Lectures on
Torus Embeddings and Applications

(*Based on joint work with Katsuya Miyake*)

By

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Tata Institute of Fundamental Research
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Introduction

In recent years, the theory of torus embeddings has been finding many applications. The point of the theory lies in its ability of translating meaningful algebra-geometric and analytic phenomena into very simple statements about the combinatorics of cones in affine space over the reals.

In different terminology, it was first introduced by Demazure [8] and then by Mumford et al. [63], Satake [57] and Miyake-Oda [40]. There is already a good and concise account on it in [63]. Nevertheless, we produce here another. For one thing, we wanted to supply the details of the partial classification of complete non-singular 3-dimensional torus embeddings announced in [40].

Besides, we wanted to make the theory, in its most general form, accessible to non-algebraic geometers in view of its possible applications in other branches of mathematics. We can state at least the main results without using algebraic geometry, although for the proof we cannot avoid using it. This was made possible by the following down-to-earth description due to Ramanan of normal affine torus embeddings over a field k of finite type as the set of unitary semigroup homomorphisms

$$U(\sigma) = \mathrm{Hom}_{\text{unit.semigr.}}(\check{\sigma} \cap M, k),$$

where $(i) N \cong \mathbb{Z}^r$ and σ is a convex rational polyhedral cone in $N_R \cong R^r$ with $\sigma \cap (-\sigma) = \{0\}, (ii)$ for the dual $\mathbb{Z}$-module M of N,

$$\check{\sigma} \cap M = \{m \in M : \langle m, y \rangle \geq 0 \text{ for all } y \in \sigma\}$$

is a finitely generated additive semigroup and (iii) k is considered to be a semigroup under the *multiplication*.

When we have a suitable collection $\triangle$ of such cones, an *r.p.p. decomposition*, then $U(\sigma)'s$ can be canonically patched together to form a normal and separated algebraic variety over k locally of finite type

$$T_N \operatorname{emb}(\triangle)$$

which has an effective action of the algebraic torus

$$T_N = N \otimes_{\mathbb{Z}} k^* = \operatorname{Hom}_{gr}(M, k^*) = k^* \times \ldots \times k^*$$

with a dense orbit. Such a variety is called a *torus embedding*, since it is a partial compactification of T_N. Conversely, we get all normal torus embeddings in this way (Theorem 4.1). Important Algebre-geometric phenomena can most often be described purely in terms of $\triangle$ (Theorems 4.2, 4.3, 4.4 and Corollary 4.5).

We may say that $\triangle$ contains all the relevant information, in a unified and globalized way, about the "exponents of monomials" necessary to describe such varieties. When an algebraic variety or a morphism can be described solely in terms of monomials, then there is a good chance that it can most effectively be described in terms of torus embeddings. For instance, a normal irreducible affine algebraic variety $V \subset \mathbb{A}_r$ defined by equation of the form

$$X_1^{a_1} X_2^{a_2} \ldots X_r^{a_r} = X_1^{b_1} X_2^{b_2} \ldots X_r^{b_r}$$

can be expressed as $U(\sigma)$ for some σ. (cf. (7.9))

We try to avoid overlaps with Demazure [8] and Mumford et al. [63] as much as possible. For later convenience for reference, we collect together in §. 4 the first main theorem in their most general form and leave their proof to §. 5. Various standard examples are collected together in §. 7.

In §. 6, we deal with torus embedding which can be embedded into projectives spaces. The results are slightly more general than those in [8] but less so than those in [63].

§. 8 and §. 9 are devoted to the classification of complete nonsingular torus embeddings. We are reduced to the classification of certain weighted circular graphs and that of weighted triangulations of the

2-sphere. As by-products, we obtain many interesting complete non-singular rational three folds. (cf. Prop. 9.4) Besides, we see that torus embeddings provides us with a good testing ground for important conjectures on birational geometry in higher dimension.

There are many basic results we left out : For the cohomology of equi variant coherent sheaves on torus embeddings as well as the description of the automorphism groups of torus embeddings, we refer the reader to Demazure [8].

Mumford et al. [63] generalizes the notion of torus embedding to that of *toroidal embeddings* and proves very important *semi-stable reduction theorem*. Torus embeddings have also been used effectively in the compactification problem of the moduli spaces by Satake [57], Hirzebruch [19], Mumford et al.[61], Namikawa [46], Nakamura [44], Rapoport [54], Oda-Seshadri [49] and Ishida [26].

Here we deal with more elementary but illustrative applications in Chapter 2.

When the ground field k is the field $\mathbb{C}$ of complex numbers or one with a *non-archimedean* rank one valuation, then $U(\sigma) = \mathrm{Hom}_{unit.semigr.}$ $(\check{\sigma} \cap M, k)$, hence $T_N\,\mathrm{emb}(\triangle)$, has the topology induced from that of k by the valuation. Let CT_N be the maximal compact torus of T_N in this topology. Then we can usually draw the picture of the quotient of a torus embedding $T_N\,\mathrm{emb}(\triangle)$ by CT_N and get better geometric insight. The quotient was introduced by Mumford et al. [61]. We call it the *manifold with corners* after Borel-Serre [4] and denote it by

$$Mc(N, \triangle) = T_N\,\mathrm{emb}(\triangle)/CT_N.$$

Using this we will be able to visualize the construction and degenerations of complex tori, Hopf surfaces and other class VII_0 surfaces introduced by Inoue. (cf. §. 11, §. 13, §. 14 and §. 15). Using Suwa's classification of hyper elliptic surfaces om [60], Tsuchihashi [62] were able to describe their degenerations and the compactification of the moduli space in terms of torus embeddings.

There are many recent results on the actions of algebraic and analytic groups other than algebraic tori on algebraic varieties and complex manifolds. See, for instance, Akao [1], Popov [52], [53], Orlik-Wagreich [50] and Ishida [25].

Complete non-singular 2-dimensional torus embeddings over $\mathbb{C}$ give rise to rational compactifications of $\mathbb{C}^2$ and $(\mathbb{C}^*)^2$. Compactifications of $\mathbb{C}^2$ were shown to be always rational by Kodaira [31] and were classified by Morrow [37]. There are, however, many non-rational, even non-algebraic, compactifications of $(\mathbb{C}^*)^2$. They were recently classified by simha [58] and Ueda [64].

These notes are based on a joint work with K. Miyake of Nagoya University and grew out of the lectures which the author gave at Tata Institute of Fundamental Research, University of Paris-Sud, Orsay, Nagoya University, Tohoku University, Instituto Jorge Juan, Harvard University and various other places. He would like to thank the mathematicians at these institutions for the hospitality and the patience shown to him. The notes taken by K.Makio at Tohoku University was very helpful.[1]

Recall that for a connected locally noetherian scheme X, its *dualizing complex* R_X is determined uniquely up to quasi-isomorphism, dimension shift and the tensor product of invertible 0_X-modules. (cf. Hartshorne, Residue and duality, Lecture Notes in Math. 20, Springer-Verlag, 1966).

Let $T = T_N$ be an algebraic torus and consider the normal torus embedding Temb($\triangle$) corresponding to an r.p.p.decomposition $(N, \triangle)$. Let us fix an orientation for each cone $\sigma \in \triangle$ and define, in case $\dim\tau - \dim\sigma = 1$, the incidence number $[\sigma : \tau] = 0, 1$ or -1 in an appropriate manner so that we have a complex

$$C(\triangle, \mathbb{Z}) = (\ldots \to 0 \to C^0 \xrightarrow{d} C^1 \to \ldots \to C^{\text{rank } N} \to 0 \to \ldots)$$

where C^j is the free $\mathbb{Z}$-module generated by the set of j-dimensional cones in $\triangle$ and where

$$d(\sigma) = \sum_{\dim\tau-\dim\sigma=1} [\sigma : \tau](\tau).$$

[1]After these notes were written, M.Ishida of Tohoku University obtained the following seemingly definitive result on the Cohen-Macaulay and Gorenstein properties in relation to torus embeddings.

Let X be a *T-invariant closed connected reduced subscheme* of $\mathrm{Temb}(\triangle)$ and consider the complex R_X of 0_X-modules defined by

$$R_X^j = \oplus\, 0_Y$$

where the direct sum is taken over all the T-invariant closed *irreducible* subvarieties of X which is of codimension j *in Temb*$(\triangle)$ and where δ : $R_X^j \to R_X^{j+1}$ is defined as follows : $X = \bigcup_{\sigma\in\sum} \overline{\mathrm{orb}(\sigma)}$ for a *star closed* subset $\sum = \{\sigma \in \triangle;\ \mathrm{orb}(\sigma) \subset X\}$. When $Y = \overline{\mathrm{orb}(\sigma)}$, then for $f_\sigma \in 0_Y$,

$$\delta(f_\sigma) = \sum [\sigma : \tau](\text{the restriction of } f_\sigma \text{ to } \overline{\mathrm{orb}(\tau)})$$

with the sum taken over $\tau \in \triangle$ with $\dim\tau - \dim\sigma = 1$.

Theorem 0.1 ((Ishida))**.** *If X is a T-invariant closed connected reduced subscheme of a normal torus embedding Temb$(\triangle)$, then R_X defined above is the dualizing complex for X.*

For $\rho \in E$, let $\sum_\rho = \{\sigma \in \sum; \sigma < \rho\}$ and consider the complex of k-modules $C(\sum_\rho, k)$ with the coboundary map induced by d.

Corollary 1 ((Ishida))**.** *Let X be as above. Then X is Cohen-Macaulay if and only if there exists ℓ such that for any $\rho \in \sum$, we have*

$$H^j(\sum_\rho, k) = 0 \quad \textit{for} \quad j \neq \ell$$

We immediately see that Tamb(δ) itself is Chohen-Maculay (cf. Remark after our Prop. 6.6 and Hochster [20]). On the other hand, if $\mathrm{Temb}(\triangle) = \mathbb{A}_r$ is the affine space, we get the results in Reisner (Cohen-Macaulay quotients of polynomial rings, Advances in Math. 21(1976), 30-43) and Hochster (Cohen-Macaulay rings, combinatorics and simplicial complexes, in *Ring Theory II*, Lecture Notes in Pure and App. Math. 26(1977). Dekker, pp. 171-223).

In a special case, the complex R_X. already appears in [44].

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Contents

Chapter 1

Torus embeddings

For simplicity, we work over an algebraically closed field k of arbi- **1**
trary characteristic. All k-algebras are commutative with unity and all k-algebra homomorphisms preserve unity. Although rather unconventional, we mean by an algebraic variety a reduced, irreducible and separated scheme over k which is only *locally* of finite type, i.e. possibly an infinite union of open subsets which are usual affine varieties of finite type.

1 Algebraic tori

In this section, we recall basic facts about algebraic tori which we use later.

We denote by k^* the multiplicative group of non-zero elements of k considered as an *algebraic group* over k. It is usually denoted by G_m and is the affine algebraic group $\mathrm{Spec}(k[t, t^{-1}])$ endowed with the comultiplication $t \longmapsto t \otimes t$ on the coordinate ring. It is more convenient to consider it as a group functor which assigns to a k-algebra B its multiplicative group B^* of units.

An *algebraic torus* over k is an algebraic group T isomorphic to a finite direct product $k^* \times \cdots \times k^*$.

Mutually dual free $\mathbb{Z}$-modules M and N with the pairing $\langle , \rangle : M \times$

$N \to \mathbb{Z}$ give rise to an algebraic torus

$$T = \mathrm{Hom}_{gr}(M, k^*) = N \otimes_{\mathbb{Z}} k^*.$$

2 Conversely, an algebraic torus T gives rise to the character group

$$M = \mathrm{Hom}_{\mathrm{alg.gr}}(T, k^*)$$

and the group of 1-parameter subgroups

$$N = \mathrm{Hom}_{\mathrm{alg.gr}}(k^*, T),$$

both written additively, together with the duality pairing

$$\langle\ ,\ \rangle : M \times N \to \mathrm{Hom}_{\mathrm{alg.gr}}(k^*, k^*) = \mathbb{Z}.$$

For m in M, we denote by $e(m) : T \to k^*$ the corresponding character. The coordinate ring $\ulcorner(0_T)$ is the group algebra of M over k with $\{e(m); m \in M\}$ forming a k-basis and with $e(m + m') = e(m)e(m')$. For n in N, the corresponding 1-parameter subgroup sends t in k^* to the element $t^{\langle ?,n\rangle}$ of $T = \mathrm{Hom}_{gr}(M, k^*)$ which maps m in M to $t^{\langle m,n\rangle}$ in k^*.

A homomorphism $f : T \to T'$ of algebraic tori correspond in one-to-one fashion with a homomorphism $f_* : N \to N'$ and a homomorphism $f^* : M' \to M$, where N' and M' are, respectively, the group of 1-parameter subgroups and character group of T'. The following fact is quite relevant to us later: f is surjective if and only if f_* has finite cokernel (resp. f^* is injective).

The main reason why things work out so well later is that algebraic tori are the only algebraic groups which, besides being connected and commutative, satisfy the following basic complete reducibility property:

3 **Theorem.** *Every algebraic representation of an algebraic torus is completely reducible, and is in fact a direct sum of one dimensional representations, i.e. characters.*

For the proof of this theorem as well as other basic facts about algebraic groups, we refer the reader to Borel [3].

2 Torus embeddings and Summaries theorem

An algebraic action of an algebraic torus T on an algebraic variety X is a morphism $T \times X \to X$ satisfying the usual axioms $(tt')x = t(t'x)$ and $ex = x$ for $t, t' \in T$, $x \in X$ and $e =$ (the identity of T). When algebraic tori T and T' act on algebraic varieties X and X', respectively, an *equivariant morphism* consists of a homomorphism $f : T \to T'$ and a morphism $\overline{f} : X \to X'$ such that $\overline{f}(tx) = f(t)\overline{f}(x)$ for $t \in T$ and $x \in X$.

Here is one of the basic results about torus actions:

Theorem (Sumihiro). *If an algebraic torus T acts algebraically on a* normal *algebraic variety over k* locally of finite type*, then X is covered by T-stable* affine *open subsets of finite type.*

We do not repeat the proof here and refer the reader to [59] or [63, I.2, Thm.5]. Note that since $\mathrm{Pic}(G)$ is finite by [55, Cor. VII 1.6], the argument in [63] can be modified to cover the general case in [59]. Ishida pointed out that if an algebraic group G acts on X locally of finite type, then X is covered by G-stable open sets of finite type to which [59] ap- **4**
plies.

Remark. As was pointed out by H.Matsumura, the assertion of the theorem is not true unless X is normal. Indeed, $T = k^*$ acts on the rational curve X with a node obtained by identifying the zero and the point at infinity of the projective line P_1. The node does not have any T-stable affine open neighborhood. When $k = \mathbb{C}$, $\mathbb{C}^*$ acts *analytically* on an elliptic curve E, which is the quotient of $\mathbb{C}^*$ by an infinite cyclic subgroup. Obviously, there is no $\mathbb{C}^*$-stable open neighborhood. (cf. §. 11).

When an algebraic torus T acts on X, not much is lost, even if we assume the action to be (even scheme-theoretically) *effective*. Indeed, since T is commutative, we can replace T by its quotient torus by the kernel of the canonical homomorphism from T to the automorphism group functor $Aut(X)$. In these notes, we are interested in almost homogeneous (or sometimes called pre-homogeneous) algebraic torus actions on irreducible algebraic varieties, i.e. those which have a dense orbit. The dense orbit is necessarily Zariski open, and is isomorphic to T when the action is effective. Thus we are led to the following:

Definition. A *torus embedding* $T \subset X$ consists of an algebraic variety X containing T as a Zariski open dense subset and an algebraic action of T on X which extends the group law of T, i.e. we have a commutative diagram

$$\begin{array}{ccc} T \times X & \longrightarrow & X \\ \cup & & \cup \\ T \times T & \longrightarrow & T. \end{array}$$

An equivariant dominant morphism from a torus embedding $T \subset X$ to another $T' \subset X'$ is a dominant morphism (i.e. with dense image) $f : X \to X'$ whose restriction to T induces a *surjective* homomorphism $f|T : T \to T'$ and which is equivariant with respect to the actions, i.e. the following diagram is commutative:

$$\begin{tikzcd} T \times X \arrow[r] \arrow[d, "(f|T)\times f"'] & X \arrow[d, "f"] \\ T' \times X' \arrow[r] & X' \end{tikzcd}$$

Thus we have the category of torus embeddings.

Although dominant morphisms look too restrictive, those are the only equivariant morphisms which can be described in terms of r.p.p. decompositions.

We will give typical examples of torus embeddings in §. 7.

3 Rational partial polyhedral decompositions

We denote by $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ the set of integers, rational numbers and real numbers, respectively. $\mathbb{Z}_0, \mathbb{Q}_0$ and $\mathbb{R}_0$ are the sets of *non-negative* elements in them, respectively.

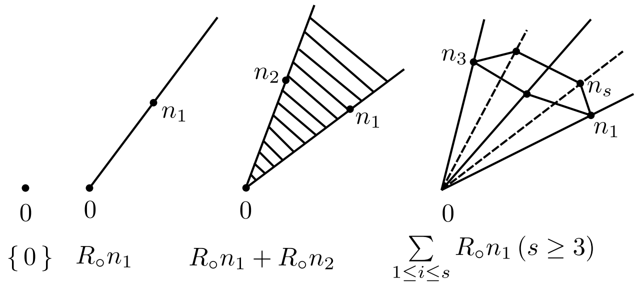
For a free $\mathbb{Z}$-module $N \cong \mathbb{Z}^r$ of rank r, let $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual $\mathbb{Z}$-module with the canonical pairing $\langle \, , \rangle : M \times N \to \mathbb{Z}$. We denote their scalar extensions to $\mathbb{R}$ by $N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$, $M_{\mathbb{R}}$ and $\langle \, , \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

Definition. A subset $\sigma \subset N_{\mathbb{R}}$ is called a strongly convex rational polyhedral cone with apex 0 (or simply a *cone* later) if $\sigma \cap (-\sigma) = \{0\}$ and there exist $n_1, \ldots, n_s$ in N such that

$$\sigma = \mathbb{R}_\circ n_1 + \cdots + \mathbb{R}_\circ n_1 = \{\, a_1 n_1 + \cdots + a_s n_s; a_1, \ldots, a_s \in \mathbb{R}_\circ\}.$$

$n_1, \ldots, n_s$, under the condition that they are irrredundant with each n_1 a **6**
primitive element of N), are uniquely determined by σ and are called the *fundamental generators* of σ. $\dim \sigma$ is the dimension of the $\mathbb{R}$-vector subspace $\sigma + (-\sigma)$ in $N_{\mathbb{R}}$ generated by σ.

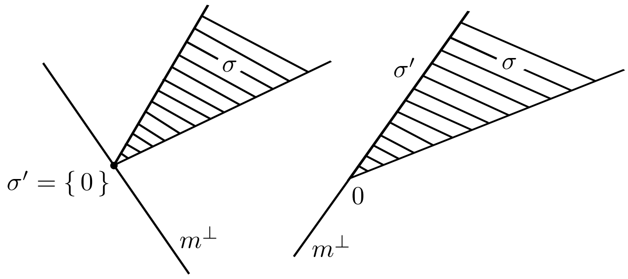
When rank $N = 3$, the following are example of cones.



Definition. Let σ be a cone in $N_{\mathbb{R}}$. A subset σ' of σ is a *face* of σ, and denoted $\sigma' < \sigma$, if there exists m in M such that $\langle m, y\rangle \geq 0$ for all $y \in \sigma$ and

$$\sigma' = \{y \in \sigma; \langle m, y\rangle = 0\} = \sigma \cap m^{\perp}.$$

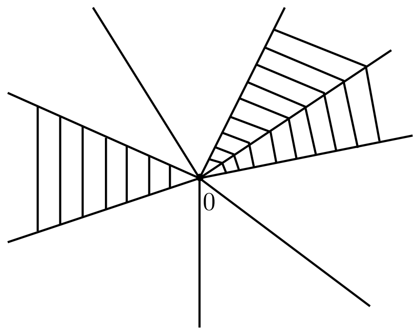
Here are examples when rank $N = 2$.



Definition. *A rational partial polyhedral decomposition* (r.p.p. decom- **7**
position, for short) is a pair (N, Δ) consisting of a free $\mathbb{Z}$module N of finite rank and a collection Δ of cones in $N_{\mathbb{R}}$ such that (i) if $\Delta \ni \sigma, \sigma > \tau$,

then $\tau \in \Delta$ and (*ii*) if σ and τ belong to Δ, then the intersection $\sigma \cap \tau$ is a face of σ as well as of τ.(N, Δ) is called a *finite rational partial polyhedral decomposition*(f. r. p. p. decomposition for short) if Δ is finite.

The relative interiors of $\sigma \in \Delta$ are disjoint and fill a part of $N_{\mathbb{R}}$. When rank $N = 2, \Delta$ looks like this:



Definition. A map $h : (N, \Delta) \to (N', \Delta')$ between r.p.p. decompositions is a $\mathbb{Z}$- homomorphism $h : N \to N'$ with *finite cokernel* such that for each $\sigma \in \Delta$, there exists $\sigma' \in \Delta'$ with the scalar extension $h : N_{\mathbb{R}} \to N_{\mathbb{R}'}$ satisfying $h(\sigma) \subset \sigma'$. Thus we have the category of r.p.p. decompositions.

Definition. A cone σ in $N_{\mathbb{R}}$ is called *simplicial* if its fundamental generators $n_1, \ldots, n_s$ are $\mathbb{R}$- linearly independent. σ is called *non- singular* if its fundamental generators form a part of a $\mathbb{Z}$-basis if N.

8 Given a strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$, we denote by $\check{\sigma}$ its dual in $M_{\mathbb{R}}$

$$\check{\sigma} = \left\{ x \in M_{\mathbb{R}}; \langle x, y \rangle \geq \text{ for all} y \in \sigma \right\},$$

where $M = N^*$ is the dual of N. $\check{\sigma}$ can be written as the set of $\mathbb{R}_\circ$-linear combinations of a finite number of elements of M and is a convex rational polyhedral cone, although it no longer satisfies the strong convexity $\check{\sigma} + (-\check{\sigma}) = \left\{ 0 \right\}$. Instead, it satisfies $\check{\sigma} + (-\check{\sigma}) M_{\mathbb{R}}$. For the general theory convex polyhedral cones, we refer the reader for instance to Grünbaum [13] and Rockafellar [56].

For a subset τ of σ, we denote

$$\tau^{\perp} = \left\{ x \in M_{\mathbb{R}}; \langle x, y\rangle = 0 \text{ for all } y \in \tau \right\}.$$

Then $y \in \sigma$ is in relative interior of σ if and only if $\check{\sigma} \cap y^{\perp} = \sigma^{\perp}$, i.e for all $x \in \check{\sigma}$ not in $\sigma^{\perp}$, we have $\langle x, y\rangle > 0$

The following propositions will be useful later.

Proposition 3.1. *Let σ be a cone in $N_{\mathbb{R}}$ and $\check{\sigma}$ be its dual in $M_{\mathbb{R}}$. Then the map*

$$\tau \longmapsto \check{\sigma} \cap \tau^{\perp}$$

is na order reversing bijection

$$\left\{ \textit{faces of } \sigma \right\} \xrightarrow{\sim} \left\{ \textit{faces of } \check{\sigma} \right\}.$$

Proof. By definition, a face of $\check{\sigma}$ is of the form $\check{\sigma} \cap y^{\perp}$ for $y \in \sigma$. y belongs to the relative interior of a face τ of σ if and only if $\check{\tau} \cap y^{\perp} = \tau^{\perp}$, i.e. $\check{\sigma} \cap \tau^{\perp}$. □

Proposition 3.2. *Let σ be a come in $N_{\mathbb{R}}$ and τ a face of σ. Then there* 9
exists $x \in \check{\sigma} \cap \tau^{\perp}$ such that

$$\check{\tau} = \check{\sigma} + \tau^{\perp} = \check{\tau} + \mathbb{R}_{\circ}(-x).$$

Proof. Since τ is a face of σ, there exists $x \in \check{\sigma}$ such that $\tau = \sigma \cap x^{\perp}$. We have inclusions $\check{\tau} \supset \check{\sigma} + \tau^{\perp} \supset \check{\sigma} + \mathbb{R}_{\circ}(-x)$ of convex polyhedral cones in $M_{\mathbb{R}}$. The dual τ of the first and the dual $\sigma \cap \mathbb{R}_{\circ}(-x)^{\vee} = \sigma \cap x^{\perp}$ of the third coincide, and we are done. □

Proposition 3.3. *The correspondence*

$$\sigma \longmapsto \check{\sigma} \cap M$$

establishes a bijection between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of subsemigroups S of M which satisfy the following properties:

(1) $S \in 0$ and is finitely generated as a semigroup.

(2) S generates M as a group.

(3) S is saturated, i.e. S contains $m \in M$ if there exists a positive integer a such that $am \in S$.

Proof. $\check{\sigma} \cap M$ obviously contains 0 and is a saturated subsemigroup. Since $\sigma \cap (-\sigma) = \{0\}$ by definition, we have $\check{\sigma} + (\check{-\sigma}) = M_{\mathbb{R}}$, hence (*ii*). The finite generation of $\check{\sigma} \cap M$ as a semigroup is what is known as Gordan's lemma and can be proved as follows : We may assume that $\check{\sigma}$ is of the form $\mathbb{R}_{\circ}m_1 + \cdots + \mathbb{R}_{\circ}m_s$ for $\mathbb{R}$-linearly independent elements $m_1, \ldots, m_S \in M$, since general $\check{\sigma}$ is a finite union of convex cones of this form by Carathoeodry's theorem (see for instance Grünbaum [13]).
10 Let $M' = M \cap (Qm_1 + \cdots + Qm_S)$. Then $\check{\sigma} \cap M = \check{\sigma} \cap M'$ and $M'' = \mathbb{Z}m_1 + \cdots + \mathbb{Z}m_s$, is a submodule of finite index in M'. Since $\check{\sigma} \cap M' = \mathbb{Z}_{\circ}m_1 + \cdots + \mathbb{Z}_{\circ}m_s$, we are done. Conversely, let S satisfy (i), (ii) and (iii). By (i), there exist elements $m_1, \ldots, m_s \in M$ such that $\mathbb{Z}_{\circ}m_1 + \cdots + \mathbb{Z}_{\circ}m_S$. Then $\check{\sigma} = \mathbb{R}_{\circ}m_1 + \cdots + \mathbb{R}_{\circ}m_s$ is a convex rational polyhedral cone in $M_{\mathscr{R}}$ satisfying $\check{\sigma} + (-\check{\sigma}) = M_{\mathbb{R}}$. Hence $\check{\sigma} = \check{\check{\sigma}}$ is a strongly convex rational polyhedral cone. It remains to show that $S = \check{\sigma} \cap M$. Again by Caratheodory's theorem, and element in $\check{\sigma} \cap M$ is a $Q_{\circ}$ linear combinations of $m_1, \ldots, m_s$. Thus a positive integral multiple of it is contained is S. Hence we are done by (iii). □

4 First main theorems

For later convenience, we state the first main theorems relating torus embeddings and r.p.p. decompositions in this section, and leave their proofs of §. 5.

The following theorem is slightly more general than those in Demazure [8, §. 4], Mumford et al.[63, I.2, Thm.6] and Miyake-Oda [40].

Theorem 4.1. *Given k, there exists an equivalence of categories*

$$\{\,r.p.p.decompositions\} \longrightarrow \left\{\begin{matrix}\text{normal and separated torus}\\ \text{embeddings over } k\\ \text{locally of finite type}\end{matrix}\right\}$$

$$(N,\Delta) \longmapsto T_N \subset T_N\,\mathrm{emb}(\Delta)$$

where $T_N = N \otimes_{\mathbb{Z}} k^* = \mathrm{Hom}_{gr}(N^*, k^*)$ and $T_N emb(\Delta)$ is obtained as the canonical patching of its affince open subsets **11**

$$\mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap N^*, k).$$

Here k is considered as a unitary semigroup under the multiplication and T_N acts on it by $(tx)(m) = t(m)x(m)$ for $t \in T_N$, $m \in N^*$ and $x \in \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap N^*, k)$.

$T_N\,\mathrm{emb}(\Delta)$ *is of finite type over k if and only if Δ is finite.*

Remark. This down-to-earth description of the functor as well as the simplified proof in §. 5 was pointed out by Ramanan.

Theorem 4.2. *Let (N,Δ) be an r.p.p decomposition.*

(i) The map

$$\sigma \longmapsto \mathrm{orb}(\sigma) = \mathrm{Hom}_{gr}(\sigma^{\perp} \cap N^*, k^*)$$

is a bijection

$$\mathrm{orb} : \Delta \xrightarrow{\sim} \left\{\, T_N - \text{orbits in } T_N\,\mathrm{emb}(\Delta)\right\},$$

such that $\mathrm{orb}(\{0\}) = T_N$ and $\dim\sigma + \dim\mathrm{orb}(\sigma) = \dim T_N$. Moreover, $\tau < \sigma$ if and only if the closure of $\mathrm{orb}(\tau)$ contains $orb(\sigma)$.

(ii) The map

$$\sigma \longmapsto U(\sigma) = \coprod_{\tau<\sigma} \mathrm{orb}(\tau) = \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap N^*, k)$$

establishes an order and intersection preserving bijection $\Delta \xrightarrow{\sim}$ **12** *$\{T_N$-stable affine open sets in $T_N\,\mathrm{emb}(\Delta)\}$. In particular, $T_n\,\mathrm{emb}(\Delta)$ is affine if and only if Δ consists of the faces of a fixed cone in $N_{\mathbb{R}}$.*

(iii) For $\sigma \in \Delta$, *let* $\bar{N}$ *be the quotient of* N *by the subgroup generated by* $\sigma \cap N$. *Then* $T_{\bar{N}} = \mathrm{Hom}_{gr}(\sigma^{\perp} \cap N^*, k^*)$. *The closure* $\overline{\mathrm{orb}(\sigma)}$ *of* $\mathrm{orb}(\sigma)$ *in* $T_N \operatorname{emb}(\Delta)$ *is normal and*

$$\overline{\mathrm{orb}(\sigma)} = \coprod_{\sigma < \tau \in \Delta} \mathrm{orb}(\tau).$$

It coincides with $T_{\overline{N}} \operatorname{emb}(\Delta)$, *where* $\bar{\Delta}$ *is the r.p.p.decomposition of* $\overline{N}_{\mathbb{R}}$ *consisting of the images* $\bar{\tau}$ *of* $\sigma < \tau \in \Delta$ *under* $N_{\mathbb{R}} \to \overline{N}_{\mathbb{R}}$.

Theorem 4.3. *Let* (N, Δ) *be an r.p.p.decomposition. Then the corresponding* $T_N \operatorname{emb}(\Delta)$ *is non-singular if and only if each* $\sigma \in \Delta$ *is non-singular, i.e. its fundamental generators of each form a part of a* $\mathbb{Z}$*-basis of* N. *In this case, the closure of each* T_N*-orbit orb* (σ) *is again non-singular.*

Remark . When $T_N \operatorname{emb}(\Delta)$ is non-singular, the collection of the sets fundamental generators of σ with running through Δ is a "fan" of Demazure [8, §. 4, $n°$.2].

Theorem 4.4. *Let* $h : (N, \Delta) \longrightarrow (N', \Delta')$ *be a map of r.p.p. decompositions, and let* $f : T_N \operatorname{emb}(\Delta) \to T_{N'} \operatorname{emb}(\Delta')$ *be the corresponding equivariant dominant morphism. Then* f *is proper if and only if for each* $\sigma' \in \Delta'$ *the set* $\left\{\sigma \in \Delta; h(\sigma) \subset \sigma'\right\}$ *is finite and* $h^{-1}(\sigma')$ *is the of its members.*

13 **Corollary 4.5.** *Let* (N, Δ) *be an r.p.p.decomposition. Then* $T_N \operatorname{emb}(\Delta)$ *is complete (i.e. proper over Spec k), if and only if* Δ *is finite and*

$$N_{\mathbb{R}} = \bigcup_{\sigma \in \Delta} \sigma.$$

5 The proof of theorems in §.4

In this section, we prove theorems stated in §. 4 in a way slightly different from Mumford et al.[63]. We informally deal with k-valued points only, although, to be rigorous, we should deal with points with values in arbitrary k-algebras.

Here is the *key observations* relating cones and normal affine torus embeddings. We were inspired by Hochster [20].

5.1 Let N and M be mutually dual free $\mathbb{Z}$-modules of finite rank with the pairing $<,>$. Let $T = T_N = N \otimes_{\mathbb{Z}} k^*$. Then the correspondence

$$\sigma \longmapsto U(\sigma) = \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap M, k^*)$$

is a bijection from the set of (strongly convex rational polyhedral) cones $\sigma \subset N_{\mathbb{R}}$ to the set of *normal affine* T_N-embeddings (of finite type) over k.

Proof. By Proposition 3.3, $\check{\sigma} \cap M$ is a finitely generated saturated subsemigroup of M which generates M as a group. Thus its semi group algebra $A = A(\sigma) = \bigoplus_{m \in \check{\sigma} \cap M} ke(m)$ is a subalgebra of finite type of the **14**
coordinate ring $k[T] = \bigoplus_{m \in M} ke(m)$ of T. It is T-stable under algebraic representation of T on $k[T]$ defined by $f(t) \longmapsto f(t't)$ for $f(t) \in k[T]$ and $t' \in T$. Since $\check{\sigma} \cap M$ generates M as a group, the quotient field of A coincides with the quotient field $k(T)$ of $k[T]$. Since $\check{\sigma} \cap M$ is saturated, A is integrally closed in $k(T)$. Indeed, the integral closure A' is of finite type over k, hence the representation of T on A' as above is also algebraic. Thus by the complete reducibility theorem, A' has a k-basis consisting of elements of the form $e(m')$ with $m' \in M$. $e(m')$ satisfies an equation

$$e(m')^c + a_1 e(m')^{c-1} + \cdots + a_c = 0$$

with a_i in A, which we may assume to be a k-multiple of an element $e(m_i), m_i \in \check{\sigma} \cap M$. Obviously there exists a non-vanishing a_i. Then we have $cm' = m_i + (c-i)m'$, i.e. $im' = m_i \in \check{\sigma} \cap M$, hence $m' \in \check{\sigma} \cap M$. A ($k$-valued) point of $U(\sigma) = \mathrm{Spec}\, A$ is a k=algebraic homomorphism $u : A \to K$, which is determined uniquely by $u(e(m)) \in k$ for $m \in M$, such that $u(e(0)) = 1$ and $u(e(m+m')) = u(e(m))u(e(m'))$, i.e., $u \circ e : M \to K$ is a homomorphism of unitary semigroups. T obviously acts on $U(\sigma)$ as in the statement of Theorem 4.1. Let $m_1, \ldots m_s$ be generators of $\check{\sigma} \cap M$ as a semigroup. Then M is generates by $m_1, \ldots, m_s$ and $-(m_1 + \cdots + m_s)$. Thus $k[T]$ is the localization of A by $e(m_1 + \cdot + m_s)$. Hence $U(\sigma)$ contains T as an open set. □

Conversely, let $T \subset \mathrm{Spec}\, A$ be a normal affine T-embedding of finite **15** type. Hence A is a k-subalgebra of finite type of $k[T]$, is normal with the quotient field $k(T)$ and is T-stable under the algebraic representations of T on $k[T]$ and is T-stable under the algebraic representations of T. Thus by the complete reducibility, it has a k-basis consisting of T-semi invariants, i.e. $A = \bigoplus_{m\in S} ke(m)$ for a finitely generated subsemigroup $S \ni 0$ of M. S generates M as a group. Indeed, for $m' \in M$, the denominator ideal of $e(m')$ is a non-zero T-stable ideal of A, hence contains some $e(m)$with $m \in S$, and $e(m)e(m') \in A$. S is obviously saturated, since A is integrally closed in $k(T)$. Hence by proposition 3.3, there exists a unique cone $\sigma \subset N_{\mathbb{R}}$ such that $S = \check{\sigma} \cap M$.

Remark. Non-normal affince T_N-embedding can also be written as

$$Spec(\bigoplus_{m\in S} ke(m))$$

for a subsemigroup $S \ni 0$ which generates M as a group. The simplest non-trivial example is the curve Spec $(k[t^2, t^3])$ with an ordinary cusp. In this case, $M = \mathbb{Z}$, and S is generates by 2 and 3. In general, it is difficult to describe such non-saturated S. For the special case of dimension one, we refer the reader to Delorme [7], Herzog [17] and Herzog-Kunz [21].

5.2 Let $h : N \to N'$ be a homomorphism with finite cokernel and let $f : T_N \to T'_N$ be the corresponding surjective homomorphism of algebraic tori. Let $T_N \subset U(\sigma)$ and $T_{N'} \subset U(\sigma')$ be the normal affine torus
16 embeddings corresponding to cones $\sigma \subset N_{\mathbb{R}}$ and $\sigma' \subset N'_{\mathbb{R}}$ as in (5.1). Then f can be extended to a unique equivariant dominant morphism $\bar{f} : U(\sigma) \to U(\sigma')$, if and only if of the scalar extension $h : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ satisfies $h(\sigma) \subset \sigma'$.

Proof. Let M and M' be the duals of N and N', respectively. Then h induces an injection $h^* : M' \to M$, which gives rise to $f : T_N = \mathrm{Hom}_{gr}(M, k^*) \to T_{N'} = \mathrm{Hom}_{gr}(M', k^*)$. Obviously, $h(\sigma \leq) \subset \sigma'$ if and only if $h^*(\check{\sigma}' \cap M') \subset \check{\sigma} \cap M$, and in this case it induces a morphism $\bar{f} : U(\sigma) = \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap M, k) \longrightarrow U(\sigma') = \mathrm{Hom}_{unit.semigr}(\check{\sigma}' \cap M', k)$. □

5.3 Let σ be a cone in $N_{\mathbb{R}}$ and let M be the dual of N. Then the corresponding normal affine T_N-embedding $U(\sigma) = \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap M, k)$ is the disjoint union of T_N-orbits

$$\mathrm{orb}(\tau) = \mathrm{Hom}_{gr}(\tau^{\perp} \cap M, k^*)$$

with τ running through the faces of σ. The closure of orb τ in $U(\sigma)$ is

$$\overline{\mathrm{orb}(\tau)} = \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap \tau^{\perp} \cap M, k)$$

and it is the disjoint union of orb τ' with τ' running through the faces of σ with $\tau < \tau'$.

Proof. The following argument, which considerably simplifies our original proof, is due to Ramanan. For simplicity, we denote $S = \check{\sigma} \cap M$. Let $g : S \to k$ be a unitary semigroup homomorphism, i.e. $g(0) = 1$ and $g(m + m')$ for $m, m' \in S$. Then S is the disjoint union of $I = g^{-1}(0)$ and $S' = g^{-1}(k^*)$. S' is a subsemigroup containing 0 of S and $S + I \subset I$. We 17
first show that a decomposition $S = S' \coprod I$ is obtained exactly by taking a face F of $\check{\sigma}$ and letting $S' = F \cap M$. Indeed, if F is a face of $\check{\sigma}$, there exists $y \in \sigma$ such that $F = \check{\sigma} \subset y^{\perp}$. Certainly, $F \cap M$ is a subsemigroup containing 0 of S, and for z and w in S with w not in $F \cap M$, $z + w$ is not in $F \cap M$. □

Conversely, let S' be a subsemigroup containing 0 of S such that its complement I satisfies $S + I \subset I$. Hence $m \in S$ is in S' of $(S + m) \cap S' \neq \phi$. Replacing $\check{\sigma}$ by its smallest face containing S', we may assume that there exists $m' \in S'$ in the relative interior of $\check{\sigma}$. We claim $(S + m) \cap \mathbb{Z}_{\circ} m' \neq \phi$ for $m \in S$, hence $m \in S'$ by assumption and $\mathbb{Z}_{\circ} m' \subset S'$. Indeed, since m' is in the relative interior of $\check{\sigma}$, we see that $\langle m', n_i \rangle > 0$ for the fundamental generators $n_1, \ldots, n_s$ of σ. Thus $S + m = \{m'' \in M; < m'' n_i > \geq < m, n_i > \text{ for } 1 \leq i \leq s\}$. Choose a positive integer a such that $a < m', n_i > \geq < m, n_i >$for $1 \leq i \leq s$. Then am' is in $S + m$.

By proposition 3.1, we know that $F = \check{\sigma} \cap \tau^{\perp}$ for a unique face τ of σ. Thus we see that $\mathrm{Hom}_{\text{unit.semigr}}(\check{\sigma} \cap M, k) = \coprod_{\tau < \sigma} \mathrm{Hom}_{\text{unit.semigr}}(\check{\sigma} \cap \tau^{\perp} \cap M)$ generates $\tau^{\perp} \cap M$ as a group, hence $\mathrm{Hom}_{\text{unit. semigr}}(\check{\sigma} \cap \tau^{\perp} \cap$

$M, k^*) = \mathrm{Hom}_{gr}(\tau^\perp \cap M, k^*) = T_N \cdot \varepsilon(\tau)$, where $\varepsilon(\tau)$ is the trivial group
18 homomorphism $\tau^\perp \cap M \to k^*$ sending every element to 1. Let $A = \bigoplus_{m \in \check{\sigma} \cap M} ke(m)$ and

$$\mathbb{P}(\tau) = \bigoplus_{\substack{m \in \check{\sigma} \cap M \\ m \notin \check{\sigma} \cap \tau^\perp \cap M}} ke(m).$$

Then $\mathbb{P}(\tau)$ is a prime ideal of A with the subring $\underset{m \in \check{\sigma} \cap \tau^\perp M}{\oplus} ke(m)$ isomorphic to $A/\mathbb{P}(\tau)$. $\mathrm{Hom}_{\text{unit. simigr}}(\check{\sigma} \cap \tau^\perp \cap M, k)$ is obviously the disjoint union of orb (τ') with τ' running through the faces of σ with $\tau < \tau'$ and is precisely (the set of k-valued points of) the closed set Spec $(A/\mathbb{P}(\tau))$.

