Lectures on
The Theorem of Browder And Novikov And Siebenmann's Thesis

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Part I

Theorem of Browder and Novikov

1 Preliminaries

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1.1 The cap-froduct The homology and the cohomology groups we use are the singular ones. Let $\mathbb{Z}$ denote the ring of integers and $\wedge$ an arbitrary commutative ring with $1 \neq 0$. For any topological space X and any integer $n \geq 0$ the set of singular n-simplices of X is denoted by $S_n(X)$. For any $s \in S_n(X)$ and any integer i satisfying $0 \leq i \leq n$ let $s(0, \ldots i)$ (resp. $s(i, \ldots, n)$) denote the element of $S_i(X)$ (resp. $S_{n-i}(X)$) got by restricting s to the front i-dimensional (resp. The rear $(n-i)$-dimensional) face of the standard n-simplex Δ_n. Let $C(X)$ denote the singular chain complex of X' over $\mathbb{Z}$ and $C = C(X) \otimes_{\mathbb{Z}} \wedge$ the chain complex of X over $\wedge$. The cochain complex of X over $\wedge$ which is defined as $\mathrm{Hom}_{\mathbb{Z}}(C(X), \wedge)$ is canonically isomorphic to $\mathrm{Hom}_{\wedge}(C(X) \otimes_{\mathbb{Z}} \wedge, \wedge)$. The boundary homomorphism δ in $C^* = \mathrm{Hom}_{\wedge}(C, \wedge)$ is given by $f = (-1)^{n-1} f \circ \partial$ for every $f \in C^n(X, \wedge) = \mathrm{Hom}(C_n, \wedge)$ where $\partial : C_n \to C_{n-1}$ is the boundary homomorphism in C. As usual C^* is considered as a chain complex with $C^*_{-n} = C^n(X, \wedge)$. The evaluation map $e : C^* \otimes_{\wedge} C \to \wedge$ is defined by $e(f \otimes c) = f(c) \forall f \in C^*_{-n}$ and $c \in C_n$ and $e | C^*_{-p} \otimes C_q = 0$ whenever $p \neq q$. Considering $\wedge$ as a chain complex (with all its elements of degree zero) it is easily seen that $e : C^* \otimes_{\wedge} C \to \wedge$ is a chain homomorphism.

For any two chain complexes A and B over $\wedge$ let $\alpha : H(A) \otimes_{\wedge} H(E) \to$ 2
$H(A \otimes B)$ be the natural map. If $x \in H_p(A)$ and $y \in H_q(B)$ and if z and z' are respectively cycles of A and B representing x and y, then $z \otimes z'$ is a cycle of $A \otimes B$ and the homology class of $z \otimes z'$ is by definition $\alpha(x \otimes y)$. Let $T : A \otimes_{\wedge} B \to B \otimes_{\wedge} A$ be the chain isomorphism given by $T(a \otimes b) = (-1)^{pq} b \otimes a \; \forall a \in A_p, b \in B_q$.

The Alexander-Whitney diagonal map $m_0 : C \to C \otimes_{\wedge} C$ is defined to be the unique $\wedge$-homomorphism satisfying $m_0(s) = \sum\limits_{i=0}^{n} s(0, \ldots, i) \otimes_{\wedge} s(i, \ldots, n) \; \forall s \in S_n(X)$. It is well-known and is not hard to check that m_0 is a chain map. We denote the composition of the chain homomorphism indicated in the following diagram

$$C^* \otimes_{\wedge} C \xrightarrow{Id_{C^*} \otimes m_0} C^* \otimes_{\wedge} C \otimes_{\wedge} C \xrightarrow{T \otimes Id_C} C \otimes_{\wedge} C^* \otimes_{\wedge} C \xrightarrow{Id_C \otimes e} C \otimes_{\wedge} \wedge = C$$

by $\bigcap : C^* \otimes_\wedge C \to C$. More explicitly this map is given by

$$\bigcap(f \otimes s) = f \cap s = \begin{cases} (-1)^{q(n-q)} f(s(n-q, \ldots, n)).s(0, \ldots, n-q) \text{ if } n \geq q \\ o \text{ if } n < q \end{cases}$$

for every $f \in C^q(X, \wedge)$ and $s \in S_n(X)$. Let $H(\cap) : H(C^* \otimes_\wedge C) \to$
3 $H(C)$ be the homomorphism induced by '$\bigcap$'. For any $a \in H^q(C^*) = H_{-q}(C^*) = H^q(X, \wedge)$ and $u \in H_n(C) = H_n(X, \wedge)$ the element $H(\bigcap) o \alpha(a \otimes u)$ is called the cap-product of a by u and is denoted by $a \cap u$.

The chain map $e : C^* \otimes_\wedge C \to \wedge$ induces a homomorphism $H(e) : H(C^* \otimes_\wedge C) \to \wedge$. For any $a \in H^q(X, \wedge)$ and $u \in H_q(X, \wedge)$ the image $H(e) o \alpha(a \otimes u)$ is known as the value of the cohomology class a on the homology class u and is denoted by $a(u)$.

1.2 The following properties of the cap-product will be needed later.

(1) $(a \cup b) \cap u = a \cap (b \cap u) \forall a \in H^p(X, \wedge)$, $b \in H^q(X, \wedge)$ and $u \in H_n(X, \wedge)$ with p, q, n arbitrary integers. Here $a \cup b$ denotes the Cup product of a and b.

(2) For any continuous map $f : Y \to X$, if the induced homomorphisms in homology and cohomology are denoted by $f_* : H(Y, \wedge) \to H(X, \wedge)$ and $f^* : H^*(X, \wedge) \to H^*(Y, \wedge)$, then for any $a \in H^q(X, \wedge)$ and $v \in H_n(Y, \wedge)$

$$f_*(f^* a \cap v) = a \cap f_*(v).$$

1.3 Poincar'e Duality When we refer to homology and cohomology groups without mentioning the coefficients we mean integer coefficients. Let M be a compact, connected, orientable manifold (without boundary) of dimension n. Then it is known that $H_n(M) \simeq \mathbb{Z}$. A choice of a gen-
4 erator u for $H_n(M)$ is known as an orientation for M. M together with a chosen orientation is called an oriented manifold and the distinguished element of $H_n(M)$ is called the fundamental class of M and is denoted by $[M]$.

Let $h : \mathbb{Z} \to \wedge$ be the obvious ring homomorphism (which sends 1 of $\mathbb{Z}$ into 1 of $\wedge$). Let $v = h_*([M])$ where $h_* : H_n(M) \to H_n(M, \wedge)$ is the homomorphism induced by h. Then Poincare duality can be stated as follows:

The map $\Delta : H^q(M, \wedge) \to H_{n-q}(M, \wedge)$ given by $\Delta(x) = x \cap v$ is an isomorphism for all q.

In case M is not necessarily orientable it is true that $H_n(M; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and if v denotes the non zero element of $H_n(M; \mathbb{Z}_2)$ then $\bigcap v : H^q(M; \mathbb{Z}_2) \to H_{n-q}(M; \mathbb{Z}_2)$ is an isomorphism for all q.

When M is compact and not necessarily connected M is orientable if and only if each of its connected components is orientable. M being compact, the number of connected components is finite and denoting them by $\{M_j\}_{j=1}^r$ we have $H_n(M) \simeq \oplus_{j=1}^r H_n(M_j)$. If each M_j is oriented and if $[M_j]$ is the fundamental class of M_j then $[M] = \sum_{j=1}^{r} [M_j] \in H_n(M) = \oplus_{j=1}^r H_n(M_j)$ is defined to be the fundamental class of M.

1.4 All the vector bundles we consider are real vector bundles. For any 5
X the trivial vector bundle of rank ℓ over X will be denoted by $\mathscr{O}_X^l$. The total space and the base space of any vector bundle ξ will be denoted by $E(\xi)$ and B_ξ respectively. To denote that ξ is of rank k we just write ξ^k. If $f : Y \to X$ is a continuous map and ξ any vector bundle over X the pull back bundle on Y is denoted by $f'(\xi)$. If ξ carries a Riemannian metric, for any $\varepsilon > 0$ the subspace of $E(\xi)$ consisting of vectors of length $\leq \varepsilon$ is denoted by $E_\varepsilon(\xi)$ and the boundary consisting of vectors of length ε is denoted by $\dot{E}_\varepsilon(\xi)$. When B_ξ is compact the Thom space ξ denoted by $T(\xi)$ is defined to be the one point compactification of $E(\xi)$. Let '∞' denote the point at infinity of $T(\xi)$. When ξ carries a Riemannian metric we can describe the Thom space alternatively as follows. Let $T_\varepsilon(\xi)$ be the quotient space got from $E_\varepsilon(\xi)$ by collapsing $\dot{E}_\varepsilon(\xi)$ to a point. The map $\beta : E_\varepsilon(\xi) \to T(\xi)$ defined by $\beta(v^\to) = \dfrac{v^\to}{\varepsilon - \|v^\to\|}$ for $v^\to \in E_\varepsilon(\xi) - \dot{E}_\varepsilon(\xi)$ and $\beta(v^\to) = \infty$ for $\overrightarrow{v} \in \dot{E}_\varepsilon(\xi)$ passes down to a homeomorphism $\Theta : T_\varepsilon(\xi) \to T(\xi)$. Compactness of B_ξ is essential for Θ to be a homeomorphism.

For any differential ($= C^\infty$) manifold M the tangent bundle of M will be denoted by τM. The word differentiable will always mean dif-

ferentiable of class C^∞ for us. For the rest of *this* sections M denotes a
6 compact, connected, oriented differential manifolds of dimension $n \geq 0$ with $[M]$ as the fundamental class. By Whitney's imbedding theorem M can be differentially imbedded in $\mathbb{R}^{n+k}$. Except when $n = 0$ the compactness of M automatically implies that $k \geq 1$. Even when $n = 0$ we can assume $k \geq 1$. Let ν be the normal bundle of this imbedding. Then $\tau_M \oplus \nu \simeq \mathscr{O}_M^{n+k}$. Since τ_M and $\mathscr{O}_M^{n+k}$ are both orientable it follows that ν is an orientable vector bundle. Identifying the tangent space to $\mathbb{R}^{n+k}$ at any point with $\mathbb{R}^{n+k}$ in the usual way and taking the usual Riemannian metric on $\tau_{\mathbb{R}^{n+k}} \simeq \mathscr{O}_{\mathbb{R}^{n+k}}^{2n+2k}$ any element of $E(\nu)$ can be thought of as a pair $(x, \frac{\mathbb{R}}{\nu})$ with $x \in M$ and $\nu^{\rightarrow} \in \mathbb{R}^{n+k}$ in a directional normal to M at x. Let $e : E(\nu) \to \mathbb{R}^{n+k}$ be defined by $e(x, \nu) = x + \nu$. $\exists$ an $\varepsilon > 0$ such that e is a diffeomorphism of the set $E_\varepsilon(\nu)$ on to a neighbourhood A of M. A is called a closed tabular neighbourhood of M. Let $\dot{A} = e(\dot{E}_\varepsilon(\nu))$. Considering S^{n+k} as the one point compactification of $\mathbb{R}^{n+k}$ we can define a map $C : S^{n+k} \to T(\nu)$. This is the map got by collapsing the complement of $A - \dot{A}$ in S^{n+k} to a point. More precisely, $C|A = \beta o e^{-1}$ and $C|(S^{n+k} - A) = \infty$.

Let $\Phi : H_n(M) \to H_{n+k}(T(\nu))$ be the Thom isomorphism [5].

Proposition 1.5. *$\Phi([M]) = C_*(\iota)$ for a generator ι of $H_{n+k}(S^{n+k})$.*

Proof. We have only to show that $C_* : H_{n+k}(S^{n+k}) \to H_{n+k}(T(\nu))$ is an isomorphism. We abbreviate $E_\varepsilon(\nu)$ by E_ε etc. Let $A_{\frac{1}{2}} = e(E_{\varepsilon/2})$. Clearly
7 $\beta|E_{\frac{\varepsilon}{2}}$ is a homeomorphism of $E_{\frac{\varepsilon}{2}}$ onto the image Γ (say). Let x be any point in M (such a point exists because dim $M \geq 0$ by assumption) and $i_x : S^{n+k} \to (S^{n+k}, S^{n+k} - x)$ and $j_x : (S^{n+k}, S^{n+k} - M) \to (S^{n+k}, S^{n+k} - x)$ the respective inclusions. Consider the following commutative diagram.

The homomorphism indicated as β_* is an isomorphism since $\beta : E_{\frac{\varepsilon}{2}} \to \Gamma$ is a homeomorphism. It follows that the monomorphism num-
8 bered ① is an isomorphism. The space $T(\nu) - M$ is contractible in itself to ∞. Hence the map $H_{n+k}(T(\nu)) \to H_{n+k}(T(\nu), T(\nu) - M)$ is an isomorphism. (The assumption $k \geq 1$ is used here). Since $H_{n+k}(T(\nu)) \simeq H_n(M) \simeq \mathbb{Z}$ we have $H_{n+k}(S^{n+k}, S^{n+k} - M)$. Since $(i_x)_*$ is an isomorphism it follows that j_* is a monomorphism and that image of j_* is a direct summand of $H_{n+k}(S^{n+k}, S^{n+k} - M)$. The groups $H_{n+k}(S^{n+k})$ and

$$
\begin{tikzcd}
 & H_{n+k}(S^{n+k}, S^{n+k} - x) & & \\
H_{n+k}(S^{n+k}) \arrow[ur, "(i_x)_*"] \arrow[r, "j_*"'] \arrow[d, "C_*"'] & H_{n+k}(S^{n+k}, S^{n+k} - M) \arrow[u, "(j_x)_*"'] \arrow[d, "C_*", "①"'] & & H_{n+k}(A_{\frac{1}{2}}, A_{\frac{1}{2}} - M) \arrow[ll, "\text{Excision}"', "\approx"] \arrow[d, "(e^{-1})_*"'] \\
H_{n+k}(T(\nu)) \arrow[r, "\approx"] & H_{n+k}(T(\nu), T(\nu) - M) & H_{n+k}(\Gamma, \Gamma - M) \arrow[l, "\approx"', "\text{Excision}"] & H_{n+k}(E_{\frac{\varepsilon}{2}}, E_{\frac{\varepsilon}{2}} - M) \arrow[l, "\approx"', "\beta_*"]
\end{tikzcd}
$$

$H_{n+k}(S^{n+k}, S^{n+k} - M)$ being both isomorphic to $\mathbb{Z}$ it follows that j_* is an isomorphism. It now follows that $C_* : H_{n+k}(S^{n+k} \to H_{n+k}(T(\nu))$ is an isomorphism.

□

1.6 The index of A 4d-dimensional manifold Let M be a compact, connected, oriented manifold of dimensional $4d$ with d an integer ≥ 0 and let $[M]$ be the fundamental class of M. The image $h_*([M])$ of the fundamental class of M under the inclusion $h : \mathbb{Z} \to \mathbb{Q}$ is called the fundamental class with coefficients in $\mathbb{Q}$ and is also denoted by $[M]$. The map$(x, y) \rightsquigarrow (x \cup y)[M]$ of $H^{2d}(M, \mathbb{Q}) \times H^{2d}(M, \mathbb{Q}) \to \mathbb{Q}$ gives a symmetric, non degenerate bilinear form $H^{2d}(M, \mathbb{Q})$. Symmetry is clear from $x \cup y = (-1)^{2d \cdot 2d} y \cup x = y \cup x$. That it is non degenerate is a consequence of Poinecare duality together with the fact that $(a, u) \rightsquigarrow a(u)$ is a bilinear non degenerate pairing of $H^{2d}(M, \mathbb{Q}) \times H_{2d}(M, \mathbb{Q}) \to \mathbb{Q}$. This
9 latter fact is embodied in the universal coefficient theorem $H^{2d}(M, \mathbb{Q}) = \hom_{\mathbb{Q}}(H_{2d}(M, \mathbb{Q}), \mathbb{Q})$. The signature (i.e. the number of $+ve$ diagonal elements minus the number of $-ve$ general elements when diagonalised over $\mathbb{Q}$) of the bilinear form $(x, y) \rightsquigarrow (x \cup y)[M]$ on $H^{2d}(M, \mathbb{Q})$ is defined to be the index of M and is denoted by $I(M)$.

In case M is also differentiable we have the following Theorem of Hirzebruch's [1].

Theorem 1.7. *Let $L_k(p_1, \ldots, p_k)$ be the multiplicative sequence of polynomials corresponding to the power series*

$$\frac{\sqrt{t}}{\tanh \sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k t^k + \cdots$$

(Here B_k is the k^{th} Bernouilli number). Then the index $I(M)$ is equal to the L-genus of M defined as $\Big\{L_d(p_1(\tau_M), \ldots, p_d(\tau_M)\Big\}([M])$, where $p_i(\tau_M)$ is the i^{th} Pontrjagin class of τ_M.

For more information about the formalism of multiplicative sequences and the correspondence between power series and multiplicative sequence the reader is referred to [1], [5].

10 *We just content ourselves with the remark that $L_k(p_1, \ldots, p_k)$ are*

universally defined polynomials (i.e. independent of M) with coefficients in the indeterminates $p_1, p_2, \ldots$. The total weight of each term of $L_k(p_1, \ldots p_k)$ is $4k$ when p_j is alloted the weight $4j$. The first two of these polynomials are $L_1(p_1) = \dfrac{1}{3}p_1$; $L_2(p_1, p_2) = \dfrac{1}{45}(7p_2 - p_1^2)$.

1.8 We will be mainly concerned with a space X which is a finite simplicial complex. Given any vector bundle ξ^k over X there exists a vector bundle η over X with $\xi \oplus \eta \simeq \mathscr{O}_X$ (of some rank). In fact $\exists$ a map $f : X \to G_{k+\ell,k}$ (the Grassmann manifold of k-planes in $\mathbb{R}^{k+\ell}$) for some ℓ such that $f!(\gamma^k) = \xi$. Here γ^k is the universal bundle on $G_{k+\ell,k}$. The space $E(\gamma^k)$ is the subspace of $G_{k+\ell,k} \times \mathbb{R}^{k+\ell}$ consisting of elements $(y, \vec{v})$ with $\vec{v} \in y$. Let $\tilde{\gamma}^\ell$ be the vector bundle on $G_{k+\ell,k}$ consisting of elements $(y, \vec{w})$ with $\vec{w} \in \mathbb{R}^{k+\ell}$ orthogonal to y. Then $\eta = f!(\tilde{\gamma}^\ell)$ satisfies $\xi \oplus \eta \simeq \mathscr{O}_x^{k+\ell}$. Two vector bundles ξ and ξ' over X are said to be stably equivalent if $\xi \oplus \mathscr{O}_X^\ell \simeq \xi' \oplus \mathscr{O}_X^{\ell'}$ for some ℓ and ℓ'. The stable class of ξ is denoted by $[\xi]$. If ξ and ξ' are stably equivalent and if η and η' are such that $\xi \oplus \eta \simeq \mathscr{O}^n$ and $\xi' \oplus \eta' \simeq \mathscr{O}^{n'}$ for some n and n' it is easy to see that η and η' are stably equivalent. The class of η is denoted by $-[\xi]$. It **11**
is known that the Pontrjagin classes of a vector bundle depend only on the stable class of the bundle. If $\bar{p}_1(\xi), \bar{p}_2(\xi), \ldots$ denote the Pontrjagin classes of some η belonging to the class $-[\xi]$ it follows that the elements $L_k, (\bar{p}_1, (\xi), \ldots, \bar{p}_k(\xi))$ depend only on the class $[\xi]$ of ξ.

Referring to the situation where M^{4d} is differentiably imbedded in $\mathbb{R}^{4d+k}$ with normal bundle ν we see that $L_k, (\bar{p}_1(\nu), \ldots, \bar{p}_K(\nu)) = L_k, (p_1(\zeta_M), \ldots, p_k(\zeta_M)) \in H^{4k'}(M, \mathbb{Q})$. Thus Hirzebruch's theorem can be rephrased in terms of the normal bundle ν as $\{L_d(\bar{p}_1(\nu), \ldots, \bar{p}_d(\nu))\}$ $([M]) = I(M)$.

2 The main Theorem

Let X be a connected finite simplicial complex with $\prod_1(X) = 0$. The theorem of Browder and Novikov deals with conditions under which X will be of the same homotopy type as a compact differentiable manifold M without boundary. Since X is simply connected if such an M exists it has to be orientable. We first state the theorem, which actually consists of two parts.

Theorem 2.1. *Let X be a connected finite simplicial complex with $\prod_1(X) = 0$. Suppose that the following two conditions are satisfied.*

12 *i) X satisfies Poincaré duality i.e. to say $\exists$ some integer n with $H_n(X) \simeq \mathbb{Z}$ and if u is a generator, $\bigcap u : H^q(X) \to H_{n-q}(X)$ is an isomorphism for all q.*

ii) $\exists$ an oriented vector bundle ξ^k over X such that $\Phi(u) \in H_{n+k}(T(\xi))$ is spherical, $\Phi : H_n(X) \to H_{n+k}(T(\xi))$ being the Thom isomorphism.

Then if n is odd X is of the same homotopy type as a compact differentiable manifold M of dimension n under a homotopy equivalence $f : M \to X$ satisfying $[f!(\xi)] = -[\tau_M]$.

The second part of the theorem is concerned with the case $n = 4d$ with d an integer > 1.

X being a finite complex we have $H^q(X, \mathbb{Q}) = H^q(X) \otimes \mathbb{Q}$ and $H_i(X, \mathbb{Q}) = H_i(X) \otimes \mathbb{Q}$. Denoting the image of u in $H_n(X, \mathbb{Q})$ under $h_* : H_n(X) \to H_n(X, \mathbb{Q})$ where $h : \mathbb{Z} \to \mathbb{Q}$ is the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$ by v we have $\cap v : H^q(X, \mathbb{Q}) \to H_{n-q}(X, \mathbb{Q})$ an isomorphism for all q. Actually $\bigcap v$ can be identified with $(\bigcap u) \otimes \mathbb{Q}$. Thus assumption *i*) actually implies Poincare duality for coefficients in $\mathbb{Q}$. Actually, it is true that assumption *i*) implies Poincare duality for any arbitrary commutative coefficient ring $\wedge$ (with $1 \neq 0$). The procedure adopted to define the index $I(M^{4d})$ in §1.6 can now be used to define the index $I(X)$ of X.

13 Assume in addition to i) and ii) we have the following valid for ξ.

iii) $I(X) = \left\{ L_d(\bar{p}_1(\xi), \ldots, \bar{p}_d(\xi)) \right\}(v)$.

Then X is of the same homotopy type as a compact differentiable manifold M of dimension $4d$ under an equivalence $f : M \to X$ satisfying $[f!(\xi)] = -[\tau_M]$.

Part *I* of these lectures is devoted to the proof of this theorem. From §1 at actually follows that the conditions *i*), *ii*), and *iii*) when $n = 4d$, are necessary for the validity of the Theorem.

From the assumption $\prod_1(X) = 0$ it follows that the integer n satisfying condition $i)$ of Theorem 2.1 has to be ≥ 3 whenever n is odd. But for $n = 3$ the condition $i)$ itself implies that X is of the same homotopy type as S^3. Moreover every vector bundle on S^3 is trivial since $\prod_2(So(k)) = 0$ for every integer $k \geq 0$. Thus for any vector bundle ξ over X and any homotopy equivalence $f : S^3 \to X$ we have $[f!(\xi)] = -[\tau_{S^3}]$. This shows that Theorem 2.1 is trivially valid for $n = 3$ and hence it only remains to prove the Theorem for $n \geq 5$. But some of the Lemmas and propositions that will be proved here are valid for $n \geq 4$, and it will be clear later when exactly we need the assumption $n > 4$.

2.2 Realizing X as a subcomplex of a simplex Δ_N for some integer N **14**
and imbedding Δ_N affinely in $\mathbb{R}^N$ we get an open set $U \supset X$ of $\mathbb{R}^N$ such that X is a deformation retract of U. Let $j : X \to U$ be the inclusion and $r : U \to X$ the retraction (i.e. $roj = Id_x$) with $jor \sim Id_U$ ($\sim$= 'homomorphic to'). Let ξ be a vector bundle on X satisfying condition $ii)$ of Theorem 2.1. Let $\xi' = r!(\xi)$. It is easy to see that ξ' can be made into a differentiable vector bundle. Actually ξ' is induced by a certain map $g : U \to G_{k+\ell,k}$ for some integer ℓ, form the universal bundle γ^k on $G_{k+\ell,k}$. Since the map g can be approximated by a differentiable map $g : U \to G_{k+\ell,k}$ with $g \sim g'$, it follows that ξ' can be made into a differentiable vector bundle. The Thom space $T(\xi')$ of ξ' is defined as follows. Introducing a fixed C^∞ Riemannian matric on ξ', let $E_1(\xi')$ be the subspace of $E(\xi')$ consisting of vectors of length ≤ 1 and $\dot{E}_1(\xi')$ the boundary of $E_1(\xi')$ consisting precisely of vectors of length 1. The space $T(\xi')$ is defined as the quotient space $E_1(\xi')/\dot{E}_1(\xi')$. In this case $T(\xi')$ is not the one point compactification of $E(\xi')$. Still we denote the point of $T(\xi')$ to which $\dot{E}_1(\xi')$ is collapsed by $''\infty''$. Clearly $T(\xi') - \infty$ is a differentiable manifold.

Since $roj = Id_X$ we have $\xi'/X = \xi$. Taking the restriction to ξ of the Riemannian metric on ξ', and realizing $T(\xi)$ as $E_1(\xi)/\dot{E}_1(\xi)$ we see **15**
that the inclusion map $h : E(\xi) \to E(\xi')$ induces a map $T(h) : T(\xi) \to T(\xi')$. The symbol Φ denotes throughout the Thom isomorphism. Let $f : S^{n+k} \to T(\nu)$ be a map such that $f^*(\iota) = \phi(u)$, ι being a generator

of $H_{n+k}(S^{n+k})$. By condition ii) such a map exists. The naturality of the Thom isomorphism yields $(T(h)of)_*(\iota) = \Phi(j_*(u))$. Denoting $T(h)of$ by f' we see that $f' : S^{n+k} \to T(\xi')$ is a map satisfying $f'_*(\iota) = \Phi(j_*(u))$. By the transverse regular approximation theorem [4], $\exists$ a differentiable map $f'' : S^{n+k} \to T(\xi')$ (whenever it makes sense i.e. on $f''^{-1}(T(\xi') - \infty))$ with $f'' \sim f'$ and f'' transverse regular on U. Clearly $f''^{-1}(U) \neq \emptyset$ for if $f''(S^{n+k}) \cap U = \emptyset$ the map $f''_* : H_{n+k}(S^{n+k}) \to H_{n+k}(T(\xi'))$ would factor through $H_{n+k}(T(\xi') - U) = 0$ (since $T(\xi') - U$ is contractible to "∞"). But $f''_*(\iota) = f'_*(\iota) = \Phi(j_*(u)) \neq 0$. Hence $M = f''^{-1}(U)$ is a differentiable manifold of codimension k in S^{n+k} with normal bundle $\nu_M \simeq f''!(\xi')$. But M need not necessarily be connected. Since $f''(\xi')$ and $\tau_{S^{n+k}}$ are orientable and since $\tau_{S^{n+k}}\Big| M \simeq \tau_M \oplus f''!(\xi')$ we see that τ_M is orientable. Since U is closed in $T(\xi)$ we have $M = f''^{-1}(U)$ closed in S^{n+k} and hence M is a compact, orientable differentiable manifold
16 of dimensional n. Choose some C^∞ Riemannian metric for ν_M. It is known that $\exists$ a tubular neighbourhood i.e. a diffeomorphism D of $E_\varepsilon(\nu)$ for some $\varepsilon > 0$ onto a closed neighbourhood B of M in S^{n+k}, and map $\bar{f} : S^{n+k} \to T(\xi')$ satisfying the following conditions:

1) $\bar{f}$ is differentiable on $\bar{f}^{-1}(T(\xi') - \infty)$ and transverse regular on U
2) $\bar{f} = f''$ on M and $\bar{f}^{-1}(U) = f''^{-1}(U) = M$
3) $\bar{f}oD$ is a bundle map of $E_\varepsilon(\nu)$ onto the image (i.e. maps the fibre of $E_\varepsilon(\nu)$ at $x \in M$ homeomorphically onto the image portion of the fibre at $f(x)$ in $E(\xi)$)
4) $\bar{f} \sim f'' : S^{n+k} \to T(\xi')$.

For a proof refer to steps 1 and 2 of the proof of Theorem 3.16 in [4].

From the compactness of M it follows that $\exists$ a $\delta > 0$ with $\bar{f}oD$ $(E_\varepsilon(\nu)) \supset E_\delta(\xi')\Big|\bar{f}(M)$. Let $\{M_i\}_{i=1,,r}$ be the connected components of M and let $A_i = \bar{f}^{-1}(E_\delta(\xi'))\Big|M_i$ and $\dot{A}_i = \bar{f}^{-1}(\dot{E}_\delta(\xi'))\Big|M_i$. We will write the same symbols A_i, $\dot{A}_i$ to denote $D^{-1}(Ai)$, $D^{-1}(Ai)$ etc. In otherwords we identify $E_\varepsilon(\nu)$ and the tubular neighbourhood B.

We now introduce the following changes in notation. We write ξ, f and u for ξ, $\overline{f}$ and $j_*(u)$. With this altered notation $f : S^{n+k} \to T(\xi)$ is a map satisfying $\Phi(u) = f_*(\iota)$, differentiable on $f^{-1}(T(\xi) - \infty)$, transverse regular on U and is also a bundle map covering $f\big|M : M \to U$ on a **17**
tubular neighbourhood of M in S^{n+k}.

