Lectures on
Curves On Rational And Unirational Surfaces

By

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Preface

These notes were prepared for my lectures at the Tata Institute from January, 1978 through March, 1978. The sections 2, 5 and 6 of Chapter I, the sections 5 and 6 of Chapter II, and the section 3 of Chapter III could not be gone into details in the lectures. A. Sathaye and N. Mohankumar pointed out some mistakes in the original text and gave me comments for improvement.

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Introduction

1. An interesting but still open problem in algebraic geometry is the following: 1

ZARISKI'S PROBLEM. *If X is an affine algebraic variety over an algebraically closed field k such that $X \times \mathbf{A}^1_k \cong \mathbf{A}^3_k$, where $\mathbf{A}^n_k$ denotes the n-dimensional affine space over k, is X isomorphic to $\mathbf{A}^2_K$?*

In considering this problem it seems important and indispensable to have algebraic (or topological) characterizations of the affine plane $\mathbf{A}^2_k$ as an algebraic variety. Several attempts have been made toward this direction (cf. [45], [32]), though the obtained characterizations are not good enough to answer the Zariski's Problem. A main motivation in writing these notes is to put together the results which have been obtained so far surrounding this problem.

The said assumption $X \times \mathbf{A}^1_k \cong \mathbf{A}^3_k$ implies the following:

(1) *X is a nonsingular affine unirational surface,*

(2) *theaffine coordinate ring A of X is a unique factorization domain whose invertible elements are constants,* i.e., $A^* = k^*$, 2

(3) *there lie sufficiently many rational (not necessarily nonsingular) curves with only one place at infinity on X.*

In looking for a criterion for X to be isomorphic to $\mathbb{A}^2$ it will be reasonable to assume that X satisfies the above two conditions (1) and (2), though the third condition has to be made more precise (or improved). A precision of the third condition above is the next condition:

(3′) *X has a nontrivial action of the additive group scheme G_a.*

Then the conditions (1), (2) and (3′) are necessary and sufficient for X to be isomorphic to $\mathbb{A}^2$ (cf. Theorem 3.1, Chapter I). When G_a acts on an affine scheme $X = \mathrm{Spec}(A)$, the G_a-action can be interpreted in terms of a locally finite iterative higher derivation on A. Indeed, several problems concerning the G_a-action, e.g. to find the subring A_0 of invariants in A and to investigate the properties of A_0 and the canonical morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ induced by the injection $A_0 \hookrightarrow A$, become easier to treat by observing the locally finite iterative higher derivation on A associated with the G_a-action. The first two sections of Chapter I are devoted to the study of locally finite (iterative) higher derivations on k-algebras.

Instead of the condition (3′) one may consider the next milder condition

3 (3′′) *Xhas an algebraic family $\mathscr{F}$ of closed curves on X parametrized by a rational curve such that a general member of $\mathscr{F}$ is an affine rational curve with only one place at infinity and that two distinct general members of $\mathscr{F}$ have no intersection on X.*

If $\mathrm{char}(k) = 0$ and X satisfies the conditions (1) and (2), the conditions (3′) and (3′′) are equivalent to each other (cf. Theorem 2.3, Chapter I); indeed, a general member of $\mathscr{F}$ is isomorphic to $\mathbb{A}^1$. However, if either one of the conditions (1) and (2) is dropped the equivalence of (3′) and (3′′) no longer holds (cf. 2.4, Chapter I).

In connection with the condition (3′′) we are interested in an algebraic family $\mathscr{F}$ on a nonsingular affine surface, whose general members are isomorphic to $\mathbb{A}^1$. We have the following result (cf. Theorem 4.1.2, Chapter I):

Let S be a nonsingular variety over k and let $f : X \to S$ be a faithfully flat, affine morphism of finite type such that every fiber of f is irreducible. Assume that the general fiber of f are isomorphic to $\mathbf{A}^1$. Then there exist a nonsingular variety S' over k and a faithfully flat, finite, radical morphism $S' \to S$ such that $X \underset{S}{\times} S'$ is an $\mathbf{A}^1$-bundle over S'. Thus, if $\mathrm{char}(k) = 0$ *then X is an $\mathbf{A}^1$-bundle over S, and if* char$(k) > 0$ *the generic fiber* f *is a purely inseparable* $k(S)$*-form of* $\mathbb{A}^1$. (cf. 4.6, Chapter I).

Weare interested especially in the case where S is the projective 4
line $\mathbb{P}^1$ over k. Affine $\mathbb{A}^1$-bundles over $\mathbb{P}^1$ are classified (cf. Theorem 5.5.4, Chapter I), while the case where the generic fiber of f is a purely inseparable $k(\mathbb{P}^1)$-form of $\mathbb{A}^1$ will be studied more closely in Chapter III in connection with unirational (irrational) surfaces defined over k.

The Zariski's problem is generalized as follows:

CANCELLATION PROBLEM. *Let* A *and* B *be* k*-algebras such that* $A[x_1, \ldots, x_n]$ *is* k*-isomorphic to* $B[y_1, \ldots, y_n]$*, where* $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_n$ *are indeterminates. Is* A *then* k*-isomorphic to* B*?*

A k-algebra A is said to be strongly n-invariant if A satisfies the condition: If any k-algebra B and a k-isomorphism θ: $A[x_1, \ldots, x_n] \xrightarrow{\sim} B[y_1, \ldots, y_n]$ then $\theta(A) = B$. The property that A is strongly 1-invariant is closely related to the property that A is not birationally ruled over k (cf. . Lemma 6.2, Chapter I), and the strong 1-invariance of a k-algebra A is studied via locally finite (iterative) higher derivation on A (cf. Lemma 6.3, Proposition 6.6.2, etc., Chapter I).

2. The significance of studying a family of (nonsingular) rational curves with only one place at infinity on a nonsingular affine rational surface may be gathered from the foregoing discussions. Several important results have been obtained in this line (cf. Abhyankar-Moh [2], Moh [38] and Abhyankar-Singh [3]).

Let$k$ be an algebraically closed field of characteristic p. Let C_0 be an 5
irreducible curve with only one place at infinity on $\mathbb{A}^2 := \mathrm{Spec}(k[x, y])$ defined by $f(x, y) = 0$. Embed $\mathbb{A}^2$ into the projective plane $\mathbb{P}^2$ as the complement of a line ℓ_0. Let C be the closure of C_0 in $\mathbb{P}^2$, let $C \cdot \ell_0 = d_0 \cdot P_0$ and let $d_1 = \mathrm{mult}_{P_0} C$. Let C_α be the curve on $\mathbb{A}^2$ defined by $f(x, y) = \alpha$ for $\alpha \in k$ and let $\Lambda(f)$ be the linear pencil on $\mathbb{P}^2$ spanned by C and $d_0\ell_0$. Then the results are stated as follows:

(i) IRREDUCIBILITY THEOREM (Moh [38]; cf. Section 1, Chapter II).

Assume that $p \times d_0$ *or* $p \times d_1$*. Then the curve* C_α *is an irreducible curve with only one place at infinity for an arbitrary constant* α

of k.

(ii) EMBEDDING THEOREM (Abhyankar-Moh [2]; cf. Section 1, Chapter II). *Assume that $p \times d_0$ or $p \times d_1$, and that C_0 is nonsingular and rational. Then there exists a biregular automorphism of $\mathbb{A}^2$ which maps C_0 into the y-axis.*

(iii) FINITENESS THEOREM (Abhyankar-Singh [3]; cf. Section 4, Chapter II). *Assume that $p = 0$.* By an embedding of C_0 into $\mathbb{A}^2$ we mean a biregular mapping ϵ of C_0 into $\mathbb{A}^2$; two embeddings ϵ_1 and ϵ_2 of C_0 into $\mathbb{A}^2$ are said to be equivalent to each other if there exists a biregular automorphism ρ of $\mathbb{A}^2$ such that $\epsilon_2 = \rho \cdot \epsilon_1$. *Then there exist only finitely many equivalence classes of embeddings of C_0 into $\mathbb{A}^2$.*

In their proofs the main roles are played by the theory of approxi-
6 materoots of polynomials, i.e., the theory of generalized Tschirnhausen transformations. We shall present more geometric proofs of these theorems (though we could not prove the third theorem in full generality), which are based on the notions of admissible data and the Euclidean transformations (as well as the (e, i)-transformations) associated with admissible data. Roughly speaking, our idea of proof is explained as follows.

Let X be a nonsingular affine rational surface defined over k and let C_0 be an irreducible closed curve on X such that C_0 has only one place at infinity. Suppose that there exists an admissible datum $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) (cf. Definition 1.2.1, Chapter II). C is then linearly equivalent to $d_0(e\ell_0 + \Gamma)$ on V, and the linear pencil Λ on V spanned by C and $d_0(e\ell_0 + \Gamma)$ has base points centered at $P_0 := C \cap \ell_0$ and its infinitely near points. If $p \times (d_0, d_1)$ the Euclidean transformation or the (e, i)-transformation of V associated with $\mathscr{D}$ plays a role of producing a new admissible datum $\widetilde{\mathscr{D}} = \{\widetilde{V}, X, \widetilde{C}, \widetilde{\ell}_0, \widetilde{\Gamma}, \widetilde{d}_0, \widetilde{d}_1, \widetilde{e}\}$ for (X, C_0) such that either $\widetilde{d}_0 < d_0$ or $\widetilde{d}_0 = d_1$ and $\widetilde{d}_1 < d_1$ and that $p \times (\widetilde{d}_0, \widetilde{d}_1)$. After the Euclidean transformations or the (e, i)-transformations associated with admissible data repeated finitely many times we reach to an admissible datum $\widehat{\mathscr{D}} = \{\widehat{V}, X, \widehat{C}, \widehat{\ell}_0, \widehat{\Gamma}, \widehat{d}_0, \widehat{d}_1, \widehat{e}\}$ for (X, C_0) such that $\widehat{d}_0 = \widehat{d}_1 = 1$. Then, by

the $(\widehat{e},\widehat{e})$-transformation of $\widehat{V}$ associated with $\widehat{\mathscr{D}}$, we obtain a nonsingular projective surface V' such that the proper transform Λ' of Λ on Λ' is free frombase points, that if Δ' is the member of Λ' corresponding to **7**
$d_0(e\ell_0 + \Gamma)$ of Λ then Δ' is the unique irreducible member of Λ', that the fibration of V' defined by Λ' has a cross-section S and $V' - S \cup \mathrm{Supp}(\Delta')$ is isomorphic to X, and that if C' is the proper transform of C on V' then $C' - C' \cap S$ is isomorphic to the given curve C_0. Retaining the notations C_0, C, ℓ_0, d_0 and d_1 as before the statement of the irreducibility theorem, $\{\mathbb{P}^2_k, \mathbb{A}^2_k, C, \ell_0, \phi, d_0, d_1, 1\}$ is an admissible datum for $(\mathbb{A}^2_k, C_0)$. Hence by the foregoing arguments we know that the curve $C_\alpha : f(x,y) = \alpha$ is an irreducible curve with only one place at infinity for every $\alpha \in k$, and that if C_0 is isomorphic to $\mathbb{A}^1_k$ then C_α is isomorphic to $\mathbf{A}^1_k$ for every $\alpha \in k$. The theorems (i) and (ii) can be proved in this fashion. The foregoing process of eliminating the base points of $\Lambda(f)$ in conjunction with Artin-Winters's theorem [7] on degenerate fibers of a curve of genus g and the Kodaira vanishing theorem by Ramanujam [46] proves the weakened version of the finiteness theorem. Sections 1 and 4 of Chapter II are devoted to the proofs of these theorems.

Furthermore, we can give a new proof of the structure theorem on the automorphism group $\mathrm{Aut}_k\, k[x,y]$ over a field of arbitrary characteristic, which is based on the foregoing arguments of eliminating the base points of the pencil $\Lambda(f)$ and an easy lemma on reducible fibers of a fibration by rational curves (cf. Sections 2 and 3 of Chapter II).

In Sections 2 and 6 of Chapter II, some related topics are discussed.
Let C_0 be a nonsingular rational curve on $\mathbf{A}^2_k := \mathrm{Spec}(k[x,y])$defined **8**
by $f(x,y) = 0$; C_0 may have one or more places at infinity. Let C_α be the curve on $\mathbf{A}^2_k$ defined by $f(x,y) = \alpha$ for $\alpha \in k$, and let $\Lambda(f)$ be the linear pencil on $\mathbf{P}^2_k$ defined by the inclusion of fields $k(f) \hookrightarrow k(x,y)$, where $\mathbf{A}^2_k$ is embedded into $\mathbf{P}^2_k$ as the complement of a line ℓ_0. Then the generic member of $\Lambda(f)$ is a rational curve if and only if f is a field generator, i.e., $k(x,y) = k(f,g)$ for some $g \in k(x,y)$ (cf. Lemma 2.4.1, Chapter II), and if f is a field generator then C_0 has at most two points (including infinitely near points) on the line ℓ_0 at infinity (cf. Lemma 2.4.2, Chapter II). In Section 6 of Chapter II the following theorem is proved:

Assume that the characteristic of k is zero. With the above notations, $f = c(x^d y^e - 1)$ after a suitable change of coordinates x, y of $k[x, y]$, where $c \in k^*$ and d and e are positive integers with $(d, e) = 1$, if and only if the following conditions are met:

(a) *f is a field generator,*

(b) *C_α has exactly two places at infinity for almost all $\alpha \in k$,*

(c) *C_α is connected for every $a \in k$.*

In Section 5 of Chapter II, we shall study the structure of the affine coordinate ring $A := k[x, y, f/g]$ of a hypersurface on $\mathbf{A}^3_k$ of the type: $gz - f = 0$, where f, $g \in k[x, y]$ and $(f, g) = 1$. Namely, we shall
9 show that the divisor class group $C\ell(A)$ and the multicativegroup A^* are completely determined if Spec(A) has only isolated singularities, and that, in case of char(k) = 0, Spec(A) has nontrivial G_a-action if and only if $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$.

3. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a nonsingular projective surface, and let $f : X \to \mathbb{P}^1$ be a surjective morphism such that a general fiber of f is an irreducible rational curve with a single cusp as its singularity. Then the generic fiber $X_{\mathscr{R}}$ of f with the unique singular point deleted off is a purely inseparable form of $\mathbb{A}^1$ over the function field $\mathscr{R} := k(\mathbb{P}^1)$, and X is a unirational surface over k. In Chapter III, we shall describe the structure of such a surface X in the case where the arithmetic genus g of X is either 1 or 2, under the additional assumption that f has a rational cross-section. When $g = 1$ then p is either 2 or 3 and X is a unirational quasi-elliptic surface; there exist $K3$-surfaces and surfaces with canonical dimension $\kappa = 1$ besides rational surfaces (cf. Theorems 2.1.1 and 2.1.2, Chapter III). When $g =$ 2 then p is either 2 or 5; if $p = 5$ there exist $K3$-surfaces and surfaces of general type besides rational surfaces (cf. Section 3, Chapter III).

Notations and conventions 10

Notations and conventions of the present notes conform to the general current practice. Therefore we shall make some additional notes below.

1. Let A be an algebra over a field k. Then A^* denotes the multiplicative group of invertible elements of A; thus k^* denotes $k - (0)$. If A is an integral domain $Q(A)$ denotes the quotient field of A. A unique factorization domain A is sometimes called a factorial domain (or ring). If $A_{\mathscr{P}}$ is factorial for every prime ideal $\mathscr{P}$ of A then A is called locally factorial. For an affine k-variety X, the affine coordinate ring of X is denoted by $k[X]$ if there is no fear of confusing $k[X]$ and a polynomial ring over k.

2. The n-dimensional affine space and projective space defined over k are denoted respectively by $\mathbb{A}^n_k$ (or $\mathbb{A}^n$) and $\mathbb{P}^n_k$ (or $\mathbb{P}^n$). We denote by $\mathbb{A}^1_*$ the k-scheme isomorphic to the underlying k-scheme of the multiplicative k-group scheme $G_m := \mathrm{Spec}(k[t, t^{-1}])$. The additive k-group scheme is denoted by G_a (or $G_{a,k}$).

3. Let V be a nonsingular projective surface defined over an algebraically closed field. Then we use the following notations:

K_V**:** a canonical divisor (or the canonical divisor class) of V.

ω_V**:** the dualizing invertible sheaf on V, i.e., $\omega_V \cong \mathscr{O}_V(K_V)$. $\chi(\mathscr{O}_V)$ (or $\chi(V, \mathscr{O}_V)$): the Euler-Poincare characteristic of V, $\chi(\mathscr{O}_V) = \sum_{i=0}^{2}(-1)^i \dim H^i(V, \mathscr{O}_V)$. 11

$P_r(V)$**:** the r-genus of V for a positive integer r, i.e., $P_r(V) = \dim H^0(V, \omega_V^{\otimes r})$.

$P_g(V)$**:** the geometric genus of V.

$P_a(V)$**:** the arithmetic genus of V, i.e., $p_a(V) = \chi(\mathscr{O}_V) - 1$.

$\kappa(V)$ **(or κ):** the canonical dimension of V, i.e., $\kappa(V) = \operatorname{Sup}_{r>0} \dim \rho_r(V)$ where ρ_r is the r-th canonical mapping of V defined by $|rK_V|$.

Let D and D' be divisors on V. Then we use the notations:

$\mathscr{O}_V(D)$**:** the invertible sheaf attached to D.

$p_a(D)$**:** the arithmetic genus of D, i.e., $p_a(D) = \dfrac{1}{2}(D \cdot D + K_V) + 1$.

$(D \cdot D')$**:** the intersection multiplicity of D and D'.

(D^2)**:** the self-intersection multiplicity of D.

$D \sim D'$**:** D is linearly equivalent to D'.

$D > 0$**:** D is an effective divisor.

$|D|$**:** the complete linear system defined by D.

$|D| - \sum m_i p_i$**:** the linear subsystem of $|D|$ consisting of members of $|D|$ which pass through the points p_i's with multiplicities $\geqq m_i$, where m_i's are positive integers.

Let C be an irreducible curve on V and let P be a point on C. Then $\mathrm{mult}_P\, C$ denotes the multiplicity of C at P.

12 4. Let $f : W \to V$ be a birational morphism of nonsingular projective surfaces. If D is an effective divisor on V then $f^*(D)$ denotes the total transform (or the inverse image as a cycle) of D by f; $f'(D)$ denotes the proper transform of D by f. If C is an irreducible curve on V, $f^{-1}(C)$ denotes the set-theoretic inverse image of C by f. On the other hand, if D' is an effective divisor on W then $f_*(D')$ denotes the direct image of D' by f as a cycle. If Λ is a linear pencil on V consisting of effective divisors then $f'\Lambda$ denotes the proper transform of Λ; namely, if $f^*\Lambda$ is the linear pencil on W consisting of the total transforms f^*D of members D of Λ then $f'\Lambda$ is the pencil $f^*\Lambda$ with all fixed components deleted off. If Λ' is a linear pencil on W consisting of effective divisors then $f_*\Lambda'$ denotes the linear pencil on V consisting of the direct images f_*D' of members D' of Λ'.

5. If $f : W \to V$ is a finite morphism of nonsingular projective surfaces then the notations $f^*(C)$, $f^{-1}(C)$ and $f_*(C')$ conform to those in the case where f is a birational morphism.

6. Let Λ be an irreducible linear pencil of effective divisors on a nonsingular projective surface V. An irreducible curve S on V is called a quasi-section if S is not contained in any member of Λ and Λ has no base points on S. A quasi-section S of Λ is called a cross-section of Λ if $(S \cdot D) = 1$ for a general member D of Λ.

7. Asurface V defined over a field k is said to be unirational over k if **13**
there exists a dominating rational mapping $f : \mathbb{P}^2_k \to V$.

8. The present notes consist of three chapters. When we refer to a result stated in the same chapter we only quote the number of the paragraph (e.g. (cf. Theorem 1.1) or (cf. 1.1)); when we refer to a result stated in other chapters we quote it with the number of chapter (e.g. (cf. Theorem (I.1.1)) or (cf. I.1.1)).

Contents

Part I

Geometry of the affine line

1 Locally nilpotent derivations

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1.1

Throughout this section, x denotes a fixed field of characteristic p. Let A be a k-algebra. A *locally finite higher derivation* on A is a set of k-linear endomorphisms $D = \{D_0, D_1, \ldots\}$ of the k-vector space A satisfying the following conditions:

(1) $D_0 =$ identity; $D_i(ab) = \sum_{j+\ell=i} D_j(a)D_\ell(b)$ for any a, b of A.

(2) For any element a of A, there exists an integer $n > 0$ such that $D_m(a) = 0$ for every integer $m \geqq n$.

The higher derivation D is called *iterative* if D satisfies the additional condition:

(3) $D_iD_j = \binom{i+j}{i}D_{i+j}$ for all $i, j \geqq 0$.

If $D = \{D_0, D_1, \ldots\}$ is a locally finite higher derivation of A, then 15
D_1 is a k-trivial derivation on A. If D is iterative, it is an easy exercise to show that:

(3-1) If the characteristic p is zero, $D_i = \dfrac{1}{i!}(D_1)^i$ for every $i > 0$;

(3-2) If p is positive, $D_i = \dfrac{(D_1)^{i_0}(D_P)^{i_1}\dots(D_{p^r})^{i_r}}{(i_0)!(i_1)!\dots(i_r)!}$, where $i = i_0 + i_1 p + \cdots + i_r p^r$ is a p-adic expansion of i.

The fact (3-1) implies that if $p = 0$ a locally finite iterative higher derivation D is completely determined by D_1, which satisfies the condition that, for any element a of A, $D^n(a) = 0$ for sufficiently large n. Such a k-trivial derivation on A is called *locally nilpotent.*

1.2

Lemma. *Let A be a k-algebra. Then the following conditions are equivalent to each other:*

(1) *D is a locally finite higher derivation on A.*

(2) *The mapping $\varphi : A \to A[t]$ given by $\varphi(a) = \sum_{i\geq 0} D_i(a)t^i$ is a homomorphism of k-algebras, where t is an indeterminate. Similarly, the following conditions are equivalent to each other:*

(1′) *D is a locally finite iterative higher derivation on A.*

(2′) *$\varphi : A \to A[t]$ defined in the above condition* (2) *is a homomorphism of k-algebras such that $(\varphi \otimes \mathrm{id.})\varphi = (\mathrm{id.} \otimes \Delta)\varphi$,*
16 *where $\Delta : k[t] \to k[t] \otimes k[t]$ is a homomorphism of k-algebras defined by* $\Delta(t) = t \otimes 1 + 1 \otimes t$ (cf. the commutative diagram below);

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\varphi"] & A[t] = A\underset{k}{\otimes}k[t] \arrow[d, "\varphi\otimes\mathrm{id.}"] \\
A\underset{k}{\otimes}k[t] \arrow[r, "\mathrm{id.}\otimes\Delta"'] & A\underset{k}{\otimes}k[t]\underset{k}{\otimes}k[t]
\end{tikzcd}$$

(3′) *a_φ : $\mathrm{Spec}(A)\underset{k}{\times}G_{a,k} \to \mathrm{Spec}(A)$ is an action of the additive k-group scheme $G_{a,k}$ on* $\mathrm{Spec}(A)$.

Proof. The equivalence of the conditions (1) and (2) is a reformulation of the definition. The equivalence of the conditions (1′), (2′) and (3′) follows easily from an equality:

$$\sum_{i,j\geq 0} D_i D_j(a) \otimes t^i \otimes t^j = \sum_{\ell\geq 0} D_\ell(a)(t\otimes 1 + 1\otimes t)^\ell .$$

□

1.3

Let $D = \{D_0, D_1, \ldots\}$ be a locally finite higher derivation on a k-algebra A. An element a of A is called *a D-constant* if $D_n(a) = 0$ for every $n > 0$, or synonymously if $\varphi(a) = a$. The set A_0 of all D-constants is clearly a k-subalgebra of A.

1.3.1

Lemma. *Let A, D and A_0 be as above. Assume that A is an integral domain. Then the following assertions hold:*

(1) *A_0 is an inert subring of A. Namely, if $a = bc$ with $a \in A_0$ and b, $c \in A$ then b, $c \in A_0$. Therefore, if A is a unique factorization* **17**
domain and A_0 is noetherian, A_0 is a unique factorization domain.

(2) A^* (:= the multiplicative group of invertible elements in A) *is contained in A_0; hence $A^* = A_0^*$.*

(3) *A_0 is integrally closed in A.*

Proof. (1) Assume that $a = bc$ with $a \in A_0$ and b, $c \in A$, then $a = \varphi(b)\varphi(c)$, whence $\deg_t \varphi(b) = \deg_t \varphi(c) = 0$. This shows that b, $c \in A_0$.

(2) Let $a \in A^*$ and let b be its inverse. Then $\varphi(a)\varphi(b) = 1$ whence $\deg_t \varphi(a) = \deg_t \varphi(b) = 0$. Hence $a \in A_0$.

(3) Assume that an element a of A satisfies a monic equation,

$$X^n + c_1 X^{n-1} + \cdots + c_n = 0 \quad \text{with} \quad c_1 \ldots, c_n \in A_0.$$

Then, by applying φ, one gets

$$\varphi(a)^n + c_1\varphi(a)^{n-1} + \cdots + c_n = 0,$$

whence follows that $\deg_t \varphi(a) = 0$. Hence $a \in A_0$.

□

1.3.2

Assume that A is an integral domain, and let K be the quotient field of A. The k-algebra homomorphism, $\varphi : A \to A[t]$ associated with a locally finite higher derivation D is naturally extended to a homomorphism $\phi : K \to K[[t]]$ by setting

$$\phi\left(\frac{a}{b}\right) = \frac{\varphi(b)}{\varphi(a)} \quad \text{for} \quad a, b \in A \quad \text{with} \quad a \neq 0.$$

18 The homomorphism ϕ defines, in turn, a k-trivial higher derivation $\overline{D} = \{\overline{D}_0 = \mathrm{id}\,., \overline{D}_1, \ldots\}$ on K such that $\phi(\lambda) = \sum\limits_{i \geq 0} \overline{D}_i(\lambda)t^i$ for $\lambda \in K$ and that $\overline{D}_i|_A = D_i$ for every $i \geq 0$. We set $K_0 := \{\lambda \in K; D_i(\lambda) = 0$ for every $i > 0\}$. Then K_0 is a sub field of K, and for $\lambda \in K$, $\lambda \in K_0$ if and only if $\phi(\lambda) = \lambda$. We have the following:

Lemma. *With the notations as above, the following assertions hold:*

(1) *K_0 is algebraically closed in K.*

(2) *$K_0 \cap A = A_0$; if D is iterative K_0 is the quotient field of A_0.*

Proof. (1) Assume that $\lambda \in K$ satisfies an algebraic equation.

$$X^n + \mu_1 X^{n-1} + \cdots + \mu_n = 0 \quad \text{with} \quad \mu_1, \ldots, \mu_n \in K_0.$$

Then, by applying ϕ, one obtains

$$\phi(\lambda)^n + \mu_1\phi(\lambda)^{n-1} + \cdots + \mu_n = 0.$$

Note that if $\phi(\lambda) \neq \lambda$ then $\phi(\lambda)$ is analytically independent over K; hence $\phi(\lambda)$ does not satisfy a nontrivial algebraic equation over K_0. Thus $\phi(\lambda) = \lambda$, i.e., $\lambda \in K_0$.

(2) The equality $K_0 \cap A = A_0$ is clear because $\phi(a) = \varphi(a)$ for $a \in A$.
19 Assume now that D is iterative. We have only to show that any $\lambda \in K_0$ is written as $\lambda = b_0/a_0$. With $a_0, b_0 \in A_0$. Write $\lambda = b/a$ with $a, b \in A$ and $a \neq 0$. Let

$$\varphi(a) = a + a_1 t + \cdots + a_m t^m \quad \text{with} \quad a_m \neq 0$$

and

$$\varphi(b) = b + b_1 t + \cdots + b_n t^n \quad \text{with} \quad b_n \neq 0.$$

Since $D_i(a_m) = D_i D_m(a) = \binom{i+m}{i} D_{i+m}(a) = 0$ for $i > 0$, we know that $a_m \in A_0$. Similarly, $b_n \in A_0$. Since $\phi(\lambda) = \lambda$ implies that $a\varphi(b) = b\varphi(a)$ we have: $n = m$ and $ab_n = ba_m$. Hence $b/a = b_n/a_m$.

□

1.3.3

If D is not iterative K_0 is not necessarily the quotient field of A_0, as is shown by

Example. Let $A := k[x, y]$ be a polynomial ring in two variables over k. Define a k-algebra homomorphism

$$\varphi : A \to A[t] \quad \text{by} \quad \varphi(x) = x + xt \quad \text{and} \quad \varphi(y) = y + yt$$

which defines a locally finite derivation D on A. With respect to this higher derivation, $A_0 = k$, while, after extending φ to a k-algebra homomorphism $\phi : k(x, y) \to k(x, y)[[t]]$, we have $\phi(y/x) = y(1 + t)/x(1 + t) = y/x$. Thus, $K_0 \neq$ the quotient field of A_0.

1.4

We prove the following:

Lemma. *Assume that A is an integral domain and that D is iterative. If there exists an element u of A such that $D_1(u) = 1$ and $D_i(u) = 0$ for all $i > 1$, then $A = A_0[u]$ and u is algebraically independent over A_0.*

Proof. For any element a of A we set $\ell(a) := \deg_t \varphi(a)$ and call it *the D-* 20
length of a. By induction on the D-length $\ell(a)$ we show that $a \in A_0[u]$. If $\ell(a) = 0$ then $a \in A_0$. Assume that $n : \ell(a) > 0$. Let $a_n := D_n(a)$. Then, as was noted in the proof of Lemma 1.3.2, $a_n \in A_0$. Since $\ell(a - a_n u^n) < n$ we know that $a - a_n u^n \in A_0[u]$, hence that $a \in A_0[u]$. Therefore we know that $A = A_0[u]$. By virtue of Lemma 1.3.2, it is clear that u is algebraically independent over A_0. □

1.5

In studying an integral domain A endowed with a locally finite iterative higher derivation, a key result is the following:

Lemma. *Assume that A is an integral domain and that D is iterative. If D is nontrivial* (i.e., $A \neq A_0$) *then there exist an element $c \neq 0$ of A_0 and an element u of A such that $A[c^{-1}] = A_0[c^{-1}][u]$, where u is algebraically independent over A_0. Conversely, if A is finitely generated over a subring A_0 the existence of elements c and u satisfying the above conditions implies that A has a locally finite iterative higher derivation.*

1.5.1

The proof of the above lemma is given in the paragraphs 1.5.1 ∼ 1.5.4. Let $A_i := \{a \in A; D_n(a) = 0 \text{ for all } n > i\}$. Then A_i is an A_0-module, and we have $A = \bigcup_{i \geq 0} A_i$. An integer n is called *a jump index* if $A_{n-1} \subsetneqq A_n$. If 1 is a jump index, let u be an element of $A_1 - A_0$ and let $c = D_1(u)$.
21 Then $c \in A_0$. The higher derivation D can be extended naturally to a locally finite iterative higher derivation on $A[c^{-1}]$ by setting $D_i(a/c^r) = D_i(a)/c^r$, with respect to which the ring of D-constants is $A_0[c^{-1}]$. Since $D_1(u/c) = 1$ and $D_i(u/c) = 0$ for all $i > 1$ we have by virtue of 1.4 that $A[c^{-1}] = A_0[c^{-1}][u]$. If the characteristic p is zero, let a be an element of A such that $s := \ell(a) > 0$ and let $u := D_{s-1}(a)$. Then $u \in A_1 - A_0$. Hence 1 is a jump index, and we have $A[c^{-1}] = A_0[c^{-1}][u]$ with $c = sD_s(a)$. Thus we may assume in the rest of the proof that the characteristic p is positive and that the first jump index is larger than 1.

1.5.2

Lemma. *With the notations and assumptions as above we have the following:*

(1) *The first jump index n is a power of p, say $n = p^r$.*

(2) *The m-th jump index is $mp^r (m = 1, 2, \ldots)$.*

Proof. (1) Let n be the first jump index, and let

$$n = n_0 + n_1 p + \cdots + n_r p^r \quad \text{with} \quad n_r \neq 0$$

be the p-adic expansion of n. Assume that n is not a power of p. Then we have: Either (i) $n_0 \geq 1$ or (ii) $n_0 = 0$ and $n_1 + \cdots + n_r \geq 2$. In case (i), $n \not\equiv 0 \pmod{p}$. Let a be an element of $A_n - A_{n-1}$, and let $a' = D_{n-1}(a)$. Then $a' \in A_1 - A_0$ because $D_1(a') = D_1 D_{n-1}(a) = nD_n(a) \neq 0$ and $D_i(a') = \binom{n+i-1}{i} D_{n+i-1}(a) = 0$ for $i > 1$. This contradicts the assumption that $n > 1$. In case (ii), let 22
a be an element of $A_n - A_{n-1}$, and let $a' = D_{p^r}(a)$. Since $D_i(a') = D_i D_{p^r}(a) = \binom{p^r+i}{i} D_{p^r+i}(a) = 0$ for $i > n - p^r$ and $n - p^r < n - 1$, we know that $a' \in A_{n-p^r} = A_0$. On the other hand, $D_{n-p^r}(a') = D_{n-p^r} D_{p^r}(a) = n_r D_n(a) \neq 0$, which implies that $a' \notin A_0$ because $n - p^r \geq 1$. This is a contradiction. Thus $n = p^r$.

(2) Let $u \in A_{p^r} - A_0$. For any integer $m \geq 1$, $u^m \in A_{mp^r} - A_{mp^t-1}$ because $\varphi(u^m) = \varphi(u)^m$ and $D_{p^r}(u)$ is the leading coefficient of a polynomial $\varphi(u)$ in t. Hence mp^r is a jump index for $m = 1, 2, \ldots$. Let q be the least jump index which is not a multiple of p^r, and let

$$dp^r < q < (d+1)p^r \quad \text{with} \quad d \geq 1.$$

Let $a \in A_q - A_{q-1}$ and let $a' = D_{dp^r}(a)$. Let $q_0 := q - dp^r < p^r$. Then $D_{q_0}(a') = D_{q_0} D_{dp^r}(a) = D_q(a) \neq 0$, which implies that $A_0 \subsetneqq A_{q_0}$. because $a' \in A_{q_0} - A_0$. This is a contradiction. Therefore, every jump index is of the form $mp^r (m = 1, 2, \ldots)$.

□

1.5.3

Proof of Lemma 1.5. *Let $u \in A_{p^r} - A_0$. First, we assert that if $D_m(u) \neq 0$ for $0 < m \leqq p^r$ then m is a power of p and $D_m(u) \in A_0$. Indeed, assume that $D_m(u) \neq 0$ for $0 < m \leqq p^r$, and let*

$$m = m_0 + m_1 p + \cdots + m_s p^s (m_s \neq 0)$$

23 *be the p-adic expansion of m. If either $m_s \geqq 2$ or $m_i \neq 0$ for $i < s$, let $a = D_{p^s}(u)$. Then $D_{m-p^s}(a) = D_{m-p^s} D_{p^s}(u) = m_s D_m(u) \neq 0$ and $D_i(a) = 0$ if $i > p^r - p^s$; hence $a \in A_{p^r - p^s} - A_0$. This is a contradiction. Thus, m is a power of p. On the other hand, $D_m(u) \in A_{p^r - m} = A_0$ since $m > 0$. The first assertion is now verified. Let c be the product of all $D_m(u) \neq 0$ for $0 < m \leqq p^r$. Since $c \in A_0$, we can extend D uniquely to $A[c^{-1}]$. Now, we assert that $A[c^{-1}] = A_0[c^{-1}][u]$. For this, we have only to show that every element a of A is contained in $A_0[c^{-1}][u]$. For $a \in A$, there exists an integer m such that $a \in A_{mp^r}$. Let $a_1 := a - D_{mp^r}(a) D_{p^r}(u) - m_u m$. Then $D_{mp^r}(a_1) = 0$, whence $a_1 \in A_{(m-1)p^r}$. By induction on m we know that $a \in A_0[c^{-1}][u]$. by virtue of Lemma 1.3.2, it is clear that u is algebraically independent over A_0.*

1.5.4

Proof continued. Conversely, assume that $A[c^{-1}] = A_0[c^{-1}][u]$ for a subring A_0 and elements $c \in A_0$ and $u \in A$, where u is algebraically independent over A_0 and A is finitely generated over A_0. Define a locally finite iterative higher derivation Δ on $A[c^{-1}]$ by a homomorphism of $A_0[c^{-1}]$-algebras $\varphi' : A[c^{-1}] \to A[c^{-1}][t']$ (t' being a variable) such that $\varphi'(u) = u + t'$. Since A is finitely generated φ' induces a homomorphism of A_0-algebras $\varphi : A \to A[t]$ with $t' = C^N t$ for a sufficiently large integer
24 N. Then it is easy to see that φ defines a locally finite iterative higher derivation D on A such that A_0 is the set of D-constants in A.

1.6

In this paragraph, we assume that A is an integral domain and A is finitely generated over k. Let D be a locally finite higher derivation.

As in 1.3, we denote by A_0, K and K_0 the subring of D-constants, the quotient field of A and the sub field of D-constants of K, respectively. Concerning a problem whether A_0 is finitely generated over k, we have only a partial result due to Zariski (cf. Nagata [41; p.52]), which is stated as follows:

Lemma. *With the notations and assumptions as above, we have:*

(1) *If* $\text{trans.deg}_k K_0 = 1$, *then* A_0 *is finitely generated over* k.

(2) *If* A *is normal and* $\text{trans.deg}_k K_0 = 2$, *then* A_0 *is finitely generated over* k.

2 Algebraic pencils of affine lines

25

In this section the ground field k is assumed to be algebraically closed.

2.1

Let A be an affine k-domain (i.e., a k-algebra which is finitely generated over k and is an integral domain), and let D be a locally finite higher derivation on A. Let A_0 be the subring of D-constants, and let K and K_0 be respectively the quotient field of A and the sub field of D-constants of K. Let $X := \text{Spec}(A)$, and let $f : X \times \mathbb{A}^1 \to X$ be the k-morphism associated with $\varphi : A \to A[t]$ (cf. 1.2). For any point P of X, denote by $C(P)$ the image $f(P \times \mathbb{A}^1)$ on X. Then $C(P)$ is either a point or a closed irreducible rational curve with one place at infinity[1]. If $A \neq A_0$ then the set $\mathscr{F} := \{C(P); P \in X, C(P) \neq \text{a point}\}$ is a family of irreducible rational curves with one place at infinity. If D is iterative f is the morphism giving rise to an action of the additive group scheme G_a (cf. 1.2) and $\mathscr{F}$ is the set of G_a-orbits; $\mathscr{F}$ contains a subset $\mathscr{F}'$ whose members are parametrized by $\text{Spec}(A_0[c^{-1}])$ (cf. 1.5). In this section, we shall study the set $\mathscr{F}$ more closely when $\dim A = 2$.

[1]An irreducible curve C on an affine variety is said to have one place at infinity if C has only one place having no center on X.

2.2

Let $Y := \mathrm{Spec}(A_0)$, and let $q : X \to Y$ be the morphism associated
26 with the inclusion $A_0 \hookrightarrow A$. Then we have: $q \cdot f = q \cdot \mathrm{pr}_1$, where $\mathrm{pr}_1 : X \times \mathbb{A}^1 \to X$ is the projection to the first factor. Hence $q(C(P)) = q(P)$ for $P \in X$; namely $C(P)$ is contained in a fiber of q. Moreover, if $K_0 = Q(A_0)$ (= the quotient field of A_0), then the general fibers of q are irreducible.

2.2.1

Lemma. *Assume that k is of characteristic zero, D is nontrivial,* $\dim A = 2$ *and A is normal. Assume, moreover, that* $K_0 = Q(A_0)$ *and* $\mathrm{trans.deg}_k\, K_0 = 1$*. Then there exist elements* $c \in A_0$ *and* $u \in A$ *such that* $A[c^{-1}] = A_0[c^{-1}][u]$*, where u is algebraically independent over* A_0*.*

Proof. Note that $Y := \mathrm{Spec}(A_0)$ is a nonsingular curve since A is normal (cf. Lemma 1.3.1, (3)). Embed X into a projective surface V as an open set; we may assume that V is nonsingular at every point of $V - X$ by desingularizing singularities of $V - X$ if necessary. Since K_0 is algebraically closed in K and $\mathrm{trans.deg}_k\, K_0 = 1$, K_0 defines an irreducible pencil Λ on V such that if C is a general member of Λ then $C \cap X$ is a fiber of q. Hence Λ has no base points on X. We may assume that Λ has no base points[2]; if necessary we can eliminate the base points of Λ by a succession of quadratic transformations with centers at points on $V - X$ and their suitable infinitely near points. Let $\widetilde{q} : V \to \widetilde{Y}$ be the morphism defined by Λ; then $\widetilde{d}|_X = q : X \to Y$. Let $Z := \widetilde{q}^{-1}(Y)$ and
27 let $\pi := \widetilde{q}|_Z$. Let S be an irreducible component of $Z - X$ such that S intersects the general fibers of $\widetilde{q}$; then S is a cross-section of π because a general fiber of q has only one place at infinity and k is of characteristic zero[1]. Moreover by Bertini's theorem the general fibers of π are nonsingular rational curves. Then, by virtue of Hironaka [22; Theorem 1.8], there exists a nonempty open set U of Y such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$.

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[2] Since V is nonsingular at every point of $V - X$, Λ is a linear pencil if Λ has base points.

[1] Indeed, $\pi|_S : S \to Y$ is a generically one-to-one mapping. Hence it is birational.

Hence, $q^{-1}(U) = \pi^{-1}(U) - S \cap \pi^{-1}(U) \cong U \times \mathbb{A}^1$. This shows that our assertion holds. □

2.2.2

Lemma 2.2.1 shows that the higher derivation D is determined uniquely on $A[c^{-1}]$ by those values $D_i(u) = g_i(u)(i > 0)$ or by $\varphi(u) = u + \sum_{i>0} g_i(u)t^i$, where $g_i(u) \in A_0[c^{-1}][u]$. Then, D is iterative as higher derivation on A if and only if $g_1(u) \in (A_0[c^{-1}])^*$ and $g_i(u) = 0$ for $i > 1$. Conversely, assume that $A[c^{-1}] = A_0[c^{-1}][u]$ for a subring A_0 and elements $c \in A_0$ and $u \in A$, where u is algebraically independent over A_0 and A is finitely generated over A_0. For any $g_i(u) \in A_0[c^{-1}][u](1 \leq i \leq n)$, not all of which are zero, we can define a locally finite higher derivation D' on $A[c^{-1}]$ by a homomorphism of $A_0[c^{-1}]$-algebras $\varphi' : A[c^{-1}] \to A[c^{-1}][t']$ (t' being a variable) given by $\varphi'(u) = u + \sum_{i=1}^{n} g_i(u)t'^i$. Since A is finitely generated over A_0 we may find an integer $N > 0$ such that **28**
the homomorphism $\varphi' : A[c^{-1}] \to A[c^{-1}][t'] \hookrightarrow A[c^{-1}][t]$ with $t' = c^N t$ gives rise to a homomorphism of A_0-algebras $\varphi : A \to A[t]$. Then φ defines a locally finite higher derivation D on A such that A_0 is the subring of D-constants in A, $Q(A_0)$ is the sub field of D-constants in $K := Q(A)$ and $\text{trans.deg}_k K_0 = 1$.

2.2.3

Note that if the curves in $\mathscr{F}$ have a point in common we have $A_0 = k$. Indeed, if $\text{trans.deg}_k Q(A_0) \geqq 1$ two curves in $\mathscr{F}$ belonging to distinct fibers of $q : X \to Y$ have no points in common. Hence, $\text{trans.deg}_k Q(A_0) = 0$, which implies that $A_0 = k$. An example of a locally finite higher derivation D, in which the curves in $\mathscr{F}$ have a point in common, is given by the following:

Example. Let A be the affine ring of the affine cone of an irreducible projective variety U. Write $A = k[Z_0, \ldots, Z_n]/(F_1, \ldots, F_m]$, where $F_1, \ldots, F_m$ are homogeneous polynomials in $k[Z_0, \ldots, Z_n]$. Define a higher derivation D' on $k[Z_0, \ldots, Z_n]$ by $D'_0 =$ i.d., $D'_1(Z_i) = Z_i$ and $D'_j(Z_i) = 0$

for $0 \leqq i \leqq n$ and $j \geqq 2$. Then D' induces a nontrivial locally finite higher derivation D on A; the set $\mathscr{F}$ consists of lines in $\mathbb{A}^{n+1}$ connecting the point $(0, \ldots, 0)$ and points of U; $A_0 = k$ and K_0 (= the subgield of D-constants in $K := Q(A)) \cong k(U)$.

2.3

An interesting problem is to ask the following: Let X be an affine surface
29 defined over k and let $\mathscr{F}$ be an algebraic family of the affine lines on X; when are all (or almost all) members of $\mathscr{F}$ of the form $C(P)$ with $P \in X$ for a locally finite (or locally finite iterative) higher derivation on the affine ring of X? A partial answer to this problem is given by the following:

Theorem. *Let A be a regular, rational, affine k-domain of dimension* 2 *and let X be the affine surface defined by A. Assume that k is of characteristic zero, that A is a unique factorization domain and that $A^* = k^*$. Let $\mathscr{F}$ be an algebraic family of closed curves on X parametrized by a rational curve such that a general member of $\mathscr{F}$ is an affine rational curve with only one place at infinity and that two distinct general members of $\mathscr{F}$ have no intersection on X. Then there exists a locally finite iterative higher derivation D on A such that almost all members of $\mathscr{F}$ are the G_a-orbits with respect to the associated G_a-action on X.*

2.3.1

The proof of the above theorem is given in the paragraphs 2.3.1 ∼ 2.3.3. Let us embed X into a nonsingular projective surface V as an open set; note that $V - X$ is of pure co-dimension 1 in V. We have then:

Lemma. *Let A, X and V be as above. If $V - X$ is irreducible then V is isomorphic to the projective plane $\mathbb{P}^2$ and $V - X$ is isomorphic to a line.*

30 *Proof.* Let V_0 be a relatively minimal rational surface dominated by V; V_0 is isomorphic to $\mathbb{P}^2$ or $F_n (n \geqq 0, n \neq 1)$; V is obtained from V_0 by a succession of quadratic transformations $V = V_r \to \ldots \to V_0$. Then $\mathrm{Pic}(V)$ is isomorphic to a free $\mathbb{Z}$-module of rank $r + 1$ or $r + 2$ according

as $V_0 \cong \mathbb{P}^2$ or $V_0 \cong F_n$. The assumption that $\mathrm{Pic}(X) = (0)$ and $V - X$ is irreducible implies that $V = V_0 \cong \mathbb{P}^2$ and $V - X$ is a line. □

2.3.2

Lemma. *Let A, X and V be as above. Then there exists an irreducible linear pencil Λ on V such that for a general member C of Λ, $C \cap X$ is a member of $\mathscr{F}$.*

Proof. Let T be a rational curve and let W be a sub variety of $X \times T$ such that if we denote by p_1 and p_2 the projections of W onto X and T respectively, then for any point t of T, $p_1 * (p_2^{-1}(t))$ is a member of $\mathscr{F}$. Since two distinct members of $\mathscr{F}$ have no intersection on X, it is easy to ascertain that $p_1 : W \to X$ is a birational morphism and general fibers of $p_2 : W \to T$ are irreducible. In other words, if we identify $k(X)$ with $k(W)$ by p_1 and $k(T)$ as a sub field of $k(W)$ by p_2, $k(T)$ is algebraically closed in $k(X)$. Hence $k(T)$ defines an irreducible linear pencil Λ on V such that for a general point t of T, the member C_t of Λ corresponding to t cuts out a member $C_t \cap X$ of $\mathscr{F}$ on X. □

2.3.3

Proof of the theorem. *By the second theorem of Bertini, a general member C of the pencil constructed in 2.3.2 has no singular points out-* 31
side base points of Λ. Since Λ has no base points on X and $C \cap X$ is a rational curve with only one place at infinity, $C \cap X$ is isomorphic to $\mathbb{A}^1$ and Λ has at most one base point which will lie on $V - X$ if it exists. Then by replacing V by the surface which is obtained from V by a succession of quadratic transformations with centers at base points (including the infinitely near base points) of Λ and replacing Λ by its proper transform, we may assume that Λ has no base points. Let $f : V \to \mathbb{P}^1$ be the morphism defined by Λ; f has a cross-section S such that $S \subset V - X$ (cf. the proof of Lemma 2.2.1). Since the general fibers of f are isomorphic to $\mathbb{P}^1$, by virtue of [22; Theorem 1.8], there exists an affine open set $U(\neq \phi)$ of $\mathbb{P}^1$ such that $f^{-1}(U) \cong U \times \mathbb{P}^1$. Then $f^{-1}(U) \cap X = f^{-1}(U) - S \cap f^{-1}(U) \cong U \times \mathbb{A}^1$.

The complement $X - f^{-1}(U) \cap X$ consists of a finitely many (mutually distinct) irreducible curves $G_1, \ldots, G_r$ which are defined by prime elements $a_1, \ldots, a_r$ of A, respectively. Then $k[f^{-1}(U) \cap X] = A[a^{-1}]$ with $a = a_1 \ldots a_r$. Let $B := k[U]$; B is a subring of $A[a^{-1}]$ such that $A[a^{-1}] = B[u]$ for some element u of A which is algebraically independent over B. Write B in the form $B = k[v, g(v)^{-1}]$ with $v \in B$ and $g(v) = \prod_{l \leqq i \leqq s} (v - \alpha_i)$ ($\alpha_1, \ldots, a_s$ being mutually distinct elements of k). Since $A^* = k^*$ and A is a unique factorization domain, $(A[a^{-1}])^*/k^*$ is a free $\mathbb{Z}$-module of rank r generated by $a_1, \ldots, a_r$. On the other hand,
32 since $A[a^{-1}] = B[u]$ we have $(A[a^{-1}])^* = B^*$. Hence we have $r = s$.

We shall show that $f(X)$ is an affine open set of $\mathbb{P}^1$. Assume the contrary: $f(X) = \mathbb{P}^1$. Here we note that $V - X$ is not irreducible. Indeed, if so, $V \cong \mathbb{P}^2$ and $V - X$ is a line by virtue of 2.3.1; then two distinct general fibers of f have to meet at a point on $V - X$ which is a contradiction. The irreducible components of $V - X$ other than the section S correspond to a finite number of points $Q_1, \ldots, Q_m$ of $\mathbb{P}^1$ by f, i.e., $f(V - X \cup S) = \{Q_1, \ldots, Q_m\}$. Then the assumption $f(X) = \mathbb{P}^1$ implies that for every $1 \leqq i \leqq m$, $f^{-1}(Q_i) \cap X$ is not empty and consists of a finite number of irreducible curves which belong to $\{G_1, \ldots, G_r\}$. We may assume that $\bigcup_{1 \leqq i \leqq m} (f^{-1}(Q_i) \cap X) = G_1 \cup \ldots \cup G_r$, with $r' \leqq r$. Let $f(G_{r'+1} \cup \ldots \cup G_r) = \{Q_{m+1}, \ldots, Q_s\}$. Then $s' = s + 1$ since U is obtained from $\mathbb{P}^1$ by deleting the points $v = \alpha_1, \ldots, v = \alpha_s$ and the points at infinity $v = \infty$; we have $s' \leqq r$ since all irreducible curves of $X - f^{-1}(U) \cap X$ are sent to the points $Q_1, \ldots, Q_s$, by f. However, this is absurd since $r = s$. Therefore $f(X)$ is an affine open set of $\mathbb{P}^1$.

Let $A_0 := k[f(X)]$. Then A_0 is a subring of A, and there exists an element a_0 of A_0 such that $U = \mathrm{Spec}(A_0[a_0^{-1}])$, $f^{-1}(U) \cap X = \mathrm{Spec}(A[a_0^{-1}])$ and $A[a_0^{-1}] = A_0[a_0^{-1}][u]$. Now define a locally finite iterative higher derivation $D = \{D_0 = id., D_1, \ldots\}$ by setting $D_i = (1/i!)D_1^i$, $D_1(b) = 0$ for any element b of A_0 and $D_1(u) = a_0^N$ for a sufficiently large integer N (cf. 1.5.4). With respect to the G_a-action on X associated with D, almost
33 all members of $\mathscr{F}$ are the G_a-orbits.

2.3.4

The assumptions on A in the statements of the theorem imply that X is isomorphic to the affine plane (cf. 3.1 below).

2.4

Let X be a nonsingular affine surface defined over k, and let $\mathscr{F}$ be an algebraic family of closed curves parametrized by a curve T such that a general member of $\mathscr{F}$ is an affine rational curve with only one place at infinity and that two distinct general members of $\mathscr{F}$ have no intersection on X. The proof of Lemma 2.3.2 slightly modified shows that there exists a nonsingular projective surface V containing X as an open set and an algebraic pencil Λ on V (whose members are parametrized by the complete normal model of T) such that almost all members of $\mathscr{F}$ are cut out on X by members of Λ; in fact, a general member of $\mathscr{F}$ is isomorphic to $\mathbb{A}^1$. Thus we may speak of $\mathscr{F}$ as *an algebraic pencil of affine lines on X parametrized by T*. Given an algebraic pencil $\mathscr{F}$ of affine lines on an affine surface, it is not necessarily true that almost all members of $\mathscr{F}$ are G_a-orbits with respect to an action of G_a on X. To construct such examples we need the following two lemmas.

2.4.1

Lemma. *Let C be a nonsingular projective curve of genus g. Let L be an ample line bundle on C and let E be a nontrivial extension of L by $\mathscr{O}_C$ (if it exists at all). Let S be the section of the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ corresponding to L and let $X = \mathbb{P}(E) - S$. Assume that the characteristic of k is zero* **34**
or $\deg L > 2g$. Then X is an affine surface such that the restriction onto X of the projection $\mathbb{P}(E) \to C$ is a surjective morphism onto C. Conversely, if an affine surface X is a $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ over C deleted a section then X is isomorphic to an affine surface constructed in the above-mentioned way.

Proof. Let L be an ample line bundle on C and let E be a nontrivial extension of L by $\mathscr{O}_C$. Assume that the characteristic of k is zero or $\deg L > 2g$. Then it is known by Giesecker [15] that E is an ample vector bundle on C and the tautological line bundle $\mathscr{O}_{\mathbb{P}(E)}(1)$ is isomorphic to $\mathscr{O}_{\mathbb{P}E}(S)$. Therefore S is an ample divisor on a nonsingular projective surface $\mathbb{P}(E)$ and $X = \mathbb{P}(E) - S$ is affine. It is clear that the restriction onto X of the projection $\mathbb{P}(E) \to C$ is a surjective morphism onto C.

Conversely, let E be a vector bundle on C of rank 2, let $\mathbb{P}(E)$ be the $\mathbb{P}^1$-bundle associated with E and let X be the $\mathbb{P}(E)$ deleted a section S. Let L' be the quotient line bundle of E corresponding to the section S and let L be the kernel of $E \to L'$. We shall show that $L' \otimes L^{-1}$ is ample and that X is isomorphic to $\mathbb{P}(E \otimes L^{-1})$ deleted the section S' corresponding to $L' \otimes L^{-1}$. Since X is affine, S is irreducible and $\mathbb{P}(E)$ is nonsingular, the section S regarded as a divisor on $\mathbb{P}(E)$ must be ample. Let $i : \mathbb{P}(E) \to \mathbb{P}(E \otimes L^{-1})$ be the canonical isomorphism. Then the section S is transformed to the section S' by i and X to the affine surface $\mathbb{P}(E \otimes L^{-1}) - S'$. Hence S' is an ample divisor on $\mathbb{P}(E \otimes L^{-1})$. Let $j : C \to \mathbb{P}(E)$ be the isomorphism sending C to S. Then $i \cdot j$ is an embedding. Taking account of the facts that $\mathscr{O}_{\mathbb{P}(E\otimes L^{-1})}(1) \cong \mathscr{O}_{\mathbb{P}(E\otimes L^{-1})}(S')$ and $(i \cdot j)^*(\mathscr{O}_{\mathbb{P}(E\otimes L^{-1})}(1)) = j^*(\mathscr{O}_{\mathbb{P}(E)}(1) \otimes L^{-1}) = L' \otimes L^{-1}$, we know that $L' \otimes L^{-1}$ is an ample divisor on C. $\square$

2.4.2

The affine ring of an affine surface observed in 2.4.1 has no nontrivial locally finite iterative higher derivation. This is an immediate consequence of

Lemma. *Let V be a variety defined over k, let L be a line bundle over V and let E be an extension of L by $\mathscr{O}_V$. Let X be the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ minus the section corresponding to L. If $H^0(X, L^{-1}) \neq 0$, X has a nontrivial G_a-action. Conversely, assume that X has a nontrivial G_a-action and that either there is no non-constant morphism from $\mathbb{A}^1$ to V or G_a acts along fibers of the canonical projection $\mathbb{P}(E) \to V$. Then $H^0(V, L^{-1}) \neq 0$.*

Proof. Let $\mathscr{U} = \{U_i\}_{iI}$ be an affine open covering of V such that $E|_{U_i}$ is $\mathscr{O}_{U_i}$-free for any $i \in I$, and let $\left\{\begin{pmatrix} a_{ji} & b_{ji} \\ 0 & 1 \end{pmatrix}\right\}$ be the transition matrices of E relative to $\mathscr{U}$, where $\{a_{ji}\}$ is the transition functions of L. X is in fact an $\mathbb{A}^1$-bundle on V with affine coordinates $\{x_i\}$ which are subject to $x_j - a_{ji}x_i + b_{ji}$ for any $i, j \in I$. If $H^0(V, L^{-1}) \neq 0$, we may assume

that there is a set of functions $\{s_i\}$ on V such that $s_i(\neq 0) \in \Gamma(V_i, \mathscr{O}_V)$ and $s_j = a_{ji}s_i$ for any $i,\ j \in I$. Define a nontrivial locally finite it- **36**
erative higher derivation $D = \{D_0, D_1, \ldots\}$ on $\Gamma(U_i, \mathscr{O}_V)[x_i]$ by $D_0 = id.$, $D_n|_{\Gamma(U_i,\mathscr{O}_V)} = 0$ for $n > 0$ and $D_n(x_i^m) = \binom{m}{n} x_i^{m-n} s_i^n$ if $m \geqq n$ and 0 otherwise, where $\Gamma(U_i, \mathscr{O}_V)[x_i]$ is the affine ring of $\pi^{-1}(U_i)$, π being the restriction onto X of the projection $\mathbb{P}(E) \to V$; D gives rise to a nontrivial G_a-action on $\pi^{-1}(U_i)$. It is now easy to ascertain that the G_a-actions defined on $\{\pi^{-1}(U_i)\}_{iI}$ patch each other on $\pi^{-1}(U_i \cap U_j)$ to give a nontrivial G_a-action on X. Assume next that X has a nontrivial G_a-action on X; by the assumption in the statement of Lemma, G_a acts along fibers of π. By virtue of 1.3.2, the G_a-invariant sub field in $k(X)$ is $k(V)$. For every $i \in I$, the G_a-action restricted on $\pi^{-1}(U_i)$ gives rise to a $\Gamma(U_i, \mathscr{O}_V)$-homomorphism $\varphi_i : \Gamma(U_i, \mathscr{O}_V)[x_i] \to \Gamma(U_i, \mathscr{O}_V)[x_i, t]$, t being an indeterminate. Write $\varphi_i(x_i) = s_i t^n +$ (terms of lower degree in t with coefficients in $\Gamma(U_i, \mathscr{O}_V)[x_i]$), where $n \geqq 1$, $s_i \neq 0$ and $s_i \in \Gamma(U_i, \mathscr{O}_V)[x_i]$. Since s_i is G_a-invariant we have $s_i \in \Gamma(U_i, \mathscr{O}_V)[x_i] \cap k(V) = \Gamma(U_i, \mathscr{O}_V)$. Moreover it is easy to see that n is independent of i and $s_j = a_{ji}s_i$ for $i,\ j \in I$. Then $\{s_i\}_{i\in I}$ gives a nonzero section of $H^0(V, L^{-1})$. Hence $H^0(V, L^{-1}) \neq 0$. □

2.4.3

By virtue of Lemma 2.4.1 and 2.4.2 we can present an example of an affine surface with an algebraic pencil of affine lines, on which there is no G_a-action such that general members of the pencil are the G_a-orbits.
We shall content ourselves with the following: **37**

Example. Let Δ be the diagonal on the surface $F_0 := \mathbb{P}^1 \times \mathbb{P}^1$, let $X := F_0 - \Delta$ and let $\pi : X \to \mathbb{P}^1$ be the restriction of the projection of F_0 onto the second factor. Consider an algebraic pencil $\mathscr{F}$ of affine lines on X consisting of fibers of π. Then there is no G_a-action on X with respect to which general members of $\mathscr{F}$ are G_a-orbits.

Proof. Let $L = \mathscr{O}_{\mathbb{P}^1}(2)$ and let $E = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)$. Then $0 \to \mathscr{O}_{\mathbb{P}^1} \to E \to L \to 0$ is a nontrivial extension; the section of $\mathbb{P}(E) = F_0$ corresponding to L is the diagonal Δ of F_0. Thus X is an affine surface of the kind treated in 2.4.1. Now, By 2.4.2 our assertion follows from the fact that $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) = 0$. □

In this example, the affine ring $A := k[X]$ has the property: $\mathrm{Cl}(A) \cong \mathbb{Z}$ and $A^* = k^*$. This remark shows that the assumption $\mathrm{Cl}(A) = 0$ in Theorem 2.3 is indispensable.

3 Algebraic characterizations of the affine plane

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3.1

As the title indicates, the purpose of this section is to find criteria for a given affine surface to be isomorphic to the affine plane; in other words, criteria for an affine k-domain to be isomorphic to a polynomial ring in two variables over k. In this section the ground field k is assumed to be algebraically closed. Firstly we shall prove:

Theorem. *Let A be an affine k-domain of dimension 2. Then A is isomorphic to a polynomial ring in two variables over k if and only if the following conditions are satisfied:*

(1) *A is a unique factorization domain.*

(2) $A^* = k^*$.

(3) *A has a nontrivial locally finite iterative higher derivation.*

3.1.1

This theorem was firstly proved by the lecturer in [32; Theorem 1] by analyzing the associated G_a-action on $\mathrm{Spec}(A)$. Recently, Nakai [44] gave an elementary proof using only the structure of an affine k-domain with a locally finite iterative higher derivation. We shall present here the proof of Nakai's. The theorem will be proved in the paragraphs 3.1.2 ∼ 3.1.4.

3.1.2

Assume that A is a unique factorization domain, $A^* = k^*$ and A has a nontrivial locally finite iterative higher derivation D. Let A_0 be the subring of D-constants in A. Then we have:

Lemma. *A_0 is a polynomial ring in one variable over k.* **39**

Proof. By virtue of Lemmas 1.3.1 and 1.6, A_0 is a normal affine k-domain such that $A_0^* = k^*$; then A_0 is a unique factorization domain (cf. 1.3.1). Lemma 1.5 shows that $\dim A_0 = 1$. Then these facts imply our assertion. $\square$

3.1.3

With the same notations as in 1.5, let $e := p^r$ and let $M_n := A_{ne}(n = 1, 2, \ldots)$, where ne is the n-th jump index. Note that if $a \in A$ has D-length n (cf. 1.4) then $D_n(a) \in A_0$. Let $I_n := \{D_{ne}(a); a \in M_n\}$ for $n \geqq 1$; then it is easy to ascertain that I_n is an ideal of A_0. By virtue of 3.1.2, we may write $I_n = a_n A_0$ for $n \geqq 1$. Let u be an element of $M_1 - A_0$ such that $D_e(u) = a_1$. We shall prove by induction on n the following assertions:

$$(1)_n : I_n = I_1^n$$
$$(2)_n : M_n = A_0 + A_0 u + \cdots + A_0 u^n$$

Firstly we shall see that $(1)_n$ implies $(2)_n$. Let $\xi \in M_n$. Then $(1)_n$ implies that $D_{ne}(\xi) = \mathrm{ca}_1^n$ for some $c \in A_0$. Hence the D-length $\ell(\xi - \mathrm{cu}^n) < ne$, i.e., $\xi \in M_{n-1} + A_0 u^n$. Thus $(2)_n$ follows from $(1)_n$. Next we shall show that $(1)_n + (2)_n \Rightarrow (1)_{n+1}$. Let ξ be an element of M_{n+1} such that $D_{(n+1)e}(\xi) = a_{n+1}$. Since $I_1^{n+1} \subseteq I_{n+1}$ we may write $a_1^{n+1} = \mathrm{ca}_{n+1}$ with $c \in A_0$. Then $\ell(c\xi - u^{n+1}) < (n+1)e$, i.e., $c\xi - u^{n+1} \in M_n$. Hence by $(2)_n$ we have:

$$c\xi = u^{n+1} + \sum_{i=0}^{n} b_i u^i \quad \text{with} \quad b_i \in A_0.$$

We shall show that $c \in A_0^* = k^*$. Assume the contrary, and let f be **40** a prime factor of c. Taking the residue classes in A/fA, which is an integral domain by virtue of 1.3.1, we have:

$$\overline{u}^{n+1} + \sum_{i=0}^{n} \overline{b}_i \overline{u}^i = 0, \quad \text{with} \quad \overline{b}_i \in A_0/fA_0 = k.$$

Since k is algebraically closed we find $\lambda \in k$ such that $\overline{u} = \lambda$. Namely, $u - \lambda = fv$ with $v \in A$; it is easy to see that $v \in M_1$ and $a_1 = D_e(u) = fD_e(v)$. This is a contradiction. Therefore $I_{n+1} = I_1^{n+1}$. Since $(1)_0$ and $(2)_0$ obviously hold, we know that $A = \bigcup_{n=1}^{\infty} M_n = A_0[u]$. Hence by virtue of 3.1.2 we know that A is a polynomial ring in two variables over k.

3.1.4

Conversely, if A is a polynomial ring in two variables over k, A satisfies the conditions (1), (2) and (3) of Theorem 3.1.

3.2

Another algebraic characterization of the affine plane is given by the following:

Theorem. *Let k be an algebraically closed field of characteristic zero and let X be a nonsingular affine surface defined by an affine k-domain A. Assume that the following conditions are satisfied:*

(1) *A is a unique factorization domain and $A^* = k^*$.*

(2) *There exist nonsingular irreducible curves C_1 and C_2 on X such that $C_1 \cap C_2 = \{v\}$, and C_1 and C_2 intersect each other transversely at v.*

41 (3) *C_1 (resp. C_2) has only one place at infinity.*

(4) *Let a_2 be a prime element of A defining the curve C_2. Then $a_2 - \alpha$ is a prime element of A for all $\alpha \in k$.*

(5) *There is a nonsingular projective surface V containing X as an open set such that the closure $\overline{C}_2$ of C_2 in V is nonsingular and $(a_2)_0 = \overline{C}_2$.*

Then X is isomorphic to the affine plane $\mathbb{A}^2$, and the curves C_1 and C_2 are sent to the axes of a suitable coordinate system of $\mathbb{A}^2$.

3.2.1

The Proof of the theorem will be given in the paragraphs 3.2.1 ~ 3.2.4. We shall begin with

Lemma. *Let the notations and assumptions be as above. Let a_1 and a_2 be prime elements of A defining the curves C_1 and C_2, respectively, and let C_2^α be the curve on X defined by $a_2 - \alpha$ for $\alpha \in k$. Then we have:*

(1) *C_1 and C_2 are rational curves.*

(2) *For every $\alpha \in k$, $C_1 \cap C_2^\alpha = \{v_\alpha\}$ and C_1 intersects C_2^α transversely at v_α.*

Proof. Let $d = a_2$ (modulo a_1A). Then d is a regular function on C_1. Let $\widetilde{C}_1$ be the complete nonsingular model of C_1, let P_∞ be the unique point of $\widetilde{C}_1$ corresponding to the unique place at infinity of C_1 and let w be the normalized discrete valuation of $k(C_1)$ determined by P_∞. Then $(d) = v + w(d)P_\infty$, whence $w(d) = -1$. Then C_1 is a rational curve. Interchanging the roles of C_1 and C_2, we know that C_2 is a rational **42**
curve as well. For every $\alpha \in k$, we have $w(d - \alpha) = w(d(1 - \alpha d^{-1})) = w(d) = -1$. Hence $(d - \alpha) = v_\alpha - P_\infty$, where $C_1 \cap C_2^\alpha = \{v_\alpha\}$; this implies that C_1 and C_2^α intersect each other transversely at v_α. □

3.2.2

Lemma. *Let A be an affine k-domain and let a be an element of $A - k$. Assume that A is a unique factorization domain, that $A^* = k^*$ and that $a - \alpha$ is a prime element of A for every $\alpha \in k$. Let $S = k[a] - 0$ and let $A' = S^{-1}A$. Then we have:*

(1) *A' is a unique factorization domain.*

(2) *$A'^* = K^*$, where $K = k(a)$.*

(3) *The quotient field $Q(A')$ of A' is a regular extension of K; therefore A' defines an affine variety defined over K with dimension one less than* $\dim A$*.*

Proof. The assertion (1) is well-known. If $A'^* \neq K^*$ there exist elements x and y of $A - k[a]$ such that $xy = \varphi(a) \neq 0$ with $\varphi(a) \in k[a]$. Then, by the assumptions that A is a unique factorization domain and $a - \alpha$ is a prime element of A for every $\alpha \in k$ we have $x, y \in k[a]$. This is a contradiction, and the assertion (2) is proved. As for the assertion (3), we have only to show that K is algebraically closed in $Q(A')$ because $\mathrm{char}(k) = 0$. Assume that f/g is algebraic over K, where f and g are mutually prime elements of A. Then there exist elements $\varphi_0, \ldots, \varphi_n$ of
43 $k[a]$ such that the greatest common divisor of $\varphi_0, \ldots, \varphi_n$ is 1 and that

$$\varphi_0(f/g)^n + \varphi_1(f/g)^{n-1} + \cdots + \varphi_n = 0.$$

Then f and g divide φ_n and φ_0 respectively. Hence f and $g \in k[a]$. Thus $f/g \in K$. □

3.2.3

Lemma. *Let the notations and assumptions be as in the statement of the theorem. Then, for almost all elements α of k, C_2^α is a rational curve with only one place at infinity.*

Proof. For a general element α of k, the principal divisor $(a_2 - \alpha)$ on V is of the form: $(a_2 - \alpha) = \overline{C}_2^\alpha + D - D'$, where $\overline{C}_2^\alpha$ is the closure of C_2^α on V, D and D' are effective divisors such that $D \geqq 0$, $D' \geqq 0$, $\mathrm{Supp}(D) \cup \mathrm{Supp}(D') \subseteq V - X$, D and D' have no common components, and D and D' are independent of α. By the condition (5) of the theorem we have $(a_2) = \overline{C}_2$- (the polar divisor), whence we can easily conclude that $(a_2)_\infty = D'$[2] and $D = 0$. Therefore, there exists a linear pencil Λ on

[2] If E is an irreducible component of $V - X$, let w be the corresponding normalized discrete valuation of $k(V)$. If $E \subset \mathrm{Supp}((a_2)_\infty)$ then $w(a_2 - \alpha) = w(a_2(1 - \alpha a_2^{-1})) = w(a_2)$. Similarly, if $E \subset \mathrm{Supp}((a_2 - \alpha)_\infty)$ then $w(a_2) = w((a_2 - \alpha)(1 + \alpha(a_2 - \alpha)^{-1})) = w(a_2 - \alpha)$. Hence $(a_2)_\infty = D'$.

V such that $\overline{C}_2$ and $\overline{C}_2^\alpha$ are members of Λ for almost all α of k; Λ has no base points on X; since $\overline{C}_2$ is a nonsingular rational curve, general members $\overline{C}_2^\alpha$ are nonsingular rational curves. Let W be the generic member of Λ; W is then a nonsingular projective curve of genus 0 defined over **44**
$K = k(a_2)$. Let C be the affine curve defined by an affine K-domain $A' = S^{-1}A$, where $S = k[a_2] - 0$ (cf. 3.2.2). Then C is an open set of W. Note that W has a K-rational point P which is provided by the sectional curve C_1. Hence W is isomorphic to $\mathbb{P}^1$ over K. Let x be an inhomogeneous coordinates of $W := \mathbb{P}^1_K$ such that $x = \infty$ at P. Then there exist irreducible polynomials $f_1, \ldots, f_n$ of $K[x]$ such that the affine ring $K[C - P]$ is $K[x, f_1^{-1}, \ldots, f_n^{-1}]$. Then $(K[x, f_1^{-1}, \ldots, f_n^{-1}])^*/K^*$ is a free $\mathbb{Z}$-module of rank n. However, since A' is a unique factorization domain and $A'^* = K^*$ we must have $(K[C-P])^*/K^* \cong \mathbb{Z}$, i.e., $n = 1$. This means that $W - C$ consists of only one K-rational prime cycle. On the other hand, P is linearly equivalent to some multiple of the K-rational prime cycle $W - C$. This implies that $W - C$ consists of only one K-rational point. Hence C is isomorphic to $\mathbb{A}^1$ over K. This implies that C_2^α is isomorphic to $\mathbb{A}^1$ for almost all α of k. □

3.2.4

The proof of the theorem. Let $\mathscr{F} := \{C_2^\alpha; \alpha \in k\}$. Then Lemma 3.2.3 shows that $\mathscr{F}$ is an algebraic pencil of affine lines parametrized by A^1; by virtue of Theorem 2.3 we can find a G_a-action on X with respect to which almost all members of $\mathscr{F}$ are G_a-orbits. Let D be the nontrivial locally finite iterative higher derivation corresponding to the G_a-action. Then the subring A_0 of D-constants in A is $k[a_2]$; in fact, if $A_0 = k[a]$ for a prime element a in A (cf. 3.1.2) then $a_2 \in k[a]$, whence follows that $k[a_2] = k[a]$ because a_2 is a prime element of A. By virtue of Theorem **45**
3.1 we know that $A = k[a_2, u]$ for some element u of A. Hence X is isomorphic to the affine plane, and the curve C_2 is identified with a axis of a coordinate system of $\mathbb{A}^2$. Write a_1 in the form

$$a_1 = g_0(a_2) + g_1(a_2)u + \cdots + g_n(a_2)u^n$$

with $g_i(a_2) \in k[a_2]$ for $1 \leqq i \leqq n$. Since C_1 meets C_2^α transversely only in one point (cf. 3.2.1) we can easily ascertain that $g_i(a_2) = 0$ for $2 \leqq i \leqq n$

and $g_1(a_2) \in k^*$. This implies that the curves C_1 and C_2 are identified with axes of a coordinate system of $\mathbb{A}^2$. Thus, Theorem 3.2 is proved.

4 Flat fibrations by the affine line

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4.1

The results of this section were worked out jointly by Kambayashi and the lecturer [28]. The goal of this section is to prove the following two theorems:

4.1.1

Theorem. *Let $\varphi : X \to S$ be an affine. faithfully flat morphism of finite type; assume that S is a locally noetherian, locally factorial, integral scheme, and that the generic fiber of φ is $\mathbb{A}^1$ and all other fibers are geometrically integral. Then X is an $\mathbb{A}^1$-bundle over S.*

4.1.2

Theorem. *Let k be an algebraically closed field, let S be a regular, integral k-scheme of finite type, and let $\varphi : X \to S$ be an affine, faithfully flat morphism of finite type. Assume that each fiber of φ is geometrically integral and the general fibers of φ are isomorphic to $\mathbb{A}^1$ over k. Then there exists a regular, integral k-scheme S' of finite type and a faithfully flat, finite, radical morphism $S' \to S$ such that $X \underset{S}{\times} S'$ is an $\mathbb{A}^1$-bundle over S'. If the characteristic of k is zero X is an $\mathbb{A}^1$-bundle over S.*

4.2

The proof of Theorem 4.1.1 will be given in the paragraphs 4.2 ~ 4.5 and the proof of Theorem 4.1.2 in the paragraphs 4.6 ~ 4.8. We shall begin with the following elementary result, which is a special case of a theorem by Nagata [40]:

Lemma . *Let σ be a discrete valuation ring and let A be a flat $\mathscr{O}$-algebra of finite type. Let K be the quotient field of $\mathscr{O}$, t a uniformisant and k the residue field, and let A_K and A_k denote $K \underset{\mathscr{O}}{\otimes} A$ and $k \underset{\mathscr{O}}{\otimes} A$, respectively. Assume that A_K and A_k are integral domains. Then we have:* **47**

(1) *If A_K is a normal ring, so is A.*

(2) *If A_K is factorial (i.e., A_K is a unique factorization domain), so is A.*

Proof. We shall prove only (2), as proof of (1) is a routine exercise. By flatness there is a natural inclusion $\mathscr{O} \subset A$, and A is in turn contained in A_K and is noetherian. Since A_k is integral, tA is a prime ideal in A and $\bigcap_{\nu \geqq 0} t^\nu A = (0)$. Let $\mathscr{P} \subset A$ be an arbitrary prime ideal of height 1. If $t \in \mathscr{P}$ then clearly $tA = \mathscr{P}$. In case $t \notin \mathscr{P}$, the ideal $\mathscr{P} A_K$ is prime of height 1 in the factorial domain $A_K = A[t^{-1}]$, whence $\mathscr{P} A_K = f A_K$, where we may and shall take $f \in A - tA$. Let $b \in \mathscr{P}$ be arbitrary, and write $b = f t^m a$ with integer m and $a \in A - tA$. If $m < 0$, then $fa = bt^{-m} \in tA$, an absurdity. Consequently, $m \geqq 0$ and $\mathscr{P} \subseteq fA$. It follows that $\mathscr{P} = fA$ because $f \in \mathscr{P}$. □

4.3

Lemma. *Let $(\mathscr{O}, t\mathscr{O})$ be a discrete valuation ring with residue field k and quotient field K. Let A be a flat $\mathscr{O}$-algebra of finite type. Assume that $A_K := K \underset{\mathscr{O}}{\otimes} A$ is K-isomorphic to a polynomial ring $K[x]$ in one variable and that $A_k := k \otimes A$ is a geometrically integral domain over k. Then A is isomorphic to a polynomial ring in one variable over $\mathscr{O}$.*

Proof. Because A is factorial by Lemma 4.2, or rather because of the simple fact that $\bigcap_{\nu \geqq 0} t^\nu A = (0)$, we may assume that $x \in A$ and x is prime to uniformisant t of $\mathscr{O}$. We may write $A = \mathscr{O}[x, y_1, \ldots, y_m]$. Since $A \subset A_K = K[x]$ there exist integers $\alpha_i \geqq 0$ such that **48**

$$t^{\alpha(i)} y_i = \varphi_i(x) := \lambda_{i0} + \lambda_{il} x + \cdots + \lambda_{ir(i)} x^{r(i)} \tag{1}$$

with $\lambda_{ij} \in \mathscr{J}$ for $1 \leqq i \leqq m$ and $0 \leqq j \leqq r(i)$, where we may assume with each i that if $\alpha_i > 0$ not all of $\lambda_{i0}, \lambda_{i1}, \ldots, \lambda_{ir(i)}$ are divisible by t. Let $\alpha_x = \max\{\alpha(1), \ldots, \alpha(m)\}$. Consider the following assertion:

$P(n)$: If $x \in A$ is found as above with $\alpha_x = n$, then there is some $x' \in A$ such that $A = \mathscr{U}[x']$.

We shall prove the assertion $P(n)$ by induction on n. $P(0)$ is obviously true. We prove $P(n)$ by assuming that $P(r)$ is true if $r < n$. By applying the canonical (reduction modulo t) homomorphism $\rho : A \to A/tA \cong A_k$ to the both hand sides of (1) for each i with $\alpha(i) = \alpha_x$, we get

$$\rho(\lambda_{i0}) + \rho(\lambda_{il})\rho(x) + \cdots + \rho(\lambda_{ir(i)})\rho(x)^{r(i)} = 0 \tag{2}$$

with at least one of the coefficients $\rho(\lambda_{ij}) \neq 0$. Since A_k is an integral domain the equation (2) is a nontrivial algebraic equation of $\rho(x)$ over k. Since A_k is geometrically integral the field k is algebraically closed in the quotient field of A_k, whence $\rho(x) \in k$. Let $\mu \in \mathscr{U}$ be such that
49 $\rho(\mu) = \rho(x)$, and write $x - \mu = t^\beta x_1$ with a positive integer β and $x_1 \in A - tA$. Then noting $\varphi_i(\mu) \in t\mathscr{U}$ and by substituting $\mu + t^\beta x_1$ for x in (1), we obtain after cancellation of t

$$t^{\alpha'(i)} y_i \in \mathscr{U}[x_1] \quad \text{for} \quad 1 \leqq i \leqq m, \quad \text{and} \quad K[x] = K[x_1]$$

where $\alpha_{x_1} = \max\{\alpha'(1), \ldots, \alpha'(m)\} < n = \alpha_x$. Since $P(\alpha_{x_1})$ is assumed to be true, the conclusion of $P(n)$ holds. □

4.4

It is easy to see, as shown in the paragraph 4.5 below, that Theorem 4.1.1 follows from 4.3 in the special case where $\dim S = 1$. In order to prove the theorem over S with $\dim S \geqq 2$ we need the following

Lemma. *Let $(A, \mathscr{M})$ be a factorial local ring of dimension $\geqq 2$ with residue field k. Let R be a flat A-algebra of finite type. Assume that $R_{\mathscr{J}} := A_{\mathscr{J}} \underset{A}{\otimes} R$ is $A_{\mathscr{J}}$-isomorphic to a polynomial ring $A_{\mathscr{J}}[t_{\mathscr{J}}]$ in one variable for every nonmaximal prime ideal $\mathscr{J}$ of A and that $\overline{R} := R/\mathscr{M}R$ is geometrically regular over k. Then R is A-isomorphic to a polynomial ring $A[t]$ in one variable over A.*

Proof. The proof consists of four steps.

