

**Lectures on
Topics In One-Parameter Bifurcation Problems**

**By
P. Rabier**

**Tata Institute of Fundamental Research
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Topics In One-Parameter Bifurcation Problems**

**By
P. Rabier**

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**Notes by
G. Veerappa Gowda**

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Preface

This set of lectures is intended to give a somewhat synthetic exposition for the study of one-parameter bifurcation problems. By this, we mean the analysis of the structure of their set of solutions through the same type of general arguments in various situations.

Chapter I includes an introduction to one-parameter bifurcation problems motivated by the example of *linear eigenvalue problems* and step by step generalizations lead to the suitable mathematical form. The *Lyapunov-Schmidt reduction* is detailed next and the chapter is completed by an introduction to the mathematical method of resolution, based on the *Implicit function theorem* and the *Morse lemma* in the simplest cases. The result by Crandall and Rabinowitz [7] about *bifurcation from the trivial branch at simple characteristic values* is given as an example for application.

Chapter II presents a *generalization of the Morse lemma* in its “weak” form to mappings from \mathbb{R}^{n+1} into \mathbb{R}^n . A slight improvement of one degree of regularity of the curves as it can be found in the literature, is proved, which allows one to include the case when the Implicit function theorem applies and is therefore important for the homogeneity of the exposition. The relationship with stronger versions of the Morse lemma is given for the sake of completeness but will not be used in the sequel.

Chapter III shows how to apply the results of Chapter II to the study of one-parameter bifurcation problems. Attention is confined to two general examples. The first one deals with problems of bifurcation from the trivial branch at a multiple characteristic value. A direct application

may be possible but, for higher non-linearities, a preliminary *change of scale* is necessary. The justification of this change of scale is given at an intuitive level only, because a detailed mathematical justification involves long and tedious technicalities which do not help much for understanding the basic phenomena, even if they eventually provide a satisfactory justification for the use of Newton diagrams (which we do not use however). The conclusions we draw are, with various additional information, those of McLeod and Sattinger [23]. The second example is concerned with a problem in which *no particular branch of solutions is known a priori*. It is pointed out that while the case of a simple singularity is without bifurcation, bifurcation does occur in general when the singularity is multiple. Also, it is shown how to get further details on the location of the curves when the results of Chapter II apply after a suitable change of scale and how this leads at once to the distinction between “turning points” and “hysteresis points” when the singularity is simple.

Chapter IV breaks with the traditional exposition of the Lyapunov-Schmidt method, of little and hazardous practical use, because its assumed data are not known in the applications while the imperfection sensitivity of the method has not been evaluated (to the best of our knowledge at least). Instead, we present a new, more general (and we believe, more realistic) method, introduced in Rabier-El Hajji [33] and derived from the “almost” constructive proofs of Chapter II. Optimal rate of convergence is obtained. For the sake of brevity, the technicalities of § 5 have been skipped but the first four sections fully develop all the main ideas.

Chapter V introduces a new method in the study of bifurcation problems in which the nondegeneracy condition of Chapter II is not fulfilled. Actually, the method is new in that it is applied in this context but similar techniques are classical in the desingularization of algebraic curves. We show how to find the local zero set of a C^∞ real-valued function of two variables (though the regularity assumption can be weakened in most of the cases) verifying $f(0) = 0$, $Df(0) = 0$, $D^2f(0) \neq 0$ but $\det D^2f(0) = 0$ (so that the Morse condition fails). This method is applied to a problem of bifurcation from the trivial branch at a geometri-

cally simple characteristic value when the nondegeneracy condition of Crandall and Rabinowitz is *not* fulfilled (i.e. the *algebraic multiplicity* as > 1). The role played by the generalized null space is made clear and the result complements Krasnoselskii's bifurcation theorem in the particular case under consideration.

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Chapter 1

Introduction to One-Parameter Bifurcation Problems

1.1 Introduction

In this, section, we introduce one-parameter bifurcation problems through the example of linear eigenvalue problems. In increasing order of generality, they first lead to non-linear eigenvalue problems, next, to problems of bifurcation from the trivial branch and finally to a large class of problems for which a general mathematical analysis can be developed. 1

1.1a Linear Eigenvalue Problems

Let X be a real vector space. Given a linear operator $L : X \rightarrow X$, we consider the problem of finding the pairs $(\lambda, x) \in \mathbb{R} \times X$ that

$$x = \lambda Lx.$$

For $\lambda = 0$, $x = 0$ is the unique solution. For $\lambda \neq 0$, and setting $\tau = 1/\lambda$, it is equivalent to

$$\tau x = Lx.$$

The values $\tau \in \mathbb{R}$ such that there exists $x \neq 0$ satisfying the above equation are called *eigen values* of L . When $\lambda \neq 0$, the corresponding value $\lambda = 1/\tau$ is called a *characteristic value* of L .

- 2 It may happen that every non-zero real number λ is a characteristic value of L . For instance, if $X = \mathcal{D}'(\mathbb{R})$ (distributions over \mathbb{R}) and L is the operator $Lx = x'' \in \mathcal{D}'(\mathbb{R})$. Then

$$x = \lambda x'' \leftrightarrow x(s) = e^{s/\sqrt{\lambda}}.$$

But this is not the case in general. In what follows, we shall assume that X is a real *Banach* space and that $L \in \mathcal{L}(X)$ is *compact*.

From the *special theory of linear operators* in Banach spaces, it is known that

- (i) The characteristic values of L form a sequence $\{\lambda_j\}_{j \geq 1}$ with no cluster point (the sequence is finite if $\dim X < \infty$).
- (ii) For every $\lambda \in \mathbb{R}$, $\text{Range}(I - \lambda L)$ is closed and $\dim \text{Ker}(I - \lambda L) = \text{codim Range}(I - \lambda L) < \infty$ (and is greater than or equal to 1 if and only if $\lambda = \lambda_j$ for some j).

Let us now set

$$H(\lambda, x) = x - \lambda Lx$$

so that the problem consists in finding the pairs $(\lambda, x) \in \mathbb{R} \times X$ such that $H(\lambda, x) = 0$. The set of solutions of this equation (*zero set* of H) is the union of the line $\{(\lambda, 0); \lambda \in \mathbb{R}\}$ (*trivial branch*) and the set $\bigcup_{j \geq 1} \{\lambda_j\} \times E_j$ where E_j denotes the eigenspace associated with the characteristic value λ_j .

- 3 Now, let us take a look at the local structure of the zero set of H around a given point $(\lambda_0, 0)$, $\lambda_0 \in \mathbb{R}$; If λ_0 is *not* a characteristic value of L , it is made up of exactly *one* curve (the trivial branch itself). If $\lambda_0 = \lambda_j$ for some $j \geq 1$, the structure changes, since there are solutions of the form (λ_j, x) , $x \in E_j$, arbitrarily close to $(\lambda_j, 0)$. The existence of *nontrivial* solutions (i.e. which do not belong to the trivial branch) around a point $(\lambda_j, 0)$ is referred to as a *bifurcation phenomenon* (here, from the trivial branch) and the points $(\lambda_j, 0)$ are called *bifurcation points*. Bifurcation can be viewed as a *breaking of smoothness* of the local zero set whereas data whereas *all the data are smooth*.

1.1b Generalization I: Problems of bifurcation from the trivial branch.

A natural extension is when the linear operator L is replaced by a mapping $T : X \rightarrow X$ (nonlinear in general) such that $T(0) = 0$. The problem becomes: Find $(\lambda, x) \in \mathbb{R} \times X$ such that

$$x = \lambda T(x).$$

Again, the pairs $(\lambda, 0) \lambda \in \mathbb{R}$ are always solutions of this equation (*trivial branch*). On the basis of the linear case, a natural question is to know whether there are “*bifurcation points*” on the trivial branch, namely solutions $(\lambda_0, 0)$ around which nontrivial solutions always exist.

Theorem 1.1 (Necessary condition). *Assume that T is differentiable at the origin and the linear operator $DT(0) \in \mathcal{L}(X)$ is compact. Then a necessary condition for $(\lambda_0, 0)$ to be a bifurcation point of the equation $x = \lambda T(x)$ is that λ_0 is a characteristic value of $DT(0)$.* 4

Proof. Write

$$T(x) = DT(0) \cdot x + o(\|x\|)$$

and let $(\lambda^{(i)}, x^{(i)})$ be a sequence tending to $(\lambda_0, 0)$ with $x^{(i)} \neq 0$ and

$$x^{(i)} = \lambda^{(i)} T(x^{(i)}).$$

Thus,

$$x^{(i)} = \lambda^{(i)} DT(0) \cdot x^{(i)} + o(\|x^{(i)}\|)$$

Dividing by $\|x^{(i)}\| \neq 0$, we get

$$\frac{x^{(i)}}{\|x^{(i)}\|} = \lambda^{(i)} DT(0) \cdot \frac{x^{(i)}}{\|x^{(i)}\|} + o(1)$$

The sequence $\left(\frac{x^{(i)}}{\|x^{(i)}\|}\right)$ is bounded. Due to the compactness of the operator $DT(0)$, we may assume, after considering a subsequence, that the right hand side tends to a limit v , which is then the limit of the sequence $\left(\frac{x^{(i)}}{\|x^{(i)}\|}\right)$ as well. Of course, $v \neq 0$ and making i tend to $+\infty$, we find

$$v = \lambda_0 DT(0) \cdot v,$$

which shows that λ_0 is a characteristic value of $DT(0)$. □

- 5 **Remark 1.1.** As T is nonlinear (a very general assumption !) it is impossible, without additional hypotheses to expect more than *local results* (in contrast to the linear case where global results are obtained).

Remark 1.2. Even when λ_0 is a characteristic value of $DT(0)$, bifurcation is not ensured. For instance, take $X = \mathbb{R}^2$ and $x = (x_1, x_2)$ with

$$T(x_1, x_2) = \begin{bmatrix} x_1 + x_2^3 \\ x_2 - x_1^3 \end{bmatrix}.$$

Here, $DT(0) = I$, whose unique characteristic value is $\lambda_1 = 1$. The equation $x = \lambda T(x)$ becomes

$$\begin{aligned} x_1 &= \lambda x_1 + \lambda x_2^3, \\ x_2 &= \lambda x_2 - \lambda x_1^3. \end{aligned}$$

Multiplying the first equation by x_2 and the second one by $-x_1$ and adding the two we get $\lambda(x_1^4 + x_2^4) = 0$. Hence for λ around 1, we must have $x = 0$ and *no bifurcation occurs*.

These nonlinear eigenvalue problems are particular cases of a more general class called *problems of bifurcation from the trivial branch*. By *definition*, a problem of bifurcation from the trivial branch is an equation of the form

$$x = \lambda Lx - \phi(\lambda, x),$$

- 6 where $L \in \mathcal{L}(X)$ and ϕ is a nonlinear operator from X to itself satisfying

$$\phi(\lambda, 0) = 0 \text{ for } \lambda \in \mathbb{R}, \quad (1.1)$$

$$\phi(\lambda, x) = 0(\|x\|), \quad (1.2)$$

for x around the origin, *uniformly with respect to λ on bounded intervals*. It is equivalent to saying that a problem of bifurcation from the trivial branch consists in finding the zero set of the mapping

$$H(\lambda, x) = x - \lambda Lx + \phi(\lambda, x)$$

From our assumptions, the pairs $(\lambda, 0)$, $\lambda \in \mathbb{R}$ are all in the zero set of H (*trivial branch*). Note, however, that our definition does *not* include

all mappings having the trivial branch in their zero set. Two reasons motivate our definition. First, problems of this type are common in the literature, for they correspond to many physical examples. Secondly, from a mathematical stand-point, their properties allow us to make a general study of their zero set. In particular, when L is *compact*, a proof similar to that of theorem 1.1 shows that λ_0 needs to be characteristic value of L for $(\lambda_0, 0)$ to be a bifurcation point. Actually, by writing the Taylor expansion of T about the origin

$$T(x) = DT(0) \cdot x + R(x),$$

with $R(x) = O(\|x\|)$, it is clear that nonlinear eigenvalue problems are a particular case of problems of bifurcation from the trivial branch in which $L = DT(0)$ and $\phi(\lambda, x) = -\lambda R(x)$.

The most famous result about problems of bifurcation from the trivial branch is a partial converse of Theorem 1.1 due to Krasnoselskii. 7

Theorem 1.2 (Krasnoselskii). *Assume that L is compact and λ_0 is a characteristic value of L with odd algebraic multiplicity, Then $(\lambda_0, 0)$ is a bifurcation point (i.e. there are solutions $(\lambda, x) \in \mathbb{R} \times X - \{0\}$ of $H(\lambda, x) = 0$ arbitrarily close to $(\lambda_0, 0)$).*

The proof of Theorem 1.2 is based on *topological degree arguments* and will not be given here (cf. [19], [27]). It is a very general result but it does not provide any information on the *structure* of the zero set of H near $(\lambda_0, 0)$, a question we shall be essentially interested in, throughout these notes.

COMMENT 1.1. (Algebraic and geometric multiplicity of a characteristic value): Let L be compact. Given a characteristic value λ_0 of L , it is well-known that the space $\text{Ker}(I - \lambda_0 L)$ is finite dimensional. Its dimension is called the *geometric multiplicity* of λ_0 .

The spectral theory of compact operators in Banach spaces (see e.g. [9]) provides us with additional information; namely, there is an integer $r \geq 1$ such that

$$(i) \quad \dim \text{Ker}(I - \lambda_0 L)^r < \infty,$$

$$(ii) X = \text{Ker}(I - \lambda_0 L)^r \oplus \text{Range}(I - \lambda_0 L)^r.$$

8 In addition, r is characterized by

$$\text{Ker}(I - \lambda_0 L)^{r'} = \text{Ker}(I - \lambda_0 L)^r \text{ for every } r' \geq r.$$

The dimension of the space $\text{Ker}(I - \lambda_0 L)^r$ is called the *algebraic multiplicity* of λ_0 . The algebraic multiplicity of λ_0 is always greater holds if and only if $r = 1$. If so, it follows from property (ii) that

$$X = \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L).$$

A typical example of this situation is when X is a Hilbert space and L is *self-adjoint*.

COMMENT 1.2. In particular, Krasnoselskii's theorem applied when $r = 1$ and $\dim \text{Ker}(I - \lambda_0 L) = 1$. For instance, this happens when L is the inverse of a *second order* elliptic linear operator associated with suitable boundary conditions and λ_0 is the "first" characteristic value of L ; this result is strongly related to the *maximum principle* through the Krein-Rutman theorem. (See e. g. [20, 35]).

COMMENT 1.3. For future use, note that the mapping ϕ is differentiable with respect to the x variable at the origin with

$$D_x \phi(\lambda, 0) = 0 \text{ for every } \lambda \in \mathbb{R}, \quad (1.3)$$

as it follows from (1.2).

9 **EXAMPLES.** The most important examples of problems of bifurcation from the trivial branch came from nonlinear partial differential equations. For instance, let us consider the model problem

$$\begin{cases} -\Delta u + \lambda u \pm u^k = 0 \text{ in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N and $k \geq 2$ is an integer.

Under general assumptions on the boundary $\partial\Omega$ of Ω , it follows from the *Sobolev embedding theorems* that $u^k \in H^{-1}(\Omega)$ for $k \leq \frac{N+2}{N-2}$ if $N > 2$, any $1 \leq k < +\infty$ for $N = 1$ and $N = 2$.

Denoting by $L \in \mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$ the inverse of the operator $-\Delta$, the problem becomes equivalent to

$$\begin{cases} u - \lambda Lu + L(u^k) = 0, \\ u \in H_0^1(\Omega). \end{cases}$$

Note that the restriction of L to the space $H_0^1(\Omega)$ is compact and the mapping

$$u \in H_0^1(\Omega) \rightarrow Lu^k \in H_0^1(\Omega)$$

is of class C^∞ . In this example, the nonlinearity u^k can be replaced by $F(u)$ (respectively $F(\lambda, u)$) where $F(u)(x) = f(x, u(x))$ (respectively $F(\lambda, u)(x) = f(\lambda, x, u(x))$) and f is a *Carathéodory* function satisfying some suitable growth conditions with respect to the second (respectively third) variable (see eg. Krasnoselskii [19], Rabier [30]).

Another example with a *non-local nonlinearity* is given by the *von Karman equations* for the study of the buckling of thin plates. The problem reads: Find u such that **10**

$$\begin{cases} u - \lambda Lu + C(u) = 0, \\ u \in H_0^2(\omega), \end{cases}$$

where ω is an open bounded subset of the plane \mathbb{R}^2 , $L \in \mathcal{L}(H_0^2(\omega))$ is compact and C is a ‘‘cubic’’ nonlinear operator. The operator L takes into account the distribution of lateral forces along the boundary $\partial\omega$, the intensity of these forces being proportional to the scalar λ . For ‘‘small’’ values of λ , the only solution is $u = 0$ but, beyond a certain critical value, nonzero solutions appear: this corresponds to the (physically observed) fact that the plate jumps out of its for sufficiently ‘‘large’’ λ (see e. g. Berger [1] Ciarlet-Rabier [6]).

Coming back to the general case, our aim is to give as precise a description as possible of the zero set of H around the point $(\lambda_0, 0)$. Assuming L is compact, we already know the answer when λ_0 is not a

characteristic value of L : the zero set coincides with the trivial branch. In any case, it is convenient to shift the origin and set

$$\lambda = \lambda_0 + \mu \quad (1.4)$$

$$\Gamma(\mu, x) = \phi(\lambda_0 + \mu, x) \quad (1.5)$$

- 11 so that the problem amounts to finding the zero set around the origin (abbreviated as *local zero set*) of the mapping

$$G(\mu, x) = x - (\lambda_0 + \mu)Lx + \Gamma(\mu, x) \quad (1.6)$$

Note that the mapping Γ verifies the properties

$$\Gamma(\mu, 0) = 0 \text{ for every } \mu \in \mathbb{R}, \quad (1.7)$$

$$\Gamma(\mu, x) = 0(\|x\|), \quad (1.8)$$

around the origin, *uniformly with respect to μ on bounded intervals*. In particular, Γ is differentiable with respect to the x -variable and

$$D_x \Gamma(\mu, 0) = 0 \text{ for every } \mu \in \mathbb{R}. \quad (1.9)$$

1.1c GENERALIZATION II :

Let us consider a problem of bifurcation from the trivial branch with compact operator $L \in \mathcal{L}(X)$, put under the form $G(\mu, x) = 0$ after fixing the real number λ_0 as described above. From (1.7)

$$D_\mu \Gamma(0) = 0. \quad (1.10)$$

Together with (1.9), we see that the (global) derivative $DG(0)$ is the mapping

$$(\mu, x) \in \mathbb{R} \times X \rightarrow (I - \lambda_0 L)x \in X. \quad (1.11)$$

Hence

$$\text{Ker } DG(0) = \mathbb{R} \times \text{Ker}(I - \lambda_0 L), \quad (1.12)$$

$$\text{Range } DG(0) = \text{Range}(I - \lambda_0 L). \quad (1.13)$$

12 As a result, $\text{Range } Dg(0)$ is *closed* and $\dim \text{Ker } DG(0) = \text{codim } \text{Range } DG(0) + 1 < \infty$ (≥ 1 , if and only if λ_0 is a characteristic value of L). In other words, $DG(0)$ is a *Fredholm operator with index 1*. Recall that a linear operator A from a Banach space \tilde{X} to a Banach space Y is said to be a *Fredholm operator* if

- (i) $\text{Range } A$ is closed,
- (ii) $\dim \text{ker } A < \infty$, $\text{codim } \text{Range } A < \infty$.

In this case, the difference

$$\dim \text{Ker } A - \text{codim } \text{Range } A$$

is called the *index* of A . For the definition and further properties of Fredholm operators, see Kato [17] or Schechter [36].

Remark 1.3. One should relate the fact that the index of $DG(0)$ is 1 to the fact that the parameter μ is *one-dimensional*. This will be made more clear in Remark 2.2 later.

Although the parameter μ has often a *physical significance* (and hence must be distinguished from the variable x for physical reasons), it is not always desirable to let it play a particular role in the *mathematical approach* of the problem.

The suitable general mathematical framework is as follows:

Let there be given two real Banach spaces \tilde{X} and Y and let $G : \tilde{X} \rightarrow Y$ be a mapping satisfying the conditions

13

$$G(0) = 0 \tag{1.14}$$

$$G \text{ is differentiable at the origin ,} \tag{1.15}$$

$$DG(0) \text{ is a Fredholm operator with index 1.} \tag{1.16}$$

Naturally, G need not be defined everywhere, but in a neighbourhood of the origin only. However, for notational convenience, *we shall repeatedly make such an abuse of notation* in the future, without further mention.

In our previous example of problems of bifurcation from the trivial branch, one has $\tilde{X} = \mathbb{R} \times X, Y = X$ and $G(\mu, 0) = 0$ for $\mu \in \mathbb{R}$. None of these assumptions is required here. In particular, nothing ensures that the trivial branch is in the local zero set of G . As a matter of fact, *no branch of solutions* (trivial or not) *is supposed to be known a priori*.

The first step of the study consists in performing the *Lyapunov-Schmidt reduction* allowing us to reduce the problem to a *finite dimensional one* and this will be done in the next section.

1.2 The Lyapunov-Schmidt Reduction

Let \tilde{X} and Y be two real Banach spaces and $G : \tilde{X} \rightarrow Y$ a mapping of class $C^m, m \geq 1$ verifying (1.14) - (1.16). Let us set

$$\tilde{X}_1 = \text{Ker } DG(0), \quad (2.1)$$

$$Y_2 = \text{Range } DG(0). \quad (2.2)$$

14

By hypothesis, Y_2 has finite codimension $n \geq 0$ as \tilde{X}_1 has finite dimension $n + 1$. Let \tilde{X}_2 and Y_1 be two topological complements of \tilde{X}_1 and Y respectively.

Remark 2.1. (*Existence of topological complements*) From the Hahn-Banach theorem, each one-dimensional subspace of \tilde{X} has a topological complement. Then, each finite dimensional subspace of \tilde{X} (direct sum of a finite number of one-dimensional subspaces) has a topological complement (the intersection of the complements of these one-dimensional subspaces). In particular, \tilde{X}_1 has a topological complement. Next, the existence of a topological complement of Y_2 is due to the fact that Y_2 is *closed* with finite codimension. Any (finite-dimensional) algebraic complement of Y_2 is closed and hence is also a topological complement. Details are given for instance, in Brezis [4].

Thus we can write

$$\tilde{X} = \tilde{X}_1 \oplus \tilde{X}_2 \quad (2.3)$$

$$Y = Y_1 \oplus Y_2. \quad (2.4)$$

Let Q_1 and Q_2 denote the (continuous) projection operators onto Y_1 and Y_2 respectively. On the other hand, for every $\tilde{x} \in \tilde{X}$, set

$$\tilde{x} = \tilde{x}_1 + \tilde{x}_2, \tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2.$$

With this notation, the equation $G(\tilde{x}) = 0$ goes over into the system

$$Q_1 G(\tilde{x}_1 + \tilde{x}_2) = 0 \in Y_1, \quad (2.5)$$

$$Q_2 G(\tilde{x}_1 + \tilde{x}_2) = 0 \in Y_2, \quad (2.6)$$

15

Now, from our assumptions, one has

$$DG(0)|_{\tilde{X}_2} \in \text{Isom}(\tilde{X}_2, Y_2). \quad (2.7)$$

Indeed, $DG(0)|_{\tilde{X}_2}$ is clearly one-to-one, onto (by definition of \tilde{X}_2 and Y_2) and continuous. As Y_2 is closed in Y , it is a Banach space by itself and the result follows from the *open mapping theorem*. Thus equation (2.6) is solved in a neighbourhood of the origin by

$$\begin{aligned} \tilde{x}_2 &= \tilde{\varphi}(\tilde{x}_1) \\ \tilde{\varphi}(0) &= 0 \end{aligned}$$

where $\tilde{\varphi} : \tilde{X}_1 \rightarrow \tilde{X}_2$ is a uniquely determined C^m mapping (Implicit function theorem). After substituting in the first equation, we find the *reduced equation*

$$Q_1 G(\tilde{x}_1 + \tilde{\varphi}(\tilde{x}_1)) = 0 \in Y_1, \quad (2.8)$$

equivalent to the original equation : $\tilde{x} \in \tilde{X}, G(\tilde{x}) = 0$, around the origin. From now on, we drop the index “1” in the notation of the generic element of the space \tilde{X}_1 . The mapping

$$\tilde{x} \in \tilde{X}_1 \rightarrow f(\tilde{x}) = Q_1 G(\tilde{x} + \tilde{\varphi}(\tilde{x})) \in Y_1, \quad (2.9)$$

whose local zero set is made up of the solution of the reduced equation (2.8) is called the *reduced mapping* (note of course that f verifies $f(0) = 0$).

Therefore, we have reduced the problem of finding the local zero set of G to finding the local zero set of the reduced mapping f in (2.9), which is of class C^m from a neighbourhood of the origin in the $(n + 1)$ -dimensional space \tilde{X}_1 into the n -dimensional space Y_1 . 16

Remark 2.2. More generally, assume $DG(0)$ is a Fredholm operator with index $p \geq 1$. The same process works; we end up with a reduced mapping from the space \tilde{X}_1 ($(n + p)$ -dimensional) into the space Y_1 (n -dimensional), so that p can be thought of as the number of “free” real variables (cf. Remark 1.3).

Two Simple Properties.

The derivative at the origin of the reduced mapping f (cf. (2.9)) is immediately found to be

$$Q_1 DG(0)(I_{\tilde{X}_1} + D\tilde{\varphi}(0)).$$

But $Q_1 DG(0) = 0$, by the definition of the space Y_1 , so that

$$Df(0) = 0 \tag{2.10}$$

Next, from the characterization of the function $\tilde{\varphi}$ and by implicit differentiation, we get

$$Q_2 DG(0)(I_{\tilde{X}_1} + D\tilde{\varphi}(0)) = 0.$$

In other words, since $Q_2 DG(0) = DG(0)$ by a definition of the space Y_2 and since $\tilde{X}_1 = \text{Ker } DG(0)$,

$$DG(0) \cdot D\tilde{\varphi}(0) = -DG(0) \cdot I_{\tilde{X}_1} = 0 \tag{2.11}$$

On the other hand, the function $\tilde{\varphi}$ takes its values in the space so that

$$D\tilde{\varphi}(0) \in \mathcal{L}(\tilde{X}_1, \tilde{X}_2),$$

and the relation (2.11) can be written as

$$DG(0)|_{\tilde{X}_2} \cdot D\tilde{\varphi}(0) = 0.$$

Thus, from (2.7)

$$D\tilde{\varphi}(0) = 0. \quad (2.12)$$

The Lyapunov-Schmidt reduction in the case of problems of bifurcation from the trivial branch.

In our examples later, we shall consider the case of problems of bifurcation from the trivial branch. This is the reason why we are going to examine the form taken by the Lyapunov-Schmidt reduction in this context. Of course, this is simply a particular case of the general method previously described.

Let $X = Y$ and consider a problem of bifurcation from the trivial branch in the form $G(\mu, x) = 0$ after fixing the real number λ_0 . As we observed earlier,

$$\tilde{X}_1 = \text{Ker } DG(0) = \mathbb{R} \times \text{Ker}(I - \lambda_0 L), \quad (2.13)$$

$$Y_2 = \text{Range } DG(0) = \text{Range}(I - \lambda_0 L). \quad (2.14)$$

Setting

$$X_1 = \text{Ker}(I - \lambda_0 L), \quad (2.15)$$

this becomes

$$\tilde{X}_1 = \mathbb{R} \times X_1, \quad (2.16)$$

18

so that any element $\tilde{x}_1 \in \tilde{X}_1$ can be identified with a pair $(\mu, x_1) \in \mathbb{R} \times X_1$. Thus given a topological complement X_2 of X_1 in the space X , we can make the choice

$$\tilde{X}_2 = \{0\} \times X_2. \quad (2.17)$$

Note that *not every complement of \tilde{X}_1 in $\mathbb{R} \times X$ is of the form (2.17)*. Nevertheless, such a choice is “standard” in the literature devoted to problems of bifurcation from the trivial branch. Writing each element $x \in X$ as a sum

$$x = x_1 + x_2, \quad x_1 \in X_1, \quad x_2 \in X_2,$$

the mapping $\tilde{\varphi}$ identifies with a mapping $\varphi(= \varphi(\mu, x_1))$ with values in X_2 , characterised by the relation

$$Q_2(I - (\lambda_0 + \mu)L)(x_1 + \varphi(\mu, x_1)) + Q_2\Gamma(\mu, x_1 + \varphi(\mu, x_1)) = 0. \quad (2.18)$$

Remark 2.3. In particular, $\varphi(\mu, 0) = 0$ since $\varphi(\mu, 0)$ is *uniquely determined* by the relation (2.18) which is satisfied with the choice $x_1 = 0, \varphi(\mu, 0) = 0$.

The reduced mapping thus takes the form

$$f(\mu, x_1) = Q_1G(\mu, x_1 + \varphi(\mu, x_1)), \quad (2.19)$$

where the mapping G is given by (1.6).

19 As

$$Q_1(I - \lambda_0L) = 0, \quad (2.20)$$

by definition of the space Y_1 , we find

$$Q_1G(\mu, x) = -\mu Q_1Lx + Q_1\Gamma(\mu, x).$$

But the relation (2.20) also shows that

$$Q_1L = \frac{1}{\lambda_0}Q_1 \quad (2.21)$$

and hence

$$Q_1G(\mu, x) = -\frac{\mu}{\lambda_0}Q_1x + Q_1\Gamma(\mu, x).$$

Therefore, dropping the index “1” in the notation of the generic element of the space X_1 , the reduced mapping takes the form

$$\begin{aligned} (\mu, x) \in \mathbb{R} \times X_1 \rightarrow f(\mu, x) = & -\frac{\mu}{\lambda_0}Q_1 \times -\frac{\mu}{\lambda_0}Q_1\varphi \\ & (\mu, x) + Q_1\Gamma(\mu, x + \varphi(\mu, x)) \in Y_1. \end{aligned} \quad (2.22)$$

1.3 Introduction to the Mathematical Method of Resolution (Implicit Function Theorem and Morse Lemma).

Since $f(0) = 0$, the first natural tool we can think of, for finding the local zero set of f , is the *Implicit function theorem*. But we already saw that $Df(0) = 0$. Hence the Implicit function theorem can be applied when $Y_1 = \{0\}$ (i.e., $n = 0$) only. In other words, one must have $Y_2 = Y$ so that $DG(0)$ is *onto*. If so, the reduced mapping f *vanishes identically* and the problem has actually already been solved while performing the Lyapunov-Schmidt reduction: the local zero set of G is given by the graph of the mapping $\tilde{\varphi}$, that is to say, the *curve*

$$\tilde{x} \in \tilde{X}_1 \rightarrow \tilde{x} + \tilde{\varphi}(\tilde{x}) \in \tilde{X}.$$

20

The reader would have noticed that since $DG(0)$ is onto, the Lyapunov - Schmidt reduction amounts to applying the Implicit function theorem to the original problem. There is more to say about this apparently obvious situation. Let us come back to the case when the parameter μ is explicitly mentioned in the expression for G (for physical reasons for instance), namely $\tilde{X} = \mathbb{R} \times X$ and $G = G(\mu, x)$. Then, for every $(\mu, x) \in \mathbb{R} \times X$, we have

$$DG(0) \cdot (\mu, x) = \mu D_\mu G(0) + D_x G(0) \cdot x \quad (3.1)$$

and there are *two* possibilities for $DG(0)$ to be onto; either

$$D_x G(0) \text{ is onto,} \quad (3.2)$$

or

$$\text{codim Range } D_x G(0) = 1 \text{ and } D_\mu G(0) \notin \text{Range } D_x G(0). \quad (3.3)$$

We shall

$$X_1 = \text{Ker } D_x G(0).$$

When condition (3.2) is fulfilled, there is an element $\xi \in X$ such that

$$D_x G(0) \cdot \xi = -D_\mu G(0).$$

Hence, $DG(0)$ is the linear mapping

$$(\mu, x) \in \mathbb{R} \times X \rightarrow D_x G(0) \cdot (x - \mu\xi) \in Y$$

21 and it follows that

$$\begin{aligned} \widetilde{X}_1 = \text{Ker } DG(0) &= \{(\mu, x) \in \mathbb{R} \times X, x - \mu\xi \in X_1\} = \\ &= \{(\mu, x) \in \mathbb{R} \times X, x = \mu\xi + x_1, x_1 \in X_1\} = \\ &= \mathbb{R}_\mu(1, \xi) \oplus (\{0\} \times X_1), \end{aligned}$$

where \mathbb{R}_μ denotes the real line with generic variable μ . As $\dim \widetilde{X}_1 = 1$, we must have $X_1 = \{0\}$ (i.e. $\text{Ker } D_x G(0) = \{0\}$) and $D_x G(0)$ is then an isomorphism (in particular, ξ is unique). Thus

$$\widetilde{X}_1 = \mathbb{R}_\mu(1, \xi),$$

a relation showing that the local zero set of G is parametrized by μ (See Figure 3.1 below)

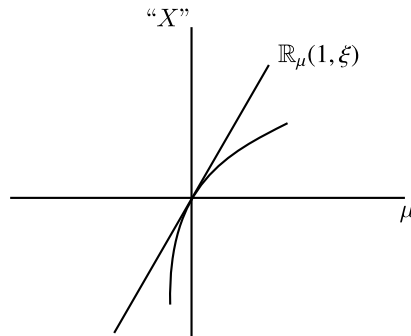


Figure 3.1: “regular point”

In this case, the origin is referred to as a “*regular point*”. It is immediately checked that this is what happens in problems of bifurcation

from the trivial branch when λ_0 is not a characteristic value of L . To sum up, in the first case, the *physical parameter* μ can also be used as a *mathematical parameter* for the parametrization of the local zero set of G . The situation is different when (3.3) holds instead of (3.2). We can write

$$Y = \mathbb{R}D_\mu G(0) \oplus \text{Range} D_x G(0),$$

whereas

$$\widetilde{X}_1 = \text{Ker} DG(0) = \{0\} \times \text{Ker} D_x G(0) = \{0\} \times X_1.$$

Since $\dim \widetilde{X}_1 = 1$, we find $\dim X_1 = 1$ and the local zero set of G is a curve parametrized by $x_1 \in X_1$. It has a “vertical” tangent at the origin, namely $\{0\} \times X_1$. Two typical situations are as follows:

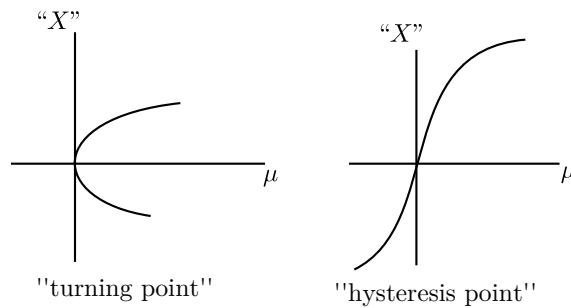


Figure 3.2:

Remark 3.1. From a geometric point of view, there is no differences between a “regular point”, a “turning point” or a “hysteresis point”, since the last two become “regular” after a change of coordinates. But there is a difference in the number of solutions of the equation $G(\mu, x) = 0$ as the parameter μ changes sign.

EXAMPLE. Given an element $y^0 \in Y$, let us consider the equation 23

$$F(x) = \mu Y^0 \tag{3.4}$$

where $F : X \rightarrow Y$ is a mapping of class $C^m (m \geq 1)$ such that $F(0) = 0$. Setting $G(\mu, x) = F(x) - \mu y^0$, we are in the first case (i.e. $D_x G(0)$ is onto

) if $D_x F(0)$ is onto and the solutions are given by

$$x = x(\mu), x(0) = 0,$$

where $x(\cdot)$ is a mapping of class C^m around the origin. Now, if $\text{codim Range } D_x F(0) = 1$ and $y^0 \notin \text{Range } D_x F(0)$, we are in the second case. The curve of solutions has a vertical tangent at the origin and it is not parametrized by μ ; as it follows from the above, a “natural” parameter is the one-dimensional variable of the space $\widetilde{X}_1 = \text{Ker } D_x F(0) \geq 1$ and $y^0 \in \text{Range } D_x F(0)$, no conclusion can be drawn as yet.

We shall now examine the case $n = 1$. Here the main tool will be the *Morse lemma*, of which we shall give two equivalent formulations.

