Lectures on
Results on Bezout's Theorem

By

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Tata Institute of Fundamental Research

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Introduction

These notes are based on a series of lectures given at the Tata Institute in November and December, 1982. The lectures are centered about my joint work with Jürgen Stückrad [85] on an algebraic approach to the intersection theory. More-over, chapter *II* and *III* also contain new results.

Today, we have the remarkable theory of W.Fulton and R. Macperson on defining algebraic intersections:

Suppose V and W are subvarieties of dimension v and w of a non-singular algebraic variety X of dimension n. Then the equivalence class $V \cdot W$ of algebraic $v + w - n$ cycles which represents the algebraic intersection of V and W is defined up to rational equivalence in X. This intersection theory produces subvarieties Y_i of $V \cap W$, cycle classes α_i on Y_i, positive integers m_i, with $\sum m_i \alpha_i$ representing $V \cdot W$, and $\deg \alpha_i \geq \deg Y_i$ even in the case $\dim(V \cap W) \neq v + w - n$.

Our object here is to give an algebraic approach to the intersection theory by studying a formula for $\deg(V)$. $\deg(W)$ in terms of algebraic data, if V and W are Gubvarieties of $X = \mathbb{P}^n_K$.

The basis of our formula is a method for expressing the intersection multiplicity of two properly intersecting varieties as the length of a certain primary ideal associated to them in a canonical way. Using the geometry of the join construction in $\mathbb{P}^{2n+1}$ over a field extension of K, we may apply this method even if $\dim(V \cap W) > \dim V + \dim W - n$. More precisely, we will prove the following statement in Chapter II:

Let X, Y be pure dimensional projective subvarieties of $\mathbb{P}^n_K$. There is a collection $\{C_i\}$ of subvarieties of $X \cap Y$ (one of which may be ϕ), including all irreducible components of $X \cap Y$, and intersection numbers, say $j(X, Y; C_i) \geq 1$, of X and Y along C_i given by the length of primary ideals, such that

$$\deg(X) \cdot \deg(Y) = \sum_{C_i} j(X, Y; C_i) \cdot \deg(C_i),$$

where we put $\deg(\phi) = 1$.

The key is that our approach does provide an explicit description of the subvarieties $C_i \subset \mathbb{P}^n$ counted with multiplicities $j(X, Y; C_i)$, which are canonically determined over a field extension of K.

In case $\dim(X \cap Y) = \dim X + \dim Y - n$, then our collection $\{C_i\}$ only consists of the irreducible components C of $X \cap Y$ and the multiplicities $j(X, Y; C)$ coincide with Weil's intersection numbers; that is, our statement also provides the classical theorem of Bezout. Furthermore, by combining our approach with the properties of reduced system of parameters, we open the way to a deeper study of Serre's observations on "multiplicity" and "length" (see: J.-P.Serre [72], p.V-20).

In 1982, W. Fulton asked me how imbedded components contribute to intersection theory. Using our approach, we are able to study some pathologies in chapter III. (One construction is due to R. Achilles). Of course, it would be very interesting to say something about how imbedded components contribute to intersection multiplicities. Also, it appears hard to give reasonably sharp estimates on the error term between deg (X). $\deg(Y)$ and $\sum j(X, Y; C_j)$. $\deg(C_j)$ or even $\sum \deg(C_j)$ where C_j runs through all irreducible components of $X \cap Y$. Therefore, we will discuss some examples, applications and problems in chapter III.

I wish to express my gratitude to the Tata Institute of Fundamental Research of Bombay, in particular to Balwant Singh, for the kind invitation to visit the School of Mathematics. Dilip P. Patil has written these notes and it is a pleasure for me to thank him for his efficiency, his remarks and for the time- consuming and relatively thankless task of writing up these lecture notes. I am also grateful for the many insightful comments and suggestions made by persons attending the lectures, including R.C. Cowsik, N. Mohan Kumar, M.P. Murthy, Dilip P. Patil, Balwant singh, Uwe Storch and J.-L. Verdier. The typists of the School of Mathematics have typed these manuscripts with care and I thank them very much.

Finally I am deeply grateful to R.Sridharan for showing me collected poems and plays of Rabindranath Tagore.

Let me finish with an example from "Stray birds":

The bird wishes it were a cloud.

The cloud wishes it were a bird.

However, all errors which now appear are due to myself.

Wolfgang Vogel

NOTATION

The following notation will be used in the sequel.

We denote the set of natural numbers (respectively, non-negative integers, integers, rational numbers) by $\mathbb{N}(resp. .\mathbb{Z}^+, \mathbb{Z}, Q$. For $n \in \mathbb{N}$, we write "$n >> 1''$ for "all sufficiently large integers n''. By a ring, we shall always mean a commutative ring with identity. All ring homomorphisms considered are supposed to be unitary and, in particular, all modules considered are unitary. If A is a ring, Spec (A) denotes the set of all prime ideals of A. For any ideal $I \subset A$ and any A-module M, if $N \subset M$ is an A-submodule then $(N \underset{M}{:} I) := \{m \in M | I \cdot m \subset N\}$.

For any field K, $\bar{K}$ denotes the algebraic closure of K and $\mathbb{P}^n_K$ denotes the projective n-space over K.

Contents

Chapter 0

Historical Introduction

A. The Classical Case

The simplest case of Bezout's theorem over an algebraically closed field **1**
is the following very simple theorem.

(0.1) Fundamental Principle

The number of roots of a polynomial $f(x)$ in one variable, counted with their multiplicities, equals the degree of $f(x)$.

This so-called fundamental theorem of algebra was conjectured by Girard (from the Netherlands) in 1629. In 1799, C.F. Gauss provided the first proof of this statement. M.Kneser [40] produced a very simple proof of this fundamental principle in 1981. This proof also yields a constructive aspect of the fundamental theorem of algebra.

The definition of this multiplicity is well-known and clear. Nowadays, the problem of determining the multiplicity of polynomial root by machine computation is also considered (see e.g. [101]).

The second simple case to consider is that of plane curves. The problem of intersection of two algebraic plane curves is already tackled by Newton; he and Leibnitz had a clear idea of 'elimination' process expressing the fact that two algebraic equations in one variable have a
common root, and using such a process, Newton observed in [53] that **2**

the abscissas (for instance) of the intersection points of two curves of respective degrees m, n are given by an equation of degree $\leq m.n$. This result was gradually improved during the 18^{th} century, until Bezout, using a refined elimination process, was able to prove that, in general, the equation giving the intersections had exactly the degree $m.\ n$; however, no general attempt was made during that period to attach an integer measuring the 'multiplicity' of the intersection to each intersection point in such a way that the sum of multiplicities should always be $m.n$ (see also [14]). Therefore the classical theorem of Bezout states that two plane curves of degree m and n, intersect in atmost $m.n$ different points, unless they have infinitely many points in common. In this form, however, the theorem was also stated by Maclaurin in his 'Geometrica Organica', published in 1720 (see [48, p. 67/68]); nevertheless the first correct proof was given by Bezout. An interesting fact, usually not mentioned in the literature, is that: In 1764, Bezout not only proved the above mentioned theorem, but also the following n-dimensional version:

(0.2)

Let X be an algebraic projective sub-variety of a projective n-space. If X is a complete intersection of dimension zero the degree of X is equal to the product of the degrees of the polynomials defining X.

3 The proof can be found in the paper [4], [5] and [6]. In his book "on algebraic equations", published in 1770, [7], a statement of this theorem can be found already in the foreword. We quote from page XII:

'Le degré de ℓ' équation finale résultante d'unnombre quelcoque d'équations complétes renfermant un pareil nombre d'inconnues, and de degrés quelconques, est égal au produit des exposants des degrés de ces équations. Théoréme dont ℓa verité n'etait connue et démontrée que pout deux équations seulement.'

The theorem appears again on page 32 as theorem 47. The special cases $n = 2, 3$ are interpreted geometrically on page 33 in section 3^0 and it is mentioned there, that these results are already known from Geometry. (For these historical remarks, see also ([61]).

Let us look at projective plane curves C defined by the equation $F(X_0, X_1, X_2) = 0$ and D defined by the equation $G(X_0, X_1, X_2) = 0$ of degree n and m, respectively, without common components. Then we get

(0.3) BEZOUT'S Theorem.

$m.n = \sum\limits_{P} i(C, D; P)$ where the sum is over all common points P of C and D and where the positive integer $i(C, D; P)$ is the intersection multiplicity of C and D at P. **4**

We wish to show that this multiplicity is defined, for instance, in terms of a resultant:

Given P, we may choose our coordinates so that at P we have $X_2 = 1$ and $X_O = X_1 = 0$. By the Preparation Theorem of Weierstrass [102] (after a suitable change of coordinates) we can write

$$F(X_0, X_1, 1) = f'(X_0, X_1) \cdot \overline{f}(X_0, X_1)$$

and

$$G(X_0, X_1, 1) = g'(X_0, X_1) \cdot \overline{g}(X_0, X_1)$$

where $f'(X_0, X_1)$ and $g'(X_0, X_1)$ are power series in X_0 and X_1 such that $f'(0, 0) \neq 0 \neq g'(0, 0)$ and where

$$\overline{f}(X_0, X_1) = X_1^e + V_1(X_0)X_1^{e-1} + \cdots + V_e(X_0)$$

and

$$\overline{g}(X_0, X_1) = X_1^\ell + W_1(X_0)X_1^{\ell-1} + \cdots + W_\ell(X_0)$$

where $V_i(X_0)$ and $W_j(X_0)$ are power series with $V_i(0) = W_j(0) = 0$. Following Sylvester [86], we define the X_1 resultant of $\overline{f}$ and $\overline{g}$, denoted by $Res_{X_1}(\overline{f}, \overline{g})$, to the $(e + \ell) \times (e + \ell)$ determinant

$$\begin{vmatrix}
1 & V_1 \cdots V_e & & \\
 & 1 V_1 \cdots V_e & & \\
 & \cdots\cdots\cdots\cdots & & \\
 & & & 1 V_1 \cdots V_e \\
1 & W_1 \cdots W_\ell & & \\
 & 1 W_1 \cdots W_\ell & & \\
 & \cdots\cdots\cdots\cdots & & \\
 & & & 1 W_1 \cdots W_\ell
\end{vmatrix}$$

with zero in all blank spaces. 5

The application of the Preparation Theorem of Weirstrass, enables us to get, from the Resultant Theorem (see e.g. [92]), that $Res_X(\overline{f},\overline{g}) = 0$. Now $Res_{X_1}(\overline{f},\overline{g})$ is a power series in X_0 and we define

$$i(C,D;P) = X_0 - \text{ order of } \operatorname{Res}_{X_1}(\overline{f},\overline{g}).$$

It is also possible to define the above multiplicity by using the theory of infinitely near singularities (see, for instance, [1], ch. *VI*).

However, Poncelet, as a consequence of his general vaque 'Principle of continuity' given in 1822, had already proposed to defined the intersection multiplicity at one point of two subvarieties U, V of complementary dimensions (see definition below) by having V (for instance) vary continuously in such a way that for some position V' of V all the intersection points with U should be simple, and counting the number of these points which collapses to the given point when V' tended to V, in such a way the total number of intersections (counted with multiplicities) would remain constant *('principle of conservation of number')*; and it is thus that Poncelet proved Bezout's Theorem, by observing that a curve C in a plane belongs to the continuous family of all curves of the same degree m, and that in that family there exist curves which degener-
6 ate into a system of straight lines, each meeting a fixed curve Γ of degree n in n distinct points. Many mathematicians in the 19th century had extensively used such arguments, and in 1912, Severi had convincingly argued for their essential correctness, see [73].

In view of our exposition below, we wish to mention that the starting point of C. Chevalley's considerations [11], [12] has been the observation that the intersection multiplicity at the origin 0 of two affine curves $f(X,Y) = 0, g(X,Y) = 0$, may be defined to be the degree of the field extension $K((X,Y)) \mid K((f,g))$, where $K((x,y))$ is the field of quotients of the ring of power series in X, Y with coefficients in the base field K, and where $K((f,g))$ is the field of quotients of the ring of those power series in X, Y which can be expressed as power series in f and g. From there C. Chevalley was led to the definition of multiplicity of a local ring with respect to a system of parameters, and then to the general notion of intersection multiplicity.

The ideal generalization of these observations would be the well-known theorem of Bezout. First we note that the degree of an algebraic projective subvariety V of a projective n-space $\mathbb{P}^n_k$ (K algebraically closed field), denoted by deg (V), is the number of points in which almost all linear subspaces $L \subset \mathbb{P}^n_k$ of dimension $n-d$ meet X, where d is the dimension of V. Let V_1, V_2 be unmixed varieties of dimensions r, s **7**
and degrees d, e in $\mathbb{P}^n_k$, respectively. Assume that all irreducible components $V_1 \cap V_2$ have dimension $= r+s-n$, and suppose that $r+s-n \geq 0$. For each irreducible component C of $V_1 \cap V_2$, define intersection multiplicity $i(V_1, V_2; C)$ of V_1 and V_2 along C. Then we should have

$$\sum_C i(V_1, V_2; C).deg(C) = d.e,$$

where the sum is taken over all irreducible components of $V_1 \cap V_2$. The hardest part of this generalization is the correct definition of the intersection multiplicity and, by way, historically it took may attempts before a satisfactory treatment was given by A. Weil [103] in 1946. Therefore the proof of Bezout's Theorem has taken three centuries and a lot of work to master it.

To get equality in the above equation, one may follow different approaches to arrive at several different multiplicity theories. At the beginning of this century, one investigated the notion of the length of a primary ideal in order to define intersection multiplicities. This multiplicity is defined as follows:

Let $V_1 = V(I_1), V_2 = V(I_2) = V(I_2) \subset \mathbb{P}^n_k$ be projective varieties defined by homogeneous ideals $I_1, I_2 \subset K[X_0, \ldots, X_n]$. Let C be an irreducible component of $V_1 \cap V_2$. Denote by $A(V_i; C)$ the local ring of V_i at C. Then we set

$$\ell(V_1, V_2; C) = \text{ the length of } A(V_1; C)/I_2.A(V_1; C).$$

For instance, this multiplicity yields the intersection multiplicity as **8**
set forth in the beginning for projective plane curves. Furthermore, this length provides the "right" intersection number for unmixed subvarieties $V_1, V_2 \subset \mathbb{P}^n_k$ with $n \leq 3$ and $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - n$ (see, e.g., W. Gröbner [26]). Therefore prior to 1928 most mathematicians hoped

that this multiplicity yield for Bezout's Theorem the correct intersection multiplicity for the irreducible components of two projective varieties of arbitrary dimensions (see, e.g. Lasker [44], Macaulay [49]). And, by the way, we want to mention that Grobner's papers [26], [29] are a plea for adoptions of the notion of intersection multiplicity which is based on this length of primary ideals. He also posed the following problem:

(0.4) Problem. What are some of the deeper lying reasons that the so-called generalized Bezout's Theorem

$$\deg(V_2 \cap V_2) = \deg(V_1) \cdot \deg(V_2)$$

is not true under certain circumstances ?

In 1928, B.L.Van der Waerden [90] studied the space curve given parametrically by $\left\{s^4, s^3t, s^3, t^4\right\}$ to show that the length does not yield the correct multiplicity, in order for Bezout's Theorem to be valid in
9 projective space $\mathbb{P}_k^n$ with $n \geq 4$ and he has written in [89, p. 770]:

"In these cases we must reject the notion of length and try to find another definition of multiplicity" (see also [64, p. 100].

We will study this example (see also [26], [50] or [32]). The leading coefficient of the Hilbert polynomial of a homogeneous ideal $I \subset K[X_0, \ldots, X_n]$ will be denoted by $h_0(I)$. Let $V = V(I)$ be a projective variety defined by a homogeneous ideal $I \subset K[X_0, \ldots, X_n]$. Then we have $\deg(V) = h_0(I)$.

(0.5) Example. Let V_1, V_2 be the subvarieties of projective space $\mathbb{P}_k^4$ with defining prime ideals:

$$\mathscr{Y}_1 = (X_0X_3 - X_1X_2, X_1^3 - X_0^2X_2, X_0X_2^2 - X_1^2X\,X_3, X_1X_3^2 - X_2^3)$$
$$\mathscr{Y}_2 = (X_0, X_3)$$

Then $V_1 \cap V_2 = C$ with the defining prime ideal $\mathscr{Y} : I(C) = (X_0, X_1, X_2, X_3)$. It is easy to see that (see, e.g. (1.42) , (iii) $h_0(\mathscr{Y}_1) = 4, h_0(\mathscr{Y}_2) = 1, h_0(\mathscr{Y}) = 1$ and therefore $i(V_1, V_2; C) = 4$. Since $\mathscr{Y}_1 + \mathscr{Y}_2 = (X_0, X_3, X_1X_2, X_1^3, X_2^3) \subset (X_0, X_3, X_1X_2, X_1^2, X_2^3) \subset (X_0, X_3, X_1, X\,X_2^3,) \subset (X_0, X_3, X_1, X_2^2,) \subset (X_0, X_1, X_2, X_3)$, we have $\ell(V_1, V_2; C) = 5$. Therefore we obtain

$\deg(V_1).\deg(V_2) = i(V_1, V_2; C).\deg(C) \neq \ell(V_1, V_2; C)\deg(C)$

Nowadays it is well-known that

$$\ell(V_1, V_2; C) = i(V_1, V_2; C)$$

if and only if the local rings $A(V_1, C)$ of V_1 at C and $A(V_2, C)$ of V_2 **10**
at C are Cohen - Macaulay rings for all irreducible components C of $V_1 \cap V_2$ where $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - n$ (see [72], p. V-20; see also (3.25)). We assume again that $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - n$. Without loss of generality, we may suppose, by applying our observations of §2 of chapter *I*, that one of the two intersecting varieties V_1 and V_2 is complete intersection, say V_1.

Having this assumption, we get that

$$\ell(V_1, V_2; C) \geq i(V_1, V_2; C)$$

for every irreducible component C (see also (3.18)). Let V_2 be a complete intersection. Then there arises another problem posed by D.A. Buchsbaum [9] in 1965.

(0.6) Problem. Is it true that $\ell(V_1, V_2; C) - i(V_1, V_2; C)$ is independent of V_2, that is, does there exist an invariant $I(A)$, of the local ring $A := A(V_1; C)$ of V_1 at C such that

$$\ell(V_1, V_2; C) - i(V_1, V_2; C) = I(A)?$$

This is not the case, however. The first counter - example is given in [95]. The theory of local Buchsbaum rings started from this negative answer to the problem of D.A. Buchsbaum. The concept of Buchsbaum rings was introduced in [82] and [83], and the theory is now developing
rapidly. The basic underlying idea of a Buchsbaum ring continues the **11**
well-known concept Cohen-Macaulay ring, its necessity being created by open questions in Commutative algebra and Algebraic geometry. For instance, such a necessity to investigate generalized Cohen Macaulay structure arose while classifying algebraic curves in $\mathbb{P}^3_k$ or while studying singularities of algebraic varieties. Furthermore, it was shown by Shiro Goto (Nihon University, Tokyo) and his colleagues that interesting and extensive classes of Buchsbaum rings do exist (see, e.g. [23]).

However, our observations from the Chapter *II* yield the intersection multiplicities by the length of well - defined primary ideals. Hence these considerations again provide the connection between the different view points which are treated in the work Lasker - Macaulay - Gröbner and Severi - van der Waerden - Weil concerning the multiplicity theory in the classical case, that is, in case $\dim(V_! \cap V_2) = \dim V_1 + \dim V_2 - n$. We want to end this section with some remarks on Buchsbaum's problem. First we give the following definition:

(0.7) Definition. Let A be a local ring with maximal ideal $\mathfrak{M}$. A sequence $\{a_1, \ldots, a_r\}$ of elements of is a $\mathfrak{M}$ is a *weak A- sequence* if for each $i = 1, \ldots, r$

$$\mathfrak{M} \cdot [(a_1, \ldots, a_{i-1}) : a_i] \subseteq (a_1, \ldots, a_{i-1})$$

for $i = 1$ we set $(a_1, \ldots, a_{i-1}) = (0)$ in A).

12 If every system of parameters of A is a weak A- sequence, we say that A is a *Buchsbaum ring*.

Note that Buchsbaum rings yields a generalization if Cohen Macaulay rings.

In connection with Buchsbaum's problem and with our observations concerning the theory of multiplicities in the paper [82], we get an important theorem (see [82]).

(0.8) Theorem. *A local ring A is a Buchsbaum ring if and only if the difference between the length and the multiplicity of any ideal q generated by a system of parameters is independent of q.*

In order to construct simple Buchsbaum rings and examples which show that the above problem is not true in general, we have to state the following lemma (see [82], [87], or [97]).

(0.9) Lemma. *Let A be a local ring. First we assume that* $\dim(A) = 1$. *The following statements are equivalent:*

(i) *A is a Buchsbaum ring.*

(ii) $\mathfrak{M}U((0)) = (0)$, *where* $U((0))$ *is the intersection of all minimal primary zero ideals belonging to the ideal* (0) *in A. Now, suppose* *that* $\dim(A) > depth\,(A) \geq 1$ *then the following statements are equivalent:*

(iii) *A is a Buchsbaum ring.*

(iv) *There exists a non-zero - divisor $x \in \mathfrak{M}^2$ such that $A/(x)$ is a Buchsbaum ring.* **13**

(v) *For every non-zero - divisor $x \in \mathfrak{M}^2$, the ring $A/(x)$ is a Buchsbaum ring.*

Applying the statements (i), (ii) of the lemma, we get the following simple examples.

(0.10) Example. Let K be any field

(1) We set $A := K[[X,Y]]/(X) \cap (X^2, Y)$ then it is not difficult to show that A is Buchsbaum non - Cohen - Macaulay ring.

(2) We set $A := K[[X,Y]]/(X) \cap (X^3, Y)$ then A is not a Buchsbaum ring.

For the view point of the theory of intersection multiplicities, we can construct the following examples by using the statements (iii), (iv) of the lemma.

(3) Take the curve $V \subset \mathbb{P}^3_k$ given parametrically by $\{S^5, S^4t, St^4, t^5\}$. Let A be the local ring of the affine cone over V at the vertex, that is, $A = K[X_0, X_1, X_2, X_3]_{(X_0,X_1,X_2,X_3)/\mathscr{Y}_V}$ where $\mathscr{Y}_V = (X_0X_3 - X_1X_2, X_0^3X_2 - X_1^4, X_0^2X_2^2 - X_1^3X_3, X_0X_2^3 - X_1^2X_3^2, X_2^4 - X_1X_3^3)$. Then A is not a Buchsbaum ring (see [62]). We get again this statement from the following explicit calculations:

Consider the cone $C(V) \subset \mathbb{P}^4_k$ with defining ideal $\mathscr{Y}_V$ and the surfaces W and W' defined by the equations $X_0 = X_3 = 0$ and $X_1 =$ **14**
$X_0^2 + X_3^2 = 0$, respectively. It is easy to see that $C(V) \cap W = C(V) \cap W' = C$, where C is given by $X_0 = X_2 = X_3 = X_4 = 0$. Some simple calculations yield:

$\ell(C(V), W; C) = 7, i(C(V), W; C) = 5$ and $\ell(C(V), W'; C) = 13,$

$i(C(V), W'; C) = 10$ and hence

$$\ell(C(V), W; C) - i(C(V), W'; C) \neq \ell(C(V), W'; C) - i(C(V), W'; C).$$

Therefore this example shows that the answer to the above problem of D.A. Buchsbaum is negative.

(4) Take the curve $V \subset \mathbb{P}^3_K$ given parametrically by $\{s^4, s^3t, st^3, t^4\}$. Let A be the local ring of the affine cone over V at the vertex, that is $A = K[X_0, X_1, X_2, X_3]_{(X_0,X_1,X_2,X_3)}\mathscr{Y}_V$, where $\mathscr{Y}_V = (X_0X_3 - X_1X_2, X_0^2X_2 - X_1^3, X_0X_2^2 - X_1^2X_3, X_1X_3^2 - X_2^3)$. Then A is a Buchsbaum ring (see e.g [83]).

(0.11) Remark. This last example has an interesting history. This curve was discovered by G. Salmon ([[67], p. 40]) already in 1849 and a little later in 1857 by J.Steiner ([[79], p. 138]) by using the theory of residual intersections. This curve was used by F.S.Macaulay ([[49], p. 98]) in 1916. His purpose was to show that not every prime ideal in a polynomial ring is perfect. In 1928, B.L. Van der Waerden [90] studied this example to show that the length of a primary ideal does not yield the correct local intersection multiplicity in order Bezout's theorem to be valid in projective space $\mathbb{P}^n_k$ with $n \geq 4$, and he has written (as cited already); "In these cases we must reject the notion of length and try to find another definition of multiplicity". As a result, the notion of intersection multiplicity of two algebraic varieties was put on a solid base by Van der Waerden for the first time, (see e.g. [88], [91], [92]). We know now that this prime ideal of *F.S*. Macaulay is not a Cohen-Macaulay ideal, but a Buchsbaum ideal (i.e., the local ring of A of example (4) is not a Cohen-Macaulay ring, but is a Buchsbaum ring). This fact motivated us to create a foundation for the theory of Buchsbaum rings. (For more specific information on Buchsbaum rings, see also the forthcoming book by *W*. Vogel with J. Stückrad.)

B. The Non-classical Case

Let $V_1, V_2 \subset \mathbb{P}^n_k$ be algebraic projective varieties. The projective dimension theorem states that every irreducible component of $V_1 \cap V_2$ has dimension $\geq \dim V_1 + \dim v_2 - n$. Knowing the dimensions of the

irreducible components of $V_1 \cap V_2$, we can ask for more precise information about the geometry of $V_1 \cap V_2$. The classical case in the first section works in case of $\dim V_1 \cap V_2 = \dim V_1 + \dim V_2 - n$. The purpose of this section is to study the non-classical case, that is $\dim V_1 \cap V_2 > \dim V_1 + \dim V_2 - n$ If V_1, V_2 are irreducible varieties, what **16**
can one say about the geometry of $V_1 \cap V_2$?. A typical question in this direction was asked by 5. Kleiman: Is the number of irreducible components of $V_1 \cap V_2$ bounded by the Bezout's number $\deg(V_1) \cdot \deg(V_2)$? A special case of this question was studied by C.G.J. Jacobi [36] already in 1836. But we want to mention that Jacobi's observations relies on a modification of an idea of Euler [16] from 1748. We would like to describe Jacobi's observation.

(0.12) JACOBI'S Example

Let F_1, F_2, F_3 be three hypersurfaces in $\mathbb{P}^3_K$. Assume that the intersection $F_1 \cap F_2 \cap F_3$ is given by one irreducible curve, say C and a finite set of isolated points, say $P_1, \ldots, P_r$. Then $\prod_{i=1}^{3} \deg(F_i) - \deg(C) \geq$ number of isolated points of $F_1 \cap F_2 \cap F_3$. The first section of this example was given by Salmon and Fielder [68] in their book on geometry, published in 1874, by studying the intersection of r hypersurfaces in $\mathbb{P}^n_k$. The assumption is again that this intersection is given by one irreducible curve and a finite set of isolated points. In 1891, M, Pieri [59] studied the intersection of two subvarieties, say M. Pieri [59] studied the intersection of two subvarieties, say V_1, V_2 of $\mathbb{P}^n_K$ assuming that $V_1 \cap V_2$ is given by one irreducible component of dimension $\dim V_1 \cap V_2$ and a finite set of isolated points. Also, it seems that a starting point of an intersection theory in the non-classical case was discovered by M.Pieri. In 1947, 56 **17**
years after M. Pieri, F. Severi [78] suggested a beautiful solution to the decomposition of Bezout's number $\deg(V_1) \cdot \deg(V_2)$ for any irreducible subvarieties V_1, V_2 of $\mathbb{P}^n_k$. Unfortunately, Sever i's solution is not true. The first counter- example was given by *R*. Lazarfeld [45] in 1981. But Lazarfeld also shows how Severi's procedure can be modified so that it does yield a solution to the stated problem.

Nowadays, we have a remarkable theory of W. Fulton and R. Mac-

Pherson on defining algebraic intersection (see, e.g. [18], [19]). Suppose V_1 and V_2 are subvarieties of dimension r and s of a non-singular algebraic variety X of dimension n. Then the equivalence class $V_1 \cdot V_2$ of algebraic $r + s - n$ cycles which represents the algebraic intersection of V_1 and V_2 is defined upto rational equivalence in X. This intersection theory produces subvarieties W_i of $V_1 \cap V_2$, cycle classes α_i on W_i positive integers m_i with $\sum m_i\alpha_i$ representing $V_1 \cdot V_2$ and $\deg \alpha_i \geq \deg W_i$ even in the case $\dim V_1 \cap V_2 \neq r + s - n$.

Our object here is to describe the algebraic approach of [85] (see also [56]) to the intersection theory by studying a formula for $\deg(V_1) \cdot \deg(V_2)$ in terms of algebraic data, if V_1 and V_2 are pure dimensional subvarieties of $\mathbb{P}^n_k$. The basis of this formula is a method (see [8], [98])
18 for expressing the intersection multiplicity of two properly intersecting varieties as the length of a certain primary ideal associated to them in a canonical way. Using the geometry of the join construction in $\mathbb{P}^{2n+1}_{\overline{K}}$ over a field extension $\overline{K}$ of K we may apply this method even if $\dim(V_1 \cap V_2) > \dim V_1 + \dim V_2 - n$. The key is that algebraic approach provides an explicit description of the subvarieties C_i and the intersection numbers $j(V_1, V_2; C_i)$ which are canonically determined over a field extension of K.