(I) Let $X := \mathrm{Spec}(R)$, $S := \mathrm{Spec}(A)$ and let $\varphi : X \to S$ be the flat morphism of finite type corresponding to the canonical injection $A \hookrightarrow R$. φ is in fact faithfully flat, and each fiber of φ is geometrically regular. φ is, therefore, smooth. Since S is normal this **50**
implies that X is normal [17; IV (6.5.4)]. Thus, R is a normal domain.

(II) Let $U := S - \{\mathscr{M}\}$. Since R is fainitely generated over A and $R_{\mathscr{J}} = A_{\mathscr{J}}[t_{\mathscr{J}}]$ for each $\mathscr{J} \in U$, there is $f_{\mathscr{J}} \in A - \mathscr{J}$ such that $R[f_{\mathscr{J}}^{-1}] = A[f_{\mathscr{J}}^{-1}][t_{\mathscr{J}}]$, whence we know that existence of an open covering $\mathscr{V} = \{V_i\}_{i \in I}$ of U such that $V_i := \mathrm{Spec}(A[f_i^{-1}])$ with $f_i \in A$ and $R[f_i^{-1}] = A[f_i^{-1}][t_i]$ for each $i \in I$. This shows that $X_U := \varphi^{-1}(U) = X \underset{S}{\times} U$ can be viewed as an $\mathbb{A}^1$-bundle over U. Set $A_i := A[f_i^{-1}]$, $A_{ij} := A[f_i^{-1}, f_j^{-1}]$ and $A_{ij\ell} := A[f_i^{-1}, f_j^{-1}, f_\ell^{-1}]$ for $i, j, \ell \in I$. Since $A_{ij}[t_i] = R[f_i^{-1}, f_j^{-1}] = A_{ij}[t_j]$ and A_{ij} is an integral domain we get $t_j = \alpha_{ji} t_i + \beta_{ji}$ with units α_{ji} in A_{ji} and $\beta_{ji} \in A_{ji}$ for each pair i, j of I. where α_{ij}'s and β_{ij}'s are subject to the relations in $A_{ij\ell}$:

$$\alpha_{\ell i} = \alpha_{\ell j} \alpha_{ji} \quad \text{and} \quad \beta_{\ell i} = \alpha_{\ell j} \beta_{ji} + \beta_{\ell j}.$$

Hence, $\{\alpha_{ij}\}$ gives rise to an invertible sheaf $L \in H^1(U, \mathscr{O}_U^*)$. However, $H^1(U, \mathscr{O}_U^*) = (0)$ because $(A, \mathscr{M})$ is a factorial domain [19; Exp. XI, 3.5 and 3.10]. Thus, by replacing $\mathscr{V}$ by a finer open covering of U if necessary, we may assume that

$$\begin{aligned} & t_j = t_i + \beta_{ji} \quad \text{with} \quad \beta_{ji} \in A_{ji} \quad \text{such that} \\ (3) \qquad & \beta_{\ell i} = \beta_{ji} + \beta_{\ell i} \quad \text{for} \quad i, j, \ell \in I. \end{aligned}$$

Hence, $\{\beta_{ij}\}$ defines an element $\xi \in H^1(U, \mathscr{O}_U)$.

(III) Consider $X_U = \varphi^{-1}(U) = X \underset{S}{\times} U$ and let $Y := X - X_U$. By the local **51** cohomology theory we have the following commutative diagram:

$$\begin{tikzcd}
H^1(X_U, \mathscr{O}_X) \arrow[r, "\sim"] & H^2_Y(X, \mathscr{O}_X) \arrow[r, "\sim"] & \varinjlim_n \operatorname{Ext}^2_R(R/\mathscr{M}^n R, R) \\
H^1(U, \mathscr{O}_S) \arrow[r, "\sim"] \arrow[u, "\theta_U"'] & H^2_{\mathscr{M}}(S, \mathscr{O}_S) \arrow[r, "\sim"] \arrow[u, "\theta_{\mathscr{M}}"'] & \varinjlim_n \operatorname{Ext}^2_A(A/\mathscr{M}^n, A) \arrow[u, "\theta_A"']
\end{tikzcd}$$

where the terms in the upper and lower rows are respectively R-modules and A-modules, and θ_U, $\theta_{\mathscr{M}}$ and θ_A are homomorphisms induced by the canonical injection $\mathscr{O}_S \hookrightarrow \varphi_* \mathscr{O}_X$; for the definitions and relevant results, see [19] or [20]. Since R is A-flat and $\varinjlim_n$ commutes with $R \underset{A}{\otimes} ?$, we have

$$\varinjlim_n \operatorname{Ext}^2_R(R/\mathscr{M}^n R, R) \cong R \underset{A}{\otimes} \varinjlim_n \operatorname{Ext}^2_A(A/\mathscr{M}^n, A)$$

and θ_A is identified with the homomorphism: $u \mapsto 1 \otimes u$ for $u \in \varinjlim_n \operatorname{Ext}^2_A(A/\mathscr{M}^n, A)$. Since R is A-flat, θ_A is then injective. The commutative diagram above shows that θ_U is injective. On the other hand X_U has an open covering $\varphi^{-1}\mathscr{V} = \{\varphi^{-1} V_i\}_{J'}$, and the element $\theta_U(\xi) \in H^1(X_U, \mathscr{O}_X)$ is represented by a Čech 1-cocycle $\{\beta_{ij}\}$ with respect to $\varphi^{-1}\mathscr{V}$. The relation (3) above implies that $\{\beta_{ij}\}$ is in fact a Čech 1-coboundary because $t_i \in \Gamma(\varphi^{-1}(V_i), \mathscr{O}_X) = A_i[t_i]$. Thus $\theta_U(\xi) = 0$, and hence $\xi = 0$ because θ_U is injective, which implies that X_U has a section and is, in fact, a trivial bundle $\mathbb{A}^1_U$.

52 (IV) Replacing $\mathscr{V}$ by a finer open cover in of U if necessary, we may assume that $\beta_{ji} = \gamma_j - \gamma_i$ with $\gamma_i \in A_i$, and for $i, j \in I$. Then $t_i - \gamma_i = t_j - \gamma_j$ for every pair i, j of I. Let $t := t_i - \gamma_i$. Then $t \in \Gamma(X_U, \mathscr{O}_X)$. On the other hand, since $\operatorname{codim}(Y, X) \geqq 2$ and R is normal, $\mathscr{O}_X$ is Y-closed [17; IV (5.10.5)]. Hence $t \in \Gamma(X_U, \mathscr{O}_X) = \Gamma(X, \mathscr{O}_X) = R$. Now, look at the A-subalgebra $A[t]$ of R, and let $Z := \operatorname{Spec}(A[t])$. Then, φ decomposes as $X \xrightarrow{\varphi_1} Z \xrightarrow{\varphi_2} S$, where φ_1 and φ_2 are

the morphisms corresponding to the injections $A \hookrightarrow A[t] \hookrightarrow R$. By step (III), $R_{\mathscr{J}} = A_{\mathscr{J}}[t]$ for each $\mathscr{J} \in U$. This implies that $\varphi_{1|U} : X_U \to \varphi_2^{-1}(U) = Z \underset{S}{\times} U$ is a U-isomorphism. Notice that $\mathscr{O}_Z$ is $Z - \varphi_2^{-1}(U)$-closed because $\operatorname{codim}(Z - \varphi_2^{-1}(U), Z) \geqq 2$ and Z is normal. Then we have:

$$A[t] = \Gamma(Z, \mathscr{O}_Z) = \Gamma(\varphi_2^{-1}(U), \mathscr{O}_Z) \xrightarrow[(\varphi_{1|U})^*]{\sim} \Gamma(X_U, \mathscr{O}_X) = R.$$

Thus $R = A[t]$.

□

4.5

Proof of Theorem 4.1.1 . *Since φ is affine, it suffices clearly to prove the theorem under the hypothesis that X and S are affine schemes. The proof consists of two steps.*

(I) *Let $A := \Gamma(S, \mathscr{O}_S)$ and $R := \Gamma(X, \mathscr{O}_X)$. The homomorphism $A \to R$ induced by φ is injective, and makes R a flat A-algebra of finite type. For each prime ideal $\mathscr{J}$ of A, let $R_{\mathscr{J}} := A_{\mathscr{J}} \underset{A}{\otimes} R$. By induction on $n :=$ height $\mathscr{J}$ we shall establish the following assertion:*

$$P(n) \begin{array}{l} : R_{\mathscr{J}} \text{ is a polynomial ring in one variable over } A_{\mathscr{J}} \\ : \text{if } \mathscr{J} \text{ is of height } n. \end{array}$$

Indeed, $P(0)$ follows from the assumption of the theorem. If $n = 1$, **53**
$A_{\mathscr{J}}$ is a discrete valuation ring, and $P(1)$ follows from Lemma 4.3. We shall prove $P(n)$ for $n \geqq 2$, assuming $P(r)$ to hold for every $r < n$. By slight abuse of notations we write R and A instead of $R_{\mathscr{J}}$ and $A_{\mathscr{J}}$, respectively. Now, A is a factorial local ring of dimension $\geqq 2$ with maximal ideal $\mathscr{M}$. By virtue of [17; II (7.1.7)] one can find a discrete valuation ring $\mathscr{U}$ such that the quotient field K of $\mathscr{U}$ agrees with that of A and that $\mathscr{U}$ dominates A. Then $\mathscr{U} \underset{A}{\otimes} R$ is a flat $\mathscr{U}$-algebra of finite type, $K \underset{\mathscr{U}}{\otimes} (\mathscr{U} \underset{A}{\otimes} R) = K \underset{A}{\otimes} R$ is a polynomial ring in one variable over K,

and $(\mathscr{U}/t\mathscr{U})\underset{\mathscr{U}}{\otimes}(\mathscr{U}\underset{A}{\otimes}R) = (\mathscr{U}/t\mathscr{U})\underset{A/\mathscr{M}}{\otimes}(R/\mathscr{M}R)$ *is geometrically integral, where* t *is a uniformisant of* $\mathscr{U}$*. By Lemma 4.3,* $\mathscr{U}\underset{A}{\otimes}R$ *is then a polynomial ring in one variable over* $\mathscr{U}$*. If follows that* $(\mathscr{U}/t\mathscr{U})\underset{A/\mathscr{M}}{\otimes}(R/\mathscr{M}R)$ *is geometrically regular and, therefore,* $R/\mathscr{M}R$ *is geometrically regular over* $A/\mathscr{M}$*. This remark and* $P(r)$ *for* $0 \leqq r < n$ *imply that* A *and* R *satisfy all assumptions in Lemma 4.4. Thus, by that Lemma we know that* R *is a polynomial ring in one variable over* A*.*

(II) *Since* R *is finitely generated over* A*, step (I) implies that for each prime ideal* $\mathscr{J}$ *of* A *there exists an element* $f \in A$ *such that* $f \notin \mathscr{J}$ *and* $R[f^{-1}]$ *is a polynomial ring in one variable over* $A[f^{-1}]$*. Thus, for the Zariski open set* $U_f := \mathrm{Spec}(A[f^{-1}]) \subseteq S$*,*
54 *an isomorphism* $X\underset{S}{\times}U_f \cong A^1\underset{\mathbb{Z}}{\times}U_f$ *obtains, and* S *is clearly covered by finitely many such* U_f*'s. This completes the proof of Theorem 4.1.1.*

4.6

Let k be a field. A k-scheme X is called a form of $\mathbb{A}^1$ over k, or simply a k-form of $\mathbb{A}^1$ if for an algebraic extension field k' of k there exists a k'-isomorphism $X\underset{k}{\times}k' \xrightarrow{\sim} \mathbb{A}^1_k\underset{k}{\otimes}k' = \mathbb{A}^1_{k'}$. When that is so, there is a purely inseparable extension field k'' of k such that $X\underset{k}{\otimes}k''$ is k''-isomorphic to $\mathbb{A}^1_{k''}$ (cf. Chapter 3, 1.2). It is easy to see that, for a k-scheme X and an algebraic extension field k' of k, X is a k-form of $\mathbb{A}^1$ if and only if $X\underset{k}{\otimes}k'$ is a k'-form of $\mathbb{A}^1$. A k-form of $\mathbb{A}^1$ is evidently an affine smooth k-scheme. A k-form of $\mathbb{A}^1$ may be characterized as a one-dimensional k-smooth scheme of geometric genus zero having exactly one purely inseparable point at infinity. For detailed study on K-forms of $\mathbb{A}^1$, see [26; §6] and [27].

4.7

A key result to prove Theorem 4.1.2 is the following

Lemma. *Let k be a field of characteristic $p \geqq 0$, let S be a geometrically integral k-scheme of finite type, and let $\varphi : X \to S$ be an affine, flat morphism of finite type. Assume that the general flbers of φ are forms of $\mathbb{A}^1$ over their respective residue fields. Then the generic fiber X_K is a K-form of $\mathbb{A}^1$, where K is the function field of S over k. If $p = 0$, X_K is K-isomorphic to $\mathbb{A}^1_K$.*

Proof. The proof consists of four steps. **55**

(I) Let $\overline{k}$ be an algebraic closure of k. Let $\overline{S} := S \underset{k}{\otimes} \overline{k}$, $\overline{X} := X \underset{k}{\otimes} \overline{k}$ and $\overline{\varphi} := \varphi \underset{k}{\otimes} \overline{k}$. Then $\overline{S}$ is an integral $\overline{k}$-scheme, and the general fibers of $\overline{\varphi}$ are $\overline{k}$-isomorphic to $\mathbb{A}^1_{\overline{k}}$. The stated conditions for φ are evidently present for $\overline{\varphi}$. Let $\overline{K} := \overline{k} \underset{k}{\otimes} K$. As remarked in 4.6, the generic fiber X_K of φ is a K-form of $\mathbb{A}^1$ if and only if the generic fiber $\overline{X}_{\overline{K}}$ of $\overline{\varphi}$ is a $\overline{K}$-form of $\mathbb{A}^1$. These observations show that in proving the lemma we may assume from the outset that k is algebraically closed and that the general fibers are k-isomorphic to $\mathbb{A}^1_k$. Furthermore, we may assume with no loss of generality that S is smooth over k because the set of all k-smooth points of S is a non-empty open set. We assume these additional conditions in the step below.

(II) Let C denote the generic fiber X_K of $\varphi \cdot C$ is an affine curve over K, whose function field $K(C)$ is a regular extension field of K [17; IV (9.7.7), III (9.2.2)]. For each positive integer n we let $K_n := K^{p^{-n}}$. If $p = 0$, K_n is understood to mean K. By virtue of [12; Th.5, p.99], there exists a positive integer N such that a complete K_N-normal model of $K_N(C) := K_N \underset{K}{\otimes} K(C)$ is smooth over K_N. We fix such an N once for all. Let S_N be the normalization of S in K_N. Since S is smooth over k and k is algebraically closed, S_N is smooth over k and the normalization morphism $S_N \to S$ is identified with the N-th power of the Frobenius morphism of S_N.

(III) Let $\widetilde{C}_N$ be a complete normal model of $K_N(C)$ over K_N. Then, $\widetilde{C}_N$ **56**
is a smooth projective curve over K_N. Thus $\widetilde{C}_N$ is a closed sub

scheme in the projective space $\mathbb{P}^m_{K_N}$ defined by a finite set of homogeneous equations $\{f_\lambda(X_0,\ldots,X_m)=0;\lambda\in\Lambda\}$. One can then find a nonempty open set U of S_N such that all the coefficients of all $f'_\lambda s$, as elements of $K_N = k(S_N)$, are defined on U. Let $\widetilde{X}_N$ be the closed sub scheme of $\mathbb{P}^m_k \times_k U$ defined by the same set of homogeneous equations $\{f_\lambda(X_0,\ldots,X_m)=0;\lambda\in\Lambda\}$, and let $\widetilde{\varphi}_N : \widetilde{X}_N \to U$ be the projection onto U. The generic fiber of $\widetilde{\varphi}_N$, which coincides with $\widetilde{C}_N$, is geometrically regular. Applying the generic flatness theorem [17; IV (6.9.1)] and the Jacobian criterion of smoothness, we may assume, by shirinking U to a smaller nonempty open set if necessary, that $\widetilde{\varphi}_N$ is smooth over U. Now, look at the morphism $\varphi_N : X_N := X \times_S U \to U$ obtained from $\varphi : X \to S$ by the base change $U \to S$. Since $\widetilde{C}_N$ is a completion of the generic fiber $C_N := C \otimes_K K_N$ of φ_N, we have a birational U-mapping $\psi_N : X_N \to \widetilde{X}_N$ such that $\varphi_N = \widetilde{\varphi}_N \cdot \psi_N$. Since ψ_N is everywhere defined on C_N, we may assume, by replacing U by a smaller nonempty open set if necessary, that $\psi_N : X_N \to \widetilde{X}_N$ is an open immersion of U-schemes.

(IV) If now suffices to show that X_K is a K-form of $\mathbb{A}^1$ under the additional hypotheses:

(i) There exists a projective smooth morphism $\widetilde{\varphi} : \widetilde{X} \to S$ and an open immersion $\psi : X \to \widetilde{X}$ such that $\varphi = \widetilde{\varphi} \cdot \psi$.

57 (ii) Every closed fiber of φ is k-isomorphic to $\mathbb{A}^1_k$.

Then, every closed fiber of $\widetilde{\varphi}$ is k-isomorphic to $\mathbb{P}^1_k$ by virtue of the conditions (i) and (ii). Since $\widetilde{\varphi}$ is faithfully flat and arithmetic genus is invariant under flat deformations ([18; Exp. 221, p.5], [17; III, §7]) we have the arithmetic genus $p_a(\widetilde{X}_k) = 0$ for the generic fiber $\widetilde{X}_K$ of $\widetilde{\varphi}$, which is a smooth projective curve defined over K. We shall next show that $\widetilde{X}_K - \psi(X_K)$ has only one point and the point is purely inseparable over K. Let η be a point on $\widetilde{X}_K - \psi(X_K)$ and let T be the closure of η in $\widetilde{X}$. Then, $T \subseteq \widetilde{X} - \psi(X)$, the restriction $\widetilde{\varphi}_T : T \to S$ of $\widetilde{\varphi}$ onto T is a dominating morphism, and $\deg \widetilde{\varphi}_T = [K(\eta) : K]$. Notice that $\widetilde{\varphi}_T$ is a generically one-to-one morphism because for each closed point P on S, $\widetilde{\varphi}_T^{-1}(P) \subseteq \widetilde{\varphi}^{-1}(P) - \psi\widetilde{\varphi}^{-1}(P) \cong P^1_k - \mathbb{A}^1_k = \{\text{one point}\}$. This implies that $\widetilde{\varphi}_T$ is a birational morphism if $p = 0$ and a radical morphism if $p > 0$. Thus, $k(\eta)$ is purely inseparable over K. If η' is a point of $\widetilde{X}_K = \psi(X_K)$ distinct from η, let T' be the closure of η' in $\widetilde{X}$. Then $T' \subseteq \widetilde{X} - \psi(X)$ and $T \neq T'$. Then, for a general closed point P on S, $\widetilde{\varphi}^{-1}(P) - \psi\varphi^{-1}(P)$ would have distinct two points, and this is a contradiction. Thus, $\widetilde{X}_k - \psi(X_K)$ has only one point, and the point is purely inseparable over K. As ψ is an open immersion, this last combined with the fact $p_a(\widetilde{X}_K) = 0$ tells us in view of 4.6 that X_K is a K-form of $\mathbb{A}^1$, as desired (cf. [26: 6.7.7]).

□

4.8

Proof of Theorem 4.1.2. *Notice that k is assumed to be algebraically closed. Using the same notations as in 4.7 (especially step (III)), we* 58
know that for a sufficiently large integer N the generic fiber of $\varphi_N : X_N \to U$ is $k(S_N)$-isomorphic to $A^1_{k(S_N)}$, where $k(S_N)$ is the function field of S_N over k. Let $S' := S_N$. Then, S' is a regular, integral k-scheme of finite type and the canonical morphism $S' \to S$ is a faithfully flat, finite, radical morphism. Let $X' := X \underset{S}{\times} S'$ and $\varphi' := \varphi \underset{S}{\times} S'$. Then φ' is a faithfully flat, affine morphism of finite type, the generic fiber of φ' is $k(S')$-isomorphic to $\mathbb{A}^1_{k(S')}$, and every fiber of φ' is geometrically integral. Thus, all conditions of Theorem 4.1.1 present for S', X' and φ'. Hence X' is an $\mathbb{A}^1$-bundle over S'. If $p = 0$ it is clear that X is an $\mathbb{A}^1$-bundle over S. This completes the proof of Theorem 4.1.2.

4.9

In the characteristic zero case we have the following, superficially stronger, version of Theorem 4.1.2:

Theorem. *Let k be a field of characteristic zero, let S be a locally factorial, geometrically integral k-scheme of finite type, and let $\varphi : X \to S$*

be a faithfully flat, affine morphism of finite type. Assume that every fiber of φ is geometrically integral. Then, the following conditions are equivalent to each other:

(i) *X is an $\mathbb{A}^1$-bundle over S.*

(ii) *For every point P on S (including the generic point) the fiber $\varphi^{-1}(P)$ above P is isomorphic to the affine line $\mathbb{A}^1_{\kappa(P)}$ over the residue field $\kappa(P)$ of P.*

59 (iii) *The general fibers of φ are k-isomorphic to $\mathbb{A}^1$.*

(iv) *The generic fiber of φ is $k(S)$-isomorphic to $\mathbb{A}^1_{k(S)}$.*

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii): Obvious. (iii) $\Longrightarrow$ (iv) follows from Lemma 4.7. (iv) $\Longrightarrow$ (i) follows from Theorem 4.1.1. □

4.10

A flat specialization of $\mathbb{A}^n (n \geqq 2)$ is not necessarily isomorphic to $\mathbb{A}^n$, as shown by the next:

Example. Let k be an algebraically closed field, and let C be a smooth affine plane curve of genus > 0 contained as a closed sub scheme in $\mathbb{A}^2_k := \mathrm{Spec}(k[x,y])$. Let $f(x,y) = 0$ be the equation of C. Let $\mathscr{U} := k[t]_{(t)}$ be the local ring of $\mathbb{A}^1_k := \mathrm{Spec}(k[t])$ at $t = 0$, let $K := k(t)$, and let $A := \mathscr{U}[x,y,z]/(tz - f(x,y))$. Let $X := \mathrm{Spec}(A)$, $S := \mathrm{Spec}(\mathscr{U})$, and let $\varphi : X \to S$ be the morphism inducted by the injection $\mathscr{U} \hookrightarrow A$. Then, φ is a faithfully flat, affine morphism of finite type, the generic fiber X_K of φ is isomorphic to $\mathbb{A}^2_K$, and the closed fiber is k-isomorphic to $C \times_k \mathbb{A}^1_k$, which is evidently not isomorphic to $\mathbb{A}^2_k$. (Flatness of φ follows from [17; IV (14.3.8)].)

4.11

In the positive characteristic case there can be a flat fibration of a curve in which every closed fiber is $\mathbb{A}^1$ and yet the generic fiber is non-isomorphic to $\mathbb{A}^1$. For instance, let

$$A := k[t] \subset R := k[t, X, Y]/(Y^P - X - tX^p)$$

be the natural inclusion, and $\varphi : X := \mathrm{Spec}(R) \to S := \mathrm{Spec}(A)$ be the 60
corresponding morphism, where k denotes an algebraically closed field of characteristic $p > 0$. In this example, the generic fiber is a purely inseparable $k(t)$-form of A^1 studied in [26; §6], [27].

5 Classification of affine $\mathbb{A}^1$-bundles over a curve 61

5.1

In this section the ground field is assumed to be algebraically closed. Let C be a nonsingular curve defined over k. An $\mathbb{A}^1$-bundle over C is a surjective morphism $f : X \to C$ from a nonsingular surface X defined over k to C such that for every point P on C there exists an open neighborhood U of P for which $f^{-1}(U) \cong U \underset{k}{\times} \mathbb{A}^1$. When a curve is fixed we denote an $\mathbb{A}^1$-bundle $f : X \to C$ simply by (X, f). Given two $\mathbb{A}^1$-bundles (X, f) and (X', f') over C, we say that (X, f) is isomorphic to (X', f') if there exists an isomorphism $\theta : X \xrightarrow{\sim} X'$ such that $f = f' \cdot \theta$. An $\mathbb{A}^1$-bundle (X, f) is said to be *affine* if the surface X is affine. The purpose of this section is to describe the set of isomorphism classes of affine $\mathbb{A}^1$-bundles over a nonsingular complete curve C, especially when $C \cong \mathbb{P}^1_k$. In the paragraphs below we let C be a nonsingular complete curve.

5.2

Lemma (cf. 2.4.1). *Let $f : X \to C$ be an affine $\mathbb{A}^1$-bundle over C. Then there exist an ample line bundle L over C and a nontrivial extension E of L by $\mathscr{O}_C$ such that X is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ minus the section S_∞ of $\mathbb{P}(E)$ corresponding to L and that f is the restriction onto X of the canonical projection $\mathbb{P}(E) \to C$.*

Proof. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an affine open covering of C such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^1$ for $i \in I$. Let $A_i := k[U_i]$ and $R_i := k[f^{-1}(U_i)]$.

Then $R_i = A_i[t_i]$, where t_i's are subject to

$$\begin{aligned} t_j &= a_{ji}t_i + b_{ji} \quad \text{with} \quad a_{ji} \in A_{ji}^* \quad \text{and} \quad b_{ji} \in A_{ji} \\ a_{\ell i} &= a_{\ell j}a_{ji} \quad \text{and} \quad b_{\ell i} = a_{\ell j}b_{ji} + b_{\ell j} \quad \text{in} \quad A_{ij\ell}, \end{aligned}$$

62 where $i, j, \ell \in I$, $A_{ij} = k[U_i \cap U_j]$ and $A_{ij\ell} = k[U_i \cap U_j \cap U_\ell]$. Let L be a line bundle over C having transition functions $\{a_{ji}\}$ with respect to $\mathscr{U}$, and let E be a rank 2 vector bundle over C having transition matrices $\left\{\begin{pmatrix} a_{ji} & b_{ji} \\ 0 & 1 \end{pmatrix}\right\}$ with respect to $\mathscr{U}$. Then E is an extension of L by $\mathscr{O}_C$; $0 \to \mathscr{O}_C \to E \to L \to 0$, and (X, f) is isomorphic to $(\mathbf{P}(E) - S_\infty, \pi)$, where S_∞ is the section of the $\mathbf{P}^1$-bundle $\mathbf{P}(E)$ corresponding to L and π is the restriction onto $\mathbf{P}(E) - S_\infty$ of the canonical projection $\mathbf{P}(E) \to C$. The assumption that X is affine implies that L is an ample line bundle and E is a nontrivial extension of L by $\mathscr{O}_C$, (cf. 2.4.1). □

5.3

Lemma. *Let (X, f) and (X', f') be affine $\mathbb{A}^1$-bundles over C. Let*

$$0 \longrightarrow \mathscr{O}_C \xrightarrow{\iota} E \xrightarrow{\rho} L \longrightarrow 0$$

$$(\textit{resp.} \qquad 0 \longrightarrow \mathscr{O}_C \xrightarrow{\iota'} E \xrightarrow{\rho'} L' \longrightarrow 0)$$

be a nontrivial extension of an ample line bundle L (resp. L') by $\mathscr{O}_C$ as constructed in 5.2 from (X, f) (resp. (X', f')). Then (X, f) is isomorphic to (X', f') if and only if there exist isomorphisms $\phi : E' \xrightarrow{\sim} E$ and $\psi : L' \xrightarrow{\sim} L$ of vector bundles over C which make the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_C & \xrightarrow{l'} & E' & \xrightarrow{\rho'} & L' & \longrightarrow & 0 \\ & & \big\downarrow \text{id.} & & \big\downarrow \phi & & \big\downarrow \psi & & \\ 0 & \longrightarrow & \mathscr{O}_C & \xrightarrow{l} & E & \xrightarrow{\rho} & L & \longrightarrow & 0 \end{array}$$

63 *Proof.* We shall prove the "only if" part only. There exists an affine open covering $\mathscr{U} = \{U_i\}_{i \in I}$ such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^1$ and $f'^{-1}(U_i) \cong$

$U_i \times \mathbb{A}^1$ for any $i \in I$. Set $A_i = k[U_i]$, $A_{ij} = k[U_i \cap U_j]$, $R_i = k[f^{-1}(U_i)] = A_i[t_i]$ and $R'_i = k[f'^{-1}(U_i)] = A_i[t'_i]$. Let

$$t_j = a_{ji}t_i + b_{ji} \quad \text{and} \quad t'_j = a'_{ji}t'_i + b'_{ji} \quad \text{with} \quad a_{ji}, a'_{ji} \in A^*_{ij}$$
$$\text{and} \quad b_{ji}, b'_{ji} \in A_{ij} \quad \text{for any pair} \quad i, j \in I.$$

Then an isomorphism $\theta : X \to X'$ with $f = f' \cdot \theta$ induces an A_i-isomorphism $\varphi_i : R'_i \to R_i$ for any $i \in I$ such that $\varphi_i = \varphi_j$ on $R'_{ij} := k[f'^{-1}(U_i \cap U_j)]$. Write $\varphi_i(t'_i) = \alpha_i t_i + \beta_i$ with $\alpha_i \in A^*_i$ and $\beta_i \in A_i$. Then it is easily ascertained that we have:

$$\begin{pmatrix} a'_{ji} & b'_{ji} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_i & \beta_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_j & \beta_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{ji} & b_{ji} \\ 0 & 1 \end{pmatrix}$$

for $i, j \in I$. Let $E|_{U_i} = \mathscr{O}_{U_i}v_i + \mathscr{O}_{U_i}w$ and $E'|_{U_i} = \mathscr{O}_{U_i}v'_i + \mathscr{O}_{U_i}w'$, where $\mathscr{O}_{U_i}w = 1(\mathscr{O}_{U_i})$ and $\mathscr{O}_{U_i}w' = l'(\mathscr{O}_{U_i})$. Define $\mathscr{O}_{U_i}$-isomorphisms $\phi_i : E'|_{U_i} \to E|_{U_i}$ and $\psi_i : L'|_{U_i} \to Z|_{U_i}$ by

$$\phi_i(v'_i) = \alpha_i v_i + \beta_i w,\ \phi_i(w') = w \quad \text{and} \quad \psi_i(\rho(v'_i)) = \alpha_i \rho(v_i).$$

Then it is easy to see that ϕ_i's and ψ_i's patch each other on $U_i \cap U_j$ to give isomorphisms of vector bundles $\phi : E' \to E$ and $\psi : L' \to L$ such that $\phi_i = \phi|_{U_i}$, $\psi_i = \psi|_{U_i}$ for $i \in I$. By construction ϕ and ψ satisfy $\phi \cdot l' = 2$ and $\psi \cdot \rho' = \rho \cdot \phi$. □

5.4

We have the following:

5.4.1

Lemma. *With the notations of 5.2, we have $L \cong \mathscr{O}_C(S_\infty \cdot S_\infty)$, where* 64
S_∞ is identified with C.

Proof. Let $V := \mathbb{P}(E)$ and $S := S_\infty$. Then we have an exact sequence,

$$0 \longrightarrow \mathscr{O}_V(-2S) \longrightarrow \mathscr{O}_V(-S) \longrightarrow \mathscr{O}_S(-S \cdot S) \longrightarrow 0.$$

Now write $E|_{U_i}\mathscr{O}_{v_i} + \mathscr{O}_{U_i}w$ as in the proof of Lemma 5.3. Then we have $v_j = a_{ji}v_i + b_{ji}w$ for $i, j \in I$. Let $M := \mathscr{O}_V(-S)/\mathscr{O}_V(-2S)$, which is viewed as a line bundle on $S \cong C$. Then $M|_{U_i} \cong \mathscr{O}_{U_i}(w/v_i)$ (modulo $(w/v_i)^2$), and $w/v_j = a_{ij}(w/v_i)$ (modulo $(w/v_i)^2$) on $U_i \cap U_j$. Therefore $M \cong L^{-1}$, and consequently we obtain $L \cong \mathscr{O}_S(S \cdot S)$. □

5.4.2

An immediate consequence of Lemma 5.2 and Lemma 5.3 is:

Lemma. *Let L be an ample line bundle over C. Then the set of isomorphism classes of affine $\mathbb{A}^1$-bundles (X, f) such that $\mathscr{O}_C(S_\infty \cdot S_\infty) \cong L$ (cf. 5.2 and 5.4.1) is isomorphic to the projective space $\mathbb{P}(H^1(C, L^{-1}))$.*

5.5

In this paragraph we assume that C is isomorphic to the projective line $\mathbb{P}^1$. Then note that any $\mathbb{P}^1$-bundle over C is isomorphic to one of F_n's ($n \geqq 0$), where $F_n = \mathbb{P}(\mathscr{O}_C \oplus \mathscr{O}_C(-n))$. We denote by B_n the unique section of F_n such that $(B_n^2) = -n$ and by ℓ a fiber of the canonical projection $F_n \to C$. If $n = 0$, F_0 has two distinct structures of $\mathbb{P}^1$-bundle and B_0 is not uniquely chosen; hence we fix a structure of $\mathbb{P}^1$-bundle on F_0 and a section B_0. With these conventions we have:

5.5.1

65 **Lemma.** *Let (X, f) be an affine $\mathbb{A}^1$-bundle over $C = \mathbb{P}^1$. Then (X, f) is isomorphic to $(F_n - S_\infty, \pi)$* (cf. 5.2)*, where $S_\infty \sim B_n + s\ell$ with $s > n$ and $L \cong \mathscr{O}_C(2s - n)$. Moreover, such n and s are uniquely determined by the $\mathbb{A}^1$-bundle (X, f).*

Proof. Since S_∞ is an ample divisor on F_n we have: $S_\infty \sim B_n + s\ell$ with $s > n$ (cf. [16]). With the notations of 5.2, we have $F_n = \mathbb{P}(E)$, where E is a nontrivial extension of L by $\mathscr{O}_C$. By virtue of 5.4.1, we have $L \cong \mathscr{O}_C((S_\infty^2))$; hence $L \cong \mathscr{O}_C(2s - n)$. Moreover, by virtue of 5.3, E and L are uniquely determined up to isomorphism. Hence n and also s are uniquely determined by (X, f). □

5.5.2

Lemma. *Let n and s be the fixed integers such that $s > n \geqq 0$. Then we have the following:*

(1) *The set of isomorphism classes of affine $\mathbb{A}^1$-bundles of the form $(F_n - S_\infty, \pi)$ with $S_\infty \sim B_n + s\ell$ is a locally closed subset $A(n,s)$ in the projective space $\mathbb{P}(H^0(\mathbb{P}^1, \mathscr{O}(2s-n-2))) = \mathbb{P}^{2s-n-2}$.*

(2) $\dim A(n,s)$ *equals* $2s-2n-1$ *if* $n > 0$ *and* $2s-2$ *if* $n = 0$.

(3) $A(0,s)$ *and* $A(1,s)$ *are dense subset of* $\mathbb{P}^{2s-2}$ *and* $\mathbb{P}^{2s-3}$*, respectively.*

Proof. Our proof consists of three steps.

(I) Let (X, f) be an affine $\mathbb{A}^1$-bundle isomorphic to $(F_n - S_\infty, \pi)$ with $S_\infty \sim B_n + s\ell$. By virtue of 5.2, (X, f) is determined by a nontrivial extension

$$0 \longrightarrow \mathscr{O}_C \xrightarrow{\iota} E \xrightarrow{\rho} \mathscr{O}_C(2s-n) \longrightarrow 0,$$

where $F_n = \mathbb{P}(E)$ and $\mathscr{O}_C(2s-n)$ gives rise to a section S_∞ of F_n. **66**
Then it is easily shown that $E = \mathscr{O}_C(s-n)e_1 \oplus \mathscr{O}_C(s)e_2$, where e_1 and e_2 constitute a basis of the decomposable rank 2 vector bundle E over C. The injection $l : \mathscr{O}_C \hookrightarrow E$ is given by elements $f \in H^0(C, \mathscr{O}_C(s-n))$ and $g \in H^0(C, \mathscr{O}_C(s))$ such that $f \neq 0$, $g \neq 0$ and $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \phi$. Such a pair (f, g) is a point of a nonempty open set U in $\{\mathbb{A}^{n-s+1} - (0)\} \times \{\mathbb{A}^{s+1} - (0)\}$. On the other hand, l determines the surjection $\rho : E \to \mathscr{O}_C(2s-n)$ uniquely up to multiplication of elements of k^* on $\mathscr{O}_C(2s-n)$; indeed, if $\mathscr{O}_C(2s-n)$ is identified with $\Lambda^2 E$ then ρ is given by $\rho(e_1) = -ge_1\Lambda e_2$ and $\rho(e_2) = fe_1\Lambda e_2$.

(II) Let

$$0 \longrightarrow \mathscr{O}_C \xrightarrow{\iota'} E' \xrightarrow{\rho'} \mathscr{O}_C(2s-n) \longrightarrow 0$$

be a nontrivial extension with $E' = \mathscr{O}_C(s-n)e'_1 \oplus \mathscr{O}_C(s)e'_2$, and let l' be determined by a pair $(f', g') \in U$. If $\phi : E' \to E$ is an

$\mathscr{O}_C$-isomorphism then ϕ is expressed in the form:

$$\phi\begin{pmatrix}e_1'\\ e_2'\end{pmatrix}=\begin{pmatrix}\alpha & h\\ 0 & \beta\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}\ \text{ if } n>0,\ \text{ where } \alpha,\beta\in k^* \text{ an } h\in H^0(C,\mathscr{O}_C(n));$$

$$\phi\begin{pmatrix}e_1'\\ e_2'\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}\ \text{ if } n=0,\ \text{ where } \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(2,k).$$

67 Then ϕ satisfies $l=\phi\cdot l'$ and $\rho\cdot\phi=\psi\cdot\rho'$ for some $\mathscr{O}_C$-isomorphism $\psi:\mathscr{O}_C(2s-n)\xrightarrow{\sim}\mathscr{O}_C(2s-n)$ if and only if we have:

(i) $f=\alpha f'$ and $g=\beta g'+hf'$ if $n>0$,

(ii) $f=\alpha f'+\gamma g'$ and $g=\beta f'+\delta g'$ if $n=0$.

Let G be an algebraic group defined by:

$$G=\left\{\begin{pmatrix}\alpha & h\\ 0 & \beta\end{pmatrix};\alpha,\beta\in k^* \text{ and } h\in H^0(C,\mathscr{O}_C(n))\right\}\ \text{ if } n>0,$$

and $G=GL(2,k)$ if $n=0$.

Then it is readily verified that the subset U of $\{\mathbb{A}^{s-n+1}-(0)\}\times\{\mathbb{A}^{s+1}-(0)\}$ is G-stable and G acts freely on U. Therefore, $A(n,s)$ is a locally closed subset of $\mathbb{P}(H^1(C,L^{-1}))$ with $L\cong\mathscr{O}_C(2s-n)$, and $A(n,s)$ is isomorphic to the quotient variety U/G. Thus we know that $\dim A(n,s)=(2s-n+2)-(n+3)=2s-2n-1$ if $n>0$, and $\dim A(0,s)=(2s+2)-4=2s-2$.

(III) Note that $\mathbb{P}(H^1(C,L^{-1}))\cong\mathbb{P}^{2s-n-2}$ where $L\cong\mathscr{O}_C(2s-n)$. By comparison of dimensions of $A(n,s)$ and $\mathbb{P}^{2s-n-2}$ we know that $A(0,s)$ and $A(1,s)$ are dense subsets of $\mathbb{P}^{2s-2}$ and $\mathbb{P}^{2s-3}$, respectively. This completes the proof of Lemma 5.5.2.

□

5.5.3

Lemma. *Let (n,s) and (n',s') be pairs of integers such that $s>n\geqq 0$, $s'>n'\geqq 0$ and $2s-n=2s'-n'$. Then the subsets $A(n,s)$ and $A(n',s')$ of $\mathbb{P}(H^1(C,\mathscr{O}_C(n-2s))$ have no intersection if $(n,s)\neq(n',s')$.*

Proof. Immediate in virtue of Lemmas 5.3 and 5.5.1. □

5.5.4

In virtue of Lemmas 5.4.2, 5.5.1, 5.5.2 and 5.5.3 we have the following: **68**

Theorem. *Let m be a positive integer. Then we have:*

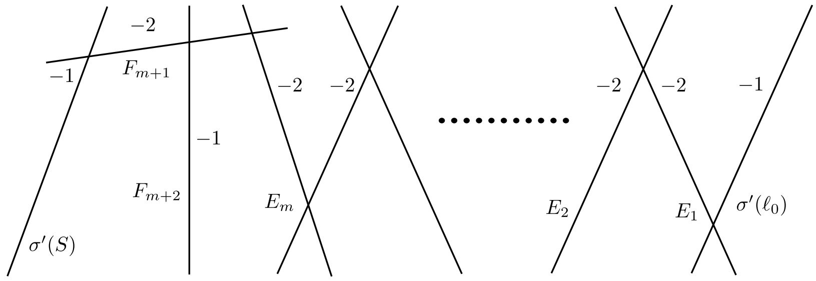
(1) *The set of isomorphism classes of affine $\mathbb{A}^1$-bundles (X, f) with $(S_\infty^2) = m$ is isomorphic to the set of k-rational points of $\mathbb{P}(H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m-2)))$.*

(2) *The projective space $\mathbb{P}(H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m-2)))$ is decomposed into a disjoint union of locally closed subsets $A(n, s)$, where (n, s) runs through all pairs of integers such that $s > n \geqq 0$ and $m = 2s - n$.*

(3) *$A(n, s)$ is isomorphic to the set of isomorphism classes of affine $\mathbb{A}^1$-bundles (X, f) over $\mathbb{P}^1$ which are of the form: $(F_n - S_\infty, \pi)$ with $S_\infty \sim B_n + s\ell$.*

5.6

Let (X, f) be an affine $\mathbb{A}^1$-bundle $(F_n - S_\infty, \pi)$ with $S_\infty \sim B_n + s\ell$ and $s > n$. Then the affine surface X has structures of $\mathbb{A}^1$-bundles other than $f : X \to \mathbb{P}^1$, as will be shown below. Let $V := F_n$ and $S := S_\infty$. Let P_0 be an arbitrary point on S, let $\sigma_1 : V_1 \to V$ be a quadratic transformation with center at P_0 and let $P_1 := \sigma_1'(S) \cap \sigma_1^{-1}(P_0)$. For $1 \leq i \leq m$, define inductively a quadratic transformation $\sigma_{i+1} : V_{i+1} \to V_i$ with center P_i and let $P_{i+1} := (\sigma_1 \dots \sigma_{i+1})'(S) \cap \sigma_{i+1}^{-1}(P_i)$, where $m = 2s - n$. Let Q be a point on $(\sigma_{m+1})^{-1}(P_m)$ other than P_{m+1} and $(\sigma_{m+1})^{-1}(P_m) \cap (\sigma_{m+1})'(\sigma_m^{-1}(P_{m-1}))$. Let $\tau : W \to V_{m+1}$ be a quadratic transformation with center at Q. Let $\sigma := (\sigma_1 \cdot \dots \cdot \sigma_{m+1} \cdot \tau) : W \to V$, let $E_i := (\sigma_{i+1} \cdot \dots \cdot \sigma_{m+1} \cdot \tau)'(\sigma_i^{-1}(P_{i-1}))$ for $1 \leq i \leq m$, and let **69**
$E_{m+1} := \tau'(\sigma_{m+1}^{-1}(P_m))$ and $E_{m+2} := \tau^{-1}(Q)$. Let ℓ_0 be the fiber of the canonical projection $F_n \to \mathbb{P}^1$ passing through P_0. Then $\sigma^{-1}(S \cup \ell_0)$ has the following configuration:



5.6.1

Let Λ be the linear subsystem of $|B_n + (s+1)\ell|$ consisting of members which pass through $P_0, \ldots, P_m, Q$ with multiplicities $\geqq 1$. Then we have the following:

Lemma. *With the notations as above, we have:*

(1) *Λ is an irreducible linear pencil.*

(2) *$S + \ell_0$ is a unique reducible member of Λ, and all other members of Λ are nonsingular rational irreducible curves.*

(3) *The proper transform Λ' of Λ by σ has no base points; E_{m+2} is a cross-section of the morphism $\Phi_{\Lambda'} : W \to \mathbb{P}^1$ defined by Λ'; $\sigma'(S) + E_{m+1} + E_m + \cdots + E_1 + \sigma'(\ell_0)$ is a member of Λ'.*

Proof. Our proof consists of two steps.

(I) Since $\dim|B_n + (s+1)\ell| = 2s - n + 3 = m + 3$ we know that $\dim \Lambda \geqq m+3-(m+2) = 1$. Let D be a reducible member (if at all) of Λ such that $D \neq S + \ell_0$, and write $D = \sum_{i=1}^{t} n_t D_t$ with irreducible
70 components D_i and integers $n_i > 0$ for $1 \leqq i \leqq t$. Then it is easy to see that one of D_i's, say D_1, is linearly equivalent to $B_n + r\ell$ with $r \geqq 0$ and $n_1 = 1$, and $D_2, \ldots, D_t$ are fibers of the canonical projection $F_n \to \mathbb{P}^1$; we have $r \leqq s$ because D is a reducible member. Then, since $m \geqq 2$, D_1 must pass through the points P_0, $P_1, \ldots, P_m$. This implies that $(D_1 \cdot S) = s+r-n \geqq m+1 = 2s-n+1$, whence $r \geqq s+1$. This is a contradiction. Hence every member

D of Λ such that $D \neq S + \ell_0$ is an irreducible curve. On the other hand, since $(D \cdot \ell) = 1$ we know that D is a nonsingular rational curve.

(II) The fact that every number D of Λ such that $D \neq S + \ell_0$ is a nonsingular irreducible curve implies the following:

(i) $\sigma'(S) + E_{m+1} + E_m + \cdots + E_1 + \sigma'(\ell_0)$ is a member of Λ'; hence $S + \ell_0$ is a (unique reducible) member of Λ.

(ii) Every member of Λ' other than $\sigma'(S) + E_{m+1} + \cdots + E_1 + \sigma'(\ell_0)$ is of the form $\sigma'(D)$ with $D \in \Lambda$.

Let D and D' be general members of Λ. Then, since $(D \cdot D') = ((B_n + (s+1)\ell)^2) = 2s - n + 2 = m + 2$ we have $(\sigma'(D) \cdot \sigma'(D')) = 0$. This implies in turn the following:

(iii) Λ' (hence Λ) is an irreducible linear pencil; Λ' has no base points at all.

(iv) E_{m+2} is a cross-section of the morphism $\Phi_{\Lambda'} : W \to \mathbb{P}^1$ defined by Λ'.

The above observations complete the proof of Lemma (5.6.1). □

5.6.2

Let $\rho : W \to Z$ be the contraction of $\sigma'(S), E_{m+1}, E_m, \ldots, E_1$ in this order, and let $T = \rho(E_{m+2})$. Since ρ contracts only curves in the member $\sigma'(S) + E_{m+1} + \cdots + E_1 + \sigma'(\ell_0)$ of Λ' we know that the proper transform of Λ' by ρ defines a structure of $\mathbb{P}^1$-bundle on Z, for which $\rho(\sigma'(D))$ $(D \in \Lambda, D \neq S + \ell_0)$ and $\rho(\sigma'(\ell_0))$ constitute the fibers of the $\mathbb{P}^1$-bundle $q : Z \to \mathbb{P}^1$, and T is a cross-section with $(T^2) = m$. Note that $X = F_n - S$ is unchanged under a birational transformation $\rho \cdot \sigma^{-1} : V \to Z$. Consequently, X has a structure of $\mathbb{A}^1$-bundle $g : X \to \mathbb{P}^1$ other than $f : X \to \mathbb{P}^1$, where $X = Z - T$ and $g := q|_X$. However, we could not determine integers n' and s' such that $Z = F_{n'}$, and $T \sim B_{n'} + s'\ell$. 71

5.7

In this paragraph we shall show that the affine surface $X = F_n - S_\infty$ constructed in 5.2 has a nontrivial G_a-action. With the notations of 5.6, choose a point P_0 so that $P_0 \notin S_\infty \cap B_n$ if $n > 0$. Take the points $P_1, \ldots, P_{m-1}$ as in 5.6, and let $\sigma_i : V_i \to V_{i-1}$ be a quadratic transformation with center at P_{i-1} for $1 \leqq i \leqq m$, where $V_0 := V$. Let $\varphi = \sigma_1 \cdot \ldots \cdot \sigma_m$, and let $E_i = (\sigma_{i+1} \cdot \ldots \cdot \sigma_m)'(\sigma_i^{-1}(P_{i-1}))$, by abuse of notations, for $1 \leqq i < m$ and $E_m = \sigma_m^{-1}(P_{m-1})$.

5.7.1

Let N be the linear subsystem of $|B_n + (s-1)\ell|$ consisting of members which pass through the points $P_0, P_1, \ldots, P_{m-2}$ with multiplicities $\geqq 1$. Then we have:

Lemma. *With the notations as above, M consists of a single member T which is a nonsingular rational irreducible curve.*

72 *Proof.* Since $\dim |B_n + (s-1)\ell| = 2s - n - 1 = m - 1$, we have $\dim M \geqq (m-1) - (m-1) = 0$. Hence M is not empty. Let D be a member of M. We shall show that D is an irreducible curve. Assume the contrary, and write $D = \sum_{i=1}^{t} n_i D_i$ with irreducible components D_i and integers $n_i > 0$ for $1 \leqq i \leqq t$. Then, as in the proof of Lemma 5.6.1, one of D_i's, say D_1, is linearly equivalent to $B_n + r\ell$ with $r \geqq 0$ and $n_1 = 1$, and $D_2, \ldots, D_t$ are fibers of the canonical projection $F_n \to \mathbb{P}^1$. Then we have $r \leqq s - 2$ since D is reducible, whence $s \geqq 2$ and $m \geqq 3$. Then D_1 must pass through the points $P_0, P_1, \ldots, P_{m-2}$. This implies that $(D_1 \cdot S) = s + r - n \geqq m - 1 = 2s - n - 1$, whence $r \geqq s - 1$. This is a contradiction. Thus every member D of M is irreducible. On the other hand, since $(D \cdot \ell) = 1$ we know that D is a nonsingular rational curve. If $\dim M > 0$, let D and D' be general members of M. Then $(D \cdot D') = ((B_n + (s-1)\ell)^2) = 2s - n - 2 = m - 2$ while $(D \cdot D')$ must be $\geqq m - 1$. This is a contradiction. Hence $\dim M = 0$. $\square$

5.7.2

Let M be the linear subsystem of $|B_n + s\ell|$ consisting of members which pass through the points $P_0, P_1, \ldots, P_{m-1}$ with multiplicities $\geqq 1$. Then we have:

Lemma. *With the notations as above, we have:*

(1) *N is an irreducible linear pencil.*

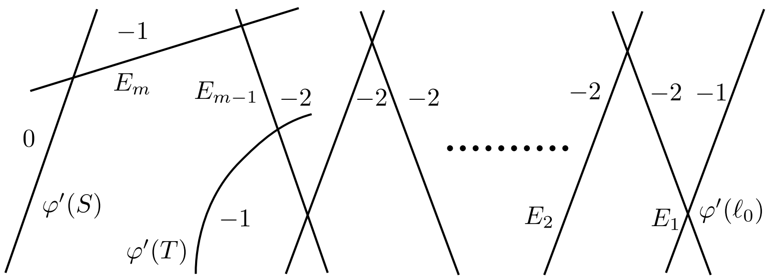
(2) *$T + \ell_0$ is a unique reducible member of N, and all other members of N are nonsingular rational irreducible curves.*

(3) *The proper transform N' of N by φ has no base points; E_m is a cross-section of the morphism $\Phi_{N'} : V_m \to \mathbb{P}^1$ defined by N';* **73**
$\varphi'(T) + E_{m-1} + \cdots + E_1 + \varphi'(\ell_0)$ is a member of N'.

Proof. All assertions can be proved in the same fashion as in the proof of 5.6.1 with slight modifications. Therefore we shall leave a proof to readers as an exercise. □

5.7.3

We have the following configuration of $\varphi^{-1}(S \cup T \cup \ell_0)$:



Note that $X = V_m - (\varphi'(S) \cup E_m \cup \ldots \cup E_1)$ and V_m has a linear pencil N' whose members are $\varphi'(D)$'s for $D \in N$ with $D \neq T + \ell_0$ and $\varphi'(T) + E_{m-1} + \cdots + E_1 + \varphi'(\ell_0)$. Therefore, it is easily seen that the affine surface X has an algebraic pencil $\mathscr{F}$ of affine lines parametrized by the affine line $\mathbb{A}^1$. Let Q_0 be the point on $\mathbb{A}^1$ corresponding to the member $(\varphi'(T) \cup \varphi'(\ell_0)) \cap X$. Then $X_0 := X - (\varphi'(T) \cup \varphi'(\ell_0))$ has an algebraic pencil of affine lines parametrized by $\mathbb{A}^1_* := \mathbb{A}^1 - \{Q_0\}$, where every member of the pencil is the affine line. Hence X_0 is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1_*$, which is trivial, i.e., $X_0 \cong \mathbb{A}^1 \times \mathbb{A}^1_*$. Then, as in Lemma 2.2.1 and Theorem 2.3, we can readily show that there exists a nontrivial G_a-action on X such that every member of $\mathscr{F}$ other than $(\varphi'(T) \cup \varphi'(\ell_0)) \cap X$ is the G_a-orbit.

6 Locally nilpotent derivations in connection with the cancellation problem

74

6.1

A k-algebra A is called *strongly n-invariant* (or *n-invariant*) if A satisfies the condition: Given a k-algebra B and indeterminates $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, if $\theta : A[X_1, \dots, X_n] \xrightarrow{\sim} B[Y_1, \dots, Y_n]$ is a k-isomorphism then we have necessarily $\theta(A) = B$ (or A is isomorphic to B under some k-isomorphism). If A is strongly n-invariant (or n-invariant) for all integers $n \geqq 1$ then A is called *strongly invariant* (or *invariant*). A problem asking whether or not a (given) k-algebra A is strongly invariant (or invariant) is called, in general, the cancellation problem. The purpose of this section is to apply the results in the previous sections to the cancellation problem. Namely, we are interested in looking for necessary or sufficient conditions for a given k-algebra to be strongly invariant, which can be written in terms of locally finite (or locally finite iterative) higher derivations.

6.2

A sufficient condition for strong 1-invariance is given, by making use of Nagata's theorem [42], in the following:

Lemma (cf. [1]). *Let A be an affine k-domain. If A is not birationally ruled over k, then A is strongly* 1*-invariant.*

Here, an affine k-domain A is said to be *birationally ruled over k* if the quotient field $Q(A)$ is a purely transcendental extension $K(t)$ in one variable over a sub field K of $Q(A)$ containing k.

6.3

Another sufficient condition for strong invariance is the following: 75

Lemma. *Let A be a k-algebra. If A has no nontrivial locally finite higher derivations then A is strongly invariant.*

Proof. [3] Assume that A is not strongly invariant. Then there exists a k-algebra $B(\neq A)$ such that $A[X_1, \dots, X_n] = B[Y_1, \dots, Y_n]$ for some integer $n \geqq 1$, where $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are algebraically independent over A and B, respectively. Let a be an element of A not in B. Then a is written as

$$a = \Sigma b_{\alpha_1 \dots \alpha_n} Y_1^{\alpha_1} \dots Y_n^{\alpha_n} = f(Y_1, \dots, Y_n) \notin B.$$

Assume that Y_1 appears in $f(Y_1, \dots, Y_n)$. Let T be an indeterminate and let ψ be a k-algebra homomorphism of $B[Y_1, \dots, Y_n]$ into $B[Y_1, \dots, Y_n, T]$ such that $\psi(Y_1) = Y_1 + T$ and $\psi(Y_i) = Y_i$ for $2 \leqq i \leqq n$. Then we can see easily that $\psi(a)$ is written as

$$\psi(a) = a + T^m g(Y_1, \dots, Y_n, T) \text{ with } g(Y_1, \dots, Y_n, T) \neq 0 \text{ and } m \geqq 1.$$

Write $g(Y_1, \dots, Y_n, T) = h(X_1, \dots, X_n, T) \in A[X_1, \dots, X_n, T]$. Let $\mu_1, \dots, \mu_n$ be a set of positive integers such that $h(T^{\mu_1}, \dots, T^{\mu_n}, T) \neq 0$. Let 2 be the canonical injection $A \hookrightarrow A[X_1, \dots, X_n]$ and let τ be a homomorphism (of A-algebras) of $A[X_1, \dots, X_n, T] = B[Y_1, \dots, Y_n, T]$ into $A[T]$ such that $\tau(X_i) = T^{\mu_i}$ for $1 \leqq i \leqq n$ and $\tau(T) = T$. Let $\rho = \tau \cdot \psi \cdot 2$. Then ρ is a k-algebra homomorphism of A into $A[T]$ such that $\rho(a) \notin A$ and ρ defines a nontrivial locally finite higher derivation 76
(cf. Lemma 1.2). □

[3] We are indebted to Y. Ishibashi for improving the original proof.

6.4

As a practical criterion for strong invariance, the next result given in 6.4.1 below is often more useful than the one given in Lemma 6.3.

6.4.1

Lemma. *Let k be an infinite field and let A be an affine k-domain satisfying the conditions:*

(1) $\mathrm{Spec}(A)(k)$ *is dense in* $\mathrm{Spec}(A)$.

(2) *There is no nonconstant k-morphism from the affine line $\mathbb{A}^1_k$ to* $\mathrm{Spec}(A)$.

Then A is strongly invariant.

The proof can be done along the same principle as in the proof of Lemma 6.3, and we shall leave it to readers.

6.4.2

The rings in the next two examples can be shown to be strongly invariant by applying Lemma 6.4.1, the first one of which was first given by Hochster [23] and discussed later by Eakin and Heinzer [13].

Example 1. Let $A_n := \mathbb{R}[X_0, \dots, X_n]/(X_0^2 + \cdots + X_n^2 - 1)$ be the affine ring of the real n-sphere for $n \geqq 1$. Then A_n is strongly invariant; a polynomial ring $A_n[t]$ in one variable over A_n is invariant; a polynomial ring $A_n[t_1, \dots, t_n]$ in n-variables over A_n is not 1-invariant if $n \neq 1, 3, 7$.

Example 2. Let k be a non-perfect field of characteristic $p > 0$, and let $A = k[X, Y]/(Y^{p^n} - X - a_1X^p - \dots - a_rX^{p^r})$, where r, $n > 0$ and
77 $a_1, \dots, a_r \in k$ with one of $a_1, \dots, a_r \notin k^p$. A is the affine ring of a Russell k-group, which will be discussed in Chapter III. Then A is strongly invariant, while, for the perfect closure k' of k, $A \underset{k}{\otimes} k'$ is not strongly invariant because $A \underset{k}{\otimes} k'$ is a polynomial ring in one variable over k'.

The second example exhibits that strong invariance is not preserved under faithfully flat ascent, while it is preserved under faithfully flat descent (cf. Miyanishi and Nakai [36]).

6.4.3

The converse of Lemma 6.3 does not hold as shown by the next

Example. Let k be an algebraically closed field and let A be the affine ring of the affine cone of a nonsingular projective variety U. Assume that there is no nonconstant k-rational mapping from $\mathbb{A}^1_k$ to U. Then A is strongly invariant, while A has a nontrivial locally finite higher derivation.

Proof. As shown in 2.2.3, A has a nontrivial locally finite higher derivation. Hence it remains to show that A is strongly invariant. Assume that we are given a k-algebra B satisfying $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$, where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are algebraically independent over A and B, respectively. Set $V := \mathrm{Spec}(A)$ and $W := \mathrm{Spec}(B)$; W (as well as V) is an affine variety defined over k because the relation $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$ implies that B is an affine k-domain; we have $V \times \mathbb{A}^n_k = W \times \mathbb{A}^n_k$. Let $q_V : V \times \mathbb{A}^n_k \to V$ and $q_W : W \times \mathbb{A}^n_k \to W$ be the canonical projections onto V and W, respectively, and let $\pi : V - \{v_0\} \to U$ be the projection of the cone to the base variety, where v_0 is the vertex of the 78
cone V. For a point w of W, $\pi q_V(q_W^{-1}(w))$ is a point u of U because of the stated assumption that there is no nonconstant k-rational mapping of $\mathbb{A}^1_k$ to U. Assume that $q_V(q_W^{-1}(w))$ is not a point. Then $q_V(q_W^{-1}(w)) = \pi^{-1}(u)$ because $q_V(q_W^{-1}(w))$ is an affine rational curve with only one place at infinity and $q_V(q_W^{-1}(w)) \subset \pi^{-1}(u)$. This implies that $q_W^{-1}(w)$ intersects the singular locus $q_V^{-1}(v_0)$ of $V \times \mathbb{A}^n_k = W \times \mathbb{A}^n_k$. Besides, it is readily shown that W has a unique singular point w_0 and $q_W^{-1}(w_0)$ is the singular locus of $W \times \mathbb{A}^n_k$; hence $q_V^{-1}(v_0) = q_W^{-1}(w_0)$. Thus, we have $w = w_0$ because $q_W^{-1}(w) \cap q_W^{-1}(w_0) \neq \phi$. If $w \neq w_0$ we have shown that $q_V(q_W^{-1}(w))$ is a point v of V, i.e., $q_W^{-1}(w) \subset q_V^{-1}(v)$. Indeed, we have $q_W^{-1}(w) = q_V^{-1}(v)$ because both $q_W^{-1}(w)$ and $q_V^{-1}(v)$ are isomorphic to $\mathbb{A}^n_k$ (cf. Ax [8]). This means that every maximal ideal of B is vertical relative to A in the terminology of [1]. Then $A = B$ by virtue of [ibid., (1.13)]. □

6.5

A necessary condition for strong invariance is given by the next

Lemma. *Let A be a k-algebra. If A has a nontrivial locally finite iterative higher derivation D then A is not strongly* 1*-invariant.*

Proof. Let $\varphi : A \to A[t]$ be the k-homomorphism associated with D (cf. 1.2). Let $B = \varphi(A)$. We shall show that $A[t] = B[t]$. Since $B[t] \subseteq A[t]$, we have only to show the following assertion by induction on n:

79 $P(n)$: If a is an element of A with D-length $\ell(a) = n$ (cf. 1.4) then $a \in B[t]$.

If $\ell(a) = 0$ then $a = \varphi(a) \in B$. Assume that $\ell(a) = n > 0$ and $P(r)$ is true for $0 \leqq r < n$. Since $\ell(D_i(a) < n)$ for $i \geqq 1$ we have $D_i(a) \in B[t]$ by virtue of $P(r)$ for $r < n$. Then, since $a = \varphi(a) - \sum_{i \geqq 1} D_i(a)t^i$ we have $a \in B[t]$. Thus, $P(n)$ is proved, and A is not strongly 1-invariant. □

6.6

In the paragraphs 6.6 and 6.7 we shall consider whether or not the converse of Lemma 6.6 is true. When A is an affine k-domain of dimension 1, this is true and was essentially proved in [1; (3.4)]. We have in fact:

6.6.1

Proposition. *Let A be an affine k-domain of dimension* 1*. Then the following conditions are equivalent to each other:*

(1) *A is strongly invariant.*

(2) *A is strongly* 1*-invariant.*

(3) *A has no nontrivial locally finite iterative higher derivation.*

Proof. (1) $\Longrightarrow$ (2) is clear; (2) $\Longrightarrow$ (3) follows from Lemma 6.5 and its proof. (3) $\Longrightarrow$ (1): It is proved in [1; (3.4)] that under the stated assumption A is either strongly invariant or A is a polynomial ring $k_0[x]$ over the algebraic closure k_0 of k in A. In the latter case A has a nontrivial locally finite iterative higher derivation. Thus we have (3) $\Longrightarrow$ (1). □

6.6.2

When $\dim A = 2$ we have the following: **80**

Proposition . *Let k be an algebraically closed field of characteristic zero, and let A be an irrational nonsingular affine k-domain of dimension 2. Then we have one of the following three cases:*

(1) *A is strongly 1-invariant.*

(2) *A has a nontrivial locally finite iterative higher derivation.*

(3) *There is a surjective morphism $\pi : \mathrm{Spec}(A) \to C$ from $\mathrm{Spec}(A)$ to a nonsingular complete curve C of genus > 0, whose general fibers are isomorphic to the affine line $\mathbb{A}^1_k$.*

Proof. Assume that A is not strongly 1-invariant. Then, by virtue of Lemma 6.2, A is birationally ruled. Set $V := \mathrm{Spec}(A)$. Since A is irrational the irregularity g of V is positive; the Albanese mapping of a nonsingular completion[4] of V induces a unique morphism $\pi : V \to C$, where $C = \pi(V)$ and C is a nonsingular (not necessarily complete) curve of genus $g > 0$; the general fibers of π are irreducible rational curves. On the other hand, since A is not strongly 1-invariant there exists an affine k-domain $B(\neq A)$ of dimension 2 such that $A[X] = B[Y]$, where X and Y are algebraically independent over A and B, respectively. Set $W := \mathrm{Spec}(B)$, and let $q_V : V \times \mathbb{A}^1_k \to V$ and $q_W : W \times \mathbb{A}^1_k \to W$ be the canonical projections from $V \times \mathbb{A}^1_k = W \times \mathbb{A}^1_k$ to V and W, respectively. **81**
For a general point w of W, $\ell_w := q_V(q_W^{-1}(w))$ is an affine rational curve with only one place at infinity. Indeed, if $q_V(q_W^{-1}(w))$ is a point v on V then $q_V^{-1}(v) = q_W^{-1}(w)$; if $q_W^{-1}(w) = q_V^{-1}(v)$ for every point w of W and a point v of V (depending on w) every maximal ideal of B is vertical relative to A, whence $A = B$ (cf. [1; (1.13)]); thus $q_V(q_W^{-1}(w))$ is not a point for some point w of W and, *a fortiori*, for a general point of W. Since $\pi(\ell_w)$ is a point on C we know that ℓ_w is contained in a fiber of π; since a general fiber of π is irreducible ℓ_w coincides with a fiber of π for a general point w of W. Moreover, since the morphism $\pi : V \to C$ defines

[4] Note that π does not depend on choices of nonsingular completions of V.

an irrational pencil on V and since an irrational pencil on a nonsingular surface has no base points, the second theorem of Bertini's tells us that ℓ_w is isomorphic to the affine line. consequently, we know that the morphism $\pi : V \to C$ is an algebraic pencil of affine lines parametrized by the curve C (cf. Section 2). If C is not complete, set $A_0 := k[C]$. Then A_0 is a k-subalgebra of A of dimension 1, and we have a nontrivial G_a-action on V with respect to which the general fibers of π are G_a-orbits (cf. Lemma 2.2.1 and Theorem 2.3). Thus we are reduced to the case (2). If C is a complete curve then we are reduced to the case (3). □

6.6.3

Lemma . *Let k be an algebraically closed field of characteristic zero and let V be a nonsingular affine surface defined over k. Assume that there exists a surjective morphism $\pi : V \to C$ from V onto a nonsingular*
82 *complete curve C, whose general fibers are isomorphic to the affine line. Then we have:*

(1) *Every irreducible component of a fiber of π is isomorphic to the affine line; if a fiber is reducible every irreducible component is a connected component.*

(2) *There exist a nonsingular projective surface $\widetilde{V}$ and a surjective morphism $\widetilde{\pi} : \widetilde{V} \to C$ such that:*

   (i) *$\widetilde{V}$ contains V as an open set, and $\widetilde{\pi}|_V = \pi$,*

   (ii) *general fibers of $\widetilde{\pi}$ are isomorphic to the projective line $\mathbb{P}^1_k$,*

   (iii) *$\widetilde{V} - V$ consists of a cross-section S and irreducible components contained in several reducible fibers of $\widetilde{\pi}$.*

Proof. Let $\widetilde{V}$ be a nonsingular projective surface containing V as an open set. Then the morphism $\pi : V \to C$ defines an irreducible pencil Λ on $\widetilde{V}$, whose base points (if at all) lie on $\widetilde{V} - V$. By replacing $\widetilde{V}$ (if necessary) by the surface which is obtained from $\widetilde{V}$ by a succession of quadratic transformations with centers at base points (including infinitely near base points) of Λ, we may assume that Λ has no base points. Let $\widetilde{\pi} : \widetilde{v} \to C$. Since a general fiber ℓ of π is isomorphic to $\mathbb{A}^1_k$

and the characteristic of k is zero, we know that a general fiber of $\widetilde{\pi}$ is isomorphic to $\mathbb{P}^1_k$ and ℓ is of the form: $\ell = \widetilde{\ell} - \widetilde{\ell} \cap S$, where $\widetilde{\ell} \cong \mathbb{P}^1_k$, S is a cross-section of $\widetilde{\pi}$ and $(\widetilde{\ell} \cdot S) = 1$. Then all assertions stated in the lemma are readily verified by looking at the fibration $\widetilde{\pi} : \widetilde{V} \to C$ and taking into account that V is an affine open set of $\widetilde{V}$, (see Chapter 2, Section 2). □

6.6.4

In the case (3) of Proposition 6.6.2 the surface $V := \mathrm{Spec}(A)$ has a **83**
structure as described in Lemma 6.6.3. We have an impression that A is strongly 1-invariant in this case. As an evidence we shall prove in the next paragraph that A is strongly 1-invariant in the simplest case; namely the case where every fiber of π is irreducible (cf. Theorem 4.9 and Lemma 5.2).

6.7

Proposition . *Let k be an algebraically closed field of characteristic zero, let C be a nonsingular complete curve of genus $g > 0$ defined over k, let L be an ample line bundle over C and let E be a nontrivial extension of L by $\mathcal{O}_C$. Let X be the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ minus the section S corresponding to L and let A be the affine ring of X. Then A is strongly* 1*-invariant.*

6.7.1

In order to prove this result we need the next

Lemma. *Let k be a field of characteristic zero and let φ be a k-automorphism of a polynomial ring $k[x, y]$ in two variables x, y over k. Assume that φ is given by $\varphi(x) = f$ and $\varphi(y) = g$ with $f, g \in k[x, y]$. Then f has the following form unless f is a polynomial in x or y alone:*

(*) $$f = ax^m + by^n + \sum_{\substack{m>i \\ n>j}} c_{ij} x^i y^j,$$

where a, b and c_{ij}'s are elements of k and $ab \neq 0$. The same assertion holds for g.

Proof. Our proof consists of four steps.

84 (I) First we shall treat the case where one of f and g, say f, is a polynomial in either one only of variables x and y, say y. Since φ is a k-automorphism of $k[x, y]$ the Jacobian determinant $\left|\dfrac{\partial(f, g)}{\partial(x, y)}\right| = -\left(\dfrac{\partial f}{\partial y}\right)\left(\dfrac{\partial g}{\partial x}\right)$ is a nonzero constant in k. Hence $\dfrac{\partial f}{\partial y} = a$ and $\dfrac{\partial g}{\partial x} = b$ are also nonzero constants in k. Thence we may write: $f = ay + c$ and $g = bx + h(y)$ with $c \in k$ and $h(y) \in k[y]$.

(II) Assume that f has the form $(*)$ and g is not a polynomial in x or y alone. Then we shall show that g has also the form $(*)$. Write

$$g = \alpha_0(y)x^u + \alpha_1(y)x^{u-1} + \cdots + \alpha_u(y) \quad (\alpha_0(y) \neq 0, u > 0)$$

where $\alpha_i(y) \in k[y]$. Since $\left|\dfrac{\partial(f, g)}{\partial(x, y)}\right|$ is a nonzero constant in k we can easily ascertain that the first derivative $\alpha'_0(y)$ is zero. Hence $\alpha_0(y)$ is a nonzero constant in k. Similarly if we write g in the form

$$g = \beta_0(x)y^v + \beta_1(x)y^{v-1} + \cdots + \beta_v(y) \quad (\beta_0(x) \neq 0, v > 0),$$

we have $\beta_0(x) \in k$. These facts imply that g has the form $(*)$

(III) It is known (cf. Chapter II, Section 3; also [43]) that any k-automorphism of $k[x, y]$ is a composite of linear automorphisms of type $(x, y) \mapsto (\alpha x + \beta y + c, \gamma x + \delta y + d)$ with $\alpha\delta - \beta\gamma \neq 0$ and de Jonquière automorphisms of type $(x, y) \mapsto (x, y + h(x))$ with $h(x) \ominus k[x]$. Using this fact we shall show that any k-automorphism of $k[x, y]$ is a composite of automorphisms, each of which is an automorphism ρ such that $\rho(x)$ or $\rho(y)$ coincides with one of x
85 and y. We shall say such an automorphism to be of type (P). Since a de Janquière automorphism is obviously of type (P) it

suffices to show that a linear automorphism is a composite of linear automorphism (P). Indeed, a linear automorphism $(x, y) \mapsto (\alpha x + \beta y + c, \gamma x + \delta y + d)$ is decomposed as follows: If $\alpha \neq 0$, $(x, y) \mapsto (x', y') = (\alpha x + \beta y + c, y)$, $(x', y') \mapsto (x', (\gamma/\alpha)x' + ((\alpha\delta - \beta\gamma)/\alpha)y' + (d - (\gamma c/\alpha)))$; if $\alpha = 0$, $(x, y) \mapsto (x', y') = (y, \gamma x + \delta y + d)$, $(x', y') \mapsto (((\beta\gamma - \alpha\delta)/\gamma)x' + (\alpha/\gamma)y' + (c - (\alpha d/\gamma)), y')$.

(IV) Write the given automorphism φ as $\varphi = \varphi_r \cdot \varphi_{r-1} \cdot \ldots \cdot \varphi_1$, where $\varphi_1, \ldots, \varphi_r$ are automorphisms of type (P). We shall prove our assertion by induction on r. If $r = 1$, φ has one of the following forms: $(x, y) \mapsto (ax + h(y), y)$, $(x, y) \mapsto (y, a_1 x + h_1(y))$, $(x, y) \mapsto (x, by + \ell(x))$ or $(x, y) \mapsto (b_1 y + \ell_1(x), x)$, where a, a_1, b, $b_1 \in k$, $h(y)$, $h_1(y) \in k[y]$ and $\ell(x)$, $\ell_1(x) \in k[x]$. Hence the assertion holds clearly. Assuming that the assertion is true when φ is a composition of less than r automorphisms of type (P) we shall consider the case where $\varphi = \varphi_r \cdot \varphi_{r-1} \cdot \ldots \cdot \varphi_1$. Let $\psi = \varphi_{r-1} \cdot \ldots \cdot \varphi_1$, and let $(\psi(x), \psi(y)) = (f_1, g_1)$ with f_1, $g_1 \in k[x, y]$. By the assumption of induction f_1 and g_1 have the form $(*)$ unless they are polynomials in x or y alone. Since φ_r is an automorphism of type (P) we have one of the following cases:

(i) $\varphi(x) = f_1$, (ii) $\varphi(x) = g_1$, (iii) $\varphi(y) = f_1$, (iv) $\varphi(y) = g_1$.

In any case we can easily ascertain the truth of our assertion in virtue of steps (I) and (II).

□

6.7.2 **86**

Proof of Proposition. *Our proof consists of three steps.*

(I) Let B be a k-algebra such that $A[T] = B[V]$, where T and V are algebraically independent over A and B, respectively. Set $Y :=$ Spec(B), and let $\pi : X \to C$ be the restriction onto X of the canonical projection $\mathbb{P}(E) \to C$. By a composition of projections

$Y \times \mathbb{A}^1_k = X \times \mathbb{A}^1_k \xrightarrow{p_1} X \xrightarrow{\pi} C$, each line $(y) \times \mathbb{A}^1_k$ with $y \in Y$ is sent to a point of C. Hence $\pi \cdot p_1$ factors as $Y \times \mathbb{A}^1_k \xrightarrow{p'_1} Y \xrightarrow{q} C$, and Y is viewed as a C-scheme by means of q. Note that q is surjective. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an affine open covering of C such that $E|_{U_i}$ is trivial for every $i \in I$. Let $\{x_i\}_{i \in I}$ be an affine coordinate system of X relative to $\mathscr{U}$; $\{x_i\}_{i \in I}$ is subject to $x_j = a_{ji}x_i + b_{ji}$ with $a_{ji} \in R^*_{ij}$ and $b_{ji} \in R_{ij}$, where $R_{ij} := k[U_i \cap U_j]$. Set $R_i := k[U_i]$ and $B_i := k[q^{-1}(U_i)]$. Then we have $R_i[x_i, T] = B_i[V]$ for every $i \in I$. Since B_i is an R_i-algebra and R_i is regular there is an element $y_i \in B_i$ such that $B_i = R_i[y_i]$ (cf. [1; (4.7)]). This implies that $q : Y \to C$ is an $\mathbb{A}^1$-bundle over C (cf. 4.9). Hence by virtue of Lemma 5.2 there exist an ample line bundle L' over C and a nontrivial extension E' of L' by $\mathscr{O}_C$ such that Y is C-isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(E')$ minus the section S' corresponding to L'.

(II) Let $\Omega^1_{X/C}$ be the $\mathscr{O}_X$-Module of 1-differential forms of X over C. Since $\Omega^1_{X/C}|_\pi - l_{(U_i)} = (dx_i)\mathscr{O}_\pi - l_{(U_i)}$ and $dx_j = a_{ji}dx_i$ we have in fact $\Omega^1_{X/C} \cong L \otimes_{\mathscr{O}_C} \mathscr{O}_X$. The relation $A[T] = B[V]$ implies

$$L \otimes_{\mathscr{O}_C} \mathscr{O}_X[T] \oplus \mathscr{O}_X[T] \cong L' \otimes_{\mathscr{O}_C} \mathscr{O}_Y[V] \oplus \mathscr{O}_Y[V].$$

87 *Hence we obtain $L \otimes_{\mathscr{O}_C} \mathscr{O}_X[T] \cong L' \otimes_{\mathscr{O}_C} \mathscr{O}_Y[V]$, or equivalently $(L \otimes_{\mathscr{O}_C} L'^{-1}) \otimes_{\mathscr{O}_C} \mathscr{O}_X[T] \cong \mathscr{O}_X[T]$. Hence we have $(L \otimes_{\mathscr{O}_C} L'^{-1}) \otimes_{\mathscr{O}_C} \mathscr{O}_X \cong \mathscr{O}_X$ by reduction modulo $T\mathscr{O}_X[T]$. Write $L \otimes_{\mathscr{O}_C} L'^{-1} = \mathscr{O}_C(D)$ for a divisor D on C. Then there exists an element h of $k(X)$ such that $\pi^{-1}(D) = (h)$. Let $\widetilde{\pi} : \mathbb{P}(E) \to C$ be the canonical projection. Then, viewing h as an element of $k(\mathbb{P}(E))$, we have $\widetilde{\pi}^{-1}(D) + mS = (h)$ for some integer m. Since $(\widetilde{\pi}^{-1}(D) + mS \cdot \ell) = ((h) \cdot \ell) = 0$ for a general fiber ℓ of $\widetilde{\pi}$ we obtain $m = 0$,* i.e., *$\widetilde{\pi}^{-1}(D) = (h)$. Now by restricting both hand sides on the section S we know that $D \sim 0$ on C. Therefore $L \cong L'$.*

(III) We have $R_i[x_i, T] = R_i[y_i, V]$ for every $i \in I$. Hence y_i is written

as

$$y_i = f_{i0}(x_i) + f_{il}(x_i)T + \cdots + f_{in}(x_i)T^n$$

with $f_{i0}(x_i), \ldots, f_{in}(x_i) \in R_i[x_i]$. *We shall show that* $n = 0$. *If otherwise, since* $K[x_i, T] = K[y_i, V]$ *with* $K := k(C)$, *Lemma 6.7.1 implies that* $f_{in}(x_i) \in K$. *Hence* $f_{in}(x_i) \in R_i[x_i] \cap K = R_i$. *Besides, since* $L' \cong L$ *we may assume, by replacing* $\mathcal{U}$ *by a finer affine open covering of* C *if necessary, that* $y_j = a_{ji}y_i + b'_{ji}$ *with* $b_{ji} \in R_{ij}$ *for any* $i, j \in I$. *Thence we know that* n *is independent of* $i \in I$ *and* $f_{jn}(x_j) = a_{ji} f_{in}(x_i)$ *for any* $i, j \in I$. *Set* $\alpha_i := f_{in}(x_i)$. *Then* $f_{jn}(x_j) = a_{ji} f_{in}(x_i)$ *for any* $i, j \in I$. *Set* $\alpha_i := f_{in}(x_i)$. *Then* $\{\alpha_i\}_{i \in I}$ *defines a nonzero element of* $H^0(C, L^{-1})$*; this contradicts the assumption that* L *is an ample line bundle over* C. *Thus,* $n =$
0. *This implies that* $y_i \in R_i[x_i]$ *for every* $i \in I$. *Hence* $B \subseteq A$. **88**
Changing the roles of x_i *and* y_i *in the above argument we have* $A \subset B$. *Consequently,* $A = B$ *and* A *is thus strongly 1-invariant.*

6.7.3

In contrast with Proposition 6.7 we have the following:

Proposition . *Let* k *be an algebraically closed field. Let* (X, f) *be an affine* $\mathbb{A}^1$*-bundle over the projective line* $\mathbb{P}^1_k$ (cf. 5.2) *and let* A *be the affine ring of* X. *Then* A *is not strongly* 1*-invariant.*

Proof. In virtue of 5.7 there exists a nontrivial G_a-action on X. Namely, A has a nontrivial locally finite iterative higher derivation. Then A is not strongly 1-invariant in virtue of Lemma 6.5. ◻

Part II

Curves on an affine rational surface

1 Irreducibility theorem

89

1.1

In this section the ground field k is assumed to be an algebraically closed field of characteristic p. Let $\mathbb{A}^2 := \mathrm{Spec}(k[x,y])$ be an affine plane over k. Fix an open immersion 1 of $\mathbb{A}^2$ into the projective plane $\mathbb{P}^2$ as a complement of the line at infinity ℓ_0. Assume that we are given an irreducible curve $C_0 : f(x,y) = 0$ ($f(x,y) \in k[x,y]$) on $\mathbb{A}^2$ with only one place at infinity. Let C be the closure of C_0 on $\mathbb{P}^2$, let $p_0 := \ell_0 \cap C$, let $d_0 = (\ell_0 \cdot C)$ (which equals the total degree of $f(x,y)$) and let d_1 be the multiplicity of C at P_0. With these notations and assumptions our ultimate goals are to prove the following theorems.