Theorem 3.1. (*Morse lemma : “strong” version*)¹: Let f be a mapping of class C^m , $m \geq 2$ on a neighbourhood of the origin in \mathbb{R}^2 with values in \mathbb{R} , such that

$$f(0) = 0.$$

$$Df(0) = 0,$$

24

$$\det D^2 f(0) \neq 0 \text{ (Morse condition).}$$

Then, there is an origin-preserving C^{m-1} local diffeomorphism ϕ in \mathbb{R}^2 with $D\phi(0) = I$ which transforms the local zero set of the quadratic form

$$\widetilde{\xi} \in \mathbb{R}^2 \rightarrow D^2 f(0).(\widetilde{\xi})^2 \in \mathbb{R},$$

into the local zero set of f . Moreover, ϕ is C^m away from the origin.

Theorem 3.1'. (*Morse lemma, “weak” version*): Let f be a mapping of class C^m , $m \geq 2$ on a neighbourhood of the origin in \mathbb{R}^2 with values in \mathbb{R} , verifying

$$f(0) = 0,$$

$$Df(0) = 0,$$

¹There is an even stronger version that we shall, however, not use here.

$$\det D^2 f(0) \neq 0 (\text{Morse condition}).$$

Then, the local zero set of f reduces to the origin if $\det D^2 f(0) > 0$ and is made up of two curves of class C^{m-1} if $\det D^2 f(0) < 0$. Moreover, these curves are of class C^m away from the origin and each of them is tangent to a different one from among the two lines of the zero set of the quadratic form

$$\tilde{\xi} \in \mathbb{R}^2 \rightarrow D^2 f(0) \cdot (\tilde{\xi})^2 \in \mathbb{R},$$

at the origin.

NOTE : We say that the curves *intersect transversally* at the origin. 25

COMMENT 3.1. By $\det D^2 f(0)$, we mean the determinant of the 2×2 matrix representing the second derivative of f at the origin for a given basis of \mathbb{R}^2 . Of course, this determinant depends on the basis (because the partial derivatives of f do) but its *signs does not* (the proof of this assertion is simple and is left to the reader): the assumptions of Theorem 3.1 and 3.1' are *intrinsically linked to f* . In short, we shall say that the quadratic form $D^2 f(0) \cdot (\tilde{\xi})^2$ is non-degenerate.

COMMENT 3.2. Theorem 3.1 implies Theorem 3.1', since the local zero set of the quadratic form

$$\tilde{\xi} \in \mathbb{R}^2 \rightarrow D^2 f(0) \cdot (\tilde{\xi})^2 \in \mathbb{R},$$

reduces to the origin if $\det D^2 f(0) > 0$ (the quadratic form is then positive or negative definite) and is made up of exactly two distinct lines if $\det D^2 f(0) < 0$. If so, the local zero set of f is the image of these two lines through the diffeomorphism ϕ : it is then made up of two curves whose tangents at the origin are the images of the two lines in question through the linear isomorphism $D_\phi(0) = I$. We shall prove Theorem 3.1' and that it implies Theorem 3.1' in the next chapter, in a more general frame work.

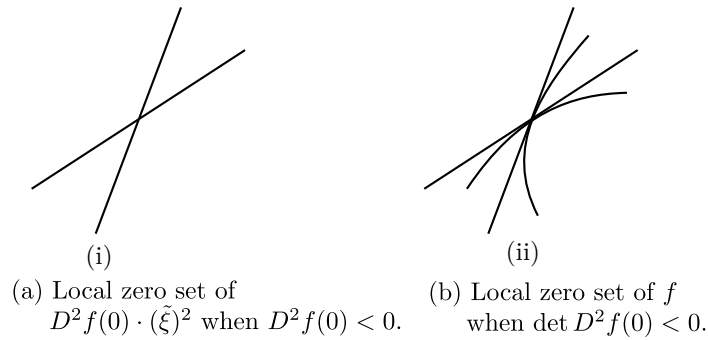


Figure 3.3:

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COMMENT 3.3. Theorem 3.1 has a generalization to mappings from $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$, $n \geq 1$. It is generally stated assuming $f \in C^m$, $m \geq 3$ and the diffeomorphism ϕ is shown to be of class C^{m-2} only (as in Nirenberg [27]). This improved version is due to Kuiper [21]. Infinite dimensional versions (cf. [14, 28, 41]) are also available in this direction.

COMMENT 3.4. In contrast, Theorem 3.1' has a generalization to mapping from $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, which we shall prove in the next chapter. This will be a basic tool in Chapter 3 where we study some general one-parameter problems.

27 **Remark 3.2.** There is also a generalization of the *strong version* to mapping from $\mathbb{R}^{n+p} \rightarrow \mathbb{R}^n$, $p \geq 1$ at the expense of losing some regularity at the origin (cf. [5]).

Application-first results. Assume $n = 1$ in the Lyapunov-Schmidt reduction and let f denote the reduced mapping. Since $\dim \tilde{X}_1 = n + 1 = 2$, $\dim Y_1 = n = 1$ and the hypotheses of the Morse lemma are *independent of the system of coordinates*, we can identify \tilde{X}_1 with \mathbb{R}^2 , Y_1 with \mathbb{R} so that f becomes a mapping from a neighbourhood of the origin in \mathbb{R}^2 with values in \mathbb{R} . The conditions $f(0) = 0, Df(0) = 0$ are automatically fulfilled (cf. (2.10)). If, in addition, $\det D^2 f(0) \neq 0$ (which requires G to be of class C^2 at least) the structure of the local zero set of

f , hence that of G follows. In particular, *bifurcation occurs as soon as* $\det D^2 f(0) = 0$.

Remark 3.3. The independence of the hypotheses of the Morse Lemma on the system of coordinates does not prove that they are independent of the spaces X_2 and Y_1 . However, such a result is true as we shall see later, in this section.

New formulation of the Morse lemma: We shall prove that the Morse condition has an *equivalent formulation* which will be basic for further generalizations.

Lemma 3.1. *Let f be as in the Morse lemma. Then the Morse condition $\det D^2 f(0) \neq 0$ is equivalent to the following property: For every $\tilde{\xi} \in \mathbb{R}^2 - \{0\}$ such that $D^2 f(0) \cdot (\tilde{\xi})^2 = 0$, the linear form $D^2 f(0) \cdot \tilde{\xi} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ is onto.* 28

Proof. As f is a mapping from \mathbb{R}^2 into \mathbb{R} , its derivative Df is a mapping from \mathbb{R}^2 into the space $\mathcal{L}(\mathbb{R}^2, \mathbb{R}) \simeq \mathbb{R}^2$. Hence

$$D^2 f(0) \in \mathcal{L}(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2, \mathbb{R})) \simeq \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$$

and saying that $\det D^2 f(0) \neq 0$ means that $\text{Ker } D^2 f(0) = \{0\}$, i.e. for every $\tilde{\xi} \in \mathbb{R}^2 - \{0\}$, $D^2 f(0) \cdot \tilde{\xi} \neq 0$. But this last condition is certainly fulfilled by those elements $\tilde{\xi}$ such that $D^2 f(0) \cdot (\tilde{\xi})^2 \neq 0$ (regardless of the condition $\det D^2 f(0) \neq 0$) and it is then not restrictive to write

$$\det D^2 f(0) \neq 0 \Leftrightarrow \{D^2 f(0) \cdot \tilde{\xi} \neq 0 \text{ for every } \tilde{\xi} \in \mathbb{R}^2 - \{0\} \text{ such that } D^2 f(0) \cdot (\tilde{\xi})^2 \neq 0\}.$$

The result follows from the obvious fact that a linear form is onto if and only if it is not the zero form. □

With the above lemma, we get equivalent version of the “weak” Morse lemma.

Theorem 3.1''. *Let f be a mapping of class, $C^m \geq 2$ on a neighbourhood of the origin in \mathbb{R}^2 with values in \mathbb{R} , verifying*

$$f(0) = 0,$$

$$Df(0) = 0,$$

29 such that the linear form $D^2 f(0) \cdot \tilde{\xi} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ is onto for every $\tilde{\xi} \in \mathbb{R}^2 - \{0\}$ with $D^2 f(0) \cdot (\tilde{\xi})^2 = 0$. Then the local zero set of the quadratic form $D^2 f(0) \cdot (\tilde{\xi})^2$ consists of exactly 0 or 2 real lines (depending on the sign of $\det D^2 f(0)$) and the local zero set of f is made up of the same number of C^{m-1} curves through the origin. Moreover, these curves are of class C^m away from the origin and each of them is tangent to a different one from among the lines in the zero of the quadratic form $D^2 f(0) \cdot (\tilde{\xi})^2$ at the origin.

Application - further details. Assume that $n = 1$ in the Lyapunov-Schmidt reduction and let f denote the reduced mapping. From the exposition of Theorem 3.1'', it does not matter if \mathbb{R}^2 and \mathbb{R} are replaced by the space \tilde{X}_1 and Y_1 respectively. Since $f(0) = 0$ and $Df(0) = 0$, it remains to check whether the Morse condition holds. By definition of f (cf.(2.9)) we first find, for every $\tilde{\xi} \in \tilde{X}_1$, that

$$Df(\tilde{x}) \cdot \tilde{\xi} = Q_1 DG(\tilde{x} + \tilde{\varphi}(\tilde{x})) \cdot (\tilde{\xi} + D\tilde{\varphi}(\tilde{x}) \cdot \tilde{\xi}).$$

Since $\tilde{\varphi}(0) = 0$ and $D\tilde{\varphi}(0) = 0$ (cf. (2.12)), we obtain

$$D^2 f(0) \cdot (\tilde{\xi})^2 = Q_1 D^2 G(0) \cdot (\tilde{\xi})^2 + Q_1 DG(0) \cdot (D^2 \tilde{\varphi}(0) \cdot (\tilde{\xi})^2).$$

But $Q_1 DG(0) = 0$, by the definition of Q_1 and hence

$$D^2 f(0) \cdot (\tilde{\xi})^2 = Q_1 D^2 G(0) \cdot (\tilde{\xi})^2.$$

a particularly simple expression in terms of the mapping G .

We now prove

30 **Theorem 3.2.** *The validity of the Morse condition for the reduced mapping f is independent of the choice of the spaces \tilde{X}_2 and Y_1 .*

Proof. Clearly, the mapping

$$\tilde{\xi} \in \tilde{X}_1 \rightarrow Q_1 D^2 G(0) \cdot (\tilde{\xi})^2 \in Y_1,$$

is independent of the space \widetilde{X}_2 and it remains to show that the required property of surjectivity is independent of Y_1 as well. Let us then assume that the property holds and let \hat{Y}_1 be another complement of Y_2 . Denoting by \hat{Q}_1 and \hat{Q}_2 the associated projection operators, we must show that the linear form $\hat{Q}_1 D^2 G(0) \cdot \tilde{\xi}$ is nonzero for every $\tilde{\xi} \in \widetilde{X}_1 - \{0\}$ such that $\hat{Q}_1 D^2 G(0) \cdot (\tilde{\xi})^2 = 0$.

First, note that \hat{Q}_1 is an *isomorphism* of Y_1 to \hat{Y}_1 . Indeed, both spaces have the same dimension and it suffices to prove that the restriction of \hat{Q}_1 to Y_1 is one-to-one. If $\hat{Q}_1 y_1 = 0$ for some $y_1 \in Y_1$, we must have $y_1 \in Y_2$ (since $\text{Ker } \hat{Q}_1 = Y_2$) and hence $y_1 \in Y_1 \cap Y_2 = \{0\}$. Next, observe that

$$\hat{Q}_1 = \hat{Q}_1 Q_1. \quad (3.6)$$

Indeed, one has $\hat{Q}_1 Q_1 = \hat{Q}_1 (I - Q_2) = \hat{Q}_1 - \hat{Q}_1 Q_2$. But $\hat{Q}_1 Q_2 = 0$ (since $\text{Ker } \hat{Q}_1 = Y_2$ again), which proves (3.6).

Let then $\tilde{\xi} \in \widetilde{X}_1 - \{0\}$ be such that $\hat{Q}_1 D^2 G(0) \cdot (\tilde{\xi})^2 = 0$.

From (3.6), this can be written as

$$\hat{Q}_1 Q_1 D^2 G(0) \cdot (\tilde{\xi})^2 = 0.$$

As $\hat{Q}_1 \in \text{Isom}(Y_1, \hat{Y}_1)$, we find

$$Q_1 D^2 G(0) \cdot (\tilde{\xi})^2 = 0.$$

31

But, from our assumptions, $Q_1 D^2 G(0) \cdot \tilde{\xi} \neq 0$. By the same argument of isomorphism, $\hat{Q}_1 Q_1 D^2 G(0) \cdot \tilde{\xi} \neq 0$ and using (3.6) again we finally see that

$$\hat{Q}_1 D^2 G(0) \cdot \tilde{\xi} \neq 0,$$

which completes the proof. \square

Practical Method: Let y^0 be any nonzero element of the space Y_1^- and let $y^* \in Y'$ (topological dual of Y) be characterized by

$$\begin{cases} \langle y^*, y \rangle = 1, \\ \langle y^*, y \rangle = 0 \text{ for every } y \in Y_2 \end{cases} \quad (3.7)$$

(The existence of such an element y^* is ensured by the Hahn-Banach theorem). Then, for every $y \in Y$.

$$Q_1 y = \langle y^*, y \rangle y^0. \quad (3.8)$$

Remark 3.4. One may object that using the Hahn-Banach theorem is “practical”. Actually, using the linear form y^* is only a convenient way to get explicit formulation of the projection operator Q_1 , which is the important thing to know in practice.

It follows, for every $\tilde{\xi} \in \tilde{X}_1$, that

$$D^2 f(0) \cdot (\tilde{\xi})^2 = Q_1 D^2 G(0) \cdot (\tilde{\xi})^2 = \langle y^*, D^2 G(0) \cdot (\tilde{\xi})^2 \rangle y^0.$$

32 Now let $(\tilde{e}_1, \tilde{e}_2)$ be a basis of \tilde{X}_1 so that $\tilde{\xi} \in \tilde{X}_1$ writes

$$\tilde{\xi} = \xi_1 e_1 + \xi_2 e_2, \xi_1, \xi_2 \in \mathbb{R}.$$

Then

$$\begin{aligned} Q_1 D^2 G(0) \cdot (\tilde{\xi})^2 &= [\xi_1^2 \langle y^*, D^2 G(0) \cdot (\tilde{e}_1)^2 \rangle + 2\xi_1 \xi_2 \langle y^*, D^2 G(0) \cdot (\tilde{e}_1, \tilde{e}_2) \rangle \\ &\quad + \xi_2^2 \langle y^*, D^2 G(0) \cdot (\tilde{e}_2)^2 \rangle] y^0 \end{aligned}$$

and the above mapping verifies the Morse condition if and only if the quadratic form

$$\begin{aligned} (\xi_1, \xi_2) \in \mathbb{R}^2 \rightarrow &[\xi_1^2 \langle y^*, D^2 G(0) \cdot (\tilde{e}_1)^2 \rangle + 2\xi_1 \xi_2 \langle y^*, D^2 G(0) \cdot (\tilde{e}_1, \tilde{e}_2) \rangle \\ &+ \xi_2^2 \langle y^*, D^2 G(0) \cdot (\tilde{e}_2)^2 \rangle] \in \mathbb{R}, \end{aligned} \quad (3.9)$$

is non-degenerate, i.e. the discriminant

$$\langle y^*, D^2 G(0) \cdot (\tilde{e}_1, \tilde{e}_2) \rangle^2 - \langle y^*, D^2 G(0) \cdot (\tilde{e}_1)^2 \rangle \langle y^*, D^2 G(0) \cdot (\tilde{e}_2)^2 \rangle \quad (3.10)$$

is non zero.

Remark 3.5. Note that the *discriminant* (3.10) is just (-1) times of the *determinant* of the quadratic form (3.9).

The example of problems of bifurcation from the trivial branch at a geometrically simple characteristic value.

33 Let $G(\mu, x) = 0$ be a problem of bifurcation from the trivial branch (cf. (1.6)) with compact operator $L \in \mathcal{L}(X)$ and nonlinear part $\Gamma \in C^m$ with $m \geq 2$, the real number λ_0 being a characteristic value of L (the obvious case when λ_0 is not a characteristic value of L has already been considered). As we know

$$\begin{aligned}\widetilde{X}_1 &= \mathbb{R} \times \text{Ker}(I - \lambda_0 L) \subset \mathbb{R} \times X, \\ Y_2 &= \text{Range}(I - \lambda_0 L) \subset X (= Y),\end{aligned}$$

so that $n = \text{codim} Y_2 = 1$ if and only if $\dim \text{Ker}(I - \lambda_0 L) = 1$ i.e. the characteristic value λ_0 is *geometrically simple*.

Since

$$G(\mu, x) = (I - (\lambda_0 + \mu)L)x + \Gamma(\mu, x),$$

we find, from (1.7) and (1.9), that

$$\begin{aligned}D_\mu^2 G(0) &= 0 \\ D_\mu D_x G(0) &= -L + D_\mu D_x \Gamma(0) = -L\end{aligned}$$

and

$$D_x^2 G(0) = D_x^2 \Gamma(0).$$

As a result, for $(\mu, x) \in \mathbb{R} \times X$

$$D^2 G(0) \cdot (\mu, x)^2 = -2\mu Lx + D_x^2 \Gamma(0) \cdot (x)^2.$$

In particular, for $x \in X_1$, one has $Lx = (1/\lambda_0)x$, so that

$$Q_1 D^2 G(0) \cdot (\mu, x)^2 = -\frac{2}{\lambda_0} \mu Q_1 x + Q_1 D_x^2 \Gamma(0) \cdot (x)^2,$$

for $(\mu, x) \in \mathbb{R} \times \text{Ker}(I - \lambda_0 L)$.

Given any $x^0 \in \text{Ker}(I - \lambda_0 L) - \{0\}$, the pair $((1, 0), (0, x^0))$ is a basis of $\mathbb{R} \times \text{Ker}(I - \lambda_0 L)$. The practical method described before leads to the examination of the sign of the determinant of the quadratic polynomial 34

$$(\mu, t) \in \mathbb{R}^2 \rightarrow \frac{-2\mu t}{\lambda_0} \langle y^*, x^0 \rangle + t^2 \langle y^*, D_x^2 \Gamma(0) \cdot (x^0)^2 \rangle, \quad (3.11)$$

where $y^* \in X'$ is some linear continuous form with null space Y_2 . The discriminant of the polynomial (3.11) is

$$\frac{4}{\lambda_0^2} \langle y^*, x^0 \rangle^2 \geq 0.$$

It is positive if and only if $\langle y^*, x^0 \rangle \neq 0$, namely $x^0 \notin Y_2$. As $\text{Ker}(I - \lambda_0 L) = \mathbb{R}x^0$ and $Y_2 = \text{Range}(I - \lambda_0 L)$, this means that $\text{Ker}(I - \lambda_0 L) \not\subset \text{Range}(I - \lambda_0 L)$. But, if so, (as $\text{codim Range}(I - \lambda_0 L) = \dim \text{Ker}(I - \lambda_0 L) = 1$, by hypothesis) we deduce that

$$X = \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L) \quad (3.12)$$

which expresses that the characteristic value λ_0 is also *algebraically simple*.

To sum up, the Morse lemma applies to problems of bifurcation from the trivial branch at a *geometrically simple* eigenvalue λ_0 if and only if λ_0 is also *algebraically simple*. Then, the local zero set of the reduced mapping and hence that of G consists of the union of the trivial branch and a *second branch* (curve of class C^{m-1} at the origin and of class C^m away from it) bifurcating from the trivial branch at the origin.

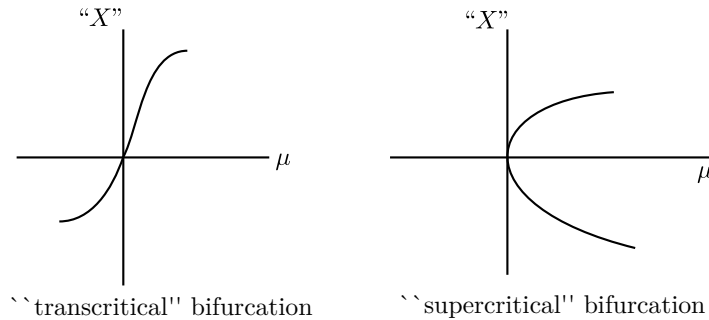


Figure 3.4: Local zero set of G .

Remark 3.6. These conclusions agree with Krasnoselskii's Theorem (Theorem 1.2) but provide much more precise information on the *structure* of the local zero set. This result was originally proved by Crandall and Rabinowitz ([7]) in a different way involving the application of

the Implicit function theorem, *after a modification of the reduced equation*. This method uses the fact that the trivial branch is in the local zero set of G explicitly. Note that the same result holds (with the same proof) when the more general conditions $\Gamma(\mu, 0) = 0$, $D_x\Gamma(0) = 0$ and $Q_1D_\mu D_x\Gamma(0) = 0$ replace (1.7)-(1.9).

Remark 3.7. The complementary case $\text{Ker}(I - \lambda_0 L) \subset \text{Range}(I - \lambda_0 L)$, namely, when the characteristic value λ_0 is *geometrically simple* in which the Morse condition fails (referred to as a “*degenerate case*”) will be considered in Chapter 5.

Chapter 2

A generalization of the Morse Lemma

As We Saw in Chapter 1, the problem is to find the local zero set of the *reduced mapping*, which is a mapping of class $C^m(m \geq 1)$ in a neighbourhood of the origin in the $(n + 1)$ -dimensional space $X_1 = \text{Ker } DG(0)$ into the n -dimensional space Y_1 (a given complement of $Y_2 = \text{Range } DG(0)$). 36

We shall develop an approach which is analogous to the one we used in the case $n = 1$ (Morse lemma). The first task is to find a suitable generalization of the *Morse condition*.

2.1 A Nondegeneracy Condition For Homogeneous Polynomial Mappings.

Let $q : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be a polynomial mapping, homogeneous of degree $k \geq 1$ (i.e. $q = (q_\alpha)_{\alpha=1,n}$ where q_α is a polynomial, homogeneous of degree k in $(n + 1)$ variables with real coefficients).

Definition 1.1. We shall say that the polynomial mapping q verifies the condition of \mathbb{R} -nondegeneracy (in short, \mathbb{R} -N.D.) if, for every non-zero solution $\tilde{\xi} \in \mathbb{R}^{n+1}$ of the equation $q(\tilde{\xi}) = 0$, the mapping $Dq(\tilde{\xi}) \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is onto.

As is homogeneous, its zero set in \mathbb{R}^{n+1} is a cone in \mathbb{R}^{n+1} with vertex at the origin. Actually, we have much more precise information. First, observe that

$$q(\tilde{\xi}) = \frac{1}{k} Dq(\tilde{\xi}) \cdot \tilde{\xi}, \quad (1.1)$$

37 for every $\tilde{\xi} \in \mathbb{R}^{n+1}$ (*Euler's theorem*). Indeed, from the homogeneity of q , write

$$q(t\tilde{\xi}) = t^k q(\tilde{\xi})$$

and differentiate both sides with respect to t ; then

$$Dq(t\tilde{\xi}) \cdot \tilde{\xi} = kt^{k-1} q(\tilde{\xi}).$$

Setting $t = 1$, we get the identity (1.1).

Let $\tilde{\xi} \in \mathbb{R}^{n+1} - \{0\}$ be such that $q(\tilde{\xi}) = 0$. Then, it follows that

$$\mathbb{R}\tilde{\xi} \subset \text{Ker } Dq(\tilde{\xi}).$$

But $Dq(\tilde{\xi}) \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is *onto* by hypothesis. Hence, $\dim \text{Ker } Dq(\tilde{\xi}) = 1$, so that

$$\text{Ker } Dq(\tilde{\xi}) = \mathbb{R}\tilde{\xi}. \quad (1.2)$$

Theorem 1.1. *Let the polynomial mapping q verify the condition ($\mathbb{R} - N.D.$). Then, the zero set of q in \mathbb{R}^{n+1} is made up of a finite number v of lines through the origin.*

Proof. Since the zero set of q is a cone with vertex at the origin, its zero set is a union of lines. To show that there is a finite number of them, it is equivalent to showing that their intersection with the unit sphere S_n in \mathbb{R}^{n+1} consists of a finite number of points. Clearly, the set

$$\{\tilde{\xi} \in S_n; q(\tilde{\xi}) = 0\},$$

38 is closed in S_n (continuity of q) and hence *compact*. To prove that it is finite, it suffices to show that it is also *discrete*. Let then $\tilde{\xi} \in S_n$ such that $q(\tilde{\xi}) = 0$. By hypothesis, $Dq(\tilde{\xi}) \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is *onto* and we know that $\text{Ker } Dq(\tilde{\xi}) = \mathbb{R}\tilde{\xi}$ (cf. (1.2)). Then, the restriction of $Dq(\tilde{\xi})$ to any complement of the space $\mathbb{R}\tilde{\xi}$ is an isomorphism to \mathbb{R}^n . In particular,

observe that the points of the sphere S_n have the following property: for every $\tilde{\zeta} \in S_n$, the tangent space $T_{\tilde{\zeta}}S_n$ of S_n at $\tilde{\zeta}$ is nothing but $\{\tilde{\zeta}\}^\perp$. hence, for every $\tilde{\zeta} \in S_n$, we can write

$$\mathbb{R}^{n+1} = \mathbb{R}\tilde{\zeta} + T_{\tilde{\zeta}}S_n.$$

In particular, we deduce

$$Dq(\tilde{\xi}) \in \text{Isom}(T_{\tilde{\xi}}S_n, \mathbb{R}^n).$$

As q is regular, the Inverse function theorem shows that there is no solution other than $\tilde{\xi}$ for the equation $q(\tilde{\xi}) = 0$ near $\tilde{\xi}$ on S_n . \square

COMMENT 1.1. The above theorem does *not* prove that there is any line in the zero set of q . Actually, the situation when $\nu = 0$ can perfectly occur.

COMMENT 1.2. It is tempting to try to get more information about the number ν of lines in the zero set of q . Of course, it is not possible to expect a formula expressing ν in terms of q but one can expect an upper bound for ν . It can be shown (under the condition (R. N.D.)) that the inequality

$$\nu \leq k^n, \tag{1.3}$$

always holds. This estimate is an easy application of the *generalized Bezout's theorem* (see e. g. Mumford [26]). Its statement will not be given here because it requires preliminary notions of algebraic geometry that are beyond the scope of these lectures. 39

We shall give a flavour of the result by examining the simplest case $n = 1$. Let q be a homogeneous polynomial of degree k in two variables. More precisely, given a basis $(\tilde{e}_1, \tilde{e}_2)$ of \mathbb{R}^2 , write

$$\tilde{\xi} = \xi_1 \tilde{e}_1 + \xi_2 \tilde{e}_2, \xi_1, \xi_2 \in \mathbb{R}.$$

then

$$\begin{cases} q(\tilde{\xi}) = \sum_{s=0}^k a_s \xi_1^{k-s} \xi_2^s, \\ a_s \in \mathbb{R}, 0 \leq s \leq k. \end{cases} \tag{1.4}$$

It is well-known that such a polynomial q can be deduced from a unique k -linear symmetric form Q on \mathbb{R}^2 by

$$q(\tilde{\xi}) = Q(\tilde{\xi}, \dots, \tilde{\xi}),$$

where the argument $\tilde{\xi} \in \mathbb{R}^2$ is repeated k times. In particular,

$$a_s = \binom{k}{s} Q(\tilde{e}_1, \dots, \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_2),$$

where the argument \tilde{e}_1 (respectively \tilde{e}_2) is repeated s times (respectively $k - 1$ times). Now, the basis $(\tilde{e}_1, \tilde{e}_2)$ can be chosen so that $a_k \neq 0$. Indeed

$$a_k = q(\tilde{e}_2, \dots, \tilde{e}_2) = q(\tilde{e}_2)$$

- 40 and \tilde{e}_2 can be taken so that $q(\tilde{e}_2) \neq 0$ (since $q \neq 0$), \tilde{e}_1 being any vector in \mathbb{R}^2 , not collinear with \tilde{e}_2 . If so, observe that the local zero set of q contains no element of the form $\xi_2 \tilde{e}_2, \xi_2 \neq 0$, since $q(\xi_2 \tilde{e}_2) = a_k \xi_2^k$. In other words, each nonzero solution of the equation $q(\tilde{\xi}) = 0$ has a nonzero component ξ_1 . Dividing then (1.4) by ξ_1^k , we find

$$q(\tilde{\xi}) = 0 \Leftrightarrow \sum_{s=0}^k a_s \left(\frac{\xi_2}{\xi_1} \right)^s = 0.$$

Setting $\tau = \frac{\xi_2}{\xi_1}$ and since τ is real whenever ξ_1 and ξ_2 are, we see that τ must be a *real root* of the polynomial

$$a(\tau) = \sum_{s=0}^k a_s \tau^s.$$

Conversely, to each real root τ of the above polynomial is associated the line $\{\xi_1 \tilde{e}_1 + \tau \xi_1 \tilde{e}_2; \xi_1 \in \mathbb{R}\}$ of solutions of the equation $q(\tilde{\xi}) = 0$. Here, the inequality $\nu \leq k$ follows from the *fundamental theorem of algebra*.

Remark 1.1. Writing

$$a(\tau) = \sum_{s=0}^k a_s \tau^s = a_k \prod_{s=1}^k (\tau - \tau_s)$$

where $\tau_s, 1 \leq s \leq k$ are the k (not necessarily distinct) roots of the polynomial $a(\tau)$ and replacing τ by ξ_2/ξ_1 with $\xi_1 \neq 0$, we find

$$q(\tilde{\xi}) = a_k \prod_{s=1}^k (\xi_2 - \tau_s \xi_1),$$

as relation which remains valid when $\xi_1 = 0$.

COMMENT 1.3. Recall that a continuous *odd* mapping defined on the sphere $S_{m-1} \subset \mathbb{R}^m$ with values in \mathbb{R}^n always vanishes at some point of S_{m-1} when $m > n$. Here, with $m = n + 1$ we deduce that $\nu \geq 1$ when k is *odd*. When k is *even*, it can be shown (cf. Buchner, Marsden and Schecter [5]) that ν is *even too* (possibly 0 however). 41

Remark 1.2. Any small perturbation of q (as a homogeneous polynomial mapping of degree k) still verifies the condition (\mathbb{R} - N.D.) and its local zero set is made of the *same number of lines* (Hint: let Q denote the finite dimensional space of homogeneous polynomials of degree k . Consider the mapping $(p, \tilde{\zeta}) \in Q \times S_n \rightarrow p(\tilde{\zeta}) \in \mathbb{R}^n$ and note that the derivative at $(q, \tilde{\xi})$ with respect to $\tilde{\zeta}$ is an isomorphism when $q(\tilde{\xi}) = 0$.)

COMMENT 1.4. Condition (\mathbb{R} -N.D.) ensures that the zero set of q in \mathbb{R}^{n+1} is made of a finite number of lines through the origin. The converse is *not true*¹. Actually, the condition (\mathbb{R} -N.D.) also shows that each line in the zero set is “simple” in the way described in Remark 1.2. When the condition (\mathbb{R} -N.D.) does not hold but the zero set of q is still made up of a finite number of lines, some of them are “multiple”, namely, split into *several lines* or else *disappear* when replacing q by a suitable small perturbation.

2.2 Practical Verification of the Condition (\mathbb{R} -N.D.).

The above considerations leave us with two basic questions:

- (i) How does one check the condition (\mathbb{R} -N.D.) for a given mapping 42

¹Incidentally, we have shown that the zero set of q is a finite union of lines, when $n = 1$, with no assumption other than $q \neq 0$.

q ?

- (ii) If the condition (\mathbb{R} -N.D.) holds, how can we compute (approximations of) the lines in the zero set of q ?

Here we shall give a *partial answer* to these questions. We begin with the case $n = 1$. As we know, in every system of coordinates $\xi = \xi_1 \tilde{e}_1 + \xi_2 \tilde{e}_2$ of \mathbb{R}^2 such that $q(\tilde{e}_2) \neq 0$ (such a system always exists when $q \neq 0$), one has

$$q(\tilde{\xi}) = a_k \prod_{s=1}^k (\xi_2 - \tau_s \xi_1), \quad (2.1)$$

where $a_k = q(\tilde{e}_2) \neq 0$ and $\tau_s, 1 \leq s \leq k$ are the k roots of the polynomial

$$a(\tau) = \sum_{s=0}^k a_s \tau^s. \quad (2.2)$$

If so, each line in the zero set of q is of the form

$$\{\xi_1 \tilde{e}_1 + \tau_s \xi_1 \tilde{e}_2; \xi_1 \in \mathbb{R}\}, \quad (2.3)$$

where τ_s is a real root of $a(\tau)$. Saying that q verifies the condition (\mathbb{R} -N.D.) amounts to saying that the derivative $Dq(\tilde{\xi}) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ is *onto* (i.e. not equal to zero) at each point $\tilde{\xi} \in \mathbb{R}^2 - \{0\}$ such that $q(\tilde{\xi}) = 0$. For $\tilde{\xi}, \tilde{\zeta} \in \mathbb{R}^2$,

$$Dq(\tilde{\xi}) \cdot \tilde{\zeta} = a_k \sum_{s=1}^k \left[\prod_{\sigma \neq s} (\xi_2 - \tau_\sigma \xi_1) \right] (\zeta_2 - \tau_s \zeta_1). \quad (2.4)$$

Since $q(\tilde{\xi}) = 0$, there is an index s_0 such that $\xi_2 - \tau_{s_0} \xi_1 = 0$. Thus

$$Dq(\tilde{\xi}) \cdot \tilde{\zeta} = a_k \left[\prod_{\sigma \neq s_0} (\tau_{s_0} \xi_1 - \tau_\sigma \xi_1) \right] (\zeta_2 - \tau_{s_0} \zeta_1).$$

- 43** Clearly, the linear form $\tilde{\zeta} \in \mathbb{R}^2 \rightarrow \zeta_2 - \tau_{s_0} \zeta_1$ is onto (i.e. non-zero). Thus, $Dq(\tilde{\xi})$ will be onto if and only if

$$a_k \prod_{\sigma \neq s_0} (\tau_{s_0} - \tau_\sigma) \xi_1 \neq 0. \quad (2.5)$$

But from the choice of the system of coordinates, we know that $\xi_1 \neq 0$. Hence, $Dq(\tilde{\xi})$ will be onto if and only if

$$\tau_\sigma \neq \tau_{s_0} \text{ for } \sigma \neq s_0,$$

i.e. τ_{s_0} is a *simple root* of the polynomial $a(\tau)$.

To sum up, when $n = 1$, the mapping q will verify the condition (\mathbb{R} -N.D.) if and only if given a system of coordinates $(\tilde{e}_1, \tilde{e}_2)$ in \mathbb{R}^2 such that $q(\tilde{e}_2) (= a_k) \neq 0$, each real root of the polynomial $a(\tau) = \sum_{s=0}^k a_s \tau^s$ is *simple*.

In the particular case when $k = 2$, $a(\tau)$ is a quadratic polynomial.

- (i) If its discriminant is < 0 , it has no real root (then, each of them is simple) : (\mathbb{R} -N.D.) *holds*.
- (ii) If its discriminant is zero, it has a double real root: (\mathbb{R} -N.D.) *fails to hold*.
- (iii) If its discriminant is > 0 , it has two simple real roots: (\mathbb{R} -N.D.) *holds*.

Note that the above results give a way for finding (approximations of) the lines in the zero set of q . It suffices to use an algorithm for the computation of the roots of the polynomial $a(\tau)$. However, it is not easy to check whether a given root of a polynomial is simple, by calculating approximations to it through an algorithm and the first question is not satisfactorily answered. 44

Observe that it is of course *sufficient*, for the condition (\mathbb{R} -N.D.) to hold, that every root (real or complex) of the polynomial $a(\tau)$ is simple.

Definition 2.1. If every root (real or complex) of $a(\tau)$ is simple, we shall say that the polynomial q satisfies the condition of \mathbb{C} -nondegeneracy (in short, \mathbb{C} -N.D.)

Definition 2.1 can be described by saying that the polynomial $a(\tau)$ and its derivative $a'(\tau)$ have no common root.

The quantity $a_1^2 - 4a_0a_2$ is the usual discriminant and, as $a_2 \neq 0$ we conclude

$$\mathcal{D} \neq 0 \Leftrightarrow a_1^2 - 4a_2a_0 \neq 0 \quad (2.8)$$

Remark 2.1. When $k = 2$, the condition (\mathbb{R} - N.D.) is characterized by saying that the discriminant is $\neq 0$ too. Hence, when $k = 2$

$$(\mathbb{R} - N.D.) \Leftrightarrow (\mathbb{C} - N.D.), \quad (2.9)$$

but this is no longer true for $k \geq 4$.

We have found that the condition ($\mathbb{C} - N.D.$) holds $\Leftrightarrow \mathcal{D} \neq 0$. The advantage of this stronger assumption is that it is immediate to obtain the discriminant \mathcal{D} in terms of the coefficients a_s 's, hence from q .