Chapter 1

Preliminary Results

A. Preliminary Definitions and Remarks

(1.1)

Let R be a noetherian ring and I be an ideal in R. The Krull-dimension, 19
$K - \dim(I)$ of I is the Krull-dimension of the ring R/I. Suppose that $I = q_1 \cap \cdots \cap q_r$ is a primary decomposition of I, where q_i is $\mathscr{Y}_i$-primary, $\mathscr{Y}_i \in \operatorname{Spec}(R)$ for $1 \leq i \leq r$. We say that q_i is a $\mathscr{Y}_i$-*primary component* of I any $\mathscr{Y}_i$ is *an associated prime of* R/I. We write $Ass(R/I) = \mathscr{Y}_1, \ldots, \mathscr{Y}_r$. Suppose that $K - \dim(I) = K - \dim(q_i)$ for $1 \leq i \leq s \leq r$. We set $U(I) := \bigcap_{i=1}^{s} q_i$. This ideal is well defined and is called the *unmixed part of* I. It is clear that $I \subset U(I)$ and $k \dim U(I)$. An ideal $I \subset R$ is called *unmixed* if and only if $I = U(I)$. A ring R is called *unmixed* if the zero ideal (0) in R is unmixed.

Let $\mathscr{Y} \in \operatorname{Spec}(R)$ and q be a $\mathscr{Y}$-primary ideal. The length of the Artinian local ring $(R/q)_{\mathscr{Y}}$ is called *the length of* q and we will denote it by $\ell_R(q)$. It is easy to see that the length of q is the number of terms in a composition series, $q = q_1 \subset q_2 \subset \cdots \subset q_\ell = \mathscr{Y}$ for q, where $q_1, \cdots, q_r$ are $\mathscr{Y}$-primary ideals.

Remark. (see [[106], Corollary 2 on *p*. 237, vol. 1]) Let $\mathfrak{M} \subset R$ be 20
a maximal ideal of R and $q \subset R$ be a $\mathfrak{M}$ primary ideal. If $q = q_1 \subset$

$q_2 \subset \cdots \subset q_\ell = \mathfrak{M}$ is a composition series for q, where $q_1, \ldots, q_\ell$ are $\mathfrak{M}$-primary ideals. Then there exist $a_i \in q_i, 2 \leq i \leq \ell$, such that

(i) $a_i \notin q_{i-1}$

(ii) $q_i = (q_{i-1}, a_i)$

(iii) $\mathfrak{M}q_i \subset q_{i-1}$ for all $2 \leq i \leq \ell$.

Proof. (i) and (ii) are easy to prove.

(iii) Replacing R by R/q we may assume that $q = 0$ and R is Artinian local. Suppose $\mathfrak{M}q_i \not\subset q_{i-1}$ for some $2 \leq i \leq \ell$. Then we get $q_{i-1} \subsetneq (q_{i-1} + \mathfrak{M}q_i) = q_i = (q_{i-1}, a_i)$. Therefore we can write $a_i = q + ma_i$ for some $q \in q_{i-1}$ and $m \in \mathfrak{M}$. Then $a_i = \dfrac{1}{(1-m)} \cdot q \in q_{i-1}$ which is a contradiction to (i). □

Let R be a semi-local noetherian ring and rad(R) be the Jacobson radical of R. An ideal $q \subset R$ is called an *ideal of definition* if $(rad(R))^n \subset q \subset rad(R)$ for some $n \in \mathbb{N}$.

(1.2) The Hilbert-samuel Function

Let R be a semilocal noetherian ring and $q \subset R$ be an ideal of definition. Let M be any finitely generated R-module. The numerical function $H^1_M(q, -) : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $H^1_M(q, n) = \ell(M/q^{n+1}M) < \infty$ is called *the Hilbert-Samuel function of* q on M. If $M = R$ we say that
21 $H^1(q, -) := H^1_R(q, -)$ is the Hilbert-Samuel function of q. If $(R, \mathscr{M})$ is a local ring then $H^1_R(-) := H^1(\mathcal{M}, -)$ is called *the Hilbert-Samuel function of R*.

The following theorem is well known(for proof, see [72] or [106]).

Hilbert-samuel Theorem.

Let R be a semilocal noetherian ring and $q \subset R$ be an ideal of definition. Let M be any finitely generated R-module. Then $H^1_M(q, -)$ is, for $n >> 1$, a polynomial $P_M(q, -)$ in n, with coefficients in Q. The degree of $P_M(q, -)$ is δ where δ = Krull dimension of $M(:= k - \dim R/ann_R)(m))$.

We will write this polynomial in the following form:

$$P_M(q,n) = e_0\binom{n+d}{d} + e_1\binom{n+d-1}{d-1} + \cdots + e_d$$

where $e_0(\geq 0).e_1,\ldots,e_d$ are integers and $d = K - \dim(r)$. The multiplicity of q on $M, e_0(q;M)$, is defined by $e_0(q;M) := e_0$. Note that $e_0(q;M) := 0$ if and only if $K - \dim(M) < K - \dim(R)$. The positive integer $e_0(q;R)$ is called the multiplicity of q. If R is local and $q = \mathscr{M}$ is the maximal ideal of R then $e_0(R) := e_0(\mathscr{M};R)$ is called the multiplicity of R.

Remark.

(i) Let $(A,\mathscr{M}) \to (B,\mathscr{N})$ be a flat local homomorphism of local rings. Assume that $\mathscr{M}B = \mathscr{N}$. Then for every $\mathscr{M}$-primary ideal q of A we have

$$e_0(q;A) = e_0(qB;B)$$

Proof. It is easy to see (see, e.g. [[34], (1.28)] that, $H^1_B(qB,t) = \ell_B$ 22
$(B/q^tB) = \ell_B(A/q^t \otimes_A B) = \ell_A(A/q^t)\ell_B(B/\mathscr{M}_B) = \ell_A(A/q^t) = H^1_A(q,t)$ for all $t \geq 0$. Therefore $e_0(qB;B) = e_0(q;A)$. □

Let $A = \underset{n\geq 0}{\oplus} A_n$ be a graded ring such that A_0 is artinian and A is generated as an A_0-algebra by r elements $\bar{x},\ldots,\bar{x}_r$. of A_1. Let $N = \underset{n\geq 0}{\oplus} N_n$ be a finitely generated graded A-module. The numerical function $H^1_A(N,-) : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $H^1_A(N,n) = \ell_{A_0}(N/N_{n+1})$ is called *the Hilbert function of* N. The following is a well-known theorem (for proof, see [55] or [72]).

Theorem HILBERT. *The function $H^1_A(N,-)$ is, for $n >> 1$, a polynomial $P_A(N,-)$ in n with coefficients in Q. The degree of $P_A(N,-)$ is $\leq r$.*

We will write this polynomial in the following form:

$$P_A(N,n) = h_0\binom{n+r}{r} + h_1\binom{n+r-1}{r-1} + \cdots + h_r,$$

where $h_0(\geq 0), h_1,\ldots,h_r$ are integers.

Note that

$$H^0_A(N,n) = \ell_{A_0}(N_n) = h_0\binom{n+r-1}{r-1} + h'_1\binom{n+r-2}{r-2} + \cdots + h'_{r-l},$$

where $h'_1, \ldots, h'_r$ are integers.

Remark.

23 (ii) From the exact sequence

$$0 \to \left(0 \underset{N}{:} \bar{X}_1\right) \to N \xrightarrow{\bar{X}} N \to N/\bar{X}_1 \to 0$$

of graded modules, it follows that

$$H^1_{A/\bar{X}_1}(M/\bar{X}_1 N, n) - H^1_{A/\bar{X}_1}\left(\left(0 \underset{N}{:} \bar{X}_1\right), n-1\right)$$
$$= H^0_A(N,n) \text{ for all } n \geq 0.$$

Therefore we have $h_0(N/\bar{X}_1 N) - h_0\left((0 \underset{N}{:} \bar{X}^1)\right) = h_0(N)$.

(iiii) Let R be a semi-local ring and $q = (x_1, \ldots, x_d) \subset R$ be an ideal of definition generated by a system of parameters $x_1, \ldots, x_d$ for R. Let M be any finitely generated R-module.

Then $H^1_M(q,n) = H^1_{gr_q}(gr_q(M), n)$ for all n. Therefore $P_M(q,n)$ $= P_{gr_q(R)}(gr_q(M), n)$ for all n and $e_0(q; M) = h_0(gr_q(m))$, where $gr_q(R) = \bigoplus_{n\geq 0} q^n/q^{n+1}$ and $gr_q(M) = \bigoplus_{n\geq 0} q^n M/q^{n+1}M$.

B. The General Multiplicity Symbol

Let R be a noetherian ring and M be any finitely generated R-module. Let $x_1, \ldots, x_d$ be a system of parameters for R. We shall now define the general multiplicity symbol, $e_R(x_1, \ldots, x_d|M)$, of $x_1, \ldots, x_d$ on M.

(1.3) Definition. Let R be a noetherian ring and M be any finitely generated R-module. Let $x_1, \ldots, x_d$ be a system of parameters for R. We

24 shall define $e_R(x_1, \ldots, x_d|M)$, by induction on d. If $d = 0$, then define $e_R(\cdot|M) = \ell_R(M) < \infty$. Assume that $d \geq 1$ and the multiplicity symbol has been defined for $s \leq d-1$ elements and all modules. Define $e_R(x_1, \ldots, x_d|M) = e_{R/x_1}(x_2, \ldots, x_d|M/x_1M) - e_{R/x_1}(x_2, \ldots, x_d|(0 \underset{M}{:} x_1))$. It is clear that $e_R(x_1, \ldots, x_d|M)$ is an integer (in fact, non-negative, see (1.9)).

(1.4) Remarks. (i) By induction on d, it follows that $e_R(x_1, \ldots, x_d|M)) = \ell(M/qM) - \ell((q_{d-1}M \underset{M}{:} x_d)/q_{d-1}M) - \sum_{k=1}^{d-1} e_{R/q_k}(x_{k+1}, \ldots, x_d|(q_{k-1}M \underset{M}{:} x_k)/q_{k-1}M)$ where $q_k = (x_1, \cdots, x_K)R, 0 \leq k \leq d-1, q = (x_1, \cdots, x_d)$.

(ii) Assume that $d \geq 2$ and $1 \leq m < d$. Then

$$e_R(x_1, \ldots, x_d|M) = \sum_{\nu} \varepsilon_\nu e_{R/q_{m-1}}(x_m, \ldots, x_d|M_\nu)$$

where $\in_\nu = \pm 1$ and M_ν are uniquely determined by M and $x_1, \ldots, x_{m-1}$.

Some Properties of the General Multiplicity Symbol

(1.5) The additive property

Let $0 \to M' \to M'' \to 0$ *be an exact sequence of finitely generated R-modules and* $x_1, \ldots, x_d$ *be a system of parameters for R. Then*

$$e_R(x_1, \ldots, x_d|M) = e_R(x_1, \ldots, x_d|M') + e_R(x_1, \ldots, x_d|M'').$$

(1.6) Corollary. *Let* $o \to M_p \to M_{p-1} \to \cdots \to M_1 \to M_0 \to 0$ *be an exact sequence of finitely generated R-modules and* $x_1, \cdots, x_d$ *be a system of parameters for R. Then* 25

$$\sum_{i=0}^{p} (-1)^i e_R(x_1, \ldots, x_d|M_i) = 0.$$

Proof. It is convenient to prove (1.5) and (1.6) simultaneously. Proof by induction on d. If $d = 0$ then

$$\sum_{i=0}^{p}(-1)^i e_R(\cdot|M_i) = \sum_{i=0}^{p}(-1)^i \ell(M_i) = 0.$$

Now suppose that $d = s + 1$, $s \geq 0$ and that (1.5) and (1.6) holds for $d = s$. We have an exact sequence

$$0 \to (0 \underset{M'}{:} x_1) \to (0 \underset{M}{:} x_1) \to (0 \underset{M''}{:} x_1) \to M'/x_1M' \\ \to M/x_1M \to M''/x_1M'' \to 0$$

Therefore by induction hypothesis, we have

$$\begin{aligned} &e_{R/x_1}(x_2, \ldots, x_d|M/x_1M) - e_{R/x_1}(x_2, \ldots, x_d|(0 \underset{M'}{:} x_1)) \\ &\quad = e_{R/x_1}(x_2, \ldots, x_d|M'/x_1M') - e_{R/x_1}(x_2, \ldots, x_d|(0 \underset{M'}{:} x_1)) \\ &\qquad + e_{R/x_1}(x_2, \ldots, x_d|M''/x_1M'') - e_{R/x_1}(x_2, \ldots, x_d)|(0 \underset{M''}{:} x_1)) \end{aligned}$$

Hence $e_R(x_1, \cdots, x_d|M) = e_R(x_1, \cdots, x_d|M') + e_R(x_1, \cdots, x_d|M")$. □

(1.7) The Exchange Property

Let M be any finitely generated R-module and $x_1, \ldots, x_d$ be a system of parameters for R. Then

$$e_R(x_1, \ldots, x_d|M) = e_R(x_{i_1}, \ldots, x_{i_d}|M)$$

26 for every permutation $(i_1, \ldots, i_d)$ of $(1, \ldots, d)$.

Proof. By remark (1.4) (ii), it is enough to prove that,

$$e_R(x_1, \ldots, x_d|M) = e_R(x_2, x_1, \ldots, x_d|M).$$

Let K be any finitely generated $R/(x_1, x_2)$ -module. Then we denote $e_{R/(x_1,x_2)}(x_3, \ldots, x_d|K)$ by $[K]$. □

Now, we have

$$e_R(x_1,\ldots,x_d|M) = [M/(x_1,x_2)M] - [(0 \underset{M/x_1M}{:} x_2)]$$
$$-[(0 \underset{M}{:} x_1)/x_2(0 \underset{M}{:} x_1)] + [(0 \underset{M}{:} x_2)] = [a] - [b] - [c] + [d],$$

where

$$[a] = [M/(x_1,x_2)M], [b] = [(0 \underset{M/x_1M}{:} x_2)], [c] = [(0 \underset{M}{:} x_1)/x_2(0 \underset{M}{:} x_1)]$$

and $[d] = [(0 \underset{M_1}{:} x_2)]$, with $M_i := (0 \underset{M}{:} x_i)$ for $i = 1, 2$.

Now,

$$(0 \underset{M/x_1M}{:} x_2) \xrightarrow{\sim} (x_1M : x_2)/x_1M, (0 \underset{M_1}{:} x_2) = (0 \underset{M}{:} x_1) \cap (0 \underset{M}{:} x_2)$$

Therefore, $[a]$ and $[d]$ are symmetric in x_1 and x_2. Thus it is enough to prove that $[b] + [c]$ is also symmetric in x_1 and x_2. Since $x_1M \subset x_1M + (0 \underset{M}{:} x_2) \cap (x_1M \underset{M}{:} x_2)$ we get by (1.5) ,

$$[b] = [x_1M + (0 \underset{M}{:} Mx_2)/x_1M] + [(x_1M \underset{M}{:} x_2)/x_1M + (0 \underset{M}{:} x_2)]$$
$$= [(0 \underset{M}{:} x_2)/x_1M \cap (0 \underset{M}{:} x_2)] + [x_1M \cap x_2M/x_1x_2M] = [e] + [f]$$

where $[e] = [(0 \underset{M}{:} x_2)/x_1M \cap (0 \underset{M}{:} x_2)]$ and $[f] = [x_1M \cap x_2M/x_1x_2M]$. 27
Clearly $[f]$ is symmetric in x_1 and x_2. Now consider $[c] + [e]$. Since $x_1M \cap (0 \underset{M}{:} x_2) = x_1(0 \underset{M}{:} x_1x_2)$ and $x_2(0 \underset{M}{:} x_2) \subset x_2(0 \underset{M}{:} x_1x_2) \subset x_1(0 \underset{M}{:} x_2)$ we get by (1.5) , $[e] + [c]$

$$= [x_2(0 \underset{M}{:} x_1x_2)/x_2(0x_{:1})] + [(0 \underset{M}{:} x_1)/x_2(0 \underset{M}{:} x_1x_2)]$$
$$+ [(0 \underset{M}{:} x_2)/x_1(0 \underset{M}{:} x_1x_2)]$$
$$= [g] + [h]$$

where

$$[g] = [x_2[(0 \underset{M}{:} x_1x_2)/x_2(0 \underset{M}{:} x_1)] \text{ and}$$

$$[h] = [(0 \underset{M}{:} x_1)/x_2(0 \underset{M}{:} x_1x_2)] + [(0 \underset{M}{:} x_2)/x_1(0 \underset{M}{:} x_1x_2)].$$

Clearly $[h]$ is symmetric in x_1 and x_2 and since $(0 \underset{M}{:} x_1x_2)/(0 \underset{M}{:} x_1) + (0 \underset{M}{:} x_2) \overset{x_2}{\underset{\sim}{\longrightarrow}} x_2(0 \underset{M}{:} x_1x_2)/x_2(0 \underset{M}{:} x_1), [g] = [(0 \underset{M}{:} x_1x_2)/x_2(0 \underset{M}{:} x_1) + (0 \underset{M}{:} x_2)]$ is also symmetric in x_1 and x_2. Therefore $e_R(x_1, \ldots, x_d|M) = [a] + [d] - [f] - [g] - [h]$ is symmetric in x_1 and x_2. This completes the proof.

(1.8)

Let M be any finitely generated R-module and $x_1, \ldots, x_d$ be a system of parameters for R. Suppose $x_i^m M = 0$ for some $1 \leq i \leq d$ and $m \in \mathbb{N}$. Then $e_R(x_1, \ldots, x_d|M) = 0$.

Proof. By (1.7) we may assume the $i = 1$. Proof by induction on m. If $m = 1$ then $M = M/x_1M$ and $(0 \underset{M}{:} x_1) = M$ and hence $e_R(x_1, \ldots, x_d|M) = e_{R/x_1}(x_2, \ldots, x_d|M) - e_{R/x_1}(x_2, \ldots, x_d|M) - e_{R/x_1}(x_2, \ldots, x_d|M) = 0$. □

Now suppose that $d = s + 1, s \geq 0$ and the result holds for $d = s$. We have by (1.5) ,

$$e_R(x_1, \ldots, x_d|M) = e_R(x_1, \ldots, x_d|x_1M) + e_R(x_1, \ldots, x_d|M/x_1M).$$

28 Since $x_1^{m-1}(x_1M) = x_1(M/x_1M) = 0$, by induction the result follows.

(1.9)

Let M be any finitely generated R-module and $x_1, \ldots, x_d$ be a system of parameters for R. Then

$$0 \leq e_R(x_1, \ldots, x_d|M) \leq \ell(M/(x_1, \ldots, x_d)M) < \infty.$$

Proof. First, by induction on d, we show $e_R(x_1, \ldots, x_d|M) \leq 0$. If $d = 0$ then $e_R(\cdot|M) = \ell_R(M) \geq 0$. Now suppose that $d = s + 1, s > 0$ and the result holds for $d = s$. □

Put $N = M/(0 \underset{M}{:} x_1^m)$. If $\mathscr{M} >> 1$, then it is easy to see that $(0 \underset{N}{:} x_1^m) = 0$. From (1.5) and (1.8), we get $e_R(x_1, \ldots, x_d|M) = e_R(X_1, \ldots, x_d|N) = e_R(x_2, \ldots, x_d|N/x_1N)$ and hence, by induction, it follows that $e_R(x_1, \ldots, x_d/M) \geq 0$. The second inequality follows from (1.4) (i).

(1.10) Corollary. *If $(x_1, \ldots, x_d)M = M$, then $e_R(x_1, \ldots, x_d|M) = 0$.*

(1.11) Proposition. *Let M be any finitely generated R-module. Let $x_1, \ldots, x_{d-1}, x$ and $x_1, \ldots, x_{d-1}, y$ be two systems of parameters for R. Then we have $e_R(x_1, \ldots, x_{d-1}, xy|M) = e_R(x_1, \ldots, x|M) + e_R(x_1, \ldots, x_{d-1}, y|M)$.*

Proof. By induction on d. Suppose $d = 1$. Then we have exact sequences

$$0 \to (xyM \underset{M}{:} y)/xM \to M/xM \xrightarrow{y} M/xyM \to M/yM \to 0$$

$$0 \to (0 \underset{M}{:} x) \to (0 \underset{M}{:} xy) \xrightarrow{x} (0 \underset{M}{:} x) \to (0 \underset{M}{:} x)/(0 \underset{M}{:} xy) \to 0$$

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Therefore we get

$$\ell(M/yM) + \ell(M/xM) = \ell(M/xyM) + \ell((xyM \underset{M}{:} y)/xM),$$

and $$\ell((0 \underset{M}{:} y)) + \ell((0 \underset{M}{:} x)) = \ell((0 \underset{M}{:} xy)) + \ell((0 \underset{M}{:} y))((0 \underset{M}{:} xy)).$$

□

Now, it is easy to see that $(0 \underset{M}{:} y)/(0 \underset{M}{:} xy) \xrightarrow{\sim} (xyM \underset{M}{:} y)/xM$ is an isomorphism. Therefore we get that

$$e_R(x|M) + e_R(y|M) = \ell(M/xM) - \ell_M((0 : x)) + \ell(M/yM) - \ell_M((0 : y))$$
$$= \ell(M/xyM) - \ell_M((0 : xy)) = e_R(xy|M).$$

Now suppose that $d = s + 1, s \geq 1$ and the result holds for $d = s$. Let $q = (x_1, \ldots, x_{d-1})$. Then, by induction, we have

$$e_R(x_1, \ldots, x_{d-1}, xy|M) = e_{R/x_1}(x_2, \ldots, x_{d-1}, xy|M/x_1M)$$

$$- e_{R/x_1}(x_2,\ldots,x_d,xy|(0 \underset{M}{:} x_1)$$

$$\begin{aligned} e_R/_{x_1}(x_2,\ldots,x_{d-1},xy|M) = {} & e_{R/x_1}(x_2,\ldots,x_{d-1},xy|M/x_1M) \\ & - e_R/_{x_1}(x_2,\ldots,x_{d-l},x|(0 \underset{M}{:} x_1) \\ & - e_{R/x_1}(x_2,\ldots,x_{d-l},y|(0 \underset{M}{:} x_1) \\ = {} & e_R(x_1,\ldots,x_{d-l},x|M) + e_R(x_1,\ldots,x_{d-1},y|M). \end{aligned}$$

(1.12) Corollary. *For any positive integers $n_1,\ldots,n_d$, we have*

30 *(i)* $e_R(x_1^{n_1},\ldots,x_d^{n_d}M) = n_1\ldots n_d e_R(x_1,\ldots,x_d|M)$

(ii) $0 \leq e_R(x_1,\ldots,x_d|M) \leq \dfrac{\ell(M/(x_1^{n_1},\ldots,x_d^{n_d})M)}{n_1\ldots n_d}$

Proof. (i) follows from ((1.7)) and (1.11). (ii) follows from (i) and (1.9) . □

(1.13) Corollary. *If $x_i^{m_M} \subset (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d)M$ for some $i \leq i \leq d$ and $m \in \mathbb{N}$, then $e_R(x_1,\ldots,x_d|M) = 0$.*

Proof. By (1.7) , we may assume that $i = 1$. If $n > m$, then $(x_1^n, x_2,\ldots, x_d)\ M = (x_2,\ldots,x_d)M$ and so, by (1.12), we get, $0 \leq e_R(x_1,\ldots,x_d|M) \leq \frac{\ell(M/(x_2,\ldots,x_d)M)}{n} \to 0$ as $n \to \infty$. Hence $e_R(x_1,\ldots,x_d|M) = 0$. □

(1.14) Proposition . *Let M be any finitely generated R-module and $x_1,\ldots,x_d$ be a system of parameters for R contained in rad (R). Then $e_r(x_1,\ldots,x_d|M) = \ell(M/(x_1,\ldots,x_d)M)$ if and only if $x_1,\ldots,x_d$ is an M-sequence, that is, $((x_1,\ldots,x_{i-1})M \underset{M}{:} x_i) = (x_1,\ldots,x_{i-1})M$ for $1 \leq i \leq d$.*

Proof. (<=) This implication follows from 1.14 (i) (=>) Proof by induction on d. Suppose $d = 1$. Then we have $\ell(M/x_1M) = e_R(x_1|M) = \ell_R(M/x_1M) - \ell_R((0 \underset{M}{:} x_1))$. Therefore, we get $\ell_R((0 \underset{M}{:} x_1)) = 0$, that is, $(0 \underset{M}{:} x_1) = 0$ □

Now suppose that $d = s + 1$ and the result holds for $d = s$.
31 Let $n_1,\ldots,n_d$ be arbitrary positive integers. Then by (1.9) and

(1.12) (i) we have

$$e_R(x_1^{n_1},\dots,x_d^{n_d}) \le l(M/(x_1^{n_1},\dots,x_d^{n_d})M) \le n_1\dots\dots n_d l$$
$$(M/(x_1,\dots,x_d)M) = n_1\dots\dots n_d e_R(x_1,\dots,x_d|M) = e_R(x_1^{n_1},\dots,x_d^{n_d})|M).$$

Put $N = M/(0 \underset{M}{:} x_1)$. Then by (1. 8) we have

$$\ell(M/(x_1^{n_1},\dots,x_d^{n_d})M = e_R(x_1^{n_1},\dots,x_d^{n_d})|M) = e_R(x_1^{n_1},\dots,x_d^{n_d})|N)$$
$$\le l(N/(x_1^{n_1},\dots,x_d^{n_d})N) = l(M/(0 \underset{M}{:} x_1) + (x_1^{n_1},\dots,x_d^{n_d})M)$$

and hence $(0 \underset{M}{:} x_1) \subset (x_1^{n_1},\dots,x_d^{n_d})M$ for arbitrary positive integers $n_1,\dots,n_d$. Then we get that $(0 \underset{M}{:} x_1) \subset \bigcap_{n\ge 0}(x_1^{n_1},\dots,x^{n_d})M \subset \bigcap_{n\ge 0} q^n M = 0$ by Krull's Intersection Theorem, where $q = (x_1,\dots,x_d)$.

Now,

$$e_{R/x_1}(x_2,\dots,x_d|M/x_1M) = e_R(x_1,\dots,x_d|M) = l$$
$$(M/(x_1,\dots,x_d)M) = l(M/x_1M/(x_2,\dots,x_d)M/x_1M).$$

Therefore, by induction, we get that $\{x_2,\dots,x_d\}$ is M/x_1M-sequence.

This completes the proof.

(1.15) Corollary. *(i) Let $(R,\mathscr{M})$ be a noetherian local ring. Then R is a Cohen-Macaulay ring if and only if there exists a system of parameters $\{x_1,\dots,x_d\}$ for R such that $e_R(x_1,\dots,x_d|R) = l(R/(x_1,\dots,x_d))$.* 32

(ii) Let $(R,\mathscr{M})$ be a noetherian local ring. Then R is a Cohen-Macaulay ring if and only if for every system of parameters $x_1,\dots,x_d$ for R, we have $e_R(x_1,\dots,x_d|R) = l(R/(x_1,\dots,x_d))$.

Proof. (i) Clear. (ii) Follows from [71, Theorem 2, VI-20] and (1.14). □

(1.16) The limit formula of Lech:

Let M be any finitely generated R-module and $x_1, \dots, x_d$ be a system of parameters for R. Let $n_1, \dots, n_d$ be positive integers. Then

$$\lim_{\min(n_i)\to\infty} \frac{l(M/(x_1^{n_1}, \dots, x_d^{n_d})M)}{n_1 \cdot \dots \cdot n_d} = e_R(x_1 \dots x_d|M).$$

Proof. Proof by induction on d. Suppose $d = 1$. Then, for any $n > 0$, we have $e_R(x^n|M) = l(M/x^nM) - l((0 \underset{M}{:} x^n))$. Choose an integer $m > 0$ such that $(0 \underset{M}{:} x^n) = (0 \underset{M}{:} x^m)$ for all $n \geq m$. Therefore, by (1.12)(i), we have $e_R(x^n|M) = ne_R(x|M) = \ell(M|x^nM) - l((0 \underset{M}{:} x^m))$ for $n \geq m$. Thus $e_R(x|M) = \frac{\ell(M/x^nM)}{n} + C/n$, where C is independent of n. In particular, we get

$$\lim_{n\to\infty} \frac{l(M/x^nM)}{n} = e_R(x|M).$$

Now suppose that $d = s + 1$, $s \geq 1$ and the result holds for $d = s$.

33 Using (1.5) and (1.8) and replacing M by $N := M/(0 \underset{M}{:} x_1^m), m >> 1$, we may assume that $(0 \underset{M}{:} x_1) = 0$. Note that $e_R(x_1, \dots, x_d|M) = e_R(x_1, \dots, x_d|N)$ and

$$\begin{aligned}
&0 \leq \ell_R(M/(x_1^{n_1}, \dots, x_d^{n_d})M) - \ell_R(N/(x_1^{n_1}, \dots, x_d^{n_d})N) \\
&= l((0 : x_1^m) + (x_\ell^{n_1}, \dots, x_d^{n_d})M/(x_1^{n_1}, \dots, x_d^{n_d})M = \ell((0 \underset{M}{:} x_1^m)/(0 : x_1^m) \\
&\qquad \cap (x_1^{n_1}, \dots, x_d^{n_d})M) \leq l((0 \underset{M}{:} x_1^m)/(x_2^{n_2}, \dots, x_d^{n_d})(0 \underset{M}{:} x_1^m)) \\
&\qquad\quad \leq n_2 \cdots n_d l(((0 \underset{M}{:} x_1^m)/(x_2, \dots, x_d).(0 \underset{M}{:} x_1^m)) = n_2 \dots n_d C,
\end{aligned}$$

where C is a positive integer which is independent of $n_1, \dots, n_d$.