Remark. Let A and $\mathbb{P}(\tau)$ be as above. Then the correspondence

$$\tau \longmapsto \mathbb{P}(\tau)$$

establishes an order preserving bijection

$$\{\text{ faces of } \sigma\} \xrightarrow{\sim} \{T_N - \text{ stable prime ideals of A}\}.$$

$\mathbb{P}(\sigma)$ is the largest T_N-stable proper ideal of A. Let $n_1, \ldots, n_s$ be the fundamental generators of σ, hence $\mathbb{R}_o n_1, \ldots, \mathbb{R}_o n_s$ are the one-dimensional faces of σ. Then $\mathbb{P}(\mathbb{R}_o n_1), \ldots, \mathbb{P}(\mathbb{R}_o n_s)$ are the T_N-stable height one prime ideals of A. The localization $A_{\mathbb{P}(\mathbb{R}_o n_i)}$ is a discrete valuation ring, hence we have a surjective homomorphism $\mathrm{ord}_i : k(T)^* \to \mathbb{Z}$, valuation ring onto $\mathbb{Z}_\circ$. Composed with $e : M \to k(T)^*$, it gives rise to a surjective homomorphism $\mathrm{ord}_i \circ e : M \to \mathbb{Z}$, which is exactly the primitive element $n_i \in N$.

19 5.4 Let $\sigma \subset N_\mathbb{R}$ be a cone and let $U(\sigma) = \mathrm{Hom}_{\text{unit.semigr}}(\check{\sigma} \cap M, k)$ be the corresponding normal affine T_N-embedding. Then the map

$$\tau \longmapsto U(\tau) = Hom_{\text{unit.semigr}}(\check{\tau} \cap M, k)$$

is a bijection

$$\{\text{faces of } \sigma\} \xrightarrow{\sim} \{T_N - \text{ stable affine open subsets of } U(\sigma)\}.$$

Moreover, we have $U(\tau) = \coprod_{\tau' < \tau} \mathrm{orb}(\tau')$.

Proof. Let $A = \bigoplus_{m \in \check{\sigma} \cap M} ke(m)$ so that $U(\sigma) = \mathrm{Spec}(A)$. If τ is a face of σ, there exists, by proposition 3.2, an element $m_0 \in \check{\sigma} \cap \tau^{\perp} \cap M$ such that $\check{\tau} \cap M = (\check{\sigma} \cap M) + \mathbb{Z}_{\circ}(-m_0)$. Hence $\bigoplus_{m \in \check{\tau} \cap M} ke(m) = A[e(m_0)^1]$, whose spectrum $U(\tau)$ is obviously a T_N-stable affine open set. □

Conversely, let spec B be a T_N-stable affine open set of $U(\sigma)$. Then by (5.1), There exists a cone $\tau \subset \sigma$ in $N_{\mathbb{R}}$ such that $\{e(m); m \in \check{\tau} \cap M\}$ form a k-basis of B. It remains to show that τ is a face of σ. The following argument is again due to Ramanan. Replacing σ by its smallest face containing τ, we may assume that there exists $n \in \tau \cap N$ in the relative interior of σ. Then $\check{\tau} \cap n^{\perp}$ is face of $\check{\sigma}$ and its intersection with $\check{\sigma}$ is exactly $\check{\sigma} \cap n^{\perp} = \sigma^{\perp}$. Thus the ideal $\mathbb{P}(\sigma)B$ generated by the prime ideal $\mathbb{P}(\sigma)$ of A is a proper ideal of B. Thus $\mathrm{orb}(\sigma) = \mathrm{spec}(A/\mathbb{P}(\sigma))$ is contained in the T_N-stable affine open set spec B. Since the closure of any T_N-orbit in $U(\sigma)$ contains $\mathrm{orb}(\sigma)$ by (5.3), any T_N-orbit in $U(\sigma)$ is contained in spec B and we are done. $U(\tau)$ is the disjoint union of $\mathrm{orb}(\tau')$, $\tau' < \tau$, by (5.3).

Combining (5.3) and (5.4), we have the following: **20**

5.5 Let $\sigma \subset N_{\mathbb{R}}$ be a cone and let $U(\sigma) = \mathrm{Hom}_{\mathrm{unit.semigr}}(\check{\sigma} \cap M, k)$ be the corresponding normal affine T_N-embedding. Then

$$\tau \longmapsto \mathrm{orb}(\tau) = \mathrm{Hom}_{gr}(\tau^{\perp} \cap M, k^*)$$

is a bijection

$$\mathrm{orb} : \{\text{faces of } \sigma\} \xrightarrow{\sim} \{T_N - \text{orbits in } U(\sigma)\}.$$

Moreover,

$$\tau \longmapsto U(\tau) = \mathrm{Hom}_{\mathrm{unit.semigr}}(\check{\tau} \cap M, k)$$

is a bijection

$$\{\text{faces of } \sigma\} \xrightarrow{\sim} \{T_N - \text{stable affine open subsets of } U(\sigma)\}.$$

For each $\tau < \sigma$, they satisfy the following properties :

(i) $\dim \tau + \dim \operatorname{orb}(\tau) = \dim T_N$, $\operatorname{orb}(\{0\}) = T_N$ and $\operatorname{orb}(\sigma)$ is the unique closed orbit of $U(\sigma)$.

(ii) $U(\tau) = \coprod_{\tau' < \tau} \operatorname{orb}(\tau)$

(iii) The closure of $\operatorname{orb}(\tau)$ in $U(\sigma)$ is

$$\overline{\operatorname{orb}(\tau)} = \coprod_{\tau < \tau' < \sigma} \operatorname{orb}(\tau') = \operatorname{Hom}_{\text{unit.semigr}}(\check{\sigma} \cap \tau^{\perp} \cap M, k).$$

It is the normal affine embedding of $\operatorname{orb}(\tau) = \operatorname{Hom}_{gr}(t^{\perp} \cap M, k^*)$ corresponding to the image of σ under the map from $N_{\mathbb{R}}$ to its quotient by the subspace $\sigma + (-\sigma)$ generated by σ.

(iv) The morphism $\rho_\tau : U(\sigma) \to \overline{\operatorname{orb}(\tau)}$ induced by the inclusion $\check{\sigma} \cap \tau^{\perp} \cap M \hookrightarrow \check{\sigma} \cap M$ is a retraction such that $U(\tau) = \rho_\tau^{-1}(\operatorname{orb}(\tau))$.

Remark. The map orb can also be described as in Mumford et al. [63] as follows : For $t \in k^*$ and $n \in N$, consider the element $\tau^{\langle ?, n \rangle}$ of $T_N =$
21 $\operatorname{Hom}_{gs}(M, k^*)$. Let $U(\tau) = \operatorname{Hom}_{\text{unit. semigr}}(\check{\sigma} \cap M, k)$ be the normal affine T_N-embedding. The limit of $t^{\langle ?, n \rangle}$ as t tends to zero exists in $U(\sigma)$ if and only if $\langle m, n \rangle \geq 0$ for all $m \in \check{\sigma} \cap M$, i.e. $n \in \sigma \cap N$. In that case, the limit is the semigroup homomorphism form $\check{\sigma} \cap M$ to k sending those m with $\langle m, n \rangle = 0$ to $1 \in k$ and those with $\langle m, n \rangle > 0$ to $0 \in k$, i.e. the identity element $\varepsilon(\tau)$ of $\operatorname{orb}(\tau) = \operatorname{Hom}_{gr}(\tau^{\perp} \cap M, k^*)$, where τ is the face of σ containing n in its relative interior, by Proposition 3.1.

5.6 Let $\sigma \subset N_{\mathbb{R}}$ correspond to the normal affine T_N-embedding $U(\sigma) = \operatorname{spec} A$. Let $\mathbb{P}(\sigma)$ be the largest T_N-stable proper ideal of A as in the remark after (5.3). Then the following are equivalent:

(i) $U(\sigma)$ is non-singular.

(ii) The local ring $A_{\mathbb{P}(\sigma)}$ is regular.

(iii) σ is non-singular, i.e. the fundamental generators of σ form a part of a $\mathbb{Z}$-basis of N.

Proof. (i) obviously implies (ii). Let us assume (ii) and show (iii). Obviously, there exist $m_1, \ldots, m_s$, s = height ($\mathbb{P}(\sigma)$), such that $\{e(m_1), \ldots, e(m_s)\}$ is a minimal set of generators of the maximal ideal of the local ring. It is easy to see that $e(m)$, $m \in M$ is contained in the local ring if and only if $m \in \check{\sigma} \cap M$. But such m can be written uniquely as $m = m_0 + a_1 m_1 + \cdots + a_s m_s$ with $a_i \in \mathbb{Z}o$ and $m_0 \in \sigma^\perp \cap M$. Hence 22
a $\mathbb{Z}$-basis of $\sigma^\perp \cap M$ together with $m_1, \ldots, m_s$ form a $\mathbb{Z}$-basis of M. Among the dual basis of N, we can choose $n_1, \ldots, n_s \in \sigma$ such that $\langle m_i, n_j \rangle = \delta_{ij}$. Then $\sigma = \mathbb{R}_o n_1 + \cdots + \mathbb{R}_o n_1$. It remains to show that (iii) implies (i). Let the fundamental generators $n_1, \ldots, n_s$ of σ be extended to a $\mathbb{Z}$-basis $n_1, \ldots, n_r$ of N. Let $m_1, \ldots, m_r$ be the dual basis of M. Then $\check{\sigma} \cap M = (\mathbb{Z}_o m_1 + \cdots + \mathbb{Z}_o m_s) + (\mathbb{Z} m_{s+1} + \cdots + \mathbb{Z} m_r)$, hence $A = k[u_1, \ldots, u_r, u_1^{-1}, \ldots u_s^{-1}]$ with $u_i = e(m_i)$, and spec A is non-singular. □

5.7 **Proof of Theorem 4.1** Let (N, Δ) be an r.p.p.decomposition. Let $M = N^*$ be the dual of N. For $\sigma \in \Delta$, let $U(\sigma) = \mathrm{Hom}_{\text{unit.semigr}}(\check{\sigma} \cap M, k)$ be the corresponding normal affine open $T_N$embedding of finite type. For $\sigma, \tau \in \Delta, \sigma \cap \tau$ is a face of σ and τ. Thus by (5.4), $U(\sigma \cap \tau)$ is canonically a T_N-stable affine open set of $U(\sigma)$ and $U(\tau)$. Thus we can paste $U(\sigma)$'s together along $U(\sigma \cap \tau)$ to obtain an irreducible normal scheme $T_N\,\mathrm{emb}(\Delta)$ locally of finite type. Obviously, T_N acts algebraically on it with the dense orbit $U(\{0\}) = T_N$. It is separated, since $U(\sigma) \cap U(\tau) = U(\sigma \cap \tau)$ is an affine open set and the coordinate ring of $U(\sigma \cap \tau)$ is generated by those of $U(\sigma)$ and $U(\tau)$. Indeed, they k-bases consisting of elements of the form $e(m)$ and $(\check{\sigma} \cap M) + (\check{\tau} \cap M) = (\sigma \cap \tau)^\vee \cap M$.

A map $h : (N, \Delta) \to (N', \Delta')$ of *r.p.p.* decompositions obviously gives rise to an equivariant dormant morphism $f : T_N\,\mathrm{emb}(\Delta) \to T_{N'}$ *emb*(Δ'). On the other hand, suppose f is given. It induces $h : N \to N'$. 23
Then for $\sigma \in \Delta$, the unique closed T_N-orbit orb (σ) of $U(\sigma)$ is mapped by f to a T'_N-orbit $f(\mathrm{orb}(\sigma)) = \mathrm{orb}(\sigma')$ with some $\sigma' \in \Delta'$. Hence by (5.5) we have $f(U(\sigma)) \subset U(\sigma')$, i.e. $h(\sigma) \subset \sigma'$ by (5.2).

Let $T \subset X$ be a normal and separated torus embedding over k. Thus there exists N such that $T = T_N$. Consider the collection of T-stable

affine open subsets of X. Since each of them is a normal affine T-embedding of finite type, there exists, by (5.1), a collection Δ of cones in $N_{\mathbb{R}}$ such that $\{U(\sigma); \sigma \in \Delta\}$ is the set of T-stable affine open subsets of X. By Sumihiro's theorem in §. 2, $U(\sigma)'s$ convex X. We now show that (N, Δ) is an r.p.p.decomposition. If σ is in Δ and τ is a face of σ , then τ is in Δ by (5.4). For σ and τ in $\Delta, U(\sigma) \cap U(\tau)$ is a T-stable affine open set, hence equals $U(\rho)$ for a $\rho \in \Delta$, since X is separated. The coordinate ring of $U(\rho)$ is generated by those of $U(\sigma)$ and $U(\tau)$. Hence looking at T-semiinvariants in them, we see that $(\check{\sigma} \cap M) + (\check{\tau} \cap M) = \check{\rho} \cap M$. Thus $\sigma \cap \tau = \rho$. Since $U(\rho)$ is an affine open subset of $U(\sigma)$ and of $U(\tau)$, we have $\rho < \sigma$ and $\rho < \tau$, again by (5.4).
X is of finite type over k if and only if Δ is finite, since each $U(\sigma)$ has only a finite number of T-stable affine open subsets by (5.4)

24 5.8 **Proof of Theorem 4.2:** We first show (*ii*). By the construction of T_N emb (Δ), it is covered by T_N-stable affine open sets $\{U(\sigma); \sigma \in \Delta\}$, and $U(\sigma) \cap U(\tau) = U(\sigma \cap \tau)$. Hence the map $\sigma \longmapsto U(\sigma)$ is injective. Let U be a T_N-stable affine open set of T_N emb (Δ). Then U is a normal affine T_N-embedding, hence by (5.1) there exists a unique cone $\sigma \subset N_{\mathbb{R}}$ such that $U = U(\rho)$.

Let σ be in Δ. Then since $U(\sigma) \cap U(\rho)$ is affine, it is equal as in (5.7) to $U(\sigma \cap \rho)$ which is a T_N-stable affine open set of $U(\sigma)$ and $U(\rho)$. Thus by (5.5), we see that $U(\rho)$ is covered by $U(\tau)$ with τ running through the elements of Δ with $\tau < \rho$.
Thus the unique closed orbit orb (ρ) of $U(\rho)$ belongs to some $U(\tau)$ with $\Delta \ni \tau < \rho$. On the other hand, ρ is then a face of τ by (5.5), and we are done.

We next show (*i*). For $\sigma \in \Delta$, orb(σ) is the unique closed orbit of $U(\sigma)$ by (5.5) (i). On the other hand, each T_N-orbit of T_N emb (Δ) is contained in a T_N-stable affine open set, hence by (5.5) is the unique closed T_N-orbit of a unique $\sigma \in \Delta$.

It remains to show (*iii*). The closure $\overline{\text{orb}(\sigma)}$ of orb(σ)in $T_N$emb (Δ) is the union of its closures in $U(\tau)$ with τ running through the elements of Δ with $\sigma < \tau$. In particular, $\overline{\text{orb}(\sigma)}$ is the disjoint union of orb(τ) with $\sigma < \tau \in \Delta$ by (5.5) (iii). $\overline{\text{orb}(\sigma)}$ is the union of normal affine embeddings of $T_{\overline{N}} = \mathrm{Hom}_{gr}(\sigma^{\perp} \cap M, k^{*})$ corresponding to the image $\overline{\tau}$ of τ under $N_{\mathbb{R}} \to \overline{N}_{\mathbb{R}}$, again by (5.5) (iii). The collection $\overline{\Delta}$ of those $\overline{\tau}'s$ is obviously an r.p.p.decomposition of $\overline{N}$.

5.9 **Proof of Theorem 4.3:** This follows easily from (5.6) and theorem 4.2 (iii). Note that if τ is non-singular and $\sigma < \tau$, then its image $\overline{\tau}$ under the map from $N_{\mathbb{R}}$ to its quotient $\overline{N}_{\mathbb{R}}$ by the subspace generated by σ is again non-singular. **25**

5.10 **Proof of Theorem 4.4:** For simplicity, let $X = T_N emb(\Delta)$ and $X' = T_N emb(\Delta')$. By the valuate criterion of properness (see for instance [10] and Mumford [39]), $f : X \to X'$ is proper if and only if it is of finite type and, moreover, each discrete valuation ring $R \supset k$ which is contained in the function field $k(X)$ of X and which dominates a local ring of X' necessarily dominates a local ring of X.

Let ord $:k(X)^{*} \to \mathbb{Z}$ be the valuation corresponding to R. Let us denote by $n = \mathrm{ord} \circ e$ its composite with $e : M \to k(X)^{*}$.
Hence n is an element of N. Each non-zero element $n \in N$ is obtained in this way. R dominates a local ring of X if and only if it contains the coordinate ring of one of one of the T_N stable affine open sets, i.e. there exists $\sigma \in \Delta$ such that $\langle m, n\rangle \geq 0$ for all $m \in \check{\sigma} \cap M$, i.e. $n \in \sigma$. Similarly, since $M' \hookrightarrow M, R$, dominates a local ring of X' if and only if there exists $\sigma' \in \Delta'$ such that $n \in \sigma'$.

f is of finite type if and only if for each $\sigma' \in \Delta'$, there exist only a finite number of $U(\sigma)$'s with $f(U(\sigma)) \subset U(\sigma')$, i.e. $\{\sigma \in \Delta; h(\sigma) \subset (\sigma)'\}$ is finite, by (5.2). The rest of the proof is obvious.

6 Projective torus embeddings

For simplicity, we restrict ourselves to complete normal torus embeddings and try to generalize Demazure's results in [8] on the ampleness of invertible sheaves to this case. Mumford et al. [63] deal more generally with T-stable fractional ideals on not necessarily complete T-embeddings. **26**

In this section, we fix an f.r.p.p.decompositioon (N, Δ) with $N_{\mathbb{R}} = \bigcup_{\sigma\in\Delta} \sigma$. Note that Δ consists of the faces of the maximal dimensional cones $\sigma \in \Delta$, i.e. $\dim \sigma = \operatorname{rank} N$. For simplicity, we denote $T = T_N$ and $X = T_N emb(\Delta)$, which is complete and normal. As before, we denote by M the $\mathbb{Z}$-module dual to N with the canonical pairing $\langle\, ,\rangle : M \times N \to \mathbb{Z}$.

As usual, a Weil divisor D on X is a finite $\mathbb{Z}$-linear combination of reduced and irreducible closed subvarieties of codimension one. D gives rise to a fractional ideal $O_X(D)$. A Cartier divisor D is a locally principal Weil divisor, which gives rise to an invertible fractional ideal $O_X(D)$. We denote by Pic (X) the group of isomorphism classes of invertible sheaves on X, i.e. the group of linear equivalence classes of Cartier divisors on X.

Let the 1-skeleton $Sk^1(\Delta) = \{\sigma_1, \ldots, \sigma_d\}$ be the set of 1-dimensional
27 cones in Δ. Let n_i be the fundamental generator of σ_i. By Theorem 4.2, $\{D_1, \ldots, D_d\}$, where $D_i = \overline{\operatorname{orb}(\sigma_i)}$, is the set of T-stable irreducible Weil divisors on X and forms a $\mathbb{Z}$-basis of the group

$$\bigoplus_{1\le i\le d} \mathbb{Z}D_i$$

of T-stable Weil divisors on X.

Proposition 6.1 ((Demazure)). *For a complete normal T-embedding X, we have exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\mathrm{div}"] \arrow[d, equal] & \bigoplus_{1\le i\le d} \mathbb{Z}D_i \arrow[r] & \left\{\begin{array}{c}\textit{linear equivalence}\\ \textit{classes of Weil}\\ \textit{divisors on } X\end{array}\right\} \arrow[r] & 0 \\
 & & \cup & \cup & \\
0 \arrow[r] & M \arrow[r, "\mathrm{div}"] & \left\{\begin{array}{c}T\textit{-stable Cartier}\\ \textit{divisors on } X\end{array}\right\} \arrow[r] & \mathrm{Pic}(X) \arrow[r] & 0
\end{tikzcd}$$

where the first arrows send $m \in M$ to

$$\mathrm{div}(m) = \sum_{1\le i\le d} \langle m, n_i\rangle D_i$$

which is the divisor of the character $e(m)$ as a rational function on X

Proof. For $m \in M$, the divisor of the rational function $e(m)$ on X is equal to $\sum_{1\leq i\leq d} \langle m, n_i\rangle D_i$ by the remark after (5.3). This vanishes if and only if $m = 0$, since X is complete. For a non-zero rational function f on X, its divisor on X is T-stable if and only if f is a T-semiinvariant. Since $X - \bigcup_{1\leq i\leq d} D_i = T$ is factorial, any Weil divisor on X is linearly equivalent to a linear combination of $D_i's$ □

Lemma 6.2. *Let $D = \sum_{1\leq i\leq d} a_i D_i$ be a T-stable Weil divisor. Then the following are equivalent.* 28

(i) D is a Cartier divisor.

(ii) D is principal on each T-stable affine open set $U(\sigma)$ of X with $\sigma \in \Delta$.

(iii) For all $\sigma \in \Delta$, there exists $m(\sigma) \in M$ such that

$$\langle m(\sigma), n_i\rangle = -a_i$$

for all n_i contained in σ.

Proof. D defines a T-stable fractional ideal $0_X(D)$ on X. The space of its sections over the affine open set $U(\sigma)$ has a k-basis consisting of $e(m)$ with m in

$$u(\sigma, D) = \{m \in M; \langle m, n_i\rangle \geq -a_i \text{ for all } n_i \in \sigma\}$$

by the complete reducibility. Thus (ii) and (iii) are obviously equivalent and imply (i). It remains to show (i) $\Longrightarrow$ (ii). Let $A(\sigma)$ be the coordinate ring of $U(\sigma)$ and let $L(\sigma)$ be the $A(\sigma)$- module of sections of $O_X(D)$ over $U(\sigma)$. Then, as is well-known (see Cartier [5], $O_X(D)$ is a Cartier divisor on $U(\sigma)$ if and only if $L(\sigma).(A(\sigma) : L(\sigma)) = A(\sigma)$. In terms of T-semi invariants in them, it amounts to

$$u(\sigma, D) + \{m' \in M; m' + \mu(\sigma, D) \in \check{\sigma} \cap m\} = \check{\sigma} \cap M.$$

Since the right hand side contains 0, there exists $m(\sigma) \in \mu(\sigma, D)$such that $-m(\sigma) + u(\sigma, D) \subset \check{\sigma} \cap M$. We thus conclude that $\mu(\sigma, D) = m(\sigma) + \check{\sigma} \cap M$. Hence $L(\sigma) = A(\sigma) \cdot e(m(\sigma))$. □

Lemma 6.3. *Let $D = \sum\limits_{1\leq i\leq d} a_i D_i$ be a T-stable Weil divisor. The frac-*
29 *tional ideal $O_X(D)$ is generated by its global sections if and only if for all $\sigma \in \Delta$, there exists $m(\sigma) \in M$ such that*

$$\langle m(\sigma), n_i\rangle \geq -a_i \qquad \text{for } 1 \leq i \leq d$$

with the equality holding if $n_i \in \sigma$.

Proof. The space of global sections of $O_X(D)$ has a k-basis consisting of $e(m)$ with m in

$$\lambda(D) = \{m \in M; \langle m, n_i\rangle \geq -a_i \qquad \text{for } 1 \leq i \leq d\}.$$

□

The sufficiency is obvious, since $m(\sigma)$ is in $\lambda(D)$ and $\mu(\sigma, D) = m(\sigma) + \check{\sigma} \cap M$. Let us assume that $O_X(D)$ is generated by its global sections. Hence, first of all, D is a Cartier divisor. Thus by lemma 6.2, there exists $m'(\sigma) \in \mu(\sigma, D)$ with $\langle m'(\sigma), n_i\rangle = -a_i$ for $n_i \in \sigma$ such that $\mu(\sigma, D) = m'(\sigma) + \check{\sigma} \cap M$ for each $\sigma \in \Delta$. On the other hand $\mu(\sigma, D) = \lambda(D) + \check{\sigma} \cap M$ by assumption. Hence there exists $m(\sigma) \in \lambda(D)$ such that $m'(\sigma, D) =\in m(\sigma) + \check{\sigma} \cap M$ and we are done.

Theorem 6.4. *Let X= Temb (Δ) be a complete normal T-embedding, and let $D = \sum\limits_{1\leq i\leq d} a_i D_i$ be a T-stable Weil divisor. Then $O_X(D)$ is an ample invertible sheaf if and only if there exists a positive integer b and, for all maximal dimensional $\sigma \in \Delta$, (a unique $m(\sigma) \in M$, such that*

$$\langle m(\sigma), n_i\rangle \geq -ba_i \qquad \text{for } 1 \leq i \leq d$$

with the equality holding if and only if n_i is in σ.

Proof. Let $O_X(D)$ be ample. Then there exists a positive integer b such that $O_X(bD)$ is very ample, hence there exists a projective embedding

$$f : X \to \mathbb{P}(\bigoplus_{m\in\lambda} ke(m))$$

30 where $\lambda(bD)$ is as in the proof of Lemma 6.3 and is finite, since X is complete. $O_X(bD)$ is generated by its global sections, hence for each $\sigma \in \Delta$ there exists $m(\sigma) \in M$ such that $\langle m(\sigma), n_i\rangle \geq -ba_i$ for $1 \leq i \leq d$ with the equality holding if n_i is in σ. If $\dim \sigma$ is maximal, i.e. σ generates N_R, then such $m(\sigma)$ is unique. We now show that $\langle m(\sigma), n_i\rangle - ba_i$ if n_i is not in a maximal dimensional σ. The restriction of f to the. T-stable affine open set $U(\sigma)$ induces an open immersion of $U(\sigma)$ into the spectrum of the k-subalgebra $k[e(m - m(\sigma)); m \in \lambda(bD)] \subset k[T]$, hence into its normalization, which corresponds by (5.1) to the cone $\sigma' = \{y \in N_{\mathbb{R}}; \langle m - m(\sigma), y\rangle \geq 0$ for all $m \in \lambda(bD)\}$. Thus by (5.4), σ is a face of σ'. Since σ is maximal dimensional, we conclude that $\sigma = \sigma'$. Hence if n_i is not in σ, there exists $m \in \lambda(bD)$ such that $0 > \langle m - m(\sigma), n_i\rangle$ and we are done. $\square$

Let us now prove the sufficiency. Suppose there exists a positive integer b and, for each maximal dimensional $\sigma \in \Delta, m(\sigma) \in M$ such that $\langle m(\sigma), n_i\rangle \geq -ba_i$ for $1 \leq i \leq d$ with the equality holding if and only if n_i is in σ. In particular, $m(\sigma)$ is in $\lambda(bD)$. Thus by Lemma 6.3, $O_X(bD)$ is generated by its global sections, and there exists a morphism

$$f : X \longrightarrow \mathbb{P}(\bigoplus_{m \in \lambda(bD)} ke(m)).$$

For each maximal dimensional $\sigma \in \Delta$, let $V(\sigma)$ be the affine subspace of **31**
the projective space whose homogeneous coordinate corresponding to $m(\sigma)$ does not vanish. To show that f is a closed immersion by possibly replacing b by its multiple, it is enough to show that

(i) $f^{-1}(V(\sigma)) = U(\sigma)$ for all maximal dimensional $\sigma \in \triangle$, and

(ii) for a multiple of b, the restriction $f|U(\sigma) : U(\sigma) \to V(\sigma)$ is a closed immersion.

For (i), it is enough to show that

$$f^{-1}(V(\sigma)) \cap U(\sigma') = U(\sigma) \cap U(\sigma')$$

for all maximal dimensional $\sigma' \in \wedge$, since X is covered by those $U(\sigma')$'s. The right hand side is contained in the left hand side and equals $U(\sigma \cap$

σ') by Theorem 4.2 (ii). The left hand side is the T-sable affine open set of $U(\sigma')$ defined by the non-vanishing of $e(m(\sigma) - m(\sigma'))$, thus it corresponds to the face $\sigma'' = \sigma' \cap (m(\sigma) - m(\sigma'))^{\perp}$ of σ'. Hence we are reduced to showing $\sigma'' = \sigma \cap \sigma'$. Both of those are faces of σ', and $\sigma \cap \sigma' \subset \sigma''$. Let n_i be in σ' but not in σ. Then $\langle m(\sigma) - m(\sigma'), n_i \rangle > 0$ by assumption, thus n_i is not in σ''.

It remains to show (*ii*). It is enough to show that as a semigroup $\check{\sigma} \cap M$ is generated by $\{m - m(\sigma); m \in \lambda(bD)\}$ by possibly taking a multiple of b. Let $m_1, \ldots, m_s$ be generators of $\check{\sigma} \cap M$ as a semigroup. Let b' be a positive integer. Thus for $n_i \in \sigma$ and $1 \leq j \leq s$, we have
32 $\langle m_j, n_i \rangle \geq 0$, hence $\langle m_j + b'm(\sigma), n_i \rangle \geq -b'ba_i$. On the other hand, for $n_i \notin \sigma$ and $1 \leq j \leq s$, we have $\langle m_j + b'm(\sigma), n_i \rangle \geq -b'ba_i$ if b' is large enough. Thus $m_j \in \lambda(b'bD)$ for $1 \leq j \leq s$ and we are done.

Remark. The inequalities in Theorem 6.4 can be interpreted as follows: The convex hull in $M_{\mathbb{R}}$ of $\{m(\sigma); \sigma \in \triangle$ maximal dimensional $\}$ has exactly d facets (i.e. codimension one faces) $F_1, \ldots, F_d$ perpendicular to $n_1, \ldots, n_d$, respectively. Moreover, $m(\sigma)$'s are exactly the vertices, and the intersection of F_{i_s} is the vertex $m(\sigma)$ if and only if $n_{i_1}, \ldots, n_{i_s}$ are the fundamental generators of σ.

Remark. We refer the reader to Mumford et al. [63] for the interpretation of these inequalities in terms of the "concavity" and the "strict concavity" of certain functions. In our language, they show in Theorem 13, p.48 that even if X is not complete, $D = \sum_{1\leq i\leq d} a_i D_i$ is ample if and only if there exists a positive integer b and, for each $\sigma \in \triangle, m(\sigma) \in M$ such that $\langle m(\sigma), n_i \rangle \geq -ba_i$ with the equality holding if and only if $n_i \in \sigma$.

Corollary 6.5 (Demazure). *Let $X = \mathrm{temb}(\Delta)$ be a complete nonsingular T-embedding, and let $D = \sum_{1\leq i\leq d} a_i D_i$. Then the following are equivalent :*

(i) D is very ample

(ii) D is ample.

33 *(iii) For each maximal dimensional $\sigma \in \Delta$, the unique element $m(\sigma) \in M$ defined by*

$$\langle m(\sigma), n_i \rangle = -a_i \quad \text{for } n_i \in \sigma$$

satisfies

$$\langle m(\sigma), n_i \rangle > -a_i \quad \text{for } n_i \notin \sigma$$

Proof. $(ii) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (iii)$. By Theorem 6.4, there exists a positive integer b and, for each maximal dimensional $\sigma \in \triangle, m'(\sigma) \in M$ such that $\langle m'(\sigma), n_i \rangle \geq -ba_i$ for $1 \leq i \leq d$ with the equality holding if and only if $n_i \in \sigma$. Since X is non-singular, the fundamental generators of σ form a $\mathbb{Z}$-basis of N. Hence $m'(\sigma) = bm(\sigma)$, where $m(\sigma)$ is as in (iii). □

It remains to show $(iii) \Rightarrow (i)$. From what we saw in the proof of Theorem 6.4, it is enough to show that for each maximal dimensional $\sigma \in \Delta, \check{\sigma} \cap M$ is generated as a semigroup by $\{m - m(\sigma); m \in \lambda(D)\}$. We may assume that $n_1, \dots, n_r$ are the fundamental generators of σ, hence form a $\mathbb{Z}$-basis of N. Since X is complete and non-singular, there exists, for each $1 \leq i \leq r$, a maximal dimensional cone $\sigma_i \in \Delta$ such that the fundamental generators of $\sigma_i \cap \sigma$ are $\{n_1, \dots, i, \dots, n_r\}$. Let the remaining fundamental generator of σ_i be n_i, with $r < i' < d$. Then $\langle m(\sigma_i), n_j \rangle \geq -a_j$ with the equality holding if and only if $j = i'$ or $1 \leq j \leq r$. Let $\{m_1, \dots, m_r\}$ be the basis of M dual to $\{n_1, \dots, n_r\}$. They generate $\check{\sigma} \cap M$. We see that $m(\sigma) = \sum_{1 \leq i \leq d} a_j m_j$ and $m(\sigma_i) - m(\sigma) =$ 34
am_i, where $a = \langle m(\sigma_i) - m(\sigma), n_i \rangle$ is positive by assumption. Since $\langle m(\sigma), n_j \rangle \geq -a_j$ for $1 \leq j \leq d$, we see easily that $\langle m_i + m(\sigma), n_j \rangle \geq -a_j$ for $1 \leq j \leq d$, i.e. $m_i + m(\sigma) \in \lambda(D)$.

Remark. We show in §. 8 that any 2-dimensional normal torus embedding of finite type is quasi-projective, using Theorem 6.4. On the other hand, there are many are non-projective 3-dimensional complete non-singular torus embeddings, as we see in §. 9.

Proposition 6.6 (Demazure). *Let $X = \mathrm{temb}(\Delta)$ be a non-singular T-embedding. Then the canonical invertible sheaf* $\det(\Omega^1_X)$ *equals*

$$\omega_X = 0_X\Big(-\sum_{1 \leq i \leq d} D_i\Big),$$

where $D_i = \overline{\mathrm{orb}(\sigma_i)}$ *and* $Sk^1(\triangle) = \{\sigma_1, \ldots, \sigma_d\}$.

Proof. For each $\sigma \in \Delta$, there exists a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N such that $n_1, \ldots, n_s$ are the fundamental generators of σ. Let $\{m_1, \ldots, m_r\}$ be the dual basis of M, and let $u_i = e(m_i)$. Then the coordinate ring of $U(\sigma)$ equals $k[u_1, \ldots, u_r, 1/u_{s+1}, \ldots, 1/u_r]$. Consider the rational section

$$\xi = (du_1/u_1) \wedge \ldots \wedge (du_r/u_r)$$

of $\det(\Omega^1_X)$. Its divisor on $U(\sigma)$ equals $-\sum_{1\leq i\leq s} \overline{\mathrm{orb}(\mathbb{R}_o n_i)}$. We are done, since for a different $\mathbb{Z}-$ basis of N, the rational section of $\det(\Omega^1_X)$ defined in this fashion for another affine open set differs from ξ only by sign. □

35 **Remark.** Mumford et al. [63, Thm.9, p.29 and Thm.14, p.52] show that any normal T-embedding $X = \mathrm{temb}(\Delta)$ is Cohen-Macaulay. Moreover, its dualizing sheaf ω_X coincides with the double dual of $\det(\Omega^1_X)$ and equals the fractional ideal associated with the Weil divisor $-\sum_{1\leq i\leq d} D_i$, where $D_i = \overline{\mathrm{orb}(\sigma_i)}$ and $Sk^1(\Delta) = \{\sigma, \ldots, \sigma_d\}$. (See the end our Introduction for a recent generalization of this result by Ishida.)

Proposition 6.7. *Let* $\mathrm{temb}(\triangle)$ *be a complete non-singular torus embedding. For a 1-codimensional cone* $\tau \in \triangle$*, there exist exactly two maximal dimensional cones* $\sigma, \sigma' \in \triangle$ *such that* $\tau < \sigma$ *and* $\tau < \sigma'$. *Let* $n_1, \ldots, n_{r-1}$ *be the fundamental generators of* τ. *Let the additional fundamental generator of* $\sigma(resp.\ .\sigma')$ *be* $n(resp.\ .n')$. *Then there exist* $a_i \in \mathbb{Z}, 1 \leq i \leq r-1$ *such that*

$$n + n' + \sum_{1\leq i\leq r-1} a_i n_i = 0.$$

Moreover, we have

$$a_i = D_1 \cdots D_i^2 \cdots D_{r-1} \quad 1 \leq i \leq r-1$$

where $D_i = \overline{\mathrm{orb}(\mathbb{R}_o n_i)}$.

Proof. Since X is complete, $N_{\mathbb{R}}$ is the union of cones in Δ. The first assertion, for which the non-singularity is not necessary, is a well-known fact ibn convex geometry. Since. X is on-singular, $\{n, n_1, \ldots, n_{r-1}\}$ and $\{n', n_1, \ldots, n_{r-1}\}$ are $\mathbb{Z}$-bases of N. Thus n' is a $\mathbb{Z}$-linear combination of
$n, n_1, \ldots, n_{r-1}$ with the coefficient of n equals -1, since σ and σ' are **36**
on the opposite side of τ. As for the assertion, it is enough to restrict ourselves to $i = 1$. The divisors $D_i = \overline{\mathrm{orb}(\mathbb{R}_o n_i)}$ $1 \leq i \leq r-1$ intersect transversally with the intersection $\mathrm{orb}(\tau) \cong \mathbb{P}_1$. Let $\{m, m_1, \ldots, m_{r-1}\}$ be the basis of M dual to $\{n, n_1, \ldots, n_{r-1}\}$. Then $\mathrm{div}(m_1)$ is the sum of $D_1 + \langle m_1, n' \rangle D'$ and a divisor disjoint from $\overline{\mathrm{orb}(\tau)}$, where $D' = \overline{\mathrm{orb}(\mathbb{R}_o n')}$. Since $\langle m_1, n' \rangle = -a_i$ and D' intersects transversally with $\overline{\mathrm{orb}(\tau)}$, we are done. $\square$

7 Example of torus embeddings and morphisms

In this section, we give typical examples of torus embeddings and equivariant dominant morphisms. We need some of them later.

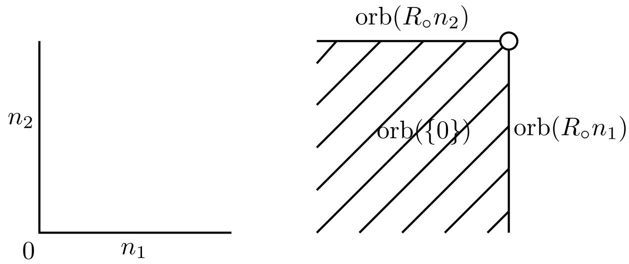
7.1 Affine spaces The r-dimensional affine space $\mathbb{A}_r = \underline{k}^r$ is obviously a $(\underline{k}^*)^r$-embedding. It corresponds to $(N, \triangle)$, where $N \cong \mathbb{Z}^r$ with a $\mathbb{Z}-$basis $\{n_1, \ldots, n_r\}$ and

$$\Delta = \{\text{the faces of } \mathbb{R}_o n_1 + \cdots + \mathbb{R}_o n_r\}.$$

7.2 More generally for $0 \leq s \leq r$, $\underline{k}^s \times (\underline{k}^*)^{r-s}$ corresponds to (N, Δ) with

$$\Delta = \{\text{the faces of } \mathbb{R}_\circ n_1 + \cdots + \mathbb{R}_\circ n_s\}.$$

7.3 $\underline{k}^r - \{0\}$ is again a $(\underline{k}^*)^r$- embedding. It corresponds to (N, Δ), where Δ consists of the proper (i.e. not equal to itself) faces of $\mathbb{R}_\circ n_1 + \cdots + \mathbb{R}_\circ n_r$. When $r = 2$, it looks like this:



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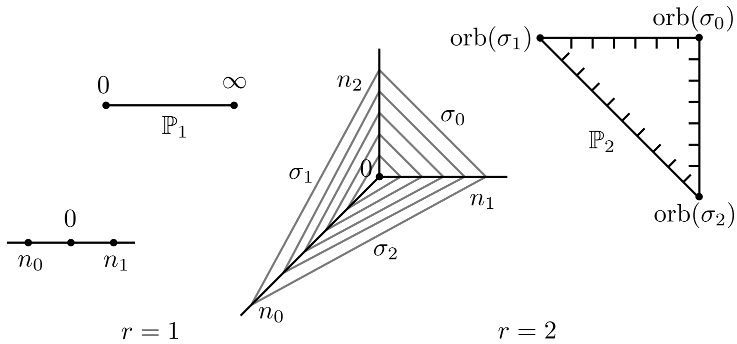
7.4 **Projective spaces:** The r-dimensional projective space $\mathbb{P}_r$ is again obviously a $(\underline{\mathrm{k}}^*)^r$-embedding. The corresponding (N, Δ) is defined as follows : $N \cong \mathbb{Z}^r$ with a $\mathbb{Z}$- basis $\{n_1, \ldots, n_r\}$. Let $n_0 = -(n_1 + \cdots + n_r)$. Then $\triangle$ consists of the faces of $\sigma_0, \ldots, \sigma_r$, where

$$\sigma_i = \mathbb{R}_\circ n_0 + \cdots + \overset{i}{\vee} + \cdots + \mathbb{R}_\circ n_r.$$



The canonical morphism $\underline{\mathrm{k}}^{r+1} - \{0\} \to \mathbb{P}_r$ is equivariant and corresponds to the homomorphism $\tilde{N} \cong \tilde{\mathbb{Z}}^{r+1} \to N$ sending elements of the basis $\{\tilde{n}_o, \ldots, \tilde{n}_r\}$ of $\tilde{N}$ to $\{n_0, \ldots, n_r\}$.

In this connection, we have the following characterization of the projective space due to Mabuchi [32].

Theorem 7.1 (Mabuchi)**.** *Let X be an r-dimensional complete non-singular T-embedding. Then the following are equivalent.*

38 *(1) $X \cong \mathbb{P}_r$ equivariantly.*

(2) The tangent bundle θ_X of X is ample.

(3) The normal bundle of each T-stable irreducible divisor (which is non-singular by Theorem 4.3) is an ample invertible sheaf.

(4) The number of T-fixed points of X (which equals the Euler number of X by Iversen [27]) is exactly $r + 1$.

(5) The number of T-stable irreducible divisors is exactly $r + 1$.

(6) Pic $(X) = \mathbb{Z}$.

Let $X = T_N \operatorname{emb}(\triangle)$. Then (5) means that $Sk^1(\Delta)$ has $r + 1$ cones. (4) means that Δ has $r + 1$ maximal dimensional cones.
(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) is easy to show in view of Proposition 6.7.
(6) $\Rightarrow$ (5) follows easily from the exact sequence in Proposition 6.1.
(5) $\Rightarrow$ (6) follows from the exact sequence and the fact that Pix(X) is torsion free. (See Demazure [8, p.566]).
(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is well-known.
(3) $\Rightarrow$ (4) is due to Mabuchi, who showed (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) as a special case of Hartshorne's conjecture [16] $(H - r)$ to the effect that an r-dimensional non-singular complete variety with the ample tangent bundle is necessarily $\mathbb{P}_r$.