Expression of the conditions ($\mathbb{R} - N.D.$) and ($\mathbb{C} - N.D.$) in any system of coordinates: We know to express the conditions ($\mathbb{R} - N.D.$) and ($\mathbb{C} - N.D.$) in a system of coordinates \tilde{e}_1, \tilde{e}_2 such that $q(\tilde{e}_2) \neq 0$. Actually, we can get such an expression in *any* system of coordinates. Indeed, assume $q(\tilde{e}_2) = 0$. Then, the coefficient a_k vanishes and we have

$$q(\tilde{\xi}) = \sum_{s=0}^{k-1} a_s \xi_1^{k-1-s} \xi_2^s. \quad (2.10)$$

It follows that the line $\{\xi_2 \tilde{e}_2, \xi_2 \in \mathbb{R}\}$ is in the zero set of q . Away from the origin on this line, the derivative $Dq(\tilde{\xi})$ must be onto (i.e. $\neq 0$). An immediate calculation shows, for $\tilde{\xi} = \xi_2 \tilde{e}_2$, that

$$Dq(\tilde{\xi}) \cdot \tilde{\zeta} = a_{k-1} \xi_2^{k-1} \zeta_1, \quad (2.11)$$

for every $\tilde{\zeta} \in \mathbb{R}^2$. As $\tilde{\xi}$ is $\neq 0$ if and only if ξ_2 is $\neq 0$, we have

$$Dq(\tilde{\xi}) \text{ is onto} \Leftrightarrow a_{k-1} \neq 0.$$

Now, for any solution $\tilde{\xi}$ of the equation $q(\tilde{\xi}) = 0$ which is not on the line $\mathbb{R}\tilde{e}_2$, we must have $\xi_1 \neq 0$. Arguing as before, we get

$$q(\tilde{\xi}) = \xi_1^k \sum_{s=0}^{k-1} a_s \left(\frac{\xi_2}{\xi_1} \right)^s$$

and all this amounts to solving the equation,

$$\sum_{s=0}^{k-1} a_s \left(\frac{\xi_2}{\xi_1} \right)^s = 0.$$

Setting $\tau = \xi_2/\xi_1$ and

$$a(\tau) = \sum_{s=0}^{k-1} a_s \tau^s$$

the method we used when $a_k \neq 0$ shows that the condition ($\mathbb{R} - N.D.$) is equivalent to the fact that *each real root* of $a(\tau)$ is simple. The condition ($\mathbb{C} - N.D.$) being expressed by saying that *every root* of $a(\tau)$ is simple, can be written as

$$\mathcal{D} \neq 0,$$

where \mathcal{D} is the discriminant of $a(\tau)$.

- 48 Remark 2.2.** As $a_k = 0$, the polynomial $a(\tau)$ must be considered as a polynomial of degree $k-1$, namely \mathcal{D} is a $(2k-3) \times (2k-3)$ determinant (instead of $(2k-1) \times (2k-1)$ when $a(\tau)$ is of degree k). If $a(\tau)$ is considered as a polynomial of degree k with leading coefficient equal to zero, the determinant we obtain is *always* zero and has no significance.

Now, we come back to the general case when n is arbitrary. Motivated by the results when $n = 1$, it is interesting to look for a generalization of the condition ($\mathbb{C} - N.D.$). Let Q be the k -linear symmetric mapping such that

$$q(\tilde{\xi}) = Q(\tilde{\xi}, \dots, \tilde{\xi}),$$

where the argument $\tilde{\xi}$ is repeated k times. Then Q has a *canonical extension* as a k -linear symmetric mapping from $\mathbb{C}^{n+1} \rightarrow \mathbb{C}^n$ (so that linear means \mathbb{C} -linear here). Indeed, \mathbb{C}^{n+1} and \mathbb{C}^n identify with $\mathbb{R}^{n+1} + i\mathbb{R}^{n+1}$ and $\mathbb{R}^n + i\mathbb{R}^n$ respectively. Now, take k elements $\tilde{\xi}^{(1)}, \dots, \tilde{\xi}^{(k)}$ in \mathbb{C}^{n+1} . These elements can be written as

$$\tilde{\xi}^{(s)} = \tilde{u}^{(s)} + i\tilde{v}^{(s)}, \tilde{u}^{(s)}, \tilde{v}^{(s)} \in \mathbb{R}^{n+1}, 1 \leq s \leq k.$$

Define the extension of Q (still denoted by Q) by

$$Q(\tilde{\xi}^{(1)}, \dots, \tilde{\xi}^{(k)}) = \sum_{j=1}^k i^{k-j} \sum_{P \in \mathcal{P}_j} Q(\tilde{u}^{(P(1))}, \tilde{u}^{(P(j))}, \tilde{v}^{(P(j+1))}, \tilde{v}^{(P(k))}),$$

where \mathcal{P}_j denotes the set of permutations of $\{1, \dots, k\}$ such that

$$P(1) < \dots < P(j)$$

and

$$P(j+1) < \dots < P(k).$$

It is easily checked that this defines an extension of Q which is \mathbb{C} -linear with respect to each argument and symmetric. An extension of q to \mathbb{C}^{n+1} is then

$$q(\tilde{\xi}) = Q(\tilde{\xi}, \dots, \tilde{\xi}). \quad (2.12)$$

Remark 2.3. In practice, if $q = (q_\alpha)_{\alpha=1, \dots, n}$ and

$$\tilde{\xi} = \xi_1 \tilde{e}_1 + \dots + \xi_{n+1} \tilde{e}_{n+1},$$

where $(\tilde{e}_1, \dots, \tilde{e}_{n+1})$ is a basis of \mathbb{R}^{n+1} with $\xi_1, \dots, \xi_{n+1} \in \mathbb{R}$, each q_α is a polynomial, homogeneous of degree k with real coefficients. The extension (2.12) is obtained by simply replacing each $\xi_j \in \mathbb{R}$ by $\xi_j \in \mathbb{C}$.

Definition 2.2. We shall say that q verifies the condition of \mathbb{C} -non-degeneracy (in short, (\mathbb{C} -N.D.)) if, for every non-zero solution $\tilde{\xi}$ of the equation $q(\tilde{\xi}) = 0$, the linear mapping $Dq(\tilde{\xi}) \in \mathcal{L}(\mathbb{C}^{n+1}, \mathbb{C}^n)$ (complex derivative) is onto.

Of course, when $n = 1$, Definition 2.2 coincided with Definition 2.1. Whenever the mapping q verifies the condition (\mathbb{C} -N.D.), it verifies the condition (\mathbb{R} -N.D.) as well: This is immediately checked after observing for $\tilde{\xi} \in \mathbb{R}^{n+1}$ that the restriction to \mathbb{R}^{n+1} of the complex derivative of q at $\tilde{\xi}$ is nothing but its real derivative. A method for checking the conditions (\mathbb{R} -N.D.) and (\mathbb{C} -N.D.) when $n = 2$ is described in Appendix 1 where we also make some comments on the general case, not quite completely solved however.

2.3 A Generalization of the Morse Lemma for Mappings from \mathbb{R}^{n+1} into \mathbb{R}^n : “Weak” Regularity Results.

- 50 From now on, the space \mathbb{R}^{n+1} is equipped with its euclidean structure. Let \mathcal{O} be an open neighbourhood of the origin in \mathbb{R}^{n+1} and

$$f : \mathcal{O} \rightarrow \mathbb{R}^n,$$

a mapping of class C^m , $m \geq 1$. Assume there is a positive integer k , $1 \leq k \leq m$ such that

$$D^j f(0) = 0 \quad 0 \leq j \leq k-1 \quad (3.1)$$

(in particular $f(0) = 0$). For every $\tilde{\xi} \in \mathbb{R}^{n+1}$, set

$$q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k. \quad (3.2)$$

Our purpose is to give a precise description of the zero set of f around the origin (local zero set). We may limit ourselves to seeking *nonzero* solution only. For this, we first perform a *transformation of the problem*. Let $r_0 > 0$ be such that the closed ball $\bar{B}(0, r_0)$ in \mathbb{R}^{n+1} is contained in \mathcal{O} . The problem will be solved if, for some r , $0 \leq r \leq r_0$ and every $0 < |t| < r$, we are able to determine the solutions of the equation

$$f(\tilde{x}) = 0, \|\tilde{x}\| = |t|.$$

It is immediate that $\tilde{\xi}$ is a solution for this system if and only if we can write

$$\tilde{x} = t\tilde{\xi}$$

- 51 with

$$\begin{cases} 0 < |t| < r, \tilde{x} \in S_n, \\ f(t\tilde{x}) = 0 \end{cases}$$

where S_n is the unit sphere in \mathbb{R}^{n+1} . Also it is not restrictive to assume that f is defined in the whole space \mathbb{R}^{n+1} (Indeed, f can always be extended as a C^m mapping outside $\bar{B}(0, r_0)$).

Now, let us define

$$g : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n,$$

by

$$g(t, \tilde{\xi}) = k \int_0^1 (1-s)^{k-1} D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^k ds, \quad (3.3)$$

a formula showing that g is of class C^{m-k} in $\mathbb{R} \times \mathbb{R}^{n+1}$.

Lemma 3.1. *Let g be defined as above. Then*

$$g(t, \tilde{\xi}) = \frac{k!}{t^k} f(t\tilde{\xi}) \text{ for } t \neq 0, \tilde{\xi} \in \mathbb{R}^{n+1}, \quad (3.4)$$

$$g(0, \tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k \text{ for } \tilde{\xi} \in \mathbb{R}^{n+1}. \quad (3.5)$$

Proof. The relation $g(0, \cdot) = q$ follows from the definitions immediately. Next for $\tilde{x} \in \mathbb{R}^{n+1}$, write the Taylor expansion of order $k-1$ of f about the origin. Due to (3.1),

$$f(\tilde{x}) = \frac{1}{(k-1)!} \int_0^1 (1-s)^{k-1} D^k f(s\tilde{x}) \cdot (\tilde{x})^k ds.$$

With $\tilde{x} = t\tilde{\xi}$, $t \neq 0$ and comparing with (3.3), we find

$$g(t, \tilde{\xi}) = \frac{k!}{t^k} f(t\tilde{\xi}).$$

□

From Lemma 3.1 the problem is equivalent to

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$$\begin{cases} 0 < |t| < r, \tilde{\xi} \in S_n, \\ g(t, \tilde{\xi}) = 0. \end{cases}$$

In what follows, we shall solve the equation (for small enough $r > 0$)

$$\begin{cases} t \in (-r, r), \tilde{\xi} \in S_n, \\ g(t, \tilde{\xi}) = 0. \end{cases} \quad (3.6)$$

Its solutions $(t, \tilde{\xi})$ with $t \neq 0$ will provide the nonzero solutions of $f(\tilde{x}) = 0$ verifying $\|\tilde{x}\| = |t|$ through the simple relation $\tilde{x} = t\tilde{\xi}$. Of course, the trivial solution $\tilde{x} = 0$ is also obtained as $\tilde{x} = 0\tilde{\xi}$ with $g(0, \tilde{\xi}) = q(\tilde{\xi}) = 0$ unless this equation has no solution on the unit sphere. Thus all the solutions of $f(\tilde{x}) = 0$ such that $0 < \|\tilde{x}\| < r$ are given by $\tilde{x} = t\tilde{\xi}$ with $(t, \tilde{\xi})$ solution of (3.6) provided that the zero set of q does not reduce to the origin.

From now on, we assume that the mapping q verifies the condition $(\mathbb{R} - N.D.)$ and we denote by $\nu \geq 0$ the number of lines in the zero set of q , so that the set

$$\{\tilde{\xi} \in S_n; q(\tilde{\xi}) = 0\}, \quad (3.7)$$

has exactly 2ν elements. We shall set

$$\{\tilde{\xi} \in S_n; q(\tilde{\xi}) = 0\} = \{\tilde{\xi}_0^1, \dots, \tilde{\xi}_0^{2\nu}\},$$

53 with an obvious abuse of notation when $\nu = 0$. This set is stable under multiplication by -1 , so that we may assume that the $\tilde{\xi}_0^j$'s have been arranged so that

$$\tilde{\xi}_0^{v+j} = -\tilde{\xi}_0^j, 1 \leq j \leq \nu. \quad (3.8)$$

Lemma 3.2. (i) Assume $\nu \geq 1$ and for each index $1 \leq j \leq 2\nu$, let $\sigma_j \subset S_n$ denote a neighbourhood of $\tilde{\xi}_0^j$. Then, there exists $0 < r \leq r_0$ such that the conditions $(t, \tilde{\xi}) \in (-r, r) \times S_n$ and $g(t, \tilde{\xi}) = 0$ together imply

$$\tilde{\xi} \in \bigcup_{j=1}^{2\nu} \sigma_j.$$

(ii) Assume that $\nu = 0$. Then, there exists $0 < r < r_0$ such that the equation $g(t, \tilde{\xi}) = 0$ has no solution in the set $(-r, r) \times S_n$.

Proof. (i) We argue by contradiction: If not, there is a sequence $(t_\ell, \tilde{\xi}_\ell)_{\ell \geq 1}$ with $\lim_{\ell \rightarrow \infty} t_\ell = 0$ and $\tilde{\xi}_\ell \in S_n$ such that

$$g(t_\ell, \tilde{\xi}_\ell) = 0$$

and

$$\tilde{\xi}_\ell \notin \bigcup_{j=1}^{2\nu} \sigma_j.$$

From the *compactness* of S_n and after considering a subsequence, we may assume that there exists $\tilde{\xi} \in S_n$ such that $\lim_{\ell \rightarrow +\infty} \tilde{\xi}_\ell = \tilde{\xi}$. By the continuity of g , $g(0, \tilde{\xi}) = 0$. As $g(0, \cdot) = q, \tilde{\xi}$ must be one of the elements $\tilde{\xi}_0^j$, which is impossible since

$$\tilde{\xi}_\ell \notin \bigcup_{j=1}^{2\nu} \sigma_j,$$

for every $\ell \geq 1$ so that the sequence $(\tilde{\xi}_\ell)$ cannot converge to $\tilde{\xi}$.

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(ii) Again we argue by contradiction. If there is a sequence $(t_\ell, \tilde{\xi}_\ell)_{\ell \geq 1}$ such that $\lim_{\ell \rightarrow +\infty} t_\ell = 0$, $\tilde{\xi}_\ell \in S_n$ and $g(t_\ell, \tilde{\xi}_\ell) = 0$, the continuity of g and the compactness of S_n show that there is $\tilde{\xi} \in S_n$ verifying $q(\tilde{\xi}) = g(0, \tilde{\xi}) = 0$ and we reach a contradiction with the hypothesis $\nu = 0$. \square

Remark 3.1. From Lemma 3.2, the equation $f(\bar{x}) = 0$ has then *no solution* $\bar{x} \neq 0$ in a sufficiently small neighbourhood of the origin when $\nu = 0$; in other words, the local zero set of f *reduces to the origin*.

We shall then focus on the main case when $\nu \geq 1$. For this we need the following lemma.

Lemma 3.3. *The mapping g verifies*

$$g \in C^{m-k}(\mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R}^n)$$

and the partial derivative $D_{\tilde{\xi}} g(t, \tilde{\xi})$ exists for every pair $(t, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^{n+1}$. Moreover

$$D_{\tilde{\xi}} g \in C^{m-k}(\mathbb{R} \times \mathbb{R}^{n+1}, \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n))$$

Proof. We already know that $g \in C^{m-k}$. Besides, the existence of a partial derivative $D_{\tilde{\xi}} g(t, \tilde{\xi})$ at every point $(t, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^{n+1}$ is obvious from the relations (3.4) and (3.5), from which we get

$$D_{\tilde{\xi}} g(t, \tilde{\xi}) = \frac{k!}{t^{k-1}} Df(t\tilde{\xi}) \text{ if } t \neq 0, \quad (3.9)$$

and

$$D_{\tilde{\xi}} g(0, \tilde{\xi}) = Dq(\tilde{\xi}) = kD^k f(0) \cdot (\tilde{\xi})^{k-1}. \quad (3.10)$$

First, assume that $k = 1$. Then

$$\begin{aligned} D_{\tilde{\xi}}g(t, \tilde{\xi}) &= Df(t, \tilde{\xi}), \\ D_{\tilde{\xi}}g(0, \tilde{\xi}) &= Df(0) \end{aligned}$$

and the assertion follows from the fact that Df is C^{m-1} , by hypothesis. Now, assume $k \geq 2$ and write the Taylor expansion of Df of order $k - 2$ about the origin. For every $\tilde{x} \in \mathbb{R}^{n+1}$ and due to (3.1)

$$Df(\tilde{x}) = \frac{1}{(k-2)!} \int_0^1 (1-s)^{k-2} D^k f(s\tilde{x}) \cdot (\tilde{x})^{k-1} ds.$$

With $\tilde{x} = t\tilde{\xi}$,

$$Df(t\tilde{\xi}) = \frac{t^{k-1}}{(k-2)!} \int_0^1 (1-s)^{k-2} D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^{k-1} ds.$$

From (3.9) and (3.10), the relation

$$D_{\tilde{\xi}}g(t, \tilde{\xi}) = k(k-1) \int_0^1 (1-s)^{k-2} D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^{k-1} ds$$

holds for every $t \in \mathbb{R}$ and every $\tilde{\xi} \in \mathbb{R}^{n+1}$. Hence the result, since the right hand side of this identity is of class C^{m-k} . \square

Finally, let us recall the so-called “strong” version of the Implicit function theorem (see Lyusternik and Sobolev [22]).

Lemma 3.4. *Let U, V and W be real Banach spaces and $F = (F(u, v))$ a mapping defined on a neighbourhood \mathcal{O} of the origin in $U \times V$ with values in W . Assume $F(0) = 0$ and*

- (i) F is continuous in \mathcal{O} ,
- (ii) the derivative $D_v F$ is defined and continuous in \mathcal{O} ,
- (iii) $D_v F(0) \in \text{Isom}(V, W)$.

Then, the zero set of F around the origin in $U \times V$ coincides with the graph of a continuous function defined in a neighbourhood of the origin in U with values in V .

Remark 3.2. In the above statement, F is *not* supposed to be C^1 and the result is *weaker* than in the usual Implicit function theorem. The function whose graph is the zero set of F around the origin is found to be poly continuous (instead of C^1). The proof of this “strong” version is the same as the proof of the usual statement after observing that the assumptions of Lemma 3.4 are sufficient to prove continuity.

We can now state an important result on the structure of the solutions of the equation (3.6).

Theorem 3.1. Assume $\nu \geq 1$; then, there exists $r > 0$ such that the equation

$$g(t, \tilde{\xi}) = 0, (t, \tilde{\xi}) \in (-r, r) \times S_n$$

is equivalent to

$$t \in (-r, r), \tilde{\xi} = \tilde{\xi}^j(t),$$

for some index $1 \leq j \leq 2\nu$ where, for each index $1 \leq j \leq 2\nu$, the function $\tilde{\xi}^j$ is of class C^{m-k} from $(-r, r)$ into S_n and is uniquely determined. In particular,

$$\tilde{\xi}^j(0) = \tilde{\xi}_0^j, 1 \leq j \leq 2\nu,$$

and

$$\tilde{\xi}^{\nu+j}(t) = -\tilde{\xi}^j(-t),$$

for every $1 \leq j \leq 2\nu$ and every $t \in (-r, r)$.

Proof. We first solve the equation $g(t, \tilde{\xi}) = 0$ around the solution $(t = 0, \tilde{\xi} = \tilde{\xi}_0^j)$ for each index $1 \leq j \leq 2\nu$ separately. Let us then fix $1 \leq j \leq 2\nu$. As we in Chapter 2, 2.1, the condition ($\mathbb{R} - N.D.$) allows us to write

$$\mathbb{R}^{n+1} = \text{Ker } Dq(\tilde{\xi}_0^j) \oplus T_{\tilde{\xi}_0^j} S_n. \quad (3.11)$$

From (3.5),

$$Dq(\tilde{\xi}_0^j) = D_{\tilde{\xi}} g(0, \tilde{\xi}_0^j)$$

and (3.11) shows that

$$D_{\tilde{\xi}}g(0, \tilde{\xi}_0^j)|_{T_{\tilde{\xi}_0^j}S_n} \in \text{Isom}(T_{\tilde{\xi}_0^j}S_n, \mathbb{R}^n). \quad (3.12)$$

Let then $\theta_j^{-1}(\theta_j = \theta_j(\xi'))$ be a chart around $\tilde{\xi}_0^j$, centered at the origin of \mathbb{R}^n (i.e. $\theta(0) = \tilde{\xi}_0^j$) and set

$$\hat{g}(t, \xi') = g(t, \theta_j(\xi')).$$

Then D_{ξ}, \hat{g} is defined around $(t = 0, \xi' = 0)$ and

$$D_{\xi}, \hat{g}(t, \xi') = D_{\tilde{\xi}}g(t, \theta_j(\xi')) \cdot D\theta_j(\xi')$$

58 From Lemma 3.3, it follows that \hat{g} and D_{ξ}, \hat{g} are of class C^{m-k} around the origin in $\mathbb{R} \times \mathbb{R}^n$. Besides,

$$\hat{g}(0) = g(0, \tilde{\xi}_0^j) = q(\tilde{\xi}_0^j) = 0$$

and combining (3.12) with the fact that $D\theta_j(0)$ is an isomorphism of \mathbb{R}^n to $T_{\tilde{\xi}_0^j}S_n$ (recall that θ_j^{-1} is a chart), one has

$$D_{\xi}, \hat{g}(0) \in \text{Isom}(\mathbb{R}^n, \mathbb{R}^n).$$

If $m - k \geq 1$, the Implicit function theorem (usual version) states that the zero set of \hat{g} around the origin is the graph of a (necessarily unique) mapping

$$t \rightarrow \xi'(t)$$

of class C^{m-k} around the origin verifying $\xi'(0) = 0$. If $m - k = 0$, the same result holds by using the “strong” version of the Implicit function theorem (Lemma 3.4). The zero of g around the point $(0, \tilde{\xi}_0^j)$ is then the graph of the mapping

$$t \rightarrow \theta_j(\xi'(t)),$$

of class C^{m-k} around the origin, which is the desired mapping $\tilde{\xi}^j(t)$. In particular, given any sufficiently small $r > 0$ and any sufficiently small neighbourhood σ_j of $\tilde{\xi}_0^j$ in S_n , there are no solutions of the equation $g(t, \tilde{\xi}) = 0$ in $(-r, r) \times \sigma_j$ other than those of the form $(t, \tilde{\xi}^j(t))$.

At this stage, note that the above property is not affected by arbitrarily shrinking $r > 0$. In particular, the same $r > 0$ can be chosen for every index $1 \leq j \leq 2\nu$. Further, applying Lemma 3.2 with disjoint neighbourhoods $\sigma_j, 1 \leq j \leq 2\nu$, we see after shrinking r again, if necessary, that there is *no* solution of the equation $g(t, \tilde{\xi}) = 0$ with $|t| < r$ outside $\bigcup_{j=1}^{2\nu} \sigma_j$. In other words, we have $\tilde{\xi}^j(t) \in \sigma_j, 1 \leq j \leq 2\nu$ and the solutions of $g(t, \tilde{\xi}) = 0$ with $|t| < r$ are exactly the 2ν pairs $(t, \tilde{\xi}^j(t)), 1 \leq j \leq 2\nu$. These pairs are distinct, since the neighbourhoods σ_j we have been taken disjoint.

Finally, writing $g(-t, \tilde{\xi}^j(-t)) = 0$ for $|t| < r$ and observing that $g(-t, -\tilde{\xi}) = g(t, \tilde{\xi})$, we find $g(t, -\tilde{\xi}^j(-t)) = 0$ for $|t| < r$. Since $-\tilde{\xi}^j(0) = -\tilde{\xi}_0^j = \tilde{\xi}_0^{\nu+j}$ for $1 \leq j \leq \nu$ (cf. (3.8)), we deduce that the function $-\tilde{\xi}^j(-t)$ has the property characterizing $\tilde{\xi}^{\nu+j}(t)$ and the proof is complete. \square

Corollary 3.1. *Under our assumptions, the equation $f(\tilde{x}) = 0$ has no solutions $(\tilde{x}) \neq 0$ around the origin in \mathbb{R}^{n+1} when $\nu = 0$. When $\nu \geq 1$ and for $r > 0$ small enough, the solutions of the equation $f(\tilde{x}) = 0$ with $\|\tilde{x}\| < r$ are given by $\tilde{x} = \tilde{x}^j(t), 1 \leq j \leq \nu$, where the functions $\tilde{x}^j \in C^{m-k}((-r, r), \mathbb{R}^{n+1})$ are defined through the functions $\tilde{\xi}, 1 \leq j \leq \nu$ of Theorem 3.1 by the formula*

$$\tilde{x}^j(t) = t\tilde{\xi}^j(t), t \in (-r, r).$$

In addition, the functions $\tilde{x}^j, 1 \leq j \leq \nu$, are differentiable at the origin with

$$\frac{d\tilde{x}^j}{dt} = \tilde{\xi}^j, 1 \leq j \leq \nu.$$

Proof. We already observed in Remark 3.1 that the local zero set of f reduces to $\{0\}$ when $\nu = 0$. Assume then $\nu \geq 1$. We know that the solution of $f(\tilde{x}) = 0$ with $\|\tilde{x}\| < r$ are of the form $\tilde{x} = t\tilde{\xi}$ with $|t| < r, \tilde{\xi} \in \mathcal{S}_n, g(t, \tilde{\xi}) = 0$. From Theorem 3.1, $r > 0$ can be chosen so that

$$\tilde{x} = \tilde{x}^j(t) = t\tilde{\xi}^j(t), 1 \leq j \leq \nu.$$

Like $\tilde{\xi}^j$, each function \tilde{x}^j is of class C^{m-k} . In addition, for $t \neq 0$,

$$\frac{\tilde{x}^j(t)}{t} = \tilde{\xi}^j(t),$$

so that $\frac{d\tilde{x}^j}{dt}(0)$ exists with

$$\frac{d\tilde{x}^j}{dt}(0) = \lim_{t \rightarrow 0} \tilde{\xi}^j(t) = \tilde{\xi}_0^j.$$

Finally, from the relation $\tilde{\xi}^{\nu+j}(t) = -\tilde{\xi}^j(t)$, for $1 \leq j \leq \nu$, we get

$$\tilde{x}^{\nu+j}(t) = \tilde{x}^j(t), 1 \leq j \leq \nu,$$

and the solution of $f(x) = 0$ are given through the first ν functions \tilde{x}^j only. \square

2.4 Further Regularity Results.

With Corollary 3.1 as a starting point, we shall now show, *without any additional assumption, that the functions \tilde{x}^j are actually of class C^{m-k+1} at the and of class C^m away from it.* This latter assertion is the simpler one to prove.

61 Lemma 4.1. *After shrinking $r > 0$ if necessary, the functions \tilde{x}^j , $1 \leq j \leq \nu$ are of class C^m on $(-r, r) - \{0\}$.*

Proof. From the relation $\tilde{x}^j(t) = t\tilde{\xi}^j(t)$, it is clear that the functions \tilde{x}^j and $\tilde{\xi}^j$ have the same regularity away from the origin. Thus, we shall show that the functions $\tilde{\xi}^j$ are of class C^m away from the origin. Recall that $\tilde{\xi}^j$ is characterized by

$$\begin{cases} \tilde{\xi}^j(t) \in S_n \\ g(t, \tilde{\xi}^j(t)) = 0, \\ \tilde{\xi}^j(0) = \tilde{\xi}_0^j, \end{cases}$$

for $t \in (-r, r)$. Also recall (cf. (3.12))

$$D_{\tilde{\xi}}g(0, \tilde{\xi}_0^j) = Dq(\tilde{\xi}_0^j) \in \text{Isom}(T_{\tilde{\xi}_0^j}S_n, \mathbb{R}^n).$$

Then, by the continuity of $D_{\tilde{\xi}}g$ (Lemma 3.3), there is an open neighbourhood σ_j of $\tilde{\xi}_0^j$ in S_n such that, after shrinking $r > 0$ if necessary,

$$D_{\tilde{\xi}}g(t, \tilde{\xi}) \in \text{Isom}(T_{\tilde{\xi}}S_n, \mathbb{R}^n) \quad (4.1)$$

for every $(t, \tilde{\xi}) \in (-r, r) \times \sigma_j$. By shrinking $r > 0$ again and due to the continuity of the function $\tilde{\xi}^j$, we can suppose that $\tilde{\xi}^j$ takes its values in σ_j for $t \in (-r, r)$. Let then $t_0 \in (-r, r)$, $t_0 \neq 0$. Thus

$$g(t_0, \tilde{\xi}^j(t_0)) = 0,$$

and, from (4.1),

$$D_{\tilde{\xi}}g(t_0, \tilde{\xi}^j(t_0)) \in \text{Isom}(T_{\tilde{\xi}^j(t_0)}S_n, \mathbb{R}^n).$$

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As $t_0 \neq 0$, $g(t, \tilde{\xi})$ is given by (3.4) for t around t_0 and $\tilde{\xi} \in \mathbb{R}^{n+1}$ and thus the mapping g has the same regularity as f (i.e. is of class C^m) around the point $(t_0, \tilde{\xi}^j(t_0))$. By the Implicit function theorem, we find that the zero set of g around $(t_0, \tilde{\xi}^j(t_0))$ in $(-r, r) \times S_n$ coincides with the graph of a unique function $\tilde{\zeta} = \tilde{\zeta}(t)$ of class C^m around t_0 , such that $\tilde{\zeta}^j(t_0) = \tilde{\xi}^j(t_0)$. But from the *uniqueness*, we must have $\tilde{\zeta}(t) = \tilde{\xi}^j(t)$ around $t = t_0$ so that the function $\tilde{\xi}^j(\cdot)$ is of class C^m around t_0 for every $t_0 \neq 0$ in $(-r, r)$. \square

To prove the regularity C^{m-k+1} of the function \tilde{x}^j at the origin, we shall introduce the mapping

$$h : \mathbb{R} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n,$$

defined by

$$h(t, \tilde{\xi}) = Df(t\tilde{\xi}) \cdot \tilde{\xi} - \int_0^1 Df(st\tilde{\xi}) \cdot \tilde{\xi} ds, \quad (4.2)$$

if $k = 1$ and by

$$h(t, \tilde{\xi}) = k \int_0^1 \frac{d}{ds} [-s(1-s)^{k-1}] D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^k ds, \quad (4.3)$$

if $k \geq 2$. Since f is of class C^m , it is clear, in any case, that $h \in C^{m-k}(\mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R}^n)$. In addition, with $t = 0$ in the definition of h ((4.3)), we find

$$h(0, \tilde{\xi}) = 0 \text{ for every } \tilde{\xi} \in \mathbb{R}^{n+1}. \quad (4.4)$$

Lemma 4.2. For any $(t, \tilde{\xi}) \in (\mathbb{R} - \{0\}) \times \mathbb{R}^{n+1}$, the partial derivative $(\partial g / \partial t)$ 63
 $(t, \tilde{\xi})$ exists and

$$\frac{\partial q}{\partial t}(t, \tilde{\xi}) = \frac{1}{t} h(t, \tilde{\xi}). \quad (4.5)$$

Proof. The existence of $\frac{\partial q}{\partial t}(t, \tilde{\xi})$ for $t \neq 0$ immediate from the relation (3.4) from which we get

$$\frac{\partial q}{\partial t}(t, \tilde{\xi}) = \frac{k!}{t^k} (Df(t\tilde{\xi}) \cdot \tilde{\xi} - \frac{k}{t} f(t\tilde{\xi})).$$

Hence

$$\frac{t \partial q}{\partial t}(t, \tilde{\xi}) = \frac{k!}{t^{k-1}} (Df(t\tilde{\xi}) \cdot \tilde{\xi} - \frac{k}{t} f(t\tilde{\xi})). \quad (4.6)$$

If $k = 1$, the relation (4.6) becomes

$$\frac{t \partial q}{\partial t}(t, \tilde{\xi}) = Df(t\tilde{\xi}) \cdot \tilde{\xi} - \frac{1}{t} f(t\tilde{\xi}).$$

Writing

$$f(t\tilde{\xi}) = t \int_0^1 Df(st\tilde{\xi}) \cdot \tilde{\xi} ds,$$

the desired relation (4.5) follows from (4.2).

Now, assume $k \geq 2$. Recall the relations

$$f(t\tilde{\xi}) = \frac{t^k}{(k-1)!} \int_0^1 (1-s)^{k-1} D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^k ds,$$

$$Df(t\tilde{\xi}) = \frac{t^{k-1}}{(k-2)!} \int_0^1 (1-s)^{k-2} D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^{k-1} ds,$$

that we have already used in the proofs of Lemma 3.1 and Lemma 3.3 respectively. After an immediate calculation, (4.6) becomes

$$t \frac{\partial q}{\partial t}(t, \tilde{\xi}) = k \int_0^1 [(k-1)(1-s)^{k-2}] D^k f(st\tilde{\xi}) \cdot (\tilde{\xi})^k ds.$$

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But

$$(k-1)(1-s)^{k-2} - k(1-s)^{k-1} = \frac{d}{ds} [-s(1-s)^{k-1}],$$

so that (4.5) follows from the definition (4.3) of h . □

Theorem 4.1. (Structure of the local zero set of f): Assume $\nu \geq 1$. For sufficiently small, $r > 0$ the local zero set of f in the ball $B(0, r) \subset \mathbb{R}^{n+1}$ consists of exactly ν curves of class C^{m-k+1} at the origin and class C^m away from the origin. These curves are tangent to a different one from among the ν lines in the zero set of $q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k$ at the origin.

