J}$ is regular, hence, strongly non-degenerate. Hence $\mathscr{J} = \mathscr{J}_1 \oplus \mathscr{J}_2 \oplus \ldots \oplus \mathscr{J}_s$ where $\mathscr{J}_i$ is an ideal and is simple with unit 1. If e is an idempotent $\neq 0, 1$ in $\mathscr{J}$ then $Q(e) = 0$ **177**
since e is not invertible, and $T(e) = 1$ since $e^2 - T(e)e + Q(e) = 0$. Then the formula $yU_x = Q(x, \overline{y})x - Q(x)\overline{y}$ in shows that $\mathscr{J} U_e = \Omega e$. In particular $\mathscr{J}_i = \mathscr{J} U_{1_i} = \Omega 1_i$. If $s > 1$ we put $u = 1_1 - 1_2$. Then $T(u) = T(1_1) - T(1_2) = 0$ so $u^2 + Q(u) = 0$. But $u^2 = 1_1 + 1_2$. Hence $Q(u) = -1$ and $1_1 + 1_2 = 1 = \sum 1_i$, which implies that $s = 2$. Thus either $\mathscr{J}$ =Jord$(Q, 1)$, Q non-degenerate, is simple or $\mathscr{J} = \Omega 1_1 \oplus \Omega 1_2$. Suppose $\mathscr{J}$ is simple and not a division algebra. Suppose first that $\mathscr{J}$ contians an idempotent $e \neq 0, 1$. Then $\mathscr{J} U_e = \Omega e$ so e is completely primitive. The same is true of $1 - e$. Hence $\mathscr{J}$ is of capcity two. Next suppose $\mathscr{J}$ contains no idempotent $\neq 0, 1$ (and is simple and not a division algebra). Then the Theorem on Minimal Inner Ideals shows that there exists an isotope of $\mathscr{J}$ which is of capacity two. Thus we have the following possibilities for $\mathscr{J}$ =Jord $(Q, 1)$, Q non-degenerate I $\mathscr{J} = \Omega 1_1 \oplus \Omega 1_2$, II $\mathscr{J}$ is a division algebra III $\mathscr{J}$ is simple and has an isotope of capacity two. Now let $\mathfrak{K}$ be an outer ideal in $\mathscr{J}$ containing 1. In case *I* it is immediate that $\mathfrak{K} = \Omega_1 1_1 \oplus \Omega_2 1_2$ where Ω_i is an outer ideal containing 1_i in Ω so Ω_i is a division algebra. In case II $\mathfrak{K}$ is a division algebra. In case III $\mathfrak{K}$ is simple by Lemma 2 of §6.

If $\mathscr{J}$ is of types 1), 3) or 4) then we have seen in §5 that satisfies the DCC for pricipal inner ideals. Also in there cases it follows from Theorem 2.2 and lemma 2 of §6 that $\mathscr{J}$ is simple. This completes the proof of the second statement of the second structure Theorem.

We now consider a special case of this theorem, namely, that in **178**
which $\mathscr{J}$ is finite dimensional over algebraically closed field Φ. The only finite dimensional quadratic Jordan division algebra over Φ is Φ itself (see §1.10). It is clear also that the field Γ in the statement of the theorem is finite dimensional over Φ, so $\Gamma = \Phi$. The simple algebra with involution over Φ are: $\Phi_n \oplus \Phi_n$ with exchange involution, Φ_n with standard involution, Φ_{2m} with the involution $J : X \to S^{-1}X^tS$ where

S =diag$\{Q, Q \ldots, Q\}$,$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In all cases it is easy to check that any outer ideals of $\mathscr{H})(\mathfrak{a}, J)$ containing 1 in $\mathscr{H}(\mathfrak{a}, J)$ coincides with $\mathscr{H}(\mathfrak{a}, J)$. The same is true of Jord $(Q, 1)$ for a non-degenerate Q. There is only one algebra of octonion's O over Φ (the split one). Since the norm form for this represents every $\rho \neq 0$ in Φ it is clear that there is only one exceptional simple quadratic Jordan algebra over Φ, namely, $\mathscr{H}(O_3)$.

The determination of the simple quadratic Jordan algebras of capacity two, which was so arduous in the general case, can be done quickly for finite dimensional algebras over an algebraically closed field. In this case $\mathscr{J} = \Phi e_1 \oplus \Phi_{e_2} \oplus \mathfrak{m}$ where the e_i are supplementary orthogonal idempotents and $\mathfrak{m} = \mathscr{J}_{12}$. If $m\epsilon\mathfrak{m}$, $m^2 = \mu e_1 + \nu e_2, \mu, \nu\epsilon\Phi$. As before, $\mu m = \mu e_i \circ m = \{\Gamma e_1 e_1 m\} = \{m^2 e_1 m\} = e_1 U_{m.m^2} = e_1 U_m V_m = m^2 U_{e_2} V_m = \nu m$. Hence $\mu = \nu$ and $m^2 = \mu 1 = -Q(m)1$ where Q is a quadratic form on $\mathfrak{m}$. We extend this to $\mathscr{J}$ by defining $Q(\alpha e_1 + \beta e_2 + m) = \alpha\beta + Q(m)$. It is easy to check that if $x = \alpha e_1 + \beta_{e_2} + m$ and
179 $T(x) = Q(x, 1)$ then $x^2 = T(x)x + Q(x)1 = 0$ and $x^3 - T(x)x^2 + Q(x) = 0$. Hence $\mathscr{J}$ =Jord $(Q, 1)$. Since $\mathscr{J}$ is simple, Q is non-degenerate.

There remains the problem of isomorphism of simple quadratic Jordan algebras with DCC for principal inner ideals. This can be discussed as in the linear case considered on pp 183 − 187 and 378 − 381 of Jacobson's book [2].

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