$(H-2)$ is known to be true. $(H-3)$ was proved recently by Mabuchi [32] and Mori-Sumihiro [41].
(6) $\Rightarrow$ (1) can also be proved directly by means of results in Mori [35], as Sumihiro and Ishida pointed out.

7.5 Products We leave the proof of the following to the reader. 39

Proposition 7.2. *Let (N', Δ') and (N'', Δ'') be r.p.p. decompositions. Then we have*

$$T_{N'} \operatorname{emb}(\Delta') \times T_{N''} \operatorname{emb}(\Delta'') = T_{N' x N''} \operatorname{emb}(\Delta' \times \Delta'')$$

where $\Delta' \times \Delta''$ is the r.p.p.decomposition of $N' \times N''$ consisting of cones

$$\sigma = \sigma' \times \sigma''$$

with σ' and σ'' running through Δ' and Δ'', respectively.

7.6 Equivariant fiber bundles More generally, we have the following description of equivariant fiber bundles.

Proposition 7.3. *Let $h : (N, \Delta) \to (N', \Delta')$ be a map of r.p.p. decompositions, and let $f : X = T_N \operatorname{emb}(\Delta) \to X' = T_N \operatorname{emb}(\Delta')$ be the corresponding equivariant dominant morphism. Consider $N'' = \ker[h : N \to N']$ and an r.p.p. decomposition (N'', Δ'') with $X'' = T_{N''} \operatorname{emb}(\Delta'')$. Then $f : X \to X'$ is an equivariant fiber bundle with the typical fiber X'' if and only if the following conditions are satisfied:*

(i) $h : N \to N'$ is surjective.

(ii) There exists a lifting $\tilde{\Delta}' \subset \Delta$ of Δ', i.e. $(N, \tilde{\triangle}')$ is an r.p.p. decomposition and for each $\sigma' \in \triangle'$, there exists a unique $\tilde{\sigma}' \in \tilde{\triangle}'$ such that h induces a bijection

$$h : \tilde{\sigma}' \xrightarrow{\sim} \sigma'.$$

40 *(iii) $\triangle$ consists of the cones*

$$\sigma = \tilde{\sigma}' + \sigma'' = \{\tilde{y}' + y''; \tilde{y}' \in \tilde{\sigma}', y'' \in \sigma''\}$$

with $\tilde{\sigma}'$ and σ'' running through $\tilde{\triangle}'$ and $\triangle''$.

Again we leave the proof to the reader. Note that the lifting $\tilde{\triangle}'$ itself defines an open set of X which is an equivariant $T_{N''}$-bundle over $X' \cdot X$ is associated to it via the $T_{N''}$-action on X''.

(7.6′) Equivariant $\mathbb{P}_r$-bundles: As a special case of (7.6), let $D'_0, \ldots, D'_r$ be T'-stable Cartier divisors on X' and let $L'_i = 0_{X'}(D'_i)$. Then the totally decomposable $\mathbb{P}_r$- bundle

$$f : X = \mathbb{P}(L'_0 \oplus \cdots \oplus L'_r) \to X'$$

has a lifting of $T_{N'}$-action determined by D'_i's Moreover, X has a fiberwise action of $(\underline{k}^*)^r$ and is a $T_{N'} \times (\underline{k}^*)^r$-embedding. A change of Cartier divisors in the linear equivalence classes gives rise to isomorphic torus embeddings.

In terms of r.p.p.decompositions, it can be described as follows: Let $N'' = \mathbb{Z}^r$ with a $\mathbb{Z}$- basis $\{\ell_1, \ldots, \ell_r\}$, and let $N = N' \times N''$. For each $0 \le i \le r$ and $\sigma' \in \triangle'$, there exists $m_i'(\sigma') \in M' = (N')^*$ such that the $A(\sigma')$-module $L_i'(U(\sigma'))$ is generated by $e(m_i'(\sigma'))$, by Lemma 6.2. Let $\ell_0 = -(\ell_1 + \cdots + \ell_r)$, and let $\tilde{\sigma}'$ be the image of σ' under the linear map $N'_{\mathbb{R}} \hookrightarrow N_{\mathbb{R}}$ sending y' to $(y', -\sum_{0 \le i \le r} \langle m_i'(\sigma'), y' \rangle \ell_i)$.
The collection $\tilde{\Delta}' = \{\tilde{\sigma}'; \sigma' \in \Delta'\}$ forms a lifting of Δ' to $N_{\mathbb{R}}$. Then 41
$X = T_N \operatorname{emb}(\triangle)$, where $\triangle$ consists of

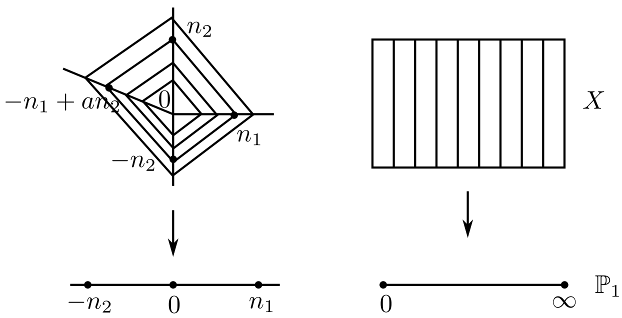
$$\sigma = \tilde{\sigma}' + \sigma''$$

with $\tilde{\sigma}'$ running through $\tilde{\Delta}'$ and σ'' running through $\triangle'' = \{$the faces of $\sigma_0'', \ldots, \sigma_r''\}$, with $\sigma_i'' = \mathbb{R}_o \ell_0 + \cdots + \overset{i}{\vee} + \cdots + \mathbb{R}_o \ell_r$.

Example. Let $X' = \mathbb{P}_1$ and $r = 1$. For $a \in \mathbb{Z}_o$, consider the rational ruled surface

$$X = \mathbb{P}(0_{\mathbb{P}_1} \oplus 0_{\mathbb{P}_1}(a))$$

which is usually denoted by F_a or Σ_a and called a Hirzebruch manifold. The corresponding r.p.p.decomposition of $N = \mathbb{Z}^2$ with a $\mathbb{Z}$- basis $\{n_1, n_2\}$ looks like this:



Remark. When E' is a T'-linearized vector bundle on a T'-embedding X', the bundle $\mathbb{P}(E')$ has a lifting of T'-action. But it is not a torus embedding unless E' is totally decomposable as above. Nevertheless, it provides us with a typical example of varieties with torus action which need not be a torus embedding. Thus it is of some interest to classify equivariant vector bundles on torus embeddings. This was partly carried out by Kaneyama [28], who, in particular, showed the existence of equivariant but not homogeneous vector bundles on $\mathbb{P}_2$.

7.7 Quotient singularities: The quotient singularities explained by Mumford et all. [63, p.16-19] fit in nicely with our formulation.

Let $N \cong \mathbb{Z}^r$ be a free $\mathbb{Z}$-module of rank r with the $\mathbb{Z}$ basis $\{n_1, \dots, n_r\}$. Let $n'_1, \dots, n'_r \in N$ be primitive elements which are $\mathbb{R}$-linearly independent in $N_{\mathbb{R}}$, and let $N' \subset N$ be the $\mathbb{Z}$-submodule of finite index generated by $n'_1, \dots, n'_r$. Consider the simplicial cone

$$\sigma = \mathbb{R}_o n'_1 + \cdots + \mathbb{R}_o n'_r \subset N_{\mathbb{R}} = N'_{\mathbb{R}}.$$

We thus have a map of r.p.p.decompositions $h : (N', \Delta) \to (N, \Delta)$, where Δ consists of the faces of σ. Hence we have an equivariant surjective morphism

$$f : T_{N'} \operatorname{emb}(\Delta) = \underline{k}^r \longrightarrow X = T_N \operatorname{emb}(\Delta),$$

which is the quotient map under the canonical action of the (scheme-theoretic) kernel $\ker[f : T_{N'} \to T_N]$. When the characteristic of the ground field k does not divide the order of N/N', then the kernel is (non-canonically) isomorphic to N/N'. Indeed, let M (resp. M') be the $\mathbb{Z}$-module dual to N (resp. N'). Then M is canonically a submodule of finite index of M', with a pairing

$$\langle , \rangle : M' \times N \longrightarrow \mathbb{Q}$$

which extends the canonical pairings $M \times N \to \mathbb{Z}$ and $M' \times N \to \mathbb{Z}$, and which, moreover, induces a non-degenerate pairing

$$\langle , \rangle : (M'/M) \times (N/N') \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Then $\ker[f : T_{N'} \to T_N] = \operatorname{Hom}_{gr}(M'/M, k^*)$, which is (non-
42 canonically) isomorphic to N/N' via the above pairing.

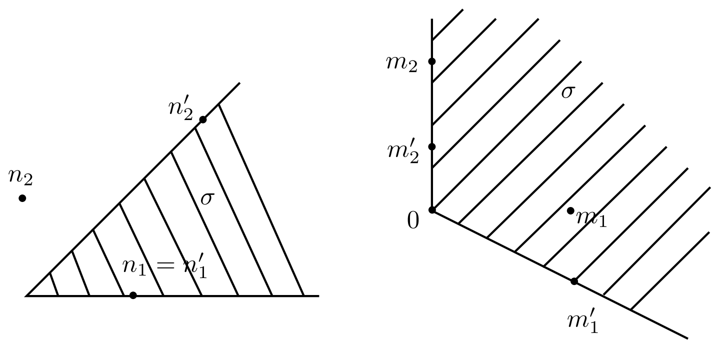
For instance, let $r = 2$. We may choose a $\mathbb{Z}$-basis $\{n_1, n_2\}$ of N so that

$$\begin{aligned} n'_1 &= n_1 \\ n'_2 &= an_1 + bn_2 \end{aligned}$$

where a, $b \in \mathbb{Z}$ with $(a, b) = 1$ and $0 \leq a < b$. Then the action of $\ker[f : T_{N'} \to T_N]$ on $\underline{k}^2$ coincides with the action of $\mathbb{Z}/b\mathbb{Z}$ on $\underline{k}^2$ with the generator acting via

$$\underline{k}^2 \ni (z, w) \longmapsto (\zeta^{-a}z, \zeta w) \in \underline{k}^2,$$

where ζ is a primitive b-th root of 1 and $z = e(m'_1)$, $w = e(m'_2)$ with $\{m'_1, m'_2\}$ the basis of M' dual to $\{n'_1, n'_2\}$.



7.8 Equivariant blowing up: The maps of the form $h : (N, \Delta' \to (N, \Delta)$ give rise to equivariant birational morphisms f of the corresponding torus embeddings. Obviously, f is an open immersion if and only if $\Delta' \subset \Delta$.

The most interesting case is when Δ' is a *subdivision* of Δ. It was shown by Mumford et all. [63, Thm.10, p.31] that is this case, f is the normalization of the blowing up along a T-stable fractional ideal. **43**

They show [ibid., Thm.11, p.32] that given Δ, there always exists a subdivision Δ' which is non-singular, i.e. any torus embedding has an *equivariant resolution of singularities* .

The *minimal resolution* of singularities of a 2-dimensional normal torus embedding can be described very simply as in [ibid., p.35-40]. Note that the singularities in this case are isolated, and the blowing up an isolated singular point is automatically normal, although its description in terms of *r.p.p.* decompositions is rather complicated and closely related to continued fractions. In this connection, we refer the reader also to Gonzalez-Sprinberg [12], who dealt with *Nash transforms* of 2-dimensional normal torus embeddings

We need later the following description of a blowing up in the non-singular case.

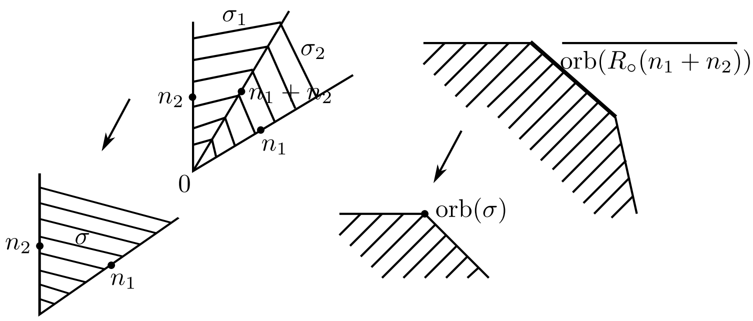
Proposition 7.4. *Let $T \subset X$ be a non-singular torus embedding corresponding to an r.p.p.decompositon (N, Δ). For $\sigma \in \Delta$, the blowing up of X along the T-stable non-singular closed subvariety $\overline{\mathrm{orb}(\sigma)}$ is a non-singular T-embedding corresponding to the subdivision (N, Δ^*) obtained from Δ by "starring at its barycenter" as follows: Let $n_1, \ldots, n_s$ be the fundamental generators of σ, and let $n_0 = n_1 + \cdots + n_s$ be the "barycenter". For $\Delta \ni \tau > \sigma$ and $1 \leq i \leq s$. Let $\tau_i \subset N_R$ be the cone*
44 *obtained from τ by replacing one of its fundamental generators n_i by n_0 and leaving the other generators as they are.*

Then

$$\Delta^* = (\Delta - \{\tau \in \Delta; \tau > \sigma\}) \cup (\bigcup_{\sigma < \tau \in \Delta} \{ \text{ the faces of } \tau_i; 1 \leq i \leq s\}).$$

Proof. $\overline{orb(\sigma)}$ is non-singular by Theorem 4.3. Obviously it is enough to describe the blowing up on the affine open set $U(\tau)$ with $U(\tau) \cap \overline{orb(\sigma)} \neq \mathbb{Q}$, i.e. $\sigma < \tau \in \Delta$ by Theorem 4.2. Let $n_1, \ldots, n_s$, with $s \leq s'$ be the fundamental generators of τ. Since τ is non-singular, they can be extended to a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N. Let $\{m_1, \ldots, m_r\}$ be the dual basis of $M = N^*$, and let $u_i = e(m_i)$. Then the coordinate ring A of $U(\tau)$ is the localization $A = k[u_1, \ldots, u_r]_{u_{s'+1} \cdots u_r}$ and the ideal of $\overline{orb(\sigma)} \cap U(\tau)$ is generated by $u_1, \ldots, u_s$. For $1 \leq i \leq s$, let $A_i = A[u_1/u_i, \ldots, u_r/u_i]$. Then the inverse image of $U(\sigma)$ in the blowing up is covered by Spec A_i with $1 \leq i \leq s$. Obviously, Spec A_i is a normal affine T-embedding corresponding to the cone $\tau_i = \mathbb{R}_o n_0 + \sum_{\substack{1 \leq j \leq s' \\ j \neq i}} \mathbb{R}_o n_j$. □

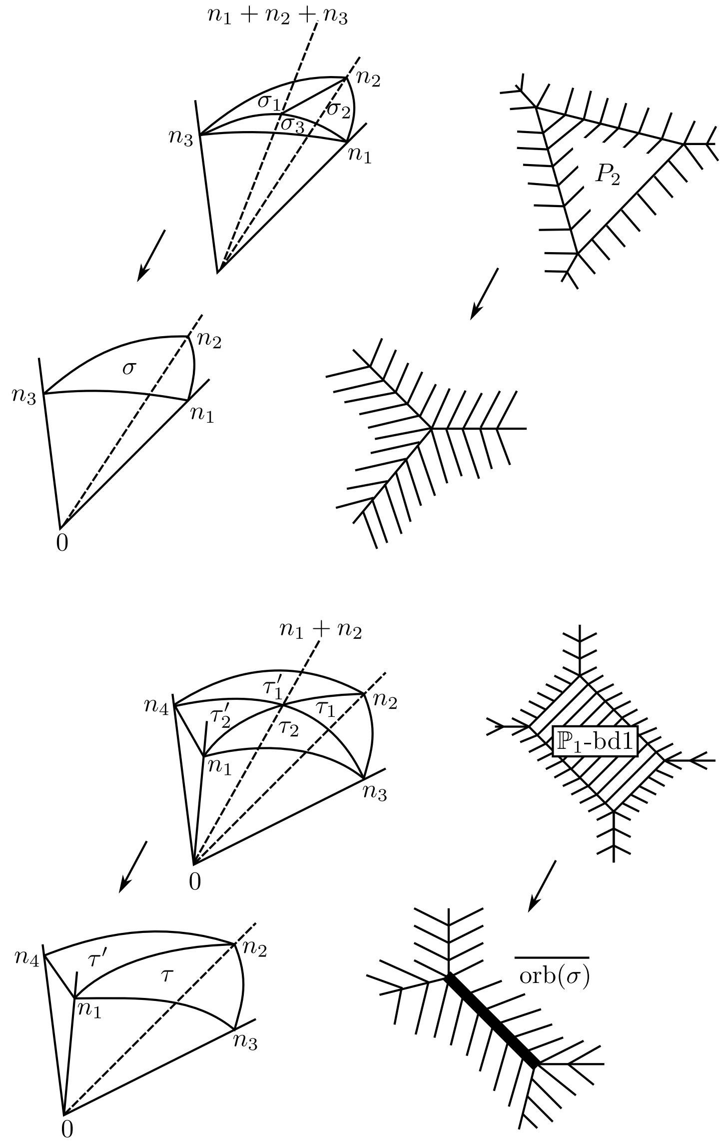
Corollary 7.5. *Let $T \subset X$ be a 2-dimensional non-singular torus embedding corresponding to an r.p.p.decompostion (N, Δ). For a 2 dimensional cone $\sigma = \mathbb{R}_o n_1 + \mathbb{R}_o n_2$, the blowing up of X along the T-fixed point ord(σ) is a non-singular T-embedding corresponding to (N, Δ^*), where Δ^* is obtained from Δ by removing σ and adding the faces of $\sigma_1 = \mathbb{R}_o(n_1 + n_2) + \mathbb{R}_o n_2$ and $\sigma_2 = \mathbb{R}_o n_1 + \mathbb{R}_o(n_1 + n_2)$.*



Corollary 7.6. *Let $T \subset X$ be a 3-dimensional non-singular torus embedding corresponding to an r.p.p.decomposition (N, Δ).*

(i) *For a 3-dimensional cone $\sigma = \mathbb{R}_o n_1 + \mathbb{R}_o n_2 + \mathbb{R}_o n_3$, the blowing up of X along the T-fixed point* $\mathrm{orb}(\sigma)$ *is a nonsingular T-embedding corresponding to (N, Δ^*), where Δ^* is obtained from Δ by removing σ and adding the faces of σ_1, σ_2 and σ_3 as in the picture below.*

(ii) *For a 2-dimensional cone $\sigma = \mathbb{R}_o n_1 + \mathbb{R}_o n_2$, let $\tau = \mathbb{R}_o n_1 + \mathbb{R}_o n_2 + \mathbb{R}_o n_3$ and $\tau' = \mathbb{R}_o n_1 + \mathbb{R}_o n_2 + \mathbb{R}_o n_4$ be the 3-dimensional cones in Δ having σ as a face (cf. Prop.6.7). Then the blowing up of X along the T-stable curve $\overline{orb(\sigma)}$ is a non singular T-embedding corresponding to (N, Δ^*), where Δ^* is obtained from Δ by removing σ, τ, τ' and adding the faces of $\tau_1, \tau_2, \tau'_1, \tau'_2$ as in the picture below.*



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47 7.9 Algebraic varieties of monomial type: As Mumford et al. [63] pointed out in the it introduction the theorey of torus embeddings is

a unified and globalized treatment of the "exponents of monomials". It can be said that when we have algebraic varieties or morphisms defined in terms of monomials only, then there is a possibility of formulating them more transparently in terms of torus embeddings. We encounter many examples in Chapter 2. Here is a motivating example:

Let $\overline{X} \subset \mathbb{A}_r = \underline{k}^r$ be a closed algebraic set defined polynomials $f_1(t), \ldots, f_\nu(t) \in k[t_1, \ldots, t_r]$, each of which is of the form

$$f(t) = t_1^{a_1} t_2^{a_r} \cdots t_r^{a_r} - t_1^{b_1} t_2^{b_r} \cdots t_r^{b_r}$$

with non-singular integers $a_1, \ldots, a_r, b_1, \ldots, b_r$. Then $\overline{X}$ is invariant the coordinatewise multiplication of elements of the group

$$\overline{T} = \overline{X} \cap (\underline{k}^*)^r,$$

This $\overline{T}$ is an algebraic subgroup of $(\underline{k}^*)^r$ but may not be an algebraic torus. It may neither be connected nor reduced. But when $\overline{T}$ is an algebraic torus, then $\overline{X}$ is a $\overline{T}$-embedding, although it may not be normal in general. Consider, for instance the rational curve with a cusp

$$\overline{X} = \{(t_1, t_2) \in \mathbb{A}_2; t_2^2 = t_1^3\}.$$

We have analogues in $\mathbb{P}_r$ or its generalization in Mori [34]

Here is a more general formulation in the affine case Let $M \cong \mathbb{Z}^r$ and **48**
let $0 \in S$ be a finitely generated sub semigroup of M which generates M as a group. Hence

$$X = \mathrm{Hom}_{\mathrm{u.s.g}}(S, \underline{k})$$

is a T-embedding with

$$T = \mathrm{Hom}_{gr}(M, \underline{k}^*).$$

Let $m_1, \ldots, m_\nu, m'_1, \ldots, m'_\nu \in S$. Then consider the quotient

$\overline{M} = M/$ (the subgroup $gen\underline{\mathrm{d}}$ by $m_1 - m'_1, \ldots, m_\nu - m'_\nu$) and the image $\overline{S}$ of S under the canonical projection $M \to \overline{M}$. Then $\overline{S}$ is the quotient of S with respect to the equivalent relation

$$s \sim s' \Leftrightarrow {}^\exists s'' \in S, a_1, \ldots, a_\nu, b_1, \ldots, b_\nu \in \mathbb{Z}_o \text{ such that}$$

$$s = s'' + \Sigma a_i m_i + \Sigma b_i m'_i$$
$$s' = s'' + \Sigma b_i m_i + \Sigma a_i m'_i$$

Let

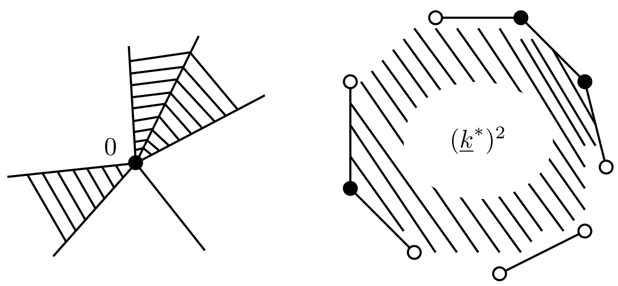
$$\overline{T} = \mathrm{Hom}_{gr}(\overline{M}, \underline{k}^*) \subset T \quad \text{algebraic subgroup}$$
$$\overline{X} = \mathrm{Hom}_{u.s.g}(\overline{S}, \underline{k}) \subset X \quad \text{closed algebraic set.}$$

Then $\overline{X}$ contains $\overline{T}$ as a dense open subset and is invariant under $\overline{T}$. If $m_1 - m'_1, \ldots, m_\nu - m'_\nu$ generates a pure subgroup of M, then $\overline{T}$ is an algebraic torus and $\overline{X}$ is a $\overline{T}$-embedding. If, moreover, $S = \check{\sigma} \cap M$ for a cone $\sigma \subset N_{\mathbb{R}}$ as in (5.1), then the *normalization* of $\overline{X}$ corresponds to the cone $\sigma \cap \overline{N}_{\mathbb{R}}$, where $\overline{N} = N \cap \{m_1 - m'_1, \ldots, m_\nu - m'_\nu\}^\perp$.

8 Torus embeddings of dimension ≤ 2

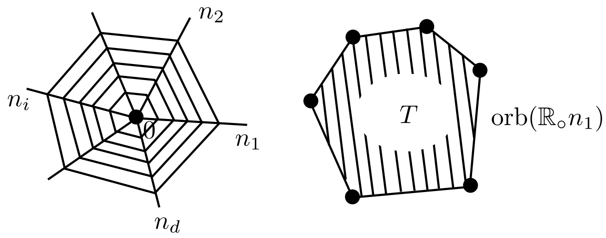
49 It is easy to see that 1-dimensional normal torus embeddings are $\underline{k}^*$, $\mathbb{A}_1 = \underline{k}$ and $\mathbb{P}_1$, up to isomorphism.

From now in this section, Let $N \cong \mathbb{Z}^2$ and $T = T_N$. A 2-dimensional normal T-embedding of finite type looks like this:



$\triangle$ Temb $(\triangle)$

In particular, a complete normal 2 dimensional T-embedding is determine by a finite cycle of primitive elements $n_1, \ldots, n_d$ in N going counterclockwise once around 0 in the given order such that det $(n_i, n_{i+1}) > 0$ for $1 \leq i \leq d$ $(n_{d+1} = n_1)$, i.e. n_{i+1} lies strictly before $-n_i$ (which may not belong to the cycle).



Proposition 8.1. *Any 2-dimensional normal torus embedding $T \subset X$ of finite is quasi-projective, i.e. can be equivariantly embedded into a projective space.* 50

Proof. Let $X = \text{Temb}\,(\Delta)$. By filling in the complement $N_{\mathbb{R}} - \bigcup_{\sigma\in\Delta} \sigma$ by appropriate cones if necessary, we can embed X equivariantly into a complete normal T-embedding. (For the existence of equivariant completions in general, we refer the reader to Sumihiro [59].) We may thus assume that X itself is complete. Let $n_1, \dots n_d$ be the fundamental generators of the 1-dimensional cones in Δ arranged, as above, in such a way that they go around 0 counterclockwise once in this order. By Theorem 6.4, X is projective if and only if there exist $a_1, \dots, a_d \in \mathbb{Z}$ such that for each 2-dimensional cone $\sigma \in \Delta$, there exists a unique $m(\sigma) \in M$ with $\langle(\sigma), n_i\rangle \geq -a_i \;\; 1 \leq i \leq d$ and with the equality holding if and any if $n_i \in \sigma$. As we remarked after the proof of Theorem 6.4, this means that the convex hull of

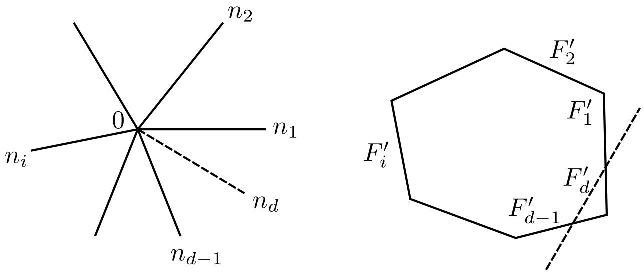
$$\{M(\sigma) : \Delta \ni \sigma \;\; \text{2-dimensional}\}$$

in $M_{\mathbb{R}}$ has exactly d edges $F_1, \dots, F_d$ going around 0 in this order with F_i perpendicular to n_i. □

It is thus enough to show, by induction on d, the existence of a convex polygon P in $M_{\mathbb{R}}$ with the vertices in $M_{\mathbb{Q}}$ such that it has exactly d edges $F_1, \dots, F_d$ going around 0 in this order with F_i perpendicular to n_i

Obviously $d \geq 3$ and the case $d = 3$ is trivial. If $d \geq 4$, there 51
certainly exists n_d, say, such that the cycle $n_1, \dots, n_{d-1}$ still defines an $f.r.p.p.$ decomposition. Let P' be a polygon with $d - 1$ edges

$F'_1, \dots, F'_{d-1}$ satisfying the required property. At the vertex of P' which is the intersection of edges F'_1 and F'_{d-1}, we can cut the corner by a line perpendicular to n_d and obtain a required P with d edges.



A complete non-singular 2-dimensional T-embedding is determined by a cycle of elements $n_1, \dots, n_d \in N$ as above with

$$\det(n_i, n_{i+1}) = 1 \quad 1 \leq i \leq d.$$

This condition is rigid enough to allow a complete classification.

Theorem 8.2. *Up to isomorphism, a complete non-singular* 2 *- dimensional T-embedding is obtained from*

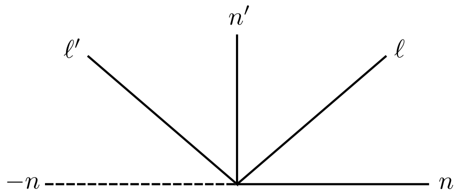
$$\mathbb{P}_2 \text{ or } F_a = \mathbb{P}(0_{\mathbb{P}_1} \oplus 0_{\mathbb{P}_1}(a)) \qquad 1 \neq a \geq 0$$

by a finite succession of blowing ups along T-fixed points. It can be
52 *transformed to $\mathbb{P}_2$ by a finite succession of blowing ups and blowing downs along T-fixed points. Given non-singular T-embeddings X and X', there exists X'' obtained both from X and from X' by a finite succession of blowing ups along T-fixed points.*

Proof. Let X =Temb (Δ) with Δ determined by a cycle $n_1, \dots, n_d \in N$ going around 0 once counterclockwise. Certainly $d \geq 3$. If $d = 3$, then $n_1 + n_2 + n_3 = 0$ and $X = \mathbb{P}_2$ (cf. Theorem 7.1). We may thus assume $d \geq 4$. □

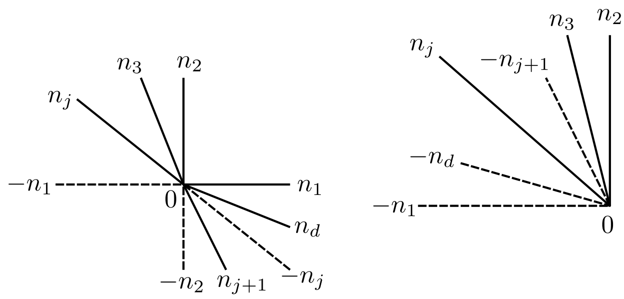
Lemma 8.3. *If $d \geq 4$, there exist i and j with $n_i + n_j = 0$.*

Sublemma 8.4. *Let $\{n, n'\}$ and $\{\ell, \ell'\}$ be $\mathbb{Z}$-bases of N. If ℓ lies in the interior of the sector $\mathbb{R}_o n + \mathbb{R}_o n'$ and if ℓ' is in the sector $\mathbb{R}_o n' + \mathbb{R}_o(-n)$, then $\ell' = n'$ or $\ell' = -n$.*



Proof of the sublemma: By assumption, there exist positive integers a, a' such that $\ell = an + a'n'$ and non-negative integers b, b' such that $\ell' = -bn + b'n'$. We are done, since $1 = \det(\ell, \ell') = ab' + a'b$ by assumption.

Proof of the lemma: Suppose the assertion of the lemma is false. By re-numbering the primitive elements if necessary, we may assume that starting counterclockwise from n_1, we have more than half of the prim- **53**
itive elements before $-n_1$, which does not coincide with any of the $n_i's$ by assumption. Let $n_j (j > 2)$ be th last primitive element before $-n_1$. Since n_{j+1} does not coincide with $-n_1$ and $-n_2$, we see by sublemma 8.4 and the convexity that n_{j+1} is is strictly between $-n_2$ and $-n_j$. Thus we conclude that $n_2, n_3, \ldots, n_j$ and $-n_{j+1}, -n_{j+2}, \ldots, -n_d, -n_1$ are all different and lie between n_2 and $-n_1$.

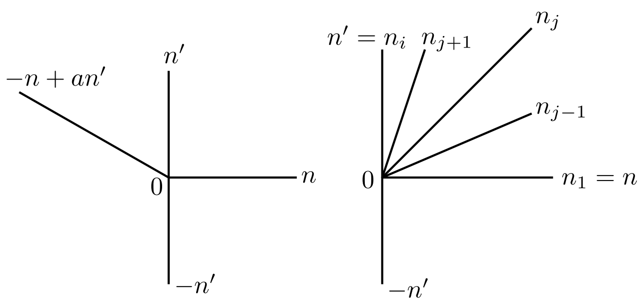


This is obviously a contradiction in view of Sublemma 8.4.

Proof of Theorem 8.2 continued: By Lemma 8.3, we may assume that $n_d + n_i = 0$ for some i. Let $n' = -n_d = n_i$ and $n = n_1$. Thus $\{n, n'\}$ is a basis of N.

If $d = 4$, we see that $n_2 = n'$ and $n_3 = -n + an'$ for some integer a. This gives rise to $X = F_a$. By symmetry, we may assume that $a \geq 0$. If $a = 1, n'$ is the sum of the adjacent primitive elements, hence by blowing down we can eliminate n'.

Thus we may assume $d \geq 5$ and at least one primitive element lies between $n = n_1$ and $n' = n_i$. It remains to show that



$d = 4$ $d \geq 5$

54

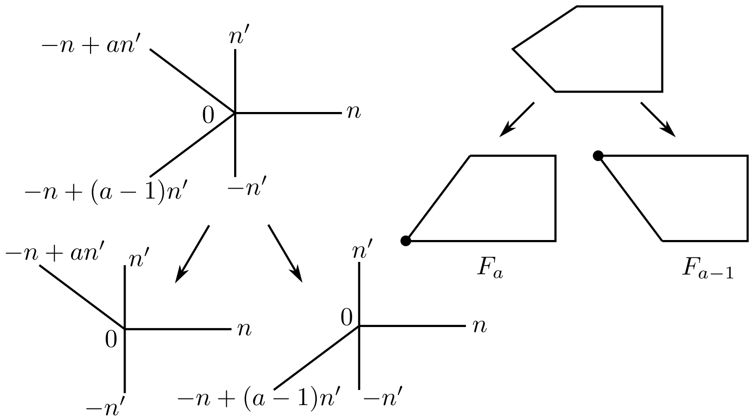
there exists $1 < j < i$ such that $n_j = n_{j-1} + n_{j+1}$. For each $1 \leq j \leq i$, there exist non -negative integers b_j, b'_j with $n_j = b_j n + b'_j n$. On the other hand for each j, there exists an integer a_j such that $n_{j-1} + n_{j+1} + a_j n_j = 0$ by Proposition 6.7. Obviously $a_j < 0$. Consider the function

$$c(j) = b_j + b'_j > 0.$$

Then $c(j-1) + c(j+1) + a_j c(j) = 0$. Moreover, since $c(1) = c(i) = 1$, there exists j so that $c(j) > c(j+1)$ and $c(j) \geq c(j-1)$. For this j, we have $(2 + a_j)c(j) > 0$ and conclude $a_j = -1$. This means that if $d \geq 5$, we can successively blow X down to some F_a. (cf. the Farey series in additive number theory.)

This fact can also be proved sing Nagata's classification [42] of relatively minimal models of rational surfaces. Note that an exceptional curve of the first kind is automatically T-stable, hence the blowing down is equivariant.

As for the second assertion of Theorem 8.2, we may assume $X = F_a$. If we insert the sum of $-n + an'$ and $-n'$ between them, then $-n + an'$ is the sum of $-n + (a-1)n'$ and n'.



This means that by blowing up a long a T-fixed point of F_a and then blowing down a T-stable curve, we get F_{a-1}. This process is usually called an *elementary transformation* of a ruled surface. By a finite repetition of this process, we transform F_a to F_1, which can be blown down to $\mathbb{P}_2$. **55**

The last assertion of Theorem 8.2 follows easily from the decomposability of birational morphisms of non-singular surfaces into blowing ups along closed points. But it can also be proved in terms of r.p.p.decompositions as follows: Let X = Temb (Δ) and X' = Temb (Δ'). Then the decomposition of N_R obtained by the intersections $\sigma \cap \sigma'$ with $\sigma \in \Delta$ and $\sigma' \in \Delta$ need not be non singular, but it has a non singular subdivision Δ''. Thus it is enough to show that a non singular subdivision Δ'' of a non-singular Δ is obtained by finite succession of operations in Corollary 7.5. This can be seen exactly as in the last part of the proof of the first assertion.

Let us now derive certain consequences from Theorem 8.2 which we need in the next section and which are also of independent interest. **56**

Let a cycle $n_1, \ldots, n_d$ of primitive elements in N determine a complete non-singular 2- dimensional torus embedding X =Temb (Δ). Let

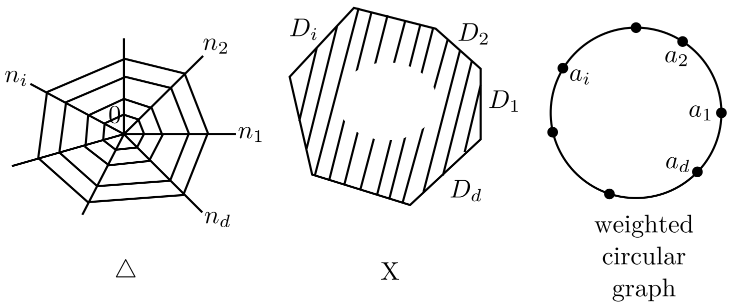
$D_i = \overline{\mathrm{orb}(\mathbb{R}_o n_i)}$. Then by Proposition 6.7, we have

$$n_{i-1} + n_{i+1} + a_i n_i = 0 \quad 1 \leq i \leq d$$

and

$$a_i = D_i^2.$$

Thus we have a cycle $D_1, \dots, D_d$ of non-singular rational curves, which can be expressed, as usual, by a circular graph with weights a_i. Note that this weighted circular graph determines the 2-dimensional T-embedding X up to isomorphism.

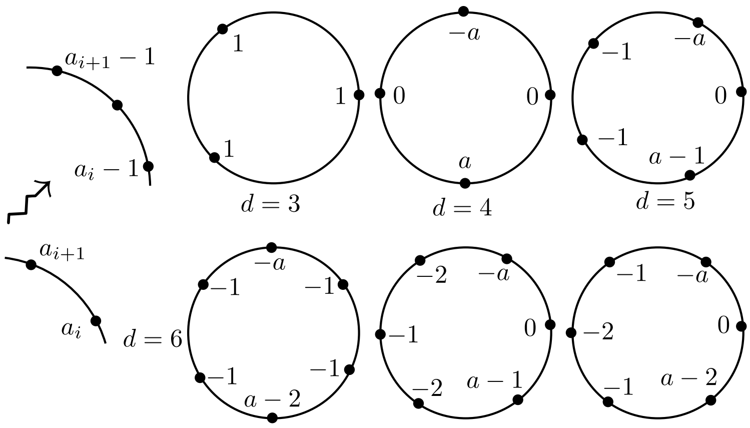


The cycle $a_1, \dots, a_d$ of integers cannot be arbitrary. For instance, we have the following necessary but not sufficient condition as a conse-
57 quence of Noether's formula $K^2 + c_2 = 12$ and $c_2 = d$ (cf. Iversen[I6]):

$$a_1 + \cdots + a_d = 3(4 - d).$$

We have the following characterization as a consequence of Theorem 8.2.

Corollary 8.5. *A weighted circular graph with d vertices corresponds to a 2-dimensional complete non-singular torus embedding if and only if it is obtained from those with* $d = 3$ *or* $d = 4$ *below by a successive application of the following operation: insert one vertex with weight* -1 *and subtract 1 from the weights of the two adjacent vertices. For* $d \leq 6$, *we have the following possible cases.*



58

Remark. A 2-dimensional complete non-singular torus embedding can be considered as a compactification of $\underline{k}^2$, the affine plane, in many different ways. The graph obtained from the corresponding wighted circular graph by removing two adjacent vertices is the graph of the complement of $\underline{k}^2$. cf. Morrow [37].

9 Complete non-singular torus embeddings in dimension 3

In this section, we give a partial classification of 3-dimensional com- 59
plete non-singular torus embeddings. As by-products we get many non-projective complete non singular threefolds and birational morphisms which cannot be written as a succession of blowing ups along non-singular centers. These are of some independent interest even if the torus action is ignored.

Torus embeddings provides us with a good testing ground for the birational geometry of non-singular varieties, since many questions are reduced to the elementary geometry of r.p.p.decompositions. Consider, for instance, the following basic question. By abuse of language, a blowing up $Y \to X$ along a non-singular center of X is also called a *blowing down of Y along a non-singular center*.

Question: Let X and X' be birational non-singular algebraic varieties.

(strong version) Does there exist X'' which can be obtained both from X and from X' by a finite succession of blowing ups along non-singular centers?

(weak version) Can X' be obtained from X by a finite succession of blowing ups and blowing along non-singular centers?

60 The answer is affirmative dimension 2, but is not known even in dimension 3. According to Hironaka [18], we have the following weaker affirmative answer in characteristic 0: There exists X'' obtained from X by a finite succession of blowing ups along non-singular centers such that there exists a *morphism* from X'' to X'.

In the case of torus embeddings, the question is completely reduced to one on non-singular r.p.p.decompositions and looks much easier. Nevertheless, we were unable to prove the following conjecture even in dimension 3. We have already shown in Theorem 8.2 that the conjecture is true in dimension 2.

Conjecture

(strong version) Let X and X' be non-singular T-embeddings. Then there exists a T-embedding X'' obtained both from X and from X' by a finite succession of T-equivariant blowing ups along T-stable non-singular centers.

(weak version) Any complete non-singular r-dimensional torus embedding is obtained from $\mathbb{P}_r$ by a finite succession of equivariant blowing ups and blowing downs along non-singular centers.

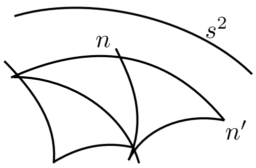
From now on, we fix $N \cong \mathbb{Z}^3$ and the ground field k.
Let $T = T_N$.

The classification of 3-dimensional complete non-singular T - embeddings X is reduced to that of f.r.p.p.decompositions Δ of $N_{\mathbb{R}} \cong \mathbb{R}^3$ such that $N_{\mathbb{R}} = \bigcup_{\sigma \in \Delta} \sigma$ and that the fundamental generators of each 3-dimensional $\sigma \in \Delta$ form a $\mathbb{Z}$-basis of N.

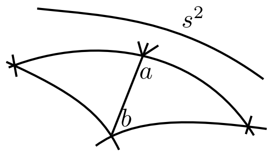
61 It is slightly more complicated to describe such Δ in a computable way than in the 2-dimensional case.

Definition. Let $S = S^2$ be a sphere in $N_{\mathbb{R}}$ centered at 0. Consider a triangulation of S by spherical triangles.