IRREDUCIBILITY THEOREM [(cf. Moh [38])] Assume that at least one of d_0 and d_1 is not divisible by p. Then the curve C_α on $\mathbb{A}^2$ defined by $f(x,y) = \alpha$ is an irreducible curve with only one place at infinity for 90
an arbitrary constant α in k.

Even in the case where d_0 and d_1 are divisible by p we can establish:

GENERIC IRREDUCIBILITY THEOREM [(cf. Ganong [14])] Let $\Lambda(f)$ be the linear pencil on $\mathbb{A}^2$ consisting of curves C_α with $\alpha \in k$.

Then the generic member of $\Lambda(f)$ is an irreducible curve with one purely inseparable place at infinity. Therefore the curve C_α is an irreducible curve with only one place at infinity for a general element α of k.

EMBEDDING THEOREM [(cf. Abhyankar-Moh [2])] Assume that C_0 is a nonsingular and rational curve, and that at least one of d_0 and d_1 is not divisible by p. Then there exists a biregular algebraic map of $\mathbb{A}^2$ onto itself which maps C_0 onto the y-axis.

1.2

In the paragraphs below we fix a nonsingular, rational, affine surface X defined over k and an irreducible closed curve C_0 on X with only one place at infinity (i.e., outside of C_0).

1.2.1

Definition. *An admissible datum for* (X, C_0) *is a set* $\mathscr{D} = \{V, U, C, \ell_0, \Gamma, d_0, d_1, e\}$ *such that:*

(1) *V is a nonsingular, rational, projective surface defined over k containing an open set U such that U is isomorphic to X over k. (Since U is affine, $V - U$ is of co-dimension 1.)*

(2) *Write $V - U := \bigcup_{i=1}^{n} \Gamma_i$ with irreducible components Γ_i. Then the following conditions hold:*

91 (i) *Γ_i is a nonsingular, rational, complete curve.*

(ii) *Γ_i intersects Γ_j transversely (if at all) in at most one point.*

(iii) *$\Gamma_i \cap \Gamma_j \cap \Gamma_\ell = \phi$ for three distinct indices.*

(iv) *$V - U$ contains no cyclic chains, i.e., there is no sequence $\{\Gamma_{i_1}, \ldots, \Gamma_{i_a}\}$ $(a \geqq 3)$ such that $\Gamma_{i_j} \cap \Gamma_{i_{j+1}} \neq \phi (1 \leqq j \leqq a-1)$ and $\Gamma_{i_a} \cap \Gamma_{i_1} \neq \phi$.*

(3) *C is an irreducible closed curve on V such that $C \cap U$ is isomorphic to C_0 by an isomorphism between U and X. (Hence $C - C_0$ consists of one point P_0, which is a one-place point.)*

(4) *C meets only one irreducible component ℓ_0 of $V - U$ at P_0. We set $d_0 := i(C, \ell_0; P_0) = (C \cdot \ell_0)$ and $d_1 :=$ the multiplicity of C at P_0.*

(5) *As a divisor on V, C is linearly equivalent to a divisor $d_0(e\ell_0 + \Gamma)$, where $e \geqq 1$ and Γ is an effective divisor such that* $\mathrm{Supp}(\Gamma) = \overline{V - (U \cup \ell_0)}$.

If there is no fear of confusion we denote $\mathscr{D}$ simply by $(V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ *by identifying U with X.*

1.2.2

Example. With the notations of 1.1, the set $\{\mathbb{P}^2, \mathbb{A}^2, C, \ell_0, \phi, d_0, d_1, 1\}$ is an admissible datum for $(\mathbb{A}^2, C_0)$. It is clear that $d_0 > d_1$ if $d_0 > 1$.

1.3

Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 > d_1 \geqq 1$. Find integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ by the following Euclidean algorithm:

$$
\begin{array}{rll}
d_0 &= q_1 d_1 + d_2 & 0 < d_2 < d_1 \\
d_1 &= q_2 d_2 + d_3 & 0 < d_3 < d_2 \\
& \cdots\cdots\cdots\cdots\cdots & \\
d_{\alpha-2} &= q_{\alpha-1} d_{\alpha-1} + d_\alpha & 0 < d_\alpha < d_{\alpha-1} \\
d_{\alpha-1} &= q_\alpha d_\alpha & 1 < q_\alpha.
\end{array}
$$

Here, we introduce the following transformation. 92

1.3.1

Definition. *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 > d_1 \geqq 1$.* The Euclidean transformation of V associated with $\mathscr{D}$ *(or simply,* the Euclidean transformation of V*) is the composition ρ of the following quadratic transformations: Let $P_0 := \ell_0 \cap C$, and let $\sigma_1 : V_1 \to V_0 := V$ be the quadratic transformation of V_0 with center at P_0. Set $C^{(1)} := \sigma_1'(C)$, $\Gamma^{(1)} := \sigma_1'(\Gamma) = \sigma_1^*(\Gamma)$, $\ell_0^{(1)} := \sigma_1'(\ell_0)$ and*

$\ell_1 = \ell_1^{(1)} := \sigma_1^{-1}(P_0)$. Let $P_1 := \ell_1 \cap C^{(1)}$, and let $\sigma_2 : V_2 \to V_1$ be the quadratic transformation of V_1 with center P_1. For $1 \leqq i \leqq N := q_1 + q_2 + \cdots + q_\alpha$, define $\sigma_i : V_i \to V_{i-1}$ inductively as follows: σ_i is the quadratic transformation of V_{i-1} inductively as follows: σ_i is the quadratic transformation of V_{i-1} with center at $P_{i-1} := \ell_{i-1} \cap C^{(i-1)}$. Let $\ell_i = \ell_i^{(i)} := \sigma_i^{-1}(P_{i-1})$, let $\ell_j^{(i)} := \sigma_i'(\ell_j^{(i-1)})$ for $0 \leqq j < i$, let $C^{(i)} := \sigma_i'(C^{(i-1)})$ and let $\Gamma^{(i)} := \sigma_i'(\Gamma^{(i-1)}) = \sigma_i^*(\Gamma^{(i-1)})$. The Euclidean transformation of V associated with $\mathscr{D}$ is the composition $\rho := \sigma_1 \ldots \sigma_N$.

1.3.2

For $0 \leqq i < N$, set $r_i := d_s$ if $q_1 + \cdots + q_{s-1} \leqq i < q_1 + \cdots + q_s$. (Set $q_0 := 0$). Then we have

Lemma (cf. Nagata [43; Prop. 4.3 *)] For $0 \leqq i < N$, P_{i+1} is an in-*
93 *finitely near point of P_i of order one, and the (effective) multiplicity of P_i on C is r_i.*

Proof. The first assertion is clear. As for the second assertion, note that we have:

$$i(C^{(i)}, \ell^{(i)}_{q_1+\cdots+q_{s-1}}; P_i) = d_{s-1} - td_s$$
$$i(C^{(i)}, \ell^{(i)}_i; P_i) = d_s$$

for $0 \leqq i < N$, where $t = i - (q_1 + \cdots + q_{s-1})$. Since $0 \leqq t < q_s$ we know that $d_s < d_{s-1} - td_s$ if $i \neq N - 1$, and that $d_{s-1} - td_s = d_s$ if $i = N - 1$. Since P_i is a one-place point of $C^{(i)}$, the smaller one of d_s and $d_{s-1} - td_s$ is the multiplicity of $C^{(i)}$ at P_i. □

1.3.3

Lemma. *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 > d_1 \geqq 1$. Let $\rho : \widehat{V} \to V$ be the Euclidean transformation of V associated with $\mathscr{D}$. Then, with the notations of 1.3.1 we have:*

(1) *$\ell_i^{(N)}$ $(0 \leqq i \leqq N)$ is a nonsingular, rational, complete curve.*

(2) *$(\ell_i^{(N)} \cdot \ell_j^{(N)}) = 1$ if $(i, j) = (q_1 + \cdots + q_{s-1}, q_1 + \cdots + q_{s-1} + q_s + 1)$ with $1 \leqq s \leqq \alpha - 1$, $(i, j) = (q_1 + \cdots + q_{\alpha-1}, q_1 + \cdots + q_\alpha)$, or $(i, j) = (q_1 + \cdots + q_{s-1} + t, q_1 + \cdots + q_{s-1} + t + 1)$ with $1 \leqq s \leqq \alpha$ and $1 \leqq t \leqq q_s - 1$; $(\ell_i^{(N)} \cdot \ell_j^{(N)}) = 0$ for every pair (i, j) $(i \neq j)$ other than those enumerated above.*

(3) *$((\ell_0^{(N)})^2) = (\ell_0^2) - q_1 - 1$ if $\alpha > 1$ and $((\ell_0^{(N)})^2) = (\ell_0^2) - q_1$ if $\alpha = 1$;
$((\ell_{q_1+\cdots+q_s}^{(N)})^2) = -2 - q_{s+1}$ for $1 \leqq s < \alpha - 1$ $((\ell_{q_1+\cdots+q_{\alpha-1}}^{(N)})^2) = -1 - q$,* **94**
and $((\ell_N^{(N)})^2) = -1$; $((\ell_{q_1+\cdots+q_{s-1}+t}^{(N)})^2) = -2$ for $1 \leqq s \leqq \alpha$ and $1 \leqq t \leqq q_s - 1$.

Proof. Follows from a straightforward computation with Lemma 1.3.2 taken into account. □

1.3.4

Set $E_0 := \ell_0^{(N)}$ and $E(s,t) := \ell_i^{(N)}$ if $i = q_1 + \cdots + q_{s-1} + t$ with $1 \leqq s \leqq \alpha$ and $1 \leqq t \leqq q_s$. Then the configuration of $\rho^{-1}(\ell_0)$ is expressed by the weighted graphs in the Figure 1, where each vertex 0 stands for an irreducible component of $\rho^{-1}(\ell_0)$ with self-intersection multiplicity as its weight and two vertices are connected by an edge if the corresponding irreducible components of $\rho^{-1}(\ell_0)$ intersect each other.

1.4

Let d_0 and d_1 be positive integers such that $d_0 > d_1$. Find integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ as in 1.3 by the Euclidean algorithm. Define an integer $a(s,t)$ $(1 \leqq s \leqq \alpha; 1 \leqq t \leqq q_s)$ inductively in the following way:

$$
\begin{aligned}
a_0 &= d_0 \\
a(1,t) &= t(a_0 - d_1) && \text{for } 1 \leq t \leq q_1 \\
a(2,t) &= a_0 + t(a(1,q') - d_2) && \text{for } 1 \leqq t \leqq q_2 \\
&\cdots\cdots
\end{aligned}
$$

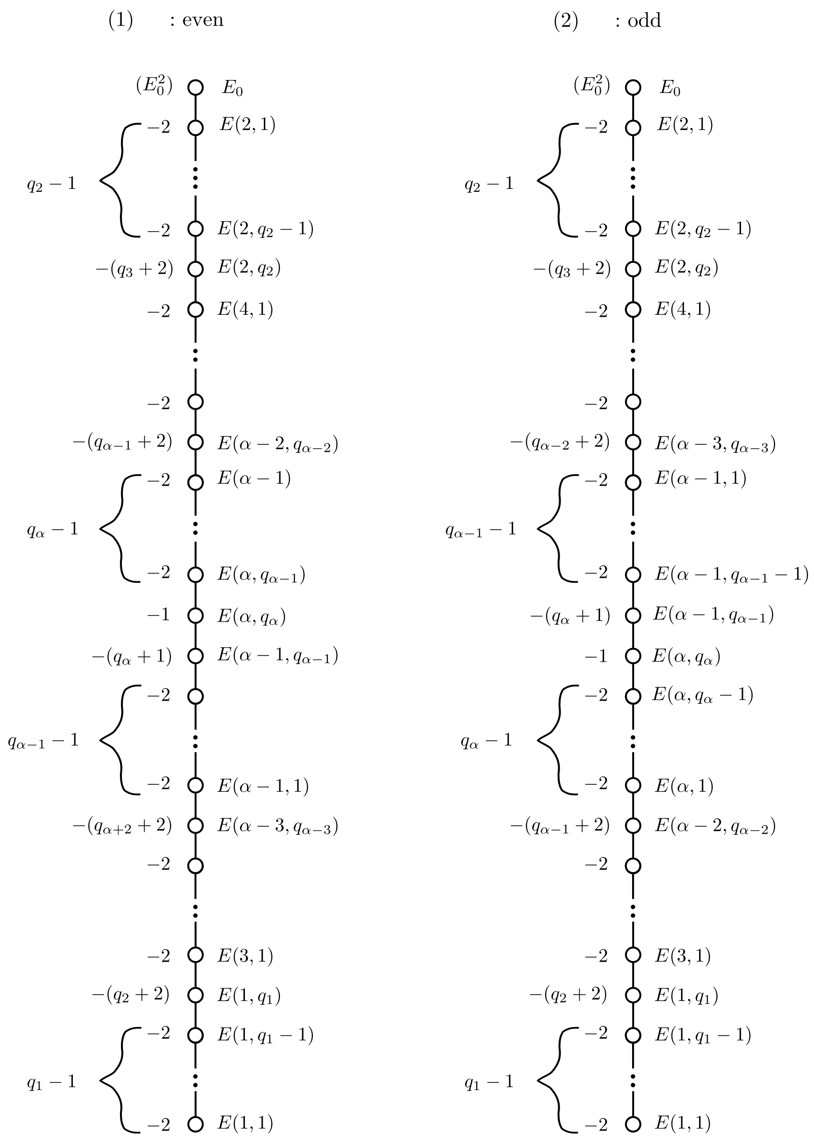
$$a(s,t) = a(s-2, q_{s-2}) + t(a(s-1, q_{s-1}) - d_s) \quad \text{for } 1 \leqq t \leqq q_s \text{ and } 2 \leqq s \leqq \alpha.$$

1.4.1

Lemma. *With the notations as above we have:*

(1) *If* $\alpha = 1$*, i.e.,* $d_2 = 0$ *then* $a(1, q_1) \geqq d_0$*;* $a(1, q_1)$

95 **Figure 1**



$\geqq d_2$ *otherwise. More precisely, if* $\alpha > 1$ *and* $q_1 \geqq 2$*, then* **96**
$a(1, q_1) > d_0$.

(2) *If* $\alpha \geqq 2$ *then* $a(s, q_s) > d_{s-1} > d_s$ *for* $2 \leqq s \leqq \alpha$*. Especially,* $a(\alpha, q_\alpha) > d_\alpha$*.*

(3) *For* $2 \leqq s \leqq \alpha$*,* $a(s, 1) > a(s-1, q_{s-1})$*.*

(4) *For* $1 \leqq s \leqq \alpha$ *and* $1 \leqq t \leqq q_s - 1$*,* $a(s, t+1) \geqq a(s, t) > 0$*.*

(5) $d_\alpha | a(s, t)$*.*

(6) $a(\alpha, q_\alpha) d_\alpha = d_0(d_0 - d_1)$*.*

Proof. (1) By definition, $a(l, q_1) = q_1(d_0 - d_1) = q_1 d_0 - d_0 + d_2 = (q_1 - 1)d_0 + d_2$. Since $d_0 > d_1$ we have either $q_1 \geqq 2$ or $q_1 = 1$ and $d_2 > 0$. If $q_1 \geqq 2$ then $a(l, q_1) \geqq d_0 + d_2 \geqq d_0$. If $q_1 = 1$ and $d_2 > 0$ then $a(l, q_1) = d_2$. If $\alpha = 1$ then $q_1 \geqq 2$. Hence $a(l, q_1) \geqq d_0$. If $\alpha \neq 1$ then $d_2 \neq 0$ and $a(l, q_1) \geqq d_2$. If $\alpha > 1$ and $q_1 \geqq 2$ then $a(l, q_1) > d_0$.

(2) If $\alpha \geqq 2$ then $a(l, q_1) \geqq d_2$ by (1). Since $a(2, q_2) = d_0 + q_2(a(l, q_1) - d_2)$, we have $a(2, q_2) \geqq d_0$. Hence $a(2, q_2) > d_1 > d_2$. If $\alpha \geqq 3$ we shall prove $a(s, q_s) > d_{s-1} > d_s$ by induction on s. For $s = 3$, $a(3, q_3) = a(1, q_1) + q_3(a(2, q_2) - d_3) > d_2 + q_3(d_2 - d_3) > d_2$. By induction on $s (\geqq 4)$, assume that $a(s-2, q_{s-2}) > d_{s-2}$ and $a(s-1, q_{s-1}) > d_{s-1}$. Then $a(s, q_s) = a(s-2, q_{s-2}) + q_s(a(s-1, q_{s-1}) - d_s) > d_{s-2} + q_s(d_{s-1} - d_s) > d_{s-2} > d_{s-1}$. Therefore, if $\alpha \geqq 2$, $a(s, q_s) > d_{s-1}$ for $2 \leqq s \leqq \alpha$. Especially $a(\alpha, q_\alpha) > d_{\alpha-1} > d_\alpha$.

(3) For $s = 2$, $a(2, 1) - a(1, q_1) = d_0 - d_2 > 0$. For $s \geqq 3$, $a(s, 1) - a(s-1, q_{s-1}) = a(s-2, q_{s-2}) - d_s > 0$ by (2).

(4) For $s = 1$, $a(1, t+1) - a(1, t) = d_0 - d_1 > 0$. Thus $a(1, t+1) > a(1, t) > 0$. For $s \geqq 2$, $a(s, t+1) - a(s, t) = a(s-1, q_{s-1}) - d_s \geqq 0$,
97 where > 0 takes place if $s \geqq 3$. Thus $a(s, t+1) \geqq a(s, t) \geqq \ldots \geqq a(s, 1) > a(s-1, q_{s-1}) \geqq \ldots > a(1, q_1) > 0$ by (3).

(5) Note that $d_\alpha | d_1, d_2, \ldots, d_\alpha$. Since $a(1, t) = t(d_0 - d_1), d_\alpha | a(1, t)$. Then $a(2, t) = d + t(a(1, q_1) - d_2)$, and $d_\alpha | a(2, t)$. Assume that $d_\alpha | a(s', t)$ for $s' < s$ and $1 \leqq t \leqq q_{s'}$. Then $a(s, t) = a(s-2, q_{s-2}) + t(a(s-1, q_{s-1}) - d_s)$, and $d_\alpha | a(s, t)$.

(6) $a(\alpha, q_\alpha)d_\alpha = a(\alpha-2, q_{\alpha-2})d_\alpha + a(\alpha-1, q_{\alpha-1})q_\alpha d_\alpha - q_\alpha d_\alpha^2$

$$\begin{aligned}
&= a(\alpha-2, q_{\alpha-2})d_\alpha + \{a(\alpha-1, q_{\alpha-1}) - d_\alpha\}d_{\alpha-1} \\
&= a(\alpha-2, q_{\alpha-2})d_\alpha + \{a(\alpha-3, q_{\alpha-3}) \\
&\qquad + a(\alpha-2, q_{\alpha-2})q_{\alpha-1} - q_{\alpha-1}d_{\alpha-1} - d_\alpha\}d_{\alpha-1} \\
&= a(\alpha-2, q_{\alpha-2})(d_\alpha + q_{\alpha-1}d_{\alpha-1}) + a(\alpha-3, q_{\alpha-3})d_{\alpha-1} - d_{\alpha-2}d_{\alpha-1} \\
&= a(\alpha-3, q_{\alpha-3})d_{\alpha-1} + \{a(\alpha-2, q_{\alpha-2}) - d_{\alpha-1}\}d_{\alpha-2}.
\end{aligned}$$

Assume by induction that

$$a(\alpha, q_\alpha)d_\alpha = a(j-2, q_{j-2})d_j + \{a(j-1, q_{j-1}) - d_j\}d_{j-1}.$$

Then, since $a(j-1, q_{j-1}) - d_j = a(j-3, q_{j-3}) + q_{j-1}a(j-2, q_{j-2}) - q_{j-1}d_{j-1} - d_j = a(j-3, q_{j-3}) + q_{j-1}a(j-2, q_{j-2}) - d_{j-2}$, we have

$$\begin{aligned}
a(\alpha, q_\alpha)d_\alpha &= a(j-2, q_{j-2})(d_j + q_{j-1}d_{j-1}) \\
&\qquad + a(j-3, q_{j-3})d_{j-1} - d_{j-2}d_{j-1} \\
&= a(j-3, q_{j-3})d_{j-1} + \{a(j-2, q_{j-2}) - d_{j-1}\}d_{j-2}.
\end{aligned}$$

Thus, $a(\alpha, q_\alpha)d_\alpha = a_0d_2 + \{a(1, q_1) - d_2\}d_1 = d_0d_2 + (q_1-1)d_0d_1 = d_0(d_0 - d_1)$.

□

1.4.2

Define positive integers $c(s,t)$ $(1 \leqq s \leqq \alpha; 1 \leqq t \leqq q_s)$ inductively in the following way:

$$\begin{aligned}
&c(1,t) = t && \text{for } 1 \leqq t \leqq q_1 \\
&c(2,t) = 1 + tc(1, q_1) && \text{for } 1 \leq t \leq q_2 \\
&\qquad \cdots\cdots\cdots \\
&c(s,t) = c(s-2, q_{s-2}) + tc(s-1, q_{s-1}) && \text{for } 1 \leq t \leq q_s \text{ and} \\
& && 2 \leqq s \leqq \alpha.
\end{aligned}$$

With the above notations, we shall show 98

Lemma. $c(\alpha, q_\alpha)d_\alpha = d_0$.

Proof. $c(\alpha, q_\alpha)d_\alpha = c(\alpha-2, q_{\alpha-2})d_\alpha + c(\alpha-1, q_{\alpha-1})q_\alpha d_\alpha$

$$\begin{aligned}
&= c(\alpha-2, q_{\alpha-2})d_\alpha + c(\alpha-1, q_{\alpha-1})d_{\alpha-1}\\
&= c(\alpha-2, q_{\alpha-2})d_\alpha + \{c(\alpha-3, q_{\alpha-3}) + c(\alpha-2, q_{\alpha-2})q_{\alpha-1}\}d_{\alpha-1}\\
&= c(\alpha-3, q_{\alpha-3})d_{\alpha-1} + c(\alpha-2, q_{\alpha-2})(d_\alpha + q_{\alpha-1}d_{\alpha-1})\\
&= c(\alpha-3, q_{\alpha-3})d_{\alpha-1} + c(\alpha-2, q_{\alpha-2})d_{\alpha-2}.
\end{aligned}$$

As in the proof of Lemma 1.4.1, (6), we can show:

$$c(\alpha, q_\alpha)d_\alpha = c(j-2, q_{j-2})d_j + c(j-1, q_{j-1})d_{j-1} \quad \text{for} \quad 3 \leqq j \leqq \alpha.$$

Thus

$$c(\alpha, q_\alpha)d_\alpha = c(1, q_1)d_3 + c(2, q_2)d_2 = q_1 d_3 + (q_1 q_2 + 1)d_2 = d_0.$$

□

1.5

Lemma. *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 > d_1 \geqq 1$. Let $\rho : \widehat{V} \to V$ be the Euclidean transformation of V associated with $\mathscr{D}$. Let $\widehat{C} := C^{(N)} = \rho'(C)$, $\widehat{\ell_0} := \ell_N^{(N)}$, $\widehat{d_0} = d_\alpha$, and let*

$$\widehat{e} = \{a(\alpha, q_\alpha)/d_\alpha + (e-1)c(\alpha, q_\alpha)d_0/d_\alpha\}$$

and

$$\widehat{\Gamma} = e(d_0/d_\alpha)E_0 + \sum_{s=1}^{\alpha}\sum_{t=1}^{q_s}\{a(s,t)/d_\alpha + (e-1)c(s,t)d_0/d_\alpha\} E(s,t) + (d_0/d_\alpha)\rho^*(\Gamma) - \widehat{e}\widehat{\ell_0},$$

where $a(s,t)$'s and $c(s,t)$'s are integers defined in 1.4. Let $\widehat{d_1}$ be the multiplicity of $\widehat{C}$ at $\widehat{P_0} := \widehat{C} \cap \widehat{\ell_0}$. Then we have:

(1) *$\widehat{D} = \{\widehat{V}, X, \widehat{C}, \widehat{\ell_0}, \widehat{\Gamma}, \widehat{d_0}, \widehat{d_1}, \widehat{e}\}$ is an admissible datum for (X, C_0) with $\widehat{d_1} \leqq \widehat{d_0} \leqq d_1 < d_0$ and $\widehat{e} \geqq 4e - 2$.*

99 (2) $(\widehat{\ell}_0^2) = -1$, *and* $\widehat{\Gamma}$ *contains no exceptional components provided* Γ *contains no exceptional components and* $(\ell_0^2) \neq q_1$ *if* $\alpha > 1$ *and* $(\ell_0^2) \neq q_1 - 1$ *if* $\alpha = 1$.

(3) *Let* Λ *be the linear pencil on* $\widehat{V}$ *spanned by* $\widehat{C}$ *and* $\widehat{d}_0(\widehat{e}\widehat{\ell}_0 + \widehat{\Gamma})$. *Then* Λ *is the proper transform by* ρ *of the linear pencil* Λ *on* V *spanned by* C *and* $d_0(e\ell_0 + \Gamma)$.

Proof. By a straightforward computation we have

$$C^{(N)} \sim d_0 E_0 + \sum_{s=1}^{\alpha} \sum_{t=1}^{q_s} a(s,t)E(s,t) + d_0 \Delta^{(N)}$$

where $\Delta^{(N)} = \rho^*((e-1)\ell_0 + \Gamma) = (e-1)\{E_0 + \sum_{s=1}^{\alpha}\sum_{t=1}^{q_s} c(s,t)E(s,t) + \rho^*(\Gamma)$ and where $(C^{(N)} \cdot \ell_N^{(N)}) = d_\alpha$, $(C^{(N)} \cdot \ell_j^{(N)}) = 0$ for $0 \leqq j < N$ and $(C^{(N)} \cdot \rho^*(\Gamma)) = 0$. Then, with $\widehat{C}, \widehat{\ell}_0, \widehat{d}_0, \widehat{e}$ and $\widehat{\Gamma}$ defined as above we have $\widehat{C} \sim \widehat{d}_0(\widehat{e}\widehat{\ell}_0 + \widehat{\Gamma})$. Note that $\rho^{-1}(X)$ is identified with X, that $\mathrm{Supp}(\widehat{\Gamma}) = \widehat{V} - (X \cup \widehat{\ell}_0)$ as is easily seen by Lemma 1.4.1, and that $\widehat{V} - X = \widehat{\ell}_0 \cup \widehat{\Gamma}$ satisfies the condition (2) of Definition 1.2.1. Thus, we know that $\widehat{\mathscr{D}} = \{\widehat{V}, X, \widehat{C}, \widehat{\ell}_0, \widehat{\Gamma}, \widehat{d}_0, \widehat{d}_1, \widehat{e}\}$ is an admissible datum for (X, C_0). It is clear that $\widehat{d}_1 \leqq \widehat{d}_0 \leqq d_1 < d_0$. Let $d_0 = b_0 d_\alpha$ and $d_1 = b_1 d_\alpha$. Then $(b_0, b_1) = 1$ and $b_0 > b_1 \geqq 1$, whence $b_0 \geqq 2$. Since $\widehat{e} = b_0(b_0 - b_1) + (e-1)b_0^2$ by virtue of Lemmas 1.4.1 and 1.4.2, we know that $\widehat{e} \geqq 4(e-1) + 2 = 4e - 2$. The assertion (2) follows from Lemma 1.3.3, and the assertion (3) is easy to prove. ◻

1.6

Definition. *Let* $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ *be an admissible datum for* (X, C_0) *with* $d_0 = d_1 \geqq 1$. *Let* $P_0 := C \cap \ell_0$, *and let* $\sigma_1 : V_1 \to V_0 := V$ *be the quadratic transformation of* V_0 *with center at* P_0. *Let* $C^{(1)} := \sigma_1'(C)$, 100 *let* $\ell_1 := \sigma_1^{-1}(P_0)$ *and let* $P_1 := C^{(1)} \cap \ell_1$. *Let* $d_1^{(1)}$ *be the multiplicity of* $C^{(1)}$ *at* P_1. *(Set* $d_1^{(0)} := d_1$*). If* $d_0 = d_1^{(0)} = d_1^{(1)}$, *let* $\sigma_2 : V_2 \to V_1$ *be the quadratic transformation of* V_1 *with center at* P_1. *Define* σ_j :

$V_j \to V_{j-1}$, $C^{(j)}$, $\ell_j := \ell_j^{(j)}$, $\ell_t^{(j)}(0 \leqq t < j)$ and $d_1^{(j)}$ inductively as follows when $1 \leq j \leq e$ and $d_0 = d_1^{(0)} = \ldots = d_1^{(j-1)}$: σ_j is the quadratic transformation of V_{j-1} with center at $P_{j-1} := C^{(j-1)} \cap \ell_{j-1}$; $C^{(j)} := \sigma'_j(C^{(j-1)})$, $\ell_j := \sigma_j^{-1}(P_{j-1})$, $\ell_t^{(j)} := \sigma'_j(\ell_t^{(j-1)})$; $d_1^{(j)}$ is the multiplicity of $C^{(j)}$ at $P_j := C^{(j)} \cap \ell_j$. For $1 \leqq i \leqq e$, if $d_0 = d_1^{(0)} = d_1^{(1)} = \ldots = d_1^{(i-1)}$, defined the (e, i)-transformation ρ of V associated with $\mathscr{D}$ (or simply, the (e, i)-transformation of V*) as the composition $\rho := \sigma_1 \ldots \sigma_i$.*

Here it should be noted that the Euclidean transformation of V associated with $\mathscr{D}$ is defined when $d_0 > d_1$, while the (e, i)-transformation of V is defined when $d_0 = d_1 = d_1^{(1)} = \ldots = d_1^{(i-1)}$ and $1 \leqq i \leqq e$.

1.7

Lemma. *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 = d_1 \geqq 1$. If the (e, i)-transformation $\rho : V_i \to V$ is defined for some i with $1 \leqq i \leqq e$ then we have the following:*

(1) *$(\ell_s^{(i)} \cdot \ell_t^{(i)}) = 1$ if $t = s + 1$ with $0 \leqq s < i$; $(\ell_s^{(i)} \cdot \ell_t^{(i)}) = 0$ for other pairs (s, t) with $s \neq t$.*

(2) *$((\ell_0^{(i)})^2) = (\ell_0^2) - 1$, $((\ell_t^{(i)})^2) = -2$ for $1 \leqq t < i$ and $((\ell_i^{(i)})^2) = -1$.*

101 (3) *$\mathscr{D}_i = \{V_i, X, C^{(i)}, \ell_i, \Gamma_i, d_0, d_1^{(i)}, (e - i)\}$ is an admissible datum for (X, C_0) when $1 \leqq i < e$, where*

$$\Gamma_i := \rho^*(\Gamma) + (e - i + 1)\ell_{i-1}^{(i)} + \cdots + e\ell_0^{(i)}.$$

If $(\ell_0^2) \neq 0$ and Γ contains no exceptional components then Γ_i contains no exceptional components. The linear pencil $\Lambda^{(i)}$ on V_i spanned by $C^{(i)}$ and $d_0((e - i)\ell_i + \Gamma_i)$ is the proper transform by ρ of the linear pencil Λ on V spanned by C and $d_0(e\ell_0 + \Gamma)$.

(4) *If $i = e$ we have $C^{(e)} \sim d_0\Gamma_e$, where*

$$\Gamma_e := \rho^*(\Gamma) + \ell_{e-1}^{(e)} + \cdots + e\ell_0^{(e)}.$$

The linear pencil $\Lambda^{(e)}$ on V_e spanned by $C^{(e)}$ and $d_0\Gamma_e$ is the proper transform by ρ of the linear pencil Λ on V spanned by C and $d_0(e\ell_0 + \Gamma)$; $\Lambda^{(e)}$ is irreducible and free from base points.

Proof. (1) and (2) follow from a straightforward computation. (3) By a direct computation again we have for $1 \leqq i \leqq e$:

$$C^{(i)} \sim d_0\{(e-i)\ell_i + (e-i+1)\ell_{i-1}^{(i)} + \cdots + e\ell_0^{(i)} + \Gamma^{(i)}\}$$

where $\Gamma^{(i)} := \rho^*(\Gamma)$, $(C^{(i)} \cdot \ell_i) = d_0$, $(C^{(i)} \cdot \ell_j^{(i)}) = 0$ for $0 \leqq j < i$ and $(C^{(i)} \cdot \Gamma^{(i)}) = 0$. Note that $\rho^{-1}(X)$ is identified with X, that $V_i - \rho^{-1}(X) = \rho^{-1}(\Gamma) \cup \ell_0^{(i)} \cup \ell_1^{(i)} \cup \ldots \cup \ell_i^{(i)}$ satisfies the condition (2) of Definition 1.2.1 and that $\mathrm{Supp}(\Gamma_i) = \overline{V_i - (X \cup \ell_i)}$. Therefore, if $1 \leqq i < e$, $\mathscr{D}_i = \{V_i, X, C^{(i)}, \ell_i, \Gamma_i, d_0, d_1^{(i)}, (e-i)\}$ is an admissible datum for (X, C_0). The other assertions are easy to prove.

(4) We have only to note that $\Lambda^{(e)}$ is irreducible. Since $C^{(e)}$ is irreducible, $\Lambda^{(e)}$ is apparently irreducible. □

1.8

We need the following auxiliary **102**

Lemma. *Let k be an algebraically closed field of characteristic p and let V be a nonsingular projective surface defined over k. Let $f : V \to B$ be a surjective morphism of V onto a nonsingular complete curve B, whose general fibers are irreducible curves. Assume that we are given a fiber $f^*(b)$ such that:*

(1) *$f^*(b) = d\Delta$, where d is the multiplicity and Δ is the reduced form, i.e., $f^*(b) = \sum_{i=1}^{n} d_i\Delta_i$ with irreducible components Δ_i then d is the greatest common divisor of $d_1, \ldots, d_n$ and $\Delta = \sum_{i=1}^{n}(d_i/d)\Delta_i$,*

(2) $\mathrm{Supp}(\Delta) = \bigcup_{i=1}^{n}\Delta_i$ *satisfies the following conditions;*

   (i) *each irreducible component Δ_i is a nonsingular, rational complete curve,*

   (ii) *Δ_i intersects Δ_j (if at all) transversely in at most one point,*

(iii) $\Delta_i \cap \Delta_j \cap \Delta_\ell = \phi$ *for three distinct indices,*

(iv) $\mathrm{Supp}(\Delta)$ *contains no cyclic chains.*

Then the multiplicity d of $f^*(b)$ is a power of the characteristic p.

Proof. Our proof consists of three steps.

(I) Set $Z := \Delta_{\mathrm{red}}$. We shall show that Z is simply connected, i.e., Z has no nontrivial unramified covering of degree prime to p. Let $\varphi : W \to Z$ be an unramified covering of degree $m > 1$ with $(m, p) = 1$. For $1 \leqq i \leqq n$, $\varphi_i := \varphi \underset{Z}{\times} \Delta_i : W_i := W \underset{Z}{\times} \Delta_i \to \Delta_i$ is
103 an unramified covering of Δ_i. Since Δ_i is isomorphic to $\mathbb{P}^1$ and Δ_i is thus simply connected, W_i is a disjoint union $W_i := \Delta_i^{(1)} \cup \ldots \cup \Delta_i^{(m)}$ of irreducible components $\Delta_i^{(j)} (1 \leqq j \leqq m)$ which are isomorphic to Δ_i. Now we shall prove our assertion by induction on the number n of irreducible components of Z. When $n = 1$ our assertion holds clearly as seen from the above remark. For $n > 1$ there exists an irreducible component of Z, say Δ_1, such that Δ_1 meets only one irreducible component of Z other than Δ_1 and $Z' = \overline{Z - \Delta_1} = \bigcup_{i=2}^{n} \Delta_i$ satisfies the same conditions (i) ~ (iv) as above for Z. Let $P := Z' \cap \Delta_1$ and let $\varphi' := \varphi \underset{Z}{\times} Z' : W' := W \underset{Z}{\times} Z' \to Z'$. Since φ' is an unramified covering of degree m we know by assumption of induction that W is a disjoint union $W := Z'^{(1)} \cup \ldots \cup Z'^{(m)}$, where $Z'^{(j)} (1 \leqq j \leqq m)$ is isomorphic to Z'. Let $\varphi^{-1}(P) := \{P^{(1)}, \ldots, P^{(m)}\}$. We may assume with no loss of generality that $P^{(j)} \in Z'^{(j)} \cap \Delta_1^{(j)}$ for $1 \leqq j \leqq m$. Since Z' and Δ_1 meet transversely each other at P and φ is unramified, $Z'^{(j)}$ and $\Delta_1^{(j)}$ meet transversely each other at $P^{(j)}$ for $1 \leqq j \leqq m$. Set $Z^{(j)} := Z'^{(j)} \cup \Delta_1^{(j)}$ for $1 \leqq j \leqq m$. Then it is easy to see that each $Z^{(j)}$ is isomorphic to Z for $1 \leqq j \leqq m$ and W is a disjoint union of $Z^{(1)}, \ldots, Z^{(m)}$. Thus, our assertion is proved.

(II) Assume that d is not a power of p, and write $d = p^\alpha d'$ with $(d', p) = 1$. Let t be a uniformisant of B at the point b, and let

B' be the complete nonsingular model of an algebraic function field $k(B)(t^{1/d'})$. The canonical morphism $\psi : B' \to B$ determined by the injection $k(B) \hookrightarrow k(B)(t^{1/d'})$ ramifies totally over the point 104
b. Let b' be the unique point of B' over b. Let V' be the normalization of $V \times_B B'$ and let $\widetilde{\psi} : V' \to V$ and $f' : V' \to B'$ be the canonical projections onto V and B', respectively. We have, thus, a commutative diagram:

$$\begin{tikzcd} V' \arrow[r, "\widetilde{\psi}"] \arrow[d, "f'"] & V \arrow[d, "f"] \\ B' \arrow[r, "\psi"] & B \end{tikzcd}$$

Let $W' := f'^*(b')$. We shall show that $\widetilde{\psi}$ maps W' onto $p^\alpha\Delta$ (considered as a closed sub scheme of V) and $\varphi' := \widetilde{\psi}|_{W'} : W' \to p^\alpha\Delta$ is a nontrivial unramified covering of degree d'. Let v be a (k-rational) point of V on Δ, and let x be an element of $\mathscr{O}_{V,v}$ such that $x = 0$ is a local equation of Δ (considered as a divisor on V). Then we have $t = ux^d$ with $u \in \mathscr{O}^*_{V,v}$. Hence we have $t^{1/d'} = (u^{1/d'})x^{p^\alpha}$ in $k(V') := k(V) \otimes_{k(B)} k(B')$. Here note that t and x cannot be chosen so that u has a d'-th root in $\mathscr{O}_{V,v}$; indeed, if possible, $k(V) \otimes_{k(B)} k(B')$ would be not an integral domain, and this contradicts the fact that $k(V)$ is a regular extension of $k(B)$. We have then:

$$\begin{aligned} V' \times_V \operatorname{Spec}(\mathscr{O}_{V,v}) &= \text{the normalization of } (V \times_B B') \times_V \operatorname{Spec}(\mathscr{O}_{V,v}) \\ &= \text{the normalization of } B' \times_B \operatorname{Spec}(\mathscr{O}_{V,v}) \\ &= \operatorname{Spec}(\mathscr{O}_{V,v}[u^{1/d'}]). \end{aligned}$$

This implies that $\widetilde{\psi} : V' \to V$ is unramified at every point of V' over v. Moreover, we know by construction that $\widetilde{\psi}^*(p^\alpha\Delta) = f'^*(b')$ at every point of V' over v. Since these assertions hold for 105
all points v of Δ we know that $\widetilde{\psi}$ maps W' onto $p^\alpha\Delta$ and $\varphi' := \widetilde{\psi}|_{W'} : W' \to p^\alpha\Delta$ is an unramified covering of degree d', where φ' is nontrivial because $W' := f'^*(b')$ is connected.

(III) Set $\varphi : \varphi'_{\mathrm{red}}$, $W := W'_{\mathrm{red}}$ and $Z := (p^{\alpha}\Delta)_{\mathrm{red}}$. Then $\varphi : W \to Z$ is a nontrivial unramified covering of degree $d' > 1$, which contradict the assertion in step (I). Consequently, d is a power of p.

□

1.9

As consequences of Lemma 1.8 we have the following results.

1.9.1

Corollary. *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 = d_1 > 1$. Assume that d_0 is not divisible by p. Then there exists an admissible datum $\mathscr{D}' = \{V', X, C', \ell'_0, \Gamma', d'_0, d'_1, e'\}$ for (X, C_0) such that:*

(1) *$\mathscr{D}'$ is obtained from $\mathscr{D}$ by some (e, i)-transformation of V associated with $\mathscr{D}$;*

(2) *$d'_0 = d_0$, $d'_1 < d'_0$ and $e' < e$;*

(3) *$(\ell'^2_0) = -1$, and Γ' contains no exceptional components provided $(\ell_0^2) \neq 0$ and Γ contains no exceptional components.*

Proof. (I) Assume that $d_0 = d_1 = d_1^{(1)} = \ldots = d_1^{(i-1)} > d_1^{(i)}$ for some i with $1 \leqq i < e$. Let $\mathscr{D}_i$ be an admissible datum for (X, C_0) obtained from $\mathscr{D}$ by the (e, i)-transformation of V associated with $\mathscr{D}$. Then $\mathscr{D}_i$ satisfies all conditions (1) ~ (3) above by virtue of Lemma 1.7.

(II) Assume that the equalities $d_0 = d_1 = d_1^{(1)} = \ldots = d_1^{(e-1)}$ hold.
106 Let $\rho : V_e \to V$ be the (e, e)-transformation of V associated with $\mathscr{D}$, and let $\Delta := \Gamma_e$. The linear pencil $\Lambda^{(e)}$ on V_e spanned by $C^{(e)}$ and $d_0\Delta$ is an irreducible pencil free from base points. Hence $\Lambda^{(e)}$ defines a fibration $f : V_e \to \mathbb{P}^1_k$, of which $d_0\Delta$ is a multiple singular fiber because $d_0 > 1$. Note that d_0 is the multiplicity of the fiber $d_0\Delta$ by virtue of Lemma 1.7 (esp. (4)) and that the

irreducible components Δ_i of Δ (i.e., $\mathrm{Supp}(\Delta) = \bigcup_{i=1}^{n} \Delta_i$) satisfy the conditions (i) ~ (iv) of Lemma 1.8. Hence, by virtue of Lemma 1.8, d_0 is a power of p, which contradicts the assumption that d_0 is not divisible by p.

□

1.9.2

Corollary. *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) with $d_0 = d_1 \geqq 1$. Assume that the (e, e)-transformation $\rho : V_e \to V$ is defined, i.e., the equalities $d_0 = d_1 = d_1^{(1)} = \ldots = d_1^{(e-1)}$ hold. Let Λ be the linear pencil on V spanned by C and $d_0(e\ell_0 + \Gamma)$. Then the generic member of Λ has only one place outside of X, which is a purely inseparable place. In other words, a general member of Λ has only one place outside of X.*

Proof. (I) Let $\Lambda^{(e)}$ be the proper transform of Λ by ρ; $\Lambda^{(e)}$ is spanned by $C^{(e)}$ and $d_0\Lambda$, where $\Delta := \Gamma_e$, and $\Lambda^{(e)}$ is an irreducible linear pencil free from base points. Set $\overline{S} := \ell_e$ (cf. 1.7). Then $(C^{(e)} \cdot \overline{S}) = d_0(\Delta \cdot \overline{S}) = d_0$, and d_0 is a power of p in virtue of Lemma 1.8. Let $f : V_e \to \mathbb{P}^1_k$ be the fibration defined by $\Lambda^{(e)}$. Let $S := \overline{S} - \{\overline{S} \cap \Delta\}$ and $T := \mathbb{P}^1_k - f(\Delta)$. Then, by restricting f onto S we have a surjective morphism $\varphi : S \to T$ of degree d_0. Choose inhomogeneous coordinates s and t on S and T respectively such **107**
that the point $C^{(e)} \cap S$ is defined by $s = 0$ and the point $f(C^{(e)})$ is defined by $t = 0$. Then φ is given by a polynomial $t = G(s)$ in s with coefficient in k and with $\deg G = d_0$. By choices of s and t we have $G(0) = 0$. Since the point $C^{(e)} \cap S$ is a one-place point of $C^{(e)}$ and $\Lambda^{(e)}$ has no base points, we conclude readily that $G(s)$ is written as $G(s) = as^{d_0}$ with $a \in k$. We may assume that $a = 1$ by substituting s for $(a^{1/d}0)s$. This implies that $\overline{f} := f|_{\overline{S}} : \overline{S} \to \mathbb{P}^1_k$ is the α-th iteration F^α of the Frobenius endomorphism F of $\overline{S}$, where $d_0 = p^\alpha$.

(II) Let $K := k(t) = k(\mathbb{P}^1)$, and let W be the generic fiber of f, i.e., $W = V^{(e)} \underset{\mathbb{P}^1}{\times} \operatorname{Spec}(K)$. Then W is a projective normal curve defined over K, and the curve $\overline{S}$ gives rise to a point $\mathscr{P}$ on W which is purely inseparable over K, as was seen in the step (I). Hence $\mathscr{P}$ is a one-place point of W. Thus the generic member of Λ has only one place outside of X.

□

The following example, which was communicated to the author by A. Sathaye, shows that a general fiber of Λ has only one place outside of X, while some special fiber has 2 or more places outside of X. Let k be a field of characteristic $p > 0$. Choose integers n, U, V such that (1) $UV = 1 + p + \cdots + p^n$ and (2) $U > V > 1$ and $LU - MV = 1$ is the unique relation with $L, M > 0$, $L < V$ and $M < U$. Then there exists a unique positive integer a such that

$$LUp^{n+1} + UV - 1 > aUV > LUp^{n+1} \quad \text{and} \quad a \not\equiv 0 \pmod{p}.$$

Consider an affine plane curve $f(x,y) = x^{Vp} + y^{U_p} + x^r y^s$, where $r =$
108 $aV - Lp^{n+1}$ and $s = Up - aU + Mp^{n+1}$. Then the curve $f(x,y)$ has the property that $f + \lambda$ has exactly one place at infinity for all except one value of λ and for the special value of λ, it has 2 or more places at infinity. Here n can be chosen to be 1 for all $p \neq 2^m - 1$ for any m, and $n \leqq 3$ otherwise. Consequently, $\deg f = Up < p^2$ in the former case and $\deg f < p^5$ in the latter case.

1.10

Corollary to Lemma 1.5 and Corollary 1.9.1. *Let* $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ *be an admissible datum for* (X, C_0) *such that at least one of* d_0 *and* d_1 *is not divisible by* p. *Then there exists an admissible datum* $\widetilde{D} = \{\widetilde{V}, X, \widetilde{C}, \widetilde{\ell}_0, \widetilde{\Gamma}, 1, 1, \widetilde{e}\}$ *for* (X, C_0) *such that:*

(1) *There exists a birational morphism* $\rho : \widetilde{V} \to V$, *which is the composition of Fuclidean transformations and the* (e,i)*-transformations associated with admissible data.*

(2) $\widetilde{C} = \rho'(C)$ *and* $\rho^{-1}(X) \cong X$.

(3) *The linear pencil $\widetilde{\Lambda}$ on $\widetilde{V}$ spanned by $\widetilde{C}$ and $\widetilde{e}\widetilde{\ell}_0 + \widetilde{\Gamma}$ is the proper transform by ρ of the linear pencil Λ on V spanned by C and $d_0(e\ell_0 + \Gamma)$.*

Proof. We shall prove the assertion by induction on d_0. If $d_0 = 1$, we have only to take $\widetilde{\mathscr{D}} = \mathscr{D}$. If $d_0 > d_1$ then the Euclidean transformation ρ_0 of V associated with $\mathscr{D}$ can be defined, and we obtain by Lemma 1.5 an admissible datum $\widehat{\mathscr{D}} = \{\widehat{V}, X, \widehat{C}, \widehat{\ell}_0, \widehat{\Gamma}, \widehat{d}_0, \widehat{d}_1, \widehat{e}\}$ for (X, C_0) such that $\widehat{d}_1 \leqq \widehat{d}_0 \leqq d_1 < d_0$ and $\widehat{d}_0$ is not divisible by p. By inductive assumption we have an admissible datum $\widetilde{\mathscr{D}} = \{\widetilde{V}, X, \widetilde{C}, \widetilde{\ell}_0, \widetilde{\Gamma}, 1, 1, \widetilde{e}\}$ 109
and a birational morphism $\rho_0 : \widetilde{V} \to \widehat{V}$ which satisfy the above conditions (1) ~ (3). Then we have only to take $\widetilde{D}$ and $\rho := \rho_1\rho_0$. If $d_0 = d_1 > 1$, Corollary 1.9.1 shows that there exists an admissible datum $\mathscr{D}' = \{V', X, C', \ell_0', \Gamma', d_0', d_1', e'\}$ such that $d_1' < d_0' = d_1 = d_0, d_0'$ is not divisible by p, and that $\mathscr{D}'$ is obtained by some (e, i)-transformation ρ' of V associated with $\mathscr{D}$. By the former case treated above we have an admissible datum $\widetilde{\mathscr{D}}$ and a birational morphism $\rho_2 : \widetilde{V} \to V'$ which satisfy the above conditions (1) ~ (3). Then we have only to take $\widetilde{\mathscr{D}}$ and $\rho := \rho_2\rho'$. □

1.11

Lemma . *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, 1, 1, e\}$ be an admissible datum for (X, C_0). Let $\rho : V_e \to V$ be the (e, e)-transformation of V associated with $\mathscr{D}$, let $\Lambda^{(e)}$ be the linear pencil on V_e spanned by $C^{(e)}$ and $\Gamma_e = \rho^*(\Gamma) + \ell_{e-1}^{(e)} + \cdots + e\ell_0^{(e)}$* (cf. Lemma 1.7) *and let $f : V_e \to \mathbb{P}_k^1$ be the fibration defined by $\Lambda^{(e)}$. Then we have the following results:*

(1) *$C^{(e)}$ is an irreducible curve which is nonsingular at $C^{(e)} \cap \ell_e$.*

(2) *ℓ_e is a cross-section of the fibration $f : V_e \to \mathbb{P}_k^1$.*

(3) *$\Lambda^{(e)}$ has no multiple members.*

(4) *Let D be a member of $\Lambda^{(e)}$ other than Γ_e. Then, D is an irreducible curve; $D_0 := D \cap X$ has only one place outside of D_0; if C_0 is nonsingular the arithmetic genus of D is equal to the geometric genus of C_0.*

(5) *If C_0 is nonsingular and rational then D is a non-singular rational*
110 *curve; X with a fibration $f_0 := f|_X : X \to \mathbb{A}^1_k := \mathbb{P}^1_k - \{f(\Gamma_e)\}$ is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k$, and hence X is isomorphic to the affine plane $\mathbb{A}^2_k$.*

Proof. It will be clear that the (e,e)-transformation $\rho : V_e \to V$ associated with $\mathscr{D}$ is defined. Lemma 1.7 tells us that $\Lambda^{(e)}$ is an irreducible pencil free from base points. Hence the general fibers of f are irreducible. Since $(C^{(e)} \cdot \ell_e) = 1$ we know that $C^{(e)}$ is an irreducible curve which is nonsingular at $C^{(e)} \cap \ell_e$ and that ℓ_e is a cross-section of f. Hence $\Lambda^{(e)}$ has no multiple members. Let D be a member of $\Lambda^{(e)}$ other than Γ_e. Then $(D \cdot \ell_e) = 1$. Since $\overline{V_e - (X \cup \ell_e)} = \mathrm{Supp}(\Gamma_e)$ (cf. the proof of Lemma 1.7) and since X is affine, D is an irreducible member. Since $D_0 = D - (D \cap \ell_e)$ and $D \cap \ell_e$ is a simple point of D, D has only one place outside of D_0. By invariance of arithmetic genera for members of a linear system we have: $p_a(D) = p_a(C^{(e)})$, which is equal to the genus of C_0 if C_0 is nonsingular. If C_0 is non-singular and rational then $p_a(D) = 0$, whence follows that D is a nonsingular rational curve and D_0 is isomorphic to the affine line $\mathbb{A}^1_k$. Furthermore, if C_0 is nonsingular and rational then $\varphi : V_e - \mathrm{Supp}(\Gamma_e) \to \mathbb{A}^1_k := \mathbb{P}^1_k - \{f(\Gamma_e)\}$ is a $\mathbb{P}^1$-bundle over $\mathbb{A}^1_k$ by virtue of Hironaka [22; Th. 1.8] and $\ell_e - (\Gamma_e \cap \ell_e)$ is a cross-section of φ, where φ is the restriction of f onto $V_e - \mathrm{Supp}(\Gamma_e)$. Hence, $f_0 := f|_X : X \to \mathbb{A}^1_k$ is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k$, and X is isomorphic to the affine plane $\mathbb{A}^2_k$ because every $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k$ is trivial. □

1.12

Corollary 1.10 combined with Lemma 1.11 implies the following

111 **Theorem.** *Let $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ be an admissible datum for (X, C_0) such that at least one of d_0 and d_1 is not divisible by p, and let Λ be the linear pencil on V spanned by C and $d_0(e\ell_0 + \Gamma)$. Let D be an arbitrary member of Λ other than $d_0(e\ell_0 + \Gamma)$ and let $D_0 := D \cap X$. Then we have the following:*

(1) *D_0 is an irreducible curve with only one place outside of D_0.*

(2) *The geometric genus of D is equal to the geometric genus of C if D is a general member of Λ and C_0 is nonsingular.*

(3) *If C_0 is nonsingular and rational D_0 is a nonsingular rational curve; X is isomorphic to the affine plane $\mathbb{A}^2_k$.*

Proof. By Corollary 1.10 there exist an admissible datum $\widetilde{\mathscr{D}} = \{\widetilde{V}, X, \widetilde{C}, \widetilde{\ell}_0, \widetilde{\Gamma}, 1, 1, \widetilde{e}\}$ for (X, C_0) and a birational morphism $\rho : \widetilde{V} \to V$ such that the linear pencil $\widetilde{\Lambda}$ on $\widetilde{V}$ spanned by $\widetilde{C}$ and $\widetilde{e}\widetilde{\ell}_0 + \widetilde{\Gamma}$ is the proper transform by ρ of Λ. Let $\widetilde{\rho} : \widetilde{V}(\widetilde{e}) \to \widetilde{V}$ be the $(\widetilde{e}, \widetilde{e})$-transformation of $\widetilde{V}$ associated with $\widetilde{D}$ and let $\widetilde{\Lambda}(\widetilde{e})$ be the proper transform of $\widetilde{\Lambda}$ by $\widetilde{\rho}$. Let $\sigma = \rho\widetilde{\rho}$. Then $L := \widetilde{\Lambda}(\widetilde{e})$ is the proper transform of Λ by σ. Let D' be the member of L corresponding to D of Λ. Then $D'_0 := D' \cap X$ is isomorphic to D_0, where X is identified with $\sigma^{-1}(X)$. The above assertions now follow from the assertions (4) and (5) of Lemma 1.11. □

1.13

Lemma. *Let $\mathscr{D}$ and Λ be as in Theorem 1.12 and let $A = \Gamma(X, \mathscr{O}_X)$. Then the following assertions hold.*

(1) *Assume that A is a factorial ring and that $A^* = k^*$. Let f be a prime element of A defining C_0, and let C_α be the curve on X* 112
defined by $f - \alpha$ for $\alpha \in k$. Then D_0 (cf. Theorem 1.12) *coincides with C_α for some $\alpha \in k$, and conversely, every C_α is of the form D_0 for some member D of Λ other than $d_0(e\ell_0 + \Gamma)$.*

(2) *If C_0 is nonsingular and rational then A is a polynomial ring in two variables over k; thence A is a factorial ring with $A^* = k^*$.*

Proof. (1) Under the assumptions of the assertion (1) we have $(f) = C - \Delta$ with a divisor Δ such that $\mathrm{Supp}(\Delta) \subset \ell_0 \cup \mathrm{Supp}(\Gamma)$. Let g be an element of $k(V)$ such that $(g) = C - d_0(e\ell_0 + \Gamma)$. Then $(f/g) = d_0(e\ell_0 + \Gamma) - \Delta$, whence f/g is an invertible element of A. Since $A^* = k^*$, $f = \lambda g$ with $\lambda \in k^*$. Hence $(f) = C - d_0(e\ell_0 + \Gamma)$. This implies that Λ is spanned by 1 and f (or more precisely, by

$d_0(e\ell_0 + \Gamma)$ and $d_0(e\ell_0 + \Gamma) + (f)$. It is now clear that the assertion (1) holds.

(2) The second assertion was proved in the assertion (3) of Theorem 1.12.

□

1.14

Theorem. *Let A be a nonsingular, rational, affine k-domain of dimension 2, and let $X := \mathrm{Spec}(A)$. Assume that the following conditions hold:*

(1) *There exists an irreducible closed curve C_0 on X, which is isomorphic to the affine line $\mathbb{A}^1$ over k.*

(2) *There exists an admissible datum $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ for (X, C_0) such that at least one of d_0 and d_1 is not divisible by p.*

113 *Then X is isomorphic to the affine plane $\mathbb{A}^2$ over k. Furthermore if f is an element of A defining the curve C_0 then $A = k[f, g]$ for some element g of A.*

Proof. The first assertion was proved in Theorem 1.12. We shall show the second assertion. By virtue of Lemma 1.13, the proof of Theorem 1.12 and Lemma 1.11, we know that X has a structure of an $\mathbb{A}^1$-bundle over $\mathbb{A}^1_k := \mathrm{Spec}(k[f])$, whose fibers are the curves C_α defined by $f - \alpha$ with $\alpha \in k$. Hence $A = k[f, g]$ for some element g of A. □

1.15

Proof of Irreducibility theorem. Set $X := \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$. Then, as seen in 1.2.2, $\mathscr{D}_0 = \{\mathbb{P}^2_k, X, C, \ell_0, \phi, d_0, d_1, 1\}$ is an admissible datum for (X, C_0); (for the notations, see the paragraph 1.1). We may assume that $d_0 > 1$; if otherwise, C is a line on $\mathbb{P}^2_k$ and the assertion of the theorem is apparently true. If $d_0 > 1$, the theorem follows from Theorem 1.12 and Lemma 1.13.

1.16

Proof of Generic irreducibility theorem. Let $\mathscr{D}_0$ be as in 1.15; we may assume that $d_0 > d_1$. Starting with the Euclidean transformation of $\mathbb{P}^2_k$ associated with $\mathscr{D}_0$ and repeating successively the Euclidean transformations or the (e, i)-transformations associated with admissible data we obtain ultimately an admissible datum $\mathscr{D} = \{V, X, C, \ell_0, \Gamma, d_0, d_1, e\}$ such that one of the following conditions holds:

(1) $d_0 = d_1 = 1$;

(2) $d_0 = d_1 > 1$ and $e = 1$.

In the case (1), we know by virtue of Lemma 1.11 that every curve $C_\alpha(\alpha \in k)$ is an irreducible curve with only one place at infinity. In **114**
the case (2), the generic irreducibility theorem follows from Corollary 1.9.2.

1.17

Proof of the Embedding theorem. Consider an admissible datum $\mathscr{D}_0$ in the paragraph 1.15. If $d_0 = 1$ the theorem holds apparently; hence we may assume that $d_0 > 1$. Then $d_0 > d_1 \geqq 1$. Now the embedding theorem follows from Theorem 1.14.

2 Linear pencils of rational curves

115

2.1

In this section the ground field k is assumed to be an algebraically closed field of characteristic p. Let V be a non-singular projective surface defined over k. We shall consider an irreducible linear pencil Λ on V satisfying the properties:

(1) *General members of* Λ *are rational curves.*

(2) *The generic member of Λ is smoothable; namely, setting $\mathscr{K}$:= the sub field of $k(V)$ corresponding to the pencil Λ, the complete normal $\mathscr{K}$-model of an algebraic function field $k(V)$ in one variable over $\mathscr{K}$ is geometrically regular over $\mathscr{K}$.* The property (2) is equivalent to saying:

(2′) *There exist a nonsingular projective surface $\widetilde{V}$ and a birational morphism $\rho : \widetilde{V} \to V$ such that general members of the proper transform $\widetilde{\Lambda}$ of Λ by ρ are nonsingular curves.*

When the field k is of characteristic zero the pencil Λ satisfies automatically the property (2); hence the condition (2) is superfluous. Moreover, we note by Tsen's Theorem that V is a rational surface if Λ has the properties (1) and (2). In the next chapter we shall consider a linear pencil having only the property (1) but not (2), in order to construct unirational, irrational surfaces.

2.2

Lemma. *Let $f : V \to B$ be a surjective morphism from a nonsingular projective surface V onto a nonsingular complete curve B such that almost all fibers are isomorphic to $\mathbb{P}^1_k$. Let $F = n_1C_1 + \cdots + n_rC_r$ be a*
116 *singular fiber of f, where C_i is an irreducible curve, $C_i \neq C_j$ if $i \neq j$, and $n_i > 0$. Then we have:*

(1) *The greatest common divisor $(n_1, \ldots, n_r)$ of $n_1, \ldots, n_r$ is 1;* $\mathrm{Supp}(F) = \bigcup_{i=1}^{r} C_i$ *is connected.*

(2) *For $1 \leqq i \leqq r$, C_i is isomorphic to $\mathbb{P}^1_k$ and $(C_i^2) < 0$.*

(3) *For $i \neq j$, $(C_i \cdot C_j) = 0$ or 1.*

(4) *For three distinct indices i, j and ℓ, $C_i \cap C_j \cap C_\ell = \phi$.*

(5) *One of C_i's, say C_1, is an exceptional component, i.e., an exceptional curve of the first kind. If $\tau : V \to V_1$ is the contraction of C_1, then f factors as $f : V \xrightarrow{\tau} V_1 \xrightarrow{f_1} B$, where $f_1 : V_1 \to B$ is a fibration by $\mathbb{P}^1$.*

(6) *If one of n_i's, say n_1, equals* 1 *then there is an exceptional component among C_i's with* $2 \leqq i \leqq n$.

See Gizatullin [16]. (I) Let $m = (n_1, \dots, n_r)$ and let $C = mD$ with $D := (n_1/m)C_1 + \cdots + (n_r/m)C_r$. Then, by the arithmetic genus formula we have:

$$p_a(C) = \{m^2(D^2) + m(D \cdot K_V)\}/2 + 1 = m(D \cdot K_V)/2 + 1 = 0.$$

Since $(D \cdot K_V)$ is an integer, either $m = 1$ or $m = 2$ and $(D \cdot K_V) = -1$. In the latter case, $p_a(D) = \{(D^2) + (D \cdot K_V)\}2 + 1 = 1/2$, which is a contradiction. Hence $m = 1$. If $r = 1$ then C is isomorphic to $\mathbb{P}^1_k$. Hence $r \geqq 2$ and we know by virtue of Zariski's connectedness theorem that $\mathrm{Supp}(F) = \bigcup_{i=1}^{r} C_i$ is connected.

(II) For each i, $n_i(C_i^2) + \sum_{i \neq j} n_j(C_i \cdot C_j) = 0$ where $(C_i \cdot C_j) > 0$ for some j because F is connected. Hence $(C_i^2) < 0$. To prove the **117**
assertions (2), (3), (4) and (5) we have only to show that one of C_i's is an exceptional component. Note that $(F \cdot K_V) = -2$ because $p_a(F) = 0$. Hence we have:

$$(*)\ \ -2 = (F \cdot K_V) = \sum_i n_i(C_i \cdot K_V) = \sum_i n_i(2p_a(C_i) - 2 - (C_i^2)),$$

where $2p_a(C_i) - 2 - (C_i^2) \geqq -1$ and the equality holds if and only if C_i is an exceptional curve of the first kind. However, it is impossible that $2p_a(C_i) - 2 - (C_i^2) \geqq 0$ for every i, as seen from the above equality $(*)$. Therefore, $2p_a(C_i) - 2 - (C_i^2) = -1$ for some i.

(III) We shall prove the assertion (6). Assume the contrary, i.e., C_1 is an exceptional component with $n_1 = 1$ and none of C_i's ($2 \leqq i \leqq r$) is an exceptional component. Then we have:

$$2p_a(C_1) - 2 - (C_1^2) = -1 \text{ and } 2p_a(C_i) - 2 - (C_i^2) \geqq 0 \text{ for } 2 \leqq i \leqq r.$$

Then we have $\sum_i n_i(2p_a(C_i) - 2 - (C_i^2)) \geqq -1$, which contradicts the equality $(*)$.

□

2.3

Lemma . *Let V be a nonsingular projective surface and let Λ be an irreducible linear pencil on V satisfying the properties* (1) *and* (2) *of 2.1. Let B be the set of points of V which are base points of Λ. Let $F := n_1C_1 + \cdots + n_rC_r$ be a reducible member of Λ such that $r \geqq 2$, where C_i is an irreducible component, $C_i \neq C_j$ if $i \neq j$, and $n_i > 0$. Then the following assertions hold:*

(1) *If $C_i \cap B = \phi$ then C_i is isomorphic to $\mathbb{P}^1_k$ and $(C_i^2) < 0$.*

118 (2) *If $C_i \cap C_j \neq \phi$ for $i \neq j$ and $C_i \cap C_j \cap B = \phi$ then $C_i \cap C_j$ consists of a single point where C_i and C_j intersect each other transversely.*

(3) *For three distinct indices i, j, ℓ, if $C_i \cap C_j \cap C_\ell \cap B = \phi$ then $C_i \cap C_j \cap C_\ell = \phi$.*

(4) *Assume that $(C_i^2) < 0$ whenever $C_i \cap B \neq \phi$. Then the set $S = \{C_i; C_i$ is an irreducible component of F such that $C_i \cap B = \phi\}$ is nonempty, and there is an exceptional component in the set S.*

(5) *With the same assumption as in* (4) *above, if a component of S, say C_1, has multiplicity $n_1 = 1$ then there exists an exceptional component in S other than C_1.*

Proof. Let $\rho : \widetilde{V} \to V$ be the (shortest) succession of quadratic transformations with centers at base points (including infinitely near base points) of Λ such that the proper transform $\widetilde{\Lambda}$ of Λ by ρ has no base points. Then, by the equivalence of the properties (2) and (2′) as explained in 2.1, general members of $\widetilde{\Lambda}$ are isomorphic to $\mathbb{P}^1_k$. The assertions (1), (2) and (3) are then apparently true. We shall prove the assertions (4) and (5), assuming that $B \neq \phi$. Let $P \in B$. Set $P_0 := P$, and let $P_1, \ldots, P_{s-1}$ exhaust infinitely near base points of Λ such that P_i

is an infinitely near point of P_{i-1} of order one for $1 \leqq i \leqq s-1$. For $1 \leqq i \leqq s$, let $\sigma_i : V_i \to V_{i-1}$ be a quadratic transformation of V_{i-1} with center at P_{i-1}, where $V_0 := V$, and let $\sigma = \sigma_1 \ldots \sigma_s$. Then σ factors ρ, i.e., $\rho = \sigma \cdot \overline{\rho}$. Let $E'_i := (\sigma_{i+1} \ldots \sigma_s)'(\sigma_i^{-1}(P_{i-1}))$ for $1 \leqq i < s$ and let $E'_s := \sigma_s^{-1}(P_{s-1})$. Let $E_i := \overline{\rho}'(F'_i)$ for $1 \leqq i \leqq s$. It is clear that $E'_i \cong E_i$
and $({E'}_i^1) = (E_i^2)$ for $1 \leq i \leqq s$, and that $(E_i^2) < -1$ for $1 \leqq i < s$ and 119
$(E_s^2) = -1$. Moreover E_s is not contained in any member of $\widehat{\Lambda}$; indeed, if otherwise, $\widetilde{\Lambda}$ would have yet a base point on E_s, which contradicts the choice of points $P_1, \ldots, P_{s-1}$. The member $\widetilde{F}$ of $\widetilde{\Lambda}$ corresponding to F of Λ may contain some (not necessarily all) of $E_1, \ldots, E_{s-1}$. After the above argument made for every point of B we know that if we write $\widetilde{F} = (n_1\widetilde{C}_1 + \cdots + n_r\widetilde{C}_r) + (m_1D_1 + \cdots + m_tD_t)$ with $\widetilde{C}_i = \rho'(C_i)$ for $1 \leqq i \leqq r$ then we have;

1° if $C_i \in S$ then $\widetilde{C}_i \cong C_i$ and $(\widetilde{C}_i^2) = (C_i^2)$,

2° if $C_i \notin S$ then $(\widetilde{C}_i^2) \leqq -2$,

3° $(D_i^2) \leqq -2$ for $1 \leqq i \leqq t$.

Then the assertions (4) and (5) follow from the assertions (5) and (6) of Lemma 2.2. □

2.4

Let $\mathbb{A}_k^2 := \mathrm{Spec}(k[x,y])$ be the affine plane, and fix an open immersion of $\mathbb{A}_k^2$ into $\mathbb{P}_k^2$ as the complement of a line ℓ_0[1]. Let f $k[x,y]$ be an irreducible element such that the curve C_0 defined by $f = 0$ is a nonsingular, rational curve. Let C be the closure of C_0 in $\mathbb{P}_k^2$, and let $d := (C \cdot \ell_0)$. Denote by $\Lambda(f)$ the linear pencil on $\mathbb{P}_k^2$ spanned by C and $d\ell_0$; $\Lambda(f)$ is an irreducible pencil determined uniquely by the inclusion $k(f) \hookrightarrow k(x,y)$.
We may ask under what conditions the pencil $\Lambda(f)$ has properties (1) 120

[1]We note that if $\mathbb{A}_k^2$ is embedded into $\mathbb{P}_k^2$ as an affine open set then the complement is a line. Indeed, let $\tau : \mathbb{A}_k^2 \to \mathbb{P}_k^2$ be such an embedding; then $\mathbb{P}_k^2 - \tau(\mathbb{A}_k^2) = \cup_{i=1}^r C_i$ with irreducible components C_i. If $r = 1$, $C_1 \sim m_1H$ where H is a line of $\mathbb{P}_k^2$. Since $\mathrm{Pic}(\mathbb{A}_k^2) = (0)$ we have $m = 1$. Assume that $r \geqq 2$ and $C_i \sim m_iH$ for $1 \leqq i \leq r$. Then there exists a nonconstant regular function f on $(\mathbb{A}_k^2)$ such that $(f) = m_1C_2 - m_2C_1$, which is a contradiction.

and (2) of the paragraph 2.1.

2.4.1

Lemma. *With the above notations, the pencil* $\Lambda(f)$ *has properties* (1) *and* (2) *of 2.1 if and only if* f *is a field generator, i.e., there exists an element* g *of* $k(x, y)$ *such that* $k(x, y) = k(f, g)$.

Proof. The properties (1) and (2) of 2.1 are equivalent to saying that an algebraic function field $k(x, y)$ in one variable over $k(f)$ has genus 0. By vartue of Tsen's theorem, this is equivalent to saying that $k(x, y)$ is a purely transcendental extension of $k(f)$. □

2.4.2

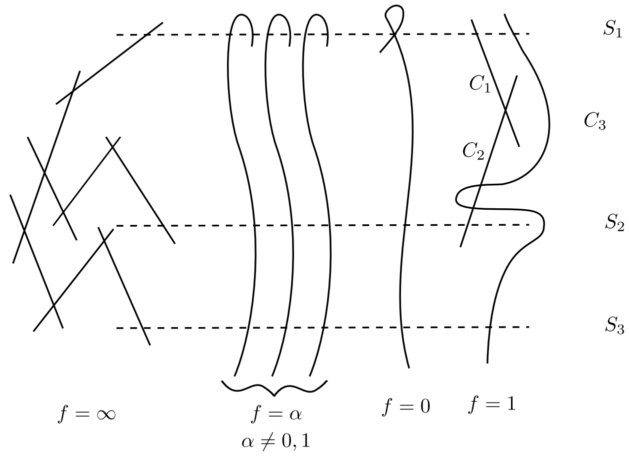
Various properties of a field generator were studied by Russel [48], [50], one of which tells us:

Lemma Russell [48; Cor. 3.7. *] Let* $f \in k[x, y]$ *be a field generator. Then there are at most two points (including infinitely near points) of* f *on the line at infinity. In particular, the degree form of* f *has at most two distinct irreducible factors.*

2.4.3

If the curve C_0 defined by $f = 0$ is isomorphic to $\mathbb{A}^1_k$, the pencil $\Lambda(f)$ satisfies the properties (1) and (2) of 2.1 under some mild restrictions, as we saw in the previous section. An example of a nonsingular, rational curve $C_0 : f = 0$, for which $\Lambda(f)$ does not satisfy the properties (1) and (2) of 2.1, is given by the following:

121 **Example.** Assume that $p \neq 2$. Let $f := xy^2(x+y)+2xy+1$. Then f is an irreducible element and the curve $C_0 : f = 0$ is a non-singular, rational curve. Moreover, if $\rho : \widetilde{V} \to \mathbb{P}^2_k$ is the shortest succession of quadratic transformations such that the proper transform $\widetilde{\Lambda}$ of $\Lambda(f)$ by ρ has no base points, then $\widetilde{\Lambda}$ is a pencil of elliptic curves with three singular fibers and $\widetilde{\Lambda}$ has the following configuration:



where;

1° two dotted lines S_2 and S_3 are cross-sections of $\widetilde{\Lambda}$; and the dotted line S_1 meets each fiber of $\widetilde{\Lambda}$ with multiplicity 2;

2° the singular fiber $f = \infty$ is a singular fiber of type B_9 (cf. Šafarevič [51; p. 172]);

3° the singular fiber $f = 0$ is a rational curve with only one (ordinary) node on the line S_1;

4° the singular fiber $f = 1$ has three irreducible components C_1, C_2 and C_3 which are nonsingular rational curves, and correspond to the curves $y = 0$, $x = 0$ and $y^2 + xy + 2 = 0$ respectively, in the decomposition $f - 1 = yx(y^2 + xy + 2)$; $(C_1^2) = -1$, $(C_2^2) = -3$ and **122**
$(C_3^2) = -2$;

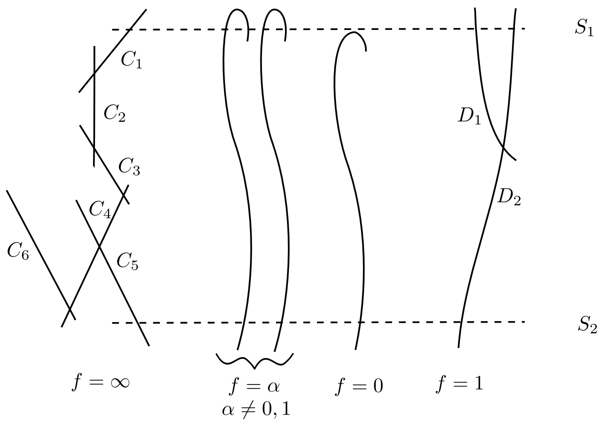
5° each fiber $f = \alpha(\alpha \in k, \alpha \neq 0, 1)$ is a nonsingular elliptic curve meeting S_1 in two distinct points;

6° $\widetilde{V}$-(the fiber $f = \infty) \cup S_1 \cup S_2 \cup S_3 \cong \mathbb{A}^2_k$.

2.4.4

Let $f \in k[x, y]$ be an irreducible element such that the curve $C_0 : f = 0$ is a nonsingular rational curve. Even if C_0 has exactly two places at infinity and f is a field generator, a curve $C_\alpha : f = \alpha (\alpha \in k)$ does not necessarily have two places at infinity as is shown by the next:

Example. Assume that $p \neq 2$. Let $f = x^2y^2 + 2xy^2 + y^2 + 2xy + 1$. Then f is an irreducible element and the curve $C_0 : f = 0$ is a nonsingular rational curve. Moreover, if $\rho : \widetilde{V} \to \mathbb{P}^2_k$ is the shortest succession of quadratic transformations such that the proper transform $\widetilde{\Lambda}$ of $\Lambda(f)$ has no base points, then $\widetilde{\Lambda}$ is a pencil of rational curves with two singular fibers and $\widetilde{\Lambda}$ has the following configuration:



123 where;

1° the dotted line S_2 is a cross-section of $\widetilde{\Lambda}$, and the dotted line S_1 meets each fiber with multiplicity 2;

2° the singular fiber $f = \infty$ is $2C_1 + 4C_2 + 2C_3 + 2C_4 + C_5 + C_6$ with $(C_1^2) = (C_4^2) = (C_5^2) = (C_6^2) = -2$, $(C_2^2) = -1$ and $(C_3^2) = -3$;

3° the singular fiber $f = 1$ is $D_1 + D_2$, where $(D_1^2) = (D_2^2) = -1$, and D_1 and D_2 correspond to the curves $y = 0$ and $x^2y + 2xy + y + 2x = 0$, respectively, in the decomposition $f - 1 = y(x^2y + 2xy + y + 2x)$;

4° the fiber $f = 0$ is a nonsingular rational curve meeting S_1 in a single point with multiplicity 2;

5° each fiber $f = \alpha(\alpha \in k, \alpha \neq 0, 1)$ is a nonsingular rational curve meeting S_1 in two distinct points;

6° $\widetilde{V}$-(the fiber $f = \infty) \cup S_1 \cup S_2 \cong \mathbb{A}^2_k$.

2.4.5

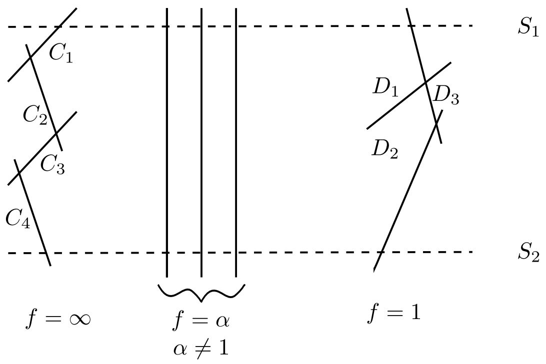
Let $f \in k[x, y]$ be an irreducible element such that:

(1) f is a field generator;

(2) every irreducible curve of the form $C_\alpha : f = \alpha$ with $\alpha \in k$ is a nonsingular rational curve with exactly two places at infinity.

Even with these conditions satisfied, there might exist a curve C_α : $f = \alpha(\alpha \in k)$ which is not connected, as is shown by the next:

Example. Let $f = y(xy + 1) + 1$. Then f is an irreducible element. If $\rho : \widetilde{V} \to \mathbb{P}^2_k$ is the shortest succession of quadratic transformations such that the proper transform $\widetilde{\Lambda}$ of $\Lambda(f)$ has no base points, then $\widetilde{\Lambda}$ is a pencil of rational curves with two singular fibers and $\widetilde{\Lambda}$ has the following **124**
configuration:



where;

1° two dotted lines S_1 and S_2 are cross-sections of $\widetilde{\Lambda}$'

2° the singular fiber $f = \infty$ is $C_1 + 3C_2 + 2C_3 + C_4$ with $(C_1^2) = -3$, $(C_2^2) = -1$ and $(C_3^2) = (C_4^2) = -2$;

3° the singular fiber $f = 1$ is $D_1 + D_2 + D_3$ with $(D_1^2) = (D_2^2) = -1$ and $(D_3^2) = -2$, where D_1 and D_2 correspond to the curves $y = 0$ and $xy + 1 = 0$, respectively, in the decomposition $f - 1 = y(xy + 1)$;

4° the fibers $f = \alpha(\alpha \neq 1, \infty)$ are nonsingular rational curves;

5° $\widetilde{V}$-(the fiber $f = \infty) \cup S_1 \cup S_2 \cup D_3 \cong \mathbb{A}_k^2$.

2.4.6

In the section 6 below we shall show the following result:

Assume that the characteristic of k is zero. Let f be an irreducible element of $k[x, y]$, and let C_α be the curve defined by $f = \alpha$ for $\alpha \in k$. Then $f = x^d y^e - 1$ for positive integers d and e such that $(d, e) = 1$, after
125 a suitable change of coordinates x and y, if the following conditions are satisfied:

1° f is a field generator;

2° C_α has exactly two places at infinity for almost all $\alpha \in k$;

3° C_α is connected for every $\alpha \in k$.

3 Automorphism theorem

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3.1

We shall begin with

Lemma cf. Nagata [43; p. 21]. *Let k be an algebraically closed field of characteristic p. Let C_0 be a closed irreducible curve on the affine plane $\mathbb{A}_k^2 :=$ Spec$(k[x, y])$ such that C_0 is defined by $f = 0$ with $f \in k[x, y]$ and that C_0 is isomorphic to the affine line $\mathbb{A}_k^1$. Fix an open immersion of $\mathbb{A}_k^2$*

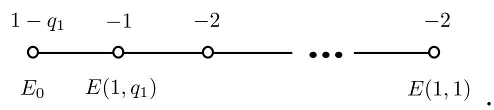
into the projective plane P_k^2, and let $\ell_0 := \mathbb{P}_k^2 - \mathbb{A}_k^2$. Let C be the closure of C_0 on $\mathbb{P}_k^2$, let $P_0 = C \cap \ell_0$, let $d_0 = (C \cdot \ell_0)$ and let d_1 be the multiplicity of C at P_0. Assume that f is a field generator (cf. 2.4.1). *Then d_0 and d_1 are divisible by $d_0 - d_1$. If either d_0 or d_1 is not divisible by p then f is a field generator, and d_0 and d_1 are divisible by $d_0 - d_1$.*

Proof. Our proof consists of three steps.

(I) We may assume with no loss of generality that $d_0 > d_1$. Let Λ_0 be an irreducible linear pencil on $\mathbb{P}_k^2$ spanned by C and $d_0\ell_0$, and let $\rho : \widetilde{V} \to P_k^2$ be the (shortest) succession of quadratic transformations such that the proper transform $\widetilde{\Lambda}$ of Λ_0 has no base points. By assumption and by virtue of 1.2.2 and 1.12 when either d_0 or d_1 is not divisible by p, the pencil $\widetilde{\Lambda}$ satisfies the properties (1) and (2) of 2.1; furthermore, the member of $\widetilde{\Lambda}$ corresponding to $d_0\ell_0$ of Λ_0 is a reducible fiber of $\widetilde{\Lambda}$.