Thus we get

$$0 \leq \frac{l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) - l(N/(x_1^{n_1}, \dots, x_d^{n_d})N)}{n_1 n_2 \cdots n_d} \leq C/n_1$$

i.e., $\displaystyle\lim_{\min(n_i)\to\infty} l(M/(x_1^{n_1}, \dots, x_d^{n_d})M) = \lim_{\min(n_i)\to\infty} l(N/(x_1^{n_1}, \dots, x_d^{n_d})N).$

This shows that we may assume $(0 \underset{M}{:} x_1) = 0$. Now by (1.12),

$$\begin{aligned} 0 \le n_1 \cdots n_d e_R(x_1, \ldots, x_d|M) &\le l(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) \\ &\le n_1 l(M/(x_1, x_2^{n_2}, \ldots, x_d^{n_d})M) \\ &= n_1 l(\bar{M}/x_2^{n_2}, \ldots, x_d^{n_d})\bar{M}) \end{aligned}$$

where $\bar{M} = M/x_1M$. Therefore, by induction, it follows that

$$\begin{aligned} &e_R(x_1, \ldots, x_d|M) \\ &\quad\le \lim_{\min(n_i)\to\infty} \frac{\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M)}{n_1 n_2 \cdot \cdots \cdot n_d} \\ &\quad\le \lim_{\min(n_i)\to\infty} \frac{\ell(\bar{M}/(x_2^{n_2}, \ldots, x_d^{n_d})M)}{n_2 \cdot \cdots \cdot n_d} = e_{R/x_1}(x_2, \ldots, x_d|\bar{M}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad = e_R(x_1, \ldots, x_d|M), \text{ since} \end{aligned}$$

$(0 \underset{M}{:} x_1) = 0$. Thus we get **34**

$$\lim_{\min(n_i)\to\infty} \frac{\ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M)}{n_1 \cdot \cdots \cdot n_d} = e_R(x_1, \ldots, x_d|M).$$

In the next proposition, we will prove that the general multiplicity symbol is nothing but the multiplicity defined in (1.2) . *Now onwards, we assume that R is semilocal noetherian.* ◻

(1.17) The Limit Formula of Samuel

Let M be any finitely generated R-module and $x_1, \ldots, x_d$ be a system of parameters for R.

Assume that $q = (x_1, \ldots, x_d)$ is an ideal of definition in R. Then $e_0(q; M) = \lim\limits_{n\to\infty} \dfrac{l(M|q^n M)}{n^d/d!} = e_R(x_1, \ldots, x_d|M)$.

For the proof of this formula, we need the following lemma.

(1.18) Lemma . *Let M be any finitely generated R-module and $q = (x_1, \ldots, x_d$ be an ideal of definition generated by a system of parameters $\{x_1, \ldots, x_d\}$ for R. Then*

$$e_0(q; M) = e_{gr_q}(R)(\bar{x}_1, \ldots, \bar{x}_d|gr_q(M)),$$

35 *where* $\bar{x}_1, \ldots, \bar{x}_d$ *are images of* $x_1, \ldots, x_d$ *in* q/q^2.

Proof. We put $A := gr_q(R), N := gr_q(M)$. By induction on d we shall prove that $h_0(N) = e_A(\bar{x}_1, \ldots, \bar{x}_d|N)$. Suppose $d = 0$, then $q = 0$, $A = R$, $N = M$ and $h_0(N) = l_R(M) = \ell_A(N) = e_A(\cdot|N)$. Now suppose that $d = s + 1$, $s \geq 0$ and the result holds for $d = s$. Then we have by induction

$$e_A(\bar{x}_1, \ldots, \bar{x}_d|N) = e_{A/\bar{x}_1}(\bar{x}_2, \ldots, \bar{x}_d|N/\bar{x}_1 N) - e_{A/\bar{x}_1}(\bar{x}_2, \ldots, \bar{x}_d|(0 \underset{N}{:} \bar{x}_1))$$
$$= h_0(N/\bar{x}_1 N) - h_0((0 \underset{N}{:} \bar{x}_1)) = h_0(N),$$

see remark (ii) in (1.2) . Also, it follows from the same remark (iii) that $e_0(q; M) = h_0(gr_g(M))) = e_{gr_q(R)}(\bar{x}_1, \ldots, \bar{x}_d|gr_q(M))$. □

Proof of (1.17) First, we prove that

$$\lim_{n\to\infty} \frac{l(M|q^n M)}{n^d/d!} \leq e_R(x_1, \ldots, x_d|M).$$

If $d = 0$ then $q = 0$ and $\lim_{n\to\infty} \frac{\ell(M|q^n M)}{n^d/d!} = \ell(M) = e_R(x_1, \ldots, x_d|M)$. Now suppose that $d \geq 1$ and put $\bar{M} = M/x_1 M, \bar{R} = R/x_1, \bar{q} = q/x_1$. The we have $\bar{M}/\bar{q}^n \bar{M} = M/(x_1 M + q^n M)$

$$\ell_{\bar{R}}(\bar{M}/\bar{q}^n M) = \ell_{\bar{R}}(M/q^n M) - \ell(x_1 M + q^n M/q^n M)$$
$$= \ell(M/q^n M) - \ell(x_1 M/x_1 M \cap q^n M)$$

Now, it is easy to see that $x_1 M/x_1 M \cap q^n M = x_1 M/x_1(q^n M \underset{M}{:} x_1) \overset{x_1}{\underset{\approx}{\longleftarrow}} M/(q^n M \underset{M}{:} x_1)$ is an isomorphism. Therefore we get

$$\ell_{\bar{R}}(\bar{M}/\bar{q}^n \bar{M}) = l(M/q^n M) - l(M/(q^n M \underset{M}{:} x_1)) \geq l(M/q^n M) - l(M/q^{n-1} M)$$
$$= H^0_{gr_q(R)}(gr_q(M), n-1) \text{ for all n.}$$

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Thus
$$e_0(\bar{q}; \bar{M}) = \lim_{n\to\infty} \frac{\ell(\bar{M}/\bar{q}^n M)}{n^{d-1}/(d-1)!} \geq e_0(q; M)$$

If $d \geq 2$, replacing M by $\overline{M}$, R by $\overline{R}$, q by $\overline{q}$, we get

$$e_0(q; M) \leq e_0(q/x_1; M/x_1M) \leq e_0(q/(x_1, x_2), M/(x_1, x_2)M) \leq \cdots \leq e_0((0); M/qM) = \ell(M/qM).$$

i.e.,
$$\lim_{n\to\infty} \frac{\ell(M/q^n M)}{n^d/d!} \leq \ell(M/(x_1, \ldots, x_d)M).$$

Now, replacing $x_1, \ldots, x_d$ by $x_1^p, \ldots, x_d^p$, we get

$$\lim_{n\to\infty} \frac{\ell(M/q^{np}M)}{(np)^d/d!} \leq \lim_{n\to\infty} \frac{\ell(M/(x_1^p, \ldots, x_d^p)^n M)}{(np)^d/d!} \leq \frac{l(M/(x_1^p, \ldots, x_d^p)^n M)}{p^d} \text{ for all } p \geq 0.$$

Hence $e_0(q; M) \leq \lim\limits_{p\to\infty} \dfrac{\ell(M/(x_1^p, \ldots, x_d^p)M)}{p^d} = e_R(x_1, \ldots, x_d|M)$ by (1.16) . It remains to prove the reverse inequality. Let $n_1, \ldots, n_d$ be positive integers. Put $A = gr_q(R), N = gr_q(M)$ and $\overline{x}_1, \ldots, \overline{x}_d$ be the image of $x_1, \ldots, x_d$ in q/q^2. Set $F := (x_1^{n_1}, \ldots, x_d^{n_d})M, K := \underset{n\geq 0}{\otimes} q^n M \cap (q^{n+1}M + F)/q^{n+1}M$ and $L = (\overline{x}_1^{n_1}, \ldots, \overline{x}_d^{n_d})N$. Then it is clear that K, L are graded A-submodules of $N, L \subset K$ and 37

$$N/K = \underset{n\geq 0}{\oplus} q^n M/q^n M \cap (q^{n+1}M + F) = \underset{n\geq 0}{\oplus} q^n M + F/q^{n+1}M + F.$$

For $n \geq n_1 + \cdots + n_d$, we have $q^n M \subset F$. Therefore we get

$$\ell_R(M(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{n\geq 0} \ell_R(q^n M + F/q^{n+1}M + F) = \ell_R(N/K) \leq \ell_R(N/L) = \ell_R(N/(\overline{x}_1^{n_1}, \ldots, \overline{x}_d^{n_d})N).$$

If $\ell_R(N/(\overline{x}_1^{n_1}, \ldots, \overline{x}_d^{n_d})N) = \ell_A(N/\overline{x}_1^{n_1}, \ldots, \overline{x}_d^{n_d}).N)$, then we get ℓ_R $(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) \leq \ell_A(N/(\overline{x}_1^{n_1}, \ldots, \overline{x}_d^{n_d})N)$ for arbitrary positive integers $n_1, \ldots, n_d$. Therefore, by (1.16) , we get

$$e_R(x_1, \ldots, x_d|M) = \lim_{\min(n_i)\to\infty} \frac{l_R(M/(x_1^{n_1}, \ldots, x_d^{n_d})M)}{n_1 \cdot \cdots \cdot n_d}$$

$$\leq \lim_{\min(n_i)\to\infty} \frac{\ell_A(N/(\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})N)}{n_1\cdot\dots\cdot n_d}$$
$$= e_A(\overline{x}_1,\dots,\overline{x}_d|N) = e_0(q;M)$$

by1.18.

Thus it is enough to prove that

$$\ell_R(N/\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})N) = \ell_A(N/\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})M).$$

Put $I = (\overline{x}_1,\dots,\overline{x}_d)\cdot A$. Since every A-module is also R-module, it follows that $\ell_A(N/\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})N) \leq l_R(N/\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})N)$ and $l_A\ ((N/\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})N/I^nN) \leq l_R((\overline{x}_1^{n_1},\dots,\overline{x}_d^{n_d})N/I^nN)$, where $n \geq n_1 + \cdots n_d$.

38 Therefore it is enough to prove that

$$\ell_R(N/I^nN) = \ell_A(N/I^nN).$$

Since $\overline{x}_1,\dots,\overline{x}_d$ annihilates $I^iN/I^{i+1}N$, it follows that $l_R(I^iN/I^{i+1}N) = l_A(I^iN/I^{i+1}N)$ for all $i \geq 0$. Therefore we get $l_R(N/I^nN) = \sum_{i=0}^{n-1} l_R\ (I^iN/I^{i+1}N) = \sum_{i=0}^{n-1} \ell_A(I^iN/I^{i+1}N) = \ell_A(N/I^nN)$.

This completes the proof of (1.17) .

(1.19) Corollary . *Let M be any finitely generated R-module and-$(x_1,\dots,x_d) = q \subset R$ be an ideal of definition generated by a system of parameters for R. Then*

$$e_R(x_1,\dots,x_d|M) = 0 \iff K-\dim(M) < \dim(R) = d.$$

In particular, $e_R(x_1,\dots,x_d|R) > 0$.

(1.20) Corollary. *Let M be any finitely generated R-module and $q = (x_1,\dots,x_d), q' = (x'_1,\dots,x'_d)$ be ideals of definitions generated by systems of parameters for R. Then*

(i) *If $q' \subset q$, then $e_R(x_1,\dots,x_d|M) \leq e_R(x'_1,\dots,x'_d|M)$ and*

(ii) *If $q' = q$, then $e_R(x_1,\dots,x_d|M) = e_R(x'_1,\dots,x'_d|M)$.*

(1.21) Corollary. *Let M be any finitely generated R-module and $q = (x_1, \ldots, x_d)$ be an ideal of definition generated by a system of parameters for R. Then*

$$e_0(q; M) = l(M/qM) - \ell((q_{d-1}M \underset{M}{:} x_d)/q_{d-1}M) - \sum_{k=l}^{d-1} e_0(q/q_k, (q_{k-1}0 \underset{M}{:} x_k)/q_{k-l}M)$$

where $q_k = (x_1, \ldots, x_k), 0 \leq k \leq d-1.$ 39

Proof. This follows from remark (1.4)(i) and ((1.17)). □

(1.22) Corollary. *Let M be any finitely generated R-module and $q = (x_1, \ldots, x_d) \subset R$ be any ideal of definition generated by a system of parameters $\{x_1, \ldots, x_d\}$ for R. Then $e_0(q; M) = \ell(M/qM) - \ell((q_{d-1} \underset{M}{:} x_d)$ $q_{d-1}M)$ if and only if x_k is not in any prime ideal $\mathscr{Y}$ belonging to Ass $(M/q_{k-1}M)$ such that $K - \dim R/\mathscr{Y} \geq d-k$, where $q_k = (x_1, \ldots, x_k), 0 \leq k \leq d-1$.*

Proof. It is easy to see that

$$\text{Ass } ((q_{k-1} \underset{M}{:} x_k)\ /q_{k-1}M) = \text{ Ass } (M/q_{k-1}M) \cap V((x_k)).$$

Therefore we get

$$K - \dim(q_{k-1}M \underset{M}{:} x_k)/q_{k-1}M) = \sup_{\mathscr{Y}_i \in Ass(M/q_{k-1}M) \cap V((x_k))} K - \dim R/\mathscr{Y}_i \ < d-k$$

if and only if $x_k \notin \mathscr{Y}$ for all $\mathscr{Y} \in Ass(M/q_{k-1}M)$ with $K - \dim R/\mathscr{Y} \geq d-k$. □

Therefore by (1.19), we get $e_0(q/q_k; (q_{k-1}M : x_k)/q_{k-1}M) = 0$ if and only if $x_k \notin \mathscr{Y}$ for all $\mathscr{Y} \in Ass(M/q_{k-1}M)$ with $K - \dim R/\mathscr{Y} \geq d-k$. Now (1.22) follows from (1.21).

(1.23) Definition. (see [2]). Let M be a finitely generated R-module. A set of elements $x_1, \ldots, x_d \in \text{Rad}(R)$ is said to be a reducing system of parameters with respect to M if

(a) $\{x_1, \ldots, x_d\}$ is system of parameters for R

(b) $e_0(q; M) = \ell(M/qM) - \ell((q_{d-1}M :_M x_d)/q_{d-1}M)$ where $q = (x_1, \ldots, x_d)$ and $q_{d-1} = (x_1, \ldots, x_{d-1})$.

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The following propositions are useful for the computation of the multiplicity

(1.24) Proposition . *Let M be any finitely generated R-module and $(x_1, \ldots, x_d) = q \subset R$ be an ideal generated by a system of parameters $x_1, \ldots, x_d$ for R. Then q can be generated by a reducing system of parameters with respect to M.*

Proof. By (1.22), it is enough to prove that $x_k \notin \mathscr{Y}$ for all $\mathscr{Y} \in$ Ass $(M/q_{k-1}M)$ such that $K - \dim R/\mathscr{Y} \geq d - k$, where $q_k = (x_1, \ldots, x_k), 0 \leq k \leq d - 1$. □

Let i be an integer with $1 \leq i \leq d$. Suppose that there exist elements $y_1, \ldots, y_{i-l}$ such that $q = (y_1, \ldots, y_{i-l}, x_i, \ldots, x_d)$ and $y_j \notin \mathscr{Y}$ for all $\mathscr{Y} \in$ Ass $(M/(y_1, \ldots, y_{j-l})M)$ with $K - \dim R/\mathscr{Y} \geq d - j$, for any $j = 1, \ldots, i - 1$.

We set $q = (y_1, \ldots, y_{i-l}, x_{i+1}, \ldots, x_d)$. It is clear that $q \subset mq + q_i$, where m= rad (R). Hence there is an element $y_i \in q$ such that $y_i \notin mq + q_i$ and $y_i \notin \mathscr{Y}$ for any $\mathscr{Y} \in$ Ass $(M/(y_1, \ldots, y_{i-1})M)$ with $R/\mathscr{Y} \geq d - i$ Since $y_1, \ldots, y_{i-1}, x_{i+1}, \ldots, x_d$ are linearly independent mod m, q, Nakayama's lemma implies $q = (y_1, \ldots, y_i, x_{i+1}, \ldots, x_d)$.

(1.25) Proposition . *Let $(R, \mathscr{M})$ be a noetherian local ring and $q =$*
41 *$(x_1, \ldots, x_d) \subset R$ be an ideal generated by a system of parameters $(x_1, \ldots, x_d)$ for R. Then we put $\mathscr{O}_0 := (0)$ and $\mathscr{O}_k := U(\mathscr{O}_{k-1}) + (x_k)$ for any $0 < k < d$. Then $e_0(q; R) = l(R/\mathscr{O}_d)$.*

Proof. From the proof of ((1.9)), we have

$$e_0(q; R) = e_0(q/x_1; R/((x_1) + (0 : x_1^n)))$$

42 for large n. Proof by induction on d. Let $d = 1$. Then it is clear that

$(0 : x_1^n) = U(0)$ for large n and $e_0((x_1); R) = \ell(R/((x_1) + (0 : x_1^n))) = \ell(R/((x_1) + U(0))) = \ell(R/\mathscr{U})$.

Now suppose that $d = s + 1, s \geq 1$ and the result holds for $d = s$. First we shall show that $U((x_1) + (0 : x_1^n)) = U((x_1) + U(0))$ for large n. Let $\mathscr{Y} \in V((x_1))$ be such that $K - \dim R/\mathscr{Y} = d - 1$. Then it is easy to see that, for large n

$$\mathscr{Y} \in \text{ Ass } (R/((x_1) + (0 : x_1^n))) \Longleftrightarrow \mathscr{Y} \in \text{ Ass } (R/((x_1) + U(0)))$$

Moreover, $((x_1) + (0 : x_1^n))_{\mathscr{Y}} = ((x_1) + U(0))_{\mathscr{Y}}$ for any $\mathscr{Y} \in$ Ass $(R/((x_1) + (0 : x_1^n))) =$ Ass $(R/((x_1) + U(0)))$ with $K - \dim R/\mathscr{Y} = d - 1$. □

Therefore $U((x_1) + (0 : x_1^n)) = U((x_1) + U(0))$ for larger n

Put $R' := R/(x_1) + (0 : x_1^n)$ for large $n, \mathscr{O}1 = (0)$ and $\mathscr{O}_k = (x_k) + U(\mathscr{O}_{k-1}$ for any $1 < k \leq d$. Then by induction we get $e_0(q; R) = e_0(q'; R') = \ell(R'/\mathcal{U})$. Now, since $\mathscr{O} = U(0) + (x_1)$, it follows that

$U(\mathscr{O}) = U(U(0) + U((x_1) + (0 : x_1^n)$ and $\mathscr{O}'_d = \mathscr{O}_d/((x_1) + (0 : x_1^n))$ for large n. Therefore $e_0(q; R) = e_0(q'; R') = \ell(R'/\mathscr{O}'d) = \ell(R/_{\mathscr{O}d})$.

(1.26) Example. Take the classical example from [[90], §11] (see also [[26], p. 180] and [[50], p. 126]).

Let V_1, V_2 and C be the subvarieties of $\mathbb{P}_k^4$ with defining prime ideals:

$$\mathscr{Y}_{v_1} = (X_1X_4 - X_2X_3, X_1^2X_3 - X_2^3, X_1X_3^2 - X_2^2X_4, X_2X_4^2 - X_3^3),$$
$$\mathscr{Y}_{v_1} = (X_1, X_4) \text{ and } \mathscr{Y}_{\mathscr{C}} = (X_1, X_2, X_3, X_4).$$

We put $A(V_1; C) := A := (K[(X_0, X_1, X_2, X_3, X_4]/\mathscr{Y}_{v_1})\mathscr{Y}_c$.

Then $\mathscr{Y}_{v_2}.A = (X_1, X_4)A$ is generated by a system of parameters X_1, X_4 for A and

$$\mathscr{Y}_{v_2} + (X_1) = (X_1, X_2X_3, X_2^2, X_2X_4^2 - X_3^3) \cap (X_1, X_2^3, X_3, X_4)$$

is a primary decomposition of $\mathscr{Y}_{v_1} + (X_1)$ in A. Therefore $U(\mathscr{Y}_{v_2} + (X_1)) = (X_1, X_2X_3, X_2^2, X_2X_4^2 - X_3^3)$. It follows from (1.25) that

$$e_0(\mathscr{Y}_{v_2}A; A) = \ell(A/(x_4) + U(\mathscr{Y}_{v_1} + (X_1)))$$

$$= \ell(A/(X_1, X_4, X_2^2, X_2X_3, X_3^2)A) = 4.$$

Also, $\ell(A/\mathscr{Y}_{v_2}) = \ell(A(X_1, X_4, X_2X_3, X_2^3, X_3^3)A) = 5$. Therefore in this example the inequality in (1.9) is a strict inequality, i.e., $e_0(A/\mathscr{Y}_{v_2}A; A) < l(A/(\mathscr{Y}_{v_2})$.

C. The HILBERT Function and the Degree

43 **(1.27) Notation.** The following notation will be used in sequel.

Let K be a field and $R := K[X_O, \ldots, X_n]$ be the polynomial ring in $(n+1)$-variables over K. Let $V(n+1, t)$ denote the K-vector space consisting all forms of degree t in $X_O, \ldots, X_n$. It is easy to see that $\dim_K V(n+1, t) = \binom{t+n}{n}$, for all $t \geq 0, n \geq 0$.

Let $I \subset R$ be a homogeneous ideal. Let $V(I, t)$ be the K-vector space consisting of all forms in $V(n+1, t)$ which are contained in I.

(1.28) Definition. The numerical function $H(I, -) : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $H(I, t) = \dim_K V(n+1, t) - \dim_K V(I, t)$ is called the *Hilbert function of* I.

General properties of the HILBERT function

Let $I, J \subset R$ be two homogeneous ideals.

(1.29)

(i) If I $\subset J$ then $H(I, t) \geq H(J, t)$ for all $t \geq 0$.

(ii) $H(I + J, t) = H(I, t) + H(J, t) - H(I \cap J, t)$ for all $t \geq 0$.

Proof. Since $V(I, t) \leq V(J, t)$ if $I \subset J$ and $V(I + J, t) = V(I, t) + V(J, t) - V(I \cap J, t)$ for all $t \geq 0$, (i) and (ii) are clear. □

Let $\varphi \varepsilon R$ be a form of degree r. Then

(1.30)

(i)
$$H(\varphi R, t) = H((0), t) - H((0), t - r)$$
$$= \binom{t+n}{n} - \binom{t-r+n}{n} \text{ for } t \geq r - n$$
$$= \binom{t+n}{n} \text{ for } 0 \leq t \leq r - n$$

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(ii)
$$H(I \cap \varphi R, t) = H(\varphi R, t) + H(I : \varphi), t - r).$$
$$= H(\varphi R, t) + H(I, t - r) \text{ if } (I : \varphi) = I.$$

(iii)
$$H(I + \varphi R, t) = H(I, t) - H(I : \varphi), t - r).$$
$$= H(I, t) - H(I : t - r) \text{ if } (I : \varphi) = I.$$

In particular, $H((0), t) = \binom{t+n}{n}$
$H((1), t) = 0$ for all $t \geq 0$.

Proof. It is easy to see that $I \cap \varphi R = (I : \varphi) \cdot \varphi R$. Therefore we get

$$\dim_K V(I \cap \varphi R, t) = \dim_K V(I : \varphi, t - r)$$

In particular (take $I = R$), $\dim_K V(I\varphi R, t) = \dim_K(R, t - r)$. From this and (1.29) all (i),(ii), (iii) are clear. □

(1.31)

Let $\mathscr{Y} \subset R$ be a homogeneous prime ideal with $K - \dim R/\mathscr{Y} = 1$ If K is algebraically closed, then $\mathscr{Y}$ is generated by n linear forms and

(i) $H(\mathscr{Y}, t) = 1$ *for all* $t \geq 0$

(ii) *For any* $r > 0, H(\mathscr{Y}^r, t) = 1 + \binom{n}{l} + \binom{n+1}{2} + \cdots + \binom{n+r-2}{r-1} = \binom{n+r-2}{r-1}$

Proof. We may assume that $X_0 \notin \mathscr{Y}$. Consider the ideal $\mathscr{Y}_* = \{f_* | f(1, X_1/X_0, \ldots, X_n/X_0), f \in \mathscr{Y}\} \subset K[X_1/X_0, \ldots, X_n/X_0]$. It is easy to see that this is a maximal ideal in $K[X_1/X_0, \ldots, X_n/X_0]$. Therefore, by Hilbert's Nullstellensatz, there exist $a_1, \ldots, a_n \in K$ such that $\mathscr{Y}_* = (X_1/X_0 - a_1, \ldots, X_n/X_0 - a_n)$. Now it is easy to see that $\mathscr{Y} = (X_1 - a_1X_0, \ldots, X_n - a_nX_0)$. To calculate $H(\mathscr{Y}, t)$ and $H(\mathscr{Y}^r, t)$, we may assume that $\mathscr{Y} = (X_1, \ldots, X_n)$ Then it is clear that $H(\mathscr{Y}, t) = 1$ for all $t \geq 0$ and since $\mathscr{Y}^r$ is generated by forms of degree r in $X_1, \ldots, X_n$ it follows that □ 45

$$H(\mathscr{Y}^r, t) = \sum_{k=0}^{r-1} \text{(forms of degree } k \text{ in } x_1, \ldots, x_n)$$
$$= \sum_{k=0}^{r-1} \binom{n+k-1}{k} = \binom{n+r-1}{r-1}$$

The following is a well-known theorem (for proof see [26], [55] or [72]).

(1.32) HILBERT-SAMUEL Theorem

Let $I \subset R$ be a homogeneous ideal. The Hilbert function $H(I, t)$, for large t is a polynomial $P(I, t)$ in t with coefficients in Q. The degree $d(0 \leq d \leq n)$ of this polynomial $P(I, t)$ is called the projective dimension or dimension of I and we will denote it by $\dim(I)$. It is well-known that $\dim(I) = K - \dim(I) - 1$. We will write the polynomial $P(I, t)$ in the following form:

$$P(I, t) = h_0(I)\tbinom{t}{d} + h_1\tbinom{t}{d-1} + \cdots + h_d,$$

46 where $h_0(I) > 0, h_1, \ldots, h_d$ are integers.

(1.33) Definition. (a) Let $I \subset R$ be a homogeneous ideal. The positive integer $h_0(I)$ is called *the degree* of I.

(b) Let $V = V(I) \subset \mathbb{P}^n_K$ be a projective variety in $\mathbb{P}^n_K$ defined by a homogeneous ideal $I \subset R$. Then $K - \dim(I)(resp.\ \dim(I)$, degree of I) is called the *Krull-dimension of V (resp. The dimension of V, the degree of V*) and we denote it by $K - \dim(V)(resp.\ \dim(V), \deg(V))$. V is called *pure dimensional or unmixed* if I is unmixed.

(1.34) Remark. In general, the degree of V is to be the number of points in which almost all linear subspaces $L^{n-\dim(V)} \subset \mathbb{P}^n_K$ meet V. By combining this geometric definition with a variant of the Hilbert polynomial, we can give our purely algebraic definition of $\deg(V)$ and open the way to the deeper study of this properties (see [50, Theorem (6.25) on p. 112]).

Some properties of the degree.

(1.35)

Let $\varphi_1, \ldots, \varphi_s \in R$ be forms of degrees $r_1, \ldots, r_s$, respectively. If $((\varphi_1, \ldots, \varphi_{i-1}) : \varphi_i) = (\varphi_1, \ldots, \varphi_{i-1})$ for any $1 \leq i \leq s$ then

$$h_0((\varphi_1, \ldots, \varphi_s)) = r_1 \ldots ..r_s.$$

Proof. Proof by induction on s. Suppose $s = 1$. Then by (1.30) (i) we have $H((\varphi_1), t) = \binom{t+n}{n} - \binom{n+r-n}{n^1} = r_1\binom{t}{n-1}) + \cdots$ for all $t \geq r_1 - n$. □

Therefore $h_0((\varphi_1)) = r_1$. Now suppose $s = p + 1, p \geq 1$ and result holds for $r = p$. Since $((\varphi_1, \ldots, \varphi_{s-1}) : \varphi_s) = (\varphi_1, \ldots, \varphi_{s-1})$ by (1.30) (iii) we have 47

$$\begin{aligned} H((\varphi_1, \ldots, \varphi_s), t) &= H((\varphi_1, \ldots, \varphi_{s-1}), t) - H((\varphi_1, \ldots, \varphi_{s-1}), t - r_s) \\ &= h_0((\varphi_1, \ldots, \varphi_{s-1}))\binom{t}{n-s+1} + \cdots - h_0((\varphi_1, \ldots, \varphi_{s-1}))\binom{t-r_s}{n-s+1} \cdots \\ &= r_s - h_0((\varphi_1, \ldots, \varphi_{s-1}))(\underset{n-s}{t}) + \cdots \text{ for all } t \geq r_s - n. \end{aligned}$$

Therefore, by induction, we get $h_0((\varphi_1, \ldots, \varphi_s)) = r_s \cdot h_0(\varphi_1, \ldots, \varphi_{s-1}) = r_1, \ldots r_s$.

(1.36)

Let $I \subset R$ be a homogeneous ideal and $\varphi \in R$ be a form of degree r. Then

(i) If $\dim(I, \varphi) = \dim(I) = \dim(I : \varphi)$ then $h_0(I, \varphi) = h_0(I) - h_0(I : \varphi))$.

(ii) If $\dim(I, \varphi) = \dim(I) > \dim(I : \varphi)$ then $h_0(I, \varphi) = h_0(I)$.