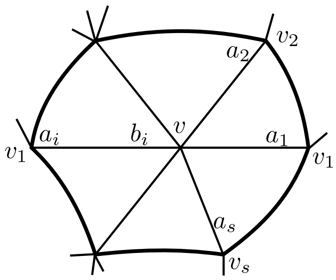
(i) By an $N-$*weighting* of the triangulation. we mean a primitive element of N attached to each spherical vertex.
(ii) By a *double* $\mathbb{Z}-$*weighting* of the triangulation, we mean a pair of integers attached to each spherical edge with one integer on the side of one vertex and with the other on the side of the other vertex. Let $v_1, \dots, v_s$ be the vertices adjacent to a vertex v of the triangulation, and let a_i be the $\mathbb{Z}$-weight on the edge vv_i which is on the side of v_i. The *weighted link* of v is the spherical polygon $v_1v_2 \cdots v_sv_1$ together with weights a_i at v_i.



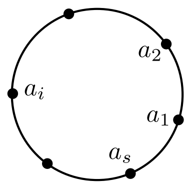
N-weighting



double $\mathbb{Z}$-weighting



weighted link of v



weighted circular graph

Let X = Temb (Δ) be a 3-dimensional complete non-singular torus **62**
embedding. If we intersect Δ with a sphere $S \subset N_R$ centered at 0, we get a triangulation of S

$$S = \bigcup_{\sigma\in\Delta}(\sigma \cap S).$$

We have a canonical N-weighting for this triangulation. Indeed, each spherical vertex is of the form

$$(\mathbb{R}_\circ n) \cap S$$

for a primitive $n \in N$, which we attach to the vertex. Obviously, Δ is determined completely by this N-weighted triangulation of S.

Definition. An N-weighted triangulation of S^2 is *admissible* if it is obtained from a complete non-singular $X = \text{Temb}\,(\Delta)$ in this way.

The admissibility means the following : For each spherical triangle, the N-weights $\{n, n', n''\}$ at the three vertices from a $\mathbb{Z}$-basis of N and the cone $\mathbb{R}_\circ n + \mathbb{R}_\circ n' + \mathbb{R}_\circ n''$ cuts out the original triangle on S.

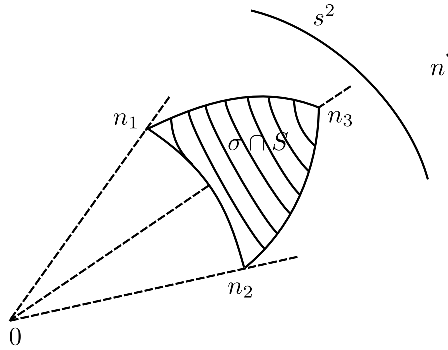
On the other hand, an admissible N-weighting for a triangulation of S gives rise to a double $\mathbb{Z}$-weighting. Indeed, each spherical edge is of the form

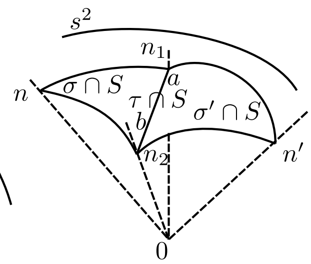
$$\tau \cap S$$

for a 2-dimensional cone $\tau \in \Delta$. Let $\{n_1, n_2\}$ be the fundamental generators of τ. By Proposition 6.7, τ is the common fact of exactly two
63
3-dimensional cones σ and σ'. Let $\{n, n_1, n_2\}$ and $\{n', n_1, n_2\}$ be the fundamental generators of σ and σ', respectively. There exist $a, b \in \mathbb{Z}$ such that

$$(*) n + n' + an_1 + bn_2 = 0.$$

We attach a pair (a, b) to the edge $\tau \cap S$, with a on the side of $(\mathbb{R}_o n_1) \cap S$ and b on the side of $(\mathbb{R}_o n_2) \cap S$.





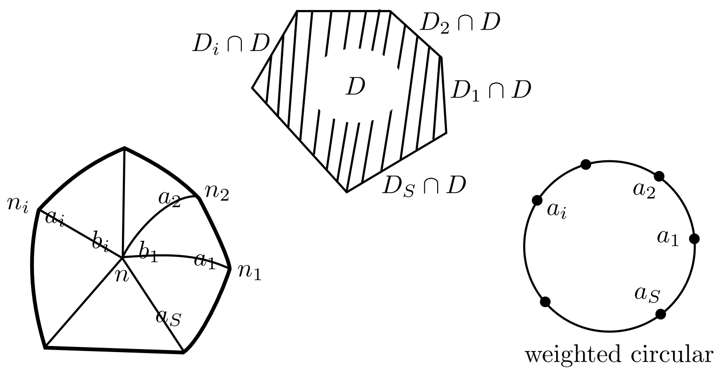
In this case, consider a vertex v with N-weight n and its link with the vertices $v_1, \ldots, v_s$ with N-weights $n_1, \ldots, n_s$ going around v in this order. Let $D = \overline{\mathrm{orb}(\mathbb{R}_o n)}$ and $D_i = \overline{\mathrm{orb}(\mathbb{R}_o n_i)}$ $1 \leq i \leq s$ be the corresponding T-stable irreducible divisors on $X = \mathrm{Temb}\,(\Delta)$. Again by Proposition 6.7, we have

$$(**) n_{i-1} + n_{i+1} + a_i n_i + b_i n = 0 \quad 1 \leq i \leq s,$$

where $n_0 = n_s$, $n_{s+1} = n_1$, and

$$a_i = D_i^2 \cdot D.$$

Since D is a 2-dimensional complete non-singular torus embedding (cf. **64**
Theorem 4.3) with stable curves $D_i \cap D$ $1 \leq i \leq s$, we see that the weighted link at v is exactly the weighted circular graph of D described immediately before Coro 8.5.



weighted link at v weighted circular graph of D

Definition. A doubly $\mathbb{Z}$-weighted triangulation of S^2 is called *admissible* if, around each vertex, the equations $(**) 1 \leq i \leq s$ in unknowns $n, n_1, \ldots, n_s$ are compatible and if, moreover, the weighted link of *each* vertex is a weighted circular graph obtained as in Corollary 8.5.

Definition. An isomorphism from an admissible N-weighted triangulation to another is an automorphism of N which induces an isomorphism from the corresponding f.r.p.p.decomposition to another. *A combinatorial isomorphism* from an admissible doubly $\mathbb{Z}$- weighted triangulation of S^2 to another is an incidence and $\mathbb{Z}$-weight preserving bijection from the set of spherical simplices of one triangulation to that of the other.

65 **Proposition 9.1.** *We have canonical bijections*

$$\left\{\begin{array}{l}\textit{isom. classes of}\\ 3-\dim \underline{1}\ \textit{com-}\\ \textit{plete n.s. torus}\\ \textit{embeddings/k}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\textit{isom. classes}\\ \textit{of admissible}\\ N\textit{-weighted}\\ \textit{triang}\underline{\textit{ns}}\textit{ of s}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{l}\textit{combinatorial}\\ \textit{isom. Classes of}\\ \textit{admissible doubly}\\ \mathbb{Z}\textit{-wei-ted triang}\underline{\textit{ns}}\\ \textit{of } S\end{array}\right\}$$

Proof. From what we have seen so far, it is enough to prove the surjectivity of the second map. Let $\{n, n', n''\}$ be a $\mathbb{Z}$ -basis of N. Pick an arbitrary triangle of an admissible doubly $\mathbb{Z}$-weighted triangulation of S, and give N-weights n, n', n'' to its three vertices in any way. Then using the equality $(*)$, we can successively determine the N-weights of the other vertices of the triangulation. This N-weighting is admissible. Indeed, for each vertex v with N-weight n, the vertices $v_1, \ldots, v_s$ with N-weights $n_1, \ldots, n_s$ of its link satisfy the equalities $(**)$. Since the double $\mathbb{Z}$-weighting is admissible, $n_1, \ldots, n_s$ go around n *once* in this order, by Proposition 6.7 applied to $N/\mathbb{Z}n$ and the images in it of $n_1, \ldots, n_s$. Thus the spherical triangles on S cut out by

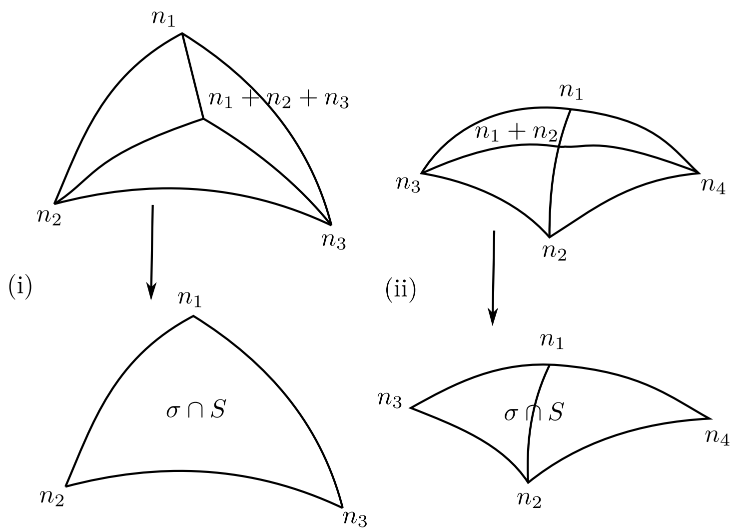
$$\mathbb{R}_\circ n + \mathbb{R}_\circ n_i + \mathbb{R}_\circ n_{i+1} \quad 1 \leq i \leq s$$

together fill up a neighborhood of $(\mathbb{R}_\circ n) \cap S$ combinatorially isomorphic to the star of v in the original triangulation which is the spherical polygon with vertices $v_1, \ldots, v_s$. Since S is simply connected, we are done. $\square$

66 **Remark.** The advantage of using admissible double $\mathbb{Z}$-weighting is the following: Once a combinatorial type of a triangulation of S^2 is given, possible admissible double $\mathbb{Z}$-weighting on it are computable in principle, although it is more convenient sometimes to use information's on the N-weights. To state the final result, however, it is less cumbersome to choose a convenient $\mathbb{Z}$-basis of N and describe one possible admissible N-weighting corresponding to it in terms of the $\mathbb{Z}$-basis.

Convention Given an admissible N-weighted triangulation of S and a vertex with the N-weight n, we call that vertex *the vertex n*, for simplicity, although n itself need not be on S.

By Corollary 7.6, the blowing up of $X = Temb(\Delta)$ along $\overline{orb(\sigma)}$ corresponds to the following subdivision of the N-weighted triangulation $\bigcup_{\tau\in\triangle}(\tau \cap S)$.



The following result, whose proof is immediate, will turn out to be **67**
useful in our classification below.

Corollary 9.2. *Given an admissible N-weighted triangulation of $S = S^2$, let n be a vertex and let $n_1, \ldots, n_2$ be the vertices of its link going around n in this order.*

(i) If $s \geq 4$, there exist i, j such that the vertices n_i, n, n_j are collinear on S, i.e are on a great circle.

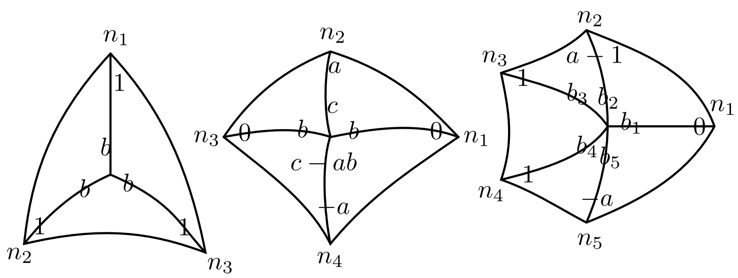
(ii) If the vertex n is *3-valent*, i.e. $s = 3$, there exists an integer b such that $n_1 + n_2 + n_3 + bn = 0$. The vertex n can be eliminated by a blowing down along a non-singular center if and only if $b = -1$.

The double $\mathbb{Z}$-weights around the vertex n are as in the picture below.

(iii) If the vertex n is *4-valent*, i.e $s = 4$, there exist integer a, b, c such that after re-numbering n_i's, we have $n_2+n_4+bn = 0$ (in particular, the vertices n_2, n, n_4 are collinear) and $n_1 + n_3 + an_2 + cn = 0$. The vertex n can be eliminated by a blowing down along a non-singular center if and only if $b = -1$. The double $\mathbb{Z}$-weights around the vertex n are as in the picture below.

(iv) If the vertex n is *5-valent*, i.e. $s = 5$, there exist integer $a, b_1, b_2, b_3, b_4, b_5$ with $b_1 = b_3 + b_4$ and $(-1)b_2 + (a-1)b_3 = (-a)b_4 + (-1)b_5$ such that, after re-numbering n_i's we have $n_1+n_3+(a-1)n_2+b_2n = 0$; $n_2 + n_4 - n_3 + b_3n = 0, n_3 + n_5 - n_4 + b_4n = 0, n_4 + n_1 - an_5 +$
68 $b_5n = 0$ and $n_2 + n_5 + b_1n = 0$ (in particular, the vertices n_2, n, n_5 are collinear) The double $\mathbb{Z}$-weights around the vertex n are as in picture below.

$$n_1 + n_2 + n_3$$



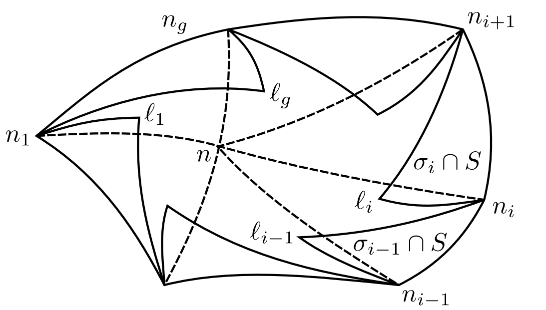
3-valent 4-valent 5-valent

To show that a given complete non-singular $X = Temb(\Delta)$ is projective, i.e. there exist integers a_i satisfying the condition (*iii*) of Corollary 6.5, we have to check many inequalities. The following sufficient condition for non projectively is very convenient in concrete applications.

Proposition 9.3. *Let* $S = \bigcup_{\sigma\epsilon\Delta}(\sigma \cap S)$ *be an admissible* N*-weighted triangulation of* $S = S^2$*. Then the corresponding complete non-singular* $Temb(\Delta)$ *is non-projective if there exists a spherical polygon* $n_1 n_2 \ldots n_g n_1$ *in the triangulation satisfying the following: For* $1 \leq i \leq g$*, let* $\sigma_i \cap S$ *be one of two triangles having* $n_i n_{i+1}$ *as edge* $(n_{g+1} = n_1)$*. Let* ℓ_i *be the remaining vertex of* $\sigma_i \cap S$*, i.e.* $\{\ell_i, n_i, n_{i+1}\}$ *are the fundamental generators of* σ_i*. There exists* $n \in S$ *not on the polygon* $n_1 n_2 \ldots n_g n_1$ *and real numbers* $\alpha_i, \beta_i, \gamma_i$ *with* $\alpha_i \geq 0$ *and* $\gamma_i > 0$ *such that* **69**

$$\ell_i + \alpha_{i+1} n_{i+1} = \beta_i n_i + \gamma_i \qquad 1 \leq i \leq g,$$

i.e. the edge $\ell_i n_{i+1}$ *intersects the great circle passing through* n_i *and* n *strictly inside the arc* $n_i n(-n_i)$*.*



Proof. If $Temb(\Delta)$ were projective, then Corollary 6.5, there exists a_i, $b_i \in \mathbb{Z}$ and $m_i = m(\sigma_i) \in M$ for $1 \leq i \leq g$ such that.

$$\langle m_i, \ell_i \rangle = -a_i, \langle m_i, n_i \rangle = -b_i, \langle m_i, n_{i+1} \rangle = -b_{i+1}$$

and

$$\langle m_{i-1}, \ell_i \rangle > -a_i, \langle m_{i-1}, n_{i+1} \rangle > -b_{i+1},$$

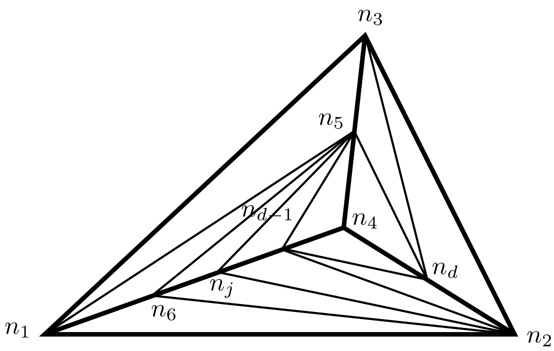
for $1 \leq i \leq g$, where $m_0 = m_g$. We see immediately that $0 < \langle m_{i-1}, \ell_i \rangle + a_i = -\alpha_{i+1}\{\langle m_{i-1}, n_{i+1} \rangle + b_{i+1}\} + \gamma_i\{\langle m_{i-1}, n \rangle - \langle m_i, n \rangle\}$, hence

$$\langle m_{i-1}, n \rangle > \langle m_i, n \rangle \qquad 1 \leq i \leq g,$$

a contradiction. □

Let $\{n, n', n''\}$ be a $\mathbb{Z}$-basis of N, and let $\ell = -(n + n' + n'')$. Let P be the surface of the tetrahedron in $N_{\mathbb{R}}$ with vertices n, n', n'', ℓ, which has 0 as the bary center. For $0 \neq y \in N_{\mathbb{R}}$, let us call $(\mathbb{R}_{\circ}y) \cap P$ the point y on P. For $d \geq 7$, consider the following subdivision P_d of P, where the bottom triangle $nn'n''$ is left as it is. **70**

$$\left.\begin{aligned}
&n_1 = n, n_2 = n',\ n_3 = n'' \\
&n_4 = \ell \\
&n_5 = n_3 + n_4 \\
&n_6 = n_1 + n_4 \\
&n_j = n_6 + (j-6)n_4\ (6 \leq j \leq d-1) \\
&n_d = n_2 + n_4
\end{aligned}\right\}$$



Proposition 9.4. *For $d \geq 7$, let Δ_d be the f.r.p.p.decomposition of N_R obtained by joining 0 with the simplices of the above subdivision p_d. Then we have the following:*

(i) $X_d = Temb(\Delta_d)$ is a non-projective complete non-singular 3 - dimensional tours embedding.

(ii) There exists a T-equivariant surjective morphism from X_d to $\mathbb{P}_3$.

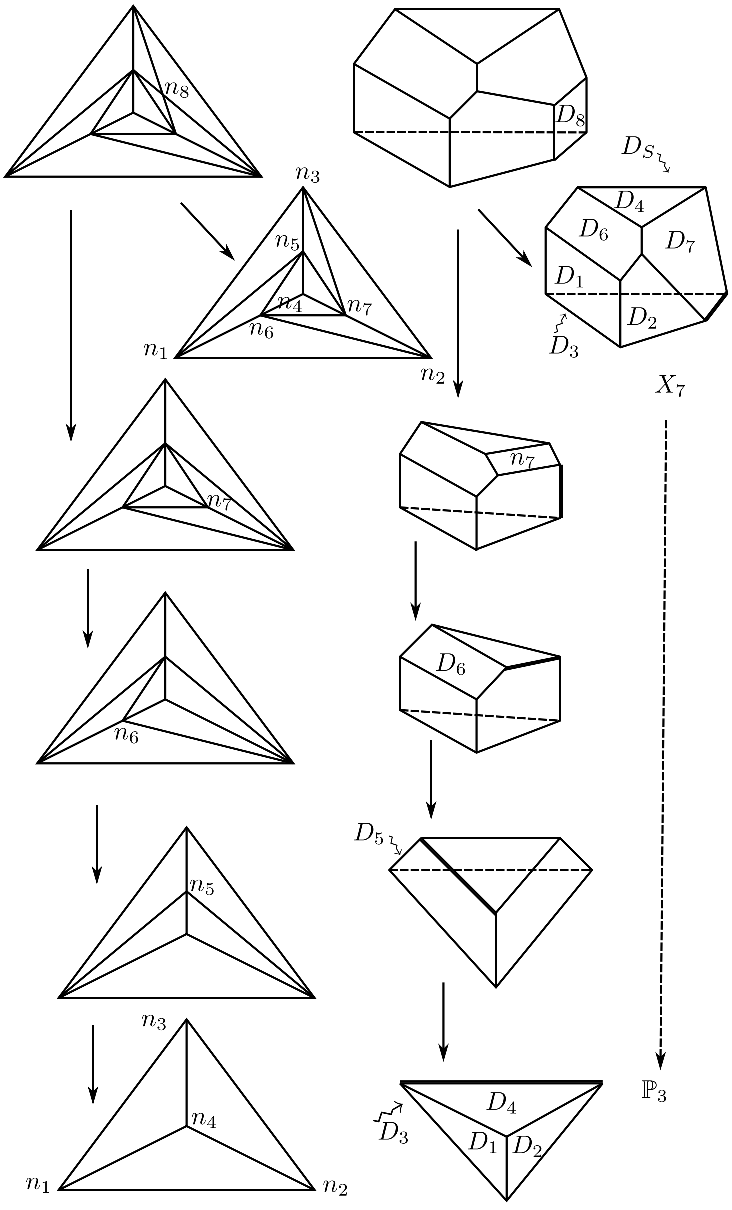
(iii) X_d cannot be (not necessarily equivariantly) blown down along any non-singular center.

(iv) *The tours embedding obtained by blowing up X_d along the curve $\overline{orb(\mathbb{R}_o n_3 + \mathbb{R}_o n_d)}$ is projective, and is obtained from $\mathbb{P}_3$ by a succession of blowing ups along T-stable curves.* 71

Proof. It is immediate to see that X_d is complete and non-singular. X_d is non-projective by proposition 9.3 applied to the polygon $nn'n''n$ of $\bigcup_{\sigma \epsilon \Delta_d} (\sigma \cap S)$, with $(\mathbb{R}_o \ell) \cap S$ to be taken as n in Proposition 9.3 (*ii*) is obvious, since P_d is a subdivision of P. (*iii*) If X_d were obtained by blowing up a non-singular variety along a non-singular center, then the exceptional divisors is automatically T-stable. But it is easy to check that X_d cannot be equivariantly blown along a non-singular center. (*iv*) is immediate. □

Remark. The case $d = 7$ is the simplest non-projective complete non-singular 3-dimensional torus embedding, which was already described in in Miyake-Oda [40]. See the next page for the picture of T-orbits under the process described in (*iv*) Demazure [8, Appendice] previously gave an example of non projective complete non-singular 3-dimensional torus embeddings which has 32 T-stable irreducible divisors instead of 7 in our case. We latter get many other non-projective examples as the by-products of our partial classification below.

72

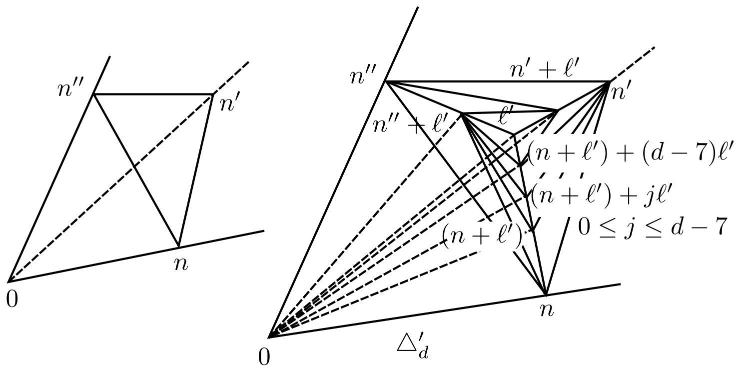


n_8
D_8
D_S
n_3
D_4
n_5
D_6
D_7
D_1
n_4
n_7
n_6
n_1
D_2
D_3
n_2
X_7
n_7
n_7
D_6
n_6
D_5
n_5
n_3
D_4
n_4
D_1
D_2
D_3
$\mathbb{P}_3$
n_1
n_2

By analogy, consider instead the 3-dimensional cone 73

$$\sigma = \mathbb{R}_o n + \mathbb{R}'_n + \mathbb{R}_o n''$$

and let $\ell' = n + n' + n''$. For $d \geq 7$, let Δ'_d be the subdivision of σ which looks like this:



Then the following example gives as answer to a question raised by Fujiki.

Corollary 9.5. *The equivariant proper morphism*

$$f : Y_d = Temb(\Delta'_d) \to U(\sigma) = k^3$$

is non-projective and cannot be written as a succession of blowing ups along non-singular centers. But the blowing up Y'_d of Y_d along

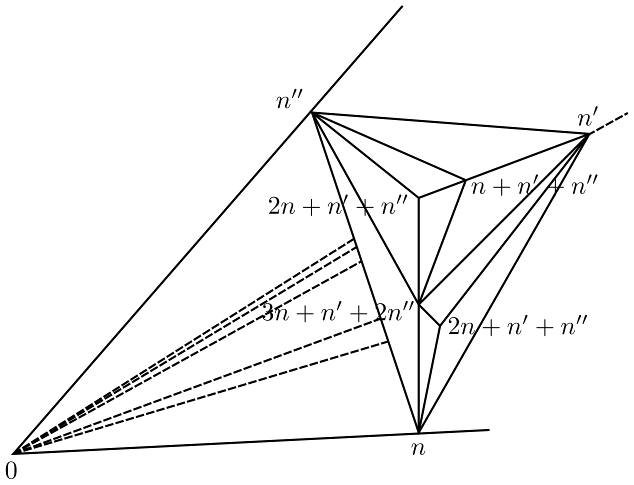
$$\overline{orb(\mathbb{R}_o n'' + \mathbb{R}_o(n' + \ell'))}$$

is obtained first by blowing up $U(\sigma)$ along the T-fixed point and then by successively blowing up along T-stable curves.

Proof. All except the non-projectivity of f are immediate. Consider the tetrahedron with vertices n, n', n'' and $\ell = -(n + n' + n'')$, and its subdivision which Δ'_d induces on the face $nn'n''$. By Proposition 9.3, we have a surjective morphism from a non-projective complete non-singular torus embedding to $\mathbb{P}_3$. Since f is obviously its restriction to 74
$U(\sigma) \subset \mathbb{P}_3$, we are done. □

It is possible to describe projective equivariant birational morphism directly in terms of the corresponding subdivision. We refer the reader to Mumford et al. [63, Chap. III, p. 152-154 in particular]. See also Namikawa [47].

Remark. On the other hand, we can show in a similar fashion that the following subdivision Δ''_d of $\sigma = \mathbb{R}_o n + \mathbb{R}_o n' + \mathbb{R}_o n''$ gives rise to a *projective* morphism $Temb(\Delta''_d) \to U(\sigma) \cong k^3$ which cannot be written as a finite succession of blowing ups along non-singular centers, but can be if $Temb(\Delta''_d)$ is blown up once along $\overline{\mathrm{orb}(\mathbb{R}_o n' + \mathbb{R}_o(3n + n' + 2n''))}$.



75 Let d be the number of spherical vertices of a triangulation of S^2. Then obviously we have

$$\begin{aligned} 3d - 6 &= \text{ the number of spherical edges} \\ 2d - 4 &= \text{ the number of spherical triangles.} \qquad (*) \end{aligned}$$

If the triangulation is of form $S = \bigcup_{\sigma\epsilon\Delta}(\sigma \cap S)$ with $X = Temb(\Delta)$ complete non-singular, then d = the cardinality of $Sk^1(\Delta)$, $3d - 6$ = the number of 2-dimensional cones in Δ, $2d - 4$ = the number of 3-dimensional cones in Δ. By Proposition 6.1, we have

$$d - 3 = \text{ the Picard number of } X.$$

Furthermore for $v \geq 3$, let p_v be the number of v-valent spherical vertices of a triangulation of S^2. Then we see that

$$\sum_{v \geq 3} (6 - v) p_v = 12. \tag{**}$$

By Proposition 9.1, the classification of complete non-singular 3-dimensional torus embedding is reduced to

(i) the classification of combinatorially different triangulations of S^2 and

(ii) the different admissible N-weightings (or admissible double $\mathbb{Z}$ – *weightings*) on them.

According to Grünbaum [13, Chap.13] [14], we have the following relevant facts: By Steinitz's theorem (which is known to be false in higher dimension) the combinatorial equivalence classes of the triangulation of S^2 are in one to one correspondence with the combinatorial types of convex simplicial polyhedral in $\mathbb{R}^3$. Given d, there seems, however, to be no formula for the number of the latter. It is empirically **76**
known for smaller values of d as follows:

d	4	5	6	7	8	9	10	11	12
	1	1	2	5	14	50	233	1249	7595

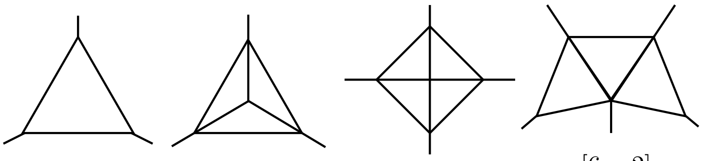
In this connection, Eberhard's theorem [13, 13.3] looks intriguingly relevant to us, in view of our description of blowing ups along non-singular centers immediately before Corollary 9.2.

Anyway for $d \leq 8$, we have the list in the next page of the combinatorially different triangulation of S^2, where we stereographically project them onto the plane from one of the spherical vertices, and we express circular arcs on the plane simply be liner arcs.

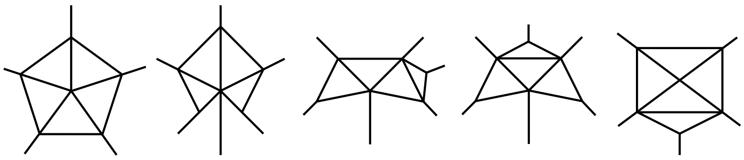
Among 7595 different triangulation for $d = 12$, we have the one induced by an icosahedron.

For the classification of complete non-singular 3-dimensional torus embeddings, it remains to find possible admissible N-weightings on them. Obviously, it is enough to find those *without any vertices which can be eliminated by blowing downs along non-singular centers*.

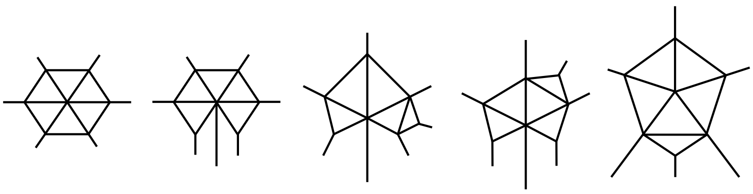
Combinatorial types of the triangulation of S^2 for $d \leq 8$ **77**



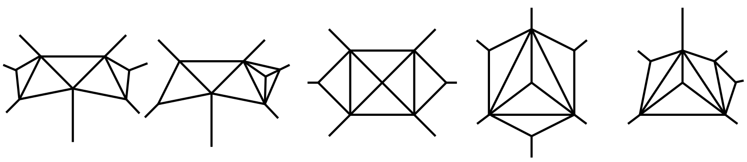
$[4-1]$ $[5-1]$ $[6-1]$ $[6-2]$



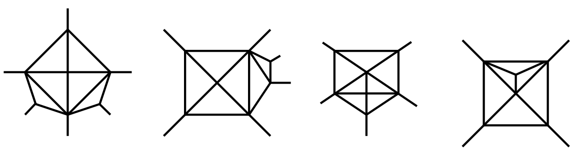
$[7-1]$ $[7-2]$ $[7-3]$ $[7-4]$ $[7-5]$



$[8-1]$ $[8-2]$ $[8-3]$ $[8-4]$ $[8-5]$

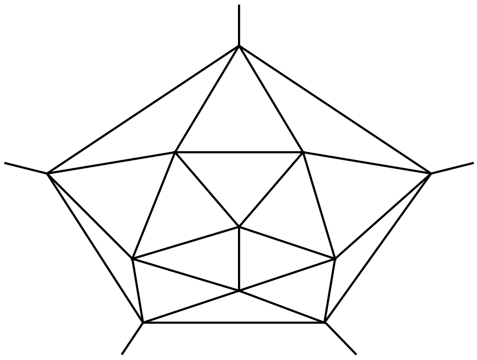


$[8-6]$ $[8-7]$ $[8-8]$ $[8-9]$ $[8-10]$



$[8-11]$ $[8-12]$ $[8-13]$ $[8-14]$

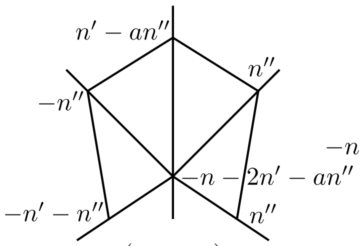
Icosahedron.



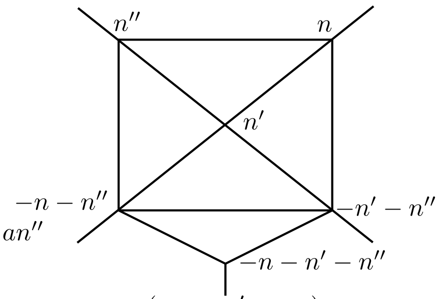
Theorem 9.6. *Let X be a 3-dimensional complete non-singular torus embedding which cannot be blown down along any no-singular center. Then X is isomorphic to one among the projective space $\mathbb{P}_3$* 78

$$\mathbb{P}_2 - \textit{bundles over } \mathbb{P}_1 \qquad \mathbb{P}(0_{\mathbb{P}_1} \oplus 0_{\mathbb{P}_1}(a) \oplus 0_{\mathbb{P}_1}(b)) \quad a, b \in \mathbb{Z}$$

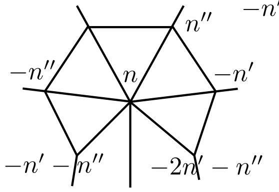
$\mathbb{P}_1$-bundles $\mathbb{P}(0_Y \oplus L)$ over complete non-singular 2-dimensional torus embeddings Y and $L \in Pic(Y)$ and, if the Picard number $d - 3 \leq 5$, $Temb(\triangle)$ corresponding to the following 13 different sequences with integral parameters of admissible N-weighted triangulations of S^2, where $\{n, n', n''\}$ is a $\mathbb{Z}$-basis of $N = \mathbb{Z}^3$.



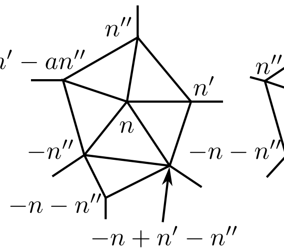
$(n \text{ at } \infty)$
$[7-2]$



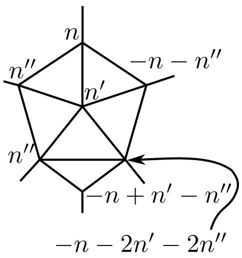
$(-n - n' \text{ at } \infty)$
$[7-5]$



$(-n-3n'-2n' \text{ at } \infty)$
$[8-2]$

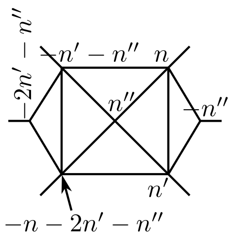


$(-n-n' \text{ at } \infty)$
$[8-5']$

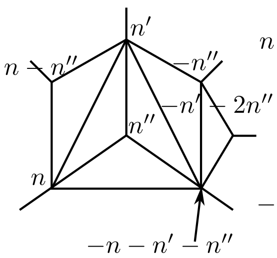


$(-n-n' \text{ at } \infty)$
$[8-5'']$

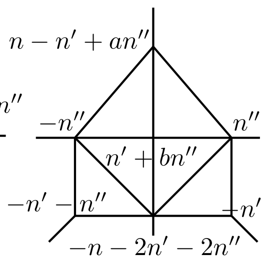
79



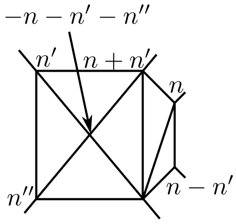
$(-n-n'-2n'' \text{ at } \infty)$
$[8-8]$



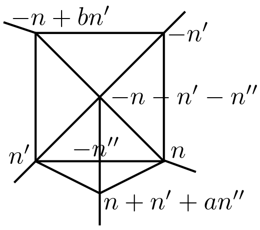
$(-n-n'-2n'' \text{ at } \infty)$
$[8-10]$



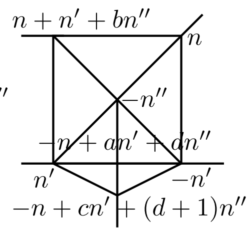
$(n \text{ at } \infty)$
$[8-11]$



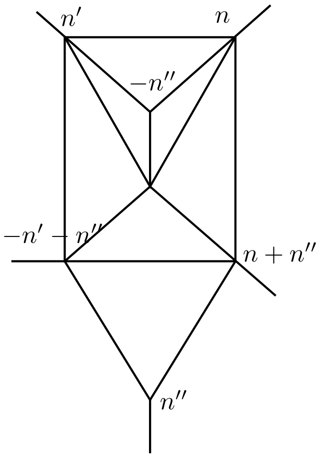
$(n-n'+n'' \text{ at } \infty)$
$[8-12]$



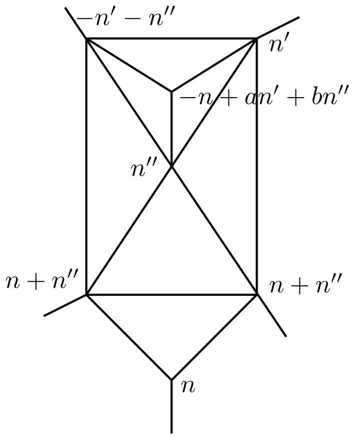
$(n'' \text{ at } \infty)$
$[8-13']$



$(n'' \text{ at } \infty)$
$[8-13'']$



$(n' + n''$ at $\infty)$
$[8 - 14']$



$(n - n' - n''$ at $\infty)$
$[8 - 14'']$

Remark. This theorem was already announced in Miyake-Oda [40] except that [8-12] was not counted there. This fact, as well as the considerable simplification of the proof below, was pointed out by Nagaya [43]. **80**

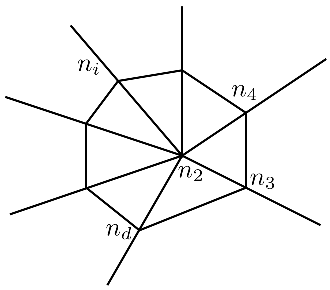
Remark. Some of the torus embeddings listed in Theorem 9.6 can be blown down along a non-singular center if the integral parameters take special values. $\mathbb{P}_3$, $\mathbb{P}_2$-bundles, $\mathbb{P}_1$-bundles, [7-2], [8-2] and [8-10] are all projective. On the other hand, [7-5] and $[8-5'']$ are X_7 and X_8 of Proposition 9.4, respectively, hence are non-projective. We can show that $[8-5'](a \neq 0)$, [8-8], [8-12], $[8-14']$ and $[8-14'']$ are non-projective, using Proposition 9.3. $[8-13']$ and $[8-13'']$ are not projective, except when they can be blown down for special values of the integral parameters. For these torus embeddings we can easily check the weak version of the conjecture at the beginning of this section.

For simplicity, we adopt the following definition.

Definition. An admissible N-weighted triangulation of S^2 is called *weakly minimal* if the corresponding complete non-singular center, i.e. the N-weighted triangulation has no 3-valent or 4-valent vertex which can be eliminated by blowing down along any non-singular center.

81 **Proof of Theorem 9.6.** If $d = 4$, then $X \cong \mathbb{P}_3$ as a special case of Theorem 7.1. in the remainder of the proof, we repeatedly use Corollary 9.2, without explicit reference, to determine admissible N-weightings or the corresponding double $\mathbb{Z}$-weightings for a given triangulation of S^2 with $d \geq 5$. The argument depends very much on the distribution of the valency of the vertices.

Lemma 9.7. *For $d \geq 5$, let $[d-1]$ be the triangulation of S^2 which looks like the picture. A weakly minimal admissible N-weighting on it is necessarily of the following form up to isomorphism and corresponds to a projective torus embedding:*



n_1 at ∞
$[d-1]$

(1) $d = 5$, and $\{n_3, n_4, n_5\}$ are collinear. In this case, the corresponding torus embedding is a $\mathbb{P}_2$-bundle over $\mathbb{P}_1$.

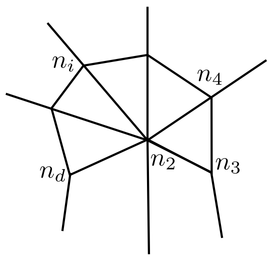
(2) $d \geq 5$, $n_1 + n_2 = 0$ and $\{n_1, n_i, n_2\}$ are collinear for all $3 \leq i \leq d$. In this case, the corresponding torus embedding is a $\mathbb{P}_1$-bundle over a 2-dimensional complete non-singular torus embedding determined by the images of $n_i, 3 \leq i \leq d$, in $N/\mathbb{Z}n_1$.

Proof. If $d = 5$, look at the 4-valent vertices n_3, n_4, n_5. Then by Corollary 9.2, either $\{n_3, n_4, n_5\}$ are collinear hence on a great circle, or $\{n_1, n_i, n_2\}$ are collinear for all $3 \leqslant i \leq 5$. Thus we easily get (1) or (2) $d = 5$ by 7.6′. □

82 If $d = 6$, the triangulation is symmetric with respect to the pairs

$\{n_1, n_2\}, \{n_3, n_5\}$, and $\{n_4, n_6\}$. Again by Corollary 9.2 applied to the 4-valent vertices, we may assume that n_1 and n_2 are antipodal, i.e. $n_1 + n_2 = 0$. Hence we get (2) $d = 6$, by 7.6′
When $d \geq 7$, look at the 4-valent vertices $n_3, \ldots, n_d$. If $\{n_1, n_i, n_2\}$ were collinear for at most one $3 \leq i \leq d$, then $\{n_3, n_4, \ldots, n_d\}$ would be on a great circle by Corollary 9.2. Since the valency of n_2 is $d - 2 \geq 5$, the weighted link of n_2 would have weight -1 at some $n_j, 3 \leqslant j \leqslant d$, which could be eliminated by blowing down. If $\{n_1, n_i, n_2\}$ were collinear for exactly two $i's$, $i = 3$ and j say, then $\{n_1, n_3, n_2, n_j\}$ would be on a great circle, and $\{n_3, n_4, \ldots, n_j\}$ and $\{n_j, n_{j+1}, \ldots, n_d\}$ would be collinear. Since n_3 and n_j would then be antipodal, and since $\{n_3, n_2, n_j\}$ would be collinear, the weighted link of n_2 would have weight -1 at some $n_i, i \neq 3, j$, which could again be eliminated. Thus we conclude that $\{n_1, n_i, n_2\}$ are collinear for more than two, hence for all $3 \leq i \leq d$, since n_1 and n_2 are necessarily antipodal. In view of 7.6′ again, we are done.