Proof. In Lemma 4.1, we proved that the functions \tilde{x}^j and $\tilde{\xi}^j$ are of class C^m away from the origin. Since $m \geq 1$, we may differentiate the identity

$$\tilde{x}^j(t) = t\tilde{\xi}^j(t),$$

to get

$$\frac{d\tilde{x}^j(t)}{dt} = t\frac{d\tilde{\xi}^j(t)}{dt} + \tilde{\xi}^j(t), 0 < |t| < r. \quad (4.7)$$

Also, we know that \tilde{x}^j is differentiable at the origin with

$$\frac{d\tilde{x}^j}{dt}(0) = \tilde{\xi}_0^j.$$

We shall prove that the function $(d\tilde{x}^j/dt)$ is of class C^{m-k} at the origin. Let $\sigma_j \subset S_n$ be the open neighbourhood of $\tilde{\xi}_0^j$ considered in Lemma 4.1, so that $D_{\tilde{\xi}}g(t, \tilde{\xi}) \in \text{Isom}(T_{\tilde{\xi}}S_n, \mathbb{R}^n)$ for every $(t, \tilde{\xi}) \in (-r, r) \times \sigma_j$. 65
In other words, the mapping $[D_{\tilde{\xi}}g(t, \tilde{\xi})|_{T_{\tilde{\xi}}S_n}]^{-1}$ is an isomorphism of \mathbb{R}^n to the space $T_{\tilde{\xi}}S_n \subset \mathbb{R}^{n+1}$ for every pair $(t, \tilde{\xi}) \in (-r, r) \times \sigma_j$ and hence can be considered as a one-to-one linear mapping from \mathbb{R}^n into \mathbb{R}^{n+1} (with range $T_{\tilde{\xi}}S_n \subset \mathbb{R}^{n+1}$). A simple but crucial observation is that the regularity C^{m-k} of the mapping $D_{\tilde{\xi}}g$ (Lemma 3.3) yields the regularity C^{m-k} of the mapping

$$(t, \tilde{\xi}) \in (-r, r) \times \sigma_j \rightarrow [D_{\tilde{\xi}}g(t, \tilde{\xi})|_{T_{\tilde{\xi}}S_n}]^{-1} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n+1}).$$

This is easily seen by considering a chart of S_n around $\tilde{\xi}_0^j$ and can be formally seen by observing that the dependence of the tangent space $T_{\tilde{\xi}}S_n$ on the variable $\tilde{\xi} \in S_n$ is C^∞ while taking the inverse of an invertible linear mapping is a C^∞ operation. Setting, for $(t, \tilde{\xi}) \in (-r, r) \times \sigma_j$,

$$\varphi^j(t, \tilde{\xi}) = [D_{\tilde{\xi}}g(t, \tilde{\xi})|_{T_{\tilde{\xi}}S_n}]^{-1} h(t, \tilde{\xi})$$

and noting that $h \in C^{m-k}(\mathbb{R} \times \mathbb{R}^{n+1}, \mathbb{R}^n)$, we deduce

$$\varphi^j \in C^{m-k}((-r, r) \times \sigma_j, \mathbb{R}^{n+1}).$$

66 Note from (4.4) that $\varphi^j(0, \tilde{\xi}) = 0$ for $\tilde{\xi} \in \sigma_j$. On the other hand, by implicit differentiation of the identity $g(t, \tilde{\xi}^j(t)) = 0$ for $0 < |t| < r$, (the chain rule applies since g is differentiable with respect to $(t, \tilde{\xi})$ at any point of $(\mathbb{R} - \{0\}) \times \mathbb{R}^{n+1}$),

$$\frac{\partial q}{\partial t}(t, \tilde{\xi}^j(t)) + D_{\tilde{\xi}} g(t, \tilde{\xi}^j(t)) \cdot \frac{d\tilde{\xi}^j}{dt}(t) = 0.$$

But $(d\tilde{\xi}^j/dt)(t) \in T_{\tilde{\xi}^j(t)} S_n$, since $\tilde{\xi}^j$ takes its values in S_n . Hence

$$\frac{d\tilde{\xi}^j}{dt}(t) = - \left[D_{\tilde{\xi}} g(t, \tilde{\xi}^j(t))|_{T_{\tilde{\xi}^j(t)} S_n} \right]^{-1} \frac{\partial q}{\partial t}(t, \tilde{\xi}^j(t)).$$

With Lemma 4.2, this yields

$$\frac{d\tilde{\xi}^j}{dt}(t) = -\frac{1}{t} \varphi^j(t, \tilde{\xi}^j(t)), 0 < |t| < r$$

and the relation (4.7) can be rewritten as

$$\frac{d\tilde{x}^j}{dt}(t) = -\varphi^j(t, \tilde{\xi}^j(t)) + \tilde{\xi}^j(t), 0 < |t| < r. \quad (4.8)$$

But (cf. Corollary 3.1)

$$\frac{d\tilde{x}^j}{dt}(0) = \tilde{\xi}_0^j,$$

whereas

$$-\varphi^j(0, \tilde{\xi}(0)) + \tilde{\xi}^j(0) = -\varphi^j(0, \tilde{\xi}_0^j) + \tilde{\xi}_0^j = \tilde{\xi}_0^j.$$

since $\varphi^j(0, \tilde{\xi}) = 0$ for $\tilde{\xi} \in \sigma_j$. Thus the identity (4.8) holds for every $t \in (-r, r)$. As its right hand side is of class C^{m-k} , we obtain

$$\frac{d\tilde{\xi}^j}{dt} \in C^{m-k}((-r, r), \mathbb{R}^n).$$

As a last step, it remains to show that the curves generated by the functions $\tilde{\xi}^j$, $1 \leq j \leq 2\nu$ have the same regularity as the functions \tilde{x}^j themselves. From an elementary result of differential geometry, it is sufficient to prove that

$$\frac{d\tilde{x}^j}{dt}(t) \neq 0, t \in (-r, r).$$

But this is immediate from (4.8) by observing that $\tilde{\xi}^j(t)$ and $\varphi^j(t, \tilde{\xi}^j(t))$ are orthogonal, since $\varphi^j(t, \tilde{\xi}^j(t)) \in T_{\tilde{\xi}^j(t)} S_n$. Hence

$$\left\| \frac{d\tilde{x}^j(t)}{dt} \right\| \geq \|\tilde{\xi}^j(t)\| = 1, t \in (-r, r)$$

and the proof is complete. \square

COMMENT 4.1. When $k = 1$, the condition ($\mathbb{R} - N.D.$) amounts to saying that $Df(0)$ is onto. In particular, $\nu = 1$ and the local zero set of f is made up of exactly one curve of class C^m away from the origin and also C^m at the origin: the conclusion is the same as while using the *Implicit function theorem*.

COMMENT 4.2. Assume now $n = 1$ and $k = 2$. The condition ($\mathbb{R} - N.D.$) is the Morse condition and we know that $\nu = 0$ or $\nu = 2$. The statement is nothing but the weak form of the Morse Lemma.

COMMENT 4.3. In the same direction see the articles by Magnus [24], Buchner, Marsden and Schechter [5] Szulkin [40] among others. Theorem 4.1 is a particular case of the study made in Rabier [29]. More generally, the following extension (which, however, has no major application in the nondegenerate cases we shall consider in Chapter 3) does not require $Df(0)$ to vanish. Such a result is important in generalizations of the desingularization process we shall describe in Chapter 5.

Theorem 4.2. Let f be a mapping of class C^m , $m \geq 1$, defined on a neighbourhood of the origin in \mathbb{R}^{n+1} with values in \mathbb{R}^n . Let us set

$$n_0 = n - \dim \text{Range} Df(0),$$

so that $0 \leq n_0 \leq n$ and the spaces $\text{Ker } Df(0)$ and $\mathbb{R}^n/\text{Range } Df(0)$ can be identified with \mathbb{R}^{n_0+1} and \mathbb{R}^{n_0} respectively. Let π denote the canonical projection operator from \mathbb{R}^n onto $\mathbb{R}^n/\text{Range } Df(0)$. We assume that

$$\begin{cases} f(0) = 0 \\ \pi D^j f(0) = 0, 1 \leq j \leq k-1 \end{cases}$$

for some $1 \leq k \leq m$ and that the mapping

$$q_0 : \tilde{\xi} \in \text{Ker } Df(0) \simeq \mathbb{R}^{n_0+1} \rightarrow q_0(\tilde{\xi}) = \pi D^k f(0) \cdot (\tilde{\xi})^k \in \mathbb{R}^n / \text{Range } Df(0) \simeq \mathbb{R}^{n_0}$$

verifies the condition ($\mathbb{R} - N.D.$). Then the zero set of the mapping consists of a finite number $\nu \leq k^{n_0}$ of lines through the origin in $\text{Ker } Df(0)$ and the local zero set of the mapping f consists of exactly ν curves of class C^m away from the origin and of class C^{m-k+1} at the origin. These ν curves are tangent to a different one from among the lines of the zero of the mapping q_0 at the origin.

For a proof, cf. [29]. Theorem 3.2.

2.5 A Generalization of the Strong Version of the Morse Lemma.

- 69 To complete this chapter, we shall see that the local zero set of f can be deduced from the zero set of the polynomial mapping $q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k$ through an origin-preserving C^{m-k+1} local diffeomorphism of the ambient space \mathbb{R}^{n+1} under some additional assumptions on the lines in the zero set of q .

Theorem 5.1. *Under the additional assumptions that $\nu \leq n+1$ and the ν lines in the zero set of q are linearly independent (a vacuous condition if $\nu = 0$), there exists an origin-preserving local diffeomorphism ϕ of class C^{m-k+1} in \mathbb{R}^{n+1} that transforms the local zero set of the mapping q into the local zero set of the mapping f . Moreover, ϕ is of class C^m away from the origin and can be taken so that $D\phi(0) \equiv I$.*

Proof. If $\nu = 0$, we can choose $\phi = I$. If $\nu \geq 1$, the local zero set of f is the image of the ν functions $\tilde{x}^j(t)$, $1 \leq j \leq \nu$, $|t| < r$. These functions

are of class C^{m-k+1} at the origin and of class C^m away from it, with $(d\bar{x}^j/dt)(0) = \tilde{\xi}_0^j$, $1 \leq j \leq \nu$.

Let us denote by $(\cdot|\cdot)$ the usual inner product of \mathbb{R}^{n+1} . We may write

$$\tilde{x}^j(t) = \alpha_j(t)\tilde{\xi}_0^j + \tilde{\zeta}^j(t),$$

where

$$\alpha_j(t) = (\tilde{x}^j(t)|\tilde{\xi}_0^j).$$

Therefore, both functions α_j and $\tilde{\zeta}^j$ are of class C^{m-k+1} at the origin and of class C^m away from it. Besides, 70

$$\begin{cases} \alpha_j(0) = 0 \in \mathbb{R}, \frac{d\alpha_j}{dt}(0) = 1, \\ \tilde{\zeta}^j(0) = 0 \in \mathbb{R}^{n+1}, \frac{d\tilde{\zeta}^j}{dt}(0) = 0 \in \mathbb{R}^{n+1}. \end{cases} \quad (5.1)$$

From (5.1), after shrinking r if necessary, we deduce that the function α_j is a C^{m-k+1} diffeomorphism from $(-r, r)$ to an open interval I_j containing zero. We shall set

$$(-\rho, \rho) = \bigcap_{j=1}^{\nu} I_j, \rho > 0$$

so that each function α_j^{-1} is well defined, of class C^{m-k+1} in $(-\rho, \rho)$ and of class C^m away from the origin,

As the ν vectors $\tilde{\xi}_0^j$, $1 \leq j \leq \nu$ are linearly independent we can find a bilinear form $a(\cdot, \cdot)$ on \mathbb{R}^{n+1} such that

$$a(\tilde{\xi}_0^i, \tilde{\xi}_0^j) = \delta_{ij} \text{ (Kronecker delta)}, 1 \leq i, j \leq \nu \quad (5.2)$$

For $\|\tilde{\xi}\| < \rho$, let us define

$$\phi(\tilde{\xi}) = \tilde{\xi} + \sum_{j=1}^{\nu} \tilde{\zeta}_0^j \alpha_j^{-1} a(\tilde{\xi}, \tilde{\xi}_0^j)$$

clearly, ϕ is of class C^{m-k+1} at the origin and of class C^m away from it and further $\phi(0) = 0$, $D\phi(0) = I$. Therefore ϕ is an origin preserving local diffeomorphism of \mathbb{R}^{n+1} having the desired regularity properties.

Let us show that ϕ transforms the local zero set of q into the local zero set of : For $|\tau| < \rho$ and $1 \leq i \leq \nu$

$$\begin{aligned} \phi(\tau \tilde{\xi}_0^i) &= \tau \tilde{\xi}_0^j + \sum_{j=1}^{\nu} \tilde{\zeta}^j(\alpha_j^{-1}(a(\tau \tilde{\xi}_0^i, \tilde{\xi}_0^j))) \\ &= \tau \tilde{\xi}_0^i + \sum_{j=1}^{\nu} \tilde{\zeta}(\alpha_j^{-1}(\tau) \delta_{ij}) = \tau \tilde{\xi}_0^i + \tilde{\zeta}^i(\alpha_i^{-1}(\tau)) \\ &= \tilde{x}^i(\alpha_i^{-1}(\tau)). \end{aligned}$$

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□

COMMENT 5.1. Assume $k = 2$, $n = 1$. Then the condition ($\mathbb{R} - NM.D.$) is equivalent to the Morse condition and $\nu = 0$ or $\nu = 2$. In any case, $\nu \leq n + 1 = 2$ and, if $\nu = 2$, the two lines are distinct. Thus, they are linearly independent and Theorem 5.1 coincides with the “strong” version of the Morse lemma (Theorem 3.1 of Chapter 1).

Remark 5.1. If the diffeomorphism ϕ is only required to be C^1 at the origin, the result is true *regardless of the number of lines* in the zero set of q ($\leq k^n$ because of the condition ($\mathbb{R} - N.D.$)) and without assuming of course that they are linearly independent. The condition also extends to mapping from \mathbb{R}^{n+p} into \mathbb{R}^n , $p \geq 1$. Under this form, there is also a *stronger result* when a *stronger assumption* holds : Assume for every $\tilde{\xi} \in \mathbb{R}^{n+1} - \{0\}$ that the derivative $Dq(\tilde{\xi})$ is onto. Then, the diffeomorphism ϕ can be taken so that

$$f(\phi(\tilde{\xi})) = q(\tilde{\xi}),$$

for $\tilde{\xi}$ around the origin (see Buchner, Marsden and Schechter [5]). For $k = 1$, the assumption reduces to saying that $Df(0)$ is onto. For $k = 2$ and $n = 1$, it reduces to the Morse condition again.

Chapter 3

Applications to Some Nondegenerate problems

The AIM of this Chapter is to show how to apply the general results of Chapter 2, 2.4, to the one-parameter nonlinear problems introduced in Chapter 1. Such an application is possible either directly or after a preliminary change of the parameter. We answer such questions as getting an upper and a nontrivial lower bound for the *number of curves*, their *regularity* and their *location* in space. We have limited ourselves to the study of two problems, presented in §§2 and 3 respectively. These examples are drawn from a global synthetic approach developed in Rabier [31], whose technicalities seem to be too tedious to be wholly reproduced here. 72

In the first section, we prove a generalization of Theorem 3.2 of Chapter 1. This generalization is of interest because it shows that when the results of Chapter 2, §4, are applied to a mapping f which is the reduced mapping of some problem posed between real Banach spaces, the assumptions on f are *independent of the Lyapunov-Schmidt reduction* used for reducing the problems to a finite-dimensional one.

The second section is devoted to problems of bifurcation from the *trivial branch* at a *multiple characteristic value*. As the Morse lemma was shown to provide the results of Crandall and Rabinowitz when the characteristic value is simple (cf. Chapter 1), the use of the extended 73

version yields again, with various additional improvements, the conclusions of the earlier work by McLeod and Sattinger in their study of the same problem ([23]). Thus, the analysis of bifurcation from the trivial branch at a simple or multiple characteristic value appears to follow from the *same general statement*, giving some homogeneity to the presentation.

In the second section, we consider another example, in which no branch of solutions is known a priori. In the simplest form of the problem, two typical situations are those when the origin is a “*turning point*” or a “*hysteresis point*”. These notions are made precise and the structure of the local zero set is also determined in the presence of a higher order singularity. In particular, it is shown that bifurcation can be expected in this case.

3.1 Equivalence of Two Lyapunov-Schmidt Reductions.

Here, we shall define and prove the equivalence of any two Lyapunov-Schmidt reductions of a given problem. The notion of equivalence is a key tool for proving a general version of Theorem 3.2 of Chapter 1. The results of this section are due to W.J. Beyn [42].

Let then \tilde{X} and Y be two real Banach spaces and $G : \tilde{X} \rightarrow Y^1$ a mapping of class C^m , $m \geq 1$, satisfying the conditions:

$$G(0) = 0, \quad (1.1)$$

$$DG(0) \text{ is a Fredholm operator with index } 1. \quad (1.2)$$

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As in Chapter 1, we shall set

$$\tilde{X}_1 = \text{Ker } DG(0), \quad (1.3)$$

$$Y_2 = \text{Range } DG(0), \quad (1.4)$$

¹Actually, the mapping G needs only to be defined in a neighbourhood of the origin.

so that Y_2 has finite codimension $n \geq 0$ and \widetilde{X}_1 has finite dimension $n + 1$ from the assumption (1.2). Given two topological complements \widetilde{X}_2 and Y_1 of \widetilde{X}_1 and Y_2 respectively, recall that Q_1 and Q_2 denote the (continuous) projection operators onto Y_1 and Y_2 respectively and that, writing $\widetilde{x} \in \widetilde{X}$ in the form $\widetilde{x} = \widetilde{x}_1 + \widetilde{x}_2$, the reduced mapping is defined by

$$\widetilde{x}_1 \in \widetilde{X}_1 \rightarrow f(\widetilde{x}_1) = Q_1 G(\widetilde{x}_1 + \widetilde{\varphi}(\widetilde{x}_1)) \in Y_1, \quad (1.5)$$

where the mapping $\widetilde{\varphi}$ with values in the space \widetilde{X}_2 is characterized by

$$Q_2 G(\widetilde{x}_1 + \widetilde{\varphi}(\widetilde{x}_1)) = 0, \quad (1.6)$$

for \widetilde{x}_1 around the origin in the space \widetilde{X}_1 (so that the reduced mapping f in (1.5) is actually defined in a neighbourhood of the origin in \widetilde{X}_1). In Theorem 1.1 below, we show that the mappings $G(\widetilde{x})$ and $f(\widetilde{x}_1) + DG(0) \cdot \widetilde{x}_2$ differ only in the context of “changes of variables” in the spaces \widetilde{X} and Y . More precisely,

Theorem 1.1. *There is a neighbourhood \widetilde{U} of the origin in the space \widetilde{X} and*

- (i) *a mapping $\tau \in C^{m-1}(\widetilde{U}, \text{Isom}(Y))$,*
- (ii) *an origin-preserving diffeomorphism $\widetilde{\rho} \in C^m(\widetilde{U}, \widetilde{X})$ with $D\widetilde{\rho}(0) = I_{\widetilde{X}}$ such that*

$$\widetilde{\tau}(\widetilde{X}) G(\widetilde{\rho}(\widetilde{x})) = f(\widetilde{x}_1) + DG(0) \widetilde{x}_2, \quad (1.7)$$

for every $\widetilde{x} \in \widetilde{U}$.

Proof. For \widetilde{x} around the origin in the space \widetilde{X} , set

$$\widetilde{R}(\widetilde{x}) = \widetilde{x}_1 + [DG(0)|_{\widetilde{X}_2}]^{-1} Q_2 G(\widetilde{x}). \quad (1.8)$$

Then, the mapping \widetilde{R} is of class C^m and $\widetilde{R}(0) = 0$, $D\widetilde{R}(0) = I_{\widetilde{X}}$ so that \widetilde{R} is an origin-preserving local C^m -diffeomorphism of the space \widetilde{X} . Besides, from (1.8), we have

$$Q_2 \widetilde{R}(\widetilde{x}) = [DG(0)|_{\widetilde{X}_2}]^{-1} Q_2 G(\widetilde{x})$$

and hence

$$DG(0)Q_2\tilde{R}(\tilde{x}) = Q_2G(\tilde{x}).$$

Setting $\tilde{\rho} = \tilde{R}^{-1}$, it follows that

$$Q_2G(\tilde{\rho}(\tilde{x})) = DG(0)\tilde{x}_2, \quad (1.9)$$

for \tilde{x} in some neighbourhood \tilde{U} of the origin in \tilde{X} . Note from (1.8) that $\tilde{\rho}$ is of the form

$$\tilde{\rho}(\tilde{x}) = \tilde{x}_1 + \tilde{\varphi}(\tilde{x}), \quad (1.10)$$

76 with $\tilde{\varphi} \in C^m(\tilde{U}, \tilde{X}_2)$. In particular, putting $\tilde{x} = \tilde{x}_1$ in (1.9) (i.e. $\tilde{x}_2 = 0$), we find that $\tilde{\varphi}(\tilde{x}_1)$ is characterized by (1.6), which agrees with our notation. Besides

$$\rho(\tilde{x}_1) = Q_1G(\tilde{\rho}(\tilde{x}_1)). \quad (1.11)$$

It is not restrictive to assume that the neighbourhood \tilde{U} is convex. Let us then define

$$\tau(\tilde{x}) = Q_1 + \tau_{12}(\tilde{x})Q_2 + Q_2 \quad (1.12)$$

where $\tau_{12}^{(\tilde{x})} \in \mathcal{L}(Y_2, Y_1)$ is given by

$$\tau_{12}(\tilde{x}) = - \left[\int_0^1 Q_1 DG(\tilde{\rho}(\tilde{x}_1 + s\tilde{x}_2)) D_{\tilde{x}_2} \tilde{\rho}(\tilde{x}_1 + s\tilde{x}_2) dx \right] \left[DG(0)|_{\tilde{x}_2} \right]^{-1}. \quad (1.13)$$

Clearly, $\tau \in C^{m-1}(\tilde{U}, \mathcal{L}(Y))$ and $\tau(0) = I_Y$ so that, after shrinking \tilde{U} if necessary, $\tau(\tilde{x})$ is an isomorphism of Y for every $\tilde{x} \in \tilde{U}$. Now, we have

$$\begin{aligned} \tau(\tilde{x})G(\tilde{\rho}(\tilde{x})) &= Q_1G(\tilde{\rho}(\tilde{x})) + \tau_{12}(\tilde{x})Q_2G(\tilde{\rho}(\tilde{x})) + Q_2G(\tilde{\rho}(\tilde{x})) = \\ &= Q_1G(\tilde{\rho}(\tilde{x})) + \tau_{12}(\tilde{x})DG(0)\tilde{x}_2 + DG(0)\tilde{x}_2. \end{aligned} \quad (1.14)$$

From (1.11) and the Taylor formula, we have

$$Q_1G(\tilde{\rho}(\tilde{x})) = f(\tilde{x}_1) + \int_0^1 Q_1 DG(\tilde{\rho}(\tilde{x}_1 + s\tilde{x}_2)) D_{\tilde{x}_2} \tilde{\rho}(\tilde{x}_1 + s\tilde{x}_2) \cdot \tilde{x}_2 dx,$$

and the assertion follows from (1.13) and (1.14). \square

By definition, $\tilde{\rho} = \tilde{R}^{-1}$ and if we denote by \tilde{R}_1 and \tilde{R}_2 the projections of the mapping \tilde{R} onto the spaces \tilde{X}_1 and \tilde{X}_2 respectively, we can then write for \tilde{x} around the origin,

$$G(\tilde{x}) = (\tau(\tilde{x}))^{-1} [f(\tilde{R}_1(\tilde{x})) + DG(0)\tilde{R}_2]. \quad (1.15)$$

Let us now consider a second Lyapunov-Schmidt reductions corresponding to a new choice \tilde{X}'_2 of the complement of \tilde{X}_1 and a write $\tilde{x} = \tilde{x}'_1 + \tilde{x}'_2$ with $\tilde{x}'_1 \in \tilde{X}_1$ and $\tilde{x}'_2 \in \tilde{X}'_2$. We obtain a new reduced equation $\hat{f} = \hat{f}(\tilde{x}'_1)$ with which Theorem 1.1 applies. Together with (1.15), we see that there is a neighbourhood \tilde{U} of the origin in the space \tilde{X} and

- (i) a mapping $\tau \in C^{m-1}(\tilde{U}, Isom(Y))$,
- (ii) an origin-preserving diffeomorphism $\tilde{\rho} \in C^m(\tilde{U}, \tilde{X})$ with onto the spaces \tilde{X}_1 and \tilde{X}_2 respectively, one has

$$\hat{f}(\tilde{x}'_1) + DG(0)\tilde{x}'_2 = \tau(\tilde{x}) [f(\tilde{\rho}_1(\tilde{x})) + DG(0)\tilde{\rho}(\tilde{x})], \quad (1.16)$$

for every $\tilde{x} \in \tilde{U}$. The linear mapping $\tau(\tilde{x})$ has a matrix representation of the form

$$\tau(\tilde{x}) = \begin{bmatrix} \hat{\tau}_{11}(\tilde{x}) & \hat{\tau}_{12}(\tilde{x}) \\ \tau_{21}(\tilde{x}) & \tau_{22}(\tilde{x}) \end{bmatrix},$$

where $\hat{\tau}_{1i} \in C^{m-1}(\tilde{U}, (Y_i, \hat{Y}_1))$, $i = 1, 2$ and $\tau_{2i} \in C^{m-1}(\tilde{U}, (Y_i, Y_2))$, $i = 1, 2$. In this notation, relation (1.16) can be rewritten as

$$\hat{f}(\tilde{x}'_1) = \hat{\tau}_{11}(\tilde{x})f(\tilde{\rho}_1(\tilde{x})) + \hat{\tau}_{12}(\tilde{x})DG(0)\tilde{\rho}_2(\tilde{x}), \quad (1.17)$$

$$DG(0)\tilde{x}'_2 = \tau_{21}(\tilde{x})f(\tilde{\rho}_1(\tilde{x})) + \tau_{22}(\tilde{x})DG(0)\tilde{\rho}_2(\tilde{x}), \quad (1.18)$$

for every $\tilde{x} \in \tilde{U}$.

Assume first that $m \geq 2$, so that the mappings τ_{21} and τ_{22} are of class C^1 at least. By differentiating (1.18) and noting that $D\tilde{\rho}(0) = I_{\tilde{x}}$, we obtain

$$DG(0)\tilde{P}'_2 = \tau_{21}(0)Df(0) + \tau_{22}(0)DG(0)\tilde{P}_2, \quad (1.19)$$

where \widetilde{P}_2 and \widetilde{P}'_2 denote the projection operators along \widetilde{X}_1 and onto the spaces \widetilde{X}_2 and \widetilde{X}'_2 respectively. When $m = 1$, the mappings τ_{21} and τ_{22} are of class C^0 only. Nevertheless, from the fact that the mappings $f \cdot \widetilde{\rho}_1$ and $DG(0) \cdot \widetilde{\rho}_2$ are of class C^1 and *vanish at the origin*, it is immediately verified that both the mapping

$$\widetilde{x}_2 \rightarrow \tau_{21}(\widetilde{x}_2)f(\widetilde{\rho}_1(\widetilde{x})),$$

and

$$\widetilde{x}_2 \rightarrow \tau_{22}(\widetilde{x})DG(0)\widetilde{\rho}_2(\widetilde{x}),$$

are differentiable at the origin and that formula (1.19) still holds. Besides, $Df(0) = 0$ (cf, Chapter 1, §2) and hence

$$DG(0)\widetilde{P}'_2 = \tau_{22}(0)DG(0)\widetilde{P}_2.$$

In particular,

$$DG(0)\widetilde{P}'_2|_{\widetilde{X}_2} = \tau_{22}(0)DG(0)|_{\widetilde{X}_2}.$$

As $DG(0)|_{\widetilde{X}_2} \in \text{Isom}(\widetilde{X}_2, Y_2)$, we get

$$\tau_{22}(0) = [DG(0)|_{\widetilde{X}_2}]^{-1} DG(0)\widetilde{P}'_2|_{\widetilde{X}_2}. \quad (1.20)$$

79 At this stage, observe that $DG(0)\widetilde{P}'_2 = DG(0)|_{\widetilde{X}'_2}$. Indeed, as \widetilde{X}_2 and \widetilde{X}'_2 are two topological complements² of the space \widetilde{X}_1 , it suffices to notice that $\widetilde{P}'_2|_{\widetilde{X}_2}$ is one-to-one and onto and use the Open mapping theorem. Therefore, writing (1.20) in the form

$$\tau_{22}(0) = [DG(0)|_{\widetilde{X}_2}]^{-1} [DG(0)|_{\widetilde{X}_2}] P'_2|_{\widetilde{X}_2},$$

it follows that $\tau_{22}(0) \in \text{Isom}(Y_2)$. After shrinking the neighbourhood \widetilde{U} if necessary, we may then suppose that $\tau_{22}(\widetilde{x}) \in \text{Isom}(Y_2)$ for every $\widetilde{x} \in \widetilde{U}$.

Now, let us take $\widetilde{x} \in \widetilde{X}_1$ (i.e. $\widetilde{x} = \widetilde{x}_1 = \widetilde{x}'_1$) in (1.17) and (1.18) to obtain

$$\hat{f}(\widetilde{x}_1) = \hat{\tau}_{11}(\widetilde{x}_1)f(\widetilde{\rho}_2(\widetilde{x}_1)) + \hat{\tau}_{12}(\widetilde{x}_1)DG(0)\widetilde{\rho}_2(\widetilde{x}_1), \quad (1.21)$$

²In particular, \widetilde{X}_2 and \widetilde{X}'_2 are closed in \widetilde{X} and hence are Banach spaces by themselves.

$$0 = \tau_{11}(\tilde{x}_1)f(\tilde{\rho}_1(\tilde{x}_1)) + \tau_{22}(\tilde{x}_1)DG(0)\tilde{\rho}_2(\tilde{x}_1), \quad (1.22)$$

From (1.22) and the above comments, we see that

$$DG(0)\tilde{\rho}_2(\tilde{x}_1) = -(\tau_{22}(\tilde{x}_1))^{-1}\tau_{21}(\tilde{x}_1)f(\tilde{\rho}_1(\tilde{x}_1)).$$

With this relation, (1.21) becomes

$$\hat{f}(\tilde{x}_1) = \left[\hat{\tau}_{11}(\tilde{x}_1) - \hat{\tau}_{12}(\tilde{x}_1)(\tau_{22}(\tilde{x}_1))^{-1}\tau_{21}(\tilde{x}_1) \right] f(\tilde{\rho}_1(\tilde{x}_1)).$$

Let us set

$$\hat{\tau}(\tilde{x}_1) = \tau_{11}(\tilde{x}_1) - \tau_{12}(\tilde{x}_1)(\tau_{22}(\tilde{x}_1))^{-1}\tau_{21}(\tilde{x}_1).$$

Clearly, $\hat{\tau}_1$ is of class C^{m-1} on a neighbourhood of the origin in the space \tilde{X}_1 with values in the space $\mathcal{L}(Y_1, \hat{Y}_1)$. In addition, $\hat{\tau}_1(0) \in \text{isom}(Y_1, \hat{Y}_1)$. Indeed, since Y_1 and \hat{Y}_1 have the same dimension n , it suffices to show that $\hat{\tau}_1(0)$ is one-to-one. Let $y_1 \in Y_1$ be such that $\hat{\tau}_1(0)y_1 = 0$, namely

$$\hat{\tau}_{11}(0)y_1 - \hat{\tau}_{12}(0)(\tau_{22}(0))^{-1}\tau_{21}(0)y_1 = 0$$

If so, we see that

$$\begin{bmatrix} \hat{\tau}_{11}(0) & \hat{\tau}_{12}(0) \\ \tau_{21}(0) & \tau_{22}(0) \end{bmatrix} \begin{bmatrix} y_1 \\ -(\tau_{22}(0))^{-1}\tau_{21}(0)y_1 \end{bmatrix} = 0$$

and hence $y_1 = 0$. It follows that $\hat{\tau}_1(\tilde{x}) \in \text{Isom}(Y_1, \hat{Y}_1)$ for \tilde{x}_1 near the origin in the space \tilde{X}_1 and we have proved

Theorem 1.2. *There is a neighbourhood \tilde{U}_1 of the origin in the space \tilde{X}_1 and*

- (i) *a mapping $\hat{\tau}_1 \in C^{m-1}(\tilde{U}_1, \text{Isom}(Y_1, \hat{Y}_1))$,*
- (ii) *an origin-preserving diffeomorphism $\tilde{\rho}_1 \in C^m(\tilde{U}_1, \tilde{X}_1)$ with $D\tilde{\rho}_1(0) = I_{\tilde{X}_1}$ such that*

$$\hat{f}(\tilde{x}_1) = \hat{\tau}_1(\tilde{x}_1)f(\tilde{\rho}_1(\tilde{x}_1)), \quad (1.23)$$

for every $\tilde{x}_1 \in \tilde{U}_1$.

Remark 1.1. It is customary to summarize Theorem 1.2 by saying that any two Lyapunov-Schmidt reductions are *equivalent*. For our purpose, this property is important because of Corollary 1.1 below.

81 Corollary 1.2. *Assume that there is an integer $k \leq m$ such that*

$$D^j f(0) = 0, 0 \leq j \leq k - 1,$$

and the mapping

$$\tilde{x}_1 \in \tilde{X}_1 \rightarrow D^k f(0) \cdot (\tilde{x}_1)^k \in Y_1, \quad (1.24)$$

verifies the condition ($\mathbb{R} - N.D.$). Then, one has

$$D^j \hat{f}(0) = 0, 0 \leq j \leq k - 1,$$

and the mapping

$$x_1 \in X_1 \rightarrow D^k \hat{f}(0) \cdot (\tilde{x}_1)^k \in \hat{Y}_1, \quad (1.25)$$

verifies the condition ($\mathbb{R} - N.D.$).

Proof. With the notation of Theorem 1.2, note that in view of $D\tilde{\rho}_1(0) = I_{\tilde{X}_1}$ that

$$D^j(f \bullet \tilde{\rho}_1)(0) = D^j f(0), 0 \leq j \leq k. \quad (1.26)$$

Assume first $k \leq m - 1$, so that, for every $0 \leq j \leq k$, $D^j \hat{f}(0)$ involves the derivatives of order $\leq k$ of the mappings $\hat{\tau}_1$ and $f \bullet \rho_1$ at the origin as it follows from (1.23). With (1.26), a simple calculation provides

$$D^j \hat{f}(0) = 0, 0 \leq j \leq k - 1, \quad (1.27)$$

$$D^k \hat{f}(0) = \hat{\tau}_1(0) D^k f(0). \quad (1.28)$$

82 If $k = m$, there is a slight difficulty in applying the same method since the mapping $\hat{\tau}_1$ is not m times differentiable. However, it is easy to obtain $D^{m-1} \hat{f}(\tilde{x}_1)$ in terms of the derivatives of order $\leq m - 1$ of the mappings $\hat{\tau}_1$ and $(f \bullet \tilde{\rho}_1)$. We leave it to the reader to check that each term of the expression is differentiable *at the origin* and the relations (1.27) and (1.28) still hold. In particular, as $\hat{\tau}_1(0) \in \text{Isom}(Y_1, \hat{Y}_1)$, it is an obvious consequence of (1.27) that the mapping (1.25) verifies the condition ($\mathbb{R} - N.D.$) as soon as the mapping (1.24) does. \square

Remark 1.2. From Corollary 1.2, assuming that the first nonzero derivative at the origin of the reduced mapping f verifies the condition $(\mathbb{R} - N.D.)$ is independent of the choice of the spaces \widetilde{X}_2 and Y_1 and hence is an *intrinsic property of the mapping G* .

3.2 Application to Problems of Bifurcation from the Trivial Branch at a Multiple Characteristic Value

According to our definitions of Chapter 1, given a real Banach space X , we are interested in finding the solutions (μ, x) near the origin of $\mathbb{R} \times X$ of an equation of the form

$$G(\mu, x) = (I - (\lambda_0 + \mu)L)x + \Gamma(\mu, x) = 0, \quad (2.1)$$

where $L \in \mathcal{L}(X)$ is *compact* and λ_0 is a *multiple* characteristic value of L , i.e.,

$$n = \dim \text{Ker}(I - \lambda_0 L) \geq 2.$$

Recall the notation (cf. Chapter 1):

$$X_1 = \text{Ker}(I - \lambda_0 L), \quad (2.2)$$

$$Y_2 = \text{Range}(I - \lambda_0 L), \quad (2.3)$$

and X_2 and Y_1 are arbitrary topological complements of X_1 and Y_2 respectively. If so, 83

$$\dim X_1 = \dim Y_1 = n. \quad (2.4)$$

Also, Q_1 and Q_2 denote the (continuous) projection operators onto the spaces Y_1 and Y_2 . The nonlinear operator Γ verifies conditions

$$\Gamma(\mu, 0) = 0, \quad (2.5)$$

(so that $D_\mu^j \Gamma(0) = 0$ for every $j \geq 0$) and

$$D_x \Gamma(0) = D_\mu D_x \Gamma(0) = 0. \quad (2.6)$$

In what follows, we shall assume that there is an integer $k \leq m$ (and necessarily ≥ 2) such that

$$\begin{aligned} Q_1 D_x^j \Gamma(0) &= 0, 0 \leq j \leq k-1, \\ Q_1 D_x^k \Gamma(0)|_{(X_1)^k} &\neq 0, \end{aligned} \quad (2.7)$$

As we saw in Chapter 1, §2, the problem amounts to finding the local zero set of the *reduced equation*

$$f(\mu, x) = -\frac{\mu}{\lambda_0} Q_1 x - \frac{\mu}{\lambda_0} Q_1 \varphi(\mu, x) + Q_1 \Gamma(\mu, x + \varphi(\mu, x)) = 0. \quad (2.8)$$

for $(\mu, x) \in \mathbb{R} \times X_1$, where the mapping φ with values in X_2 is characterized by the properties

$$\begin{cases} Q_2 G(\mu, x + \varphi(\mu, x)) = 0, \\ \varphi(0) = 0, \end{cases} \quad (2.9)$$

84 for (μ, x) near the origin of $\mathbb{R} \times X_1$.

Remark 2.1. To avoid repetition, we shall not state the corresponding properties of the local zero set of the mapping G . The reader can check without any difficulty that *all* the results about the local zero set of the reduced mapping f (number of curves, regularity, location in the space) remain valid ad concerns the local zero set of G .

As already seen in Chapter 1 in a general setting, the derivatives $Df(0)$ and $D\varphi(0)$ vanish. An elementary calculation thus provides

$$D^2 f(0) \cdot (\mu, x)^2 = -\frac{2\mu}{\lambda_0} Q_1 x + Q_1 D_x^2 \Gamma(0) \cdot (x)^2 \epsilon Y_1, \quad (2.10)$$

for $(\mu, x) \in \mathbb{R} \times X_1$. Note, in particular, that $D^2 f(0) = 0$ if $f X_1 \subset Y_2$ and $k \geq 3$ simultaneously.

THE CASE $k = 2$

If $k = 2$, one has $D^2 f(0) \neq 0$ and the mapping (2.10) must verify the condition ($\mathbb{R} - N.D.$). An *implicit necessary condition* for this is

$$X = X_1 \oplus Y_2 (= \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L)). \quad (2.11)$$

Recall that the necessity of such a decomposition was already noticed when $n = 1$ in Chapter 1. In other words, it is necessary that the algebraic multiplicity of λ_0 equals its geometric multiplicity.