(iii) If $(I : \varphi) = I$, then $h_0(I, \varphi) = r.h_0(I)$.

Proof. This follows from (1.30) □

(1.37)

Let $I \subset R$ be a homogeneous ideal. Then

$$h_0(I) = h_0(U(I))$$

Proof. Suppose $\dim(I) = d$. We may assume that $I \not\subseteq U(I)$. Then we have $I = U(I) \cap J$ where $J \subset R$ is a homogeneous ideal with $\dim(J) < \dim(U) = \dim(I) = d$. Therefore from (1.29) (ii), we get $h_0(I) = h_0(U(I))$. ◻

(1.38)

48 Let $\mathscr{Y} \subset R$ be a homogeneous prime ideal and $q \subset R$ be a homogeneous $\mathscr{Y}$-primary ideal. Then

$$h_0(q) = l(q).h_0(\mathscr{Y}).$$

Proof. Let $q = q_1 \subset q_2 \subset \cdots \subset q_\ell = \mathscr{Y}$ be a composition series for q. It is enough to prove that

$$h_0(q_i) = h_0(q_{i+1}) + h_0(\mathscr{Y}) \text{ for any } 1 \leq i \leq \ell - 2.$$

◻

We assume $i = 1$. There exist forms $\varphi_1, \ldots, \varphi_s$ such that $q_2 = (q_1, \varphi_1, \ldots, \varphi_s)$. By using remark in (1.1) to the $\mathscr{Y}R_{\mathscr{Y}}$- primary ideal $qR_{\mathscr{Y}} \subset R_{\mathscr{Y}}$, it follows the $\mathscr{Y}\varphi_i \subset q_1$ for all i= $1, \ldots, s$ and there exist forms α_i and $\beta_i, 2 \leq i \leq r$ such that

(i) $\beta_i \notin \mathscr{Y}$ for all $2 \leq i \leq s$.

(ii) $\alpha_i\varphi_i - \beta_i\varphi_1 \in q_1$ for all $2 \leq i \leq s$

Therefore $(q_1 : \varphi_1) = \mathscr{Y}$ and since $\mathscr{Y} \underset{+}{\subset} ((q_1\varphi_1, \ldots, \varphi_i) : \varphi_{i+1})$ the homogeneous ideals $((q_1\varphi_1, \ldots, \varphi_i) : \varphi_{i+1})$ have dimension $< d$, for any $1 \leq i \leq s - 1$. Therefore from (1.36) (i), (1.36) (ii), we get $h_0(q_2) = h_0((q_1\varphi_1, \ldots, \varphi_{s-1}) = h_0((q_1\varphi_1, \ldots, \varphi_{s-2}) = \cdots = h_0(q_1) - h_0(\mathscr{Y})$.

(1.39)

Let $\mathscr{Y}_1 \neq \mathscr{Y}_2$ be two homogeneous ideals in R and let q_i be two homogeneous $\mathscr{Y}_i$- primary ideals for $i = 1, 2$ If $\dim q_1 = \dim q_2$ then $h_0(q_1 \cap q_2) = h_0(q_1) + h_0(q_2)$.

Proof. Since $\mathscr{Y}_1 \neq \mathscr{Y}_2$, it follows that $\dim(q_1 + q_2) < \dim q_1 = \dim q_2$. Therefore, from (1.29) (ii), we have

$$h_0(q_1 \cap q_2) = h_0(q_1) + h_0(q_2).$$

□

(1.40)

Let $I \subset R$ be a homogeneous ideal. Then 49

$$h_0(I) = h_0(U(I)) = \sum l(q).h_0(\mathscr{Y}).$$

where q runs through all $\mathscr{Y}$-primary components of I with $\dim(q) = \dim(I)$.

Proof. This follows from (1.37) . (1.39) and (1.38) . □

(1.41)

Let $\overline{I} \subset K[X_0, \ldots, X_{n-1}]$ be a homogeneous ideal of dimension d with the Hilbert function $H(\overline{I}, t) = h_0\binom{t}{d} + h_1\binom{t}{d-1} + \cdots h_d$ for $t >> 1$. Let $I^* \subset K[X_0, \ldots, X_n]$ be the homogeneous ideal generated by $\overline{I}$ Then $\dim(I^*) = \dim(I) + 1 = d + 1$ and the Hilbert function of I^* is given by

$H(I^*, t) = h_0\binom{t}{d+1} + (h_0 + h_1)\binom{t}{d} + \cdots + (h_d + h_d + 1)$ for $t >> 1$.

Proof. Every form $\varphi \in I^*$ of degree t can be written uniquely in the form

$$\varphi = \varphi_t + \varphi_{t-1} X_n + \cdots + X_n^t$$

where $\varphi_t, \varphi_{t-1}, \ldots$ are forms of degrees $t, t-1, \ldots$ in $\overline{I}$.

Therefore $V(I^*, t) = \sum_{k=0}^{t} V(\overline{I}, k)$ and hence □

$$
\begin{aligned}
H(I^*, t) &= \binom{t+n}{n} - \sum_{k=0}^{t} [\binom{t+n}{n-1}) - H(\bar{I}, k)]. \\
&= \sum_{k=0}^{t} H(\bar{I}, k), \text{ since } \sum_{k=0}^{t} \binom{t+n}{n-1}) = \binom{t+n}{n} \\
&= h_0 \sum_{k=0}^{t} \binom{k}{d} + h_1 \sum_{k=0}^{t} \binom{k}{d-1} + \cdots + h_d \sum_{k=0}^{t} \binom{k}{0} \\
&= h_0 \binom{t+l}{d+l} + h_1 \binom{t+1}{d} + \cdots + h_d \binom{t+1}{l} \\
&= h_0 \binom{t}{d+1} + (h_0 + h_1) \binom{t}{d} + \cdots + (h_{d-1} + h_d) \binom{t}{l} + (h_d + h_{d+1})
\end{aligned}
$$

(1.42)

50 Let $I \subset K[X_0, \ldots, X_n]$ be a homogeneous ideal of dimension $d(0 \leq d \leq n-1)$. Put $\bar{I} = I \cap K[X_0, \ldots, X_{n-1}]$

$I_1 = \{\varphi \in K[X_0, \ldots, X_{n-1}] | \varphi_i$ is a form such that $\varphi_0 + \varphi_1 X_n \in I$ for some form $\varphi \in K[X_0, \ldots, X_{n-1}]\}$

$I_1 = \{\varphi_i \in K[X_0, \ldots, X_{n-1}] | \varphi_i$ is a form and $\varphi_0 + \varphi_1 X_n + \cdots + \varphi_i X_n^i \in I$ for some forms $\varphi_0, \ldots, \varphi_{i-1} \in K[X_0, \ldots, X_{n-1}]\}$ for $i \geq 1$.

Then it is clear that

$$\bar{I} \subset I_1 \subset I_2 \subset \cdots \subset I_r = I_{r+1} = \ldots \text{ for some } r \geq 1.$$

Therefore, we get

$$\dim V(I, t) = \dim V(I, t) + \sum_{k=1}^{t} \dim V(I_k, t-k) \text{ for all } t \geq 0$$

and hence

$$H(I, T) = \binom{t+n}{n} - \binom{t+n-1}{n-1} + H(\bar{I}, t) - \sum_{k=1}^{t} [\binom{t+n-1}{n-1}) - H(I_k, t-k)]$$

$$= H(\bar{I}, t) - \sum_{k=1}^{t} H(I_k, t-k) \text{ for all } t \geq 0.$$

51

(1.43) Example. (i) Let $\mathscr{Y}$ be the prime ideal

$(X_0X_2 - X_1^2, X_1X_2 - X_0X_3, X_2^2 - X_1X_3) \subset K[X_0, X_1, X_2, X_3]$.

Following the notation of (1.42) , it is easy to see that

$$\bar{\mathscr{Y}} = (X_0X_2 - X_1^2)$$
$$\mathscr{Y}_1 = \mathscr{Y}_2 = \ldots (X_0, X_1).$$

Therefore, by (1.35) , we get

$H(\bar{\mathscr{Y}}, t) = 2t + 1$ and $H(\mathscr{Y}_1, t) = H(\mathscr{Y}_2, t) = \cdots = 1$ for all $t \geq 0$.

Hence by (1.42)

$$H(\mathscr{Y}, t) = H(\bar{\mathscr{Y}}, t) + \sum_{k=0}^{t} H(\mathscr{Y}_k, t-k) = 3t + 1.$$

Therefore $h_0(\mathscr{Y}) = 3$

(ii) Let $\mathscr{Y}$ be the prime ideal $(X_0X_2 - X_1^2, X_2^2 - X_0X_3) \subset K[X_0, X_1, X_2, X_3]$. Then $\bar{\mathscr{Y}} = (X_0X_2 - X_1^2), H(\bar{\mathscr{Y}}, t) = 2t + 1$, for all $t \geq 0$. $\mathscr{Y}_1 = \mathscr{Y}_2 = \ldots = (X_0, X_1^2)$,

$$H(\mathscr{Y}_1, t) = H(\mathscr{Y}_2, t) = \begin{cases} 1 \text{ for } t = 0 \\ 2 \text{ for all } t \geq 1 \end{cases}$$

Therefore, by (1.42) , we get

$$H(\mathscr{Y}, t) = H(\bar{\mathscr{Y}}, t) + \sum_{k=0}^{t} H(\mathscr{Y}_k, t-k) = \begin{cases} 1 \text{ for } t = 0 \\ 4t \text{ for all } t \geq 1. \end{cases}$$

Hence $h_0(\mathscr{Y}) = 4$. 52

(iii) Let $\mathscr{Y}$ be the prime ideal $(X_0^2X_2 - X_1^3, X_0X_3 - X_1X_2, X_0X_2^2X_1^2X_3, X_1X_2^3 - X_2^3) \subset K[X_0, X_1, X_2, X_3]$

Then

$$\overline{\mathscr{Y}} = (X_0^2X_2 - X_1^3), H(\overline{\mathscr{Y}}) = \begin{cases} 1 \text{ for } t = 0 \\ 3t \text{ for all } t \geq 1. \end{cases}$$

$$\mathscr{Y}_1 = (X_0X_1^2), H(\mathscr{Y}_1, t) = \begin{cases} 1 \text{ for } t = 0 \\ 2 \text{ for all } t \geq 1. \end{cases}$$

$\mathscr{Y}_2 = \mathscr{Y}_3 = \cdots = (X_0, X_1), H(\mathscr{Y}_2, t) = H(\mathscr{Y}_3, t) = \ldots = 1$ for all $t \geq 0$. Therefore, by (1.42) , we get

$$H(\mathscr{Y}, t) = H(\overline{\mathscr{Y}}, t) + \sum_{k=0}^{t} H(\mathscr{Y}_k, t-k) = \begin{cases} 1 & \text{for } t = 0 \\ 4 & \text{for } t \geq 1. \\ 4t + t & \text{for } t =\geq 2. \end{cases}$$

Hence $H_0(\mathscr{Y}) = 4$.

(iv) Let $\mathscr{Y} \subset K[X_0, X_1, X_2, X_3] = R$ be the prime ideal in example (iii) above and $\mathscr{Y} = (X_1, X_4) \subset K[X_0, X_1, X_2, X_3]$.

Then $q := (\mathscr{Y} + \mathscr{Y}') = (X_0, X_3, X_1, X_2, X_1^3, X_2^3)$ is (X_0, X_1, X_2, X_3)-primary ideal and it is easy to see that $l(R/q) = 5$. Therefore, by (1.38) , we have $h_0(q) = \ell(R/q) \cdot h_0((X_0, X_1, X_2, X_3)) = 5$ and from example (iii) $h_0(\mathscr{Y}) = 4$. This shows that

$$5 = h_0(q) \neq h_0(\mathscr{Y}) \cdot h_0(\mathscr{Y}') = 4.$$

D. Miscellaneous Results

53 Now we collect some results which will be used in the next sections. Let K be a field.

(1.44) Proposition . *Let A be a finitely generated K-algebra. Then $U = \{\mathscr{Y} \in \operatorname{Spec}(A) | A_{\mathscr{Y}}$ is Cohen-Macaulay $\}$ is a non-empty Zariski-open subset of* $\operatorname{Spec}(A)$.

For the proof of this proposition, we need the following lemma.

(1.45) Lemma . *Let A be a finitely generated K-alegbra and $\mathscr{Y} \in$ Spec(A) If $A_{\mathscr{Y}}$ is Chone-Macaulay, then there exists a maximal ideal m of A containing $\mathscr{Y}$ such that $A_{\mathscr{M}}$ is Chone-Macaulay.*

Proof. Proof by induction on $d := \dim(A)$. □

Case(i): ht $\mathscr{Y} = 0$. In this case, $\mathscr{Y}$ is a minimal prime ideal of A. If $d = K - \dim A = o$, then there is nothing to prove. Now suppose that $d = s + 1, s \geq 0$ and the result holds for $d = s$. Replacing A by A_f for some $f \notin \mathscr{Y}$, we may assume that Ass $(A) = \{\mathscr{Y}\}$ and $K - \dim A > O$. Then depth $(A) > 0$. Let $x \in A$ be a non-zero- divisor in A and q be a minimal prime ideal of xA. Then by Krull's *PID* ht $q = 1$ and hence $\mathscr{Y} \subset q$.

Put $A' = A/(\text{x})$ and $q' = qA$. Then A'_p is Cohen Macaulay and $htq' = 0$. Therefore,by induction, there exists a maximal ideal $\mathscr{M}'$ of A' with $q' \subset \mathscr{M}'$ and $A'm$ is Cohen-Macaulay. Then $\mathscr{M} = \mathscr{M}' \cap A$ is a **54**
maximal ideal of A containing $q \supset \mathscr{Y}$ and A_m is Cohen-Macaulay.

Case (ii): ht $\mathscr{Y} = r > 0$.

Since $A_{\mathscr{Y}}$ is Cohen-Macaulay of dimension r there exist $x_1, \ldots x_r$ in $\mathscr{Y}$ such that $\{x_1, \ldots x_r\}$is an $A_{\mathscr{Y}}$- sequence.

By replacing A by A_f for some $f \notin \mathscr{Y}$, we may assume that $\{x_1, \ldots x_r\}$ is an A-sequence and $\mathscr{Y}$ is a minimal prime ideal of $(x_1, \ldots x_r)$. Put $A' = A/(x_1, \ldots x_r)$ and $\mathscr{Y}' = \mathscr{Y}A'$. Then ht $\mathscr{Y}' = 0$ and $A'_{\mathscr{Y}'}$ is Cohen-Macaulay; therefore, by case(i), there exists a maximal ideal m' of A' such that $m' \supset \mathscr{Y}'$ and $A_{m'}$ is Cohen-Macaulay. Then $m = m' \cap A$ is a maximal ideal of A with $m \supset \mathscr{Y}$ and since $\{x_1, \ldots x_r\}$ is an A-sequence, it follows that $A_{\mathscr{M}}$ is Cohen-Macaulay.

Proof of Proposition (1.44). Clearly $U \neq \phi$. Let $\mathscr{Y} \in U$ shall show that there exists $f \notin \mathscr{Y}$such that $D(f) = \{q \in$ Spec $(A) | f \notin q\} \subset U$, that is, A_f is Cohen-Macaulay for some $f \notin \mathscr{Y}$. By (1.45), we may assume that $\mathscr{Y} = \mathscr{M}$ is a maximal ideal of A. Replacing A by A_f for some $f \notin \mathscr{M}$ we may assume that Ass $(A) = \mathscr{Y}_1, \ldots, \mathscr{Y}_r$ with $\mathscr{Y}_i \subset m, i \leq i \leq r$. Since A_m is Cohen- Macaulay, we have d :=ht

$m = \dim A_m = \dim(A/\mathscr{Y}_i)_m$ for all $1 \leq i \leq r$. Therefore

$$\dim A = \sup_{1 \leq i \leq r} \dim A/\mathscr{Y}_i = \sup_{1 \leq i \leq r} \dim(A/\mathscr{Y}_i)_{\mathscr{M}} = d$$

55 and there exist $x_1, \ldots, x_d \in m$ such that $\{x_1, \ldots, x_d\}$ is an A_m-sequence. Further, replacing A by A_f for some $f \notin m$, we may assume that $\{x_1, \ldots, x_d\}$ is an A-sequence. This shows that A is Cohen-Macaulay.

(1.46) Proposition.

1. *Let $L|K$ be a field extension. Let $I \subset K_0[X_0, \ldots, X_n] =: R$ be a homogeneous ideal. Put $\bar{R} = L[X_0, \ldots, X_n]$. Then $h_0(I) = h_0(I\bar{R})$.*
2. *Let A be a finitely generated K-algebra and $I \subset A$ be an unmixed ideal. Let $x \in A$ be such that $K - \dim(A/(I, x)) = K - \dim(A/I) - 1$. Then*
$$\mathrm{Rad}(U((I, x))) = \mathrm{Rad}(I, x).$$
3. *Let $V = V(I) \subset \mathbb{P}^n_K$ be a projective variety defined by the homogeneous ideal $I \subset K[X_0, \ldots, X_n] =: R$. Let C be an irreducible subvariety of V with the defining prime ideal $\mathscr{Y}$. Let $A = (R/I)_{\mathscr{Y}}$ be the local ring of V at C. If V is pure dimensional, then*
$$K - \dim(A) = K - \dim(V) - K - \dim(C).$$

Proof. 1. Clear.

2. Put $d := K - \dim(A/I)$. It is enough to prove that, for every minimal prime ideal q of (I, x)
$$K - \dim(A/q) = d - 1.$$

56 Since I is unmixed $d = K - \dim(A/I) = K - \dim\,(A/\mathscr{Y})$ for every $\mathscr{Y} \in$ Ass (A/I). Let $(I, x) \subset q \subset A$ be a minimal prime ideal of (I, x). Then there exists a minimal prime ideal $\mathscr{Y}$ of I such that $\mathscr{Y} \subsetneq q$ and by Krull's Principal Ideal Theorem, we have ht $q/\mathscr{Y} = 1$. Therefore; since A is a finitely generated K-algebra, we get
$$K - \dim A/q = K - \dim A/\mathscr{Y} - htq/\mathscr{Y} = d - 1.$$

3. Let $I \subset q \subset R$ be a minimal prime ideal of I such that $K-\dim (R/q)_{\mathscr{Y}} = K-\dim (A)$. Then, since I is unmixed and R/q is a finitely generated K-algebra, we get

$$K-\dim(V) = K-\dim R/I = K-\dim R/q = K-\dim(R/\mathscr{Y}) \\ + K-\dim(R/q)\mathscr{Y} = K-\dim(C) + K-\dim(A).$$

□

(1.47) Proposition. *Assume that K is algebraically closed. Let $L|K$ and $L'|K$ be field extensions and A, B be finitely generated K-algebras. Then*

(i) (a) $L \underset{K}{\otimes} L'$ is an integral domain.

(b) $K-\dim (A \underset{K}{\otimes} B) = K\text{-dim } (A) + K\text{-dim } (B)$ and if A and B are integral domain then $A \underset{K}{\otimes} B$ is an integral domain.

(c) Put $A_L := L \underset{K}{\otimes} A$. Then K-dim A_L =K-dim A and if A is an integral domain then A_L is an integral domain.

(ii) There is a one-one correspondence between the isolated prime ideals of A and the isolated prime ideals of $A_L = L \underset{K}{\otimes} A$ which preserves 57
K-dimensions.

(iii) (a) If A is unmixed then $A_L = L \underset{K}{\otimes} A$ is unmixed.

(b) If A and B are unmixed then $A \underset{K}{\otimes} B$ is unmixed.

(iv) (a) If A and B are Cohen-Macaulay then $A \underset{K}{\otimes} B$ is Cohen-Macaulay.

(b) Let $\mathscr{Y} \in \operatorname{Spec}(A)$ and $q \in \operatorname{Spec}(B)$. If $A_{\mathscr{Y}}$ and B_q are Cohen-Macaulay then $A_{\mathscr{Y}} \underset{K}{\otimes} B_q$ is Cohen-Macaulay.

Proof. (i) (a) We may assume that L is finitely generated over K. Let $\{x_1, \ldots, x_n\} \subset L$ be a separating transcendence basis of $L|K$ (since K is algebraically closed it exists). Put $L_1 := K(x_1, \ldots, x_n)$. Then $L|L_1$ is separable and hence $L = L_1(\alpha) = L_1[x]/(f(x))$, where

$f(x)$ is the irreducible polynomial of over L_1. Sine $L_1 \underset{K}{\otimes} L' \simeq S^{-1}(L'[x_1,\ldots,x_n])$, where $S = K[x_1,\ldots,x_n]$ - $0, L_1 \underset{K}{\otimes} L'$ is an integral domain with quotient field $E = L'(x_1,\ldots,x_n)$. Now, note that since K is algebraically closed in L', it is easy to see that $L_1 = E(x_1,\ldots,x_n)$ is algebraically closed in E. Then $L \underset{K}{\otimes} L' = L \underset{L_1}{\otimes} L_1 L_1 \underset{L_1}{\otimes} L' \supset L \underset{L_1}{\otimes} E = L_1[x]/(f(x)) \underset{L_1}{\otimes} E = E[x]/(f(x))$ is an integral domain.

58 (b) By Normalization Lemma, we have

$$K - \dim(A \underset{K}{\otimes} B) = K - \dim(A) + K - \dim(B).$$

Let L (resp. L') be the quotient field of A (resp. B).

Then $A \underset{K}{\otimes} B \subset L \underset{K}{\otimes} L'$ which is an integral domain by (a).

(c) Similar to (b).

(ii) Let $\mathscr{Y} \in \operatorname{Spec}(A)$. Then by ($i$) ($c$) $\mathscr{Y} A_L \in \operatorname{Spec}(A_L)$ and $K - \dim(\mathscr{Y}) = K - \dim(\mathscr{Y} A_L)$. It is easy to see that $\mathscr{Y}$ is isolated if and only if $\mathscr{Y} A_L$ is isolated. Therefore $\mathscr{Y} \leftrightarrow \mathscr{Y} A_L$ is, as required $a 1-1$ correspondence.

(iii) (a) Let $\mathscr{Y} \in \operatorname{Ass}(A)$. Then by $(i)(c) K - \dim \mathscr{Y} A_L = K - \dim \mathscr{Y} = K - \dim A = K - \dim A_L$. Therefore it is enough to prove that Ass $(A_L) = \{\mathscr{Y} A_L | \mathscr{Y} \in \operatorname{Ass}(A)\}$, which follows from ($ii$).

(b) Let $\mathscr{Y} \in \operatorname{Ass}(A)$ and $q \in \operatorname{Ass}(B)$. Then by ($ii$) ($b$) $(\mathscr{Y}, q)$ is a prime ideal in $A \underset{K}{\otimes} B =: C$ and $K - \dim(\mathscr{Y}, q)C = K - \dim \mathscr{Y} + K - \dim q = K - \dim A + K - \dim B = K - \dim(A \underset{K}{\otimes} B)$.

Therefore it is enough to prove that

$$\operatorname{Ass}(C) = \{(\mathscr{Y}, q).C | \mathscr{Y} \in \operatorname{Ass}(A), q \in (B)\}.$$

Let $P \in \operatorname{Ass}(C)$. Then since C is flat over A and B it follows that $P \cap A = \mathscr{Y} \in \operatorname{Ass}(A)$ and $P \cap B = q \in \operatorname{Ass}(B)$.

59 By replacing A by $A_{\mathscr{Y}}$ we may assume that A is local with maximal ideal $\mathscr{Y} \in \text{Ass}\,(A)$. Since A is unmixed $A_{\mathscr{Y}}$ is unmixed and therefore $A_{\mathscr{Y}}$ is artinian.

Now there exists a coefficient field L of A containing K and $L \underset{K}{\otimes} B \to A \underset{K}{\otimes} B$ is an integral extension. It follows from (a) that $B_L := L \underset{K}{\otimes} B$ is unmixed and by (ii) $qB_L \varepsilon$ Ass (B_L). If $(\mathscr{Y}, q)\, C \subsetneq P)$ then $qB_L \subsetneq P \cap B_L$ because $B_L \to A \underset{K}{\otimes} B$ is an integral extension.

Since $A \underset{K}{\otimes} B$ is a free B_L-module it follows that $P \cap B_L \in \text{Ass}\,(B_L)$.

This contradicts the fact that B_L is unmixed. Therefore $P = (\mathscr{Y}, q) \cdot C$.

(iv) (a) Let K-dim $A = r$ and K-dim $B = s$. Then we have K-dim $A \underset{K}{\otimes} B$ = K-dim $A + K - \dim B = r + s$. Let $\{a_1, \ldots, a_r\}$ (resp. $\{b_1, \ldots, b_s\}$) be an A-sequence (resp. B-sequence). Then, since K is a field, it is easy to see that $\{a_1 \otimes 1, \ldots, a_r \otimes 1, \otimes b_1, \ldots, 1 \otimes b_s\}$ is an $(A \underset{K}{\otimes} B)$-sequence of length $r + s$. Therefore $A \underset{K}{\otimes} B$ is Cohen-Macaulay.

(b) It is easy to see that $A_{\mathscr{Y}} \underset{K}{\otimes} B_q \xrightarrow{\sim} S^{-1}(A \underset{K}{\otimes} B)$, where S is the multiplicative set $(A - \mathscr{Y}) \underset{K}{\otimes} (B - q)$ in $A \underset{K}{\otimes} B$. By (1.44) there exist $f \in A - \mathscr{Y}$ and $g \in B - q$ such that A_f and B_f are Cohen-Macaulay. Therefore by (a) $A_f \underset{K}{\otimes} B_g$ is Cohen-Macaulay. Since $A_{\mathscr{Y}} \underset{K}{\otimes} B_g \xrightarrow{\sim} S^{-1} \underset{K}{\otimes} B)$ is a localization of $A_f \underset{K}{\otimes} B_g$ it follows that $A_{\mathscr{Y}} \underset{K}{\otimes} B_g$ is Cohen-Macaulay.

□

Chapter 2

The Main Theorem

IN THIS CHAPTER, we state and prove the Main Theorem. *Throughout this chapter K denotes an algebraically closed field and $\mathbb{P}^n_K$ the projective n-space over K.* 60

A. The Statement of the Main Theorem

(2.1) Main theorem

Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional subvarieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $K[X_0, \ldots, X_n]$. There exists a collection $\{C_i\}$ of irreducible subvarieties of $V_1 \cap V_2$ (one of which may be ϕ) such that

(i) For every $C_i \in \{C_i\}$ there are intersection numbers, say $j(V_1, V_2; C_i) \geq 1$ of V_1 and V_2 along C_i given by the lengths of certain well-defined primary ideals such that

$$\deg(V_1) \cdot \deg(V_2) = \sum_{C_i \in \{C_i]\}} j(V_1, V_2; C_i) \cdot \deg(C_i),$$

where we put $\deg(\phi) = 1$.

(ii) If $C \subset V_1 \cap V_2$ is an irreducible component of $V_1 \cap V_2$ then $C_i \in \{C_i\}$.

(iii) For every $C_i \in \{C_i\}$

$$\dim(C_i) \geq \dim(V_1) + \dim(V_2) - n.$$

61 In order to prove the main theorem (2.1) we need some preliminary results.

B. The Join-procedure

The following notation will be used in the sequel.

(2.2)

Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional subvarieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and $I_2 \subset R_0 := K[X_0, \ldots,]$.

We introduce two copies $R_i := K[X_{i0}, \ldots, X_{in}]$, $i = 1, 2$ of R_0 and denote I'_i the homogeneous ideal in R_i corresponding to $I_i, i = 1, 2$.

Put $N := 2(n+1) - 1, R := K[X_{ij} | i = 1, 2; 0 \leq n]$ and $\tau :=$ the diagonal ideal in R generated by $\{X_{ij} - X_{2j} | 0 \leq j \leq n\}$.

We introduce new independent variables U_{kj} over K, $0 \leq j, k \leq n$. Let $\bar{K}$ be the algebraic closure of $K(U_{kj} | 0 \leq j, k \leq n)$. Put $\bar{R} := \bar{K}[X_{ij} | i = 1, 2; 0 \leq j \leq n]$. Then we introduce so called *generic linear forms* $\ell_o, \ldots, \ell_n$:

$$\ell_k := \sum_{j=0}^{n} U_{kj}(X_{1j} - X_{2j}), \text{ for } 0 \leq k \leq n \text{ in } \bar{R}.$$

Note that since $\tau\bar{R}$ is generated by $(n+1)$ - elements and $\ell_0, \ldots, \ell_n \in \bar{R}$, it is clear that $\tau\bar{R} = (\ell_0, \ldots, \ell_n)\bar{R}$.

Let $J(V_1, V_2)$ be the *join-variety* defined by $(I'_1 + I'_2)\bar{R}$ in $\mathbb{P}^N_{\bar{K}}$.

62 **(2.3) Lemma.** 1. *The ideal* $(I'_1 + I'_2)\bar{R}$ *is unmixed and hence* $U((I'_1 + I'_2)\bar{R} = (I'_1 + I'_2)\bar{R}$.

2.
$$\begin{aligned} K-\dim(J(V_1, V_2)) &= K-\dim(I'_1 + I'_2)\bar{R} \\ &= K-\dim(I_1) + K-\dim(I_2) = \dim(V_1) + \dim(V_2) + 2. \end{aligned}$$
$$K-\dim((I'_1 + I'_2)\bar{R} + \tau\bar{R}) = K-\dim(I_1 + I_2) = \dim V_1 \cap V_2 + 1.$$

3. *There is a one-one correspondence between the isolated prime ideals of $(I_1 + I_2)$ in R_0 and the isolated prime ideals of $(I'_1 + I'_2)\overline{R} + \tau\overline{R}$ in $\overline{R}$ and that this correspondence preserves dimensions and degrees.*

4. *For every irreducible component C of $V_1 \cap V_2$*

$$\dim C \geq \dim V_1 + \dim V_2 - n.$$

5. $$\begin{aligned}\deg(V_1).\deg(V_2) &= h_0(I_1).h_0(I_2) = h_0((I'_1 + I'_2)\overline{R})\\ &= h_0((I'_1 + I'_2)\overline{R}.h_0(\tau R).\end{aligned}$$

Proof.