Lemma 9.8. *For $d \geq 6$, let $[d-2]$ be the triangulation of S^2 which looks like the picture below. A weakly minimal admissible N-weighting on it satisfies the following conditions: $\{n_3, n_4, \ldots, n_d\}$ are collinear,* **83**
$n_1, n_2 = n_3 + n_d$. There exists $4 \leq j \leq d-3$, such that $\{n_j, n_1, n_{j+2}\}$ and $\{n_j, n_2, n_{j+2}\}$ are collinear.



n_1 at ∞)
$[d-2]$

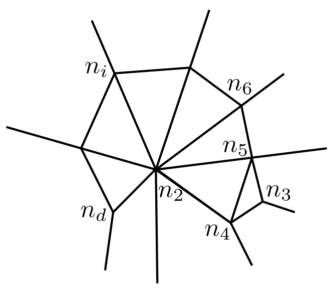
The corresponding torus embeddings are always projective.
In particular, we have $d \geq 7$, and the only possible weakly minimal admissible N-weightings for $d = 7, 8$ are [7-2] and [8-2] in Theorem 9.6 up to isomorphism.

Proof. If $d = 6$, look at the weighted link of the 5-valent vertex n_2. Its weight at n_3 and n_6 cannot be -1, since otherwise n_3 or n_6 can be eliminated. Thus $\{n_3, n_2, n_6\}, \{n_1, n_4, n_2\}$ and $\{n_1, n_5, n_2\}$ are collinear again by Corollary 9.2. This means that n_1 and n_2 are antipodal, a contradiction to the strong convexity of cones.

Let $d \geq 7$. Since n_1 and n_2 cannot be antipodal, there can be at most one $4 \leq i \leq d-1$ such that $\{n_1, n_i, n_2\}$ are collinear, hence on a great circle. If there were one such i, the weighted link of n_2 would have weight -1 at some $n_j, 3 \leq j \leq d, j \neq i$, but $\{n_3, n_4, \ldots, n_i\}$ and $\{n_i, n_{i+1}, \ldots, n_d\}$ would be collinear, a contradiction. Thus we conclude that $\{n_3, n_4, \ldots, n_d\}$ are collinear. Now look at the weighted link of
84 n_1 ($resp. n_2$). Then weight -1 is possible only at n_2 ($resp. n_1$). Thus $n_1 + n_2 = n_3 + n_d$, and together with $n_3, \ldots, n_d$ it lies on a great circle. Moreover, $\{n_j, n_1, n_{j+2}\}$ and $\{n_j, n_2, n_{j+2}\}$ are collinear for some $4 \leq j \leq d-1$. If $d = 7$ or 8, there is only one possible such weighted link. The projectivity of the corresponding torus embeddings follows easily from Corollary 9.5 in view of the projectivity of 2-dimensional torus embeddings (Proposition 8.1) and the fact that $n_1 + n_2 = n_3 + n_d, n_3 \ldots, n_d$ are on a great circle. □

Lemma 9.9 (Nagaya)**.** *For $d \geq 7$, the triangulation $[d-3]$ below has no weakly minimal admissible N-weightings.*

Proof. Suppose there were a weakly minimal admissible N-weighting. Since n_1 and n_2 cannot be antipodal, there is at most one $5 \leq j \leq d-1$ such that $\{n_1, n_j, n_2\}$ are collinear, hence on a great circle.



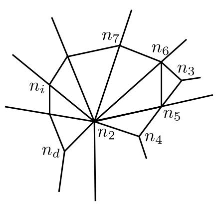
n_1 at ∞)
$[d-3]$

If there were one such j, look at the 4-valent vertex n_4. $\{n_1, n_4, n_5\}$ could not be collinear, since otherwise they would be on a great circle disjoint from the great circle above. Hence $\{n_2, n_4, n_3\}$ would be collinear. Then look at the weighted link of n_5. The weight -1 could not be at n_3 nor n_4. If $\{n_1, n_5, n_4\}$ were collinear, then they would be on a great circle disjoint from another great circle, a contradiction. Thus **85**
$\{n_2, n_5, n_3\}$ would be collinear, hence n_2, n_3 are antipodal, a contradiction again. We thus conclude that $\{n_1, n_i, n_2\}$ are not collinear for all $5 \le i \le d-1$, hence $\{n_5, n_6, \dots, n_d\}$ are collinear. Look at the vertex n_4 again. If $\{n_1, n_4, n_5\}$ were collinear, the $\mathbb{Z}$- weight of the edge n_2n_4 at n_4 would coincide with that of n_3n_4 at n_4, which is 1. Then look at the weighted link of n_2. There would exist $6 \le j \le d$ such that $\{n_4, n_2, n_j\}$ are collinear. Thus it has weight -1 at some n_k, $6 \le k \le d$, $k \neq j$, 4, 1, 5, a contradiction.

Thus $\{n_2, n_4, n_3\}$ are collinear. Look at the weighted link of n_5. Since it cannot have weight -1 at n_3 and n_4, either (i)$\{n_1, n_5, n_4\}$ are collinear, i.e. on a great circle, hence $\{n_4, n_1, n_5\}$ are collinear, or (ii) $\{n_2, n_5, n_3\}$ are collinear, hence n_2, n_3 are antipodal, and $\{n_2, n_1, n_3\}$ are collinear. In both cases, the weighted link of n_1 would have weight -1 at some n_k, $6 \le k \le d$, a contradiction, since $\{n_5, n_6, \dots, n_d\}$ are collinear. □

Lemma 9.10 (Nagaya). *For $d \ge 7$, the triangulation $[d-4]$ below has no weakly minimal admissible N-weighting.*

Proof. Suppose the contrary. Since the weighted link of n_5 cannot have weight -1 at n_3 and n_4, $\{n_3, n_5, n_4\}$ are collinear. Look at the weighted **86**
link of n_6. Since its weight at n_3 cannot be -1, we have three possibilities:



n_1 at ∞)
$[d-4]$

(i) $\{n_1, n_6, n_5\}$ are collinear, hence on a great circle. Since $\{n_1, n_5, n_6\}$ are then collinear, the weighted link of n_5 has weight -1 at n_4, a contradiction.

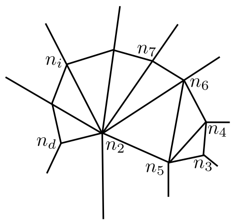
(ii) $\{n_3, n_6, n_7\}$ are collinear. In this case, the weighted link of n_5 and the $\mathbb{Z}$-weights around n_5 can be determined completely by means of the compatibility in Corollary 9.2. We see easily that the weighted link of n_1 has positive weights at n_4 and n_5, a contradiction.

(iii) $\{n_2, n_6, n_3\}$ are collinear. Again using the compatibility of the $\mathbb{Z}$-weights around n_5, we easily see that $\{n_4, n_2, n_6\}$ are collinear, hence n_3 and n_4 are antipodal. Thus $\{n_3, n_1, n_4\}$ are also collinear. Since n_2, n_2 cannot be antipodal, $\{n_1, n_i, n_2\}$ can be collinear for at most one $5 \leqslant i \leqslant d-1$. Suppose $\{n_l, n_j, n_2\}$ were collinear, hence on a great circle. If $j = 5$, then n_3 could be eliminated. If $6 \leqslant j$, then a pair n_3, n_4 of antipodal points would lie strictly on one side of the great circle, a contradiction. Thus $\{n_1, n_i, n_2\}$ are not collinear for any $5 \leq i \leq d-1$, hence $\{n_6, n_7, \dots, n_d\}$ are
87 collinear. Look at the weighted link of n_2. Since $\{n_4, n_2, n_6\}$ are assumed to be collinear, there exists $7 \leq k \leq d$ such that its weight at n_k is -1, a contradiction.

□

Lemma 9.11 (Nagaya)**.** *For $d \geq 8$, the triangulation $[d-6]$ below has no weakly minimal admissible N-weighting.*

Proof. Suppose the contrary. Since n_1, n_2 cannot be antipodal, $\{n_1, n_i, n_2\}$ can be collinear for at most one $6 \leq i \leq d-1$.



n_1 at ∞)
$[d-6]$

(1) Suppose for one $6 \leq j \leq d-1$, $\{n_1, n_j, n_2\}$ were collinear, hence on a great circle. Then $\{n_j, n_{j+1}, \ldots, n_d\}$ are collinear. Since $\{n_1, n_2, n_j\}$ are collinear, the consideration of the weighted link of n_2 shows that $j = d-1$. Look at the weighted link of n_4. Since $\{n_1, n_4, n_5\}$ cannot be on a great circle, we see that $\{n_3, n_4, n_6\}$ are collinear. Look then at the weighted link of n_5. From what we have seen so far, we can conclude that $\{n_3, n_5, n_6\}$ are collinear, hence n_3, n_6 are antipodal and are strictly on one side of the great circle passing through n_1, n_2, n_{d-1}, a contradiction.

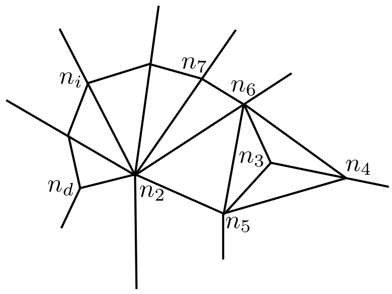
(2) We thus conclude that $\{n_1, n_i, n_2\}$ are not collinear for all $6 \leq i \leq d-1$, hence $\{n_6, n_7, \ldots, n_d\}$ are collinear.

Look at the weighted link of n_4. □ **88**

(2-i) If $\{n_1, n_4, n_5\}$ are not collinear, then $\{n_3, n_4, n_6\}$ are collinear. The weighted link of n_5 forces $\{n_3, n_5, n_6\}$ to be collinear, hence n_3, n_6 are antipodal and $\{n_3, n_1, n_6\}$ are collinear. Look at the weighted link of n_1 and n_2 simultaneously. Their weights at $n_i, 7 \leq i \leq d$, coincide. $(2-i-a)$ Suppose $\{n_j, n_2, n_k\}, 6 \leq j < k \leq d$, were collinear. Since n_3, n_4 are antipodal, $n_2 n_j n_{j+1} \ldots n_k n_2$ cannot be a great circle, hence n_j, n_k are antipodal. The sequence of weights of the weighted link of n_2 at $n_{j+1}, \ldots, n_{k-1}$ is a part of that of the weighted link of n_1 at n_7, $n_8, \ldots, n_d, n_1, n_5$. We easily see that this is a contradiction. $(2-i-b)$ If $\{n_1, n_2, n_j\}, 6 \leq j \leq d$, are collinear, then $n_1 n_j n_2 n_1$ is a great circle, which determines a hemisphere containing a pair n_3, n_6 of antipodal points, a contradiction. (2-i-c) Suppose $\{n_5, n_2, n_j\}, 6 \leq j \leq d$, are collinear. Then the consideration of the weights at $n_{j+1}, \ldots, n_d, n_1$ of the weighted link of n_2 show that $j = d$. Look then at the weighs at $n_6, n_7, \ldots, n_{d-1}$ of the weighted link of n_2. There would be -1 somewhere, a contradiction.
$(2-ii)$ Suppose $\{n_1, n_4, n_5\}$ are collinear, hence on a great circle and $\{n_4, n_1, n_5\}$ are collinear. Look at the weighted link of n_1 and n_2. As in $(2-i)$, we arrive at a contradiction.

Lemma 9.12 (Nagaya)**.** *For $d \geq 8$, the triangulation $[d-7]$ has no* **89**
weakly minimal admissible N-weighting.

Proof. Suppose the contrary. Again $\{n_1, n_i, n_2\}$ can be collinear for at most one $6 \leq i \leq d-1$.



n_1 at ∞
$[d-7]$

(1) Suppose $\{n_1, n_j, n_2\}$, $6 \leq j \leq d-1$, are collinear, hence on a great circle. Hence $\{n_1, n_2, n_j\}$, $\{n_6, n_7, \ldots, n_j\}$ and $\{n_j, n_{j+1}, \ldots, n_d\}$ are collinear. Looking at the weighted link of n_2, we again see that $j = d-1$, $\{n_6, \ldots, n_{d-1}\}$ are collinear and $n_1 n_2 n_{d-1} n_1$ is a great circle. Look then at the weighted link of n_4. Obviously, $\{n_5, n_4, n_6\}$ are not collinear, hence $\{n_1, n_4, n_3\}$ are collinear and $\{n_4, n_5, n_6\}$ are not. At 5-valent n_5, $\{n_1, n_5, n_3\}$ are necessarily collinear, but we have a contradiction, since then n_1, n_3 are antipodal and $n_1 n_2 n_{d-1} n_1$ is a great circle.

(2) We conclude that $\{n_1, n_i, n_2\}$ are not collinear for all $6 \leq i \leq d-1$, hence $\{n_6, n_7, \cdots, n_d\}$ are collinear. Look at the weighted link of n_4.

$(2-i)$ If $\{n_5, n_4, n_6\}$ are collinear, hence on a great circle, the compatibility of the $\mathbb{Z}$-weights around 5-valent n_5 shows that the weighted link of n_2 has a positive weight at n_5.

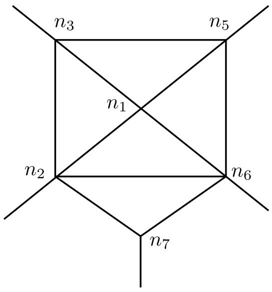
90 $(2-ii)$ If $\{n_5, n_4, n_6\}$ are not collinear, then $\{n_1, n_4, n_3\}$ are collinear. At 5-valent n_5, we see that $\{n_1, n_5, n_3\}$ are collinear, hence n_1, n_3 are antipodal and $\{n_1, n_6, n_3\}$ are collinear, Again by the compatibility of the $\mathbb{Z}$ weights around n_5, we see that the weighted link of n_2 has a positive weight at n_5.

In both cases $(2-i)$ and $(2-ii)$, look at the weighted link of n_2. Since it has a positive weight ar n_5, $\{n_5, n_2, n_k\}$ are collinear for some

$7 \leq k \leq d$, hence has weight -1 at some n_1, $6 \leq i \leq d$, $i \neq k$, a contradiction. $\square$

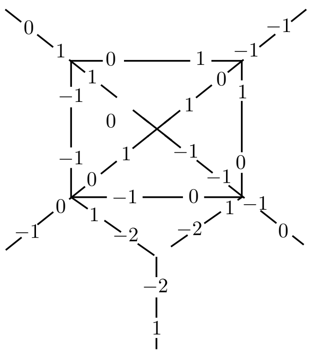
Lemma 9.13. *A weakly minimal admissible N-weighting for the triangulation [7-5] is necessarily of the form in Theorem 9.6 up to isomorphism.*

Proof. By symmetry, we may assume that $\{n_3, n_1, n_6\}$ are collinear. Look at the weighted link of n_2.



n_4 at ∞)
[7 − 5]

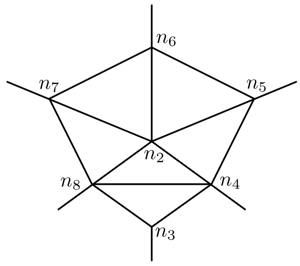
Since the weights at n_1 and n_7 cannot be -1, $\{n_1, n_2, n_7\}$ are necessarily collinear. Since its weight at n_3 is then -1, $\{n_2, n_3, n_5\}$ are collinear. Thus $\{n_3, n_4, n_7\}$ and $\{n_1, n_5, n_4\}$ are collinear for the same reason. Looking then at the weighted link of n_6 , we see that $\{n_5, n_6, n_7\}$ are collinear. From what we have seen so far, an admissible double $\mathbb{Z}$ weighting is uniquely determined as in the picture, by Cor.9.2.



91 $n = n_5$, $n' = n_1$, $n'' = n_3$ form a $\mathbb{Z}$-basis of N. The double $\mathbb{Z}$ weighting successively determines the other N-weights as follows: $n_2 = -nn''$, $n_4 = -n - n'$, $n_6 = -n' - n''$ and $n_7 = -n - n''$. □

Lemma 9.14. *A weakly minimal admissible N-weighting on the triangulation [8-5] is either* $[8 - 5']$ *or* $[8 - 5'']$ *of Theorem 9.6 up to isomorphism.*

Proof. Neither $\{n_1, n_4, n_2\}$ nor $\{n_1, n_8, n_2\}$ are collinear, since otherwise the $\mathbb{Z}$ weights around n_3 would be -1.



n_1 at ∞)
$[8 - 5]$

In particular, n_1, n_2 are not antipodal. Thus $\{n_1, n_i, n_2\}$ are collinear for at most one $5 \le i \le 7$. If there were no such i, then $n_4n_5n_6n_7n_8n_4$ would necessarily be a great circle and the weighted link of n_4 would have weight -1 at n_3, a contradiction. By symmetry, it thus suffices to consider the following two cases:

1. $\{n_1, n_5, n_2\}$ are collinear, hence $\{n_5, n_6, n_7, n_8\}$ are collinear. Look at the weighted link of n_4. We easily see that $\{n_3, n_4, n_5\}$ are
92 collinear. Look then at the weighted link of n_2. If $\{n_5, n_2, n_7\}$ were collinear, then $n_5n_2n_7n_1n_5$ would be a great circle, thus $\{n_2, n_7, n_1\}$ would be collinear, a contradiction. Thus we conclude that $\{n_6, n_2, n_8\}$ are collinear. In particular, n_6, n_8 are antipodal, hence $\{n_6, n_1, n_8\}$ are also collinear. Then the weighted link of n_1 has necessarily weights $-2, -1, -2$ at n_3, n_4, n_5, respectively. Looking at the $\mathbb{Z}$ weights around n_4 determined so far, we see that the weighted

link of n_8 has weight 0 at n_4, hence $\{n_2, n_8, n_3\}$ are collinear. The collinearity for each vertex thus determined gives rise to a unique admissible double $\mathbb{Z}$ weighting with one integral parameter, which determines the admissible N-weighting $[8-5']$ in Theorem 9.6, if we let $n = n_2, n' = n_5, n'' = n_6$.

2. $\{n_1, n_6, n_2\}$, hence $\{n_4, n_5, n_6\}$ and $\{n_6, n_7, n_8\}$ are collinear. Look at the weighted link of n_2. We see that $\{n_5, n_2, n_7\}$ are collinear. Look then at the weighted link of n_4 and n_8. By symmetry and because of the fact that their weights at n_3 cannot be -1, we have four possibilities:

□

(i) $\{n_1, n_4, n_8\}$ are collinear, hence are on a great circle. Look at the weighted link of n_1. Its weights at n_5, n_6, n_7 are necessarily. -2, $-1, -2$. By the compatibility of the $\mathbb{Z}$ weights around the 4-valent vertex n_5, we have a contradiction.

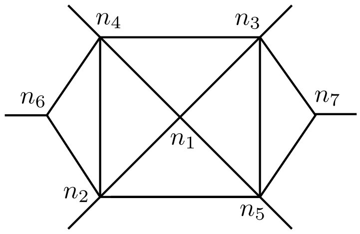
(ii) $\{n_2, n_4, n_3\}$ and $\{n_2, n_8, n_3\}$ are collinear, hence n_2, n_3 are antipodal.
We see then that $\{n_3, n_1, n_6\}$ are collinear. Then the weighted link **93**
of n_1 leads us to a contradiction.

(iii) $\{n_3, n_4, n_5\}$ and $\{n_3, n_8, n_7\}$ are collinear, hence n_3, n_6 are antipodal. Again we see that $\{n_3, n_1, n_6\}$ are collinear, a contradiction.

(iv) $\{n_3, n_4, n_5\}$ and $\{n_3, n_8, n_2\}$ are collinear. Then by the compatibility of the $\mathbb{Z}$weights around the vertices n_7 and n_8, we see that the weighted link of n_1 has weight $-3, -1, -2, -1, 1, 0$ at n_3, n_4, n_5, n_6, n_7, n_8, respectively, hence in particular $\{n_3, n_1, n_7\}$ are collinear. An admissible double $\mathbb{Z}$ -weighting is uniquely determined by these considerations and gives rise to the admissible N-weighting $[8-5'']$ of Thm 9.6.

Lemma 9.15. *A weakly minimal N-weighting on the triangulation [8-8] is of the form in Theorem 9.6 up to isomorphism.*

Proof. By symmetry, we may assume that $\{n_4, n_1, n_5\}$ are collinear. Looking at the weighted link of n_2 and of n_3, we see that $\{n_1, n_2, n_6\}$ and $\{n_1, n_3, n_7\}$ are collinear.



n_8 at ∞)
[8 − 8]

Look then at the weighted link of n_4 and of n_5. $\{n_2, n_4, n_8\}$ are not collinear, since otherwise they would be on a great circle, hence a contradiction for the weighted link of n_2 would result. Similarly, we see that $\{n_3, n_5, n_8\}$ are not collinear. By symmetry, we thus have three possibilities:

94 1. $\{n_1, n_4, n_6\}$ and $\{n_1, n_5, n_7\}$ are collinear. Then $\{n_1, n_6\}$ and $\{n_1, n_7\}$ are pairwise antipodal, a contradiction.

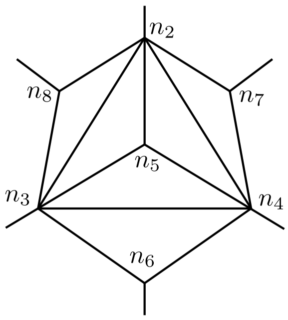
2. $\{n_3, n_4, n_6\}$ and $\{n_2, n_5, n_7\}$ are collinear. Let the weighted link of n_2 have weight b at n_6. The collinearity we have so far then determines all the $\mathbb{Z}$-weights around n_2 in terms of b. But the compatibility for them leads to $b = -1$, a contradiction.

3. $\{n_3, n_4, n_6\}$ and $\{n_1, n_5, n_7\}$ are collinear. Let the weighted link of n_2 have weight b at n_6. Then the compatibility of the $\mathbb{Z}$ weights around n_2 shows that $\{n_5, n_8, n_6\}$ are collinear. Hence the weighted link of n_8 necessarily has weights $-2, -1$ at n_7, n_3, n_4, respectively. In this way, an admissible double $\mathbb{Z}$ -weighting is uniquely determined and gives rise to the admissible N-weighting [8-8] of Thm 9.6.

□

Lemma 9.16. *The triangulation [8-9] has no weakly minimal admissible N-weighting.*

Proof. Suppose the contrary. Look at the weighted link of n_1. By symmetry, it is enough to consider three cases.



n_1 at ∞)
[8 − 9]

1. $\{n_2, n_1, n_3\}$ are collinear, hence on a great circle. Looking at the weighted link of n_2, n_3 and n_4, we immediately get a contradiction. **95**

2. $\{n_2, n_1, n_6\}$ are collinear. In this case, the weighted link of n_1 has weights -1 at n_7 and n_8, a contradiction.

3. $\{n_6, n_1, n_7\}$ are collinear. The same argument applies to the other 6 valent vertices n_2, n_3 and n_4. But for the weighted link of n_2, $\{n_5, n_2, n_8\}$ cannot be collinear, since otherwise we would easily conclude that $\{n_3, n_2, n_4\}$ are also collinear hence on a great circle, a contradiction by (1) applied to n_5 instead of n_1. Taking into account the similar conclusion for n_3, we need, by symmetry, to consider the following three cases:

□

(i) $\{n_5, n_2, n_7\}$ and $\{n_5, n_3, n_6\}$ are collinear. In this case, the consideration of the weighted link of n_2 and n_3 leads us to a contraction for the weighted link of n_1

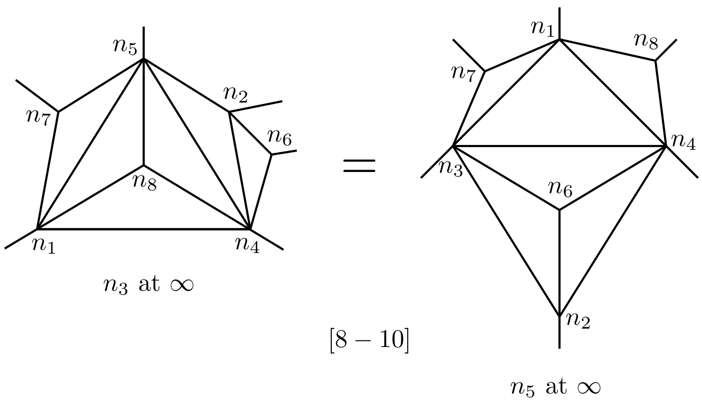
(ii) $\{n_5, n_2, n_7\}$ and $\{n_6, n_3, n_8\}$ are collinear.

(iii) $\{n_7, n_2, n_8\}$ and $\{n_6, n_3, n_8\}$ are collinear.

In cases (*ii*) and (*iii*), the $\mathbb{Z}$- weights around n_6, n_7, n_8 are necessarily $0, 0, -2$ a contradiction for the weighted link of n_1.

Lemma 9.17. *A weakly minimal admissible N-weighting for the triangulation [8-10] is necessarily of the form in Theorem 9.6 up to isomorphism.*

Proof. Note first that the triangulation can be written in a more symmetric form as on the right hand side.



96 Look at the weighted link of n_1. Then $\{n_7, n_1, n_8\}$. are collinear. Look then at the weighted link of n_2. Then $\{n_3, n_2, n_4\}$ are not collinear, hence $\{n_5, n_2, n_6\}$ are collinear. Indeed, if otherwise, $\{n_3, n_2, n_4\}$ would be a great circle. Then looking at the weighted link of n_3 and n_4, we would get a contradiction for the weighted link of n_1.

If $\{n_5, n_3, n_6\}$ or $\{n_5, n_4, n_6\}$ were collinear, then n_5, n_6 would be antipodal, and we would get a contradiction for the weighted link of n_1.

If $\{n_2, n_3, n_4\}$ or $\{n_2, n_4, n_3\}$ were collinear, then they would be on a great circle, a contradiction again.

Thus for the weighted link of n_3, either $\{n_2, n_3, n_7\}$ or $\{n_1, n_3, n_6\}$ are collinear. We have a similar conclusion for the weighted link of n_4. By symmetry, we need to consider the following three cases:

1. $\{n_1, n_3, n_6\}$ and $\{n_1, n_4, n_6\}$ are collinear. We get a contradiction 97
   for the weighted link of n_1.

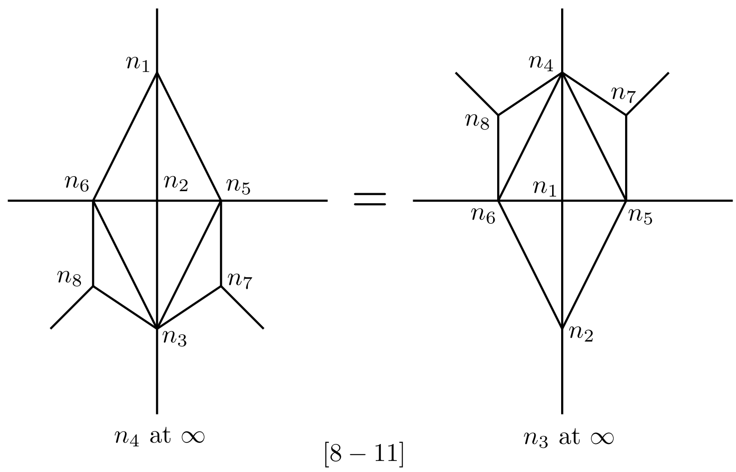
2. $\{n_2, n_3, n_7\}$ and $\{n_2, n_4, n_8\}$ are collinear. In this case, we have a contradiction for the $\mathbb{Z}$-weights around n_7, n_8, and n_2.

3. $\{n_1, n_3, n_6\}$ and $\{n_2, n_4, n_8\}$ are collinear. In this case, we can determine all the double $\mathbb{Z}$-weights uniquely from the collinearity conditions so far. We get the unique admissible N-weighting described in Theorem 9.6.

□

Lemma 9.18. *A weakly minimal admissible N-weighting for the triangulation [8-11] is of the form in Theorem 9.6 up to isomorphism.*

Proof. Look at the weighted link of n_1 and n_2. By symmetry, we need to consider the following three cases:



$[8-11]$

1. $\{n_4, n_1, n_2\}$ and $\{n_1, n_2, n_3\}$ are collinear, hence $\{n_1, n_2, n_3, n_4\}$ are on a great circle. We easily get a contradiction for the weighted link of n_5.

2. $\{n_4, n_1, n_2\}$ and $\{n_5, n_2, n_6\}$ are collinear. In this case, n_1, n_5, n_7
and n_1, n_6, n_8 are necessarily collinear. The consideration of the
98 weighted link of n_3 shows that $\{n_5, n_3, n_6\}$ are collinear, hence
n_5, n_6 are antipodal.

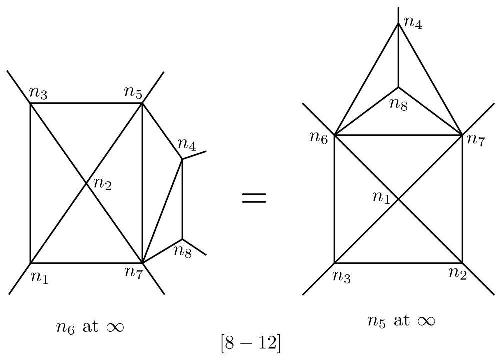
3. $\{n_5, n_1, n_6\}$ and $\{n_5, n_2, n_6\}$ are collinear, hence n_5, n_6 are again antipodal.

□

Thus in cases (2) and (3), n_5 and n_6 are antipodal, hence $\{n_5, n_3, n_6\}$ and $\{n_5, n_4, n_6\}$ are collinear. Look at the weighted link of n_5. Since $\{n_3, n_5, n_4\}$ cannot be on a great circle, and since the situation is symmetric with respect to n_3 and n_4, we may assume that $\{n_2, n_5, n_7\}$ are collinear. Looking at the weighted link of n_4 and the $\mathbb{Z}$-weights around n_7, n_8 and n_1, we conclude that $\{n_2, n_6, n_8\}$ are collinear. From what we have seen so far, we get a unique admissible double $\mathbb{Z}$-weighting with two integral parameters, which gives rise to the admissible N-weighting described in Theorem 9.6.

Lemma 9.19 (Nagaya)**.** *A weakly minimal admissible N-weighting on the triangulation [8-12] is of the form in Theorem 9.6 up to isomorphism.*

Proof. The triangulation can again be written in a more symmetric form as on the right hand side.



$[8-12]$

99 By symmetry, we may assume that $\{n_3, n_1, n_7\}$ are collinear. Look at the weighted link of n_2 and of n_3. We need to consider three cases:

1. If $\{n_3, n_2, n_7\}$ are collinear, then n_3, n_7 are antipodal, hence $\{n_3, n_5, n_7\}$ and $\{n_3, n_6, n_7\}$ are collinear. Looking at the weighted link of n_5, we see that $\{n_5, n_4, n_8\}$ are collinear, a contradiction for the weighted link of n_6.

2. If $\{n_1, n_2, n_5\}$ and $\{n_1, n_3, n_5\}$ are collinear, then n_1, n_5 are antipodal, hence $\{n_1, n_6, n_5\}$, $\{n_1, n_7, n_5\}$ and $\{n_3, n_5, n_7\}$ are collinear. Looking at the weighted link of n_5, we see that $\{n_5, n_4, n_8\}$ are collinear, again a contradiction for the weighted link of n_6.

3. Let $\{n_1, n_2, n_5\}$ and $\{n_2, n_3, n_6\}$ be collinear. If $\{n_6, n_4, n_7\}$ were collinear hence on a great circle, the weighted link of n_7 would have weight -2 at n_2. Moreover, the weighted link of n_5 would have weight 1 at n_4, hence $\{n_3, n_5, n_4\}$ would be collinear and the weight at n_2 would be -1. Since the weighted link of n_1 has weight 0 at n_2, we would violate the compatibility for the $\mathbb{Z}$-weights around n_2.

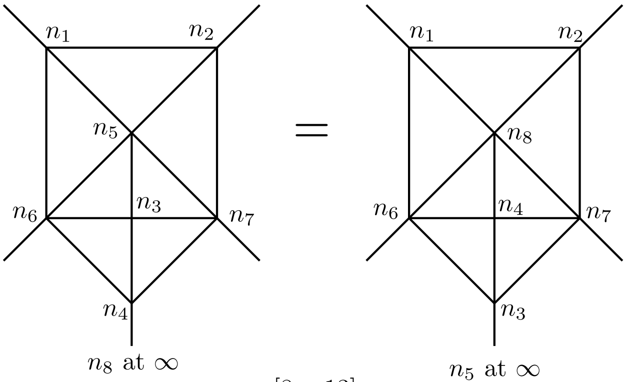
$\square$

Thus we conclude that $\{n_3, n_1, n_7\}$, $\{n_1, n_2, n_5\}$, $\{n_2, n_3, n_6\}$ and $\{n_5, n_4, n_8\}$ are collinear. We then have three possibilities for the weighted link of n_5. If $\{n_3, n_5, n_7\}$ were collinear, then n_3, n_7 would be antipodal, a contradiction, since we are in case (1). If $\{n_2, n_5, n_6\}$ were collinear, then $\{n_1, n_2, n_5, n_6\}$ would be on a great circle, hence n_2, n_6 would be antipo- 100
dal. By symmetry, we are in case (1), a contradiction. Thus $\{n_3, n_5, n_4\}$ are collinear.

We then four possibilities for the weighted link of n_6. If $\{n_3, n_6, n_4\}$ were collinear, then n_3, n_4 would be antipodal, hence $\{n_1, n_7, n_4\}$ would be collinear, a contradiction for the weighted link of n_7. If $\{n_5, n_6, n_8\}$ were collinear, then n_5, n_8 would be antipodal, hence $\{n_5, n_7, n_8\}$ would be collinear. Looking at the weighted link of n_6 and n_7, we would get a contradiction for the $\mathbb{Z}$- weights around n_1. If $\{n_4, n_6, n_7\}$ were collinear hence on a great circle, $\{n_4, n_7, n_6\}$ would be collinear. Looking at the weighted link of n_6 and n_7, we would again get a contradiction for the $\mathbb{Z}$-weights around n_1.

Thus $\{n_1, n_6, n_8\}$ are collinear. These collinearity conditions determine a unique admissible double $\mathbb{Z}$-weighting, which gives rise to the N-weighting [8-12] in Theorem 9.6.

Lemma 9.20. *A weakly minimal admissible N-weighting on the triangulation [8-13] is either [8-13] or* $[8-13'']$ *of Theorem 9.6 up to isomorphism.*



$[8-13]$

101 *Proof.* Look at the weighted link of n_1 and n_2. By symmetry, it suffices to consider three cases:

1. $\{n_2, n_1, n_6\}$ and $\{n_1, n_2, n_7\}$ are collinear. Look at the weighted link of n_5. By symmetry, we may assume that $\{n_2, n_5, n_6\}$, hence $\{n_4, n_3, n_5\}$ are collinear. Thus n_2, n_6 are antipodal, and $\{n_2, n_8, n_6\}$ are collinear. Looking at the weighted link of n_8, we see that $\{n_3, n_4, n_8\}$ are collinear. We have two possibilities for the weighted link of n_6, but by symmetry with respect to $\{n_3, n_4\}$, we may assume that $\{n_3, n_6, n_8\}$ are collinear. Thus n_3, n_8 are antipodal, hence $\{n_3, n_7, n_8\}$ are collinear. These collinearity conditions determine a unique admissible double $\mathbb{Z}$ -weighting with two integral parameter which gives rise to the N-weighting $[8-13']$ of Theorem 9.6.

2. $\{n_5, n_1, n_8\}$ and $\{n_5, n_2, n_8\}$ are collinear. In this case n_5, n_8 are antipodal, hence $\{n_5, n_6, n_8\}$ and $\{n_5, n_7, n_8\}$ are collinear. Looking at the weighted link of n_6 and n_7, we see that $\{n_6, n_3, n_7\}$ and $\{n_6, n_4, n_7\}$ are collinear, hence n_6, n_7 are antipodal and $\{n_6, n_5, n_7\}$ and $\{n_6, n_8, n_7\}$ are collinear. These collinearity conditions determine a unique admissible double $\mathbb{Z}$-weighting with four integral parameters, which gives rise to the N-weighting $[8-13'']$ of Theorem 9.6.

   It remains to show that the following cannot happen:

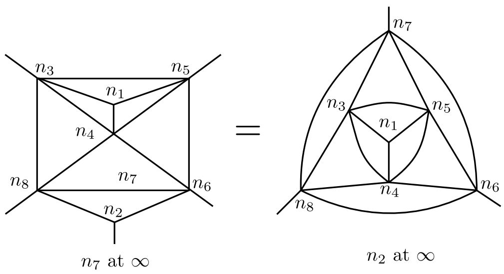
3. $\{n_5, n_1, n_8\}$ and $\{n_1, n_2, n_7\}$ are collinear.

□

By symmetry with respect to $\{n_3, n_4\}$ and our argument (1), (2) applied to n_3, n_4 instead of n_1, n_2, we may assume that $\{n_6, n_3, n_7\}$ and $\{n_3, n_4, n_8\}$ are collinear. Looking at the weighted link of n_5 and n_6, we see then that $\{n_2, n_5, n_3\}$ and $\{n_1, n_6, n_4\}$ are collinear. Let a (resp b) be the $\mathbb{Z}$-weight of the edge n_3n_5 at n_3 (resp. n_1n_6 at n_1). Then the $\mathbb{Z}$-weights around 4-valent vertices n_1, n_2, n_3, n_4 are determined in terms of a and b. We see that $a, b \neq 0, -1$. Looking at the compatibility of the $\mathbb{Z}$-weights around n_7 and n_8, we see that $\{n_4, n_7, n_5\}$ and $\{n_2, n_8, n_6\}$ are collinear. Then we have $a(b+1) = b(a+1) = -1$, a contradiction, since $a \neq -1$ and $b \neq -1$. **102**

Lemma 9.21. *A weakly minimal admissible N-weighting on the triangulation [8-14] is either* $[8-14']$ *or* $[8-14'']$ *of Theorem 9.6 up to isomorphism.*

Proof. The triangulation can be written in a more symmetric from again.



$[8-14]$

103 First look at the 5-valent n_3, n_4, n_5 around n_1 (1). If $\{n_3, n_4, n_5\}$ are collinear, then they are on a great circle. Look at the weighted link of n_6. Because of the 3-valent vertex $n_2, \{n_5, n_6, n_8\}$ and $\{n_4, n_6, n_7\}$ are not collinear $\{n_7, n_6, n_8\}$ are not collinear, since otherwise they would be on a great circle disjoint from the other great circle n_3, n_4, n_5, n_3. Thus by symmetry we may assume that $\{n_2, n_6, n_5\}$ are collinear. Look at the weighted link of n_7. If $\{n_2, n_7, n_5\}$ were collinear, then n_2, n_5 would be antipodal, a contradiction, since n_2 is not on the great circle $n_3n_4n_5n_3$. For the same reason as above, we thus conclude that $\{n_2, n_7, n_3\}$ and $\{n_2, n_8, n_4\}$ are collinear. These collinearity condition determine a unique admissible double $\mathbb{Z}$-weighting with two integral parameters, which gives rise to the N-weighting $[8-14'']$ of Theorem 9.6.

(2) By (1) and the symmetry around n_1, we need to consider two cases:

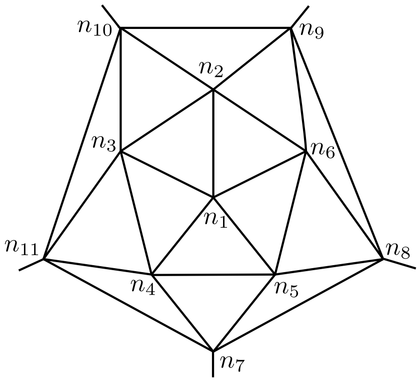
(i) $\{n_1, n_3, n_8\}, \{n_1, n_4, n_8\}$ and $\{n_1, n_5, n_6\}$ are collinear. Let b be the $\mathbb{Z}$-weight around n_1. Then the weighted link of n_3 is completely determined in terms of b.

By the compatibility of the $\mathbb{Z}$-weights around n_3, the weighted link of n_8 has weight 0 at n_3, hence $\{n_4, n_8, n_7\}$ are collinear, a contradiction.

104 (ii) $\{n_1, n_3, n_7\}$, $\{n_1, n_4, n_8\}$ and $\{n_1, n_5, n_6\}$ are collinear. Again let b be the $\mathbb{Z}$-weighted around n_1. The collinearity condition so far determine some the the double $\mathbb{Z} - weights$. By our argument (1) applied to n_2 instead of n_1, we conclude easily that $\{n_2, n_8, n_4\}$, $\{n_2, n_6, n_5\}$ and $\{n_2, n_7, n_3\}$ are collinear. We thus get a unique admissible double $\mathbb{Z}$-weighting with an integral parameter, which gives to the N-weighting $[8-14']$ of Theorem 9.6. □

We thus conclude the proof of Theorem 9.6.

Remark. By repeated application of Corollary 9.2 we determined all admissible N-weightings on the *icosahedral* triangulation of S^2. Apparently, there are 32 different admissible N-weightings, some of which integral parameters, as in the table below, although some of them might be redundant because of the symmetric nature of the triangulation.



n_7 at ∞

Some of them are projective and some others are not. The weak version of our conjecture at the beginning of this section holds for all of them.