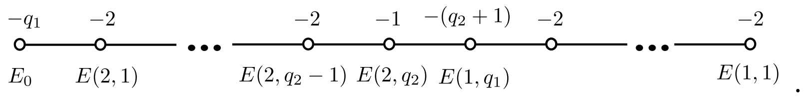
(II) As in 1.3, find integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ by the Euclidean algorithm with respect to d_0 and d_1. To obtain the morphism ρ we **127**
have to start with the Euclidean transformation $\sigma : V \to \mathbb{P}_k^2$ associated with an admissible datum $\{\mathbb{P}_k^2, \mathbb{A}_k^2, C, \ell_0, \phi, d_0, d_1, 1\}$. Let Λ be the proper transform of Λ_0 by σ and let F be the member of Λ corresponding to the member $d_0\ell_0$ of Λ_0. Then F has the weighted graph as given in 1.3, Figure 1, and Λ has a unique base point lying on the curve $E(\alpha, q_\alpha)$ but not on other curves of the weighted graph. We shall now apply Lemma 2.3 to the present V, Λ and F.

(III) **Case 1.** $\alpha = 1$, i.e., $d_0 = q_1 d_1$ with $q_1 \geqq 2$. Then the weighted graph of F is:



Now, by virtue of Lemma 2.3, we know that $(E_0^2) = 1 - q_1 = -1$, i.e., $q_1 = 2$. Then $d_0 - d_1 = d_1$; hence $d_0 - d_1$ divides d_0 and d_1.

Case 2. $\alpha = 2$, i.e., $d_0 = q_1 d_1 + d_2$ and $d_1 = q_2 d_2$ with $q_2 \geqq 2$. Then the weighted graph of F is:



Again by Lemma 2.3 we conclude that $q_1 = 1$. Hence $d_0 - d_1 = d_2$. Then d_2 divides d_0 and d_1.

Case 3. $\alpha \geqq 3$. By Lemma 2.3 we know that $(E_0^2) = -q_1 = -1$. Then we can contract the curves E_0, $E(2,1), \ldots, E(2,q_2-1)$ in this order; in each step of the contractions we obtain a nonsingular
128 projective surface V', a linear pencil Λ' and a singular fiber F' of Λ' to which Lemma 2.3 can be applied. However, after contracting the curve $E(2,q_2-1)$, the proper transform E of $E(2,q_2)$ has self-intersection number $(E^2) = -q_3$ if $\alpha = 3$ and $(E^2) = -(q_3+1)$ if $\alpha \geqq 4$. Note that $q_3 \geqq 2$ if $\alpha = 3$ and $q_3 \geqq 1$ if $\alpha \geqq 4$. Hence $(E^2) \leqq -2$ if $\alpha \geqq 3$. This is a contradiction. Consequently, we know that $\alpha \leqq 2$ and we are done.

$\square$

3.2

Corollary (Abhyankar-Moh [2]). *Let k be a field of characteristic p. Let φ and ψ be nonconstant polynomials of degree m and n in t with coefficients in k. Let $\rho : \mathbb{A}_k^1 = \mathrm{Spec}(k[t]) \to \mathbb{A}_k^2 = \mathrm{Spec}(k[x,y])$ be a morphism defined by $\rho^*(x) = \varphi(t)$ and $\rho^*(y) = \psi(t)$, and let $f(x,y)$ be an irreducible polynomial in $k[x,y]$ defining the curve $\rho(\mathbb{A}_k^1)$. Assume that $k[t] = k[\varphi,\psi]$ and that f is a field generator. Then either m divides n or n divides m. If $G.C.D(m,n)$ is not divisible by p then f is a field generator, and either m divides n or n divides m.*

Proof. We may assume that k is algebraically closed, by substituting an algebraic closure of k for k. We may also assume that $m > n$. Fix a homogeneous coordinate (X,Y,Z) on $\mathbb{P}_k^2$, and let ℓ_0 be the line $Z = 0$. Let $\mathbb{A}_k^2 := \mathbb{P}_k^2 - \ell_0$, and let $x := X/Z$ and $y := Y/Z$. Then a mapping

$t \mapsto (x = \varphi, y = \psi)$ maps isomorphically $\mathbb{A}^1_k$ to a curve C_0 on $\mathbb{A}^2_k$. Let C be the closure of C_0 on $\mathbb{P}^2_k$ and let $P_0 := C \cap \ell_0$. Now write:

$$\varphi(t) := a_m t^m + \cdots + a_0$$
$$\psi(t) := b_n t^n \cdots + b_0$$

with $a_m b_n \neq 0$. Then the point P_0 is $(a_m, 0, 0)$, and the curve C is **129**
expressed locally at P_0 in the following way:

$$X = a_m + a_{m-1}\tau + \cdots + a_0\tau^m$$
$$Y = \tau^{m-n}(b_n + b_{n-1}\tau + \cdots + b_0\tau^n)$$
$$Z = \tau^m$$

where $\tau = t^{-1}$. Then $m = (C \cdot \ell_0)$, and $m - n$ is the multiplicity of C at P_0. By virtue of Lemma 3.1 we know that n divides m. □

3.3

Let k be a field of arbitrary characteristic p, and let $k[x, y]$ be a polynomial ring in two variables x and y over k. We denote by $\mathrm{Aut}_k\, k[x, y]$ the group of k-automorphisms of $k[x, y]$. A k-automorphism ξ (σ or τ, resp.) of $k[x, y]$ is called a *linear* (*affine* or *de Jonquière*, resp.) transformation if ξ (σ or τ, resp.) has the following expression:

$\xi(x) = \alpha x + \beta y$, $\xi(y) = \alpha' x + \beta' y$ with $\alpha, \beta, \alpha', \beta' \in k$ such that $\alpha\beta' \neq \alpha'\beta$;

$\sigma(x) = \alpha x + \beta y + \gamma$, $\sigma(y) = \alpha' x + \beta' y + \gamma'$ with $\alpha, \beta, \gamma, \alpha'$, $\beta', \gamma' \in k$ such that $\alpha\beta' \neq \alpha'\beta$;

$\tau(x) = \alpha x + f(y)$, $\tau(y) = \beta y + \gamma$, where $f(y) \in k[y]$ and α, β, $\gamma \in k$ with $\alpha\beta \neq 0$.

We denote by $GL(2, k)$ (A_2 or J_2, resp.) the subgroup of all linear (affine or de Jonquière, resp.) transformations in $\mathrm{Aut}_k\, k[x, y]$. A k-automorphism ρ of $k[x, y]$ is called *tame* if ρ is an element of the subgroup generated by A_2 and J_2. An easy consequence of Corollary 3.2 is the following:

3.3.1

AUTOMORPHISM THEOREM (cf. Nagata [43]; Abhyankar-Moh
130 **[2] and many others).** Let k be a field of arbitrary characteristic. Then every k-automorphism of $k[x, y]$ is tame.

Proof. Let ρ be a k-automorphism of $k[x, y]$ and let

$$\begin{aligned}\rho(x) &:= f(x, y) = f_0(x, y) + f_1(x, y) + \cdots + f_m(x, y)\\ \rho(y) &:= g(x, y) = g_0(x, y) + g_1(x, y) + \cdots + g_n(x, y)\end{aligned}$$

where $f_i(x, y)$ and $g_j(x, y)$ are the i-th and the j-th homogeneous parts of f and g, respectively, for $0 \leqq i \leqq m$ and $0 \leqq j \leqq n$, and where $f_m(x, y) \neq 0$ and $g_n(x, y) \neq 0$. After a suitable change of coordinates x and y by a linear transformation we may assume that $f_m(x, 0) \neq 0$ and $g_n(x, 0) \neq 0$. Let $\varphi(x) := f(x, 0)$ and $\psi(x) := g(x, 0)$. Then $k[x] = k[\varphi, \psi]$. Let $\tau : \mathbb{A}^1_k = \mathrm{Spec}(k[x]) \to \mathbb{A}^2_k = \mathrm{Spec}(k[x', y'])$ be a morphism defined by $\tau^*(x') = \varphi(x)$ and $\tau^*(y') = \psi(x)$, let $C_0 = \tau(\mathbb{A}^1_k)$, and let $f'(x', y')$ be an irreducible element in $k[x', y']$ defining C_0. Then f' is a field generator; this is clear because ρ is an automorphism of $k[x, y]$. By Corollary 3.2 we conclude that either $m|n$ or $n|m$. Besides, it is easily ascertained that if $mn > 1$ then $f_m(x, y) = \alpha\lambda^m$ and $g_n(x, y) = \beta\lambda^n$ for a common linear factor λ in x and y and for $\alpha, \beta \in k^*$[2]. Assuming that $m = nd > 1$, define
131 a k-automorphism σ of $k[x, y]$ by $\sigma(x) = x - \gamma y^d$ and $\sigma(y) = y$, where $\alpha = \gamma\beta^d$. Then $\sigma\rho(x)$ has degree smaller than m. Thus, we can finish our proof by induction on $\max(m, n)$. □

3.3.2

More precisely, we know the following structure theorem on $\mathrm{Aut}_k\, k[x, y]$:

---

[2]Since the curve $f(x, y) = 0$ is isomorphic to $\mathbb{A}^1_k$, we know that $f_m(x, y) = \lambda_1^m$ for a linear factor λ_1; similarly, $g_n(x, y) = \lambda_2^n$. Since the curves $f(x, y) = 0$ and $g(x, y) = 0$ intersect in a single point transversely on $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$, we know by Bezout's Theorem that $\lambda_2 = \gamma\lambda_1$ with $\gamma \in k^*$ unless $mn = 1$.

Lemma (cf. Nagata [43; Th. 3.3 *; Kambayashi [25])]* $\mathrm{Aut}_k\, k[x, y]$ *is an amalgamated product of* A_2 *and* J_2*. Namely, if* $\sigma_i \in A_2 - J_2$ $(1 \leqq i \leqq r-1)$*,* $\tau_j \in J_2 - A_2 (1 \leqq j \leqq r)$*, then* $\tau_1\sigma_1\tau_2\sigma_2 \ldots \tau_{r-1}\sigma_{r-1}\tau_r \notin A_2$. 131

For the convenience of readers, we shall give a (sketchy) proof in the next paragraph.

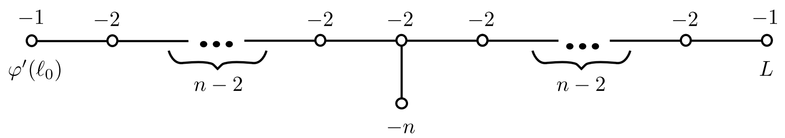
3.4

Let $\tau : \tau(x) = \alpha x + f(y)$ and $\tau(y) = \beta y + \gamma$ be a de Jonquière transformation of $k[x, y]$. τ defines a birational automorphism T of $\mathbb{P}^2_k$ by setting: $T^*(X) = \alpha X Z^{n-1} + F(Y, Z)$, $T^*(Y) = \beta Y Z^{n-1} + \gamma Z^n$ and $T^*(Z) = Z^n$, where $x = X/Z$, $y = Y/Z$, $n := \deg_y f(y)$ and $F(Y, Z) := Z^n f(Y/Z)$.[3] We assume that $n > 1$. Then it is easy to see that $P_0 : (X, Y, Z) = (1, 0, 0)$ is a unique fundamental point of T on $\mathbb{P}^2_k$ and the line $\ell_0 : Z = 0$ is a unique fundamental curve of T on $\mathbb{P}^2_k$. Let φ_1 be a quadratic transformation with center at P_0. Now eliminating fundamental points (including infinitely near fundamental points) of T by the (shortest) succession of quadratic transformations, which start with φ_1, we have a nonsingular projective surface V and birational morphisms $\varphi, \psi : V \to \mathbb{P}^2_k$ such that $T = \psi \cdot \varphi^{-1}$.

3.4.1

Lemma. *With the above notations we have:*

(1) $\varphi^{-1}(\ell_0)$ *has the following weighted graph:*



where each vertex stands for a nonsingular rational curve; the vertex with weight $-n$ *corresponds to the proper transform of* $\varphi^{-1}(P_0)$ *by* $\varphi\varphi_1^{-1}$. 132

[3]If $n = \deg_y f(y)$, τ is called a de Jonquière transformation of degree n.

(2) *Let L be the curve with weight -1 other than $\varphi'(\ell_0)$. Then $\psi(L)$ is the line at infinity of a new projective plane $\mathbb{P}^2_k$.*

(3) *The point $Q := (1,0,0)$ on the new projective plane $\mathbb{P}^2_k$ is a unique fundamental point of T^{-1}.*

Proof. Straightforward computation. See also Russell [48; 4.2]. □

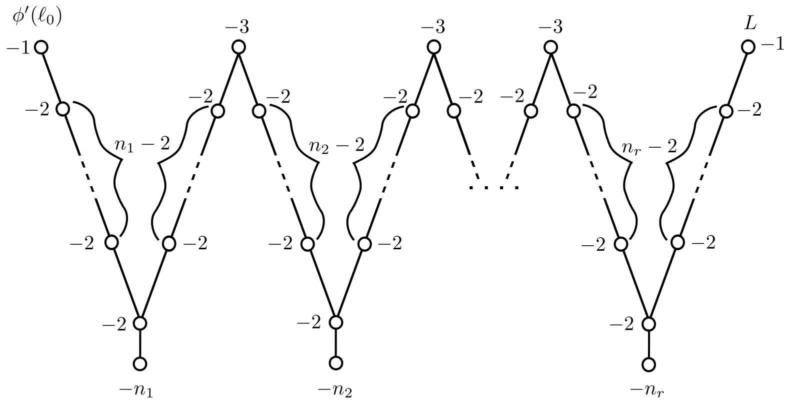
3.4.2

Note that if $\sigma : \sigma(x) = \alpha x + \beta y + \gamma$ and $\sigma(y) = \alpha' x + \beta' y + \gamma'$ is an affine transformation not in J_2, i.e., $\alpha' \neq 0$ then the associated biregular automorphism $\Sigma : (X, Y, Z) \mapsto (\alpha X + \beta Y + \gamma Z, \alpha' X + \beta' Y + \gamma' Z, Z)$ of $\mathbb{P}^2_k$ maps the point $(1,0,0)$ to $(\alpha, \alpha', 0)$ which is distinct from $(1,0,0)$. With this remark in mind we can easily show:

Lemma. *Let $\sigma_i \in A_2 - J_2 (1 \leqq i \leqq r-1)$, let $\tau_j \in J_2 - A_2 (1 \leqq j \leqq r)$ and let $\rho : \tau_1\sigma_1 \ldots \tau_{r-1}\sigma_{r-1}\tau_r$. Let n_j be the degree of τ_j for $1 \leqq j \leqq r$. Let $\Sigma_i (1 \leqq i \leqq r-1)$, $T_j (1 \leqq j \leqq r)$ and R be birational automorphisms of $\mathbb{P}^2_k$ associated with σ_i, τ_j and ρ, respectively. Then, by elimination of indeterminacy of a birational automorphism ρ, we obtain a nonsingular projective surface W and birational morphisms ϕ, $\psi : W \to \mathbb{P}^2_k$ such that:*

(1) *$R = \psi \cdot \phi^{-1}$, where $R := T_r\Sigma_{r-1}T_{r-1} \ldots \Sigma_1 T_1$;*

(2) *$\phi^{-1}(\ell_0)$ has the following weighted graph:*



where, if L is the curve corresponding to the vertex with weight -1 other than $\phi'(\ell_0)$, then $\psi(L)$ is the line at infinity of a new projective plane $\mathbb{P}^2_k$; **133**

(3) $n_j > 1$ *for* $1 \leqq j \leqq r$.

3.4.3

By virtue of Lemma 3.4.2, it is clear that $\rho \notin A_2$; if $\rho \in A_2$ then ℓ_0 would not be a fundamental curve. Therefore we completed a proof of Lemma 3.3.2.

3.5

We shall prove in this paragraph the following:

Theorem Igarashi-Miyanishi [24]. *Let k be a field of characteristic zero.*[4] *Let F be a finite subgroup of order n of $\mathrm{Aut}_k\, k[x,y]$. Then there exists an element ρ of $\mathrm{Aut}_k\, k[x,y]$ such that $\rho^{-1}F\rho$ is contained in $GL(2,k)$.*

The proof will be given in the subparagraphs 3.5.1 ~ 3.5.5.

3.5.1

Lemma. *With the notations and assumptions as above, if F is contained in A_2, then F is conjugate to a finite subgroup of $GL(2,k)$.* **134**

Proof. It suffices to show that F has a fixed point on the affine plane $\mathbb{A}^2_k$. Indeed, let ρ be an affine transformation defined by

$$\rho\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

[4]The theorem holds for an arbitrary characteristic p if $(n,p) = 1$. Indeed, Lemmas 3.5.1 and 3.5.2 hold true with this condition, while Lemmas 3.5.3 and 3.5.4 hold true without any restriction.

Then $\rho^{-1}F\rho \subset GL(2,k)$ if and only if the point $(x = s, y = t)$ on $\mathbb{A}^2_k$ is fixed under F. Each $\sigma \in A_2$ has a matrix representation:

$$\sigma\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha(\sigma) & \beta(\sigma) & a(\sigma) \\ \gamma(\sigma) & \delta(\sigma) & b(\sigma) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Let $\ell(\sigma) = {}^t(a(\sigma), b(\sigma))$ and let $M(\sigma)$ be an invertible matrix such that

$$M(\sigma) = \begin{pmatrix} \alpha(\sigma) & \beta(\sigma) \\ \gamma(\sigma) & \delta(\sigma) \end{pmatrix}.$$

For σ, $\tau \in A_2$, we have $\ell(\sigma\cdot\tau) = \ell(\sigma) + M(\sigma)\ell(\tau)$ and $M(\sigma\cdot\tau) = M(\sigma)\cdot M(\tau)$. Then $\ell_0 = {}^t(s_0, t_0)$ is a fixed point of F if and only if $\ell(\sigma) = \ell_0 - M(\sigma)\ell_0$ for every σ of F. Set $\ell_0 := \left\{\sum_{\tau\in F}\ell(\tau)\right\}/n$. Since

$$\frac{1}{n}\sum_{\tau\in F}\ell(\sigma\cdot\tau) = \ell(\sigma) + M(\sigma)\left(\frac{1}{n}\sum_{\tau\in F}\ell(\tau)\right),$$

135 we have then $\ell(\sigma) = \ell_0 - M(\sigma)\ell_0$. Hence ℓ_0 gives rise to a point Q of $\mathbb{A}^2_k$ fixed under F. □

3.5.2

Lemma. *With the notations and assumptions as above, if $F \subset J_2$, then F is conjugate to a finite subgroup of $GL(2,k)$.*

Proof. Each element $\sigma \in J_2$ acts on $\mathbb{A}^2_k$ in the following way:

$$\sigma(x) = \alpha(\sigma)x + f_\sigma(y) \quad \text{and} \quad \sigma(y) = \beta(\sigma)y + \gamma(\sigma),$$

where $f_\sigma(y) \in k[y]$; $\alpha(\sigma), \beta(\sigma), \gamma(\sigma) \in k$; $\alpha(\sigma)\cdot\beta(\sigma) \neq 0$. For $\sigma, \tau \in J_2$, we have:

$$\alpha(\sigma\cdot\tau) = \alpha(\sigma)\cdot\alpha(\tau), \beta(\sigma\cdot\tau) = \beta(\sigma)\cdot\beta(\tau), \gamma(\sigma\cdot\tau) = \gamma(\tau) + \beta(\tau)\gamma(\sigma)$$

and $f_{\sigma\cdot\tau}(y) = f_\tau(\beta(\sigma)y + \gamma(\sigma)) + \alpha(\tau)f_\sigma(y)$.

Let J_2^0 be the subgroup of J_2 such that $\gamma(\sigma) = 0$. Let $\epsilon = \left\{\sum_{\tau\in F}\gamma(\tau)\right\}/n$. Then $\gamma(\sigma) = \epsilon - \beta(\sigma)\epsilon$ for every σ of F. Replacing y by $y - \epsilon$, we may assume that F is contained in J_2^0. We shall now look for a polynomial $g(y) \in k[y]$ such that $\sigma(x + g(y)) = \alpha(\sigma)(x + g(y))$ for every $\sigma \in F$. If such a polynomial exists, we have $\rho^{-1}\sigma\rho(x) = \alpha(\sigma)x$ and $\rho^{-1}\sigma\rho(y) = \beta(\sigma)y$ for every $\sigma \in F$, setting $\rho(x) = x + g(y)$ and $\rho(y) = y$. Namely, $F \subset GL(2, k)$. Now $g(y)$ satisfies $\sigma(x + g(y)) = \alpha(\sigma)(x + g(y))$ for every $\sigma \in F$ if and only if $f_\sigma(y) = \alpha(\sigma)g(y) - g(\beta(\sigma)y)$ for every $\sigma \in F$. Write $f_\sigma(y)$ in the form:

$$f_\sigma(y) = \frac{\alpha(\sigma)}{\alpha(\sigma\cdot\tau)} f_{\sigma\cdot\tau}(y) - \frac{1}{\alpha(\tau)} f_\tau(\beta(\sigma)y).$$

Then $n f_\sigma(y) = \alpha(\sigma)\sum_{\tau\in F}\frac{f_{\sigma\cdot\tau}(y)}{\alpha(\sigma\cdot\tau)} - \sum_{\tau\in F}\frac{f_\tau(\beta(\sigma)y)}{\alpha(\tau)}$. Set $g(y) := \frac{1}{n}\sum_{\tau\in F}\frac{f_\tau(y)}{\alpha(\tau)}$. **136**
Then $f_\sigma(y) = \alpha(\sigma)g(y) - g(\beta(\sigma)y)$ for every $\sigma \in F$. This completes the proof. □

3.5.3

Lemma. *Let F be a finite subgroup of* $\mathrm{Aut}_k\, k[x, y]$. *Let σ be an element of F. Then there exists an element ρ in* $\mathrm{Aut}_k\, k[x, y]$ *such that $\rho^{-1}\sigma\rho$ is contained in either A_2 or J_2 and $\rho^{-1}(F\cap(A_2\cup J_2))\rho \subset \rho^{-1}F\rho\cap(A_2\cup J_2)$, where $A_2 \cup J_2$ is the set-theoretic union of A_2 and J_2 in* $\mathrm{Aut}_k\, k[x, y]$.

Proof. Our proof consists of three steps.

(I) By virtue of Lemma 3.3.2 we may write σ in one of the following ways:

(i)$_r$ $\sigma = \sigma_1\tau_1 \ldots \tau_{r-1}\sigma_r\tau_r$, where $\sigma_i \in A_2 - A_2 \cap J_2 (1 \leqq i \leqq r)$, $\tau_j \in J_2 - A_2 \cap J_2 (1 \leqq j \leqq r - 1)$ and $\tau_r \in J_2$;

(ii)$_r$ $\sigma = \tau'_1\sigma'_1 \ldots \sigma'_{r-1}\tau'_r\sigma'_r$, where $\tau'_i \in J_2 - A_2 \cap J_2 (1 \leqq i \leqq r)$, $\sigma'_j \in A_2 - A_2 \cap J_2 (1 \leqq j \leqq r - 1)$ and $\sigma'_r \in A_2$.

We shall prove the following assertions for every $r \geqq 1$:

$(1)_r$ if σ is written in the way $(\mathrm{i})_r$ then there exists an element ρ of $\mathrm{Aut}_k\, k[x,y]$ such that $\rho^{-1}\sigma\rho$ is written in the way $(\mathrm{ii})_{r-1}$ and $\rho^{-1}(F \cap (A_2 \cup J_2))\rho \subset \rho^{-1}F\rho \cap (A_2 \cup J_2)$;

$(2)_r$ If σ is written in the way $(\mathrm{ii})_r$ then there exists an element ρ of $\mathrm{Aut}_k\, k[x,y]$ such that $\rho^{-1}\sigma\rho$ is written in the way $(\mathrm{i})_{r-1}$
137 and $\rho^{-1}(F \cap (A_2 \cup J_2))\rho \subset \rho^{-1}F\rho \cap (A_2 \cup J_2)$; where $(1)_1$ and $(2)_1$ are understood, respectively as:

$$(1)_1\ \sigma \in A_2; \quad (2)_1\ \sigma \in J_2.$$

It is apparent that the assertion in Lemma follows from the above assertions.

(II) *Proof of the assertion* $(1)_r$. Since $\sigma^n = 1$ for some integer $n > 0$, we have:

$$\sigma^n = \underbrace{(\sigma_1\tau_1 \ldots \sigma_r\tau_r) \ldots (\sigma_1\tau_1 \ldots \sigma_r\tau_r)}_{n\text{-times}} = 1.$$

Since $(\tau_1 \ldots \sigma_r\tau_r)(\sigma_1\tau_1 \ldots \sigma_r\tau_r) \ldots (\sigma_1\tau_1 \ldots \sigma_r\tau_r) = \sigma_1^{-1} \in A_2$, Lemma 3.3.2 implies that $\tau_r \in A_2 \cap J_2$; indeed, if $\tau_r \notin A_2$ we would have a contradiction. If $r = 1$ then $\sigma = \sigma_1\tau_1 \in A_2$, i.e., $(1)_1$ holds. If $r > 1$, we know again by Lemma 3.3.2 that $\sigma_r\tau_r\sigma_1 \in A_2 \cap J_2$ because

$$\begin{aligned}(\tau_1 \ldots \tau_{r-1})(\sigma_r\tau_r\sigma_1)(\tau_1 \ldots \tau_{r-1}) \ldots (\sigma_r\tau_r\sigma_1)(\tau_1 \ldots \tau_{r-1})\\ = (\sigma_r\tau_r\sigma_1)^{-1} \in A_2.\end{aligned}$$

Let $h = \sigma_r\tau_r\sigma_1$. Then $\sigma_r\tau_r = h\sigma_1^{-1}$, and $\sigma_1^{-1}\sigma\sigma_1 = \tau_1\sigma_2 \ldots \sigma_{r-1}\tau_{r-1}h$. Thus $\sigma_1^{-1}\sigma\sigma_1$ has an expression as in $(\mathrm{ii})_{r-1}$. We shall show that $\sigma_1^{-1}(F \cap (A_2 \cup J_2))\sigma_1 \subset \sigma_1^{-1}F\sigma_1 \cap (A_2 \cup J_2)$. Let σ_0 be an element of $F \cap (A_2 \cup J_2)$. Since $\sigma \cdot \sigma_0 \in F$ and $(\sigma \cdot \sigma_0)^m = 1$ for some integer $m > 0$, we have:

$$(\sigma \cdot \sigma_0)^m = \underbrace{(\sigma_1\tau_1 \ldots \sigma_r\tau_r\sigma_0) \ldots (\sigma_1\tau_1 \ldots \sigma_r\tau_r\sigma_0)}_{m\text{-times}} = 1.$$

Since $(\tau_1\sigma_2 \ldots \sigma_r\tau_r\sigma_0)(\sigma_1\tau_1 \ldots \sigma_r\tau_r\sigma_0) \ldots (\sigma_1\tau_1 \ldots \sigma_r\tau_r\sigma_0) =$
138 $\sigma_1^{-1} \in A_2$ and since $\sigma_r\tau_r \in A_2 - A_2 \cap J_2$, Lemma 3.3.2 implies
that $\sigma_0 \notin J_2 - A_2 \cap J_2$, i.e., $\sigma_0 \in A_2$. Since this is true for every element σ_0 of $F \cap (A_2 \cup J_2)$ we know that $F \cap (A_2 \cup J_2) = F \cap A_2$. Then $\sigma_1^{-1}(F \cap A_2)\sigma_1 \subset \sigma_1^{-1}F\sigma_1 \cap (A_2 \cup J_2)$ because $\sigma_1 \in A_2$. Thus we have only to take σ_1 as ρ.

(III) *Proof of the assertion* $(2)_r$. A proof will be only sketchy because it is similar to the proof of $(1)_r$. Since $\sigma^n = 1$ for some integer $n > 0$ we conclude by Lemma 3.3.2 that $\sigma'_r \in A_2 \cap J_2$. If $r = 1$ then $\sigma = \tau'_1\sigma'_1 \in J_2$, i.e., $(2)_1$ holds. If $r > 1$, Lemma 3.3.2 again implies that $\tau'_r\sigma'_r\tau'_1 \in A_2 \cap J_2$. Let $h' = \tau'_r\sigma'_r\tau'_1$. Then $\tau'_r\sigma'_r = h'{\tau'}_1^{-1}$ and ${\tau'}_1^{-1}\sigma\tau'_1 = \sigma'_1\tau'_2 \ldots \tau'_{r-1}\sigma'_{r-1}h'$. Thus ${\tau'}_1^{-1}\sigma\tau'_1$ has an expression as in $(\mathrm{i})_{r-1}$. We shall show that ${\tau'}_1^{-1}(F \cap (A_2 \cup J_2))\tau'_1 \subset {\tau'}_1^{-1}F\tau'_1 \cap (A_2 \cup J_2)$. Let σ_0 be an arbitrary element of $F \cap (A_2 \cup J_2)$. Since $\sigma \cdot \sigma_0 \in F$ and $(\sigma \cdot \sigma_0)^m = 1$ for some integer $m > 0$, a similar argument as in step (II) shows that $\sigma_0 \in J_2$. Hence $F \cap (A_2 \cup J_2) = F \cap J_2$. Then ${\tau'}_1^{-1}(FJ_2)\tau'_1 \subset {\tau'}_1^{-1}F\tau'_1 \cap (A_2 \cup J_2)$ because $\tau'_1 \in J_2$. Thus we have only to take τ'_1 as ρ.

□

3.5.4

Lemma. *Let F be a finite subgroup of* $\mathrm{Aut}_k\, k[x,y]$. *Then there exists an element* ρ *in* $\mathrm{Aut}_k\, k[x,y]$ *such that* $\rho^{-1}F\rho$ *is contained in either* A_2 *or* J_2.

Proof. Lemma 3.5.3 tells us the following: If F has an element σ not in A_2 and J_2 then there exists an element ρ of $\mathrm{Aut}_k\, k[x,y]$ such that $|\rho^{-1}F\rho \cap (A_2 \cup J_2)| > |F \cap (A_2 \cup J_2)|$. Hence, by substituting a suitable conjugate of F for F, we may assume that $F \subset A_2 \cup J_2$. If F is not
contained in A_2 or J_2, then there exist two elements α and β in F such 139
that $\alpha \in A_2 - A_2 \cap J_2$ and $\beta \in J_2 - (A_2 \cap J_2)$. Then Lemma 3.3.2 implies that $\alpha \cdot \beta$ is not of finite order. This contradicts the fact that $\alpha \cdot \beta \in F$. Hence either $F \subset A_2$ or $F \subset J_2$. □

3.5.5

Lemma. *3.5.4 combined with Lemmas 3.5.1 and 3.5.2 completes a proof of Theorem 3.5.*

3.6

In the present and the next paragraphs we shall apply Theorem 3.5, in order to obtain two partial answers of the following:

CONJECTURE. *Let k be an algebraically closed field, and let A be a regular k-subalgebra of a polynomial ring $k[x, y]$ such that $k[x, y]$ is a flat A-module of finite type. Then A is a polynomial ring over k.*

The first result is stated as follows:

Proposition . *Let k be an algebraically closed field of characteristic zero. Let X be a nonsingular affine surface and let $f : \mathbb{A}^2_k \to X$ be an étale finite surjective morphism. Then f is an isomorphism.*

This result will be proved in the subparagraphs 3.6.1 ~ 3.6.4.

3.6.1

Definition. *Let X and Z be nonsingular varieties defined over k and let $h : Z \to X$ be an étale finite morphism. A pair (Z, h) is called* a Galois covering of X with group F *if there exist a finite group F acting freely on*
140 *Z and a k-isomorphism $\varphi : Z/F \xrightarrow{\sim} X$ between the quotient variety Z/F and X such that $h = \varphi q$, where $q : Z \to Z/F$ is the canonical quotient morphism.*

3.6.2

Lemma. *Let $f : X \to Y$ be an étale finite morphism of a nonsingular variety X onto a nonsingular variety Y. Then there exist an étale finite morphism $h : Z \to X$ of a nonsingular variety Z onto X, a finite group G and a subgroup H of G such that:*

(1) *$g : f \cdot h : Z \to Y$ is a Galois covering with group G;*

(2) *$h : Z \to X$ is a Galois covering with group H.*

Proof. Let $n := [k(X) : k(y)]$ and let $\widetilde{S} := X\underset{Y}{\times}X\underset{Y}{\times}\dots\underset{Y}{\times}X$ be the n-th iterated fiber product of X over Y. Let F be the closed subset of $\widetilde{S}$ consisting of all n-tuples $(x_1, \dots, x_n)$ in which two or more of x_i's coincide with each other. Let $Z := \widetilde{S} - F$, and let S_n be the symmetric group on n letters. Then S_n acts freely on Z. Let $h : Z \to X$ be the projection onto the first factor, and let S_{n-1} be the symmetric group on $n-1$ letters, which we let act on Z in such a way that

$$\sigma(x_1, \dots, x_n) = (x_1, \sigma(x_2, \dots, x_n)) \quad \text{for} \quad \sigma \in S_{n-1}.$$

Then $S_{n-1} \subset S_n$. Since $h^{-1}(x)$ consists of $(n-1)!$ points for every $x \in X$, h is an étale finite morphism. Thus $q := f \cdot h : Z \to Y$ is also an étale finite morphism. It is obvious that $Y \cong Z/S_n$ with the quotient morphism $q : Z \to Y$ and $X \cong Z/S_{n-1}$ with the quotient morphism $h : Z \to X$. We have now only to set $G := S_n$ and $H := S_{n-1}$. □

3.6.3

Lemma. *With the notations of 3.6.2, if X is simply connected, i.e., X* **141**
has no nontrivial étale finite coverings, then $f : X \to Y$ is a Galois covering.

Proof. Since X is simply connected, $h : Z \to X$ splits. Namely there exists a regular cross-section $s : X \to Z$ such that the morphism $H \times X \to Z$ defined by $(h, x) \mapsto hs(x)$ is an isomorphism. Therefore the number of connected components of Z, which are all isomorphic to X, is the order $|H|$ of H. Let $X_0 := s(X)$ and let F be the subgroup of G consisting of all elements g of G such that $g(X_0) = X_0$. If X_1 is a connected component of Z distinct from X_0 and if g is an element of G such that $g(X_0) = X_1$, then $g_1 F$ is the set of all elements of G which send X_0 to X_1. Hence $Z \cong X \times G/F$, and $|G/F| = |H|$. Therefore the morphism $q|_{X_0} : X_0 \to Y$ has degree $|G|/|H| = |F|$. Since F acts freely on X_0, we know that a pair $(X_0, g|_{X_0})$ is a Galois covering of Y with group F. Finally, since $X \cong X_0$, $f : X \to Y$ is a Galois covering with group F. □

3.6.4

Proof of Proposition 3.6. *Since $\mathbb{A}^2_k$ is simply connected (if the characteristic of k is zero), we know by applying Lemma 3.6.3 that $f : \mathbb{A}^2_k \to X$ is a Galois covering with group F. But since every finite subgroup of $\mathrm{Aut}_k\, k[x, y]$ has a fixed point on $\mathbb{A}^2_k$ by virtue of Theorem 3.5, we know that F can not act freely on $\mathbb{A}^2_k$. Therefore, $F \cong (1)$, and f is an isomorphism. This completes a proof of Proposition 3.6.*

3.7

142 Another application of Theorem 3.5 is the following proposition which is a slight improvement of Serre's result (cf. Lemma 3.7.1 below):

Proposition. *Let k be a field of characteristic zero, and let F be a finite subgroup of $\mathrm{Aut}_k\, k[x, y]$. Then the following conditions are equivalent to each other:*

(1) *$k[x, y]^F$ is a regular ring;*

(2) *$k[x, y]^F$ is a polynomial ring over k;*

where $k[x, y]^F$ is the invariant subring of $k[x, y]$ with respect to F.

A proof of proposition will be given in the subparagraphs 3.7.1 ∼ 3.7.3.

3.7.1

Lemma (Serre [53; Th.1]). *Let F be a finite subgroup of $GL(n, k)$ and let $k[x_1, \ldots, x_n]^F$ be the invariant subring of $k[x_1, \ldots, x_n]$ for F.*[5] *Then the following are equivalent:*

(1) *$k[x_1, \ldots, x_n]^F$ is a polynomial ring over k;*

(2) *F is generated by pseudo-reflections.*

*(*An element f of $GL(n, k)$ is called *a pseudo-reflection* if rank $(I - f) \leqq 1$.*)*

[5] The present and the next lemmas hold for a field k of arbitrary characteristic p, if we assume that $(|F|, p) = 1$.

3.7.2

Lemma (Serre [53; Th.1']). *Let S be a regular local ring with maximal ideal $\underline{m}_S$ and let F be a finite subgroup of* $\mathrm{Aut}(S)$. *Let S^F be the invariant local subring of S for F. Suppose that:*

(1) *S^F is a noetherian ring;*

(2) *S is of finite type over S^F;*

(3) *$S/\underline{m}_S = S^F/\underline{m}_S \cap S^F = k$;* **143**

(4) *the action of F on $S/\underline{m}_S$ is trivial.*

Let $\epsilon : F \to \mathrm{Aut}_k(\underline{m}_S/\underline{m}_S^2)$ be the canonical homomorphism from F to $\mathrm{Aut}_k(\underline{m}_s/\underline{m}_S^2)$. Then the following are equivalent:

(i) *S^F is a regular local ring;*

(ii) *$\epsilon(F)$ is generated by pseudo-reflections.*

3.7.3

Proof of Proposition 3.7. *(2) $\Rightarrow$ (1) is clear. We shall show (1) $\Rightarrow$ (2). By virtue of Theorem 3.5 there exists an element ρ of $\mathrm{Aut}_k\, k[x,y]$ such that $\rho^{-1}F_\rho$ is a finite subgroup of $GL(2,k)$. Then $k[x,y]^F = \rho\,(k[x,y]^{\rho^{-1}F_\rho})$ and $k[x,y]^{\rho^{-1}F_\rho}$ is a regular ring. Hence we may assume that F is a finite subgroup of $GL(2,k)$. Let (x,y) be the maximal ideal of $k[x,y]$ generated by x and y. Let S be the localization of $k[x,y]$ with respect to the ideal (x,y) and let $\underline{m}_S$ be the maximal ideal of S. We can view F as a finite subgroup of $\mathrm{Aut}(S)$ in a natural way, and it is easy to see that $S^F = S \cap Q(k[x,y]^F)$, where $Q(k[x,y]^F)$ is the quotient field of $k[x,y]^F$. Since $k[x,y]$ is a $k[x,y]^F$-module of finite type and F fixes the ideal (x,y), we know that S is a finitely generated S^F-module. Thus S and S^F satisfy the condition (2) and also the other conditions of Lemma 3.7.2. By virtue of Lemma 3.7.2, $\epsilon(F)$ is generated by pseudo-reflections in $\mathrm{Aut}_k(\underline{m}_S/\underline{m}_S^2)$. Now it is easily seen that the action of $\epsilon(F)$ on the k-vector space $\underline{m}_S/\underline{m}_S^2$ coincides with that of F on the vector space $kx+ky$. Therefore F is generated by pseudo-reflections. By virtue of Lemma 3.7.1, we know that $k[x,y]^F$ is a polynomial ring over k.*

4 Finiteness theorem

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4.1

Throughout this section the ground field k is assumed to be algebraically closed field of characteristic zero. Let C_0 be a nonsingular irreducible affine curve of genus $q > 0$ with only one place at infinity. By *an embedding* of C_0 into the affine plane $\mathbb{A}_k^2$ we mean a biregular mapping $\epsilon : C_0 \to \mathbb{A}_k^2$. Fix an open immersion of $\mathbb{A}_k^2$ into the projective plane $\mathbb{P}_k^2$ as a complement of a line ℓ_0. Let $C(\epsilon)$ be the closure of $\epsilon(C_0)$ in $\mathbb{P}_k^2$. Let $P_0(\epsilon) := C(\epsilon) \cap \ell_0$, let $d_0(\epsilon) := (C(\epsilon) \cdot \ell_0)$ and let $d_1(\epsilon)$ be the multiplicity of $C(\epsilon)$ at $P_0(\epsilon)$. Then $d_0(\epsilon) > d_1(\epsilon)$; indeed, if otherwise, we have $d_0(\epsilon) = 1$ and hence $g = 0$. By abuse of (and for the sake of simplicity of) the notations, we denote $\epsilon(C_0)$, $C(\epsilon)$, $P_0(\epsilon)$, $d_0(\epsilon)$ and $d_1(\epsilon)$ by C_0, C, P_0, d_0 and d_1 if an embedding $\epsilon : C \to \mathbb{A}_k^2$ is given and if there is no fear of confusion. Find integers $d_2, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ as in 1.3 by the Fuclidean algorithm with respect to d_0 and d_1. It should be noted that integers α, $d_1, \ldots, d_\alpha$ and $q_1, \ldots, q_\alpha$ may change depending on choice of embeddings $\epsilon : C \to \mathbb{A}_k^2$. We shall first show the following:

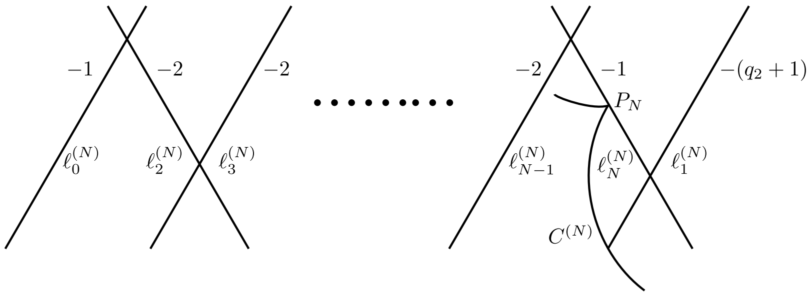
Lemma. *Given an embedding $\epsilon : C_0 \to \mathbb{A}_k^2$ there exists a birational automorphism ρ of $\mathbb{P}_k^2$, which induces a biregular automorphism ρ_0 of $\mathbb{A}_k^2$, such that, with respect to an embedding $\rho_0 \cdot \epsilon : C_0 \to \mathbb{A}_k^2$, one of the following conditions holds:*

(i) $\alpha \geqq 3$;   (ii) $\alpha = 2$ *and* $q_1 \geqq 2$;   (iii) $\alpha = 1$ *and* $q_1 \geqq 3$.

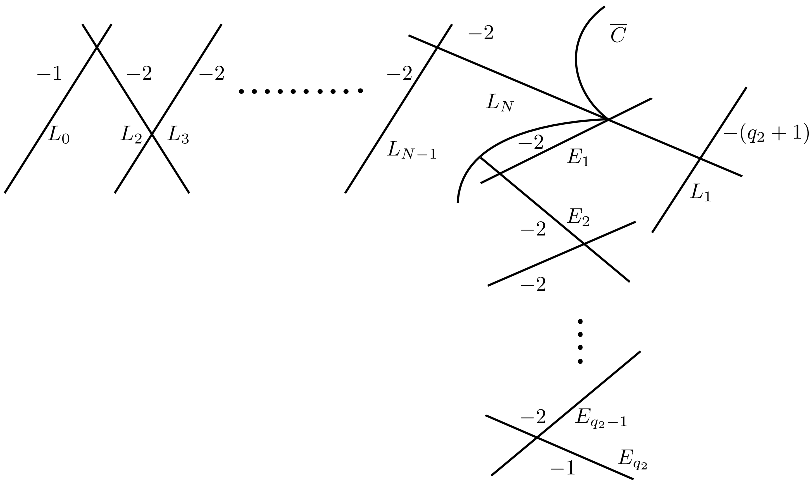
145 *Proof.* We have only to show that if either $\alpha = 2$ and $q_1 = 1$ or $\alpha = 1$ and $q_1 = 2$ with respect to a given embedding ϵ there exists a birational automorphism ρ of $\mathbb{P}_k^2$, which induces a biregular automorphism ρ_0 of $\mathbb{A}_k^2$, such that, with respect to an embedding $\rho_0 \cdot \epsilon : C_0 \to \mathbb{A}_k^2$, one of the conditions (i) ~ (iii) holds.

(I) **Case :** $\alpha = 2$ and $q_1 = 1$. We have then $d_0 = (q_2 + 1)d_2$ and $d_1 = q_2 d_2$ with $q_2 \geqq 2$. Let $\sigma : V_0 \to \mathbb{P}_k^2$ be the Euclidean transformation of $\mathbb{P}_k^2$ associated with an admissible datum

$\{\mathbb{P}^2_k, \mathbb{A}^2_k, C, \ell_0, \phi, d_0, d_1, 1\}$ and $(\mathbb{A}^2_k, \epsilon(C_0))$ (cf. 1.3.1 for definition and notations). Then $\sigma^{-1}(\ell_0 \cup C)$ has the following configuration:



where $N := q_2 + 1$, $P_N := \ell_N^{(N)} \cap C^{(N)}$, $d_2 = (\ell_N^{(N)} \cdot C^{(N)})$ and $e := \operatorname{mult}_{P_N} C^{(N)} \leqq d_2$. Let $\tau_1 : V_1 \to V_0$ be the quadratic transformation with center at $Q_0 := P_N$ and let Q_1 be a point on $\tau_1^{-1}(P_N)$ other than the points $\tau_1'(C^{(N)}) \cap \tau_1^{-1}(Q_0)$ and $\tau_1'(\ell_N^{(N)}) \cap \tau_1^{-1}(Q_0)$. For $2 \leqq i \leqq q_2 - 1$, define $\tau_i : V_i \to V_{i-1}$ and a point Q_i inductively as follows: $\tau_i : V_i \to V_{i-1}$ is the quadratic transformation with center at Q_{i-1}, and Q_i is a point on $\tau_i^{-1}(Q_{i-1})$ other than the point $\tau_i'(\tau_{i-1}^{-1}(Q_{i-2})) \cap \tau_i^{-1}(Q_{i-1})$. Let $\tau_{q_2} : W := V_{q_2} \to V_{q_2-1}$ be the **146**
quadratic transformation with center at Q_{q_2-1}. Let $\tau := \tau_1 \ldots \tau_{q_2}$, let $L_i := \tau'(\ell_i^{(N)})$ for $0 \leqq i \leqq N$, let $E_j (1 \leqq j \leqq q_2)$ be the proper transform of $\tau_j^{-1}(Q_{j-1})$ on W, and let $\overline{C} := \tau'(C^{(N)})$. We have then the following configuration:



where $(\overline{C} \cdot L_N) = d_2 - e$ and $(\overline{C} \cdot E_1) = e$. Let $\varphi : W \to \mathbb{P}^2_k$ be the contraction of curves $L_0, L_2, \ldots, L_{N-1}, L_N, E_1, E_2, \ldots, E_{q_2-1}$ and L_1 in this order, let $\ell'_0 := \varphi(E_{q_2})$ and let $C' := \varphi(\overline{C})$. Then a birational automorphism $\varphi \cdot (\sigma\tau)^{-1}$ is biregular on $\mathbb{A}^2_k$; indeed, $\varphi \cdot (\sigma\tau)^{-1}|_{\mathbb{A}^2_k}$ is a de Jonquière transformation φ_0 of $\mathbb{A}^2_k$ of degree N (cf. 3.4.1). By a straightforward computation we can easily verify that:

(1) $C' - (C' \cap \ell'_0) \cong C_0$;

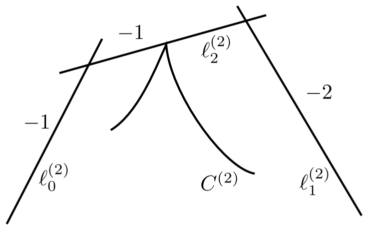
(2) $(C' \cdot \ell'_0) = (q_2 + 1)d_2 - e = d_0 - e$;

(3) $\mathrm{mult}_{p'} C' = q_2 d_2 - e = d_1 - e$, where $P'_0 := C' \cap \ell'_0$.

Let $\epsilon' : \varphi_0 \cdot \epsilon : C_0 \to \mathbb{A}^2_k$. Then ϵ' is an embedding of C_0 into
147 $\mathbb{A}^2_k$ with $(C' \cdot \ell'_0) = d_0 - e < d_0$ and $\mathrm{mult}_{P'_0} C' = d_1 - e < d_1$. By induction on d_0, we can show that there exists a birational automorphism ρ of $\mathbb{P}^2_k$, which induces a biregular automorphism ρ_0 of $\mathbb{A}^2_k$, such that, with respect to an embedding $\rho_0 \cdot \epsilon : C_0 \to \mathbb{A}^2_{k'}$ either one of the conditions (i) $\sim$ (iii) holds or $\alpha = 1$ and $q_1 = 2$. In the latter case d_0 becomes smaller than the original one.

(II) **Case:** $\alpha = 1$ and $q_1 = 2$. Let $\sigma : V_0 \to \mathbb{P}^2_k$ be the Euclidean transformation of $\mathbb{P}^2_k$ associated with an admissible datum $\{\mathbb{P}^2_k, \mathbb{A}^2_k, C,$

$\ell_0, \phi, d_0, d_1, 1\}$. Since $d_0 = 2d_1$ we have the following configuration of $\sigma^{-1}(\ell_0 \cup C)$:



Then by the same argument as in step (II) we can show the existence of a birational automorphism ρ of $\mathbb{P}_k^2$, which induces a biregular automorphism ρ_0 of $\mathbb{A}_k^2$, such that, with respect to an embedding $\rho_0 \cdot \epsilon : C_0 \to \mathbb{A}_k^2$, either one of the conditions (i) ~ (iii) holds or $\alpha = 2$ and $q_1 = 1$ with d_0 smaller than the original one.

(III) By steps (I) and (II) we can show the existence of a birational automorphism ρ as claimed in Lemma.

□

4.2

With the notations and assumptions of 4.1, choose an embedding ϵ :
$C_0 \to \mathbb{A}_k^2$ for which one of the conditions (i) ~ (iii) of Lemma 4.1 **148**
holds. Let $\sigma : V_0 \to \mathbb{P}_k^2$ be the Euclidean transformation of $\mathbb{P}_k^2$ associated with an admissible datum $\mathscr{D} := \{\mathbb{P}_k^2, \mathbb{A}_k^2, C, \ell_0, \phi, d_0, d_1, 1\}$ for $(\mathbb{A}_k^2, \epsilon(C_0))$. Then, looking at the weighted graph of $\sigma^{-1}_{(\ell_0)}$ (cf. Figure 1 of 1.3.4 as well as the figures given in the proof of Lemma 3.1), we know that $\sigma^{-1}(\ell_0)$ has an exceptional curve of the first kind other than $E(\alpha, q_\alpha)$ if and only if $\alpha \geqq 3$ and $q_1 = 1$. When $\alpha \geqq 3$ and $q_1 = 1$, by virtue of 1.3.4, 1.4 and 1.5 we can readily show the following assertions:

1° The curves $E_0 := \sigma'(\ell_0)$, $E(2, 1), \ldots, E(2, q_2 - 1)$ can be contracted successively in this order; let $\tau : V_0 \to V$ be the contraction of these curves.

2° $(\tau(E(2,q_2))^2) = -(q_3+1) \leqq -2$ if $\alpha \geqq 4$ and $(\tau(E(2,q_2))^2) = -q_3 \leqq -2$ if $\alpha = 3$.

3° $a_0 = a(2,1) = \ldots = a(2,q_2) = d_0$.

4° Let $\overline{C} := \tau(C^{(N)})$, $\overline{\ell}_0 := \tau(\ell_N^{(N)})$, $\overline{E}(s,t) := \tau(E(s,t))$ for $1 \leqq s \leqq \alpha$, $1 \leqq t \leqq q_s$ and $(s,t) \neq (2,1), \ldots, (2, q_2 - 1)$, $\overline{d}_0 := d_\alpha$, $\overline{d}_1 :=$ the multiplicity of $\overline{C}$ at the point $\overline{C} \cap \overline{\ell}_0$, $\overline{e} := a(\alpha, q_\alpha)/d_\alpha$, and

$$\overline{\Gamma} := (d_0/d_\alpha)\overline{E}(2,q_2) + \sum_{\substack{s=1 \\ s\neq 2}}^{\alpha} \sum_{t=1}^{q_s} (a(s,t)/d_\alpha)\overline{E}(s,t) - \overline{e}\overline{\ell}_0.$$

Then $\overline{\mathscr{D}} := \{V, \mathbb{A}_k^2, \overline{C}, \overline{\ell}_0, \overline{\Gamma}, \overline{d}_0, \overline{d}_1, \overline{e}\}$ is an admissible datum for $(\mathbb{A}_k^2, \epsilon(C_0))$ such that $\mathrm{Supp}(\overline{\Gamma})$ has no exceptional components and that the divisor $\overline{d}_0(\overline{e\ell}_0 + \overline{\Gamma})$ contains $\overline{E}(2,q_2)$ with multiplicity d_0.

4.3

Assume that an embedding $\epsilon : C_0 \to \mathbb{A}_k^2$ is chosen so that one of the
149 conditions (i) ~ (iii) of Lemma 4.1 holds. Define an irreducible linear pencil Λ as follows; if $\alpha \leqq 2$ or $q_1 \geqq 2$, Λ is the linear pencil on $\mathbb{P}_k^2$ spanned by C and $d_0\ell_0$; if $\alpha \geqq 3$ and $q_1 = 1$, Λ is the linear pencil on V spanned by $\overline{C}$ and $\overline{d}_0(\overline{e\ell}_0 + \overline{\Gamma})$. Now eliminating the base points of Λ by a succession of the Euclidean transformations and the (e,i)-transformations associated with suitable admissible data for $(\mathbb{A}_k^2, \epsilon(C_0))$, we obtain a nonsingular projective surface W and a surjective morphism $f : W \to \mathbb{P}_k^1$ such that:

1° The fibers of f are irreducible, except only one fiber Δ which corresponds to the member $d_0\ell_0$ (or $\overline{d}_0(\overline{e\ell}_0 + \overline{\Gamma})$) of Λ.

2° General fibers of f are nonsingular curves of genus g, where g is the genus of the given curve C_0.

3° f is a relatively minimal fibration, i.e., each fiber does not contain exceptional components.

4° If $\Delta := \sum_{i=1}^{r} n_i C_i$ with irreducible components C_i and integers $n_i > 0$ then the greatest common divisor of $n_1, \ldots, n_r$ is equal to 1 and at least one of n_i's is equal to d_0.

[For the proof of the assertions 1° and 2°, see Corollary 1.10 and Lemma 1.11; for the proof of the assertion 3°, see Lemmas 1.5 and 1.7; the assertion 4° follows from the choice of Λ and the fact that f has a regular cross-section.]

4.4

According to Artin-Winters [7], we shall call any collection T of integers

$$T := \{r, m_{ij}, k_i, n_i; i, j = 1, \ldots, r\},$$

up to permutation of indices, *a fiber type of genus g* if there exist a nonsingular projective surface V defined over k, a surjective morphism **150**
f of V onto a nonsingular complete curve B whose general members are nonsingular irreducible curves of genus g, and a reducible fiber Δ of f such that:

(1) $\Delta := \sum_{i=1}^{r} n_i C_i$, C_i being its irreducible component,

(2) $m_{ij} = (C_i \cdot C_j)$ and $k_i = (C_i \cdot K_V)$ for $i, j = 1, \ldots, r$, where K_V is a canonical divisor of V.

The integers n_i are called the *multiplicities* of a fiber type T of genus g. A fiber type $T = \{r, m_{ij}, k_i, n_i; i, j = 1, \ldots, r\}$ of genus g is called *relatively minimal* if $m_{ii} \neq -1$ or $k_i \neq -1$ for $i = 1, \ldots, r$; T is called *reduced* if the greatest common divisor of $n_1, \ldots, n_r$ is equal to 1. Now we can state the following results.

4.4.1

Lemma (Artin-Winters [7; Cor. 1.7]). *Assume that $g \geqq 2$. Then there exists an integer $N(g)$ depending only on g such that the multiplicities*

$n_i \leqq N(g)$ for every relatively minimal fiber type $T = \{r, m_{ij}, k_i, n_i; i, j = 1, \ldots, r\}$ of genus g.

4.4.2

Lemma (Kodaira [29; p. 123], Šafarevič [51; p.171]). *Assume that $g = 1$. Then the multiplicities $n_i \leqq 6$ for every reduced relatively minimal fiber type $T = \{r, m_{ij}, k_i, n_i; i, j = 1, \ldots, r\}$ of genus 1.*

4.5

As a consequence of the observations made in the paragraphs 4.3 and 4.4, we have:

151 **Theorem.** *Let the notations and assumptions be as in 4.1. Assume that $g > 0$. Then there are only finitely many possible pairs (d_0, d_1), for each of which there exists an embedding $\epsilon : C_0 \to \mathbb{A}_k^2$ such that $d_0 = (C \cdot \ell_0)$ and $d_1 = \mathrm{mult}_{P_0} C$* (cf. 4.1) *and that one of the conditions* (i) ∼ (iii) *of Lemma 4.1 holds.*

Proof. By virtue of the assertion 4° of 4.3, the singular fiber Δ has an irreducible component whose multiplicity in Δ is d_0. Then Lemmas 4.4.1 and 4.4.2 imply that there are finitely many possible values of d_0 (and therefore, of d_1 because $0 < d_1 < d_0$). □

4.6

In the remaining paragraphs of this section we shall prove the following:

Theorem. *Let the notations and assumptions be as in 4.1. Assume that $q > 0$. Then there are finitely many embeddings $\epsilon : C_0 \to \mathbb{A}_k^2$, up to biregular automorphisms of $\mathbb{A}_k^2$, such that:*

(1) *The curve C (:= the closure of $\epsilon(C_0)$ in $\mathbb{P}_k^2$) is smoothable by the Euclidean transformation of $\mathbb{P}_k^2$ associated with an admissible datum $\{\mathbb{P}_k^2, \mathbb{A}_k^2, C, \ell_0, \phi, d_0, d_1, 1\}$ for $(\mathbb{A}_k^2, \epsilon(C_0))$.*

(2) *One of the following conditions holds:*

(i) $\alpha = 2$ *and* $q_1 \geqq 2$*;*

(ii) $\alpha = 1$ *and* $q_1 \geqq 3$.

More precisely, if two embeddings $\epsilon, \epsilon' : C_0 \to \mathbb{A}^2_k$ *satisfying the conditions* (1) *and* (2) *above have the same value of* d_0 *then there exists an affine automorphism* ρ_0 *of* $\mathbb{A}^2_k$ *such that* $\epsilon' = \rho_0 \cdot \epsilon$*. We* **152**
shall note that this result is a special case of Finiteness Theorem due to Abhyankar-Singh [3]; we also note that the condition (1) *above is fulfilled if G.C.D.*$(d_0, d_1) = 1$.

4.7

Let $\epsilon : C_0 \to \mathbb{A}^2_k$ be an embedding of C_0 into $\mathbb{A}^2_k$ for which the condition (2) above holds. Let $\sigma : V_0 \to \mathbb{P}^2_k$ be the Euclidean transformation of $\mathbb{P}^2_k$ associated with the admissible datum $\{\mathbb{P}^2_k, \mathbb{A}^2_k, C, \ell_0, \phi, d_0, d_1, 1\}$ for $(\mathbb{A}^2_k, \epsilon(C_0))$, let $C' := \sigma'(C)$ and let ℓ be a line on $\mathbb{P}^2_k$ different from the line $\ell_0 := \mathbb{P}^2_k - \mathbb{A}^2_k$. Then we have:

Lemma. *With the notations as above and as in 1.3.4, we have:*

$$C' - \sigma^*(\ell) \sim (d_0 - d_1 - 1)\sigma^*(\ell) + \Delta,$$

where

$$\Delta := \begin{cases} d_1 E_0 & \text{if } \alpha = 1, \\ d_1 E_0 + \sum_{i=1}^{r} \sum_{t=1}^{q_{2i}} (d_{2i-1} - t d_{2i}) E(2i, t) & \text{if } \alpha = 2r \text{ and } r \geqq 1, \\ d_1 E_0 + \sum_{i=1}^{r} \sum_{t=1}^{q_{2i}} (d_{2i-1} - t d_{2i}) E(2i, t) & \text{if } \alpha = 2r + 1 \text{ and } r \geqq 1. \end{cases}$$

Proof. By virtue of Lemma 1.5 and its proof, we have:

$$C' \sim d_0 E_0 + \sum_{s=1}^{\alpha} \sum_{t=1}^{q_s} a(s, t) E(s, t)$$

$$\sigma^*(\ell) \sim \sigma^*(\ell_0) = E_0 + \sum_{s=1}^{\alpha} \sum_{t=1}^{q_{2s}} c(s, t) E(s, t),$$

where the integers $a(s,t)$ and $c(s,t)$ are defined in 1.4. Hence we have:

$$C' - (d_0 - d_1)\sigma^*(\ell) \sim d_1E_0 + \sum_{s=1}^{\alpha}\sum_{t=1}^{q_s} b(s,t)E(s,t),$$

153 where $b(s,t) := a(s,t) - (d_0 - d_1)c(s,t)$ is defined as follows:

$$\begin{aligned}
b(1,t) &= (d_0 - d_1)t - (d_0 - d_1)t = 0 && \text{for } 1 \leqq t \leqq q_1\\
b(2,t) &= d_1 + t(b(1,q_1) - d_2) = d_1 - td_2 && \text{for } 1 \leqq t \leqq q_2,\\
&\cdots\cdots\cdots\cdots\\
b(s,t) &= b(s-2,q_{s-2}) + t(b(s-1,q_{s-1}) - d_s) && \text{for } 1 \leqq t \leqq q_s\\
& && \text{and } 2 \leqq s \leqq \alpha.
\end{aligned}$$

Thence we have: $b(2i,t) = d_{2i-1} - td_{2i}$ for $1 \leqq i \leqq r$ and $1 \leqq t \leqq q_{2i}$, and $b(2i+1,t) = 0$ for $0 \leqq i \leqq r(2i+1 \leqq \alpha)$ and $1 \leqq t \leqq q_{2i+1}$, where $r = \left[\dfrac{\alpha}{2}\right]$. Thus we obtain our assertion. □

4.8

According to Ramanujam [46], an effective divisor D on a nonsingular projective surface V_0 defined over k is called *numerically connected* if for every decomposition $D = D_1 + D_2$ with $D_i > 0 (i = 1,2)$ we have $(D_1 \cdot D_2) > 0$. We shall show:

Lemma. *The divisor* $(d_0 - d_1 - 1)\sigma^*(\ell) + \Delta$ *is numerically connected provided* $q_1 \geqq 2$.

A proof of the lemma will be given in the subparagraphs 4.8.1 ∼ 4.8.3.

4.8.1

Lemma. *Let D be an effective divisor on a nonsingular projective surface V defined over k. Write $D := \sum_{i=1}^{r} m_iD_i$ with irreducible components D_i and integers $m_i > 0$. Assume that $(D_i^2) = -\alpha_i$ for $1 \leqq i \leqq r$ and*

154 $(D_i \cdot D_j) = \delta_{j,i+1}$ (Kronecker's delta) *for* $1 \leqq i, j \leqq r$ *and* $i \neq j$. *Let* $D_1 := \sum_{i=1}^{r} x_i D_i$ *with* $0 \leqq x_i \leqq m_i (1 \leqq i \leqq r)$, *and let* $D_2 := D - D_1$. *Then we have:*

$$\begin{aligned}(D_1 \cdot D_2) &= (\alpha_1 - 1)x_1^2 + \sum_{i=2}^{r-1} (\alpha_i - 2)x_i^2 + (\alpha_r - 1)x_r^2 + \sum_{i=1}^{r-1} (x_i - x_{i+1})^2 \\ &+ (m_2 - \alpha_1 m_1)x_1 + \sum_{i=2}^{r-1} (m_{i-1} - \alpha_i m_i + m_{i+1})x_i + (m_{r-1} - \alpha_r m_r)x_r.\end{aligned}$$

Proof. A straightforward computation. □

4.8.2

Lemma. *With the notations as in 4.7, let* $D := (d_0 - d_1 - 1)\sigma^*(\ell) + \Delta$ *and let*

$$D_1 := y\sigma^*(\ell) + x_0 E_0 + \sum_{i=1}^{r} \sum_{t=1}^{q_{2i}} x(2i, j)E(2i, j) \quad \text{and} \quad D_2 := D - D_1,$$

where we assume that $D_i > 0$ *for* $i = 1, 2$. *Then we have:*

$$(D_1 \cdot D_2) = -2y^2 + (q_1 - 2)x_0^2 + (x_0 - y)^2 + (d_0 - 1)y - x_0 + Q$$

where $Q := 0$ *if* $\alpha = 1$*;*

$$\begin{aligned}Q : = &\sum_{i=1}^{r-1} q_{2i+1} x(2i, q_{2i})^2 + x(2r, q_{2r} - 1)^2 \\ &+ \{(x_0 - x(2, 1))^2 + \sum_{t=1}^{q_2 - 1} (x(2, t) - x(2, t + 1))^2\} \\ &+ \sum_{i=2}^{r-1} \left\{ (x(2i - 2, q_{2i-2}) - x(2i, 1))^2 + \sum_{t=1}^{q_{2i}-1} (x(2i, t) - x(2i, t + 1))^2 \right\} \\ &+ \left\{ (x(2r - 2, q_{2r-2}) - x(2r, 1))^2 + \sum_{t=1}^{q_{2r}-2} (x(2r, t) - x(2r, t + 1))^2 \right\}\end{aligned}$$

if $\alpha = 2r$ *and* $r \geqq 1$*;*

$$Q := \sum_{i=1}^{r} q_{2i+1} x(2i, q_{2i})^2 + \{(x_0 - x(2,1))^2 + \sum_{t=1}^{q_2-1} (x(2,t) - x(2,t+1))^2\}$$
$$+ \sum_{i=2}^{r} \{(x(2i-2, q_{2i-2}) - x(2i,1))^2 + \sum_{t=1}^{q_{2i}-1} (x(2i,t) - x(2i,t+1))^2\}$$

155 *if* $\alpha = 2r + 1$ *and* $r \geqq 1$.

Proof. Note that $(\sigma^*(\ell)^2) = 1$, $(\sigma^*(\ell) \cdot E_0) = 1$ and $(\sigma^*(\ell) \cdot E(2i,t)) = 0$ for $1 \leqq i \leqq r$ and $1 \leqq t \leqq q_{2i}$. Then we obtain our assertion by applying Lemma 4.8.1 and taking account of 1.3.3 and 1.3.4. ◻

4.8.3

Proof of Lemma 4.8. *Regarding* $(D_1 \cdot D_2)$ *as a function of variables* y, x_0 *and* $x(2i,t)$*'s, we shall estimate the smallest value of* $(D_1 \cdot D_2)$ *when the variables* y, x_0 *and* $x(2i,t)$*'s take integral values in the domain A:*

$0 \leqq y \leqq d_0 - d_1 - 1$; $\quad 0 \leqq x_0 \leqq d_1$; $\quad 0 \leqq x(2i,t) \leqq d_{2i-1} - td_{2i}$
$(1 \leqq i \leqq r; 1 \leqq t \leqq q_{2i})$.

By virtue of Lemma 4.8.2, $(D_1 \cdot D_2)$ *is written in the form:*

$$(D_1 \cdot D_2) = -y^2 + (d_0 - 1 - 2x_0)y + (q_1 - 1)x_0^2 - x_0 + Q,$$

which, viewed as a function in y *only, has the smallest value at* $y = 0$ *or* $y = d_0 - d_1 - 1$ *whenever values of* x_0 *and* $x(2i,t)$*'s* $(1 \leqq i \leqq r; 1 \leqq t \leqq q_{2i})$ *are fixed in the domain A. If* $y = 0$ *we have:*

$$(D_1 \cdot D_2) = x_0\{(q_1 - 1)x_0 - 1\} + Q.$$

Consider first the case where $\alpha = 1$*. Then* $q_1 \geqq 3$ *as assumed. Since*
156 $D_1 > 0$*, i.e.,* $x_0 \neq 0$ *and* x_0 *takes an integral value, we know that* $(D_1 \cdot D_2) > 0$ *for* $0 < x_0 \leqq d_1$*. Assume that* $\alpha \geqq 2$*. Since* $q_1 \geqq 2$ *as assumed and* x_0 *takes an integral value, we know that* $(D_1 \cdot D_2) \geqq Q \geqq 0$*, and that* $(D_1 \cdot D_2) = Q$ *if and only if either* $x_0 = 0$ *or* $q_1 = 2$ *and* $x_0 = 1$*.*

Besides, by virtue of Lemma 4.8.2, $Q = 0$ *if and only if* $x_0 = x(2i, t) = 0$ *for* $1 \leqq i \leqq r$ *and* $1 \leqq t \leqq q_{2i}$. *Therefore we have* $(D_1 \cdot D_2) > 0$ *because* $D_1 > 0$. *If* $y = d_0 - d_1 - 1$ *we obtain* $(D_1 \cdot D_2) > 0$ *by interchanging the roles of* D_1 *and* D_2. *Hence* $(D_1 \cdot D_2) > 0$ *for every decomposition* $D = D_1 + D_2$ *with* $D_i > 0 (i = 1, 2)$. *This completes a proof of Lemma 4.8.*

4.9

We shall next consider the case where $q_1 = 1$ and $\alpha \geqq 3$. We shall use the notations of the paragraph 4.2. Thus, $\tau : V_0 \to V$ is the contraction of curves $E_0, E(2, 1), \ldots, E(2, q_2 - 1)$. Let $L := \tau_* \sigma^*(\ell)$ and $\overline{E}_0 := \overline{E}(2, q_2)$.

4.9.1

Lemma. *We have:* $\overline{C} - L \sim (d_2 - 1)L + \overline{\Delta}$, *where*

$$\overline{\Delta} := \begin{cases} d_3\overline{E}_0 & \text{if } \alpha = 3, \\ d_3\overline{E}_0 + \sum_{i=2}^{r} \sum_{t=1}^{q_{2i}} (d_{2i-1} - td_{2i})\overline{E}(2i, t) & \text{if } \alpha \geqq 4. \end{cases}$$

Proof. Immediate from Lemma 4.7. □

4.9.2

Lemma. *Let* $\overline{D} := (d_2 - 1)L + \overline{\Delta}$ *and let*

$$\overline{D}_1 := \overline{y}L + \overline{x}_0\overline{E}_0 + \sum_{i=2}^{r} \sum_{t=1}^{q_{2i}} \overline{x}(2i, t)\overline{E}(2i, t) \quad \textit{and} \quad \overline{D}_2 := \overline{D} - \overline{D}_1,$$

where we assume that $\overline{D}_i > 0$ *for* $i = 1, 2$. *Then we have:*

$$(\overline{D}_1 \cdot \overline{D}_2) = -(q_2 + 2)\overline{y}^2 + (q_3 - 1)\overline{x}_0^2 + (\overline{x}_0 - \overline{y})^2 + (d_0 - (q_2 + 1))\overline{y} - \overline{x}_0 + \overline{Q},$$

where **157**

$\overline{Q} := 0$ *if* $\alpha = 3$;

$$\begin{aligned}\overline{Q} := &\sum_{i=2}^{r-1} q_{2i+1}\overline{x}(2i, q_{2i})^2 + \overline{x}(2r, q_{2r} - 1)^2 \\ &+ \{(\overline{x}_0 - \overline{x}(4,1))^2 + \sum_{t=1}^{q_4 - 1}(\overline{x}(4,t) - \overline{x}(4,t+1))^2\} \\ &+ \sum_{i=3}^{r-1}\{(\overline{x}(2i-2, q_{2i-2}) - \overline{x}(2i,1))^2 + \sum_{t=1}^{q_{2i}-1}(\overline{x}(2i,t) - \overline{x}(2i,t+1))^2\} \\ &+ \{(\overline{x}(2r-2, q_{2r-2}) - \overline{x}(2r,1))^2 + \sum_{t=1}^{q_{2r}-2}(\overline{x}(2r,t) - \overline{x}(2r,t+1))^2\}\end{aligned}$$

if $\alpha = 2r \geqq 4$*;*

$$\begin{aligned}\overline{Q} := &\sum_{i=2}^{r} q_{2i+1}\overline{x}(2i, q_{2i})^2 + \{(\overline{x}_0 - \overline{x}(4,1))^2 + \sum_{t=1}^{q_4 - 1}(\overline{x}(4,t) - \overline{x}(4,t+1))^2\} \\ &+ \sum_{i=3}^{r}\{(\overline{x}(2i-2, q_{2i-2}) - \overline{x}(2i,1))^2 + \sum_{t=1}^{q_{2i}-1}(\overline{x}(2i,t) - \overline{x}(2i,t+1))^2\}\end{aligned}$$

if $\alpha = 2r + 1 \geqq 4$*.*

Proof. Note that $(L^2) = q_2+1$, $(L \cdot \overline{E}_0) = 1$, $(L \cdot \overline{E}(2i,t)) = 0$ for $2 \leqq i \leqq r$ and $1 \leqq t \leqq q_{2i}$, and $(\overline{E}_0^2) = -q_3$ if $\alpha = 3$ and $(\overline{E}_0^2) = -(q_3 + 1)$ if $\alpha \geqq 4$. Then our assertion follows from Lemma 4.8.1. $\square$

4.9.3

Lemma. *The divisor* $\overline{D} := (d_2 - 1)L + \overline{\Delta}$ *is numerically connected.*

Proof. Regarding $(\overline{D}_1 \cdot \overline{D}_2)$ as a function of variables $\overline{y}$, $\overline{x}_0$ and $\overline{x}(2i,t)$'s $(2 \leqq i \leqq r; 1 \leqq t \leqq q_{2i})$, we shall estimate the smallest value of $(\overline{D}_1 \cdot \overline{D}_2)$ when the variables $\overline{y}$, $\overline{x}_0$ and $\overline{x}(2i,t)$'s take integral values in the domain $\overline{A}$:

$$0 \leqq \overline{y} \leqq d_2 - 1; \quad 0 \leqq \overline{x}_0 \leqq d_3; \quad 0 \leqq \overline{x}(2i,t) \leqq d_{2i-1} - td_{2i}$$

158 for $2 \leqq i \leqq r$ and $1 \leqq t \leqq q_{2i}$.

By virtue of Lemma 4.9.2, $(\overline{D}_1 \cdot \overline{D}_2)$ is written in the form:

$$(\overline{D}_1 \cdot \overline{D}_2) = -(q_2+1)\overline{y}^2 + \{d_0 - (q_2+1) - 2\overline{x}_0\}\overline{y} + q_3\overline{x}_0^2 - \overline{x}_0 + \overline{Q},$$

which, viewed as a function only in $\overline{y}$, has the smallest value at $\overline{y} = 0$ or $\overline{y} = d_2 - 1$. If $\overline{y} = 0$ we have:

$$(\overline{D}_1 \cdot \overline{D}_2) = \overline{x}_0(q_3\overline{x}_0 - 1) + \overline{Q}.$$

Since $\overline{x}_0$ takes an integral value, we know that $(\overline{D}_1 \cdot \overline{D}_2) \geqq \overline{Q} \geqq 0$. If $\alpha = 3$, then $q_3 \geqq 2$ and $(\overline{D}_1 \cdot \overline{D}_2) = \overline{x}_0(q_3\overline{x}_0 - 1) = 0$ if and only if $\overline{x}_0 = 0$, i.e., $\overline{D}_1 = 0$. Thus $(\overline{D}_1 \cdot \overline{D}_2) > 0$ if $\alpha = 3$. Assume that $\alpha \geqq 4$. Then, by virtue of Lemma 4.9.2, $\overline{Q} = 0$ if and only if $\overline{x}_0 = \overline{x}(2i, t) = 0$ for $2 \leqq i \leqq r$ and $1 \leqq t \leqq q_{2i}$, i.e., $\overline{D}_1 = 0$. Hence $(\overline{D}_1 \cdot \overline{D}_2) > 0$. If $\overline{y} = d_2 - 1$ we obtain $(\overline{D}_1 \cdot \overline{D}_2) > 0$ by interchanging the roles of $\overline{D}_1$ and $\overline{D}_2$. Therefore we know that $(\overline{D}_1 \cdot \overline{D}_2) > 0$ for every decomposition $\overline{D} = \overline{D}_1 + \overline{D}_2$ with $\overline{D}_i > 0$ for $i = 1, 2$. □

4.10

Lemma. *With the notations of 4.1, let $\epsilon : C_0 \to \mathbb{A}_k^2$ be an embedding such that one of the following conditions holds:*

(i) $\alpha \geqq 2$ *and* $q_1 \geqq 2$*;*

(ii) $\alpha = 1$ *and* $q_1 \geqq 3$*.*

Let $\sigma : V_0 \to \mathbb{P}_k^2$ be the Euclidean transformation of $\mathbb{P}_k^2$ associated with an admissible datum $\{\mathbb{P}_k^2, \mathbb{A}_k^2, C, \ell_0, \phi, d_0, d_1, 1\}$ for $(\mathbb{A}_k^2, \epsilon(C_0))$, let $C' := \sigma'(C)$ and let ℓ be a line on $\mathbb{P}_k^2$ different from the line ℓ_0. Then we **159**
have:

$$\dim_k H^0(C', \mathscr{O}_{C'}(\sigma^*(\ell) \cdot C')) = 3.$$

Proof. Consider an exact sequence

$$0 \to \mathscr{O}_{V_0}(-C' + \sigma^*(\ell)) \to \mathscr{O}_{V_0}(\sigma^*(\ell)) \to \mathscr{O}_{C'}(\sigma^*(\ell) \cdot C') \to 0.$$

Thence we obtain an exact sequence

$$0 \to H^0(V_0, \mathscr{O}_{V_0}(-C' + \sigma^*(\ell))) \to H^0(V_0, \mathscr{O}_{V_0}(\sigma^*(\ell))) \to$$

$$H^0(C', \mathscr{O}_{C'}(\sigma^*(\ell)\cdot C')) \to H^1(V_0, \mathscr{O}_{V_0}(-C' + \sigma^*(\ell))).$$

By virtue of Lemmas 4.7 and 4.8 we know that $C' - \sigma^*(\ell) \sim D := (d_0 - d_1 - 1)\sigma^*(\ell) + \Delta$ and D is numerically connected. Since V_0 is a nonsingular projective rational surface we have: $H^1(V_0, \mathscr{O}_{V_0}) = (0)$. Hence we have:

$$\dim_k H^1(V_0, \mathscr{O}_{V_0}(-D)) = \dim_k H^0(D, \mathscr{O}_D) - 1,$$

where $\dim_k H^0(D, \mathscr{O}_D) = 1$ by virtue of Ramanujam's theorem [46; Lemma 3]. Thus we know that $H^1(V_0, \mathscr{O}_{V_0}(-C' + \sigma^*(\ell))) = (0)$. Since $H^0(V_0, \mathscr{O}_{V_0}(-C' + \sigma^*(\ell))) = (0)$ clearly, we obtain:

$$H^0(C', \mathscr{O}_{C'}(\sigma^*(\ell)\cdot C')) \cong H^0(V_0, \mathscr{O}_{V_0}(\sigma^*(\ell))) \cong H^0(\mathbb{P}^2_k, \mathscr{O}_{\mathbb{P}^2}(1)).$$

Therefore we have $\dim_k H^0(C', \mathscr{O}_{C'}(\sigma^*(\ell)\cdot C')) = 3$. □

Remark. If $q_1 = 1$ and $\alpha \geqq 3$, let $\tau : V_0 \to V$ be the contraction of curves $E_0, E(2,1), \ldots, E(2, q_2 - 1)$, let $\overline{C} = \tau(C')$ and let $L := \tau_*\sigma^*(\ell)$ (cf. 4.9). Then we obtain:

$$\begin{aligned}\dim_k H^0(\overline{C}, \mathscr{O}_{\overline{C}}(L\cdot\overline{C})) &= \dim_k H^0(V, \mathscr{O}_{\mathscr{V}}(L)) \geqq \dim_k H^0(V_0, \mathscr{O}_{V_0}(\sigma^*(\ell)) \\ &= \dim_k H^0(\mathbb{P}^2_k, \mathscr{O}_{\mathbb{P}^2}(1)) = 3.\end{aligned}$$

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4.11

Proof of Theorem 4.6. *Let $\epsilon : C_0 \to \mathbb{A}^2_k$ be an embedding satisfying the conditions (1) and (2) as stated in Theorem 4.6. With the notations of 4.1 and 4.7, we know that:*

1° *The curve C' is a normalization of the curve C which is of qenux $g > 0$; set $\widetilde{C}(\epsilon) := C'$ and $\delta(\epsilon) := \sigma^*(\ell)\cdot C'$.*

2° *Let $\widetilde{P}(\epsilon)$ be the (unique) point of $\widetilde{C}(\epsilon)$ dominating the point P_0 of C. Then $\delta(\epsilon) \sim d_0\widetilde{P}(\epsilon)$, $\delta(\epsilon)$ is an effective divisor such that $|\delta(\epsilon)|$ has no base points and* $\dim|\delta(\epsilon)| = 2$ *(cf. . Lemma 4.10).*

3° *Let* $f(\epsilon) : \widetilde{C}(\epsilon) \xrightarrow{\pi(\epsilon)} \overline{\epsilon(C_0)} \hookrightarrow \mathbb{P}^2_k$ *be the morphism defined from the embedding* ϵ*, where* $\pi(\epsilon) := \sigma|_{C'}$ *and where* $\overline{\epsilon(C_0)} = C$ *is the closure of* $\epsilon(C_0)$ *in* $\mathbb{P}^2_k$*. Then* $f(\epsilon)$ *is a morphism defined by* $|\delta(\epsilon)|$ *with respect to a suitable basis of* $|\delta(\epsilon)|$.