1. Follows from (1.47).

2. Follows from (1.47).

3. We have a ring homomorphism

$$\varphi : \overline{R} \to \overline{R}_0 : \overline{K}[X_0, \ldots, X_n]$$

given by $X_{ij} \to X_j$ for $i = 1, 2$ and every $0 \leq j \leq n$. It is easy to see that Ker $\varphi = \tau\overline{R}, \varphi^{-1}(\mathscr{Y}) = (\mathscr{Y}' + \tau)\overline{R}$. where $\mathscr{Y}'$ is the prime ideal in $\overline{R}_1$ corresponding to the prime ideal $\mathscr{Y}$ of 63
$\overline{R}_\circ$ and $\varphi^{-1}((I_1 + I_2)\overline{R}_\circ) = (I'_1 + I'_2)\overline{R} + \tau\overline{R}$. Therefore $\overline{R}/(I'_1 + I'_2)\overline{R} + \tau\overline{R} \xrightarrow{\sim} = \overline{R}_0/(I_1 + I_2)\overline{R}_0$ and $\mathscr{Y} \leftrightarrow \mathscr{Y}' + \tau\overline{R}$ gives $1 - 1$ correspondence between the isolated prime ideals of $(I_1 + I_2)\overline{R}_0$ and the isolated prime ideals of $(I'_1 + I'_2)\overline{R} + \tau\overline{R}$ in $\overline{R}$. It is clear that this correspondence preserves the dimension and degree. Now (iii) follows from (1.47).

4. This follows from (iii) and the fact that, every isolated prime ideal of $(I'_1 + I'_2)\overline{R}$ has Krull dimension $K - \dim(I'_1) + K - \dim(I'_2) = \dim V_1 + \dim V_2 + 2$ (see (i) and (ii)).

   Therefore it follows that every isolated prime ideals of $(I'_1 + I'_2)\overline{R} +$ 64
$\tau\overline{R}$ has Krull dimension $\geq \dim(V_1) + \dim(V_2) + 2 - (n + 1) = \dim V_1 + \dim V_2 - n + 1$.

5. We have $h_0(\tau\overline{R}) = 1$ by (1.34). Therefore we only have to prove $h_0(I_1).h_0(I_2) = h_0((I'_1 + I'_2)\overline{R})$. We have $\overline{R}/(I'_1 + I'_2)\overline{R} \xrightarrow{\sim} \overline{R}_1/I'_1 \underset{K}{\otimes} \overline{R}_2/I'_2 \simeq \overline{R}_0/I_1 \underset{K}{\otimes} \overline{R}_0/I_2$. Therefore $H((I'_1+I'_2)\overline{R}, t) = \sum_{i+j=t} H(I_1\overline{R}_0, i). H(I_2\overline{R}_0, j)$ for all $t \geq 0$. Choose an integer r such that the Hilbert functions $H(I_1\overline{R}_0, i) =: H_i$ and $H(I_2\overline{R}_0, i) =: H'_i$ are given by polynomials h_i and h'_1, respectively for $i > r$. Then

$$\sum_{t}^{i=0} H_i \cdot H'_{t-i} = \sum_{i=0}^{t} h_i \cdot h'_{t-i} + \sum_{i=0}^{r}(H_i - h_i) \cdot h'_{t-i} + \sum_{i=t-r}^{t} h_i(H'_{t-i} - h'_{t-i})$$

for $n >> 0 (n > 2r)$.

□

Therefore it follows from (1.32) that

$$\begin{aligned}\sum_{i=0}^{t} H_i \cdot H'_{t-i} &= h_0(I_1\overline{R}_0) \cdot h_0(I_2\overline{R}_0) \cdot [\sum_{i=0}^{t} \binom{i}{d_1}\binom{t-i}{d_2}] + \quad \text{(other terms)} \\ &= h_0(I_1\overline{R}_0) \cdot h_0(I_2\overline{R}_0)\binom{t}{d_1 + d_2 + 1} \\ &\qquad + \text{terms with degree (in t)} \leq d_1 + d_2.\end{aligned}$$

Therefore we get

$$\begin{aligned}h_0((I'_1 + I'_2)R) &= h_0(I_1\overline{R}_0).h_0(I_2\overline{R}_0) \\ &= h_0(I_1).h_0(I_2)\end{aligned}$$

by 1.46.

It is clear that Lemma (2.3), the Join-Procedure in $\mathbb{P}^N_K$, reduces our considerations to the case that one variety is a complete intersection of degree 1.

To calculate $h_0((I'_1+I'_2)\overline{R})$, we will study the sum ideal $(I'_1+I'_2)\overline{R}+\tau\overline{R}$ and the radical (denoted by $\mathrm{Rad}(\cdots)$) of this ideal.

(2.4) Notation. The following notation will be used in the sequel:

$$\begin{aligned} \delta :&= K-\dim((I_1'+I_2')\overline{R}) &&= \dim V_1 + \dim V_2 + 2 \\ d :&= K-\dim((I_1'+I_2')\overline{R}+\tau\overline{R} &&= K-\dim((I_1+I_2)R_0) \\ & &&= \dim(V_1\cap V_2)+1. \end{aligned}$$

Let $\mathscr{Y}_{i,j}$ be the minimal prime ideals of $(I_1'+I_2')\overline{R}+\tau\overline{R}$ of Krull dimension $j, 0\le t\le j\le d\le\delta$. We thus put:

(*) $\operatorname{Rad}((I_1'+I_2')\overline{R}+\tau\overline{R}) = \mathscr{Y}_{1,d}\cap\ldots\cap\mathscr{Y}_{m_d,d}\cap\ldots\cap\mathscr{Y}_{1,t}\cap\ldots\cap\mathscr{Y}_{m_t,t}$,

where $m_d\ge 1, m_{d-1},\ldots,m_t\ge 0$ are integers, and where we set $m_j=0$ **65**
for an integer $t\le j\le d-1$ if $(*)$ has no prime ideal of Krull dimension j.

(2.5) Remark. From 2.3 (iii) it follows that the prime ideals $\mathscr{Y}_{i,j}$ in $(*)$ of 2.4 are in $1-1$ correspondence with the irreducible components of $V_1\cap V_2$ and that this correspondence preserves the dimension and the degree.

(2.6) Lemma. *Let C be an irreducible component of $V_1\cap V_2$ and $\mathscr{Y}_{i,j}$ be the prime ideal corresponding to C in $(*)$ of 2.4. Put $\overline{A}=(\overline{R}/(I_1'+I_2')\overline{R})_{\mathscr{Y}_{i,j}}$, the local ring of the join-variety $J(V_1,V_2)$ at $\mathscr{Y}_{i,j}$. Then*

(i) $K-\dim(\overline{A}) = K-\dim(\overline{R}/(I_1'+I_2')\overline{R}) - K-\dim(\mathscr{Y}_{i,j}) = \delta - j.$

(ii) Let $\mathscr{Y}\subset\overline{R}$ be a prime ideal. Then

$\mathscr{Y}\in Ass(\overline{R}/(I_1'+I_2')\overline{R}+(\ell_0,\ldots,\ell_k)\overline{R})$ with $\mathscr{Y}\subset\mathscr{Y}_{ij}$ and $K-\dim(\mathscr{Y})=\ell$ if and only if $\mathscr{Y}\overline{A}\in Ass(\overline{A}/(\ell_0,\ldots,\ell_k)\overline{A})$ and $K-\dim(\mathscr{Y}\overline{A})=\ell-j$.

Proof. Follows from 2.3 (i) and 1.45. □

(2.7) Proposition. *(i) For any $\delta-d$ generic linear forms, say $\ell_0,\ldots,\ell_{\delta-d-1}$ we have*

$$K-\dim((I_1'+I_2')\overline{R}+(\ell_0,\ldots,\ell_{\delta-d-1})\overline{R}=d$$

(if $\delta=d$ then we set $(\ell_0,\ldots,\ell_{\delta-d-1})\overline{R}=(0)$).

(ii) $\delta - t - 1 \leq n$ *and equality holds if and only if* $t = \dim(V_1) + \dim(V_2) - n + 1$. **66**

Proof. (i) Assume that there exists k such that $0 \leq k \leq \delta - d - 1$ and $\ell_k \in \mathscr{Y}$ for some $\mathscr{Y} \in \text{Ass}\,(\bar{R}/(I_1' + I_2')\bar{R} + (ell_0, \ldots, \ell_{k-1})\bar{R})$ with $K - \dim(\mathscr{Y} = \delta - K)$. Let $k + 1 \leq m \leq n$. Let φ_m be te automorphism of $\bar{K}$ over K given by $\varphi_m(U_{k\ell}) = U_{m\ell}, \varphi_m(U_{m\ell}) = U_{k\ell}$ and $\varphi_m(U_{p\ell}) = U_{p\ell}$ for all $0 \leq p(\neq k, m) \leq n$ and $0 \leq \ell \leq n$. Now, since $\mathscr{Y}$ is defined over $K_1 = \overline{K(U_{pj}|0 \leq p \leq k-1, o \leq j \leq n)}$ and $\varphi_m(K_1) \subset K_1$, we get $\varphi_m(\ell_k) = \ell_m \in \mathscr{Y}$ and therefore $(I_1' + I_2')\bar{R} + (\ell_0, \ldots, \ell_n)\bar{R} \subset \mathscr{Y}$. Therefore we get $d = K - \dim((I_1' + I_2')\bar{R} + \tau\bar{R}) \geq K - \dim(\mathscr{Y}) = \delta - k$, that is, $\delta - d - 1 \geq k \geq \delta - d$ which is absurd. This proves (i).

(ii) From 2.3 (i) and 2.5 we get $t \geq \dim(V_1) + \dim(V_2) - n + 1 \geq \delta - n - 1$. Therefore $\delta - t - 1 \leq n$ and equality holds if and only if

$$t = \dim(V_1) + \dim(V_2) - n + 1.$$

□

(2.8) Proposition. *Let C be an irreducible component of $\underline{V_1} \cap \underline{V_2}$ and $\mathscr{Y}_{i,j}$ be the prime ideal corresponding to C in $(*)$ of 2.4. Let $\bar{A} = (\bar{R}/(I_1' + I_2')\bar{R})_{\mathscr{Y}_{i,j}}$ be the local ring of the join-variety $J(V_1, \underline{V_2})$ at $\mathscr{Y}_{i,j}$. Then $\{\ell_0, \ldots, \ell_{\delta-j-1}\}$ is a reducing system of parameters for $\bar{A}$.*

67 *Proof.* In view of 2.6 (ii) it is enough to prove: □

(i) For every $1 \leq k \leq \delta - j - 1$,

$\ell_{k-1} \neq \mathscr{Y}$ for all $\mathscr{Y} \in \text{Ass}\,(\bar{R}/(I_1' + I_2')\bar{R} + (\ell_0, \ldots, \ell_{k-2})\bar{R})$ with $\mathscr{Y} \subset \mathscr{Y}_{i,j}$ and $k - \dim(\mathscr{Y}) \geq \delta - k$.

(ii) $\ell_{\delta-j-1} \notin \mathscr{Y}$ for all $\mathscr{Y} \in Ass(\bar{R}/(I_1' + I_2')\bar{R} + (\ell_0, \ldots, \ell_{\delta-j-2})\bar{R})$ with $\mathscr{Y} \subset \mathscr{Y}_{i,j}$ and $k - \dim(\mathscr{Y}) = K - \dim((I_1' + I_2')\bar{R}(\ell_0, \ldots, \ell_{\delta-j-2})\bar{R})$

Proof of (i) : Suppose for some $1 \leq k \leq \delta - j - 1, \ell_{k-1} \in \mathscr{Y}$ for some $\mathscr{Y} \in \text{Ass}\,(\bar{R}/(I_1' + I_2')\bar{R})$ with $\mathscr{Y} \subset \mathscr{Y}_{i,j}$ and $K - dim(\mathscr{Y}) \geq \delta - k$. Then from the proof of 2.7(i) we get

$$(I_1' + I_2')\bar{R} + \tau \subset \bar{R} \subset \mathscr{Y} \subset \mathscr{Y}_{i,j}.$$

Therefore $\mathscr{Y} = \mathscr{Y}_{i,j}$ and $K - \dim(\mathscr{Y}) = j \geq \delta - k$. This shows that $j \geq \delta - k \geq \delta - \delta + j + 1 = j + 1$ which is absurd.

Proof of (ii) : From (i) we get

$K - dim(I'_1 + I'_2)\overline{R} + (\ell_0, \ldots, \ell_{\delta-j-2}\overline{R}) = \delta - (\delta - j - 1) = j + 1$. If $l_{\delta-j-1} \in \mathscr{Y}$ for same $\mathscr{Y} \in Ass(\overline{R}/(I'_1 + I'_2)\overline{R} + (\ell_o, \ldots, \ell_{\delta-j-2})\overline{R})$ with $\mathscr{Y} \subset \mathscr{Y}_{i,j}$. Then by the same argument in (i) above we get $\mathscr{Y} = \mathscr{Y}_{i,j}$. Therefore $K - dim(\mathscr{Y}) = j$. This proves (ii).

C. Step I of the Proof

Step I. *In this step, we define the intersection numbers* $j(V_1, V_2; C)$ *of* V_1 *and* V_2 *along* C*, where* C *is an irreducible component of* $V_1 \bigcap V_2$ *with* $\dim(C) = dim(V_1 \bigcap V_2)$.

The following notation will be used in the sequel. **68**

$$[(I'_1 + I'_2)\overline{R}]_{-1} := (I'_1 + I'_2)\overline{R}$$
$$[(I'_1 + I'_2)\overline{R}]_k := \cup([(I'_1 + I'_2)\overline{R}]_{k-1}) + \ell_k\overline{R}$$

for any $0 \leq k \leq \delta - d - 1$.

(2.9) Remarks. (i) $(I'_1 + I'_2)\overline{R} \subset (I'_1 + I'_2)\overline{R} + (\ell_0, \ldots, \ell_k)\overline{R} \subset [(I'_1 + I'_2)\overline{R}]_k$ for every $0 \leq k \leq \delta - d - 1$.

(ii) It follows from the lemma 2.7 (i) and the repeated application of 1.46(ii) that

$$\begin{aligned} \mathrm{Rad}(\cup([(I'_1 + I'_2)_{k-1}) &= \mathrm{Rad}(\cup(\cup([I'_1 + I'_2]) + l_{k-2}\overline{R}) \\ &= \mathrm{Rad}(\cup([I'_1 + I'_2]_{k-2}) + l_{k-1}\overline{R}) \\ &= \ldots \\ &= \mathrm{Rad}(I'_1 + I'_2)\overline{R} + (\ell_0, \ldots, \ell_{k-1})\overline{R} \end{aligned}$$

for every $0 \leq k \leq \delta - d$.

(iii) From (ii), we get

$$(\cup([(I'_1 + I'_2)\overline{R}]_{k-l}) : \ell_k) = \cup([(I'_1 + I'_2)\overline{R}]_{k-l})$$

for every $0 \leq k \leq \delta - d - 1$.

Now we study the primary decomposition of $\cup([(I'_1 + I'_2)\bar{R}]_{\delta-d-l})$ in the following lemma.

(2.10) Lemma. (i) *The primary decomposition of*

$$U \cup ([(I'_1 + I'_2)\bar{R}]_{\delta-d-1})$$

69 *is given by*

$$\cup([(I'_1 + I'_2)\bar{R}]_{\delta-d-l}) = q_{l,d} \bigcap \dots q_{m_d}, d \bigcap \mathscr{O}_1$$

where $q_{i,d}$ are primary ideals belonging to the prime ideals $\mathscr{Y}_{i,d}$ in $(*)$ of (2.4) $1 \leq i \leq m_d$ and $\mathscr{O}$ is the intersection of all other primary component of $\cup([I'_1 + I'_2)\bar{R}]_{\delta-d-1})$.

(ii)
$$\deg(V_1) \cdot \deg(V_2) = h_o((I'_1 + I'_2)\bar{R}) = h_o(\cup([I'_1 + I'_2)\bar{R}]_{\delta-d-1}))$$
$$= \sum_{i=1}^{m_d} (\textit{length of} q_{i,d}) \cdot h_o(\mathscr{Y}_{i,d}) + h_o(\mathscr{O})$$

(iii) *Every prime ideals $\mathscr{Y}_{i,j}$ in $(*)$ of (2.4) with $t \leq j \leq d-1$ contains $\mathscr{O}_1$. In particular, if $V_1 \cap V_2$ has an irreducible component of Krull dimension $\leq d-1$ then $\mathscr{O}_1 \neq \bar{R}$.*

(iv) *Every associated prime ideal $\mathscr{Y}$ of $\mathscr{O}_1$ has Krull dimension d.*

(v) *The diagonal ideal $\tau\bar{R}$ is not contained in any associated prime of $\mathscr{O}_1$.*

Proof. (i) From (2.7)(i) and (2.9)(ii), we have

$$K-\dim(\cup([I'_1 + I'_2)\bar{R}]_{\delta-d-1})) =$$
$$K-\dim(I'_1 + I'_2)\bar{R} + (l_0, \cdots, l_{\delta-d-1})\bar{R}) = d$$

and

$$\text{Rad }(\cup([I'_1 + I'_2)\bar{R}]_{\delta-d-1})) =$$
$$\text{Rad }((I'_1 + I'_2)\bar{R} + (\ell_0, \dots, \ell_{\delta-d-1})\bar{R}) \subset \mathscr{Y}_{i,d}$$

for every $1 \leq i \leq m_d$. Therefore $\mathscr{Y}_{i,d}$ is associated to $\cup([I'_1 + I'_2)\bar{R}]_{\delta-d-1})$ for every $1 \leq i \leq m_d$.

70 (ii) From (1.40) and (2.9)(iii), (1.36) (iii), we get

$$\begin{aligned} h_o(\cup([I_1' + I_2')\bar{R}]_{\delta-d-1})) &= h_o([I_1' + I_2')\bar{R}]_{\delta-d-2}) + \ell_{\delta-d-1}\bar{R})) \\ &= h_0(U([I_1' + I_2')\bar{R}]_{\delta-d-2})) \\ &= \ldots = h_0(I_1' + I_2')\bar{R}) = \deg(V_1)\cdot\deg(V_2) \end{aligned}$$

(iii) We have from the proof of (i) that

$$\text{Rad } (\cup([I_1' + I_2')\bar{R}]_{\delta-d-1})) = \mathscr{Y}_{1,d} \cap \cdots \cap \mathscr{Y}_{m_d,d} \cap \text{Rad}(\mathscr{O}) \subset \mathscr{Y}_{i,j}$$

for every $t \leq j \leq d-1$ and for all i. Therefore we get $\mathscr{O}_1 \subset \text{Rad}(\mathscr{O}_1) \subset \mathscr{Y}_{i,j}$ for every $t \leq j \leq d-1$ and for all i.

(iv) Clear.

(v) If $\tau\bar{R} \subset \mathscr{Y}$ for some $\mathscr{Y} \in Ass(\bar{R}/\mathscr{O}_1)$ then $(I_1' + I_2')\bar{R} + \tau\bar{R} \subset \text{Rad}((I_1' + I_2')\bar{R})\bar{R} + \tau\bar{R}) \subset \mathscr{Y}$. Therefore $\mathscr{Y} = \mathscr{Y}'_{i,d}$ because $K - \dim(\mathscr{Y}) = d$ (from (iv)). This is contradiction!

□

(2.11) Definition. Let $C \subset V_1 \cap V_2$ be an irreducible component with $\dim C = \dim(V_1 \cap V_2) = d-1$. Let $\mathscr{Y}_{i,d}$ be the prime ideal in $(*)$ of (2.4) corresponding C (see (2.5)). We define the *intersection number* $j(V_1, V_2; C)$ *of* V_1 *and* V_2 *along* C to be the length of the corresponding $\mathscr{Y}_{i,d}$-primary component $q_{i,d-1}$ of $\cup([(I_1' + I_2')\bar{R}]_{\delta-d-l})$.

From (2.10)(i) it is clear that, for every irreducible component C of $V_1 \cap V_2$ with $\dim(C) = \dim(V_1 \cap V_2)$ the intersection number $j(V_1, V_2; C)$ of V_1 and V_2 along C is defined and $j(V_1, V_2; C) \geq l$.

(2.12) Remarks. (i) It follows (2.10)(ii) and the definition (2.11) that 71

$$\deg(V_1)\cdot\deg(V_2) = \sum_C j(V_1, V_2; C)\cdot\deg(C) + h_0(\mathscr{O}_1),$$

where C runs through all irreducible components of $V_1 \cap V_2$ with $\dim(C) = \dim(V_1 \cap V_2)$.

(ii) If $t = d$, then our algorithm stops.

Assume that $t < d$, that is, $V_1 \bigcap V_2$ has irreducible components of Krull dimension $\leq d - 1$. Therefore by (2.10) (iii) $\mathscr{O} \neq \bar{R}$. In the next step we apply modified procedure to study $h_0(\mathscr{O}_1)$.

D. Step II of the Proof

Step II. *In the step, we define the intersection numbers $j(V_1, V_2; C)$ of V_1 and V_2 along C in the following two cases:*

(i) If C is an irreducible component of $V_1 \cap V_2$ of Krull dimension $\leq d - 1$.

(ii) Certain imbedded irreducible subvarieties C of $V_1 \cap V_2$ with $t \leq K - \dim(C) \leq d - 1$.

(2.13)

From (2.10)(ii) we have $(l_{\delta-d}, \ldots, l_n)\bar{R} \not\subset \mathscr{Y}$ for every prime ideal $\mathscr{Y} \in Ass(\bar{R}/\mathscr{O}_1)$. It follows from the proof of the proposition (2.7)(i) that $\ell_r \notin \mathscr{Y}$ for every $\mathscr{Y} \in Ass(\bar{R}/\mathscr{O}_1)$ with $n \geq r \geq \delta - d$. Consider $l_{\delta-d}$, we have

Therefore $K - \dim(\mathscr{O}_1 + l_{\delta-d}\bar{R}) = K - \dim(\mathscr{O}_1) - 1 = d - 1$ and
72 $(\mathscr{O}_1 : l_{\delta-d'}) = \mathscr{O}_1$

Now we study the primary decomposition of the ideal $U(\mathscr{O}_1 + l_{\delta-d'}\bar{R})$. Every primary component q of $U(\mathscr{O}_\ell + \ell_{\delta-d'}\bar{R})$ belongs to one of the following three cases:

Case (1). q is $\mathscr{Y}$- primary such that there is a prime ideal $\mathscr{Y}_{i,j}$ in $(*)$ of (2.4), $t \leq j \leq d - 1$ with $\mathscr{Y} = \mathscr{Y}_{i,j}$.

Case (2). q is $\mathscr{Y}$-primary such that there is a prime ideal $\mathscr{Y}_{ij}$ in $(*)$ of (2.4) with $\mathscr{Y}_{i,j} \subset \mathscr{Y}$.

Case (3). q is $\mathscr{Y}$-primary such that $\mathscr{Y}_{i,j} \not\subset \mathscr{Y}$ for all prime ideals $\mathscr{Y}_{i,j}$ in $(*)$ of (2.4)

Let $\cup(\mathscr{O}_\ell + l_{\delta-d'}\overline{R}) = \bigcap q_1 \cap \bigcap q_2 \cap \bigcap q_3$ be the primary decomposition of $U(\mathscr{O}_1 + \ell_{\delta-d'}\overline{R})$, where q_1, q_2 and q_3 run through the primary components of $U(\mathscr{O}_1 + \ell_{\delta-d'}\overline{R})$ which appear in case (1), case (2) and case (3), respectively. If there is no primary component in case (1), case (2) or case (3) then we set $\bigcap q_i = \overline{R}$ for $i = 1, 2$ or 3. We put $\mathscr{O}_2 := \bigcap q_3$. We then have

(2.14) Lemma. *(i) If $V_1 \cap V_2$ has irreducible components of Krull dimension $d-1$, then $\mathscr{Y}_1$ runs through prime ideals $\mathscr{Y}_{i,d-1}$ $1 \leq i \leq m_{d-1}$ in $(*)$ of (2.4).*

(ii) $h_0(\mathscr{O}_1) = \sum_{q_1}(\textit{length of } q_1)\cdot h_0(\mathscr{Y}_1) + \sum_{q_2}(\textit{length of } q_2) h_0(\mathscr{Y}_2) + h_o(\mathscr{O}_2).$

(iii) Every prime ideal $\mathscr{Y}_{i,j}$ in $(*)$ of (2.4) with $t \leq j \leq d-2$ contains **73**
$\mathscr{O}_2$. In particular, if $V_1 \cap V_2$ has an irreducible component of Krull dimension $\leq d-2$ then $\mathscr{O}_2 \neq \overline{R}$.

(iv) Every associated prime ideal $\mathscr{Y}$ of $\mathscr{O}_2$ has Krull dimension $d-1$.

(v) The diagonal ideal $\tau\overline{R}$ is not contained in any associated prime ideal of $\mathscr{O}_2$.

Proof. (i) From (2.10)(iv) and (2.10)(iii), we have $\mathscr{O}_1 \subset \mathscr{Y}_{i,d-1}$ for every $1 \leq i \leq m_{d-1}$. Therefore $(\mathscr{O}_1 + \ell_{\delta-d}\overline{R}) \subset \mathscr{Y}_{i,d-1}$ for every **74**
$1 \leq i \leq m_{d-1}$. Since $K-\dim(\mathscr{O}_1 + l_{\delta-d}\overline{R}) = d-1 = K-\dim(\mathscr{Y}_{i,d-1})$ it follows that $\mathscr{Y}_{i,d-1}$ is associated to $(\mathscr{O}_1 + l_{\delta-d}\overline{R})$ for every $1 \leq i \leq m_{d-1}$.

(ii) From (1.36) (iii) and (1.40) , we get

$$\begin{aligned} & h_0(\mathscr{O}_1) = h_0(\mathscr{O}_1 + l_{\delta-d}\overline{R}) = h_0(U(\mathscr{O}_1 + l_{\delta-d}\overline{R})) \\ = & \sum_{q_1}(\text{length of } q_1) h_0(\mathscr{Y}_1) + \sum_{q_2}(\text{length of } q_2)\cdot h_0(\mathscr{Y}_2) + h_0(\mathscr{O}_2) \end{aligned}$$

(iii) From (2.10) (iii), we have $(\mathscr{O}_1) \subset \mathscr{Y}_{i,j}$ for every $t \leq j \leq d-2$. Therefore $(\mathscr{O}_1 + \ell_{\delta-d}\overline{R}) \subset \mathscr{Y}_{i,j}$ for every $t \leq j \leq d-2$. Now it follows from (1.45) (ii) that

$$\begin{aligned}&\mathrm{Rad}(\cup(\mathscr{O}_1 + l_{\delta-d}\bar{R}))\\ &\quad= \mathscr{Y}_{i,d-1}\bigcap\cdots\bigcap\mathscr{Y}_{m_{d-l},d-l}\bigcap\cap\mathscr{Y}_2\bigcap\mathrm{Rad}(\mathscr{O}_2)\\ &\qquad= \mathrm{Rad}(\mathscr{O}_1 + l_{\delta-d}\bar{R})\subset\mathscr{Y}_{i,j}\text{ for every } t\le j\le d-2\end{aligned}$$

Therefore $\mathscr{O}_2 \subset \mathrm{Rad}(\mathscr{O}_2) \subset \mathscr{Y}_{i,j}$ for every $t \le j \le d-2$.

(iv) Clear.

(v) If $\tau\bar{R} \subset \mathscr{Y}$ for some $\mathscr{Y} \in Ass(\bar{R}/\mathscr{O}_2)$ then $(I'_1 + I'_2) + \tau \subset \mathscr{Y}$. Therefore $\mathscr{Y}_{i,j} \subset \mathscr{Y}$ for some $t \le j \le d$ and some i. This is a contradiction (see (i)).

□

(2.15) Definition. (a) Let $C \subset V_1 \cap V_2$ be an irreducible component with $K-\dim(C) = d-1$. Let $\mathscr{Y}_{i,d-1}$ be the prime ideal in $(*)$ of (2.4) corresponding to C (see (2.5)). We define *the intersection*
75 *number* $J(V_1, V_2; C)$ *of* V_1 *and* V_2 *along* C to be the length of the corresponding $\mathscr{Y}_{i,d-1}$-primary component $q_{i,d-1}$ of $\cup(\mathscr{O}_1 + l_{\lambda-d}\bar{R})$ (see (2.14)(i)).

From (2.14)(i), it is clear that, for every irreducible component C of $V_1 \bigcap V_2$ with $K-\dim(C) = d-1$ the intersection number $j(V_1, V_2; C)$ of V_1 and V_2 along C is defined and $j(V_1, V_2; C) \ge 1$.

From (2.3)(iii), it follows that the prime ideals $\mathscr{Y}_2$ which appear in case (2) of (2.13) corresponds to certain imbedded irreducible subvariety of $V_1 \cap V_2$.