2.3 We choose ι as the fundamental class $\left[S^{n+k}\right]$. Then each $(A_i, \dot{A}_i)$ receives the induced orientation $\left[A_i, \dot{A}_i\right]$. Denoting by ν_i the restriction of ν to M_i and by $\Phi_i : H_n(M_i) \to H_{n+k}(T(\nu_i))$ the Thom isomorphism, let $\psi_i : H_n(M_i) \to H_n(A_i, A_i - M_i)$ be the unique isomorphism making the following diagram commutative.

$$\begin{tikzcd} H_n(M_i) \arrow[rr, "\Phi_i"] \arrow[d, "\psi_i"'] & & H_{n+k}(T(\nu_i)) \arrow[d, "\approx"', "(\mathrm{incln})_*"] \\ H_{n+k}(A_i, A_i - M_i) \arrow[rr, "\approx", "\text{Excision}"'] & & H_{n+k}(T(\nu_i), T(\nu_i) - M_i) \end{tikzcd}$$

The homomorphisms $(j_i)_* : H_{n+k}(A_i, \dot{A}_i) \to H_{n+k}(A_i, A_i - M_i)$ induced by inclusions are isomorphisms (since $\dot{A}_i$ is a deformation retract of $A_i - M_i$). We choose orientations $[M_i]$ for M_i by the requirement that $\psi_i([M_i]) = (j_i)_*\left(\left[A_i, \dot{A}_i\right]\right)$

Lemma 2.4. *The map $f : M \to U$ is of degree 1 i.e. to say $f_*([M]) = u$ with $[M] = \sum [M_i]$.*

Proof. Let $\psi : H_n(U) \to H_{n+k}(E_\delta(\xi), E_\delta(\xi) - U)$ be the isomorphism **18**
making the square. □

$$\begin{tikzcd} H_n(U) \arrow[r, "\psi"] \arrow[d, "\Phi"'] & H_{n+k}(E_\delta(\xi), E_\delta(\xi) - U) \arrow[d, "\approx"', "\text{excision}"] \\ H_{n+k}(T(\xi)) \arrow[r, "\approx"] & H_{n+k}(T(\xi), T(\xi) - U) \end{tikzcd}$$

commutative. Naturality of the Thom isomorphism together with the fact that $f\big|B$is a "bundle map" yield the following commutative diagram.

$$
\begin{tikzcd}
 & H_{n+k}(S^{n+k}) \arrow[r, "f_*"] \arrow[d, "j_*"] & H_{n+k}(T(\xi)) \arrow[d, "(j_\xi)_*"] \\
 & H_{n+k}(S^{n+k}, S^{n+k} - M) \arrow[r, "f_*"] & H_{n+k}(T(\xi), T(\xi) - U) \\
\bigoplus_{i=1}^{r} H_{n+k}(A_i, \dot{A}_i) \arrow[r, "(\oplus j_i)_*"] & \bigoplus_{i=1}^{r} H_{n+k}(A_i, A_i - M_i) = H_{n+k}(\bigcup_{i=1}^{r} A_i, \bigcup_{i=1}^{r} (A_i - M_i)) \arrow[r, "f_*"] \arrow[u, "e_*\text{(excision)}"] & H_{n+k}(E_\delta(\xi), E_\delta(\xi) - U) \arrow[u, "(e_\xi)_*\text{(excis)}", "\approx"'] \\
 & \bigoplus_{i=1}^{r} H_n(M_i) \arrow[r, "f_*"] \arrow[u, "\oplus\psi_i"] & H_n(U) \arrow[u, "\psi"]
\end{tikzcd}
$$

Diagram 2.

Let $f_*[M] = du$. We have to show that $d = 1$. We have $(e_*^{-1})j_*$ 19
$[S^{n+k}] = \sum_i (j_i)_*\left(\left[A_i, \dot{A}_i\right]\right)$. To show that $d = 1$ it suffices to show that $\psi f_*[M] = \psi(u)$. From Diagram 2 we have

$$\begin{aligned}\psi f_*[M] &= f_*\left(\sum \psi_i[M_i]\right) = f_*\left(\sum (j_i)_*([A_i, \dot{A}_i])\right)\\ &= f_*(e_*)^{-1} j_*[S^{n+k}] = (e_{\xi_*})^{-1}(j_\xi)_*(f_*[S^{n+k}])\\ &= \left(e_{\xi_*}\right)^{-1}\left(j_\xi\right)_*(\Phi(u)).\end{aligned}$$

But by definition of ψ we have $\psi(u) = (e_{\xi_*})^{-1}(j_\xi)_*\Phi(u)$.

We change our notations again and write $f : M \to X$ for the map of rof where $r : U \to X$ is the homotopy equivalence chosen already and write u for the original generator of $H_n(X)$. Then f is of degree 1. The homomorphism $H_q(M) \to H_q(X)$ induced by f is denoted by f_q.

Lemma 2.5. *There exist homomorphism $g_q : H_q(X) \to H_q(M)$ with $f_q o g_q = Id_{H_q(X)}$ and hence $H_q(M) = Kar\ f_q \oplus g_q(H_q(X))$.*

Proof. For any $x \in H_q(X)$ let $\gamma \in H^{n-q}(X)$ be the element $\Delta^{-1}(x)$ where $\Delta : H^{n-q}(X) \to H_q(X)$ is the Poincare isomorphism. Setting $g_q(x) = f^*(\gamma) \cap [M]$ we have $f_q g_q(x) = f_*(f^*(\gamma) \cap [M]) = \gamma \cap f_*[M] = \gamma \cap u = x$. □

The proof of this lemma uses only two facts : ⓐX satisfies Poincare 20
duality and ⓑ$f : M \to X$ is a map of degree 1.

Let η' be a bundle over X (of rank ℓ' say) such that $\xi \otimes \eta' \simeq \mathcal{O}_X^{k+\ell'}$. Let $\eta = \eta' \otimes \mathcal{O}_X^{k+n}$. Then $[\eta] = [\eta'] = -[\xi]$ and

$$\begin{aligned}f!(\eta) = f!(\eta') \oplus \mathcal{O}_M^{k+n} &\simeq f!(\eta') \oplus \tau_M^n + \nu_M^k \simeq \oplus f!(\eta') \oplus f!(\xi)\\ &\simeq \tau_M^n \oplus f!(\eta' \oplus \xi) \simeq \tau_M^n \oplus \mathcal{O}_M^{k+\ell'}.\end{aligned}$$

Denoting $k + \ell'$ by ℓ we have the following situation: $\exists$ a vector bundle η of rank $n + \ell$ on X with $[\eta] = -[\xi]$ and a map $f : M \to X$ of degree 1 satisfying $f!(\eta) \approx \tau_M^n \otimes \mathcal{O}_M^\ell$. Without loss of generality we can assume $\ell' \geq 1$. Our aim is to surgerize M finitely many times and obtain a connected simply connected manifold M' together with a map $f' : M' \to X$ inducing isomorphisms in homology and further satisfying

$f'!(\xi) \approx \tau^n_{M'} \oplus \mathscr{O}^\ell_{M'}$. In this is done the theorem is proved since f' will then be a homotopy equivalence by a theorem of J.H.C. Whitehead and the relation $f'!(\xi) = \tau^n_{M'} \otimes \mathscr{O}^\ell_{M'}$ implies $[f'!(\xi)] = -[\tau^n_{M'}]$. In case n is odd and ≥ 5 we will be able to achieve this using conditions i) and ii) and when $n = 4d$ with d an integer > 1 we will also need condition iii) to do the same.

3 Surgery or Spherical modification

21 The unit *disk* $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \middle| \sum_{i=1}^n x_i^2 \leq 1\right\}$ in $\mathbb{R}^n$ is denoted by D^n and the unit open ball $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \middle| \sum_{i=1}^n x_i^2 < 1\right\}$ by B^n. For any real number $t > 0$ the closed disk and the open ball of radius t are denoted by tD^n and tB^n respectively. All the manifolds we consider are oriented C^∞ manifolds. We use the letter V to denote a compact manifold without boundary, of dimension $n \geq 1$.

Definition 3.1. *Given an orientation preserving differentiable imbedding $\varphi : S^q \times \frac{3}{2}D^{n-q} \to V$ with $n > q \geq 0$ let $\chi(V, \varphi)$ denote the quotient manifold obtained from the disjoint union $V - \varphi(S^q \times \frac{1}{2}D^{n-q}) U \frac{3}{2}B^{q+1} \times S^{n-q-1}$ by identifying $\varphi(x, t, y)$ with $(tx, y) \forall x \in S^q$, $y \in S^{n-q-1}$ and $\frac{1}{2} < t < 3/2$.*

It is easy to check that $\chi(V, \varphi)$ is Hausdorff. Since $\varphi(x, ty) \rightsquigarrow (tx, y)$ is a diffeomorphism for $x \in S^q$, $y \in S^{n-q-1}$ and $\frac{1}{2} < t < 3/2$ it follows that $\chi(V, \varphi)$ is a C^∞-manifold. It is clearly compact and oriented. The manifold $\chi(V, \varphi)$ is said to be got from V by a surgery of type $(q+1, n-q)$.

Two compact if oriented manifolds V and V' are said to be χ-equivalent if $\exists$ a finite sequence of manifolds $V_1 = V_1, V_2, \ldots, V_r = V'$ such that V_{i+1} is got from V_i by a surgery.

22 **Lemma 3.2.** *Suppose V has s connected components with $s \geq 2$ and $\varphi : S^o \times D^n \to V$ an orientation preserving imbedding which carries the*

two components of $S^o \times D^n$ into distinct components of V. Then $\chi(V, \varphi)$ has exactly $(s-1)$ connected components.

Proof. Trivial for $n \geq 2$. For $n = 1$ we have to use the fact that every component of V is diffeomorphic to S'.

Using conditions i) and ii) of Theorem 2.1. we obtained a compact oriented manifold M of dimension n, a vector bundle η of rank $(n+\ell)$ on X with $[\eta] = -[\xi]$ and a map $f : M \to X$ of degree 1 satisfying $f!(\eta) \approx \tau^n_M \oplus \mathscr{O}^\ell_M$. Let $\varphi : S^q \times \frac{3}{2}D^{n-q} \to M$ be an orientation preserving imbedding with $n > q \geq 0$. Assume further that $f \circ \varphi(S^q \times \frac{3}{2}D^{n-q}) = x^*$, a chosen base point for X. Let $M' = \chi(M, \varphi)$ and let $f' : M' \to X$ be defined as follows. Setting $M_o = M - \varphi(S^q \times B^{n-q})$ the map f' is given by $f'|M_o = f|M_o$ and $f'|\varphi'(D^{q+1} \times S^{n-q-1}) = x^*$ where $\varphi' : D^{q+1} \times S^{n-q-1} \to M'$ denotes the imbedding induced by the inclusion $D^{q+1} \times S^{n-q-1} \to \frac{3}{2}B^{q+1} \times S^{n-q-1}$. Clearly f' is well defined and continuous. □

Lemma 3.3. *The map $f' : M' \to X$ is of degree 1.*

Proof. Consider the following commutative diagram. 23

$$\begin{tikzcd}
H_n(M) \arrow[r, "j_*"] & H_n(M, \varphi(S^q \times D^{n-q})) \arrow[r, "f_*"] & H_n(X, x^*) \\
 & H_n(M_o, \varphi(S^q \times S^{n-q-1})) \arrow[u, "e_*"', "\approx"] \arrow[ur, "f_*"] \arrow[d, "\approx"', "e'_*"] & \\
H_n(M') \arrow[r, "j'_*"] & H_n(M', \varphi'(D^{q+1} \times S^{n-q-1})) \arrow[uur, "f'_*"'] &
\end{tikzcd}$$

Diagram 3.

Here j_*, j'_*, e_* and e'_* are homomorphisms induced by the respective inclusions. The maps e_* and e'_* are isomorphisms by excision and homotopy. That f' is of degree 1 now follows from $e'^{-1}_* j'_*[M'] = e^{-1}_* j_*[M]$.

Suppose M is not connected. Choosing $\varphi : S^o \times \dfrac{3}{2}D^n$ such that the two components of $S^o \times \dfrac{3}{2}D^n$ go into distinct components of M let $M' = \chi(M, \varphi)$. Since X is connected it follows that $f \circ \varphi : S^o \times \dfrac{3}{2}D^n \to X$ is homotopic to constant map. By homotopy extension property we can choose a map $g : M \to X$ with $g \sim f$ and $g|\varphi(S^o \times \dfrac{3}{2}D^n) = x^*$. Then
24 clearly g is of degree 1 and $g!(\eta) \approx \tau^n_M \oplus \mathscr{O}^\ell_M$. Thus we can without loss of generality assume that f itself satisfies the condition $f\varphi(S^o \times \dfrac{3}{2}D^n) = x^*$. Let $f' : M' \to X$ be the associated map i.e. $f'|M_o = f|M_o$ and $f'|\varphi'(D' \times S^{n-1}) = x^*$. $\square$

Lemma 3.4. *$f' : M' \to X$ is of degree 1 and $f'!(\eta) \approx \tau^n_{M'} \oplus \mathscr{O}^\ell_{M'}$.*

Proof. That f' is of degree 1 follows from Lemma 3.3 Let $T_M = \tau^n_M \oplus \mathscr{O}^\ell_{M'}$ and $T_M, = \tau^n_{M'} \otimes \mathscr{O}^\ell_{M'}$ and $\psi : T_M \to f!(\eta)$ a bundle isomorphism. Our aim is to get a bundle isomorphism $\psi' : T_{M'} \to f'!(\eta)$. Since $T_{M'}|M_o = T_M|M_o$ and $f'|M_o = f|M_o$ we can take $\psi' = \psi$ on $T_{M'}|M_o$. We denote the image of $S^o \times D^n$ by φ in M by $\operatorname{im}\varphi$ and the image of $D^1 \times S^{n-1}$ under φ' in M' by $\operatorname{im}\varphi'$. We identify $T_{M'}|\operatorname{im}\varphi = \tau_\varphi, (D^1 \times S^{n-1})$ with $(\tau_{\frac{3}{2}B^1 \times \frac{3}{2}B^2}\Big|D^1 \times S^{n-1}) \oplus \mathscr{O}^{\ell-1}_{D' \times S^{n-1}}$. Let $w_1, \ldots, w_{n+\ell}$ be a trivialization of $\tau_{\frac{3}{2}B^1 \oplus \frac{3}{2}B^n} \oplus \mathscr{O}^{\ell-1}_{\frac{3}{2}B^1 \times \frac{3}{2}B^n}$ and take the induced trivialization of $T'_M|\operatorname{im}\varphi'$ to identify it with $D^1 \times S^{n-1} \times \mathbb{R}^{n+\ell}$. Let $e_1, \ldots, e_{n+\ell}$ be a basis of the fibre of η at x and let $u_1, \ldots, u_{n+\ell}$ be the pull back trivialisation of $f'!(\eta)|\operatorname{im}\varphi'$. Using this trivialization we identify $f'!(\eta)|\operatorname{im}\varphi'$ with $D^1 \times S^{n-1} \times \mathbb{R}^{n+\ell}$. The map $\psi : T_{M'}|$ Bdry $M_o \to f'!(\eta)|$ Bdry M_o then corresponds to an orientation preserving bundle map $\psi : S^o \times S^{n-1} \times \mathbb{R}^{n+\ell} \to S^o \times S^{n-1} \times$
25 $\mathbb{R}^{n+\ell}$ and thus to a continuous map $\Theta : S^o \times S^{n-1} \to GL_+(n+\ell, \mathbb{R})$ given by $\psi(x, \overrightarrow{v}) = (x, \Theta(x)\overrightarrow{v})\ \forall \overrightarrow{v} \in \mathbb{R}^{n+\ell}$. To get a bundle map $T_{M'} \to f'!(\eta)$ extending $\psi' : T_{M'}\Big|M_o \to f'!(\eta)\Big|M_o$ it suffices to get a continuous extension of Θ into a map $D^1 \times S^{n-1} \to GL_+(n+\ell, \mathbb{R})$. But we know that ψ comes from a bundle map $T_M|\operatorname{im}\varphi \to f!(\eta)|\operatorname{im}\varphi$. Since $f|\varphi(S^o \times D^n) = x^*$ the trivialization $u_1, \ldots, u_{n+\ell}$ of $T'_M|$ Bdry $M_o = T_M|$ Bdry M_o extends to a trivialization of $f!(\eta)|\operatorname{im}\varphi$. Also $T_M|\operatorname{im}\varphi = \tau_{\varphi(S^o \times D^n)} \oplus$

$\mathscr{O}^{\ell}_{\varphi(S^o\times D^n)}$ can be identified with $\left(\tau_{\frac{3}{2}B^1\times\frac{3}{2}B^n}\oplus\mathscr{O}^{\ell-1}_{\frac{3}{2}B^1\times\frac{3}{2}B^n}\right)\Big|S^o\times D^n$. Thus the trivialization $w_1,\ldots,w_{m+\ell}$ extends to a trivialization of $T_M|\operatorname{im}\varphi$. Using these trivializations we see that ψ corresponds to a bundle map $S^o\times D^n\times\mathbb{R}^{n+\ell}\to S^o\times D^n\times\mathbb{R}^{n+\ell}$. In otherwords $\exists$ an extension $\overline{\Theta}$ of Θ into a map $S^o\times D^n\to GL_+(n+\ell,\mathbb{R})$. Since $GL_+(n+\ell,\mathbb{R})$ is connected and D^n contractible it follows that $\exists$ a map $D^1\times D^n\to GL_+(n+\ell,\mathbb{R})$ extending $\overline{\Theta}$. This complete the proof of Lemma 3.4. □

As an immediate consequence of lemmas 3.2 and 3.4 we get the following:

Proposition 3.5. *There exists a connected, compact, oriented C^∞ manifold M' which is χ-equivalent to M and a map $f':M'\to X$ of degree 1 with $f'!(\eta)\approx T_{M'}=\tau^n_{M'}\oplus\mathscr{O}^{\ell}_{M'}$.* 26

We now change our notations. We replace M' by M and f' by f. Thus M is connected and $f:M\to X$ is of degree 1 with $f!(\eta)\approx\tau^n_M\oplus\mathscr{O}^{\ell}_M$.

Let $\varphi:S^q\times\dfrac{3}{2}D^{n-q}\to M$ be an orientation preserving imbedding where $n>q\geq 1$ and let us assume $f\varphi(S^q\times\dfrac{3}{2}D^{n-q})=x^*$. Let $f':M'=\chi(M,\varphi)\to X$ be the associated map. In general $f'!(\eta)$ need not be isomorphic to $\tau^n_m,\oplus\mathscr{O}^{\ell}_{M'}$. Consider the following alteration of the map φ. Let $\alpha:S^q\to So(n-q)$ be a C^∞ map and let $\varphi_\alpha:S^q\times\dfrac{3}{2}D^{n-q}\to M$ be given by $\varphi_\alpha(x,y)=\varphi(x,\alpha(x)y)\ \forall(x,y)\in S^q\times\dfrac{3}{2}D^{n-q}$. Clearly φ_α is an imbedding, also satisfying, $f\varphi_\alpha(S^q\times\dfrac{3}{2}D^{n-q})=x^*$. Let $f'_\alpha:M'_\alpha=\chi(M,\varphi_\alpha)\to X$ be the associated map. The sets $\varphi(S^q\times D^{n-q})$ and $\varphi'(D^{q+1}\times S^{n-q-1})$ (and similarly $\varphi_\alpha(S^q\times D^{n-q})$ and $\varphi'_\alpha(D^{q+1}\times S^{n-q-1})$) are denoted by $\operatorname{im}\varphi$ and $\operatorname{im}\varphi'$ respectively (similarly by $\operatorname{im}\varphi_\alpha$ and $\operatorname{im}\varphi'_\alpha$ respectively). Let ψ' be defined to be ψ on $T_{M'}|M_o=T_M|M_o$ into $f'!(\eta)|M_o=f!(\eta)|M_o$. Let $e_1,\ldots,e_{n+\ell}$ be a fixed basis of the fibre 27
of η at x^* and $u_1,\ldots,u_{n+\ell}$ the pull back trivialization of $f!(\eta)|\operatorname{im}\varphi$. Then $v_i=\psi^{-1}(u_i)$ constitute a trivialization of $T_{M'}|\operatorname{im}\varphi=T_M|\operatorname{im}\varphi$ and there exists a bundle isomorphism $T_{M'}\to f'!(\eta)$ extending ψ' if and only if

the trivialization $v_1, \ldots, v_{n+\ell}$ of $T_{M'}|BdryM_o$ extends to a trivialization of $T_{M'}|\operatorname{im}\varphi'$. We identify $T_{M'}|\operatorname{im}\varphi'$ with

$$\left(\tau_{\frac{3}{2}B^{q+1}\times\frac{3}{2}B^{n-q}} \oplus \mathscr{O}^{\ell-1}_{\frac{3}{2}B^{q+1}\times\frac{3}{2}B^{n-q}}\right)\Big|D^{q+1}\times S^{n-q-1}.$$

Let $w_1, \ldots, w_{n+\ell}$ be any trivialization of

$\mathscr{L}_{\frac{3}{2}B^{q+1}\times\frac{3}{2}B^{n-q}} = (\tau \oplus \mathscr{O}^{\ell-1})_{\frac{3}{2}B^{q+1}\times\frac{3}{2}B^{n-q}}$. Then we get a continuous map $\Theta : S^q \times S^{n-q-1} \to GL_+(n+\ell, \mathbb{R})$ given by $v(x,y) = \Theta(x,y)w(x,y)$ $\forall(x,y) \in S^q \times S^{n-q-1}$. If there is an extension of Θ into a continuous map $D^{q+1} \times S^{n-q-1} \to GL_+(n+\ell, \mathbb{R})$ then $v_1, \ldots, v_{n+\ell}$ can be extended to a trivialization of $T_{M'}\big|\operatorname{im}\varphi'$. But since $T_M|\operatorname{im}\varphi$ is identifiable with $(\tau \oplus \mathscr{O}^{\ell-1})_{\frac{3}{2}B^{q+1}\times\frac{3}{2}B^{n-q}}$ we see that Θ admits of an extension $\overline{\Theta} : S^q \times D^{n-q} \to GL_+(n+\ell, \mathbb{R})$. Hence $\Theta : S^q \times S^{n-q-1} \to GL_+(n+\ell, \mathbb{R})$ admits
28 of an extension $D^{q+1} \times S^{n-q-1} \to GL_+(n+\ell, \mathbb{R})$ whenever $\overline{\Theta}$ admits of an extension $D^{q+1} \times D^{n-q} \longrightarrow GL_+(n+\ell, \mathbb{R})$. Choosing a fixed point $y_0 = S^{n-q-1}$ the obstruction to the existence of such an extension is given by the homotopy class of the map $\gamma : S^q \longrightarrow GL_+(n+\gamma, \mathbb{R})$ where $\gamma(x) = \Theta(x, y_0)$. Let us denote this obstruction class by $\gamma(\varphi) \in \Pi_q(GL_+(n+\ell, \mathbb{R}))$. Let the obstruction class for the imbedding φ_α be denoted by $\gamma(\varphi_\alpha)$.

Lemma 3.6. *The obstruction $\gamma(\varphi_\alpha)$ depends only on $\gamma(\varphi)$ and the homotopy class (α) of α in $\Pi_q(S0(n-q))$. More precisely identifying $\pi_q(SO(n-q))$ with $\pi_q(GL_+(n-q), \mathbb{R})$ we have $\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$ where $s_* : \pi_q(GL_+(n-q, \mathbb{R})) \to \pi_q(GL_+(n+\ell, \mathbb{R}))$is the map induced by the inclusion $s : GL_+((n-q), \mathbb{R}) \to GL_+(n+\ell, \mathbb{R})$.*

Proof. Suppose $\varepsilon_1, \ldots, \varepsilon_{n+\ell}$ is any trivialisation of $T_{M'}\big|\operatorname{im}\varphi'$ and suppose $\lambda : S^q \times S^{n-q-1} \to GL_+(n+\ell, \mathbb{R})$ the map given by $v(x,y) = \lambda(x,y) \in (x,y)$ $\forall(x,y) \in S^q \times S^{n-q-1}$. Then $\exists$ a counts map $P : D^{q+1} \times S^{n-q-1} \to GL_+(n+\ell, \mathbb{R})$ such that $\Theta(x,y) = \lambda(x,y)p(x,y)$. Actually P is the transformation relating the frame $\varepsilon(x,y)$ to $v'(x,y)$. Hence the homotopy class of $A|S^q \times yo$ is the same as that of $\lambda\big|S^q \times yo$. Now let $\Phi' : D^{q+1} \times (D^{n-q} - \{0\}) \to M' \times \mathbb{R}$ be the map given by $\Phi'(x,y) =$
29 $(\varphi'(x, \frac{y}{\|y\|}), \|y\| - 1)$. Choosing some trivialisation $C_0, C_1 \ldots, C_{\ell-1}$ of

$\mathscr{O}^{\ell}_{\operatorname{im}\varphi'}$ we see that

$$\frac{\partial\Phi'}{\partial\xi} = \left(\frac{\partial\Phi'}{\partial x_1}, \ldots, \frac{\partial\Phi'}{\partial x_{q+1}}, \ldots, \frac{\partial\Phi'}{\partial y_{n-q}}, C_1, \ldots, C_{\ell-1}\right)$$

can be chosen as a trivialization for $T_{M'}\big|\operatorname{im}\varphi'$. Thus the obstruction $\gamma(\varphi)$ is the class of the continuous map $\gamma(x)$ given by $\gamma(x) = \left\langle \frac{\partial\Phi'}{\partial\xi}, v\right\rangle(x)$, the matrix of v w.r.t the basis $\frac{\partial\Phi'}{\partial\xi}$. The obstruction $\gamma(\varphi_\alpha)$ is the homotopy class of the map $\gamma_\alpha(x) = \left\langle \frac{\partial\Phi'_\alpha}{\partial\xi}, v\right\rangle(x)$ where Φ'_α is defined similar to Φ' using φ'. It is easily seen that we have $\frac{\partial\Phi'\alpha}{\partial x_i} = \frac{\partial\Phi'}{\partial x_i} + \sum\limits_k \frac{\partial\Phi'}{\partial y_k} a_{ki}$ (for some a_{ki}) $\frac{\partial\Phi'_\alpha}{\partial y_j} = \sum\limits_k \frac{\partial\Phi'}{\partial y_k} A_{kj}$ where $(A_{kj}(x)) = \alpha(x)$. If, for every $0 \leq t \leq 1$ the frame $\left(\frac{\partial\Phi'_\alpha}{\partial\xi}\right)_t$ is defined by $\left(\frac{\partial\Phi'_\alpha}{\partial x_i}\right)_t = \frac{\partial\Phi'}{\partial x_i} + t\sum\limits_k \frac{\partial\Phi'}{\partial y_k}\, a_{ki}(i = 1, 2, \ldots q + 1)$ $\left(\frac{\partial\Phi'_\alpha}{\partial y_j}\right)_t = \frac{\partial\Phi'_\alpha}{\partial y_j}(j = 1, 2, \ldots n - q)$ and $(C_\mu)_t = C_\mu(= 1, 2, \ldots \ell - 1)$.

We see that $\gamma^t_\alpha(x) = \left\langle\left(\dfrac{\partial\Phi'_\alpha}{\partial\xi}\right)_t, v\right\rangle(x)$ gives a homotopy between the map $\gamma^0_\alpha(x) = \gamma(x).\ s(x)$ where $s : GL_+(n - q, \mathbb{R}) \to GL_+(n + \ell, \mathbb{R})$ is the inclusion and $\gamma^1_\alpha(x) = \gamma_\alpha(x)$. Thus the homotopy class $[\gamma_\alpha]$ is the same **30**
as $[\gamma] + s_*(\alpha)$. Thus is to say $\gamma(\varphi_\alpha) = \gamma(\varphi) + s_*(\alpha)$.

Perhaps we should have remarked earlier that while dealing with oriented bundles the trivializations are supposed to be those belonging to the orientation class. Since $s_* : \prod_q(SO(n - q)) \to \prod_q(SO(n + \ell))$ is surjective for $q < n - q$ we have the following: □

Proposition 3.7. *If $q < \dfrac{n}{2}$ $\exists$ a C^∞map $\alpha : S^q \to SO(n - q)$ such that $f'_\alpha : M'_\alpha = \chi(M, \varphi_\alpha) \to X$ satisfies $f'_\alpha!(\eta) \approx \tau^n_{M'_\alpha} \oplus \mathscr{O}^{\ell}_{M'_\alpha}$.*

Let now V be connected of dimension $n \leq 4$ and v^* some chosen base point in V. Choose some base point P^* in S^1 and let $\varphi : S^1 \times \dfrac{3}{2}D^{n-1} \to V$ be an orientation preserving imbedding such that $\varphi(p^*, 0) = v^*$ and $\varphi\big|S^1 \times 0$ represents $\lambda \in \prod_1(V, v^*)$. Let $V' = \chi(V, \varphi)$ and let $V_\circ$ and $\varphi' : D^2 \times S^{n-2} \to V'$ have their usual meanings i.e. $V_\circ = V - \varphi(S^1 \times B^{n-1})$ and φ' is the imbedding of $D^2 \times S^{n-2}$ into V' induced by the inclusion

of $D^2 \times S^{n-2}$ in $\dfrac{3}{2}B^2 \times S^{n-2}$. Choose some fixed $z^* \in S^{n-2}$ and choose $v'^* = \varphi(p^*, z^*) = \varphi'(p^*, z^*)$ as the base point of V'. Let σ be the path in
31 *V given by $\sigma(t) = \varphi(p^*, tz^*)$; it is a path joining v^* to v'^* in V and let $\sigma_* : \prod_1(V, v^*) \to \prod_1(V, v'^*)$ be the isomorphism induced by σ.*

Lemma 3.8. *Let $N(\lambda)$ be the normal subgroup of $\prod_1(V, v'^*)$ generated by $\sigma_*(\lambda)$. Then $\prod_1(V', v'^*)$ is isomorphic to $\prod_1(V, v'^*)/N(\lambda)$.*

Proof. Let $j_* : (V_\circ, v'^*) \to (V, v'^*)$ be the inclusion. We claim that $j_* : \prod_1(V_\circ, v'^*) \to \prod_1(V, v'^*)$ is an isomorphism.

In fact if $\Theta : (S^1, p^*) \to (V, v'^*)$ is any map and $\bar{\Theta} : (S^1, p^*) \to (V, v'^*)$ a map homotopic to Θ and transverse regular on $\varphi(S^1 \times 0)$ (such a map exists since $v'^* \notin \varphi(S^1 \times 0)$), since Codim $\varphi(S^1 \times 0)$ in V is ≥ 2 (actually Codim $\varphi(S^1 \times 0)$ in $V \geq 3$). We see that $\bar{\Theta}(S^1) \cap \varphi(S^1 \times 0) = \phi$. Choosing a deformation retraction $r : S^1 \times (D^{n-1} - 0) \to S^1 \times S^{n-2}$ we see that $r' = \varphi r \varphi^{-1} : \varphi(S^1 \times (D^{n-1} - 0)) \to \varphi(S^1 \times S^{n-2})$ is a deformation retraction and that $r'\bar{\Theta}$ is a map homotopic to $\bar{\Theta}$ and satisfying $r'\bar{\Theta}(S^1) \subset V_\circ$. Thus j_* is onto. Also if $\psi : (S^1, p^*) \to (V_\circ, v'^*)$ is a map such that $j\psi$ is homotopic to a constant map then $\exists$ an extension (also denoted by ψ) of ψ into a map $\psi : D^2 \to V$ with $\psi(0) = v'^*$. We can get a map $\bar{\psi}$ with $\bar{\psi}|S^1 \cup 0 + \psi|s^1 \cup 0$ and $\bar{\psi}$ transverse regular on $\varphi(S^1 \times 0)$. Since Codim of $\varphi(S^1 \times 0)$ in $V \geq 3$ we see that $\bar{\psi}(D^2) \cup \varphi(S^1 \times 0) = \phi$ and an argument
32 similar to the one above yields a homotopy of $\psi : (S^1, p^*) \to (V_\circ, v^*)$ with the constant map, taking place on $V_\circ$ itself. This show that j_* is a monomorphism.

We have $V' = V_\circ \cup \operatorname{im}\varphi'$ (as usual $\operatorname{im}\varphi' = \varphi'(D^2 \times S^{n-2})$) with $V_\circ \cap \operatorname{im}\varphi' = \varphi(S^1 \times S^{n-2}) = \varphi'(S^1 \times S^{n-2})$. Clearly $V_\circ$, $\operatorname{im}\varphi'$ and $V_\circ \cap \operatorname{im}\varphi'$ are connected. Lemma 3.8 follows immediately from Van Kampen theorem. also, clearly V' is connected.