Now, let ϵ *and* ϵ' *be embeddings of* C_0 *into* $\mathbb{A}^2_k$ *satisfying the conditions (1) and (2) as stated in Theorem 4.6 and having the same value of* d_0*. Then* $\epsilon' \cdot \epsilon^{-1} : \epsilon(C_0) \to \epsilon'(C_0)$ *induces an isomorphism* $h : \widetilde{C}(\epsilon) \to \widetilde{C}(\epsilon')$ *such that* $h(\widetilde{P}(\epsilon)) = \widetilde{P}(\epsilon')$ *and* $(\epsilon' \cdot \epsilon^{-1}) \cdot \pi(\epsilon) = \pi(\epsilon') \cdot h$ *on* $\widetilde{C}(\epsilon) - \{\widetilde{P}(\epsilon)\}$*. Since* $\delta(\epsilon) \sim d_0\widetilde{P}(\epsilon)$ *and* $\delta(\epsilon') \sim d_0\widetilde{P}(\epsilon')$*, we know that* $\delta(\epsilon) \sim h^*\delta(\epsilon')$*. This implies by virtue of the above assertions* 2° *and* 3° *that there exists a biregular (hence, linear) automorphism* ρ *of* $\mathbb{P}^2_k$ *such that* $\rho \cdot \pi(\epsilon) = \pi(\epsilon') \cdot h$ *and* $\rho(\ell_0) = \ell_0$*;*

$$\begin{tikzcd}
\widetilde{C}(\epsilon) \arrow[rr, "\pi(\epsilon)"] \arrow[dd, "h"'] & & \overline{\epsilon(C_0)} \arrow[r, hook] \arrow[dd, "\epsilon'\cdot\epsilon^{-1}"] & \mathbb{P}^2_k \arrow[dd, "\rho"] \\
& C_0 \arrow[ur, "\epsilon"] \arrow[dr, "\epsilon'"'] & & \\
\widetilde{C}(\epsilon') \arrow[rr, "\pi(\epsilon')"'] & & \overline{\epsilon'(C_0)} \arrow[r, hook] & \mathbb{P}^2_k
\end{tikzcd}$$

Let $\rho_0 = \rho|_{\mathbb{A}^2_k}$*. Then it is clear that* $\rho_0 \cdot \epsilon = \epsilon'$. **161**

5 Simple birational extensions of a polynomial ring $k[x,y]$ **162**

5.1

Let k be an algebraically closed field of characteristic p and let $k[x,y]$ be a polynomial ring over k in two variables x and y. Let f and g be two elements of $k[x,y]$ without common nonconstant factors, and let $A := k[x,y,f/g]$. In this section we shall consider the structures of the

affine k-domain A under an assumption that $V := \mathrm{Spec}(A)$ has only isolated singularities. In the paragraphs 5.2 ~ 5.9 we shall describe how V is obtained from $\mathbb{A}_k^2 := \mathrm{Spec}(k[x,y])$, and see that if V has only isolated singularities V is a normal surface whose singular points (if any) are rational double points (cf. Theorem 5.9). The divisor class group $C\ell(V)$ can be explicitly determined (cf. Theorem 5.11); we obtain, therefore, necessary and sufficient conditions for A to be a unique factorization domain. If g is irreducible and if the curves $f = 0$ and $g = 0$ on $\mathbb{A}_k^2$ meet each other then A is a unique factorization domain if and only if the curves $f = 0$ and $g = 0$ meet in only one point where both curves intersect transversely. We shall consider, in the paragraphs 5.13 and 5.14 a problem: When is every invertible element of A constant? (cf. Theorem 5.14). In the remaining paragraphs 5.16 ~ 5.23, assuming that k is of characteristic zero, we shall look for a necessary and sufficient condition for A to have a nontrivial locally nilpotent k-derivation (cf. Theorem 5.23). An affine k-domain of type A as above was studied by Russell
163 [49] and Sathaye [52] in connection with the following result:

Assume that A is isomorphic to a polynomial ring over k in two variables. In a polynomial ring $k[x,y,z]$ over k in three variables x, y and z, let $u := gz - f$. Then there exist two elements v, w of $k[x,y,z]$ such that $k[x,y,z] = k[u,v,w]$.

5.2

Let $k[x,y,z]$ be a polynomial ring over k in three variables x, y and z, and let $\mathbb{A}_k^3 := \mathrm{Spec}(k[x,y,z])$. Let V be an affine hyper surface on $\mathbb{A}_k^3$ defined by $gz - f = 0$, and let $\pi : V \to \mathbb{A}_k^2 := \mathrm{Spec}(k[x,y])$ be the projection: $\pi(x,y,z) = (x,y)$. Let F and G be respectively the curves $f = 0$ and $g = 0$ on $\mathbb{A}_k^2$. Then we have:

Lemma. (1) *For each point $P \in F \cap G$, $\pi^{-1}(P)$ is isomorphic to the affine line $\mathbb{A}_k^1$.*

(2) *If Q is a point on G but not on F, then $\pi^{-1}(P) = \phi$.*

Proof. Straightforward. □

5.3

The Jacobian criterion of singularity applied to the hyper surface V shows us the following:

Lemma. *Let P be a point on F and G. Then the following assertions hold:*

(1) *If P is a singular point for both F and G then every point of $\pi^{-1}(P)$ is a singular point of V.*

(2) *If P is a singular point of F but not a singular point of G then the point $(P, z = 0)$ is the unique singular point of V lying on $\pi^{-1}(P)$.*

(3) *If P is a singular point of G but not a singular point of F then V* **164**
is nonsingular at every point of $\pi^{-1}(P)$.

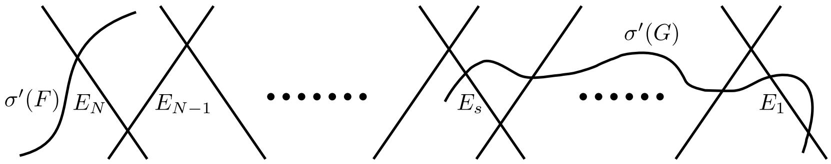
(4) *If P is a nonsingular point of both F and G and if $i(F, G; P) \geqq 2$ then the point $(P, z = \alpha)$ is the unique singular point of V lying on $\pi^{-1}(P)$, where $\alpha \in k$ satisfies: $\dfrac{\partial f}{\partial x}(P) = \alpha \dfrac{\partial g}{\partial x}(P)$ and $\dfrac{\partial f}{\partial y}(P) = \alpha \dfrac{\partial g}{\partial y}(P)$. If $i(F, G; P) = 1$ then V is nonsingular at every point of $\pi^{-1}(P)$.*

We assume, from now on, that V has only isolated singularities. Hence, if $P \in F \cap G$, either F or G is nonsingular at P. Furthermore, we assume that $F \cap G \neq \phi$. When $F \cap G = \phi$ then $A = k[x, y, 1/g]$ and A is a unique factorization domain.

5.4

Let P be a point on F and G. We shall first consider the case where F is nonsingular at P but G singular at P. Let $P_1 := P$ and let ν_1 be the multiplicity of G at P_1. Let $\sigma_1 : V_1 \to V_0 := \mathbb{A}^2_k$ be the quadratic transformation with center at P_1, let $P_2 := \sigma_1'(F) \cap \sigma_1^{-1}(P_1)$ and let ν_2 be the multiplicity of $\sigma_1'(G)$ at P_2. For $i \geqq 1$ we define a surface V_i, a point P_{i+1} on V_i and an integer ν_{i+1} inductively as follows: When V_{i-1}, P_i and ν_i are defined, let $\sigma_i : V_i \to V_{i-1}$ be the quadratic transformation

of V_{i-1} with center at P_{i-1}, let $P_{i+1} := (\sigma_1 \ldots \sigma_i)'(F) \cap \sigma_i^{-1}(P_i)$ and let ν_{i+1} be the multiplicity of $(\sigma_1 \ldots \sigma_i)'(G)$ at P_{i+1}. Let s be the smallest integer such that $\nu_{s+1} = 0$, and let $N : \nu_1 + \cdots + \nu_s$. We shall simply say that $P_1, \ldots, P_s$ *are all points of* G *on the curve* F *over* P_1 *and* $\nu_1, \ldots, \nu_s$ *are the respective multiplicities of* G *at* $P_1, \ldots, P_s$. Let $\sigma : V_N \to V_0$
165 be the composition of quadratic transformations $\sigma := \sigma_1 \ldots \sigma_N$ and let $E_i := (\sigma_{i+1} \ldots \sigma_N)'\sigma_i^{-1}(P_i)$ for $1 \leqq i \leqq N$. In a neighborhood of $\sigma^{-1}(P_1)$, $\sigma^{-1}(F \cup G)$ has the following configuration:



(Fig 1)

If $g = cg_1^{\beta_1} \ldots g_n^{\beta_n} (c \in k^*)$ is a decomposition of g into n distinct irreducible factors, let G_j be the curve $g_j = 0$ on $V_0 := \mathbb{A}_k^2$ for $1 \leqq j \leqq n$. Let $\nu_i(j)$ be the multiplicity of G_j at the points P_i for $1 \leqq i \leqq s$ and $1 \leqq j \leqq n$. Then it is clear that $\nu_i = \beta_1 \nu_i(1) + \cdots + \beta_n \nu_i(n)$ for $1 \leqq i \leqq s$.

5.5

We have the following:

Lemma. *With the same assumption and notations as in 1.3,* V *is isomorphic, in a neighborhood of* $\pi^{-1}(P_1)$*, to* V_N *with the curves* $E_1, \ldots, E_{N-1}$ *and* $\sigma'(G)$ *deleted off.*

Proof. Let $\mathscr{O} := \mathscr{O}_{V_0, P_1}, \widetilde{V}_0 := \mathrm{Spec}(\mathscr{O})$ and $\widetilde{V} := V \underset{V_0}{\times} \widetilde{V}_0$. Since the curve F is nonsingular at P_1 there exist local parameters u, v of V_0 at P_1 such that $v = f$. Let $g(u, v) = 0$ be a local equation of G at P_1. Then $\widetilde{V} = \mathrm{Spec}(\mathscr{O}[v/g(u, v)])$. Note that V is nonsingular in a neighborhood of $\pi^{-1}(P_1)$ (cf. 1.2). Hence there exist a nonsingular projective surface $\overline{V}$ and a birational mapping $\varphi : V \to \overline{V}$ such that φ is an open immersion in
166 a neighborhood of $\pi^{-1}(P_1)$ and a birational mapping $\overline{\pi} := \pi \cdot \varphi^{-1} : \overline{V} \to$

$\mathbb{P}_k^2$ is a morphism, where V_0 is embedded into the projective plane $\mathbb{P}_k^2$ as a complement of a line. Since $\pi(\pi^{-1}(P_1)) = P_1$ we know that $\overline{\pi}$ is factored by the quadratic transformation of $\mathbb{P}_k^2$ with center at P_1. Hence we know that $\pi : V \to V_0$ is factored by $\sigma_1 : V_1 \to V_0$, i.e., $\pi : V \xrightarrow{\pi_1} V_1 \xrightarrow{\sigma_1} V_0$.

Set $v = uv_1$, $u = vu_1$, $q(u, uv_1) = u^{\nu_1} g_1(u, v_1)$ and $g(vu_1, v) = v^{\nu_1} g_1'(u_1, v)$. Then $V_1 \times_{V_0} \widetilde{V}_0 = \mathrm{Spec}(\mathscr{O}[v_1]) \cup \mathrm{Spec}(\mathscr{O}[u_1])$; $\sigma_1^{-1}(P_1)$ and $\sigma_1'(G)$ are respectively defined by $u = 0$ and $g_1(u, v_1) = 0$ on Spec $(\mathscr{O}[v_1])$, and by $v = 0$ and $g_1'(u_1, v) = 0$ on $\mathrm{Spec}(\mathscr{O}[u_1])$. Since

$$\begin{aligned}\widetilde{V} &:= V \times_{V_0} \widetilde{V}_0 = V \times_{V_1} (V_1 \times_{V_0} \widetilde{V}_0) = V \times_{V_1} \mathrm{Spec}(\mathscr{O}[v_1]) \cup V \times_{V_1} \mathrm{Spec}(\mathscr{O}[u_1]) \\ &= \mathrm{Spec}(\mathscr{O}[v_1, v_1/u^{\nu_1 - 1} g_1(u, v_1)]) \cup \mathrm{Spec}(\mathscr{O}[u_1, 1/v^{\nu_1 - 1} g_1'(u_1, v)])\end{aligned}$$

and since v is an invertible function on $\mathrm{Spec}(\mathscr{O}[u_1, 1/v^{\nu_1 - 1} g_1'(u_1, v)])$, we know that:

(1) $\widetilde{V} = \mathrm{Spec}(\mathscr{O}[v_1, v_1/u^{\nu_1 - 1} g_1(u, v_1)])$;

(2) $\widetilde{\pi} := \pi \times_{V_0} \widetilde{V}_0 : \widetilde{V} \to \widetilde{V}_0$ is a composition of $\widetilde{\pi}_1 := \pi_1 \times_{V_0} \widetilde{V}_0 : \widetilde{V} \to \widetilde{V}_1 := \mathrm{Spec}(\mathscr{O}[v_1])$ and $\widetilde{\sigma}_1 := \sigma_1|_{\widetilde{V}_1} : \widetilde{V}_1 \to \widetilde{V}_0$;

(3) if $Q \in (\sigma_1^{-1}(P_1) \cup \sigma_1'(G)) - \sigma_1'(F)$ then $\widetilde{\pi}_1^{-1}(Q) = \phi$.

Set $v_1 = uv_2, \ldots, v_{s-1} = uv_s$ and $g_1(u, v_1) = u^{\nu_2} g_2(u, v_2), \ldots, g_{s-1}(u, v_{s-1}) = u^{\nu_s} g_s(u, v_s)$. Set $\widetilde{V}_2 := \mathrm{Spec}(\mathscr{O}[v_2]), \ldots, \widetilde{V}_s := \mathrm{Spec}(\mathscr{O}[v_s])$. **167**
Then, by the same argument as above, we know that the following assertions hold for $2 \leqq i \leqq s$:

(1)′ $\widetilde{V} = \mathrm{Spec}(\mathscr{O}[v_i, v_i/u^{\nu_1 + \cdots + \nu_i - i} g_i(u, v_i)])$;

(2)′ $\widetilde{\pi} : \widetilde{V} \to \widetilde{V}_0$ is a composition of a morphism $\widetilde{\pi}_i : \widetilde{V} \to \widetilde{V}_i$ and $\widetilde{\sigma}_1 \cdot \widetilde{\sigma}_2 \cdot \ldots \cdot \widetilde{\sigma}_i : \widetilde{V}_i \to \widetilde{V}_0$, where $\widetilde{\sigma}_i := \sigma_i|_{\widetilde{V}_i} : \widetilde{V}_i \to \widetilde{V}_{i-1}$; moreover, $\widetilde{\pi}_{i-1} = \widetilde{\sigma}_i \cdot \widetilde{\pi}_i$;

(3)′ if $Q \in (\sigma_i^{-1}(P_i) \cup (\sigma_1 \ldots \sigma_i)'(G)) - (\sigma_1 \ldots \sigma_i)'(F)$ then $\widetilde{\pi}_i^{-1}(Q) = \phi$.

When $i = s$, the proper transform $(\sigma_1 \dots \sigma_s)'(G)$ of G on V_s does not meet the proper transform $(\sigma_1 \dots \sigma_s)'(F)$of F on $\widetilde{V}_s$ (cf. the definition of s in 5.4). Therefore, in virtue of $(3)'$ above, we know that $g_s(u, v_s)$ is an invertible function on $\widetilde{V}$, where $g_s(u, v_s) = 0$ is the equation of the proper transform $(\sigma_1 \dots \sigma_s)'(G)$ of G on $\widetilde{V}_s$. Thus, $\widetilde{V} = \mathrm{Spec}(\mathscr{O}[v_s, v_s/u^{N-s}])$.

Furthermore, set $v_s = uv_{s+1}, \dots, v_{N-1} = uv_N$ and $\widetilde{V}_{s+1} = \mathrm{Spec}(\mathscr{O}[v_{s+1}]), \dots, \widetilde{V}_N = \mathrm{Spec}(\mathscr{O}[v_N])$. Then it is easy to see that the following assertions hold for $s + 1 \leqq i \leqq N$:

$(1)''$ $\widetilde{V} = \mathrm{Spec}(\mathscr{O}[v_i, v_i/u^{N-i}])$;

$(2)''$ $\widetilde{\pi}_s : \widetilde{V} \to \widetilde{V}_s$ is a composition of a morphism $\widetilde{\pi}_i : \widetilde{V} \to \widetilde{V}_i$ and $\widetilde{\sigma}_{s+1} \cdot \ldots \cdot \widetilde{\sigma}_i : \widetilde{V}_i \to \widetilde{V}_s$, where $\widetilde{\sigma}_i := \sigma_i|_{\widetilde{V}_i} : \widetilde{V}_i \to \widetilde{V}_{i-1}$ and $\widetilde{\pi}_{i-1} = \widetilde{\sigma}_i \cdot \widetilde{\pi}_i$.

Then $\widetilde{V} \cong \widetilde{V}_N = \mathrm{Spec}(\mathscr{O}[v_N])$. Hence, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to V_N with the curves $E_1, \dots, E_{N-1}$ and $\sigma'(G)$ deleted off. In particular, $\pi^{-1}(P_1) = \epsilon := E_N - E_N \cap E_{N-1}$. $\square$

5.6

168 Assume that we are given two curves (not necessarily irreducible) F, G on a nonsingular surface V_0 and a point $P_1 \in F \cap G$ at which one of F and G, say F, is nonsingular. Let $P_1, P_2, \dots, P_s$ be all points of G on F over P_1, and let $\nu_1, \dots, \nu_s$ be the multiplicities of G at $P_1, \dots, P_s$, respectively. Let $N := \nu_1 + \cdots + \nu_s$. As explained in 5.4, define $\sigma : V_N \to V_0$ as a composition of quadratic transformations with centers at N points $P_1, \dots, P_N$ on F, each $P_i (2 \leqq i \leqq N)$ being infinitely near to P_{i-1} of order one. We call $\sigma : V_N \to V_0$ *the standard transformation of* V_0 *with respect to a triplet* (P_1, F, G). The configuration of $\sigma^{-1}(F \cup G)$ in a neighborhood of $\sigma^{-1}(P_1)$ is given by the Figure 1 in 5.4. With the notations in the Figure 1, we obtain a new surface V by deleting $E_1, \dots, E_{N-1}$ from V_N. We then say that V is obtained from V_0 by *the standard process of the first kind with respect to* (P_1, F, G). On the other hand, note that $(E_i^2) = -2$ for $1 \leqq i \leqq N - 1$. Hence we obtain a new normal surface V' from V_N by contracting $E_1, \dots, E_{N-1}$ to a point Q_1 on V' which is a rational double point (cf. Artin [5; Theorem 2.7]). We then

say that V' is obtained from V_0 by *the standard process of the second kind with respect to* (P_1, F, G).

5.7

We shall next consider the case where, at a point $P_1 \in F \cap G$, the curve G is nonsingular. Indeed, we prove the following:

Lemma. *With the assumption as above, let V' be the surface obtained from $V_0 := \mathbb{A}_k^2$ by the standard process of the second kind with respect to (P_1, G, F). Then, in a neighborhood of $\pi^{-1}(P_1)$, V is isomorphic to* 169
V' with the proper transform of G deleted off. If either F is singular at P_1 or $i(F, G; P_1) \geqq 2$, V has a unique rational double point on $\pi^{-1}(P_1)$.

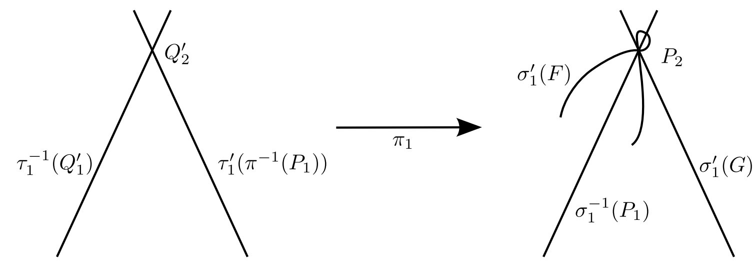
Proof. Let $P_1, P_2, \ldots, P_r$ be all points of F on G over P_1, and let $\mu_1, \ldots, \mu_r$ be the multiplicities of F at $P_1, \ldots, P_r$, respectively. Let $M := \mu_1 + \cdots + \mu_r$. We prove the assertion by induction on M. Note that $M = 1$ if and only if $i(F, G; P_1) = 1$. It is then easy to see that V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to a surface V_1' obtained as follows: Let $\sigma_1 : V_1 \to V_0$ be the quadratic transformation of $V_0 := \mathbb{A}_k^2$ with center at P_1, and let $V_1' := V_1 = \sigma_1'(G)$. Now, assume that $M > 1$. Since G is nonsingular at P_1 there exist local parameters u, v of V_0 at P_1 such that $v = g$. Let $f(u, v) = 0$ be a local equation of F at P_1. Then, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to an affine hyper surface $vz = f(u, v)$ in the affine 3-space $\mathbb{A}_k^3$. There exists only one singular point $Q_1' : (u, v, z) = (0, 0, \alpha)$ of V lying on $\pi^{-1}(P_1)$, where $\alpha = \dfrac{\partial f}{\partial v}(0, 0)$. Note that if $\alpha \neq 0$ then $A := k\left[x, y, \dfrac{f}{g}\right] = k\left[x, y, \dfrac{f - \alpha g}{g}\right]$ and $i(F, G; P_1) = i(H, G; P_1) = M$, where H is the curve on $\mathbb{A}_k^2$ defined by $f = \alpha g$. Replacing f by $f - \alpha g$ we may assume, from the outset and without loss of generality, that $\alpha = 0$. Then we have $\mu_1 \geqq 2$. Let $\rho_1 : W_1 \to \mathbb{A}_k^3$ be the quadratic transformation of $\mathbb{A}_k^3$ with center at the curve $\pi^{-1}(P_1) : u = v = 0$, let V_1' be the proper transform of V on W_1, and let $\tau_1 := \rho_1|_{V_1'} : V_1' \to V$ be the restriction of ρ_1 onto V_1'.

Set $v = uv_1$, $u = vu_1$ and $f(u, uv_1) = u^{\mu_1} f_1(u, v_1)$, $f(vu_1, v) = v^{\mu_1} \widetilde{f_1}(u_1, v)$. Then V_1' is given by $v_1 z = u^{\mu_1 - 1} f_1(u, v_1)$ with respect to 170 the coordinate system (u, v_1, z) and by $z = u^{\mu_1 - 1}\widetilde{f_1}(u_1, v)$ with respect to the coordinate system (u_1, v, z). By construction of V_1', V_1' dominates the surface V_1 obtained from V_0 by the quadratic transformation σ_1 with center at P_1;

$$\begin{tikzcd} V_1' \arrow[r, "\tau_1"] \arrow[d, "\pi_1"] & V \arrow[d, "\pi"] \\ V_1 \arrow[r, "\sigma_1"] & V_0 \end{tikzcd}$$

The proper transform $\tau_1'(\pi^{-1}(P_1))$ of $\pi^{-1}(P_1)$ on V_1' is given by $u = v_1 = 0$; the curve $\tau_1^{-1}(Q_1')$ is given by $u = z = 0$; $\tau_1 : V_1' - \tau_1^{-1}(Q_1') \xrightarrow{\sim} V - \{Q_1'\}$; the singular point of V_1' is possibly $Q_2' : (u, v_1, z) = (0, 0, 0)$.

The morphism $\pi_1 : V_1' \to V_1$ is isomorphic at every point of $\tau_1^{-1}(Q_1')$ $-\{Q_2'\}$. Indeed, if $v_1 \neq 0$ and ∞, π_1 is given by $(u, v_1, z) = (u, v_1, u^{\mu_1 - 1}$ $f_1(u, v_1)/v_1) \mapsto (u, v_1)$ which is clearly isomorphic; if $v_1 = \infty$, π_1 is given by $(u_1, v, v^{\mu_1 - 1}\widetilde{f_1}(u_1, v)) \mapsto (u_1, v)$ which is isomorphic as well. Under the morphism π_1, $\tau_1^{-1}(Q_1')$ corresponds to $\sigma_1^{-1}(P_1)$ and τ_1' $(\pi^{-1}(P_1))$ to the point P_2; moreover $\pi_1^{-1}(P_2) = \tau_1'(\pi^{-1}(P_1))$;



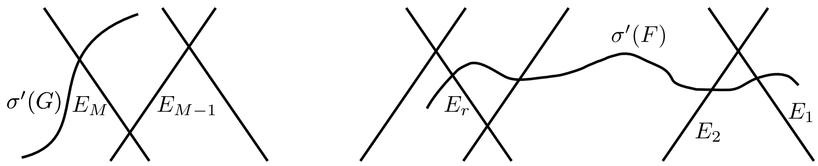
171 Note that the following assertions hold:

(1) V_1' is isomorphic, in a neighborhood of $\pi_1^{-1}(P_2)$, to an affine hyper surface $v_1 z = u^{\mu_1 - 1} f_1(u, v_1)$ on $\mathbb{A}_k^3$;

(2) in a neighborhood of P_2, $\sigma_1'(G)$ is defined by $v_1 = 0$ and $\sigma_1'(F)$ is defined by $f_1(u, v_1) = 0$;

(3) $P_2, \ldots, P_r$ are all points of the curve $F_1 : u^{\mu_1 - 1} f_1(u, v_1) = 0$ on $\sigma_1'(G)$ over P_2, and the sum of multiplicities of the curve F_1 at $P_2, \ldots, P_r$ is $M - 1$.

Let V_1'' be the surface obtained from V_1' by the standard process of the second kind with respect to a triplet $(P_2, \sigma_1'(G), F_1)$. Then, by the assumption of induction applied to V_1', we know that, in a neighborhood of $\tau_1'(\pi^{-1}(P_1)) = \pi_1^{-1}(P_2)$, V_1' is isomorphic to V_1'' with the proper transform of $\sigma_1'(G)$ on V_1'' deleted off. Let $\rho : V_M \to V_1$ be the standard transformation of V_1 with respect to a triplet $(P_2, \sigma_1'(G), F_1)$. Then $\sigma_1 \cdot \rho$ is clearly the standard transformation $\sigma : V_M \to V_0$ with respect to a triplet (P_1, G, F);



(Fig 2)

where, in the Figure 2, we have:

1° $E_1 = \rho'(\sigma_1^{-1}(P_1))$;

2° the surface V_1' is obtained by contracting $E_2, \ldots, E_{M-1}$ to a point Q_2' and by deleting the proper transform of $\sigma'(G)$; under this contraction, say φ, we have $\varphi(E_1) = \tau_1^{-1}(Q_1')$ and $\varphi(E_M - E_M \cap \sigma'(G)) = \tau_1'(\pi^{-1}(P_1))$. 172

It is now easy to see that V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to the surface V' with the proper transform of G deleted off, where V' is obtained from V_M by contracting $E_1, \ldots, E_{M-1}$. Hence, the unique singular point of V lying on $\pi^{-1}(P_1)$ is a rational double point. □

5.8

Let $P_1 \in F \cap G$, and assume that G is nonsingular at P_1. Let $P_1, P_2, \ldots, P_r$ be all points of F on G over P_1, and let $\mu_1, \mu_2, \ldots, \mu_r$ be the multiplicities of F at $P_1, P_2, \ldots, P_r$, respectively. If $f = cf_1^{\alpha_1} \ldots f_m^{\alpha_m} (c \in k^*)$ is a decomposition of f into distinct irreducible factors, let $F_j (1 \leq j \leq m)$ be the curve on V_0 defined by $f_j = 0$. Let $\mu_i(j)$ be the multiplicity of F_j at P_i for $1 \leqq i \leqq r$ and $1 \leqq j \leqq m$. Then it is clear that $\mu_i = \alpha_1 \mu_i(1) + \cdots + \alpha_m \mu_i(m)$ for $1 \leqq i \leqq r$.

5.9

As a consequence of Lemmas 5.5 and 5.7, we have the following:

Theorem. *Assume that V has only isolated singularities. Let W be the surface obtained from $V_0 := \mathbb{A}_k^2$ by the standard processes of the first (or the second) kind at every point of $F \cap G$. Then V is isomorphic to the surface W with the proper transform of G on W deleted off. The surface V is, therefore, a normal surface whose singular points (if any) are rational double points.*

5.10

In the paragraphs 5.10 ~ 5.12 we shall study the divisor class group
173 $C\ell(V)$. Let $g = cg_1^{\beta_1} \ldots g_n^{\beta_n} (c \in k^*)$ be a decomposition of g into distinct irreducible factors, and let G_j be the curve $g_j = 0$ on V_0 for $1 \leqq j \leqq n$. Assume that $F \cap G \neq \phi$. Let $F \cap G = \{P_1^{(1)}, \ldots, p_1^{(e)}\}$. For $1 \leqq \ell \leqq e$, either F is nonsingular at $P_1^{(\ell)}$ but G is singular, or G is nonsingular at $P_1^{(\ell)}$. We may assume that F is nonsingular at $P_1^{(1)}, \ldots, P_1^{(a)}$ but G is singular, and G is nonsingular at $P_1^{(a+1)}, \ldots, P_1^{(e)}$. (The number a may be 0). For $1 \leqq \ell \leqq a$, let $P_1^{(\ell)}, \ldots, P_{s_\ell}^{(\ell)}$ be all points of G on F over $P_1^{(\ell)}$, and let $\nu_i^{(\ell)}(j)$ be the multiplicity of G_j at $P_i^{(\ell)}$ for $1 \leqq i \leqq s_\ell$ and $1 \leqq j \leqq n$; let $N^{(\ell)}(j) = \nu_1^{(\ell)}(j) + \cdots + \nu_{s_\ell}^{(\ell)}(j)$, let $\nu_i^{(\ell)} = \beta_1 \nu_i^{(\ell)}(1) + \cdots + \beta_n \nu_i^{(\ell)}(n)$ and let $N^{(\ell)} = \beta_1 N^{(\ell)}(1) + \cdots + \beta_n N^{(\ell)}(n)$. For $a + 1 \leqq \ell \leqq e$, let $P_1^{(\ell)}, \ldots, P_{r_\ell}^{(\ell)}$ be all points of F on G over $P_1^{(\ell)}$, and let $\mu_i^{(\ell)}$ be the multiplicity of F at

$P_i^{(\ell)}$ for $1 \leqq i \leqq r_\ell$. Let $M^{(\ell)} = \mu_1^{(\ell)} + \cdots + \mu_{r_\ell}^{(\ell)}$. Since G is nonsingular at $P_1^{(\ell)}$ for $a + 1 \leqq \ell \leqq e$, there exists a unique $G_j (1 \leqq j \leqq n)$ such that $P_1^{(\ell)}, \ldots, P_{r_\ell}^{(\ell)}$ lie on G_j. Then We set $M^{(\ell)}(j) = M^{(\ell)}$ and $M^{(\ell)}(j') = 0$ for $j' \neq j$. Let $\epsilon^{(\ell)} := \pi^{-1}(P_1^{(\ell)})$ for $1 \leqq \ell \leqq e$.

5.11

The structure of the divisor class group $C\ell(V)$ is given by the following:

Theorem. *With the notations as above, the divisor class group $C\ell(V)$ is isomorphic to:*

$$\{\mathbb{Z}\epsilon^{(1)} + \cdots + \mathbb{Z}\epsilon^{(e)}\}/\{\sum_{\ell=1}^{a} N^{(\ell)}(j)\epsilon^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j)\epsilon^{(\ell)}; 1 \leqq j \leqq n\}.$$

Proof. Embed $V_0 := \mathbb{A}_k^2$ into the projective plane $\mathbb{P}_k^2$ as a complement 174
of a line ℓ_∞. For $1 \leqq \ell \leqq e$, let $E_1^{(\ell)}, \ldots, E_q^{(\ell)}$ be all exceptional curves which arise from the standard transformation of V_0 with respect to a triplet $(P_1^{(\ell)}, F, G)$ (or $(P_1^{(\ell)}, G, F)$), where $q = N^{(\ell)}$ (or $M^{(\ell)}$). Let $\tau : W \to \mathbb{P}_k^2$ be the composition of standard transformations of $\mathbb{P}_k^2$ with respect to triplets $(P_1^{(\ell)}, F, G)$ for $1 \leqq \ell \leqq a$ and triplets $(P_1^{(\ell)}, G, F)$ for $a + 1 \leqq \ell \leqq e$. Then it is easy to see that the divisor

$$(g_j)_W - \left\{ \sum_{\ell=1}^{a} N^{(\ell)}(j) E_{N^{(\ell)}}^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j) E_{M^{(\ell)}}^{(\ell)} \right\} \quad (1 \leqq j \leqq n)$$

has support on $\tau'(G_j)$, $\tau'(\ell_\infty)$, $E_1^{(\ell)}, \ldots, E_{q-1}^{(\ell)}$ ($q = N^{(\ell)}$ or $M^{(\ell)}$) for $1 \leqq \ell \leqq e$. Hence we have:

$$(g_j)_V = \sum_{\ell=1}^{a} N^{(\ell)}(j)\epsilon^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j)\epsilon^{(\ell)} \sim 0$$

as a divisor on V for $1 \leqq j \leqq n$.

Now, let C be an irreducible curve on V such that $\pi(C)$ is not a point, and let the closure of $\pi(C)$ on V_0 be defined by $h = 0$ with $h \in k[x, y]$. Then, by considering the divisor $(h)_W$ on W, we easily see that C is linearly equivalent to an integral combination of $\epsilon^{(1)}, \ldots, \epsilon^{(e)}$. Hence, by setting

$$\mathfrak{g} := \{\mathbb{Z}\overline{\epsilon}^{(1)} + \cdots + \mathbb{Z}\overline{\epsilon}^{(e)}\} / \left\{\sum_{\ell=1}^{a} N^{(\ell)}(j)\overline{\epsilon}^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j)\overline{\epsilon}^{(\ell)}; 1 \leqq j \leqq n\right\}$$

we have a surjective homomorphism:

$$\theta : \mathfrak{g} \to C\ell(V); \theta(\overline{\epsilon}^{(\ell)}) = \epsilon^{(\ell)} (1 \leqq \ell \leqq e).$$

175 We shall show that θ is an isomorphism. Assume that $\operatorname{Ker}\theta \neq (0)$, and let $d_1\overline{\epsilon}^{(1)} + \cdots + d_e\overline{\epsilon}^{(e)}$ be a nonzero element of $\operatorname{Ker}\theta$. Then $d_1\epsilon^{(1)} + \cdots + d_e\epsilon^{(e)} = (t)_V$ on V, where $t \in k(V)$ such that $t \notin k$. Then we may write $(t)_{V_0} = \sum_i m_i C_i$ with irreducible curves C_i on V_0 and nonzero integers m_i. Let $t_i \in k[x, y]$ be such that C_i is defined by $t_i = 0$, and write:

$$(t_i)_V = \pi'(C_i) + \sum_{\ell=1}^{e} b_{i\ell}\epsilon^{(\ell)} \quad \text{with} \quad b_{i\ell} \in \mathbb{Z}.$$

Then, since $t = C\prod_i t_i^{m_i}$ with $c \in k^*$ we have:

$$(t)_V = \sum_i \{m_i\pi'(C_i) + \sum_{\ell=1}^{e} m_i b_{i\ell}\epsilon^{(\ell)}\} = \sum_{\ell=1}^{e} d_\ell \epsilon^{(\ell)}.$$

Hence we know that $\pi'(C_i) = \phi$ for every i. This implies that every C_i must coincide with one of G_j's $(1 \leqq j \leqq n)$, i.e., $(t)_V$ is an integral combination of $(g_j)_V$'s. Hence $d_1\overline{\epsilon}^{(1)} + \cdots + d_e\overline{\epsilon}^{(e)} = 0$ in $\mathfrak{g}$. This is a contradiction. □

5.12

The affine k-domain $A = k[x, y, \frac{f}{g}]$ is a unique factorization domain if and only if $C\ell(V) = (0)$. We have the following two consequences of 5.11.

5.12.1

Corollary. *With the notations of 5.10, if $e > n$ then A is not a unique factorization domain.*

5.12.2

Corollary. *Assume that g is irreducible and that $F \cap G \neq \phi$. Then A is a unique factorization domain if and only if the curves F and G meet each other in only one point where they intersect transversely.*

5.13

Let A^* be the group of all invertible elements of $A = k\left[x, y, \dfrac{f}{g}\right]$. Then 176
A^* contains $k^* = k - (0)$ as a subgroup. By virtue of Miyanishi [32; Remark 2, p.174] we know that A^*/k^* is a free $\mathbb{Z}$-module of finite rank and A^* is isomorphic to a direct product of k^* and A^*/k^*. The purpose of the present and the next paragraphs is to determine the group A^*/k^*. Let H be the subgroup of $\mathbb{Z}\epsilon^{(1)} + \cdots + \mathbb{Z}\epsilon^{(e)}$ generated by

$$\left\{\sum_{\ell=1}^{a} N^{(\ell)}(j)\epsilon^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j)\epsilon^{(\ell)}; 1 \leqq j \leqq n\right\}.$$

Let $T_1, \ldots, T_n$ be n-indeterminates, and let $\eta : \mathbb{Z}^{(n)} := \mathbb{Z}T_1 + \cdots + \mathbb{Z}T_n \to H$ be a homomorphism such that, for $1 \leqq i \leqq n$,

$$\eta(T_i) = \sum_{\ell=1}^{a} N^{(\ell)}(j)\epsilon^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j)\epsilon^{(\ell)}.$$

Let L be the kernel of η, and define a homomorphism $\xi : L \to K^*$ (where $K = k(x, y)$) by

$$\xi(\gamma_1 T_1 + \cdots + \gamma_n T_n) = g_1^{\gamma_1} \ldots g_n^{\gamma_n}, \quad \text{where} \quad \gamma_i \in \mathbb{Z}.$$

Then we have the following:

Lemma. *The homomorphism ξ induces an isomorphism $\overline{\xi} : L \xrightarrow{\sim} A^*/k^*$.*

Proof. (1) Since $(g_i)_V = \sum_{\ell=1}^{a} N^{(\ell)}(j)\epsilon^{(\ell)} + \sum_{\ell=a+1}^{e} M^{(\ell)}(j)\epsilon^{(\ell)} = \eta(T_i)$ for $1 \leqq i \leqq n$, we have:

$$\eta(\gamma_1 T_1 + \cdots + \gamma_n T_n) = (g_1^{\gamma_1} \ldots g_n^{\gamma_n})_V.$$

Therefore, if $\gamma_1 T_1 + \cdots + \gamma_n T_n \in L$ then $g_1^{\gamma_1} \ldots g_n^{\gamma_n}$ is an invertible
177 element of A, which is a constant if and only if $\gamma_1 = \ldots = \gamma_n = 0$. Thus, ξ induces a monomorphism $\overline{\xi}$ from L into A^*/k^*.

(2) Let t be a non-constant invertible element of A. Write $(t)_{V_0} = \sum_i m_i C_i$ with irreducible curves C_i and nonzero integers m_i. Let C_i be defined by $t_i = 0$ with $t_i \in k[x, y]$. As in the proof of 5.11, write:

$$(t_i)_V = \pi'(C_i) + \sum_{\ell=1}^{e} b_{i\ell}\epsilon^{(\ell)} \quad \text{with} \quad b_{i\ell} \in \mathbb{Z}.$$

Then we have:

$$(t)_V = \sum_i \left\{ m_i\pi'(C_i) + \sum_{\ell=1}^{e} m_i b_{i\ell}\epsilon^{(\ell)} \right\} = 0.$$

Hence we have $\pi'(C_i) = \phi$ for every i. This implies that C_i must coincide with one of G_j's. Hence we could write:

$$(t)_{V_0} = \sum_{j=1}^{n} m_j G_j$$

where m_j may be zero. Then $t = cg_1^{m_1} \ldots g_n^{m_n}$ with $c \in k^*$. It is then clear that $m_1 T_1 + \cdots + m_n T_n \in L$ and $\xi(m_1 T_1 + \cdots + m_n T_n) = t/c$. Therefore, $\overline{\xi} : L \to A^*/k^*$ is an isomorphism.

□

5.14

By virtue of 5.11 and 5.13, we have the following:

Theorem. *Assume that* V *has only isolated singularities. Then we have the following exact sequence of* $\mathbb{Z}$*-modules:*

$$0 \to A^*/k^* \to \mathbb{Z}^{(n)} \to \mathbb{Z}^{(e)} \to C\ell(V) \to 0,$$

where $\mathbb{Z}^{(r)}$ *stands for a free* $\mathbb{Z}$*-module of rank* r*;* n *is the number of distinct irreducible factors of* g*;* e *is the number of distinct points of* $F \cap G$. **178**

5.15

Remarks. (1) It is clear from 5.14 that if g is irreducible then $A^* = k^*$.

(2) $\operatorname{rank}(C\ell(V)) - \operatorname{rank}(A^*/k^*) = e - n$.

(3) Though we proved Theorem 5.14 under the assumption that $F \cap G \neq \phi$ it is clear that the theorem is valid also in the case $F \cap G = \phi$.

5.16

From now on in the remaining paragraphs of this section we assume that the characteristic of k is zero. Assume that $A := k[x, y, \frac{f}{g}]$ is normal and A has a nontrivial locally nilpotent k-derivation D (cf. (1.1)). By virtue of 1.2 we know that D defines a nontrivial action of the additive group scheme $G_{a,k}$ on V and *vice versa*. Then we have the following:

Lemma cf. 1.3.1, 1.6. *The subring* A_0 *of* D*-constants is a finitely generated, normal, rational* k*-domain of dimension* 1.

Proof. The fact that A_0 is rational over k follows from Lüroth's theorem. □

5.17

By virtue of the previous lemma we may write $A_0 = k\left[t, \dfrac{1}{h(t)}\right]$ with $h(t) \in k[t]$; $U := \operatorname{Spec}(A_0)$ is an open set of the affine line $\mathbb{A}^1_k$. Let

$q : V \to U$ be the morphism defined by the canonical inclusion $A_0 \hookrightarrow A$. For almost all elements α of k such that $h(\alpha) \neq 0$, the fiber $q^{-1}(\alpha)$ is a G_a-orbit with respect to the G_a-action on V corresponding to D, and
179 hence $q^{-1}(\alpha)$ is isomorphic to the affine line $\mathbb{A}^1_k$. Let $\rho : V' \to V$ be the minimal resolution of singularities of V[6]. As we saw in 5.9, singular points of V are rational double points. Hence, ρ is a composition of quadratic transformations with centers at singular points. Let $q' := q \cdot \rho : V' \to U$. Almost all fibers of q' are therefore isomorphic to the affine line $\mathbb{A}^1_k$. Now we shall prove the following:

Lemma. *There exists a nonsingular projective surface W and a surjective morphism p[7]$: W \to \mathbb{P}^1_k$ satisfying the following conditions:*

(1) *Almost all fibers of p are isomorphic to $\mathbb{P}^1_k$.*

(2) *There exists an open immersion $l : V' \to W$ such that $p \cdot l = \bar{l} \cdot q'$, where $\bar{l} : U \hookrightarrow \mathbb{P}^1_k$ is the canonical open immersion via $U \hookrightarrow \mathbb{A}^1_k := \mathrm{Spec}(k[t])$.*

(3) *The fibration p has a cross-section S such that $S \subset W - l(V')$.*

Proof. Let $\overline{V}$ be a nonsingular projective surface containing V' as an open set. Then, a sub field $k(t)$ of $k(V') = k(\overline{V})$ defines a linear pencil $\overline{\Lambda}$ of effective divisors on $\overline{V}$ such that a general member of $\overline{\Lambda}$ cuts out a general fiber of q' on V'. The base points of $\overline{\Lambda}$ are situated on $\overline{V} - V'$. Let $\theta : W \to \overline{V}$ be the shortest succession of quadratic transformations of $\overline{V}$ with centers at the base points of $\overline{\Lambda}$ such that the proper transform
180 Λ of $\overline{\Lambda}$ by θ has no base points, and let $p : W \to \mathbb{P}^1_k$ be the morphism defined by Λ. Since V' is naturally embedded into W as an open set, let $l : V' \to W$ be the natural open immersion. Then it is not hard to see that $p : W \to \mathbb{P}^1_k$ and $l : V' \to W$ satisfy the conditions (1), (2) and (3) of Lemma. □

[6] For the existence of the minimal resolution of singularities of V, we refer to Lipman [30; Th. 4.1].

[7] Since k is assumed to be of characteristic zero there would not be a confusion of notations.

5.18

Lemma 2.2 applied to the fibration $p : W \to \mathbb{P}^1_k$ implies the following:

Lemma. *Write* $W - l(V') = \bigcup_{i=1}^{r} C_i$ *with irreducible curves* C_i*. Then we have:*

(1) *Every* C_i *is isomorphic to* $\mathbb{P}^1_k$.

(2) *For* $i \neq j$, C_i *and* C_j *meet each other (if at all) in a single point where they intersect transversely.*

(3) *For three distinct indices* i, j *and* ℓ, $C_i \cap C_j \cap C_\ell = \phi$.

(4) $\bigcup_{i=1}^{r} C_i$ *does not contain any cyclic chains.*

Proof. Note that one of C_i's is the cross-section S and the other components are contained in the fibers of p. Noting that S is isomorphic to $\mathbb{P}^1_k$, we obtain readily the above assertions from Lemma 2.2. ◻

5.19

Let $V_0 := \mathrm{Spec}(k[x,y])$, and let F, G be as in 5.2. Let $G_j (1 \leq j \leq n)$ be as in 5.10. Embed V_0 into the projective plane $\mathbb{P}^2_k$ as the complement of a line ℓ_∞, and let $\overline{F}$, $\overline{G}$, $\overline{G}_j (1 \leqq j \leqq n)$ be the closures of F, G, G_j in $\mathbb{P}^2_k$, respectively. Let $\tau : Z \to \mathbb{P}^2_k$ be a composition of the standard transformations of $\mathbb{P}^2_k$ with respect to triplets (P, F, G) (or (P, G, F)), where P runs over all points of $F \cap G$. Then we know that V' is embedded 181
into Z as an open set. We may assume, by replacing W if necessary by a surface which is obtained from W by a succession of the quadratic transformations, that there exists a birational morphism $\varphi : W \to Z$ such

that we have the following commutative diagram:

$$\begin{tikzcd}
 & W \arrow[r, "p"] \arrow[dl, "\varphi"'] & \mathbb{P}^1_k \\
Z \arrow[dd, "\tau"] & V' \arrow[l, hook'] \arrow[u, hook, "l"'] \arrow[r, "q'"] \arrow[d, "\rho"] & U \arrow[u, hook, "\bar{l}"'] \\
 & V \arrow[d, "\pi"] & \\
\mathbb{P}^2_k & V_0 \arrow[l, hook'] &
\end{tikzcd}$$

5.20

Lemma. (1) *With the notations of 5.19, $(\tau\varphi)'(\overline{G}_j)$ is contained in a fiber of p for $1 \leqq j \leqq n$; in particular, $(\tau\varphi)'(\overline{G}_j)$ is isomorphic to $\mathbb{P}^1_k$.*

(2) *Let $P_1 \in F \cap G$. Assume that F is nonsingular at P_1 but G is singular at P_1. Then, with the notations of the Figure 1 of 5.4, $\varphi'(E_1), \ldots, \varphi'(E_{N-1})$ are contained in one and only one fiber of p.*

Proof. (1) We know by virtue of 5.17 that if λ is a general member of p then $\lambda_{V'} := l^{-1}(\lambda \cap l(V'))$ is isomorphic to the affine line $\mathbb{A}^1_k$; we also know that $\pi'(G_j) = \phi$ for $1 \leqq j \leqq n$. Hence $\lambda_{V'} \cap (\pi\rho)'(\overline{G}_j) = \phi$. This implies that if $(\tau\varphi)'(\overline{G}_j) \cap \lambda \neq \phi$ then λ meets $(\tau\varphi)'(\overline{G}_j)$ at
182 some of finitely many points of $(\tau\varphi)'(\overline{G}_j)$ which are independent of choice of λ. However, this is impossible because λ is a general member of an irreducible linear pencil on W free from base points. Hence $(\tau\varphi)'(\overline{G}_j) \cap \lambda = \phi$. This implies that $(\tau\varphi)'(\overline{G}_j)$ is contained in a fiber of p for $1 \leqq j \leqq n$. The fact that $(\tau\varphi)'(\overline{G}_j)$ is isomorphic to $\mathbb{P}^1_k$ follows from Lemma 2.2.

(2) By construction of V and V' (cf. 5.9) we know that $\varphi'(E_i) \cap l(V') = \phi$ for $1 \leqq i \leqq N-1$. Hence, for each i with $1 \leqq i \leqq N-1$, a general fiber λ of p meets $\varphi'(E_i)$ at some of finitely many points of $\varphi'(E_i)$ which are independent of choice of λ. By the same reason as in (1) above we know that $\varphi'(E_i)$ is contained in a fiber of p. Since $\varphi'(E_1), \ldots, \varphi'(E_{N-1})$ are connected they are contained in one and only one fiber of p.

□

Note that, with the notations of the assertion (2) above, a general fiber λ of p may intersect $\varphi'(E_N)$.

5.21

Lemma. (1) *For* $1 \leqq j \leqq n$, G_j *has only one place at infinity; every singular point of* G_j *is a one-place point.*

(2) *For distinct* i, j $(1 \leqq i, j \leqq n)$, $G_i \cap G_j = \phi$.

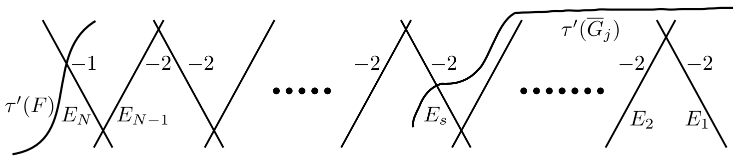
Proof. Let Λ_Z be the linear pencil on Z defined by a subfield $k(t) = k(\mathbb{P}^1)$ of $k(Z) = k(W)$, the inclusion $k(t) \hookrightarrow k(Z)$ corresponding to p. A general member of Λ_Z cuts out on V' a curve of the form $\lambda_{V'}$, where λ is a fiber of p. Hence, if Λ_Z has base points they are centered at a point on $\tau'(\ell_\infty)$. If φ is not an isomorphism, we may assume without loss of generality that φ is the shortest succession of quadratic transformations with centers at base points of Λ_Z (including infinitely near base points) such that the proper transform of Λ_Z by φ has no base points. Then every **183**
singular point of $G_j (1 \leqq j \leqq n)$ lies on the curve F; indeed, if otherwise, $(\tau\varphi)'(\overline{G}_j)$ has a singular point, which contradicts Lemma 5.20, (1). Now, if G_j has two or more places at infinity then $W - l(V')$ would contain a cyclic chain because $l(V') \cap ((\tau\varphi)'(\overline{G}_j) \cup \varphi^{-1}(\tau'(\ell_\infty))) = \phi$, which contradicts Lemma 5.18. Thus, $G_j (l \leqq j \leqq n)$ has only one place at infinity. If G_j has a singular point P_1 which is not a one-place point, then $P_1 \in G_j \cap F$ as remarked above and, with the notations of the Figure 1 of 5.4, $(\tau\varphi)'(\overline{G}_j) \cup \varphi'(E_1) \cup \ldots \cup \varphi'(E_{N-1})$ would contain a cyclic chain. Since $(\tau\varphi)'(\overline{G}_j)$ and $\varphi'(E_i) (1 \leqq i \leqq N-1)$ are contained in $W - l(V')$ this is a contradiction to Lemma 5.18. Thus, every singular

point of G_j is a one-place point. Similarly, if $G_i \cap G_j \neq \phi (i \neq j)$ then $W - l(V')$ would contain a cyclic chain. Thus, $G_i \cap G_j = \phi$ for $i \neq j$. □

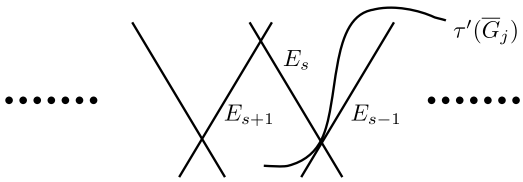
5.22

Lemma. *For $1 \leqq j \leqq n$, the curve G_j is nonsingular.*

Proof. As remarked in the proof of Lemma 5.21, if P is a singular point of G_j then $P \in F \cap G_j$. Then, in a neighborhood of $\tau^{-1}(P)$, $\tau^{-1}(F \cup G_j)$ must have the following configuration as in the Figure 1 of 5.4:



184 where $\varphi'(E_1), \ldots, \varphi'(E_{N-1})$ and $(\tau\varphi)'(\overline{G}_j)$ belong to the same fiber of p. Note that $N \geqq s + 1$ since P is a singular point of G_j and that $(\tau\varphi)'(\overline{G}_j)$ intersects $\varphi'(E_s)$ transversely in one point. Assume that $\nu_b \geqq 2$ and $\nu_{b+1} = \ldots = \nu_s = 1$ (cf. 5.4 for the notations). Such b exists because P is a singular point of G_j and $(\varphi'(E_s) \cdot (\tau\varphi)'(\overline{G}_j)) = 1$. Then it is not hard to show that $s = b + 1$ and we have the configuration:



where $\tau'(\overline{G}_j)$ touches E_{s-1} with $(\tau'(\overline{G}_j) \cdot E_{s-1}) = \nu_b - 1 \geqq 1$. This contradicts Lemma 2.2. Therefore, the curve G_j is nonsingular. □

5.23

Theorem. *Assume that V has only isolated singularities. Then A has a nontrivial locally nilpotent k-derivation if and only if we have $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$.*

Proof. Assume that $g \in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$. Then $D = g\dfrac{\partial}{\partial x}$ is a nontrivial locally nilpotent k-derivation on A. We shall prove the converse. With the notations of 5.16 ~ 5.22, $G_j (1 \leqq j \leqq n)$ is a nonsingular rational curve with only one place at infinity (cf. 5.20, 5.21 and 5.22). Hence, G_j is isomorphic to the affine line **185**
$\mathbb{A}^1_k$. By virtue of the Embedding theorem of Abhyankar-Moh (cf. 1.1), we may assume that $g_1 = y$ after a suitable change of coordinates x, y of $k[x, y]$. Then, for $2 \leqq j \leqq n$, g_j is written in the form $g_j = c_j + yh_j$ with $c_j \in k$ and $h_j \in k[x, y]$ because $G_j \cap G_1 = \phi$ (cf. 5.21, (2)). On the other hand, by virtue of the Irreducibility theorem (cf. 1.1), the fact that G_j has only one place at infinity implies that the curve $g_j = \alpha$ on $\mathbb{A}^2_k$ is irreducible for every $\alpha \in k$. Therefore, h_j is a constant $\in k$. Thus $g \in k[y]$. □

5.24

We know by virtue of Theorem 1.3.1 that A is isomorphic to a polynomial ring over k if and only if A satisfies the following conditions:

(1) A is a unique factorization domain,

(2) $A^* = k^*$,

(3) A has a nontrivial locally nilpotent k-derivation.

The condition (1) above can be described as follows:

Lemma. *Assume that* $A := k[x, y, \dfrac{f}{g}]$ *satisfies the conditions* (2) *and* (3) *above. We may assume that* $g \in k[y]$ *after a suitable change of coordinates* x, y *of* $k[x, y]$. *Write:* $f(x, y) = a_0(y) + a_1(y)x + \cdots + a_r(y)x^r$ *with* $a_i(y) \in k[y]$ $(0 \leqq i \leqq r)$. *Then* A *is a unique factorization domain if and only if* $a_1(y)$ *is a unit modulo* $gk[x, y]$ *and* $a_i(y)$ *is nilpotent modulo* $gk[x, y]$ *for* $2 \leqq i \leqq r$.

Proof. Assume that A is a unique factorization domain. With the notations of 5.10, we have $a = 0$ because every $G_j (1 \leqq j \leqq n)$ is nonsingular **186**
and $G_j \cap G_i = \phi$ if $i \neq j$. By virtue of 5.14, we have $e = n$. Theorem 5.11

then implies that every G_j intersects F transversely. This is easily seen to be equivalent to the condition on $f(x, y)$ in the above statement. The "if" part of Lemma will be clear by the above argument and Theorem 5.11. $\square$

5.25

Finally, we shall prove the following:

Theorem cf. Russell [49] and Sathaye [52] in case $m = 1$**; cf. Wright [56] in case** $m > 1$**.** *Let k be an algebraically closed field of characteristic zero and let $k[x, y]$ be a polynomial ring over k in two variables x and y. Let f and g be two nonzero elements of $k[x, y]$ such that:*

(1) *f and g have no nonconstant common factors;*

(2) *let $B := k[x, y, w]/(gw^m - f)$ with a variable w and an integer $m \geqq 1$; then B is isomorphic to a polynomial ring over k. Then there exist $\varphi, \psi \in k[x, y, w]$ such that $k[x, y, w] = k[\varphi, \psi, gw^m - f]$.*

Proof. We shall prove the theorem only in the case where $m > 1$; for the case where $m = 1$, see the original proofs. Our proof consists of four steps.

(I) Let $A := k[x, y, z]/(gz - f)$. Let $V := \mathrm{Spec}(A)$ and $W := \mathrm{Spec}(B)$. By assigning x, y, w^m to x, y, z, respectively we have an inclusion $A \hookrightarrow B$, which defines in turn a morphism $g : W \to V$. Let $\mathfrak{g}$ be the group of m-th roots of unity; $\mathfrak{g}$ is identified with a cyclic group $\mathbb{Z}_m$ of order m. Note that $\mathfrak{g}$ acts on W via $(x, y, w) \mapsto (x, y, \xi w)$ for
187 $\xi \in \mathfrak{g}$. It is readily ascertained that A is the subring of $\mathfrak{g}$-invariants in B and that the morphism $q : W \to V$ is the quotient morphism for the above-defined action of $\mathfrak{g}$ on W.

(II) For the moment, assume only that W is nonsingular. By applying the Jacobian criterion of singularity to W we easily see that;

1° the curve F on $\mathbb{A}^2_k := \mathrm{Spec}(k[x, y])$ defined by $f = 0$ is a nonsingular curve;

2° let G be the curve on $\mathbb{A}_k^2$ defined by $g = 0$; then, if G intersects F at a point P, either G is singular at P or G intersects F transversely at P.

This implies by virtue of 5.3 that if W is nonsingular then V is nonsingular as well. Let $\pi : V \to \mathbb{A}_k^2$ be a morphism defined by $(x, y, z) \mapsto (x, y)$. Note that $(\pi q)^{-1}(Q) \neq \phi$ and $\pi^{-1}(Q) \neq \phi$ for every point Q on F, and that the proper transform $\pi'(F)$ of F on V is defined by $z = 0$. Moreover, note that the morphism $q : W \to V$ is a finite morphism, which is unramified at every point P of V with $P \notin \pi'(F)$ and totally ramified on the curve $\pi'(F)$.

(III) Assume now that W is isomorphic to the affine plane $\mathbb{A}_k^2$. Then $\mathfrak{g}$ is a finite subgroup of $\mathrm{Aut}_k W$. Since V is nonsingular as seen in the step (II), Proposition 3.7 implies that V is isomorphic to the affine plane $\mathbb{A}_k^2$ as well. We shall show that F is isomorphic to the affine line $\mathbb{A}_k^1$. Write $W := \mathrm{Spec}(k[u, v])$. Since $\mathfrak{g}$ is conjugate to a finite subgroup of $GL(2, k)$ (cf. 3.5) and since a finite subgroup of $GL(2, k)$ isomorphic to $\mathbb{Z}_m$ is diagonalizable, we may assume that $\mathfrak{g}$ acts on W via

$$\xi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi^i & 0 \\ 0 & \xi^j \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $\xi \in \mathfrak{g}$ and $i, j \in \mathbb{Z}_m$. Since $q : W \to V$ is totally ramified **188**
over $\pi'(F)$, every point of the ramification locus of q is fixed by $\mathfrak{g}$. Hence either $i = 0$ or $j = 0$. We may assume that $i = 0$. Then the curve R on W defined by $v = 0$ is the ramification locus of q. Since $\pi'(F) = q(R)$ and $\pi'(F)$ is isomorphic to R, we know that $\pi'(F)$ is isomorphic to the affine line. Therefore, F is an irreducible nonsingular rational curve with only one place at infinity (cf. the step (II)). Thus, F is isomorphic to the affine line $\mathbb{A}_k^1$.

(IV) By virtue of the Embedding theorem (cf. 1.1) we may assume that $f = x$. On the other hand, since V is isomorphic to the affine plane $\mathbb{A}_k^2$, we know by virtue of Lemma 5.24 and its proof that the curve G is nonsingular, each connected component of G is isomorphic to $\mathbb{A}_k^1$ and F intersects each connected component of

G transversely at a single point. Therefore, we may assume that $g \in k[y]$ (cf. 1.3.2). Then it is easily verified that $k[x, y, w] = k[y, z, gw^m - f]$.

□

6 Certain affine plane curves with two places at infinity

189

6.1

The results of this section were worked out jointly by T. Sugie and the lecturer (cf. Miyanishi and Sugie [37]). Throughout the section the ground field k is assumed to be an algebraically closed field of characteristic zero. Our ultimate purpose is to prove the following:

Theorem. *Let f be an irreducible element of a polynomial ring $k[x, y]$ and let C_α be the curve on $\mathbb{A}^2_k := \mathrm{Spec}(k[x, y])$ defined by $f = \alpha$ for $\alpha \in k$. The, after a suitable change of coordinates x, y of $k[x, y]$, $f = c(x^d y^e - 1)$ for $c \in k^*$ and positive integers d and e with $(d, e) = 1$ if and only if the following conditions are satisfied:*

(1) *f is a field generator* (cf. 2.4.1).

(2) *C_α has exactly two places at infinity for almost all $\alpha \in k$.*

(3) *C_α is connected for every $\alpha \in k$.*

6.2

Let V be a nonsingular projective surface defined over k and let Λ be an irreducible linear pencil on V whose general members are rational curves. Let B be the set of (ordinary) base points of Λ. *We assume that each point of B is a one-place point of a general member of* Λ. A reducible member Δ of Λ is said to be *linear* if the following conditions are satisfied;

(i) every irreducible component of Δ is isomorphic to $\mathbb{P}^1_k$,

(ii) two distinct irreducible components of Δ meet each other (if at all) transversely in a single point, 190

(iii) three distinct irreducible components of Δ have no points in common,

(iv) the weighted graph of Δ is a linear chain.

An irreducible component D of a linear reducible member Δ of Λ is called *a terminal component* if D meets only one irreducible component of Δ other than D. An irreducible curve S on V is called *a quasi-section of* Λ if S is not contained in any member of Λ and Λ has no base points on S; a quasi-section of Λ is called *a section of* Λ if $(C \cdot S) = 1$ for a general member C of Λ.

6.3

Lemma. *With the notations and assumptions of 6.2, let* $\Delta := n_0D_0 + n_1D_1 + \cdots + n_rD_r$ *be a linear reducible member of* Λ *with irreducible components* D_i *and integers* $n_i > 0$. *Assume that the following conditions are satisfied:*

(1) $D_0 \cap B = \{P\}$ *and* $P \notin D_i$ *for* $1 \leqq i \leqq r$*;*

(2) $(D_0^2) = p > 0$ *and* D_0 *is not a terminal component of* Δ*;*

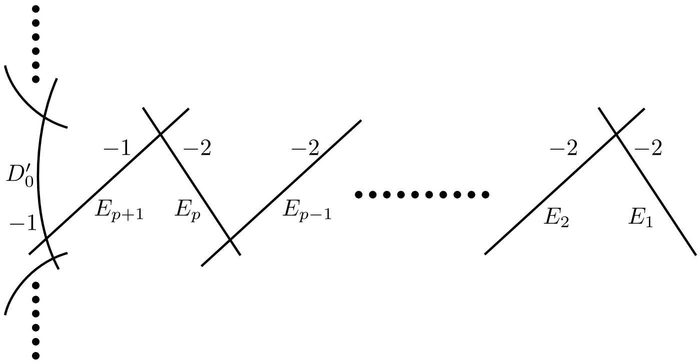
(3) $(D_i^2) < 0$ *for* $1 \leqq i \leqq r$ *and* $(D_i^2) < -1$ *whenever* $D_i \cap B = \phi$.

Then the multiplicity n_0 *of* D_0 *in* Δ *is equal to* 1. *Furthermore,* $(C \cdot D_0) = i(C, D_0; P) = p + 1$.

Proof. Our proof consists of six steps.

(I) Let C be a general member of Λ. Let $e := (C \cdot D_0) = i(C, D_0; P)$ and $\nu := \mathrm{mult}_P C$. Let $P_0 := P, P_1, \ldots, P_P$ be points on D_0 over P_0, where P_i is an infinitely near point of P_{i-1} of order one for 191

$1 \leqq i \leqq p$.[8] Let $\sigma : V' \to V$ be a succession of quadratic transformations of V with centers at $P_0, \ldots, P_p$, let $C' := \sigma'(C)$ and let $D'_0 := \sigma'(D_0)$. Then $\sigma^{-1}(D_0)$ has the configuration as follows:



(II) Note that P is a one-place point of C. We shall show that C' meets E_{p+1}. Assume the contrary, i.e., $E_{p+1} \cap C' = \phi$. Let Λ' be the proper transform of Λ by σ and let Δ' be the member of Λ' corresponding to Δ. Then it is easily ascertained that:

1° Λ' is spanned by Δ' and C';

2° D'_0 and E_{p+1} are irreducible components of Δ';

3° there exist no base points of Λ' on D'_0 and E_{p+1}.

Let $\tau : V' \to \overline{V}$ be the contraction of D', let $\overline{\Lambda}$ be the proper transform of Λ' and $\overline{\Delta} := \tau_*(\Delta')$ be the member of $\overline{\Lambda}$ corresponding to Δ'. Then $\overline{\Delta}$ has three irreducible components meeting each other
192 in one point, which is not a base point of $\overline{\Lambda}$. This is a contradiction (cf. 2.3, (3)). Thus we know that $C' \cap E_{p+1} \neq \phi$.

(III) With the notations of the step (II), we shall show that E_{p+1} is not a component of Δ'. Assume the contrary, and let $Q := C' \cap E_{p+1}$.

[8]Let $\sigma_1 : V_1 \to V$ be the quadratic transformation of V with center at P. Then $P_1 = \sigma'_1(D_0) \cap \sigma_1^{-1}(P_0)$. For $2 \leqq i \leqq p$, define inductively the quadratic transformation $\sigma_i : V_i \to V_{i-1}$ of V_i with center at P_{i-1}. Then $P_i = (\sigma_1 \ldots \sigma_i)'(D) \cap \sigma_i^{-1}(P_{i-1})$.

If $Q \neq D'_0 \cap E_{p+1}$ we would have a contradiction by contracting D'_0 (cf. 2.3, (3)). Hence $Q = D'_0 \cap E_{p+1}$. However this is again a contradiction because of the condition (3) above (cf. 2.3, (4)).

(IV) We shall show that $Q := C' \cap E_{p+1}$ is distinct from $D'_0 \cap E_{p+1}$ and $E_P \cap E_{p+1}$. Indeed, if $Q = D'_0 \cap E_{p+1}$ we have a contradiction because of the condition (3) above (cf. 2.3, (4)). Assume that $Q = E_P \cap E_{p+1}$. Note that $E_1, \ldots, E_P$ are contained in one and only one member of Λ' other than Δ' because $E_i \cap \mathrm{Supp}(\Delta') = \phi$ for $1 \leqq i \leqq p$. Then Q is a base point of Λ'. This is a contradiction because $Q \notin \mathrm{Supp}(\Delta')$.

(V) From the above arguments we know that E_{p+1} is a quasi-section of Λ' such that $(\Delta' \cdot E_{p+1}) = n_0$. Assume that $n_0 > 1$. Then we have a ramified covering $E_{p+1} \to \mathbb{P}^1_k$ of degree n_0, which ramifies totally over at least three points of $\mathbb{P}^1_k$. By Hurwitz's formula, we have:

$$-2 \geqq -2n_0 + 3(n_0 - 1) = n_0 - 3.$$

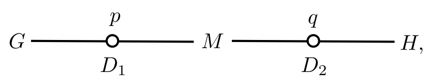
This is a contradiction. Hence we obtain $n_0 = 1$.

(VI) Since P is a one-place point of C, the fact that E_{p+1} is a quasi-section of Λ' implies that $e = (p + 1)\nu$ and $\nu = (C' \cdot E_{p+1})$. Since 193
$\nu = n_0 = 1$ we know that $e = p + 1$.

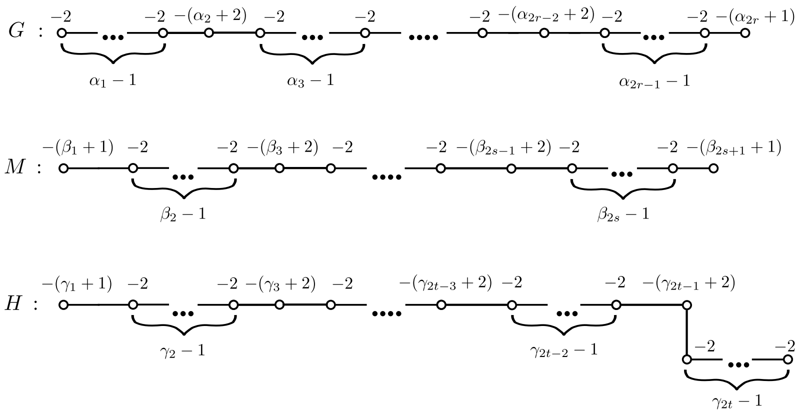
□

6.4

In the paragraphs 6.4 ∼ 6.6, let the notations and assumptions be as in 6.2. Assume furthermore that Λ has a linear reducible member Δ whose weighted graph is the following linear chain:



where G, M and H are given respectively by



α_i, β_j and γ_ℓ being positive integers for $1 \leqq i \leqq 2r$, $1 \leqq j \leqq 2s+1$ and $1 \leqq \ell \leqq 2t$. G, M and H are called respectively *the left, the middle and the right branches* of the weighted graph of Δ. The absence of the left branch G (or the middle branch M, or the right branch H, resp.) is denoted by $G = \phi$ (or $M = \phi$, or $H = \phi$, resp.). In the above graph the components of Δ with self-intersection multiplicities p and q are denoted respectively by D_1 and D_2.

6.5

194 **Lemma.** *Let the notations and assumptions be as in 6.2 and 6.4. Let $n_i (i = 1, 2)$ be the multiplicity of D_i in Δ. Assume that $B = \{P_1, P_2\}$ with $P_i \in D_i$ and $P_i \notin$ the components of Δ other than D_i for $i = 1, 2$, that $n_1 \neq 1$ and $n_2 \neq 1$, and that either $p \leqq 0$ or $q \leqq 0$. Then the following assertions hold true:*

(1) *Either $p \geqq 0$ or $q \geqq 0$. Thus, in the assertions below we assume that $q \geqq 0$ and $p \leqq 0$.*

(2) *If $p < 0$ and $q \geqq 0$ then $H = \phi$.*

(3) *If $p = 0$ and $q \geqq 0$ then either $G = \phi$ or $H = \phi$.*

Proof. Let C be a general member of Λ and let $e_i := (C \cdot D_i) = i(C, D_i; P_i)$ and $\nu_i := \mathrm{mult}_{P_i} C$ for $i = 1, 2$.

(1) The assertion (1) follows from Lemma 2.3, (4).

(2) Consider first the case where $p < 0$ and $q > 0$. Assume that $H \neq \phi$. Then D_2 is not a terminal component of Δ. Lemma 6.3 then tells us that $n_2 = 1$, which contradicts the assumption. Hence $H = \phi$. Consider next the case where $p < 0$ and $q = 0$. Assume that $H \neq \phi$. Let $\sigma : V' \to V$ be the quadratic transformation of V with center at P_2. Let $\Lambda' := \sigma'(\Lambda)$, $C' := \sigma'(C)$, $D'_2 := \sigma'(D_2)$, $E := \sigma^{-1}(P_2)$, $Q := E \cap C'$ and $\Delta' :=$ the member of Λ' corresponding to Δ. Then Λ' is spanned by C' and Δ'. We shall show that $Q \notin D'_2$ and $E \not\subset \mathrm{Supp}(\Delta')$, which imply that $e_2 = \nu_2 = n_2$ and that E is a quasi-section of Λ' with $(C' \cdot E) = \nu_2$. If $Q \in D'_2$ then we would have a contradiction by Lemma 2.3, (4), regardless of whether or not $E \subset \mathrm{Supp}(\Delta')$. Thus $Q \notin D'_2$. If $E \subset \mathrm{Supp}(\Delta')$ then we would have a contradiction by contracting D'_2 as in the proof of Lemma 6.3, because D_2 is not a terminal **195**
component of Δ. Thus $E \not\subset \mathrm{Supp}(\Delta')$. Since every member of Λ' has a one-place point on E and the characteristic of k is zero, E must be a cross-section of Λ', i.e., $\nu_2 = n_2 = 1$. This is a contradiction. Hence we have $H = \phi$.

(3) Assume that $G \neq \phi$ and $H \neq \phi$. We shall first consider the case where $p = 0$ and $q > 0$. Let $\sigma : V' \to V$ be the quadratic transformation of V with center at P_1. Let $\Lambda' := \sigma'(\Lambda)$, $C' := \sigma'(C)$, $D'_1 := \sigma'(D_1)$, $E := \sigma^{-1}(P_1)$, $Q := E \cap C'$ and $\Delta' :=$ the member of Λ' corresponding to Δ. Then Λ' is spanned by C' and Δ'. We shall show that $Q \notin D'_1$ and $E \not\subset \mathrm{Supp}(\Delta')$, which imply that $e_1 = \nu_1 = n_1$ and that E is a quasi-section of Λ' with $(C' \cdot E) = \nu_1$. If $Q \in D'_1$, Lemma 6.3[9] applied to $(V', \Lambda', \Delta', D'_2 := \sigma'(D_2), P_2)$ instead of $(V, \Delta, \Delta, D_0, P)$ implies that $n_2 = 1$, which contradicts the assumption. Thus $Q \notin D'_1$. If $E \subset \mathrm{Supp}(\Delta')$ we would have a contradiction by contracting D'_1 as in the proof of Lemma 6.3,

[9] If $E \not\subset \mathrm{Supp}(\Delta')$ we can apply Lemma 6.3 in the stated form because Δ' is linear. However, if $E \subset \mathrm{Supp}(\Delta')$, we have to strengthen Lemma 6.3 so as to apply it to the present situation. However, this is a very easy task; the given proof works without any modification.

because D_1 is not a terminal component of Δ. Since every member of Λ' has a one-place point on E and the characteristic of k is zero, E must be a cross-section of Λ', i.e., $\nu_1 = n_1 = 1$. This is a contradiction. Therefore, we know that either $G = \phi$ or $H = \phi$. consider next the case where $p = q = 0$. Let $\sigma : V' \to V$ be the composition of quadratic transformations of V with centers at P_1 and P_2. Let $\Lambda' := \sigma'(\Lambda)$, $C' := \sigma'(C)$, $D_i' := \sigma'(D_i)$, $E_i := \sigma^{-1}(P_i)$, $Q_i := E_i \cap C'$ and $\Delta' :=$ the member of Λ' corre-
196 sponding to Δ, where $i = 1, 2$. We have only to show that either $Q_1 \notin D_1'$ and $E_1 \not\subset \mathrm{Supp}(\Delta')$ or $Q_2 \notin D_2'$ and $E_2 \notin \mathrm{Supp}(\Delta')$; in either case we get a contradiction. If $Q_i \in D_i' (i = 1, 2)$ we have a contradiction by Lemma 2.3 (4), regardless of whether or not $E_i \subset \mathrm{Supp}(\Delta')(i = 1, 2)$. Thus, either $Q_1 \notin D_1'$ or $Q_2 \not\subset D_2'$. Assume that $Q_1 \notin D_1'$. Then $E_1 \not\subset \mathrm{Supp}(\Delta')$, for, if otherwise, we would have a contradiction by contracting D_1' because D_1 is not a terminal component of Δ. Similarly, we have $E_2 \not\subset \mathrm{Supp}(\Delta')$ if $Q_2 \notin D_2'$.

□

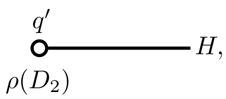
6.6

Lemma. *Let the notations and assumptions be as in 6.2 and 6.4. Assume that $B = \{P_2\}$ with $P_2 \in D_2$ and $P_2 \notin$ the components of Δ other than D_2. Assume that the multiplicity n_2 of D_2 in Δ is not equal to* 1. *Then the following assertions hold true:*

(1) $p < 0$, *and if $p \neq -1$ then $q \geqq 0$.*

(2) *If $p \leqq -2$ then $H = \phi$.*

(3) *If $p = -1$ then either $H = \phi$ or there exists a contraction ρ of V onto a nonsingular projective surface W such that $\rho_*\Lambda$ is spanned by $\rho_*(C)$ and $\rho_*(\Delta)$ and that $\rho_*(\Delta)$ has the following weighted graph:*



where H is the same graph as in 6.4 and $q' = q + 1$ or $q + \alpha_1 + 1$; in the second case, any component in the graphs G and M as well as D_1 has multiplicity > 1 in Δ.

Proof. Let $e_2 := (C \cdot D_2) = i(C, D_2; P_2)$ and $\nu_2 := \mathrm{mult}_{P_2} C$.

(1) Since $D_1 \cap B = \phi$ we have $p = (D_2^2) < 0$ (cf. 2.3, (1)). If $p \neq -1$ **197**
we must have $q \geqq 0$ by virtue of Lemma 2.3, (4).

(2) Assume that $H \neq \phi$. Then D_2 is not a terminal component of Δ. Since $n_2 \neq 1$ we must have $q \leqq 0$ (cf. Lemma 6.3). Since $q \geqq 0$ as shown above, we know that $q = 0$. The same argument as used to prove the assertion (2) of Lemma 6.5 leads us to a contradiction. Hence $H = \phi$.

(3) Assume that $H \neq \phi$. Since D_1 is an exceptional component of Δ, D_1 is contractible. After the contraction of D_1, if there exists a contractible component in the graphs G and M, it must be either one of the components with weights $-(\alpha_{2r} + 1)$ and $(-(\beta_1 + 1)$; the weights becoming $-\alpha_{2r}$ and $-\beta_1$ respectively after the contraction of D_1, we must have $\alpha_{2r} = 1$ or $\beta_1 = 1$. If $\beta_1 = 1$ for instance, the β_2 components in the graph M with weights $-(\beta_1 + 1), -2, \ldots, -2$ respectively are contractible. After the contraction of these β_2 components, the component with weight $-(\alpha_{2r} + 1)$ in the graph G has a (new) weight $-(\alpha_{2r} - \beta_2)$, and the component with weight $-(\beta_3 + 2)$ in the graph M has a weight $-(\beta_3 + 1)$. If there exists still a contractible component in the graphs G and M after the contraction of D_1 and β_2 components in M, it must be the component with weight $-(\alpha_{2r} + 1)$ in the graph G, i.e., we must have $\alpha_{2r} = \beta_2 + 1$. Repeat the above argument, and let $\rho : V \to W$ be the contraction of all possible components in the graphs G and M. The contraction ρ is uniquely determined. It is clear that the proper transform $\rho_*(\Lambda)$ of Λ by ρ is spanned by $\rho(C)$ and $\rho_*(\Delta)$, and that $\rho_*(\Delta)$ has the following weighted graph:

$$G' \;\frac{\quad\quad}{}\!\!\overset{q'}{\underset{\rho(D_2)}{\circ}}\!\!\frac{\quad\quad}{}\; H,$$

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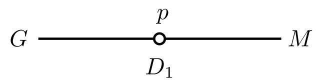
where G' is the graph similar to G and obtained in the above-explained way by the contraction ρ from the graph:



and where $q' \geqq q$, the inequality $q' > q$ taking place only if all components of the graph M are contracted by ρ. Note that $\rho_*(\Lambda)$ has only one base point $\rho(P_2)$, that $e_2 = (\rho(C) \cdot \rho(D_2))$ and $\nu_2 = \mathrm{mult}_{\rho(P_2)}\, \rho(C)$ and that n_2 is the multiplicity of $\rho(D_2)$ in $\rho_*(\Delta)$. If $G' \neq \phi$ then $H = \phi$ by virtue of the assertion (2) above. It is easily verified that the case $G' = \phi$ occurs only in one of the following four cases:

1° $s = r$; $\beta_1 = 1, \beta_2 = \alpha_{2r-1}, \beta_3 = \alpha_{2r-1}, \ldots, \beta_{2r-1} = \alpha_3, \beta_{2r} = \alpha_2, \beta_{2r+1} = \alpha_1 + 1$; $q' = q + 1$,

2° $s = r$; $\beta_1 = 1, \beta_2 = \alpha_{2r-1}, \beta_3 = \alpha_{2r-1}, \ldots, \beta_{2r-1} = \alpha_3, \beta_{2r} = \alpha_2 - 1, \beta_{2r+1} = 1$; $q' = q + \alpha_1 + 1$,

3° $s = r-1$; $\alpha_{2r} = 1, \alpha_{2r-1} = \beta_1 - 1, \alpha_{2r-2} = \beta_2, \ldots, \alpha_3 = \beta_{2r-3}, \alpha_2 = \beta_{2r-2}, \alpha_1 = \beta_{2r-1} - 1$; $q' = q + 1$,

4° $s = r-1$; $\alpha_{2r} = 1, \alpha_{2r-1} = \beta_1 - 1, \alpha_{2r-2} = \beta_2, \ldots, \alpha_3 = \beta_{2r-3}, \alpha_2 = \beta_{2r-2} + 1, \beta_{2r-1} = 1$; $q' = q + \alpha_1 + 1$.

The last assertion is clear because $n_2 > 1$ and the base point P_2 lies on D_2 but not on the other components of Δ.