(b) Let $C \subset V_1 \cap V_2$ be an irreducible subvarieties of $V_1 \cap V_2$ corresponding to the prime ideal $\mathscr{Y}_2$ which appear in case (2) of (2.13) . We define *the intersection number* $j(V_1, V_2; C)$ *of* V_1 *and* V_2 *along* C *to be the length of the corresponding* $\mathscr{Y}_2$ *-primary component* q_2 *of* $U(\mathscr{O}_1 + l_{\delta-d}\bar{R})$ *(see (2.13)). It is clear that* $j(V_1, V_2; C) \ge 1$.

(2.16) Remarks. (i) Put $c_1 := \sum_{q_2}(\text{length of } q_2)h_0(\mathscr{Y}_2)$, where q_2 runs through all primary components of $U(\mathscr{O}_1 + l_{\delta-d}\bar{R})$ which appears in case (2) of (2.13) . Then it follows from (2.12)(i) and (2.14)(ii) that

$$\deg(V_1).\deg(V_2) = \sum_C j(V_1, V_2; C).\deg(C) + c_1 + h_0(\mathscr{O}_2).$$

where C runs through all irreducible components of $V_1 \bigcap V_2$ with $d - 1 \leq K - \dim(C) \leq d$.

(ii) If $t = d - 1$, then our algorithm of step II stops.

(2.17)

Assume that $t < d - 1$, that is $V_1 \bigcap V_2$ has irreducible component of Krull dimension $\leq d-2$. Therefore by (2.14)(iii), $\mathscr{O}_2 \neq \overline{R}$. Then we again apply the above procedure to the ideal $\mathscr{O}$. In general, the application of our algorithm to the ideal $\mathscr{O}_s \leq s \leq d-t$ is given by the following considerations.

Suppose the ideals $\mathscr{O}_2, \cdots, \mathscr{O}_s, 2 \leq s \leq d-t$ are already defined. Consider the ideal $(\mathscr{O}_s l_{\delta-d+s-1}\overline{R})$. Then we have

(i) $K - \dim(\mathscr{O}_s + \ell_{\delta-d+s-1}\overline{R}) = K - \dim(\mathscr{O}_s + l_{\delta-d+s-1}\overline{R}) = d - s$ *and* $(\mathscr{O}_s : \ell_{\delta-d+s-1}) = \mathscr{O}_s$

Let $(\mathscr{O}_s + \ell_{\delta-d+s-1}\overline{R}) = \bigcap q_1 \cap \bigcap q_2 \cap \bigcap q_3$ be the primary decomposition of $U(\mathscr{O}_s + \ell_{\delta-d+s-1}\overline{R})$, which appear in case (1), case (2) and case (3) of (2.13) , respectively. We put $\mathscr{O}_{s+1} := \bigcap q_3$. Then we have **76**

(ii) If $V_1 \bigcap V_2$ has irreducible components of K-dimension $d - s$ then $\mathscr{Y}_1$ runs through the prime ideals $\mathscr{Y}_{i,d-s}$ in $(*)$ of (2.4).

(iii)
$$\begin{aligned} h_0(\mathscr{O}_s) &= h_0(\mathscr{O}_s + l_{\delta-d+s-1}\overline{R}) = h_0(U(\mathscr{O}_s + l_{\delta-d+s-1}\overline{R})) \\ &= \sum_{q_1}(\text{length of } (q_1) \cdot h_0(\mathscr{Y}_1) \\ &+ \sum_{q_2}(\text{length of } (q_2).h_0(\mathscr{Y}_2) + h_0(\mathscr{O}_{s+1}). \end{aligned}$$

We put $c_s = \sum_{q_2}(\text{ length of } (q_2).h_0(\mathscr{Y}_2)$, where q_2 runs through all primary components of $U(\mathscr{O}_s + \ell_{\delta-d+s-1}\overline{R})$ which appear in case (2) of (2.13) .

(iv) Every prime ideal ${}_{i,j}$ in $(*)$ of (2.4) with $t \leq j \leq d-s-1$ contains $\mathscr{O}_{s+1}$. In particular, if $V_1 \bigcap V_2$ has an irreducible of Krull dimension $\leq d-s-1$ then $\mathscr{O}_{s+1} \neq \bar{R}$.

(v) Every associated prime $\mathscr{Y}$ of $\mathscr{O}_{s+1}$ has Krull dimension $d-s$.

(vi) The diagonal ideal $\tau\bar{R}$ is not contained in any associated prime ideal of $\mathscr{O}_{s+1}$

In any case, our algorithm of Step II stops if we have constructed
77 the ideal $\mathscr{O}_{d-t+1}$. We obtain this ideal by studying the primary decomposition of $U(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R})$. Therefore the last step yields the following result:

$$
\begin{aligned}
h_0(\mathscr{O}_{d-t}) &= h_0(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R}) = h_0(\cup(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R}))\\
&= \sum_{i=0}^{m_t} \text{(length of } q_{i,t}) h_0(\mathscr{Y}_{i,t}) + c_{d-t} + h_0(\mathscr{O}_{d-t+1})
\end{aligned}
$$

where $q_{i,t}$ is the $\mathscr{Y}_{i,t}$-primary component of $U(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R})$ for all $1 \leq i \leq m_t$ and $c_{d-t} = \sum\limits_{q_2}$ (length of q_2) $h_0(\mathscr{Y}_2)$, where q_2 runs through all primary components of $U(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R})$ which appear in case (2) of (2.13) .

Summarizing all these we have:

(2.18)

(i) For every irreducible component C of $V_1 \bigcap V_2$ we have defined the intersection number $j(V_1, V_2; C)$ of V_1 and V_2 along C. Moreover, $j(V_1, V_2; C) \geq 1$ and $j(V_1, V_2; C)$ is the length of the corresponding $\mathscr{Y}_{i,j}$-primary component $q_{i,j}$ of $\cup(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R}), t \leq j \leq d$ (see (2.17) (ii)).

(ii) We have collected certain imbedded irreducible subvarieties of $V_1 \bigcap V_2$ corresponding to the primary components of $\cup(\mathscr{O}_{d-t}+\ell_{\delta-t-1}\bar{R})$, $1 \leq s \leq d-t$ which appear in case (2) of (2.13) . For

every imbedded irreducible subvariety C of $V_1 \bigcap V_2$ in this collection we have defined the intersection number $j(V_1, V_2; C)$ of V_1 and V_2 along C. Moreover, $j(V_1, V_2; C) \geq 1$ and $j(V_2, V_2; C)$ is the length of the corresponding $\mathscr{Y}_2$-primary componentq_2 of **78**
$U(\mathscr{O}_{d-t} + l_{\delta-t-1}\overline{R}), 1 \leq s \leq d-t$ which appear in case (2) of (2.13)
.

(iii) It follows from (2.16)(i) and (2.17) (iii),(vii) that

$$
\begin{aligned}
&\deg(V_1) \cdot \deg(V_2) \\
&\quad = \sum_C j(V_1, V_2; C) \deg(C) + c_1 + c_2 + \cdots c_{d-t} + h_0(\mathscr{O}_{d-t+1}),
\end{aligned}
$$

where C runs through all irreducible components of $V_1 \bigcap V_2$. We put $c(V_1, V_2) := c_1 + c_2 + \cdots c_{d-t} + h_0(\mathscr{O}_{d-t+1})$. This $c(V_1, V_2)$ is called the *correction term.*

(iv) If $\delta - t - 1 = n$ then $\mathscr{O}_{d-t+1} = \overline{R}$.

Proof. If $\delta - t - 1 = n$ then $(I_1' + I_2')\overline{R} + \tau\overline{R} \subset \mathscr{O}_{d-t+1}$. Therefore, if $\mathscr{O}_{d-t+1} \neq \overline{R}$ then for every associated some prime ideal $\mathscr{Y}$ of $\mathscr{O}_{d-t+l}$ contains some prime ideal $\mathscr{Y}_{i,j}$ in $(*)$ of (2.4). This is a contradiction (see (2.17)). ◻

We note the following important observation from Step II.

(2.19) Lemma. *Let C be an irreducible component of $V_1 \bigcap V_2$. Let $\mathscr{Y}_{i,j}$ be the prime ideal corresponding to C in $(*)$ of (2.4). Let $\overline{A} = (\overline{R}/(I_1', I_2')\overline{R})_{\mathscr{Y}_{i,j}}$ be the local ring of the join-variety $J(V_1, V_2)$ at $\mathscr{Y}_{i,j}$. Then we have*

(i) $\mathscr{O}_{k+1}\overline{A} = U(\mathscr{O}_k + \ell_{\delta-d+k-1}\overline{A})$ *for every* $0 \leq k \leq d-j-1$, *where* $\mathscr{O} := U([(I_1', I_2')]_{\delta-d-1})$.

(ii) $(\mathscr{O}_{d-j} + l_{\delta-j-1})\overline{A} = U(\mathscr{O}_{d-j} + l_{\delta-j-1})\overline{A} = \underline{q}_{i,j}\overline{A}$ *where* $q_{i,j}$ *is the* $\mathscr{Y}_{i,j}$*-primary component of* $U(\mathscr{O}_{d-j} + l_{\delta-j-1}\overline{R})$

(iii) $\mathscr{U}_{d-j+1}\overline{A} = \overline{A}$.

Proof. (i) From (2.17) (i), we have 79

$$K - \dim U(\mathscr{O}_k + \ell_{\delta-d+k-1}\overline{R}) = K - \dim(\mathscr{O}_{k+1}) = d - k \text{ and}$$
$$\mathscr{Y} \in \operatorname{Ass}(\mathscr{O}_{k+1} \Longleftrightarrow \mathscr{Y} \in \operatorname{Ass}(U(\mathscr{O}_k + \ell_{\delta-d+k-1}\overline{R})) \text{ and } \mathscr{Y}_{p,\ell} \nsubseteq \mathscr{Y}$$

for all p and $t \leq \ell \leq d$.

Therefore, $\mathscr{Y} \in \operatorname{Ass}(\mathscr{O}_{k+1})$ and $\mathscr{Y} \subset \mathscr{Y}_{i,j} \Longleftrightarrow \mathscr{Y} \in \operatorname{Ass}(U(\mathscr{O}_k + \ell_{\delta-d+k-1}\overline{R}))$ and $\mathscr{Y} \underset{+}{\subset} \mathscr{Y}_{i,j}$. This shows that $\mathscr{O}_{k+1}\overline{A} = U(\mathscr{O}_k + \ell_{\delta-d+k-1})\overline{A}$ for every $0 \leq k \leq d - j - 1$.

(ii) It follows from (2.17))(i) and (2.17) (ii) that $K - \dim(\mathscr{O}_{d-j} + \ell_{\delta-j-1}\overline{R})) = j$ and $\mathscr{Y}_{i,j} \in \operatorname{Ass}(U(\mathscr{O}_{d-j} + \ell_{\delta-j-1}\overline{R})) \subset \operatorname{Ass}(_{d-j} + \ell_{\delta-j-1}\overline{R}$. Therefore $\mathscr{O}_{d-j} + \ell_{\delta-j-1})\overline{A} = U(\mathscr{O}_{d-j} + \ell_{\delta-j-1})\overline{A} = q_{i,j}\overline{A}$, where $q_{i,j}$ is the $\mathscr{Y}_{i,j}$-primary component of $U(\mathscr{O}_{d-j} + \ell_{\delta-j-1}\overline{R})$.

(iii) Since $K - \dim(\mathscr{O}_{d-j+1}) = j = K - \dim(\mathscr{Y}_{i,j})$, it follows from the proof of (i) that $\operatorname{Ass}(\mathscr{O}_{d-j+1}\overline{A} = \phi$. Therefore $\mathscr{O}_{d-j+1}\overline{A} = \overline{A}$.

□

(2.20) Corollary.

$$e_0((\ell_0, \ldots, \ell_{\delta-j-1})\overline{A}; \overline{A} = \ell(\overline{A}/(\mathscr{O}_{d-j} + \ell_{\delta-j-1})\overline{A}) = j(V_1, V_2; C)$$

Proof. This follows from (2.9), (2.19)(i) and (ii), (1.25) and (2.18) (i).

□

E. Step III of the Proof

80 **Step III.** *In this step we collect certain imbedded irreducible subvarieties of $V_1 \cap V_2$ with $t - s \leq K - \dim(C) < t$, where $s = n - \delta + t + 1 \geq 0$ (see (2.7)(ii)).*

(2.21)

From (2.7)(ii), we have $\delta - t - 1 \leq n$. If $\delta - t - 1 = n$ then $\mathscr{O}_{d-t+1} = \overline{R}$ (see (2.18) (iv)) and our algorithm stops.

Assume that $\delta - t - 1 < n$ and $\mathscr{O}_{d-t+1} \neq \overline{R}$. Put $\delta - t - 1 + s = n$ for some $s > 0$. To calculate $h_0(\mathscr{O}_{d-t+1})$, we study the primary decomposition of the ideal $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R})$. Every primary component q of $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R}$ belongs to one of the following two cases:

Case (1). q is $\mathscr{Y}$-primary such that there is a prime ideal $\mathscr{Y}_{i,j}$ in (∗) of (2.4) such that $\mathscr{Y}_{i,j} \underset{+}{\subset} \mathscr{Y}$.

Case (2). q is $\mathscr{Y}$-primary such that the $\mathscr{Y}_{i,j} \not\subset \mathscr{Y}$ for all prime ideals $\mathscr{Y}_{i,j}$ in (*) of (2.4).

Let $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R}) = \cap q_1 \cap \bigcap q_2$ be the primary decomposition of $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R})$, where q_1 and q_2 run through the primary components of $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R}$ which appear in case (1) and case (2), respectively. We put $\mathscr{O}_{d-t+2} := \cap q_2$. Then we have

$$\begin{aligned} h_0(\mathscr{O}_{d-t+1}) &= h_0(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R}) \\ &= \sum_{q_1} (\text{ length of } q_1) h_0(\mathscr{Y}_1) + h_0(\mathscr{O}_{d-t+2}). \end{aligned}$$

where q_1 runs through the primary components of $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R})$ which appear in case (1).

From (2.3)(iii), it follows that the prime ideals $\mathscr{Y}_1$ which appear in case (2) corresponds to certain imbedded irreducible subvarieties of $V_1 \cap V_2$. **81**

(2.22) Definition. Let $C \subset V_1 \cap V_2$ be an irreducible subvariety of $V_1 \cap V_2$ corresponding to the prime ideal $\mathscr{Y}_1$ which appear in case (1) of ((2.21)). We define *the intersection number* $j(V_1, V_2; C)$ *of* V_1 *and* V_2 *along* C to be the length of the corresponding $\mathscr{Y}_1$-primary component q_1 of $U(\mathscr{O}_{d-t+1} + \ell_{\delta-t}\overline{R})$ (see (2.21)). It is clear that $j(V_1, V_2; C) \geq 1$. Assume $\mathscr{O}_{d-t+2} = \overline{R}$; then our algorithm of step III stops.

(2.23)

If $\mathscr{O}_{d-t+2} \neq \overline{R}$, then we repeat the above procedure to the ideal $\mathscr{O}_{d-t+2}$ by using $\ell_{\delta-t+1}$.

In general, the application of our algorithm to the ideal $\mathscr{O}_{d-t+k}, 1 \leq k \leq s = n - \delta + t + 1$ is given by the following considerations:

Suppose the ideals $\mathscr{O}_{d-t+2}, \ldots, \mathscr{O}_{d-t+k}$ are already defined for $2 \leq k \leq s$. Then consider the ideal $(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R})$. Then we have

$$K - \dim(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R}) = K - \dim U(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}) = t - k$$

and $(\mathscr{O}_{d-t+k} : \ell_{\delta-t+k-1}) = \mathscr{O}_{\underline{d}-t+k}$.

Let $U(\mathscr{O}_{d-t+k}+\ell_{\delta-t+\underline{k}-1}\overline{R}) = \cap q_1 \cap \cap q_2$ be the primary decomposition of $U(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R})$, where q_1 and q_2 are the primary components

82 of $U(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R})$ which appear in case (1) and case (2) of (2.21) , respectively. We put $\mathscr{O}_{d-t+k+1} = \cap q_2$. Then we have

$$\begin{aligned} h_0(\mathscr{O}_{d-t+k} &= h_0(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R}) = h_0(U(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R})) \\ &= \sum_{q_1} \text{ length of } q_1)\ h_0(\mathscr{Y}_1) + h_0(\mathscr{O}_{d-t+k+1}) \text{ for } 2 \leq k \leq s. \end{aligned}$$

In any case, our algorithm stops if we have used all generic linear forms $\ell_0, \ldots, \ell_n$. Therefore $\mathscr{O}_{d-t+s+1} = \overline{R}$, where $s = n - \delta + t + 1$. Therefore the last step yields:

$$\begin{aligned} h_0(\mathscr{O}_{d-t+s}) &= h_0(\mathscr{O}_{d-t+s} + \ell_n\overline{R} = h_0(U(\mathscr{O}_{d-t+s} + \ell_n\overline{R})) \\ &= \sum_{q_1}(\text{ length of } q_1).h_0(\mathscr{Y}_1) \end{aligned}$$

where q_1 runs through all $\mathscr{Y}_1$-primary components of $U(\mathscr{O}_{d-t+s} + \ell_n\overline{R}$ (Note that all primary components of $U(\mathscr{O}_{d-t+s} + \ell_n\overline{R})$ appear in case (1) of (2.21)).

(2.24) Remark. Note that $K - \dim \mathscr{O}_{d-t+k} = t - k$ for every $1 \leq k \leq s$. Therefore, in this step, we have collected certain imbedded irreducible subvarieties C of $V_1 \cap V_2$ corresponding to the primary components of $U(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R}), 1 \leq k \leq s$, which appear in case (1) of (2.21) . For every imbedded irreducible subvariety C of $V_1 \cap V_2$ in this collection, we have defined the intersection number $j(V_1, V_2; C)$ of V_1 and V_2 along C. Moreover, $j(V_1, V_2; C) \geq 1$, $j(V_1, V_2; C)$ is the length of the corresponding $\mathscr{Y}_1$-primary component q_1 of $U(\mathscr{O}_{d-t+k} + \ell_{\delta-t+k-1}\overline{R})$, $1 \leq k \leq s = n - \delta + t - 1$ which appear in case (1) of (2.21) .

Proof of the Main Theorem(2.1).

(i) Let $\{C_i\}$ be the collection of irreducible subvarieties of $V_1 \cap V_2$ 83
consisting of irreducible subvarieties of $V_1 \cap V_2$ which are collected in (2.18) (i), (2.18) (ii) and (2.23). Then the intersection numbers $j(V_1, V_2; C_i) \geq 1$ of V_1 and V_2 along C_i are defined and $j(V_1, V_2; C_i)$ are the lengths of certain well-defined primary ideals. It follows from (2.18) (iii) and (2.23) that

$$\deg(V_1). \deg(V_2) = \sum_{C_i} j(V_1, V_2; C_i) \deg(C_i)$$

(ii) It is clear from (2.18) (i) that every irreducible component of $V_1 \cap V_2$ belongs to our collection $\{C_i\}$.

(iii) Let $C_i \in \{C_i\}$. Then it follows (2.18) (ii) and (2.23) that

$$K - \dim(C_i) \geq t - s = t - (n - \delta + t + 1) = \delta - n - 1$$

that is, $$\dim(C_i) \geq \dim(V_1) + \dim(V_2) - n.$$

This completes the proof of the main theorem (2.1).

We have the following generalization of the main theorem (2.1).

(2.25) The General Main Theorem

Let $V_1 = V(I_1), \ldots, V_r = V(I_r), r \geq 2$ be pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals $I_1, \ldots, I_r \subset K[X_0, \ldots, X_n]$. There exists a collection $\{C_i\}$ of irreducible subvarieties of $V_1 \cap \cdots \cap V_r$ (one of which may be ϕ) such that

(i) For every $C_i \in \{C_i\}$ there are intersection number, say $j(V_1, \ldots, V_r;$ 84
$C_i) \geq 1$, of $V_1, \ldots, V_r$ along C_i given by the lengths of certain well-defined primary ideals such that

$$\prod_{i=1}^{r} \deg(V_i) = \sum_{C_i \in \{C_i\}} j(V_1, \ldots, V_r; C_i). \deg(C_i),$$

where we put $\deg\ (\phi) = 1$.

(ii) If $C \subset V_1 \cap \cdots \cap V_r$ is an irreducible component of $V_1 \cap \cdots \cap V_r$ then $C_i \in \{C_i\}$.

(iii) For every $C_i \in \{C_i\}$,

$$\dim(C_i) \geq \sum_{i=1}^{r} \dim(V_i) - (r-1).n.$$

Proof. Proof of this theorem is very similar to the proof of that in case $r = 2$ (see (2.1)). Therefore we omit the proof. (For details, see Patil-Vogel [56, Main theorem (1.2)]). □

F. Consequences

In the following, we list some immediate consequences of the main theorem some of which are already known.

A typical classical result in this direction says that if $V_1, \ldots, V_r, r \geq 2$ are pure dimensional subvarieties in $\mathbb{P}^n_K$, and $\sum\limits_{i=1}^{r} \dim(V_i) = (r-1).n$, and $\bigcap\limits_{i=1}^{r} V_i$ is finite set of isolated points, then $\bigcap\limits_{i=1}^{r} V_i$ contains atmost $\prod\limits_{i=1}^{r} \deg(V_i)$ points. The following corollary (2.26) strengthens this to allow arbitrary intersections.

(2.26) Corollary (Refined Bezout's Theorem)

85 Let $V_1, \ldots, V_r \subset \mathbb{P}^n_K, r \geq 2$ be pure dimensional projective varieties in $\mathbb{P}^n_K$. Let $Z_1, \ldots, Z_m$ be the irreducible components of $\bigcap\limits_{i=1}^{r} V_r$. Then

$$\prod_{i=1}^{r} \deg(V_i) \geq \sum_{i=1}^{m} \deg(Z_i) \geq m.$$

This refined Bezout's theorem was developed by W.Fulton and R.MacPherson (see [19], [18]) to give an affirmative answer the following question asked by S.Kleiman in 1979.

(2.27) Corollary (Kleiman's Question)

Let $V_1, \ldots, V_r \subset \mathbb{P}^n_K$, $r \geq 2$ be pure dimensional projective varieties in $\mathbb{P}^n_K$. Then the number of irreducible components of $\bigcap_{i=1}^{r} V_i$ is bounded by the Bezout's number $\prod_{i=1}^{r} \deg(V_i)$.

The first proof is given in [17, §7.6] (see also [18]). A second proof (see [17]) was suggested by a construction of Deligne used to reduce another intersection question in projective space to an intersection with a linear factor (see also the method used in [98 Lemma on p.127] and (2.3) (v)). A new interpretation of the refined Bezout's theorem was given by R.Lazarsfeld [45].

The following Corollary (2.28) strengthens the refined Bezout's theorem (2.26) .

(2.28) Corollary . *Let $V_1, \ldots, V_r \subset \mathbb{P}^n_K, r \geq 2$ be pure dimensional projective varieties in $\mathbb{P}^n_K$. Then* **86**

$$\prod_{i=1}^{r} \deg(V_i) \geq \sum_{C} j(V_1, \ldots, V_r; C) \deg(C)$$

where C runs through all irreducible components of $V_1 \cap \cdots \cap V_r$.

The following corollary (2.29) gives a generalization of $C.G.$Jacobi's observation (see the Historical Introduction, [36], [16] and [60]).

(2.29) Corollary . *Let $F_1, \ldots, F_n$ be any hypersurfaces in $\mathbb{P}^n_K$ of degrees $d_1, \ldots, d_n$, respectively. Assume that $\bigcap_{i=1}^{n} F_i$ contains a finite set of isolated points,say $P_1, \ldots, P_s$. Then we get*

$$\prod_{i=1}^{n} d_i - \sum_{C} \deg(C) \geq \prod_{i=1}^{n} d_i - \sum_{C} j(F_1, \ldots, F_n; C). \deg(C) \geq s.$$

where C runs through all irreducible components of $V := \bigcap_{i=1}^{n} F_i$ with $\dim(C) \geq 1$.

(2.30)

Analyzing these results and their proofs, one might be tempted to ask the following question:

Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1, and $I_2 \subset K[X_n, \ldots, X_n]$. We consider a primary decomposition of

$$I_1 + I_2 = q_1 \cap \cdots \cap q_m \cap q_{m+1} \cap \cdots \cap q_\ell$$

where q_i is $\mathscr{Y}_i$-primary and $\mathscr{Y}_1, \ldots, \mathscr{Y}_m$ are the minimal prime ideals of $I_1 + I_2$. Then

87 **Question 1.** If $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - n$ is then $\deg(V_1)$. $\deg(V_2) \geq \ell - 1$?

Question 2. If $\dim(V_1 \cap V_2) > \dim(V_1) + \dim(V_2) - n$ is then $\deg(V_1)$. $\deg(V_2) \geq \ell$?

Question 3. If $\deg(V_1) \deg(V_2) > m$, is then $\deg(V_1) \deg(V_2) \geq \ell$?

However, these questions have negative answers, as we will show by examples (3.14) in the next chapter.

Chapter 3

Examples, Applications and Problems

A. Examples

In this section, we shall illustrate the proof of the main theorem (2.1) **88**
by describing some examples.

First, we would like to make the following definitions.

We preserve the notation of Chapter II

(3.1) Definition. Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and $I_2 \subset R_0 := K[K_0, \ldots, X_n]$.

(a) An irreducible subvariety $C \subset V_1 \cap V_2$ is said to be an *imbedded component of* $V_1 \cap V_2$, if the defining prime ideal $\mathscr{Y}(C) = \mathscr{Y}$ of C is an imbedded prime ideal of $(I_1 + I_2)$.

(b) An irreducible subvariety $C \subset V_1 \cap V_2$ is called a *geometric imbedded component of* $V_1 \cap V_2$, if

(i) the defining prime ideal $\mathscr{Y}(C) = \mathscr{Y}$ of C is not associated to $I_1 + I_2$ and

(ii) C does yield a contribution to Bezout's number $\deg(V_1) \cdot \deg(V_2)$, that is, C belongs to our collection $\{C_i\}$ of the main theorem (2.1) .

In the following examples, we use the notation of Chapter II. For simplicity, we put $X_{1j} = X_j$ and $X_{2j} = Y_j$ for all $0 \leq j \leq n$.

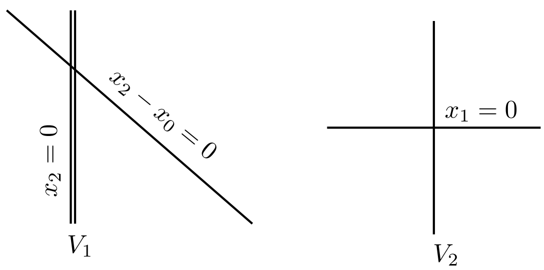
89 **(3.1) Example.** Let V_1 and V_2 be two hypersurfaces in $\mathbb{P}^2_K$ defined by $F_1 := X_2^2(X_2 - X_0) = 0$ and $F_2 := X_1X_2 = 0$. Put $I_1 = (F_1)$ and $I_2 = (F_2)(\subset K[X_0, X_1, X_2])$. It is easy to see that:

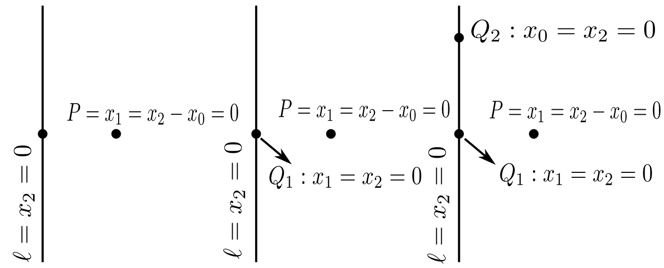
(i) The primary decomposition of $I_1 + I_2$ is given by $I_1 + I_2 = (X_2) \cap (X_1, X_2 - X_0) \cap (X_1, X_2^2)$ and therefore $\operatorname{Rad}(I_1 + I_2) = (X_2) \cap (X_1, X_2 - X_0)$.

(ii) (a) The *set-theoretic intersection* $V_1 \cap V_2$ of V_1 and V_2 is precisely the line $\ell : X_2 = 0$ and the isolated point $p : X_1 = X_2 - X_0 = 0$.

(b) The *ideal-theoretic intersection of* V_1 and V_2 is precisely the line $\ell : X_2 = 0$, the isolated point $P : X_1 = X_2 - X_0 = 0$ and the imbedded point $Q_1 : X_1 = X_2 = 0$.

(c) The *geometric intersection of* V_1 and V_2 is precisely the line $\ell : X_2 = 0$, the isolated point $P : X_1 = X_2 - X_0 = 0$ and two imbedded points $Q_1 : X_1 = X_2 = 0, Q_2 : X_0 = X_2 = 0$.