As already remarked earlier by us Theorem 2.1 needs to be proved only when $n \geq 5$. We have already obtained a compact, connected, oriented C^∞ manifold M of dimension n and a map $f : M \to X$ of degree 1 with $f!(\eta) \simeq \tau^n_M \oplus \mathscr{O}^\ell_M$. (Refer Proposition 3.5.) ◻

Proposition 3.9. *There exists a connected simply connected manifold M' which is χ-equivalent to M and map $f' : M' \to X$ of degree 1 satisfying $f'!(\eta) \simeq \tau^n_{M'} \oplus \mathscr{O}^\ell_{M'}$.*

Proof. Choose some base point $m^* \in M$. We can without loss of generality assume that $f(m^*) = x^*$ for otherwise we can change f to a homotopic map satisfying this condition. Since M is a compact manifold $\prod_1(M, m^*)$ is finitely generated. Let $\lambda_1, \ldots, \lambda_r$ be generators for $\prod_1(M, m^*)$. We can get an imbedding $\varphi : S^1 \to M$ representing λ_1 (for this $n \geq 3$ is sufficient). Since M is oriented the normal bundle of φ in M is trivial and hence it can be extended into an orientation 33
preserving diffeomorphism $\varphi : S^1 \times \dfrac{3}{2}D^{n-1} \to M$. Since X is simply connected we have $f \circ \varphi$ homotopic to the constant map. By changing f if necessary to a homotopic map we can assume $f\varphi(S^1 \times \dfrac{3}{2}D^{n-1}) = x^*$. Now let $M'_\varphi = \chi(M, \varphi)$ and $f'_\varphi : M'_\varphi \to X$ be the map associated to f. By proposition 3.7 $\exists$ a C^∞ map $\alpha : S^1 \to SO(n-1)$ such that $f'_\alpha : M'_\alpha = M'_{\varphi_\alpha} = \chi(M, \varphi_\alpha) \to X$ satisfies $f'_\alpha!(\eta) \simeq \tau^n_{M'_\alpha} \oplus \mathscr{O}^\ell_{M'\alpha}$ and is of degree 1. The map $\varphi_\alpha | S^1 \times 0$ is the same as $\varphi | S^1 \times 0 = \varphi : S^1 \to M$. Hence $\varphi_\alpha \big| S^1$ represents the same element as φ i.e. λ_1. By Lemma 3.8 it follows that $\prod_1(M'_\alpha)$ is isomorphic to $\prod_1(M)/$ (Normal $s \cdot g$ generated by λ_1) and hence $\prod_1(M'_\alpha)$ is generated by $(r-1)$ elements. It now follows that after a finite number of surgeries we can get a connected, simply connected manifold M' and a map $f' : M' \to X$ satisfying the requirements of the proposition. $\square$

Remark. For applying lemma 3.8 we only nee that $\dim M = n \geq 4$. Moreover we have so far used only conditions i) and ii) of Theorem 2.1.

4 Effect of surgery on homology

Let A and B be any two connected, simply connected topological spaces and q an integer ≥ 2. Suppose $h : A \to B$ is a continuous map such that $h_* : H_i(A) \to H_i(B)$ is an isomorphism for $i < q$ and an epimorphism for $i = q$. Denote the Kernel of $h_q : H_q(A) \to H_q(B)$ by K_q.

Lemma 4.1. *Any $x \in K_q$ can be represented by a map $\Theta : S^q \to A$ (i.e.* 34
$\Theta_*(i_q) = x$ where i_q is a generator of $H_q(S^q)$) with $ho\Theta$ homotopic to a constant map.

Proof. Without loss of generality we can assume h to be an inclusion map, for otherwise, we replace h by the inclusion of A into the mapping cylinder of h. For the proof of Lemma 4.1 we use the Relative Hurewicz Theorem. Since $h_* : H_i(A) \to H_i(B)$ is an isomorphism for $i < q$ and an epimorphism for $i = q$ it follows from the exact homology sequence of the pair (B, A) that $H_i(B, A) = 0$ for $i \leq q$. Hence by the relative Hurewicz Theorem $\prod_i(B, A) = 0$ for $i \leq q$ and $\rho : \prod_{q+1}(B, A) \xrightarrow{\approx} H_{q+1}(B, A)$ where ρ is the Hurewicz homomorphism. Now consider the following diagram.

$$\begin{tikzcd}
\prod_{q+1}(B,A) \arrow[r, "\partial"] \arrow[d, "\rho", "\approx"'] & \prod_q(A) \arrow[r, "h_*"] \arrow[d, "\rho"] & \prod_q(B) \arrow[r] \arrow[d, "\rho"] & \prod_q(B,A) = 0 \arrow[d, "\rho"] \\
H_{q+1}(B,A) \arrow[r, "\partial"] & H_q(A) \arrow[r, "h_*"] & H_q(B) \arrow[r] & H_q(B,A) = 0
\end{tikzcd}$$

Diagram 4

The maps indicated by ρ are the Hurewicz homomorphisms. If $x \in K_q$ then $\exists\, y \in H_{q+1}(B, A)$ such that $\partial y = x$.

35 Let $y^1 \in \prod_{q+1}(B, A)$ be given by $\rho^{-1}(y)$. The element $z \in \prod_q(A)$ given by $z = \partial y^1$ satisfies $\rho(z) = x$ and $h_*(z) = h_*(\partial y^1) = 0$. Hence if $\Theta : S^q \to A$ represents $z \in \prod_q(A)$ then Θ satisfies the requirements of the Lemma. □

Lemma 4.2. *Suppose ν is a vector bundle of rank $(n - q)$ over S^q which is stably trivial. If $2q < n$ then ν itself is trivial.*

Proof. Let ν be determined by the element μ of $\prod_{q-1}(SO(n-q))$. Stable triviality of ν implies that $\exists$ an integer $r \geq n-q$ such that $s_*(\mu) = 0$ where $s_* : \prod_{q-1}(SO(n - q)) \to \prod_{q-1}(SO(r))$ is the homomorphism induced by the inclusion $SO(n - q) \to (SO(r))$. But if $2q < n$ the map s_* is an isomorphism. Hence $\mu = 0$.

Let V be a compact, connected, oriented C^∞ manifold with $\prod_1(V) = 0$ of dimension n and let B be any connected, simply connected space. Let $h : V \to B$ be a continuous map with $h_* : H_i(V) \to H_i(B)$ an isomorphism for $i < q$ and an epimorphism for $i = q$ where $q \geq 2$.

Further assume $\exists$ a vector bundle ζ on B with $[h:(\zeta)] = [\zeta_V]$. Denote the Kernel of h_q by K_q. $\square$

Lemma 4.3. *If $2q < n$ any $x \in K_q$ can be represented by a C^∞ imbedding $\varphi : S^q \to V$ whose normal bundle ν_φ is trivial and which further satisfies $h \circ \varphi \sim$ constant map.*

Proof. By Lemma 4.1 $\exists$ a map $\Theta : S^q \to V$ representing x such that $h \circ \Theta$ is homotopically trivial. If $2q < n$ $\exists$ a C^∞ imbedding $\varphi : S^q \to V$ with $\Theta \sim \varphi$. We have $\tau_V|\varphi(S^q) \simeq \tau_{\varphi(S^q)} \oplus \nu_\varphi$ where ν_φ is the normal bundle of the imbedding φ. Since $\tau_{\varphi(S^q)} \oplus \mathscr{O}_{\varphi(S^q)} \simeq \mathscr{O}^{q+1}_{\varphi(S^q)}$, we see that $[\tau_V|\varphi(S^q)] = [\nu_\varphi]$. But $[\tau_V|\varphi(S^q)] = [h!(\zeta)|\varphi(S^q)]$. Since $h \circ \varphi$ is homotopically trivial by construction we see that ν_φ is stably trivial. Now Lemma 4.2 yields that ν_φ itself is trivial. 36

Assume $2q < n$. Let $x \in K_q$ and let $\varphi : S^q \to V$ be a C^∞ imbedding representing x. Since the normal bundle ν_φ is trivial we can extend φ into a orientation preserving imbedding $\varphi : S^q \times \frac{3}{2}D^{n-q} \to V$. Since $h \circ \varphi$ is homotopic to the constant map, changing h in its homotopy class we may assume $h \circ \varphi =$ Const b^*. Let $V' = \chi(V, \varphi)$ and $h' : V' \to B$ the associated map i.e. to say $h'|V_\circ = h|V_\circ$ and $h'|\operatorname{im}\varphi' = b^*$ where $V_\circ, \operatorname{im}\varphi$ and $\operatorname{im}\varphi'$ have their customary meanings. $\square$

Proposition 4.4. *$h'_* : H_i(V') \to H_i(B)$ is an isomorphism for $i < q$ and the Kernel K'_q of $h'_q = H_q(V') \to H_q(B)$ is isomorphic to $K_q|(x)$, whenever $2q < n - 1$.*

Proof. Consider the following commutative diagram. 37

$$
\begin{tikzcd}
H_i(S^q \times D^{n-q}) \arrow[r, "\varphi_*"] & H_i(V) \arrow[r, "j_*"] \arrow[d, "h_*"] & H_i(V, \operatorname{im}\varphi) \arrow[r, "\partial"] \arrow[d, "e", "\approx"'] & H_{i-1}(S^q \times D^{n-q}) \\
 & H_i(B) & H_i(Vo, BdryVo) \arrow[l] & \\
H_i(D^{q+1} \times S^{n-q-1}) \arrow[r] & H_i(V') \arrow[r, "j'_*"'] \arrow[u, "h'_*"] & H_i(V', \operatorname{im}\varphi') \arrow[r, "\partial"] \arrow[u, "e'", "\approx"'] & H_{i-1}(D^{q+1} \times S^{n-q-1})
\end{tikzcd}
$$

Diagram 5

Since by assumption $2q < n-1$, whenever $1 \leq i \leq q$ we have $H_i(S^q \times D^{n-q}) = 0 = H_i(D^{q+1} \times S^{n-q-1})$ and hence

$j_*^{-1} \circ e^{-1} \circ e' j'_* = H_i(V') \to H_i(V)$ will then be an isomorphism satisfying commutativity in

$$\begin{tikzcd} H_i(V') \arrow[rr, "\approx"] \arrow[dr, "h'"'] & & H_i(V) \arrow[dl, "h_*"] \\ & H_i(B) & \end{tikzcd}$$

This shows that h'^* is an isomorphism for $i < q$. When $i = q$ Diagram 5 yields the following diagram. 38

$$\begin{tikzcd} \mathbb{Z}_1 \arrow[r, "\varphi_*", "\rightsquigarrow x"'] & H_q(V) \arrow[r, "j_*"] \arrow[d, "h_q"] & H_q(V, \operatorname{im}\varphi) \arrow[r] \arrow[d, "\approx"', "e"] & 0 \\ & H_q(B) & H_q(V_o,\ \text{Bdry}\ V_o) & \\ 0 \arrow[r] & H_q(V') \arrow[r, "j'_*"'] \arrow[u, "h'_q"] & H_q(V', \operatorname{im}\varphi') \arrow[r] \arrow[u, "\approx", "e'"'] & 0 \end{tikzcd}$$

Diagram 6.

The map φ_* is given by $\varphi_*(1) = x$. We get an isomorphism of $H_q(V))/(x)$ (induced by j_*) with $H_q(V, \operatorname{im}\varphi)$ and then we see that $\exists$ an isomorphism $H_q(V)/(x) \xrightarrow{\approx} H_q(V')$ making

$$\begin{tikzcd} H_q(V)/(x) \arrow[rr, "\approx"] \arrow[dr, "\text{got from}\ \ h_*"'] & & H_q(V') \arrow[dl, "h'_*\ \ \text{commutative}"] \\ & H_q(B) & \end{tikzcd}$$

This proves that $K'_q \approx K_q/(x)$.

Assuming conditions i) and ii) of Theorem 2.1 with $n \geq 4$ we have obtained a compact, connected oriented C^∞ manifold M of dimension n with $\prod_1(M) =$ and a map $f : M \to X$ of degree 1 satisfying $f!(\eta) \approx \tau_M^n \oplus \mathscr{O}_M^\ell$. 39 $\square$

Proposition 4.5. *There exists a connected, simply connected manifold M' which is χ equivalent to M and a map $f' = M' \to X$ of degree 1 such that $f'!(\eta) \approx \tau^n_{M'} \oplus \mathscr{O}^\ell_M$, and $f'_* : H_i(M') \to H)_i(X)$ an isomorphism for $i < \dfrac{n}{2}$.*

Proof. For $n = 4$ there is nothing to prove for $f : M \to X$ already satisfies the requirements of the proposition. Since M is compact the homology groups $H_i(M)$ are all finitely generated. For $n \geq 5$ Proposition 4.5 is a consequence of this fact, Lemma 4.3 and Propositions 4.4 and 3.7. □

Remark 4.5′. *If $f'_* : H_q(M') \to H_q(X)$ also is an isomorphism for $q = [\dfrac{n}{2}]$ then $f' : M' \to X$ will be a homotopy equivalence. To show this we have only to show that $f'_* : H_i(M') \to H_i(X)$ is an isomorphism for every i. As already proved (Lemma 2.5) the fact that f' is of degree 1 implies that $f'_* : H_i(M') \to H_i(X)$ is onto for every i. Let $a \in H_i(M')$ be such that $f'_*(a) = 0 (i > q)$. Let $\alpha = \triangle^{-1}(a) \in H^{n-1}(M')$. Since $i > q$ we have $n - i \leq q$. Since $f'_* : H_j(M') \to H_j(X)$ is an isomorphism for $j \leq q$ we have $f'^* : H^j(X) \to H^j(M')$ an isomorphism for $j \leq q$ by the Universal Coefficient Theorem. Hence α can be written as $f'^*(\beta)$ for a unique $\beta \in H^{n-i}(X)$. Then if $x = \beta \cap u \in H_i(X)$ by the definition of g given in Lemma 2.5, we have $g(x) = a$. But $H_i(M') = \ker f'_* \oplus g_i H_i(X)$ (direct sum). This implies $a = 0$ and hence f'_* an isomorphism for all i.*

Let A be any connected topological space satisfying Poincare duality with $u \in H_n(A) \simeq \mathbb{Z}$ as the fundamental class.

Definition 4.6. *Let $a \in H_i(A)$ and $b \in H_{n-i}(A)$. The homology intersection of a and b, denoted by $a.\ b$ is defined as follows: We identify $H_0(A)$ with $\mathbb{Z}$ with any element (i.e. pt) w of A as a generator. Let $\alpha = \Delta^{-1}(a)$ and $\beta = \Delta^{-1}(b)$ where Δ is the Poincare isomorphism. Then $\alpha \cup \beta \in H^n(A)$. The homology intersection $a.\ b$ is that integer which satisfies $(\alpha \cup \beta) \cap u = (a.b)w$. Because of (1) §1.2 we see that $a.\ b$ can also be defined as the value $(\alpha \cup \beta)[u]$ of $\alpha \cap \beta$ on the homology class u.*

Let V be a compact, connected, simply connected C^∞ manifold of dimension $n \geq 4$ and let $q = [\dfrac{n}{2}]$.

Lemma 4.7. *Let $a \in H_q(V)$ and suppose $\exists\ b \in H_{n-q}(V)$ such that $a.b = 1$. Suppose also that a is represented by an imbedding $\phi : S^q \times \frac{3}{2}D^{n-q} \to V$ (i.e. $\varphi|S^q \times 0$ represents a).*

Let $V' = \chi(V, \varphi)$. Then Rank $H_q(V') <$ Rank $H_q(V)$ and $H_i(V') \approx H_i(V)$ for $i < q$. 41

Proof. Let $V_\circ$, $\operatorname{im}\varphi$ and $\operatorname{im}\varphi'$ have their customary meanings. By excision and homotopy we have $H_i(V, V_\circ) \underset{\approx}{\overset{\varphi_*}{\longleftarrow}} H_i(S^q \times D^{n-q}, S^q \times S^{n-q})$.

Also

$$H_i(S^q \times D^{n-q}, S^q \times S^{n-q-1}) = \begin{cases} \mathbb{Z} \text{ if } i = n-q \text{ or } n \\ 0 \text{ otherwise} \end{cases}$$

□

From the homology exact sequence of the pair $(V, V_\circ)$ we see that $H_i(V_\circ) \xrightarrow{(i_\circ)_*} H_i(V)$ is an isomorphism whenever $i \neq n-q$ and n. (Here $i_\circ : V_\circ \to V$ denotes the inclusion). Also we have the following exact sequence:

$$0 \to H_{n-q}(V_\circ) \to H_{n-q}(V) \xrightarrow{j_*} H_{n-q}(V, V_\circ) \simeq \mathbb{Z} \xrightarrow{\partial} H_{n-q-1}(V) \to \cdots$$

The homomorphism j_*: $H_{n-q}(V) \to H_{n-q}(V, V_\circ)$ can more explicitly be described as follows. Identifying $H_{n-q}(V, V_\circ)$ with $H_{n-q}(S^q \times D^{n-q}, S^q \times S^{n-q-1})$ we see that $\varphi(x_0 \times D^{n-q})$ with x_0 some fixed base point in S^q, is a generator for the group $H_{n-q}(V, V_\circ) \simeq \mathbb{Z}$. Denoting this generator by 1 we have $j_*(y) = \pm a.\ y1$. In fact the intersection number of $\varphi(S^q \times 0)$ with $\varphi(x_0 \times D^{n-q})$ being clearly $\pm$ 1 we have $j_*(y) = \pm a.\ y1$.

The existence of an element $b \in H_{n-q}(V)$ with $a.b = 1$ ensures that $j_* : H_{n-q}(V) \to \mathbb{Z}$ is an epimorphism and hence we have the exact sequence 42

$$0 \to H_{n-q}(V_\circ) \to H_{n-q}(V) \xrightarrow{j_*} \mathbb{Z} \to 0.$$

In particular Rank $H_{n-q}(V_\circ) <$ Rank $H_{n-q}(V)$

We have $V' = V_\circ \cup D^{q+1} \times S^{n-q-1}$ with $V_\circ \cap D^{q+1} \times S^{n-q-1} = S^q \times S^{n-q-1}$. Letting $j_1 \colon S^q \times S^{n-q-1} \to D^{q+1} \times S^{n-q-1}$ and $i' = V_\circ \to V'$ denote the respective inclusions we have the Mayer-Vietais sequence.

$$H_i(S^q \times S^{n-q-1}) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_i(D^{q+1} \times S^{n-q-1}) \oplus H_i(V_\circ)$$
$$\xrightarrow{\varphi'_* + i'_*} H_i(V') \to H_{i-1}(S^q \times S^{n-q-1})$$

It follows that if $1 < i < n-q-1$ we have

$$H_i(V_\circ) \xrightarrow{i'_*} H_i(V').$$

Also if $i = 1$ and $i < n-q-1$ we have the exact sequence

$$0 \to 0 \oplus H_1(V_\circ) \xrightarrow{i'_*} H_1(V') \to \mathbb{Z} \xrightarrow{(-j_1)_* \oplus \varphi_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi'_* + i'_*} \mathbb{Z}$$

The map $(-j_1)_* \oplus \varphi_*$ carries $1 \in \mathbb{Z} = H_0(S^q \times S^{n-q-1})$ into $(-1, 1)$ of $\mathbb{Z} \oplus \mathbb{Z}$ and hence a monomorphism. Therefore $H_1(V_\circ) \xrightarrow{i'_*} H_1(V')$ is also an isomorphism in this case. Thus we see that if $i < n-q-1$ then
43 $H_i(V_\circ) \xrightarrow{i'_*} H_i(V')$ is an isomorphism. We now consider the two cases $n = 2q+1$ and $n = 2q$ separately.

Case (1)$n = 2q+1$. Then $q = n-q-1$. We have already proved that $H_i(V_\circ) \xrightarrow{(i_\circ)_*} H_i(V)$ is an isomorphism for $i \neq n-q$ and n. The Mayer-Victoris sequence for $i = q$ yields the exact sequence $H_q(S^q \times S^q) \xrightarrow{(-j)_* \oplus \varphi_*} H_q(D^{q+1} \times S^q) \oplus H_q(V_\circ) \to H_q(V') \to 0$. Writing $H_q(S^q \times S^q)$ as $\mathbb{Z} \oplus \mathbb{Z}$ we see that $(-j_1)_* \oplus \varphi_*$ carries $(1, 0)$ of $\mathbb{Z} \oplus \mathbb{Z}$ into $(i_{\circ *}^{-1}, (a)$ of $H_q(D^{q+1} \times S^q) \oplus H_q(V_\circ)$ and $(0, 1)$ into $(-1, 0)$. Since the intersection number $a \cdot b = 1$ we see that a has to be of infinite order and the above sequence now yields $H_q(V') \simeq H_q(V_\circ)/(a)$. Observing that $(i_\circ)_* : H_q(V_\circ) \to H_q(V)$ is an isomorphism we see that Rank $H_q(V') <$ Rank $H_q(V)$. Actually $H_q(V') \simeq H_q(V)/(a)$.

Case (2)$n = 2q$. As already verified $H_i(V_\circ) \xrightarrow{i'_*} H_i(V')$ is an isomorphism for $i < n-q-1 = q-1$. Also $H_i(V_\circ) \xrightarrow{(i_\circ)_*} H_i(V)$ is an isomorphism for $i \neq q$ and n. Combining these $H_i(V) \xrightarrow{i'_* \circ (i_\circ)_*^{-1}} H_i(V')$ is an isomorphism for $i < q-1$. For $i = q-1$ the Mayer-Victoris sequence yields the exact sequence

$$H_{q-1}(S^q \times S^{q-1}) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_{q-1}(D^{q+1} \times S^{q-1}) \oplus H_{q-1}(V_\circ) \\ \to H_{q-1}(V') \to 0.$$

But $H_{q-1}(S^q \times S^{q-1}) \simeq \mathbb{Z}, H_{q-1}(D^{q+1} \times S^{q-1}) \simeq \mathbb{Z}$ and the map **44**
$(-j_1)_* \oplus \varphi_*$ carries 1 of $H_{q-1}(S^q \times S^{q-1})$ into $(-1,0)$. Hence $i'_* : H_{q-1}(V_\circ) \to H_{q-1}(V')$ is an isomorphism. Since $(i_\circ)_* : H_{q-1}(V_\circ) \to H_{q-1}(V)$ is also an isomorphism we have $H_{q-1}(V) \xrightarrow{i'_* \cdot (i_\circ)_*^{-1}} H_{q-1}(V')$ an isomorphism. For $i = q$ the Mayer-Victoris sequence yields

$H_q(S^q \times S^{q-1}) \to 0 \oplus H_q(V_\circ) \to H_q(V') \to H_{q-1}(S^q \times S^{q-1}) \xrightarrow{\text{'mono'}} H_{q-1}(D^{q+1} \times S^{q-1}) \oplus H_{q-1}(V_\circ)$.

The map $H_{q-1}(S^q \times S^{q-1}) \xrightarrow{(-j_1)_* \oplus \varphi_*} H_{q-1}(D^{q+1} \times S^{q-1}) \oplus H_{q-1}(V_\circ)$ which carries the generator 1 of $H_{q-1}(S^q \times S^{q-1})$ into $(-1,0)$ is clearly a monomorphism. Hence $H_q(S^q \times S^{q-1}) \to H_q(V_\circ) \xrightarrow{i'_*} H_q(V') \to 0$ is exact. It follows that Rank $H_q(V') <$ Rank $H_q(V_\circ)$. The map composite $H_q(S^q \times S^{q-1}) \to H_q(V_\circ)$ carries the generator of $H_q(S^q \times S^{q-1})$ into 'a', an element of infinite order. As already verified Rank $H_q(V_\circ) <$ Rank $H_q(V)$ (since $q = n-q$, and we actually verified Rank $H_{n-q}(V_\circ) <$ Rank $H_{n-q}(V)$).

This completes the proof of Lemma 4.7

5 Proof of the main theorem for $n = 4d > 4$

We have already obtained a compact, connected, simply connected C^∞ **45**
manifold M of dimension $4d$ and a map $f\colon M \to X$ of degree 1 satisfying $f!(\eta) \simeq \tau_M^n \oplus \mathscr{O}_M^\ell$ and $f_* : H_i(M) \to H_i(X)$ an isomorphism $\forall i < 2d$. (Proposition 4.5).

Let $K_{2d} =$ Ker $f_{2d} : H_{2d}(M) \to H_{2d}(X)$.

Lemma 5.1. *K_{2d} is a free abelian group.*

Proof. Since $H_{2d}(M)$ is finitely generated and K_{2d} a direct summand of $H_{2d}(M)$ (Lemma 2.5) it follows that K_{2d} is finitely generated. To prove that K_{2d} is free it therefore suffices to prove that K_{2d} is torsion free. We write g for $2d$ for simplicity. If possible let $x \in K_q$ be any torsion element and let $x^1 \in H^q(M)$ correspond to x under Poincare duality i.e. $x^1 \cap [M] = x$. x^1 is then a torsion element of $H^q(M)$. By the Universal Coefficient Theorem for cohomology we have the following commutative diagram. □

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{q-1}(M),\mathbb{Z}) \arrow[r, "\beta"] & H^q(M) \arrow[r, "\alpha"] & \mathrm{Hom}(H_q(M),\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{q-1}(X),\mathbb{Z}) \arrow[r, "\beta"] \arrow[u, "\mathrm{Ext}(f_*,Id_{\mathbb{Z}})", "\approx"'] & H^q(X) \arrow[r, "\alpha"] \arrow[u, "f_*"] & \mathrm{Hom}(H_q(X),\mathbb{Z}) \arrow[r] \arrow[u, "\mathrm{Hom}(f_*,Id_{\mathbb{Z}})"'] & 0
\end{tikzcd}$$

Diagram 7

46 Clearly, $\mathrm{Hom}(H_q(M),\mathbb{Z})$ is torsion free. Also for any finitely generated abelian group A the group $\mathrm{Ext}(A,\mathbb{Z})$ is a torsion group. It follows that $\beta(\mathrm{Ext}(H_{q-1}(M),\mathbb{Z}))$ is precisely the torsion subgroup of $H^q(M)$. Hence $\exists$ an element $y^1 \in \mathrm{Ext}(H_{q-1}(M),\mathbb{Z})$ with $\beta(y^1) = x^1$. Since f_*: $H_i(M) \to H_i(X)$ is an isomorphism for $i \leq q-1$ we have

$$\mathrm{Ext}(f_*, Id_{\mathbb{Z}}) : \mathrm{Ext}(H_{q-1}(X),\mathbb{Z}) \to \mathrm{Ext}(H_{q-1}(M),\mathbb{Z})$$

an isomorphism. Let $z^1 \in H^q(X)$ be given by $z^1 = \beta \circ (\mathrm{Ext}(f_*, Id_{\mathbb{Z}})^{-1}(y'))$. Then clearly $f^*(z^1) = x^1$. Our aim is to show that K_q has no torsion, or that $x = 0$. For this it suffices to show that $x^1 = 0$ since $\cap[M] = \Delta : H^q(M) \to H_q(M)$ is an isomorphism. Now consider the element $z^1 \cap u \in H_q(X)$. Since f is of degree 1 we have $f_*([M]) = u$. We have

$$0 = f_*(x) = f_*(x^1 \cap [M]) = f_*(f^*(z^1) \cap [M]) = z^1 \cap f_*[M] = z^1 \cap u.$$

But by assumption $\cap u$: $H^q(X) \to H_q(X)$ is an isomorphism. Hence $z^1 = 0$ and therefore $x^1 = f^*(z^1) = 0$. This completes the proof of Lemma 5.1.

For the rest of §5 we denote $2d$ by q.

Let $H_q(M) = K_q \oplus gH_q(X)$ be the splitting given by Lemma 2.5

Lemma 5.2. *For any $a \in K_q$ and any $b \in gH_q(X)$ the intersection number $a \cdot b = 0$. Also if $b_1 = g(c_1)$ and $b_2 = g(c_2)$ with $c_1, c_2 \in H_q(X)$ then the intersection number $b_1 \cdot b_2$ is the same is $c_1 \cdot c_2$.*

Proof. Let $b = g(c)$ with $c \in H_q(X)$ (c is unique since g is a mono). 47
Let $\gamma \in H^q(X)$ be such that $\gamma \cap u = c$. Then by the very definition of g we have $b = f^*(\gamma) \cap [M]$. To prove that $a \cdot b = 0$ it suffices to verify that $f_*((\alpha \cup f^*(\gamma)) \cap [M]) = 0$ with $\alpha \in H^q(M)$ satisfying $\alpha \cap [M] = a$. Since $q = 2d$ we have $\alpha \cup f^*(\gamma) = f^*(\gamma) \cup \alpha$. Hence $f_*((\alpha \cup f^*\gamma) \cap [M]) = (-1)^{q \cdot q} f_*((f^*\gamma \cup \alpha)[M]) = f_*(f^*\gamma \cap (\alpha \cap [M]))$ (since $q = 2d$) $= f_*(f^*\gamma \cap a) = \gamma \cap f_*(a) = 0$ since $f_*(a) = 0$. Choosing γ_1, γ_2 in $H^q(X)$ with $\gamma_1 \cap u = c_1$, $\gamma_2 \cap u = c_2$ we have $b_1 = f^*(\gamma_1) \cap [M]$ and $b_2 = f^*(\gamma_2) \cap [M]$. Now

$$f_*((f^*\gamma_1 \cup f^*\gamma_2) \cap [M]) = f_*(f^*(\gamma_1 \cup \gamma_2) \cap [M]) = (\gamma_1 \cup \gamma_2) \cap f_*([M])$$
$$= (\gamma_1 \cup \gamma_2) \cap u.$$

From this the equality $b_1 \cdot b_2 = c_1 \cdot c_2$ follows.

Denoting by $T_q(M)$ and $T_q(X)$ respectively the torsion subgroup of $H_q(M)$ and $H_q(X)$ we have $H_q(M)/_{T_q(M)} \simeq K_q \oplus \dfrac{H_q(X)}{T_q(X)}$. (because of Lemma 5.1). Lemma 5.2 precisely states that we can find bases for K_q and $\dfrac{H_q(X)}{T_q(X)}$ such that the matrix A_M of the intersection bilinear form on $H_q(M)/T_q(M)$ take the form $\left(\begin{smallmatrix} A_K & 0 \\ 0 & A_X \end{smallmatrix}\right)$ where A_K and A_X are the matrices 48
of the form restricted to K_q and $H_q(X)/T_q(X)$. Also the lemma asserts that the restriction of the intersection bilinear form on $H_q(M)/T_q(M)$ to $H_q(X)/T_q(X)$ agrees with the intersection bilinear form on $H_q(X)/T_q(X)$ got from the fact that X satisfies Poincare duality. Since intersection by definition corresponds to cup-product under Poincare duality we see that the signature of A_M is the same as the index of the manifold $I(M)$ defined in 1.6 and similarly signature if A_X is $I(X)$. Let us denote the signature of A_K by $I(K)$. Then we have $I(X) + I(K) = I(M)$. □

Lemma 5.3. *$I(K)$ is zero.*

Proof. The assumption iii) of Theorem 2.1 is actually used in concluding that $I(K) = 0$. We have a map $f : M \to X$ of degree 1 with

$f!(\eta) = \tau^n_M \oplus \mathscr{O}^\ell_M$. Also $[\eta] = -[\xi]$. By Hirzobruch's Index Theorem $I(M) = \left\{L_d(p_1(\tau^n_M), \ldots, p_d(\tau^n_M))\right\}[M]$.