Indeed, assume $X_1 \cap Y_2 \neq \{0\}$. The line $\{(\mu, 0), \mu \in \mathbb{R}\}$ (trivial branch) **85** is clearly in the zero set of the mapping (2.10) and its derivative at any point of this line is

$$(\mu', x') \in \mathbb{R} \times X_1 \rightarrow -\frac{2\mu}{\lambda_0} Q_1 x' \in Y_1, \quad (2.12)$$

whose range is that of $Q_1|_{X_1}$. But, from the assumption $X_1 \cap Y_2 \neq \{0\}$, we deduce that $\dim \text{Ker } Q_1|_{X_1} \geq 1$. Hence $\dim \text{Range } Q_1|_{X_1} \leq n - 1$ and the mapping (2.12) is not onto.

Now, from the results of §1, we see that the validity (or the failure) of the condition (\mathbb{R} -N.D.) is independent of the choice of the complements X_2 of X_1 and Y_1 of Y_2 . From (2.11), we may choose $X_1 = Y_1$ and the mapping (2.10) becomes

$$D^2 f(0) \cdot (\mu, x)^2 = -\frac{2\mu x}{\lambda_0} + Q_1 D_x^2 \Gamma(0) \cdot (x)^2 \in X_1, \quad (2.13)$$

for $(\mu, x) \in \mathbb{R} \times X_1$. Its derivative at any point $(\mu, x) \in \mathbb{R} \times X_1$ is the linear mapping

$$(\mu', x') \in \mathbb{R} \times X_1 \rightarrow 2 \left(-\frac{\mu}{\lambda_0} x' - \frac{\mu'}{\lambda_0} x + Q_1 D_x^2 \Gamma(0) \cdot (x, x') \right) \in X_1. \quad (2.14)$$

It is clearly onto at each point of the form $(\mu, 0)$ with $\mu \neq 0$. Whether or not it is onto at the other nonzero solutions of the equation $D^2 f(0) \cdot (\mu, x)^2 = 0$ has to be checked in each particular problem separately. Simple finite-dimensional examples show that this hypothesis is quite realistic.

Remark 2.2. It is not restrictive to limit ourselves to finite dimensional **86** examples. Indeed, the mapping (2.13) involves *finite dimensional terms only*; in particular, the term $Q_1 D_x^2 \Gamma(0) \cdot (x)^2$ is nothing but the second derivative with respect to x of the mapping $Q_1 \Gamma|_{\mathbb{R} \times X_1}$ at the origin.

When the results of Chapter 2, §4 hold, the largest number ν of curves (which are of class C^{m-1} at the origin and of class C^m away from it) is 2^n since $k = 2$. As k is even, ν must be even too (cf. Chapter 2, §1). The trivial branch being in the local zero set of f , existence of a second curve is then ensured, namely, bifurcation *does occur*.

COMMENT 2.1. When $n = 1$ and $Q_1 D_x^2 \Gamma(0)|_{(X_1)^2} \neq 0$ it is immediate that the nontrivial curve in the zero set of f has a nonvertical tangent at the origin. Therefore, bifurcation occurs *transcritically* which means that the nontrivial curve in question is located on *both sides* of the axis $\{0\} \times X_1$ in $\mathbb{R} \times X_1$. The situation is different when $n \geq 2$ and there may be curves in the local zero set of f which are tangent to some lines of the hyperplane $\{0\} \times X_1$ of $\mathbb{R} \times X_1$ at the origin, as in the example below. However, this cannot happen if the hypothesis $Q_1 D_x^2 \Gamma(0)|_{(X_1)^2} \neq 0$ is replaced by the stronger one $Q_1 D_x^2 \Gamma(0) \cdot (x)^2 \neq 0$ for every $x \in X_1 - \{0\}$ (note that the two assumptions are equivalent when $n = 1$).

EXAMPLE. Take $X = \mathbb{R}^2, L = I$ and $\lambda_0 = 1$. Thus, $X_1 = X = \mathbb{R}^2$, $Q_1 = I$ and $X_2 = \{0\}$. Writing $x = (x_1, x_2), x_1, x_2 \in \mathbb{R}$ and with

$$\Gamma(\mu, x) = \Gamma(x) = \begin{bmatrix} x_2^2 + x_1^3 \\ x_1 x_2 \end{bmatrix},$$

we find

$$D^2 f(0) \cdot (\mu, x)^2 = 2 \begin{bmatrix} -\mu x_1 + x_2^2 \\ -\mu x_2 + x_1 x_2 \end{bmatrix}.$$

This mapping verifies the condition ($\mathbb{R} - N.D.$) and its zero set is made up of the 4 lines: $\mathbb{R}(1, 0, 0)$ (trivial branch), $\mathbb{R}(0, 1, 0)$, $\mathbb{R}(1, 1, 1)$ and $\mathbb{R}(1, 1, -1)$. The zero set of f is made up of the 4 curves: $(\mu, 0, 0)$ (trivial branch), $(x_1^2, x_1, 0)$, $(x_1, x_1, \sqrt{x_1^2 - x_1^3})$ and $(x_1, x_1, -\sqrt{x_1^2 - x_1^3})$.

The curve $(x_1^2, x_1, 0)$ is tangent to the plane $\{0\} \times X_1$ at the origin and located in the half-space $\mu \geq 0$ (cf. Fig. 2.1)

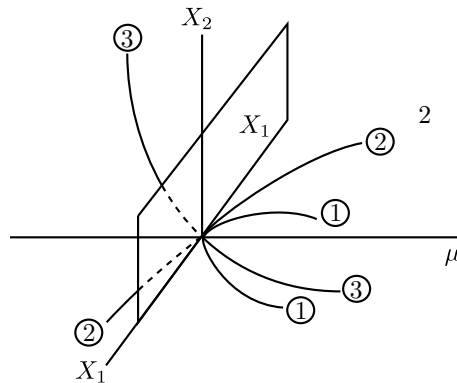


Figure 2.1:

COMMENT 2.2. The same picture as above describes the local zero set of G : the only modification consists in replacing the hyperplane $\{0\} \times X_1$ of $\mathbb{R} \times X_1$ by the hyperplane $\{0\} \times X$ in $\mathbb{R} \times X$. 88

THE CASE $k \geq 3$.

If $k \geq 3$, the mapping (2.10) is

$$D^2 f(0) \cdot (\mu, x)^2 = -\frac{2\mu}{\lambda_0} Q_1 x, \quad (2.15)$$

because $Q_1 D_x^2 \Gamma(0) = 0$. When $X_1 \not\subset Y_2$ (which can be expected in most of the applications), one has $D^2 f(0) \neq 0$ so that the mapping (2.15) should verify the condition $(\mathbb{R} - N.D.)$. Unfortunately, *this is never the case* when $n \geq 2$ (as assumed throughout this section). To see this, note that the local zero set of the mapping (2.15) contains the pair $(0, x)$ for any $x \in X_1$. Its derivative at such a point is given by

$$(\mu', x') \in \mathbb{R} \times X_1 \rightarrow -\frac{2\mu'}{\lambda_0} Q_1 x \in Y_1.$$

Its range is the space $\mathbb{R} Q_1 x$ of dimension ≤ 1 and it cannot be onto for $n \geq 2$ (note that it is onto when $n = 1$ and $X = X_1 \oplus Y_2$ so that there is no contradiction with the results of Chapter 1).

However, it is possible to overcome the difficulty by performing a change of scale in our initial problem. The idea is as follows. For any odd integer p , the mapping

$$\eta \in \mathbb{R} \rightarrow \eta^p \in \mathbb{R},$$

89 is a C^∞ homeomorphism. One can then wonder whether there is a “suitable” choice of the odd integer $p \geq 3$ such that, setting $\mu = \eta^p$, the results of Chapter 2, §4 apply to the problem: Find (η, x) around the origin of $\mathbb{R} \times X$ such that

$$G(\eta^p, x) = (I - (\lambda_0 + \eta^p)L)x + \Gamma(\eta^p, x) = 0. \quad (2.16)$$

The Lyapunov-Schmidt reduction of this new problem yields a reduced equation

$$g(\eta, x) = 0,$$

for (η, x) around the origin in $\mathbb{R} \times X_1$, where the mapping is easily seen to be

$$g(\eta, x) = f(\eta^p, x),$$

the mapping f being the reduced mapping in (2.8) of the problem (2.1). In particular, the mapping which is found through the Implicit function theorem in the Lyapunov-Schmidt reduction is exactly $\varphi(\eta^p, x)$ where φ is characterized by (2.9) and the solutions to the problem (2.1) are the pairs

$$\{(\eta^p, x + \varphi(\eta^p, x)) \in \mathbb{R} \times X, (\eta, x) \in \mathbb{R} \times X_1, g(\eta, x) = 0\}. \quad (2.17)$$

Intuitively, the choice of p can be made by examining the function

$$g(\eta, x) = -\frac{\eta^p}{\lambda_0} Q_1 x - \frac{\eta^p}{\lambda_0} Q_1 \varphi(\eta^p, x) + Q_1 \Gamma(\eta^p, x + \varphi(\eta^p, x)). \quad (2.18)$$

90 As $D\varphi(0) = 0$, the leading term in the expression $-(\eta^p/\lambda_0)Q_1 x - (\eta^p/\lambda_0)Q_1 \varphi(\eta^p, x)$ is $-(\eta^p/\lambda_0)Q_1 x$. On the other hand, in analogy with the case $k = 2$, it might be desirable that the term $Q_1 D_x^k \Gamma(0) \cdot (x)^k$ as well as the first nonzero derivative of the term $-(\eta^p/\lambda_0)Q_1 x$ at the origin be in the expression of the first nonzero derivative of g at the origin. The later

is of order $p + 1$ whereas the term $D_x^k \Gamma(0) \cdot (x)^k$ does not appear before differentiating k times. A “good” relation thus seems to be $p + 1 = k$, namely $p = k - 1$.

This turns out to be the right choice of p , which can actually be found after eliminating all the other values on the basis of natural mathematical requirements instead of using the above intuitive arguments. Of course, a mathematical method for finding the proper change of parameter may have some importance in problems in which there is no a priori guess of what it should be. Incidentally, one can also provide a complete justification of the use of Newton diagrams in the change of scale as is done, for instance, in Sattinger [34]. However, the study is technical and too long to be presented here (cf. Rabier [31]).

As we have assumed that p is odd for the reasons explained above, we must suppose that k is even. A similar method will be analysed later on when k is odd.

The case k even: We need to find the first nonzero derivative of the mapping of (2.18) at the origin in $\mathbb{R} \times X_1$ when $p = k - 1$. We can already guess that it will be of order k but we have to determine its explicit expression in terms of the data. With this aim, we first prove 91

Lemma 2.1. *Around the origin in $\mathbb{R} \times X$, one has*

$$Q_1 \Gamma(\eta^{k-1}, x) = \frac{1}{k!} Q_1 D_x^k \Gamma(0) \cdot (x)^k + o(|\eta| + \|x\|)^k,$$

where $\|\cdot\|$ denote the norm of the space X .

Proof. From the assumption $k \geq 3$, $D\Gamma(0) = 0$ and $Q_1 D^2 \Gamma(0) = 0$, since we already saw that $D_\mu^2 \Gamma(0) = 0$, $D_\mu D_x \Gamma(0) = 0$ (cf. (2.5) - (2.6)). Thus, the Taylor formula about the origin yields

$$Q_1 \Gamma(\mu, x) = \sum_{j=3}^k \frac{1}{j!} Q_1 D^j \Gamma(0) \cdot (\mu, x)^j + o(|\mu| + \|x\|)^k,$$

For any j with $3 \leq j \leq k$, one has

$$Q_1 D^j \Gamma(0) (\mu, x)^j = \sum_{i=0}^j \binom{j}{i} \mu^{j-i} Q_1 D_\mu^{j-i} D_x^i \Gamma(0) \cdot (x)^i.$$

But, from (2.5) $D_\mu^j \Gamma(0) = 0$ for every j and the index i actually runs over the set $\{1, \dots, j\}$. Also, for $j < k$, it follows, from (2.7), that i runs over the set $1, \dots, j-1$ only. To sum up,

$$Q_1 \Gamma(\mu, x) = \frac{1}{k!} Q_1 D_x^k \Gamma(0) \cdot (x)^k + \sum_{j=3}^k \frac{1}{j!} \sum_{i=1}^{j-1} \binom{j}{i} \mu^{j-i} Q_1 D_\mu^{j-i} D_x^i \Gamma(0) \cdot (x)^i + o((|\mu| + \|x\|)^k).$$

Replacing μ by η^{k-1} , we shall see that each term in the sum is of order $o(|\eta| + \|x\|)^k$, except for $(1/k!) Q_1 D_x^k \Gamma(0) \cdot (x)^k$. This is obvious as concerns the remainder which becomes $o((|\eta|^{k-1} + \|x\|)^k)$ and, for $3 \leq j \leq k$ and $1 \leq i \leq j-1$, each term in the sum is of order

$$O(|\eta|^{(k-1)(j-i)} \|x\|^i).$$

92 Replacing $|\eta|$ and $\|x\|$ by $|\eta| + \|x\|$, it is *a fortiori* of order

$$O((|\eta| + \|x\|)^{(k-1)(j-i)+1}).$$

As $j-i \geq 1$ and $j-i = 1$ if and only if $i = j-1$, we have, for $i = j-1$

$$(k-1) + (j-1) \geq k+1$$

because $j \geq 3$. On the other hand, if $j-i \geq 2$ and since $i \geq 1$

$$(k-1)(j-i) + i \geq 2(k-1) + 1 = 2k-1 \geq k+1.$$

In any case, each term is of order

$$O((|\eta| + \|x\|)^{k+1}),$$

which completes the proof. \square

Proposition 2.1. *The first nonzero derivative of the mapping*

$$(\eta, x) \in \mathbb{R} \times X_1 \rightarrow g(\eta, x) = f(\eta^{k-1}, x) = -\frac{\eta^{k-1}}{\lambda_0} Q_1 x - \frac{\eta^{k-1}}{\lambda_0} Q_1 \varphi(\eta^{k-1}, x) \quad (2.19)$$

$$+ Q_1\Gamma(\eta^{k-1}, x + \varphi(\eta^{k-1}, x)) \in Y_1$$

at the origin is of order k and its value at the point $(\eta, x) \in \mathbb{R} \times X_1$ repeated k times (which determines it completely) is

$$D^k g(0) \cdot (\eta, x)^k = -\frac{k!}{\lambda_0} \eta^{k-1} Q_1 x + Q_1 D_x^k \Gamma(0) \cdot (x)^k. \quad (2.20)$$

Proof. As $D\varphi(0) = 0$, one has

$$\varphi(\mu, x) = o(|\mu| + \|x\|),$$

around the origin. In particular, replacing μ by η^{k-1}

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$$\varphi(\eta^{k-1}, x) = o(|\eta| + \|x\|), \quad (2.21)$$

so that the term $(\eta^{k-1}/\lambda_0)Q_1\varphi(\eta^{k-1}, x)$ is of order

$$(\eta^{k-1}/\lambda_0)Q_1\varphi(\eta^{k-1}, x) = O(|\eta|^{k-1} o(|\eta| + \|x\|)) = o(|\eta| + \|x\|)^k. \quad (2.22)$$

Next, from Lemma 2.1,

$$\begin{aligned} Q_1\Gamma(\eta^{k-1}, x + \varphi(\eta^{k-1}, x)) &= \frac{1}{k!} Q_1 D_x^k \Gamma(0) \cdot (x + \varphi(\eta^{k-1}, x))^k \\ &+ o((|\eta| + \|x + \varphi(\eta^{k-1}, x)\|)^k). \end{aligned} \quad (2.23)$$

But

$$|\eta| + \|x + \varphi(\eta^{k-1}, x)\| = O(|\eta| + \|x\| + \|\varphi(\eta^{k-1}, x)\|).$$

Using (2.21), we see that

$$o(|\eta| + \|x\| + \|\varphi(\eta^{k-1}, x)\|) = o(|\eta| + \|x\|) + o(|\eta| + \|x\|) = o(|\eta| + \|x\|).$$

Thus, the remainder in (2.23) is

$$o(|\eta| + \|x + \varphi(\eta^{k-1}, x)\|)^k = o([O(|\eta| + \|x\|)]^k) = o((|\eta| + \|x\|)^k),$$

so that

$$Q_1\Gamma(\eta^{k-1}, x + \varphi(\eta^{k-1}, x)) = \frac{1}{k!} Q_1 D_x^k \Gamma(0) \cdot (x + \varphi(\eta^{k-1}, x))^k + o((|\eta| + \|x\|)^k). \quad (2.24)$$

Now, from (2.21) again and due to the k -linearity of the derivative $D_x^k \Gamma(0)$, we deduce

$$Q_1 D_x^k \Gamma(0) \cdot (x + \varphi(\eta^{k-1}, x))^k = Q_1 D_x^k \Gamma(0) \cdot (x)^k + o(|\eta| + \|x\|)^k.$$

94 By putting this expression in (2.24) and together with (2.22), the mapping g (2.19) can be rewritten as

$$g(\eta, x) = -\frac{\eta^{k-1}}{\lambda_0} Q_1 x + \frac{1}{k!} Q_1 D_x^k \Gamma(0) \cdot (x)^k + o((|\eta| + \|x\|)^k). \quad (2.25)$$

This relation shows that the first $k - 1$ derivatives of g vanish at the origin. Besides, from Taylor's formula, we have

$$g(\eta, x) = \frac{1}{k!} D^k g(0) \cdot (\eta, x)^k + o(|\eta| + \|x\|)^k,$$

which, on comparing with (2.25), yields

$$D^k g(0) \cdot (\eta, x)^k = -\frac{k!}{\lambda_0} \eta^{k-1} Q_1 x + Q_1 D_x^k \Gamma(0) \cdot (x)^k.$$

□

Remark 2.3. We leave it to the reader to check that *the same result holds* if we replace conditions (2.7) by

$$\begin{aligned} D_x^j \Gamma(0)|_{(X_1)^j} &= 0, 0 \leq j \leq k-1, \\ Q_1 D_x^k \Gamma(0)|_{(X_1)^k} &\neq 0. \end{aligned} \quad (2.26)$$

More generally, it still holds under *some combination* of the hypotheses (2.7) and (2.26): Denoting by k the order of the first non-zero derivative with respect to x of the mapping $Q_1 \Gamma|_{\mathbb{R} \times X_1}$ at the origin (suppose to be finite), set

$$k_1 = \min\{0 \leq j \leq k, Q_1 D_x^j \Gamma(0) \neq 0\}, \quad (2.27)$$

$$k^1 = \min\{0 \leq j \leq k, D_x^j \Gamma(0)|_{(X_1)^j} \neq 0\}, \quad (2.28)$$

95 so that $2 \leq k_1, k^1 \leq k$. Then, it can be shown (cf. [31]) that Proposition 2.7 remains true when

$$k \leq k_1 + k^1 - 2. \tag{2.29}$$

Note that $k = k_1$ in our assumptions whereas $k = k^1$ under the hypotheses (2.26), two cases when the criterion (2.29) is satisfied.

When the mapping $D^k g(0) \cdot (\eta, x)^k$ in (2.20) verifies the condition ($\mathbb{R} - N.D.$), the results of Chapter 2, §4 give the structure of the local zero set of g (2.19). We shall derive the structure of the local zero set of f but, first observe that

Proposition 2.2. *The mapping $D^k g(0) \cdot (\eta, x)^k$ in (2.20) can verify the condition ($\mathbb{R} - N.D.$) only if the following two implicit conditions are fulfilled:*

- (i) $X = X_1 \oplus Y_2 (= \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L))^3$,
- (ii) $Q_1 D_x^k \Gamma(0) \cdot (x)^k \neq 0$ for every $x \in X_1 - \{0\}$.

Proof. (i) Assume that $X_1 \cap Y_2 = \{0\}$. The trivial branch $\{(\eta, 0); \eta \in \mathbb{R}\}$ is in the zero set $D^k g(0) \cdot (\eta, x)^k$ and its derivative at any of its nonzero points is

$$(\eta', x') \in \mathbb{R} \times X_1 \rightarrow -\frac{k!}{\lambda_0} \eta^{k-1} Q_1 x' \in Y_1,$$

whose range is $\text{Range } Q_1|_{X_1} \subseteq Y_1$ since $\text{Ker } Q_1|_{X_1} \neq \{0\}$ from the assumption $X_1 \cap Y_2 \neq \{0\}$. 96

(ii) Assume there is a nonzero $x \in X_1$ such that $Q_1 D_x^k \Gamma(0) \cdot (x)^k = 0$. Then, the line $\mathbb{R}(0, x)$ is in the zero set of $Q_1 D^k g(0) \cdot (\eta, x)^k$ and its derivative at $(0, x)$ is

$$(\eta', x') \in \mathbb{R} \times X_1 \rightarrow k Q_1 D_x^k \Gamma(0) \cdot (x^{k-1}, x') \in Y_1,$$

whose null-space contains the trivial branch and the line $\mathbb{R}(0, x)$: its range is then at most $(n - 1)$ -dimensional. □

³Namely, the algebraic multiplicity of λ_0 equals its geometric multiplicity.

COMMENT 2.3. Assume that the mapping $D^k g(0) \cdot (\eta, x)^k$ (1.20) verifies the condition ($\mathbb{R} - N.D.$). From our analysis of §1, this assumption is *independent of the choice of the complements* X_2 and Y_1 of X_1 and Y_2 respectively. In the applications, it follows from Proposition 2.2 that we can take $X_1 = Y_1$ so that the mapping g becomes (note that $Q_1\varphi = 0$ in this case)

$$g(\eta, x) = -\frac{\eta^{k-1}}{\lambda_0}x + Q_1\Gamma(\eta^{k-1}, x + \varphi(\eta^{k-1}, x)) \quad (2.30)$$

and one has

$$D^k g(0) \cdot (\eta, x)^k = -k! \frac{\eta^{k-1}}{\lambda_0}x + Q_1 D_x^k \Gamma(0) \cdot (x)^k,$$

for every $(\eta, x) \in \mathbb{R} \times X_1$.

COMMENT 2.4. From (ii) of Proposition 2.2, none of the lines of the zero set of $D^k g(0) \cdot (\eta, x)^k$ is “vertical” i.e. lies on the hyperplane $\{0\} \times X_1$ of $\mathbb{R} \times X_1$. In other words, if $t \rightarrow (\eta(t), x(t))$ is any curve in the local zero set of g , one has

$$\frac{d\eta}{dt}(0) \neq 0. \quad (2.32)$$

COMMENT 2.5. The trivial branch is in the zero set of the mapping $D^k g(0) \cdot (\eta, x)^k$ (and *also* in the local zero set of g since $\varphi(\mu, 0) = 0$; cf. Chapter 1). As k is even, we know that the number ν of lines must be even too ($\leq k^n$). Thus $\nu \geq 2$ and *bifurcation occurs* (as in the case $k = 2$). In view (2.32), each curve is located on both sides of the hyperplane $\{0\} \times X_1$ (i.e. bifurcation occurs *transcritically*).

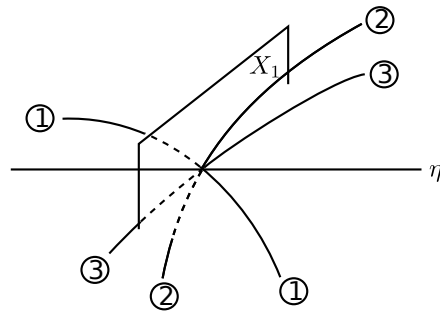


Figure 2.2: Local zero set of g .

We now pass to the description of the local zero set of f : it is made up of the ν curves

$$t \rightarrow (\mu(t), x(t)), \tag{2.33}$$

where $\mu(t) = \eta^{k-1}(t)$ and $t \rightarrow (\eta(t), x(t))$ is one of the ν curves in the local zero set of g , hence of class C^{m-k+1} at the origin and of class C^m away from it. Of course, the ν curves in (2.33) are *distinct* since the mapping $\eta \rightarrow \eta^{k-1}$ is a homeomorphism. Also, it is clear that $(d\mu/dt)(0) = 0$ so that

$$\frac{d}{dt}(\mu(t), x(t))|_{t=0} = (0, \frac{dx}{dt}(0)). \tag{2.34}$$

But, for each curve $(\eta(t), x(t))$ in the local zero set of g which is *distinct from the trivial branch*, we must have

$$\frac{dx}{dt}(0) \neq 0,$$

because different curves in the local zero set of g have different tangents at the origin. Thus, for each curve $(\mu(t), x(t))$ in the local zero set of f which is *distinct from the trivial branch*, one has

$$\frac{d}{dt}(\mu(t), x(t))|_{t=0} = (0, \frac{dx}{dt}(0)) \neq 0.$$

As a result and further since the functions $\mu(t) = \eta^{k-1}(t)$ and $x(t)$ are of class C^{m-k+1} at the origin and of class C^m away from it, we may deduce that the nontrivial curves in the local zero set of f are also of

class C^{m-k+1} at the origin and of class C^m away from it. Besides, due to (2.34), they are tangent to the hyperplane $\{0\} \times X_1$ of $\mathbb{R} \times X_1$ at the origin. Finally, as the sign of $\mu(t) = \eta^{k-1}(t)$ changes as that $\eta(t)$ does, bifurcation remains transcritical (cf. Comment 2.5) as in the case when $k = 2$ and the condition $Q_1 D_x^2 \Gamma(0) \cdot (x) \neq 0$ for every $x \in X_1 - \{0\}$ holds (cf. Comment 2.1)⁴.

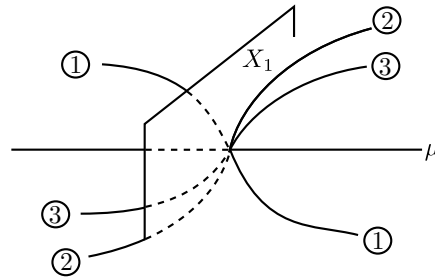


Figure 2.3: Local zero set of f (k even)

Remark 2.4. It is possible for several curves in the local zero set of f to have the same tangent at the origin: This will happen if and only if several lines in the zero set of the mapping $D^k g(0) \cdot (\eta, x)^k$ (2.20) have the same projection onto X_1 (along the η -axis). If this is the case, that is one more reason for the assumptions of Chapter 2, §4 to fail when the parameter μ is unchanged, for, when the curves are found through Theorem 4.1 of Chapter 2, they must have different tangents at the origin.

100 *The case k odd:* When k is odd, the mapping $\eta \rightarrow \eta^{k-1}$ is no longer a homeomorphism. However, by performing the change $\mu = \eta^{k-1}$, we shall find all solutions in the local zero set of f associated with $\mu \geq 0$. In order to get the solutions associated with $\mu \leq 0$, it suffices to perform the change $\mu = -\eta^{k-1}$.

The method is quite similar and we shall set

$$g_\sigma(\eta, x) = f(\sigma\eta^{k-1}, x) = -\frac{\sigma\eta^{k-1}}{\lambda_0} Q_1 x - \frac{\sigma\eta^{k-1}}{\lambda_0} Q_1 \varphi(\sigma\eta^{k-1}, x) +$$

⁴Recall however that this condition is not required when $k = 2$.

$$+ Q_1 \Gamma(\sigma \eta^{k-1}, x + \varphi(\sigma \eta^{k-1}, x)), \quad (2.35)$$

where $\sigma = \pm 1$. Arguing as in Proposition 2.1, we find that the first nonzero derivative of the mapping g_σ at the origin is of order k with

$$D^k g_\sigma(0) \cdot (\eta, x)^k = -\frac{\sigma k!}{\lambda_0} \eta^{k-1} Q_1 x + Q_1 D_x^k \Gamma(0) \cdot (x)^k \in Y_1, \quad (2.36)$$

for every $(\eta, x) \in \mathbb{R} \times X_1$. Again, two implicit condition for the results of Chapter 2, §4 to be available are

$$X = \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L) (= X_1 \oplus Y_2),$$

(i.e. the algebraic and geometric multiplicities of the characteristic value of λ_0 coincide) and

$$Q_1 D_x^k \Gamma(0) \cdot (x)^k \neq 0 \text{ for every } x \in X_1 - \{0\}.$$

Again, from §1 the mapping $D^k g(0)(\eta, x)^k$ verifies the condition (\mathbb{R} - $N.D$) or not independently of the choice of X_2 and Y_1 so that we can take $X_1 = Y_1$ in the applications. With this choice, we get the simplified expressions 101

$$g_\sigma(\eta, x) = -\frac{\sigma \eta^{k-1}}{\lambda_0} x + Q_1 \Gamma(\sigma \eta^{k-1}, x + \varphi(\sigma \eta^{k-1}, x)), \quad (2.37)_\sigma$$

$$D^k g_\sigma(0) \cdot (\eta, x)^k = -\frac{\sigma k! \eta^{k-1}}{\lambda_0} x + Q_1 D_x^k \Gamma(0) \cdot (x)^k \in X_1, \quad (2.38)_\sigma$$

for every $(\eta, x) \in \mathbb{R} \times X_1$. Also, none of the curves in the local zero set of g_σ has a “vertical” tangent at the origin and each curve in the local zero set of f is of the form

$$t \rightarrow (\mu(t), x(t)),$$

with $\mu(t) = \sigma \eta^{k-1}(t)$ and $t \rightarrow (\eta(t), x(t))$ is one of the curves in the local zero set of g_σ (thus of class C^{m-k+1} at the origin and of class C^m away from it). Each non-trivial curve in the local zero set of f is then also of class C^{m-k+1} at the origin and of class C^m away from it (the argument is the same as when k is even) and it is tangent to the hyperplane $\{0\} \times X_1$ at the origin. For $\sigma = 1$ (respectively -1), the corresponding curves of

the local zero set of f are located in the half space $\mu \geq 0$ (respectively $\mu \leq 0$) because, contrary to what happens when k is even, $\sigma\eta^{k-1}(t)$ does not change sign here.

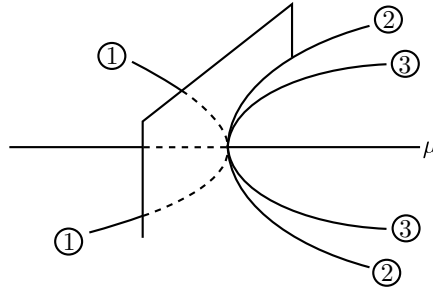


Figure 2.4: Local zero set of f (k odd).

102 The differences with the case when k is even are as follows: (i) Both mapping $(2.38)_\sigma$ must verify the condition $(\mathbb{R} - N.D.)$ for $\sigma = 1$ and $\sigma = -1$. Changing η into $e^{i\pi(k-1)}\eta$, it is easy to see that this is the case if one of them verifies the condition $(\mathbb{C} - N.D.)$, because this property is invariant under \mathbb{C} -linear changes of variables.

(ii) Bifurcation (i.e. existence of nontrivial curves) is *not ensured* (an example was given in Chapter 1). However, bifurcation occurs if η is odd (Theorem 1.2 of Chapter 1). This means that for at least one of the values $\sigma = 1$ or $\sigma = -1$, the zero set of the mapping $D^k g_\sigma(0) \cdot (\eta, x)^k$ contains at least one nontrivial line.

103 **Remark 2.5.** Theorem 1.2 of Chapter 1 (Krasnoselskii's theorem) is based on topological degree arguments. Is there however a purely algebraic proof of the above statement?

(iii) Let $\nu_\sigma, \sigma = \pm 1$ denote the number of curves in the local zero set of the mapping g_σ . Then, *a priori*, the number of curves in the local zero set of f is $\nu_1 + \nu_{-1}$. Actually, ν_1 and ν_{-1} must be *odd* and the number of *distinct* curves in the local zero set of f is $(\nu_1 + \nu_{-1})/2$ (hence $\leq k^n$ again since $\nu_\sigma \leq k^n$). Indeed, it is clear from the evenness of $k - 1$ that when a curve $(\eta(t), x(t))$ is in the local zero set of g_σ , the curve $(-\eta(t), x(t))$

is also in it. Both provide the same curve $(\mu(t), x(t)) = (\sigma\eta^{k-1}(t), x(t))$ in the local zero set of f and the two vectors $((d\eta/dt)(0)(dx/dt)(0))$ and $(-(d\eta/dt)(0), (dx/dt)(0))$ are not collinear if and only if $(dx/dt)(0) \neq 0$, because $(d\eta/dt)(0)$ is $\neq 0$ as we observed earlier⁵. Now the condition $(dx/dt)(0) \neq 0$ is fulfilled by all the curves in the local zero set of g_σ , except the trivial branch (recall that the correspondence between the curves in the local zero set of g_σ and their tangents at the origin is one-to-one). Hence, each non-trivial curve in the local zero set of f is provided by *two* distinct non-trivial curves in the local zero set of g_1 or by two distinct non-trivial curves in the local zero set of g_{-1} . Thus, ν_1 and ν_{-1} must be odd and the number of non-trivial curves in the local zero set of f is

$$\frac{\nu_1 - 1}{2} + \frac{\nu_{-1} - 1}{2} = \frac{\nu_1 + \nu_{-1}}{2} - 1.$$

Adding the trivial branch, we find $(\nu_1 + \nu_{-1})/2$ to be the number of distinct curves in the local set of f . Exactly $(\nu_1 - 1)/2$ of them are supercritical (i.e. located in the half-space $\mu \geq 0$ of $\mathbb{R} \times X_1$) and exactly $(\nu_{-1} - 1)/2$ are subcritical (i.e. located in the half space $\mu \leq 0$ of $\mathbb{R} \times X_1$). One (the trivial branch) is *transcritical*. Of course, it may happen that $\nu_1 = 0$ or $\nu_{-1} = 0$ (or both). 104

Remark 2.6. Recall that in both the case (k even and k odd), we made the *a priori* assumption $X_1 \not\subset Y_2$ (and it turned out that the conditions $X_1 \cap Y_2 = \{0\}$ was necessary). If $X_1 \subset Y_2$, the trick of changing the parameter μ into η^{k-1} , $\sigma = \pm 1$ does not work: Indeed, the first nonzero derivative of g or g_σ at the origin remains of order k and its value at the point $(\eta, x) \in \mathbb{R} \times X_1$ repeated k times reduces to

$$(\eta, x) \in \mathbb{R} \times X_1 \rightarrow Q_1 D_x^k \Gamma(0) \cdot (x)^k \in Y_1.$$

This mapping *never verifies the condition* ($\mathbb{R} - N.D.$) because of the trivial branch in its zero set at which its derivative vanishes.

Remark 2.7. We leave it to the reader to check that changing the parameter μ into $a\eta^{k-1}$, $a \neq 0$, when k is even is equivalent to changing it into

⁵In comment 2.4 and when k is even, but the argument can be repeated when k is odd.

105 η^{k-1} (i.e. $a = 1$) as we have done. No more generality will be reached either by changing μ into $a\eta^{k-1} + o(\eta^{k-1})$, $a \neq 0$ and no other exponent $p \neq k-1$ can be used in the change $\mu = \eta^p$ (because the condition $(\mathbb{R} - N.D.)$ always fails with such a choice). Similarly, when k is odd, changing μ into $a\eta^{k-1} + o(\eta^{k-1})$, $a > 0$ is equivalent to changing μ into η^{k-1} and changing μ into $a\eta^{k-1} + o(\eta^{k-1})$ is equivalent to changing μ into $-\eta^{k-1}$

To complete this section, we come back to the case $n = 1$ for comparisons and further comments. In Chapter 1, we have solved the problem without any assumption on the derivatives $Q_1 D_x^j \Gamma(0)|_{(X_1)^j}$ and the condition $(\mathbb{R} - N.D.)$ was seen to be equivalent to the fact that the algebraic multiplicity of λ_0 equals its geometric multiplicity i.e.

$$X = \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L) (= X_1 \oplus Y_2). \quad (2.39)$$

Nevertheless, if there is an integer $3 \leq k \leq m^6$ such that

$$\begin{aligned} Q_1 D_x^j \Gamma(0) &= 0, 0 \leq j \leq k-1, \\ Q_1 D_x^k \Gamma(0)|_{(X_1)^k} &\neq 0, \end{aligned} \quad (2.40)$$

106 the trick of changing the parameter μ into η^{k-1} (k even) or $\sigma\eta^{k-1}$, $\sigma = \pm 1$ (k odd) works as when n is ≥ 2 and the analysis we have made in this section can be repeated with no modification. In particular, condition (2.39) remains necessary. Here, it is also a sufficient condition for the mapping $D^k g(0) \cdot (\eta, x)^k$ (2.20) (or $D^k g_\sigma(0) \cdot (\eta, x)^k$ ((2.38) $_\sigma$)) to verify the condition $(\mathbb{R} - N.D.)$. Indeed, as X_1 and Y_1 are one-dimensional, we can write $x \in X_1$ in the form

$$x = tx^0,$$

for some $t \in \mathbb{R}$, where x^0 is a given non-zero element of X_1 . Also let y^0 be a given non-zero element of Y_1 ⁷ and consider the linear continuous form $y^* \in X' (= Y')$ characterized by (cf. Chapter 1, §3)

$$\begin{cases} \langle y^*, y^0 \rangle = 1, \\ \langle y^*, y \rangle = 0 \text{ for every } y \in Y_2. \end{cases}$$

⁶If $k = 2$, see Comment 2.1.