Since all the vertices are 5-valent, these admissible N-weightings are automatically weakly minimal. In the table below let $\{n, n'n''\}$ be a $\mathbb{Z}$-basis of N and $a, b, c, d\epsilon, \mathbb{Z}$

	(1)	(2)	(3)	(4)	(5)
n_1	$-n + bn' + cn''$	$-n' + bn''$	$n - n' + an''$	$-n + (a+1)n' + bn''$	$n - an' - n''$
n_2	$-n' + an''$	$-n + an' + cn''$	$-n'$	n'	n
n_3	$-n''$	n''	n''	$-n''$	n'
n_4	$n' - n''$	n	n	$-n + an' + cn''$	$n' - n''$
n_5	n'	$n - n''$	$n - n''$	$-n + an' + (c+1)n''$	$-n''$
n_6	n''	$-n''$	$-n''$	n''	$-n'$
n_7	$n - n'$	$n + n'$	$n + n'$	$-n'$	$-n + n' + bn''$
n_8	n	$n + n - n''$	n'	$-n' + n''$	$-n$
n_9	$n - n' + n''$	$n' - n''$	$-n' + bn' + cn''$	n	$cn - n' + n''$
n_{10}	$n - 2n' + an''$	$-n + (a+1)n' + cn''$	$-n + (b+1)n'(c+1)n''$	$n - n - n''$	n''
n_{11}	$n - n' - n''$	$n'n''$	$n' + n''$	$-n' - n''$	$-n + n' + (b+1)n''$
n_{12}	$n - n'$	n'	$2n' + n''$	$n - n'$	$-n + n''$

	(6)	(7)	(8)	(9)
n_1	$-n + bn' + cn''$	$-n' + bn' + cn''$	$-n + bn + an''$	$n' - n''$
n_2	n'	n''	$n' - n''$	$-n + bn' + (c-1)n''$
n_3	n''	$-n''$	$-n + bn' + (c+1)n''$	n'
n_4	$-n + (b-1)n' + (a+c)n''$	$-n + bn' + (c-1)n''$	$-n' + (a+1)n''$	n
n_5	$-n' + (a-1)n''$	$n' - n''$	$-n' + n''$	$n - n''$
n_6	$-n''$	n'	$-n''$	$-n''$
n_7	$n' + an''$	$-n''$	$n - n' + an''$	$n - n' + an''$
n_8	$n - 2n''$	$n - n''$	$n - n'$	$-n' + an''$
n_9	$n + n' - n''$	n	$-n' + n' + n''$	$n + (b-1)n' + (a+c)n''$
n_{10}	n	$n - n' + (a+1)n''$	n'	$-n + bn' + cn''$
n_{11}	$-n' + (a+1)n''$	$-n' - n''$	n''	n''
n_{12}	$n - n''$	$n - n' + an''$	n	$-n + (a+1)n''$

	(10)	(11)	(12)	(13)	(14)
n_1	$n - n' + an''$	$an - n' - (b+1)n''$	$an - n' - n''$	$-n + an' + bn''$	$n - n' + an''$
n_2	$-n' + (a+1)n''$	$-n - bn''$	$n - n'$	$-n + (b+1)n''$	n
n_3	n	n	$-n$	n'	n''
n_4	$n - n''$	$n - n''$	$-n''$	$n' - n''$	$-n'$
n_5	$-n''$	$-n''$	$n - n''$	$-n''$	$-n' - n''$
n_6	$-n' + an''$	$-n$	n	$-n'$	$-n''$
n_7	$n' - n''$	$n' - n''$	$n' - n''$	n	$-n$
n_8	$-n + bn' + cn''$	$-n + n' + bn''$	n'	$n - n'(b+1)n''$	$-n - n''$
n_9	$-n + bn' + (c+1)n''$	$-n + n'(b+1)n''$	$n' + n''$	$n - n' - bn''$	$-n + n' - n''$
n_{10}	n''	n''	n''	n''	n'
n_{11}	n'	n'	$-n + n' + bn''$	$n + n'$	$-n + bn'n''$
n_{12}	$-n + (b+1)n' + cn''$	$-n + 2n + bn''$	$-n + 2n' + bn''$	$n + n''$	$-n + n'$

	(15)	(16)	(17)	(18)	(19)
n_1	$n - n' + an''$	$n + n'an''$	$n + n' + an''$	$n + n' + an''$	n
n_2	n	n	n'	n	$n - n' - (a-1)n''$
n_3	$-n''$	n''	n''	n''	n''
n_4	$-n' - n''$	$n' + n''$	$n + n''$	$n' + n''$	$n + n' - n''$
n_5	$-n'$	n'	n	n'	$-n' - 2n''$
n_6	n''	$-n''$	$-n''$	$-n''$	$-n''$
n_7	$-n - n' - n''$	$-n + bn'$	$-n'$	$-n$	$n' - n''$
n_8	$-n$	$-n - n''$	$-n' - n''$	$-n - n''$	$-n + bn' + (a-1)n''$
n_9	$n + n' + n''$	$n - n'$	$-n + bn' - n''$	$n - n' - n''$	$-n'$
n_{10}	n'	$-n' + n''$	$-n$	$-n'$	$-n + (a+1)n''$
n_{11}	$-n - n''$	$-n + bn' + n''$	$-n - n'$	$-n + n'' + b(n' + n'')$	n'
n_{12}	$-n + n' + b(n + n' + n')$	$-n'$	$-n + (b-1)n' - n''$	$-n' - n''$	$-n + bn' + an''$

	(20)	(21)	(22)	(23)	(24)	(25)
n_1	$-n' + an''$	n''	n	n	$n + n' + an''$	$-n + an' + (b+1)n''$
n_2	$-n - 2n'$	$-n - n' + (a+1)n''$	$-n - n + an''$	$-n - n' + (a+1)n''$	$n + n''$	$-n + an' + bn'' + c(n' - n'')$
n_3	n''	$-n'$	$-n''$	$-n''$	$-n''$	$n' - n''$
n_4	$n + n''$	$n - n' - an''$	$n + n'$	$n + n'$	$n + 2n' + an''$	n'
n_5	n	n	$n' + n''$	$n' + n''$	$n' + n''$	n''
n_6	$-n''$	n'	n''	n''	n''	$-n' + n''$
n_7	$2n + n' + n''$	$-n''$	n'	n'	n'	n
n_8	$n + n'$	$n + n' - n''$	$-n + n' + an''$	$-n + n' + an''$	$-n$	$n - n' + (d+1)n''$
n_9	$n' - n''$	$2n' - n''$	$-n + an''$	$-n + an''$	$-2n - n'$	$-n'$
n_{10}	$-n - n'$	$-n + an''$	$n' - n''$	$n' - 2n''$	$-n - n'$	$-n''$
n_{11}	n'	$-n + (a-1)n''$	$n + 2n' - n''$	$n + 2n' - n''$	$n' - n''$	$n - n''$
n_{12}	$-n + n' - n''$	$n' - n''$	$2n - n''$	$n' - n''$	$-n + n' - n''$	$n - n + dn''$

	(26)	(27)	(28)	(29)	(30)	(31)	(32)
n_1	$-n + an' + bn''$	$-n + 2n''$	$n' - n''$	$n' + 2n''$	$n - n' + 2n''$	$3n' - n''$	$n - n' - n''$
n_2	$-n' + cn''$	$-n - n' + n''$	$-n - n''$	$-n + 2n''$	$2n + n''$	$-n + n' + 2n''$	$n - n'$
n_3	$-n + an' + (b + 1)n''$	$-n'$	n'	$n' + n''$	$-n + n''$	$2n' - n''$	n
n_4	n'	$n - n' - n''$	$n - n''$	n	n''	$n - n'$	$n - n''$
n_5	$n' - n''$	n''	$n - 2n''$	$n + n''$	$n + 2n''$	n	$n - n' - 2n''$
n_6	$-n''$	n'	$-n'$	n''	$n + n''$	n'	$-n' - n''$
n_7	n	$2n - n'$	$-n' + 2n''$	$-n - n'n''$	$n' + 2n''$	$-n - n' + n''$	$n + n' - n''$
n_8	$n - n''$	n	$-n' + n''$	$-n' + n''$	$n' + n''$	$n + n''$	n'
n_9	$n - n' + dn''$	$2n' - n''$	$-n$	$-n - n' + n''$	n'	n''	$-n - n' - n''$
n_{10}	$-n' + (c + 1)n''$	$-n$	$n' + n''$	$-n + n''$	n	$-n + 2n''$	$n + n''$
n_{11}	n''	$-n''$	n	n'	$-n - n''$	$n' - n''$	$2n + n' - n''$
n_{12}	$n - n' + (d + 1)n''$	$-n + n' - n''$	$-n + n''$	$-n - n'$	$-n - n' - n''$	$-n - n' + 2n''$	n''

Chapter 2

Applications

Mumford et al. [63, Chap. 2] generalized the notion of tours embedding **111**
to that of *toroidal embeddings*. Using this they proved, among other things, the important semi-stable reduction theorem is characteristic 0. Here we are concerned with more elementary applications of tours embedding.

In this chapter we restrict ourselves to tours embedding over the field $\mathbb{C}$ of complex numbers, although some of the result have interesting analogues over fields with non-archimedean rank one valuation.

10 Manifolds with corners associated to torus embeddings

Let $U(1) = \{z\epsilon\mathbb{C};\ |\ z\ |= 1\}$ be the I-dimensional unitary group, i.e. I-dimensional real torus. Then the r-dimensional algebraic torus $T = (\mathbb{C}^*)^r$ has the r-dimensional real torus $CT = (U(1))^r$ as the maximal compact subgroup such that

$$T/CT \cong (\mathbb{R}_{>0})^r \cong \mathbb{R}^r$$

where $\mathbb{R}_{>0}$ is the multiplicative group of positive real numbers.

Here is a coordinates-free description, which will be useful for torus embeddings. Recall that $\mathbb{R}_0$ is the set of non-negative real numbers. We have the valuation

$$\mathrm{ord} : \mathbb{C} \xrightarrow{\;|\;\;|\;} \mathbb{R}_0 \underset{\sim}{\xrightarrow{-\log}} \mathbb{R} \cup \{\infty\}$$

112 which induces a homomorphism

$$\mathrm{ord} : \mathbb{C}^* \xrightarrow{\;|\;\;|\;} \mathbb{R}_{>0} \underset{\sim}{\xrightarrow{-\log}} \mathbb{R}.$$

Let $N \cong \mathbb{Z}^r$ be a free $\mathbb{Z}$-module of rank r and let M be its dual. Then as in §. 1

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{gr}(M, \mathbb{C}^*)$$

is an r-dimensional algebraic torus over $\mathbb{C}$.

Definition. We denote by CT_N the compact (real) torus

$$CT_N = N \otimes_{\mathbb{Z}} U(1) = \mathrm{Hom}_{gr}(M, U(1)).$$

The valuation ord applied to the second factor thus induces a surjection

$$\begin{aligned}\mathrm{ord} : T_N = \mathrm{Hom}_{gr}(M, \mathbb{C}*) &\xrightarrow{\;|\;\;|\;} \mathrm{Hom}_{gr}(M, \mathbb{R}_{>0}) \\ &\underset{\sim}{\xrightarrow{-\log}} \mathrm{Hom}_{gr}(M, \mathbb{R}) = N_{\mathbb{R}}\end{aligned}$$

whose kernel is CT_N, thus we have

$$\mathrm{ord} : T_N/CT_N \underset{\sim}{\to} N_{\mathbb{R}}.$$

Let $(N, \triangle)$ be an r.p.p.decomposition. Then CT_N acts on the corresponding tours embedding $T_N\,\mathrm{emb}(\triangle)$ endowed with the *classical (Hausdorff) topology* instead of the Zariski topology. We then adopt the following definition for the notion introduced by Mumford et al. [61, Chap. 1, §. 1].

Definition. We denote by

$$\mathrm{ord} : T_N\,\mathrm{emb}(\triangle) \to Mc(N, \triangle) = T_N\,\mathrm{emb}(\triangle)/CT_N$$

the quotient space with respect to the classical topology and call it the *manifold with corners* associated to $T_N\,\mathrm{emb}(\triangle)$

113 It is indeed a manifold with corners in the usual sense (cf. Borel-Serre [4]) if T_N emb($\triangle$) is non-singular. By Theorem 4.2, T_N emb($\triangle$) is the union of affine open sets

$$U(\sigma) = \mathrm{Hom}_{unit.semigr}(\check{\sigma} \cap M, \mathbb{C})$$

with σ running through $\triangle$, hence $Mc(N, \triangle)$ is the union of its open sets

$$\begin{aligned} \mathrm{ord} : U(\sigma)/CT_N &\xrightarrow[\sim]{\quad |\ \ | \quad} \mathrm{Hom}_{u.s.g}(\check{\sigma} \cap M, \mathbb{R}_o) \\ &\xrightarrow[\sim]{-\log} \mathrm{Hom}_{u.s.g}(\check{\sigma} \cap M, \mathbb{R} \cup \{\infty\}). \end{aligned}$$

The second description shows that $U(\sigma)/CT_N$ is isomorphic to $(\mathbb{R}_o)^s \times (\mathbb{R}_{>_o})^{r-s}$ for some s if σ is non-singular.

Proposition 10.1. *For an r.p.p.decomposition* $(N, \triangle)$ *the associated manifold with corners* $Mc(N, \triangle)$ *has an action of* $N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$ *and is an equivariant partial compactification of* $N_{\mathbb{R}}$ *with the orbit decomposition.*

$$Mc(N, \triangle) = \coprod_{\sigma \epsilon \triangle} \mathrm{orb}(\sigma)/CT_N$$

such that

$$\mathrm{orb}(\sigma)/CT_N \xrightarrow[\sim]{\mathrm{ord}} N_{\mathbb{R}}/(\mathbb{R} - \textit{subspace generated by } \sigma).$$

In particular

$$\dim_{\mathbb{C}} \mathrm{orb}(\sigma) = \dim_{\mathbb{R}} \mathrm{orb}(\sigma)/CT_N = \mathrm{rank} N - \dim \sigma.$$

Proof. The first part is obvious, since $Mc(N, \triangle)$ has an action of $T_N/CT_N = N_{\mathbb{R}}$. Since $\mathrm{orb}(\sigma) = \mathrm{Hom}_{gr}(\sigma^{\perp} \cap M, \mathbb{C}^*)$ by Theorem 4.2, we see that

$$\mathrm{orb}(\sigma)/CT_N \xrightarrow[\sim]{\mathrm{orb}} \mathrm{Hom}_{gr}(\sigma^{\perp} \cap M, R) = N_R/(R - \text{ subspace } gen^d \text{ by } \sigma).$$

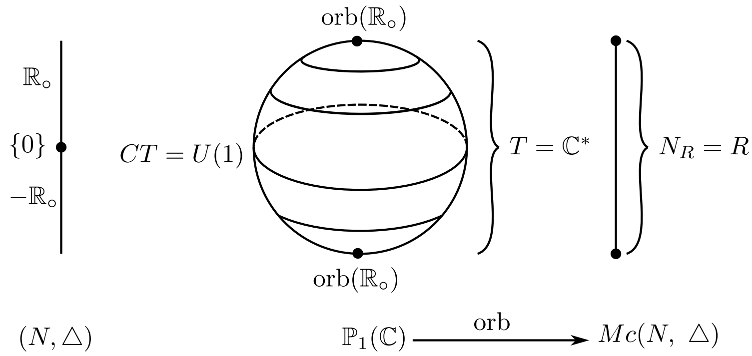
A map h:$(N, \triangle) \to (N', \triangle')$ of $r.p.p.$ decompositions obviously gives 114
rise to an equivariant continuous map

$$Mc(h) : Mc(N, \triangle) \to Mc(N', \triangle')$$

$\square$

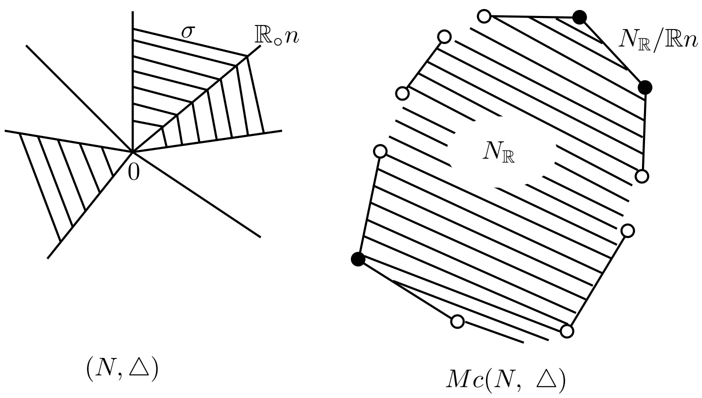
Example (Mumford). Consider the projective line $\mathbb{P}_1(\mathbb{C})$, which is the 2-sphere in the classical topology. It corresponds to the r.p.p. decomposition $(N, \triangle) = (\mathbb{Z} \cdot \{\mathbb{R}_\circ, \{0\}, -\mathbb{R}_\circ\})$.

$CT_N = U(1)$, and the orbits under it are the circles of the same latitude. The quotient $Mc(N, \triangle)$ is the closed interval, a compactification of the open interval isomorphic to $N_{\mathbb{R}} = \mathbb{R}$.



$$(N, \triangle) \qquad \mathbb{P}_1(\mathbb{C}) \xrightarrow{\text{ord}} Mc(N, \triangle)$$

Example. For a 2-dimensional $(N, \triangle)$, $Mc(N, \triangle)$ looks like this



115 We encounter many other examples later in §§. 11, 13, 14, 15. Since

the decomposition of $Mc(N, \triangle)$ into $N_{\mathbb{R}}$-orbits faithfully reflects the incidence among the T_N-orbits of T_N emb $(\triangle)$ the pictures we can draw of $Mc(N, \triangle)$ in dimensions 2 and 3 will give us a good geometric insight.

Remark. Over a field k with a non-archimedean rank one valuation

$$\mathrm{ord} : k \to \mathbb{R} \cup \{\infty\},$$

we have an obvious analogue of $Mc(N, \triangle)$.

Although the following description of the fundamental group has nothing to do with the manifolds with corners, we state it here, since it will be very convenient later.

Proposition 10.2 (Mumford)**.** *Let $(N, \triangle)$ be an r.p.p.decomposition and let T_N emb$(\triangle)$ be the corresponding torus embedding over $\mathbb{C}$ endowed with the classical topology. Then we have the following canonical description of the fundamental group.*

$$\pi_1(T_N emb(\triangle)) = N/(\textit{the subgroup gen}\underline{^{d}} \bigcup_{\sigma\in\triangle}(\sigma \cap N)).$$

Proof. For simplicity, let $T = T_N$ and $X = T_N$ emb$(\triangle)$. We have a well-known isomorphism

$$N \xrightarrow{\sim} \mathrm{Hom}_{gr}(U(1), T) = \pi_1(T)$$

which sends $n \in N$ to the loop $U(1) \ni u \to u^{<?,\, n>} \in \mathrm{Hom}_{gr}(M, \mathbb{C}^*) = T$, where $\langle , \rangle : M \times N \to \mathbb{Z}$ is the dual pairing as before. By Theorem 4.2, X **116**
is covered by $U(\sigma) = \mathrm{Hom}_{u.s.g}(\check{\sigma} \cap M, \mathbb{C}) \supset T$ with σ running through $\triangle$. By generalized van Kampen's theorem in Olum [48], it is thus enough to show that the canonical homomorphism

$$N = \pi_1(T) \to \pi_1(U(\sigma))$$

is surjective with the subgroup N' generated by $\sigma \cap N$ as the kernel. Note that N' is a direct summand of N. $\square$

Since $U(\sigma)$ is non-canonically isomorphic to the product of the algebraic torus $(N/N') \otimes_{\mathbb{Z}} \mathbb{C}^*$ and the lower dimensional $T_{N'}$-embedding corresponding to σ considered as a cone in N'_R, it is obviously enough to show that $U(\sigma)$ is simply connected if σ generates $N_{\mathbb{R}}$. First of all, $N = \pi_1(T)$ is mapped surjectively onto $\pi_1(U(\sigma))$, since $U(\sigma) - T$ is of real codimension ≥ 2 in $U(\sigma)$, by the so-called general position argument. On the other hand, for $n \in \sigma \cap N$ and $0 \leq \varepsilon \leq 1$, the map

$$U(1) \ni u \mapsto (\varepsilon u)^{<?,n>} \in \mathrm{Hom}_{u.s.g}(\check{\sigma} \cap M, \mathbb{C}) = U\sigma$$

establishes a homotopy which kills n in $\pi_1(U(\sigma))$ as ε goes to 0, since $< m, n > \geq 0$ for $m \in \check{\sigma} \cap M$. Since $\sigma \cap N$ generates N as a group, we are done.

For illustrative examples, we refer the reader to §§. 13, 14.

11 Complex tori

117 The following "multiplicative" formulation of complex tori and polarizations was given by Tate in his lecture in July 1967. It was used effectively by Mumford et al. [61]. In this way, we can also formulate their analogues in the non-archimedean case, or even in the ideal-adic case. (cf. Morikawa [36], McCabe [33], Mumford [38] and Gerritzen [11].)

Given a g-dimensional complex torus X, there exists a symmetric $g \times g$ matrix $\tau = (\tau_{jk})$ with the positive definite imaginary part $\mathrm{im}(\tau)$ such that

$$X = \mathbb{C}^g/(\mathbb{Z}^g + \mathbb{Z}\tau_1 + \cdots + \mathbb{Z}\tau_g)$$

where $\tau_1, \ldots, \tau_g$ are the row vectors of τ. We have a surjective homomorphism

$$\mathbb{C}^g \to (\mathbb{C}^*)^g$$

sending $(z_1, \ldots, z_g)$ to $(\exp(2\pi i z_1), \ldots, \exp(2\pi i z_g))$.
Let Γ be the image of the lattice, which is a free abelian subgroup of $(\mathbb{C}^*)^g$ of rank g such that

$$X = (\mathbb{C}^*)^g/\Gamma.$$

This description is very efficient, since it gets rid of the redundancy $\mathbb{Z}^g$. It also fits in nicely with torus embeddings and will give us good insight into the construction of degenerating families. Here is the *coordinate-free* approach:

Let $N \cong \mathbb{Z}^g$ be a free abelian group of rank g with the dual M. As before, let $T = T_N = \mathrm{Hom}_{gr}(M, \mathbb{C}^*)$ be the algebraic torus. **118**

Definition. A *period* for $T = \mathrm{Hom}_{gr}(M, \mathbb{C}^*)$ is a homomorphism

$$q : \Gamma \to T, \quad \gamma \mapsto q^\gamma$$

from a free $\mathbb{Z}$-module Γ of rank g, which is injective with compact cokernel $X = T/q^\Gamma$, a complex torus.

Composed with the valuation $\mathrm{ord}(?) = -\log|?| : \mathbb{C}^* \to \mathbb{R}$, q gives rise to a homomorphism

$$\mathrm{ord} \circ q : \Gamma \to N_{\mathbb{R}} = Mc(N, \{0\}).$$

The injectivity and the compactness of the cokernel of q are equivalent to those of $\mathrm{ord} \circ q$.

For $\gamma \in \Gamma$ and $m \in M$, let $Q(\gamma, m) = q^\gamma(m) \in \mathbb{C}^*$. Thus we have a non-degenerate biadditive pairing

$$Q : \Gamma \times M \to \mathbb{C}^*.$$

Definition. For a period $q : \Gamma \to T = \mathrm{Hom}_{gr}(M, \mathbb{C}^*)$, let

$$\hat{q} : M \to \hat{T} = \mathrm{Hom}_{gr}(\Gamma, \mathbb{C}^*)$$

be the period given by $\hat{q}^m(\gamma) = q^\gamma(m)$. We call $\hat{X} = \hat{T}/\hat{q}^M$ the *dual complex torus*.

Recall that $m \in M$ gives rise to the character $T \ni t \mapsto e(m)(t) \in \mathbb{C}^*$ (cf. §. 1). Consider the *formal Laurent series*

$$\theta(t) = \sum_{m \in M} a_m e(m)(t)$$

for $t \in T$ and $a_m \in \mathbb{C}$. The following is well-known:

Proposition 11.1. *The ring $Ho1(T)$ of holomorphic functions consists* 119
of the Laurent series which are convergent everywhere on T. The units in $Ho1(T)$ are of the form

$$ae(m) \quad \textit{for} \quad a \in \mathbb{C}^* \quad \textit{and} \quad m \in M.$$

It is also well-known that the *positive divisors* on the complex torus $X = T/q^\Gamma$ are in one to one correspondence with the Γ-invariant positive divisors on T, which are of the form $\mathrm{div}(\theta)$ for $0 \neq \theta \in Ho1(T)$. From what we saw above, the Γ-invariance means that there exist maps

$$c : \Gamma \to \mathbb{C}^* \quad \text{and} \quad \Gamma \to M$$

such that

$$\theta(q^\gamma t)c(\gamma)t^{\alpha(\gamma)} = \theta(t) \text{ for all } \gamma \in \Gamma. \tag{*}$$

Such $\theta(t)$ is called a *theta function*. We see immediately the following:

Lemma 11.2. *The pair (α, c) satisfies the following conditions and is called a theta type for the period q.*

(i) $\alpha : \Gamma \to M$ is a Q-symmetric homomorphism, i.e.

$$\alpha(\gamma + \gamma') = \alpha(\gamma) \cdot \alpha(\gamma') \quad \textit{and} \quad Q(\gamma', \alpha(\gamma)) = Q(\gamma, \alpha(\gamma')).$$

(ii) $c : \Gamma \to \mathbb{C}^*$ is a quadratic character with respect to $Q(?, \alpha(?))$, i.e.

$$c(\gamma + \gamma')/c(\gamma)c(\gamma') = Q(\gamma, \alpha(\gamma')).$$

Given theta functions $\theta(t)$ and $\theta'(t)$ with the theta types (α, c) and
120 (α', c'), respectively, we see that $\theta(t)\theta'(t)$ is of the theta type $(\alpha + \alpha', cc')$. Moreover, $\mathrm{div}(\theta)$ and div (θ') give rise to linearly equivalent divisors on X if and only if there exists $m \in M$ such that

$$\alpha' = \alpha \quad \text{and} \quad c' = c \cdot Q(?, m).$$

Thus we conclude:

Proposition 11.3. *We have an isomorphism*

$$Pic(X) = \{theta\ types\ (\alpha, c)\}/\{(0, Q(?, m)); m \in M\}$$

and an exact sequence

$$0 \to \hat{X} \to \ Pic\ (X) \to \mathrm{Hom}_{Q-symm}(\Gamma, M) \to 0.$$

Given a theta type (α, c), let $\Gamma(\alpha, c)$ denote the corresponding line bundle on X.

Proposition 11.4. *Given a theta type (α, c) for q, we have a homomorphism*

$$\Lambda(L(\alpha, c)) : X \to \hat{X}$$

induced by $\circ\alpha : T = \mathrm{Hom}_{gr}(M, \mathbb{C}^*) \to \hat{T} = \mathrm{Hom}_{gr}(\Gamma, \mathbb{C}^*)$.

(1) $\alpha : \Gamma \to M$ is injective if and only if ord $\circ Q(?, \alpha(?)) : \Gamma \times \Gamma \to \mathbb{R}$ is non-degenerate. In this case, $\Lambda(\alpha, c))$ is an isogeny whose degree is equal to the order $\deg(\alpha)$ of coker $[\alpha : \Gamma \to M]$, and $L(\alpha, c)$ is called *non-degenerate*.
(2) If, moreover, ord $\circ Q(?, \alpha(?)) : \Gamma \times \Gamma \to \mathbb{R}$ is positive definite, then $L(\alpha', c)$ is an ample line bundle, and α is called a *polarization*. In this case, the theta functions of the theta type (α, c) form a $\mathbb{C}$-vector space of dimension $\deg(\alpha)$. The polarization is *principal* if and only if **121**

$$\alpha : \Gamma \xrightarrow{\sim} M, \quad \text{i.e.} \quad \Lambda(L(\alpha, c)) : X \xrightarrow{\sim} \hat{X}.$$

When we have a principal polarization α on X, we can identify Γ with M via α. Thus a principally polarized g-dimensional complex torus X is determined by a symmetric biadditive map

$$Q : M \times M \to \mathbb{C}^*$$

with $P = \mathrm{ord} \circ Q : M \times M \to \mathbb{R}$ positive definite so that

$$X = T/\{Q(m, ?); m \in M\}.$$

Let us fix $M = \mathbb{Z}^g$ with the $\mathbb{Z}$-basis $\{m_1, \ldots, m_g\}$. Let

$$PdSym(M \times M, \mathbb{C}^*) = \left\{\begin{matrix}\text{symmetric biadditive } Q : M \times M \to \mathbb{C}^* \\ \text{with ord} \circ Q \text{ positive definite}\end{matrix}\right\}$$

We then have a *versal family* over $PdSym(M \times M, \mathbb{C}^*)$ of principally polarized g-dimensional complex tori, whose total space is the quotient of $T \times PdSym(M \times M, \mathbb{C}^*)$, with $T = \mathrm{Hom}_{gr}(M, \mathbb{C}^*)$, by the action of M defined by

$$(t, Q) \mapsto (Q(m, ?)t, Q) \quad \text{for} \quad m \in M.$$

$Pdsym(M \times M, \mathbb{C}^*)$ is an open subset of

Sym $(M \times M, \mathbb{C}^* = \{\text{symmetric biadditive } Q : M \times M \to \mathbb{C}^*\}$, which, by component wise multiplication, is a $g(g+1)/2$-dimensional algebraic torus over $\mathbb{C}$ with the character group

122 $S^2(M)$ = the symmetric product of M of degree 2 and the group of one parameter subgroups

$$\mathrm{Sym}(M \times M, \mathbb{Z}) = \{\text{symmetric bilinear maps} M \times M \to \mathbb{Z}\}.$$

As usual, let S_g be the Siegel upper half plane of complex symmetric gxg matrices $\tau = (\tau_{jk})$ with the positive definite imaginary part $Im(\tau) > 0$. Then an element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the Siegel modular group $Sp_g(\mathbb{Z})$ acts on S_g via $\tau \mapsto (A\tau + B)(C\tau + D)^{-1}$. The map $\tau \mapsto Q$ defined by

$$Q(m_j, m_k) = \exp(2\pi i \tau_{jk})$$

establishes an embedding

$$S_g / \{\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in Sp_g(\mathbb{Z})\} \xrightarrow{\sim} PdSym(M \times M, \mathbb{C}^*) \subset Sym(M \times M, \mathbb{C}^*).$$

The first step in compactifying the moduli space of principally polarized complex tori in Mumford et al. [61] is to choose a nice $Sym(M \times M, \mathbb{C}^*)$-embedding corresponding to an *admissible* r.p.p.decomposition $(Sym(M \times M, \mathbb{Z}, \triangle)$ which is invariant under the canonical action of $\mathrm{Aut}(M) \cong GL_g(\mathbb{Z})$ with $\triangle/\mathrm{Aut}(M)$ finite and fills up the convex hull in $Sym(M \times M, \mathbb{R})$ of the set of positive semi-definite integral forms.

The Delony-Voronoi decomposition is such a decomposition and was used by Namikawa [46] for his Delony-Voroni compactification of

123 the moduli space, over which he could even construct a family of what he calls *stable quasi-abelian varieties*. Here is a brief coordinate-free account of the part relevant to us:

Let $P : M \times M \to \mathbb{R}$ be a positive semi-definite bilinear form. P defines a pseudo-norm $\|x\|_p = P(x,x)^{1/2}$ on $M_{\mathbb{R}}$. For $x \in M_{\mathbb{R}}$ let

$$w(x,P) = \{m \in M; \|x-m\|_P = \min_{m' \in M} \|x-m'\|_P\}.$$

The convex hull $D(x,P)$ in $M_{\mathbb{R}}$ of $w(x,P)$ is an (unbounded if P is not definite) polyhedron in $M_{\mathbb{R}}$ and is called a *Delong cell*. The set $Del(M_{\mathbb{R}}, P)$ of Delony cells is the *Delony decomposition* and is invariant under the translation action of M.

On the other hand,

$$V(x,P) = \{y \in N_{\mathbb{R}}; \langle m', y\rangle + P(m', m'+2m) \geq \\ \text{for all } \ m' \in M \text{ and all} m \in w(x,P)\}$$

is a polyhedron in $N_{\mathbb{R}}$ and is called a *Voronoi cell*. Let $P^* : M_{\mathbb{R}} \to N_{\mathbb{R}}$ be the canonical linear map defined by $x' \mapsto P(x', ?)$. Then $V(x,P)$ is the image under $-2P^*$ of $\{x' \in M_{\mathbb{R}}; \|x'-m\|_P = \min_{m' \in M} \|x'-m'\|_p$ for all $m \in w(x,P)\}$. The set $\mathrm{Vor}(N_{\mathbb{R}}, P)$ of Voronoi cells is the *Voronoi decomposition* of the image of $P^* : M_{\mathbb{R}} \to N_{\mathbb{R}}$ and is invariant under the translation action of $2P^*(M)$. This is the original definition by Voronoi and is different from the usual one, for instance in Oda-Seshadri [49].

Definition. Positive semi-definite bilinear forms P, $P' : M \times M \to \mathbb{R}$ 124
are *equivalent* if

$$\mathrm{Del}(M_{\mathbb{R}}, P) = \mathrm{Del}(M_{\mathbb{R}}, P').$$

An equivalence class $\sum$ of positive semi-definite forms is a polyhedral cone, a *Delony-Voronoi cone*, in $Sym(M \times M, \mathbb{R})$. Let Del-Vor be the set of such equivalence classes. Then we have an admissible r.p.p.decomposition

$$(\mathrm{Sym}(M \times M, \mathbb{Z}, \ \text{Del-Vor}),$$

the *Delony-Voronoi decomposition*, which is $\mathrm{Aut}(M)$-invariant with Del-Vor/$\mathrm{Aut}(M)$ finite and which fills up the convex hull of the set of positive semi-definite integral forms.

On the other hand, for a Delony-Voronoi cone $\sum$ and a Delony cell D of the Delony decomposition common to forms in $\sum$,

$$\sigma(\sum, D) = (y, P) \in N_{\mathbb{R}} \times \sum ; \langle m', y\rangle + P(m', m' + 2m) \geq 0 \\ \text{for all } m' \in M \text{ and all } m \in M \cap D \Big\}$$

is a polyhedral cone in $N_{\mathbb{R}} \oplus Sym(M \times M, \mathbb{R})$. The set Mixed of these cones, *mixed cones*, gives rise to an r.p.p.decomposition

$$(N \oplus \mathrm{Sym}(M \times M, \mathbb{Z}),\ \text{Mixed})$$

called the *mixed decomposition*.

The second projection induces a map of r.p.p.decompositions

$$p_2 : (N \oplus\ \mathrm{Sym}\ (M \times M, \mathbb{Z}), Mixed) \to (Sym(M \times M, \mathbb{Z}), Del - vor).$$

125 Thus we have an equivariant morphism of torus embeddings

$$\rho : \underset{\substack{\| \\ \mathscr{X}}}{T_{N\oplus\ \mathrm{Sym}\ (M\times M,\ \mathbb{Z}}\ \mathrm{emb}(\mathrm{Mixed})} \to \underset{\substack{\| \\ \mathscr{B}}}{T_{\mathrm{Sym}\ (M\times M, \mathbb{Z}} emb(Del - Vor)}$$

a "family of semi-universal coverings" of Namikawa [46, §. 8].

For various reasons (cf. ibid. (13.8)), it is more convenient to consider the *level* 2ν *action* $(\nu \geq 1)$ of the lattice M on $\mathscr{B}$. It is induced by the action of M on $N \oplus Sym(M \times M, \mathbb{Z})$ by

$$(y, P) \mapsto (y + P(2\nu m, ?), P) \quad \text{for} \quad m \in M$$

which preserves the mixed decomposition. On $T\times$ PdSym $(M\times M, \mathbb{C}) \subset \mathscr{B}$, it coincides with the action

$$(t, Q) \mapsto (Q(2\nu m, ?)t, Q) \quad \text{for} \quad m \in M.$$

Let $\mathscr{X}^\circ \subset \mathscr{X}$ be the interior of the closure of PdSym $(M \times M, \mathbb{C}^*)$. Then $\mathscr{B}^\circ = \rho^{-1}(\mathscr{X}^\circ)$ is the inverse image under ord of the open set in $Mc(N\oplus$ Sym $(M\times M, \mathbb{Z})$, Mixed) consisting of $N_{\mathbb{R}}\times$ PdSym $(M\times M, \mathbb{R})$ and the boundary at infinity.

The level 2ν action of M induced on the open set, hence also its action on $\mathscr{B}^\circ$, is properly discontinuous and fixed point free. Thus we have a family

$$\omega^{(\nu)} : a^{(\nu)} = B^\circ/(\text{level } 2\nu \text{ action of } M) \to \mathscr{X}^\circ$$

of "stable quasi-abelian varieties of dimension g and of degree ν" of Namikawa [46, §. 13].

We now look more specifically at *one parameter degenerations* of principally polarized g-dimensional complex tori which was studied in detail by Mumford [38], Nakamura [44], [45] and Namikawa [46, §. 17]. See also Namikawa [47] for toroidal degenerations. **126**

Definition. We denote by $U = \{\lambda \in \mathbb{C}; |\lambda| < 1\}$ the open unit disk and $U^* = U - \{0\}$ the punctured disk.

It will be more convenient later to think of U^* and U as open sets in $T_{\mathbb{Z}\ell}$ emb$(\{\mathbb{R}_0\ell, \{0\}\}) = \mathbb{C}$ for a free $\mathbb{Z}$-module $\mathbb{Z}\ell$ of rank one with the base ℓ with

$$U = \mathrm{ord}^{-l}(\mathbb{R}_{>0}\ell \amalg \{\infty\})$$
$$U^* = \mathrm{ord}^{-l}(\mathbb{R}_{>0}\ell)$$

where $Mc(\mathbb{Z}\ell, \{\mathbb{R}_o\ell, \{0\}\}) = \mathbb{R}\ell \amalg \{\infty\}$.

As before, we fix $M \cong \mathbb{Z}^g$ and its dual N and let $T = T_N = \mathrm{Hom}_{gr}(M, \mathbb{C}^*)$. Given a family $\pi^* : Y^* \to U^*$ of principally polarized g-dimensional complex tori, we are interested in extending it to a nice family $\pi : Y \to U$ which is proper and flat.

The semi-stable reduction theorem allows us to replace U by its covering ramified at 0 and reduces the problem to the case of *unipotent monodromy*. (For details see Namikawa [46, §. 17]). We may thus assume that there exists a *homomorphic* map $U \ni \lambda \longmapsto Q_\lambda \in$ (the closure of PdSym$(M \times M, \mathbb{C}^*$ in Sym$(M \times M, \mathbb{C}^*))$ with $\mathrm{ord}^0\, Q_\lambda$ positive definite for $\lambda \neq 0$, and a positive semi-definite $B \in Sym(M \times M, \mathbb{Z})$ such **127**
that the restriction of $\mathrm{ord}^0\, Q_0$ to $\{x \in M_R; B(x, x') = 0 \text{ for all } x' \in M_R\}$ is positive definite and that Y^* is the quotient of $T \times U^*$ under the action of M defined for $m \in M$ by

$$\tilde{f}_m(t, \lambda) = (Q_\lambda(m, ?)\lambda^{B(m,?)}, t, \lambda).$$

Let $\tilde{N} = N \oplus \mathbb{Z}\ell$ and consider the action of M on $\tilde{N}$ defined for $m \in M$ by

$$\tilde{h}_m(n) = n \text{ for } n \in N$$
$$\tilde{h}_m(\ell) = \ell + B(m, ?).$$

Then $\tilde{f}_m$ is the composite of the translation by $(Q_\lambda(m, ?), 1) \in T_{\tilde{N}} = T \times \mathbb{C}^*$ and the automorphism of $T_{\tilde{N}}$ as an algebraic group induced by $\tilde{h}_m$.

To make sure that there is a nice open set V below, let $(\tilde{N}, \tilde{\triangle})$ be an r.p.p.decomposition invariant under $\tilde{h}_m$ for all $m \in M$ such that the union of cones in $\tilde{\triangle}$ coincides with

$$\{0\} \cup \{B^*(M_{\mathbb{R}}) \times \mathbb{R}_{>0}\ell\},$$

where $B^* : M_{\mathbb{R}} \to N_{\mathbb{R}}$ is the linear map defined by $B^*(x) = B(x, ?)$. Such $\tilde{\triangle}$ can be most easily described as the join of 0 with a polyhedral decomposition

$$\tilde{\triangle} \cap (B^*(M_{\mathbb{R}}) + \ell)$$

of the affine subspace $B^*(M_{\mathbb{R}})+\ell$ by bounded polyhedral invariant under the translation by the lattice $\{B(m, ?); m \in M\}$. It is a special case of
128 torus embeddings over a discrete valuation ring in Mumford et al. [63, Chap.IV, §. 3].

Let $V \subset T_{\tilde{N}}\,\mathrm{emb}(\tilde{\triangle})$ be the inverse image under ord of the union of $N_{\mathbb{R}} \times \mathbb{R}_{>0}\ell$ and the boundary at infinity of $Mc(\tilde{N}, \tilde{\triangle})$. Then there exists a unique extension of the action of M onto V, denoted again by $\tilde{f}_m$ for $m \in M$, which is properly discontinuous and fixed point free. Then

$$\pi : Y = V/\{\tilde{f}_m; m \in M\} \to U$$

is the sought-for extension of the family $\pi^* : Y^* \to U^*$.

An example of such $\tilde{\triangle}$ was obtained by Namikawa [46] and Nakamura [44], which is of the following form:

$$\tilde{\triangle} = \{\tilde{\sigma}(D); D \in Del(M_{\mathbb{R}}, B)\},$$

i.e. of the *Voronoi type*, where

$$\tilde{\sigma}(D) = \{(y, w\ell \in \tilde{N}_{\mathbb{R}} = \tilde{N}_{\mathbb{R}} \oplus \mathbb{R}\ell; \; \begin{matrix} 2\langle m', y\rangle + wB(m', m' + 2m) \geq 0 \\ \text{for all } m' \in M \text{ and } m \in M \cap D \end{matrix} \right\}.$$

For the relation of all these to the compactification of the generalized Jacobian varieties, we refer the reader to Namikawa [46, §.18] and Nakamura [44] in the $\mathbb{C}$ case and to Oda-Seshadri [49] and Ishida [26] in the general case.