But $L_d(p_1(\tau^n_M), \ldots, p_d(\tau^n_M)) = L_d(p_1(f!(\eta)), \ldots, p_d(f!(\eta))$ (since $L_k(p_1(\lambda), \ldots, p_k(\lambda))$ for any vector bundle λ depends only on the stable class of λ). Hence

$$\begin{aligned} I(M) &= \{L_d(p_1(f!(\eta), \ldots, p_d(f!(\eta))\}[M] \\ &= \{L_d(p_1(\eta), \ldots, p_d(\eta))\}(f_*[M]) \\ &= \{L_d(\overline{p_1}(\xi), \ldots, \overline{p_1}(\xi))\}(u) \\ &= I(X) \text{ by assumption } (iii). \end{aligned}$$

This proves that $I(K) = 0$.

49 Denote the group $H^q(M)/T^q(M)$ (where $T^q(M)$ is the torsion of $H^q(M)$) By $B^q(M)$ and similarly the group $H_q(M)/T_q(M)$ by $B_q(M)$. Choosing any basis $x_1, \ldots, x_r$ for B^q we see that $y_i = x_i \cap [M]$ (actually $\cap[M] : H^q(M) \to H_q(M)$ gives a well determined isomorphism also denoted by $\cap M$ of $B^q(M)$ onto $B_q(M)$) form a basis for $B_q(M)$. Since $B^q(M) \simeq \hom(B_q(M), \mathbb{Z})$ we can get elements $y^1_i, \ldots, y^1_r$ in B^q such that $y^1_i(y_j) = \delta_{ij}$. The bilinear form $(x, y) \rightsquigarrow (x \cup y)[M]$ on B^q is easily seen to have determinant ± 1, for $(y^1_j \cup x_i)[M] = y^1_j(y_i) = \delta_{ij}$. It follows that A_M has determinant ± 1. Similarly A_X has determinant ± 1. It follows that A_K has determinant ± 1. $\square$

Lemma 5.4. *If B is a symmetric non-degenerate bilinear form on a finitely generated free abelian group H. with determinant ± 1 and if the signature of B is Zero then $\exists x \neq 0$ in H such that $B(x, x) = 0$.*

A proof of this can be found in [6]. As a corollary we see that if $K_q \neq 0 \exists$ an element $a \neq 0$ in K_q such that $a.\ a = 0$. Moreover we can choose 'a' to be indivisible in K_q. Then $K_q|(a)$ is free and hence we can find a basis of the form $a, b_2, \ldots, b_r$ for K_q. Since A_k has determinant ± 1 and $a.\ a = 0$ we cannot have $a \cdot b_j = 0\ \forall j$. If $j_1, \ldots, j_r$, are the indices in $(2, \ldots, r)$ with $a.b_j \neq 0$ then $\underset{i=1,\ldots r'}{g.c.d}(a.b_{j_i})$ has to be 1 for otherwise this greatest common divisor will divide determinant of A_K.

50 *Hence $\exists$ integers m_{j_i} such that $\sum\limits_{i=1}^{r'} m_{j_i}(a.b_{j_i}) = 1$. The element $b \in K_q$*

given by $b = \sum_{i=1}^{r'} m_{j_i}(b_{j_i})$ satisfies a. $b = 1$.

Lemma 5.5. *If $d > 1$ there exists an imbedding φ: $S^q \to M^{4d} (q = 2d)$ representing a and further satisfying $f \circ \varphi \sim \tilde{x}^*$ (where $\tilde{x}^*$ is the constant map $S^q \to x$ carrying the whole of S^q into x^*.)*

Proof. It is for the proof of this lemma that we need d to be 1. By Lemma 4.1 $\exists$ a continuous map $\Theta : S^q \to M$ representing 'a' and satisfying $f \circ \Theta \sim \tilde{x}^*$. We use the fact that M is simply connected. Also since M is of dimension $4d$ with d an integer > 1 it follows from Lemma 6 of [6] that $\exists$ a C^∞ imbedding $\varphi : S^q \to M$ with $\varphi \sim \Theta$. This proves Lemma 5.5. ◻

Remark. It is not true that a continuous map Θ: $S^2 \to V^4$ is homotopic to a C^∞ imbedding even if V^4 is a compact, simply connected C^∞ manifold (if dimension 4). An example is given by Kervaire and Milnor in [3].

Lemma 5.6. *For any C^∞ imbedding $\varphi : S^q \to M$ representing 'a' and satisfying $f \circ \varphi \sim \tilde{x}^*$ the normal bundle ν_φ is trivial.*

Proof. We have $\tau_M | \varphi(S^q) \simeq \tau^q_{\varphi(S^q)} \oplus \nu^q_\varphi$. Since M and S^q are orientable it follows that ν_φ in orientable. Also from $f!(\eta)|\varphi(S^q) \simeq (\tau^n_M \oplus \mathscr{O}^\ell_M)|\varphi(S^q)$, 51
we have

$$f!(\eta)|\varphi(S^q) \simeq \tau^q \varphi(S^q) \oplus \nu^q_\varphi \oplus \mathscr{O}^\ell_{\varphi(S^q)} \approx \tau_{\varphi(S^q)} \oplus \mathscr{O}_{\varphi(S^q)} \oplus \nu^q_\varphi \oplus \mathscr{O}^{\ell-1}_{\varphi(S^q)}$$
$$\simeq \mathscr{O}^{q+1}_{\varphi(S^q)} \oplus \nu_\varphi \oplus \mathscr{O}^{\ell-1}_{\varphi(S^q)} \simeq \nu_\varphi \oplus \mathscr{O}^{q+\ell}_{\varphi(S^q)}.$$

But since $f \circ \varphi \sim \tilde{x}^*$ we have $f!(\eta)|\varphi(S^q) \simeq \mathscr{O}^{2q+\ell}_{\varphi(S^q)}$.

Thus $\nu_\varphi \otimes \mathscr{O}^{q+\ell}_{\varphi(S^q)} \simeq \mathscr{O}^{2q+\ell}_{\varphi(S^q)}$. Thus ν_φ is stably trivial. If $\nu \in \Pi_{q-1}(SO_q)$ is the element corresponding to the bundle ν_φ on S^q we have $S_*(\nu) = 0$ where s_*: $\Pi_{q-1}(SO_q) \to \Pi_{q-1}(SO_{2q+\ell})$ is the homomorphism induced by the inclusion. Since $\Pi_{q-1}(SO_{q+1}) \to \Pi_{q-1}(SO_{2q+\ell})$ is an isomorphism it follows that $i_*(\nu) = 0$ where i_*: $\Pi_{q-1}(SO_q) \to \Pi_{q-1}(SO_{q+1})$

is induced by the inclusion. Since $SO_{q+1}/SO_q = S^q$ we have a fibration of SO_{q+1} by SO_q as the fibre and S^q as the base. Consider the corresponding exact sequence

$$\Pi_q(S^q) \xrightarrow{\partial} \Pi_{q-1}(SO_q) \xrightarrow{i_*} \Pi_{q-1}(SO_{q+1}).$$

∂ carries a generator of $\Pi_q(S^q)$ into the element τ of $\Pi_{q-1}(SO_q)$ corre-
52 sponding to the tangent bundle of S^q. Since $i_*(\nu) = 0$ it follows that $\nu - k\tau$ for some integer k. The map which assigns to an isomorphism class λ of an orientable vector bundle of rank q over S^q its Euler class $\chi(\lambda)$ defines a homomorphism χ: $\Pi_{q-1}(SO_q) \to H^q(S^q)$. For the tangent bundle τ of S^q the class $\chi(\tau)$ is known to be twice a generator of $H^q(S^q)$. (That $q = 2d$ is even, we use here). Thus the composition $\Pi_q(S^q) \xrightarrow{\partial} \Pi_{q-1}(SO_q) \xrightarrow{\chi} H^q(S^q)$ is a monomorphism and any element in the image of ∂ is zero if and only if its Euler class is zero. The Euler class of the normal bundle of the imbedding φ representing 'a' can be identified with $a \cdot a$ times a generator of $H^q(S^q)$. For, given a normal vector field with a finite number of zeros on $\varphi(S^q)$ we can deform $\varphi(S^q)$ along these vectors to obtain a new imbedding which intersects $\varphi(S^q)$ at only finitely many places. The multiplicity of each such intersection is equal to the index of the corresponding zero of the normal vector field. $\square$

Remark. A more 'formal' proof for the fact that $\chi(\nu_\varphi) = a \cdot a$ times a generator of $H^q(S^q)$ can be given as follows.

Denoting the imbedded manifold $\varphi(S^q)$ by S^q itself, let Φ: $H^i(S^q) \to H^{q+i}(T(\nu))$ be the Thom isomorphism. If $U = \Phi(1) \in H^q(T(\nu))$ then the Euler class of ν can be defined by $\chi(\nu) = \Phi^{-1}(\bigcup \cup \bigcup)$. [5]. Taking a tubular neighbourhood A of S^q in M and collapsing the exte-
53 rior of A to a point we get a map $C : M \to T(\nu)$. If $\gamma \in H^q(M)$ is the class which corresponds to 'a' under Poincare duality (i.e. $\gamma \cap [M] = a$) it is known that $C^*(\cup) = \gamma$ [9]. Hence $C^*(\bigcup \cup \bigcup) = \gamma \cup \gamma = a \cdot a[M]$ by

the definition of the intersection number. But from the diagram

$$\begin{tikzcd}
H^q(S^q) \arrow[r, "\Phi", "\approx"'] & H^{2q}(T(\nu)) \arrow[dl, "C^*"'] & H^{2q}(T(\nu), T(\nu) - S^q) \arrow[l, "\approx"'] \arrow[d, "\approx"', "\text{excision}"] \\
H^{2q}(M) & H^{2q}(M, M - S^q) \arrow[l, "j_*"] \arrow[r, "\text{excision}", "\approx"'] & H^{2q}(A, A - S^q)
\end{tikzcd}$$

We see that $H^{2q}(M, M - S^q) \simeq \mathbb{Z}$. Taking any *pt* $x \in S^q$ we have the triangle:

$$\begin{tikzcd}
\mathbb{Z} \simeq H^{2q}(M) & H^{2q}(M, M - S^q) \arrow[l, "j_*"'] \\
& H^{2q}(M, M - x) \arrow[ul, "\approx"] \arrow[u]
\end{tikzcd}$$

Hence $H^{2q}(M, M-x) \simeq \mathbb{Z}$ has to be a direct summand of $H^{2q}(M, M-S^q)$ which is also $\simeq \mathbb{Z}$. It follows that $j^* : H^{2q}(M, M - S^q) \approx H^{2q}(M)$. Examining the diagram again we see that $C^* = H^{2q}(T(\nu)) \approx H^{2q}(M)$. Hence $\bigcup \cup \bigcup = a \cdot a$ times a generator of $H^{2q}(T(\nu))$ and $\Phi^{-1}(\bigcup \cup \bigcup) = a \cdot a$ times a generator of $H^q(S^q)$.

We ar now almost at the end of the proof of Theorem 2.1 for the case $n = 4d$. Choosing an indivisible $a \neq 0$ in K_q with $a \cdot a = 0$ we saw that **54**
$\exists\ b \in K_q$ with $a \cdot b = 1$. The existence of such an 'a' was guaranteed by Lemma 5.4. From Lemma 5.5 and 5.6 we see that $\exists$ an orientation preserving imbedding $\varphi : S^q \times \frac{3}{2}D^q \to M$ with $f \circ \varphi \sim \tilde{x}^*$ and representing 'a'. Let now $M' = \chi(M, \varphi)$ and $f' : M' \to X$ the associated map which is constructed after altering f in its homotopy class so as to satisfy $f \circ \varphi = x^*$. By Lemma 3.3 f' is of degree 1. To get an isomorphism τ^n_M, $\oplus \mathscr{O}^\ell_M, \to f'!(\eta)$ we had an obstruction $\gamma \in \Pi_q(SO_{n+\ell})$ and when φ was replaced by φ_α given by $\varphi_\alpha(x, y) = \varphi(x, \alpha(x)y)$ with $\alpha : S^q \to SO_q$ a C^∞ map then the new obstruction γ_α satisfied the relation $\gamma_\alpha = \gamma + s_*(\alpha)$ where $S_* : \Pi_q(SO_q) \to \Pi_q(SO_{n+\ell})$ is the homomorphism induced by the inclusion. (Lemma 3.6). Since q is even the homomorphism

$\Pi_q(SO_q) \to \Pi_q(SO_{q+1})$ is onto [8]. Also $\Pi_q(SO_{q+1}) \to \Pi_q(SO_{n+\ell})$ is onto. Thus there exists an α such that $f'_\alpha : M' = \chi(M, \varphi_\alpha) \to X$ satisfies the condition $f'\alpha!(\eta) \simeq \tau^n_M, \oplus \mathscr{O}^\ell_M$, in addition to being of degree 1. Thus without loss of generality we can assume that f' itself was 'good' in the sense that $f'!(\eta) \simeq \tau^n_M, \mathscr{O}^\ell_M$. Denoting the inclusions of M_o in M and M' respectively by i and i' we have the following commutative diagram for every integer j.

$$\begin{tikzcd}
H_j(M) \arrow[dr, "f_*"] & \\
H_j(M_o) \arrow[u, "i_*"] \arrow[d, "i'_*"'] & H_j(X) \\
H_j(M') \arrow[ur, "f'_*"'] &
\end{tikzcd}$$

55 By Case 2 of Lemma 4.7 we have $i_* : H_j(M_o) \to H_j(M)$ and $i'_* : H_j(M_o) \to H_j(M')$ to be isomorphisms for $j < q$. Since $f_* : H_j(M) \to H_j(X)$ is an isomorphism for $j < q$ it follows that $f' : H_j(M') \to H_j(X)$ is an isomorphism for $j < q$. Also by the same lemma $RK\ H_q(M') < RKH_q(M)$. If K'_q denotes the Kernel of $f'_q = H_q(M') \to H_q(X)$ we have K'_q free and of rank $<$ rank of K_q. It follows that after a finite number of spherical modifications we can obtain a manifold M'' and a map $f'' : M'' \to X$ with deg $f'' = 1$, $f''!(\eta) \simeq \tau^n_{M''} \oplus \mathscr{O}^\ell_{M''}$ and $K''_q = \ker f''_q = 0$. It follows from the Remark 4.5$'$ that $f'' : M'' \to X$ is a homotopy equivalence. This completes the proof of the main theorem for $n = 4d > 4$.

6 Proof of the main theorem for $n = 2q + 1$

Throughout §6 we will assume $n = 2q+1$ with q an integer ≥ 2. Let $W =$
56 W^{2q+2} be a compact orientable topological manifold of dimension $2q+2$ with boundary bW. Let F be any fixed field. The semi-characteristic

$e^*(bW; F)$ of bW with respect to F is defined to the residue class $\sum_{i=0}^{q}$ Rank $H_i(bW; F)$ modulo 2. Let ρ_F be the rank of the bilinear pairing $H_{q+1}(W; F) \otimes H_{q+1}(W; F) \to F$ given by the intersection number and $e(W)$ the Euler characteristic of W,

Lemma 6.1. *We have* $e^*(bW; F) + e(W) \equiv \rho_F (\mod 2)$.

Proof. Consider the homology exact sequence of the pair (W, bW) with coefficients in F,

$$\begin{aligned} H_{q+1}(W; F) &\xrightarrow{j_*} H_{q+1}(W, bW; F) \\ &\xrightarrow{\partial} H_q(bW; F) \\ &\to \cdots \to H_0(W; bW; F) \to 0. \end{aligned}$$

□

By Poincare-Lefschetz duality if $z \in H_{q+1}(W, bW; F)$ is such that $x \cdot Z = 0\ \forall\ x \in H_{q+1}(W; F)$ then $Z = 0$. It follows from this remark and the relation $x \cdot y = x \cdot j_*(y)$ for any $x, y \in H_{q+1}(W; F)$ that $\ker j_*$ is precisely the nullity of the intersection bilinear form on $H_{q+1}(W; F)$. Hence

$$\begin{aligned} \rho_F &= \dim H_{q+1}(W; F) - \dim \ker j_* = \dim \operatorname{im} j_* = \dim \ker \partial \\ &= \dim H_{q+1}(W, bW; F) - \dim \operatorname{im} \partial \end{aligned}$$

Denoting the dimensions of $H_j(W : F)$ and $H_j(W, bW; F)$ by $b_j(W; F)$ and $b_j(W, bW; F)$ respectively we have

$$\rho_F = b_{q+1}(W, bW; F) - b_q(bW; F) + b_q(W; F) - b_q(W, bW; F) + \cdots .$$

But $b_i(W, bW; F) = b_{2q+2-i}(W; F)$ by Poincare-Lefschetz duality. 57
Thus $\rho_F \equiv e^*(bW; F) + e(W) (\mod 2)$.

Let V be a compact connected oriented C^∞ manifold of dimension $n = 2q + 1$ and let $a \in H_q(V)$ be any torsion element $\neq 0$. Suppose further $\varphi : S^q \times \frac{3}{2} D^{n-q} \to V$ is an orientation preserving imbedding representing the homology class 'a'. Let $V' = \chi(V, \varphi)$.

Lemma 6.2. *If q is even we have an exact sequence*

$$0 \to \mathbb{Z} \to H_q(V') \to H_q(V)/(a) \to 0$$

where (a) is the subgroup generated by a in $H_q(V)$

Proof. As usual let $V_\circ = V - \varphi(S^q \times B^{q+1})$ and let $\varphi' \colon D^{q+1} \times S^q \to V'$ be the imbedding induced by the inclusion of $D^{q+1} \times S^q$ in $\frac{3}{2}B^{q+1} \times S^q$. We then have the following commutative diagram with exact horizontal rows. $\square$

$$\begin{tikzcd}
\mathbb{Z} \simeq H_q(S^q \times D^{q+1}) \arrow[r, "\varphi_*"] & H_q(V) \arrow[r] & H_q(V, \varphi(S^q \times D^{q+1})) \arrow[r] & 0 \\
 & & H_q(V_\circ, \varphi(S^q \times S^q)) \arrow[u, "\approx"'] & \\
\mathbb{Z} \simeq H_q(D^{q+1} \times S^q) \arrow[r, "\varphi'_*"] & H_q(V') \arrow[r] & H_q(V', \varphi'(D^{q+1} \times S^q)) \arrow[r] & 0
\end{tikzcd}$$

Diagram 8

58 Since $\varphi_*(1) = a$ by assumption it follows that $H_q(V', \varphi'(D^{q+1} \times S^q))H_q(V, \varphi(S^q \times D^{q+1})) \simeq H_q(V)/(a)$. To prove Lemma 6.2 we have only to show that $\varphi'_* : \mathbb{Z} \to H_q(V')$ is a monomorphism. Since 'a' is a torsion element to show that φ'_* is a monomorphism we have only to prove that $b_q(V', \mathbb{Q}) \not\equiv b_q(V, \mathbb{Q})(\mod 2)$ where $b_q(V, \mathbb{Q})$ is the q^{th} Bettinumber of V i.e. the rank of $H_q(V, \mathbb{Q})$. Since $H_i(V) \simeq H_i(V, \varphi(S^q \times D^{q+1})) \simeq H_i(V_\circ, \varphi(S^q \times S^q)) \simeq H_i(V', \varphi'(S^{q+1} \times S^q)) \simeq H_i(V')$ for $i < q$ the statement $b_q(V', \mathbb{Q}) \not\equiv b_q(V, \mathbb{Q})(\mod 2)$ will follow if we show that $\sum\limits_{i=0}^{q} b_i(V', \mathbb{Q}) + \sum\limits_{i=0}^{q} b_i(V, \mathbb{Q}) \not\equiv 0(\mod 2)$.

Let $W = I \times V \bigcup\limits_{\varphi} D^{q+1} \times D^{q+1}$ be the topological manifold got as follows. We take the disjoint union of $I \times V$ and $D^{q+1} \times D^{q+1}$ and identify the points of $S^q \times D^{q+1}$ with their images under φ in $V \times 1$. Then W is a compact orientable manifold of dimension $2q + 2$ with boundary consisting of the disjoint union of V and V'. Hence by Lemma 6.1 we have $e^*(bW;\mathbb{Q}) + e(W) \equiv \rho(\mod 2)$ where ρ is the rank of the intersection bilinear pai ring $H_{q+1}(W,\mathbb{Q}) \times H_{q+1}(W,\mathbb{Q}) \to \mathbb{Q}$. Since q is even, this intersection bilinear pairing is skew symmetric and hence ρ is even. But

$$e^*(bW;\mathbb{Q}) \equiv \sum_{i=0}^{q} b_i(V',\mathbb{Q}) + \sum_{i=0}^{q} b_i(V,\mathbb{Q}) \ (\text{mod } 2).$$

Also W is of the same homotopy type as the space got from V by attaching D^{q+1} by means of $\varphi|S^q \times 0$ and hence $e(W) = e(V) + (-1)^{q+1}$. Since V is of dimension $2q + 1$ by Poincare duality we have $e(V) \equiv 0$ 59
$(\mod 2)$ and hence the relation $e^*(bW;\mathbb{Q}) + e(W) \equiv 0(\mod 2)$ yields $\sum\limits_{i=0}^{q} b_i(V'\mathbb{Q}) + \sum\limits_{i=0}^{q} b_i(V,\mathbb{Q}) + (-1)^{q+1} \equiv (\mod 2)$ or $\sum\limits_{i=0}^{q} b_i(V'\mathbb{Q}) + \sum\limits_{i=0}^{q} b_i(V,\mathbb{Q}) \not\equiv 0(\mod 2)$. This completes the proof of Lemma 6.2.

We now consider the case when q is odd. Let d be the order of 'a'. Since $a \neq 0$ and is a torsion element of $H_q(V)$, d is an integer > 1. Now suppose the imbedding φ: $S^q \times \frac{3}{2} D^{q+1} \to V$ representing 'a' is replaced by φ_α given by $\varphi_\alpha(x, y) = \varphi(x, \alpha(x).y)$ with $\alpha : S^q \to SO_{q+1}$ a C^∞ map satisfying $s_*(\alpha) = 0$ where $s_* : \Pi_q(SO_{q+1}) \to \Pi_q(SO_{2q+1+\ell})$ is the homomorphism induced by the inclusion $s : SO_{q+1} \to SO_{2q+1+\ell}$. Let y^* be a base point chosen once for all and let $j : SO_{q+1} \to S^q$ be the map given by $j(w) = wy^*$. (We consider y^* as a column vector in $\mathbb{R}^{q+1}$ and the matrix w operates on the right on y^*). We want to study the q^{th} homology of $V'_\alpha = \chi(V, \varphi_\alpha)$. Clearly the manifold $V_\circ = V - \varphi_\alpha(S^q \times B^{q+1})$ is independent of α and the meridian $\varphi_\alpha(y^* \times S^q)$ of the torus $\varphi_\alpha(S^q \times S^q) =$ Bdry $V_\circ$ as a point set does not depend on α, hence its homology class ε' in $H_q(V_\circ)$ does not depend on α. On the other hand the homology class ε_α of $\varphi_\alpha(S^q \times y^*)$ in $H_q(V_\circ)$ does depend on α. Let ε be the homology class of $\varphi(S^q \times y^*)$ in $H_q(V_\circ)$. Then we have

$$\varepsilon_\alpha = \varepsilon + j_*(\alpha)\varepsilon' \text{ where } j_* : \Pi_q(SO_{q+1}) \to \Pi_q(S^q) \simeq \mathbb{Z}$$

is the homomorphism induced by j. 60

We claim that $\exists$ an integer d'_α such that $d\varepsilon_\alpha = d'_\alpha \varepsilon'$ in $H_q(V_\circ)$. Actually in the homology exact sequence

$$\to H_{q+1}(V_\circ) \xrightarrow{i_*} H_{q+1}(V) \to H_{q+1}(V, V_\circ) \xrightarrow{\partial} H_q(V_\circ) \xrightarrow{i_*} H_q(V) \to$$

identifying $H_{q+1}(V, V_\circ)$ with $\mathbb{Z} \simeq H_{q+1}(S^q \times D^{q+1}, S^q \times S^q)$ by excision we saw that the homomorphism $H_{q+1}(V) \to H_{q+1}(V, V_\circ)$ was given by $x \rightsquigarrow \pm$ a.x(Refer to the proof of Lemma 4.7). Since 'a' is a torsion element we have $a.x = 0$ and hence

$$0 \to \mathbb{Z} \simeq H_{q+1}(V, V_\circ) \xrightarrow{\partial} H_q(V_\circ) \xrightarrow{i_*} H_q(V) \to \cdots$$

is exact. ∂ carries the generator $\varphi(y^* \times D^{q+1})$ of the relative group $H_{q+1}(V, V_\circ)$ into ε' in $H_q(V_\circ)$. The element $d\varepsilon$ of $H_q(V_\circ)$ gets mapped into $da = 0$ by i_* and hence $\exists$ an integer d' such that $d\varepsilon = d'\varepsilon'$. From $\varepsilon_\alpha = \varepsilon + j_*(\alpha)\varepsilon'$ we have $d\varepsilon_\alpha = d\varepsilon + dj_*(\alpha)\varepsilon' = (d' + dj_*(\alpha))\varepsilon'$. Thus $d'_\alpha = d' + dj_*(\alpha)$ satisfies the requirement $d\varepsilon_\alpha = d'_\alpha\varepsilon'$. Let a'_α be the element $(i'_\alpha)_*(\varepsilon') \in H_q(V'_\alpha)$ where $i'_\alpha : V_\circ \to v'_\alpha$ is the inclusion. Then from the exact sequence

$$H_{q+1}(V', V_\circ) \xrightarrow{\partial} H_q(V_\circ) \xrightarrow{(j'_\alpha)_*} H_q(V'_\alpha) \to 0$$

61 we see that $(i'_\alpha)_*(d_\alpha\varepsilon') = (i'_\alpha)_*(d\varepsilon_\alpha) = 0$ since ∂ carries the generator $\varphi'_\alpha(D^{q+1} \times y^*)$ of the relative group $H_{q+1}(V', V_\circ)$ into the element $\varepsilon_\alpha \in H_q(V_\circ)$ represented by $\varphi_\alpha(S^q \times y^*)$. It follows that a'_α is of order $|d'. + dj_*(\alpha)|$ with d' = the order of $a' \in H_q(V')$ represented by $\varphi'(y^* \times S^q)$.

Identifying the stable group $\Pi_q(SO_{2q+1+\ell})$ with $\Pi_q(SO_{q+2})$ there is an exact sequence associated with the fibration $SO_{q+2}/SO_{q+1} = S^{q+1}$:

$$\Pi_{q+1}(S^{q+1}) \xrightarrow{\partial} \Pi_q(SO_{q+1}) \xrightarrow{s_*} \Pi_q(SO_{q+2}).$$

The composition $\Pi_{q+1}(S^{q+1}) \xrightarrow{\partial} \Pi_q(SO_{q+1}) \xrightarrow{j_*} \Pi_q(S^q)$ (for q odd) carries a generator of $\Pi_{q+1}(S^{q+1})$ into twice a generator of $\Pi_q(S^q)$. It follows that $j_*(\alpha)$ with $\alpha \in \ker s_*$ can take any even value. (+ ve or − ve). Thus if d' is not divisible by d we can choose an $\alpha \in \operatorname{Ker} s_*$ such that the order $|d'_\alpha|$ of a'_α satisfies $|d'_\alpha| < d$. Thus we have proved the following

Lemma 6.3. *Let q be odd and > 1 and $\varphi : S^q \times \frac{3}{2}D^{q+1} \to V$ an orientation preserving imbedding representing a torsion element $a \in H_q(V)$ of order $d > 1$. Then the element $a' \in H_q(V')$ represented by $\varphi'(y^* \times S^q)$ is of finite order; moreover if d' is the order of a' and if* 62
d' is not divisible by d then $\exists$ an $\alpha \in \operatorname{Ker} s_*$ such that the element a'_α in $H_q(V'_\alpha) = H_q(\chi(V, \varphi_\alpha))$ represented by $\varphi'_\alpha(y^* \times S^q)$ has order strictly less than that of a in $H_q(V)$.

Next we deal with the case when d' is divisible by d. We recall the definition of linking numbers [Siefert-Threlfall [7]] Let $\lambda \in H_p(V)$ and $\mu \in H_{n-p-1}(V)$ be torsion classes in the respective groups. Associated with the coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{h} \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we have the exact homology sequence

$$\to H_{p+1}(V; \to \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} H_p(V) \xrightarrow{h_*} H_p(V; \mathbb{Q}) \to \cdots$$

(h is the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$). Since λ is a torsion element we have $h_*(\lambda) = 0$. Therefore $\exists \nu \in H_{p+1}(V; \mathbb{Q}/\mathbb{Z})$ such that $\partial(\nu) = \lambda$. The pairing $(\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ defined by multiplication gives an intersection pairing $H_{p+1}(V; \mathbb{Q}/\mathbb{Z}) \otimes H_{n-p-1}(V) \to \mathbb{Q}/\mathbb{Z}$. We denote this pairing by a dot $'.'$.

Definition 6.4. *The linking number $L(\lambda, \mu)$ is the rational number modulo 1 defined by $L(\lambda, \mu) = \nu.\mu$. This linking number is well-defined and satisfies the relation $L(\mu, \lambda) + (-1)^{p(n-p-1)} L(\lambda, \mu) = 0$ [Ref: Siefert-Threlfall [7]].*

Lemma 6.5. *$L(a, a) = \pm d'd(\mod 1)$. (This lemma is valid even if d'* 63
is not divisible by d. In fact when d' is divisible by d this lemma asserts that $L(a, a) = 0$).

Proof. We have $d\varepsilon - d'\varepsilon' = 0$ in $H_q(V_\circ)$. Therefore the cycle $d\varphi(S^q \times y^*) - d'\varphi'(y^* \times S^q)$ bounds a chain C in $V_\circ$. Let $C_1 = \varphi(y^* \times D^{q+1})$ be the cycle in $\varphi(S^k \times D^{k+1}) \subset V$ with boundary $\varphi(y^* \times S^q)$. The chain

$C + d'C_1$ has boundary $d\varphi(S^q \times y^*)$. Hence $\dfrac{C + d'C_1}{d}$ has boundary $\varphi(S^q \times y^*)$. Also $\varphi(S^q \times 0)$ represents the same class $a \in H_q(V)$ as $\varphi(S^q \times y^*)$. Taking the intersection of $\dfrac{C + d'C_1}{d}$ with $\varphi(S^k \times 0)$ we get $\pm d'/d$ since C is disjoint from $\varphi(S^k \times 0)$ and C_1 has intersection number ± 1 with $\varphi(S^k \times 0)$. Thus $L(a, a) = \pm d'/d(\mod 1)$. ◻

Lemma 6.6. *Let $V = V^{2q+1}$ be a compact oriented C^∞ manifold with $q > 1$ odd, and $f = V \to X$ a map of degree 1 satisfying the following conditions.*

(1) $f_* : H_i(V) \to H_i(X)$ is an isomorphism for $i < q$

(2) $k_q = \ker f_* : H_q(V) \to H_q(X)$ is a torsion group. Suppose further that $L(a, a) = 0\ \forall\ a \in k_q$. Then K_q is a direct sum of a finite number of copies of $\mathbb{Z}_2 = \mathbb{Z}/_{2\mathbb{Z}}$.