(a) Set-theoretic (b) Ideal-theoretic (c) Geometric

(iii) $$\begin{aligned}\operatorname{Rad}((I_1' + I_2')\overline{R} + \tau\overline{R}) &= (X_2, Y_2, X_0 - Y_0, X_1 - Y_1)\\ &\quad\cap (X_2 - X_0, X_1, Y_1, X_0 - Y_0, X_2 - Y_2)\\ &= \mathscr{Y}_{1,2} \cap \mathscr{Y}_{1,1}\end{aligned}$$ 90

where $\mathscr{Y}_{1,2} := (X_2, Y_2, X_0 - Y_0, X_1 - Y_1)$
and $\mathscr{Y}_{1,1} = (X_2 - X_0, X_1, Y_1, X_0 - Y_0, X_2 -_2)$.

$$\begin{aligned}\delta &= \dim(V_1) + \dim(V_2) + 2 = 1 + 1 + 2 = 4\\ d &= \dim(V_1 \cap V_2) + 1 = 1 + 1 = 2\\ t &= 1. \text{ Therefore } \delta - d - 1 = 1.\end{aligned}$$

Following the proof of Step I in Chapter II, we get:

(iv) $$\begin{aligned}U([(I_1' + I_2')\overline{R}]_1) &= (I_1' + I_2')\overline{R} + (\ell_0, \ell_1)\overline{R}\\ &= q_{1,2} \cap \mathscr{O}_1\end{aligned}$$

where $q_{1,2} \;\underline{=}\; (X_2^2, Y_2, \ell_0, \ell_1)$ is the $\mathscr{Y}_{1,2}$- primary component of $U([(I_1' + I_2')\overline{R}]_1)$ and

$$\mathscr{O}_1 = (X_2 - X_0, Y_1, \ell_0, \ell_1) \cap (X_2^2, Y_1, \ell_0, \ell_1) \cap (X_2 - X_0, Y_2, \ell_0, \ell_1).$$

(v) Let C_1 be the irreducible component of $V_1 \cap V_2$ corresponding to the prime ideal $\mathscr{Y}_{1,2}$. Then the defining prime ideal of C is 91

$\mathscr{Y}(C) = (X_2) \subset K[X_0, X_1, X_2]$ and

$$j(V_1, V_2; C) = \text{ length of } q_{1,2} = \ell((\overline{R}/q_{1,2})_{\mathscr{Y}1,2}) = 2$$

Note that $t < d$; therefore $\mathscr{O}_1 \neq \bar{R}$.

Now following the proof of step II in Chapter II, we get:

(vi) $U(\mathscr{O}_1, \ell_2) = q_{1,1} \cap q_2 \cap q_2'$ where $q_{1,1} = (X_2 - X_0, Y_1, \ell_0, \ell_1, \ell_3)$ is the $\mathscr{Y}_{1,1}$-primary component of $U(\mathscr{O}_1, \ell_2)$ and $q_2 = (X_2^2, Y_1, \ell_0, \ell_1, \ell_2)$ (resp. $q_2' = (X_2 - X_0, Y_2, \ell_0, \ell_1, \ell_2)$ is $\mathscr{Y}_2 = (X_1, Y_1, X_2, Y_2, X_0 - Y_0)$ (resp. $\mathscr{Y}_2{}' = (X_0, Y_0, X_2, Y_2, X_1, -Y_1)$) -primary component of $U(\mathscr{O}_1, \ell_2)$

(vii) Let C_2 be the irreducible component of $V_1 \cap V_2$ corresponding to the prime ideal $\mathscr{O}_{1,1}$ and let C_3, C_4 be irreducible subvarieties of $V_1 \cap V_2$ corresponding to the prime ideals $\mathscr{Y}_2, \mathscr{Y}_2{}'$, respectively. Then the defining prime ideals of C_2, C_3 and C_4 are $(X_2 - X_0, X_1)$, (X_1, X_2^2) and $(X_2 - X_0, X_2)$ respectively and

$$j(V_1, V_2; C_2) = \text{length of } q_{1,1} = 1$$
$$j(V_1, V_2; C_3) = \text{lent of } q_2 = 2$$
$$j(V_1, V_2; C_4) = \text{Lent of } q_3' = 1$$

Note that $\mathscr{O}_{d-t+1} = \mathscr{O}_2 = \bar{R}$ and therefore there is no step III in this example

(viii) (a) The required collection $\{C_i\}$ of irreducible subvarieties of $V_1 \cap V_2$ is:

$C_1 : X_2 = 0$ the line ℓ with $j(V_1, V_2; C_1) = 2$.

$C_2 : X_1 = X_2 - X_0 = 0$(the isolated point P) with $j(V_1, V_2; C_2) = 1$.

$C_3 : X_1 = X_2 = 0$(the imbedded point Q_1) with $j(V_1, V_2; C_3) = 2$.

$C_4 : X_0 = X_2 = 0$(the geometric imbedded point Q_2)with

$$j(V_1, V_2; C_4) = 1.$$

92 (b) From (1.35) , we have $\deg(V_1) = 3, \deg(V_2) = 2$ and $\deg(C_i) =$

1 for all $i = 1, \dots, 4$. therefore we get

$$6 = \deg(V_1).\deg(V_2) = \sum_{i=1}^{4} j(V_1, V_2; C_i)\deg(C_i) = 6.$$

(3.3) Example. Let V be the non-singular curve in $\mathbb{P}^3_K$, parametrically given by $\{s^4, s^3t, st^3, t^4\}$ (see [[26], p.180], [[50], p.126]; [[90], §11] and (0.5))

It is easy to see that the prime ideal of V is

$$I = (X_0X_3 - X_1X_2, X_0^2X_2 - X_1^3, X_1X_3^2 - X_3^2, X_0X_2^2 - X_1^2X_3) \subset K[X_0, X_1, X_2, X_3].$$

Let $V_1 \subset \mathbb{P}^3_K$ be the defined by $X_0 = X_1 = 0$. Then $I_1 = (X_0, X_1) \subset K[X_0, X_1, X_2, X_3]$ is the prime ideal of V_1. It is easy to see that:

(i) $(I + I_1) = (X_0, X_1, X_2^3)$ is $\mathscr{Y} = (X_0, X_1, X_2,)$ -primary; therefore the intersection $V \cap V_1$ has precisely one isolated point $p : X_0 = X_1 = X_2 = 0$.

(ii) $\mathrm{Rad}((I' + I_1')\overline{R} + \tau\overline{R}) = \mathscr{Y}_{1,1}$, where

$\mathscr{Y}_{1,1} = (X_0, X_1, X_2, Y_0, Y_1, Y_2, X_3 - Y_3)$

$\delta = \dim V + \dim V_1 + 2 = 1 + 1 + 2 = 4$

$d = \dim(V \cap V_1) + 1 = 1, t = 1$. Therefore $t = d$ and $\delta - d - 1 = 2$. **93**
Following the proof of Step I in Chapter II, we get:

(iii) $U([I' + I_1')\overline{R}]_2 = q_{1,1} \cap \mathscr{O}_1$ where $q_{1,1}$ is $\mathscr{Y}_{1,1}$ primary component of $U([I' + I_1')\overline{R}]_2)$ and $((I' + I')\overline{R} + (\ell_0, \ell_1, \ell_2)\overline{R})_{\mathscr{Y}_{1,1}} = U([I' + I_1')\overline{R}]_2)_{\mathscr{Y}_{1,1}}$. Therefore $(q_{1,1})_{\mathscr{Y}_{1,1}} = (X_0, X_1, X_2^3, Y_0, Y_1, X_2 - Y_2, X_3 - Y_3)\overline{R}_{\mathscr{Y}_{1,1}}$

(iv) Let C_1 be the irreducible component of $V \cap V_1$ corresponding to the prime ideal $\mathscr{Y}_{1,1}$. Then the defining prime ideal of C_1 is $(X_0, X_1, X_2) = \mathscr{Y}$ and

$$j(V, V_1; C_1) \text{ length } (q_{1,1}) = 3$$

Note that $t = d = 1$ therefore their is no Step II in this example. Following Step III in Chapter II, we get $K - \dim U(\mathscr{O}_1, \ell_3) = K \dim(\mathscr{O}_1, \ell_3) = K - \dim(\mathscr{O}_1) - 1 = 0$ Therefore $q := U(\mathscr{O}_1, \ell_3) = (\mathscr{O}_1, \ell_3)$ is primary ideal corresponding to the homogeneous maximal ideal $(X_0, Y_0, X_1, Y_1, X_2, Y_2, X_3, Y_3) \subset \bar{R}$ This primary ideal q gives the empty subvariety ϕ in our collection.

(v) (a) The required collection $\{C_i\}$ of irreducible subvarieties of $V \cap V_1$ is:
$C_1 : X_0 = X_1 = X_2 = 0$ (the isolated point P) with $j(V, V_1, C_1) = 3$, ϕ: the empty subvariety with $j(V, V_1; \phi) =$ length of (q)

94 (b) From (1.43) (iii) and (1.35)), we have $\deg(V) = 4$ and $\deg(V_1) = 1, \deg(C_1) = 1$.

(c) Therefor, from the main theorem (2.1) we get

$$\begin{aligned} 4 = \deg(V).\deg(V_1) &= j(V, V_1; C_1) \deg C_1 + j(V, V_1, \phi).\deg(\phi) \\ &= 3 + j(V, V_1, \phi). \end{aligned}$$

This shows that $j(V, V_1, \phi) =$ length of $(q) = 1$, so that $q = (\mathscr{O}_1, \ell_3)$ is the homogeneous maximal ideal $(X_0, X_1, X_2, X_3, Y_0, Y_1 Y_2, Y_3)$.

The line V_1 is a tangent line to V at P whose intersection multiplicity with V at P is 3. In general, the non-singular curves $C_d \subset \mathbb{P}^3_K$ defined parametrically by

$$\left\{s^d, s^{d-1}t, st^{d-1}, t^d\right\}, d \geq 4,$$

are of degree d and have a tangent line with a contact of order $d - 1$

(3.4) Remark. The empty subvariety ϕ is geometric imbedded component of $V \cap V_1$ in example (3.3).

(3.5) Example. Let $V \subset \mathbb{P}^3_K$ be the non-singular curve of example (3.3) and $V_2 \subset \mathbb{P}^3_K$ be the line defined by $X_0 = X_2 = 0$. Then $I_2 = (X_0, X_2) \subset K[X_0, X_1, X_2, X_3]$ be the prime ideal of V_2. It is easy to see that:

(i) The primary decomposition of $I + I_2$ is given by $I + I_2 = (X_0, X_1, X_2)$
95 $\cap\, (X_0, X_1, X_1^3, X_3^2, X_1^2 X_3)$ and therefore $\mathrm{Rad}(I + I_2) = (X_0, X_1, X_2)$.

(ii) The set-theoretic intersection $V \cap V_2$ of V and V_2 has precisely one isolated point.

(iii) $\mathrm{Rad}((I' + I'_2)\bar{R} + \tau\bar{R}) = \mathscr{Y}_{1,1}$ where

$\mathscr{Y}_{1,1} = (X_0, X_1, X_2, Y_0, Y_1, Y_2, X_3 - Y_3)$.

$\delta = \dim V + \dim V_2 + 2 = 1 + 1 + 2 = 4$

$d = \dim(V \cap V_2) + 1 = 1, t = 1$. Therefore $t = d$ and $\delta - d - 1 = 2$.

Following the proof of step I is Chapter II, we get:

$$U([(I' + I'_2)\bar{R}]_2) = q_{1,1} \cap \mathscr{O}_1$$

where $q_{1,1}$ is $\mathscr{Y}_{1,1}$ -primary component of $U([(I' + I'_2)\bar{R}]_2)$ and

$$\begin{aligned}(q_{1,1})_{\mathscr{Y}\,1,1} U([(I' + I'_2)\bar{R}]_2)_{\mathscr{Y}_{1,1}} &= (I' + I'_2)\bar{R} + (\ell_0, \ell_1, \ell_2)\bar{R}_{\mathscr{Y}_{1,1}} \\ &= (X_0, X_1, X_2, Y_0, Y_1, Y_2, X_3 - Y_3).\end{aligned}$$

(iv) Let C_1 be the irreducible component of $V_1 \cap V_2$ corresponding to the prime ideal $\mathscr{Y}_{1,1}$. Then the defining prime ideal of C_1 is $(X_0, X_1, X_2) = \mathscr{Y}$ and $j(V, V_2; C_1) =$ length of $q_{1,1} = 1$ Note that $t = d = 1$; therefore there is no Step II in this example. Following Step III in Chapter II, we get

$K - \dim(\mathscr{O}_1, \ell_3) = K - \dim U(\mathscr{O}_1, \ell_3) = K - \dim(\mathscr{O}_1) - 1 = 0.$

Therefore $q := U(\mathscr{O}_1, \ell_3) = (\mathscr{O}_1, \ell_3)$ is primary ideal corresponding to the homogeneous maximal ideal $(X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2, Y_3) \subset \bar{R}$. This primary ideal q gives empty subvariety ϕ in our collection

(v) (a) The required collection $\{C_i\}$ of irreducible subvarieties of $V \cap V_2$ is: **96**

$C_1 : X_0 = X_1 = X_2 = 0$ (the isolated point P) with $j(V, V_2; C_1) = 1$.
ϕ: the empty subvariety with $j(V, V_2; \phi) =$ length of q

(b) We have from example(3.3) $\deg(V) = 4$ and from (1.35)

$$\deg(V_2) = 1, \deg(C_1) = 1$$

(c) Therefore from the main theorem we get

$$\begin{aligned}4 = \deg(V).\deg(V_2) = {} & j(V, V_1, V_2; C_1)\deg(C_1) \\ & + j(V, V_1, V_2; \phi)\deg(\phi) \\ = {} & 1 + j(V, V_1, V_2; \phi)\end{aligned}$$

This shows that $j(V, V_1, V_2; \phi)$ = length of $q = 3$.

(3.6) Remark. The empty subvariety ϕ is imbedded component of $V \cap V_2$ in example (3.5)

(3.7) Example. Let V_1 and V_2 be two hypersurfaces in $\mathbb{P}^2_K$ defined by $F_1 := X_0X_1(X_0 - 2X_1) = 0$ and $F_2 := X_0X_1(X_1 - 2X_0) = 0$. put $I_1 = (F_1)$ and $I_2 = (F_2)(\subset K[X_0, X_1, X_2])$. It is easy to see that:

(i) The primary decomposition of $I_1 + I_2$ is given by $I_1 + I_2 = (X_0) \cap (X_1) \cap (X_0^2, X_1^2)$ and therefore $\mathrm{Rad}(I_1 + I_2) = (x_0) \cap (X_1)$.

(ii) (a) The set-theoretic intersection $V_1 \cap V_2$ of V_1 and V_2 is precisely the two lines $\ell_0 : X_0 = 0$ and $\ell_1 : X_1 = 0$.

97 (b) The ideal-theoretic intersection of V_1 and V_2 is precisely the two lines $\ell_0 : X_0 = 0 \ell_1 : X_1 = 0$ and the imbedded point $P : X_0 = X_1 = 0$

(iii) $\mathrm{Rad}((I'_1 + I'_2)\bar{R} + \tau\bar{R}) = \mathscr{Y}_{1,2} \cap \mathscr{Y}_{2,2} \cap \mathscr{Y}_{1,1}$ where

$\mathscr{Y}_{1,2} = (X_0, Y_0, X_1 - Y_1, X_2 - Y_2), \mathscr{Y}_{2,2} = (X_1, Y_1, X_0 - Y_0, X_2 - Y_2)$ and

$\mathscr{Y}_{1,1} = (X_0, X_1, Y_0 - Y_1, X_2 - Y_2)$.

$\delta = \dim V_1 + \dim V_2 + 2 = 1 + 1 + 2 = 4$

$d = \dim(V_1 \cap V_2) + 1 = 2, t = 1$ therefore $\delta - d - 1 = 1$

Following the proof of Step I in Chapter II, we get:

(iv) $\cup([(I'_1 + I'_2)\bar{R}]_1) = q_{1,2} \cap q_{2,2} \cap \mathscr{O}_1$, where $q_{1,2}$(resp. $q_{2,2}$) is $\mathscr{Y}_{1,2}$ -($\mathscr{Y}_{2,2}$)- primary component of$U([(I'_1 + I'_2)\bar{R}]_1)$ In fact, $q_{1,2} = (X_0, Y_0, \ell_0, \ell_1), q_{2,2} = (X_1, Y_1, \ell_0, \ell_1)$

(iv) Let C_1 and C_2 be irreducible components of $V_1 \cap V_2$ corresponding to the prime ideals $\mathscr{Y}_{1,2}$ and $\mathscr{Y}_{2,2}$. Then the defining prime ideals of C_1 and C_2 are (X_0) and (X_1),respectively and $j(V_1, V_2; C_1)$ = length of $q_{1,2} = 1$, $j(V_1, V_2; C_2)$ = length of $q_{2,2} = 1$,,

Note that $t < d$: therefore $\mathscr{O}_1 \neq \overline{R}$.

Following the proof of step II is Chapter II we get:

(v) $q := U(\mathscr{O}_1, \ell_2) = (\mathscr{O}_1, \ell_2)$ is $\mathscr{Y}_{1,1}$ - primary ideal. Let C_3 be the irreducible component of $V_1 \cap V_2$ corresponding to the prime ideal $\mathscr{Y}_{1,1}$. Then the defining prime ideal of C_3 is (X_0, X_1) and $j(V_1, V_2; C_3)$ = length of (q).

(vi) (a) The required collection $\{C_i\}$ is: **98**

$C_1 : X_0 = 0$ (the line ℓ_0) with $j(V_1, V_2; C_1) = 1$

$C_2 : X_1 = 0$ (the line ℓ_1) with $j(V_1, V_2; C_2) = 1$

$C_3 : X_0 = X_1 = 0$(the imbedded point P) with $j(V_1, V_2; C_3) = \ell(q)$

(b) From (1.35) , we have $\deg(V_1) = \deg(V_2) = 3$ and $\deg C_i = 1$ for all $i = 1, 2, 3$. **99**

(c) from the main theorem (2.1) , we get $9 = \deg(V_1) \cdot \deg(V_2) = \sum_{i=1}^{3} j(V_1, V_2; C_i) \deg(C_i) = 2 + j(V_1, V_2; C_3)$.

Therefore $\ell(q) = j(V_1, V_2; C_3) = 7$

This example was also studied by W. Fulton and R. MacPherson in [[19], p.10]

(3.8) Example. Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals $I_1 = (X_4, X_3^3 - X_1X_2(X_2 - 2X_1))$ and $I_2 = (X_3, X_4^3 - X_1X_2(X_1 - 2X_2)) \subset K[X_0, X_1, X_2, X_3, X_4]$. Following the proof of the main theorem (2.1) it is easy to see that:

(a)The required collection $\{C_i\}$ is given by:.

$C_1 : X_1 = X_3 = X_4 = 0$ (the line) with $j(V_1, V_2; C_1) = 1$

$C_2 : X_1 = X_3 = X_4 = 0$ (the line) with $j(V_1, V_2; C_2) = 1$

$C_3 : X_1 = X_2 = X_3 = X_4 = 0$ (the imbedded point) with $j(V_1, V_2; C_3) = 7$.

(b) From (1.35) we have $\deg(V_1) = \deg(V_2) = 3$ and $\deg(C_i) = 1$for $i = 1, 2, 3$. Therefore

$$9 = \deg(V_1)\deg(V_2) = \sum_{i=1}^{3} j(V_1, V_2; C_i)\deg(C_i).$$

(3.9) Remark. The example (3.8) was also studied by W. Fulton and R. MacPherson, (see [[19],p.10]) This example illuminates our problem 6 below as follows:

Use the diagram for $X := V_1$ and $Y := V_2$

$$\begin{array}{ccc} X \cap Y & \longrightarrow & \Delta \\ \downarrow & & \downarrow \\ X \times Y & \longrightarrow & \mathbb{P}^4 \times \mathbb{P}^4 \end{array}$$

the origin P is a so-called distinguished variety in the theory of Fulton and MacPherson, its contribution to the multiplicity is 3, each line also contributes 3 to the Bezout's number $\deg(X).\deg(Y) = 9$.

In view of the problem 6 below, we want to consider another diagram. Using the diagram

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \Delta_{\mathbb{P}^4} & \longrightarrow & \mathbb{P}^4 \times \mathbb{P}^4 \end{array}$$

then we get the intersection numbers 7,1,1 (7 at the point P) by apply-
100 ing the theory of Fulton and MacPherson. Our method also assigns the multiplicity 7 to the origin P and 1 to each line.

A simpler example in $\mathbb{P}^2_K$ is the following: Let X and Y be given by $X_1X_2 = 0$ and $X_1 = 0$,resp. Then $X \cap Y$ is the line $X_1 = 0$. Applying again the theory of Fulton and MacPherson, we will construct intersection from the diagram

$$\begin{array}{ccc} X \cap Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}^2_K \end{array}$$

Then only Y is a distinguished variety and counts twice. Construct intersection from the diagram

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \Delta_{\mathbb{P}^2} & \longrightarrow & \mathbb{P}^2 \times \mathbb{P}^2 \end{array}$$

Then Y and the origin are distinguished varieties each contributes 1. In our theory, the origin is also a so-called distinguished variety and its contribution to the multiplicity is 1, the line $X_1 = 0$ also contributes 1 to the Bezout's number $\deg(X)\deg(Y) = 2$.

Therefore we want to study the following example:

(3.10) Example. Let V_1 and V_2 be two hypersurfaces in $\mathbb{P}^2_K$ defined by 101
$F_1 := X_1X_2$ and $F_2 := X_1^n = 0$

Following the proof of the main theorem (2.1) , it is easy to see that:

(a) The required collection $\{C_i\}$ is:

$C_1 : X_1 = 0$(the line) with $j(V_1, V_2; C_1) = n$

$C_2 : X_1 = X_2 = 0$(the imbedded points) with $j(V_1, V_2; C_2) = n$

(b) From ((1.35)), we have $\deg(V_1) = 2$, $\deg(V_2) = n$ and $\deg(C_i) = 1$ for $i = 1, 2$.

Therefore $2n = \deg(V_1).\deg(V_2) = \sum\limits_{i=1}^{2} j(V_1, V_2; C_i)\deg(C_i)$

(3.11) Example. Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two subvarieties in $\mathbb{P}^n_K$ with $V_1 \cap V_2 = \phi$ (for example,the lines $\ell_0 : X_0 = X_1 = 0$ and $\ell_1 : X_2 = X_3 = 0$ in $\mathbb{P}^3_K$). Following the proof of the main theorem (2.1) , it is easy to see that: $d = 0, \delta - d - 1 = \delta - 1 \leq n$ and $U([I_1' + I_2']_{\delta-1}) = q$ is primary for the homogeneous maximal ideal $(X_0, \ldots, X_n, Y_0, \ldots, Y_n)$. Therefore the required collection $\{C_i\}$ is just the empty subvariety ϕ and by the main Theorem (2.1)), we get

$$j(V_1, V_2; \phi) = \deg(V_1) \cdot \deg(V_2).$$

(3.12) Remark. Let $V_1 and V_2$ be as in example (3.11). Let $C(V_1)$ (resp. $C(V_2)$) be the projective cone over V_1 (resp. over V_2) in $\mathbb{P}_k^{n+1}$. Then $C(V_1) \cap C(V_2)$ is given by one point say $P : X_0 = \cdots = X_n = 0$. It is possible to show that

$$j(V_1, V_2; \phi) = j(C(V_1), C(V_2); P).$$

102 which does provided a geometrical interpretation of the intersection number $j(V_1, V_2; \phi)$.

(3.13)

Let V_1, V_2 and $V_3 \subset \mathbb{P}_K^3$ be three hypersurfaces defined by $F_1 := X_0X_1$, $F_2 : X_0X_2$ and $F_3 : X_0X_3$, respectively. Put $I_1 = (F_1), I_2 = (F_2)$ and $I_3 = (F_3)(\subset K[X_0, X_1, X_2, X_3])$. It is easy to see that:

(i) The primary decomposition of $I_1 + I_2 + I_3$ is given by $I_1 + I_2 + I_3 = (X_0) \cap (X_1, X_2, X_3)$ and $\text{Rad}(I_1 + I_2 + I_3) = (X_0) \cap (X_1, X_2, X_3)$. Therefore the intersection $V_1 \cap V_2 \cap V_3$ is precisely one surface $C : X_0 = 0$ and the isolated point $P : X_1 = X_2 = X_3 = 0$.

Note that we cannot apply the main theorem (2.1) to this example but we can apply the general main theorem (2.25) . We preserve the notation of [56]. For simplicity, put $X_{ij} = X_j, X_{2j} = Y_j and X_{3j} = Z_j$ for $j = 0, \ldots, 3$.

(ii) $\text{Rad}((I'_1 + I'_2 + I'_3)\overline{R} + \tau\overline{R}) = \mathscr{Y}_{1,3} \cap \mathscr{Y}_{1,1}$ where $\mathscr{Y}_{1,3} = (X_0, Y_0, Z_0, X_1 - Y_1, X_1 - Z_1, X_2 - Z_2, X_3 - Y_3, X_3, -Z_3)$ and

$\mathscr{Y}_{1,1} = (X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3, X_0, -Y_0, X_0, -Z_0)$ and

$\delta = \dim V_1 + \dim V_2 + \dim V_3 + 3 = 2 + 2 + 2 + 3 = 9,$

$\delta = \dim(V_1 \cap V_2 \cap V_3) + 1 = 3, t = 1$. Therefore $\delta - d - 1 = 9 - 3 - 1 = 5$

Following the proof of Step I of [[56], (2.1)], we get:

(iii) $U([(I'_1 + I'_2 + I'_3)\overline{R}]_5) = q_{1,3} \cap \mathscr{O}_1$ where $q_{1,3} = (X_0, \underline{Y_0}, Z_0, \ell_0, \ldots, \ell_5)$ is the $\mathscr{O}_{1,3}$ primary component of $U([(I'_1 + I'_2 + I'_3)\overline{R}]_5)$

103 (iv) Let C_1 be the irreducible component of $V_1 \cap V_2 \cap V_3$ corresponding to the prime ideal $\mathscr{O}_{1,3}$. Then the defining prime ideal of C_1 is (X_0) and $j(V_1, V_2, V_3; c_1)$ = length of $q_{1,3} = 1$. Following the proof of Step II of [56], (2.1) , we get:

(v) $\cup(\mathscr{O}_1, \ell_6) = q_1^1 \cap q_1^2 \cap q_1^3 \cap \mathscr{O}_2$ where q_1^1 (resp. q_1^2, q_1^3) is $(X_1, Y_0, Z_0) + \tau\overline{R}$ (resp. $(X_0, Y_2, Y_0) + \tau\overline{R}(X_0, Y_0, Z_3) + \tau\overline{R}$)-primary components of $U(\mathscr{O}_1, \ell_6)$. Let C_2, C_3andC_4 be irreducible subvarieties of $V_1 \cap V_2 \cap V_3$ corresponding to the prime ideals $(X_1, Y_0, Z_0) + \tau\overline{R}(X_0, Y_2, Z_0) + \tau\overline{R}$ and $(X_0, Y_0, Z_3) + \tau\overline{R}$ respectively. Then the defining prime ideals of C_2, C_3 and C_4 are $(X_0, X_1), (X_0, X_2)$ and (X_0, X_3), respectively and $j(V_1, V_2, C_i) = 1$ for $i = 2, 3, 4$.

(vi) $\cup(\mathscr{O}_2, \ell_7) = q_1^1 \cap q_1^2 \cap q_1^3 \cap q_1^4$ where
$q_1^1 = (X_0, Y_2, Z_3) + \tau\overline{R} q_1^2 = (X_1, Y_0, Z_3) + \tau\overline{R} q_1^3 = (X_0, Y_2, Z_0) + \tau\overline{R}$ and

$q_1^4 = (X_0, Y_2, Z_3) + \tau\overline{R}$

Note that $d - t + 1 = 3$ and $\mathscr{O}_{d-t+1} = \mathscr{O}_3 = \overline{R}$. Let C_5, C_6, C_7 and C_8 be irreducible. subvarieties of $V_1 \cap V_2 \cap V_3$ corresponding to q_1^1, q_1^2, q_1^3 and q_1^4, respectively. Then the defining prime ideals of C_5, C_6, C_7 and C_8 are $(X_0, X_2, X_3), (X_0, X_1, X_3), (X_0, X_1, X_2)$, and (X_1, X_2, X_3) respectively and $j(V_1, V_2, V_3, C_i) = 1$ for $i = 5, 6, 7, 8$.

(vii) (a) The required collection $\{C_i\}$ is:

$$C_1 : X_0 = 0 \quad \text{(the surface)}$$
$$C_2 : X_0 = X_1 = 0$$
$$C_3 : X_0 = X_2 = 0$$
$$C_4 : X_0 = X_3 = 0$$
$$C_5 : X_0 = X_2 = X_3 = 0$$
$$C_6 : X_0 = X_1 = X_3 = 0$$
$$C_7 : X_0 = X_1 = X_2 = 0$$
$$C_8 : X_1 = X_2 = X_3 = 0 \quad \text{(the isolated point)}$$

and $j(V_1, V_2; C_i) = 1$ for all $i = 1, \ldots, 8$. 104

(b) From (1.35) , we have

$\deg(V_i) = 2$ for all $i = 1, 2, 3$ and $\deg(C_i) = 1$ for all $i = 1, \ldots, 8$.

Therefore we get $8 = \prod_{i=1}^{3} \deg(V_i) = \sum_{i=1}^{8} j(V_1, V_2; C_i) \deg(C_i)$.

(3.14) Examples. (i) Let $V_1 = V(I_1)$ and $V_2 = V(I_2) \subset \mathbb{P}^7_K$ be defined by $I_1 = (X_0, X_1) \cap (X_2, X_3) \cap (X_4, X_5) \cap (X_6, X_7)$ and $I_2 = (X_0 + X_2, X_4 + X_6)$. Then the primary decomposition of $I_1 + I_2$ is given by

$$\begin{aligned} I_1 + I_2 = (X_0, X_1, X_2, X_4 + X_6) \cap (X_0, X_2, X_3, X_4 + X_6) \\ \cap (X_4, X_5, X_6, X_0 + X_2) \cap (X_4, X_6, X_7, X_0 + X_2) \\ \cap (X_0^2, X_2^2, X_0 + X_2, X_1, X_3, X_4 + X_6)(X_4^2, X_6^2, X_4 + X_6, X_5, X_7, X_0, +X_2). \end{aligned}$$

Using the notation of (2.30) we have

$$m = 4, \ell = 6.$$

We also have $3 = \dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - 7 = 5 +$
105 $5 - 7 = 3$ and from (1.40) and (1.35) $\deg(V_1) = 4, \deg(V_2) = 1$. Therefore $4 = \deg(V_1) \deg(V_2) \ngeq \ell - 1 = 5$. This example shows that Question 1 of (2.30) is not true in general.