⁷From (2.39), the choice $x^0 = y^0$ is available and brings slight simplifications in the formulae.

This linear form allows us to express the projection operator Q_1 onto the space Y_1 . More precisely

$$Q_1 y = \langle y^*, y \rangle y^0,$$

for every $y \in X (= Y)$. Then, for k even, we find from (2.20) that

$$D^k g(0) \cdot (\eta, x)^k = \left[-\frac{k! \eta^{k-1}}{\lambda_0} \langle y^*, x^0 \rangle + t^k \langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle \right] y^0$$

and proving that this mapping verifies the condition ($\mathbb{R} - N.D.$) amounts to proving that the real-valued mapping of the two real variables η, t given by

$$(\eta, t) \in \mathbb{R}^2 \rightarrow -\frac{k! \eta^{k-1}}{\lambda_0} \langle y^*, x^0 \rangle + t^k \langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle \in \mathbb{R},$$

does the same, which is *always* the case because of the relations $\langle y^*, x^0 \rangle \neq 0$ (since $X_1 \cap Y_2 = \{0\}$) and $\langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle \neq 0$ (since $Q_1 D_x^k \Gamma(0)|_{(X_1)^k} \neq 0$). Besides, as expected, the zero set of $D^k g(0) \cdot (\eta, x)^k$ is the union of the trivial branch and exactly *one nontrivial line*, namely that line for which

$$t = \left[k! \frac{\langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle}{\lambda_0 \langle y^*, x^0 \rangle} \right]^{1/k-1} \eta, \eta \in \mathbb{R}.$$

Repeating the arguments we used in the case $n \geq 2$, we see that the corresponding non-trivial branch in the local zero set of f is tangent to the axis $\{0\} \times X_1$ at the origin and located on both sides of it: we are in the presence of a phenomenon of *transcritical bifurcation*.

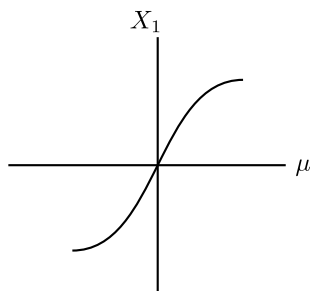


Figure 2.5: Local zero set of f when $n = 1, k$ even ≥ 4

Analogously, when k is odd, one has (cf. ((2.38) $_{\sigma}$))

$$D^k g_{\sigma}(0) \cdot (\eta, x)^k = \left[-k! \frac{\sigma \eta^{k-1} t}{\lambda_0} \langle y^*, x^0 \rangle + t^k \langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle \right] y^0.$$

108 If so, because $k-1$ is even, the local zero set of this mapping *reduces to the trivial branch* if

$$\operatorname{sgn} \left[\frac{\langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle}{\lambda_0 \langle y^*, x^0 \rangle} \right] = \operatorname{sgn}(-\sigma)$$

and is the union of the trivial branch and *two* nontrivial lines, namely, those for which

$$t = \pm \left[k! \sigma \frac{\langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle}{\lambda_0 \langle y^*, x^0 \rangle} \right]^{1/k-1} \quad \eta, \eta \in \mathbb{R},$$

if

$$\operatorname{sgn} \left[\frac{\langle y^*, D_x^k \Gamma(0) \cdot (x^0)^k \rangle}{\lambda_0 \langle y^*, x^0 \rangle} \right] = \operatorname{sgn} \sigma.$$

They both provide the *same* curve in the local zero set of f , located on one side of the axis $\{0\} \times X_1$. The bifurcation is either *supercritical* or *subcritical* depending on which among the mappings $D^k g_1(0) \cdot (\eta, x)^k$ and $D^k g_{-1}(0) \cdot (\eta, x)^k$ possesses the non-trivial lines in its zero set

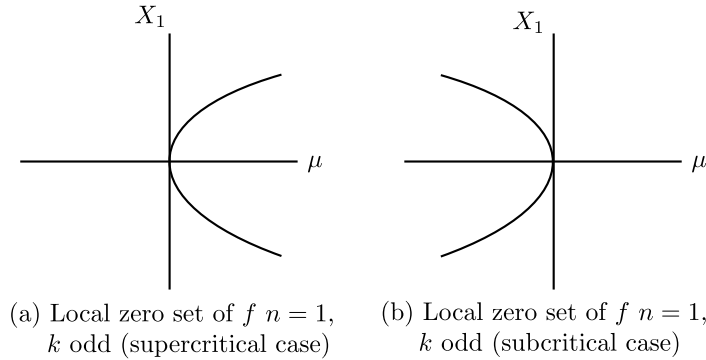


Figure 2.6:

109 Remark 2.8. Though the assumptions of this section are stronger than those we made in Chapter 1 when $n = 1$, the regularity of the non-trivial curve in the local zero set of f is only found to be of class C^{m-k+1} at the origin (but we know it is actually of class C^{m-1}). In contrast, the location of this curve with respect to the axis $\{0\} \times X_1$ could not be derived from the analysis of Chapter 1 because it clearly depends on the evenness of k , which was ignored in Chapter 1.

Remark 2.9. The assumption (2.40) can be replaced by

$$\begin{aligned} D_x^j \Gamma(0)|_{(X_1)_j} &= 0, 0 \leq j \leq k-1, \\ Q_1 D_x^k \Gamma(0)|_{(X_1)^k} &\neq 0, \end{aligned} \quad (2.41)$$

or by a suitable combination (2.40) and (2.41) such as in Remark 2.3.

3.3 Application to a Problem with no Branch of Solutions Known a Priori.

We shall here consider the problem of finding the “small” solution $(\mu, x) \in \mathbb{R} \times X$ an equation of the form

$$F(x) = \mu y^0 \quad (3.1)$$

where F is a mapping of class C^m , $m \geq 1$, in a neighbourhood of the origin in the Banach space X , with values in another Banach space Y , such that $F(0) = 0$ and where y^0 is a given element of Y . In order to deal with a situation essentially different from that of §2 we shall assume **110**

$$y^0 \notin \text{Range} D_x F(0). \quad (3.2)$$

First, we write the problem in the form

$$G(\mu, x) = 0,$$

with

$$G(\mu, x) = F(x) - \mu y^0. \quad (3.3)$$

Due to (3.2), the space $Y_2 = \text{Range}DG(0)$ is

$$Y_2 = \text{Range}DG(0) = \mathbb{R}y^0 \oplus \text{Range}D_xF(0) \quad (3.4)$$

whereas, setting

$$X_1 = \text{Ker}D_xF(0),$$

one has

$$\widetilde{X}_1 = \text{Ker}DG(0) = \{0\} \times X_1. \quad (3.6)$$

As required in these notes, we shall assume that $DG(0) \in \mathcal{L}(\mathbb{R} \times X, Y)$ is a Fredholm operator with index 1, which will be for instance, the case, if $D_xF(0)$ is a Fredholm operator with index 0. As usual, n will denote the codimension of the space Y_2 .

When $n = 0$, the same problem has already been encountered in Chapter 1. The situation was simple because the Implicit function theorem applied so that the local zero set of G was found to be made up of exactly one curve of class C^m . Recall that this curve has a “vertical” tangent (i.e. the one - dimensional space $\{0\} \times X_1$) at the origin and we mentioned that two typical cases were those when the origin is either a “turning point” or a “hysteresis point” (cf. Fig. 3.2 of Chapter 1). A detailed explanation of these phenomena will be given later on. For the time being, we shall assume $n \geq 1$. Performing the Lyapunov-Schmidt reduction as described in Chapter 1, §2 and setting $\widetilde{x} = (\mu, x)$, we get a reduced equation of the general form

$$f(\widetilde{x}) = Q_1G(\widetilde{x} + \widetilde{\varphi}(\widetilde{x})),$$

where \widetilde{x} belongs to some neighbourhood of the origin in the space \widetilde{X}_1 and Q_1 denotes the projection operator onto some given complement of the space Y_2 . Because of (3.6), the variable $\widetilde{x} \in \widetilde{X}_1$ identifies itself with the variable $x \in X_1$. Provided we choose the complement \widetilde{X}_2 of the space \widetilde{X}_1 of the form

$$\widetilde{X}_2 = \mathbb{R} \times X_2,$$

where X_2 is a given topological complement of X_1 , the mapping $\widetilde{\varphi}$ can be identified with a pair

$$\widetilde{\varphi}(x) = (\mu(x), \varphi(x)) \in \mathbb{R} \times X_2,$$

of mappings of class C^m around the origin.

As $Q_1 y^0 = 0$ (cf. (3.4)), the reduced equation takes the form

$$f(x) = Q_1 F(x + \varphi(x)) = 0. \quad (3.7)$$

The results of Chapter 2, §4 will be available if the first nonzero derivative of the above mapping at the origin verifies the condition ($\mathbb{R} - N.D.$). A general frame work in which it takes a very simple expression is as follows: Consider the integer $2 \leq k \leq m$ characterized by

$$\begin{cases} Q_1 D_x^j F(0)|_{(X_1)^j} = 0, 0 \leq j \leq k-1, \\ Q_1 D_x^k F(0)|_{(X_1)^k} \neq 0, \end{cases} \quad (3.8)$$

(if such a k exists of course) and let k_1 and k^1 be defined by

$$k_1 = \min\{0 \leq j \leq k, Q_1 D_x^j F(0) \neq 0\}, \quad (3.9)$$

$$k^1 = \min\{0 \leq j \leq k, D_x^j F(0)|_{(X_1)^j} \neq 0\}, \quad (3.10)$$

so that $2 \leq k_1, k^1 \leq k$. Under the condition⁸

$$k \leq k_1 + k^1 - 2, \quad (3.11)$$

the first non-zero derivative of the mapping f in (3.7) at the origin is of order k and its value at the point $x \in X_1$ repeated k times is

$$D^k f(0) \cdot (x)^k = Q_1 D_x^k F(0) \cdot (x)^k. \quad (3.12)$$

The reader can easily check this assertion in the two frequently encountered particular cases $k = k_1$ or $k = k^1$, namely when

$$Q_1 D_x^j F(0) = 0, 0 \leq j \leq k-1,$$

or

$$D_x^j F(0)|_{(X_1)^j} = 0, 0 \leq j \leq k-1.$$

⁸Observe that none of the integers k, k_1 and k^1 depends on the choice of the space Y_1 .

For a general result, see [31]. When (3.11) holds and the mapping (3.12) verifies the condition ($\mathbb{R} - N.D.$) (a property which is easily seen to be independent of the choice of the space Y_1) the largest number of curves in the local zero set of f - hence of G - is k^n . Note here that

$$n = \dim X_1 - 1 (= \dim \text{Ker} D_x F(0) - 1). \quad (3.13)$$

Of course, the curves are of class C^{m-k+1} at the origin and of class C^m away from it.

Remark 3.1. In contrast to the case $n = 0$, it is quite possible that the origin is an *isolated solution*. For instance, let $X = Y = \mathbb{R}^3$ and choose F as

$$F(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 \\ 0 \\ x_3 \end{pmatrix}$$

while y^0 is taken as

$$y^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The only solution of the equation $F(x_1, x_2, x_3) = \mu y^0$ is $\mu = x_1 = x_2 = x_3 = 0$. Nevertheless, (3.8) is fulfilled with $k = 2$ and (3.11) holds. The mapping (3.13) associated with this example is

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \rightarrow x_1^2 + x_2^2 + x_3^2 \in \mathbb{R}$$

114 and verifies the condition ($\mathbb{R} - N.D.$) trivially.

Observe, however, that existence of at least one curve of solution is generated by comment 1.3 of Chapter 2 when k is *odd*.

The analysis we have made does not provide any information on the *location* of the curves. Incidentally, such information was obtained in the problem we considered in the previous section after we had to make a change of parameter to find the structure of its set of solutions. Here, one has no such motivation indeed since a satisfactory answer has already been given to this question but it is not without interest to see

where a change of parameter could lead us: setting $\mu = \eta^p$ for some integer $p \geq 2$, we have to solve the problem

$$\hat{G}(\eta, u) = G(\eta^p, x) = F(x) - \eta^p y^0 = 0. \quad (3.14)$$

The main difference with the case when the parameter μ is unchanged is that (compare with (3.4) and (3.6))

$$\hat{Y}_2 = \text{Range } D\hat{G}(0) = \text{Range } D_x F(0), \quad (3.15)$$

$$\widetilde{\hat{X}}_1 = \text{Ker } D\hat{G}(0) = \mathbb{R} \times \text{Ker } D_x F(0) = \mathbb{R} \times X_1. \quad (3.16)$$

Thus, the mapping $D\hat{G}(0)$ will be a Fredholm operator with index 1 *if and only if* the mapping $D_x F(0)$ is a *Fredholm operator with index 0* (namely, $\text{Range } D_x F(0)$ must be closed). Also, with the previous definition of n ($= \text{codim } Y_2$), one has

$$\text{codim } \hat{Y}_2 = n + 1 (= \hat{n}) \geq 1, \quad (3.17)$$

$$\dim \text{Ker } \widetilde{\hat{X}} = n + 2 (= \hat{n} + 1) \geq 2. \quad (3.18)$$

We shall denote by \hat{Y}_1 any complement of \hat{Y}_2 and call \hat{Q}_1 and \hat{Q}_2 115 the projection operators onto \hat{Y}_1 and \hat{Y}_2 respectively. Performing the Lyapunov-Schmidt reduction of the problem (3.14), we find a reduced equation of the general form

$$\hat{g}(\hat{x}) = \hat{Q}_1 \hat{G}(\hat{x} + \hat{\varphi}(\hat{x})).$$

Here, the variable \hat{x} of the space $\widetilde{\hat{X}}_1 = \mathbb{R} \times X_1$ is the pair $(\eta, x) \in \mathbb{R} \times X_1$ and the mapping $\hat{\psi}$ takes its values in some topological complement $\widetilde{\hat{X}}_2$ of the space $\widetilde{\hat{X}}_1$. Choosing $\widetilde{\hat{X}}_2 = \{0\} \times X_2$ where X_2 is any topological complement of X_1 , the mapping $\hat{\psi}$ identifies itself with a mapping $\hat{\psi}$ with values in X_2 so that

$$\hat{g}(\eta, x) = \hat{Q}_1 F(x + \hat{\varphi}(\eta, x)) - \eta^p \hat{Q}_1 y^0$$

(note that $\hat{Q}_1 y^0 \neq 0$ since $y^0 \notin \text{Range } D_x F(0)$ by hypothesis). At this stage, it is necessary for a better understanding to examine how the mapping $\hat{\varphi}$ is related to the variable μ through the change $\mu = \eta^p$. To this

end, write each element $x \in X$ in the form $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. The equation $G(\mu, x) = 0$ becomes equivalent to the system

$$\begin{aligned}\hat{Q}_1 G(\mu, x_1 + x_2) &= 0, \\ \hat{Q}_2 G(\mu, x_1 + x_2) &= 0.\end{aligned}$$

As $\hat{Q}_2 D_x G(0) = \hat{Q}_2 D_x F(0) \in \text{Isom}(X_2, \hat{Y}_2)$, the second equation is solved by the Implicit function theorem and is equivalent to

$$x_2 = \hat{\varphi}(\mu, x_1),$$

- 116** where $\hat{\varphi}$ is mapping of class C^m around the origin in $\mathbb{R} \times X_1$ with values in X_2 , uniquely determined by the condition $\hat{\varphi}(0) = 0$. Referring to the first equation, we find a reduced equation

$$\tilde{f}(\mu, x_1) = \hat{Q}_1 G(\mu, x_1 + \hat{\varphi}(\mu, x_1))$$

for (μ, x_1) around the origin in $\mathbb{R} \times X_1$. Dropping the index “1” in the variable $x_1 \in X_1$ and using (3.3), we deduce

$$\hat{f}(\mu, x) = \hat{Q}_1 G(\mu, x + \hat{\varphi}(\mu, x)), \quad (3.19)$$

for (μ, x) around the origin of $\mathbb{R} \times X_1$. Note that this reduction *differs* from the Lyapunov-Schmidt reduction as it was described in Chapter 1 because the space \hat{Y}_2 is *not* the range of the *global* derivative $DG(0)$ but only the range of the *partial* derivative $D_x G(0)$. In particular, $D\hat{\varphi}(0) \neq 0$ since $D_\mu \hat{\varphi}(0) \neq 0$ in general. However, the local zero set of \hat{f} (3.19) immediately provides the local zero set of G and a simple verification shows that the mapping $\hat{\psi}$ and $\hat{\varphi}$ are linked through the relation

$$\hat{\psi}(\eta, x) = \hat{\varphi}(\eta^p, x).$$

It follows that

$$\hat{g}(\eta, x) = \hat{f}(\eta^p, x) = \hat{Q}_1 F(x + \hat{\varphi}(\eta^p, x)) - \eta^p \hat{Q}_1 y^0.$$

Again, the problem now is to find the first non-zero derivative of \hat{g} at the origin. Of course, it will depend on p but since the condition

($\mathbb{R} - N.D.$) must hold under general hypotheses, it is possible to show that the *only* available value of p is $p = \hat{k}$ where the integer $2 \leq \hat{k} \leq m$ is characterized by

$$\begin{cases} \hat{Q}_1 D_x^j F(0)|_{(X_1)^j} = 0, 0 \leq j \leq \hat{k} - 1, \\ \hat{Q}_1 D_x^{\hat{k}} F(0)|_{(X_1)^{\hat{k}}} \neq 0. \end{cases} \quad (3.20)$$

The process allowing the selection of p should be described in a general framework rather than on this particular example but even so, it remains quite technical and will not be presented here (see [31] for details).

The mapping \hat{g} corresponding to the choice $p = \hat{k}$ is

$$\hat{g}(\eta, x) = \hat{f}(\eta^{\hat{k}}, x) = \hat{Q}_1 F(x + \hat{\varphi}(\eta^{\hat{k}}, x)) - \eta^{\hat{k}} \hat{Q}_1 y^0. \quad (3.21)$$

Setting

$$\hat{k}_1 = \min\{0 \leq j \leq \hat{k}, \hat{Q}_1 D_x^j F(0) \neq 0\}, \quad (3.22)$$

$$\hat{k}^1 = \min\{0 \leq j \leq \hat{k}, D_x^j F(0)|_{(X_1)^j} \neq 0\}, \quad (3.23)$$

one has $2 \leq \hat{k}_1, \hat{k}^1 \leq \hat{k}$. Clearly, none of the integers \hat{k}, \hat{k}_1 and \hat{k}^1 depends on the choice of the space \hat{Y}_1 and, under the condition

$$\hat{k} \leq \hat{k}_1 + \hat{k}^1 - 2, \quad (3.24)$$

the first non-zero derivative of the mapping \hat{g} (3.21) at the origin is of order \hat{k} with

$$D^{\hat{k}} \hat{g}(0) \cdot (\eta, x)^{\hat{k}} = \hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} - \hat{k}! \eta^{\hat{k}} \hat{Q}_1 y^0 \epsilon \hat{Y}_1, \quad (3.25)$$

for every $(\eta, x) \in \mathbb{R} \times X_1$. The reader can check there formulae in the particular cases $\hat{k}_1 = \hat{k}$ or $\hat{k}^1 = \hat{k}$, namely

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$$\hat{Q}_1 D_x^j F(0) = 0, 0 \leq j \leq \hat{k} - 1,$$

or

$$D_x^j F(0)|_{(X_1)^j} = 0, 0 \leq j \leq \hat{k} - 1.$$

In the general case, see [31]. If the mapping (2.25) verifies the condition $(\mathbb{R} - N.D.)$ and \hat{k} is odd, the local zero set of \hat{g} provides that of \hat{f} (3.19) immediately. If \hat{k} is even the local zero set of \hat{g} gives the elements of the local zero set of \hat{f} associated with $\mu \geq 0$. So as to get the elements associated with $\mu \leq 0$, the change $\mu = -\eta^{\hat{k}}$ is necessary too. When \hat{k} is even, we must then examine both the mappings

$$\hat{g}_\sigma(\eta, x) = \hat{f}(\sigma\eta^{\hat{k}}, x) = \hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} - \hat{k}! \sigma \eta^{\hat{k}} \hat{Q}_1 y^0 \in \hat{Y}_1, \quad (3.26)_\sigma$$

where $\sigma = \pm 1$ and we have

$$D^{\hat{k}} g_\sigma(0) \cdot (\eta, x)^{\hat{k}} = \hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} - \hat{k}! \sigma \eta^{\hat{k}} \hat{Q}_1 y^0, \quad ((3.27)_\sigma)$$

for $(\eta, x) \in \mathbb{R} \times X_1$.

Assume first that \hat{k} is odd and the mapping (3.25) verifies the condition $(\mathbb{R} - N.D.)$. Arguing as in §1, we see that its zero set contains no “vertical” line (i.e. no line contained in the hyperplane $\{0\} \times X_1$). Therefore, if $t \rightarrow (\eta(t), x(t))$ is one of the curves in the local zero set of \hat{g} , one has

$$\frac{d\eta}{dt}(0) \neq 0,$$

119 so that the curve is located on both sides of the hyperplane $\{0\} \times X_1$ in $\mathbb{R} \times X_1$.

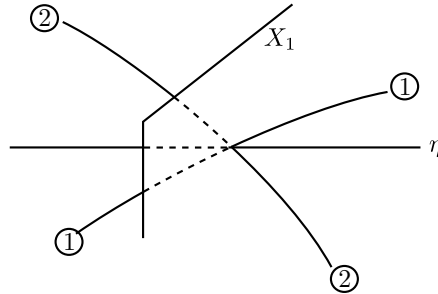


Figure 3.1: Local zero set of \hat{g} (\hat{k} odd).

As \hat{k} is odd and the mapping $n \rightarrow \eta^{\hat{k}}$ is a homeomorphism of \mathbb{R} , the corresponding curve $(\mu(t), x(t))$ with $\mu(t) = \eta^{\hat{k}}(t)$ in the local zero set of

\hat{f} is also located on both sides of the hyperplane $\{0\} \times X_1$ in $\mathbb{R} \times X_1$. Of course,

$$\frac{d}{dt}(\mu(t), x(t))|_{t=0} = (0, \frac{dx}{dt}(0)).$$

But $\frac{dx}{dt}(0) \neq 0$, because the non-zero vector, $(\frac{d\eta}{dt}(0), \frac{dx}{dt}(0))$ is in the zero set of the mapping (3.25), which does not contain the line $(\eta, 0)$ ⁹ As a result, each curve $(\mu(t), x(t))$ in the local zero set of \hat{f} is tangent to the hyperplane $\{0\} \times X_1$ at the origin. Replacing the hyperplane $\{0\} \times X_1$ in $\mathbb{R} \times X_1$ by the hyperplane $\{0\} \times X$ in $\mathbb{R} \times X$, these statements remain readily valid as concerns the local zero set of the mapping G . 120

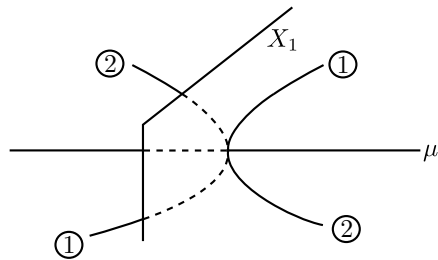


Figure 3.2: Local zero set of \hat{f} (\hat{k} odd).

Assume next that \hat{k} is *even* and the both mappings $((3.26)_\sigma)$ verify the condition $(\mathbb{R} - N.D.)$. As before, if $t \rightarrow (\eta(t), x(t))$ is one of the curves in the local zero set of \hat{g}_1 or \hat{g}_{-1} , one has

$$\frac{d\eta}{dt}(0) \neq 0, \frac{dx}{dt}(0) \neq 0.$$

In particular, the corresponding curve $(\mu(t), x(t))$ with $\mu(t) = \sigma\eta^{\hat{k}}(t)$ in the local zero set of \hat{f} is tangent to the hyperplane $\{0\} \times X_1$ at the origin. But, as $\sigma\eta^{\hat{k}}(t)$ does *not* change sign for t varying around 0, this curve is *located on one side* of the hyperplane $\{0\} \times X_1$ in $\mathbb{R} \times X_1$. Again, replacing the hyperplane $\{0\} \times X_1$ in $\mathbb{R} \times X_1$ by the hyperplane $\{0\} \times X$ in $\mathbb{R} \times X$, these statements remain readily valid as concerns the local zero 121

⁹See Proposition 3.1 later

set of the mapping G .

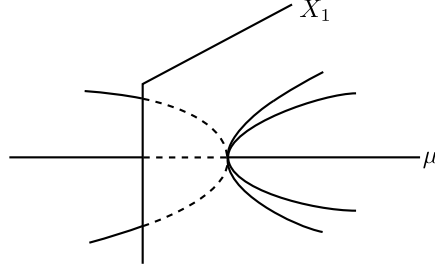


Figure 3.3: Local zero set of \hat{f} (\hat{k} even)

We are now going to relate the assumptions we make for solving the problem after changing the parameter μ into $\eta^{\hat{k}}$ or $\sigma\eta^{\hat{k}}$ to the assumptions we made earlier to solve it without changing μ . First, observe that the integer \hat{k} characterized by (3.18). Indeed, \hat{k} is independent of the choice of the space \hat{Y}_1 and from the definitions, it is immediate that \hat{Y}_1 can be chosen so that

$$\hat{Y}_1 \supset Y_1 \tag{3.28}$$

where Y_1 is any space chosen for defining k . By the same arguments, note that

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$$\hat{k}_1 \leq k_1, \hat{k}^1 \leq k^1, \tag{3.29}$$

(cf. (3.9)-(3.10) and (3.22)-(3.23)). But we shall see that the integers k and \hat{k} must coincide for the mappings (3.25) or (3.27) $_{\sigma}$ to verify the condition ($\mathbb{R} - N.D.$) when $n \geq 1$. This is proved in

Proposition 3.1. *Assume \hat{k} is odd (resp. even). Then, the mapping (3.25) (resp. (3.27) $_1$ and (3.27) $_{-1}$) erifies the condition ($\mathbb{R} - N.D.$) is and only if*

$$\hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} \neq 0, \tag{3.30}$$

for every $x \in X_1 - \{0\}$ and the mapping

$$x \in X_1 \rightarrow Q_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} \in Y_1, \tag{3.31}$$

verifies the condition ($\mathbb{R} - N.D.$). In particular, when $n \geq 1$, one must have $\hat{k} = k$. When $n = 0$, the condition (3.30) holds by the definition of \hat{k} , k is not defined and $Y_1 = \{0\}$ so that the mapping (3.31) verifies the condition ($\mathbb{R} - N.D.$) trivially.

Proof. We prove the equivalence when \hat{k} is odd. When \hat{k} is even, the proof is identical and left to the reader.

The condition (3.30) is necessary. Indeed, if $\hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} = 0$ for some $x \in X_1 - \{0\}$, the line $\{0\} \times \mathbb{R}x$ is in the zero set of the mapping (3.25) whose derivative at $(0, x) \in \mathbb{R} \times X_1$ is

$$(\eta', x') \in \mathbb{R} \times X_1 \rightarrow \hat{k} \hat{Q}_1 D_x^{\hat{k}} F(0) \cdot ((x)^{\hat{k}-1}, x') \in \hat{Y}_1.$$

Its null-space contains the two-dimensional subspace $\mathbb{R}(1, 0) \oplus (\{0\} \times \mathbb{R}x)$ and hence its range is of dimension $\leq \hat{n} - 1$ so that the condition ($\mathbb{R} - N.D.$) fails. 123

Now, we prove that the mapping (3.31) verifies the condition ($\mathbb{R} - N.D.$). As our assumptions are independent of the space \hat{Y}_1 , we can suppose that

$$\hat{Y}_1 = \mathbb{R}y^0 \oplus Y_1. \quad (3.32)$$

If so, $\hat{Q}_1 y^0 = y^0$, $Q_1 \hat{Q}_1 = Q_1$ and the operator $\hat{Q}_1 - Q_1$ is the projection onto the space $\mathbb{R}y^0$ associated with the decomposition $Y = \mathbb{R}y^0 \oplus (Y_1 \oplus \text{Range } D_x F(0))$. Then, the mapping (3.25) becomes

$$(\eta, x) \in \mathbb{R} \times X_1 \rightarrow \hat{k}! \eta^{\hat{k}} y^0 + \hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} \in \hat{Y}_1.$$

Let now x be a non-zero element of the zero set of the mapping (3.31) so that $\hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} = (\hat{Q}_1 - Q_1) D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}}$, and, from the above, the right hand side is collinear with y^0 . Therefore

$$\hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} = \lambda y^0,$$

for some real number λ . As \hat{k} is odd, there is $\eta \in \mathbb{R}$ such that $\hat{k}! \eta^{\hat{k}} = -\lambda$ and the pair $(\eta, x) \in \mathbb{R} \times X_1$ is in the zero set of the mapping (3.25). Let $y \in Y_1$ be given. In particular, $y \in \hat{Y}_1$ (cf. 3.32) and there is a pair $(\eta', x') \in \mathbb{R} \times X_1$ such that

$$\hat{k}(\hat{k}!) \eta'^{\hat{k}-1} \eta' y^0 + \hat{k} \hat{Q}_1 D_x^{\hat{k}} F(0) \cdot ((x)^{\hat{k}-1}, x') = y.$$

Taking the projection onto Y_1 , we find

$$\hat{k}Q_1D_x^{\hat{k}}F(0) \bullet ((x)^{\hat{k}-1}, x') = y,$$

124 which proves our assertion.

Conversely, let $(\eta, x) \in \mathbb{R} \times X_1$ be a non-zero element of the zero set of the mapping (3.25). Observe that $\eta \neq 0$ from (3.30) and it is obvious that $x \neq 0$. Taking the projection onto Y_1 , we find that x is in the zero set of the mapping (3.31). Let $\hat{y} \in \hat{Y}_1$ be given and set $y = Q_1\hat{y}$. By hypothesis, there exists $x' \in X_1$ such that $\hat{k}Q_1D_x^{\hat{k}}F(0) \bullet ((x)^{\hat{k}-1}, x') = y$. As $(\hat{Q}_1 - Q_1)$ is the projection onto the space $\mathbb{R}y^0$ and by definition of y , we find

$$\hat{k}\hat{Q}_1D_x^{\hat{k}}F(0) \bullet ((x)^{\hat{k}-1}, x') = \hat{y} + \lambda y^0,$$

for some real number λ . Setting

$$\eta' = -\lambda/\hat{k}(\hat{k}!) \eta^{\hat{k}-1},$$

it is clear that

$$\hat{k}(\hat{k}!)^{\hat{k}-1} \eta' y^0 + \hat{k}\hat{Q}_1D_x^{\hat{k}}F(0) \bullet ((x)^{\hat{k}-1}, x') = \hat{y},$$

so that the mapping (3.25) verifies the condition ($\mathbb{R} - N.D.$).

The end of our assertion is obvious. If $n \geq 1$, we already know that $\hat{k} \leq k$ and, if $\hat{k} < k$, one has

$$Q_1D_x^{\hat{k}}F(0)|_{(X_1)^{\hat{k}}} = 0,$$

by definition of k , which contradicts the fact that the mapping (3.31) verifies the condition ($\mathbb{R} - N.D.$). When $n = 0$, the space \hat{Y}_1 is one-dimensional (i.e. $\hat{n} = 1$) and the condition (3.30) is then automatically satisfied by definition of \hat{k} . □

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To sum up, the local zero set of the mapping G (3.3) can be found after changing the parameter μ into $\eta^{\hat{k}}$ under the conditions

- a) $D_xF(0)$ is a Fredholm operator with index 0,
- b) $\hat{k} \leq \hat{k}_1 + \hat{k}^1 - 2$,

c) $\hat{Q}_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} \neq 0$ for every $x \in X_1 - \{0\}$,

d) the mapping

$$x \in X_1 \rightarrow Q_1 D_x^{\hat{k}} F(0) \cdot (x)^{\hat{k}} \in Y_1,$$

verifies the condition ($\mathbb{R} - N.D.$).

In any case, these assumptions are *stronger* than those made when the local zero set of G is determined without changing the parameter μ (see in particular (3.29) and recall that \hat{k} must equal k when $n \geq 1$) and the conclusions are *not as precise*: The number of curves is found to be $\leq \hat{k}^{\hat{n}} = \hat{k}^{n+1}$. When $n \geq 1$, this upper bound is nothing but k^{n+1} (but we know that it is actually k^n) and \hat{k} when $n = 0$ (but we know that there is exactly one curve). The regularity we obtain is of class $C^{m-\hat{k}+1}$ at the origin and of class C^m away from it, which agrees with the previous result if $n \geq 1$ (since $\hat{k} = k$) but is not as good when $n = 0$ (the curve is actually of class C^m at the origin). However, the additional information which may be of interest is the *location* of the curves in the space $\mathbb{R} \times X$. In particular, when $n = 0$, the reader can immediately deduce, from the appropriate discussion given before, that the origin is a “*turning point*” when \hat{k} is *even* and a “*hysteresis point*” when \hat{k} is *odd*. When $n \geq 1$, bifurcation may occur but the origin can be an isolated solution as well (see the example given in Remark 3.1). 126

Chapter 4

An Algorithm for the Computation of the Branches

IN THIS CHAPTER, we describe an algorithm for finding the local zero set of a mapping f verifying the assumptions of Chapter 2. When f is *explicitly known*, we obtain an iterative scheme with optimal rate of convergence. However, in the applications to one parameter problems in Banach spaces (such as those described in Chapter 3 for instance) f is the reduced mapping which is known *only theoretically*, because the mapping $\tilde{\varphi}$ coming from the Lyapunov-Schmidt reduction (cf. Chapter 1, §2) involved in the definition of f is found through the *implicit function theorem*. 127

What is explicitly available in this case is only a sequence (f_ℓ) of mappings tending to f in some appropriate way and the algorithm must be modified accordingly. Again, optimal rate of convergence can be established but the proof is more delicate. For the sake of brevity, we shall only outline the technical differences which occur. The exposition follows Rabier-E1-Hajji [33]. Full details can be found in Appendix 2. This chapter is completed by an explicit description of the algorithm in the case of problems of bifurcation from the trivial branch. A comparison with the classical “*Lyapunov-Schmidt method*” is given as a conclusion.

4.1 A Short Review of the Method of Chapter II.

- 128 In Chapter 2, we considered the problem of finding the local zero set of a mapping f of class $C^m, m \geq 1$, from \mathbb{R}^{n+1} into \mathbb{R}^n (recall that it is not restrictive to assume that f is defined everywhere) satisfying the following condition: There is an integer $1 \leq k \leq m$ such that

$$D^j f(0) = 0 \quad 0 \leq j \leq k-1,$$

and the mapping

$$\tilde{\xi} \in \mathbb{R}^{n+1} \rightarrow q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k \in \mathbb{R}^n,$$

verifies the condition ($\mathbb{R} - N.D.$). Under this assumption, the set

$$\{\tilde{\xi} \in \mathcal{S}_n; q(\tilde{\xi}) = 0\}$$

is finite and consists of the 2ν elements $\tilde{\xi}_0^j, 1 \leq j \leq 2\nu$. Of course, as the origin is an isolated solution of the equation $f(\tilde{x}) = 0$ when $\nu = 0$, it is not restrictive to limit ourselves to the case $\nu \geq 1$ for defining the algorithm. This assumption will be implicitly made through-out this chapter.

Setting, for $(t, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^{n+1}$

$$g(t, \tilde{\xi}) = \begin{cases} \frac{k!}{t^k} f(t\tilde{\xi}) & \text{if } t \neq 0, \\ q(\tilde{\xi}) & \text{if } t = 0, \end{cases}$$

- 129 it was observed for $r > 0$ that finding the solution \tilde{x} of the equation $f(\tilde{x}) = 0$ with $0 < \|\tilde{x}\| = t < r$ was equivalent to finding \tilde{x} of the form $\tilde{x} = t\tilde{\xi}$ where

$$\begin{aligned} 0 < |t| < r, \tilde{\xi} \in \mathcal{S}_n, \\ g(t, \tilde{\xi}) &= 0. \end{aligned}$$

Next, an essential step consisted in proving for $r > 0$ small enough that the above equation was equivalent to solving the problem

$$0 < |t| < r, \tilde{\xi} \in \bigcup_{j=1}^{2\nu} \sigma_j,$$

$$g(t, \tilde{\xi}) = 0,$$

where, for each $1 \leq j \leq 2\nu$, σ_j denotes an *arbitrary* neighbourhood of $\tilde{\xi}_0^j$ in S_n . Taking the σ_j 's disjoint, the problem reduces to the study of 2ν independent equations

$$\begin{aligned} 0 < |t| < r, \tilde{\xi} \in \sigma_j, \\ g(t, \tilde{\xi}) = 0, \end{aligned}$$

for $1 \leq j \leq 2\nu$ where actually, ν of them (properly selected) are sufficient, to provide all the other ν solutions because of symmetry properties. From a practical point of view, it remains to solve the equation $g(t, \tilde{\xi}) = 0$ for $|t|$ small enough and $\tilde{\xi} \in S_n$ around one of the points $\tilde{\xi}_0^j$.