64 **Remark.** When stating this lemma we have a complex X satisfying the conditions of Theorem 2.1 in our mind. In particular X satisfies Poincare duality and it is only this that is needed for the validity of Lemma 6.6.

Proof. Since X satisfies Poincare duality for integer coefficients it follows that X satisfies Poincare duality for coefficients in any arbitrary commutative ring. Using the fact that f is of degree 1, monomorphisms $g_j : H_j(X) \to H_j(V)$ were constructed satisfying $H_j(V) = \ker f_j \oplus g_j(H_j(x))$ for every j [Lemma 2.5]. The same procedure can be adopted to define monomorphisms $g_{j,\wedge} : H_j(X, \wedge) \to H_j(V, \wedge)$ for any commutative coefficient ring and we still have $H_j(V, \wedge) = \ker f_{j,\wedge} \oplus g_{j,\wedge}(H_j(X, \wedge))$. Also the exact sequences in homology corresponding to the exact coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ give rise to a commutative diagram. ◻

$$\begin{tikzcd}
H_{q+1}(V, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial"] & H_q(V; \mathbb{Z}) \arrow[r, "h_*"] & H_q(V, \mathbb{Q}) \arrow[r] & {} \\
H_{q+1}(X, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[u, "g_q"'] & H_q(X, \mathbb{Z}) \arrow[r, "h_*"] \arrow[u, "g_q"'] & H_q(X, \mathbb{Q}) \arrow[r] \arrow[u, "g_q"'] & {}
\end{tikzcd}$$

Let $T_q(V)$ and $T_q(X)$ denote the torsion subgroups of $H_q(V)$ and $H_q(X)$ respectively. Then from assumption (2) we have $T_q(V) = K_q \oplus gT_q(X)$. For any $b, b^1 T_q(V)$ let $L(b, b^1)$ denote their linking number. Then since q is odd we have $L(b, b^1) = L(b^1, b)$. According to Poincaré duality theorem for torsion group [7, p. 245] L defines a non degenerate pairing $T_q(V) \otimes T_q(V) \to \mathbb{Q}/\mathbb{Z}$. We claim that $L|K_q \otimes K_q$ gives a non degenerate pairing $K_q \otimes K_q \to \mathbb{Q}/\mathbb{Z}$. Let $b \in K_q$ satisfy $L(b, b^1) = 0\ \forall\ b^1 \in K_q$. We have to show that $L(b, c) = 0 \forall c \in T_q(V)$. Since $T_q(V) = K_q \oplus gT_q(X)$ we have only to prove that $L(b, y) = 0\ \forall\ y \in gT_q(X)$. Let $y^1 \in T_q(X)$ be such that $g(y^1) = y$. Then $h_*(y^1) = 0$ (since y^1 is a torsion element) and therefore $\exists Z^1 \in H_{q+1}(X, \mathbb{Q}/\mathbb{Z})$ such that $\partial Z^1 = y^1$. The element $Z \in H_{q+1}(V, \mathbb{Q}/\mathbb{Z})$ given by $Z = g(Z^1)$ satisfies $\partial Z = y$. Now $L(b, y) = L(y, b) = Z.b$ (this intersection is the one corresponding to the pairing $(\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$). Thus we have only to verify $K_q.g(H_{q+1}(X, \mathbb{Q}/\mathbb{Z}) = 0$. This can be proved in a way similar to Lemma 5.2. Thus $L|K_q \otimes K_q \to \mathbb{Q}/\mathbb{Z}$ gives a nondegenerate pairing. 65

We now claim that every element $a \in K_q$ is of order 2. In fact for any $b \in K_q$ we have $0 = L(a + b, a + b) = L(a, b) + L(b, a) = L(2a, b)$. Hence $2a = 0$. This completes the proof of Lemma 6.6.

Lemma 6.7. *Let $f : V \to X$ be of degree 1 satisfying the following conditions.* 66

1) *$f_* : H_i(V) \to H_i(X)$ an isomorphism for every $i < q$*

2) *$K_q = \ker f_q : H_q(V) \to H_q(X)$ a direct sum of a finite number of copies of $\mathbb{Z}_2$ and that $\forall\ a \in K_q$ the linking number $L(a, a) = 0$.*

Suppose $\varphi : S^q \times \frac{3}{2} D^{q+1} \to V$ is an imbedding representing $a \neq 0$ in K_q. Then for the manifold $V' = \chi(V, \varphi)$ the Bettinumber $b_q(V'; \mathbb{Z}_2)$ (i.e. the dimension of $H_q(V'; \mathbb{Z}_2)$) satisfies $b_q(V'; \mathbb{Z}_2) \not\equiv b_q(V; \mathbb{Z}_2) (\bmod 2)$.

Proof. Let $W = I \times V \cup_\varphi D^{q+1} \times D^{q+1}$ as in the proof of Lemma 6.2. By Lemma 6.1 we have $e^*(V' : \mathbb{Z}_2) + e^*(V; \mathbb{Z}_2) + e(W) \equiv \rho(\bmod 2)$ where ρ is the rank of the intersection bilinear $H_{q+1}(W; \mathbb{Z}_2)$. If we show that ρ is even then as in the proof of Lemma 6.2 it will follow that $b_q(V'; \mathbb{Z}_2) \not\equiv b_q(V; \mathbb{Z}_2) (\bmod 2)$. Thus we have only to show that ρ

is even. If for every $x \in H_{q+1}(W;\mathbb{Z}_2)$ the intersection $x \cdot x$ is zero then ρ will be even. Thus we have only to show that $x{\cdot}x = 0 \;\forall\; x \in H_{q+1}(W;\mathbb{Z}_2)$. In the homology exact sequence for the pair (W, V) with $\mathbb{Z}_2$ coefficients

$$H_{q+1}(V;\mathbb{Z}_2) \xrightarrow{j_*} H_{q+1}(W;\mathbb{Z}_2) \to H_{q+1}(W,V;\mathbb{Z}_2) \xrightarrow{\partial}\to H_q(V;\mathbb{Z}_2)$$

67 the group $H_{q+1}(W,V;\mathbb{Z}_2) \simeq \mathbb{Z}_2$ with $\varphi(D^{q+1} \times y^*)$ as generator and ∂ carries it into $a \neq 0$ in $H_q(V;\mathbb{Z}_2)$. Actually if we use $\mathbb{Z}_2$ coefficients and take the kernel $K_q(\mathbb{Z}_2)$ of $f_* : H_q(V;\mathbb{Z}_2) \to H_q(X,\mathbb{Z}_2)$ it will be isomorphic to K_q since K_q is a direct sum of a finite number of copies of $\mathbb{Z}_2$ and $f_* : H_j(V) \to H_j(X)$ is an isomorphism for $j < q$. Hence $\partial : H_{q+1}(W,V;\mathbb{Z}_2) \to H_q(V;\mathbb{Z}_2)$ is a monomorphism and therefore $j_* : H_{q+1}(V;\mathbb{Z}_2) \to H_{q+1}(W;\mathbb{Z}_2)$ is onto. It is clear that $x.x = 0$ for elements of the form $x = j_*(y)$ with $y \in H_{q+1}(V;\mathbb{Z}_2)$ because a cycle representing y can be deformed in W so as not to intersect V. This completes the proof of Lemma 6.7. $\square$

Now we go to the proof of Theorem 2.1 when $n = 2q + 1$ with $q \geq 2$. We have already obtained a connected simply connected, compact oriented C^∞ manifold M of dimension n and a map $f : M \to X$ of degree 1 satisfying $f : (\eta) \simeq \tau^n_M \oplus \mathscr{O}^\ell_M$ and $f_* : H_j(M) \to H_j(X)$ isomorphism for $j < q$. Let K_q be the Kernel of $f_q : H_q(M) \to H_q(X)$. Let $K_q = F_q \mathscr{O} T(K_q)$ with F_q free and $T(K_q)$ the torsion subgroup of K_q. Choose an element 'a' forming part of a basis for F_q. As an easy consequence of Poincare duality we get an element $b \in H_{q+1}(M)$ such that $a.b = 1$. By Lemma 4.3 $\exists a C^\infty$ imbedding $\varphi : S^q \to M$ representing 'a' with trivial normal bundle ν_φ and further satisfying $f \circ \varphi \sim \tilde{x}^*$ (the constant map). Extending φ to an orientation preserving imbedding $\varphi : S^q \times \frac{3}{2} D^{q+1} \to$
68 M and performing survey we get a manifold $\chi(M,\varphi) = M'$ and a map $f' : M' \to X$ of degree 1 with $f'_* : H_j(M') \to H_j(X)$ isomorphisms for $j < q$ and $K'_q = \ker f'_q : H_q(M') \to H_q(X)$ isomorphic to $K_q/(a)$. (Refer to case (i) of Lemma 4.7). Changing φ to φ_α if necessary for a suitable C^∞ map $\alpha : S^q \to SO_{q+1}$ we may assume $f'!(\eta) \simeq \tau^n_{M'} \oplus \mathscr{O}^\ell_{M'}$ (Proposition 3.7). Applying surgery successively to 'kill' elements of a basis of F_q we get a connected, simply connected compact oriented

C^∞ manifold M'' and a map $f'' : M'' \to X$ of degree 1 satisfying the following conditions:

1) $f''_* : H_j(M'') \to H_j(X)$ is an isomorphism $\forall j < q$ and $K''_q = \ker f''_q : H_q(M'') \to H_q(X)$ is precisely the torsion subgroup of K_q.

2) $f''!(\eta) \simeq \tau^n_{M''} \oplus \mathscr{O}^\ell_{M''}$.

Thus changing notations we may assume that the original $f : M \to X$ itself satisfied the condition that K_q is a torsion group. Now assume q even. Choosing an element $a \neq 0$ in K_q and applying surgery to 'kill' a' (this is possible because of Lemma 4.3) we introduce an additional $\mathbb{Z}$ to the kernel, but the torsion subgroup of the Kernel becomes $K_q/(a)$. (Refer to Lemma 6.2) But by our earlier remarks we can successfully apply surgery to kill $\mathbb{Z}$. In other words by two suitable surgeries on M we can get a compact, oriented, connected, simply connected C^∞ **69**
manifold M' and a map $f^1 : M^1 \to X$ of degree 1 with $f' : (\eta) \approx \tau^n_{M'} \oplus \mathscr{O}^\ell_{M^1}, f^1_* : H_j(M) \to H_j(X)$ isomorphism for $j < q$ and $K'_q = \ker f'_q : H_q(M^1) \to H_q(X)$ definitely smaller than K_q. Iteration of this procedure a finite number of times proves Theorem 2.1 for $n = 2q + 1$ with q even.

We have still to consider the case q odd. If $a \neq 0$ in K_q is of order d when we perform surgery by means of an imbedding $\mathscr{O} : S^q \times \dfrac{3}{2} D^{q+1} \to M$ representing 'a' and get $f^1 : M^1 = \chi(M, \varphi) \to X$ we introduce a new element of finite order in the kernel of f^1. To get $f^1 : (\eta) \simeq \tau^n_{M^1} \oplus \mathscr{O}^\ell_{M^1}$ we may have to alter φ into φ_α for a suitable $\alpha : S^q \to SO_{q+1}$ and this can be done by Proposition 3.7. We can assume that φ itself satisfied this requirement also. However if we change again φ to φ_α with $\alpha \in \operatorname{Ker} s_*$ there is no obstruction to getting an isomorphism of $f^1_{\alpha !}(\eta)$ with $\tau^n_{M'_\alpha} \oplus \mathscr{O}^\ell_{M^1_\alpha}$. It is this freedom of choice of α in $\operatorname{Ker} s_*$ that helps in proving Theorem 2.1 for $n = 2q + 1$ with q odd > 1. If the order d^1 of $a^1 \in H_q(M^1)$ represented by $\varphi^1(y^* \times S^q)$ is not divisible by d then for a suitable $\alpha \in \operatorname{Ker} s_*$ the element $a^1_\alpha \in H_q(M^1_\alpha)$ will have order strictly less than d (Lemma 6.3). It follows now from Lemma 6.5 and **70**
6.6 that we can get a manifold M'' which is χ−-equivalent to M and a map $f'' : M'' \to X$ satisfying the following conditions.

1. M'' is connected, simply connected and f'' is of degree 1.

2. $f''_* : H_j(M'') \to H_j(X)$ is an isomorphism for $j < q$; the kernel K''_q of $f''_q : H_q(M'') \to H_q(X)$ is a direct sum of a finite number of copies of $\mathbb{Z}_2$.

3. $f''!(\eta) \simeq \tau^n_{M''} \oplus \mathscr{O}^\ell_{M''}$.

Lemma 6.7 coupled with the observations made above helps in getting a manifold M''' which is connected and simply connected and χ-equivalent to M'' and a map $f''' : M''' \to X$ with $f'''_* : H_j(M''') \to H_j(X)$ isomorphism for $j \quad q$ and $f'''!(\eta) \simeq \tau^n_{M'''} \oplus \mathscr{O}^\ell_{M'''}$. From the remark 4.5 it follows that f''' is a homotopy equivalence. This completes the proof of Theorem 2.1.

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Part II

Siebenmann's Theorem

1 The Assumption of simple-connectedness in the browder-novikov theorem

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In this section we will illustrate by examples that simple connectedness of X and condition (iii) are essential for the validity of Theorems 2.1 of Part I. We first construct a compact, connected combinational manifold Y of dimension 12 with $\pi_1(Y) = 0$ and satisfying condition (ii) of Theorem 2.1 which however is not of homotopy type of any close C^∞ manifold. Since Y is an orientable ($\pi_1(Y) = 0$) compact manifold condition (i) is automatically satisfied. This example thus illustrates that condition (iii) of theorem 2.1 (part I) is not redundant. Let k be any integer ≥ 1 and $\pi^k S^1$ the cartesian product of k copies of the circle. We will show that $X = Y \times \pi^k S^1$ satisfies condition (ii), and in case k is divisible by 4 satisfies condition (iii) as well. However form Siebenmann's Theorem (which will be stated later) it follows that X is not of the homotopy type of any closed C^∞ manifold.

1.1 The symmetric 8×8 matrix given below is a unimodular matrix of signature 8.

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Denote the (i, j) th entry of this matrix by C_{ij}. It is known that one 73
can choose C^∞ imbeddings $f_i : S^5 \times 0 \to bD^{12} = S^{11} (i = 1, \ldots, 8)$ with disjoint images such that the linking number $L(f_i(S^5 \times 0), f_j(S^5 \times 0))$ of $f_i(S^5 \times 0)$ and $f_j(S^5 \times 0)$ in bD^{12} for $i \neq j$ are C_{ij}. Moreover, for each i we can choose f_i so that $\equiv$ a differentiably imbedded disk $D_i'^6$ in D^{12} which bounds $f_i(S^5 \times 0)$. A tubular neighbourhood of $f_i(S^5)$ in bD^{12} can be got as the restriction of a tubular neighbourhood of $D_i'^6$ in D^{12}. In otherwords $\equiv$ C^∞ imbeddings $g_i : D^6 \times D^6 \to D^{12}$ such that

$g_i(S^5 \times D^6) \subset bD^{12}$, $g_i(S^5 \times 0 = f_i)$ and $D_i'^6 = g_i(D^6 \times 0)$. We can choose these g_i such that $g_i(S^5 \times D^6)$ are pair-wise disjoint in bD^{12}. Let $\wedge : S^5 \to S0_6$ be a C^∞ map representing the element $\partial i_6 \in \pi_5(S0_6)$ where $\partial i_6 \in \pi_6(S^6)$ is a generator and λ is the boundary homomorphism in the exact sequence $\pi_6(S^6) \to \pi_5(S0_6) \to \pi_5(S0_7)$ corresponding to the fibration $S0_7/S0_6 = S^6$. Let $\varphi_i : S^5 \times D^6 \to bD^{12}$ be defined by $\varphi_i(x, y) = g_i(x, \alpha(x)y)$. Let $D_i^6 \times D_i^6 (i = 1, \ldots 8)$ be eight disjoint copies of $D^6 \times D^6$ and let $S_i^5 \times D_i^6$ be the submanifold $S^5 \times D^6$ of $D_i^6 \times D_i^6$. Let $W^{12} = D^{12} + (\varphi_1^6) + \cdots + (\varphi_8^6)$ be the compact C^∞ manifold with boundary got from the disjoint union $D^{12} U(U_i D_i^6 \times D_i^6)$ by identifying points of $S_i^5 \times D_i^6$ with their images under φ_i and then rounding off the corners. We claim that W^{12} is a manifold with boundary, with $H_6(W^{12})$ free of rank 8 and having the given matrix as intersection matrix for a suitable choice of a basis for $H_6(W^{12})$. In W^{12} the image of $D_i^6 \times 0$ also a disk bounding
74 $f_i(S^5 \times 0)$ and $\sum_i^6 = D_i'^6 U(D_i^6 \times 0)$ is a differentiably imbedded sphere in W^{12} whose normal bundle corresponds to the element $\partial\iota_6 \in \pi_5(S0_6)$. The classes corresponding to $\sum_i^6$ form a basis for $H_6(W^{12})$ since the classes corresponding to $D_i^6 \times 0$ form a basis for $H_6(W^{12}, D^{12})$. The intersections of $\sum_i^6$ and $\sum_j^6$ in W^{12} are precisely those of $D_i'^6$ and $D_j'^6$ in D^{12} which by definition are the linking numbers $L(f_i(S^5 \times 0), f_j(S^5 \times 0))$. Hence $\sum_i^6 . \sum_j^6 = C_{ij}$ for $i \neq j$. Also if $k_* : \pi_5(S0_6) \to \pi_5(S^5)$ is the map induced by $\varphi \xrightarrow{k} x_\circ.\rho(x_\circ$ a fixed element in $S^5)$ of $S0_6$ in S^5 then it is known that $k_*\partial\iota_6 = \pm 2\iota_5$ (i_5 a generator for $\pi_5(S^5)$). Also $k_*(\partial\iota_6)$ is precisely the Euler class of the normal bundle of each $\sum_i^6$ in W^{12}, and this as we have seen already (Refer to proof of Lemma 5.6, Part I) is the self intersection $\sum_i^6 \cdot \sum_i^6$ times a generator of $\pi_5(S^5)$. Thus by proper choice of $\iota_6 \in \pi_6(S^6)$ we see that $\sum_i^6 \cdot \sum_i^6$ can be made equal to 2. Since the matrix we started with is a unimodular matrix it follows that the boundary ∂W is a homotopy sphere [12]. Hence by Smale [10] W is actually a combinatorial S^{11}. By attaching the cone over S^{11} to W by a PL-isomorphism we get a closed combinatorial manifold Y^{12}. Clearly W is 5-connected and since Y^{12} is got by attaching a 12-cell to W it follows that Y is also 5-connected and that $H_6(W) \simeq H_6(Y^{12})$ under the map induced by the inclusion $W \to Y$. It follows that Y is a

5-connected combinatorial manifold of dimension 12, having the given matrix as intersection matrix for a suitable choice of basis for $H_6(Y)$. **75**

Lemma 1.2. *Y is not homotopy type of any compact C^∞ manifold.*

Proof. For if Y were of the homotopy type a compact C^∞ manifold there should exist classes $p_i H^{4i}(Y;\mathbb{Z})(i = 1,2,3)$ such that $\{L_3(p_1, p_2, p_3)\}[Y] = \left\{\frac{1}{3^3.5.7}(62p_3 - 13p_2p_1 + 2p_1^3)\right\}[Y] = 8$. Since $H^4(Y;\mathbb{Z}) = 0$ and $H^8(Y;\mathbb{Z}) = 0$ the above implies that $\exists$ a class $p_3 \in H^{12}(Y;\mathbb{Z}) \simeq \mathbb{Z}$ such that $\frac{1}{3^3.5.7}62p_3[Y] = 8$. This in turn means the existence of an integer ℓ_3 such that $62\ell_3 = 3^3.5.7.8$. This is impossible since the prime 31 does not divide $3^3.5.7.8$. □

Lemma 1.3. *Let ξ be the tr ivial line bundle over Y. Then for the Thom space $T(\xi)$ of ξ the homology $H_{13}(T(\xi))$ has a spherical generator.*

(This observation is due to A. Vasqez.)

Proof. Y is a 5-connected polyhedron with $H_6(Y)$ free abelian of rank $8, H_{12}(Y) \simeq \mathbb{Z}; H_j(Y) = 0$ for all other $j \geq 1$. Thus a 'homology decomposition' [2] for Y will be $(S^6V \ldots VS^6)\underset{h}{U}E^{12}$ where the wedge is a 8 fold wedge and to it is attached a 12-cell by means of a map $h : S^{11} \to S^6 \ldots VS^6$ representing the so called k-invariant or the dual Postnikov invariant. The Thom space $T(\xi)$ of ξ is homotopy equivalent to the suspension $\sum(YU\, 'a')$ of the disjoint union of Y and a point 'a'. Hence $T(\xi) \sim S^1V(S^7V \ldots VS^7)\underset{g}{U}e^{13}$ (we use '$\sim$' to mean homotopy equivalence) where $g : S^{12} \to S^1VS^7V \ldots VS^7$ is some map. It is known that $\pi_{12}(S^7) = 0$ [4]. By a theorem of Hilton [3] **76**
it follows that $\pi_{12}(S^1VS^7V \ldots VS^7) = 0$. This shows that g is homotopically trivial and hence $T(\xi) \sim S^1V(S^7V \ldots VS^7)VS^{13}$. The inclusion of S^{13} in $S^1V(S^7V \ldots VS^7)VS^{13}$ followed by a homotopy equivalence $f : S^1V(S^7V \ldots VS^7)VS^{13} \to T(\xi)$ represents a generator of $H_{13}(T(\xi))$. □

Lemma 1.4. *Let V be a closed, connected, orientable combinatorial manifold satisfying condition (ii) of Theorem 2.1 (Part I). Then $V \times S^1$*

also satisfies condition (ii). If $\dim .V = 4d - 1$ *then* VS^1 *also satisfies condition (iii).*

Proof. Let $\dim V = n$ and let ξ^k be an orientable vector bundle of rank k on V with $H_{n+k}(T(\xi)) \simeq \mathbb{Z}$ with a spherical generator, say represented by the map $f : S^{n+k} \to T(\xi)$. Choose any orientable vector bundle η of rank ℓ over S^1 with a spherical generator for $H_{\ell+1}(T(\eta)) \simeq \mathbb{Z}$ represented by $g : S^{\ell+1} \to T(\eta)$. Such a bundle exists since S^1 is a C^∞ manifold. (In fact the trivial line bundle itself satisfies this condition). Let $\zeta \times \eta$ be the cartesian product bundle on $V \times S^1$. Choosing fixed Riemannian metrics for ζ and η denote the associated unit disk bundles by A_ξ and A_η and let $\dot{A}_\xi$ and $\dot{A}_\eta$ be the boundaries of A_ξ and A_η respectively. Then $T(\xi) = A_\xi/\dot{A}_\xi$ and $T(\eta) = A_\eta/\dot{A}_\eta$. For the bundle $\xi \times \eta$ with the cartesian product Riemannian metric we have $A_{\xi\times\eta} = A_\xi \times A_\eta$ and $\dot{A}_{\xi\times\eta} = A_\xi \times \dot{A}_\eta \cup \dot{A}_\xi \times A_\eta$. Choosing the respective points at ∞ as base points in $T(\xi)$ and $T(\eta)$ let $T(\xi)\#T(\eta) = \dfrac{T(\xi) \times T(\eta)}{T(\xi)VT(\eta)}$. The canonical projections $\varepsilon_\xi : A_\xi \to T(\xi)$
77 and $\varepsilon_\eta : A_\eta \to T(\eta)$ yield the map $\varepsilon_\xi \times \varepsilon_\eta : A_\xi \times A_\eta \to T(\xi) \times T(\eta)$. If $p : T(\xi) \times T(\eta) \to T(\xi)\#T(\eta)$ is the canonical map then $po(\varepsilon_\xi \times \varepsilon_\eta) : A_\xi \times A_\eta \to T(\xi/\#)Tm$ yields a $(1-1)$ onto map of $\dfrac{A_\xi \times A_\eta}{A_\xi \times \dot{A}_\eta \cup \dot{A}_\xi \times A_\eta} \to T(\xi)\#T(\eta)$. The compactness of the spaces involved shows that the map $T(\xi \times \eta) \to T(\xi)\#T(\eta)$ thus obtained is a homeomorphism. Clearly the map $f\#g : S^{n+k}\#S^{\ell+1} = S^{n+1+k+\ell} \to T(\xi)\#T(\eta)$ represents a generator of $H_{n+1+k+\ell}(T(\xi \times \eta))$. □

Suppose $n = 4d - 1$. Choose a basis $X_1, \dots, X_r$ for $H^{2d-1}(V; \mathbb{Q})$. By Poincare duality $\exists$ a basis $Y_1, ., Y_r$ for $H^{2d}(V; \mathbb{Q})$ such that $X_i.\ Y_j = \delta_{ij}$. Then for $H^{2d}(V \times S^1; \mathbb{Q})$ the elements $X_1 \otimes s, \dots, X_r \otimes s; Y_1 \otimes 1, \dots, Y_r \otimes 1$ where $s \in H^1(S^1, \mathbb{Q})$ is a generator form a basis. With respect to this basis the intersection matrix is $2d\left\{\overbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}^{2d}\right.$. Hence the signature of the manifold $V \times S^1$ is 0. Choosing η to be the trivial line bundle on S^1 we have $L_d(\bar{p}_1(\xi \times \eta), \dots, \bar{p}_d(\xi \times \eta))[V \times S^1] = L_d(\bar{p}_1(\xi) \otimes 1, \dots, \bar{p}_d(\xi) \otimes 1)[V \times S^1] = 0$.

It follows from Lemmas 1.3 and 1.4 that $X^{12+k} = Y^{12} \times \pi^k S^1$ satisfies conditions (i) and (ii) of Theorem 2.1 (Part I) and also (iii) in case $k \geq 1$ is divisible by 4. From Siebenmann's Theorem stated below and Lemma 1.2 it will follow that none of the manifolds $X^{12+k}(k \geq 0)$ is of the homotopy type of a compact C^∞ manifold.

Let π be any multiplicative group and $\mathbb{Z}(\pi)$ the group ring of π over $\mathbb{Z}$. Two finitely generated projective $\mathbb{Z}(\pi)$-modules P_1 and P_2 are said to be equivalent if $\exists$ finitely generated free $\mathbb{Z}(\pi)$ -modules F_1 and F_2 with **78**
$P_1 \oplus F_1 \simeq F_2 \oplus F_2$. The set of equivalence classes of finitely generate a projective modules is denoted by $\tilde{K}_\circ(\mathbb{Z}(\pi))$; it is an abelian group under the operation induced by the direct sum operation on projective modules.

Theorem 1.5 (Siebenmann)**.** *Let X be a finite complex such that $X \times S^1$ is of the homotopy type of a compact, connected, C^∞ manifold V^{n+1} without boundary of dimension $n+1$ with $n \geq 5$. Suppose $\mathbb{Z}(\pi)$ is Noetherian and that $\tilde{K}_\circ(\mathbb{Z}(\pi)) = 0$ where $\pi = \pi_1(X)$. Choosing a homotopy equivalence $\theta : V \to X \times S^1$ and denoting the projection onto the second factor $X \times S^1 \to S^1$ by p_2 let W be the covering of V got as the pull back of the covering $\mathbb{R} \xrightarrow{(Exp 2\pi i)} S^1$ by means of the map $p_2.\theta : V \to S^1$. Then W with the natural differential structure it acquires as a covering manifold of V, is diffeomorphic to $N^n \times \mathbb{R}$ with $N = N^n$ a compact C^∞ manifold without boundary, of dimension n.*

Remark. As W is of the homotopy type of $X \times \mathbb{R}$ or X it follows that X is of the homotopy type of N. If π is free abelian of rank $\ell < \infty$ we have $\mathbb{Z}(\pi)_{\simeq}\mathbb{Z}[x_1, \ldots, x, x_1^{-1}, \ldots, x_\ell^{-1}]$ where $x_1, x_2, \ldots x_\ell$ are ℓ-indeterminates over $\mathbb{Z}$ and in this case $\mathbb{Z}(\pi)$ is Noetherian and $\tilde{K}_\circ \mathbb{Z}(\pi)) = 0$. It is now clear that none of the manifolds $X^{12+k} = Y^{12} \times \pi^k S^1$ is of the homotopy type of any compact C^∞ manifold without boundary.

The theorem remains true if we drop the assumption that $\mathbb{Z}(\pi)$ is Noetherian. We give some more details on this in §3. The assumption $\tilde{K}_\circ \mathbb{Z}(\pi))$ is however essential. An example of a group with $\tilde{K}_\circ \mathbb{Z}(\pi)) \neq 0$ **79**
is the cyclic group or order 23. (See D.S. Rim [9]).

The rest of Part II deals with the Proof of Theorem 1.5. Let $f : V \to S^1$ pe a C^∞ approximation to $p_2 \circ \theta$ with $f \sim p_2 \circ \theta : V \to S^1$ (we use

'∼' to mean 'homotopic'). We denote the map $Exp(2\pi i) : \mathbb{R} \to S^1$ by q and let $p : W \to V$ denote the covering mapping. By definition W is the inverse image of the covering $q : \mathbb{R} \to S^1$ by means of the map $p_2 \circ \theta : V \to S^1$. Since $f \sim p_2 \circ \theta \exists$ a map $F : W \to \mathbb{R}$ making the following diagram commutative. Moreover F is C^∞.

$$\begin{tikzcd} W \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "q"] \\ V \arrow[r, "f"] & S^1 \end{tikzcd}$$

Diagram 1

By Sard's Theorem $\exists$ a regular value $a \in S^1$ for f and without loss of generality we can assume $1 \in S^1$ to be a regular value for f. Then any integer is a regular value of F.

2 The existence of arbitrary small 0 and 1-Neighbourhood of '∞' and '$-\infty$'

Definition 2.1. *A C^∞ sub-manifold $M = M^{n+1}$ of dimension $n+1$ with boundary bM, of W is said to be a 0-nbd of ∞ (respy "$-\infty$") if*

(1) M is a closed subset of W

(2) $\exists$ integers $m_1 < m_2$ with $F^{-1}[m_1, \infty) \supset M \supset F^{-1}[M_2, \infty)$
$\left\{ \text{respy } F^{-1}(-\infty, M_1] \subset M \subset F^{-1}(-\infty, M_2] \right\}$

80 *(3) bM is compact; M and bM are connected.*

M is said to be a $1-$nbd of ∞ (respy "$-\infty$") if it is already a $0-$nbd of ∞ (respy "$-\infty$") and the maps $\pi_1(b, M) \to \pi_1(M)$, $\pi_1(M) \to \pi_1(W)$ induced by the respective inclusions are isomorphisms.

Definition 2.2. *By the statement "arbitrary small 0 (or 1)-nbds of ∞ (respy $-\infty$)" we mean that given any compact set $K \subset W$ $\exists$ a 0 (or 1) $-$nbd M of ∞ (respy $-\infty$) with $M \subset W - K$.*

Let J denote an infinite cyclic group and let x be a generator of J. The Deck transformation group of the covering $\mathbb{R} \xrightarrow{q} S^1$ can be identified with J with x acting as the homeomorphism $r \to r + 1$ of $\mathbb{R}$ onto itself. Since $W \xrightarrow{p} V$ is the pull back of the covering space $\mathbb{R} \xrightarrow{q} S^1$ the Deck transformation group of the covering $W \xrightarrow{p} V$ is also J and we denote the homeomorphism of W which corresponds to the generator x by α.