(ii) Let $V_1 = V(I_1)$ and $V_2 = V(I_2) \subset \mathbb{P}^4_K$ be defined by $I_1 = (X_0, X_1) \cap (X_2, X_3) \cap (X_0 + X_2, X_4)$ and $I_2 = (X_0 + X_2)$. Then the primary decomposition of $I_1 + I_2$ is given by

$$\begin{aligned} I_1 + I_2 = (X_0 + X_2, X_4) \cap (X_0, X_1, X_2) \cap (X_0, X_2, X_3) \\ \cap (X_0^2, X_2^2, X_0 + X_2, X_1, X_3). \end{aligned}$$

Therefore $m = 1$ and $\ell = 4$

Also $2 = \dim(V_1 \cap V_2) > \dim V_1 + \dim V_2 - 4 = 2 + 3 - 4 = 1$ and $\deg(V_1) = 3, \deg(V_2) = 1$ by (1.39) and (1.34). Therefore $3 = \deg(V_1) \deg(V_2) \ngeq \ell = 4$.

This example shows that Question 2 of (2.30) is not true in general.

(iii) Let $V_1 = V(I_1)$ and $V_2 = V(I_2) \subset \mathbb{P}^7_K$ be defined by $I_1 = (X_0^2, X_1) \cap (X_2, X_3) \cap (X_4, X_5) \cap (X_6, X_7)$ and $i_2 = (X_0 + X_2, X_4 + X_6)$. The primary decomposition of $I_1 + I_2$ is given by

$$\begin{aligned} I_1 + I_2 = (X_0^2, X_1, X_0 + X_2, X_4 + X_6) \cap (X_0, X_2, X_3, X_4 + X_6) \\ \cap (X_4, X_5, X_6, X_0 + X_2) \cap (X_4, X_6, X_7, X_0 + X_2) \cap \\ (X_0^3, X_0^3, X_1, X_3, X_0 + X_2, X_4, +X_6) \cap (X_4^3, X_6^2, X_5, X_7, X_0 + X_2, X_4 + X_6) \end{aligned}$$

Therefore $m = 4$ and $\ell = 6$. From (1.40) and (1.35) we have $\deg(V_1) = 5, \deg(V_2) = 1$. Therefore $\deg(V_1) \cdot \deg(V_2) > m$ but $\deg(V_1) \cdot \deg(V_2) \not\geq \ell$. This example shows that Question 3 of (2.30) is not true in general.

B. Applications of the Main Theorem

The purpose of this section is to show that the main theorem (2.1) also yields Bezout's Theorem. **106**

We preserve the notation of Chapter II. In addition, the following notation will be used in sequel.

(3.15) Notation. Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $R_0 := K[X_0, \ldots, X_n]$. Let C be an irreducible component of $V_1 \cap V_2$ with the defining prime ideal $\mathscr{Y}(C) = \mathscr{Y}$. Let $q(C) = q$ be the primary component of $I_1 + I_2$. we put:

$$\ell(V_1, V_2; C) := \text{ length of } q = \ell((R_0/I_1 + I_2)\mathscr{Y}) \text{ and}$$
$$e(V_1, V_2; C) := e_0(q(R_0)_{\mathscr{Y}}; (R_0)_{\mathscr{Y}}).$$

Using (2.3) (iii), we get the isolated prime ideal P(resp. $\bar{P}$) of $(I_1' + I_2' + \tau)R$ (resp. $(I_1' + I_2' + \tau)\bar{R}$)$-$ Let Q (resp. $\bar{Q}$) be the P (resp. $\bar{P}$)$-$primary component of $(I_1' + I_2' + \tau)\, R$ (resp. $(I_1' + I_2' + \tau)\bar{R}$). Let $A = (R/(I_1' + I_2'))_P$ (resp. $\bar{A} = (\bar{R}/(I_1' + I_2'))_{\bar{P}}$ be the local ring of the join-variety $J(V_1, V_2)$ (resp. $\bar{J}(V_1, V_2)$) at P (resp. $\bar{P}$).

(3.16) Remarks. (i) Note that $\bar{P} = \mathscr{Y}_{i,j}$ for some prime ideal $\mathscr{Y}_{i,j}$ in $(*)$ of (2.4), where $j = K - \dim(C)$.

(ii)
$$e_0(\tau A; A) = e_0(QA; A) \text{ and}$$
$$e_0(\tau\bar{A}; \bar{A}) = e_0(\bar{Q}\bar{A}; A)$$

(iii)
$$e_0(\tau A; A) = e_0(\tau\bar{A}; \bar{A}) \text{ and}$$
$$e_0(QA; A) = e_0(\bar{Q} \cdot \bar{A}; \bar{A})$$

107 *Proof.* (ii) We have $(QA)^n = ((I'_1 + I'_2 + \tau)A)^n = (\tau A)^n$ for all $n \geq 1$, so that $\ell(A/(QA)^n) = \ell(\bar{A}/(\tau\bar{A})^n)$ for all $n \geq 1$. Therefore $E_0(QA, A) = e_0(\tau A, A)$. Similarly, $e_0(\bar{Q}.\bar{A}; \bar{A}) = e_0(\tau\bar{A}; \bar{A})$.

(iii) This follows from the remark (i) of (1.2) . □

(3.17) Remark. If C is a proper component of $V_1 \cap V_2$, that is, C is irreducible and $\dim(C) = \dim(V_1) + \dim(V_2) - n$,then the *Weil's intersection multiplicity symbol* $i(V_1, V_2; C)$ of V_1 and V_2 along C is given by
$$i(V_1, V_2; C) = e_0(\tau A; A)$$

Proof. See [[69] ; ch. II, §5, a] □

(3.18) Lemma. *Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $R_0 := K[X_0, \dots, X_n]$. Let C be an irreducible component of $V_1 \cap V_2$. Then $j(V_1, V_2; C) = e_0((\ell_0, \dots, \ell_{\delta-j-1})\bar{A}; \bar{A}) = \ell_0, \dots, \ell_{\delta-j-1})\bar{A}) - \ell(\dfrac{(\ell_0, \dots, \ell_{\delta-j-2}) : \ell_{\delta-j-1}}{(\ell_0, \dots, \ell_{\delta-j-2})\bar{A}})$ where $j = K - \dim(C)$.*

Proof. This follows from (2.20), (2.8) and (1.23). □

(3.19) Corollary.

$$J(V_1, V_2; C) = \ell(V_1, V_2; C) + \ell(\tau\bar{A}/(\ell_0, \dots, \ell_{\delta-j-1})\bar{A}) - \ell(\frac{(\ell_0, \dots, \ell_{\delta-j-2}) : \ell_{\delta-j-1}}{(\ell_0, \dots, \ell_{\delta-j-2})\bar{A}})$$

108 *Proof.* We have $\ell(V_1, V_2; C) = \ell(\bar{A}/\tau\bar{A})$. Therefore this corollary follows from (3.18). □

We put

$$K_1 := \ell(\tau\overline{A}/(\ell_0, \dots \ell_{\delta-j-1})A)$$

and

$$K_2 := \frac{((\ell_0, \dots, \ell_{\delta-j-2}) : \ell_{\delta-j-1})\overline{A}}{(\ell_0, \dots, \ell_{\delta-j-2})\overline{A}}$$

(3.20) Corollary. *(i) Suppose that the local rings $A(V_i; C)$ of V_i at C are Cohen-Macaulay for $i = 1, 2$. Then $j(V_1, V_2; C) \geq \ell(V_1, V_2; C)$ and equality holds if and only if $\tau\overline{A} \subset (\ell_0, \dots, \ell_{\delta-j-1})\overline{A}$*

(ii) If $\tau\overline{A} \subset (\ell_0, \dots, \ell_{\delta-j-1})\overline{A}$ then $j(V_1, V_2; C) \leq \ell(V_1, V_2; C)$ and equality holds if and only if $A(V_1; C)$ and $A(V_2; C)$ are Cohen-Macaulay. (Note that $\tau\overline{A} \subset (\ell_0, \dots, \ell_{\delta-j-1})\overline{A}$ when $\dim C = \dim V_1 + \dim V_2 - n$*).*

Proof. (i) From the proposition (3.21) below, it follows that $\overline{A}$ is Cohen -Macaulay. Therefore (i) results from (3.19).

(ii) Follows from (3.19) and the following proposition (3.21).

We study the connecting between the Cohen -Macaulay properties of $A(V_i; C)$ and $\overline{A}$ in the following proposition. ◻

(3.21) Proposition. *The notations being the same as (2.2) and (3.18). The following conditions are equivalent:*

(i) $A(V_1; C)$ and $A(V_2; C)$ are Cohen -Macaulay **109**

(ii) $\overline{A}$ is Cohen-Macaulay.

(iii) $(\overline{K}[X_{10}, \dots, X_{1n}, X_{20}, \dots, X_{2n}]/(I_1' + I_2'))_{\mathscr{Y}(C)' + \mathscr{Y}(C)''}$ is Cohen-Macaulay, where $\mathscr{Y}(C)'$; and $\mathscr{Y}(C)''$ are prime ideals in R_1 and R_2, respectively corresponding to $\mathscr{Y}(C)$.

Proof. (ii) ⇒ (iii). Since $\mathscr{Y}(C)' + \mathscr{Y}(C)'' \subset \mathscr{Y} + \tau$, the local ring of (iii) is a localization of $\overline{A}$ and hence Cohen -Macaulay.

(iii) ⇒ (i). This is proved by *R*. Achilles. This proof is not so easy, since he uses Samuel's techniques on the extension of fields of definition (see [[69], ch. II, §1, No. 3 and 4]). Therefore, for the proof, see the forthcoming thesis (Promotion *B*) of R. Achilles.

(i) $\Rightarrow$ (ii). By (1.44), there exist elements $f \in A(V_1)\, g \in A(V_2)$ such that $A(V_1)_f$ and $A(V_2)_g$ are Cohen-Macaulay ($A(V_i)$ denote the coordinates ring of $V_i, i = 1, 2$). It follows immediately from (1.47) (iv)(a) that

$$\square \qquad A(V_1)_f \underset{K}{\otimes} A(V_2)_g \text{ is Cohen -Macaulay.}$$

Now, put $S = \{f^n \underset{K}{\otimes} g^m | n, m \in \mathbb{N}\}$. Then S is a multiplicative set in $A(V_1) \underset{K}{\otimes} A(V_2)$ and it is easy to see that $S^{-1}(A(V_1) \underset{K}{\otimes} A(V_2)) \xrightarrow{\sim} A(V_1)_f \underset{K}{\otimes} A(V_2)_g$. Therefore

$$\begin{aligned} &\bar{K} \underset{K}{\otimes} A(V_1))_f \underset{\bar{K}}{\otimes} (\bar{K} \underset{K}{\otimes} A(V_2))_g \xrightarrow{\sim} (\bar{K}[X_{10}, \ldots, X_{1n}]/I_1')_f \underset{\bar{K}}{\otimes} \\ &(\bar{K}[X_{20}, \ldots, X_{2n}])/I_2')_g \xrightarrow{\sim} (\bar{K}[X_{10}, \ldots, X_{1n}, x_{20}, \ldots, X_{2n}]/(I_1' + I_2'))_{fg} \end{aligned}$$

110 is Cohen-Macaulay. Note that $f.g \notin \mathscr{Y}(C) + \tau$, therefore $\bar{A}$ is a localization of $(\bar{K}[X_{10}, \ldots, X_{1n}, X_{20}, \ldots, X_{2n}]/I_1' + I_2')_{fg}$ and hence $\bar{A}$ is Cohen -Macaulay.

(3.22) Proposition. *Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $R_0 = K[X_0, \ldots, X_n]$. Let C be an irreducible component of $V_1 \cap V_2$. Then*

$$j(V_1, V_2; C) = e_0(\tau\bar{A}, \bar{A}) = e_0(\bar{Q}.\bar{A}; \bar{A}).$$

In particular, if C is a proper component of $V_1 \cap V_2$, that is, $\dim(C) = \dim(V_1) + \dim(V_2) - n$, *then*

$$j(V_1, V_2; C) = e_0(\tau\bar{A}, \bar{A}) = i(V_1, V_2; C).$$

Proof. In view of (3.16) (ii) and (3.18), it is enough to prove that : $e_0((\ell_0, \ldots, \ell_{\delta-j-1})\bar{A}; \bar{A}) = e_0(\tau\bar{A}; \bar{A})$. Since $\ell_0, \ldots, \ell_{\delta-j-1}$ are generic linear forms we see from the proof of [[51], Theorem [69]] that $\ell_0, \ldots, \ell_{\delta-j-1}$ is a "superficial sequence" of order 1 for $\tau\bar{A} = (\ell_0, \ldots, \ell_n)\bar{A}$. Therefore from [[51], Theorem [71]] we get

$$\square \qquad e_0((\ell_0, \ldots, \ell_{\delta-j-1})\bar{A}; \bar{A}) = e_0((\ell_0, \ldots, \ell_n)\bar{A}; \bar{A}) = e_0(\tau\bar{A}; \bar{A}).$$

(3.23) Remark. Proposition (3.22) does yield a connection between our definition of intersection multiplicity and Samuel's observations on improper components given in his thesis (see: *J*. Math.Pures Appl. (9), 30 (1951)-274 in particular chapter *V*, section 2), see also [[69], ch. II, §5, No. 9].

(3.24) Proposition . *Let* $V_1 = V(I_1)$ *and* $V_2 = V(I_2)$ *be two pure dimensional projective varieties in* $\mathbb{P}^n_K$ *defined by homogeneous ideals* I_1 *and* I_2 *in* $R_0 := K[X_0, \ldots, X_n]$. *Suppose that* $\dim(V_1 \cap V_2) =$ **111**
$\dim(V_1) + \dim(V_2) - n$. *Then* $\deg(V_1) \cdot \deg(V_2) = \sum_C j(V_1, V_2; C) \deg(C)$, *where C runs through all irreducible components of* $V_1 \cap V_2$.

For the proof of this proposition, we need the following lemma.

(3.25) Lemma . *Let* $V_1 = V(I_1)$ *and* $V_2 = V(I_2)$ *be two pure dimensional projective varieties in* $\mathbb{P}^n_K$ *defined by homogeneous ideals* I_1 *and* I_2 *in* $R_0 := K[X_0, \ldots, X_n]$. *Then the following conditions are equivalent:*

(i) $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - n$

(ii) *Every irreducible component of* $V_1 \cap V_2$ *has dimension* $\dim V_1 + \dim V_2 - n$.

(iii) $\dim((I'_1 + I'_2 + \tau\overline{R}) = \dim((I'_1 + I'_2)\overline{R}) - (n + 1)$.

Proof. We prove (i) ⇒ (iii) ⇒ (ii) □

From (i) we have $\dim((I'_1 + I'_2 + \tau)\overline{R}) = \dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - n = \dim((I'_1 + I'_2)\overline{R} - 1 - n$, that is, we have (iii).

(iii) ⇒ (ii) Follows from the fact that $(I'_1 + I'_2)\overline{R}$ is unmixed (see (2.3)(i)) and (1.46) (ii). (ii) ⇒ (i) is trivial.

Proof of Proposition (3.24). From Lemma (3.25), we have $t = d$ and $\delta - d - 1 = n$. Hence we get from the Step I of our proof of the main **112**
theorem (2.1) that

$$\cup([(I'_1 + I'_2)\overline{R}]_n) = q_{1,d} \cap \cdots \cap q_{m_d,d} \cap \mathscr{O}_1.$$

Therefore, since $\mathcal{O}_1 \not\subset_{i,d}$ for any $1 \leq i \leq m_d$, we get $\mathcal{O} = \bar{R}$. Hence $\{C | C$ an irreducible component of $V_1 \cap V_2\}$ is the required collection in the main theorem (2.1) . Then (3.24) follows from the main theorem (2.1) .

The following proposition yields a new and simple proof of the well-known Bezout's theorem.

(3.26) Proposition (Bezout's Theorem)

Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $R_0 := K[X_0, \ldots, X_n]$. Suppose that $\dim(V_1 \cap V_2) = \dim(V_1 + \dim(V_2) - n$. Then $\deg(V_1).\deg(V_2) = \sum_C i(V_1, V_2; C) \cdot \deg(C)$ where C runs through all irreducible components of $V_1 \cap V_2$.

Proof. Follows from (3.24) and (3.22). □

(3.27) Remark. In [72] Serre gave an elegant formula for the intersection multiplicity with correction terms to the naive guess which takes only the length of primary ideals (see our discussion of chapter 0, section A). In a sense this Tor-formula of Serre explains why the naive guess fails. Another explanation is given by W. Fulton and R. MacPherson in [[19], §4]. Also, our approach does give the reason for this phenomenon. Our correction term is given by K_2 (See the notation after the proof of Corollary (3.19)). Roughly speaking, our construction
113 shows that we have to drop the imbedded components. Furthermore, we open the way to deeper study by applying our results (3.19) and (3.22). For example, it follows immediately from Corollary (3.20) (ii) the well-known fact that

$$\deg(V_1) \cdot \deg(V_2) \leq \deg(V_1 \cap V_2)$$

when $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - n$. In case $\dim(V_1 \cap V_2) > \dim V_1 + \dim V_2 - n$, we obtain the following results.

(3.28) Proposition. *Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $R_0 := K[X_0, \dots, X_n]$. Assume that the local rings $A(V_1; C)$ and $A(V_2; C)$ of V_1 and V_2 at C are Cohen -Macaulay for all irreducible components C of $V_1 \cap V_2$ with $\dim C = \dim(V_1 \cap V_2)$. Then*

$$\deg(V_1) \cdot \deg(V_2) \geq \deg(V_1 \cap V_2)$$

Proof. From the main theorem (2.1) , we get:

$$\begin{aligned}\deg(V_1) \cdot \deg(V_2) &= \sum_{C_i} j(V_1, V_2; C_1) \cdot \deg(C_i) \\ &\geq \sum_{C} j(V_1, V_2; C) \deg(C) \\ &\geq \sum_{C} \ell(V_1, V_2; C) \deg(C) \quad \text{by} \quad (3.20)\ (i) \\ &= \deg(V_1 \cap V_2) \qquad \text{by } (1.40)\end{aligned}$$

where C runs through all irreducible components of $V_1 \cap V_2$. with **114**
$\dim(C) = \dim(V_1 \cap V_2)$. □

(3.29) Corollary. *With the same assumption as in (3.28), we have* $\deg(V_1) \cdot \deg(V_2) - \deg(V_1 \cap V_2) \geq \sum_C \deg(C) \geq$ *number of irreducible components of* $V_1 \cap V_2$ *with* $\dim C < \dim(V_1 \cap V_2)$.

(3.30) Proposition. *Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_k$ defined by homogeneous ideals I_1 and I_2 in $R_0 := K[X_0, \dots, X_n]$. Assume that the local rings $A(V_1; C)$ and $A(V_2; C)$ of V_1 and V_2 at C are Cohen-Macaulay for all irreducible components C of $V_1 \cap V_2$. Then* $\deg(V_1) \cdot \deg(V_2) \geq \sum_C \ell(V_1, V_2, ; C) \deg(C)$, *where C runs through all irreducible components of $V_1 \cap V_2$, and equality holds if and only if $j(V_1, V_2; C) = \ell(V_1, V_2 : C)$ for all irreducible components C of $V_1 \cap V_2$ and $\{C_i\} = \{C | C$ an irreducible component of $V_1 \cap V_2\}$.*

(3.31) Corollary. *With the same assumption as in (3.30), we have* $\deg(V_1) \cdot \deg(V_2) - \deg(V_1 \cap V_2) \geq \sum \ell(V_1, V_2; C) \deg(C)$, *where C runs* *through all irreducible components of* $V_1 \cap V_2$ *with* $\dim(C) < \dim(V_1 \cap V_2)$.

C. Problems

Let $V_1 = V(I_1)$ and $V_2 = V)I_2)$ be two pure dimensional projective varieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $R_0 := K[X_0, \ldots, X_n]$.

(3.32) The Main Problem

115 Analyzing the proof of the main theorem (2.1) and the example (3.14), one might be tempted to ask the following question:

Let $C \subset V_1 \cap V_2$ be an irreducible subvariety corresponding to an imbedded prime ideal $\mathscr{Y}$ belonging to $I_1 + I_2$. If $\dim C \geq \dim V_1 + \dim V_2 - n$, then C belongs to our collection $\{C_i\}$ of the main theorem (2.1) . However, this is not so, as we will show by the following example:

The construction is due to *R.* Achilles.

Let V_1 and V_2 ve two surfaces in $\mathbb{P}^4_K$ given by the following ideals

$$\begin{aligned} I_1 &= (X_0, X_1) \cap (X_0, X_2) \cap (X_2, X_3) \text{ and} \\ I_2 &= (X_1, X_4) \cap (X_0^2, X_0 + X_2). \end{aligned}$$

Then we have the following primary decomposition of $I_1 + I_2$: $I_1 + I_2 = (X_0, X_2) \cap (X_0, X_1, X_4) \cap (X_1, X_2, X_3, X_4) \cap (X_0X_2, X_0 + X_2, X_1, X_3)$.

Applying proposition (1.46), (ii) it is not too difficult to show that the collection $\{C_i\}$ of irreducible subvarieties of $V_1 \cap V_2$ is given by:

$C_1 : X_0 = X_2 = 0$	with	$j(V_1, V_2; C_1) = 2$
$C_2 : X_0 = X_1 = X_4 = 0$	with	$j(V_1, V_2; C_2) = 1$
$C_3 : X_0 = X_1 = X_2 = 0$	with	$j(V_1, V_2; C_3) = 2$
$C_4 : X_0 = X_2 = X_3 = 0$	with	$j(V_1, V_2; C_4) = 2$
$C_5 : X_0 = X_1 = X_2 = X_4 = 0$	with	$j(V_1, V_2; C_5) = 1$
$C_6 : X_1 = X_2 = X_3 = X_4 = 0$	with	$j(V_1, V_2; C_6) = 1$

Therefore, the imbedded point $X_0 = X_1 = X_2 = X_3 = 0$ of $V_1 \cap V_2$ **116**
is not an element of $\{C_i\}$.

Note that, the correction term $c(V_1, V_2) > 0$ (See notation of (2.8), (ii)).

Of course, it would be very interesting to say something about how imbedded components contribute to the Bezout's number $\deg(V_1) \cdot \deg(V_2)$. This is our main problem.

(3.33) Problem 1. Give reasonably sharp estimate between $\deg(V_1) \cdot \deg(V_2)$ and $\sum\limits_C j(V_1, V_2; C) \deg(C)$ where C runs through all irreducible components of $V_1 \cap V_2$

(3.34) Remark. It seems to us a rather hard question to give reasonably sharp estimates on the error term between $\deg(V_1) \deg(V_2)$ and $\sum\limits_C j(V_1, V_2; C) \deg(C)$ or even $\sum\limits_C \deg(C)$.

In 1982, Lazarsfeld was able to show that if one intersects a linear space L in $\mathbb{P}^n_K$ with a subvariety $V \subset \mathbb{P}^n_K$(V is irreducible and non- degenerate) and $C_1, \ldots, C_r$ are the irreducible components of $L \cap V$, then

$$\sum_{i=1}^{r} \deg C_i \leq \deg(V) - e$$

where $e = \dim(L \cap V) - \dim V + \operatorname{codim} L$.

His proof is rather complicated. Therefore we want to pose our **117**
second problem.

(3.35) Problem 2. Would our methods yield similar results as in (3.34)?

(3.36) Problem 3. (a) If $c(V_1, V_2) > 0$ then is it true that: $\deg(V_1) \cdot \deg(V_2) \geq$ number of associated primes of $I_1 + I_2$?

(b) Assume that V_1 and V_2 are reduced. If $\deg(V_1) d \cdot \deg(V_2) >$ number of irreducible components of $V_1 \cap V_2$ then is it true that:

$\deg(V_1) \cdot \deg(V_2) \geq$ number of associated primes of $I_1 + I_2$?

(3.37) Remark. The assumption "reduced" is necessary in Problem 3(b) (see example (3.14) (iii)).

(3.38) Problem 4. Let C be an irreducible component of $V_1 \cap V_2$ with $K-\dim(C) = j, t \leq j \leq d$. Give geometric (or algebraic) interpretation of the property:

$$\tau\overline{A} \subset (\ell_0, \ldots, \ell_{\delta-j-1})\overline{A}.$$

In connection with Corollary (3.20), we want to pose the following problem:

(3.39) Problem 5. Let C be an irreducible component of $V_1 \cap V_2$. If $j(V_1, V_2; C) \geq \ell(V_1, V_2; C)$ or $j(V_1, V_2; C) \leq \ell(V_1, V_2; C)$,then describe the structure of the local rings $A(V_1; C)$ and $A(V_2; C)$ of V_1 and V_2 at C.

118 **(3.40) Remarks.** (i) If $A(V_1, C)$ and $A(V_2; C)$ are Cohen -Macaulay. then $j(V_1, V_2; C) \geq \ell(V_1, V_2; C)$.

(ii) If $\dim V_1 \cap V_2 = \dim(V_1) + \dim(V_2) - n$, then $j(V_1, V_2; C) = \ell(V_1, V_2; C)$ if and only if $A(V_1; C)$ and $A(V_2; C)$ are Cohen-Macaulay.

(3.41) Problem 6. Give the connection between our approach and function's approach to the intersection theory in [18] or [19] (see our remark on example (3.8)).

(3.42) Problem 7. Is it possible to give an extension of our approach to the intersection theory for pure dimensional subvarieties $V_1, \ldots, V_r, r \geq 2$ of an ambient smooth variety Z?

(3.43) Problem 8. (David Buchsbaum) Describe all intersection numbers of V_1 and V_2 along C as Euler -Poincare Characteristic.

(3.44) Problem 9. (David Eisenbud) Assume that V_1 and V_2 are irreducible subvarieties of$\mathbb{P}^n_K$. Suppose that $V_1 \subseteq V_2$. Then describe all elements of the collection $\{C_i\}$ of the main theorem (2.1) .

In connection with this problem, we want to study the following example.

(3.45) Example. Let $V_1 = V_2$ be defined by the equation $X_1^2 + X_2^2 - X_0^2 =$ **119**
0 in $\mathbb{P}^2_K$. Then it is not hard to show that our collection $\{C_i\}$ of the main theorem (2.1) is given by:

$C_1 : X_1^2 + X_2^2 - X_0^2 \underline{=} 0$ and two imbedded points, say C_2 and C_3, which are defined over $\bar{K}$.

Therefore we get

$$4 = \deg(V_1)\deg(V_2) = \sum_{i=1}^{3} j(V_1, V_2; C_1)\deg(C_i).$$

Hence $j(V_1, V_2; C_1) = 1$ for all $i = 1, 2, 3$.

(3.46) Proposition. *Let V_1, V_2, V_3 be pure dimensional varieties in $\mathbb{P}^n_K$. Then*

$$\begin{aligned}\prod_{i=1}^{3}\deg(V_i) &= \sum_{D\subset V_1\cap V_2V_3} j(V_1, V_2, V_3; D)\deg(D)\\ &= \sum_{C\subset V_1\cap V_2}\left[j(V_1, V_2; C)\sum_{E\subset V_3} j(C, V_3; E)\deg(E)\right]\end{aligned}$$

where D runs through the collection $\{D_i\}$ of the general main theorem (2.25) , C runs through the collection $\{C_i\}$ of the main theorem (2.1) for V_1 andV_2 and E runs through the collection $\{E_i\}$ of the main theorem (2.1) for C and V_3.

Proof. Immediate from main theorem (2.1) and (2.25) . □

(3.47) Problem 10. Let $V_1, \ldots, V_s$ be pure dimensional subvarieties of $\mathbb{P}^n_K$. Let C an irreducible component of $V_1 \cdots V_S$, $s \geq 1$. Then is it true that

$$j(V_1, \ldots, V_S; C) \geq \prod_{i=1}^{s} J(V_i, C; C)?$$

Also give a characterization for the equality. **120**

(3.48) Remark. In 1937, 0. Zariski proved the following statement (see: Trans. Amer. Math. Soc. 41(1937), 249-265): if the origin is an m_i-fold point of n hypersurfaces $F_1, \ldots, F_n$ of $\mathbb{P}^n_k$ and it is an isolated point of intersection of these n varieties, then the intersection multiplicity at the origin is not less than $m_1 m_2 \cdots m_n$, by assuming that the hypersurfaces $F_1, \ldots, F_n$ have only a finite number of common points. Other proofs have been given, for example, by 0. Perron (see:Bayer. Akad. Wiss. Math. Natur. K1. Sitzungsber. Jahrgang 1954, 179-199) or by H. Gigl (see: Monatsh. math. 60(1956), 198-204). Also Zariski's theorem is a special case of a theorem given by D.G. Northcott (see: Quart. j. math. Oxford Ser. (2)4 (1953), 67-80) or by W. Vogel (see: Monatsh. math. 71(1967), 238-247) as an illustration of the general theory which was developed in these papers. Studying our problem (3.47), we want to give an extension of these observations. In the meanwhile, R. Achilles proved the above inequality. The characterization of the equality is yet open. We want to conclude these notes with the following conjecture:

Conjecture. *Let X and Y be two pure dimensional subschemes of* $\mathbb{P}^n_K$ *given by the ideals $I(X)$ and $I(Y)$. Then* $\deg(X).\deg(Y) \geq$ *number of prime ideals $\mathscr{Y}$ belonging to $I(X) + I(Y)$ such that* $\dim \mathscr{Y} \geq \dim X + \dim Y - n$.

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