4.2 Equivalence of Each Equation with a Fixed Point Problem.

Since the specific value of the index j is not important, we shall denote any one of the points $\tilde{\xi}_0^j$ by $\tilde{\xi}_0$. Given $\delta > 0$, we call Δ the closed ball 130 in \mathbb{R}^{n+1} with centre $\tilde{\xi}_0$ and radius $\delta/2$. The *diameter* δ of the ball Δ is always supposed to satisfy the condition

$$0 < \delta < 1. \tag{2.1}$$

Finally, we denote by C the spherical cap centered at $\tilde{\xi}_0$ (playing the role of the neighbourhood σ_j) defined by

$$C = \Delta \cap S_n \tag{2.2}$$

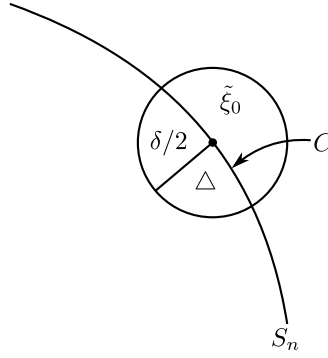


Figure 2.1:

We now establish two simple but crucial geometric properties. As in Chapter 2, $\|\cdot\|$ denotes the euclidean norm.

Lemma 2.1. (i) For every $\tilde{\xi} \in \Delta$, every $\tilde{\zeta} \in C$ and every $\tilde{\tau} \in T_{\tilde{\zeta}}S_n$, one has

$$\|\tilde{\xi} - \tilde{\tau}\| \geq 1 - \delta > 0.$$

131 (ii) For every $\tilde{\xi} \in C$ and every $\tilde{\zeta} \in C$, one has

$$\mathbb{R}^{n+1} = \mathbb{R}\tilde{\xi} \oplus T_{\tilde{\zeta}}S_n.$$

Proof. Recall first, for $\tilde{\zeta} \in S_n$, that

$$T_{\tilde{\zeta}}S_n = \{\tilde{\zeta}\}^\perp.$$

To prove (i), we begin by writing

$$\|\tilde{\xi} - \tilde{\tau}\| \geq \|\tilde{\zeta} - \tilde{\tau}\| - \|\tilde{\xi} - \tilde{\zeta}\|$$

Now, since $\tilde{\zeta}$ and $\tilde{\tau}$ are orthogonal and $\tilde{\zeta} \in C \subset S_n$, we have

$$\|\tilde{\zeta} - \tilde{\tau}\|^2 = \|\tilde{\zeta}\|^2 + \|\tilde{\tau}\|^2 = 1 + \|\tilde{\tau}\|^2.$$

Thus, as $\delta < 1$

$$\|\tilde{\xi} - \tilde{\tau}\| > 1 - \|\tilde{\xi} - \tilde{\zeta}\| \geq 1 - \delta > 0.$$

Next, to prove (ii), it suffices to show that $\mathbb{R}\tilde{\xi} \cap T_{\tilde{\zeta}}S_n = \{0\}$, namely that $\tilde{\xi} \notin T_{\tilde{\zeta}}S_n = \{\tilde{\zeta}\}^\perp$. But a simple calculation shows that $\tilde{\xi} \in C$ and $\tilde{\zeta} \in C$ are never orthogonal where

$$\delta < 2\sqrt{2 - \sqrt{2}}$$

and hence when $\delta < 1$.

Since the mapping q satisfies the condition ($\mathbb{R} - N.D.$) and $\tilde{\xi}_0$ is one of the points $\tilde{\xi}_0^j$, we know that

$$Dq(\tilde{\xi}_0)|_{T_{\tilde{\xi}_0}S_n} \in \text{Isom}(T_{\tilde{\xi}_0}S_n, \mathbb{R}^n).$$

□

Lemma 2.2. *After shrinking $\delta > 0$ if necessary, one has*

$$Dq(\tilde{\xi})|_{T_{\tilde{\xi}}S_n} \in \text{Isom}(T_{\tilde{\xi}}S_n, \mathbb{R}^n),$$

for every $\tilde{\xi} \in C$. Besides, setting

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$$A(\tilde{\xi}) = (Dq(\tilde{\xi})|_{T_{\tilde{\xi}}S_n})^{-1} \in \mathcal{L}(\mathbb{R}^n, T_{\tilde{\xi}}S_n) \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n+1}), \quad (2.3)$$

one has

$$A \in C^0(C, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n+1})).$$

Proof. Let B_1 be the open unit ball in \mathbb{R}^n

$$B_1 = \{\tilde{\xi}' \in \mathbb{R}^n; \|\tilde{\xi}'\| < 1\}.$$

By identifying \mathbb{R}^{n+1} with the product $\mathbb{R}^n \times \mathbb{R}\tilde{\xi}_0$, the spherical cap C is homeomorphic to the closed ball $\bar{B}_{d/2} \subset \mathbb{R}^n$ centred at the origin with diameter

$$d = \delta \left(1 - \frac{\delta^2}{16}\right)^{\frac{1}{2}} < 1.$$

This homeomorphism is induced by a C^∞ -mapping

$$\tilde{\xi}' \in B_1 \rightarrow \theta(\tilde{\xi}') = (\tilde{\xi}', (1 - \|\tilde{\xi}'\|^2)^{\frac{1}{2}} \tilde{\xi}_0) \in \mathbb{R}^{n+1}$$

It is immediately checked that

$$D\theta(\tilde{\xi}') \in \text{Isom}(\mathbb{R}^n, T_{\theta(\tilde{\xi}')} S_n) \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n+1}), \quad (2.4)$$

for every $\tilde{\xi}' \in B_1$.

The mapping $q \bullet \theta$ is of class C^∞ from B_1 into \mathbb{R}^n and by differentiating

$$D(q \bullet \theta)(\tilde{\xi}') = Dq(\theta(\tilde{\xi}')) \cdot D\theta(\tilde{\xi}').$$

133 Since the mapping $D\theta(\tilde{\xi}')$ takes its values in the space $T_{\theta(\tilde{\xi}')} S_n$, this can be rewritten as

$$D(q \bullet \theta)(\tilde{\xi}') = Dq(\theta(\tilde{\xi}'))|_{T_{\theta(\tilde{\xi}')} S_n} \cdot D\theta(\tilde{\xi}'). \quad (2.5)$$

In particular, for $\tilde{\xi}' = 0$

$$D(q \bullet \theta)(0) = Dq(\tilde{\xi}_0)|_{T_{\tilde{\xi}_0} S_n} \cdot D\theta(0).$$

As $Dq(\tilde{\xi}_0)|_{T_{\tilde{\xi}_0} S_n} \in \text{Isom}(T_{\tilde{\xi}_0} S_n, \mathbb{R}^n)$ and $D\theta(0) \in \text{Isom}(\mathbb{R}^n, T_{\tilde{\xi}_0} S_n)$, we deduce

$$D(q \bullet \theta)(0) \in \text{Isom}(\mathbb{R}^n).$$

By continuity, we may assume that $d > 0$ (or, equivalently, $\delta > 0$) has been chosen small enough for $D(q \bullet \theta)(\tilde{\xi}')$ to be an isomorphism for every $\tilde{\xi}' \in \bar{B}_{d/2}$. Together with (2.4), relation (2.5) shows that

$$Dq(\theta(\tilde{\xi}'))|_{T_{\theta(\tilde{\xi}')} S_n} = D(q \bullet \theta)(\tilde{\xi}') \cdot [D\theta(\tilde{\xi}')]^{-1} \in \text{Isom}(T_{\theta(\tilde{\xi}')} S_n, \mathbb{R}^n). \quad (2.6)$$

Hence

$$Dq(\tilde{\xi})|_{T_{\tilde{\xi}} S_n} \in \text{Isom}(T_{\tilde{\xi}} S_n, \mathbb{R}^n),$$

for every $\tilde{\xi} \in C$ since θ is a bijection from $\bar{B}_{d/2}$ to C . Now, taking the inverse in (2.6) and according to the definition of A in (2.3), we get

$$A(\theta(\tilde{\xi}')) = [Dq(\theta(\tilde{\xi}'))|_{T_{\theta(\tilde{\xi}')} S_n}]^{-1} = D\theta(\tilde{\xi}') \cdot (D(q \bullet \theta)(\tilde{\xi}'))^{-1}.$$

134 The continuity of the mappings D_θ (with values in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n+1})$) and $D(q \bullet \theta)$ (with values in $\text{Isom}(\mathbb{R}^n)$) and the continuity of the mapping $L \rightarrow L^{-1}$ in the set $\text{Isom}(\mathbb{R}^n)$ shows that the mapping $A \bullet \theta$ is continuous on the ball $\bar{B}_{d/2}$. The continuity of A follows since θ is a homeomorphism between $\bar{B}_{d/2}$ and C . \square

Remark 2.1. With an obvious modification of the above arguments, we see that the mapping $A \bullet \theta$ is actually C^∞ around the origin of \mathbb{R}^n . As θ^{-1} is a chart of the sphere S_n centered at the point $\tilde{\xi}_0$, this means that A is of class C^∞ on a neighbourhood of $\tilde{\xi}_0$ in the sphere S_n . More generally, replacing the mapping $q(\tilde{\xi})$ by $q(t, \tilde{\xi})$ a similar proof shows that the mapping

$$(t, \tilde{\xi}) \rightarrow [D_{\tilde{\xi}}g(t, \tilde{\xi})|_{T_{\tilde{\xi}}S_n}]^{-1}$$

is of class C^ℓ in a neighbourhood of $(0, \tilde{\xi}_0)$ in $\mathbb{R} \times S_n$ whenever $D_{\tilde{\xi}}g$ is of class C^ℓ . This result was used in the proof of Theorem 4.1 of Chapter 2 (with $\ell = m - k$).

Let us now fix $0 < \delta < 1$ such that Lemma 2.2 holds. Given a triple $(t, \tilde{\zeta}_0, \tilde{\xi}) \in \mathbb{R} \times C \times \Delta$, the mapping

$$M(t, \tilde{\zeta}_0, \tilde{\xi}) = \tilde{\xi} - A(\tilde{\zeta}_0) \cdot g(t, \tilde{\xi}) \in \mathbb{R}^{n+1}, \quad (2.7)$$

is well-defined and we have (recalling that $g(0, \tilde{\xi}_0) = q(\tilde{\xi}_0) = 0$)

$$M(0, \tilde{\zeta}_0, \tilde{\xi}_0) = \tilde{\xi}_0. \quad (2.8)$$

Taking $\tilde{\zeta} = \tilde{\zeta}_0$ and $\tilde{\tau} = A(\tilde{\zeta}_0) \cdot g(t, \tilde{\xi})$ in Lemma 2.1(i), we get

$$\|M(t, \tilde{\zeta}_0, \tilde{\xi})\| \geq 1 - \delta > 0.$$

Thus, the mapping

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$$N(t, \tilde{\zeta}_0, \tilde{\xi}) = \frac{M(t, \tilde{\zeta}_0, \tilde{\xi})}{\|M(t, \tilde{\zeta}_0, \tilde{\xi})\|} \in S_n, \quad (2.9)$$

is well-defined in $\mathbb{R} \times C \times \Delta$ and

$$N(0, \tilde{\zeta}_0, \tilde{\xi}_0) = \tilde{\xi}_0. \quad (2.10)$$

Theorem 2.1. Let $(t, \tilde{\zeta}_0) \in \mathbb{R} \times C$ be given.

- (i) Let $\tilde{\xi} \in C$ be such that $g(t, \tilde{\xi}) = 0$. Then $\tilde{\xi}$ is a fixed point of the mapping $N(t, \tilde{\zeta}_0, \cdot)$

(ii) Conversely, let $\tilde{\xi} \in \Delta$ be a fixed point of the mapping $N(t, \tilde{\zeta}_0, \cdot)$. Then, $\tilde{\xi} \in C$ and $g(t, \tilde{\xi}) = 0$.

Proof. Part (i) is obvious since, for $\tilde{\xi} \in C$ such that $g(t, \tilde{\xi}) = 0$ one has $N(t, \tilde{\zeta}_0, \tilde{\xi}) = \tilde{\xi} / \|\tilde{\xi}\| = \tilde{\xi}$. Conversely, the mapping $N(t, \tilde{\zeta}_0, \cdot)$ takes its values in S_n so that any fixed point $\tilde{\xi} \in \Delta$ of the mapping $N(t, \tilde{\zeta}_0, \cdot)$ must belong to $\Delta \cap S_n = C$. Next, the relation $N(t, \tilde{\zeta}_0, \tilde{\xi}) = \tilde{\xi}$ also becomes

$$\tilde{\xi}(1 - \|\tilde{\xi} - A(\tilde{\zeta}_0) \cdot g(t, \tilde{\xi})\|) = A(\tilde{\zeta}_0) \cdot g(t, \tilde{\xi}).$$

But the left hand side is collinear with $\tilde{\xi}$, while the right hand side belongs to the space $T_{\tilde{\zeta}_0} S_n$. By lemma 2.1(ii), both sides must vanish. In particular, $A(\tilde{\zeta}_0) \cdot g(t, \tilde{\xi}) = 0$ and hence $g(t, \tilde{\xi}) = 0$ since the linear operator $A(\tilde{\zeta}_0)$ is one-to-one. \square

Remark 2.2. As announced, the problem reduces to finding the fixed points of a given mapping. We emphasize that it is essential, in practice, that the vector $\tilde{\zeta}_0$ be allowed to be arbitrary in C . Indeed, the “natural” choice $\tilde{\zeta}_0 = \tilde{\xi}_0$ is *not realistic* in the applications because $\tilde{\xi}_0$ is *not known* exactly in general.

4.3 Convergence of the Successive Approximation Method.

From Theorem 2.1, a constructive method for finding the solutions of the equation $g(t, \tilde{\xi}) = 0$ in $\mathbb{R} \times C$ consists in applying the successive approximation method to the mapping

$$\tilde{\xi} \in \Delta \rightarrow N(t, \tilde{\zeta}_0, \tilde{\xi}) \in C, \quad (3.1)$$

where $\tilde{\zeta}_0 \in C$ is arbitrarily fixed. In what follows, we prove after shrinking δ if necessary that there is $r > 0$ such that the mapping (3.1) is a *contraction* of Δ into itself for every pair $(t, \tilde{\zeta}) \in (-r, r) \times C$.

As a preliminary step, let us observe, by the continuity of the mapping A (Lemma 2.2) and by the continuity of g and $D_{\tilde{\zeta}}g$ on $\mathbb{R} \times \mathbb{R}^{n+1}$ (cf. Chapter 2, Lemma 3.3), that the mapping M is continuous on $\mathbb{R} \times C \times \Delta$

and its partial derivative $D_{\bar{\xi}}M$ exists and is continuous on $\mathbb{R} \times C \times \Delta$. More precisely, for every $\tilde{h} \in \mathbb{R}^{n+1}$,

$$D_{\bar{\xi}}M(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h} = \tilde{h} - A(\tilde{\zeta}_0) \cdot D_{\bar{\xi}}g(t, \tilde{\xi}) \cdot \tilde{h}.$$

In particular, with $(t, \tilde{\zeta}_0, \tilde{\xi}) = (0, \tilde{\xi}_0, \tilde{\xi}_0)$ we find:

$$D_{\bar{\xi}}M(t, \tilde{\xi}_0, \tilde{\xi}_0) \cdot \tilde{h} = \tilde{h} - A(\tilde{\xi}_0) \cdot D_{\bar{\xi}}g(0, \tilde{\xi}_0) \cdot \tilde{h}. \quad (3.2)$$

But $D_{\bar{\xi}}g(0, \tilde{\xi}_0) = Dg(\tilde{\xi}_0)$ and by definition of A , it follows that

$$D_{\bar{\xi}}M(0, \tilde{\xi}_0, \tilde{\xi}_0)|_{T_{\tilde{\xi}_0}S_n} = 0.$$

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Next, taking $\tilde{h} = \tilde{\xi}_0$ in (3.2) we find

$$D_{\bar{\xi}}M(0, \tilde{\xi}_0, \tilde{\xi}_0) \cdot \tilde{\xi}_0 = \tilde{\xi}_0 - A(\tilde{\xi}_0) \cdot D_{\bar{\xi}}q(0, \tilde{\xi}_0) \cdot \tilde{\xi}_0 = \tilde{\xi}_0,$$

because $D_{\bar{\xi}}q(0, \tilde{\xi}_0) = kq(\tilde{\xi}_0) = 0$. The space \mathbb{R}^{n+1} being the orthogonal direct sum of the spaces $\mathbb{R}\tilde{\xi}_0$ and $T_{\tilde{\xi}_0}S_n$, these relations mean that the partial derivative $D_{\bar{\xi}}M(0, \tilde{\xi}_0, \tilde{\xi}_0)$ is nothing but the *orthogonal projection* onto the space $\mathbb{R}\tilde{\xi}_0$, namely

$$D_{\bar{\xi}}M(0, \tilde{\xi}_0, \tilde{\xi}_0) \cdot \tilde{h} = (\tilde{\xi}_0|\tilde{h})\tilde{\xi}_0,$$

for every $\tilde{h} \in \mathbb{R}^{n+1}$ where $(\cdot|\cdot)$ denotes the canonical inner product of \mathbb{R}^{n+1} .

As a result, the mapping N is continuous on $\mathbb{R} \times C \times \Delta$ and its partial derivative $D_{\bar{\xi}}N$ exists and is continuous on $\mathbb{R} \times C \times \Delta$. We now proceed to show that

$$D_{\bar{\xi}}N(0, \tilde{\xi}_0, \tilde{\xi}_0) = 0.$$

An elementary computation provides

$$D_{\bar{\xi}}N(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h} = \frac{D_{\bar{\xi}}M(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h}}{\|M(t, \tilde{\zeta}_0, \tilde{\xi})\|} - \frac{(D_{\bar{\xi}}M(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h}|M(t, \tilde{\zeta}_0, \tilde{\xi}))}{\|M(t, \tilde{\zeta}_0, \tilde{\xi})\|^3} M(t, \tilde{\zeta}_0, \tilde{\xi}).$$

With $(t, \tilde{\zeta}_0, \tilde{\xi}) = (0, \tilde{\xi}_0, \tilde{\xi}_0)$ and since $M(0, \tilde{\xi}_0, \tilde{\xi}_0) = \tilde{\xi}_0$ and $D_{\bar{\xi}}M(0, \tilde{\xi}_0, \tilde{\xi}_0) \cdot \tilde{h} = (\tilde{\xi}_0|\tilde{h})\tilde{\xi}_0$, we get

$$D_{\bar{\xi}}N(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h} = (\tilde{\xi}_0|\tilde{h})\tilde{\xi}_0 - ((\tilde{\xi}_0|\tilde{h})\tilde{\xi}_0|\tilde{\xi}_0)\tilde{\xi}_0 = 0.$$

Theorem 3.1. *Let $0 < \gamma < 1$ be a given constant. After shrinking $\delta > 0$ if necessary, there exists $r > 0$ such that the mapping $N(t, \tilde{\zeta}, \cdot)$ is Lipschitz-continuous with constant γ from the ball Δ into itself for every pair $(t, \tilde{\zeta}) \in [-r, r] \times C$.* 138

Proof. From the relation $D_{\tilde{\xi}} N(0, \tilde{\xi}_0, \tilde{\xi}_0) = 0$ and the continuity of $D_{\tilde{\xi}} N$ in $\mathbb{R} \times C \times \Delta$, we deduce that there is a neighbourhood of $(0, \tilde{\xi}_0, \tilde{\xi}_0)$ in $\mathbb{R} \times S_n \times \mathbb{R}^{n+1}$ in which $\|D_{\tilde{\xi}} N(t, \tilde{\zeta}_0, \tilde{\xi})\|$ is bounded by γ . Clearly, after shrinking δ (hence Δ and C simultaneously), the neighbourhood in question can be taken as the product $[-r, r] \times C \times \Delta$ for $r > 0$ small enough. Hence, by Taylor's formula (since Δ is convex)

$$\|N(t, \tilde{\zeta}_0, \tilde{\xi}) - N(t, \tilde{\zeta}_0, \tilde{\zeta})\| < \gamma \|\tilde{\xi} - \tilde{\zeta}\|$$

for every $(t, \tilde{\zeta}_0) \in [-r, r] \times C$ and every $\tilde{\xi}, \tilde{\zeta} \in \Delta$. In other words, for every $(t, \tilde{\zeta}_0) \in [-r, r] \times C$, the mapping $N(t, \tilde{\zeta}, \cdot)$ is Lipschitz-continuous with constant γ in the ball Δ . The property is obviously not affected by *arbitrarily shrinking* $r > 0$. Let us then *fix* δ as above, which determines the ball Δ and the spherical cap C . We shall complete the proof by showing, for $r > 0$ small enough, that the mapping $N(t, \tilde{\xi}_0, \cdot)$ maps the ball Δ into itself for every point $(t, \tilde{\zeta}_0) \in [-r, r] \times C$. Indeed, for $\tilde{\xi} \in \Delta$ and since $N(0, \tilde{\zeta}_0, \tilde{\xi}_0) = \tilde{\xi}_0$ for every $\tilde{\zeta}_0 \in C$ (cf. (2.10)), one has

$$\begin{aligned} \|N(t, \tilde{\zeta}_0, \tilde{\xi}) - \tilde{\xi}_0\| &= \|N(t, \tilde{\zeta}_0, \tilde{\xi}) - N(0, \tilde{\zeta}_0, \tilde{\xi}_0)\| \leq \\ &\leq \|N(t, \tilde{\zeta}_0, \tilde{\xi}) - N(t, \tilde{\zeta}_0, \tilde{\xi}_0)\| + \|N(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N(0, \tilde{\zeta}_0, \tilde{\xi}_0)\|. \end{aligned}$$

139 The first term is bounded by $\gamma \|\tilde{\xi} - \tilde{\xi}_0\|$, thus by $\gamma\delta/2$. Finally, due to the *uniform continuity* of the function $N(\cdot, \cdot, \tilde{\xi}_0)$ on compact sets, we deduce that $r > 0$ can be chosen so that the second term is bounded by $(1 - \gamma)\delta/2$ and our assertion follows. \square

4.4 Description of the Algorithm.

The algorithm of computation of the solutions of the equation $g(t, \tilde{\xi}) = 0$ in $[-r, r] \times C$ is immediate from Theorem 3.1 : fixing $t \in [-r, r]$ and

choosing an arbitrary element $\tilde{\xi}_0 \in C$, we define the sequence $(\tilde{\zeta}_\ell)$ through the formula

$$\tilde{\zeta}_{\ell+1} = N(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell), \ell \geq 0. \quad (4.1)$$

This sequence tends to the unique fixed point $\tilde{\xi} \in \Delta$ of the mapping $N(t, \tilde{\zeta}_0, \cdot)$ which actually belongs to C and is a solution of the equation $g(t, \tilde{\xi}) = 0$ (Theorem 2.1).

For $t \neq 0$, the sequence

$$\tilde{x}_\ell = t\tilde{\zeta}_\ell$$

tends to $\tilde{x} = t\tilde{\xi}$, one among the 2ν solutions with euclidean norm $|t|$ of the equation $f(\tilde{x}) = 0$. Of course, for $t = 0$, the sequence $(\tilde{\zeta}_\ell)$ tends to $\tilde{\xi}_0$, unique solution in C of the equation $g(0, \tilde{\xi}) (= q(\tilde{\xi})) = 0$.

Remark 4.1. Incidentally, observe that the above algorithm allows to find arbitrarily close approximations of $\tilde{\xi}_0$ when only a “rough” estimate (i. e. $\tilde{\zeta}_0$) of it is known.

Since our algorithm uses nothing but the contraction mapping principle, the rate of convergence is *geometrical*. More precisely 140

$$\|\tilde{\zeta}_\ell - \tilde{\xi}\| < \gamma^\ell \|\tilde{\zeta}_0 - \tilde{\xi}\|,$$

for every $\ell \geq 0$, where $0 < \gamma < 1$ is the constant involved in Theorem 3.1. Thus as the points $\tilde{\zeta}_0$ and $\tilde{\xi}$ belong to the ball Δ , one has $\|\tilde{\zeta}_0 - \tilde{\xi}\| < \delta < 1$ and hence

$$\|\tilde{\zeta}_\ell - \tilde{\xi}\| \leq \gamma^\ell \delta \leq \gamma^\ell,$$

for $\ell > 0$. Multiplying these inequalities by $|t|$, we get

$$\|\tilde{x}_\ell - \tilde{x}\| \leq \gamma^\ell \|\tilde{x}_0 - \tilde{x}\| \leq |t| \gamma^\ell, \quad (4.2)$$

for $\ell \geq 0$.

Let us now make the scheme (4.1) explicit. As

$$g(t, \tilde{\xi}) = \frac{k!}{t^k} f(t\tilde{\xi}) \text{ for } t \neq 0,$$

and by the definition of the mapping N in (2.9), we get

$$\begin{cases} \tilde{\zeta}_{\ell+1} = \frac{\tilde{\zeta}_\ell}{\|\tilde{\zeta}_\ell\|}, \\ \tilde{\zeta}_\ell = \tilde{\zeta}_\ell - \frac{k!}{k} A(\tilde{\zeta}_0) \cdot f(t\tilde{\zeta}_\ell), \ell \geq 0. \end{cases} \quad (4.3)$$

A scheme for the computation of the sequence (\tilde{x}_ℓ) can be immediately derived. Since $\tilde{x}_0 = t\tilde{\zeta}$, the iterate $\tilde{x}_{\ell+1} = t\tilde{\zeta}_{\ell+1}$ is defined by

$$\begin{cases} \tilde{x}_{\ell+1} = |t| \frac{\tilde{x}_\ell}{\|\tilde{x}_\ell\|}, \\ \tilde{x}_\ell = \tilde{x}_\ell - \frac{k!}{k-1} A(\tilde{\zeta}_0) \cdot f(\tilde{x}_\ell), \ell \geq 0. \end{cases} \quad (4.4)$$

141 **Remark 4.2.** When $t = 0$, the sequence $(\tilde{\zeta}_\ell)$ is defined by

$$\begin{cases} \tilde{\zeta}_{\ell+1} = \frac{\tilde{\zeta}_\ell}{\|\tilde{\zeta}_\ell\|}, \\ \tilde{\zeta}_\ell = \tilde{\zeta}_\ell - A(\tilde{\zeta}_0) \cdot q(\tilde{\zeta}_\ell), \ell \geq 0. \end{cases}$$

Of course, the sequence (\tilde{x}_ℓ) is the constant one $\tilde{x}_\ell = 0$.

Remark 4.3. (Practical method): Note that the computation of the term

$$A(\tilde{\zeta}_0) \cdot f(\tilde{x}_\ell),$$

does not require the explicit form of the linear operator $A(\tilde{\zeta}_0)$. Indeed, by means of classical algorithm for solving linear systems, it is obtained as the solution $\tilde{\tau} \in T_{\tilde{\zeta}_0} S_n = \{\tilde{\zeta}_0\}^\perp$ of the equation

$$Dq(\tilde{\zeta}_0) \cdot \tilde{\tau} = f(\tilde{x}_\ell). \quad (4.5)$$

From the fact that the components of the mapping q are polynomials and hence are completely determined by a finite number of coefficients (depending on the derivative $D^k f(0)$), the linear mapping $Dq(\tilde{\zeta}_0)$ is completely determined by $\tilde{\zeta}_0$ and these coefficients. The tangent space $T_{\tilde{\zeta}_0} S_n = \{\tilde{\zeta}_0\}^\perp$ also is completely determined by $\tilde{\zeta}_0$, which makes it possible to solve the equation (4.5) in practice.

Remark 4.4. The case $k = 1$ is the case when the Implicit function theorem applies. The mapping q is the linear mapping $Df(0)$ and the vector $\tilde{\xi}_0$ is a normalized vector of the one-dimensional space $\text{Ker } Df(0)$. The sequence (\tilde{x}_ℓ) is defined by the simple relation (after choosing $\tilde{x}_0 = t\tilde{\xi}_0$)

$$\begin{cases} \tilde{x}_{\ell+1} = |t| \frac{\tilde{x}_\ell}{\|\tilde{x}_\ell\|}, \\ \tilde{x}_\ell = \tilde{x}_\ell - (Df(0)|_{\{\tilde{\xi}_0\}^\perp})^{-1} f(\tilde{x}_\ell), \ell \geq 0, \end{cases}$$

where $\tilde{\xi}_0 \in S_n$ is “close enough” to $\tilde{\xi}_0$. This sequence tends to one of the (two) solutions with euclidean norm $|t|$ of the equation $f(\tilde{x}) = 0$.

4.5 A Generalization to the Case When the Mapping f is not Explicitly Known.

For the reasons we explained at the beginning of this chapter, it is important to extend the algorithm described in §4 to the case when the mapping f is not explicitly known but a sequence (f_ℓ) tending to f is available.

More precisely, the mapping f verifying the same assumptions as before, we shall assume that there is an open neighbourhood \mathcal{O} of the origin in \mathbb{R}^{n+1} such that f is the limit of (f_ℓ) in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$ (equipped with its usual Banach space structure). Besides, we shall assume

$$D^j f_\ell(0) = 0, 0 \leq j \leq k-1, \quad (5.1)$$

$$D^k f_\ell(0) = D^k f(0), \quad (5.2)$$

so that *the first k derivatives of f and f_ℓ at the origin coincide for every $\ell \geq 0$.*

Remark 5.1. The conditions (5.1)-(5.2) may seem very restrictive. However, they are quite adapted to the applications to one-parameter problems in Banach spaces, as we shall see later on.

If this is the case, the mapping

$$\tilde{\xi} \in \mathbb{R}^{n+1} \rightarrow q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k \in \mathbb{R}^n, \quad (5.3)$$

can as well be defined through any of the mappings f_ℓ , namely

$$q(\tilde{\xi}) = D^k f_\ell(0)(\tilde{\xi})^k, \quad (5.4)$$

for every $\ell \geq 0$. As usual, we shall assume that it verifies the condition ($\mathbb{R} - N.D.$).

In analogy with §§1 and 2. let us define, for every $\ell \geq 0$,

$$g_\ell(t, \tilde{\xi}) = \frac{k!}{t^k} f_\ell(t\tilde{\xi}) \text{ for } t \neq 0, \quad (5.5)$$

$$g_\ell(0, \tilde{\xi}) = q(\tilde{\xi}), \quad (5.6)$$

$$M_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) = \tilde{\xi} - A(\tilde{\zeta}_0) \cdot g_\ell(t, \tilde{\xi}), \quad (5.7)$$

$$N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) = \frac{M_\ell(t, \tilde{\zeta}_0, \tilde{\xi})}{\|M_\ell(t, \tilde{\zeta}_0, \tilde{\xi})\|}. \quad (5.8)$$

Of course, the key point is that a *uniform choice* of $0 < \delta < 1$ can be made for defining the mapping M_ℓ and N_ℓ in (5.7)-(5.8) because the way of shrinking δ in §2 before Theorem 2.1 depends only upon *geometric properties of the sphere S_n* and upon *the mapping q* (Lemma 2.2) which is equivalently defined through f or any of the f_ℓ 's.

144 The natural question we shall answer is to know whether $0 < \delta < 1$ and $r > 0$ can be found so that the sequence

$$\begin{cases} \tilde{\zeta}_0 \in C, \\ \tilde{\zeta}_{\ell+1} = N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell), \ell \geq 0, \end{cases} \quad (5.9)$$

is well defined and tends to a (unique) fixed point of the mapping $N(t, \tilde{\zeta}_0, \cdot)$. In such a case, the algorithm problem is solved through Theorem 2.1 and the review given in §1.

We shall only sketch the proof of the convergence of the scheme (5.9), in which the main ideas are the same as before. The interested reader can refer to Appendix 2 for a detailed exposition.

The first step consists in proving, for any $r > 0$ and $\delta > 0$ small enough, the the sequence (g_ℓ) (respectively $D_{\tilde{\xi}} g_\ell$) tends to g (respectively $D_{\tilde{\xi}} g$) in the space $C^0([-r, r] \times \Delta, \mathbb{R}^n)$ (resp. $C^0[-r, -r] \times C \times \Delta, \mathcal{Z}(\mathbb{R}^{n+1}, \mathbb{R}^n)$). This follows from the relation $D^k f_\ell(0) = D^k f(0)$ for

every $\ell \geq 0$ and the convergence of (f_ℓ) to f in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$. As a result and by the continuity of A , the sequence (M_ℓ) and (N_ℓ) (respectively $(D_{\tilde{\xi}}M_\ell)$ and $(D_{\tilde{\xi}}N_\ell)$) tend to M and N (respectively $D_{\tilde{\xi}}M$ and $D_{\tilde{\xi}}N$) in the space $C^0([-r, r] \times C \times \Delta, \mathbb{R}^{n+1})$ (respectively $C^0([-r, r] \times C \times \Delta, \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n))$).

Theorem 5.1. *Let $0 < \gamma < 1$ be a given constant. After shrinking $\delta > 0$ if necessary, there exists $r > 0$ such that the mapping $N_\ell(t, \tilde{\zeta}_0, \cdot)$ is Lipschitz-continuous with constant γ from the ball Δ into itself for every pair $(t, \tilde{\zeta}_0) \in [-r, r] \times C$ and every $\ell \geq 0$.*

SKETCH OF THE PROOF: Because the result needs to be uniform with respect to the index ℓ , the proof given in Theorem 3.1 cannot be repeated. Showing that $\|D_{\tilde{\xi}}N_\ell\|$ is bounded by γ on a neighbourhood of the point $(0, \tilde{\xi}_0, \tilde{\xi}_0)$ in $\mathbb{R} \times C \times \Delta$ by a continuity argument does not ensure that the neighbourhood in question is independent of ℓ . Instead, we establish an estimate of $\|D_{\tilde{\xi}}N_\ell(t, \tilde{\zeta}_0, \tilde{\xi})\|$. In Appendix 2, this estimate is found to be¹

$$\begin{aligned} \|D_{\tilde{\xi}}N_\ell(t, \tilde{\zeta}_0, \tilde{\xi})\| &\leq \frac{3\delta}{(1-\delta)^3} + \frac{2\|D_{\tilde{\xi}}g_\ell(0, \tilde{\xi}_0)\|}{(1-\delta)}\omega(\delta) + \\ &+ 2 \left[\frac{\|A(\tilde{\xi}_0)\| + \omega(\delta)}{(1-\delta)} \right] \left[\frac{3}{(1-\delta)^2} \|g_\ell(t, \tilde{\xi})\| + \|D_{\tilde{\xi}}g_\ell(t, \tilde{\xi}) - D_{\tilde{\xi}}g_\ell(0, \tilde{\xi}_0)\| \right], \end{aligned}$$

for every $(t, \tilde{\zeta}_0, \tilde{\xi}) \in [-r, r] \times C \times \Delta$ and every $\ell > 0$, where

$$\omega(\delta) = \sup_{\tilde{\zeta}_0 \in C} \|A(\tilde{\zeta}_0) - A(\tilde{\xi}_0)\|.$$

By the compactness of C and the continuity of A , $\omega(\delta)$ is well - defined and tends to 0 as δ tends to 0. Using the convergence results mentioned before, it is now possible to parallel the proof of Theorem 3.1 to get the desired conclusion.

¹Recall that $D_{\tilde{\xi}}g_\ell(0, \tilde{\xi}_0) = Dg(\tilde{\xi}_0)$ for every $\ell \geq 0$ but this form is not quite appropriate in the formula.

From Theorem 5.1, the sequence

$$\tilde{\zeta}_{\ell+1} = N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell), \ell \geq 0, \quad (5.11)$$

is well-defined in the ball Δ but it is less obvious that in §4 than it tends to a fixed point of the mapping $N(t, \tilde{\zeta}_0, \cdot)$ in Δ .