Lemma 2.3. *Let σ be any are in V and $w_\circ \in W$ any point with $p(w_\circ) = \sigma(0)$. Let $\tau^{w_\circ}$ be the unique lift of σ such that $\tau^{w_\circ}(0) = w_\circ$.The variation* $\mathrm{Max}_{t,t' \in [0,1]} |F\tau^{w_\circ}(t) - - - - F\tau^{w_\circ}(t')|$ *of F on $\tau^{w_\circ}$ depends only on σ and not on the lift $w_\circ$ of $\sigma(0)$.*

This quantity which depends only on σ we refer to as the "variation of F on σ" and denote it by $V_F(\sigma)$.

Proof. Suppose $w'_\circ$, is any other element of W with $p(w'_\circ) = \sigma(0)$, then $w'_\circ = \alpha^k w_\circ$, for some integer k. The unique lift $\tau^{w'_\circ}$ of such that $\tau^{w'_\circ} = w'_\circ$ is given by $\tau^{w'_\circ}(t) = \alpha^k \tau^{w_\circ}(t)$. Because of the commutativity of diagram 1 we have **81**

$$F\tau^{w'_\circ}(t) = k + F\tau^{w_\circ}(t)$$

for all $t \in [0, 1]$. The lemma follows. □

Lemma 2.4. *There exists a constant $C > 0$ such that any two points of V can be joined by means of an are σ such that the variation $V_F(\sigma)$ of F on σ is less than C.*

Proof. For any $v \in V$ $\exists$ an arcwise connected open *ndb* U_v of v in V such that $p^{-1}(U)_v$ decomposes into a disjoint union of open sets $\{W_v^j\}$ each of which gets mapped homeomorphically onto U_v by the restriction of p. We can choose another arcwise connected open set U'_v containing v such that $\overline{U'_v} \subset U_v$. Then each of the sets

$W_v'^j = W_v^j \cap p^{-1}(U'_v)$ gets mapped homeomorphically by p onto U'_v and $W_v'^j = p^{-1}(U'_v) \cap W_v^j$ is compact since $\overline{U'}$ is compact, being a closed

subset of the compact space V. The argument used in lemma 2.3 can be used to show that $\underset{w,w^1\in\bar{w}'j}{\text{Max}}|F(w)-F(w^1)|$ is finite and depends only on U' (finiteness being a consequence of the compactness of $\bar{W'}^{j}$). We may call the above quantity the variation of F on U' or $\bar{U}'$. Compactness of V implies the existence of a finite number of sets $U'_{v_1},\ldots,U'_{v_r}$ covering V. Writing U'_i for U'_{v_i} and denoting the variation of F on U'_i by C_i let C be any constant $> C_1+\cdots+C_r$. Then C satisfies the requirement of the Lemma. For if $v_\circ, v_1$ are any two points of V, since V is arcwise connected we can find distinct indices $j_1,\ldots,j_s$, in $1,2,\ldots,r$ such that
82 $v_\circ \in U'_{j_1}$ and $v_1 \in U'_{j_\ell}$ and $U'_{j_\mu} \cap U'_{j_{\mu+1}} \neq \not\phi$. Choosing point $v'_\mu U'_{\mu+1}$ and joining $v_\circ to v'_1$ by an arc in $U_{j_1}; v'_1$ to v'_2 by an arc in U'_{j_2} and so on we get an are σ joining $v_\circ$ to v_1 such that $V_F(\sigma)\leq C_{j_1}+..+C_j<C$. ◻

Lemma 2.5. *a constant $\alpha>0$ with the following property: For every $v\in V$ $\exists$ a loop θ_v at v in V such that the loop $f\theta_v$ represents the positive generator of $\pi_1(S^1, f(v))$, and $V_F(\theta)<d$.*

Proof. Choose a point $v_\circ\in V$ and any loop $\theta_{v_\circ}$ at $v_\circ$ such that $f\theta_{v_\circ}$ represents the positive generator of $\pi_1(S^1, f(v_\circ))$. Let e be the variation of F on $\theta_{v_\circ}$ and $C>0$ the constant of Lemma 2.4. Then $d=2C+e$ satisfies the requirement of Lemma 2.5. For given any $v\in V\exists$ a path σ^v in V such that $\sigma^v(0)=v$, $\sigma^v(1)=v_\circ$ and $V_F(\sigma^v)<C$. If we define θ_v for any $v\neq v_\circ$ by $\theta_v=\sigma^v\theta_{v_\circ}(\sigma)^{v-1}$ then clearly $f\theta_v$ represents the positive generator of $\pi_1((S^1), f(v))$ and $V_F(\theta)_v<C+e+C=2C+e=d$. ◻

According to our choice of d we have $d>c$.

Lemma 2.6. *Let w be any element of $F^{-1}[\ell+d,\infty)$ with ℓ any real number and $v=p(w)$. For any integer $k\geq 0$ let τ_k be the unique lift of θ_v^k satisfying $\tau_k(0)=w$. Then the path τ_k lies in $F^-1[\ell,\infty)$ and $F(\tau_k(1))=k+F(w)$.*

Proof. The $F(\tau_k(1))=k+F(w)$ follows from the fact that $f\circ\theta_v^k$ represents the element k. (+ ve generator) of $\pi_1(S^1 f(v))$. The τ_k lies in
83 $F^{-1}[\ell,\infty)$ is proved by induction on k. For $k=0$ there is nothing to prove. Assume $k\geq 1$ and the lemma valid for $(k-1)$ instead of k. Let μ be the lift of θ_v with initial point $\mu(0)=\tau_{k-1}(1)$. Then

$F_\mu(0) = (k-1) + F(w) \geq (k-1) + \ell + d$. Since the variation of F on $\theta_v < d$ we have $F\mu(t) \geq (k-1) + \ell \forall t \in [0,1]$. Since $k \geq 1$ this implies $F\mu(t) \geq \ell$. Now τ_k is precisely the product $\tau_{k-1}.\mu$ and whenever $t \leq \dfrac{1}{2}$, $F\tau_k(t) = F\tau_{k-1}(2t) \geq \ell$ (by induction hypothesis) and if $t \leq \dfrac{1}{2}$, $F\tau_k(t) = F\mu(2t-1) \geq (k-1) + \ell$ (by what is proved above). This shows that τ_k lies in $F^{-1}[\ell, \infty)$. ◻

Proposition 2.7. *There exist arbitrary small 0-neighbourhoods of '∞' (resp. $-\infty$) in W.*

Proof. We prove the assertion for ∞, the proof for "$-\infty$" being similar is left out. Let K be any compact subset of W. $\exists$ an integer ℓ such that $F^{-1}[\ell, \infty) \subset W - K$. Since ℓ is a regular value of F we see that $F^{-1}[\ell, \infty)$ is a C^∞ submanifold of W, with boundary $F^{-1}(\ell)$. Let d be the constant of Lemma 2.5 (which as commented earlier has been chosen to be $> C$ the constant of Lemma 2.4) ◻

Claim: Any two points $w_\circ, w_1$ of $F^{-1}[\ell + 2d, \infty)$ can be joined by means of a path in $F^{-1}[\ell, \infty)$.

Let $p(w_\circ) = v_\circ$, $p(w_1) = v_1$. By Lemma 2.4 $\exists$ an arc σ in V such that $\sigma(0) = v_\circ$, $\sigma(1) = v_1$ and $V_F(\sigma) < C$. Let τ be the unique lift of σ with initial point $\tau(0) = w_\circ$. The $\tau(1)$ and w_1 are points on the same fibre of W and hence $F(w_1) = k + F(\tau(1))$ for a certain integer k. It follows that $\sigma^1 = \theta^k_{v_\circ}$. σ is a path joining $v_\circ$ to v_1 in V whose lift τ^1 with initial point $\tau^1(0) = w_\circ$ satisfies $\tau^1(1) = w_1$. We now consider the cases 84
$k \geq 0$ and $k < 0$ separately. Case (i) $k \geq 0$. Since $V_F(\sigma) < C < d$ and $F(\tau(0)) = F(w_\circ) \geq \ell + 2d$ it follows that $F(\tau(t)) > \ell + d$. From Lemma 2.6 we now have $F(\tau^1(t)) \geq \ell \forall t \in [0,1]$. Case (ii) $k < 0$. The path $(\tau^1)^{-1}$ is the composition $(\tau_{-k}).\ \tau^{-1}$ where τ_{-k} is the lift of $\theta^k_{v_\circ}$ having as initial point $\tau_{-k}(0) = w_1$. Now, by assumption $F(w_1) \geq \ell + 2d$ and $-k > 0$. From Lemma 2.6 we see that τ_{-k} is an arc in $F^{-1}[\ell, \infty)$. Since τ (and hence τ^{-1} also) is an arc in $F^{-1}[\ell + d, \infty)$ we see that $(\tau^1)^{-1} = \tau_{-k}.\tau^{-1}$ is an arc in $F^{-1}[\ell, \infty)$ and hence τ^1 too is an $F^{-1}[\ell, \infty)$.

This completes the proof of the claim. Now it is clear that $F^{-1}[\ell, \infty)$ has only one non-compact connected component say M' and a finite number of compact connected components. Since $M' \supset F^{-1}[\ell + 2d, \infty)$

it follows that the boundary bM' of M' lies in $F^{-1}[\ell,\infty)-F^{-1}(\ell+2d,\infty)$ and is therefore compact. If bM' were connected then M' itself would be a $0-nbd$ of ∞. Suppose bM' is not connected. Choosing a smooth path in M' from one component of bM' to another meeting bM' orthogonally and only at the end points and removing the interior of a tubular neighbourhood of the path we get a connected C^∞ submanifold M'' of W with

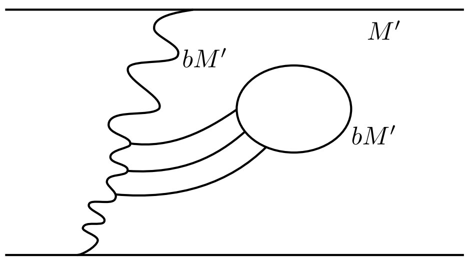


Diagram 2

85 bM'' compact and bM'' having one component less than bM'. Refer to Diagram 2. Since there are only a finite number of components after a finite number of such operations we get a connected C^∞ submanifold M of W with bM compact and connected. Further $M \supset F^{-1}[m,\infty)$ for some integer m since the original M' contained $F^{-1}[\ell+2d,\infty)$. Thus M is a $0-nbd$ of ∞.

Lemma 2.8. *Let M^{n+1} be a C^∞ submanifold of W^{n+1} with boundary $bM = N$ and let M and N be connected. Let M be a closed subset of W. Suppose the homomorphism $\pi_1(N) \to \pi_1(W)$ induced by the inclusion is an isomorphism. This $\pi_1(M) \to \pi_1(W)$ induced by the inclusion of M in W is also an isomorphism.*

Proof. Let $i : N \to M$ and $j : M \to W$ be the respective inclusions. Then $j \circ i : N \to W$ induced an isomorphism $(j \circ i)_* : \pi_1(N) \to \pi_1(W)$ by our hypothesis. Since $(joi)_* = j_* \circ i_*$ it follows that $j_* : \pi_1(M) \to \pi_1(W)$ is an epimorphism. To show that $j_* : \pi_1(M) \to \pi_1(W)$ is an isomorphism it therefore suffices to prove that j_* is a monomorphism. Since $\dim M = n+1$ and $n \geq 5$ any element of $\pi_1(M)$ can be represented by a C^∞ imbedding $\varphi : S^1 \to \operatorname{Int} M$ (in fact for this assertion to be valid

it suffices that $n + 1 \geq 3$). Suppose $\alpha \in \pi_1(M)$ is such that $j_*(\alpha) = 0$ and suppose $\varphi : S^1 \to \text{Int } M$ represents α. From $j_*(\alpha) = 0$ it follows that $\exists$ a map $h : D^2 \to W$ extending φ. Since $\phi(S^1) \cap N = \phi$ we can approximate h by a C^∞ map $\theta : D^2 \to W$ such that $\theta/S^1 = \varphi$ and θ is **86**
transverse regular on N. Then $D^2 \cap \theta^{-1}(N)$ consists of a finite number of disjoint simple closed curves (each one of them is a C^∞ imbedded S^1) in the interior of D^2. Take an inner most curve C. Now $\theta | C \to W$ admits of an extension $\theta : \triangle \to W$ where $\triangle$ is the closed region (inner most) bounded by C. Thus $\theta | C$ represents the trivial elements of $\pi_1(W)$ and $\theta(C) \subset N$. Since $\pi_1(N) \to \pi_1(W)$ is an isomorphism it follows that $\exists$ a map $\lambda : \triangle \to N$ with $\lambda | C = \theta | C$. (Refer to diagram 3). Now using the fact that N is collared in M it is easy to get a map $\theta' : D^2 \to W$ with the following properties:

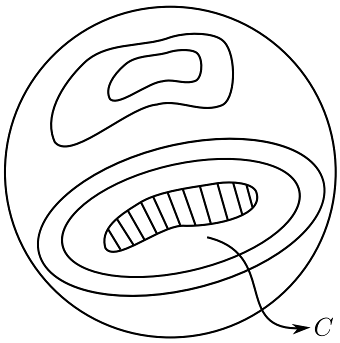


Diagram 3

(1) $\theta' | S^1 = \varphi$

(2) $\exists$ a *nbdA* of $\triangle$ in D^2 with A disjoint from the curves of $\theta^{-1}(N) \cap D^2$ different from C such that $\theta'(A) \cap = \phi$ and $\theta' | D^2 - A = \theta | D^2 - A$.

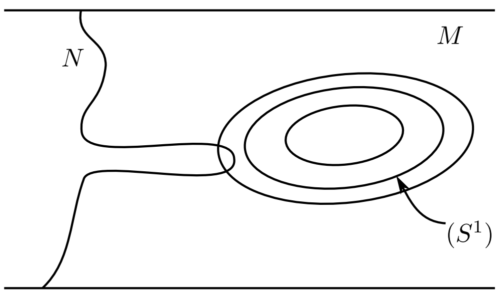
For this θ' we have $\theta'^{-1}(N) \cap D^2$ consisting precisely of the curves in $\theta^{-1}(N) \cap D^2$ excepting C. Repeating this argument a finite number of times we finally get a map $\Phi : D^2 \to W$ such that $\Phi S^1 = \varphi$ and $\Phi^{-1}(N) \cap D^2 = \emptyset$. Since $\varphi(S') \subset \text{Int } M$ and since D^2 is connected we should have $\Phi(D^2) \subset \text{Int } M$, for otherwise $D^2 \cap \Phi^{-1}$ (Int M) and $D^2 \cap \Phi^{-1}(W - M)$ **87** will be non void disjoint open sets of D^2. This means that $\alpha \in \pi_1(M)$ is the zero element and hence $\pi_1(M) \to \pi_1(W)$ is a monomorphism. $\square$

Proposition 2.9. *There exist arbitrary small 1-neighbourhoods of "∞".*

In the proof of this lemma we use a result in group theory which we state below without proof.

Lemma 2.10. *Suppose G and H are finitely presentable group and $G \xrightarrow{h} H \to 1$ is an exact sequence. Then the Kernel of h is the normal subgroup in G generated (as a normal subgroup) by a finite number of elements.*

We now go to the proof of proposition 2.9. We have $\pi_1(W) \simeq \pi_1(X)$ and by assumption X is a finite polyhedron. It follows that $\pi_1(W)$ is finitely presentable. Let M' with $N' = bM'$ be a zero neighbourhood of ∞ with $M' \subset W - K$. Choosing a base point $w_\circ \in \operatorname{Int} M'$ and a small "contractible open set 0" in $\operatorname{Int} M'$ as the "new base point" we can represent a finite system of generators $\alpha_1, \ldots, \alpha_r$ of $\pi_1(W)$ by disjoint C^∞ imbeddings $\varphi_i : S^1 \to W (i = 1, \ldots r)$ with the base point of S^1 going into 0. To represent each α_i by a C^∞ imbeddings we need that $\dim W \geq 3$ and also to get the imbedding to have disjoint images we need $\dim W \geq 3$. But hypothesis $\dim W \geq 6$. By choosing $w_\circ$ properly we can assume that $\varphi_i(S^1) \subset \operatorname{Int} M'$ for every i.



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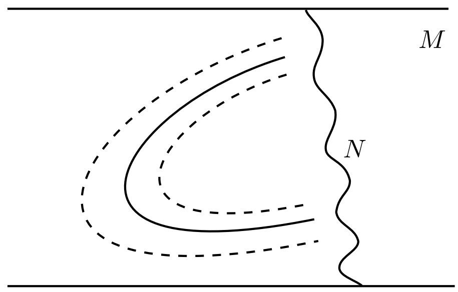
The normal bundle of φ_i has a section for every i. Let U_i be an open tubular neighbourhood of $\varphi_i(S^1)$ for every i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Define $M'' = M' - U_i U_i$. Then M'' is still connected though

$bM'' = N''$ is not in general. By choosing C^∞ paths in M'' meeting the components of bM'' only at the end points and orthogonally and removing the interiors of tubular neighbourhoods of these paths one gets a zero -neighbourhood $M''' \subset W - K$. Sections of the normal bundles $bU_i \to \varphi_i(S^1)$ yield elements $\alpha'_1, \ldots, \alpha'_r \in \pi_1(bM'')$ which map onto $\alpha_1, \ldots, \alpha_r \in \pi_1(W)$. (Refer to diagram 4). Thus $\pi(N''/ \to \pi_1(W)$ is onto, where $N''' = bM'''$. We denote (M''', N''') again by (M, N) and may assume(by Lemma 2.10) that $\pi_1 N \to \pi_1 W$ is the normal closure in $\pi_1 N$ of a finite number of elements $\beta_1, \ldots, \beta_k$. Choose C^∞ imbedding $\varphi_i : S^1 \to N$ with base point of S^1 going into some contractible **89**
open set B of N such that φ_i represents β_i. $(i = 1, \ldots, k)$. It is given that φ_i represents the zero element in $\pi_1 W$. Hence there exists a map which can be assumed to be a C^∞ imbedding $\varphi_i : D^2 \to W$ extending $\varphi_i : S^1 \to N$. By translating M if necessary by a deck transformation we can assume that the images $\varphi_i(D^2)$ all lie in $W - K$. We can get a tubular neighbourhood of $\varphi_i(S)^1$ in N as the restriction to $\varphi_i(S)^1$ of a tubular neighbourhood of $\varphi_i(D)^2$ in W. We may assume that these tubular neighbourhood are disjoint, and that their intersections with N are tubular neighbourhood of $\varphi_i(D)^2 \cap N$. Let $C \subset D^2$ be an inner most simple closed component curve of $\varphi_i^{-1}(N)$ for some i, and let D be the region of D^2 bounded by C. Then φ_i (int D) $\cap N = \theta$.

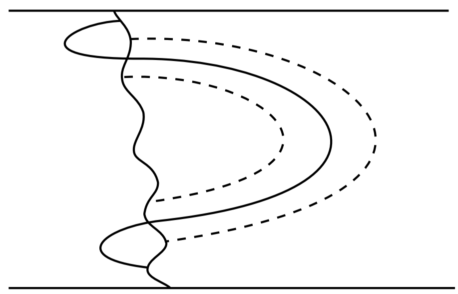
There are two cases :

If φ_i (Int D) $\subset W - M$ then add the tubular neighbourhood of $\varphi_i(D)$ to M. That is to say, a handle $D^2 \times D^{n-1}$ is attached to M. (Refer to diagram 5′).



If φ_i(Int D) $\subset$ int M delete from M the tubular neighbourhood of **90**

$\varphi_i(D)$ (Refer to diagram 5″).



The new manifold M' with boundary N' is still a 0-neighbourhood of α. Moreover, $\pi_1 N'$ is a quotient of $\pi_1 N$ and the kernel of $\pi_1 N' \to \pi_1 W$ is still (normally) generated by the classes of $\varphi_j(S^1)$, $j = 1, \ldots, K$ with $j \neq i$ if $C = bD^2$ for φ_i. But the φ_j extend to $\varphi_j = D^2 \to M'$ with $\varphi_j\varphi_j(D)^2 \cap N'$ consisting of one less component curve than the original intersection. After a finite number of such steps, one reaches a 0-neighbourhood M, $bM = N$, such that $\pi_1 N \to \pi_1 W$ is an isomorphism. By Lemma 2.8, (M, N) is then a 1-neighbourhood.

3 The Existence of Arbitrary small k. Neighbourhoods of "∞" and "$-\infty$" for $2 \leq k \leq n-2$

Definition 3.1. *Let k be an integer ≥ 2. A k-neighbourhood of ∞ (respy $-\infty$) in W is a 1-neighbourhood M of ∞ (respy $-\infty$) satisfying the following additional condition:*

91 *Denoting the universal covering of M by $\tilde{M}$ with $p : \tilde{M} \to M$ the projection, let $\tilde{N} = p^{-1}(N)$ where $N = bM$. The condition to be satisfied is : $H_i(\tilde{M}, \tilde{N}) = 0$ for $i \leq k$.*

Remark. Since $\pi_1(N) \to \pi_1(M)$ induced by the inclusion is an isomorphism it follows that $p : \tilde{N} \to N$ is the universal covering of N.

Proposition 3.2. *There exist arbitrary small k-neighbourhoods of ∞ (respy '$-\infty$') for any integer k such that $2 \leq k \leq n-2$.*

We prove this proposition for $k = 2$ first and then proceed by induction on k. It will be clear from the proof why we are forced to give a proof for $k = 2$ separately.

Lemma 3.3. *If M is a 0 (respy 1) neighbourhood of '∞' then $M_\circ = \overline{W - M}$ is a 0 (respy 1) neighbourhood of '$-\infty$'.*

Proof. Clearly the boundary of $M_\circ$ is the same as that of M. Thus $bM_\circ = bM = N$ is compact and connected. If $m_1 < m_2$ are integers such that $F^{-1}[m_1, \infty) \supset M \supset F^{-1}[m_2, \infty)$ then clearly $F^{-1}(-\infty, m_1] \subset M_\circ \subset F^{-1}(-\infty, m_2]$. Let a, b be any two points in $M_\circ$. We will show that there is an arc in $M_\circ$ joining a and b. Since W is arcwise connected $\exists$ an arc σ in W with $\sigma(o) = a$ and $\sigma(1) = b$. If the arc σ lies in $M_\circ$ there is nothing to prove. If not $\exists$ real numbers $t_\circ$ and t_1 such that $\tau(t) \in M_\circ$ $\forall\ t \leq t_\circ$ and $\sigma(s) \in M_\circ$ $\forall\ s \geq t_1$ and $\sigma(t_\circ) \in N$, $\sigma(t_1) \in N$. Choosing an arc in N joining $\sigma(t_\circ)$ and $\sigma(t_1)$ we see that a and b can be joined by means of an arc in $M_\circ$. Thus $M_\circ$ is a 0-neighbourhood of '$-\infty$'. **92**
If M is a 1-neighbourhood of ∞ then $\pi_1(bM) = \pi_1(bM_\circ) = \pi_1(N) \to \pi_1(W)$ is an isomorphism and from Lemma 2.8 it follows that $M_\circ$ is a 1-neighbourhood. ◻

Lemma 3.4. *If M is a 1-neighbourhood of ∞ in W, then $H_j(\tilde{M})$ is a finitely generated $\mathbb{Z}(J)$-module.*

For this we shall use assumption that $\mathbb{Z}(\pi)$ is a noetherian ring. By an example of J. Stallings the above lemma is definitely false without this hypothesis. However, we really only need that if (M, N) is a $(k-1)$-neighbourhood, then $H_k(\tilde{M}, \tilde{N})$ is finitely generated. In the general case ($\mathbb{Z}(\pi)$ not necessarily noetherian) one proved that (M, N) is dominated by a finite complex pair. It is then an exercise to deduce from this the finite generation of $H_k(\tilde{M}, \tilde{N})$.

Proof. Let $N = bM$ and $M_\circ = \overline{W - M}$. By lemma 3.3, $M_\circ$ is a 1-neighbourhood of "$-\infty$". If W is the universal covering of W with $p : \tilde{W} \to W$ the projection there $\tilde{M} = p^{-1}(M)$. $\tilde{M}_\circ = p^{-1}(M_\circ)$ and $\tilde{N} = p^{-1} = p^{-1}(M \cap M_\circ) = \tilde{M} \cap \tilde{M}_\circ$ are respectively the universal covering of M, $M_\circ$ and N. This is so because $\pi_1(N) \to \pi_1(W)$,

$\pi_1(M) \to \pi_1(W)$ and $\pi_1(M_\circ) \to \pi_1(W)$ induced by the respective inclusions are isomorphisms. From the Mayer-Vietoris sequence

$$H_j(\tilde{N}) \to H_j(\tilde{M}_\circ) \oplus H_j(\tilde{M}) \to H_j(\tilde{W})$$

which is sequence of $\mathbb{Z}\pi$-modules it will follow that $H_j(\tilde{M})$ is finitely generated over $\mathbb{Z}(\pi)$ if we show that $H_2(\tilde{N})$ and $H_2(\tilde{W})$ are finitely gen-
93 erated over $\mathbb{Z}(\pi)$. Since N is smooth and compact, choosing a triangulation of N we see that the chain groups of $\tilde{N}$ with the lifted triangulation are finitely generated over $\mathbb{Z}\pi$. From the fact that $\mathbb{Z}\pi$ is noetherian again it follows that all the homology groups of $\tilde{N}$ are finitely generated $\mathbb{Z}\pi$-modules. Also W is of the homotopy type of the finite polyhedron X and the same argument as above yields that all the homology groups of W are finitely generated $\mathbb{Z}\pi$-modules. □

Lemma 3.5. *There exist arbitrary small 2- neighbourhoods of "∞".*

Proof. Let M' with $bM' = N'$ be a 1-neighbourhood of ∞ with $M' \subset W - K$. By Lemma 3.4, $H_2(M')$ is finitely generated over $\mathbb{Z}(\pi)$. Let $\alpha_1, \dots, \alpha_r$ be a system of generators over $\mathbb{Z}(\pi)$ for $H_2(\tilde{M}') = \pi_2(\tilde{M}') \sim \pi_2(M')$. Choosing a small contractible open set in Int M' as the base point represent the elements α_i by C^∞ imbeddings φ_i: $S^2 \to \text{Int } M'$, with disjoint images and the base point of S^2 going into the chosen contractible open set. For this to be possible we need that $\dim M' \geq 5$ but by assumption $\dim M' = n + 1 \geq 6$. Let M be formed from M' as explained below: Choose closed tubular neighbourhoods T_i of $\varphi_i(S^2)$ in Int M' with $T_i \cap T_j = \phi$ whenever $i \neq j$. Choose C^∞ paths σ_i from N' to bT_i (the boundary of T_i) meeting N' and bT_i transversally and at the end points only. These paths can be chosen to be mutually disjoint, and tubular neighbourhoods Γ_i of σ_i can be chosen to be mutually disjoint.
94 Let $M = M' - \bigcup_{i=1}^{r} \text{Int} \cup T_i \text{ Int } \Gamma_i$. Then clearly M is a 0-neighbourhood of ∞. We claim that M is a 2-neighbourhood of ∞. First of all, if $N = bM$ it is clear that $N = N' \# bT_1 \# \cdots \# bT_r$ (connected sum). Also bT_1 is an $(n-2)$ sphere bundle over S^2 with $n \geq 5$ and hence $\pi_1(bT_i) = 1$. By Van Kampen we see that $\pi_1(N) \simeq \pi_1(N')$, under an isomorphism making the following diagram commutative: □

$$\begin{tikzcd}
\pi_1(N) \arrow[rr, "\simeq"] \arrow[d, "j_*"] & & \pi_1(N') \arrow[d, "j'_*"] \\
\pi_1(M) \arrow[rr, "\mu_*"] \arrow[dr, "i_*"] & & \pi_1(M) \arrow[dl, "i'_*"] \\
& \pi_1(W) &
\end{tikzcd}$$

Diagram 6

Here the homomorphisms indicated by i_*, j_*, i'_*, j'_* and μ_* are all induced by inclusions and the isomorphism $\pi_1(N) \to \pi_1(N')$ is got from Van Kampen's theorem. It follows that $i_* o j_*$ is an isomorphism since i'_* and j'_* are. Lemma 2.8 now implies that M is a 1-neighbourhood of ∞.

Assertion: $\pi_2(N) \xrightarrow{j_*} \pi_2(M)$ is an epimorphism.

To prove this it suffices to show that $\pi_2(N) \xrightarrow{\mu_* \circ j_*} \pi_2(M')$ is an epimorphism and that μ_*: $\pi_2 \to \pi_2(M')$ is an isomorphism. Let $\nu_i \in \pi_1$ **95**
$(So(n-1))$ be the element corresponding to the normal bundle of $\varphi_1(S^2)$ in Int M'. As $S_* : \pi_1\ (So(n-2)) \to \pi_1\ (SO(n-1))$ is an isomorphism for $n \geq 5$ we see that γ_i can be written as $\gamma_i + \mathscr{O}^1$ is a trivial line bundle. Hence there exists a non zero cross-section for the associated sphere bundle. Using this cross-section we see that $\exists$ an element in $\pi_2(bT_i)$ which represents the element $\alpha_i \in \pi_2(M')$ under the inclusion $bT_i \to M'$. It now follows that $\pi_2(N) \xrightarrow{\mu_* \circ j_*} \pi_2(M')$ is an epimorhism.

This in particular gives: $\pi_2(M) \xrightarrow{\mu_*} \pi_2(M')$ is an epimorphism. To complete the proof of the assertion we only to show that μ_* is a monomorphism. Let $x \in \pi_2(M)$ be such that $\mu_*(x) = 0$ and let $\theta : S^2 \to M$ be a C^∞ imbedding representing x. The fact that $\mu_*(x) = 0$ implies that $\exists$ a C^∞ map $\varphi : D^3 \to M'$ extending θ. We can get φ so as to be transverse regular on $\cup\varphi_i(S^2)$ (since $\theta(S^2) \cap \varphi_i(S^2) = \phi$). The condition $n + 1 \geq 6(n + 1 = \dim M')$ implies that $\varphi(D^3)$ is then disjoint from $\cup\varphi_i(S^2)$. By a further deformation we can make $\varphi(D^3)$ go into M.

Now, $\pi_2(N) \xrightarrow{j_*} \pi_2(M)$ being an epimorphism we have $\pi_2(\tilde{N}) \xrightarrow{j_*} \pi_2(\tilde{M})$ also an epimorphism and hence $\pi_2(\tilde{M}, \tilde{N}) = 0$. The simply connectedness of $\tilde{M}$ and $\tilde{N}$ now yields by the Relative Hurewicz Theorem $H_2(\tilde{M}, \tilde{N}) = \pi_2(\tilde{M}, \tilde{N}) = 0$. This completes the proof that M is a 2-neighbourhood.

96 We now proceed to the proof of proposition 3.2 for an arbitrary k satisfying $3 \leq k \leq n-2$. Assume by induction that arbitrary small $(k-1)$ neighbourhoods of ∞ exist.

Lemma 3.6. *Suppose M is any $(k-1)$-neighbourhood of ∞. Let $N = bM$. Then*

(1) $H_k(\tilde{M}\tilde{N})$ is a finitely generated $\mathbb{Z}(\pi)$-module.