Example 11.1. *(1) Elliptic curves ($g = 1$)* In this case the versal family is already a one-parameter family. With the canonical principal polarization, an elliptic curve X over $\mathbb{C}$ is of the following form: Let $N = M = T = \mathbb{Z}$. A period

$$q : \mathbb{Z} \to \mathbb{C}^* = T$$

is determined by the image $\lambda = q^l \exp(2\pi i \tau) \in \mathbb{C}^*$ with $0 < |\lambda| < 1$, i.e. **129**
$\mathrm{im}(\tau) > 0$. Hence

$$X = X_\lambda = \mathbb{C}^*/\lambda^{\mathbb{Z}}$$

where $\lambda^{\mathbb{Z}} = \{\lambda^a; a \in \mathbb{Z}\}$. Dividing everything out by $CT = U(1)$, we get

$$\mathrm{ord} \circ q : \mathbb{Z} \to \mathbb{R} = N_{\mathbb{R}}.$$

As before, let $U = \{\lambda\mathbb{C}; |\lambda| < 1\}$ and $U^* = U - \{0\}$. Then we have a versal family

$$\pi^* : Y^* = (\mathbb{C}^* \times U^*)/\tilde{f}^{\mathbb{Z}} \to U^*$$

of elliptic curves $X_\lambda = (\pi^*)^{-1}(\lambda)$, where $\tilde{f}$ is the automorphism of $\mathbb{C}^* \times U^*$ defined by $\tilde{f}(t, \lambda) = (\lambda t, \lambda)$. This can be extended to a family

$$\pi : Y \to U$$

whose fiber over $\lambda = 0$ is the rational curve with a node obtained by identifying 0 and ∞ of $\mathbb{P}_1$. Indeed, let $\tilde{N} = N \oplus \mathbb{Z}\ell$ with the $\mathbb{Z}$-basis $\{n, \ell\}$. $\mathbb{C}^* \times U^*$ is the inverse image under ord of the upper half plane $\mathbb{R} \times \mathbb{R}_{>0}\ell$ of $Mc(\tilde{N}, \{0\}) = \tilde{N}_{\mathbb{R}}$. Let $\tilde{X} = T_{\tilde{N}} emb(\tilde{\Delta})$, where $\tilde{\Delta}$ consists of the faces of

$$\tilde{\sigma}_\nu = \mathbb{R}_o(n + \nu\ell) + \mathbb{R}_o(n + (\nu + 1)\ell)$$

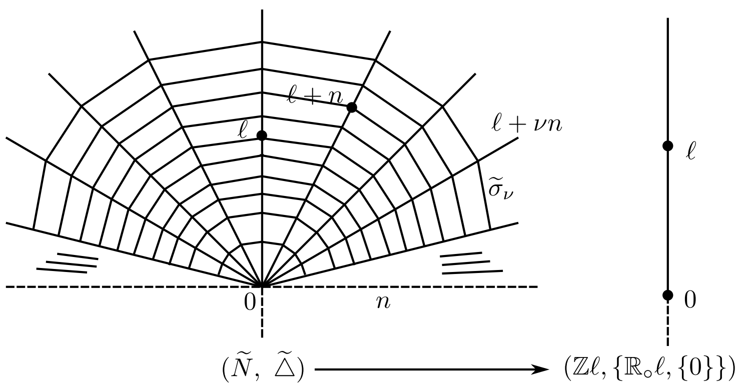
with ν running through $\mathbb{Z}$. Let $\tilde{h}$ be the automorphism of $\tilde{N}$ defined by $\tilde{h}(n) = n$ and $\tilde{h}(\ell) = \ell + n$. Obviously $\tilde{h}$ preserves $\tilde{\triangle}$, hence defines an 130
automorphism $\tilde{f}$ of $\tilde{X}$. $Mc(\tilde{N}, \triangle)$ looks like the picture below and $Mc(\tilde{h})$ (i) acts as the translation by wn for points of $N_{\mathbb{R}} + w\ell$ and (ii) transforms the boundary components $orb(\tilde{\sigma}_\nu)/CT_{\tilde{N}}$ and $\mathrm{orb}(\mathbb{R}_o(n + \nu\ell))/CT_{\tilde{N}}$ at infinity to the next ones corresponding to $\nu + 1$.

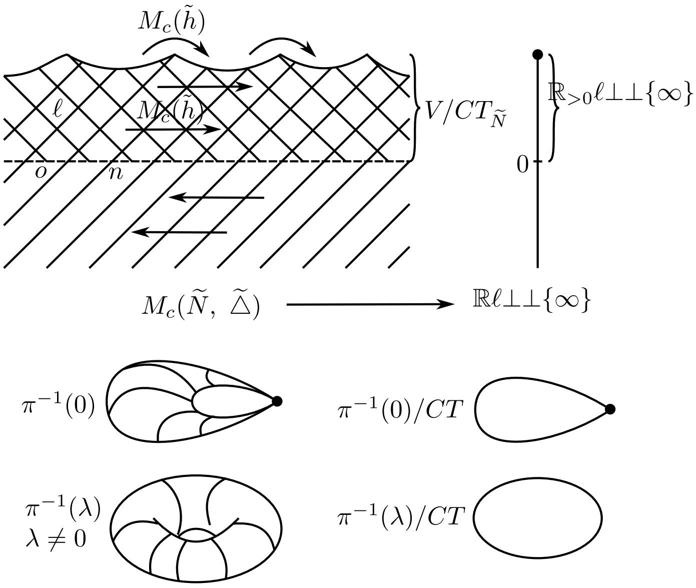
Since the group generated by $Mc(\tilde{h})$ thus acts properly discontinuously without fixed point on the "upper half", the group generated by $\tilde{f}$ also acts properly discontinuously without fixed point on the open set $V \subset \tilde{X}$ which is the inverse image under ord of the upper half of $Mc(\tilde{N}, \tilde{\triangle})$. Then the horizontal projection induces

$$\pi : Y = V/\tilde{f}^{\mathbb{Z}} \to U.$$

Note that $\pi^{-1}(0)/CT$ is the circle obtained by identifying the two end points of a closed interval.

We refer the reader to Mumford et al. [61, Chap. 1. §. 4] for the certification of universal elliptic curves.





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(2) Principally polarized 2-dimensional complex tori

Let $M = \mathbb{Z}^2$ with a $\mathbb{Z}-$ basis $\{m_1, m_2\}$ and let N the dual of M with the dual basis $\{n_1, n_2\}$. Then the Delony-Voronoi decomposition of Sym$(M \times M, \mathbb{R})$ consists of the Aut(M)-translates of

$$\sum_3 = \mathbb{R}_\circ \ell_1 + \mathbb{R}_\circ \ell_2 + \mathbb{R}_\circ \ell_3$$

and its faces $\sum_2 = \mathbb{R}_\circ \ell_1 + \mathbb{R}_\circ \ell_2$, $\sum_1 = \mathbb{R}_\circ \ell_2$ and $\sum_0 = 0$, where $\ell_1, ell_2, \ell_3 \in$ Sym $M \times M, \mathbb{Z}$ are defined as follows:

$$\begin{aligned}
&\ell_1(m_1, m_1) = 1, \quad \ell_1(m_2, m_2) = 0, \quad \ell_1(m_1, m_2) &= 0,\\
&\ell_2(m_1, m_1) = 0, \quad \ell_1(m_2, m_2) = 0, \quad \ell_2(m_1, m_2) &= 0,\\
&\ell_3(m_1, m_1) = 1, \quad \ell_1(m_2, m_2) = 1, \quad \ell_3(m_1, m_2) &= -1,
\end{aligned}$$

(cf. Namikawa [, 2.7)]). In $S^2(M)$, $\{m_1^2 + m_1 m_2, m_2^2 + m_1 m_2, -m_1 m_2\}$ 132
form the $\mathbb{Z}$-basis dual to $\{\ell_1, \ell_2, \ell_3\}$. Thus the parameter space $\mathfrak{X}^\circ$ of the family of semi-universal coverings is covered by the Aut(M)-translates of the open set isomorphic to the interior of the closure in $\mathbb{A}_3$ of $\{(\lambda_1, \lambda_2,$

$\lambda_3) \in \mathbb{A}_3; \log|\lambda_1|\log|\lambda_2| + \log|\lambda_2|\log|\lambda_3| + \log|\lambda_3|\log|\lambda_1| > 0$, and $|\lambda_1\lambda_3| < 1\}$.

We now look at semi-stable one parameter degenerations of 2 - dimensional principally polarized complex tori. It is well-known that, up to the Aut(M)-equivalence, positive semi-definite $B \in Sym(M \times M, \mathbb{Z})$ is of the following from:

$$
\begin{array}{llll}
(1) & B = 0 & & \\
(2) & B(m_1, m_1) = 0, & B(m_2, m_2) = b, & B(m_1, m_2) = 0 \\
(3) & B(m_1, m_1) = a, & B(m_2, m_2) = b, & B(m_1, m_2) = 0 \\
(4) & B(m_1, m_1) = a + c, & B(m_2, m_2) = b + c, & B(m_1, m_2) = -c
\end{array}
$$

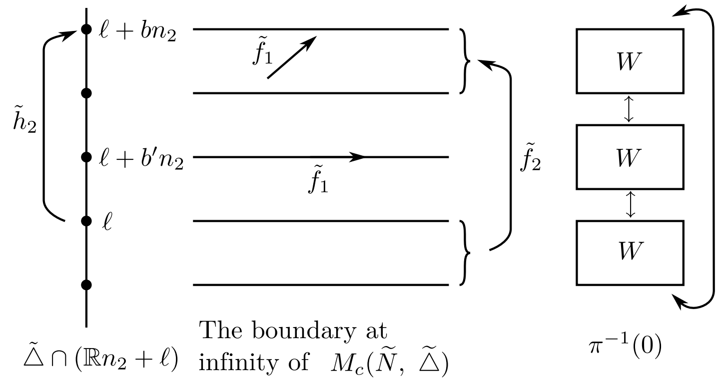
for positive integers a, b, c. For simplicity, let $\tilde{h}_1 = \tilde{h}_{m_1}$ and $\tilde{h}_2 = \tilde{h}_{m_2}$. They are the identity on $N_{\mathbb{R}}$.

In case (1), a decomposition $\tilde{\Delta}$ satisfying our requirements is necessarily of the form $\tilde{\Delta} = \{\{0\}, \mathbb{R}_o\ell\}$.

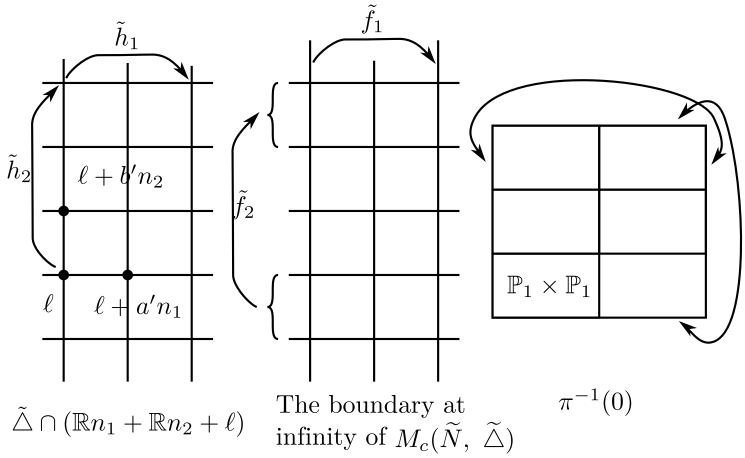
Hence $V = T \times U$ and $\pi^{-1}(0)$ is the complex torus

$$T/\{Q_0(m, ?); m \in M\}.$$

In case (2). $\tilde{h}_1(\ell) = \ell$ and $\tilde{h}_2(\ell) = \ell + bn_2$. For any divisor b'
133 of b, the decomposition of $\mathbb{R}n_2 + \ell$ into "intervals" of length b' as in picture below gives rise to $\tilde{\Delta}$ satisfying our requirements. Consider the $\mathbb{P}_1$-bundle W over the elliptic curve $\mathbb{C}^*/Q_0(m_1, m_1)^{\mathbb{Z}}$ obtained by dividing $\mathbb{C}^* \times \mathbb{P}_1$ out by the group generated by the automorphism $(t_1, t_2) \mapsto (Q_0(m_1, m_1)t_1, Q_0(m_1, m_2)t_2)$. Then by Theorem 4.2 (iii), we see easily that $\pi^{-1}(0)$ is a cycle of b/b' copies of W obtained by identifying the 0-section of one W with the ∞-section of the next. If Y is required to be non-singular, take $b' = 1$ and get the Voronoi type degeneration of Namikawa and Nakamura.



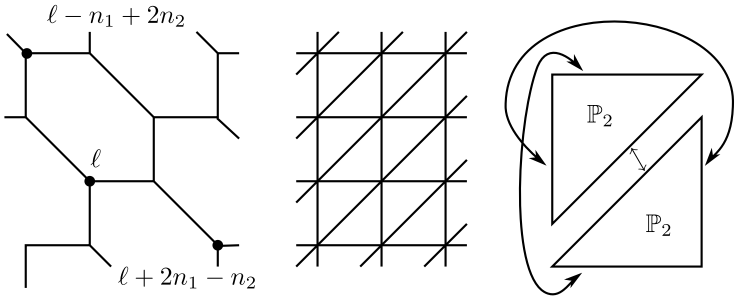
In case (3), $\tilde{h}_1(\ell) = \ell + an_1$ and $\tilde{h}_2(\ell) = \ell + bn_2$. Again for divisors a' and b' of a and b, take the decomposition of $\mathbb{R}n_1 + \mathbb{R}n_2 + \ell$ into "rectangles" of size $a' \times b'$ as in the picture below (or a suitable sub division of it, if Y is required to be non-singular). Without any subdivision, $\pi^{-1}(0)$ consists of $(a/a')(b/b')$ copies of $\mathbb{P}_1 \times \mathbb{P}_1$ glued along $0 \times \mathbb{P}_1$, $\infty \times \mathbb{P}_1$, $\mathbb{P}_1 \times 0$, $\mathbb{P}_1 \times \infty$'s like a "toroidal net", again by Theorem 4.2 (iii). If $a' = b' = 1$, we again get the Voronoi type degeneration **134**



In case (4), $\tilde{h}_1(\ell) = \ell + (a+c)n_1 - cn_2$ and $\tilde{h}_2(\ell) = \ell - cn_1 + (b+c)n_2$. There are many different decompositions of $\mathbb{R}n_1 + \mathbb{R}n_2 + \ell$ which give rise to allowable $\tilde{\Delta}$'s. The "Namikawa decompositions" in Oda-Seshadri

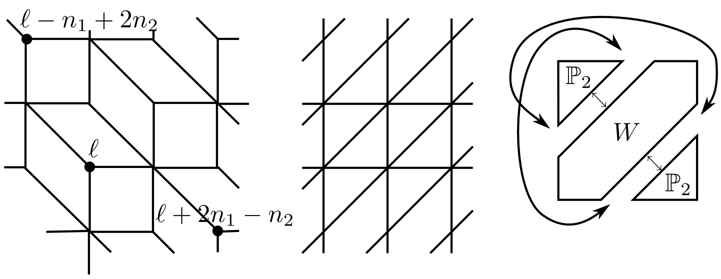
[49], among which is the Voronoi type decomposition, are examples.

For instance, let $a = b = c = 1$, and consider the two decompositions below.



$\tilde{\triangle} \cap (\mathbb{R}n_1 + \mathbb{R}n_2 + \ell)$ The boundary at infinity of $M_c(\widetilde{N}, \widetilde{\triangle})$ $\pi^{-1}(0)$

It is the Voronoi type decomposition and $\pi^{-1}(0)$ consists of two copies of $\mathbb{P}_2$ glued along the three coordinates axes, by Theorem 4.2 (iii) and (7.4).



$\tilde{\triangle} \cap (\mathbb{R}n_1 + \mathbb{R}n_2 + \ell)$ The boundary at infinity of $M_c(\widetilde{N}, \widetilde{\triangle})$ $\pi^{-1}(0)$

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In this case, $\pi^{-1}(0)$ is obtained by gluing together, as in the picture above, two copies of $\mathbb{P}_2$ and one W obtained by blowing $\mathbb{P}_2$ up along the three coordinates vertices, by Theorem 4.2 (iii) and Proposition 6.7. Note that Y is non-singular in this case and each of the six rational curves on W has the self-intersection number -1. This is Deligne's example described in Mumford [38].

12 Compact complex surfaces of class VII

Kodaira [30] classified non-singular compact complex surfaces into seven classes. Among then, the last class VII has been the least understood until recently. **136**

Definition. (Kodaira [30, II, Thm 26]) A non-singular compact complex surface X is said to be of class VII (resp. of class VII_0) if the first Betti number $b_1(X) = 1$ (resp. $b_1(X) = 1$ and X is minimal, i.e. without exceptional curves of the first kind).

As usual, let

$$b_i(X) = \dim_{\mathbb{C}} H^i(X, \mathbb{C})$$
$$h^{p,q}(X) = \dim_{\mathbb{C}} H^q(\Omega_X^p)$$

and let $c_1(x)$ and $c_2(X)$ be the Chern classes of X.
Moreover, let b^+ and b^- be the number of positive and negative eigenvalues of the cup products on $H^2(X, \mathbb{R})$.

Then Kodaira [30, I, Thm 3] showed that a class VII surface X has the following numerical characters:

$$h^{0,1} = 1, h^{1,0} = h^{2,0} = 0$$
$$b_1 = 1, b^+ = 0$$
$$-c_1^2 = c_2 = b_2 = b^-$$

Definition. An, r-dimensional compact complex manifolds X is called a *Hopf manifold* if its universal covering manifold $\tilde{X}$ is isomorphic to $\mathbb{C}^r - \{0\}$. If, moreover, $\pi_1(X) \cong \mathbb{Z}$, then X is called a *primary* Hopf manifold. (See e.g. [1] for a generalization, the Calabi-Eckmann manifold.)

Kodaira [30, III, Thm 41] gave a topological characterization of 2-dimensional Hopf manifolds, i.e. Hopf surfaces : $b_2 = 0$ and $\pi_1(X)$ contains an infinite cyclic subgroup of finite index. **137**

Again according to Kodaira [30, II, Thm 30], a primary Hopf surface X is of the following form :

$$X = (\mathbb{C}^2 - \{0\})/\gamma^{\mathbb{Z}},$$

where $\gamma^{\mathbb{Z}}$ is the group of automorphisms generated by γ of the form

$$\gamma(z, z') = (\lambda z + \mu(z')^b, \lambda' z')$$

with a positive integer b and $\lambda, \lambda', \mu \in \mathbb{C}$ satisfying

$$0 < |\lambda| \leq |\lambda'| < 1$$
$$(\lambda - (\lambda')^b)\mu = 0.$$

We have the following characterizations of some of the class VII_0 surfaces:

Theorem. (Kodaira [K3, II, §. 9 & §. 10]) *X is a class VII_0 surface with $b_2 = 0$ and containing at least one curve if and only if X is either*

a Hopf surface, or

a certain elliptic surface over $\mathbb{P}_1$ with at most non-reduced non-singular fibers, which is obtained from the product of $\mathbb{P}_1$ and an elliptic curve by a finite succession of "logarithmic transformations".

138 **Theorem (Inoue [22]).** *X is a class VII_0 surface with $b_2 = 0$ containing no curve, and with a line bundle L such that $H^0(\Omega^1_X \otimes L) \neq 0$ if and only if X is isomorphic to one of the surfaces*

$$S_M, S^+_{N,p,q,r} \quad \textit{and} \quad S^-_{p,q,r}$$

constructed by Inoue.

All these class VII_0 surface satisfy $b_2 = 0$. Recently, Inoue [23], [24] and Kato [29] constructed series of example with $b_2 \neq 0$. As we see in §. 14 and §. 15, these can most easily be described in terms of torus embeddings.

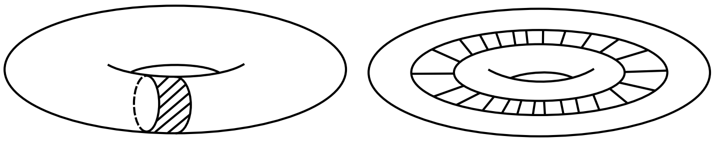
In this connection, the following result of Kato [29] is of utmost importance:

Definition. A *global spherical shell* in an r-dimensional compact complex manifold X is an open submanifold isomorphic to

$$\{z = (z_1, \ldots, z_r)\varepsilon\mathbb{C}^r; 1 - \varepsilon < |z| < 1 + \varepsilon\}$$

with $0 < \varepsilon < 1$ and $|z|^2 = |z_1|^2 + \cdots + |z_r|^2$, such that the complement in X is *connected*.

An elliptic curve, for instance, contains global spherical shells.



Theorem. (Kato [29]) *Let $r \geq 2$. If an r-dimensional compact complex manifold X contains a global spherical shell, then X is a deformation of (hence is diffeomorphic to) the blowing up of a primary Hopf manifold along a finite number of points. In particular, X is non-Kähler and $\pi_1(X) = \mathbb{Z}$.* 139

As we see in §. 14 and §. 15, Kato could also show that all the examples of Inoue with $b_2 \neq 0$ dealt with in §. 14 and §. 15 contain global spherical shells, construct a versal family of deformations for the Inoue surfaces in §. 14 (cf. Theorem 14.2), and construct may new class VII_0 surfaces with $b_2 \neq 0$ which

contain global spherical shells (cf. Remark after Thm. 14.1).

13 Hopf surfaces and their degeneration

In this section, we deal with those Hopf surfaces which can be described in terms of torus embeddings. Although we restrict ourselves to the complex case, the non-archimedean analogue might be interesting to formulate. (cf Gerritzen-Grauert [15]).

Let us first look at the primary Hopf surface

$$X \doteq (\mathbb{C}^2 - \{0\})/\gamma^{\mathbb{Z}}$$

where $\gamma(z, z') = (\lambda z, \lambda' z')$ for $(z, z') \in \mathbb{C}^2 - \{0\}$ and $\lambda, \lambda' \varepsilon \mathbb{C}$ with $0 < |\lambda| < 1$, $0 < |\lambda'| < 1$.

Since γ is defined in terms of monomials, X can be described by
140 means of torus embeddings as follows: Let $N \cong \mathbb{Z}^2$ with a $\mathbb{Z}$-basis $\{n, n'\}$, and consider the r.p.p.decomposition

$$\triangle = \{\mathbb{R}_o n, \mathbb{R}_o n', \{0\}\}.$$

Then obviously $T_N \operatorname{emb}(\triangle) \cong \mathbb{C}^2 - \{0\}$, which is simply connected for instance by Proposition 10.2. γ is the translation action by $(\lambda, \lambda') = \lambda^{\langle ?, n\rangle} \lambda'^{\langle ?, n'\rangle} \in T_N$. Thus it induces the translation action $\overline{\gamma}$ by $(-\log|\lambda|)$ $n + (-\log|\lambda'|)n' \in N_R$ on the manifold with corners $Mc(N, \triangle)$. As in the picture below, the action of $\overline{\gamma}^{\mathbb{Z}}$ on $Mc(N, \triangle)$ is properly discontinuous and fixed point free, and has a compact fundamental domain. Thus the action of $\gamma^{\mathbb{Z}}$ on $T_N \operatorname{emb}(\triangle)$ has the same properties, and we conclude that

$$X = T_N \operatorname{emb}(\triangle)/\gamma^{\mathbb{Z}}$$

is a non-singular compact complex surface with $\pi_1(X) = \mathbb{Z}$. Moreover, X contains mutually disjoint elliptic curves

$$\begin{aligned} E &= \operatorname{orb}(\mathbb{R}_o n')/\gamma^{\mathbb{Z}} \cong \mathbb{C}^*/\lambda^{\mathbb{Z}} \\ E' &= \operatorname{orb}(\mathbb{R}_o n')/\gamma^{\mathbb{Z}} \cong \mathbb{C}^*/\lambda'^{\mathbb{Z}}. \end{aligned}$$

By Proposition 6.6, the canonical bundle of X is obviously

$$\omega_X = 0_X(-E - E').$$

Since $E \cdot E' = 0$ and, furthermore, $E^2 = E'^2 = 0$, as we see below, we have $b_2 = -c_1^2 = 0$. Indeed, the vertical projection $N \to \mathbb{Z}n$ induces

$$\begin{array}{ccc} \operatorname{orb}(\mathbb{R}_o n') \subset & T_N \operatorname{emb}(\triangle) - \operatorname{orb}(\mathbb{R}_o n) & \longrightarrow \mathbb{C}^* \\ & \wr\| & \\ & \mathbb{C}^* \times \mathbb{C} & \end{array}$$

141 on which γ acts by $\gamma(z, z') = (\lambda z, \lambda' z')$. Dividing these out by $\lambda^{\mathbb{Z}}$, we get

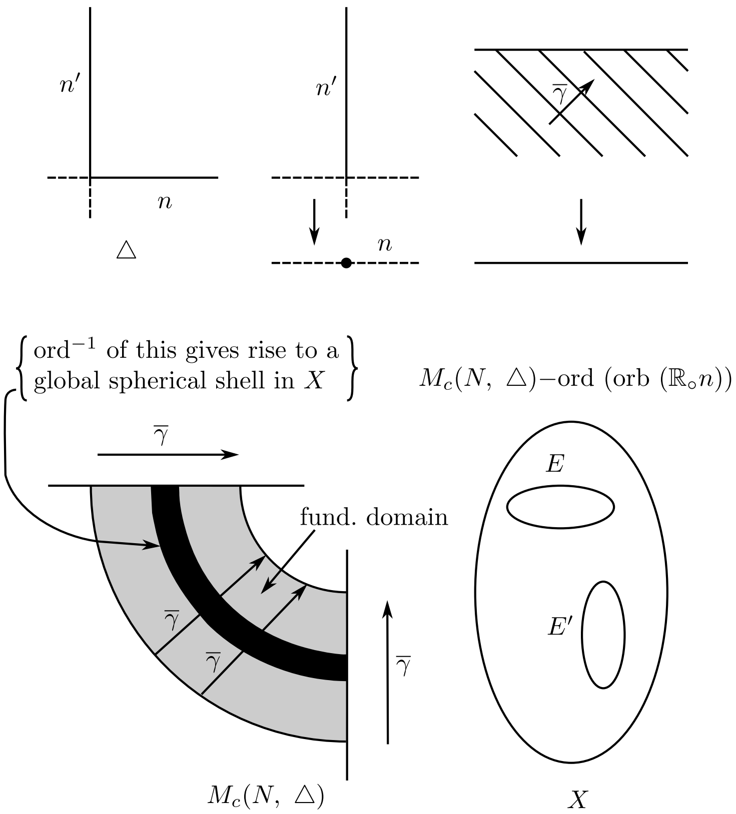
$$E \subset X - E \longrightarrow \mathbb{C}^*/\lambda^{\mathbb{Z}}.$$

$X - E'$ is obviously the total space $V(L)$ of a line bundle L of degree 0 on the elliptic curve $\mathbb{C}^*/\lambda^{\mathbb{Z}}$ and E is its 0-section. Thus

$$E^2 = -\deg L = 0.$$

X is thus a rather curious compactification of $V(L)$.

X has a global spherical shell indicated in the picture.



For a positive integer a, let $\overline{F}_a$ be the non-projective algebraic surface with an ordinary double curve obtained by identifying the 0-section and the ∞-section of the rational ruled surface 142

$$F_a = \mathbb{P}(0_{\mathbb{P}_1} \otimes 0_{\mathbb{P}_1}(a))$$

by means of the map $(z, 0) \mapsto (\phi(z), \infty)$ for $\phi \in \mathrm{Aut}\,(\mathbb{P}_1)$. Assuming that the coefficients of ϕ as a linear fractional transformation are small, Kodaira [30, III, Thm 45] showed:

Theorem. *There exists a complex analytic family over the unit disk*

$$\pi(a) : Y(a) \longrightarrow U = \{\lambda \in \mathbb{C}; |\lambda| < 1\}$$

such that

$$\pi(a)^{-1}(0) = \overline{F}_a$$

and $\pi(a)^{-1}_{(\lambda)}$ *is a Hopf surface for* $\lambda \neq 0$.

Using torus embeddings, we prove this theorem in the special case where ϕ is the identity. The proof can be modified so that it works for $\phi(z) = cz^{\pm}$, $c \in \mathbb{C}^*$.

For this purpose, let us first consider the following Hopf surface: As before, let $N \cong \mathbb{Z}^2$ with a basis $\{n, n'\}$. For a positive integer a, let

$$\triangle(a) = \{\mathbb{R}_o n, \mathbb{R}_o(-n + an'), \{0\}\}.$$

Then by Proposition 10.2, we see that

$$\pi_1(T_N \operatorname{emb}(\triangle(a)) \cong \mathbb{Z}/a\mathbb{Z}.$$

143 In fact the universal covering space is $T_{N'} \operatorname{emb}(\triangle(a)) \cong \mathbb{C}^2 - \{0\}$, where $N' \subset N$ is the subgroup generated by n and $-n + an'$. By (7.6′), we also see that $T_N \operatorname{emb}(\triangle(a))$ is the complement of the 0-section and the ∞-section of F_a.

For λ in the punctured unit disk U^*, consider the translation action γ_λ on $T_N \operatorname{emb}(\triangle(a))$ of

$$\lambda^{<?,n'>} \in T_N.$$

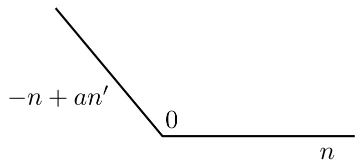
Then the induced action $\overline{\gamma}_\lambda$ on $Mc(N, \triangle(a))$ is the translation by $(-\log|\lambda|)n'$, hence is properly discontinuous, fixed point free with a compact fundamental domain.

Thus we see that

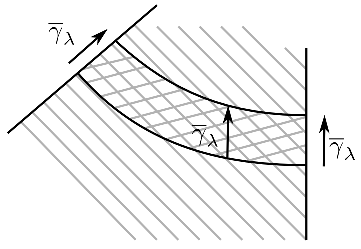
$$X_\lambda(a) = T_N \operatorname{emb}(\triangle(a))/\gamma_\lambda^{\mathbb{Z}}$$

is a Hopf surface with

$$\pi_1(X_\lambda(a)) \cong \mathbb{Z}\ x\ \mathbb{Z}/a\mathbb{Z}.$$



$\triangle(a)$



$M_c(N,\ \triangle(a))$

We then consider $\tilde{N} = N \oplus \mathbb{Z}\ell$ and its automorphism $\tilde{\gamma}$ defined by

$$\tilde{h}(n) = n, \quad \tilde{h}(n') = n', \quad \tilde{h}(\ell) = \ell + n'.$$

Let $(\tilde{N}, \tilde{\triangle}(a))$ be the r.p.p.decomposition consisting of the faces of **144**
σ_ν, σ'_ν with ν running through $\mathbb{Z}$, where

$$\begin{aligned}
\sigma_\nu &= \mathbb{R}_o n + \mathbb{R}_o(\ell + \nu n') + \mathbb{R}_o(\ell + (\nu - 1)n') \\
\sigma'_\nu &= \mathbb{R}_o(-n + an') + \mathbb{R}_o(\ell + \nu n') + \mathbb{R}_o(\ell + (\nu - 1)n').
\end{aligned}$$

Thus $\tilde{\triangle}(a) \cap N_\mathbb{R} = \triangle(a)$ and $\tilde{\triangle}(a)$ induces on the affine subspace $N_\mathbb{R} + \ell$ the polyhedral decomposition $\tilde{\triangle}(a) \cap (N_\mathbb{R} + \ell)$ as in the picture below. $\tilde{h}$ preserves $\tilde{\triangle}(a)$ and gives rise to an automorphism $\overline{\gamma}$ of non-singular $T_{\tilde{N}}\ \mathrm{emb}(\tilde{\triangle}(a))$.

Note that the boundary at infinity of $Mc(\tilde{N}, \tilde{\triangle}(a))$ consists of two "walls" $\mathrm{orb}(\mathbb{R}_o n)/CT_{\tilde{N}}$ and $\mathrm{orb}(\mathbb{R}_o(-n + an'))/CT_{\tilde{N}}$ as well as the "roof" as in the picture.

The horizontal projection induces

$$\tilde{\pi}(a) : T_{\tilde{N}}\ \mathrm{emb}(\tilde{\triangle}(a)) \to T_{\mathbb{Z}\ell}(\{\mathbb{R}_o\ell, \{0\}\}) = \mathbb{C}.$$

Let V be the inverse image of the unit disk U by $\tilde{\pi}(a)$.
Thus $\mathrm{ord}(V)$ is the upper half of $Mc(\tilde{N}, \tilde{\triangle}(a))$.

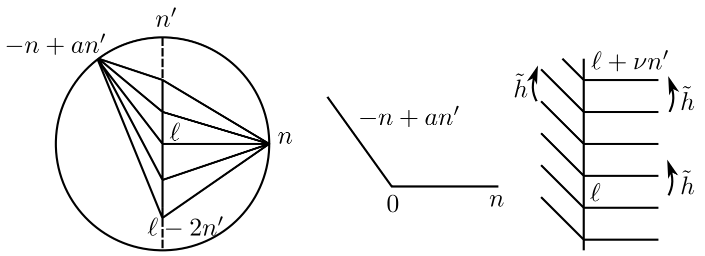
The action of $\overline{\tilde{\gamma}}^{\mathbb{Z}}$ on V is properly discontinuous and fixed point free, $\tilde{\gamma}$ coincides with γ_λ above on $\tilde{\pi}(a)^{-1}(\lambda) = T_N\,\mathrm{emb}(\triangle(a))x\{\lambda\}$. Hence

$$\pi(a) : Y(a) = V/\tilde{\gamma}^{\mathbb{Z}} \to U$$

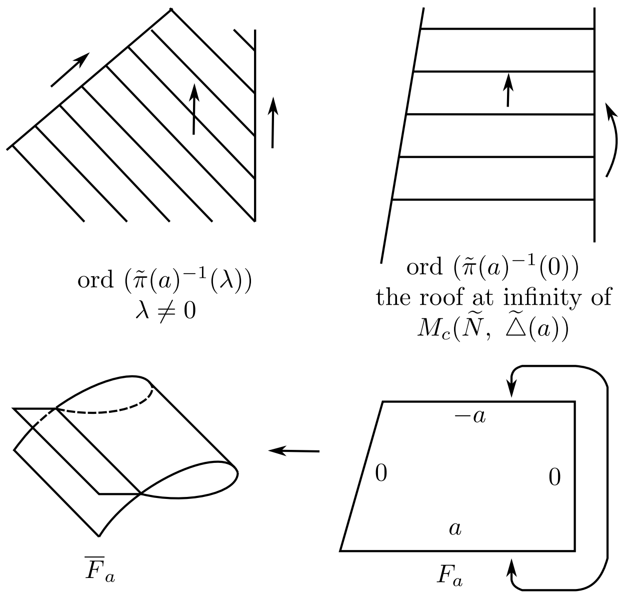
is the sougth-for family with

$$\pi(a)^{-1}(0) = \overline{F}_a \ \text{ and } \ \pi(a)^{-1}(\lambda) = X_\lambda(a) \ \text{ for } \ \lambda \neq 0$$

by Theorem 4.2 (iii) and (7.6′).



$S^2 \cap \tilde{\triangle}(a)$ looked at from above the north pole. $\tilde{\triangle}(a) \cap N_{\mathbb{R}}$ $\tilde{\triangle}(a) \cap (N_{\mathbb{R}} + \ell)$



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14 Inoue's examples of class VII_0 surfaces with $b_2 = 1$

Torus embeddings are very convenient to describe Inoue's examples of class VII_0 surfaces with $b_2 \neq 0$. In this section we describe the first series of his examples in [23]. Besides, we will be able to describe their degeneration easily in terms of torus embeddings. We also sketch Kato's generalization of Inoue's construction which provides us with many new 146
examples.

As before, let $N \cong \mathbb{Z}^2$ with a basis $\{n, n'\}$. Consider the r.p.p. decomposition $(N, \triangle)$, where

$$\triangle = \{\mathbb{R}_o n' \text{ and the faces of } \sigma_\nu \text{ for } \nu \in \mathbb{Z}\}$$
$$\sigma_\nu = \mathbb{R}_o(n + \nu n') + \mathbb{R}_o(n + (\nu - 1)n').$$

By Proposition 10.2, $T_N \operatorname{emb}(\triangle)$ is simply connected. Let h be the automorphism of N defined by

$$h(n) = n + n', \quad h(n') = n'.$$

h obviously preserves $\triangle$ and gives rise to the automorphism h_* of T_N emb($\triangle$).

For $\lambda \in U^* = \{\lambda \in \mathbb{C}; 0 < |\lambda| < 1\}$, consider the automorphism

$$\gamma_\lambda = \text{ (the translation by } \lambda^{\langle ?, n\rangle}) \circ h_*$$

of T_n emb($\triangle$). For $(z, z') \in T_N$, we see that

$$\gamma_\lambda(z, z') = (\lambda z, zz'),$$

hence

$$\gamma_\lambda^\nu(z, z') = (\lambda^\nu z, \lambda^{\nu(\nu-1)/2} z^\nu z') \text{ for } \nu \in \mathbb{Z}.$$

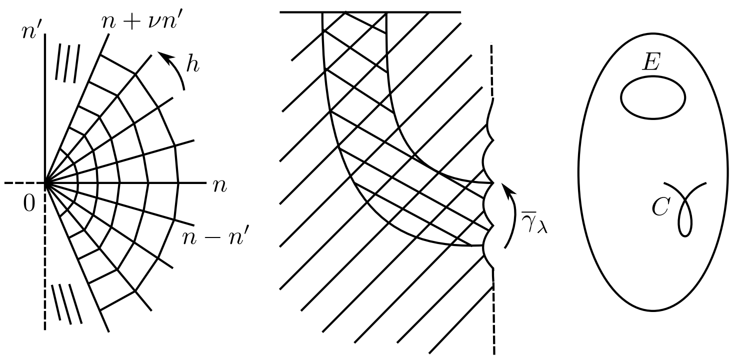
γ_λ induces an automorphism $\overline{\gamma}_\lambda$ of $Mc(N, \triangle)$

$$\overline{\gamma}_\lambda = \text{(the translation by } (-\log|\lambda|)n) \circ Mc(h).$$

As we see easily in the picture, the action of $\overline{\gamma}_\lambda^{\mathbb{Z}}$, hence that of $\gamma_\lambda^{\mathbb{Z}}$, is properly discontinuous, fixed point free and with a compact fundamental domain. Thus

$$X_\lambda = T_N \operatorname{emb}(\triangle)/\gamma_\lambda^{\mathbb{Z}}$$

147 is a non-singular compact complex surface with $\pi_1(X_\lambda) = \mathbb{Z}$, in particular $b_1 = 1$.



$\triangle$ $\qquad$ $M_c(N, \triangle)$ $\qquad$ X_λ

Theorem 14.1 (Inoue [23]). *For $\lambda \in U^*$, the surface X_λ is of class VII_0 with $b_2 = 1 \cdot X_\lambda$ has exactly two irreducible curves, an elliptic curve E and a rational curve C with a node, such that*

$E^2 = -1$, $C^2 = 0$, $E \cdot C = 0$ and C homologous to zero

$H^2(X_\lambda, \mathbb{Q})$ is 1-dimensional and is generated by E.

Proof. $\mathrm{orb}(\mathbb{R}_o n') = \mathbb{C}^*$ is certainly $\gamma_\lambda^{\mathbb{Z}}$-invariant and $E = \mathrm{orb}(\mathbb{R}_o n')/\gamma_\lambda^{\mathbb{Z}} \cong \mathbb{C}^*/\lambda^{\mathbb{Z}}$ is an elliptic curve. On the other hand, the closure of $\mathrm{orb}(\mathbb{R}_o(n + \nu n')$ is $\mathbb{P}_1$ for $\nu \in \mathbb{Z}$, and their union C^* is $\gamma_\lambda^{\mathbb{Z}}$-invariant. $C = C^*/\gamma_\lambda^{\mathbb{Z}}$ is a rational curve obtained from $\mathbb{P}_1$ by identifying 0 and ∞. Since $\mathbb{C}^*$ is the fiber of the equivariant morphism

$$T_N\,\mathrm{emb}(\triangle) \longrightarrow T_{\mathbb{Z}n}\,\mathrm{emb}(\{\mathbb{R}_o n, \{0\}\})$$

induced by the vertical projection, we see that $C^2 = 0$. Certainly, C and E are disjoint, hence $C \cdot E = 0$. Let us now look at **148**

$$\begin{array}{ccc} \mathrm{orb}(\mathbb{R}_o n') \subset T_N\,\mathrm{emb}(\{\mathbb{R}_o n', \{0\}\}) & \longrightarrow & T_{\mathbb{Z}n}\,\mathrm{emb}(\{0\}) = \mathbb{C}^* \\ \| & & \\ \mathbb{C}^* \times \mathbb{C} & & \end{array}$$

on which $\gamma_\lambda^{\mathbb{Z}}$ acts via $\gamma_\lambda^\nu(z, z') = (\lambda^\nu z, \lambda^{\nu(\nu-1)/2} z^\nu z')$. Dividing these out, we have

$$E \subset X_\lambda - C \longrightarrow \mathbb{C}^*/\lambda^{\mathbb{Z}},$$

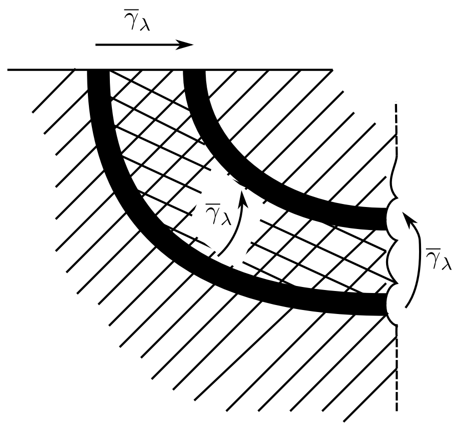
which makes $X_\lambda - C$ the total space $V(L)$ of a line bundle L of degree 1 over $\mathbb{C}^*/\lambda^{\mathbb{Z}}$ and E its 0-section. Hence $E^2 = -\deg L = -1$.
By Proposition 6.6, we see that the canonical bundle is

$$\omega_{X_\lambda} = 0_{X_\lambda}(-E - C),$$

hence $b_2 = -c_1^2 = -(-E - C)^2 = 1$. Thus $H^2(X_\lambda, \mathbb{Q})$ is necessarily spanned by E and C is homologous to zero. □

Suppose there were an irreducible curve $D \neq E, C$.
Since $C \cdot D = 0$, D would be contained in $X_\lambda - C = V(L)$. Thus $E \cdot D$, which is non-negative by assumption, would be the degree of the pull-back of L^{-1} by $D \to V(L) \to \mathbb{C}^*/\lambda^{\mathbb{Z}}$, a contradiction. X_λ is minimal, since it thus has no exceptional curve of the first kind.