□

6.7

199 In the paragraphs 6.7 ~ 6.17 we shall prove the "if" part of Theorem 6.1. Thus the conditions (1) ~ (3) of the theorem are always assumed to be satisfied. A useful remark is that we may replace f by $f - \alpha$ for a general element $\alpha \in k$ if necessary, because if $f - \alpha = c(x^d y^e - 1)$ for

$c \in k^*$ and coordinates x, y of $\mathbb{A}^2_k$ then $f = c'(x'^d y'^e - 1)$ for $c' \in k^*$ and coordinates x', y' of $\mathbb{A}^2_k$. Thus we assume once for all that the curve C_0 on $\mathbb{A}^2_k$ defined by $f = 0$ has exactly two places at infinity. Embed $\mathbb{A}^2_k := \mathrm{Spec}(k[x, y])$ into $\mathbb{P}^2_k$ as the complement of a line ℓ_0, and let C be the closure of C_0 on $\mathbb{P}^2_k$. We shall first prove the following:

Lemma. *Assume that C intersects ℓ_0 in only one point P_0. Let $d_1 := \mathrm{mult}_{P_0} C$. Then there exists a birational automorphism ρ of $\mathbb{P}^2_k$ such that ρ induces a biregular automorphism on $\mathbb{A}^2_k := \mathbb{P}^2_k - \ell_0$ and that the proper transform C' of C by ρ intersects ℓ_0 in two distinct points with $(C' \cdot \ell_0) \leqq d_1$.*

Proof. Our proof consists of four subparagraphs 6.7.1 ~ 6.7.4. □

6.7.1

Set $d_0 := (C \cdot \ell_0)$. Let Λ be a linear pencil on $\mathbb{P}^2_k$ spanned by C and $d_0\ell_0$. Let $V_0 := \mathbb{P}^2_k$ and let $\sigma_1 : V_1 \to V_0$ be the quadratic transformation of V_0 with center at P_0. Let $\ell_0^{(1)} := \sigma_1'(\ell_0)$, $\ell_1 := \sigma_1^{-1}(P_0)$, $C^{(1)} := \sigma_1'(C)$ and $\Lambda^{(1)} := \sigma_1'(\Lambda)$. *We shall show that $d_0 > d_1$.* Assume the contrary: $d_0 = d_1$. Then the linear pencil $\Lambda^{(1)}$ is spanned by $C^{(1)}$ and $d_0\ell_0^{(1)}$, and since $(C^{(1)} \cdot \ell_0^{(1)}) = d_0 - d_1 = 0$ the pencil has no base points. Hence $\Lambda^{(1)}$ defines a fibration $\varphi_1 : V_1 \to \mathbb{P}^1_k$ whose general fibers are isomorphic to $\mathbb{P}^1_k$. Then $d_0 = 1$ by virtue of Lemma 2.2, (1). However this is
impossible because C has two distinct places on ℓ_0. Therefore we know **200**
that $d_0 > d_1$.

6.7.2

We shall prove the following assertion:

Either $C^{(1)}$ intersects ℓ_1 in two distinct points, or there exists a birational automorphism ρ of $\mathbb{P}^2_k$ such that ρ induces a biregular automorphism on $\mathbb{A}^2_k := \mathbb{P}^2_k - \ell_0$ and that $(C' \cdot \ell_0) \leqq d_1 < d_0$ where C' is the proper transform of C by ρ.

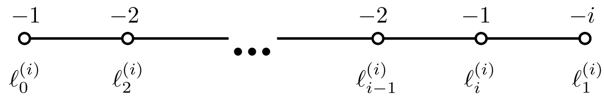
Proof. Our proof consists of four steps.

(I) Assume that $C^{(1)}$ intersects ℓ_1 in a single point P_1. Then $P_1 = \ell_0^{(1)} \cap \ell_1$ because $d_0 > d_1$. Let $\sigma_2 : V_2 \to V_1$ be the quadratic transformation of V_1 with center at P_1. Let $\ell_0^{(2)} := \sigma_2'(\ell_0^{(1)})\ell_1^{(2)} := \sigma_2'(\ell_1)$, $\ell_2 = \ell_2^{(2)} := \sigma_2^{-1}(P_1)$, $C^{(2)} := \sigma_2'(C^{(1)})$ and $\Lambda^{(2)} := \sigma_2'(\Lambda^{(1)})$. Let $d_2 := \mathrm{mult}_{P_1} C^{(1)}$. Then it is easy to see that $\Lambda^{(2)}$ is spanned by $C^{(2)}$ and $d_0\ell_0^{(2)} + (d_0 - d_1)\ell_1^{(2)} + (2d_0 - d_1 - d_2)\ell_2^{(2)}$, where $2d_0 - d_1 - d_2 > 0$ because $\ell_0^{(2)} \cap \ell_1^{(2)} = \phi$. Note that $d_2 \leqq d_1$ and $d_2 \leqq d_0 - d_1$ because $(C^{(1)} \cdot \ell_0^{(1)}) = d_0 - d_1$ and $(C^{(1)} \cdot \ell_1) = d_1$. If $d_0 - d_1 > d_2$ then $(C^{(2)} \cdot \ell_0^{(2)}) = d_0 - d_1 - d_2 > 0$, $(C^{(2)} \cdot \ell_2^{(2)}) = d_2 > 0$ and $((\ell_0^{(2)})^2) = -1$. However this is impossible by virtue of Lemma 2.3, (4). Hence we have $d_2 = d_0 - d_1 \leqq d_1$. By virtue of Lemma 2.3, (4) we know that $C^{(2)}$ intersects ℓ_2 in a single point Q. Indeed, if otherwise $C^{(2)}$ intersects ℓ_2 in two distinct points Q and Q', where neither Q nor Q' lie on $\ell_0^{(2)}$; then contract $\ell_0^{(2)}$ and blow up the points Q and Q'; this operation leads us to a
201 contradiction. If $Q \neq \ell_1^{(2)} \cap \ell_2$ then $d_1 = d_2$, whence $d_0 = 2d_1$. If $d_1 > d_2$ then $Q = P_2 := \ell_1^{(2)} \cap \ell_2$.

(II) Write $d_1 = q_2 d_2 + d_3$ with integers q_2, d_3 such that $0 \leqq d_3 < d_2$ and $q_2 \geqq 1$. For $2 \leqq i \leqq q_2 + 1$ define $V^{(i)}$, σ_i, $\ell_j^{(i)} (0 \leqq j \leqq i)$, $C^{(i)}$, $\Lambda^{(i)}$ and P_i inductively as follows: Let $\sigma_i : V^{(i)} \to V^{(i-1)}$ be the quadratic transformation of $V^{(i-1)}$ with center at P_{i-1} and let $\ell_j^{(i)} := \sigma_i'(\ell_j^{(i-1)})$ for $0 \leqq j \leqq i - 1$, $\ell_i^{(i)} = \ell_i := \sigma_i^{-1}(P_{i-1})$, $C^{(i)} := \sigma_i'(C^{(i-1)})$, $\Lambda^{(i)} := \sigma_i'(\Lambda^{(i-1)})$ and $P_i := \ell_1^{(i)} \cap \ell_i$. By induction on $i (2 \leqq i \leqq q_2 + 1)$ we shall show the following assertions:

$A_1(i)$: $\Lambda^{(i)}$ is spanned by $C^{(i)}$ and $d_0\ell_0^{(i)} + (d_0 - d_1)\ell_1^{(i)} + d_0(\ell_2^{(i)} + \cdots + \ell_i^{(i)})$,

$A_2(i)$: $(C^{(i)} \cdot \ell_j^{(i)}) = 0$ if $0 \leqq j \leqq i - 1$ and $j \neq 1$; $(C^{(i)} \cdot \ell_1^{(i)}) = d_1 - (i-1)d_2$; $(C^{(i)} \cdot \ell_i^{(i)}) = d_2$; $\cup_{j=0}^{i} \ell_j^{(i)}$ has the following weighted graph:

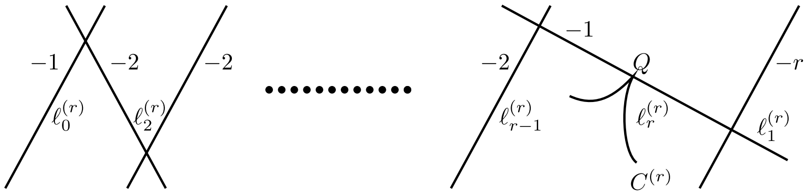


$A_3(i)$: $C^{(i)}$ intersects ℓ_i in a single point Q, where $Q = P_i$ if either $2 \leqq i \leqq q_2$ or $i = q_2 + 1$ and $d_3 > 0$.

Indeed, the assertions $A_1(2) \sim A_3(2)$ are verified in the step (I) above. Assuming that $A_1(j) \sim A_3(j)$ are verified for $2 \leqq j < i$ we shall prove $A_1(i) \sim A_3(i)$. Let $\mu := \mathrm{mult}_{P_{i-1}} C^{(i-1)}$. Then $\mu \leqq d_2$ and $\mu \leqq d_1 - (i-2)d_2$, and $\Lambda^{(i)}$ is spanned by $C^{(i)}$ and $d_0\ell_0^{(i)} + (d_0 - d_1)\ell_1^{(i)} + d_0(\ell_2^{(i)} + \cdots + \ell_{i-1}^{(i)}) + (2d_0 - d_1 - \mu)\ell_i^{(i)}$, where $2d_0 - d_1 - \mu > 0$ because $\ell_1^{(i)} \cap \ell_{i-1}^{(i)} = \phi$. Suppose that $d_2 > \mu$. Then **202**
the contraction of $\ell_0^{(i)}, \ell_2^{(i)}, \ldots, \ell_{i-2}^{(i)}$ leads us to a contradiction by virtue of Lemma 2.3, (4). Hence $d_2 = \mu$ and $2d_0 - d_1 - \mu = d_0$. Thus $A_1(i)$ is proved. By virtue of Lemma 2.3, (4) again, we know that $C^{(i)}$ intersects ℓ_i in a single point Q. Indeed, if otherwise $C^{(i)}$ intersects ℓ_i in two distinct points Q and Q', where neither Q nor Q' lie on $\ell_{i-1}^{(i)}$; then contract $\ell_0^{(i)}, \ldots, \ell_{i-1}^{(i)}$ and blow up the points Q and Q'; this operation leads us to a contradiction. If $Q \neq P_i$ then $(C^{(i)} \cdot \ell_1^{(i)}) = d_1 - (i-2)d_2 - \mu = d_1 - (i-1)d_2 = 0$, i.e., $i = q_2 + 1$ and $d_3 = 0$. Hence if either $2 \leqq i \leqq q_2$ or $i = q_2 + 1$ and $d_3 > 0$ then $Q = P_i$ and $(C^{(i)} \cdot \ell_1^{(i)}) = d_1 - (i-1)d_2$. Therefore, $A_2(i)$ and $A_3(i)$ are proved.

(III) We shall show that $d_3 = 0$. Assume the contrary: $d_3 > 0$. Set $r := q_2 + 1$. Let $\sigma_{r+1} : V_{r+1} \to V_r$ be the quadratic transformation of V_r with center at P_r, and let $\ell_j^{(r+1)} := \sigma'_{r+1}(\ell_j^{(r)})$ for $0 \leqq j \leqq r$, $\ell_{r+1} := \sigma_{r+1}^{-1}(P_r)$, $C^{(r+1)} := \sigma'_{r+1}(C^{(r)})$ and $\Lambda^{(r+1)} := \sigma'_{r+1}(\Lambda^{(r)})$. Let $\nu := \mathrm{mult}_{P_r} C^{(r)}$. Then $\nu \leqq d_3 < d_2$ because $(C^{(r)} \cdot \ell_1^{(r)}) = d_3 < d_2$, and $\Lambda^{(r+1)}$ is spanned by $C^{(r+1)}$ and $d_0\ell_0^{(r+1)} + (d_0 - d_1)\ell_1^{(r+1)} + d_0(\ell_2^{(r+1)} + \cdots + \ell_r^{(r+1)}) + (2d_0 - d_1 - \nu)\ell_{r+1}$ with $2d_0 - d_1 - \nu > 0$. Since $(C^{(r+1)} \cdot \ell_r^{(r+1)}) = d_2 - \nu > 0$, $(C^{(r+1)} \cdot \ell_{r+1}) = \nu > 0$ and $((\ell_1^{(r+1)})^2) = -(r+1) < -2$ the contraction of $\ell_0^{(r+1)}, \ell_2^{(r+1)}, \ldots, \ell_{r-1}^{(r+1)}$ leads us to a contradiction by virtue of Lemma 2.3, (4). Hence $d_3 = 0$.

Thence $d_0 = rd_2$ and $d_1 = (r-1)d_2$. We reached to the following configuration:



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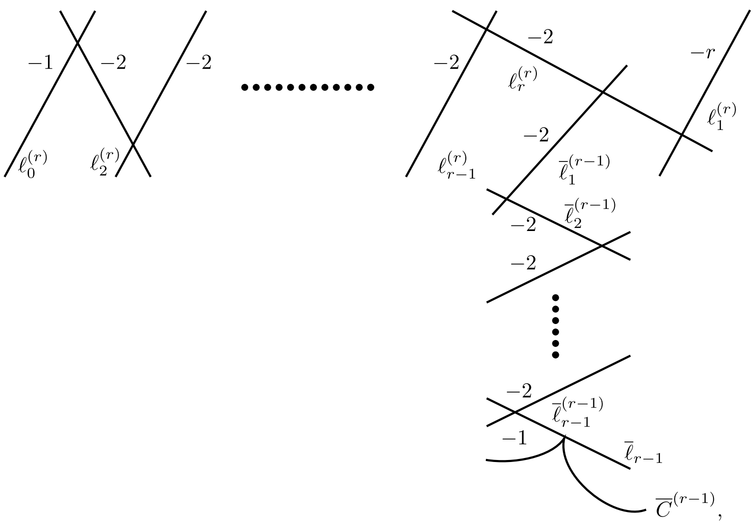
(IV) We set $\overline{V}_0 := V_r$, $\overline{\ell}_0 := \ell_r$, $\overline{C} := C^{(r)}$, $\overline{P}_0 := Q$ and $\overline{\Lambda} := \overline{\Lambda}^{(0)} := \Lambda^{(r)}$. Let $\overline{\sigma}_1 : \overline{V}_1 \to \overline{V}_0$ be the quadratic transformation of $\overline{V}_0$ with center at $\overline{P}_0$, and let $\overline{\ell}_0^{(1)} := \overline{\sigma}_1'(\overline{\ell}_0)$, $\overline{\ell}_1^{(1)} := \overline{\sigma}_1^{-1}(\overline{P}_0)$, $\overline{C}^{(1)} = \overline{\sigma}'(\overline{C})$ and $\overline{\Lambda}^{(1)} = \overline{\sigma}_1'(\overline{\Lambda})$. By abuse of notations, we denote $\overline{\sigma}_1'(\ell_j^{(r)})$ by $\ell_j^{(r)}$ again for $0 \leqq j \leqq r$. Let $\mu_0 := \mathrm{mult}_{\overline{P}_0} \overline{C}$. Then $\mu_0 \leqq d_2$, and $\overline{\Lambda}^{(1)}$ is spanned by $\overline{C}^{(1)}$ and $d_0\ell_0^{(r)} + (d_0 - d_1)\ell_1^{(r)} + d_0(\ell_2^{(r)} + \cdots + \ell_r^{(r)}) + (d_0 - \mu_0)\overline{\ell}_1$. If $\mu_0 < d_2$ the contraction of $\ell_0^{(r)}, \ell_2^{(r)}, \ldots, \ell_{r-1}^{(r)}$ leads us to a contradiction by Lemma 2.3, (4). Thus $\mu_0 = d_2$. If $r > 2$, $\overline{C}^{(1)}$ intersects $\overline{\ell}_1$ in a single point $\overline{Q}_1$; indeed, if otherwise, the contraction of $\ell_0^{(r)}, \ell_2^{(r)}, \ldots, \ell_r^{(r)}$ and blowings-up of two points in $\overline{C}^{(1)} \cap \overline{\ell}_1$ leads us to a contradiction by Lemma 2.3, (4). For $1 \leqq i \leqq r-2$, assume that we obtained inductively $\overline{V}_i$, $\overline{\sigma}_i$, $\overline{\ell}_j^{(i)}(0 \leqq j \leqq i)$, $\ell_s^{(r)}(0 \leqq s \leqq r)$, $\overline{C}^{(i)}$ and $\overline{\Lambda}^{(i)}$, where;

(1) $\overline{\sigma}_i : \overline{V}_i \to \overline{V}_{i-1}$ is the quadratic transformation of $\overline{V}_{i-1}$,

(2) $\overline{\Lambda}^{(i)}$ is spanned by $\overline{C}^{(i)}$ and $d_0\ell_0^{(r)} + (d_0 - d_1)\ell_1^{(r)} + d_0(\ell_2^{(r)} + \cdots + \ell_r^{(r)}) + (d_0 - d_2)\overline{\ell}_1^{(i)} + (d_0 - 2d_2)\overline{\ell}_2^{(i)} + \cdots + (d_0 - id_2)\overline{\ell}_i^{(i)}$,

(3) $\overline{C}^{(i)}$ intersects $\overline{\ell}_i := \overline{\ell}_i^{(i)}$ in a single point $\overline{P}_i$ with $(\overline{C}^{(i)} \cdot \overline{\ell}_i) = d_2$ and $\mu_i = \mathrm{mult}_{\overline{P}_i} \overline{C}^{(i)}$, where $\overline{P}_i \notin \overline{\ell}_{i-1}^{(i)}$.

204 Let $\overline{\sigma}_{i+1} : \overline{V}_{i+1} \to \overline{V}_i$ be the quadratic transformation of $\overline{V}_i$ with center at $\overline{P}_i$, and let $\overline{\ell}_j^{(i+1)} := \overline{\sigma}'_{i+1}(\overline{\ell}_j^{(i)})$ for $0 \leqq j \leqq i$, $\overline{\ell}_{i+1} := \overline{\ell}_{i+1}^{(i+1)} := \overline{\sigma}_{i+1}^{-1}(\overline{P}_i)\overline{C}^{(i+1)} := \overline{\sigma}'_{i+1}(\overline{C}^{(i)})$ and $\overline{\Lambda}^{(i+1)} := \overline{\sigma}'_{i+1}(\overline{\Lambda}^{(i)})$. By abuse of notations we denote $\overline{\sigma}'_{i+1}(\ell_s^{(r)})$ by $\ell_s^{(r)}$ again for $0 \leqq s \leqq r$. Then $\mu_i \leqq d_2$, and $\overline{\Lambda}^{(i+1)}$ is spanned by $\overline{C}^{(i+1)}$ and $d_0\ell_0^{(r)} + (d_0 - d_1)\ell_1^{(r)} + d_0(\ell_2^{(r)} + \cdots + \ell_r^{(r)}) + (d_0 - d_2)\overline{\ell}_1^{(i+1)} + \cdots + (d_0 - id_2)\overline{\ell}_i^{(i+1)} + (d_0 - id_2 - \mu_i)\overline{\ell}_{i+1}$. If $\mu_i < d_2$ the contraction of $\ell_0^{(r)}$, $\ell_2^{(r)}, \ldots, \ell_r^{(r)}, \overline{\ell}_1^{(i+1)}, \ldots, \overline{\ell}_{i-1}^{(i+1)}$ leads us to a contradiction by virtue of Lemma 2.3, (4). Hence $\mu_i = d_2$. If $i \leqq r - 3$, $\overline{C}^{(i+1)}$ intersects $\overline{\ell}_{i+1}$ in a single point $\overline{P}_{i+1} (\notin \overline{\ell}_i^{(i+1)})$; indeed, if otherwise, the contraction of $\ell_0^{(r)}$, $\ell_2^{(r)}, \ldots, \ell_r^{(r)}, \overline{\ell}_1^{(i+1)}, \ldots, \overline{\ell}_i^{(i+1)}$ and blowings-up of two points in $\overline{C}^{(i+1)} \cap \overline{\ell}_{i+1}$ lead us to a contradiction by Lemma 2.3, (4). continuing the above argument we obtain the following configuration on $\overline{V}^{(r-1)}$:



where: **205**

(i) $\overline{\Lambda}^{(r-1)}$ is spanned by $\overline{C}^{(r-1)}$ and $d_0\ell_0^{(r)} + (d_0 - d_1)\ell_1^{(r)} + d_0(\ell_2^{(r)} +$

$$\cdots + \ell_r^{(r)}) + (d_0 - d_2)\overline{\ell}_1^{(r-1)} + \cdots + (d_0 - (r-1)d_2)\overline{\ell}_{r-1}^{(r-1)},$$

(ii) $(\overline{C}^{(r-1)} \cdot \overline{\ell}_{r-1}) = d_2$ and $(\overline{C}^{(r-1)} \cdot \overline{\ell}_{r-2}^{(r-1)}) = 0$.

Let $\sigma : \overline{V}^{(r-1)} \to \mathbb{P}_k^2$ be the composition $\sigma := (\sigma_1 \ldots \sigma_r \cdot \overline{\sigma}_1 \ldots \overline{\sigma}_{r-1})$ and let $\tau : \overline{V}^{(r-1)} \to \mathbb{P}_k^2$ be the contraction of $\ell_0^{(r)}, \ell_2^{(r)}, \ldots, \ell_r^{(r)}, \overline{\ell}_1^{(r-1)}, \ldots,$ $\overline{\ell}_{r-2}^{(r-1)}$ and $\ell_1^{(r)}$ in this order. Then $\rho := \tau \cdot \sigma^{-1}$ is a birational automorphism on $\mathbb{P}_k^2$ such that ρ induces a biregular automorphism on $\mathbb{A}_k^2 := \mathbb{P}_k^2 - \ell_0$ and that $(C' \cdot \ell_0) = (\overline{C}^{(r-1)} \cdot \overline{\ell}_{r-1}) = d_2 \leqq d_1$, where C' is the proper transform of C by ρ. This completes a proof of 6.7.2. □

6.7.3

We shall prove the following assertion:

Assume that $C^{(1)}$ intersects ℓ_1 in two distinct points P_1 and P_1'. Then there exists a birational automorphism ρ of $\mathbb{P}_k^2$ such that ρ induces a biregular automorphism on $\mathbb{A}_k^2 := \mathbb{P}_k^2 - \ell_0$ and that C' intersects ℓ_0 in two distinct points, where C' is the proper transform of C by ρ. Moreover, $(C' \cdot \ell_0) \leqq d_1$.

Proof. Our proof consists of four steps.

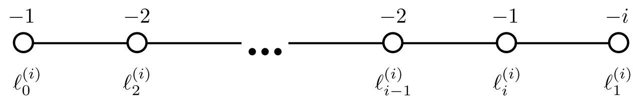
(I) One of P_1 and P_1', say P_1, must be the point $\ell_0^{(1)} \cap \ell_1$; indeed, if otherwise, the pencil $\Lambda^{(1)}$ spanned by $C^{(1)}$ and $d_0\ell_0^{(1)} + (d_0 - d_1)\ell_1$ has no base points on $\ell_0^{(1)}$, which is a contradiction by virtue of Lemma 2.3, (1) because $((\ell_0^{(1)})^2) = 0$. Moreover, both P_1 and P_1' are one-place points of $C^{(1)}$. Let $\mu_1 := i(C^{(1)}, \ell_1; p_1)$ and $\mu_1' :=$
206 $i(C^{(1)}, \ell_1; p_1')$. Then $d_1 = \mu_1 + \mu_1'$. *We shall show that* $\mathrm{mult}_{P_1} C^{(1)} = (C^{(1)} \cdot \ell_0^{(1)}) = d_0 - d_1 \leqq \mu_1$. Indeed, let $\sigma_2 : V_2 \to V_1$ be the quadratic transformation of V_1 with center at P_1, and let $\ell_j^{(2)} := \sigma_2'(\ell_j^{(1)}) (j = 0, 1)$, $\ell_2 := \ell_2^{(2)} := \sigma_2^{-1}(P_1)$, $C^{(2)} := \sigma_2'(C^{(1)})$, and $\Lambda^{(2)} := \sigma_2'(\Lambda^{(1)})$. Let $\nu_1 := \mathrm{mult}_{P_1} C^{(1)}$. Then $\nu_1 \leqq (C^{(1)} \cdot \ell_0^{(1)})$, $\nu_1 \leqq \mu_1$, and $\Lambda^{(2)}$ is spanned by $C^{(2)}$ and $d_0\ell_0^{(2)} + (d_0 - d_1)\ell_1^{(2)} + (2d_0 - d_1 - \nu_1)\ell_2$. Since $(C^{(2)} \cdot \ell_0^{(2)}) = (C^{(1)} \cdot \ell_0^{(1)}) - \nu_1$, $(C^{(2)} \cdot \ell_2) =$

$\nu_1 > 0$ and $((\ell_1^{(2)})^2) = -2$, the equality $\nu_1 = (C^{(1)} \cdot \ell_0^{(1)})$ is implied by Lemma 2.3, (4).

(II) Set $d_2 := \nu_1$[10] and write $d_1 - \mu_1' = \mu_1 = qd_2 + d_3'$ with integers q, d_3' such that $0 \leqq d_3' < d_2$ and $q \geqq 1$. For $2 \leqq i \leqq q+1$ define $V^{(i)}$, σ_i, $\ell_j^{(i)} (0 \leqq j \leqq i)$, $C^{(i)}$, $\Lambda^{(i)}$ and P_i inductively as follows: Let $\sigma_i : V^{(i)} \to V^{(i-1)}$ be the quadratic transformation of $V^{(i-1)}$ with center at P_{i-1}, and let $\ell_j^{(i)} := \sigma_i'(\ell_j^{(i-1)})$ for $0 \leqq j \leqq i-1$, $\ell_i := \ell_i^{(i)} := \sigma_i^{-1}(P_{i-1})$, $C^{(i)} := \sigma_i'(C^{(i-1)})$, $\Lambda^{(i)} := \sigma_i'(\Lambda^{(i-1)})$ and $P_i : \ell_1^{(i)} \cap \ell_i$. By induction on $i (2 \leqq i \leqq q+1)$ we can show the following assertions:

$A_1'(i)$: $\Lambda^{(i)}$ is spanned by $C^{(i)}$ and $d_0\ell_0^{(i)} + (d_0 - d_1)\ell_1^{(i)} + d_0(\ell_2^{(i)} + \cdots + \ell_i^{(i)})$,

$A_2'(i)$: $(C^{(i)} \cdot \ell_j^{(i)}) = 0$ if $0 \leqq j \leqq i-1$ and $j \neq 1$; $i(C^{(i)}, \ell_1^{(i)}; P_i) = \mu_1 - (i-1)d_2$; $(C^{(i)} \cdot \ell_i) = d_2$; $\displaystyle\bigcup_{j=0}^{i} \ell_j^{(i)}$ has the following weighted graph:

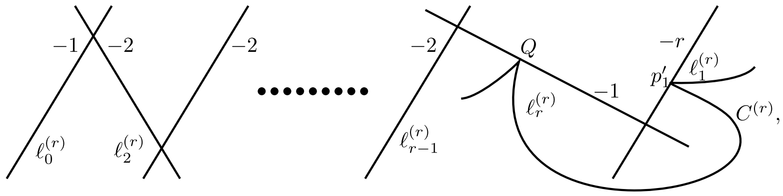


$A_3'(i)$: $C^{(i)}$ intersects ℓ_i in a single point Q, where $Q = P_i$ if either **207**
$2 \leqq i \leqq q$ or $i = q+1$ and $d_3' > 0$.

The proof is the same as that of the step (II) of 6.7.2 up to a slight modification caused by difference of the situations. Hence we leave the readers a task to reproduce it.

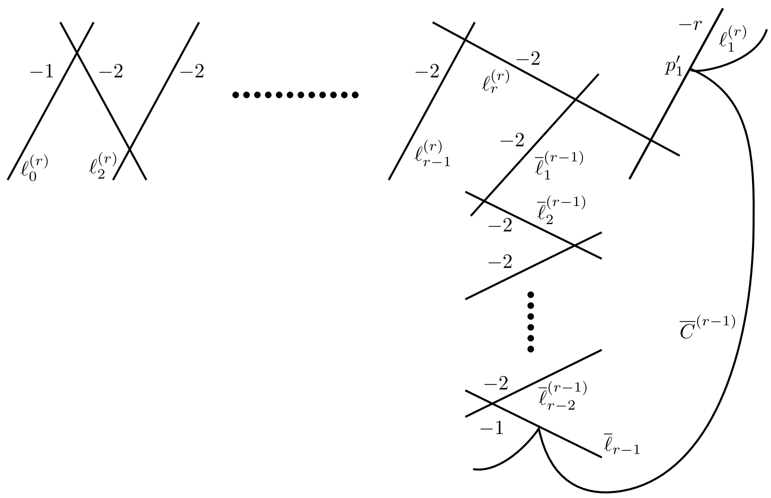
(III) By the same argument as in the proof of the step (III) of 6.7.2, we can show that $d_3' = 0$. Then, setting $r = q+1$, we have $\mu_1 = (r-1)d_2$, $d_1 = (r-1)d_2 + \mu_1'$ and $d_0 = rd_2 + \mu_1'$, where $\mu_1' > 0$. We have the following configuration on $V^{(r)}$:

[10]Note that $d_0 = d_1 + d_2$ and $d_2 < d_1 = \mu_1 + \mu_1'$.



where $(C^{(r)} \cdot \ell_r^{(r)}) = d_2$ and $(C^{(r)} \cdot \ell_1^{(r)}) = \mu'_1$.

(IV) Starting with the quadratic transformation of $V^{(r)}$ with center at Q and following the argument in the step (IV) of the proof of 6.7.2 we obtain the surface $\overline{V}^{(r-1)}$ and the configuration on it: See the next page, where $(\overline{C}^{(r-1)} \cdot \overline{\ell}_{r-1}) = d_2$ and $(\overline{C}^{(r-1)} \cdot \ell_1^{(r)}) = \mu'_1$. Let $\sigma, \tau : \overline{V}^{(r-1)} \to \mathbb{P}^2_k$ be as defined as in the step (IV) of 6.7.2 and let $\rho = \tau \cdot \sigma^{-1}$. Then ρ is a birational automorphism of $\mathbb{P}^2_k$ such that ρ induces a biregular automorphism on $\mathbb{A}^2_k := \mathbb{P}^2_k - \ell_0$ and that $(C' \cdot \ell_0) = d_2 + \mu'_1 \leqq d_1$, where C' is the proper transform of C by ρ. Apparently, C' intersects ℓ_0 in two distinct points. This completes a proof of 6.7.3.



(The configuration in the step (IV) of 6.7.3)

□

6.7.4

It is now apparent that we can finish our proof of Lemma 6.7 by induction on $d_0 := (C \cdot \ell_0)$ and by making use of 6.7.2 and 6.7.3. As the proofs of 6.7.2 and 6.7.3 indicate, we have the following remark:

Remark. Let C, ℓ_0, ρ and C' be as in Lemma 6.7. Let $d_0 := (C \cdot \ell_0)$ and $d'_0 := (C' \cdot \ell_0)$. Let Λ be the linear pencil on $\mathbb{P}^2_k$ spanned by C and $d_0\ell_0$, and let Λ' be the linear pencil on $\mathbb{P}^2_k$ spanned by C' and $d'_0\ell_0$. Then Λ' is the proper transform of Λ by ρ. In particular, if f' is an irreducible element of $k[x, y]$ defining $C' \cap \mathbb{A}^2_k$ and C'_α is the curve on $\mathbb{A}^2_k$ defined by $f' = \alpha$ for $\alpha \in k$, then f' and C'_α's ($\alpha \in k$) satisfy the conditions (1), (2), **209**
(3) of Theorem 6.1.

Thus, *we assume hereafter that C intersects ℓ_0 in two distinct points*, each of which is, therefore, a one-place point of C.

6.8

Let $C \cap \ell_0 = \{P, Q\}$, let $d_0 := i(C, \ell_0; P)$ and let $e_0 := i(C, \ell_0; Q)$. We may assume that $d_0 \leqq e_0$.

6.8.1

Lemma. *With the notations as above, we have $d_0 = \mathrm{mult}_P\, C$ and $e_0 = \mathrm{mult}_Q\, C$.*

Proof. Let $\mu := \mathrm{mult}_P\, C$ and $\nu := \mathrm{mult}_Q\, C$. Let $\sigma_1 : V_1 \to V_0 := \mathbb{P}^2_k$ be the quadratic transformation of V_0 with centers at P and Q, and let $\ell_0^{(1)} := \sigma'_1(\ell_0)$, $E_1 := \sigma_1^{-1}(P)$ and $F_1 := \sigma_1^{-1}(Q)$. Then, since $C \sim (d_0 + e_0)\ell_0$ we have: $C^{(1)} := \sigma'_1(C) \sim (d_0+e_0)\ell_0^{(1)} + (d_0+e_0-\mu)E_1 + (d_0+e_0-\nu)F_1$. If $d_0 > \mu$ or $e_0 > \nu$ we have a contradiction by virtue of Lemma 2.3, (4) because $(C^{(1)} \cdot \ell_0^{(1)}) = d_0 + e_0 - (\mu + \nu) > 0$, $(C^{(1)} \cdot E_1) = \mu > 0$, $(C^{(1)} \cdot F_1) = \nu > 0$ and $((\ell_0^{(1)})^2) = (E_1^2) = (F_1^2) = -1$. Thus $d_0 = \mu$ and $e_0 = \nu$. □

6.8.2

By substituting $f - \alpha(\alpha \in k)$ for f if necessary we may (hence shall) assume hereafter that if $\sigma : W \to \mathbb{P}_k^2$ is the shortest composition of quadratic transformations by which the proper transform $\sigma'\Lambda$ of the pencil Λ on $\mathbb{P}_k^2$ spanned by C and $(d_0 + e_0)\ell_0$ has no base points the member of $\sigma'\Lambda$ corresponding to C is irreducible (cf. a remark at the beginning of 6.7). Then we have the following:

210 **Lemma.** *Let* $P_1 := C^{(1)} \cap E_1$, $Q_1 := C^{(1)} \cap F_1$, $\mu_1 := \mathrm{mult}_{P_1} C^{(1)}$ *and* $\nu_1 := \mathrm{mult}_{Q_1} C^{(1)}$. *Then either* $d_0 = 1$ *or* $e_0 > d_0 = \mu_1 \geqq \nu_1$.

Proof. Let $\sigma_2 : V_2 \to V_1$ be the quadratic transformation of V_1 with centers at P_1 and Q_1, and let $\ell_0^{(2)} = \sigma_2'(\ell_0^{(1)})$, $E_1^{(2)} := \sigma_2'(E_1)$, $F_1^{(2)} := \sigma_2'(F_1)$, $E_2 := \sigma_2^{-1}(P_1)$, $F_2 := \sigma_2^{-1}(Q_1)$ and $C^{(2)} := \sigma_2'(C^{(1)})$. Then, since $C^{(1)} \sim (d_0 + e_0)\ell_0^{(1)} + e_0 E_1 + d_0 F_1$, we have:

$$C^{(2)} \sim (d_0 + e_0)\ell_0^{(2)} + e_0 E_1^{(2)} + d_0 F_1^{(2)} + (e_0 - \mu_1)E_2 + (d_0 - \nu_1)F_2,$$

where we must have $e_0 \geqq \mu_1$ and $d_0 \geqq \nu_1$. If $e_0 > \nu_1$ and $d_0 > \mu_1$ the contraction of $\ell_0^{(2)}$ leads us to a contradiction by Lemma 2.3, (4). Hence either $e_0 = \nu_1$ or $d_0 = \mu_1$. If $e_0 = \nu_1$ then $d_0 = \nu_1$. Hence F_2 is a quasi-section of the pencil $(\sigma_1\sigma_2)'\Lambda$. Since every member of $(\sigma_1\sigma_2)'\Lambda$ has a one-place point on F_2 and since the characteristic of k is zero, we conclude that $d_0 = (C^{(2)} \cdot F_2) = 1$. Thus, if $d_0 > 1$ then $e_0 > \nu_1$ and $d_0 = \mu_1$; moreover, we have $e_0 > d_0$ because if $e_0 = d_0 (= \mu_1)$ then E_2 is a quasi-section of $(\sigma_1\sigma_2)'\Lambda$ and thence we conclude that $e_0 = d_0 = 1$. □

6.8.3

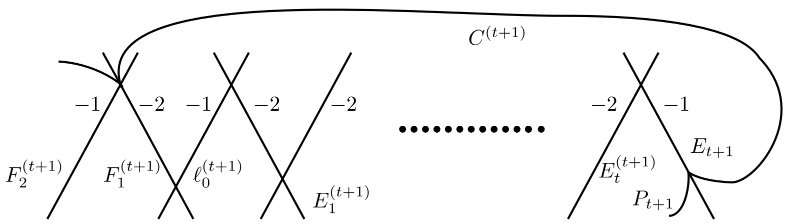
Assume now that $d_0 > 1$. Let $P_2 := C^{(2)} \cap E_2$ and let $P_2, \ldots, P_{t+1}$ be the points of $C^{(2)}$ over P_2, P_i being infinitely near to P_{i+1} of order one, such that if μ_i is the multiplicity of $C^{(2)}$ at P_i we have $d_0 = \mu_1 = \ldots = \mu_t > \mu_{t+1}$. Then we have the following:

Lemma. *With the notations as above, we have* $e_0 - td_0 > 0$.

211 *Proof.* If $t = 1$ we have nothing to show. Assume that $t \geqq 2$. For $2 < i \leqq t+1$, define V_i, σ_i, $\ell_0^{(i)}$, $E_j^{(i)}(1 \leqq j \leqq i)$, $F_j^{(i)}(j = 1, 2)$, $C^{(i)}$ and $\Lambda^{(i)}$ inductively as follows: Let $\sigma_i : V_i \to V_{i-1}$ be the quadratic transformation of V_{i-1} with center at P_{i-1}, and let $\ell_0^{(i)} := \sigma_i'(\ell_0^{(i-1)})$, $E_j^{(i)} := \sigma_i'(E_j^{(i-1)})$ for $1 \leqq j < i$, $E_i := E_i^{(i)} := \sigma_i^{-1}(P_{i-1})$, $F_j^{(i)} := \sigma_i'(F_j^{(i-1)})$ for $j = 1, 2, C^{(i)} := \sigma_i'(C^{(i-1)})$ and $\Lambda^{(i)} := \sigma_i'(\Lambda^{(i-1)})$ (where $\Lambda^{(2)} := (\sigma_1\sigma_2)'\Lambda$). Assume that for $2 \leqq i \leqq t$, $\Lambda^{(i)}$ is spanned by $C^{(i)}$ and $(d_0+e_0)\ell_0^{(i)} + d_0F_1^{(i)} + (d_0-\nu_1)F_2^{(i)} + e_0E_1^{(i)} + (e_0-d_0)E_2^{(i)} + \cdots + (e_0-(i-1)d_0)E_i^{(i)}$, where $e_0 > (i-1)d_0$, $(C^{(i)} \cdot E_{i-1}^{(i)}) = 0$ and $C^{(i)} \cdot E_i^{(i)} = d_0 \cdot P_i$. Then it is easy to see that $\Lambda^{(i+1)}$ is spanned by $C^{(i+1)}$ and $(d_0+e_0)\ell_0^{(i+1)} + d_0F_1^{(i+1)} + (d_0-\nu_1)F_2^{(i+1)} + e_0E_1^{(i+1)} + (e_0-d_0)E_2^{(i+1)} + \cdots + (e_0-id_0)E_{i+1}^{(i+1)}$, where $e_0 \geqq id_0$, $(C^{(i+1)} \cdot E_i^{(i+1)}) = 0$ and $(C^{(i+1)} \cdot E_{i+1}^{(i+1)}) = \mu_i = d_0$. If $e_0 = id_0$, then E_{i+1} is a quasi-section of $\Lambda^{(i+1)}$. Since every member of $\Lambda^{(i+1)}$ has a one-place point on E_{i+1}, we have $d_0 = 1$, which contradicts the assumption. Hence $e_0 > \mathrm{id}_0$. In particular, we know by induction on $2 \leqq i \leqq t$ that $\Lambda^{(t+1)}$ is spanned by $C^{(t+1)}$ and $(d_0+e_0)\ell_0^{(t+1)} + d_0F_1^{(t+1)} + (d_0-\nu_1)F_2^{(t+1)} + e_0E_1^{(t+1)} + (e_0-d_0)E_2^{(t+1)} + \cdots + (e_0-td_0)E_{t+1}^{(t+1)}$, where $e_0 > td_0$. □

6.8.4

With the notations in 6.8.3, it is easily checked that we have the following configuration:



where $(C^{(t+1)} \cdot F_2^{(t+1)}) = \nu_1$, $(C^{(t+1)} \cdot F_1^{(t+1)}) = e_0 - \nu_1$, $(C^{(t+1)} \cdot E_{t+1}) = d_0$ 212
and $\mu_{t+1} = \mathrm{mult}_{P_{t+1}} C^{(t+1)}$ with $e_0 > \nu_1$ and $d_0 > \mu_{t+1}$. Now let $\tau : V_{t+1} \to V$ be the contraction of $F_2^{(t+1)}$, $\ell_0^{(t+1)}$, $E_1^{(t+1)}, \ldots, E_t^{(t+1)}$, and let

$F_0 := \tau(F_1^{(t+1)})$, $E_0 := \tau(E_{t+1})$, $\overline{C} := \tau(C^{(t+1)})$ and $\overline{\Lambda} := \tau_*(\Lambda^{(t+1)})$. Then it is easy to show the following assertions:

(1) $\overline{\Lambda}$ is spanned by $\overline{C}$ and $d_0F_0 + (e_0 - td_0)E_0$, where $e_0 > td_0$,

(2) $(E_0^2) = 0$, $(F_0^2) = t$ and $(E_0 \cdot F_0) = 1$,

(3) $\overline{C} \cdot E_0 = d_0 \cdot P_0$ and $\overline{C} \cdot F_0 = e_0 \cdot Q_0$, where $P_0 \notin F_0$ and $Q_0 \notin E_0$,

(4) $d_1 := \mathrm{mult}_{P_0} \overline{C} = \mu_{t+1} < d_0$ and $e_1 := \mathrm{mult}_{Q_0} \overline{C} = \nu_1 < e_0$, where $e_0 > d_0 \geqq e_1$ (cf. 6.8.2).

6.9

In the paragraphs 6.9 ~ 6.13 we assume that $d_0 > 1$ and use the notations set forth in the assertions (1) ~ (4) of 6.8.4. Find integers $d_2, \ldots, d_m$ and $p_1, \ldots, p_m$ by the Euclidea algorithm with respect to d_0 and d_1:

$$\begin{aligned} d_0 &= p_1d_1 + d_2 & \qquad & 0 < d_2 < d_1 \\ d_1 &= p_2d_2 + d_3 & & 0 < d_3 < d_2 \\ &\cdots\cdots & & \\ d_{m-2} &= p_{m-1}d_{m-1} + d_m & & 0 < d_m < d_{m-1} \\ d_{m-1} &= p_md_m & & 1 < p_m \end{aligned}$$

Similarly, find integers $e_2, \ldots, e_n$ and $q_1, \ldots, q_n$ by the Euclidean algorithm with respect to e_0 and e_1 :

$$\begin{aligned} e_0 &= q_1e_1 + e_2 & \qquad & 0 < e_2 < e_1 \\ e_1 &= q_2e_2 + e_3 & & 0 < e_3 < e_2 \\ &\cdots\cdots & & \\ e_{n-2} &= q_{n-1}e_{n-1} + e_n & & 0 < e_n < e_{n-1} \\ e_{n-1} &= q_ne_n & & 1 < q_n \end{aligned}$$

213 As in 1.4, define an integer $a(i, j)(1 \leqq i \leqq m; 1 \leqq j \leqq p_i)$ inductively as follows:

$$a_0 = e_0 - td_0$$

$$
\begin{aligned}
a(1, j) &= j(a_0 - d_1) && \text{for } 1 \leqq j \leqq p_1 \\
a(2, j) &= a_0 + j(a(1, p_1) - d_2) && \text{for } 1 \leqq j \leqq p_2 \\
&\cdots\cdots \\
a(i, j) &= a(i-2, p_{i-2}) + j(a(i-1, p_{i-1}) - d_i) && \text{for } 1 \leqq j \leqq p_i \\
& && \text{and } 2 \leqq i \leqq m.
\end{aligned}
$$

Similarly, define an integer $b(i, j)(1 \leqq i \leqq n; 1 \leqq j \leqq q_i)$ inductively as follows:

$$
\begin{aligned}
b_0 &= d_0 \\
b(1, j) &= j(b_0 - e_1) && \text{for } 1 \leqq j \leqq q_1 \\
b(2, j) &= b_0 + j(b(1, q_1) - e_2) && \text{for } 1 \leqq j \leqq q_2 \\
&\cdots\cdots \\
b(i, j) &= b(i-2, q_{i-2}) + j(b(i-1, q_{i-1}) - e_i) && \text{for } 1 \leqq j \leqq q_i \\
& && \text{and } 2 \leqq i \leqq n.
\end{aligned}
$$

Then we have the following:

Lemma. *With the notations as above, we have:*

$$a(m, p_m)d_m = d_0(a_0 - d_1) \quad \textit{and} \quad b(n, q_n)e_n = e_0(b_0 - e_1).$$

In particular, $a(m, p_m) \neq 1$ *and* $b(n, q_n) \neq 1$. **214**

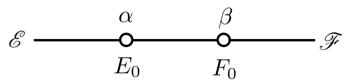
Proof. The first equalities are obtained by straightforward computations (cf. the proof of Lemma 1.4.1, (6)). As for the second assertion, assume that $a(m, p_m) = 1$. Then $d_m \geqq d_0$, which is absurd because $d_0 > d_1 \geqq d_m$. Hence $a(m, p_m) \neq 1$. Similarly, $b(n, q_n) \neq 1$. □

6.10

Set $M : p_1 + \cdots + p_m$. Let $P_0, P_1, \ldots, P_{M-1}$ be the points of $\overline{C}$ over P_0, P_i being infinitely near to P_{i-1} of order one for $1 \leqq i \leqq M - 1$. Let $\sigma_i : V_i \to V_{i-1}$ be the quadratic transformation of V_{i-1} with center at P_{i-1} for $1 \leqq i \leqq m$.[11] The composition $\rho_1 = \sigma_1 \ldots \sigma_M : V_M \to V_0 := V$

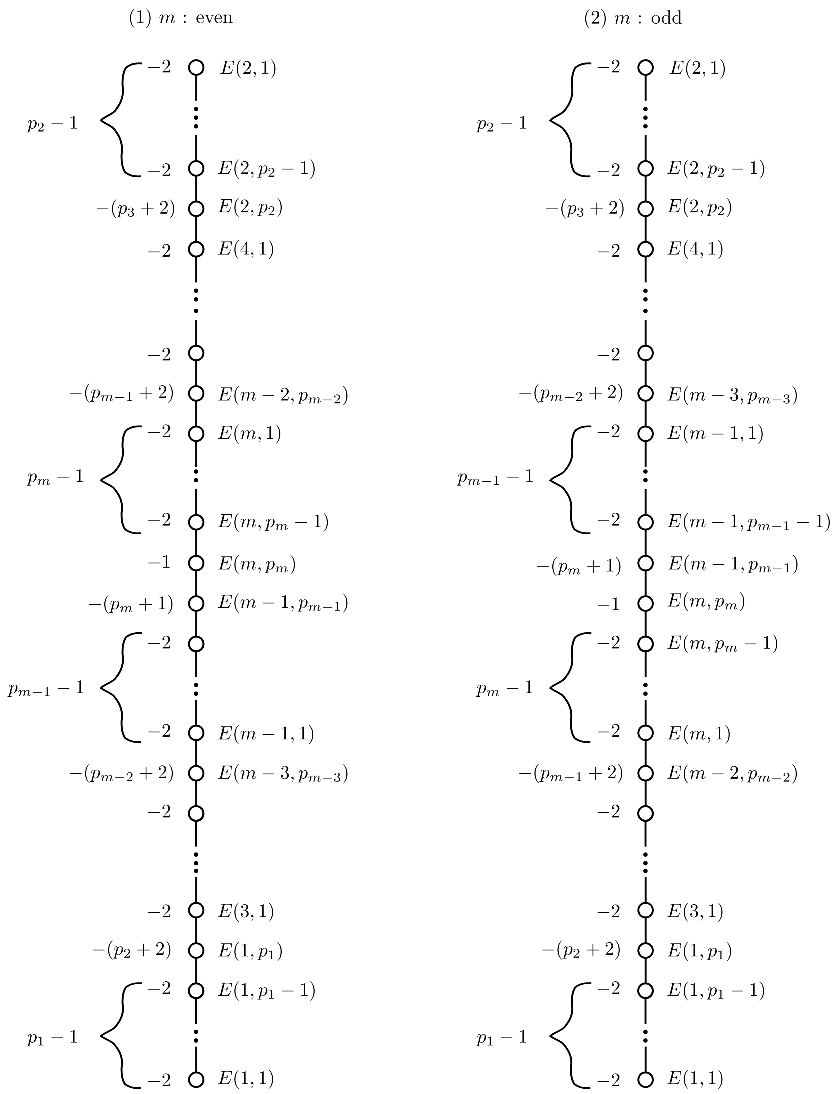
[11] By abuse of (and also for the sake of simplifying) the notations, we use these notations though they overlap in part those introduced in 6.8.1 ~ 6.8.3.

is called *the Euclidean transformation with respect to* $(\overline{C}, P_0)$ (cf. 1.3.1). Let $\rho_2 : V_{M+N} \to V_M$ be the Euclidean transformation with respect to $(\rho_1'(\overline{C}), \rho_1^{-1}(Q_0))$, where $N = q_1 + \cdots + q_n$. Let $W := V_{M+N}$ and let $\rho := \rho_1\rho_2 : W \to V$ be the composition of ρ_1 and ρ_2. Let $C' := \rho'(\overline{C})$ and $\Lambda' := \rho'(\overline{\Lambda})$. By abuse of notations, we denote $\rho'(E_0)$ and $\rho'(F_0)$ by E_0 and F_0 again, respectively. Then it is by a straightforward computation that we obtain the following weighted graph of $\rho^{-1}(E_0 \cup F_0)$:



where $\mathscr{E}$ and $\mathscr{F}$ are the graphs similar to that in the Figure 1 of 1.3.4 and where $\alpha = -(p_1 + 1)$ if $m > 1$ and $\alpha = -p_1$ if $m = 1$,

Figure 2 : The weighted graph $\mathscr{E}$



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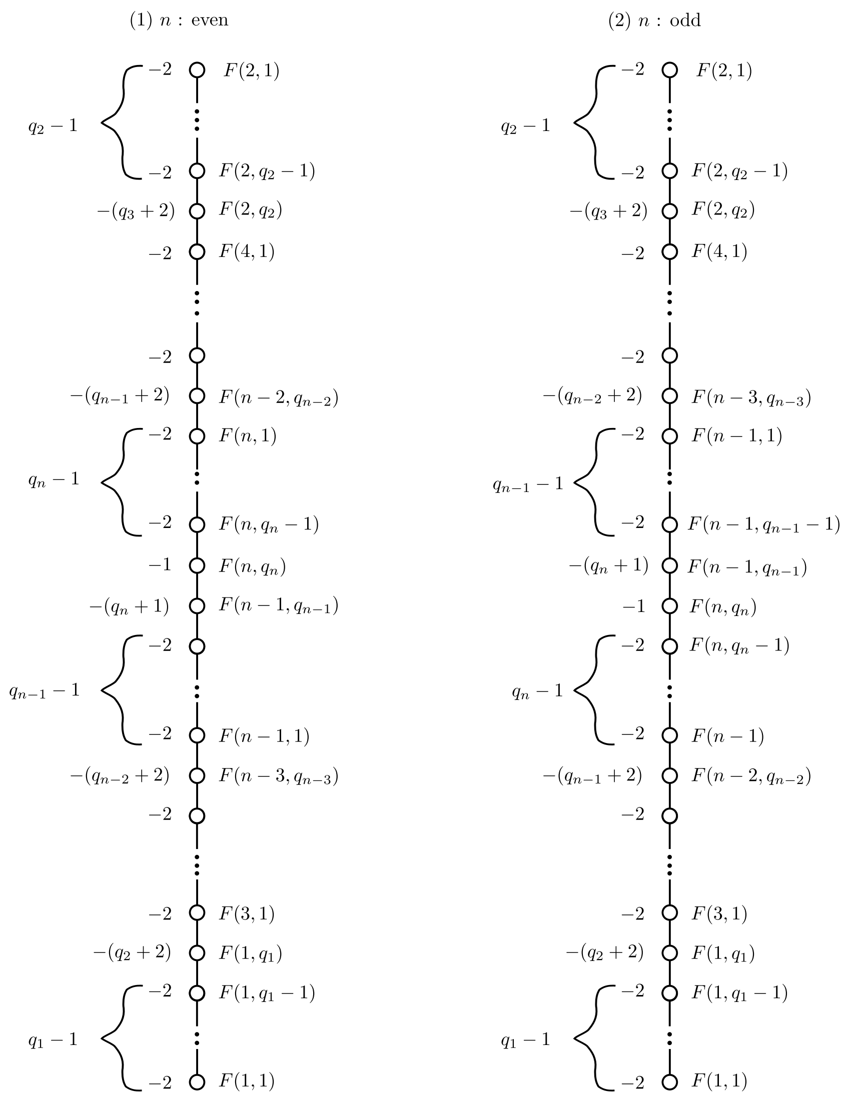
where $E(2,1)$ is linked to E_0, C' intersects $E(m,p_m)$ but not other components, and $(C' \cdot E(m,p_m)) = d_m$.

Figure 3 : The weighted graph $\mathscr{F}$



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where $F(2,1)$ is linked to F_0, C' intersects $F(n, q_n)$ but not other
217 components, and $(C' \cdot F(n, q_n)) = e_n$. $\beta = t - (q_1 + 1)$ if $n > 1$ and $\beta = t - q_1$ if $n = 1$. In order to keep the notations in accordance with the

present ones, we shall write down the graphs $\mathscr{E}$ and $\mathscr{F}$ in the Figures 2 and 3, which are given in the next two pages.

6.11

With the notations of 6.9 and 6.10, we have the following:

Lemma. (1) *The linear pencil Λ' is spanned by C' and*

$$\Delta' := a_0E_0 + b_0F_0 + \sum_{i=1}^{m}\sum_{j=1}^{p_i} a(i,j)E(i,j) + \sum_{i=1}^{n}\sum_{j=1}^{q_i} b(i,j)F(i,j)$$

(2) *$a_0 > 0$ and $a(i,j) \geqq 0$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq p_i$; moreover, if $E(i,j)$ lies between E_0 and $E(m,p_m)$ (excluding $E(m,p_m)$) in the graph $\mathscr{E}$ then the multiplicity $a(i,j) > 0$.*

(3) *$b_0 > 0$ and $b(i,j) \geqq 0$ for $1 \leqq i \leqq n$ and $1 \leqq j \leqq q_i$; moreover, if $F(i,j)$ lies between F_0 and $F(n,q_n)$ (excluding $F(n,q_n)$) in the graph $\mathscr{F}$ then the multiplicity $b(i,j) > 0$.*

Proof. (1) By a straightforward computation we obtain $C' \sim \Delta'$. By the assumption at the beginning of 6.8.2, C' is an irreducible member of Λ'. Hence the assertion (1) holds.

(2) Since Λ' consists of effective divisors we know that $a(i,j) \geqq 0$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq p_i$. Besides, $a_0 = e_0 - td_0 > 0$ (cf. 6.8.3). If $E(i,j) \neq E(m,p_m)$, then $(C' \cdot E(i,j)) = 0$, which implies that $E(i,j)$ is an irreducible component of a member of Λ'. Especially, if $E(i,j)$ lies between E_0 and $E(m,p_m)$ in the graph $\mathscr{E}$, it is readily seen that $E(i,j)$ is connected to E_0 through the components of Δ'. Hence $E(i,j)$ is a component of Δ', i.e., $a(i,j) > 0$.

(3) The assertion (3) is proved by the same argument as above. **218**

□

6.12

Lemma. (1) *Assume that* $a(1,1) = 0$. *Then* $a(m, p_m) = 0$ *and* $a(i, j) = 0$ *whenever* $E(i, j)$ *lies between* $E(1,1)$ *and* $E(m, p_m)$ *in the graph* $\mathscr{E}$; *such* $E(i, j)$*'s with* $a(i, j) = 0$ *(excluding* $E(m, p_m)$*) are contained in one and only one member of* Λ'; $E(m, p_m)$ *is a cross-section of* Λ', *esp.* $d_m = 1$.

(2) *Assume that* $b(1,1) = 0$. *Then* $b(n, q_n) = 0$ *and* $b(i, j) = 0$ *whenever* $F(i, j)$ *lies between* $F(1,1)$ *and* $F(n, q_n)$ *in the graph* $\mathscr{F}$; *such* $F(i, j)$*'s with* $b(i, j) = 0$ *(excluding* $F(n, q_n)$*) are contained in one and only one member of* Λ'; $F(n, q_n)$ *is a cross-section of* Λ', *esp.* $e_n = 1$.

Proof. We shall prove only the assertion (1) because the assertion (2) is proved in a similar fashion. The assumption $a(1,1) = 0$ implies that $a(m, p_m) = 0$ (cf. Lemma 6.9) and that $E(1,1)$ is contained in a member of Λ' other than Δ' (and C', of course). If $E(i, j)$ ($\neq E(m, p_m)$) lies between $E(1,1)$ and $E(m, p_m)$ in the graph $\mathscr{E}$ then it is readily seen that $E(i, j)$ is contained in the same member of Λ' as $E(1,1)$ is, which implies $a(i, j) = 0$. Moreover, we know that $E(m, p_m)$ is a quasi-section of Λ'. Since $(C' \cdot E(m, p_m)) = d_m$ and every member of Λ' has a one-place point on $E(m, p_m)$, we know that $d_m = 1$. ◻

6.13

We shall prove the following:

Lemma. *With the notations as above, we have:*

219 (1) $a(1,1) = b(1,1) = 0$,

(2) $d_0 = e_1$, $d_1 = e_2$, $n = m + 1$ *and* $(d_0, e_0) = 1$.

Proof. Our proof consists of four subparagraphs 6.13.1 ∼ 6.13.4. ◻

6.13.1

Assume that $a(1,1) > 0$ and $b(1,1) > 0$. Then it is clear from the arguments in the previous paragraphs that the following assertions hold true:

1° $a(i,j) > 0$ for every pair (i,j) with $1 \leqq i \leqq m$ and $1 \leqq j \leqq p_i$; similarly, $b(i,j) > 0$ for every pair (i,j) with $1 \leqq i \leqq n$ and $1 \leqq j \leqq q_i$,

2° $a(m, p_m) \neq 1$ and $b(n, q_n) \neq 1$ (cf. Lemma 6.9),

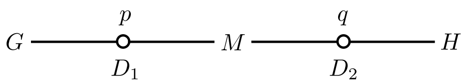
3° the set B' of base points of the pencil Λ' consists of two points P' and Q' lying on $E(m, p_m)$ and $F(n, q_n)$, respectively, such that $P' \notin$ the components of Δ' other than $E(m, p_m)$ and $Q' \notin$ the components of Δ' other than $F(n, q_n)$,

4° all components of Δ' except $E(m, p_m)$, $F(n, q_n)$ and F_0 have self-intersection multiplicities $\leqq -2$.

Since Λ' is a linear pencil of rational curves as assumed, Lemma 2.3, (4) implies that $\beta = -1$, i.e., F_0 is contractible. Let $\tau_1 : W \to W_1$ be the contraction of the components $F_0, F(2,1), \ldots, F(2, q_2 - 1)$. Then $(\tau_1(E_0)^2) = \alpha + q_2$ and $((\tau_1 F(2, q_2))^2) = -(q_3 + 1) \leqq -2$; a unique contractible component of $\tau_{1^*}(\Delta')$ is $\tau_1(E_0)$, i.e., $\alpha + q_2 = -1$. Let $\tau_2 : W_1 \to W_2$ be the contraction of $\tau_1(E_0), \tau_1(E(2,1)), \ldots, \tau_1(E(2, p_2 - 1))$. Then $((\tau_2\tau_1 F(2, q_2))^2) = -(q_3 - p_2 + 1)$ and $((\tau_2\tau_1 E(2, p_2))^2) = -(p_3 + 1)$; a unique contractible component of $(\tau_2\tau_1)_*\Delta'$ is $\tau_2\tau_1 F(2, q_2)$. We repeat the contractions of this kind as far as we can. Let $\tau : W \to$ **220**
Z be the contraction of all possible components of Δ' lying between $E(m, p_m)$ and $F(n, q_n)$ of the weighted graph of Δ' (excluding $E(m, p_m)$ and $F(n, q_n)$). Then the pencil $\tau_*\Lambda'$ (= the proper transform of Λ' by τ), which is spanned by $\tau(C')$ and $\tau_*\Delta'$, satisfies the same properties as the pencil observed in 6.4.

6.13.2

Set $D_1 := \tau(E(m, p_m))$, $D_2 := \tau(F(n, q_n))$, $p := (D_1^2)$ and $q := (D_2^2)$. Write the weighted graph of $\tau_*(\Delta')$ in the form:



where:

1° G coincides with the subgraph of $\mathscr{E}$ between $E(1, 1)$ and $E(m, p_m)$ (excluding $E(m, p_m)$); hence $G \neq \phi$,

2° H coincides with the subgraph of $\mathscr{F}$ between $F(1, 1)$ and $F(n, q_n)$ (excluding $F(n, q_n)$); hence $H \neq \phi$,

3° M is the weighted graph of the images by τ of the components of Δ' which lie between $E(m, p_m)$ and $F(n, q_n)$ in the weighted graph of Δ' (excluding $E(m, p_m)$ and $F(n, q_n)$); M might be empty,

4° either $p \leqq 0$ or $q \leqq 0$.

Only the assertion 4° needs a proof. Assume that $p > 0$ and $q > 0$. Then $M = \phi$. However, by the contraction τ, either the component of Δ' next to $E(m, p_m)$ and not belonging to G (i.e., $E(m, p_m - 1)$ if m is even;
221 $E(m-1, p_{m-1})$ if m is odd and $m > 1$; E_0 if $m = 1$) or the component of Δ' next to $F(n, q_n)$ and not belonging to H (i.e., $F(n, q_n - 1)$ if n is even; $F(n-1, q_{n-1})$ if n is odd and $n > 1$; F_0 if $n = 1$) is contracted last. Then $p = 0$ or $q = 0$, which is a contradiction. Hence either $p \leqq 0$ or $q \leqq 0$. Now, noting that $a(m, p_m) \neq 1$ and $b(n, q_n) \neq 1$ (cf. 6.13.1, 2°), we know by Lemma 6.5 that either $G = \phi$ or $H = \phi$. This is a contradiction. Therefore we have either $a(1, 1) = 0$ or $b(1, 1) = 0$.

6.13.3

Assume that $a(1, 1) = 0$ and $b(1, 1) > 0$. Then the following assertions hold true:

1° $a(i, j) > 0$ if $E(i, j)$ lies between E_0 and $E(m, p_m)$ (excluding $E(m, p_m)$) in the graph $\mathscr{E}$ and $a(i, j) = 0$ if otherwise; $b(i, j) > 0$ for every pair (i, j) with $1 \leqq i \leqq n$ and $1 \leqq j \leqq q_i$, 2° set $D_1 := F_0$, $D_2 := F(n, q_n)$, $p := \beta = (D_1^2)$ and $q = -1 = (D_2^2)$; then Δ' has the following weighted graph:

$$G \;\frac{\qquad}{}\; \underset{D_1}{\overset{p}{\circ}} \;\frac{\qquad}{}\; M \;\frac{\qquad}{}\; \underset{D_2}{\overset{q}{\circ}} \;\frac{\qquad}{}\; H,$$

where:

(i) G is the weighted graph consisting of E_0 and components of $\mathscr{E}$ lying between E_0 and $E(m, p_m)$ (excluding $E(m, p_m)$),

(ii) M coincides with the subgraph of $\mathscr{F}$ between F_0 and $F(n, q_n)$ (excluding $F(n, q_n)$),

(iii) H coincides with the subgraph of $\mathscr{F}$ between $F(1, 1)$ and $F(n, q_n)$ (excluding $F(n, q_n)$); hence $H \neq \phi$;

3° the set B' of base points of Λ' consists of a single point Q' on D_2 but not on the other components of Δ',

4° $b(n, q_n) \neq 1$ (cf. 6.13.1, 2°), **222**

5° the multiplicity $a(m, p_m - 1) = d_m = 1$ if m is even; $a(m - 1, p_{m-1}) = d_m = 1$ if m is odd and $m > 1$; $a_0 = d_1 = 1$ if $m = 1$ (cf. Lemma 6.12, (1)).

We shall apply Lemma 6.6 to the present situation. First, we know that $p = -1$, i.e., $t = q_1$, because $q = -1$ (cf. Lemma 6.6, (1) or (2)). Since $H \neq \phi$ and $b(n, q_n) \neq 1$, Lemma 6.6, (3) implies that any component in the graphs G and M as well as D_1 has multiplicity > 1 in Δ'. However this contradicts the assertion 5° above.

Therefore, the case where $a(1, 1) = 0$ and $b(1, 1) > 0$ does not occur. Similarly, we can show that the case where $a(1, 1) > 0$ and $b(1, 1) = 0$ does not occur.

6.13.4

We have thus proved that $a(1, 1) = b(1, 1) = 0$. Hence Λ' has no base points, and $E(m, p_m)$ and $F(n, q_n)$ are cross-sections of Λ', i.e., $d_m =$

$e_n = 1$. By definition of $a(1,1)$ and $b(1,1)$ (cf. 6.9), the equalities $a(1,1) = b(1,1) = 0$ imply that $e_0 = td_0 + d_1$ and $d_0 = e_1$, where $t = q_1$ if $n > 1$ and $t = q_1 - 1$ if $n = 1$[12]. If $n = 1$ and $t = q_1 - 1$ then $d_1 = e_1 = d_0$, which is a contradiction as $d_0 > d_1$. Hence, $n > 1$ and $t = q_1$. Then, since $e_0 = q_1 e_1 + d_1$, we have $d_1 = e_2$. This implies that $n = m + 1$, $d_0 = e_1, d_1 = e_2, \ldots, d_m = e_n$, and $p_1 = q_2, \ldots, p_m = q_n$. In particular, $(d_0, e_0) = e_n = 1$. This completes a proof of Lemma 6.13.

6.14

Returning to the situation of 6.8.1, we shall assume in this paragraph that $d_0 = 1$. Let $\tau : V_1 \to V$ be the contraction of $\ell_0^{(1)}$, and let $E_0 := \tau(E_1)$,
223 $F_0 := \tau(F_1)$, $\overline{C} := \tau(C^{(1)})$ and $\overline{\Lambda} := \tau_*(\sigma_1'\Lambda)$. Then we have:

(1) $\overline{\Lambda}$ is spanned by $\overline{C}$ and $e_0 E_0 + F_0$,

(2) $(E_0^2) = (F_0^2) = 0$ and $(E_0 \cdot F_0) = 1$; thence V is isomorphic to $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ whose two distinct fibrations by $\mathbb{P}_k^1$ are given by the linear pencils $|E_0|$ and $|F_0|$,

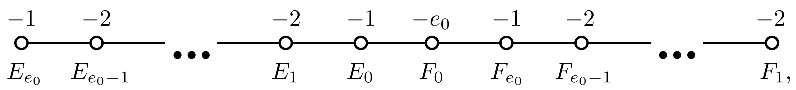
(3) $\overline{C} \cdot E_0 = P_0$ and $\overline{C} \cdot F_0 = e_0 \cdot Q_0$, where $P_0, Q_0 \neq E_0 F_0$.

We shall show the following:

Lemma. (4) *$\overline{C}$ is nonsingular.*

(5) *Let $\rho : W \to V$ be the shortest composition of quadratic transformations by which the proper transform $\Lambda' := \rho'(\overline{\Lambda})$ of $\overline{\Lambda}$ by ρ has no base points. Then we have:*

(i) $\rho^{-1}(E_0 \cup F_0)$ *has the following weighted graph:*



where, by abuse of notations, we denote $\rho'(E_0)$ and $\rho'(F_0)$ by E_0 and F_0, respectively,

[12]Since F_0 must be an exceptional component of Δ' we have $\beta = -1$.

(ii) Λ' *is spanned by* $C' := \rho'(\overline{C})$ *and* $\Delta' := \sum_{i=0}^{e_0-1}(e_0 - i)E_i + F_0$,

(iii) E_{e_0} *and* F_{e_0} *are cross-sections of* Λ',

(iv) $F_1, \ldots, F_{e_0-1}$ *are contained in one and only one member of* Λ'.

Proof. (4) Let E be a member of $|E_0|$ such that $Q_0 \in E_0$. Then, since E_0 is isomorphic to $\mathbb{P}^1_k$ and $(\overline{C} \cdot E) = 1$, $\overline{C}$ is non-singular at Q_0. $\overline{C}$ is apparently nonsingular at other points.

(5) follows from a straightforward computation.

□

6.15

Let W, Λ', C' and Δ' be as in 6.10 (and 6.11) or 6.14. Since Λ' has **224** no base points, Λ' defines a surjective morphism $\varphi : W \to \mathbb{P}^1_k$ whose fibers are members of Λ'. Set $S_1 := E(m, p_m)$ and $S_2 := F(n, q_n)$ if $d_0 > 1$; set $S_1 := E_{e_0}$ and $S_2 := F_{e_0}$ if $d_0 = 1$. Then both S_1 and S_2 are cross-sections (cf. Lemmas 6.12 and 6.14). When $d_0 > 1$, let R_1 be the union of $E(i, j)$'s which lie between $E(1, 1)$ and $E(m, p_m)$ in the graph $\mathscr{E}$ (with $F(1, 1)$ included and $E(m, p_m)$ excluded), and let R_2 be the union of $F(i, j)$'s which lie between $F(1, 1)$ and $F(n, q_n)$ in the graph $\mathscr{F}$ (with $F(1, 1)$ included and $F(n, q_n)$ excluded). Note that $R_1 \neq \phi$ and $R_2 \neq \phi$. When $d_0 = 1$ and $e_0 > 1$, let R be the union of $F_1, \ldots, F_{e_0-1}$. Let Γ_1 and Γ_2 be the fibers of φ containing R_1 and R_2, respectively if $d_0 > 1$; let Γ be the fiber of φ containing R if $d_0 = 1$ and $e_0 > 1$. Set $U := W - (R_1 \cup R_2 \cup S_1 \cup S_2 \cup \mathrm{Supp}(\Delta'))$ if $d_0 > 1$; set $U := W - (R \cup S_1 \cup S_2 \cup \mathrm{Supp}(\Delta'))$ if $d_0 = 1$ and $e_0 > 1$; set $U := W - (S_1 \cup S_2 \cup \mathrm{Supp}(\Delta'))$ if $d_0 = e_0 = 1$. Then U is isomorphic to the affine plane $\mathbb{A}^2_k$. We shall prove the next:

Lemma. *With the above notations, we have:*

(4) *If* $d_0 > 1$ *then* $\Gamma_1 = \Gamma_2$*; we set* $\Gamma := \Gamma_1 = \Gamma_2$.

(5) *If either* $d_0 > 1$ *or* $d_0 = 1$ *and* $e_0 > 1$ *then* Γ *has exactly two irreducible components* C_1 *and* C_2 *other than those contained in* R_1 *or* R_2 *(or* R *if* $d_0 = 1$ *and* $e_0 > 1$*).*

(6) *If $d_0 = e_0 = 1$, the fibration φ has only one reducible fiber Γ which has two irreducible components.*

225 *Proof.* Our proof consists of four steps.

(I) We shall prove first the following assertion:

□

The fibration φ has one and only one fiber $\varphi^{-1}(Q)$ $(Q \in \mathbb{P}^1_k)$ such that $\varphi^{-1}(Q) \cap U$ is reducible; then $\varphi^{-1}(Q) \cap U$ consists of two irreducible components.

Proof. Let $Q_1, \ldots, Q_s$ be the points of $\mathbb{P}^1_k$ such that $\varphi^{-1}(Q_i) \cap U$ is reducible for $1 \leqq i \leqq s$, and let $Q_{s+1}, \ldots, Q_t$ be the points of $\mathbb{P}^1_k$ such that $\varphi^{-1}(Q_i)$ is reducible but $\varphi^{-1}(Q_i) \cap U$ is irreducible for $s + 1 \leqq i \leqq t$. We may assume that $\varphi^{-1}(Q_1), \ldots, \varphi^{-1}(Q_s)$ and $\varphi^{-1}(Q_{s+1}), \ldots, \varphi^{-1}(Q_t)$ exhaust all fibers of φ having respective properties. Then $\varphi^{-1}(Q_i)$'s $(1 \leqq i \leqq t)$ and $\varphi^{-1}(Q_\infty) := \Delta'$ are all reducible fibers of φ. For $1 \leqq i \leqq s$, let n_i be the number of irreducible components of $\varphi^{-1}(Q_i) \cap U$. On the other hand, write $U := \mathrm{Spec}(A)$ and $U - \left(\bigcup_{i=1}^{t} \varphi^{-1}(Q_i) \cap U\right) := \mathrm{Spec}(B)$. Then, since U is isomorphic to $\mathbb{A}^2_k$, we know that A is a unique factorization domain and $A^* = k^*$. Hence, by a similar argument as in (2.3.3), we know that B^*/k^* is a free z-module of rank $n_1 + \cdots + n_s + (t - s)$. Since $\varphi^{-1}(\mathbb{P}^1_k - \{Q_1, \ldots, Q_t, Q_\infty\})$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1_k - \{Q_1, \ldots, Q_t, Q_\infty\}$ and $U - \left(\bigcup_{i=1}^{t} \varphi^{-1}(Q_i) \cap U\right) = \varphi^{-1}(\mathbb{P}^1_k - \{Q_1, \ldots, Q_t, Q_\infty\}) - (S_1 \cup S_2)$, we know that $U - (\bigcup_{i=1}^{t} \varphi^{-1}(Q_i) \cap U)$ is isomorphic to $\mathbb{A}^1_* \times (\mathbb{P}^1_k\{Q_1, \ldots, Q_t, Q_\infty\})$, where $\mathbb{A}^1_* = \mathbb{A}^1_k$-(one point). Then by virtue of the unit theorem (cf. Sweedler [54]), B^*/k^* is a free z-module of rank $1 + t$. Hence we obtain

$$n_1 + \cdots + n_s + (t - s) = 1 + t, \quad n_i \geqq 2 (1 \leqq i \leqq s)$$

226 whence follows that $s = 1$ and $n_1 = 2$.

(II) Assume that $d_0 > 1$ and $\Gamma_1 \neq \Gamma_2$. Then, each of Γ_1 and Γ_2 has two irreducible components other than those contained in R_1 and R_2. Suppose that Γ_1 has only one irreducible component C_1 other than those contained in R_1. Then the multiplicity of C_1 in Γ_1 is 1 and $(C_1 \cdot S_2) = 1$. Since the components of Γ_1 contained in R_1 are not exceptional components, Lemma 2.2 tells us that not only C_1 is an exceptional component of Γ_1 but also Γ_1 contains another exceptional component. This is a contradiction. Therefore, we know that $\Gamma_1 \cap U$ and $\Gamma_2 \cap U$ are reducible. But this contradicts the assertion proved in the step (I) above. Hence $\Gamma_1 = \Gamma_2$. By the same argument, we can show that Γ has two irreducible components C_1 and C_2 other than those contained in R provided $d_0 = 1$ and $e_0 > 1$.

(III) We shall show that if $d_0 > 1$ then $\Gamma(:= \Gamma_1 = \Gamma_2)$ has two irreducible components C_1 and C_2 other than those contained in R_1 or R_2. Assume the contrary, i.e., $\Gamma \cap U$ in irreducible. Then, the assertion proved in the step (I) implies that there exists a reducible fiber $\varphi^{-1}(Q)$ of the form: $\varphi^{-1}(Q) = L_1 + L_2$, where $L_1 \cong L_2 \cong \mathbb{P}^1_k$, $(L_1 \cdot L_2) = (L_1 \cdot S_1) = (L_2 \cdot S_2) = 1$, $(L_1 \cdot S_2) = (L_2 \cdot S_1) = 0$ and $(L_1^2) = (L_2^2) = -1$. Then $L_1 \cap U$ and $L_2 \cap U$ are isomorphic to the affine line $\mathbb{A}^1_k$; moreover, they satisfy the conditions (1) $\sim$ (5) of Theorem 3.2, Chapter I. Hence, after a suitable change of coordinates x, y of U : Spec$(k[x, y])$, we may assume that $L_1 \cap U$ and $L_2 \cap U$ are the x-axis and the y-axis, respectively; namely, $\varphi^{-1}(Q) \cap U$ is defined by $xy = 0$. Then $\Gamma \times_W U$ (with scheme struc- **227**
ture) is isomorphic to Spec$(k[x, y]/(xy - c))$ for $c \in k^*$, which is reduced. However, we shall show that $\Gamma \times_W U$ is not reduced. Indeed, let C_1 be the unique irreducible component of Γ which is not contained in R_1 and R_2. Then, since the components in R_1 and R_2 are not exceptional components, C_1 must be an exceptional component of Γ. Then the multiplicity of C_1 in Γ is larger than 1 because $R_1 \neq \phi$, $R_2 \neq \phi$ and C_1 connects R_1 to R_2. Thus, we get a contradiction, and proved that Γ has two irreducible components C_1 and C_2 other than those contained in R_1 or R_2.

(IV) If $d_0 = e_0 = 1$, the assertion proved in the step (I) implies that there exists a fiber $\varphi^{-1}(Q) = L_1 + L_2$ such that $L_1 \cong L_2 \cong \mathbb{P}^1_k$, $(L_1 \cdot L_2) = (L_1 \cdot S_1) = (L_2 \cdot S_2) = 1$, $(L_1 \cdot S_2) = (L_2 \cdot S_1) = 0$ and $(L_1^2) = (L_2^2) = -1$.

□

6.16

In this paragraph we shall derive a consequence of Lemma 6.15, which also completes a proof of the "if" part of Theorem 6.1.

Lemma. *Let f be an irreducible element of $k[x, y]$ satisfying the conditions* (1), (2) *and* (3) *of Theorem 6.1. Then f is written in the form $f = c(x^d y^e - 1)$ after a suitable change of coordinates x, y of $k[x, y]$, where $c \in k^*$, and d and e are positive integers such that* $(d, e) = 1$.

Proof. With the notations of 6.15, let Γ be the unique fiber of φ such that $\Gamma \cap U$ is reducible; as a matter of fact, $\Gamma \cap U$ consists of two irreducible
228 components. Let C_1 and C_2 be irreducible components of Γ such that $C_i \cap U \neq \phi$ for $i = 1, 2$, and let d and e be multiplicities of C_1 and C_2 in Γ, respectively. Since $\Gamma \cap U$ is connected C_1 and C_2 intersect each other transversely in a single point on U. Furthermore, $C_1 \cap U$ and $C_2 \cap U$ are isomorphic to the affine line. We shall show the latter assertion only in the case $d_0 > 1$ as the remaining cases ($d_0 = 1$ and $e_0 > 1$; $d_0 = e_0 = 1$) can be treated in a similar fashion. Since $R_1 \neq \phi$ and $R_1 \cap R_2 = \phi$, either one of C_1 and C_2, say C_1, intersects a component in R_1. Then $C_2 \cap R_1 = \phi$, for otherwise $C_1 \cup C_2 \cup R_1$ would contain a cyclic chain. The same reasoning implies that $C_2 \cap R_2 = \phi$ if $C_1 \cap R_2 \neq \phi$. Hence, if $C_1 \cap R_2 \neq \phi$ then $C_2 \cap R_i = \phi$ for $i = 1, 2$, i.e., $C_2 \subset U$, which is absurd as U is affine. Thus $C_1 \cap R_2 = \phi$ and $C_2 \cap R_2 \neq \phi$. Moreover, C_i intersects R_i in a single point for $i = 1, 2$, for otherwise $C_i \cup R_i$ would contain a cyclic chain. Since $C_i \cong \mathbb{P}^1_k$, we finally know that $C_i \cap U$ ($i = 1, 2$) is isomorphic to the affine line $\mathbb{A}^1_k$. Now, by virtue of Theorem 3.2, Chapter I we may assume that $C_1 \cap U$ and $C_2 \cap U$ are defined by $x = 0$ and $y = 0$, respectively, after a suitable change of coordinates x, y of $k[x, y]$. Then it is clear that $\Gamma \times_W U$ (as a k-scheme) is defined by

$x^d y^e = 0$ on $U = \mathbb{A}^2_k := \mathrm{Spec}(k[x, y])$. By construction of Λ' (or φ) we know that C_0 is defined by $f := x^d y^e - c = 0$ for $c \in k^*$. Since C_0 is irreducible we must have $(d, e) = 1$. Apparently we can write f in the form $f = c(x^d y^e - 1)$ after a suitable change of coordinates x, y of $k[x, y]$. ◻

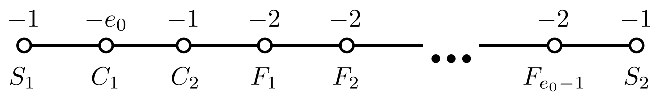
6.17

In this paragraphs we shall show that we may choose variables x, y of 229
$k[x, y]$ so that $d = d_0$ and $e = e_0$. In case $d_0 = e_0 = 1$, this was proved in the course of proving Lemma 6.15. In the remaining cases our assertion follows from the next:

Lemma. (1) *Assume that $d_0 = 1$ and $e_0 > 1$. Then we have:*

$$\Gamma = C_1 + e_0 C_2 + (e_0 - 1)F_1 + \cdots + F_{e_0 - 1},$$

where $(C_1^2) = -e_0$, $(C_2^2) = -1$, and $S_1 \cup S_2 \cup \mathrm{Supp}(\Gamma)$ has the weighted graph:



(2) *Assume that $d_0 > 1$. Then we have:*

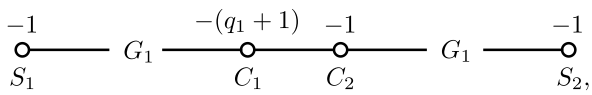
$$\Gamma = d_0 C_1 + e_0 C_2 + z_1 + z_2,$$

where:

1° *Z_i is an effective divisor such that $\mathrm{Supp}(Z_i) = R_i$ for $i = 1, 2$;*

2° *$(C_1^2) = -(q_1 + 1)$ and $(C_2^2) = -1$;*

3° *$S_1 \cup S_2 \cup \mathrm{Supp}(\Gamma)$ has the weighted graph:*



G_i being the weighted graph of the irreducible components contained in R_i for $i = 1, 2$.