(2) $\exists$ another $(k-1)$-neighbourhood M_1 of ∞ with $M_1 \subset M$ satisfying the following additional condition:

The homomorphism $H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{M}.\tilde{N})$ induced by the inclusion $(\tilde{U}, \tilde{N}) \subset (\tilde{M}, \tilde{N})$ is an epimorphism, where $U = \overline{M - M_1}$ and $\tilde{U}$ is the inverse image of U by the covering map $p \colon \tilde{M} \to M$.

Proof of (1). By Lemma 3.4 we have $H_j(\tilde{M})$ finitely generated over $\mathbb{Z}(\pi)$ for every j. Also since N is compact $H_j(\tilde{N})$ is finitely generated over $\mathbb{Z}(\pi)$. The exactness of $H_k(\tilde{M}) \to H_k(\tilde{M}, \tilde{N}) \to H_{k-1}(\tilde{N})$ together with Noetherian nature of $\mathbb{Z}(\pi)$ now yield the finite generation of $H_k(\tilde{M}, \tilde{N})$ over $\mathbb{Z}(\pi)$.

Proof of (2). Let $C_1, \ldots, C_\lambda$ be a finite set of generators for $H_k(\tilde{M}, \tilde{N})$. There exists a compact set $\tilde{K}_1$ in $\tilde{M}$ such that $\exists$ integral singular cycles representing $C_1, \ldots, C_\lambda$ with their supports contained in $\tilde{K}_1$. Let $K_1 = p(\tilde{K}_1)$. By the inductive assumption regarding existence of arbitrary small $(k-1)$-neighbourhoods of ∞ we can find a $(k-1)$-neighbourhood M_1 of ∞ with $M_1 \subset W - K_1$ and $M_1 \subset M$. Then clearly $U = \overline{M - M_1}$ satisfies the condition $U \supset K_1$ and thus the chosen cycles representing
97 $C_1, \ldots, C_\lambda$ are cycles of $(\tilde{U}, \tilde{N})$. Hence $H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{M}, \tilde{N})$ is onto.

Remark A. *For the pair $(\tilde{U}, \tilde{N})$ we have $H_i(\tilde{U}, \tilde{N}) = 0$ for $i < k-1$.*

Proof. Let $N_1 = bM_1$. We have $H_i(\tilde{M}, \tilde{U}) \underset{\simeq}{\longleftrightarrow} H_i(\tilde{M}_i, \tilde{N}_i)$ by excision. Now from the homology exact sequence of the triple (M, U, N) written below:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(\tilde{M}, \tilde{U}) \arrow[r] & H_i(\tilde{U}, \tilde{N}) \arrow[r] & H_i(\tilde{M}, \tilde{N}) \arrow[r] & H_i(\tilde{M}, \tilde{U}) \arrow[r] & \cdots \\
 & H_{i+1}(M_1, N_1) \arrow[u, "\approx"', "\text{excision}"] & & & H_i(\tilde{M}_1, \tilde{N}_1) \arrow[u, "\approx"', "\text{excision}"] &
\end{tikzcd}$$

and the fact that M_i is a $(k-1)$-neighbourhood of ∞ we see that $H_i(\tilde{U}, \tilde{N}) \to H_i(\tilde{M}, \tilde{N})$ for $i < k-1$. Since M itself is a $(k-1)$-neighbourhood we have $H_i(\tilde{U}, \tilde{N}) = 0$ for $i < k-1$. $\square$

Remark B. *The homomorphisms $\pi_1(N) \to \pi_1(U)$ and $\pi_1(N_1) \to \pi_1(U)$ induced by the inclusions are isomorphisms.*

The proof of this is similar to the proof of Lemma 2.8 and hence is omitted.

For completing the proof the proposition 3.2 we need the following two propositions which we state without proof.

Proposition 3.7. *Suppose U is a compact orientable C^∞ manifold of dimension $n+1$ with $n \geq 5$ and suppose $bU = N \cup N_1$ a disjoint union of two open and closed, connected submanifolds of bU. If the homomorphisms $\pi_1(N) \to \pi_1(U)$ and $\pi_1(N_1) \to \pi_1(U)$ induced by the inclusions are isomorphisms and if $H_i(\tilde{U}, \tilde{N}) = 0$ for $i \leq k-2 < n-2$ then (U, N)* 98
has a handle decomposition with handles of type $k-1, k, \ldots, n-1$.

In other words U has a presentation of the form

$$U = I \times N + \varphi_1^{k-1} + \cdots + \varphi_{\alpha_{k-1}}^{k-1} + \varphi_1^k + \varphi_{\alpha_k}^k + \cdots + \chi_1^{n-1} + \cdots + \chi_{\alpha_{n-1}}^{n-1}.$$

The proof is essentially given in [5], Lemma 1.

Proposition 3.8. *Let X and Y be closed C^∞ submanifolds of a C^∞ manifold N, where $\dim X + \dim Y = \dim N > 4$, and $2 < \dim Y \leq \dim N - 2$. Suppose that $\pi_1(N - Y) \to \pi_1 N$ induced by the inclusion is an isomorphism. (This is a restriction only if $\dim Y = \dim N - 2$). Suppose that X*

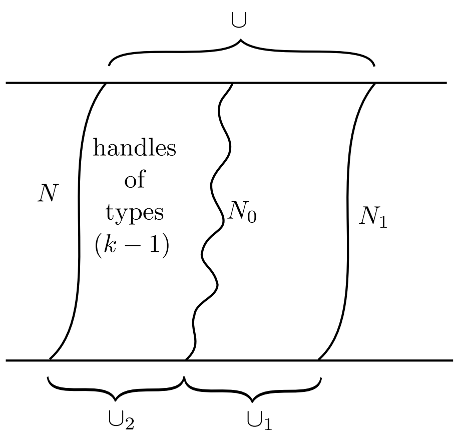
and Y can be lifted to closed submanifolds $\tilde{X}$ and $\tilde{Y}$ of $\tilde{N}$, the universal covering of N, and that

$$\tilde{X}_i . \tau \tilde{Y}_j = 0$$

(where denotes the homology intersection number) for all $\tau \in \pi$ and all connected components $\tilde{X}_i$, $\tilde{Y}_j$ of $\tilde{X}$ and $\tilde{Y}$. Then X is isotopic in N to a submanifold X_1 such that $X_1 \cap Y = \phi$, or equivalently Y is isotopic in N to a submanifold Y_1 such that $X \cap Y_1 = \phi$.

This proposition is essentially due to Whitney.

As remarked already proposition 3.2 is proved by induction on k for k in the range $3 \leq k \leq n-2$. Assume arbitrary small $(k-1)$-neighbourhoods of ∞ exist. Let K be any compact subset of W and let M be any $(k-1)$-neighbourhood of ∞ with $M \subset W - K$. By Lemma
99 3.6 $\exists$ a $(k-1)$-neighbourhood of ∞ say M_1 with $M_1 \subset M$ such that the homomorphism $H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{M}, \tilde{N})$ induced by inclusion is onto, where $U = \overline{M - M_1}$ and $bM = N$, $bM_1 = N_1$. From Remark A following Lemma 3.6 we have $H_i(\tilde{U}, \tilde{N}) = 0$ for $i < k-1$ and by Remark B the homomorphisms $\pi_1(N) \to \pi_1(U)$, $\pi_1(N_1) \to \pi_1(U)$ induced by the respective inclusions are isomorphisms. Hence by proposition 3.7 we have a handle decomposition for (U, N) with handles of type $k-1$, $k, \ldots, n-1$. Let $U_\circ$ be the union of $I \times N$ together with handles of type $k-1$ (Refer to diagram 7) and $N_\circ$ the right hand boundary of $U_\circ$. Let $U_1 = \overline{U - U_\circ}$.



Convention: In future when we are in a situation of the form $A \subset B$ or $(A, A') \subset (B, B')$ with A, A', B, B' topological spaces by the homomorphism $\pi_1(A) \to \pi_1(B)$ or $H_j(A) \to H_j(B)$ or $H_j(A, A') \to H_j(B, B')$ we mean the one induced by the inclusion. When $k > 3$ we see from **100**
Van Kampen theorem that $\pi_1(N) \to \pi_1(U_\circ)$ is an isomorphism. When $k = 3$ we first observe that the 2-handles φ_i^2 are attached by means of trivial maps to $1 \times N$. In fact $\varphi_1^2(S^1 \times 0)$ bounds a disk in W and as M is a 1-neighbourhood we have $\pi_1(N) \to \pi_1(W)$ an isomorphism. Now an application of Van Kampen immediately yields $\pi_1(N) \to \pi_1(U_\circ)$ is an isomorphism. Using the 'dual' handle decomposition for $U_\circ$ and the fact that $k \leq n-2$ we see that $\pi_1(N_\circ) \to \pi_1(U_\circ)$ is an isomorphism, again by applying Van Kampen. To get U_1 we attach handles of type $k, \ldots, n-1$ to $U_\circ$. It follows that whenever $k \geq 3$ the homomorphism $\pi_1(N_\circ) \to \pi_1(U_1)$ is actually an isomorphism. Now choose any α in $H_k(\tilde{M}, \tilde{N})$. By our choice of M_1 we have $H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{M}, \tilde{N})$ epimorphism. Choose any $\beta \in H_k(\tilde{U}, \tilde{N})$ getting mapped onto α. By excision $H_k(\tilde{U}, \tilde{U}_\circ) \simeq H_k(\tilde{U}_1, \tilde{N}_\circ)$ the isomorphism being a $\mathbb{Z}(\pi)$-isomorphism since the maps induced by the various inclusions, namely $N \to U_\circ$; $N_\circ \to U_\circ$ and $N_\circ \to U_1$ are isomorphisms on π_1. Let γ be the image of β under the composition of the maps

$$H_k(\tilde{U}, \tilde{N}) \xrightarrow{(\text{ inc } ln)_*} H_k(\tilde{U}, \tilde{U}_\circ) \xleftarrow[\text{excision}]{\simeq} H_k(\tilde{U}_1, \tilde{N}_\circ).$$

Since $(U_1, N_\circ)$ has a handle decomposition with handles of type $k, \ldots,$ $n-1$ we that $H_i(\tilde{U}_1 \tilde{N}_\circ) = 0$ for $i \leq k-1$ and by Relative Hurewicz theorem $\pi_k(\tilde{U}_1, \tilde{N}_\circ) \simeq H_k(\tilde{U}_1, \tilde{N}_\circ)$. But $\pi_k(\tilde{U}_1, \tilde{N}_\circ) \simeq \pi_k(U_1, N_\circ)$. Thus $\pi_k(U_1, N_\circ) \simeq H_k(\tilde{U}_1, \tilde{N}_\circ)$.

Claim: The element γ can be represented by a C^∞ imbedding φ : **101**
$(D^k, S^{k-1}) \to (U_1, N_\circ)$.

Now, γ is homologous to $\sum a_i D_i^k$ with $a_i \in \mathbb{Z}(\pi)$ and D_i^k the k-cell of the i-th hankle of type k. D_i^k is a differentiably imbedded k-cell in U_1 with boundary S_i^{k-1} in $N_\circ$. Let $a_i = \sum\limits_{\sigma \in \pi} a_i^\sigma \sigma$ with $a_i^{(\sigma)} \in \mathbb{Z}$ and $a_i^{(\sigma)} = 0$ for almost all σ. We can assume that all the $S_i^{k-1} D^{n-k+1}$ intersect a contractible open set in $N_\circ$ which can be chosen as the "base point" for homotopy considerations. Let $\ell_i = \sum_\sigma |a_i^\sigma|$. Let us take ℓ_i distinct points $x_1, \dots, x_{\ell_i}$ in D_i^{n-k+1}. Form connected sum of $D_i^k \times x_1, \dots, D_i^k \times x_{\ell_i}$ along paths in $N_\circ$ representing the σ's for which $a_i^{(\sigma)} \neq 0$. This operation will give a C^∞ imbedding θ_i: $(D^k, S^{k-1}) \to (U_1, N_\circ)$ representing $a_i D_i^k$. Forming connected sum of the various $\theta_i(D^k)$ along trivial arcs in $N_\circ$ gives a C^∞ imbedding $\varphi : (D^k, S^{k-1}) \to (U_1, N_\circ)$ representing γ.

Let (D^k, S_j^{n-k+1}) be the boundaries of the right hand disks D_j^{n-k+2} corresponding to he handles of type $(k-1)$.

Claim: Let $\tilde{\varphi}(S^{k-1})$ and $\tilde{S}_j^{n-k+1}$ be arbitrary lifts of $\varphi(S^{k-1})$ and S_j^{n-k+1} to $N_\circ$. Then for any $\tau \in \pi$ the homology intersection $\tilde{\varphi}(S^{k-1}).\ \tau\tilde{S}_j^{n-k+1}$ in $\tilde{N}_\circ$ is zero.

102 Actually $\tilde{\varphi}(S^{k-1}) \cdot_{\tilde{N}_\circ} \tau S_j^{n-k+1}$ is the same $\beta.\tau\{\tilde{S}_j^{n-k+1}$, this later intersection being the one associated to the pair $H_k(\tilde{U}, \tilde{N})$ and $H_{n-k+1}(\tilde{U})$. But $\left\{\tilde{S}_j^{n-k+1}\right\} = 0$ in $H_{n-k+1}(\tilde{U})$ since $\tilde{S}_j^{n-k+1}$ bounds a disk in $\tilde{U}$.

We now want to apply proposition 3.8 to $\varphi(S^{k-1}) = X$ and $Y = US_j^{n-k+1}$ which are submanifolds of $N_\circ$. To be able to apply proposition 3.8 we need to have $n-k+1 \leq n-2$ and $\pi_1(N_\circ - Y) \to \pi_1(N_\circ)$ an isomorphism. The condition $n-k+1 \leq n-2$ gives $k \geq 3$. This is precisely the reason why we had to prove the existence of 2-neighbourhoods separately. We have already seen that $\pi_1(N) \to \pi_1(U_\circ)$ and $\pi_1(N_\circ) \to \pi_1(U_\circ)$ are isomorphisms. Since $\pi_1(N_\circ) \to \pi_1(W)$ is an isomorphism, it follows that $\pi_1(U_\circ) \to \pi_1(W)$is an isomorphism and hence $\pi_1(N_\circ) \to \pi_1(W)$ an isomorphism. Let $\varphi_j(D^{k-1} \times D^{n-k+2})$ denote the handles of type $k-1$. Then the inclusion $N_\circ - U\varphi_j(B^{k-1} \times S^{n-k+1}) \to N_\circ - US_j^{n-k+1}$ is a homotopy equivalence, and $N - U\varphi_j(S^{k-2} \times B^{n-k+2}) = N_\circ - U\varphi_j(B^{k-1} \times S^{n-k+1})$. Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_1(N - U\varphi_j(S^{k-2} \times B^{n-k+2})) \arrow[r] \arrow[dd, equal] & \pi_1(N) \arrow[dr] & \\
 & & \pi_1(W) \\
\pi_1(N - U\varphi_j(B^{k-1} \times S^{n-k+1})) \arrow[r] & \pi_1(N_\circ - Y) \arrow[r] & \pi_1(N_\circ) \arrow[u]
\end{tikzcd}$$

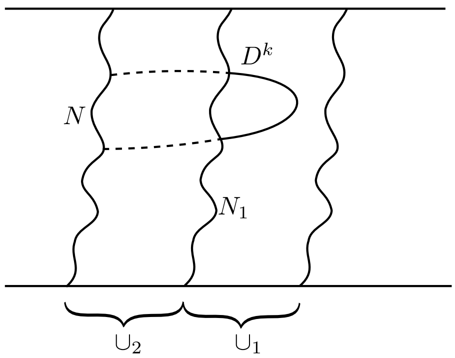
Diagram 8

The map 103

$$\pi_1(N - U_j\varphi_j(S^{k-2} \times B^{n-k+2})) \to \pi_1 N$$

is an isomorphism because it factors through $\pi_1(N - U_j\ \varphi_j(S^{k-2} \times B^{n-k+2})) \to \pi_1(N - U_j\varphi_j(S^{k-2} \times 0)) \to \pi_1 N$, where the first map is induced by a homotopy equivalence, and the second is also an isomorphism since codim $S^{k-2} = n - k + 2 \geq 3$.

Thus proposition 3.8 can be applied and it yields the following conclusion. The imbedding φ can be so chosen that $\varphi(S^{k-1}) \cap Y = \phi$. It now follows from Morse theory that $\varphi(S^{k-1})$ is diffeotopic in $U_\circ$ to an imbedding $\varphi' : S^{k-1} \to N$. Actually one gets a C^∞ imbedding Φ: $S^{k-1} \times I \to U_\circ$ extending φ i.e $\Phi|S^{k-1} \times 0 = \varphi$ and satisfying $\Phi(S^{k-1} \times I) \subset N$. Taking the diffeotopy together with the imbedding φ: $(D^k, S^{k-1}) \to (U_1, N_\circ)$ we get an imbedding φ: $(D^k, S^{k-1}) \to (U, N)$. (See diagram 9).



The homology class in $H_k(\tilde{U}, \tilde{N})$ represented by φ clearly gets mapped into the homology class γ represented by φ in $H_k(\tilde{U}_1, \tilde{N}_\circ)$ under the composition $H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{U}, \tilde{U}_\circ) \xleftarrow[\simeq]{\text{excision}} H_k(\tilde{U}_1, \tilde{N}_\circ)$. From the exact sequence of the triple $\tilde{U}$, $\tilde{U}_\circ$, $\tilde{N}$ we have

$$H_k(\tilde{U}_\circ, \tilde{N}) \to H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{U}, \tilde{U}_\circ) \text{ exact .}$$

But $H_k(\tilde{U}_\circ, \tilde{N}) = 0$ since the handle decomposition of $(U_\circ, N)$ we have, **104**
consists only of handles of type $(k-1)$. Thus $H_k(\tilde{U}, \tilde{N}) \to H_k(\tilde{U}, \tilde{U}_\circ)$ is a monomorphism and hence β is the only element of $H_k(\tilde{U}, \tilde{N})$ getting mapped into γ. It follows that the class in $H_k(\tilde{U}, \tilde{N})$ represented by φ: $(D^k, S^{k-1}) \to (U, N)$ is β.

Let A be the union of a tubular neighbourhood of $\varphi(D^k)$ in M together with a tubular neighbourhood of N in M. Define M' to be $\overline{M - A}$. Let $N' = bM'$.

Claim: *M' is a $(k-1)$-neighbourhood of ∞ with $H_k(\tilde{M}', \tilde{N}') \simeq H_k(\tilde{M}, \tilde{N})/(\alpha)$ as a $\mathbb{Z}(\pi)$-module.* Here (α) denotes the $\mathbb{Z}(\pi)$-submodule of $H_k(\tilde{M}, \tilde{N})$ generated by α.

Clearly M' is a 0-neighbourhood of ∞ and from Van Kampen's theorem we see that for k satisfying $3 \leq k \leq n-2$ $\pi_1(N') \to \pi_1(N)$ and $\pi_1(M') \simeq \pi_1(M)$ where the latter isomorphism is induced by the inclusion. Also the isomorphism $\pi_1(N') \to \pi_1(N)$ makes the diagram

$$\begin{tikzcd}
\pi_1(N') \arrow[r, "\simeq"] \arrow[d, "(\text{incln})_*"] & \pi_1(N) \arrow[d, "(\text{incln})_*", "\approx"'] \\
\pi_1(M') \arrow[r, "\approx", "(\text{incln})_*"'] & \pi_1(M)
\end{tikzcd}$$

commutative and hence $\pi_1(N') \to \pi_1(M')$ is an isomorphism. It follows tht M' is a 1-neighbourhood of ∞ . From the homology sequence of the
105 triple $(\tilde{M}, \tilde{A}, \tilde{N})$ where $\tilde{A} = p^{-1}(A)$ with $p : \tilde{M} \to M$ the covering map,
we have the following diagram with the horizontal row exact.

$$\begin{tikzcd}
H_{i+1}(\tilde{M}, \tilde{A}) \arrow[r] & H_i(\tilde{A}, \tilde{N}) \arrow[r] & H_i(\tilde{M}, \tilde{N}) \arrow[r] & H_i(\tilde{M}, \tilde{A}) \arrow[r] & {} \\
H_{i+1}(\tilde{M}', \tilde{N}') \arrow[u, "\approx \text{ excision}"] & & & H_i(\tilde{M}', \tilde{N}') \arrow[u, "\approx"', "\text{excison}"] &
\end{tikzcd}$$

Diagram 10

Now, $H_i(\tilde{A},\tilde{N}) = 0$ for $i \neq k$ and $H_k(\tilde{A},\tilde{N}) = \mathbb{Z}(\pi)$ and the map $H_i(\tilde{A},\tilde{N}) \to H_i(\tilde{M},\tilde{N})$ carries 1 of $\mathbb{Z}(\pi)$ into α. It follows that $H_i(\tilde{M}',\tilde{N}') = 0$ for $i \leq k-1$ and that $H_k(\tilde{M}',\tilde{N}') \simeq H_k(\tilde{M},\tilde{N})/(\alpha)$.

By Lemma 3.6 we have $H_k(\tilde{M},\tilde{N})$ finitely generated over $\mathbb{Z}(\pi)$. Choose a finite system of generators $\alpha_1,\ldots,\alpha_r$ and apply the above procedure to $\alpha = \alpha_1$. Then we get a $(k-1)$- neighbourhood M' such that $H_k(\tilde{M}',\tilde{N}')$ is generated by the images of $\alpha_2,\ldots,\alpha_r$ under the isomorphism $H_2(\tilde{M}',\tilde{N}') \simeq H_2(M,N)/(\alpha_1)$. By interating this procedure a finite number of times we finally arrive at a k-neighbourhood M'' of ∞. Clearly $M'' \subset M \subset W-K$. This completes the proof of Proposition 3.2.

4 The existence of arbitrary small $(n-1)$- neighbourhoods of "∞"

So far we have not used the hypothesis $\tilde{K}_\circ(\mathbb{Z}(\pi)) = 0$ any where. It is in the construction of arbitrary small $(n-1)$-neighbourhoods of ∞ that we use this hypothesis.

Lemma 4.1. *Let M be any $(n-2)$-neighbourhood of ∞ and let $N = bM$.* 106
Then the homology. $H_*(\tilde{M},\tilde{N})$ is the homology of a $\mathbb{Z}(\pi)$-chain complex of the form

$$0 \to \tilde{C}_{n-1} \xrightarrow{d} \tilde{C}_{n-2} \to 0$$

where $\tilde{C}_{n-1}$ and $\tilde{C}_{n-2}$ are free but not necessarily finitely generated $\mathbb{Z}(\pi)$-modules.

Proof. Pick a sequence of $(n-2)$-neighbourhoods

$$M = M_\circ \supset M_1 \supset ..M_r \supset M_{r+1} \ldots.$$

such that $\bigcup\limits_{r\geq 1} U_r = M$ where $U_r = \overline{M_{r-1} - M_r}$. □

We know that $\exists$ Morse functions $\lambda_r : U_r \to [r-1,r]$ with critical points of index $(n-2)$ and $(n-1)$ only, having the components of bU_r for level manifolds $\lambda_r^{-1}(r-1)$ and $\lambda_r^{-1}(r)$ of λ_r. Thus U_1 is homotopically equivalent to a space of the form $N \underset{\{f_i\}}{U} \underset{i\in I_1}{e_i^{n-2}} \underset{\{g_j\}}{U} \underset{j\in J_1}{e_j^{n-1}}$ means of attaching a finite number of $(n-2)$ cells and then a finite number of $(n-1)$ cells, under a homotopy equivalence which is the identity on N. Choose a triangulation L of N. By the cellular approximation theorem to each of the characteristic maps f_i corresponds a homotopic cellular map f_i': $S^{n-3} \to L^{n-3} \subset L$. Thus $N \underset{\{f_i\}}{U} e_i^{n-2}{}_{i \in I_1}$ is homotopy equivalent to the CW-complex $F = N \underset{\{f_i'\}}{U} e_i^{n-2}{}_{i \in I_1}$ under an equivalence θ which is identity

107 on N. Replacing the maps $\theta \circ g_j$ by cellular maps $g_j' : S^{n-2} \to F$ we get a CW-complex $K_1 = F \underset{\{g_j'\}}{U} e_j^{n-1}{}_{j \in j_1}$ and a homotopy equivalence $h_1 : U_1 \to K_1$ which is identity on N. Also K_1 contains L as a subcomplex. Using the Morse function λ_2 we see that $U_1 \cup U_2$ is of the homotopy type of a space of the form $U_1 \underset{\{f_i\}}{U} e_i^{n-2}{}_{i \in I_2} \underset{\{g_j\}}{U} e_j^{n-1}{}_{j \in I_2}$ under an equivalence which is identity on U_1. Taking cellular approximations f_i' to $h_1 \circ f_i$ and attaching $n-2$ cells by means of f_i' to K_1 we get a CW-complex F_2 and a homotopy equivalence $U_1 \underset{\{f_i\}}{U} e_i^{n-2}{}_{i \in I_2} \xrightarrow{\theta_2} F_2 = K_1 \underset{\{f_i\}}{U} e_i^{n-2}{}_{i \in I_2}$ extending h_1. Taking cellular approximations g_j' to $\theta_2 \circ g_j$ and attaching $(n-1)$ cells to F_2 by means of the maps g_j' we get a CW-complex K_2 containing K_1 as a subcomplex and a homotopy equivalence h_2: $U_1 \cup U_2 \to K_2$ extending h_1. Proceeding thus we construct a sequence of CW-complexes $L \subset K_1 \subset K_2 \subset K_3 \ldots$ and homotopy equivalences $h_r : \underset{j}{U} U_j {}_{r}^{1} \to K_r$ such that h_r is an extension of h_{r-1} and $h_1 = Id$ on $N = L$. Let $K = \underset{r}{U} K_r {}_1$ provided with the "union topology" i.e. to say a set in K is closed if and only if its intersection with each K_r is closed in K_r. Then h: $M \to K$ defined by $h|U_1 \ldots U_r = h_r$ is seen to be a homotopy

108 equivalence, because fo J.H.C. Whitchead's theorem. In fact it is easy to see that h induces isomorphisms of homotopy groups and Whitehead's theorem asserts that a map of CW-complexes inducing isomorphisms of homotopy groups is a homotopy equivalence. Since the cells of K that are not in L are either of dimension $n-2$ or of dimension $n-1$, we have proved Lemma 4.1.

Corollary 4.2. *$H_{n-1}(\tilde{M}, \tilde{N})$ is a finitely generated projective $\mathbb{Z}(\pi)$ - mod-*

ule.

The proof for the finite generation of $H_{n-1}(\tilde{M}, \tilde{N})$ over $\mathbb{Z}(\pi)$ is the same as that of (1) of Lemma 3.6. Since $H_{n-1}(\tilde{M}, \tilde{N}) = 0$ for $i \leq n-2$ we see that $d : \tilde{C}_{n-1} \to \tilde{C}_{n-2}$ has to be onto. The free nature of C_{n-2} implies $\tilde{C}_{n-1}$ $\operatorname{Ker} d \oplus \tilde{C}_{n-2}$. Now $H_{n-1}(\tilde{M}, \tilde{N}) \simeq \operatorname{Ker} d$ is a direct summand of the free module $\tilde{C}_{n-1}$ hence projective.

For any integer $e \geq 0$ let $\sum\limits_{e} \mathbb{Z}(\pi)$ denote the direct sum of e copies of $\mathbb{Z}(\pi)$. Since $\tilde{K}_\circ(\mathbb{Z}(\pi)) = 0$ it follows that $\exists$ an integer $e \geq 0$ such that $H_{n-1}(\tilde{M}, \tilde{N}) \oplus \sum\limits_{e} \mathbb{Z}(\pi)$ is a free $\mathbb{Z}$-module of finite rank. Let the rank of $H_{n-1}(\tilde{M}, \tilde{N}) + \sum\limits_{e} \mathbb{Z}(\pi)$ be r.

Lemma 4.3. *Given any compact set K of $W \exists$ an $(n-2)$ neighbourhood M of ∞ with $M \subset W - K$ such that $H_{n-1}(\tilde{M}, \tilde{N})$ is a free $\mathbb{Z}(\pi)$-module of finite rank, where $N = bM$.*

Proof. Choose any $(n-2)$-neighbourhood M' of ∞ with $M' \subset W - K$, and let $N' = bM'$. $\square$

By corollary 4.2, $H_{n-1}(M', N')$ is a finitely generated projective $\mathbb{Z}(\pi)$- module and hence $\exists$ an integer $e \geq 0$ such that $H_{n-1}(\tilde{M}', \tilde{N}') + \sum\limits_{e} \mathbb{Z}(\pi)$ is free over $\mathbb{Z}(\pi)$ of finite rank say r. We can find an $(n-2)$- **109**
neighbourhood M'' of ∞ with $M'' \subset M'$ and $H_{n-1}(\tilde{U}, \tilde{N}') \to H_{n-1}(\tilde{M}', \tilde{N}')$ onto, where $U = \overline{M' - M''}$(see 2, Lemma 3.6). By Proposition 3.7, (U, N') has a handle decomposition consisting of handles of type $(n-2)$ and $(n-1)$ only. Without even changing M' we can introduce e pairs of mutually cancelling handles of type $(n-2)$ and $(n-1)$. Let M be formed by removing from M' the union of the interiors of tubular neighbourhoods of the e newly introduced handles of type $n-2$ and a tubular neighbourhood of N', and let $N = bM$.

Claim: M is an $(n-2)$-neighbourhood of ∞ such that $H_{n-1}(\tilde{M}, \tilde{N})$ is a free $\mathbb{Z}(\pi)$-module of rank r.

Let A be the union of the closures of the tubular neighbourhoods removed and let $\tilde{A} = p^{-1}(A)$. Using Van Kampen and the fact that $n-2 \geq 3$ we see that M is a 1-neighbourhood of ∞. Also $H_i(\tilde{A}, \tilde{N}') = 0$

for $i \neq n-2$ and $H_{n-2}(\tilde{A}, \tilde{N}') = \sum\limits_{e} \mathbb{Z}(\pi)$. From the homology exact sequence of the triple $(\tilde{M}', \tilde{A}, \tilde{N}')$,

$$\begin{tikzcd}
H_j(\tilde{A}, \tilde{N}') \arrow[r] & H_j(\tilde{M}', \tilde{N}') \arrow[r] & H_j(\tilde{M}', \tilde{A}) \arrow[r] & H_{j-1}(\tilde{A}, \tilde{N}') \arrow[r] & \cdots \\
 & & H_j(\tilde{M}, \tilde{N}) \arrow[u, "\text{a excision}"'] & &
\end{tikzcd}$$

we see that $H_i(\tilde{M}, \tilde{N}) = 0$ for $i \leq n-2$ and that $H_{n-1}(\tilde{M}, \tilde{N}) = H_{n-1} +$
110 $\sum\limits_{e} \mathbb{Z}(\pi)$. But by the choice of e, this is a free $\mathbb{Z}(\pi)$-module of rank r. This completes the proof of Lemma 4.3.