Remark. Kato [29] looks at X_λ this way: it is obtained, from ord^{-1} of the shaded area in $Mc(N, \triangle)$, by identifying via γ_λ the two spherical shells which is ord^{-1} of the two thick arcs in the picture. 149



$M_c(N, \ \triangle)$

In this way, we need to consider not T_N emb(Δ) but the simpler torus embedding

$$Z = T_N \operatorname{emb}(\{\mathbb{R}_\circ n, \mathbb{R}_\circ(n+n'), \mathbb{R}_\circ n', \{0\}\}$$

obt ained by blowing up $\mathbb{C}^2$ along 0. Hence $Z = V_1 \cup V_2$ with $V_1 = \mathbb{C}^2$ and $V_2 = \mathbb{C}^2$ with coordinates (z, ζ) and (z', ζ'), respectively, glued along $(\mathbb{C}^*)^2$ via

$$z' = z\zeta \text{ and } \zeta' = \zeta^{-1}.$$

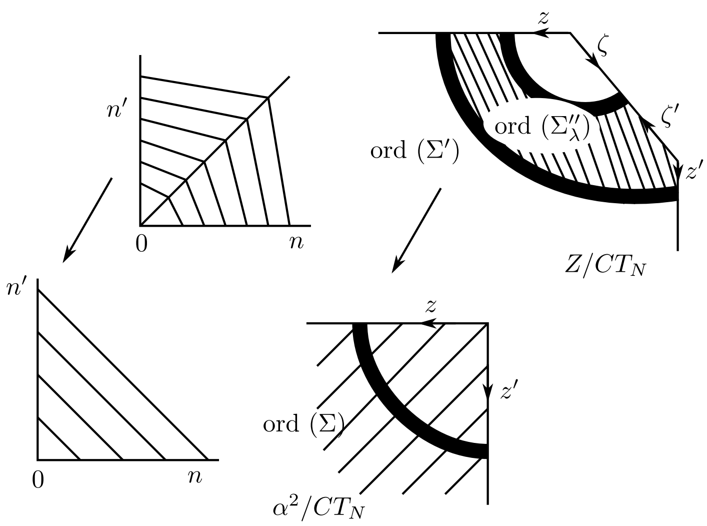
Let $\phi_\lambda : \mathbb{C}^2 \to V_1 \subset Z$be defined by

$$\phi_\lambda(z, z') = (\lambda z, z'/\lambda).$$

Consider, for $\varepsilon > 0$ small, the spherical shell

$$\sum = \{(z, z') \in \mathbb{C}^2; 1 - \varepsilon < (|z|^2 + |z'|^2)^{1/2} < 1 + \varepsilon\}$$

and its inverse image $\sum'$ by the blowing up $z \to \mathbb{C}^2$. Let $\sum''_\lambda = \phi_\lambda(\sum)$. Then $\sum'$ and $\sum''_\lambda$ are spherical shells whose images under ord are the thick arcs in the picture. Thus ϕ_λ induces an isomorphism $\psi_\lambda : \sum' \xrightarrow{\sim} \sum''_\lambda$.



Then X_λ is obtained from the inverse image in Z of the shaded area between the thick arcs ord($\sum'$) and ord($\sum''_\lambda$) by identifying $\sum'$ and $\sum_{\lambda''}$ via ψ_λ. The common image of $\sum'$ and $\sum''_\lambda$ in X_λ is obviously a global spherical shell. **150**

This process was generalized by Kato [29] to produce new class VII_0 surfaces with global spherical shells. Instead of Z, he takes Z' obtained from $\mathbb{C}^2$ by a finite succession of blowing ups each time along a point over the previous center.

Kato's formulation also enabled him to construct a versal family of deformations of Inoue's example X_λ. As before, let U^* be the punctured unit disk and let $B \subset \mathbb{C}^2$ be a small open ball centered at 0 with the coordinate (τ, τ').

Theorem 14.2 (Kato [29]). *There exists a versal family of deformations*

$$\mathscr{X} = \{X_{\lambda,\tau,\tau'}; (\lambda, \tau, \tau') \in U^* \times B\} \longrightarrow U^* \times B$$

and families of curves

$$\mathscr{C} = \{C_{\lambda,\tau,\tau'}\} \quad \textit{and} \quad \mathscr{E} = \{E_{\lambda,\tau,\tau'}\}$$

in $\mathscr{X}$ such that

(i) *$X_{\lambda,0,0} = X_\lambda$ is Inoue's example with $C_{\lambda,0,0} = C$ and $E_{\lambda,0,0} = E$ of Theorem 14.1.*

(ii) *For $\tau' \neq 0$, $E_{\lambda,0,\tau'}$ is empty and $X_{\lambda,0,\tau'}$ is a new class VII_0 surface with a unique irreducible curve $C_{\lambda,0,\tau'}$ rational with a node of self-intersection number 0.*

151 *For different $\tau' \neq 0's, X_{\lambda,0,\tau'}$ are all isomorphic.*

(iii) *For $\tau \neq 0$, $X_{\lambda,\tau,\tau'}$ is blowing up of a primary Hopf surface along a point.*

Here is a sketch of the construction: Let $Z = V_1 \cup V_2$ and $\sum'$ be as in the Remark above. Consider

$$\phi_{\lambda,\tau,\tau'} : \mathbb{C}^2 \longrightarrow V_1 \subset Z$$

defined by $\phi_{\lambda,\tau,\tau'}(z, z') = (\lambda(z_\tau), (z\zeta + \tau')/\lambda)$.
It again induces an isomorphism

$$\psi_{\lambda,\tau,\tau'} : \sum{}' \xrightarrow{\sim} \sum_{\lambda,\tau,\tau'}{}'' = \phi_{\lambda,\tau,\tau'}(\sum{}').$$

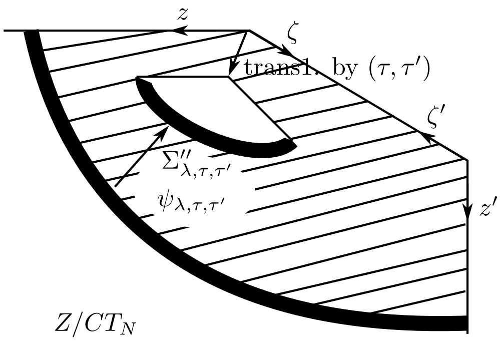
Then take ord^{-1} of the shaded area in the picture of Z/CT_N. Let $X_{\lambda,\tau,\tau'}$ be the manifold obtained from it by identifying $\sum'$ and $\sum_{\lambda,\tau,\tau'}$ via $\psi_{\lambda,\tau,\tau'}$. We then let $\mathscr{C}$ and $\mathscr{E}$ be the images in $\mathscr{X}$ of

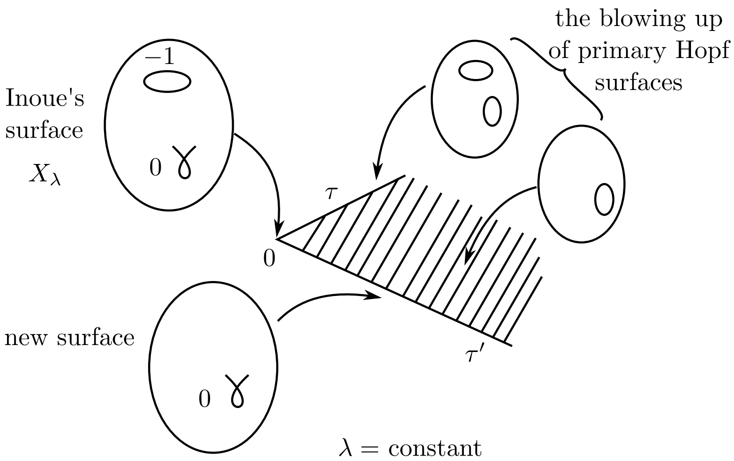
$$\{(z', \zeta', \lambda, \tau, \tau') \in V_2 \times U^* \times B : z'\zeta' + \lambda' + \lambda\tau/(\lambda - 1) = 0\}$$

and

$$\{(z, \zeta', \lambda, \tau, \tau') \in V_1 \times U^* \times B; \zeta = \tau' = 0\},$$

respectively.





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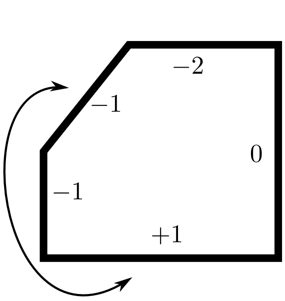
We now look at the degeneration of Inoue's surface X_λ as λ tends to 0.

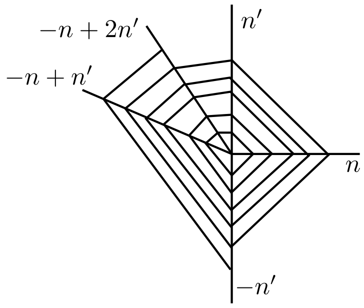
Theorem 14.3. *There exists a proper flat family*

$$\Pi : \lambda \longrightarrow U = \{\lambda \in \mathbb{C}; |\lambda| < 1\}$$

with Y non-singular such that for $\lambda \neq 0$ $\quad \Pi^{-1}(\lambda)$ is Inoue's surface X_λ, and that $\Pi^{-1}(0)$ *is a non-normal surface obtained by identifying two* $\mathbb{P}'_1$*s on the 2-dimensional complete non-singular torus embedding W as in the picture.*



W



The r. p. p. decomposition for W

Proof. Let $\widetilde{N} = N \otimes \mathbb{Z}\ell$ and consider the automorphism $\widetilde{h}$ of $\widetilde{N}$ defined by

$$\widetilde{h}(n) = n + n', \quad \widetilde{h}(n'), \quad \widetilde{h}(\ell) = \ell + n.$$

153 Let $(\widetilde{N},\ \widetilde{\Delta})$ be the r.p.p.decomposition, where $\widetilde{\Delta}$ consists of the faces of

$$\mathbb{R}_o\widetilde{h}^{\nu}(\ell) + \mathbb{R}_o\widetilde{h}^{\nu+1}(\ell) + \mathbb{R}_o\widetilde{h}^{\nu-1}(n)$$
$$\mathbb{R}_o\widetilde{h}^{\nu+1}(\ell) + \mathbb{R}_o\widetilde{h}^{\nu-1}(n) + \mathbb{R}_o\widetilde{h}^{\nu}(n)$$
$$\mathbb{R}_o\widetilde{h}^{\nu}(\ell) + \mathbb{R}_o\widetilde{h}^{\nu+1}(\ell) + \mathbb{R}_o n'$$

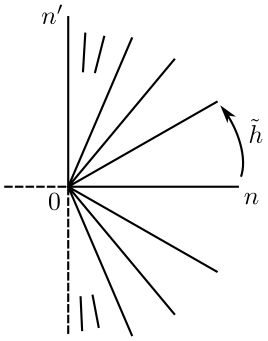
with ν running through $\mathbb{Z}$. Then $\widetilde{\Delta} \cap N_{\mathbb{R}}$ coincides with Δ of Theorem 14.1, and $\widetilde{\Delta}$ inducers the polyhedral decomposition $\widetilde{\Delta} \cap (N_{\mathbb{R}} + \ell)$ as in the picture below. Note that

$$\widetilde{h}^{\nu}(\ell) = \ell + \nu n + (\nu(\nu-1)/2)n'.$$

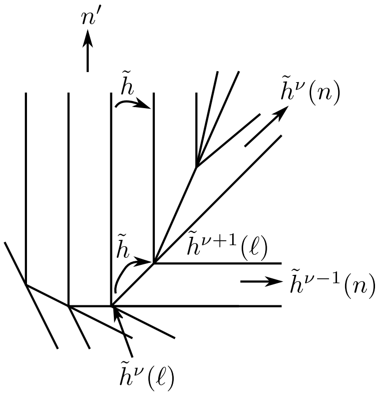
Obviously, $\widetilde{h}$ preserves $\widetilde{\Delta}$ and gives rise to an automorphism $\widetilde{\gamma}$ of $T_{\widetilde{N}}$ emb$(\widetilde{\Delta})$. The horizontal projection induces $\widetilde{\Pi}\ :\ T_{\widetilde{N}}\,\mathrm{emb}(\widetilde{\Delta}) \longrightarrow T_{\mathbb{Z}\ell}$ emb$(\{\mathbb{R}_o\ell, \{0\}\}) = \mathbb{C}$. Again let V be the inverse image of the unit disk U under $\widetilde{\pi}$. Thus ord (V) is the "upper half" of $Mc(\widetilde{N}, \widetilde{\Delta})$. The action of $\widetilde{\gamma}^{\mathbb{Z}}$ on V is properly discontinuous and fixed point free. Again by Theorem 4.2 (iii) and Proposition 6.7, we see that

$$\pi : Y = V/\widetilde{\gamma}^{\mathbb{Z}} \longrightarrow U$$

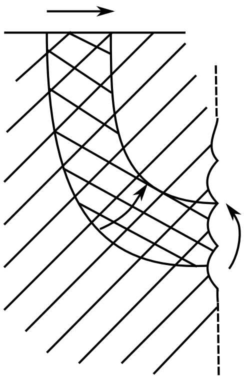
is the sought-for family



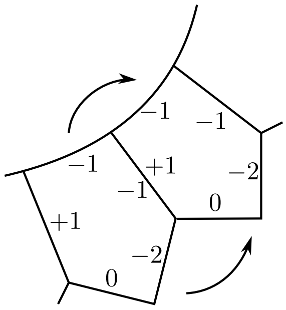
$\widetilde{\triangle} \cap N_{\mathbb{R}} = \triangle$



$\widetilde{\triangle} \cap (N_{\mathbb{R}} + \ell)$



ord $(\widetilde{\pi}^{-1}(\lambda))$
$\lambda \neq 0$



ord $(\widetilde{\pi}^{-1}(0))$, the roof at infinity of $M_c(\widetilde{N},\ \widetilde{\triangle})$

□

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15 Hilbert modular surfaces and class VII_0 surfaces

Inoue [24] constructed another series of examples of class VII_0 surfaces with $b_2 \neq 0$, using the minimal desingularization by Hirzebruch [19] of neighborhoods of cusps of the Hilbert modular surfaces. Again torus embeddings are very convenient to describe them.

We first describe the relevant part of Hirzebruch's theory in terms of torus embeddings. (See also Cohn [6].) For the complete description of compactified Hilbert modular surfaces, we refer the reader to Mumford et al. [61, Chap. 1, §. 5]. See also Rapoport [54] for the case of totally real fields in general.

155 Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field. Then we have two embeddings of K into $\mathbb{R}$

$$K \ni \xi \mapsto \xi \in \mathbb{R} \quad \text{and} \quad K \ni \xi \mapsto \xi' \in \mathbb{R}$$

so that we have a canonical isomorphism of $\mathbb{R}$-algebras

$$\mathbb{R} \otimes_{\mathbb{Q}} K \ni a \otimes \xi \mapsto (a\xi, a\xi') \in \mathbb{R}^2$$

with which we identify them from now on.

Definition. For a $\mathbb{Z}$–lattice N in K, let

$$\begin{aligned} U_N &= \{\text{positive units } u \text{ of } k \text{ such that } u_N = N\} \\ U_N^+ &= \{\text{totally positive units } u \text{ of } k \text{ such that } uN = N\}. \end{aligned}$$

Then it is known (see, for instance Hirzebruch [19]) that U_N and U_N^+ are *infinite cyclic groups* with

$$[U_N : U_N^+] = 1 \text{ or } 2.$$

We have the canonical identification

$$N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} K = \mathbb{R}^2$$

with N lying irrationally in $\mathbb{R}^2$ with respect to the coordinate system of $\mathbb{R}^2$. Consider the convex hull $\sum'_N$ of

$$N \cap (\mathbb{R}_{>0})^2$$

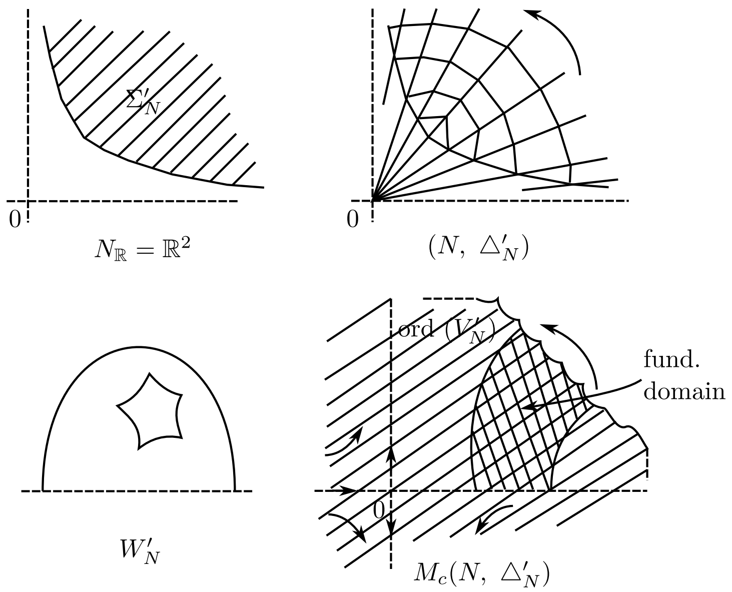
and the infinite r.p.p.decomposition (N, Δ'_N), where Δ'_N is the decomposition of the first quadrant $(\mathbb{R}_{>0})^2$ into sectors by rays joining 0 and the points in $N \cap \partial \sum'_N$.

Thus $Mc(N, \Delta'_N)$ is the union of $N_{\mathbb{R}}$ and the infinite chain of intervals at infinity.

156 The action of U_N^+ on N by multiplication certainly preserves Δ'_N, hence we have an action of U_N^+ on $T_N\,\mathrm{emb}(\Delta'_N)$. Let V'_N be ord^{-1} of the union of the first quadrant $(\mathbb{R}_{>0})^2$ and the boundary at infinity of $Mc(N, \Delta'_N)$. Then the action of U_N^+ on $\mathrm{ord}(V'_N)$, hence that on V'_N itself, is properly discontinuous and fixed point free. Thus the quotient

$$W'_N = V'_N / U_N^+$$

(or, more generally, the quotient by a subgroup of U_N^+ of finite index) is non-singular, and contains a finite cycle of $\mathbb{P}'_1 s$. Such W'_N appears as a neighborhood of the minimal resolution of a cusp of a Hilbert modular surface.



Inoue [24] obtains a new class VII_0 surface by gluing two appropri- 157
ate such W'_N's together. In our formulation, this amounts to applying the process above to the first and the fourth quadrant simultaneously:

Theorem 15.1 (Inoue)**.** *Let N be a $\mathbb{Z}$-lattice in a real quadratic field K. Let $\sum'_N$ (resp. $\sum''_N$) be the convex hull of $N \cap (\mathbb{R}_{>0})^2$ (resp. $N \cap (\mathbb{R}_{>0} \times \mathbb{R}_{<0})$). Consider the infinite r.p.p.decomposition (N, Δ_N), where Δ_N is the decomposition of $\mathbb{R}_{>0} \times \mathbb{R}$ into sectors by rays joining 0 and*

$$(N \cap \partial \sum_N{}') \cup (N \cap \partial \sum_N{}'').$$

Let V_N be ord^{-1} *of the union of $\mathbb{R}_{>0} \times \mathbb{R}$ and the boundary at infinity of $Mc(N, \triangle_N)$. Then the following hold:*

(i) $X_N = V_N/U_N^+$ *is a non-singular compact surface of class VII_0 with exactly $b_2 \neq 0$ irreducible curves, which form two disjoint cycles of rational curves.*

(ii) If $[U_N : U_N^+] = 2$, then

$$\hat{X}_N = V_N/U_N$$

is also a non-singular compact complex surface of class VII_0 with exactly. $b_2 \neq 0$ irreducible curves, which form a cycle of rational curves.

Here is a sketch of the proof:

158 From the picture below, the action of U_N^+ (or U_N in case (*ii*)) on V_N is easily seen to be properly discontinuous, fixed point free and with a compact fundamental domain.

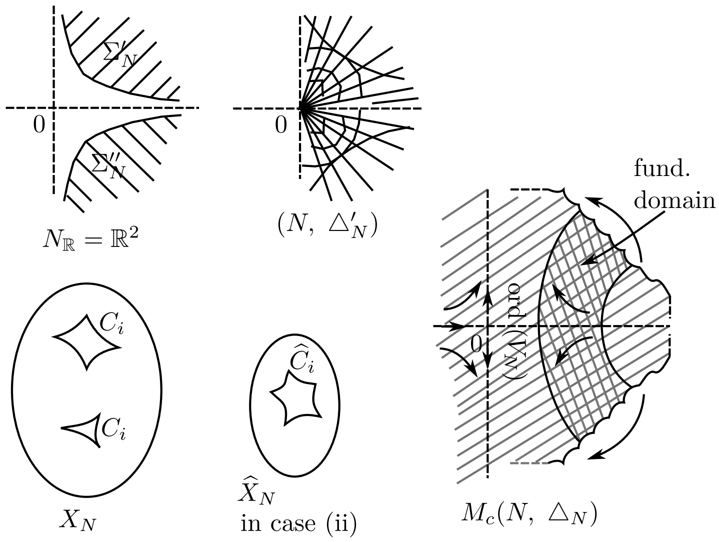
Note that in case (*ii*), the multiplication by an element of U_N not in U_N^+ interchanges the first and the fourth quadrant, hence Δ_N is in a sense "symmetric".

ord^{-1} of the two infinite chains of the boundary gives rise to two disjoint cycles

$$C = C_0 + C_1 + \cdots + C_{r-1}$$
$$C = D_0 + D_1 + \cdots + D_{r-1}$$

of rational curves in X_N. In case (*ii*), we have $r = s$ and the images of C and D in $\hat{X}_N$ coincide. We thus have a cycle of rational curves

$$\hat{C} = \hat{C}_0 + \hat{C}_1 + \cdots \hat{C}_{r-1}$$



We refer reader to Inoue [24] for the proof of the facts: 159

$$H_1(X_N, \mathbb{Z}) \cong \mathbb{Z}$$
$$H_2(X_N, \mathbb{Z}) \cong \mathbb{Z}^{\oplus(r+s)}$$

$C_0, \ldots, C_{r-1}$ and $D_0, \ldots, D_{s-1}$ are the only irreducible curves in X_N. In particular, X_N and $\hat{X}_N$ are minimal. Note that from the picture, we easily see that ord^{-1} of the positive half of the abscissa in $Mc(N, \Delta_N)$ gives rise to a CT_N-bundle over S^1 in X_N, and that $X_N - C - D$ is homeomorphic to the product of $\mathbb{R}$ and the bundle.

Remark. As Kato [29] has shown, X_N contains a global spherical shell, hence is deformation of the blowing up of a primary Hopf surface along a finite number of points. Indeed, ord^{-1} of a tubular neighborhood of the boundary of our fundamental domain in the picture gives rise to one

To study W'_N, X_N and $\hat{X}_N$ in more detail, the "modified" continued fraction expansion is very convenient as in Hirzebruch [19] and Inoue[24].

Definition. Let ξ be a real number. Then the modified continued fraction expansion

$$\xi = [[e_0, e_1, \ldots]]$$

is defined as follows: For non-negative integers ν determine ξ_ν and e_ν inductively by

$$\begin{aligned}&\xi_0 = \xi\\ &e_\nu \quad \text{is the integer with } e_\nu - 1 < \xi_\nu \leq e_\nu\\ &\xi_\nu = e_\nu - 1/\xi_{\nu+1}.\end{aligned}$$

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We call ξ_ν's the *intermediate terms* of the expansion of ξ.

Then $e_\nu \geq 2$ for $\nu > 0$ with no infinite consecutive equality $e_\nu = 2$ allowed when the expansion is infinite. As in the theory of usual continued fractions, we can show that ξ is a *irrational quadratic number* if and only if its expansion is eventually periodic, i.e. periodic from certain ν on. (cf. Perron [51]).

As before, we identify an element $\omega \in K$ with its canonical image (ω, ω') via $K \hookrightarrow N_{\mathbb{R}} = \mathbb{R}^2$. Then the continued fraction expansion of ω and the convex geometry in $\mathbb{R}^2$ have a very close relationship. To be able to describe degenerations of Inoue's surfaces X_N, we need the following slight amplification of the results pointed out by Hirzebruch [19] and Cohn [6].

Proposition 15.2. *(1) Up to the multiplication of a totally positive element of K, a $\mathbb{Z}$-lattice N is of the form*

$$N = \mathbb{Z} + \mathbb{Z}\omega$$

with $\omega > 1 > \omega' > 0$, i.e. ω reduced.

161 *(2) $\omega \in K$ is reduced if and only if it has a purely periodic continued fraction expansion*

$$\omega = [[\overline{a_0, a_1, \ldots, a_{r-1}}]]$$

(with the smallest period r).

(3) For reduced ω, let ω_ν be the intermediate terms of the expansion of ω with the smallest period r. Then

$$u = 1/(\omega_1\omega_2 \cdots \omega_r)$$

is a generator of $U^+_{\mathbb{Z}+\mathbb{Z}\omega}$, and $\{n_\nu\}_{\{\nu\in\mathbb{Z}\}}$ are the consecutive elements of

$$(\mathbb{Z}+\mathbb{Z}\omega)\cap\partial\Sigma'_{\mathbb{Z}+\mathbb{Z}\omega},$$

where

$$n_{qr+j} = (\omega_{j+1}\cdots\omega_r)u^{q+1} \text{ for } q\in\mathbb{Z} \text{ and } 0\le j<r$$

and

$$n_{\nu-1}+n_{\nu+1} = a_\nu n_\nu.$$

(4) *For ω reduced, $1/\omega$ has the expansion of the form*

$$1/\omega = [[1, e, \overline{a^*_0,\dots,a^*_{s-1}}]],$$

where s is the smallest period and $e-1<\omega/(\omega-1)<e$. Let

$$\omega^* = [[\overline{a^*_0,\dots,a^*_{s-1}}]], i.e. \omega/(\omega-1) < e-1/\omega^*$$

and let ω^*_ν be the intermediate terms of the expansion of ω^*.

Then $u = 1/(\omega_1\omega_2\cdots\omega_r) = 1/(\omega^*_1\omega^*_2\cdots\omega^*_s)$, and $\{n^*_\nu\}_{\nu\in\mathbb{Z}}$ are the consecutive elements of $(\mathbb{Z}+\mathbb{Z}\omega)\cap\partial\sum''_{\mathbb{Z}+\mathbb{Z}\omega}$, where

$$n^*_{qs+j} = u^q(\omega-1)/(\omega^*_0\dots\omega^*_j) \text{ for } q\in\mathbb{Z} \text{ and } 0\le j<s$$

and

$$n^*_{\nu-1}+n^*_{\nu+1} = a^*_\nu n_\nu.$$

Corollary 15.3. *For $N=\mathbb{Z}+\mathbb{Z}\omega$ with ω reduced, the decomposition Δ_N* **162**
of $\mathbb{R}_{>0}\times\mathbb{R}$ consists of the faces of

$$\sigma_\nu = \mathbb{R}_o n_\nu + \mathbb{R}_o n_{\nu+1}$$
$$\sigma^*_\nu = \mathbb{R}_o n^*_\nu + \mathbb{R}_o n^*_{\nu+1}$$

for $\nu\in\mathbb{Z}$, with n_ν and n^*_ν as in Proposition 15.2.

Combined with Proposition 6.7, this implies the following:

Proposition 15.4. *Let $N = \mathbb{Z} + \mathbb{Z}\omega$ with ω reduced and let ω^* be as in Proposition 15.2 (4) with*

$$\omega = [[\overline{a_0, \ldots, a_{r-1}}]]$$
$$\omega^* = [[\overline{a^*_0, \ldots, a^*_{s-1}}]]$$

in the smallest periods r and s. Then Inoue's surface X_N has two cycles of rational curves

$$C = C_0 + C_1 + \cdots + C_{r-1}$$
$$C = D_0 + D_1 + \cdots + D_{s-1}$$

and $\hat{X}_N$ a cycle of rational curves

$$\hat{C} = \hat{C}_0 + \hat{C}_1 + \cdots + \hat{C}_{r-1}$$

with the following properties:

(i) *If $r = 1$ or $s = 1$, then C, $\hat{C}$ or D is an irreducible rational curve with one node with*

$$C^2 = -a_0 + 2, \quad \hat{C}^2 = -a_0, +2, \quad D^2 = -a^*_0 + 2.$$

(ii) *If $r \geq 2$ or $s \geq 2$, then C_i, $\hat{C}_i$ or D_i are non-singular rational curves with*

$$C^2_i = -a_i, \hat{C}^2_i = -a_i \qquad i = 0, \ldots, r-1$$
$$D^2_i = -a^*_i \qquad i = 0, \ldots, s-1.$$

163 To illustrate the relationship between the continued fraction expansion of ω and the r.p.p.decomposition (N, Δ_N) with $N = \mathbb{Z}+\mathbb{Z}\omega$, we now sketch the proof of Proposition 15.2 except fo the eventual periodicity of the expansion and the fact that

$$u = 1/(\omega_1 \cdots \omega_r) = 1/(\omega^*_1 \cdots \omega^*_s)$$

generates U^+_N. Hopefully, it will explain the way we number the element of $N \cap (\partial \sum'_N \cup \partial \sum''_N)$.

The proof of the following inducted step is obvious.

Lemma 15.5. *Let $\xi \in K$ with $\xi > \xi' > 0$. For the integer e with $e-1 < \xi < e$, let $\xi = e - 1/\eta$. Then $\eta > 1$, $\eta > \eta' > 0$ and $1/\eta \in \mathbb{Z} + \mathbb{Z}\xi$. If $\xi > \xi' > 1$, then ξ is in the interior of the convex hull of $(\mathbb{Z} + \mathbb{Z}\xi) \cap (\mathbb{R}_{>0})^2$.*

If $\xi > 1 > \xi' > 0$, i.e. ξ is reduced, then $e \geq 2$ and η is also reduced. Moreover, ξ, 1 and $1/\eta$ are consecutive elements in the intersection of $\mathbb{Z}+\mathbb{Z}\xi$ with the boundary of the convex hull $\sum'_{\mathbb{Z}+\mathbb{Z}\xi}$ of $(\mathbb{Z}+\mathbb{Z}\xi) \cap (\mathbb{R}_{>0})^2$.

By repeated application of Lemma 15.5 to the intermediate terms of the expansion of ω, we get:

Lemma 15.6. *Let $\omega \in K$ with $\omega > \omega' > 0$ with the expansion*

$$\omega = [[a_0, a_1, \ldots\ldots]]$$

and the intermediate term ω_ν. Let t be the smallest integer such that $1 > \omega'_t$. Then (i) the expansion is periodic from $\nu = t$ on. Let r be the smallest period. Thus $\omega_{\nu+r} = \omega_\nu$ and $a_{\nu+1} = a_\nu$ for $\nu \geq t$. Let **164**

$$\zeta_{-1} = 1 \text{ and } \zeta_\nu = 1/(\omega_0\omega_1 \cdots \omega_\nu) \text{ for } \nu \geq 0.$$

Then (ii) $\{\zeta_\nu\}_{\nu \geq t-1}$ are consecutive elements of the intersection of $\mathbb{Z}(1/\omega)+\mathbb{Z}$ with the boundary of the convex hull of $(\mathbb{Z}(1/\omega)+\mathbb{Z}) \cap (\mathbb{R}_{>0})^2$. Moreover,

$$\zeta_{\nu+1} + \zeta_{\nu-1} = a_\nu \zeta_\nu \text{ for } \nu \geq t.$$

(iii) $1/(\omega_{t+1} \cdots \omega_{t+r})$ belongs to $U^*_{\mathbb{Z}+\mathbb{Z}\omega}$.

In particular, we get (1), (2) and a part of (3) of Proposition 15.2, since

$$\omega(\mathbb{Z}(1/\omega) + \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}\omega.$$

Let us now prove (4). Let $\xi = 1/\omega$ with the intermediate terms ξ_ν and with the expansion $\xi = [[e_0, e_1, \ldots]]$. Since ω is reduced, hence $\xi' > 1 > \xi_0 > 0$ by assumption, we have $e_0 = 1$ and $1/\xi_1 \in \mathbb{R}_{>0} \times \mathbb{R}_{<0}$. Since $0 < e_1 - \xi_1 = 1/\xi_2 < 1$ and $1/\xi'_2 = e_1 - \xi'_1 > e_1 \geq 2$, we see that $\xi_1 = \omega/(\omega-1)$, $\xi_2 = \omega^*$ and $e_1 = e$ of (4). We now apply (3) to ω^*, taking $u = 1/(\omega^*_1 \cdots \omega^*_s)$ for granted. Thus

$$(\omega^*_{j+1} \cdots \omega^*_s)/(w^*_1 \cdots \omega^*_s)^{q+1} = \omega^* n^*_{qs+j}/(\omega-1)$$

for $q \in \mathbb{Z}$ and $0 \leq j < s$ are the consecutive elements of

$$(\mathbb{Z} + \mathbb{Z}\omega^*) \cap \partial\Sigma'_{\mathbb{Z}+\mathbb{Z}\omega^*}.$$

165 *On the other hand, we have*

$$\begin{aligned} \mathbb{Z} + \mathbb{Z}\omega &= \omega(\mathbb{Z} + \mathbb{Z}(1/\omega)) = \omega(\mathbb{Z} + \mathbb{Z}(1/\xi_1)) \\ &= \omega(\mathbb{Z}(1/\xi_1) + \mathbb{Z}(1/\xi_1\xi_2)) = ((\omega - 1)/\omega^*)\{\mathbb{Z} + \mathbb{Z}\omega^*\}. \end{aligned}$$

Since $(\omega - 1) > 0 > (\omega' - 1)$, we see that $\{n^*_\nu\}_{\nu\in\mathbb{Z}}$ are the consecutive elements of

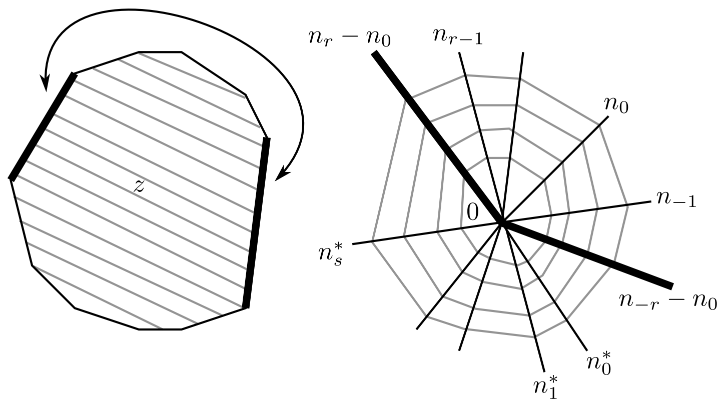
$$(\mathbb{Z} + \mathbb{Z}\omega) \cap \partial\Sigma''_{\mathbb{Z}+\mathbb{Z}\omega} = \Big((\omega - 1/\omega^*)\Big)\{(\mathbb{Z} + \mathbb{Z}\omega^*) \cap \partial\Sigma'\mathbb{Z} + \mathbb{Z}\omega^*\}.$$

We now describe degenerations of Inoue's surface X_N.

Proposition 15.7 (Makio)**.** *For reduced ω K, let $N = \mathbb{Z} + \mathbb{Z}\omega$. Then there exists a proper and flat family*

$$\pi : Y \to \mathbb{C}$$

with Y normal such that $\pi^{-1}(\lambda) = X_N$ for all $\lambda \neq 0$ and that $\pi^{-1}(0)$ is a non-normal surface obtained by identifying two $\mathbb{P}'_1$s on the 2 - dimensional complete normal T_N-embedding Z as in the picture, where n_ν and n^*_ν are as in Prop. 15.2.



The r.p.p. decomposition for Z

166 **Remark.** Unlike Theorem 14.3, Y and Z may be singular in general. They are non-singular if $r = 1$. For each specific N, we can certainly replace Y by a blow up to obtained non-singular Y and $\pi^{-1}(0)$ consisting of non-singular components crossing normally. As is obvious from the construction of Y below, there, many other choices for Y.

Proof. As before, let $\tilde{N} = N \oplus \mathbb{Z}\ell$. Left the action of U_N^* on N to $\tilde{N}$ by letting it act as the identity for ℓ. By Proposition 15.2, we have

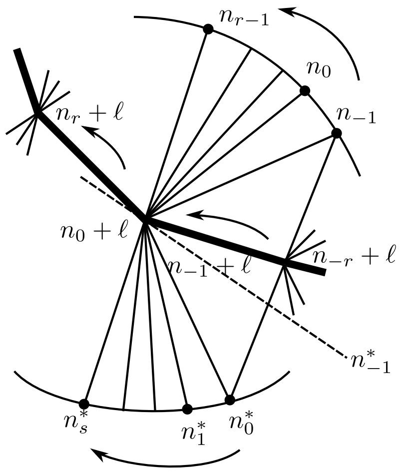
$$n_0 = 1, \quad n_{-1} = \omega, \quad n_{-1}^* = \omega - 1.$$

This fact guarantees that $(\tilde{N}, \tilde{\Delta})$ is a U_N^*-invariant r.p.p.decomposition, where $\tilde{\Delta}$ consists of the faces of

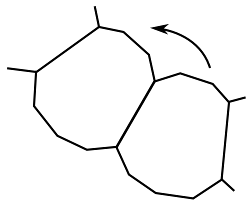
$$\begin{array}{lr}
\mathbb{R}_o(n_{ir} + \ell) + \mathbb{R}_o n_{ir+v-1} + \mathbb{R}_o n_{ir+v} & i \in \mathbb{Z}, \quad 0 \leq v < r \\
\mathbb{R}_o(n_{ir} + \ell) + \mathbb{R}_o n^*_{is+v} + \mathbb{R}_o n^*_{is+v+1} & i \in \mathbb{Z}, \quad 0 \leq v < s \\
\mathbb{R}_o(n_{ir} + \ell) + \mathbb{R}_o(n_{(i+1)r} + \ell) + \mathbb{R}_o n_{(i+1)r-1} & i \in \mathbb{Z} \\
\mathbb{R}_o(n_{ir} + \ell) + \mathbb{R}_o(n_{(i+1)r} + \ell) + \mathbb{R}_o n^*_{(i+1)s} & i \in \mathbb{Z}.
\end{array}$$

Seen from above the north pole, the decomposition induced by $\tilde{\Delta}$ on a sphere in $\tilde{N}_{\mathbb{R}}$ centered at 0 looks like the picture below.

Let B be ord^{-1} of the union in $Mc(\tilde{N}, \tilde{\Delta})$ of $(\mathbb{R}_{>0} \times \mathbb{R}) \times \mathbb{R}\ell$ and the boundary at infinity. Then B is U_N^+-invariant and the action is properly discontinuous without fixed point. By Theorem 4.2 (iii), we are done.



$\widetilde{\triangle}\cap$ (a sphere)



The "roof" at ∞ of $M_c(\widetilde{N},\widetilde{\triangle})$

□

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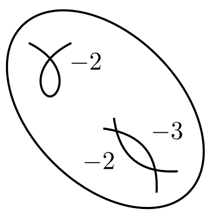
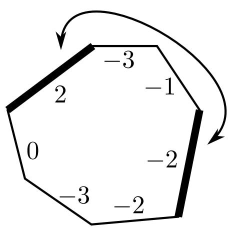
Examples. (1) If $\omega = 2+\sqrt{3} = [[\overline{4}]]$, then $\omega^* = 1+\sqrt{3}/3 = [[\overline{2,3}]]$. Hence X_N is as in the picture. We have

$$n_1 = -\omega+4, n_0 = 1, n_{-1} = \omega, n_{-2} = 4\omega-1$$
$$n_2^* = 2\omega-7, n_1^* = \omega-3, n_0^* = \omega-2, n_{-1}^* = \omega-1, n_{-2}^* = 2\omega-1.$$

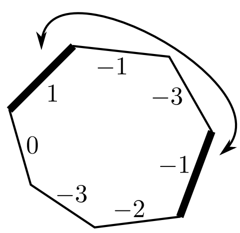
Since $r=1$, Y and Z of Prop. 15.7 are non-singular. Taking the U_N^+-translates of $\mathbb{R}_o(n_0+\ell)+\mathbb{R}_o n_o+\mathbb{R}_o n_1$ and $\mathbb{R}_o(n_0+\ell)+\mathbb{R}_o(n_1+\ell)+\mathbb{R}_o n_1$ instead of $\mathbb{R}_o(n_0+\ell)+\mathbb{R}_o(n_1+\ell)+\mathbb{R}_o n_0$ and $\mathbb{R}_o(n_0+\ell)+\mathbb{R}_o n_{-1}+\mathbb{R}_o n_0$, we get a different degeneration Z'



X_N



Z



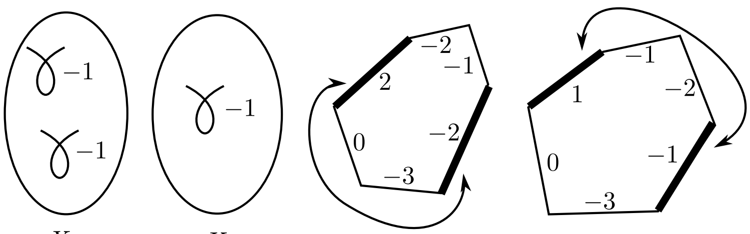
Z'

(2) Let $\omega = (3+\sqrt{5})/2 = [[\overline{3}]] = \omega^*$. Then U_N is generated by 168
$(-1+\sqrt{5})/2$. Kato showed that the only possible configurations of curves for a minimal surface with $b_2 = 1$ and with a global spherical shell are that of $\hat{X}_N$ here as well as those in Theorem 14.2 (i) and (ii).

In this case, we have

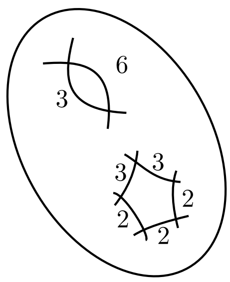
$$n_1 = 3-\omega, n_0 = 1, n_{-1} = \omega$$
$$n_1^* = 2\omega - 5, n_0^* = \omega - 2, n_{-1}^* = \omega - 1.$$

Again Y and Z of Prop. 15.7 are non-singular in this case. By a modification as in (1) above, we get a different Z'.



X_N $\quad$ X_N $\quad$ Z $\quad$ Z'

(3) If $\omega = (3+\sqrt{7})/2 = [[3,\overline{6}]]$, then $\omega^* = (5+\sqrt{7})/3 = [[3,3,\overline{2},2,2]]$.



X_N

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