230 A proof will be given in the subparagraphs 6.17.1 ~ 6.17.4 below. The facts which we frequently use in the course of a proof are the followings: Let V be a nonsingular projective surface, let $\varphi : V \to \mathbb{P}^1_k$ be a surjective morphism whose general fibers are isomorphic to $\mathbb{P}^1_k$ and let $\Gamma := n_1C_1 + \cdots + n_rC_r$ be a reducible fiber of φ. Let $\tau : V \to W$ be a contraction of several components contained in Γ, where W is nonsingular. Then, in the fiber $\tau_*(\Gamma)$ of the fibration $\psi : W \to \mathbb{P}^1_k$ with $\varphi = \psi \cdot \tau$ the following assertions hold true (cf. Lemma 2.2):

(A) No three distinct components of $\tau_*(\Gamma)$ have a point in common,

(B) Let S be a cross-section of φ. Then no two distinct components of $\tau_*(\Gamma)$ have a point in common on $\tau(S)$.

In each stage of proof where we proceed on *reduction ad absurdum*, if we obtain a situation contrary to the assertion (A) (or (B), resp.) we shall say that we obtain a contradiction of type (A) (or (B), resp.).

6.17.1 A proof of the first assertion of the lemma.

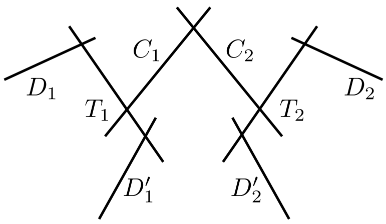
(I) Assume that $d_0 = 1$ and $e_0 > 1$. By virtue of Lemma 6.15, (2), Γ has exactly two irreducible components C_1 and C_2 other than those contained in R, one of which, say C_1, has multiplicity 1 in Γ and intersects S_1 transversely. Since $\Gamma \cap U$ is connected, C_1 and C_2 intersect each other in a single point on U. Then $C_1 \cap R = \phi$ and C_2 intersects some component T in R. Since those components
231 contained in R are not exceptional components of Γ, either one of C_1 and C_2 is an exceptional component. We shall show that C_2 is so. Indeed, if C_1 is an exceptional component, so is C_2 by virtue of Lemma 2.2, (6).

(II) We shall show that $T = F_1$. Suppose that $T = F_i (i \neq 1, e_0 - 1)$. Then since $(T^2) = -2$ three components $\tau(C_1)$, $\tau(F_{i-1})$ and $\tau(F_{i+1})$ of $\tau_*\Gamma$ have a point in common after the contraction τ of C_1 and T, which is a contradiction of type (A). If $T = F_{e_0-1}$ and $e_0 > 2$ then two components $\tau(C_1)$ and $\tau(F_{e_0-2})$ have a point in common on $\tau(S_2)$ after the contraction τ of C_2 and T, which is a contradiction of type (B). Hence $T = F_1$ and $S_1 \cup S_2 U\, \mathrm{Supp}(\Gamma)$ has the weighted graph as given in the statement, where $(C_0^2) = -e_0$ because C_1 has self-intersection multiplicity 0 after the contraction of $C_2, F_1, \ldots, F_{e_0-1}$. Once we know the way of contracting Γ to a single (irreducible) curve, it is an easy task to write down Γ in the form as given in the statement.

6.17.2

In the remaining of the paragraph 6.17 we assume that $d_0 > 1$. By virtue of Lemma 6.15, (2) and the proof of Lemma 6.16 we know that Γ has exactly two irreducible components C_1 and C_2 other than those contained in $R_1 \cup R_2$ such that C_1 and C_2 intersect each other transversely in a single point on U and that $C_i \cap R_i \neq \phi (i = 1, 2)$, $C_1 \cap R_2 = \phi$ and $C_2 \cap R_1 = \phi$. Let T_i be a unique irreducible component of R_i such that $(R_i \cdot C_i) > 0$ for $i = 1, 2$. Let G_i be the weighted graph of R_i for $i = 1, 2$. Note that for $i = 1, 2, R_i \neq \phi$ and R_i contains no exceptional components. This implies that either one of C_1 and C_2 is an exceptional component, **232**
i.e., $(C_1^2) = -1$ or $(C_2^2) = -1$. We shall show that $T_1 = E(1, 1)$ and $T_2 = F(1, 1)$. In order to do so we shall consider several possible cases separately. To avoid the tedious lengthiness a proof will not exceed a sketchy one.

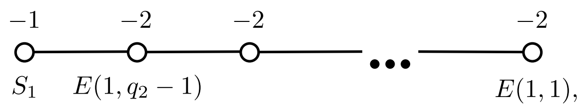
6.17.2.1 The case where T_1 and T_2 are non-terminal components in the graphs G_1 and G_2, respectively. Let D_1 and D_1' (or, D_2 and D_2', resp.) be components in R_1 (or R_2, resp.) which are linked to T_1 (or T_2, resp.) in the graph G_1 (or G_2, resp.). Then we have the configuration as follows:

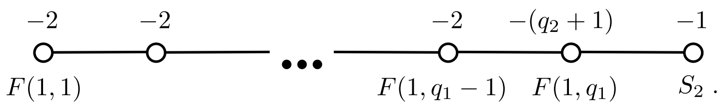


Suppose that $(C_1^2) = -1$. Then either one of T_1 and C_2 becomes contractible after contracting C_1, i.e., $(T_1^2) = -2$ or $(C_2^2) = -2$. If T_1 is so the contraction τ of C_1 and T_1 gives out three components $\tau(D_1)$, $\tau(D_1')$ and $\tau(C_2)$ of $\tau_*(\Gamma)$ having a point in common, a contradiction of type (A). If $(C_2^2) = -2$ then either one of T_1 and T_2 becomes contractible after contracting C_1 and C_2, i.e., $(T_1^2) = -3$ or $(T_2^2) = -2$. If $(T_1^2) = -3$ the contraction of C_1, C_2 and T_1 leads us to a contradiction of type (A). If $(T_2^2) = -2$ the contraction of C_1, C_2 and T_2 leads us to a contradiction of
233 type (A). Thus the assumption $(C_1^2) = -1$ ends up with a contradiction. Similarly, we can show the impossibility of the assumption $(C_2^2) = -1$.

6.17.2.2 The case where T_1 is a terminal component in the graph G_1 and T_2 is a non-terminal component in the graph G_2. Let D_2 and D_2' be as in 6.17.2.1.

(I) Firstly we shall consider the case $m = 1$. Then $R_1 \cup S_1$ and $R_2 \cup S_2$ have the following weighted graphs (cf. 6.10 and 6.13.4):





Hence $(T_2^2) = -2$. If $(C_2^2) = -1$ then the contraction τ of C_2 and T_2 turns out three components $\tau(D_2)$, $\tau(D_2')$ and $\tau(C_1)$ of $\tau_*(\Gamma)$ having a point in common, a contradiction of type (A). Hence $(C_2^2) \neq -1$ and $(C_1^2) = -1$. Then $T_1 = E(1, 1)$. Indeed, if $T_1 = E(1, q_2 - 1)$ and $q_2 \geqq 3$ then the contraction τ of C_1 and T_2 gives out two components $\tau(C_2)$ and $\tau(E(1, q_2 - 2))$ of $\tau_*(\Gamma)$ having a point in common on $\tau(S_1)$, a contradiction of type (B). Since C_2 becomes contractible after the contraction of C_1, $E(1, 1), \ldots, E(1, q_2 - 1)$ we have $(C_2^2) = -(q_2 + 1)$. Then by the contraction τ of C_1, $E(1, 1), \ldots, E(1, q_2 - 1)$, C_2 and T_2 we have two components $\tau(D_2)$ and $\tau(D'_2)$ of $\tau_*(\Gamma)$ possessing a point in common on $\tau(S_1)$, a contradiction of type (B). Thus, this case is impossible.

(II) Now we assume that $m > 1$. Looking into the graph of G_1 which is the subgraph of $\mathscr{E}$ (cf. Figure 2 of 6.10) consisting of components $E(i, j)$'s between $E(1, 1)$ and $E(m, p_m)$ (with $E(m, p_m)$ ex- **234**
cluded) we know that the contraction of all possible components in $C_1 \cup R_1$ cannot reduce $C_1 \cup R_1$ to a point. Let $D_1^{(1)}, \dots, D_1^{(r-1)}$ and $D_1^{(r)} (r \geqq 1)$ be the components in R_1 such that:

1° $T_1 = D_1^{(1)}$, and $D_1^{(i)}$ is linked to $D_1^{(i+1)}$ in the graph G_1 for $1 \leqq i \leqq r - 1$,

2° $((D_1^{(i)})^2) = -2$ if $i < r$ and $((D_1^{(r)})^2) \neq -2$.

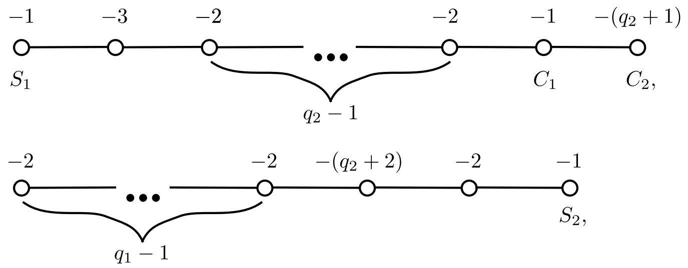
Suppose that $(C_2^2) = -1$. Then we have either $(T_2^2) = -2$ or $(C_1^2) = -2$. If $(T_2^2) = -2$ the contraction of C_2 and T_2 leads us to a contradiction of type (A). If $(C_1^2) = -2$ then T_2 becomes contractible after the contraction of $C_2, C_1, D_1^{(1)}, \dots, D_1^{(r-1)}$, i.e., $(T_2^2) = -(r + 2)$. The contraction τ of $C_2, C_1, D_1^{(1)}, \dots, D_1^{(r-1)}$ and T_2 gives out three components $\tau(D_2), \tau(D'_2)$ and $\tau(D_1^{(r)})$ of $\tau_*(\Gamma)$ having a point in common, a contradiction of type (A). Hence $(C_2^2) \neq -1$ and $(C_1^2) = -1$.

(III) We shall show that $T_1 = E(1, 1)$. Indeed, assume the contrary: $T_1 \neq E(1, 1)$. If $r \geqq 2$ then the contraction τ of C_1 and $D_1^{(1)}$ gives out two components $\tau(D_1^{(2)})$ and $\tau(C_2)$ of $\tau_*(\Gamma)$ having a point in common on $\tau(S_1)$, a contradiction of type (B). Hence $r = 1$, $(C_2^2) = -2$ and either $(T_1^2) = -3$ (where $T_1 = D_1^{(r)}$) or $(T_2^2) = -2$. If $(T_2^2) = -2$ then the contraction of C_1, C_2 and T_2 leads us to a contradiction of type (A). If $(T_1^2) = -3$ and R_1 contains at least two components (i.e., R_1 has a component D'_1 such that $D'_1 \neq T_1$ and $(D'_1 \cdot T_1) = 1$) then the contraction τ of C_1,
C_2 and T_1 gives out two components $\tau(D'_1)$ and $\tau(T_2)$ possessing **235**
a point in common on $\tau(S_1)$, a contradiction of type (B). The only remaining case is: $r = 1$, $R_1 = T_1$ and $(T_1^2) = -3$. Then

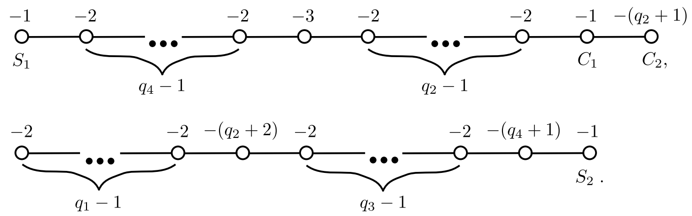
$(T_2^2) = -3$. However, the contraction τ of C_1, C_2, T_1 and T_2 gives out two components $\tau(D_2)$ and $\tau(D_2')$ having a point in common on $\tau(S_1)$, a contradiction of type (B). Therefore, $T_1 = E(1,1)$.

(IV) C_2 becomes contractible by contracting C_1, $D_1^{(1)}, \ldots, D_1^{(r-1)}$ and either one of $D_1^{(r)}$ and T_2 becomes contractible by contracting C_2 further. If this is T_2 the contraction of C_1, $D_1^{(1)}, \ldots, D_1^{(r-1)}$, C_2 and T_2 turns out a contradiction of type (A). If this is $D_1^{(r)}$ and if there exists a sequence of components $D_1^{(r+1)}, \ldots, D_1^{(t-1)}, D_1^{(t)}$ in R_1 such that $D_1^{(i)}$ is linked to $D_1^{(i+1)}$ for $r \leqq i \leqq t-1$ and that $((D_1^{(i)})^2) = -2$ if $r < i < t$ and $((D_1^{(t)})^2) \neq -2$ then T_2 becomes contractible after the contraction of $C_1, D_1^{(1)}, \ldots, D_1^{(r-1)}, C_2, D_1^{(r)}, \ldots, D_1^{(t-1)}$, while we obtain a contradiction of type (A) by contracting T_2 further. The remaining cases are the next two: (1) $m = 2$ and $p_2 = 2$, or (2) $m = 3$ and $p_2 = 1$. In each of the cases we have the following weighted graphs of $S_1 \cup R_1 \cup C_1 \cup C_2$ and $S_2 \cup R_2$:

(1)



(2)



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However, it is easy to see that both cases end up with contradictions. Thus, this case is impossible.

6.17.2.3 The case where T_1 is a non-terminal component in the graph G_1 and T_2 is a terminal component in the graph G_2. The same arguments as in 6.17.2.2 with slight modifications show that this case is impossible. The details are left to the readers.

6.17.2.4 The case where both T_1 and T_2 are terminal components in the graphs G_1 and G_2, respectively. We shall show that $T_1 = E(1,1)$ and $T_2 = F(1,1)$.

(I) We shall first show that $T_1 = E(1,1)$. Assume the contrary. Then R_1 has at least two components. Suppose that $(C_1^2) = -1$. Then $(T_1^2) \neq -2$, for otherwise the contraction τ of C_1 and T_1 would lead us to a contradiction of type (B). Hence $(C_2^2) = -2$. Let $D_2^{(1)}, \ldots, D_2^{(s)} (s \geqq 1)$ be the components of R_2 such that:

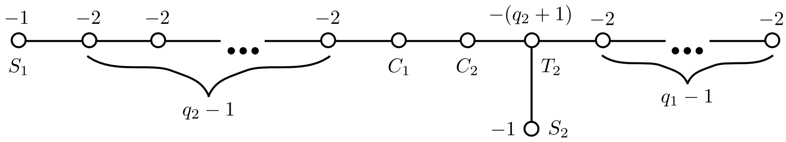
(1) $T_2 = D_2^{(1)}$, and $D_2^{(i)}$ is linked to $D_2^{(i+1)}$ in the graph G_2 for $1 \leqq i \leqq s-1$;

(2) $((D_2^{(i)})^2) = -2$ for $i < s$ and $((D_2^{(s)})^2) \neq -2$.

[It is easy to ascertain the existence of such components by look- 237
ing into the graph G_2.] Then T_1 becomes contractible after the contraction of C_1, C_2, $D_2^{(1)}, \ldots, D_2^{(s-1)}$, though we reach to a contradiction of type (B) by contracting T_1 further. Hence $(C_1^2) \neq -1$ and $(C_2^2) = -1$. Let $D_2^{(1)}, \ldots, D_2^{(s)}$ be as above. Then C_1 becomes contractible by contracting $C_2, D_2^{(1)}, \ldots, D_2^{(s-1)}$ and either one of T_1 and $D_2^{(s)}$ becomes contractible by contracting C_1 further. If T_1 is so then we obtain a contradiction of type (B) by contracting $C_2, D_2^{(1)}, \ldots, D_2^{(s-1)}, C_1$ and T_1. If $D_2^{(s)}$ is so and if there exists a sequence of components $D_2^{(s+1)}, \ldots, D_2^{(u)}$ in R_2 such that $D_2^{(i)}$ is linked to $D_2^{(i+1)}$ for $s \leqq i \leqq u-1$ and that $((D_2^{(i)})^2) = -2$ if $s < i < u$ and $((D_2^{(u)})^2) \neq -2$ then T_1 becomes contractible by contracting C_2, $D_2^{(1)}, \ldots, D_2^{(s-1)}, C_1, D_2^{(s)}, \ldots, D_2^{(u-1)}$, though we obtain a contradiction of type (B) by contracting T_1 further. The remaining cases are the next two: (1) $n = 2$ and $q_2 = 2$ or (2) $n = 3$ and

$q_1 = 1$. However, R_1 consists of only one components in both cases (cf. 6.10 and 6.13.4), a contradiction. Hence $T_1 = E(1,1)$.

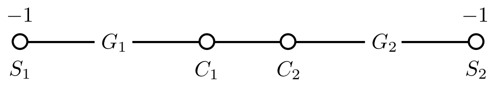
(II) We shall next show that $T_2 = F(1,1)$. Assume the contrary: $T_2 \neq F(1,1)$. We shall treat the case $m = 1$ first. If $m = 1$ the weighted graph of $\mathrm{Supp}(\Gamma)$ is given as follows:



where $q_2 \geqq 2$. If $(C_1^2) = -1$ then $(C_2^2) = -(q_2+1)$, which is absurd
238 because there is no contractible components left in Γ after C_2 is contracted. Hence $(C_1^2) \neq -1$ and $(C_2^2) = -1$. Then $(C_1^2) = -2$ and $q_1 = 1$, whence $T_2 = F(1,1)$, a contradiction. We shall now assume that $m > 1$. Suppose that $(C_2^2) = -1$. Then $(T_2^2) \neq -2$, for otherwise we would obtain a contradiction of type (B). Hence $(C_1^2) = -2$. Let $D_1^{(1)}, \ldots, D_1^{(r)}$ be a sequence of components in R_1 as in 6.17.2.2. Then T_2 becomes contractible after contracting C_2, C_1, $D_1^{(1)}, \ldots, D_1^{(r-1)}$, though we obtain a contradiction of type (B) by contracting T_2 further. Therefore, $(C_2^2) \neq -1$ and $(C_1^2) = -1$. Let $D_1^{(1)}, \ldots, D_1^{(r)}$ be as above. Then C_2 becomes contractible after the contraction of $C_1, D_1^{(1)}, \ldots, D_1^{(r-1)}$, and either one of $D_1^{(r)}$ or T_2 becomes contractible by contracting C_2 further. If this is T_2 then we obtain a contradiction of type (B) because $T_2 \neq F(1,1)$ implies that R_2 has at least two components. If this is $D_1^{(r)}$ and if there exists a sequence of components $D_1^{(r+1)}, \ldots, D_1^{(t)}$ in R_1 as chosen in 6.17.2.2 then T_2 becomes contractible after contracting $C_1, D_1^{(1)}, \ldots, D_1^{(r-1)}, C_2, D_1^{(r)}, \ldots, D_1^{(t-1)}$, though we obtain a contradiction of type (B) by contracting T_2 further. The remaining cases are: (1) $m = 2$ and $p_2 = 2$, or (2) $m = 3$ and $p_2 = 1$ (cf. the step (IV) of 6.17.2.2). These two cases are easily seen to end up with contradictions. Hence $T_2 = F(1,1)$.

6.17.3

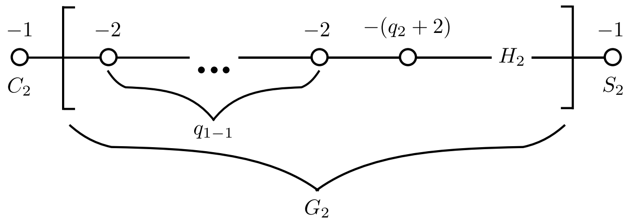
By virtue of 6.17.2 we have the following weighted graph of $S_1 \cup S_2 \cup \mathrm{Supp}(\Gamma)$:



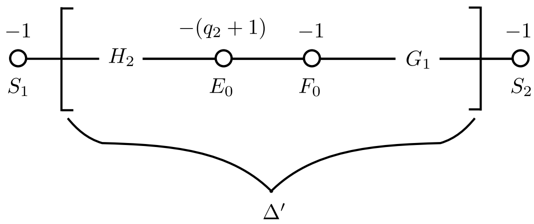
If $(C_1^2) = -1$ then $(C_2^2) = -(q_2 + 1)$. However, by writing down 239
concretely the weighted graph of Supp(Γ) and performing the contraction of all possible components of Γ we obtain readily a contradiction. We omit the details. Hence $(C_2^2) = -1$. Again by writing down concretely the weighted graph of Supp(Γ) we know that $(C_1^2) = -(q_1 + 1)$ and Γ can be, in fact, reduced to a single (irreducible) component with self-intersection multiplicity 0 by contracting all possible components of Γ.

6.17.4

The graph G_2 is written as:



Then, looking into the graphs $\mathscr{E}$ and $\mathscr{F}$, we know that the weighted graph of Δ' is given by



By comparing the weighted graphs of Supp(Γ) and Supp(Δ') we know that the weighted graph of Supp(Δ') is obtained from that of Supp (Γ) by contracting C_2 and $(q_1 - 1)$ components in G_2 which are linked successively to C_2. Since the multiplicities of E_0 and F_0 in Δ' are e_2 and
240 $d_0 = e_1$, respectively, we know that C_1 has multiplicity d_0 in Γ and the component in G_2 with weight $-(q_2 + 2)$ has multiplicity e_2 in Γ. Then it is apparent that C_2 has multiplicity $e_2 + q_1 e_1 = e_0$ in Γ. This completes a proof of Lemma 6.17.

6.18

The “only if” part of Theorem 6.1 is easy to prove. So we omit a proof. We shall finish this section by noting that if f is as in the statement then there exists a nontrivial action of the multiplicative group scheme $G_{m,k}$ on $\mathbb{A}^2_k := \mathrm{Spec}(k[x, y])$ such that C'_α's are $G_{m,k}$-orbits for almost all $\alpha \in k$. Indeed if f is written as $f : c(x^d y^e - 1)$ we have only to define an action of $G_{m,k}$ on $\mathbb{A}^2_k$ via: ${}^t x : t^{-e} x$ and ${}^t y = t^d y$ for $t \in G_{m,k}$.

Part III

Unirational surfaces

1 Review on forms of the affine line over a field

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1.1

Throughout this section the ground field k is assumed to be a nonperfect field of characteristic $p > 0$. We denote by k_s and $\overline{k}$ the algebraic separable closure and the algebraic closure of k, respectively. For an integer n we denote by k^{p^n} the sub field $\{\lambda^{p^n} | \lambda \in k\}$ of $\overline{k}$. An irreducible nonsingular affine curve X defined over k is said to be *a k-form of the affine line* $\mathbb{A}^1_k$ if $X \underset{k}{\otimes} k'$ is k'-isomorphic to $\mathbb{A}^1_{k'}$, for some algebraic extension k' of k. It is a well-known fact that X is k-isomorphic to $\mathbb{A}^1_k$ if k' is taken to be separable over k. Thus, we have only to consider the case where k' is purely inseparable over k. We shall recall several results from [26] and [27] which we need in the subsequent sections.

1.2

Let X be a k-form of $\mathbb{A}^1_k$ and let C be a complete normal model of $k(X)$. Then C has only one place P_∞ outside of X which is possibly singular. The k-genus of C is called *the genus of* X. The function field $k(X$ is a 242
k-form of the rational function field $k(t)$, i.e., $k(X) \underset{k}{\otimes} k' \cong k'(t)$ for some

algebraic extension k' of k. Conversely, given a k-form K of the rational function field $k(t)$, let C be a complete k-normal model. Then K is the function field of a k-form of $\mathbb{A}^1_k$ if and only if C has at most one singular place. If C has a unique singular place P_∞, $X := C - P_\infty$ is a nontrivial k-form of $\mathbb{A}^1_k$ (cf. [26; 6.7]). If C is nonsingular C is k-isomorphic to $\mathbb{P}^1_k$ except possibly when $p = 2$ (cf. [ibid., 6.7.7]); in case $p = 2$, if C has a k-rational point then C is k-isomorphic to $\mathbb{P}^1_k$; if P_∞ is any point purely inseparable over k on C then $X := C - P_\infty$ is a nontrivial k-form of $\mathbb{A}^1_k$.

1.3

Let a be an element of $k - k^p$ and let n be a positive integer. Let $\varphi : \mathbb{P}^1_k \to \mathbb{P}^n_k$ be the embedding of $\mathbb{P}^1_k$ into $\mathbb{P}^n_k$ given by $t \mapsto (1, t, \ldots, t^{p^n-1}, t^{p^n} - a)$, where t is an inhomogeneous parameter of $\mathbb{P}^1_k$. Let P_∞ be the point of $\mathbb{P}^1_k$ defined by $t^{p^n} = a$. Denote by $X_{a,n}$ the image $\varphi(\mathbb{P}^1_k - \{P_\infty\})$. Then we have:

Lemma (cf. [26; Th. 6.8.1]. (i) *Every k-rational k-form of $\mathbb{A}^1_k$ is k-isomorphic to $\mathbb{A}^1_k$ of $X_{a,n}$ for suitable $a \in k - k^p$ and $n \in \mathbb{Z}^+$.*

(ii) *$X_{a,n}$ is a k-rational k-form of $\mathbb{A}^1_k$ not k-isomorphic to $\mathbb{A}^1_k$.*

(iii) *$X_{a,n}$ is k-isomorphic to $X_{b,m}$ if and only if $m = n$ and there exist α, β, γ, δ in k^{p^n} such that $\alpha\delta - \beta\gamma \neq 0$ and $(\alpha a + \beta)/(\gamma a + \delta) = b$.*

1.4

243 **Lemma (cf. [ibid.; 6.8.2 f.]).** *A k-form X of $\mathbb{A}^1_k$ of k-genus 1 which has a k-rational point is k-birationally equivalent to an affine plane k-curve of one of the following types:*

(1) *$p = 3$; $y^2 = x^3 + \gamma$ with $\gamma \in k - k^3$.*

(2) *$p = 2$; $y^2 = x^3 + \beta x + \gamma$ with $\beta, \gamma \in k$ such that $\beta \notin k^2$ or $\gamma \notin k^2$.*

Let $P_\infty := (x = -\gamma^{1/3}, y = 0)$ in the first case and $P_\infty := (x = \beta^{1/2}, y = \gamma^{1/2})$ in the second case. Let C be a complete k-normal model of $k(X)$. Then X is k-isomorphic to $C - P_\infty$.

1.5

A k-form X of $\mathbb{A}^1_k$ is said to be *hyperelliptic* if the complete k-normal model of $k(X)$ is hyperelliptic.

1.5.1

Lemma (cf. [27; Th. 2.2]). *Let k be a separably closed*[1] *nonperfect field of characteristic $p > 2$. Then, a hyperelliptic k-form of $\mathbb{A}^1_k$ of k-genus $g \geqq 2$ is k-birationally equivalent to an affine plane curve of the type*

$$y^2 = x^{p^m} - a, \quad \text{where} \quad a \in k - k^p,$$

with $g = (p^m - 1)/2$. Conversely, the complete k-normal model C of every such plane curve has a unique singular point P_∞, and $C - P_\infty$ is a k-form of $\mathbb{A}^1_k$ of k-genus $(p^m - 1)/2$.

1.5.2

Lemma (cf. [ibid.; Th. 2.3]). *Let k be a separably closed nonperfect* 244
field of characteristic 2. Then a hyperelliptic k-form of $\mathbb{A}^1_k$ of k-genus $g \geqq 2$ is k-birationally equivalent to an affine plane curve of one of the following types:

(A) $y^2 + (x^{2^i} + a)^{2^\ell} y + b = 0$, *where* $i \geqq 0$, $\ell \geqq 0$, $a \in k$, $b \in k - \{0\}$*;* $a \notin k^2$ *if* $i > 0$, $b \notin k^2$ *if* $\ell > 0$*; and* $g = 2^{i+\ell} - 1$.

(B) $y^2 = x(x + \alpha)^{2g} + E(x)$, *where* $\alpha \in \overline{k}$, $(x + \alpha)^{2g} \in k[x]$, $E(x) \in k[x]$ *is an even polynomial of degree* $2g + 2$, *and* $E(\alpha) \notin k^2$ *in case* $\alpha \in k$.

Conversely, the k-normal completion of every curve of type (A) *of type* (B)*, minus its unique singularity, is a k-form of $\mathbb{A}^1_k$; of k-genus $= g$ in case* (A)*, of k-genus $\leqq g$ in case* (B)*.*

[1] Instead of assuming the separable closedness on k it suffices to assume that a k-form of $\mathbb{A}^1_k$ has a k-rational point.

1.5.3

Lemma (cf. [ibid.; Th. 2.4]). *The k-forms of $\mathbb{A}^1_k$ of genus 2 exist only if the characteristic p of the separably closed ground field k is either 2 or 5. Such a k-form is k-birationally equivalent to one of the following k-normal affine plane curves:*

(I) *In case $p = 2$:*

$$C : y^2 = x(x+\alpha)^4 + E(x)$$

where $\alpha^4 \in k$, $E(x) \in k[x]$ is even of degree 6, and either $\alpha \notin k$ or $E(\alpha) \notin k^2$.

(II) *In case $p = 5$:*

$$D : y^2 = x^5 + a, \quad a \in k - k^5.$$

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1.6

A k group scheme G is called a k-group of Russell type if $G \underset{k}{\otimes} k'$ is k'-isomorphic to the additive group scheme $G_{a,k'}$. The underlying k-scheme $\overline{G}$ of a k-group scheme of Russell type is clearly a k-form of $\mathbb{A}^1_k$.

1.6.1

Lemma (cf. Russell [47]; [26; 2.7]). *A k-group scheme of Russell type is a k-closed subgroup scheme of G_a^2 whose underlying scheme is given in $\mathbb{A}^2_k$ by an equation*

$$y^{p^n} = a_0 x + a_1 x^p + \cdots + a_r x^{p^r}, \quad a_i \in k (0 \leqq i \leqq r),$$

where $a_0 \neq 0$ and $a_i \notin k^p$ for some $1 \leqq i \leqq r$.

1.6.2

Lemma (cf. Russell [47]; [26; 6.9.1]). *Let X be a k-form of $\mathbb{A}^1_k$ and let C be the k-normal completion of X. Assume that X has a k-rational point P_0. Then the following conditions are equivalent to each other:*

(i) *X has a k-group structure with P_0 as the neutral point.*

(ii) *X is isomorphic to the underlying scheme of a k-group of Russell type.*

(iii) $\mathrm{Aut}_{k_s}(C\underset{k}{\otimes}k_s)$ *is an infinite group.*

1.6.3

Remark. With the notations of 1.6.2, $\mathrm{Aut}_{k_s}(C\underset{k}{\otimes}k_s) = \mathrm{Aut}_{k_s}(X\underset{k}{\otimes}k_s)$ if X is not k-rational (cf. [27; 3.1.1]). The function field $k(G)$ of a k-group G of Russell type is rational if and only if $p = 2$ and G is k-isomorphic to 246
an affine plane curve

$$y^2 = x + ax^2 \quad \text{with} \quad a \in k - k^2.$$

If $p > 2$, the underlying k-scheme of a k-group of Russell type is not hyperelliptic (cf. [27; Cor. 3.3.2]).

2 Unirational quasi-elliptic surfaces

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2.1

Throughout this section, the ground field k is assumed to be an algebraically closed field of characteristic $p > 0$. A nonsingular projective surface X defined over k is called *a quasi-elliptic surface* if there exists a morphism $f : X \to C$ from X to a nonsingular projective curve C such that almost all fibers of f are irreducible singular rational curves of arithmetic genus 1 (cf. [9], [39]). According to Tate [55], such surfaces can occur only in the case where the characteristic p is either 2 or 3, and almost all fibers of f have single ordinary cusps. Thus, the generic fiber $X_{\mathscr{R}}$ of f, minus the unique singular point, is a $\mathscr{R}$-form of $\mathbb{A}^1$ of $\mathscr{R}$-genus 1, where $\mathscr{R}$ is the function field of C over k. On the other hand, X is unirational over k if and only if C is a rational curve. *We assume in this section that every quasi-elliptic surface has a rational cross-section*, i.e., there is a rational mapping $s : C \to X$ such that $f \cdot s = \mathrm{id}_C$. Our ultimate purpose is to prove the following two theorems.

2.1.1

Theorem . *Let k be an algebraically closed field of characteristic 3. Then any unirational quasi-elliptic surface with a rational cross-section defined over k is birational to a hyper-surface in $\mathbb{A}^3_k : t^2 = x^3 + \varphi(y)$ with $\varphi(y) \in k[y]$ of degree prime to 3. Let $K := k(t, x, y)$ be the algebraic function field of an affine hypersurface of the above type, let $m = [\frac{d}{6}]$ and let $\widehat{H}$ be the (nonsingular) minimal model of K when K is not ra-*
248 *tional over k*[2]*. Moreover, if $d \geqq 7$ assume that the following conditions hold:*

(1) *For every root α of $\varphi'(y) = \dfrac{d\varphi}{dy} = 0$, $v_\alpha(\varphi(y) - \varphi(\alpha)) \leqq 5$, where v_α is the $(y - \alpha)$-adic valuation of $k[y]$ with $v_\alpha(y - \alpha) = 1$.*

(2) *If, moreover, $\varphi(y) - \varphi(\alpha) = a(y - \alpha)^3+$(terms of higher degree in $y - \alpha$) for some root α of $\varphi'(y) = 0$ and $a \in k^*$ then $v_\alpha(\varphi(y) - \varphi(\alpha) - a(y - \alpha)^3) \leqq 5$.*

Then we have the following:

(i) *If $m = 0$, i.e., $d \leqq 5$, then K is rational over k. If $d \geqq 7$, K is not rational over k, and the minimal model $\widehat{H}$ exists.*

(ii) *If $m = 1$, i.e., $7 \leqq d \leqq 11$, then $\widehat{H}$ is a $K3$-surface.*

[2]Note that if K is ruled and unirational then K is rational. Hence if K is not rational K has the minimal model.

(iii) *If $m > 1$, i.e., $d \geqq 13$, then $p_a(\widehat{H}) = p_g(\widehat{H}) = m$, $\dim H^1(\widehat{H}, \mathscr{O}_{\widehat{H}}) = 0$, the r-genus $P_r(\widehat{H}) = r(m-1)+1$ for every positive integer r and the canonical dimension $\kappa(H) = 1$.*

2.1.2

Theorem . *Let k be an algebraically closed field of characteristic 2. Then any unirational quasi-elliptic surface with a rational cross-section defined over k is birational to a hyper-surface in $\mathbb{A}^3_k : t^2 = x^3 + \psi(y)x + \phi(y)$ with $\phi(y), \psi(y) \in k[y]$. Conversely, let $K := k(t, x, y)$ be an algebraic function field of dimension 2 generated by t, x, y over k such that $t^2 = x^3 + \phi(y)$ with $\phi(y) = y\varphi(y)^2 \in k[y]$*[3] *and $d = \deg_y \varphi$. Let $m = [\dfrac{d}{3}]$. Assume moreover that, for every $\alpha \in k$, if we write $\phi(y+\alpha) = \sum_{i \geqq 0} a_i y^i$ then one of a_1, a_3 and a_5 is nonzero. Then we have the following:*

(i) *If $m = 0$, i.e., $0 \leq d \leqq 2$, K is rational over k. If $m > 0$, K is not rational over k, and the minimal model $\widehat{H}$ of K over k exists.* **249**

(ii) *If $m = 1$, i.e., $3 \leqq d \leqq 5$, then $\widehat{H}$ is a $K3$-surface.*

(iii) *If $m > 1$, i.e., $d \geqq 6$, then $p_a(\widehat{H}) = p_q(\widehat{H}) = m$, $\dim H^1(\widehat{H}, \mathscr{O}_{\widehat{H}}) = 0$, the r-genus $P_r(\widehat{H}) = r(m-1)+1$ for every positive integer r and the canonical dimension $\kappa(\widehat{H}) = 1$.*

Proofs of both theorems will be given after some preparations on double coverings.

2.2

Let X, with $f : X \to C$, be a quasi-elliptic surface defined over k and let $\mathscr{R}$ be the function field of C over k. Then the generic fiber $X_{\mathscr{R}}$ of f is an irreducible normal projective curve over $\mathscr{R}$ with arithmetic genus $p_a(X_{\mathscr{R}}) = 1$ and geometric genus 0. Hence $X_{\mathscr{R}}$ has only one singular point, whose multiplicity is 2. Let $\mathscr{R}_s$ be a separable algebraic closure

[3]With no loss of generality, we may write $\phi(y)$ in the form $\phi(y) = y\varphi(y)^2$.

of $\mathscr{R}$. By Chevalley [12; Th. 5, p.99], $X \underset{\mathscr{R}}{\otimes} \mathscr{R}_s$ is then a normal projective curve of arithmetic genus 1. This implies that the characteristic p of k must be either 2 or 3 by virtue of Tate [55], and that the singular point of $X_{\mathscr{R}}$ is a one-place point of multiplicity 2, which is rational over a purely inseparable extension of $\mathscr{R}$. Therefore, general fibers of f have single ordinary cusps.

Let Γ be the closure in X of the unique singular point of $X_{\mathscr{R}}$. Let $f_\Gamma : \Gamma \to C$ be the restriction of f onto Γ. Since the singular point of $X_{\mathscr{R}}$ is a one-plance point, f_Γ is a generically one-to-one morphism. Hence deg f_Γ is a power p^n of the characteristic p, and, for a fiber $f^{-1}(p)$ of f
250 such that $f^{-1}(P)$ meets Γ at a simple point of Γ, the intersection number $(\Gamma \cdot f^{-1}(P))$ must be 2 or 3 because $\Gamma \cap f^{-1}(P)$ is an ordinary cusp. Hence, $n = 1$ and $(\Gamma \cdot f^{-1}(P)) = p$. On the other hand, Γ is a nonsingular curve. Indeed, if Γ has a singular point Q, then $(\Gamma \cdot f^{-1}(f(Q))) \geqq 4$, which contradicts the fact that $(\Gamma \cdot f^{-1}(f(Q))) = p \leqq 3$.

Assume that f has a rational cross-section D; by virtue of [17, IV (2.8.5)] D is in fact extended to a regular cross-section of f. This is equivalent to saying that the generic fiber $X_{\mathscr{R}}$ of f has a $\mathscr{R}$-rational point. With the unique singular point deleted off, $X_{\mathscr{R}}$ becomes a $\mathscr{R}$-form of the affine line $\mathbb{A}^1$ of $\mathscr{R}$-genus 1 with a $\mathscr{R}$-rational point. Such a form is birationally equivalent to one of the following affine plane curves (cf. 1.4):

(i) If $p = 3$, $t^2 = x^3 + \gamma$ with $\gamma \in \mathscr{R} - \mathscr{R}^3$.

(ii) If $p = 2$, $t^2 = x^3 + \beta x + \gamma$ with $\beta, \gamma \in \mathscr{R}$ such that $\beta \notin \mathscr{R}^2$ or $\gamma \notin \mathscr{R}^2$.

The surface X is unirational over k if and only if C is rational. Indeed, the "only if" part is apparent by Lüroth's theorem; the "if" part is also easy to see: If $p = 3$, $\mathscr{R}^{1/3} \underset{\mathscr{R}}{\otimes} k(X)$ is rational over k, and if $p = 2$, $\mathscr{R}^{1/2} \underset{\mathscr{R}}{\otimes} k(X)$ is rational over k. Thus, if $f : X \to C$ is a unirational quasi-elliptic surface defined over k with a rational cross-section, the function field $\mathscr{R}$ of C over k is a rational function field $k(y)$ over k, and X is k-birationally equivalent to one of the following hypersurfaces in the affine 3-space $\mathbb{A}^3_k$:

(i)′ If $p = 3$, $t^2 = x^3 + \varphi(y)$ with $\varphi(y) \in k[y]$, where $d := \deg_y \varphi$ is prime to 3. 251

(ii)′ If $p = 2$, $t^2 = x^3 + \psi(y)x + \phi(y)$ with $\phi(y), \psi(y) \in k[y]$.

2.3

We shall recall and apply the canonical divisor formula for elliptic or quasi-elliptic fibrations (cf. [10]). Let $f : X \to C$ be a morphism form a nonsingular projective surface X to a nonsingular projective curve C such that almost all fibers of f are irreducible curves of arithmetic genus 1. A fiber $f^{-1}(P)$ of f is called *a reducible fiber* of f if $f^{-1}(P)$ has either not less than two (distinct) irreducible components or a single irreducible component with multiplicity $\geqq 2$; a fiber $f^{-1}(P)$ is called *a multiple fiber* if, when we write $f^{-1}(P)$ in the form $f^{-1}(P) = \sum_i n_i C_i$ with irreducible components C_i and positive integers n_i, the greatest common divisor q of n_i's is greater than 1. Then, writing $m_i = n_i/q$, $\sum_i m_i C_i$ is called *the reduced form* of a multiple fiber $f^{-1}(P)$. On the other hand, an elliptic or quasi-elliptic fibration $f : X \to C$ is said to be *relatively minimal* if each fiber of f contains no exceptional components. Given an elliptic or quasi-elliptic fibration $f : X \to C$ we can always find a fibration $f_0 : X_0 \to C$ such that $f_0 \cdot \sigma = f$, where $\sigma : X \to X_0$ is the contraction of exceptional components contained in the fibers. With these definitions set down, we have the following:

2.3.1

Lemma (cf. [9], [10]). *Let $f : X \to C$ be a relatively minimal elliptic or quasi-elliptic surface. Let $\{m_i Z_i; i \in I\}$ be the set of all multiple fibers of f, where Z_i is the reduced form. Then we have:*

$$\omega_X \cong f^*(S) \otimes \mathscr{O}_X(\sum_i a_i Z_i) \quad \textit{and} \quad S \cong \mathscr{O}_C \otimes L^{-1},$$

where: (i) $0 \leqq a_i \leqq m_i - 1$, (ii) *$L$ is an invertible sheaf on C defined by* 252 *either* $f_*\omega_X \cong \omega_C \otimes L^{-1}$ *or* $R^1 f_*\mathcal{O}_X \cong L \oplus T$, T *being a torsion sheaf on* C. *Let* t *be the length of* T. *Then we have*

$$\deg(S) = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C) + t.$$

For a point P *on* C, $T_P \neq 0$ *if and only if* $H^0(f^{-1}(P), \mathcal{O}_X) \supsetneqq k$, *which implies that* $f^{-1}(P)$ *is an exceptional multiple fiber*[4].

2.3.2

A key result in proving the stated theorems is the following

Lemma. *Let* $f : X \to C$ *be a unirational quasi-elliptic surface with a regular cross-section* D. *Assume that* X *is relatively minimal. Then the following results hold:*

(1) f *has no multiple fibers.*

(2) $\chi(\mathcal{O}_X) = -(D^2)$.

(3) *If* $\chi(\mathcal{O}_X) \leqq 1$, X *is rational over* k*; if* $\chi(\mathcal{O}_X) = 2$ *then* X *is a* $K3$-*surface; if* $\chi(\mathcal{O}_X) \geqq 3$ *then* $p_a(X) = p_q(X)$, $\dim H^1(X, \mathcal{O}_X) = 0$, *the* r-*genus* $P_r(X) = r(\chi(\mathcal{O}_X)-2)+1$, *and the canonical dimension* $\kappa(X) = 1$.

Proof. (1) is obvious because f has a cross-section.

(2) Since there are no multiple fibers in the fibration f, $a_i = 0$ for every i and $t = 0$ in the canonical divisor formula in 2.3.1. Since C is a rational curve we know that $\omega_X \cong f^*\mathcal{O}_C(\chi(\mathcal{O}_X) - 2)$. Since D
253 is a cross-section of f, the arithmetic genus formula on X applied to a nonsingular rational curve D tells us:

$$-2 = (D^2) + (D \cdot K_X) = (D^2) + \chi(\mathcal{O}_X) - 2.$$

Hence, $\chi(\mathcal{O}_X) = -(D^2)$

[4] a wild fiber, in other words (cf. [10]).

(3) For a positive integer r, the r-genus $P_r(X)$ is given by

$$P_r(X) = \dim H^0(X, \omega_X^{\otimes r}) = \dim H^0(C, \mathscr{O}_C(r(\chi(\mathscr{O}_X) - 2))).$$

If $\chi(\mathscr{O}_X) \leqq 1$, $P_r(X) = 0$ for every $r > 0$, whence $P_{12}(X) = 0$. This implies that X is ruled (cf. [10]). Since X is unirational, X is rational. If $\chi(\mathscr{O}_X) = 2$, we have $\omega_X \cong \mathscr{O}_X$. Then X is a $K3$-surface (cf. [9], [10]). If $\chi(\mathscr{O}_X) \geqq 3$, $P_r(X) = r(\chi(\mathscr{O}_X) - 2) + 1$. Hence the canonical dimension $\kappa(X)$ is equal to 1, and $p_g(X) = \chi(\mathscr{O}_X) - 1 = p_a(X)$. Therefore, $\dim H^1(X, \mathscr{O}_X) = 0$.

□

2.3.3

Corollary. *Let $f : X \to C$ be a relatively minimal, unirational, quasi-elliptic surface defined over k with a regular cross-section. If X is not rational over k then X is a minimal (nonsingular) model.*

Proof. Set $e := \chi(\mathscr{O}_X) - 2$. Then $K_X \sim e f^{-1}(P)$ for a point on $C \cong \mathbb{P}^1_k$. If X is not rational over k we know by 2.3.2 that $e \geqq 0$. Then the canonical linear system $|K_X|$ has no fixed components, which implies that X contains no exceptional curve of the first kind. X is therefore a minimal nonsingular model. □

2.3.4

Lemma. *Let $f : X \to C$ be a relatively minimal quasi-elliptic surface with a rational cross-section, and let $D = \sum_i n_i E_i$ be a reducible fiber* 254 *(having not less than two components). Then every component E_i is a nonsingular projective rational curve with $(E_i^2) = -2$.*

Proof. For every i, $(E_i \cdot D) = n_i(E_i^2) + \sum_{j \neq i} n_j(E_i \cdot E_j) = 0$. Since D is connected, $(E_i^2) < 0$ for every i. Since $K_X \sim f^*(S)$ for some divisor S on C as we have seen in 2.3.1, $(E_i \cdot K_X) = 0$. Then $p_a(E_i) = \frac{1}{2}(E_i^2) + 1 \geqq 0$, whence $(E_i^2) = -2$ and $p_a(F_i) = 0$. □

2.4

Throughout the paragraphs 2.4 ~ 2.7, we shall assume that k is an algebraically closed field of characteristic $p \geqq 0$. Let $\varphi(y)$ be a polynomial in y of degree $d \geqq 3$ with coefficients in k such that $A(x, y) := x^3 + \varphi(y)$ is an irreducible polynomial. We assume that $\varphi(y)$ contains no monomial terms of degree congruent to zero modulo 3 if $p = 3$, and that $\varphi(y)$ contains no monomial terms of degree congruent to zero modulo 2 if $p = 2$. Consider a hyper surface $t^2 = A(x, y)$ in the projective 3-space $\mathbb{P}^3_k$, which is birational to a double covering of $F_0 := \mathbb{P}^1_k \times \mathbb{P}^1_k$. Let $K := k(t, x, y)$. Let H_0 be the normalization of F_0 in K, and let $\rho_0 : H_0 \to F_0$ be the normalization morphism. With the above notations and assumptions we shall show the following:

2.4.1

Lemma. *Let Q be a point on H_0, and let $P := \rho_0(Q)$. If P is not a singular point of C then Q is a simple point of H_0, where C is a closed irreducible curve on F_0 defined by the equation $A(x, y) = 0$.*

2.4.2

In order to prove the above lemma we need

255 **Lemma.** *Let $A(x, y)$ be a nonzero irreducible polynomial in $k[x, y]$ such that $A(0, 0) = 0$, and let U be a hyper surface in the affine (t, x, y)-space $\mathbb{A}^3_k$ defined by $t^e = A(x, y)$ $(e \geqq 2)$, which is viewed as an e-ple covering of the (x, y)-plane $\mathbb{A}^2_k$. Then the point $Q := (t = 0, x = 0, y = 0)$ is a normal point on U if there are no irreducible curves D on $\mathbb{A}^2_k$ such that D passes through the point $P := (x = 0, y = 0)$, and that $\dfrac{\partial A}{\partial x}$ and $\dfrac{\partial A}{\partial y}$ vanish on D.*

Proof. Since U is a hyper surface in $\mathbb{A}^3_k$, the local ring $\mathscr{O} := \mathscr{O}_{Q,U}$ is a Cohen-Macaulay ring of dimension 2. By Serre's criterion of normality (cf. [17; IV (5.8.6)]), $\mathscr{O}$ is a normal ring if $\mathscr{O}_{\mathscr{B}}$ is regular for any prime ideal $\mathscr{B}$ of height 1 of $\mathscr{O}$. Let $\mathscr{J} = \mathscr{B} \cap k[x, y]$. Then $\mathscr{J}$ defines an irreducible curve D on $\mathbb{A}_k^2$ passing through the point P. If $\mathscr{O}_{\mathscr{B}}$ is not regular, the Jacobina criterion of singularity tells us that $\dfrac{\partial A}{\partial x}$ and $\dfrac{\partial A}{\partial y}$ vanish on D. However, this contradicts our assumption. ◻

2.4.3

Proof of Lemma 2.4.1 in case in case $p \neq 2$. Let $U_1 := \rho_0^{-1}(F_0 - (x = \infty) \cup (y = \infty))$, $U_2 := \rho_0^{-1}(F_0 - (\xi = \infty) \cup (y = \infty))$, $U_3 := \rho_0^{-1}(F_0 - (x = \infty) \cup (\eta = \infty))$, and $U_4 := \rho_0^{-1}(F_0 - (\xi = \infty) \cup (\eta = \infty))$, where $\xi = 1/x$ and $\eta = 1/y$. Then we can show:

Lemma. *Each of $\mathbb{U}_i$'s $(1 \leqq i \leqq 4)$ is isomorphic to a hyper-surface V_i in $\mathbb{A}_k^3$ defined by the following equation:* (1) $t^2 = x^3 + \varphi(y)$ *for* V_1*;* (2) $t^2 = x + x^4\varphi(y)$ *for* v_2*;* (3) *for* V_3, $t^2 = x^3y^d + \psi(y)$ *if* $d \equiv 0(\mod 2)$ *and* $t^2 = x^3y^{d+1} + y\psi(y)$ *if* $d \equiv 1(\mod 2)$*;* (4) *for* V_4, $t^2 = xy^d + x^4\psi(y)$ *if* $d \equiv 0(\mod 2)$ *and* $t^2 = xy^{d+1} + x^4y\psi(y)$ *if* $d \equiv 1(\mod 2)$*, where* $\psi(y) = y^d\varphi(1/y)$ *with* $\psi(0) \neq 0$*. With the notations of 2.4.1, Q is a simple* **256**
point of H_0 if P is not a singular point of C.

Proof. It is not hard to see that U_i is the normalization of V_i in K for $1 \leqq i \leqq 4$, whence follows that $U_i = V_i$ if V_i is normal. We shall show that each of V_i's is a normal hyper surface. Let $Q := (t = \gamma, x = \beta, y = \alpha)$ be a point of V_i. (1) Q is a singular point of V_1 only if $\gamma = 3\beta^2 = \varphi'(\alpha) = 0$. In case $p \neq 3$, the singular locus of V_1 is of co-dimension 2 at Q. Since V_1 is a complete intersection at Q, the Serre's criterion of normality shows that Q is a normal point. Hence V_1 is normal. In case $p = 3$, apply Lemma 2.4.2 to a triple covering $x^3 = t^2 - \varphi(y)$, $Q = (t = 0, x = \beta, y = \alpha)$ and $P := (t = 0, y = \alpha)$, noting that $\varphi'(y) \neq 0$. Q is then a normal point, and V_1 is therefore normal. (2) It is easy to see that $V_2 - (x = 0)$ is isomorphic to $V_1 - (x = 0)$ by a birational mapping $(t, x, y) \mapsto (t/x^2, 1/x, y)$, and that V_2 is nonsingular at every point on the curve $x = 0$. (3) $V_3 - (y = 0)$ is isomorphic to $V_1 - (y = 0)$ by birational mappings $(t, x, y) \mapsto (t/y^{d/2}, x, 1/y)$ if $d \equiv 0(\mod 2)$ and $(t, x, y) \mapsto (t/y^{(d+1)/2}, x, 1/y)$ if $d \equiv 1(\mod 2)$; and a point of V_3 lying on the curve $y = 0$ is a singular point only if $t = 0$ and $\psi(0) = 0$ when $d \equiv 0(\mod 2)$, and $(d+1)x^3y^d + \psi(y) + y\psi'(y) = 0$ when $d \equiv 1(\mod 2)$. However, this is

impossible because $\psi(0) \neq 0$. (4) $V_4 - (x = 0)$ is isomorphic to $V_3 - (x = 0)$ by a mapping $(t, x, y) \mapsto (t/x^2, 1/x, y)$; $V_4 - (y = 0)$ is isomorphic to $V_2 - (y = 0)$ by birational mappings $(t, x, y) \mapsto (t/y^{d/2}, x, 1/y)$ if $d \equiv 0(\mod 2)$ and $(t, x, y) \mapsto (t/y^{(d+1)/2}, x, 1/y)$ if $d \equiv 1(\mod 2)$.
257 This implies that the singular locus of V_4 is of co-dimension 2 if it is not empty. Hence by Serre's criterion of normality we know that V_4 is normal. The last assertion is now easy to see if one notes that Q is a singular point of H_0 only if $t = 0$. □

2.4.4

Proof of Lemma 2.4.1 in case $p = 2$. Since we assumed that $\varphi(y)$ has no monomial terms of degree congruent to zero modulo 2, we may write $\varphi(y)$ in the form: $\varphi(y) = y\varphi_1(y)^2$, where $d_1 := \deg_y \varphi_1 > 0$. Then we can show the following

Lemma. *Define U_i's $(1 \leqq i \leqq 4)$ as in 2.4.3. Then each of U_i's is isomorphic to a hyper surface V_i in $\mathbb{A}^3_k$ defined by the following equation:* (1) $t^2 = x^3 + \varphi(y)$ *for* V_1; (2) $t^2 = x + x^4\varphi(y)$ *for* V_2; (3) $t^2 = x^3y^{d+1} + y\psi_1(y)^2$ *for* V_3; (4) $t^2 = xy^{d+1} + x^4y\psi_1(y)^2$ *for* V_4, *where* $\psi_1(y) = y^{d_1}\varphi_1(1/y)$ *with* $\psi_1(0) \neq 0$. *With the notations of 2.4.1, Q is a simple point if P is not a singular point of C.*

Proof. We shall prove only the last assertion since the remaining assertions can be proved in a similar fashion as in 2.4.3 by applying Lemma 2.4.2. Let $Q := (t = \gamma, x = \beta, y = \alpha)$ be a point of V_i $(1 \leqq i \leqq 4)$. (1) If $Q \in V_1$, Q is a singular point only if $\beta = \varphi_1(\alpha) = 0$, whence $\gamma = 0$. (2) If $Q \in V_2$, Q is a simple point. (3) If $Q \in V_3$ and $\alpha = 0$, Q is a simple point because $\psi_1(0) \neq 0$. (4) If $Q \in V_4$ and $Q \notin V_2, V_3$ then $\alpha = \beta = \gamma = 0$. In any case, Q is a singular point of H_0 only if $\gamma = 0$. Thus P is a singular point of C. □

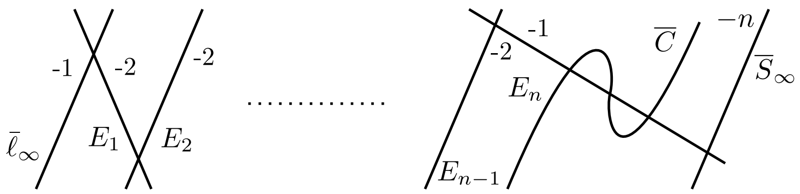
2.5

258 The equation $A(x, y) = 0$ defines a closed irreducible curve C on F_0. By Jacobian criterion of singularity, the singular points of C are the points $P := (x = \beta, y = \alpha)$, lying on the affine part $\mathbb{A}^2_k := F_0 - (x = \infty) \cup (y =$

∞), such that $3\beta^2 = \varphi'(\alpha) = \beta^3 + \varphi(\alpha) = 0$, and the point $P_\infty := (x = \infty, y = \infty)$. C is defined by an equation $\eta^d + \xi^3\psi(\eta) = 0$ locally at P_∞, where $\xi = 1/x$, $\eta = 1/y$ and $\psi(\eta) = \eta^d\varphi(1/\eta)$ with $\psi(0) \neq 0$. Hence, P_∞ is a cuspidal singular point with multiplicity $(\underbrace{3, 3, \ldots, 3}, 1, \ldots)$[5] if $d = 3n+1$ and $(\underbrace{3, 3, \ldots, 3}, 2, 1, \ldots)$ if $d = 3n+2$; P_∞ is a tacnodal singular point with three simple points in the n-th (infinitely near) neighborhood of P_∞ if $d = 3n$.

2.5.1

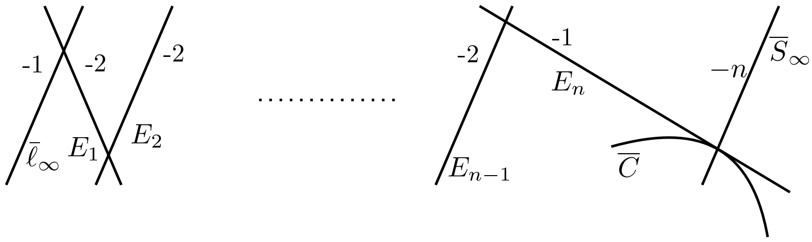
Here we introduce the following notations: Consider a fibration $\mathscr{F} := \{\ell_\alpha : \ell_\alpha$ is defined by $y = \alpha\}$ on F_0 defined by the second projection $p_2 : F_0 \to \mathbb{P}^1_k$. We denote by ℓ_∞ the fibre $y = \infty$, and by S_∞ the cross-section $x = \infty$. We denote by ℓ a general fiber of $\mathscr{F}$. Let $\overline{\sigma} : \overline{F} \to F_0$ be the shortest succession of quadratic transformations with centers at the singular points of C and its infinitely near singular points, by which the proper transform $\overline{C} := \overline{\sigma}'(C)$ of C on $\overline{F}$ becomes nonsingular. Let $\overline{S}_\infty := \overline{\sigma}'(S_\infty)$, and let $\overline{\ell}_\infty := \overline{\sigma}'(\ell_\infty)$. The following figures will indicate the configuration of $\overline{\sigma}^{-1}(\ell_\infty \cup C \cup S_\infty)$ on $\overline{F}$:



(Fig. 1)

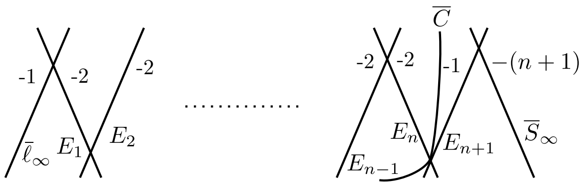
where $d = 3n$ ($n > 0$) and $p \neq 3$;

[5]By this notation we mean that P_∞ is a point with multiplicity 3, the infinitely near point of C in the first neighborhood of P_∞ (which is a single point in this case) has multiplicity 3, etc...



(Fig. 2)

259 where $d = 3n + 1$ $(n > 0)$ and $(\overline{C}, E_n) = 3$;



(Fig. 3)

where $d = 3n + 2 (n > 0)$ and $(\overline{C} \cdot E_{n+1}) = 2$.

2.5.2

Since $(A)_\infty|_{F_0} = 3S_\infty + d\ell_\infty$ we have:

$$
\begin{aligned}
(\overline{\sigma}^* A) &= \overline{C} + 3(E_1 + 2E_2 + \cdots + nE_n) + D - 3 \\
&\qquad (\overline{S}_\infty + E_1 + 2E_2 + \cdots + nE_n) - d(\bar{\ell}_\infty + E_1 + \cdots + E_n) \\
&= \overline{C} - 3\overline{S}_\infty - d\overline{\sigma}^*(\ell_\infty) + D, \quad \text{if} \quad d = 3n \\
&\text{or} \quad d = 3n + 1;
\end{aligned}
$$

$$
\begin{aligned}
(\overline{\sigma}^* A)\overline{C} &+ 3(E_1 + 2E_2 + \cdots + nE_n) + (3n+2)E_{n+1} + D - 3(\overline{S}_\infty + E_1 + \\
&\qquad 2E_2 + \cdots + nE_n + (n+1)E_{n+1}) - d(\bar{\ell}_\infty + E_1 + \cdots + E_{n+1}) \\
&= \overline{C} - 3\overline{S}_\infty - E_{n+1} - \overline{\sigma}^*(\ell_\infty) + D, \ \text{if } d = 3n + 2,
\end{aligned}
$$

where D is an effective divisor with support in the union $\mathscr{E}$ of exceptional curves which arise from the quadratic transformations with centers at the singular points and their infinitely near singular points of C in 260
the affine part $\mathbb{A}^2_k \subset F_0$.

2.5.3

We may write $(\overline{\sigma}^*A)$ uniquely in the form $(\overline{\sigma}^*A) = \overline{B} - 2\overline{Z}$ where $\overline{B}$ is a divisor whose coefficient at each prime divisor is 0 or 1 and where $\overline{Z}$ is some divisor. If $p \neq 2$, $\overline{B}$ is *the branch locus* of a double covering $\overline{\rho} : \overline{H} \to \overline{F}$, where $\overline{H}$ is the normalization of $\overline{F}$ in K and $\overline{\rho}$ is the normalization morphism (cf. [4]). In order to write down $\overline{B}$ we consider the following six cases separately.

2.5.3.1 If $d = 6m$ (i.e., $d = 3n$ with $n = 2m$) then we have:

$$\overline{B} = \overline{C} + \overline{S}_\infty + D_1$$
$$\overline{Z} = 2\overline{S}_\infty + 3m(\overline{\ell}_\infty + E_1 + \cdots + E_n) - D_2,$$

where D_1 and D_2 are the divisors determined uniquely by the conditions that D_1 is an effective divisor whose coefficient at each prime divisor is 0 or 1, $D_2 \geqq 0$, $D_1 + 2D_2 = D$, and $\mathrm{Supp}(D_1) \cup \mathrm{Supp}(D_2) \subset \mathscr{E}$.

2.5.3.2 If $d = 6m + 1$ (i.e., $d = 3n + 1$ with $n = 2m$) then we have:

$$\overline{B} = \overline{C} + \overline{S}_\infty + (\overline{\ell}_\infty + E_1 + \cdots + E_n) + D_1$$
$$\overline{Z} = 2\overline{S}_\infty + (3m + 1)(\overline{\ell}_\infty + E_1 + \cdots + E_n) - D_2,$$

where D_1 and D_2 are divisors chosen as in 2.5.3.1.

2.5.3.3 If $d = 6m + 2$ (i.e., $d = 3n + 2$ with $n = 2m$) then we have:

$$\overline{B} = \overline{C} + \overline{S}_\infty + E_{n+1} + D_1$$
$$\overline{Z} = 2\overline{S}_\infty + E_{n+1} + (3m + 1)(\overline{\ell}_\infty + E_1 + \cdots + E_{n+1}) - D_2,$$

where D_1 and D_2 are divisors chosen as in 2.5.3.1. 261

2.5.3.4 If $d = 6m + 3$ (i.e., $d = 3n$ with $n = 2m + 1$) then we have:

$$\overline{B} = \overline{C} + \overline{S}_\infty + (\overline{\ell}_\infty + E_1 + \cdots + E_n) + D_1$$
$$\overline{Z} = 2\overline{S}_\infty + (3m + 2)(\overline{\ell}_\infty + E_1 + \cdots + E_n) - D_2,$$

where D_1 and D_2 are divisors chosen as above.

2.5.3.5 If $d = 6m + 4$ (i.e., $d = 3n + 1$ with $n = 2m + 1$) then we have:

$$\overline{B} = \overline{C} + \overline{S}_\infty + D_1$$
$$\overline{Z} = 2\overline{S}_\infty + (3m + 2)(\overline{\ell}_\infty + E_1 + \cdots + E_n) - D_2,$$

where D_1 and D_2 are divisors chooser as above.

2.5.3.6 If $d = 6m + 5$ (i.e., $d = 3n + 2$ with $n = 2m + 1$) then we have:

$$\overline{B} = \overline{C} + \overline{S}_\infty + (\overline{\ell}_\infty + E_1 + \cdots + E_n) + D_1$$
$$\overline{Z} = 2\overline{S}_\infty + (3m + 3)(\overline{\ell}_\infty + E_1 + \cdots + E_{n+1}) - D_2,$$

where D_1 and D_2 are divisors chosen as above.

2.6

Let $\sigma : F \to \overline{F}$ be the shortest succession of quadratic transformations of $\overline{F}$ such that if one writes $((\overline{\sigma}\sigma)^* A)$ in the form $((\overline{\sigma}\sigma)^* A) = B - 2Z$ with divisors B and Z uniquely determined as in 2.5.3, every irreducible component of B is a connected component of $\mathrm{Supp}(B)$, i.e., $\mathrm{Supp}(B)$ is nonsingular. Let H be the normalization of F in K, and let $\rho : H \to F$ be the normalization morphism. We have a commutative diagram below:

$$\begin{tikzcd}
H \arrow[r, "\tau"] \arrow[d, "\rho"] & \overline{H} \arrow[r, "\overline{\tau}"] \arrow[d, "\overline{\rho}"] & H_0 \arrow[d, "\rho_0"] & \\
F \arrow[r, "\sigma"] & \overline{F} \arrow[r, "\overline{\sigma}"] & F_0 \arrow[r, "p_2"] & \mathbb{P}^1_k,
\end{tikzcd}$$

262 where τ and $\overline{\tau}$ are the canonical morphisms which make each of squares commutative. The following result is well-known (cf. [4]):

2.6.1

Lemma. *If $p \neq 2$ then H is a nonsingular projective surface defined over k.*

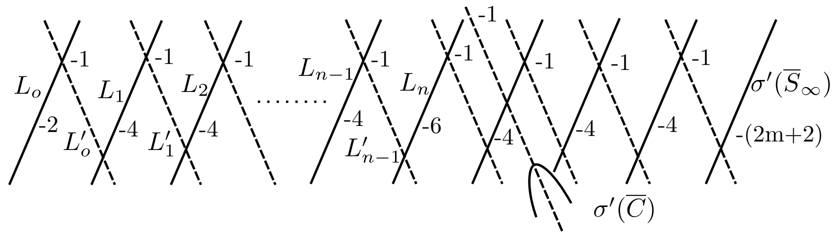
Proof. Let Q be a point of H, and let $P := \rho(Q)$. Let $\widetilde{\mathscr{O}} := \mathscr{O}_{Q,H}$ and let $\mathscr{O} := \mathscr{O}_{P,F}$. We shall show that $\widetilde{\mathscr{O}}$ is regular for every k-rational point Q. If $(\overline{\sigma}\sigma)(P)$ is not a singular point of C, $(\overline{\tau}\tau)(Q)$ is a simple point of H_0 (cf. Lemma 2.4.1). Hence Q is a simple point. Consider the case where $(\overline{\sigma}\sigma)(P)$ is a singular point of C. If $P \in \mathrm{Supp}(B)$, we may write $(\overline{\sigma}\sigma)^*A = hg^2$, where $h \in \mathscr{O}$ and $g \in k(x, y)$ such that $h = 0$ is a local equation of the irreducible component B_1 of B on which P lies. Since B_1 is nonsingular, h with some element h_1 of $\mathscr{O}$ form a regular system of parameters of $\mathscr{O}$. Then t/g and h_1 form a regular system of parameters of $\widetilde{\mathscr{O}}$. If $P \notin \mathrm{Supp}(B)$, $(\overline{\sigma}\sigma)^*A = g^2u$, where $g \in k(x, y)$ and u is a unit of $\mathscr{O}$. Then there are two distinct points on H above P, one of which is Q. Then Q is a simple point since $[K : k(x, y)] = 2$. □

2.6.2

Lemma. *Assume that $p = 2$. Let Q be a point of H, and let $P := \rho(Q)$. Then Q is a simple point if* (1) $(\overline{\sigma}\sigma)(P)$ *is not a singular point of C or if (2) $(\overline{\sigma}\sigma)(P) = P_\infty$ (cf. 2.5).*

Proof. The first case follows from Lemma 2.4.1. Consider the case (2). **263**
As in 2.4.4 we may write $\varphi(y) = y\varphi_1(y)^2$ with $d_1 = \deg_y \varphi_1(y)$ and $d = 2d_1 + 1$. We consider the following three cases separately: (I) $d_1 = 3m$, (II) $d_1 = 3m + 1$ and (III) $d_1 = 3m + 2$. □

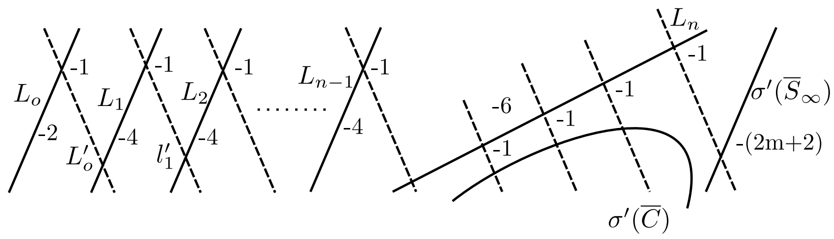
Case (I): $d_1 = 3m$. Then $d = 6m + 1$. The configuration of $(\overline{\sigma}\sigma)^{-1}(\ell_\infty \cup C \cup S_\infty)$ is easily obtained from the Figure 2 (where $n = 2m$):



(Fig. 4)

where the lines represent nonsingular projective rational curves and the numbers attached to lines are self-intersection multiplicities; where solid lines (including $\sigma'(\overline{C})$) are contained in B, while the broken lines are not contained in B; where $L_0 = \sigma'(\overline{\ell}_\infty)$, $L'_0 = \sigma^{-1}(\overline{\ell}_\infty \cap E_1)$, $L_i = \sigma'(E_i)$ and $L'_i = \sigma^{-1}(E_i \cap E_{i+1})$ for $1 \leqq i \leqq n$, $L_n = \sigma'(E_n)$ and the remaining unnamed lines arise from the quadratic transformations with centers at $E_n \cap \overline{S}_\infty$ and its infinitely near points. Note that each broken line has self-intersection multiplicity -1 and meets transversely two irreducible components of B. Let L be one of broken lines, and let B_1 and B_2 be
264 irreducible components of B which meet L. Let $\widehat{\tau} : F \to \widehat{F}$ be the contraction of L, and let $\widehat{P} := \widehat{\tau}(L)$, $\widehat{B}_1 := \widehat{\tau}(B_1)$ and $\widehat{B}_2 := \widehat{\tau}(B_2)$. Let $u = 0$ and $v = 0$ be local equations of $\widehat{B}_1$ and $\widehat{B}_2$ at $\widehat{P}$ on $\widehat{F}$. Let $\widehat{A}$ be the inverse image of $A(x, y)$ on $\widehat{F}$. Then $\widehat{A} = uvTg^2$, where $u, v, T \in \mathscr{O}_{\widehat{P},\widehat{F}}$, $T(\widehat{P}) \neq 0$ and $g \in k(x, y)$. If $P \in L$ and $P \neq L \cap B_2$, then $(\overline{\sigma}\sigma)^* A = u_1 T(vg)^2$ with $u_1 = u/v$, and $\mathscr{O}_{Q,H} \cong \mathscr{O}_{P,F}[z]/(z^2 - u_1 T)$. If $P = L \cap B_2$ then $(\overline{\sigma}\sigma)^* A = v_1 T(ug)^2$ with $v_1 = v/u$, and $\mathscr{O}_{Q,H} \cong \mathscr{Q}_{P,F}[z]/(z^2 - v_1 T)$. Hence if $P \in L$, $\mathscr{O}_{Q,H}$ is regular. If P lies on an irreducible component B_1 of B then $(\overline{\sigma}\sigma)^* A = ug^2$, where $g \in k(x, y)$ and $u = 0$ is a local equation of B_1 at P. Hence $\mathscr{O}_{Q,H} \cong \mathscr{O}_{P,F}[z]/(z^2 - u)$, and $\mathscr{O}_{Q,H}$ is regular.

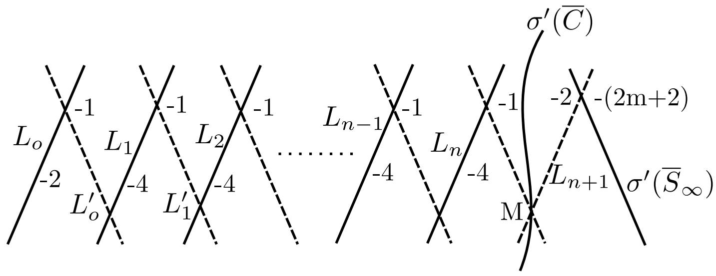
Case (II): $d_1 = 3m + 1$. Then $d = 6m + 3 = 3n$ with $n = 2m + 1$. The configuration of $(\overline{\sigma}\sigma)^{-1}(\ell_\infty \cup C \cup S_\infty)$ obtained from the Figure 1 is:



(Fig. 5)

where we use the same notation as in the Figure 4. Here, note again that each broken line has self-intersection multiplicity -1 and meets transversely two irreducible components of B. We can use the same argument as in the case (I) to show that $\mathscr{O}_{Q,H}$ is regular.

Case (III): $d_1 = 3m + 2$. Then $d = 6m + 5 = 3n + 2$ with $n = 2m + 1$. **265**
The configuration of $(\overline{\sigma}\sigma)^{-1}(\ell_\infty \cup C \cup S_\infty)$ obtained from the Figure 3 is:



(Fig. 6)

where $L_{n+1} = \sigma'(E_{n+1})$. Here all broken lines except L_{n+1} have self-intersection multiplicities -1 and meet transversely two irreducible components of B. Thus, if $P \notin L_{n+1}$ we can apply the argument in the case (I) to show that $\mathscr{O}_{Q,H}$ is regular. Consider the case where $P \in L_{n+1}$. Let $\widehat{\tau} : F \to \widehat{F}$ be the contraction of M and L_{n+1}, and let $\widehat{E_n} := \widehat{\tau}(L_n)$, $\widehat{C} := \widehat{\tau}(\sigma'(\overline{C}))$ and $\widehat{S}_\infty := \widehat{\tau}(\sigma'(\overline{S}_\infty))$. Let $u = 0$ and $v = 0$ be local equations of $\widehat{E_n}$ and $\widehat{S}_\infty$ respectively at the point $\widehat{P} := \widehat{E_n} \cap \widehat{S}_\infty$ on $\widehat{F}$. Then the

inverse image $\widehat{A}$ of $A(x,y)$ on $\widehat{F}$ is written in the form $\widehat{A} = uv(u^2+v^3T)g^2$ with suitable choice of u and v, where $T \in \mathscr{O}_{\widehat{P},\widehat{F}}$, $T(\widehat{P}) \neq 0$ and $g \in k(x,y)$. Then it is easy to show that $(\overline{\sigma}\sigma)^*A = u_2(u_2v+T)(gv^3)^2$ if $P \in M$ and $P \notin L_{n+1}$, $(\overline{\sigma}\sigma)^*A = (u_1+v_2T)(gv^2u_1)^2$ if $P \in L_{n+1}$ and $P \notin \sigma'(\overline{S}_\infty)$, and $(\overline{\sigma}\sigma)^*A = v_1(1+uv_1^3T)(gu^2)^2$ if $P = \sigma'(\overline{S}_\infty) \cap L_{n+1}$, where $u_1 = u/v$,
266 $v_1 = v/u$, $u_2 = u_1/v$ and $v_2 = v/u_1$. Hence $\mathscr{O}_{Q,H} \cong \mathscr{O}_{P,F}[z]/(z^2+u_1+v_2T)$ if $P \in L_{n+1}$ and $P \notin \sigma'(\overline{S}_\infty)$, and $\mathscr{O}_{Q,H} \cong \mathscr{O}_{P,F}[z]/(z^2 + v_1(1 + uv_1^3T))$ if $P = \sigma'(\overline{S}_\infty) \cap L_{n+1}$, whence follows that $\mathscr{O}_{Q,H}$ is regular.

2.6.3

In 2.9.3 below we prove that H is a nonsingular projective surface defined over k.

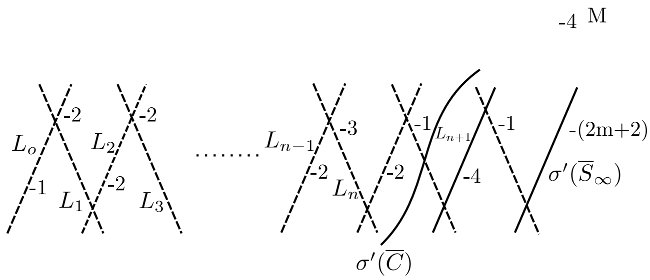
2.6.4

In the case where $p \neq 2$ it is easily seen that the configuration of $(\overline{\sigma}\sigma)^{-1}(\ell_\infty \cup C \cup S_\infty)$ is the following (cf. 2.5):

Case $d = 6m (m > 0)$. Figure 1 with $\overline{\ell}_\infty, F_1, \ldots, E_n$ replaces by broken lines.

Case $d = 6m + 1 (m > 0)$. Figure 4.

Case $d = 6m + 2 (m > 0)$. Then $d = 3n + 2$ with $n = 2m$.

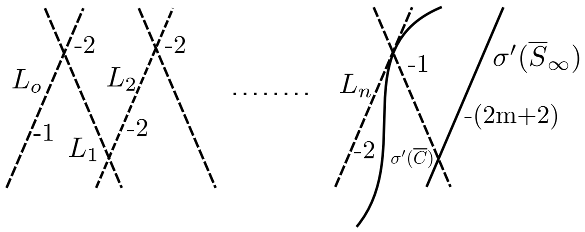


(Fig. 7)

where $L_0 = \sigma'(\overline{\ell}_\infty)$ and $L_i = \sigma'(E_i)$ for $1 \leqq i \leqq n+1$.

Case $d = 6m + 3$. Figure 5.

Case $d = 6m + 4$. Then $d = 3n + 1$ with $n = 2m + 1$.



(Fig. 8)

where $L_0 = \sigma'(\overline{\ell}_\infty)$, $L_i = \sigma'(E_i)$ for $1 \leqq i \leqq n$, and $(L_n \cdot \sigma'(\overline{C})) = 2$. **267**

Case $d = 6m + 5$. Figure 6.

2.7

We *assume* in this paragraph that H is nonsingular if $p = 2$ (cf. 2.6.3).

2.7.1

Lemma. (1) *Let D_1 and D_2 be divisors on F. Then* $(\rho^*(D_1) \cdot \rho^*(D_2)) = 2(D_1 \cdot D_2)$.