Remark 4.4. If M is any $(n-2)$-neighbourhood of ∞ and if M_1 is another $(n-2)$-neighbourhood of ∞ with $M_1 \subset M$ and $H_{n-1}(\tilde{U}, \tilde{N}) \to H_{n-1}(\tilde{M}, \tilde{N})$ onto, (where U=$\overline{M - M_1}$) then $H_{n-1}(\tilde{U}, \tilde{N}) \to H_{n-1}(\tilde{M}, \tilde{N})$ and $H_{n-1}(\tilde{M}_1, \tilde{N}_1) = H_{n-2}(\tilde{U}, \tilde{N})$.

Proof. In the homology exact sequence of the triple $(\tilde{M}, \tilde{U}, \tilde{N})$ we have $H_n(\tilde{M}_1, \tilde{N}_1) = 0$ by Lemma 4.1. By assumption $H_{n-1}(\tilde{U}, \tilde{N}) \to H_{n-1}(\tilde{M}, \tilde{N})$ is an epimorphism. It is now immediate that $H_{n-1}(\tilde{U}, \tilde{N}) \simeq H_{n-1}(\tilde{M}, \tilde{N})$ and that $H_{n-1}(\tilde{M}_1, \tilde{N}_1) \simeq H_{n-2}(\tilde{U}, \tilde{N})$.

Let M be an $(n-2)$-neighbourhood of ∞ with $H_{n-1}(\tilde{M}, \tilde{N})$ a free $\mathbb{Z}(\pi)$-module of finite rank (say r). We can find a translate M_1 of M by a Deck transformation such that $M_1 : M$ and $H_{n-1}(\tilde{U}, \tilde{N}) \to H_{n-1}(\tilde{M}, \tilde{N})$ onto, where $U = \overline{M - M_1}$. We have to only choose the translate M_1 so as not to intersect the compact set got as the projection by p of the union of supports of singular cycles (integral) representing a basis for $H_{n-1}(\tilde{M}, \tilde{N})$ over $\mathbb{Z}(\pi)$ (See 2 of Lemma 3.6). Corresponding to any handle decomposition of (U, N) with only handles of type $(n-2)$ and $n-1$ we get a chain complex $0 \to \tilde{C}_{n-1} \xrightarrow{d} \tilde{C}_{n-2} \to 0$ whose homology will precisely be $H_*(\tilde{U}\tilde{N})$.

$$
\begin{tikzcd}
H_n(\tilde{M},\tilde{U}) \arrow[r, "\partial"] & H_{n-1}(\tilde{U},\tilde{N}) \arrow[r] & H_{n-1}(\tilde{M},\tilde{N}) \arrow[r] & H_{n-1}(\tilde{M},\tilde{U}) \arrow[r] & H_{n-2}(\tilde{U},\tilde{N}) \arrow[r] & 0 \\
H_n(\tilde{M}_1,\tilde{N}_1) \arrow[u, "\text{a excision}"'] & & & H_{n-1}(\tilde{M}_1,\tilde{N}_1) \arrow[u, "\text{excision}", "a"'] & &
\end{tikzcd}
$$

□

111 For the modules $\tilde{C}_{n-1}$, $\tilde{C}_{n-2}$ the cells corresponding to handles of type $(n-1)$ and $(n-2)$ respectively form a basis over $\mathbb{Z}(\pi)$.

Proposition 4.5. *There exists a handle decomposition for (U, N) with $2m$ handles of type $(n-2)$ and $2m$ handles of type $(n-1)$ (where m is a certain integer $\geq r$) such that the boundary operator $C_{n-1} \xrightarrow{d} C_{n-2}$ with reference to the basis given by the handles has a matrix of the form $\begin{pmatrix} X & 0 \\ 0 & S^{-1}T \end{pmatrix}$, where S and T are $m \times m$ invertible matrices over $\mathbb{Z}(\pi)$ and X is the $m \times m$ matrix $\begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}$.*

Proof. By remark 4.4 we have $H_{n-1}(\tilde{U}, \tilde{N}) \simeq H_{n-1}(\tilde{M}, \tilde{N})$ and $H_{n-2}(\tilde{U}, \tilde{N}) \simeq H_{n-1}(\tilde{M}_1, \tilde{N}_1)$. Since M_1 is a translate of M we have $H_{n-1}(\tilde{M}, \tilde{N}) \simeq H_{n-1}(\tilde{M}_1, \tilde{N}_1)$ and by our choice of M, $H_{n-1}(\tilde{M}, \tilde{N})$ is a free $\mathbb{Z}(\pi)$-module of rank r. The pair (U, N) has a handle decomposition with only handles of type $n-2$ and $n-1$. Choose one such and let $0 \to \tilde{B}_{n-1} \xrightarrow{d} \tilde{B}_{n-2} \to 0$ be the complex corresponding to the chosen handle decomposition, giving the homology of the pair $(\tilde{U}, \tilde{N})$. Here $\tilde{B}_{n-1}$ and $\tilde{B}_{n-2}$ are free $\mathbb{Z}(\pi)$-modules of finite rank. Since the homology of the complex B is the same as $H_*(\tilde{U}, \tilde{N})$ we get the following exact sequence.

$$0 \to Imd \to \tilde{B}_{n-2} \to H_{n-2}(\tilde{U}, \tilde{N}) \underset{r}{\xrightarrow{\simeq}} \mathbb{Z}(\pi) \to 0.$$

It follows that Imd is finitely generated and $\mathbb{Z}(\pi)$-projective. Adding a finite number of pairs of mutually cancelling handles if necessary we can assume that imd is a free $\mathbb{Z}(\pi)$-module. (Here we use the fact that
112 Imd is stably free since $\tilde{B}_{n-2}$ is free of finite rank). Also we have the exact sequence $0 \to H_{n-1}(\tilde{U}, \tilde{N}) \simeq \sum_r \mathbb{Z}(\pi) \to \tilde{B}_{n-1} \xrightarrow{d} Imd \to 0$. If the rank of the free $\mathbb{Z}(\pi)$-module Imd is k then it follows that both $\tilde{B}_{n-1}$ and $\tilde{B}_{n-2}$ have rank m where $m = k + r$ and that $\exists$ bases $u_1, \ldots u_m$ of $\tilde{B}_{n-1}$ and $v_1, \ldots v_m$ of $\tilde{B}_{n-2}$ satisfying $du_1 = \cdots = du_r = 0$; $du_{r+1} = v_{r+1}, \ldots, du_m = v_m$. Thus the matrix of d with reference to the bases $u_1, \ldots u_m$ and $v_1, \ldots, v_m$ of $\tilde{B}_{n-1}$ and $\tilde{B}_{n-2}$ respectively is $X = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}$. Let $e_1^{n-1}, \ldots e_m^{n-1}$ and $e_1^{n-2}, \ldots, e_m^{n-2}$ be the natural bases for $\tilde{B}_{n-1}$ and

$\tilde{B}_{n-2}$ given by the handles and let the matrix of d with reference to this natural pair of bases be A. Now add m pairs of mutually cancelling handles of types $n-2$ and $n-1$. With respect to the handle decomposition of (U, N) thus obtained the chain modules $\tilde{C}_{n-1}$ and $\tilde{C}_{n-2}$ are both free $\mathbb{Z}(\pi)$-modules of rank $2m$ and the matrix of d with reference to the natural pair of bases constituted by the handles is $\left(\begin{smallmatrix} A & 0 \\ 0 & I_m \end{smallmatrix}\right)$. If $e^{n-1}_{m+1}, \dots, e^{n-1}_{2m}$ and $e^{n-2}_{m+1}, \dots, e^{n-2}_{2m}$ are the elements of $\tilde{C}_{n-1}$ and $\tilde{C}_{n-2}$ respectively, corresponding to the newly attached m pairs of mutually cancelling handles then $u_1, \dots u_m$; $e^{n-1}_{m+1}, \dots, e^{n-1}_{2m}$ and $v_1, \dots v_m$; $e^{n-2}_{m+1}, \dots e^{n-2}_{2m}$ form bases for $\tilde{C}_{n-1}$ and $\tilde{C}_{n-2}$ with reference to which the matrix of d is $\left(\begin{smallmatrix} X & 0 \\ 0 & I_m \end{smallmatrix}\right)$. Now, there exist elements $S, TGL(m, \mathbb{Z}(\pi))$ such that $X = SAT^{-1}$. The matrices $\left(\begin{smallmatrix} S & 0 \\ 0 & S^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} T^{-1} & 0 \\ 0 & T \end{smallmatrix}\right)$ are products of elementary matrices in **113**
$GL(2m, \mathbb{Z}(\pi))$, and we have

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} X, & 0 \\ 0 & S^{-1}T \end{pmatrix}$$

Thus to prove proposition 4.5 it suffices to prove the following. □

Lemma 4.6. *One can change the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ of d by left or right multiplication by elementary matrices by performing an isotopy of the attaching map of the handles.*

Proof. Let $U = I \times N + \varphi^{n-2}_1 + \cdots + \varphi^{n-2}_{2m} + \varphi^{n-1}_1 + .. + \varphi^{n-1}_{2m}$ be the handle decomposition which gives the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ for d. For each i such that $1 \leq i \leq 2m$ let Y_i be the right hand boundary of $I \times N + \varphi^{n-2}_1 + \cdots + \varphi^{n-2}_{2m} +$ all the handles of type $(n-1)$ except the i th. First we prove the lemma for left multiplication by elementary matrices. We actually show that by an isotopy of φ_i into Y_i one can change de^{n-1}_i by any $\sum\limits_{j \neq i} x_j de^{n-1}_j$ with arbitrary $x_j \in \mathbb{Z}(\pi)$. For this it suffices to prove the same assertion for $x_j de^{n-1}_j$ for a particular $j \neq i$ and $x_j \epsilon \pm \pi$. Now $\varphi_j(S^{n-2} \times *)$ with $*$ any point on bD^2, is isotopic to the trivial imbedding in Y_i for $i \neq j$, because $\varphi_j(S^{n-2} \times *)$ bounds a cell on the boundary of the handle φ_j. Perform "connected sum" of φ_i and φ_j along an arc representing x_j and take it as the new φ'_j. For proving the lemma for multiplication on the right by an elementary matrix we look at the dual handle decomposition. **114**

Let $U = I \times N_1 + \varphi_1^{*2} + \cdots + \varphi_{2m}^{*2} + \varphi_1^{*3} + \cdots + \varphi_{2m}^{*3}$ be the dual handle decomposition. Let $0 \to \tilde{C}_3 \xrightarrow{d^*} \tilde{C}_2 \to 0$ be the chain complex corresponding to this handle decomposition. With respect to the canonical bases of $\tilde{C}_3$ and $\tilde{C}_2$ constituted by the handles of type 3 and 2 respectively, the matrix of d^* is the same as $\pm\left(\begin{smallmatrix} A^* & 0 \\ 0 & I_m \end{smallmatrix}\right)$ where $A^* = (a^*_{ij})$ with $a^*_{ij} = \overline{a}_{ji} \ldots$. Here (a_{ij}) is the matrix A and for each $a \in \mathbb{Z}(\pi)$, $\overline{a}$ is the element which corresponds to a under the map which carries any $x \in \pi$ into the dement $\pm x^{-1}$. (The sign depending on whether x preserves $(+)$ or reverses $(-)$ an orientation of $\tilde{U}$). Choose listings of 3 and 2 cells for the dual decomposition $\tilde{\varepsilon}_1^3, \ldots \tilde{\varepsilon}_{2m}^3; \tilde{\varepsilon}_1^2, \ldots, \tilde{\varepsilon}_{2m}^2$ so as to satisfy $\tilde{e}_i^{n-2}.\tilde{\varepsilon}_j^3 = \delta_{ij}$; $\tilde{e}_i^{n-1}.\tilde{\varepsilon}_j^2 = \delta_{ij}$ and $\tilde{e}_i^{n-1}.\ \sigma\tilde{\varepsilon}_j^2 = \delta_{ij}\delta_{\sigma,1}$ for every $\sigma \in \pi$. Using the formula $\tilde{\varepsilon}_k^3.d\tilde{e}_i^{n-1} = d^*\tilde{\varepsilon}_k^3.\tilde{e}_i^{n-1}$ (up to a sign which depends only on n and not on i and k) it is easy to see that the matrix of d^* with reference to the pair of bases constituted by $\tilde{\varepsilon}_1^3, \ldots, \tilde{\varepsilon}_{2m}^3$ and $\tilde{\varepsilon}_1^2, \ldots, \tilde{\varepsilon}_{2m}^2$ is precisely $\left(\begin{smallmatrix} A^* & 0 \\ 0 & I \end{smallmatrix}\right)$(up to sign). Now, by what we have proved already, this handle decomposition of (U, N_1) can be altered so as to alter the matrix $\left(\begin{smallmatrix} A^* & 0 \\ 0 & I \end{smallmatrix}\right)$ be left multi-
115 plication by an elementary matrix. Now, taking the dual of the altered handle decomposition we get a handle decomposition for (U, N) which alters the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ be right multiplication by an elementary matrix. This proves Lemma 4.6. $\square$

We choose a handle decomposition for (U, N) of the type mentioned in Proposition 4.5. Then the Kernel of d: $\tilde{C}_{n-1} \to \tilde{C}_{n-2}$ is the free $\mathbb{Z}(\pi)$-module of rank r with the elements $\tilde{e}_1^{n-1}, \ldots, \tilde{e}_r^{n-1}$ corresponding to the first r handles of type $(n-1)$.

Assertion. *Any one of the elements $\tilde{e}_i^{n-1}(1 \leq i \leq r)$ can be represented by a C^∞ imbedding $\theta_i : (D^{n-1}, S^{n-2}) \to (U, N)$.*

In fact $de_i^{n-1} = 0$ implies that any lifting $\tilde{\varphi}_i(S^{n-2} \times *)$ of $\varphi_i(S^{n-2} \times *)$ has trivial homology intersection in $N_\circ$ with any lifting $\tilde{\varphi}_j(* \times S^2)$ of any of the transverse 2-spheres of the handles of type $n-2$. (Here $N_\circ$ is the right hand boundary of $I \times N + \sum\limits_{j=1}^{2m} \varphi_j^{n-2}$). Now use Proposition 3.8 with $X = \sum\limits_{j=1}^{2m} \varphi_j(* \times S^2)$ and $Y = \sum\limits_{i=1}^{2m} \varphi_i(S^{n-2} \times *)$. The condition $\pi(N_\circ - Y) \to \pi_1 N_\circ$ an isomorphism is satisfied because of the following diagram (where as above N_1 is the right boundary of U):

$$\begin{tikzcd}
\pi_1(N_\circ - Y) \arrow[r, "\simeq"] \arrow[d] & \pi_1(N_1 - U_{i=1}^{2m}\varphi_i^*(D^{n-1}\times S^1)) \arrow[r, phantom, "\simeq"] & \pi_1 N_1 \arrow[d, "\simeq"] \\
\pi_1 N_\circ \arrow[rr, "\simeq"'] & & \pi_1 W
\end{tikzcd}$$

The "upper" horizontal isomorphisms are obvious. The isomor- **116**
phism $\pi_1 N_1 \to \pi_1 W$ follows from the fact that (M_1, N_1) is a 1-neighbourhood. The "bottom" horizontal map is also an isomorphism because $\pi_1 N_\circ \to \pi_1 U_1$ is an isomorphism ($U_1 = I \times N_\circ +$ (handles of type $n-1$).) $\pi_1 U_1 \to \pi_1 U$ is also an isomorphism since $U = U_1 +$ (handles of type 3), and $\pi_1 U \to \pi_1 W$ has been noted to be an isomorphism before. (Recall Lemma 2.8.) Using proposition 3.8 as before we see that we can find C^∞ imbeddings $\theta_i : (D^{n-1}, S^{n-2}) \to (U, N)$ representing $\tilde{e}_i^{n-1} \in H_{n-1}(\tilde{U}, \tilde{N})$. Let B the union of tabular neighbourhoods of $\theta_i(D^{n-1})$ and N in M and let $M' = \overline{M - B}$. By Van Kampen it is easy to see that $\exists$ an isomorphism $\pi_1(N) \to \pi_1(N')$ where $N' = bM'$ and that the inclusion $M' \to M$ induces an isomorphism $\pi_1(M') \to \pi_1(M)$. Also the isomorphism $\pi_1(N) \to \pi_1(N')$ makes the diagram.

$$\begin{tikzcd}
\pi_1(N) \arrow[r] \arrow[dd, "\approx"] & \pi_1(M) \arrow[dd, "\simeq"'] \arrow[dr, "\simeq"] & \\
& & \pi_1(W) \\
\pi_1(N') \arrow[r] & \pi_1(M') \arrow[ur] &
\end{tikzcd}$$

commutative. It follows that M' is a 1-neighbourhood. Now from the homology exact sequence of the triple $\tilde{M}, \tilde{B}, \tilde{N}$ it follows that $H_i(\tilde{M}', \tilde{N}') = 0$ for $i \leq n-2$ and $H_{n-1}(\tilde{M}', \tilde{N}') \simeq H_{n-1}(\tilde{M}, \tilde{N})/(e_1, \ldots, e_r) = 0$. Thus starting from any $(n-2)$ neighbourhood M of ∞ with $H_{n-1}(\tilde{M}, \tilde{N})$
free of rank r over $\mathbb{Z}(\pi)$ we have constructed a $(n-1)$ neighbourhood **117**
M' of ∞ with $M' \subset M$.

Proposition 4.7. *There exist arbitrary small* $(n-1)$*-neighbourhoods of* ∞.

5 Completion of the proof of siebenmann's theorem

Lemma 5.1. *Suppose M and M_1 are two $(n-1)$ neighbourhoods of ∞ with $M \supset M_1$ and $bM_1 = \phi$. Then $U = \overline{M - M_1}$ is a h-cobordism between bM and bM_1.*

Proof. Denote bM and bM_1 by N and N_1 respectively. Then as already observed $\pi_1(N) \to \pi_1(U)$, $\pi_1(N_1) \to \pi_1(U)$ are isomorphisms. (Remark B after Lemma 3.6). Since M and M_1 are $(n-1)$-neighbourhoods we have $H_i(\tilde{M}, \tilde{N}) = 0 = H_i(\tilde{M}_1, \tilde{N}_1)$ for all i. In fact by Lemma 4.1, $H_*(\tilde{M}, \tilde{N})$ or $H_*(\tilde{M}_1, \tilde{N}_1)$ is the homology of a complex of the form $C \to \tilde{B}_{n-1} \to \tilde{B}_{n-2} \to 0$. Thus $H_i(\tilde{M}, \tilde{N}) = 0$ for $i > n$ and by definition of an $(n-1)$-neighbourhood of ∞ we have $H_i(M, N) = 0$ for $i \leq n-1$. From the homology exact sequence of the triple $(\tilde{M}, \tilde{U}, \tilde{N})$ we see immediately that $H_j(\tilde{U}, \tilde{N}) = 0$ for every j. Thus to prove Lemma 5.1 it only remains to show that $H_i(\tilde{U}, \tilde{N}_1) = 0$ for every j.

$$\begin{tikzcd} H_i(\tilde{U},\tilde{N}) \arrow[r] & H_i(\tilde{M},\tilde{N}) \arrow[r] & H_i(\tilde{M},\tilde{U}) \arrow[r] & H_{i-1}(\tilde{U},\tilde{N}) \arrow[r] & H_{i-1}(\tilde{M},\tilde{N}) \arrow[r] & \cdots \\ & & H_i(\tilde{M}_1,\tilde{N}_1) \arrow[u, "\text{excision}", "u"'] & & & \end{tikzcd}$$

□

118 For the pair (U, N) we have a handle decomposition with handles of type $n-2$ and $n-1$ only. If $0 \to \tilde{C}_{n-1} \xrightarrow{d} \tilde{C}_{n-2} \to 0$ is the corresponding complex given the homology of $(\tilde{U}, \tilde{N})$, from the fact that $H_i(\tilde{U}, \tilde{N}) = 0$ $\forall i$ it follows that d is an isomorphism. If we use the dual handle decomposition for (U, N_1) the homology $H_*(\tilde{U}, \tilde{N}_1)$ will be the homology of a complex of the form $0 \to \tilde{C}_3 \xrightarrow{d^*} \tilde{C}_2 \to 0$. If $A = (a_{ij})$ is the matrix of d with respect to the bases constituted by the handles of type $(n-2)$ and $(n-1)$, then as already seen the matrix of d^* with respect to the bases constituted by the handles of type 3 and 2 in the dual decomposition is $A^* = (a^*_{ij})$ (up to sign) where $a^*_{ij} = \overline{a_{ji}}$. It follows that if d is an isomorphism so is d^*. Hence $H_*(\tilde{U}, \tilde{N}_1) = 0$.

Proposition 5.2. *Let M be any $(n-1)$-neighbourhood of ∞ in W. Then M is diffeomorphic to $N \times [0, \infty)$ where $N = bM$.*

The proof of this proposition uses the S-cobordism theorem of Barden-Mazur-Stallings [5], [6] or [8]. Let U be a h-cobordism between two compact, connected oriented C^∞ manifolds V^n and V'^n of dimension $n \geq 5$. Using the isomorphisms $\pi_1(V) \to \pi_1(U)$ and $\pi_1(V') \to \pi_1(U)$ we identify all the three groups $\pi_1(V)$, $\pi_1(U)$ and $\pi_1(V')$ and abstractly denote any one of them by π. Let $\tau(U, V) \in Wh(\pi)$ denote the torsion of the pair (U, V). We now state the S-cobordims theorem which actually consists of two parts.

119 **S-Cobordism Theorem:** (1) *The inclusion of V in U cab be extended into a diffeormorphism of $V \times I$ onto U if and only if $\tau(U, V) = 0$.*

(2) *Given a compact, connected C^∞ manifold V^n of dimension $n \geq 5$ and any $\tau \in Wh(\pi)$ where $\pi = \pi_1(V)$, there exists a h-cobordism U between V and a certain V' such that $\tau(U, V) = \tau$.*

For more information about torsion and the Whitehead group $Wh(\pi)$ refer to [1], [5] or [13]. We list below some known properties of torsion that we need for the proof of Proposition 5.2.

The symbols V, V', V_1, V'_1 etc. are used to denote connected, compact, C^∞ manifolds. Let U_1 be a h-cobrdism between V_1 and V'_1, and

U_2 a h-cobordism between V_2 and V'_2. Let $g : V_2 \to V'_1$ be a diffeomorphism of V_2 onto V'_1. Let $U = U_1 \underset{g}{\cdot} U_2$ be the differential manifold got from the union of U_1 and U_2 by identifying V_2 with V'_1 by means of the diffeomorphism g. The groups $\pi_1(V_1)$, $\pi_1(U_1)$ and $\pi_1(V'_1)$ are all identified as explained already and let π_1 denote any one of them. Let π_2 have a similar meaning with respect to V_2, U_2 and V'_2 (i.e. $\pi_2 = \pi_1(V_2)$ etc.). The diffeomorphism g induces an isomorphism $g*$: $\pi_2 \to \pi_1$. If $\tau_1 = \tau(U_1, V_1) \in Wh(\pi_1)$ and $\tau_2 = \tau(U_2, V_2) \in Wh(\pi_2)$ then $U = U_{1\dot{g}}U_2$ is a h-cobordism between V_1 and V'_2 satisfying $\tau(U, V_1) = \tau_1 + g_*(\tau_2)$. In particular if U_1 is a h-cobordism between V and V' and if U_2 is a h-cobordism between V' and a certain V'' such that $\tau(U', V') = -\tau(U, V)$ then U_1. U_2 is diffeomorphic to $V \times I$ whenever $\dim V (= \dim V') \geq 5$. If **120**
U is a h-cobordism between V and V' with torsion $\tau(U, V)$, we can construct a h-cobordism U^{-1} from V' to some V'' with torsion $\tau(U^{-1}, V') = -\tau(U, V)$. (Use part (2) of the S-cobordism theorem). Then, pasting U and U^{-1} along V' by the identity mapping, the h-cobordism UU^{-1} form V to V'' has torsion $\tau(U, V) + \tau(U^{-1}, V') = 0$. It follows by part (1) of the S-cobordism theorem that $U \cup U^{-1}$ is diffeomorphic to $V - I$ and in particular that V and V'' are diffeomorphic. The formation of products of h-cobordisms satisfies the following associativity rule. Let $U_i (i = 1, 2, 3)$ be a h-cobordism between V_i and V'_i and let $g : V_2 \to V'_1; h : V_3 \to V'_2$ be diffeomorphisms. Then $\exists$ a diffeomorphism $\alpha : (U_1 \underset{g}{\cdot} U_2) \underset{h}{\cdot} U_3 \to U_1 \underset{g}{\cdot} (U_2 \underset{h}{\cdot} U_3)$ extending the identity map of V_1. Also if U is a h-cobordism between V and $V' \exists$ a diffeomorphism β: $U \to U.V' \times I$ with $\beta | V = Id_V$ and $\beta(v') = (v', 1)\ \forall\, v' \in V'$. (This is a consequence of the fact that V' is differentiably collared in U). For the proof of Proposition 5.2 we need the following Lemma on infinite products of h-cobordisms.

Lemma 5.3. *For every integer $k \geq 1$ let U_k be a h-cobordism between V_k and V'_k and let $V'_k = V_{k+1}$. If $\dim V_1 \geq 5$ then the infinite product $U_1.U_2.U_3.\ldots$ is diffeomorphic to $V_1 \times [0, \infty)$.*

Proof. As observed already $\exists$ diffeomorphisms β_k: $U_k \to U_K.V'_k I$ with **121**
$\beta_k | V_k = Id_{V_k}$ and $\beta_k(v') = (v', 1)\ \forall\, v' \subset V'_k$. Hence the infinite product $U_1.U_2.U_3.\ldots.$ is also diffeomorphic to the infinite product $U_1.V'_1 \times I$.

$U_2.\ V_2' \times I.\ U_3.\ V_3' \times I.\ldots.$ For every integer $k \geq 1$ the product $U_k^{-1}.U_{k-1}^{-1}.\ldots U_1^{-1}.\ U_1.\ldots U_k$ is a h-cobordism with torsion zero. Therefore $\exists$ a diffeomorphism. $\theta_k\colon V_k' \times I \to U_k^{-1}\ldots.U_1^{-1}.\ U_1.\ldots U_k$ satisfying $\theta_k(v', 0) = v'$ of the left hand boundary of $U_k^{-1}.\ldots.U_1^{-1}.\ U_1\ldots U_k$. The map $v' \to \theta_k(v', 1)$ is a diffeomorphism g_k of V_k' onto the right hand boundary of $U_k^{-1}.\ldots U_1^{-1}.\ U_1.\ldots U_k$. Now it is clear that the product $U_1.\ V_1' \times I.\ U_2.\ V_2' \times I.\ U_3.\ V_3' \times I.\ U_4.\ldots.$ is diffeomorphic to the product

$$U_1.(U_1^{-1}.U_1) \underset{g_1}{\cdot} U_2.(U_2^{-1}.U_1^{-1}.U_1.U_2) \underset{g_2}{\cdot} U_3.(U_3^{-1}.U_2^{-1}.U_1^{-1}.U_1.U_2.U_3) \underset{g_3}{\cdot} U_4\ldots.$$

Also it is clear that the diffeomorphism $g_k\colon V_k' \to V_k'$ is homotopic to the identity map of V_k' and hence $g_{k*}\colon \pi \to \pi$ is the identity map. Since product formation of h-cobordisms is an associative operation we have

$$U_1.(U_1^{-1}.U_1) \underset{g_1}{\cdot} U_2.(U_2^{-1}.U_1^{-1}.U_1.U_2) \underset{g_2}{\cdot} \ldots. \text{ diffeomorphic to}$$

$$U_1.U_1^{-1}.(U_1 \underset{g_1}{\cdot} U_2.U_2^{-1}.U_1^{-1}).(U_1.U_2 \underset{g_2}{\cdot} U_3.U_3^{-1}.U_2^{-1}.U_1^{-1}).\ldots.$$

122 Denoting the products $U_1.\ldots U_k$; $U_1\ldots U_k \underset{g_k}{\cdot} U_{k+1}.U_{k+1}^{-1}\ldots U_1^{-1}$ and $U_{k+1}.\ U_{k+1}^{-1}.\ U_k^{-1}.\ldots U_1^{-1}$ by A_k; B_k and C_k respectively we have $\tau(B_k, V_1) = \tau(A_k, V_1) + (g_k)_*\ (\tau(C_k, V_{k+1})) = \tau(A_k, V_1) + \tau(C_k, V_{k+1})$ since g_{k*} is the identity map. But $\tau(A_k, V_1) + \tau(C_k, V_{k+1}) = 0$. Hence the inclusion of V_1 into B_k as the left hand boundary extends to a diffeomorphism of $V_1 \times I$ onto B. It follows that the product

$$(U_1.U_1^{-1}).(U_1 \underset{g_1}{\cdot} U_2.U_2^{-1}.U_1^{-1}).U_1.U_2 \underset{g_2}{\cdot} U_3.U_3^{-1}.U_2^{-1}.U_1^{-1}\ldots.$$

is diffeomorphic to $V_1 \times [0, \infty)$. This completes the proof of Lemma 5.3. We now take up the proof of Proposition 5.2. Let M be any $(n-1)$-neighbourhood of ∞ in W. The Deck transformation group of the covering $W \xrightarrow{p} V$ is the same as that of $\mathbb{R} \xrightarrow{q} S^1$. Let α denote the diffeomorphism of W which corresponds to translation by $+1$ of $\mathbb{R}$ on itself, under the isomorphism between the Deck transformation groups. Choose an integer $\ell > 1$ such that $N \cap \alpha^\ell N = \phi(N = bM)$. Let $M_k =$

$\alpha^{k\ell} M$ for each integer $k \geq 0$ and $N_k = bM_k$. We have $M_\circ = M$, $M_k \supset M_{k+1}$ and $N_k \cap N_{k+1} = \phi$. Let $U_k = \overline{M_{k-1} - M_k}$ for any $k \geq 1$. We then have $\bigcup\limits_{k\ 1} U_k = M$. By Lemma 5.1, U_k is a h-cobordism between N_{k-1} and N_k. By Lemma 5.3 it now follows that M is diffeomorphic to $N \times [0, \infty)$. Actually the inclusion of N into M extends to a diffeomorphism of $N \times [0, \infty)$ onto M. ◻

Theorem 5.4. *Let M be any $(n-1)$-neighbourhood of ∞ in W. Then W is diffeomorphic to $N \times \mathbb{R}$ where $N = bM$.*

Proof. For the integer ℓ having the same meaning as above we see that **123** $\alpha^{-\ell} N \cap N = \phi$. It follows that for every integer $k \geq 0$ if we define M_{-k} by $M_{-k} = \alpha^{-k\ell} M$ then $N_{-k} \cap N_{-k-1} = \phi \forall k \geq 0$, where $N_{-k} = bM_{-k}$. Also $M_{-k} \supset M_{-k-1}$. Now, if $U'_k = \overline{M_{k-1} - M_k}$ for each $k \geq 1$, by Lemma 5.1, U'_k is a h-cobordism between N_{-k} and N_{-k+1}. It is clear that if $M' = \overline{W - M}$, then M' is the infinite product of the h-cobordisms U'^{-1}_k and by arguments used in the proof of Lemma 5.3 we see that the inclusion map of N into M' can be extended into a diffeomorphism of $N \times (-\infty, 0]$ onto M'. This, combined with Proposition 5.2 gives Theorem 5.4. ◻

This completes the proof of Siebenmann's Theorem.

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