(2) *Let D be an irreducible component of B. Then $\rho^*(D) = 2\Delta$, where Δ is a nonsingular curve. If $D \cong \mathbb{P}^1_k$, so is Δ.*

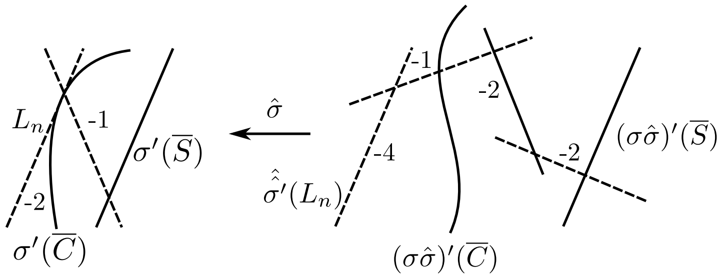
(3) *Assume that $p \neq 2$. Let D be a curve on F such that $D \cong \mathbb{P}^1_k$ and $D \not\subset \mathrm{Supp}(B)$.*

(i) *If $D \cap \mathrm{Supp}(B) = \phi$ then $\rho^*(D) = D_1 + D_2$ where $D_1 \cong D_2 \cong \mathbb{P}^1_k$, $D_1 \cap D_2 = \phi$ and $(D_1^2) = (D_2^2) = (D^2)$.*

(ii) *If D meets exactly two irreducible components B_1 and B_2 of B transversely and if $D \cap B_1 \neq D \cap B_2$ then $\rho^{-1}(D)$ is irreducible and isomorphic to $\mathbb{P}^1_k$.*

(iii) *Let $D = L_n$ in the Figure 8. Then $\rho^*(D) = D_1 + D_2$, where $D_1 \cong D_2 \cong \mathbb{P}^1_k$, $(D_1^2) = (D_2^2) = -3$ and $(D_1 \cdot D_2) = 1$.*

Proof. (1) and (2) are well-known (cf. [4]) and easy to prove. (3) (i): $\rho^{-1}(D)$ is an unramified covering of $D \cong \mathbb{P}^1_k$ of degree 2. Since $p \neq 2$ and D is simply connected, we have $\rho^*(D) = D_1 + D_2$ with $D_1 \cong D_2 \cong \mathbb{P}^1_k$ and $D_1 \cap D_2 = \phi$. (ii): Let $\rho^*(B_1) = 2\Delta_1$ and $\rho^*(B_2) = 2\Delta_2$. Then $\Delta_1 \cong \Delta_2 \cong \mathbb{P}^1_k$. Since $(\rho^*(D)\cdot\Delta_1) = (\rho^*(D)\cdot\Delta_2) = 1$ and since every point except $D \cap B_1$ and $D \cap B_2$ is not branched, we know that $\rho^{-1}(D) \cap \Delta_1$ and
268 $\rho^{-1}(D) \cap \Delta_2$ are simple points of $\rho^{-1}(D)$, and that $\rho^{-1}(D)$ is a nonsingular irreducible curve. Then, by Hurwitz's formula, $\rho^{-1}(D)$ is isomorphic to $\mathbb{P}^1_k$. (iii): Let $P := L_n \cap \sigma'(\overline{C})$. By the quadratic transformations $\widehat{\sigma}$ with centers at P and a point of $\sigma'(\overline{C})$ infinitely near P we have the following configuration:



This implies that $\rho^*(L_n) = D_1 + D_2$, where $D_1 \cong D_2 \cong \mathbb{P}^1_k$. On the other hand, 2.5.3.5 implies that $\overline{\sigma}^*A = u(v + u^3T)g^2$, where u, v, $T \in \mathscr{O}_{\overline{P},\overline{F}}$ with $\overline{P} := \overline{S}_\infty \cap \overline{C}$, $T(\overline{P}) \neq 0$, $g \in k(x, y)$ and where $u = 0$ and $v = 0$ are local equations of $\overline{S}_\infty$ and E_n. Then $(\overline{\sigma}\sigma)^*A = (v_1 + u^2T)(gu)^2$ locally at P, where $v_1 = v/u$; and $\mathscr{O}_{Q,H} \cong \mathscr{O}_{P,F}[z]/(z^2 - (v_1 + u^2T))$ with $Q = \rho^{-1}(P)$. Since L_n is defined by $v_1 = 0$, $\rho^{-1}(L_n)$ is defined by $z^2 = u^2T$. Thus, $(D_1 \cdot D_2) = 1$. Since $(D_1^2) = (D_2^2)$ and $(\rho^*(L_n)^2) = -4$, we have $(D_1^2) = (D_2^2) = -3$. □

2.7.2

Let $q := (p_2\overline{\sigma}\sigma\rho) : H \to \mathbb{P}^1_k$, $\widetilde{C} := \sigma'(\overline{C})$ and $\widetilde{S}_\infty := \sigma'(\overline{S}_\infty)$. Since $\overline{C}, \overline{S}_\infty \subset \mathrm{Supp}(\overline{B})$ we have $\widetilde{C}, \widetilde{S}_\infty \subset \mathrm{Supp}(B)$. Hence $\rho^{-1}(\widetilde{C}) = 2\Gamma$ and $\rho^{-1}(\widetilde{S}_\infty) = 2\Sigma_\infty$ with nonsingular curves Γ and Σ_∞ on H (cf. 2.7.1, (2)). We have then the following:

Lemma. *Assume that H is nonsingular if $p = 2$. Then $q : H \to \mathbb{P}^1_k$ is* 269
an elliptic or quasi-elliptic fibration with regular cross-section Σ_∞. The fibration q is elliptic if $p \neq 2, 3$; and q is quasi-elliptic if $p = 2$ or 3. Moreover we have:

(1) *If $p = 3$, Γ is the locus of movable singular points of q.*

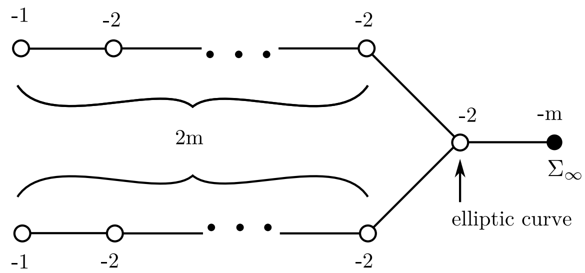
(2) *If $p = 2$, let S_0 be the cross-section of $\mathscr{F}$ defined by $x = 0$, and let $\Delta := \rho^{-1}((\overline{\sigma}\sigma)'S_0)$. Then Δ is the locus of movable singular points of q.*

Proof. Let ℓ be a general member of $\mathscr{F}$, and let $\widetilde{\ell} = (\overline{\sigma}\sigma)'(\ell)$. Since $(\widetilde{\ell} \cdot \widetilde{S}_\infty) = 1$ we have $(\rho^*(\widetilde{\ell}) \cdot \Sigma_\infty) = 1$ which implies that $\rho^{-1}(\widetilde{\ell})$ is irreducible and $\rho^{-1}(\widetilde{\ell}) \cap \Sigma_\infty$ is a simple point of $\rho^{-1}(\widetilde{\ell})$. Since $\rho^{-1}(\widetilde{\ell}) - \rho^{-1}(\widetilde{\ell}) \cap \Sigma_\infty$ is isomorphic to a curve $t^2 = x^3 + \varphi(\alpha)$ for some $\alpha \in k$, $p_a(\rho^{-1}(\widetilde{\ell})) = 1$. Thus q is an elliptic or quasi-elliptic fibration with regular cross-section Σ_∞. $\rho^{-1}(\widetilde{\ell})$ has a unique singular point $(t = 0, x = -\varphi(\alpha)^{1/3})$ if $p = 3$; $(t = \varphi(\alpha)^{1/2}, x = 0)$. Thus, q is quasi-elliptic if $p = 2$ or 3; and Γ is the locus of movable singular points of q if $p = 3$, and Δ is the locus of movable singular points of q if $p = 2$. (It will be easy to see that Δ is irreducible). □

2.7.3

Let $q^{-1}(\infty)$ be the fiber of q corresponding to $y = \infty$. To illustrate $q^{-1}(\infty) \cup \Sigma_\infty$ we shall define *the weighted graph of* $q^{-1}(\infty) \cup \Sigma_\infty$ in the following way: Assign a vertex $\circ$ (or $\circ$, resp.) to each irreducible component T of $q^{-1}(\infty) \cup \Sigma_\infty$ such that $\rho(T) \not\subset \mathrm{Supp}(B)$ (or $\rho(T) \subset \mathrm{Supp}(B)$, resp.); the weight is (T^2); join two vertices by a single edge like $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ$ (or a double edge like $\circ\!\!=\!\!\!=\!\!\!=\!\!\circ$) if the corresponding irreducible components meet each other transversely in one point (or, touch 270
in one point with multiplicity 2, resp.). By virtue of 2.6.4 and 2.7.1 we have the following weighted graphs of $q^{-1}(\infty) \cup \Sigma_\infty$ when $p \neq 2$:

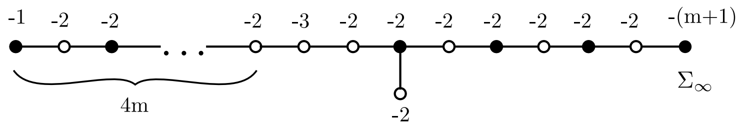
Case $d = 6m(m > 0)$ and $p \neq 3$.



(Fig. 9)

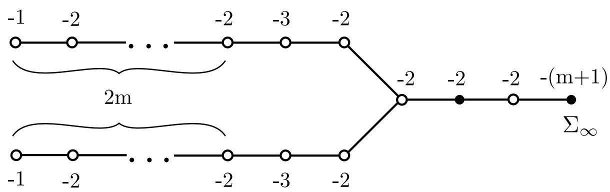
where all components of $q^{-1}(\infty) \cup \Sigma_\infty$ except one elliptic component are nonsingular projective rational curves; in the cases given below all components are nonsingular projective rational curves.

Case $d = 6m + 1 (m > 0)$.



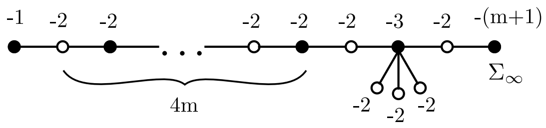
(Fig. 10)

Case $d = 6m + 2 (m > 0)$.



(Fig. 11)

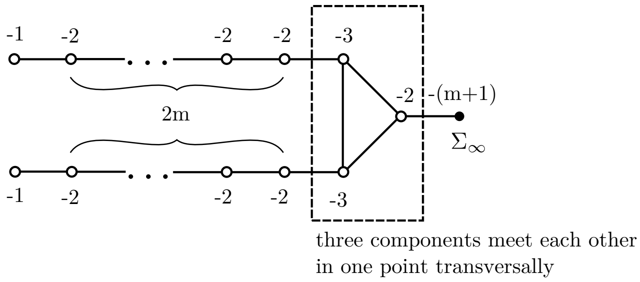
Case $d = 6m + 3 (m \geqq 0)$ and $p \neq 3$.



(Fig. 12)

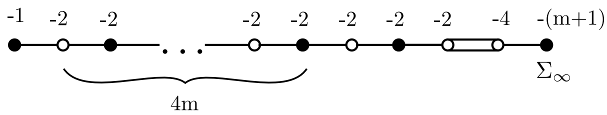
271

Case $d = 6m + 4 (m \geqq 0)$.



(Fig. 13)

Case $d = 6m + 5 (m \geqq 0)$



(Fig. 14)

2.7.4

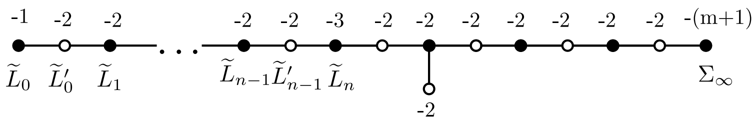
Lemma. *Assume that $p = 2$ and H is nonsingular. Then the weighted graph of $q^{-1}(\infty) \cup \Sigma_\infty$ is given as follows:*

Case $d = 6m + 1 (m > 0)$. **Figure 10.**

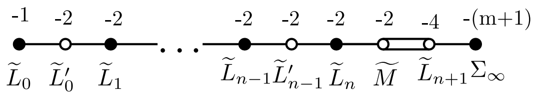
Case $d = 6m + 3 (m \geqq 0)$. **Figure 12.**

Case $d = 6m + 5 (m \geqq 0)$. **Figure 14.**

Proof. Lemma follows from 2.3.4 and 2.7.2. We shall only indicate how to use these results. **Case** $d = 6m+1$. With the notations of 2.6.2, for all solid lines L, $\rho^{-1}(L) = 2\widetilde{L}$ with $\widetilde{L} \cong \mathbb{P}^1_k$ and $2(\widetilde{L}^2) = (L^2)$; for all broken lines L with $(L^2) = -1$, $\widetilde{L} := \rho^{-1}(L)$ is irreducible. Thus the weighted graph of $q^{-1}(\infty) \cup \Sigma_\infty$ is:



272 where $\bullet$ represents a nonsingular projective rational curve. Let ν be the contraction of $\widetilde{L}_0$. If $\nu(\widetilde{L}'_0) \neq \mathbb{P}^1_k$, $\nu(q^{-1}(\infty))$ would be a reducible fiber of a relatively minimal quasi-elliptic fibration. Then, by lemma 2.3.4, $\nu(\widetilde{L}'_0) \cong \mathbb{P}^1_k$, which is a contradiction. Hence $\nu(\widetilde{L}'_0) \cong \mathbb{P}^1_k$ and $(\nu(\widetilde{L}'_0)^2) = -1$, whence $\nu(\widetilde{L}'_0)$ is contractible. Repeating this argument for $\widetilde{L}_0, \widetilde{L}'_0, \ldots, \widetilde{L}'_{n-1}$ we can see that they are all isomorphic to $\mathbb{P}^1_k$. Let π be the contraction of $\widetilde{L}_0, \ldots, \widetilde{L}'_{n-1}$. Then $\pi(\widetilde{L}_n) \cong \mathbb{P}^1_k$ and $(\pi(\widetilde{L}_n)^2) = -2$. Hence $\pi(q^{-1}(\infty))$ is a reducible fiber of a relatively minimal quasi-elliptic fibration. Then the remaining components are all isomorphic to $\mathbb{P}^1_k$ by virtue of 2.3.4. The **case** $d = 6m + 3$ can be treated in the same fashion as above. **Case** $d = 6m+5$. The weighted graph of $q^{-1}(\infty) \cup \Sigma_\infty$ is:



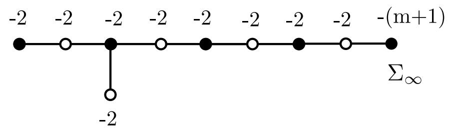
where represents a curve isomorphic to $\mathbb{P}^1_k$; $\widetilde{L}_{n+1} := \rho^{-1}(L_{n+1})$ is reduced (and irreducible) because $(\widetilde{L}_{n+1} \cdot \Sigma_\infty) = 1$; $\widetilde{M} := \rho^{-1}(M)$ touches $\widetilde{L}_{n+1}$ in one point with multiplicity 2. The foregoing argument shows that $\widetilde{L}_0, \ldots, \widetilde{L}_n$, $\widetilde{M}$ are isomorphic to $\mathbb{P}^1_k$ and contractible. Let π be the contraction of those curves. Then $\pi(\widetilde{L}_{n+1})$ is an irreducible member of a relatively minimal quasi-elliptic fibration. Hence $\pi(\widetilde{L}_{n+1})$ has one cusp, and $\widetilde{L}_{n+1}$ is a nonsingular projective rational curve. □ **273**

2.7.5

By contracting all possible exceptional components of $q^{-1}(\infty)$, the image of $q^{-1}(\infty) \cup \Sigma_\infty$ has the following weighted graph (or configuration); the type of a singular fiber according to Šafarevič [51] is also given:

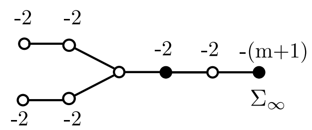
Case $d = 6m$. o (elliptic curve)

Case $d = 6m + 1$.



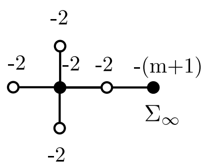
B_{10}

Case $d = 6m + 2$.



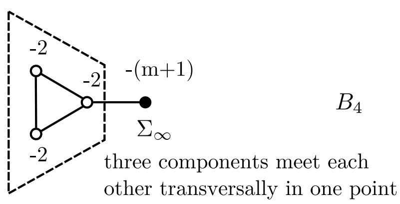
B_8

Case $d = 6m + 3$.



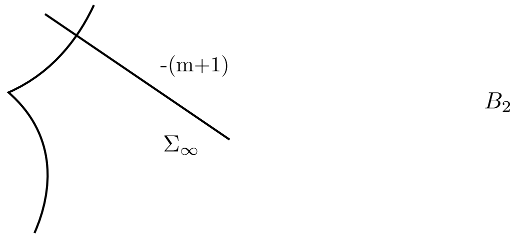
B_6

Case $d = 6m + 4$. **274**



B_4

Case $d = 6m + 5$.



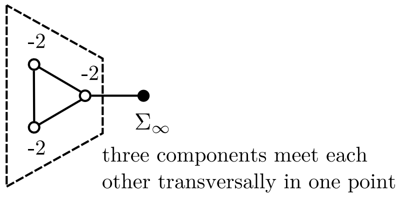
2.8

We shall proceed to a proof of Theorem 2.1.1. It is easy to see that K is rational over k if $d = 0$, 1 or 2. We shall therefore assume that $d > 3$.

2.8.1

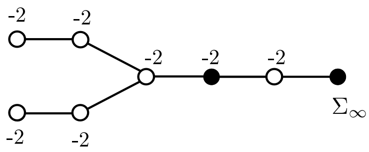
Consider reducible fibers of $q : H \to \mathbb{P}^1_k$ other than $q^{-1}(\infty)$. Such a fiber $q^{-1}(\alpha) := (\overline{\sigma}\sigma\rho)^{-1}(\ell_\alpha)$ has more than two reducible components, and $\ell_\alpha \cap C$ is a singular point of C, whence $\varphi'(\alpha) = 0$. Conversely, for a root α of $\varphi'(y) = 0$, $q^{-1}(\alpha)$ is a reducible fiber of q (cf. 2.5). Let α be a root of $\varphi'(y) = 0$, let $P := (x = -\varphi(\alpha)^{1/3}.y = \alpha)$ be the corresponding singular point of C, and let $e = v_\alpha(\varphi(y) - \varphi(\alpha))$. The condition (1) of Theorem 2.1.1 tells us that $e = 2$, 3, 4 or 5, while the condition (2) asserts that
275 the case $e = 3$ can be reduced to the case $e = 4$ or 5 by a birational transformation $(t, x, y) \mapsto (t, x + a^{1/3}(y - \alpha), y)$ which is biregular at P. P is then a cuspidal singular point of C with multiplicity $(2, 1, \ldots)$ if $e = 2$; $(3, 1, \ldots)$ if $e = 4$; $(3, 2, 1, \ldots)$ if $e = 5$. Now the weighted graph of $q^{-1}(\alpha) \cup \Sigma_\infty$ is given as follows by making a similar argument as in 2.5:

Case $e = 2$.

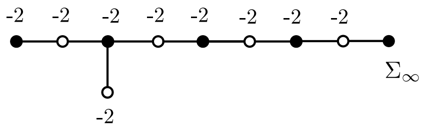


(cf. 2.7.1, (3) (iii))

Case $e = 4$.



Case $e = 5$.



Notice that $q^{-1}(\alpha)$ contains no exceptional components. The type of a singular fiber according to Šafarevič [51] is B_4 if $e = 2$; B_8 if $e = 4$; B_{10} if $e = 5$.

2.8.2

As shown in 2.8.1, $q^{-1}(\infty)$ is the only singular fiber in the fibration q, **276**
which contains exceptional components. By a contraction $\widehat{\tau} : H \to \widehat{H}$ of all exceptional components in $q^{-1}(\infty)$ we get a relatively minimal quasi-elliptic surface $\widehat{q} : \widehat{H} \to \mathbb{P}^1_k$ with $\widehat{q}\widehat{\tau} = q$, for which $\widehat{\Sigma}_\infty := \widehat{\tau}(\Sigma_\infty)$ is a regular cross-section with $(\widehat{\Sigma}^2_\infty) = -(m+1)$. (Since we are dealing with the case $p = 3$, look at only the cases $d = 6m + 1$, $d = 6m + 2$, $d = 6m + 4$ and $d = 6m + 5$). Now, Theorem 2 follows immediately from 2.2, 2.3.2 and 2.3.3.

2.9

We shall now prove Theorem 2.1.2. The first assertion follows from 2.2. So, we shall prove the second assertion. To be in accordance with the notations in the paragraphs 2.4 ~ 2.7 we start with an equation $t^2 = x^3 + \varphi(y)$, where $\varphi(y) = y\varphi_1(y)^2$, $d := \deg_y \varphi$ and $d_1 := \deg_y \varphi_1$. Hence $d = 2d_1 + 1$. We have only to consider the cases $d = 6m+1$, $d = 6m+3$ and $d = 6m+5$. Moreover, since K is easily seen to be rational over k if $d = 1$ we assume that $d \geqq 3$.

2.9.1

Write

$$\varphi_1(y) = a(y-\alpha_1)^{r_1} \dots (y-\alpha_s)^{r_s},$$

where $a \in k^*$, $\alpha_i \in k$, and α_i's are mutually distinct. The assumption in Theorem 2.1.2 implies that $r_i \leqq 2$ for every i. For if $r_i \geqq 3$, $\varphi(y+\alpha_i)$ starts with a term of degree $\geqq 6$. The singular points of C lying on the affine part $F_0 - (x = \infty) \cup (y = \infty)$ are given by $(x = 0, y = \alpha_i)$ for $1 \leqq i \leqq s$. Let $P := (t = 0, x = 0, y = \alpha)$ with $\varphi_1(\alpha) = 0$. By a birational transformation $(t, x, y) \mapsto (t, x, y + \alpha)$ which is biregular at P,
277 H_0 is a hyper surface in $\mathbb{A}^3_k$ defined locally at P by one of the following equations:

(i) $t^2 = x^3 + y^2\delta(y)$ if $\alpha \neq 0$ and $r = 1$

(ii) $t^2 = x^3 + y^3\delta(y)$ if $\alpha = 0$ and $r = 1$

(iii) $t^2 = x^3 + y^4\delta(y)$ if $\alpha \neq 0$ and $r = 2$

(iv) $t^2 = x^3 + y^5\delta(y)$ if $\alpha = 0$ and $r = 2$,

where $\delta(y) \in k[y]$, $\delta(0) \neq 0$, and $\delta(y)y^{2r+\epsilon} = \varphi(y+\alpha)$ with $\epsilon = 0$ or 1 according as $\alpha \neq 0$ or $\alpha = 0$.

2.9.2

Now write

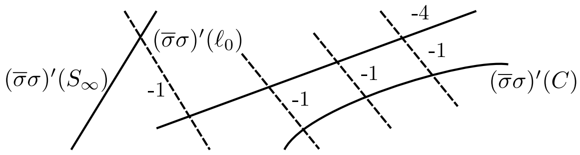
$$\delta(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \cdots \quad \text{with} \quad a_0 \neq 0.$$

The case (i) above is now reduced to the case (ii) or (iv) by a birational transformation $(t, x, y) \mapsto (t + a_0^{1/2}y + a_2^{1/2}y^2, x, y)$ which is biregular at P. Namely, if $a_1 \neq 0$ we have the case (ii); if $a_1 = 0$ and $a_3 \neq 0$ we have case (iv). (Note that $a_1 = a_3 = 0$ does not occur). Similarly, the case (iii) is reduced to the case (iv). Thus, in order to look into the singularity of P, we have only to consider the cases (ii) and (iv).

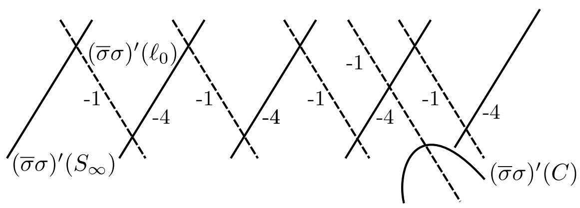
2.9.3

Let ℓ_0 be the member of $\mathscr{F}$ passing through the point $(x = 0, y = \alpha)$. The configuration of $(\overline{\sigma}\sigma)^{-1}(\ell_0 \cup C \cup S_\infty)$ is given as follows:

Case (ii):



Case (iv): 278



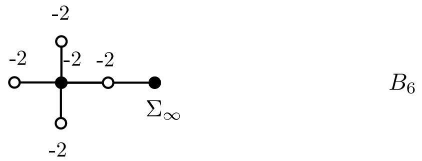
The meanings of solid or broken lines are the same as in 2.6.2. By the same argument as in 2.6.2 (especially the proof of Case (I) there) we can show:

Lemma. *Let Q be a point on H such that $(\overline{\sigma}\sigma\rho)(Q) = (x = 0, y = \alpha)$. Then Q is a simple point. Therefore H is nonsingular.*

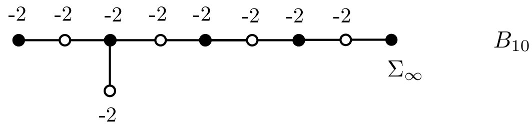
2.9.4

Now, the weighted graph of $q^{-1}(\alpha) := \rho^{-1}((\overline{\sigma}\sigma)'(\ell_0))$ is given as follows:

Case (ii).



Case (iv).



(For the proof, see 2.7.4). Thus, $q^{-1}(\alpha)$ contains no exceptional components. The proof of Theorem 2.1.2 is now completed as in 2.8.2. (Consider only the cases $d = 6m + 1$, $d = 6m + 3$ and $d = 6m + 5$).

3 Unirational surface with a pencil of quasi-hyper-elliptic curves of genus 2 (in characteristic 5)

279

3.1

Throughout this section the ground field k is assumed to be an algebraically closed field of characteristic 5. A nonsingular projective surface X is said to have *a pencil of quasi-hyper-elliptic curves of genus* 2 if there exists a surjective morphism $f : X \to C$ from X to a nonsingular projective curve C such that almost all fibers of f are irreducible singular curves of arithmetic genus 2. *We assume that $f : X \to C$ has a rational cross-section*. The purpose of this section is to prove the following:

Theorem . *Let k be an algebraically closed field of characteristic 5. Then any unirational surface X with a pencil of quasi-hyperelliptic curves of genus 2 defined over k is birationally equivalent to a hyper surface in $\mathbb{A}^3_k : t^2 = x^5 + \varphi(y)$ with $\varphi(y) \in k[y]$, provided X has a rational cross-section. Conversely, let $K := k(t, x, y)$ be the function field of an affine hyper surface of the above type. Assume that $\varphi(y)$ satisfies the conditions:*

(1) *$\varphi(y)$ has no terms of degree multiples of 5,*

(2) *every root of $\varphi'(y)\left(= \dfrac{d\varphi}{dy}\right)$ is at most a double point.*

Let $d := \deg_y \varphi$, $m := [d/10]$ and X the nonsingular minimal model if K is not rational. Then the structure of X is determined as follows:

Case $m = 0$. 280

d	1	2	3	4	6	7	8	9
$p_a(X)$	0	0	0	1	1	2	2	2
(K_X^2)				0	0	1	1	1
structure	rational surface			unirational K3-surface		unirational surface of general type		

Case $m > 0$.

d	$10m+1$	$10m+2$	$10m+3$	$10m+4$	$10m+6$	$10m+7$	$10m+8$	$10m+9$
$p_a(X)$	$4m$	$4m$	$4m$	$4m+1$	$4m+1$	$4m+2$	$4m+2$	$4m+2$
(K_X^2)	$8m-4$	$8m-4$	$8m-3$	$8m-2$	$8m-2$	$8m$	$8m$	$8m$

The surface X is then a unirational surface of general type.

A proof which is given in the subsequent paragraphs will be not more than a sketchy one, as the arguments are similar to the ones in the previous section.

3.2

Let $f : X \to C$ be as in 3.1. Let $\mathscr{R}$ be the function field of C over k and let $X_{\mathscr{R}}$ be the generic fiber of f. Then $X_{\mathscr{R}}$ is an irreducible normal projective curve with $p_a(X_{\mathscr{R}}) = 2$. Let $\mathscr{R}_s$ be a separable algebraic closure of $\mathscr{R}$. by Chevalley [12; Th. 5, p.99], $X \underset{\mathscr{R}}{\otimes} \mathscr{R}_s$ is a normal projective curve of arithmetic genus 2. Hence, every singular point of $X_{\mathscr{R}}$ is a one-place point and is rational over a purely inseparable extension of $\mathscr{R}$. If the
281 geometric genus of $X_{\mathscr{R}}$ equals 1 then $X_{\mathscr{R}}$ has a single ordinary cusp of multiplicity 2 as its unique singularity. However, this is impossible by virtue of Tate [55] because the characteristic of k is 5. Thus the geometric genus of $X_{\mathscr{R}}$ is 0 and $X_{\mathscr{R}}$ has either two ordinary cusps of multiplicity 2 or a single cuspidal point of multiplicity $(2, 2, 1, \ldots)$ as its singularity. We shall see that the former case does not occur. Indeed, let Q be one of two ordinary cusps, let Γ be the closure of Q in X and let $f_\Gamma : \Gamma \to C$ be the restriction of f onto Γ. Since Q is a one-place point of $X_{\mathscr{R}}$, f_Γ is a generically one-to-one morphism. Hence deg f_Γ is a power p^n of the characteristic p of k. For a point P of C such that $f^{-1}(P)$ meets Γ at a simple point of Γ, we have $(f^{-1}(P) \cdot \Gamma) = 2$ or 3 because $f^{-1}(P) \cap \Gamma$ is an ordinary cusp of multiplicity 2 on $f^{-1}(P)$. This is a contradiction because $p = 5$. Therefore we know that $X_{\mathscr{R}}$ is an irreducible normal projective curve of arithmetic genus 2 and geometric genus 0 and with a single cuspidal point of multiplicity $(2, 2, 1, \ldots)$ as its unique singularity. This implies that $X_{\mathscr{R}}$, minus the unique singular point, is a $\mathscr{R}$-form of the affine line $\mathbb{A}^1$ of $\mathscr{R}$-genus 2. We *assume* that $f : X \to C$ has a rational cross-section, *viz.* $X_{\mathscr{R}}$ has a $\mathscr{R}$-rational point. Then, by virtue of Lemma 1.5.1, $X_{\mathscr{R}}$ is $\mathscr{R}$-birationally equivalent to an affine plane curve:

$$t^2 = x^5 + a \quad \text{with} \quad a \in \mathscr{R} - \mathscr{R}^5. \tag{1}$$

The surface X is unirational over k if and only if C is rational over k.
282 Indeed, the "only if" part follows from the Lüroth's theorem; the "if" part holds because $\mathscr{R}^{1/5} \underset{\mathscr{R}}{\otimes} k(X)$ is rational over k. Now *assume* that X is unirational over k and write $\mathscr{R} := k(y)$. Then X is k-birationally equivalent to a hyper surface in $\mathbb{A}^3_k$:

$$t^2 = x^5 + \varphi(y) \quad \text{with} \quad \varphi(y) \in k[y], \tag{2}$$

where $d := \deg_y \varphi$ is prime to 5.

This proves the first assertion in the theorem. Conversely, let $K := k(t, x, y)$ be the function field of a hyper surface (2) and let X be a nonsingular projective model of K; we denote by X a nonsingular minimal model of K if K is not rational over k. We may assume without loss of generality that the following conditions are satisfied:

(i) $\varphi(y)$ has no terms of degree multiples of 5.

(ii) Let α and $\beta(\in k)$ satisfy $\varphi'(\alpha) = 0$ and $\beta^5 + \varphi(\alpha) = \sigma$. Then $e := v_\alpha(\varphi(y) - \varphi(\alpha))$ satisfies $2 \leqq e \leqq 9$ and $e \neq 5$.

Indeed, a birational transformation of the type $(t, x, y) \mapsto (t, x + \rho(y), y)$ annihilates the terms of degree multiples of 5 in $\varphi(y)$. With the notations of (ii), the hyper surface is written as

$$t^2 = (x-\beta)^5 + (y-\alpha)^e \varphi_1(y) \quad \text{with} \quad \varphi_1(\alpha) \neq 0.$$

If $e \geqq 10$ the degree of $\varphi(y)$ drops by a birational transformation $(t, x, y) \mapsto (t/(y-\alpha)^5, (x-\beta)/(y-\alpha)^2, y)$. Hence we may assume that $e \leqq 9$. If $e = 5$ a birational transformation $(t, x, y) \mapsto (t, (x-\beta) + \gamma(y-\alpha), y)$ with $\gamma^5 = \varphi_1(0)$ enables us to assume. $e \geqq 6$.

3.3

Set $A(x, y) := x^5 + \varphi(y)$. Embed $\mathbb{A}_k^2 := \mathrm{Spec}(k[x, y])$ into $F_0 := \mathbb{P}_k^1 \times \mathbb{P}_k^1$ **283**
as the complement of two lines $(x = \infty) \cup (y = \infty)$ and let C be the curve on F_0 defined by $A(x, y) = 0$. Let H_0 be the normalization of F_0 in $K := k(t, x, y)$ and let $\rho_0 : H_0 \to F_0$ be the normalization morphism. Then ρ_0 is a double covering with C contained in the branch locus. We shall look for a de singularization of H_0. As is well-known (cf. 2.5 and 2.6), a de singularization of H_0 is obtained as follows. Let $\overline{\sigma} : \overline{F} \to F_0$ be the shortest succession of quadratic transformations with centers at singular points of C such that $\overline{C} := \overline{\sigma}'(C)$ is nonsingular, let $\overline{H}$ be the normalization of $\overline{F}$ in K and let $\overline{\rho} : \overline{H} \to \overline{F}$ be the normalization morphism. Write $(\overline{\sigma}^* A)$ in the form:

$$(\overline{\sigma}^* A) = \overline{B} - 2\overline{Z},$$

where $\overline{B}$ is a divisor whose coefficient at each prime divisor is 0 or 1 and $\overline{Z}$ is some divisor. Then $\overline{B}$ is the branch locus of a double covering $\overline{\rho} : \overline{H} \to \overline{F}$. If $\overline{B}$ is nonsingular then $\overline{H}$ is nonsingular. If $\overline{B}$ is singular, let $\sigma : F \to \overline{F}$ be the shortest succession of quadratic transformations with centers at singular points of $\mathrm{Supp}(\overline{B})$ such that, if one writes $((\overline{\sigma}\sigma)^* A) = B - 2Z$ with divisors B and Z as above, B is nonsingular, *viz.* every irreducible component of $\mathrm{Supp}(B)$ is a connected component. Let H be the normalization of F in K and let $\rho : H \to F$ be the normalization morphism. Then ρ is a double covering with branch locus B, and H is nonsingular. Thus H is a de singularization of H_0; we have a commutative diagram

$$\begin{tikzcd} H \arrow[r, "\tau"] \arrow[d, "\rho"] & \overline{H} \arrow[r, "\overline{\tau}"] \arrow[d, "\overline{\rho}"] & H_0 \arrow[d, "\rho_0"] \\ F \arrow[r, "\sigma"] & \overline{F} \arrow[r, "\overline{\sigma}"] & F_0, \end{tikzcd}$$

284 where τ and $\overline{\tau}$ are the canonical morphisms induced by the normalizations. Set $\pi := \sigma \cdot \rho$. The curve $\overline{B}$ is said to have *a negligible singularity* at a point P if one of the following conditions is satisfied:

(1) $\overline{B}$ is nonsingular at P,

(2) P is a double point,

(3) P is a triple point with at most a double point (not necessarily ordinary) infinitely near.

If $\overline{B}$ has only negligible singularities then some numerical invariants of H are computable at the stage of $\overline{F}$. Namely we have:

Lemma Artin [4]. *Assume that $\overline{B}$ has only negligible singularities. Then the following assertions hold true.*

(1) $\pi^*(K_{\overline{F}} + \overline{Z})$ *is a canonical divisor* K_H *on* H.

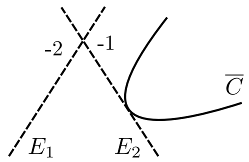
(2) $p_a(H) = 2p_a(F_0) + p_a(\overline{Z}) = p_a(\overline{Z})$.

(3) $(K_H^2) = 2((K_{\overline{F}} + \overline{Z})^2)$.

3.4

We shall look into singular points of the curve C on F_0. Let $P := (x = \beta, y = \alpha)$ be a singular point of C lying on the affine part $\mathbb{A}_k^2 : F_0 - (x = \infty) \cup (y = \infty)$. Then $\varphi'(\alpha) = \beta^5 + \varphi(\alpha) = 0$. Conversely, every root of $\varphi'(y) = 0$ gives rise to a singular point of C lying on $\mathbb{A}_k^2$. Let $P := (x = \beta, y = \alpha)$ be such a singular point and let $e := V_\alpha(\varphi(y) - \varphi(\alpha))$. Then **285**
P is a one-place point of C, and we have $2 \leqq e \leqq 9$ and $e \neq 5$ as we assumed in 3.2. Let D_P be the contribution by P in the effective divisor $\overline{\sigma}^*(C) - \overline{C}$, and write $D_P = D_P^{(1)} + 2D_P^{(2)}$, where $D_P^{(1)} \geqq 0$, $D_P^{(2)} \geqq 0$ and every component of $D_P^{(1)}$ has multiplicity 1. Let $D_P^{(3)}$ be the contribution by P in the effective divisor $K_{\overline{F}} - \overline{\sigma}^*(K_{F_0})$. In the following, we shall compute the values $\mu_P := \dfrac{1}{2}(D_P^{(2)} \cdot D_P^{(2)} - D_P^{(3)})$ and $\nu_P := ((D_P^{(2)} - D_P^{(3)})^2)$ for each type of a singular point P of C.

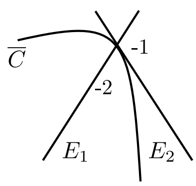
Case $e = 2$. Then P has multiplicity $(2, 2, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the configuration



with $(E_2 \cdot \overline{C}) = 2$

(cf. 2.6.2 for the conventions on the broken lines and solid curve (or line)). Hence $D_P = 2E_1 + 4E_2$, $D_P^{(1)} = 0$, $D_P^{(2)} = E_1 + 2E_2$ and $D_P^{(3)} = E_1 + 2E_2$; $\mu_P = \nu_P = 0$.

Case $e = 3$. Then P has multiplicity $(3, 2, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the configuration:

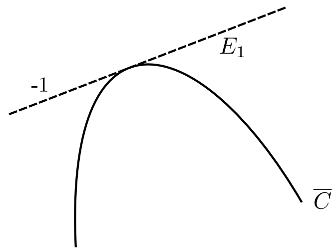


with $(E_2 \cdot \overline{C}) = 2.$

Then we have: $D_P = 3E_1 + 5E_2$, $D_P^{(1)} = E_1 + E_2$, $D_P^{(2)} = E_1 + 2E_2$, **286**

$D_P^{(3)} = E_1 + 2E_2$ and $\mu_P = \nu_P = 0$.

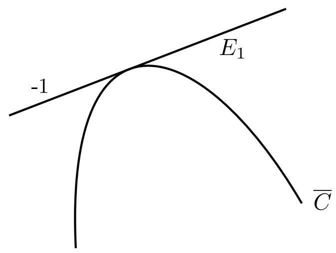
Case $e = 4$. P has multiplicity $(4, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the following configuration:



with $(E_1 \cdot \overline{C}) = 4$.

Then we have: $D_P = 4E_1$, $D_P^{(1)} = 0$, $D_P^{(2)} = 2E_1$, $D_P^{(3)} = E_1$ and $\mu_P = \nu_P = -1$.

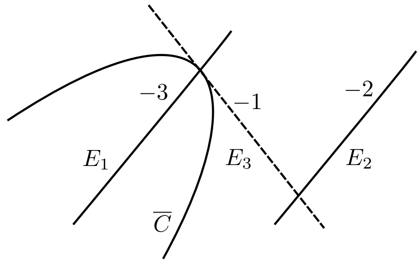
Case $e = 6$. P has multiplicity $(5, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the configuration:



with $(E_1 \cdot \overline{C}) = 5$.

Then we have: $D_P = 5E_1$, $D_P^{(1)} = E_1$, $D_P^{(2)} = 2E_1$, $D_P^{(3)} = E_1$ and $\mu_P = \nu_P = -1$.

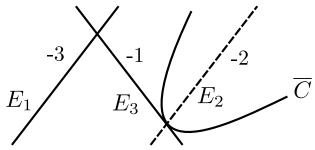
Case $e = 7$. P has multiplicity $(5, 2, 2, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the configuration:



with $(E_3 \cdot \overline{C}) = 2.$

Then we have: $D_P = 5E_1 + 7E_2 + 14E_3$, $D_P^{(1)} = E_1 + E_2$, $D_P^{(2)} =$ **287**
$2E_1 + 3E_2 + 7E_3$, $D_P^{(3)} = E_1 + 2E_2 + 4E_3$ and $\mu_P = \nu_P = -2$.

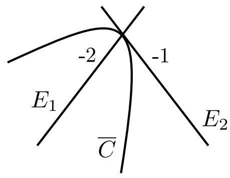
Case $e = 8$. P has multiplicity $(5, 3, 2, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the configuration:



with $(E_3 \cdot \overline{C}) = 2.$

Then we have: $D_P = 5E_1 + 8E_2 + 15E_3$, $D_P^{(1)} = E_1 + E_3$, $D_P^{(2)} = 2E_1 + 4E_2 + 7E_3$, $D_P^{(3)} = E_1 + 2E_2 + 4E_3$ and $\mu_P = \nu_P = -2$.

Case $e = 9$. P has multiplicity $(5, 4, 1, \ldots)$ and $\overline{\sigma}^{-1}(P)$ has the configuration:



with $(E_2 \cdot \overline{C}) = 4.$

Then we have: $D_P = 5E_1 + 9E_2$, $D_P^{(1)} = E_1 + E_2$, $D_P^{(2)} = 2E_1 + 4E_2$, $D_P^{(3)} = E_1 + 2E_2$, $\mu_P = -1$ and $\nu_P = -2$.

We denote by D, $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, μ and ν the sum $\sum\limits_P D_P$, $\sum\limits_P D_P^{(1)}$, $\sum\limits_P D_P^{(2)}$, $\sum\limits_P D_P^{(3)}$, $\sum\limits_P \mu_P$ and $\sum\limits_P \nu_P$, respectively, where P runs through all singular points of C lying on $\mathbb{A}_k^2$.

3.5

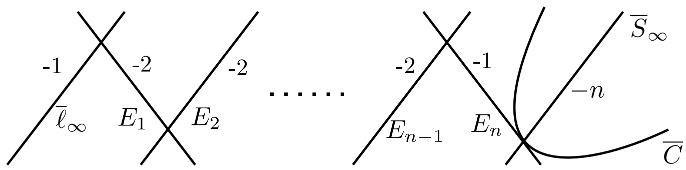
288 Now we shall turn to the singular points of C outside of $\mathbb{A}_k^2$. It is easy to see that C has only one point Q outside of $\mathbb{A}_k^2$ and C is given locally at Q by

$$\eta^d + \xi^5\psi(\eta) = 0; \quad Q = (\xi = 0, \eta = 0)$$

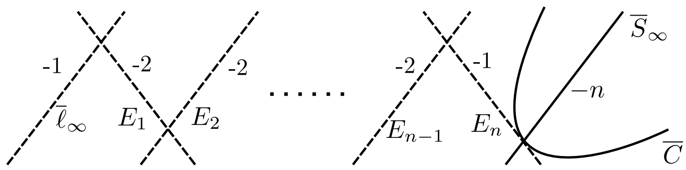
where $x = 1/\xi$, $y = 1/\eta$ and $\psi(\eta) = \eta^d\varphi(1/\eta)$ with $\psi(0) \neq 0$. We introduce the following notations: Consider a fibration $\mathscr{F} = \{\ell_\alpha : \ell_\alpha$ is defined by $y = \alpha\}$ on F_0. We denote by ℓ_∞ the fiber $y = \infty$ and by S_∞ the cross-section $x = \infty$. In the following we shall compute concretely $(\overline{\sigma}^*A)$, $\overline{B}$, $\overline{Z}$, $K_{\overline{F}}$, $K_{\overline{F}} + \overline{Z}$, $p_a(\overline{Z})$ and $((K_{\overline{F}} + \overline{Z})^2)$.

3.5.1

Case $d = 5n + 1$. Then Q is a singular point of multiplicity $(\underbrace{5, \ldots, 5}_{n}, 1, \ldots)$ and $\overline{\sigma}^{-1}(\ell_\infty \cup S_\infty \cup C)$ has the configuration below in a neighborhood of $\overline{\sigma}^{-1}(Q)$:



with $(E_n \cdot \overline{C}) = 5$ if $n = 2m$;



with $(E_n \cdot \overline{C}) = 5$ if $n = 2m + 1$, where $\overline{\ell}_\infty := \overline{\sigma}'(\ell_\infty)$ and $\overline{S}_\infty := \overline{\sigma}'(S_\infty)$.

289 We have:

$$(\overline{\sigma}^*A) = \overline{C} + 5(E_1 + 2E_2 + \cdots + nE_n) + D$$
$$- 5(\overline{S}_\infty + E_1 + 2E_2 + \cdots + nE_n) - d(\overline{\ell}_\infty + E_1 + \cdots + E_n)$$
$$= \overline{C} - 5\overline{S}_\infty - d\,\overline{\sigma}^*(\ell_\infty) + \Gamma$$

and

$$E_{\overline{F}} \sim -2(\overline{S}_\infty + E_1 + 2E_2 + \cdots + nE_n) - 2(\overline{\ell}_\infty + E_1 + \cdots + E_n)$$
$$+ (E_1 + 2E_2 + \cdots + nE_n) + D^{(3)}$$
$$= -\overline{S}_\infty - \overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + D^{(3)}.$$

Hence we have:

3.5.1.1 **Case** $d = 10m + 1$ $(n = 2m)$.

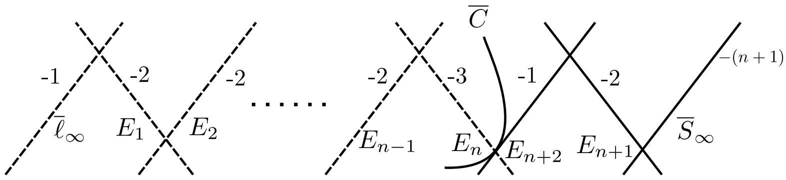
$$\overline{B} = \overline{C} + \overline{S}_\infty + \overline{\ell}_\infty + E_1 + \cdots + E_n + D^{(1)}$$
$$\overline{Z} = 3\overline{S}_\infty + (5m + 1)\overline{\sigma}^*(\ell_\infty) - D^{(2)}$$
$$K_{\overline{F}} + \overline{Z} \sim 2\overline{S}_\infty + (5m - 1)\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) + (D^{(3)} - D^{(2)})$$
$$= \overline{S}_\infty + (3m - 1)\overline{\sigma}^*(\ell_\infty)$$
$$+ \{2m\overline{\ell}_\infty + (2m - 1)E_1 + \cdots + E_{2m-1}\} + (D^{(3)} - D^{(2)})$$
$$p_a(\overline{Z}) = \frac{1}{2}(\overline{Z} \cdot K_{\overline{F}} + \overline{Z}) + 1 = 4m + \mu$$
$$((K_{\overline{F}} + \overline{Z})^2) = 2m - 2 + \nu.$$

3.5.1.2 **Case** $d = 10m + 6$ $(n = 2m + 1)$.

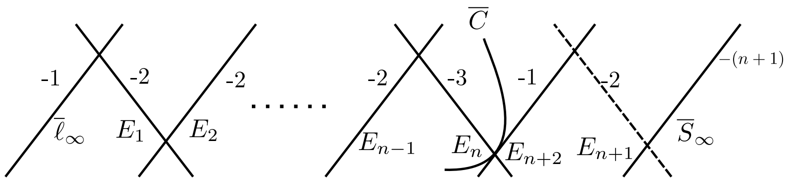
$$\overline{B} = \overline{C} + \overline{S}_\infty + D^{(1)}$$
$$\overline{Z} = 3\overline{S}_\infty + (5m + 3)\overline{\sigma}^*(\ell_\infty) - D^{(2)}$$
$$K_{\overline{F}} + \overline{Z} \sim 2\overline{S}_\infty + (5m + 1)\overline{\sigma}^*(\ell_\infty) - \overline{\alpha}^*(S_\infty) + (D^{(3)} - D^{(2)})$$
$$= \overline{S}_\infty + 3m\overline{\sigma}^*(\ell_\infty) + \{(2m + 1)\overline{\ell}_\infty + 2mE_1 + \cdots + E_{2m}\}$$
$$+ (D^{(3)} - D^{(2)})$$
$$p_a(\overline{Z}) = 4m + 1 + \mu$$
$$((K_{\overline{F}} + \overline{Z})^2) = 2m - 2 + \nu.$$

3.5.2

290 **Case** $d = 5n + 2$. Then Q has multiplicity $(\underbrace{5, \ldots, 5}_{n}, 2, 2, 1, \ldots)$ and $\overline{\sigma}^{-1}(\ell_\infty \cup S_\infty \cup C)$ has the following configuration in a neighborhood of $\overline{\sigma}^{-1}(Q)$:



with $(\overline{C} \cdot E_{n+2}) = 2$ if $n = 2m$;



with $(\overline{C} \cdot E_{n+2}) = 2$ if $n = 2m + 1$. We have:

$$
\begin{aligned}
(\overline{\sigma}^* A) = {} & \overline{C} + 5(E_1 + 2E_2 + \cdots + nE_n) + (5n+2)E_{n+1} + (10n+4)E_{n+2} + \\
& D - 5(\overline{S}_\infty + E_1 + 2E_2 + \cdots + nE_n + (n+1)E_{n+1} + (2n+1)E_{n+2}) \\
& \quad - d(\overline{\ell}_\infty + E_1 + E_2 + \cdots + E_n + E_{n+1} + 2E_{n+2}) \\
= {} & \overline{C} - 5\overline{S}_\infty - d\overline{\sigma}^*(\ell_\infty) - 3E_{n+1} - E_{n+2} + D
\end{aligned}
$$

and

$$
\begin{aligned}
K_{\overline{F}} \sim {} & -2\overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + E_1 + 2E_2 + \cdots + nE_n \\
& \qquad + (n+1)E_{n+1} + (2n+2)E_{n+2} + D^{(3)} \\
= {} & -\overline{S}_\infty - \overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + E_{n+2} + D^{(3)}.
\end{aligned}
$$

291 Hence we have:

3.5.2.1 Case $d = 10m + 2$ $(n = 2m)$.

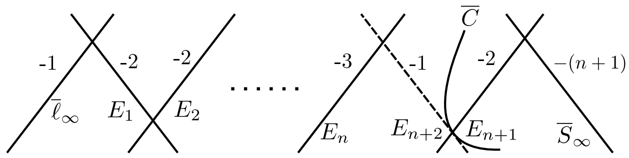
$$\begin{aligned}
\overline{B} &= \overline{C} + \overline{S}_\infty + E_{n+1} + E_{n+2} + D^{(1)} \\
\overline{Z} &= 3\overline{S}_\infty + (5m+1)\overline{\sigma}^*(\ell_\infty) + 2E_{n+1} + E_{n+2} - D^{(2)} \\
K_{\overline{F}} + \overline{Z} &\sim 2\overline{S}_\infty + (5m-1)\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) \\
&\qquad + 2E_{n+1} + 2E_{n+2} + (D^{(3)} - D^{(2)}) \\
&= \overline{S}_\infty + (3m-1)\overline{\sigma}^*(\ell_\infty) + \{2m\overline{\ell}_\infty + (2m-1)E_1 + \cdots \\
&\qquad + E_{2m-1} + E_{2m+1} + E_{2m+2}\} + (D^{(3)} - D^{(2)}) \\
p_a(\overline{Z}) &= 4m + \mu \\
((K_{\overline{F}} + \overline{Z})^2) &= 2m - 2 + \nu.
\end{aligned}$$

3.5.2.2 Case $d = 10m + 7$ $(n = 2m + 1)$.

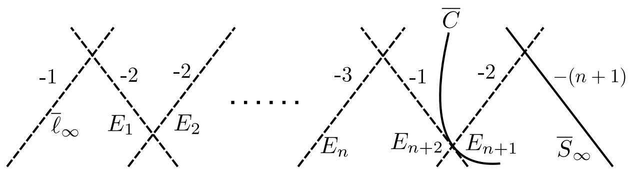
$$\begin{aligned}
\overline{B} &= \overline{C} + \overline{S}_\infty + (\overline{\ell}_\infty + E_1 + E_2 + \cdots + E_n) + E_{n+2} + D^{(1)} \\
\overline{Z} &= 3\overline{S}_\infty + (5m+4)\overline{\sigma}^*(\ell_\infty) + E_{n+1} - D^{(2)} \\
K_{\overline{F}} + \overline{Z} &\sim 2\overline{S}_\infty + (5m+2)\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) + E_{n+1} \\
&\qquad + E_{n+2} + (D^{(3)} - D^{(2)}) \\
&= \overline{S}_\infty + (3m+1)\overline{\sigma}^*(\ell_\infty) \\
&\qquad + \{(2m+1)\overline{\ell}_\infty + 2mE_1 + \cdots + E_{2m}\} + (D^{(3)} - D^{(2)}) \\
p_a(\overline{Z}) &= 4m + 2 + \mu \\
((K_{\overline{F}} + \overline{Z})^2) &= 2m - 1 + \nu.
\end{aligned}$$

3.5.3

Case $d = 5n + 3$. Then Q has multiplicity $(\underbrace{5, \ldots, 5}_{n}, 3, 2, 1, \ldots)$ and $\overline{\sigma}^{-1}(\ell_\infty \cup S_\infty \cup C)$ has the configuration below in a neighborhood of $\overline{\sigma}^{-1}(Q)$:



292 with $(\overline{C} \cdot E_{n+2}) = 2$ if $n = 2m$;



with $(\overline{C} \cdot E_{n+2}) = 2$ if $n = 2m+1$. We have:

$$\begin{aligned}(\overline{\sigma}^* A) = \overline{C} &+ 5(E_1 + 2E_2 + \cdots + nE_n) + (5n+3)E_{n+1} \\ &\qquad + (10n+5)E_{n+2} + D \\ &- 5(\overline{S}_\infty + E_1 + 2E_2 + \cdots + nE_n + (n+1)E_{n+1} + (2n+1)E_{n+2}) \\ &- d(\overline{\ell}_\infty + E_1 + E_2 + \cdots + E_n + E_{n+1} + 2E_{n+2}) \\ = \overline{C} &- 5\overline{S}_\infty - d\overline{\sigma}^*(\ell_\infty) - 2E_{n+1} + D\end{aligned}$$

and

$$\begin{aligned}K_{\overline{F}} &\sim -2\overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + E_1 + 2E_2 + \cdots + nE_n + (n+1) \\ &\qquad E_{n+1} + (2n+2)E_{n+2} + D^{(3)} \\ &= -\overline{S}_\infty - \overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + E_{n+2} + D^{(3)}.\end{aligned}$$

Hence we have:

3.5.3.1 Case $d = 10m + 3$ $(n = 2m)$.

$$\begin{aligned}\overline{B} &= \overline{C} + \overline{S}_\infty + \overline{\ell}_\infty + E_1 + \cdots + E_n + E_{n+1} + D^{(1)} \\ \overline{Z} &= 3\overline{S}_\infty + (5m+2)\overline{\sigma}^*(\ell_\infty) + E_{n+1} - E_{n+2} - D^{(2)} \\ K_{\overline{F}} + \overline{Z} &\sim 2\overline{S}_\infty + 5m\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) + E_{n+1} + (D^{(3)} - D^{(2)})\end{aligned}$$

$$= \overline{S}_\infty + (3m-1)\overline{\sigma}^*(\ell_\infty) + \Big\{(2m+1)\overline{\ell}_\infty + 2mE_1 + \cdots + E_{2m} + E_{2m+1} + E_{2m+2}\Big\} + (D^{(3)} - D^{(2)})$$

$$p_a(\overline{Z}) = 4m + \mu$$

$$((K_{\overline{F}} + \overline{Z})^2) = 2m - 2 + \nu.$$

3.5.3.2 Case $d = 10m + 8$ $(n = 2m + 1)$. 293

$$\overline{B} = \overline{C} + \overline{S}_\infty + D^{(1)}$$

$$\overline{Z} = 3\overline{S}_\infty + (5m+4)\overline{\sigma}^*(\ell_\infty) + E_{n+1} - D^{(2)}$$

$$K_{\overline{F}} + \overline{Z} \sim 2\overline{S}_\infty + (5m+2)\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) + E_{n+1} + E_{n+2} + \Big(D^{(3)} - D^{(2)}\Big)$$

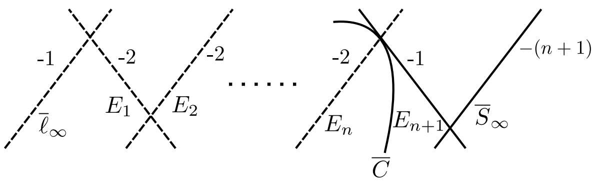
$$= \overline{S}_\infty + (3m+1)\overline{\sigma}^*(\ell_\infty) + \Big\{(2m+1)\overline{\ell}_\infty + 2mE_1 + \cdots + E_{2m}\Big\} + \Big(D^{(3)} - D^{(2)}\Big)$$

$$p_a(\overline{Z}) = 4m + 2 + \mu$$

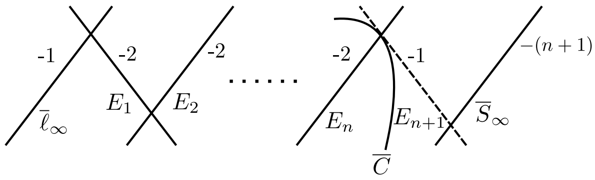
$$((K_{\overline{F}} + \overline{Z})^2) = 2m - 1 + \nu.$$

3.5.4

Case $d = 5n + 4$. Then Q has multiplicity $(\underbrace{5, \ldots, 5}_{n}, 4, 1, \ldots)$ and $\overline{\sigma}^{-1}$ $(\ell_\infty \cup S_\infty \cup C)$ has the configuration below in a neighborhood of $\overline{\sigma}^{-1}(Q)$:



with $(\overline{C} \cdot E_{n+1}) = 4$ if $n = 2m$;



with $(\overline{C} \cdot E_{n+1}) = 4$ if $n = 2m + 1$. We have:

$$\begin{aligned}(\overline{\sigma}^* A) &= \overline{C} + 5(E_1 + 2E_2 + \cdots + nE_n) + (5n+4)F_{n+1} + D \\ &\quad - 5(\overline{S}_\infty + E_1 + 2E_2 + \cdots + nE_n + (n+1)E_{n+1}) \\ &\quad - d(\overline{\ell}_\infty + E_1 + E_2 + \cdots + E_n + E_{n+1}) \\ &= \overline{C} - 5\overline{S}_\infty - d\overline{\sigma}^*(\ell_\infty) - E_{n+1} + D\end{aligned}$$

294 and

$$\begin{aligned}K_{\overline{F}} &\sim -2\overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + E_1 + 2E_2 + \cdots + \\ &\qquad nE_n + (n+1)F_{n+1} + D^{(3)} \\ &= -\overline{S}_\infty - \overline{\sigma}^*(S_\infty) - 2\overline{\sigma}^*(\ell_\infty) + D^{(3)}.\end{aligned}$$

Hence we have:

3.5.4.1 **Case** $d = 10m + 4$ $(n = 2m)$.

$$\begin{aligned}\overline{B} &= \overline{C} + \overline{S}_\infty + E_{n+1} + D^{(1)} \\ \overline{Z} &= 3\overline{S}_\infty + (5m+2)\overline{\sigma}^*(\ell_\infty) + E_{n+1} - D^{(2)} \\ K_{\overline{F}} + \overline{Z} &\sim 2\overline{S}_\infty + 5m\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) + E_{n+1} + (D^{(3)} - D^{(2)}) \\ &= \overline{S}_\infty + 3m\overline{\sigma}^*(\ell_\infty) + \{2m\overline{\ell}_\infty + (2m-1)E_1 + \cdots + E_{2m-1}\} \\ &\qquad + (D^{(3)} - D^{(2)}) \\ p_a(\overline{Z}) &= 4m + 1 + \mu \\ ((K_{\overline{F}} + \overline{Z})^2) &= 2m - 1 + \nu.\end{aligned}$$

3.5.4.2 **Case** $d = 10m + 9$ $(n = 2m + 1)$.

$$\overline{B} = \overline{C} + \overline{S}_\infty + \overline{\ell}_\infty + E_1 + \cdots + E_n + D^{(1)}$$

$$\begin{aligned}
\overline{Z} &= 3\overline{S}_\infty + (5m+5)\overline{\sigma}^*(\ell_\infty) - D^{(2)} \\
K_{\overline{F}} + \overline{Z} &\sim 2\overline{S}_\infty + (5m+3)\overline{\sigma}^*(\ell_\infty) - \overline{\sigma}^*(S_\infty) + (D^{(3)} - D^{(2)}) \\
&= \overline{S}_\infty + (3m+1)\overline{\sigma}^*(\ell_\infty) + \Big\{(2m+2)\overline{\ell}_\infty + (2m+1) \\
&\qquad E_1 + \cdots + 2E_{2m} + E_{2m+1}\Big\} + (D^{(3)} - D^{(2)}) \\
p_a(\overline{Z}) &= 4m + 2 + \mu \\
((K_{\overline{F}} + \overline{Z})^2) &= 2m - 2 + \nu.
\end{aligned}$$

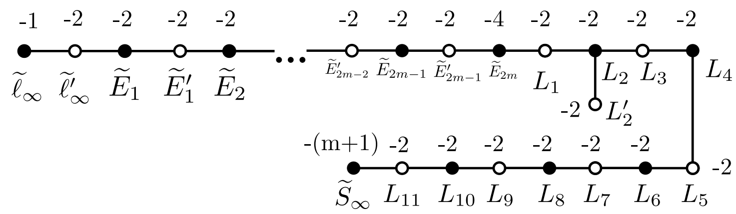
3.6

Next we shall consider the nonsingular minimal model $\widehat{H}$ of K. In the remaining paragraphs of this section we shall assume for the sake of simplicity that $D^{(2)} = p^{(3)}$. In view of 3.4, this is equivalent to assuming that $\nu_\alpha(\varphi'(y)) \leqq 2$ for every root α of $\varphi'(y) = 0$, and this implies that $\mu = \nu = 0$. We shall consider first *the case* $m \geqq 1$. We know in view of 3.4 and 3.5 that $\overline{B}$ has negligible singularities; this implies that $p_a(H) = p_a(\overline{Z}) > 0$ (because $m \geqq 1$) and $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z})$ (cf. Lemma 3.3); in particular, H is not rational over k and $\widehat{H}$ exists. In each of the cases enumerated below the results are obtained by straightforward computations. So, the details will be omitted. **295**

3.6.1

Case $d = 10m + 1$. The following assertions hold true:

(1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



where $\pi^*(\overline{\ell}_\infty) = 2\widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_1) = \widetilde{\ell}'_\infty + 2\widetilde{E}_1 + \widetilde{E}'_1$, $\pi^*(E_i) =$

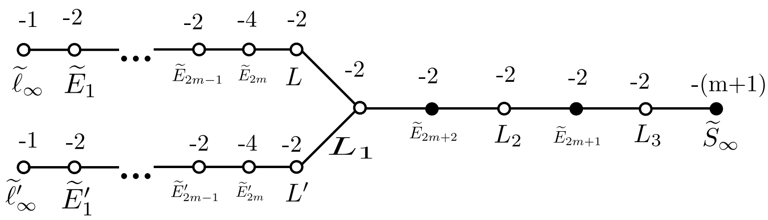
$\widetilde{E}'_{i-1} + 2\widetilde{E}_i + \widetilde{E}'_i$ for $2 \leqq i \leqq 2m-1$, $\pi^*(E_{2m}) = \widetilde{E}'_{2m-1} + 2\widetilde{E}_{2m} + 6L_1 + 5L'_2 + \Sigma_{i=2}^{11}(12-i)L_i$ and $\pi^*(\overline{S}_\infty) = 2\widetilde{S}_\infty + L_1 + L'_2 + 2(\Sigma_{i=2}^{11} L_i)$.

(2) $\pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + (3m-1)\pi^*\overline{\sigma}^*(\ell_\infty) + \{4m\overline{\ell}_\infty + (4m-1)\widetilde{\ell}'_\infty +$
296 $(4m-2)\overline{E}_1 + \cdots + 2\widetilde{E}_{2m-1} + \widetilde{E}'_{2m-1}\}$. Since $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z})$ we know by (1) and (2) above that $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty$, $\widetilde{\ell}'_\infty$, $\widetilde{E}_1$, $\widetilde{E}'_1, \ldots, \widetilde{E}_{2m-1}$ and $\widetilde{E}'_{2m-1}$. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}} + \overline{Z})^2) + 4m = (4m-4) + 4m = 8m-4$.

3.6.2

Case $d = 10m + 2$. The following assertions hold true:

(1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



where $\pi^*(\overline{\ell}_\infty) = \widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_i) = \widetilde{E}_i + \widetilde{E}'_i$ for $1 \leqq i \leqq\leqq 2m-1$, $\pi^*(E_{2m}) = \widetilde{E}_{2m} + \widetilde{E}'_{2m} + L + L' + L_1$, $\pi^*(E_{2m+2}) = L + L' + 2L_1 + 2\widetilde{E}_{2m+2} + L_2$, $\pi^*(E_{2m+1}) = L_2 + 2\widetilde{E}_{2m+1} + L_3$ and $\pi^*(\widetilde{S}_\infty) = 2\widetilde{S}_\infty + L_3$.

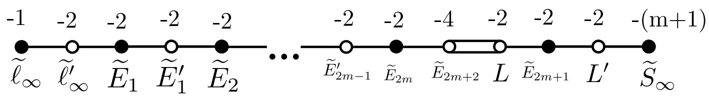
(2) $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + (3m-1)(\overline{\sigma}\pi)^*(\ell_\infty) + \{2m\widetilde{\ell}_\infty + (2m-1)\widetilde{E}_1 + \cdots + \widetilde{E}_{2m-1}\} + \{2m\widetilde{\ell}'_\infty + (2m-1)\widetilde{E}'_1 + \cdots + \widetilde{E}'_{2m-1}\} + 2\widetilde{E}_{2m+1} + 2\widetilde{E}_{2m+2} + L + L' + L_1 + 2L_2 + L_3$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty, \widetilde{E}_1, \ldots, \widetilde{E}_{2m-1}$ and $\widetilde{\ell}'_\infty, \widetilde{E}'_1, \ldots, \widetilde{E}'_{2m-1}$. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}} + \overline{Z})^2) + 4m = (4m-4) + 4m = 8m-4$.

3.6.3

Case $d = 10m + 3$. The following assertions hold true:

(1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



where $\pi^*(\overline{\ell}_\infty) = 2\widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_1) = \widetilde{\ell}'_\infty + 2\widetilde{E}_1 + \widetilde{E}'_1$, $\pi^*(E_i) =$ **297**
$\widetilde{E}'_{i-1} + 2\widetilde{E}_i + \widetilde{E}'_i$ for $2 \leqq i \leqq 2m-1$, $\pi^*(E_{2m}) = \widetilde{E}'_{2m-1} + 2\widetilde{E}_{2m}$, $\pi^*(E_{2m+2}) = \widetilde{E}_{2m+2} + L$, $\pi^*(E_{2m+1}) = L + 2\widetilde{E}_{2m+1} + L'$ and $\pi^*(\overline{S}_\infty) = L' + 2\widetilde{S}_\infty$.

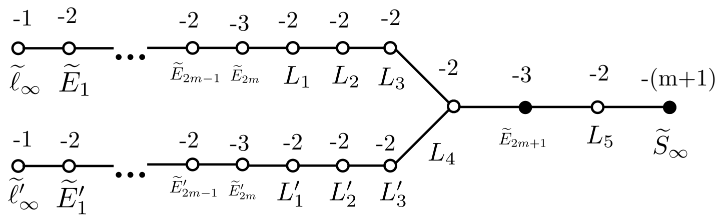
(2) $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + (3m-1)(\overline{\sigma}\pi)^*(\ell_\infty) + \{(4m+2)\widetilde{\ell}_\infty + (4m+1)\widetilde{\ell}'_\infty + 4m\widetilde{E}_1 + (4m-1)\widetilde{E}'_1 + \cdots + 3\widetilde{E}'_{2m-1} + 2\widetilde{E}_{2m}\} + \widetilde{E}_{2m+2} + 2L + 2\widetilde{E}_{2m+1} + L'$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty$, $\widetilde{\ell}'_\infty$, $\widetilde{E}_1, \ldots, \widetilde{E}_{2m}$. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}} + \overline{Z})^2) + (4m+1) = (4m-4) + (4m+1) = 8m-3$.

3.6.4

Case $d = 10m + 4$. The following assertions hold true:

(1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



where $\pi^*(\overline{\ell}_\infty) = \widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_i) = \widetilde{E}_i + \widetilde{E}'_i$ for $1 \leqq i \leqq 2m-1$, $\pi^*(E_{2m}) = \widetilde{E}_{2m} + \widetilde{E}'_{2m} + (\Sigma_{i=1}^3(L_i + L'_i)) + L_4$, $\pi^*(E_{2m+1}) = (\Sigma_{i=1}^3 i(L_i + L'_i)) + 4L_4 + 2\widetilde{E}_{2m+1} + L_5$ and $\pi^*(\overline{S}_\infty) = L_5 + 2\widetilde{S}_\infty$.

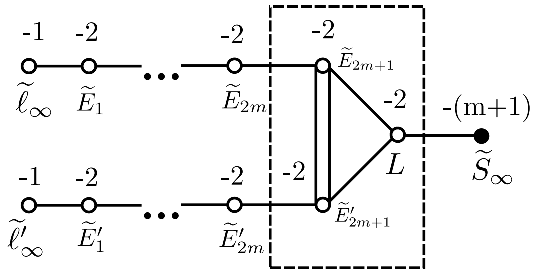
(2) $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + 3m(\overline{\sigma}\pi)^*(\ell_\infty) + \{2m\widetilde{\ell}_\infty + (2m-1)\widetilde{E}_1 + \cdots + \widetilde{E}_{2m-1}\} + \{2m\widetilde{\ell}'_\infty + (2m-1)\widetilde{E}'_1 + \cdots + \widetilde{E}'_{2m-1}\}$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty, \widetilde{E}_1, \ldots, \widetilde{E}_{2m-1}$ and $\widetilde{\ell}'_\infty$, $\widetilde{E}'_1, \ldots, \widetilde{E}'_{2m-1}$. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}}+\overline{Z})^2)+4m = (4m-2)+4m = 8m-2$.

3.6.5

Case $d = 10m + 6$. The following assertions hold true:

298 (1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



there components meet in one point with $(\tilde{E}_{2m+1} \cdot \tilde{E}'_{2m+1}) = 2$ and $(\tilde{E}_{2m+1} \cdot L) = (\tilde{E}'_{2m+1} \cdot L) = 1$

where $\pi^*(\overline{\ell}_\infty) = \widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_i) = \widetilde{E}_i + \widetilde{E}'_i$ for $1 \leqq i \leqq 2m$, $\pi^*(E_{2m+1}) = \widetilde{E}_{2m+1} + \widetilde{E}'_{2m+1} + L$ and $\pi^*(\overline{S}_\infty) = 2\widetilde{S}_\infty + L$.

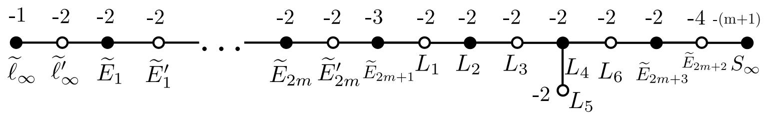
(2) $K_H \sim \pi^*(K_{\overline{F}}+\overline{Z}) \sim \pi^*(\overline{S}_\infty)+3m(\overline{\sigma}\pi)^*(\ell_\infty)+\{(2m+1)\widetilde{\ell}_\infty+2m\widetilde{E}_1+\cdots+\widetilde{E}_{2m}\}+\{(2m+1)\widetilde{\ell}'_\infty+2m\widetilde{E}'_1+\cdots+\widetilde{E}'_{2m}\}$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty, \widetilde{E}_1, \ldots, \widetilde{E}_{2m}$ and $\widetilde{\ell}'_\infty$, $\widetilde{E}'_1, \ldots, \widetilde{E}'_{2m}$. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}}+\overline{Z})^2)+(4m+2) = (4m-4)+(4m+2) = 8m-2$.

3.6.6

Case $d = 10m + 7$. The following assertions hold true:

(1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



where $\pi^*(\overline{\ell}_\infty) = 2\widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_1) = \widetilde{\ell}'_\infty + 2\widetilde{E}_1 + \widetilde{E}'_1$, $\pi^*(E_i) = \widetilde{E}'_{i-1} + 2\widetilde{E}_i + \widetilde{E}'_i$ for $2 \leqq i \leqq 2m$, $\pi^*(E_{2m+1}) = \widetilde{E}'_{2m} + 2\widetilde{E}_{2m+1} + 2(L_1 + L_2 + L_3 + L_4) + L_5 + L_6$, $\pi^*(E_{2m+3}) = 2\widetilde{E}_{2m+3} + L_1 + 2L_2 + 3L_3 + 4L_4 + 2L_5 + 3L_6$, $\pi^*(E_{2m+2}) = \widetilde{E}_{2m+2}$ and $\pi^*(\overline{S}_\infty) = 2\widetilde{S}_\infty$. **299**

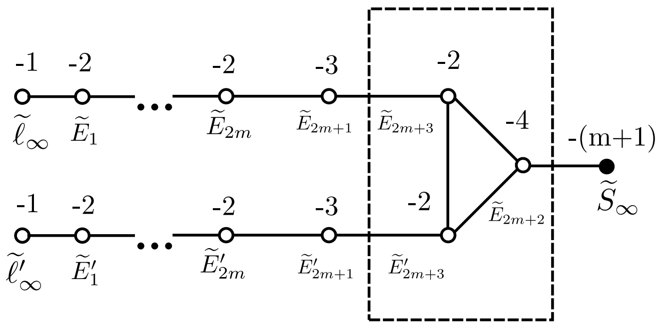
(2) $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + (3m+1)(\overline{\sigma}\pi)^*(\ell_\infty) + \{(4m+2)\widetilde{\ell}_\infty + (4m+1)\widetilde{\ell}'_\infty + 4m\widetilde{E}_1 + \cdots + 2\widetilde{E}_{2m} + \widetilde{E}'_{2m}\}$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty$, $\widetilde{\ell}'_\infty$, $\widetilde{E}_1$, $\widetilde{E}'_1, \ldots, \widetilde{E}_{2m}$ and $\widetilde{E}'_{2m}$. Hence $(K^2_{\hat{H}}) = 2((K_{\overline{F}} + \overline{Z})^2) + (4m+2) = (4m-2) + (4m+2) = 8m$.

3.6.7

Case $d = 10m + 8$. The following assertions hold true:

(1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



three components meet each other transversely in one point where $\pi^*(\overline{\ell}_\infty) = \widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_i) = \widetilde{E}_i + \widetilde{E}'_i$ for $1 \leqq i \leqq 2m+1$ or $i = 2m+3$, $\pi^*(E_{2m+2}) = \widetilde{E}_{2m+2}$ and $\pi^*(\widetilde{S}_\infty) = 2\widetilde{S}_\infty$.

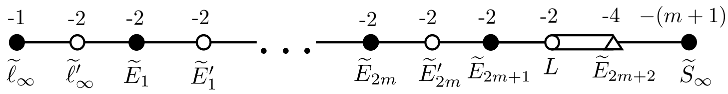
(2) $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + (3m+1)(\overline{\sigma}\pi)^*(\ell_\infty) + \{(2m+1)\widetilde{\ell}_\infty + 2m\widetilde{E}_1 + \cdots + \widetilde{E}_{2m}\} + \{(2m+1)\widetilde{\ell}'_\infty + 2m\widetilde{E}'_i + \cdots + \widetilde{E}'_{2m}\}$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty, \widetilde{E}_1, \ldots, \widetilde{E}_{2m}$ and $\widetilde{\ell}'_\infty$, $\widetilde{E}'_1, \ldots, \widetilde{E}'_{2m}$. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}} + \overline{Z})^2) + (4m+2) = (4m-2) + (4m+2) = 8m$.

3.6.8

Case $d = 10m + 9$. The following assertions hold true:

300 (1) $(\overline{\sigma}\pi)^{-1}(\ell_\infty \cup S_\infty)$ has the next weighted graph:



where Δ stands for an irreducible rational curve $\widetilde{E}_{2m+2}$ with an ordinary cusp P of multiplicity 2, L intersects $\widetilde{E}_{2m+2}$ at the cusp point with $(L \cdot \widetilde{E}_{2m+2}) = 2$ and $\widetilde{S}_\infty$ intersects $\widetilde{E}_{2m+2}$ transversely at a simple point; and where $\pi^*(\overline{\ell}_\infty) = 2\widetilde{\ell}_\infty + \widetilde{\ell}'_\infty$, $\pi^*(E_1) = \widetilde{\ell}'_\infty + 2\widetilde{E}_1 + \widetilde{E}'_1$, $\pi^*(E_i) = \widetilde{E}'_{i-1} + 2\widetilde{E}_i + \widetilde{E}'_i$ for $2 \leqq i \leqq 2m$, $\pi^*(E_{2m+1}) = \widetilde{E}'_{2m} + 2\widetilde{E}_{2m+1} + L$, $\pi^*(E_{2m+2}) = L + 2\widetilde{E}_{2m+2}$ and $\pi^*(\overline{S}_\infty) = 2\widetilde{S}_\infty$.

(2) $K_H \sim \pi^*(K_{\overline{F}} + \overline{Z}) \sim \pi^*(\overline{S}_\infty) + (3m+1)(\overline{\sigma}\pi)^*(\ell_\infty) + \{(4m+4)\widetilde{\ell}_\infty + (4m+3)\widetilde{\ell}'_\infty + (4m+2)\widetilde{E}_1 + \cdots + 4\widetilde{E}_{2m} + 3\widetilde{E}'_{2m} + 2\widetilde{E}_{2m+1} + L\}$.

Then $\widehat{H}$ is obtained from H by contracting $\widetilde{\ell}_\infty, \widetilde{\ell}'_\infty, \widetilde{E}_1, \widetilde{E}'_1, \ldots, \widetilde{E}_{2m}$, $\widetilde{E}'_{2m}, \widetilde{E}_{2m+1}$ and L. Hence $(K^2_{\widehat{H}}) = 2((K_{\overline{F}} + \overline{Z})^2) + (4m+4) = (4m-4) + (4m+4) = 8m$.

3.7

Next we shall consider *the case* $m = 0$ and assume that $D^{(2)} = D^{(3)}$. In principle we follow the arguments and computations done in 3.5 and 3.6. More precisely, the configurations of $\overline{\sigma}^{-1}(\ell_\infty \cup S_\infty \cup C)$ in a neighborhood of $\overline{\sigma}^{-1}(Q)$ are those in 3.5 up to the following modifications:

If $d = 1, 2, 3, 4$ then omit $\overline{\ell}_\infty$, $E_1, \ldots, E_{n-1}$, put $n = 0$ and set anew $\overline{\ell}_\infty := E_0$ and $(\overline{\ell}^2_\infty) = (E_n^2)+1$; if $d = 6, 7, 8, 9$ then omit $\overline{\ell}_\infty$, $E_1, \ldots, E_{n-1}$, put $n = 1$ and set anew $\overline{\ell}_\infty := E_0$ and $(\overline{\ell}^2_\infty) = (E^2_{n-1}) + 1$. The expressions of $\overline{B}$, $\overline{Z}$, $K_{\overline{F}} + \overline{Z}$, $p_a(\overline{Z})$ and $((K_{\overline{F}} + \overline{Z})^2)$ are obtained from those in 3.6 by due modifications.

3.7.1

Case $d = 1$. Then K is apparently rational over k. **301**

3.7.2

Case $d = 2$ (cf. 3.5.2.1 and 3.6.2). Then $p_a(H) = 0$ and $K_H \sim (\overline{\sigma}\pi)^*$ $(S_\infty - \ell_\infty)$. Hence the bigenus $p_2(H) = 0$. Thus H is rational over k by Castelnuovo's criterion of rationality.

3.7.3

Case $d = 3$ (cf. 3.5.3.1 and 3.6.3). Then $p_a(H) = 0$ and $K_H \sim 2\widetilde{S}_\infty + L' - \widetilde{E}_2 - L$. Let $\rho : H \to Y$ be the contraction of $\widetilde{S}_\infty$ and L'. Then $K_Y \sim -\rho(E_2) - \rho(L)$. Hence $p_a(Y) = P_2(Y) = 0$. Thus Y is rational over k, and so is H.

3.7.4

Case $d = 4$ (cf. 3.5.4.1 and 3.6.4). Then $p_a(H) = 1$ and $K_H \sim 2\widetilde{S}_\infty + L_5$. Let $\rho : H \to Y$ be the contraction of $\widetilde{S}_\infty$ and L_5. Then $K_Y \sim 0$, which implies that Y is a K3-surface and $Y \cong H$ (the nonsingular minimal model of K over k).

3.7.5

Case $d = 6$ (cf. 3.5.1.2 and 3.6.5). Then $p_a(H) = 1$ and $K_H \sim 2\widetilde{S}_\infty + L$. Let $\rho : H \to Y$ be the contraction of $\widetilde{S}_\infty$ and L. Then $K_Y \sim 0$, which implies that Y is a K3-surface.

3.7.6

Case $d = 7$ (cf. 3.5.2.2 and 3.6.6). Then $p_a(H) = 2$ and $K_H \sim \pi^*(\overline{S}_\infty + \overline{\ell}_\infty) + (\overline{\sigma}\pi)^*(\ell_\infty) = 2\widetilde{\ell}_\infty + \widetilde{\ell}'_\infty + 2\widetilde{S}_\infty + (\overline{\sigma}\pi)^*(\ell_\infty)$. Let $\rho : H \to Y$ be the contraction of $\widetilde{S}_\infty$, $\widetilde{\ell}_\infty$ and $\widetilde{\ell}'_\infty$. Then Y is a minimal surface with $K_Y \sim \rho_*((\overline{\sigma}\pi)^*(\ell_\infty))$. Hence $(K_Y^2) = 1$. Then Y is a surface of general type.

3.7.7

Case $d = 8$ (cf. 3.5.3.2 and 3.6.7). Then $p_a(H) = 2$ and $K_H \sim 2\widetilde{S}_\infty +$
$\widetilde{\ell}_\infty + \widetilde{\ell}'_\infty + (\overline{\sigma}\pi)^*(\ell_\infty)$. Let $\rho : H \to Y$ be the contraction of $\widetilde{S}_\infty$, $\widetilde{\ell}_\infty$ and
302 $\widetilde{\ell}'_\infty$. Then Y is a minimal surface with $K_Y \sim \rho_*((\overline{\sigma}\pi)^*(\ell_\infty))$ and $(K_Y^2) = 1$.
Hence Y is a surface of general type.

3.7.8

Case $d = 9$ (cf. 3.5.4.2 and 3.6.8). Then $p_a(H) = 2$ and $K_H \sim 2\widetilde{S}_\infty + 4\widetilde{\ell}_\infty + 3\widetilde{\ell}'_\infty + 2\widetilde{E}_1 + L + (\overline{\sigma}\pi)^*(\ell_\infty)$. Let $\rho : H \to Y$ be the contraction of $\widetilde{S}_\infty$, $\widetilde{\ell}_\infty$, $\widetilde{\ell}'_\infty$, $\widetilde{E}_1$ and L. Then Y is a minimal surface with $K_Y \sim \rho_*((\overline{\sigma}\pi)^*(\ell_\infty))$ and $(K_Y^2) = 1$. Hence Y is a surface of general type.

3.8

Now it is clear that Theorem 3.1 is proved in the arguments of the foregoing paragraphs. In order to show that there exists a unirational surfaces of general type in characteristic $p > 5$ we shall state the next result without proof.

Proposition. *Let k be an algebraically closed field of characteristic $p > 2$. Let $K : k(t, x, y)$ be the algebraic function field of a hyper surface in $\mathbb{A}_k^3$:*

$$t^2 = x^p + y^{p+1} + y^{p-1} + y^{p-2} + \cdots + y^2 + y.$$

Then K is rational over k if $p = 3$ and irrational over k if $p \geqq 5$. Let X be the nonsingular minimal model of K over k if $p \geqq 5$. Then X is a unirational $K3$-surface if $p = 5$, and X is a unirational surface of *general type with* $p_a(X) = (p-1)(p-3)/8$ *and* $(K_X^2) = (p-5)^2/2$ *if* $p \geqq 7$.

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