146 Note from the uniform convergence of the sequence (N_ℓ) to N and Theorem 5.1 that the mapping $N(t, \tilde{\zeta}_0, \cdot)$ is also Lipschitz-continuous with constant γ so that it has a *unique* fixed point $\tilde{\xi}$ in Δ . The problem is to prove that the sequence (5.11) tends to $\tilde{\xi}$. As it will provide us with useful information, we now establish this convergence result. Let $\tilde{\zeta} \in \Delta$ be any cluster point of the sequence $(\tilde{\zeta}_\ell)$ (whose existence follows from the compactness of Δ) and $(\tilde{\zeta}_{\ell_i})$ a subsequence with limit $\tilde{\zeta}$. Using the convergence of N_ℓ to N , we shall prove that $\tilde{\xi} = \tilde{\zeta}$ if we show that the sequence $(\tilde{\zeta}_{\ell_i})$ and $\tilde{\zeta}_{\ell_i+1}$ have the same limit. This will follow from the relation

$$\lim_{\ell \rightarrow +\infty} \|\tilde{\zeta}_\ell - \tilde{\zeta}_{\ell+1}\| = 0. \quad (5.12)$$

Once (5.12) is established, we immediately deduce that the sequence $(\tilde{\zeta}_\ell)$ has $\tilde{\xi}$ as a *unique cluster* point, whence the convergence of the *whole* sequence $(\tilde{\zeta}_\ell)$ to $\tilde{\xi}$ follows by a classical result, in topology, for compact sets.

To prove (5.12), write

$$\begin{aligned} \tilde{\zeta}_{\ell+1} - \tilde{\zeta}_\ell &= N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell) - N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_{\ell-1}) = \\ &= N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell) - N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_{\ell-1}) + \\ &= +N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_{\ell-1}) - N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_{\ell-1}). \end{aligned}$$

From Theorem 5.1, we find

$$\|\tilde{\zeta}_{\ell+1} - \tilde{\zeta}_\ell\| < \gamma \|\tilde{\zeta}_\ell - \tilde{\zeta}_{\ell-1}\| + \|N_\ell - N_{\ell-1}\|_{\infty, [-r, r] \times C \times \Delta}. \quad (5.13)$$

147 Due to the convergence of the sequence (N_ℓ) to N , given any $\epsilon > 0$, one has

$$\|N_\ell - N_{\ell-1}\|_{\infty, [-r, r] \times C \times \Delta} < \epsilon,$$

for ℓ large enough, say $\ell \geq \ell_0 \geq 1$. From (5.13),

$$\|\tilde{\zeta}_{\ell+1} - \tilde{\zeta}_\ell\| \leq \gamma \|\tilde{\zeta}_\ell - \tilde{\zeta}_{\ell-1}\| + \epsilon$$

for $\ell \geq \ell_0$. Define the sequence (for $\ell \geq \ell_0$)

$$\begin{aligned} a_{\ell+1} &= \gamma a_\ell + \epsilon \\ a_{\ell_0} &= \|\widetilde{\zeta}_{\ell_0} - \widetilde{\zeta}_{\ell_0-1}\|. \end{aligned}$$

By an immediate induction argument, it can be seen that

$$\|\widetilde{\zeta}_\ell - \widetilde{\zeta}_{\ell-1}\| \leq a_\ell$$

for every $\ell \geq \ell_0$. But the sequence $(a_\ell)_{\ell \geq \ell_0}$ tends to the real number $a = \epsilon/(1 - \gamma)$ and, for $\ell \geq \ell_n$ large enough, say $\ell \geq \ell_1$, we have

$$a_\ell \leq \frac{2\epsilon}{1 - \gamma}.$$

Therefore

$$\|\widetilde{\zeta}_\ell - \widetilde{\zeta}_{\ell-1}\| \leq \frac{2\epsilon}{1 - \gamma},$$

for $\ell \geq \ell_1$ and (5.12) follows, since $\epsilon > 0$ is arbitrarily small.

COMMENT 5.1. Setting

$$\widetilde{x}_\ell = t\widetilde{\zeta}_\ell, \quad (5.14)$$

the sequence (\widetilde{x}_ℓ) tends to $\widetilde{x} = t\widetilde{\xi}$, one of the 2ν solutions of the equation $f(\widetilde{x}) = 0$ with norm $|t|$. Here, it is not possible to ascertain that the rate of convergence of (\widetilde{x}_ℓ) to \widetilde{x} is geometrical because it depends on the *rate of convergence of the sequence (N_ℓ) to N in the space $C^0([-r, r] \times C \times \Delta, \mathbb{R}^{n+1})$* , as it appears in (5.13). 148

However, if this convergence is geometrical too, it is immediate, by (5.13)-(5.14), that there are constant $0 < \gamma' < 1$ and $K > 0$ such that

$$\|\widetilde{x}_\ell - \widetilde{x}\| \leq K|t|(\gamma')^\ell \quad (5.15)$$

for every $\ell \geq 0$. The geometrical convergence of the sequence (N_ℓ) to N in the space $C^0([-r, r] \times C \times \Delta, \mathbb{R}^{n+1})$ is ensured if, for instance, the sequence (f_ℓ) tends to f geometrically in the space $C^k(\mathcal{O}, \mathbb{R}^n)$. Unfortunately, this assumption is *beyond the best result* that the natural hypothesis ' $f \in C^k$ ' provides in the applications we have in mind, (if

$f \in C^m$ with $m > k$, the geometrical convergence in $C^k(\bar{\mathcal{O}}, \mathbb{R}^n)$ raises no problem) namely, the geometrical rate of convergence of (f_ℓ) to f in the space $C^{k-1}(\bar{\mathcal{O}}, \mathbb{R}^n)$. If so, it can be shown (cf. Appendix 2) that the rate of convergence of the sequence (\bar{x}_ℓ) to \bar{x} remains geometrical: More precisely, there are constants $0 < \gamma' < 1$ and $K > 0$ such that

$$\|\bar{x}_{\ell-\bar{x}}\| \leq K(\gamma')^\ell \quad (5.16)$$

for every $\ell \geq 0$ (note the difference with (5.15)).

149 Remark 5.2. Recall that the convergence of the sequence (f_ℓ) to f in the space $C^k(\bar{\mathcal{O}}, \mathbb{R}^n)$ is assumed throughout this chapter. In the result quoted above, this assumption is *not* dropped but *no special rate of convergence* is required in this space.

As in §4, a scheme for the computation of \bar{x} in terms of the sequence (f_ℓ) can be easily obtained from (5.11) and (5.14): After the choice of $\bar{x}_0 = t\bar{\zeta}_0$, the sequence

$$\begin{cases} \bar{x}_{\ell+1} = |t| \frac{\bar{x}_\ell}{\|\bar{x}_\ell\|_\ell}, \\ \bar{x}_\ell = \bar{x}_\ell - \frac{k!}{\ell^{k-1}} A(\bar{\zeta}_0) f_\ell(\bar{x}_\ell), \ell \geq 0. \end{cases} \quad (5.17)$$

4.6 Application to One-Parameter Problems.

The scheme (5.17) is especially suitable for the computation of the branches of solution in one parameter problems as defined in general in Chapter 1. For the sake of clarity, we shall first describe a more abstract situation: Let \tilde{Z} be a real Banach space with norm $\|\cdot\|$, $\bar{B}(0, \rho)$ the closed ball with radius $\rho > 0$ centered at the origin in \tilde{Z} and $\Phi(= \Phi(\bar{x}, \bar{z})) \in C^k(\bar{\mathcal{O}} \times \bar{B}(0, \rho), \tilde{Z})$, $k \geq 1$ (note that $\bar{\mathcal{O}} \times \bar{B}(0, \rho) = \bar{\mathcal{O}} \times B(0, \rho)$) so that $C^k(\bar{\mathcal{O}} \times \bar{B}(0, \rho), \tilde{Z}) = C^k(\bar{\mathcal{O}} \times B(0, \rho), \tilde{Z})$ but this *not* a normed space when \tilde{Z} is infinite dimensional since $\bar{B}(0, \rho)$ is not compact) with

$$\begin{aligned} \Phi(0) &= 0, \\ D_{\bar{z}}\Phi(0) &= 0. \end{aligned}$$

Given any constant $0 < \beta < 1$ and after shrinking $\rho > 0$ and the neighbourhood \mathcal{O} if necessary, we may assume that

$$\|D_{\bar{z}}\Phi(\bar{x}, \bar{z})\| \leq \beta,$$

for every $(\bar{x}, \bar{z}) \in \bar{\mathcal{O}} \times \bar{B}(0, \rho)$. Applying the mean value theorem, we see that the mapping $\Phi(\bar{x}, \cdot)$ is a contraction with constant β from $\bar{B}(0, \rho)$ into \bar{Z} . In addition, after possibly shrinking the neighbourhood \mathcal{O} once again, the mapping $\Phi(\bar{x}, \cdot)$ maps the ball $\bar{B}(0, \rho)$ into itself for every $\bar{x} \in \bar{\mathcal{O}}$. Indeed, given $(\bar{x}, \bar{z}) \in \bar{\mathcal{O}} \times \bar{B}(0, \rho)$, we have

$$\begin{aligned} \|\Phi(\bar{x}, \bar{z})\| &\leq \|\Phi(\bar{x}, \bar{z}) - \Phi(\bar{x}, 0)\| + \|\Phi(\bar{x}, 0)\| \\ &\leq \beta\|\bar{z}\| + \|\Phi(\bar{x}, 0)\| \\ &\leq \beta\rho + \|\Phi(\bar{x}, 0)\|, \end{aligned}$$

and the result follows from the fact that $\|\Phi(\bar{x}, 0)\|$ can be made less than $(1 - \beta)\rho$ for every $\bar{x} \in \bar{\mathcal{O}}$ by the continuity of Φ at the origin.

Therefore, for every $\bar{x} \in \bar{\mathcal{O}}$, the sequence

$$\begin{cases} \tilde{\varphi}_0(\bar{x}) = 0 \\ \tilde{\varphi}_{\ell+1}(\bar{x}) = \Phi(\bar{x}, \varphi_\ell(\bar{x})), \ell \geq 0, \end{cases} \quad (6.1)$$

is well defined, each mapping $\tilde{\varphi}_\ell$ being in the space $C^k(\bar{\mathcal{O}}, \bar{Z})$ with values in $\bar{B}(0, \rho)$ and verifies

$$\tilde{\varphi}_\ell = 0, \ell \geq 0.$$

The sequence $(\tilde{\varphi}_\ell)$ also tends (pointwise) to the mapping $\tilde{\varphi}$ characterized by

$$\tilde{\varphi}(\bar{x}) = \Phi(\bar{x}, \tilde{\varphi}(\bar{x})),$$

for every $\bar{x} \in \bar{\mathcal{O}}$. In particular

$$\tilde{\varphi}(0) = 0.$$

Actually, the mapping $\tilde{\varphi}$ is of class C^k in $\bar{\mathcal{O}}$. Indeed, for every $\bar{x} \in \bar{\mathcal{O}}$, $\tilde{\varphi}(\bar{x})$ is a solution for the equation

$$\bar{z} - \Phi(\bar{x}, \bar{z}) = 0.$$

As $\tilde{\varphi}(\tilde{x}) \in \overline{B}(0, \rho)$, one has $\|D_{\tilde{z}}\Phi(\tilde{x}, \tilde{\varphi}(\tilde{x}))\| \leq \beta < 1$ and hence

$$I - D_{\tilde{z}}\Phi(\tilde{x}, \tilde{\varphi}(\tilde{x})) \in \text{Isom}(\tilde{Z}).$$

Thus, the mapping $\tilde{\varphi}(\cdot)$ is of class C^k around \tilde{x} from the Implicit function theorem, which shows that $\tilde{\varphi} \in C^k(\overline{\mathcal{O}}, \tilde{Z})$.

Naturally, one may expect the convergence of the sequence $(\tilde{\varphi}_\ell)$ to be better than pointwise. Indeed, it so happens that the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^k(\overline{\mathcal{O}}, \tilde{Z})$, the convergence being *geometrical in the space $C^{k-1}(\overline{\mathcal{O}}, \tilde{Z})$* (See Appendix 2).

Let now $F(= F(\tilde{x}, \tilde{Z})) \in C^k(\overline{\mathcal{O}} \times \overline{B}(0, \rho), \mathbb{R}^n)$ be such that

$$F(0) = 0.$$

The mapping

$$\tilde{x} \in \overline{\mathcal{O}} \rightarrow f(\tilde{x}) = F(\tilde{x}, \tilde{\varphi}(\tilde{x})) \in \mathbb{R}^n, \quad (6.2)$$

is of class C^k in $\overline{\mathcal{O}}$ and

$$f(0) = 0.$$

The same property holds for each term of the sequence

$$\tilde{x} \in \overline{\mathcal{O}} \rightarrow f_\ell(\tilde{x}) = F(\tilde{x}, \tilde{\varphi}_{\ell+k}(\tilde{x})), \ell \geq 0. \quad (6.3)$$

152 The sequence (f_ℓ) tends to f in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$, the rate of convergence being *geometrical* in the space $C^{k-1}(\overline{\mathcal{O}}, \mathbb{R}^n)$. Also

$$D^j f_\ell(0) = D^j f(0), 0 \leq j \leq k, \quad (6.4)$$

for every $\ell \geq 0$.

Remark 6.1. The definition of the mapping f_ℓ has been through the term $\tilde{\varphi}_{\ell+k}$ instead of the “natural one” $\tilde{\varphi}_\ell$ in order that property (6.4) holds. More generally, the mapping f_ℓ can be defined through the term $\tilde{\varphi}_{\ell+k}$, with $k' \geq k$.

By choosing $\tilde{x}_0 = t\tilde{\zeta}_0$, the scheme (5.17) becomes

$$\begin{cases} \tilde{x}_{\ell+1} = |t| \frac{\tilde{x}_\ell}{\|\tilde{x}_\ell\|_\ell}, \\ \tilde{x}_\ell = \tilde{x}_\ell - \frac{k!}{k-1} A(\tilde{\zeta}_0) F(\tilde{x}_\ell, \tilde{\varphi}_{\ell+k}(\tilde{x}_\ell)). \end{cases}$$

In the above formula, $\tilde{\varphi}_{\ell+k}(\tilde{x}_\ell)$ coincide with the element $\tilde{z}_{\ell+k, \ell} \in \tilde{Z}$ defined for every $\ell \geq 0$ by

$$\begin{cases} \tilde{z}_{0, \ell} = 0, \\ \tilde{z}_{j+1, \ell} = \Phi(\tilde{x}_\ell, \tilde{z}_{j, \ell}), 0 \leq j \leq \ell + k - 1. \end{cases}$$

To sum up, starting from an arbitrary element $\tilde{\zeta}_0 \in S_n$ “close enough” to $\tilde{\xi}_0$ and setting $\tilde{x}_0 = t\tilde{\zeta}_0$ ($|t| > 0$ small enough), the sequence (\tilde{x}_ℓ) is defined by 153

$$\begin{cases} \tilde{z}_{0, \ell} = 0, \\ \tilde{z}_{j+1, \ell} = \Phi(\tilde{x}_\ell, \tilde{z}_{j, \ell}), 0 \leq j \leq \ell + k - 1, \\ \tilde{x}_\ell = \tilde{x}_\ell - \frac{k!}{k-1} A(\tilde{\zeta}_0) F(\tilde{x}_\ell, \tilde{z}_{\ell+k, \ell}), \\ \tilde{x}_{\ell+1} = |t| \frac{\tilde{x}_\ell}{\|\tilde{x}_\ell\|}, \ell \geq 0. \end{cases} \quad (6.5)$$

The rate of convergence of the sequence (\tilde{x}_ℓ) is then *geometrical*.

Remark 6.2. This algorithm is a “two-level” iterative process since computing the element $\tilde{x}_{\ell+1}$ from the element \tilde{x}_ℓ requires $\ell + k - 1$ iterates. Formally, “one-level” algorithms are easy to derive from the scheme (6.5) but *no convergence has been proved yet*.

The application to one-parameter problems in Banach spaces is now an immediate consequence. The notation and hypothesis being the same as in Chapter 1, it has been shown that finding the local zero set of the mapping $G(= G(\tilde{x}))$ is equivalent to the system

$$\begin{aligned} Q_1 G(\tilde{x}_1 + \tilde{x}_2) &= 0, \\ Q_2 G(\tilde{x}_1 + \tilde{x}_2) &= 0. \end{aligned}$$

Since $D_{\tilde{x}} G(0)|_{\tilde{X}_2} \in \text{Isom}(\tilde{X}_2, \tilde{Y}_2)$, the system can be rewritten as

$$Q_1 G(\tilde{x}_1 + \tilde{x}_2) = 0,$$

$$\tilde{x}_2 - (DG(0)|_{\tilde{X}_2})^{-1} Q_2 G(\tilde{x}_1 + \tilde{x}_2) = \tilde{x}_2.$$

154 This is nothing but

$$\begin{aligned} F(\tilde{x}_1, \tilde{x}_2) &= 0, \\ \Phi(\tilde{x}_1, \tilde{x}_2) &= \tilde{x}_2, \end{aligned}$$

where

$$F(\tilde{x}_1, \tilde{x}_2) = Q_1 G(\tilde{x}_1 + \tilde{x}_2), \quad (6.6)$$

$$\Phi(\tilde{x}_1, \tilde{x}_2) = \tilde{x}_2 - [DG(0)|_{\tilde{X}_2}]^{-1} Q_2 G(\tilde{x}_1 + \tilde{x}_2). \quad (6.7)$$

As $\tilde{X}_1 \simeq \mathbb{R}^{n+1}$, $Y_1 \simeq \mathbb{R}^n$ and setting $\tilde{Z} = \tilde{X}_2$, it is immediately seen that the mappings F and Φ verify the required conditions (and, of course, the mapping $\tilde{\varphi}$ of this section coincides with that of Chapter 1). Dropping the subscript 1 in the variable \tilde{x}_1 , the scheme (6.5) can be written as follows: choosing an inner product in the space \tilde{X}_1 (on which depends the unit sphere in \tilde{X}_1) and setting $\tilde{x}_0 = t\tilde{\zeta}_0$ where $\tilde{\zeta}$ is taken “close enough” to a given normalised element $\tilde{\xi}_0$ at which the mapping $q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k$ vanishes (f being the reduced mapping here), the sequence (\tilde{x}_ℓ) is defined by

$$\left\{ \begin{array}{l} \tilde{z}_{0,\ell} = 0 \in \tilde{X}_2, \\ \tilde{z}_{j+1,\ell} = \tilde{z}_{j,\ell} - [DG(0)|_{\tilde{X}_2}]^{-1} \cdot Q_2 G(\tilde{x}_\ell + \tilde{z}_{j,\ell}) \in \tilde{X}_2, 0 \leq j \leq \ell + k - 1, \\ \tilde{x}_\ell = \tilde{x}_\ell - \frac{k!}{\ell^{k-1}} A(\tilde{\zeta}_0) \cdot Q_1 G(\tilde{x}_\ell + \tilde{z}_{\ell+k,\ell}) \in \tilde{X}_1, \\ \tilde{x}_{\ell+1} = |t| \frac{\tilde{x}_\ell}{\|\tilde{x}_\ell\|} \in \tilde{X}_1, \ell \geq 0. \end{array} \right. \quad (6.8)$$

155 **Remark 6.3.** Since the choice of the inner product in the space \tilde{X}_1 affects the unit sphere, the mapping A also depends on the particular choice of the inner product.

COMMENT 6.1. The scheme (6.8) provides an approximation \tilde{x}_ℓ of the solution $\tilde{x} \in \tilde{X}_1$ to the reduced equation $f(\tilde{x}) = Q_1 G(\tilde{x} + \tilde{\varphi}(\tilde{x})) = 0$ with norm $|t|$ that we have selected among the 2ν ones. The corresponding solution to the initial problem (namely $\tilde{x} + \tilde{\varphi}(\tilde{x})$) is then approximated by

$\tilde{x}_\ell + \tilde{\varphi}_{\ell+k}(\tilde{x}_\ell)$ which is none other than $\tilde{x}_\ell + \tilde{z}_{\ell+k,\ell}$. Besides, the sequence $(\tilde{x}_\ell + \tilde{z}_{\ell+k,\ell})$ converges to $\tilde{x} + \tilde{\varphi}(\tilde{x})$ geometrically. Indeed, the sequence (\tilde{x}_ℓ) tends to \tilde{x} geometrically and denoting $\|\cdot\|$ the norm in the space \tilde{X} (hence in \tilde{X}_2) one has

$$\|\tilde{z}_{\ell+k,\ell} - \tilde{\varphi}(\tilde{x})\| \leq \|\tilde{\varphi}_{\ell+k}(\tilde{x}_\ell) - \tilde{\varphi}(\tilde{x}_\ell)\| + \|\tilde{\varphi}(\tilde{x}_\ell) - \tilde{\varphi}(\tilde{x})\|.$$

The first term in the right hand side is bounded by the norm $\|\tilde{\varphi}_{\ell+k} - \tilde{\varphi}\|_{\infty, \tilde{\mathcal{O}}}$ which tends to 0 geometrically, since the sequence $\tilde{\varphi}_{\ell+k}$ tends to $\tilde{\varphi}$ geometrically in the space $C^{k-1}(\tilde{\mathcal{O}}, \tilde{X}_2)$ and hence in $C^0(\tilde{\mathcal{O}}, \tilde{X}_2)$. As for the second term, it is bounded by $\|D\tilde{\varphi}\|_{\infty, \tilde{\mathcal{O}}}\|\tilde{x}_\ell - \tilde{x}\|$ for the mapping $\tilde{\varphi}$ is at least of class C^1 in $\tilde{\mathcal{O}}$ and thus in the ball of radius $|t| = \|\tilde{x}_\ell\| = \|\tilde{x}\|$ centered at the origin of \mathbb{R}^{n+1} (more rapidly, one can say that it is not restrictive to assume that $\tilde{\mathcal{O}}$ is convex).

Application to the Problems of Birurcation from the trivial Branch.

When the analysis of Chapter 3, §2 is available, we can apply the scheme (6.8) to problem of bifurcation from the trivial branch: Find $(\mu, x) \in \mathbb{R} \times X$ around the origin such that **156**

$$G(\mu, x) = (I - (\lambda_0 + \mu)L)x + \Gamma(\mu, x) = 0, \quad (6.9)$$

where λ_0 is a characteristic value of L with (algebraic and geometric) multiplicity $n > 1$. Here, we shall take

$$\begin{aligned} X_1 &= Y_1 = \text{Ker}(I - \lambda_0 L), \\ X_2 &= Y_2 = \text{Range}(I - \lambda_0 L), \end{aligned}$$

This choice being possible precisely because the algebraic and geometric multiplicities of λ_0 are the same. It follows that $\tilde{X}_1 = \mathbb{R} \times X_1$ while $\tilde{X}_2 = X_2$.

Under the assumptions of Chapter 3, §2 and when $k = 2$, no change of the parameter μ is required. The variable $\tilde{x} \in \tilde{X}_1$ is the pair $(\mu, x) \in \mathbb{R} \times X_1$ and the mapping q becomes (cf, Chapter 3 §2, relation (2.13))

$$(\mu, x) \in \mathbb{R} \times X_1 \rightarrow q(\mu, x) = -\frac{2\mu}{\lambda_0}x + Q_1 D_x^2 \Gamma(0) \cdot (x)^2 \in X_1.$$

Let

$$\widetilde{zeta}_0 = \frac{(\mu_0, x_0)}{\|(\mu_0, x_0)\|}, \quad (6.10)$$

be “close enough” to some selected element of the zero set of q . Clearly, we can assume that $\|(\mu_0, x_0)\| > 0$ is as small as desired. Further, by changing (μ_0, x_0) into a non-zero collinear element (with small norm) the definition of $\widetilde{\zeta}_0$ is unchanged, unless the orientation of (μ_0, x_0) is reversed and, in such a case, $\widetilde{\zeta}_0$ is transformed into $-\widetilde{\zeta}_0$. This explains, in the relation $(\mu_0, x_0) = t\widetilde{\zeta}_0 = \|(\mu_0, x_0)\|\widetilde{\zeta}_0$, that the quantity t is $\|(\mu_0, x_0)\|$ and hence positive. In the scheme (6.8) as well as in the general one (5.17) from which it is derived, it is also possible to limit ourselves to the case $t > 0$, provided we change $\widetilde{\zeta}_0$ into $-\widetilde{\zeta}_0$ when $t (= \pm\|\widetilde{x}_0\|$ in any case) should change sign. But this did not lead to a simple exposition at that time, which is the reason why this property was not mentioned earlier. Thus, starting with the point (μ_0, x_0) and defining $\widetilde{\zeta}_0$ by (6.10), the scheme (6.8) will provide the only solution of the reduced equation of (6.9) which is “close” to (μ_0, x_0) and has the same norm as (μ_0, x_0) . The corresponding curve is obtained by moving (μ_0, x_0) along the line it generates in $\mathbb{R} \times \widetilde{X}_1$. Denoting by z (instead of \widetilde{z}) the generic variable of the space $X_2 (= \widetilde{X}_2)$, the explicit form of the scheme (6.8) is

$$\begin{cases} z_{0,\ell} = 0 \in X_2, \\ z_{j+1,\ell} = [(I - \lambda_0 L)|_{X_2}]^{-1} \cdot [-\mu Q_2 L z_{j,\ell} + Q_2 \Gamma(\mu_\ell, x_\ell + z_{j,\ell})] \in X_2, 0 \leq j \leq \ell + 1, \\ (\mu_\ell, x_\ell) = (\mu_\ell, x_\ell) - \frac{2}{\|(\mu_0, x_0)\|} A(\widetilde{\zeta}_0) \cdot \left[-\frac{\mu}{\lambda_0} x_\ell + Q_1 \Gamma(\mu_\ell, x_\ell + z_{\ell+2,\ell}) \right] \in \mathbb{R} \times X_1, \\ (\mu_{\ell+1}, x_{\ell+1}) = \|(\mu_0, x_0)\| \frac{(\mu_\ell, x_\ell)}{\|(\mu_\ell, x_\ell)\|} \in \mathbb{R} \times X_1, \ell \geq 0. \end{cases} \quad (6.11)$$

According to Comment 6.1, the approximation of the corresponding solution of the equation (6.9) associated with the iterate (μ_ℓ, x_ℓ) is

$$(\mu_\ell, x_\ell + z_{\ell+2,\ell}).$$

Under the assumption of Chapter 3, § 2 and when $k \geq 3$, a preliminary change of parameter μ into η^{k-1} when k is even, into $\sigma\eta^{k-1}$, $\sigma = \pm 1$ when k is odd, is necessary. We shall consider the case k even only, leaving it to the reader to make the obvious modification when k is odd. The variable $\widetilde{x} \in \widetilde{X}_1$ is the pair $(\eta, x) \in \mathbb{R} \times X_1$ and the mapping q

becomes (cf, Chapter 3, §2)

$$(\eta, x) \in \mathbb{R} \times X_1 \rightarrow q(\eta, x) = -\frac{k! \eta^{k-1}}{\lambda_0} x + Q_1 D_x^k \Gamma(0) \cdot (x)^k \in X_1$$

Let

$$\tilde{\zeta}_0 = \frac{(\eta_0, x_0)}{\|(\eta_0, x_0)\|},$$

be close enough to some selected element of the zero set of q . Taking $\|(\eta_0, x_0)\|$ sufficiently small and for the same reasons as when $k = 2$, the explicit form of the scheme (6.8) we obtain is (denoting by z instead of \bar{z}) the variables of the space $X_2 = \bar{X}_2$

$$\left\{ \begin{array}{l} z_{0,\ell} = 0 \in X_2, \\ z_{j+1,\ell} = [(I - \lambda_0 L)|_{X_2}]^{-1} \cdot [-\eta_\ell^{k-1} Q_2 L z_{j,\ell} + Q_2 \Gamma(\eta_\ell^{k-1}, x_\ell + z_{j,\ell})] \in X_2, \\ \quad 0 \leq j \leq \ell + k - 1, \\ \overline{(\eta_\ell, x_\ell)} = (\eta_\ell, x_\ell) - \frac{k!}{\|(\eta_\ell, x_\ell)\|^{k-1}} A(\tilde{\zeta}_0) \cdot [-\frac{\eta_\ell^{k-1}}{\lambda_0} x_\ell + Q_1 \Gamma(\eta_\ell^{k-1}, x_\ell + z_{\ell+k,\ell})] \\ \quad \in \mathbb{R} \times X_1, \\ (\eta_{\ell+1}, x_{\ell+1}) = \|(\eta_0, x_0)\| \frac{\overline{(\eta_\ell, x_\ell)}}{\|(\eta_\ell, x_\ell)\|} \in \mathbb{R} \times X_1, \ell \geq 0. \end{array} \right. \quad (6.12)$$

According to Comment 6.1, the approximation of the corresponding solution of the equation (6.9) in which μ is replaced by η^{k-1} is

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$$(\eta_\ell, x_\ell + z_{\ell+k,\ell}). \quad (6.13)$$

Thus, the approximation of the corresponding solution of equation (6.9) itself is

$$(\eta_\ell^{k-1}, x_\ell + z_{\ell+k,\ell}),$$

whose convergence is geometrical (since that of the sequence (6.13)) is geometrical from Comment (6.1).

Remark 6.4. On comparison with the classical *Lyapunov-Schmidt Method*, the above algorithm has the disadvantage of being a “two - level” iterative process (cf. Remark 6.2). However, it has two basic advantages: The first one is that it does not require the zero set of the

mapping q to be known *exactly*, an assumption unrealistic in the applications. The Lyapunov-Schmidt method, which consists in finding the curves of solutions tangent to any given line in the zero set of q by an iterative process *uses this particular line* explicitly (see e.g. [2]). Of course, one can expect that the Lyapunov-Schmidt method will nevertheless be satisfactory if this line is replaced by a good approximation to it (though we are not aware of a proof of this). In the process described above, a *rough estimate* (represented by $\tilde{\zeta}_0$) is sufficient because the algorithm makes the corrections itself. It then appears more “stable” than the Lyapunov-Schmidt method.

160 The second advantage is obvious in so far as the schemes (6.11)-(6.12) are particular cases of the general algorithm (6.8) that can be used when *no branch of solutions is known a priori*, while the Lyapunov-Schmidt method is *limited to problems of bifurcation from the trivial branch* or, at the best, to problems in which one of the branches is known explicitly. We leave it to the reader to derive from (6.8) an iterative scheme associated with the problem of Chapter 3 § 3 (in which no change of parameter is required in practice).

Chapter 5

Introduction to a Desingularization Process in the Study of Degenerate Cases

LET $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ BE as in Chapter 2, namely such that the first $k - 1$ derivatives of f vanish at the origin and the k^{th} does not ($k > 1$). In chapter 2, we have solved the problem of finding the local zero set of f when the polynomial mapping

$$\tilde{\xi} \in \mathbb{R}^{n+1} \rightarrow q(\tilde{\xi}) = D^k f(0) \cdot (\tilde{\xi})^k \in \mathbb{R}^n,$$

verifies the condition ($\mathbb{R} - N.D.$). The next step consists in examining what happens when the mapping q does *not* verify the nondegeneracy condition in question and the aim is to find hypotheses as general as possible ensuring that the local zero set of f is still a finite union of curves through the origin. Problems in which the associated mapping q does not verify the condition ($\mathbb{R} - N.D.$) will be referred to as *degenerate*.

In the examples studied in Chapter 3, we have already encountered degenerate problems but, in the particular context of problems of bifurcation from the trivial branch, we were able to overcome the difficulty

by performing a suitable change of the parameter. However, with or without change of the parameter, the condition

$$X = \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L),$$

- 162 namely, the equality of the algebraic and geometric multiplicities of the characteristic value λ_0 , was always seen to be *necessary* for the condition ($\mathbb{R} - N.D.$) to hold. The failure of the above condition is then sufficient for the problem to be degenerate and no solution to it has been found by changing the parameter.

Before any attempt to develop a general theory, it is wise to examine a few particular cases. The most elementary example is when $k = 1$: The mapping q is the linear mapping $Df(0)$ and it does not verify the condition ($\mathbb{R} - N.D.$) if and only if $Df(0) \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is not onto. This implies that the zero set of q , here $\text{Ker } Df(0)$, is a subspace of \mathbb{R}^{n+1} with dimension at least two. Hence, it does not consist of a finite number of lines, the very argument that allowed us to characterize the local zero set of f as a finite union of curve in Chapter 2. Note however that the somewhat disappointing situation we have just described cannot occur if $n = 1$, for, otherwise, the derivative $Df(0)$ would vanish identically and k could not be 1. More generally and regardless of the (finite) value of k , it has been shown when $n = 1$ that the zero set of q is *always* the union of at most k lines through the origin, though the condition ($\mathbb{R} - N.D.$) holds if and only if the real roots of a certain polynomial in one variable with real coefficients are simple. Acutally, the examination of the general case when n is arbitrary but $k \geq 2$ shows that the failure of the condition ($\mathbb{R} - N.D.$) *does not necessarily* imply that the zero set of q is made of infinitely many lines. In other words, it is reasonable to expect the local zero set of f to be made of a finite number of curves under hypotheses more general than those of Chapter 2.

- 163 Assuming then that the zero set of q is made of a finite number of lines and selecting an element $\tilde{\xi} \in \mathbb{R}^{n+1} - \{0\}$ such that $q(\tilde{\xi}) = 0$ and $Dq(\tilde{\xi})$ is not onto, the analysis of the structure of the local zero set of f “along” the line $\mathbb{R}\tilde{\xi}$ is often referred to as the “*analysis of bifurcation near the degenerate eigenray $\mathbb{R}\tilde{\xi}$* ”. The almost inevitable tool here seems to be the topological degree theory (see e.g. Shaerer [39]) but the information

obtained is necessarily vague and of an essentially theoretical character. In *non-linear eigenvalue problems*, another approach, more technical, has been developed by Dancer [8]. Again, the results lose most of their accuracy (on comparison with the nondegenerate case) and the method does not seem to have been extended to other problems.

In this chapter, we present an analytic approach which provides results as precise as in the nondegenerate case. The study is limited to the - apparently - simple situation when $n = 1$ and $k = 2$. The general theory is still incomplete but the same method can almost readily be employed and will provide quite precise informations in each particular problem. The open questions that remain in the general case are internal to the method and to its links to singularity theory¹ (when $n = 1$ and $k = 2$, see Rabier [32]). Also, only a few of the theoretical results that could be derived from it have been investigated so far. As an example of such a result, we give a statement that complements Krasnoselskii's theorem (Theorem 1.2 of Chapter 1) in a particular case as well as Crandall and Rabinowitz's study of bifurcation from the trivial branch at a simple characteristic value. 164

In contrast to Chapter 2, we shall not here bother about formulating the weakest possible regularity assumptions: The mapping f under study will be of class C^∞ although this hypothesis is clearly unnecessary in most of the chapter. The exposition follows Rabier [32].

5.1 Formulation of the Problem and Preliminaries.

Let f be a C^∞ real-valued mapping of two variables defined on a neighbourhood of the origin such that $f(0) = 0$, $Df(0) = 0$ and $\det D^2f(0) \neq 0$. Then, the Morse lemma (Theorem 3.1, 1 and 2 of Chapter 1) states that the local zero set of f reduces to the origin if $\det D^2f(0) > 0$ and is made of exactly two curves of class C^∞ intersecting transversally at the origin if $\det D^2f(0) < 0$. But it does not say anything if $\det D^2f(0) = 0$. If this determinant vanishes *because* $D^2f(0) = 0$, we can examine higher order derivatives of f at the origin and the structure of the local zero set

¹However, significant progress has since been made.

of f may follow from an application of Theorem 3.1 of Chapter 2 with $n = 1$. If $\det D^2 f(0) = 0$ but $D^2 f(0) \neq 0$, no information is available and, indeed, elementary examples show that the situation may differ considerably: Denoting by (u, v) the variable in the plane \mathbb{R}^2 , the mappings

$$f(u, v) = u^2 + v^4,$$

$$f(u, v) = u^2 - v^4$$

$$f(u, v) = (u - v)^2,$$

all verify the condition $f(0) = 0$, $Df(0) = 0$, $\det D^2 f(0) = 0$ and $D^2 f(0) \neq 0$. Their local zero sets are, respectively: the origin, the union of the two tangent parabolas $u = v^2$ and $u = -v^2$, and the line $u = v$. These three cases are by no means exhaustive, as it is seen with the mapping

$$f(u, v) = v^2 - \sin\left(\frac{1}{u}\right)e^{-1/u^2}.$$

whose local zero set is picture in Figure 1.1 below.

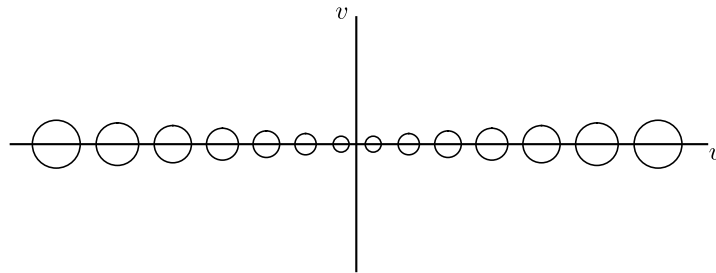


Figure 1.1:

166 This explains why it is not possible to find an exhaustive classification of the local zero sets of such mappings *in geometrical terms*. Nevertheless, a partial classification is possible. It will be obtained by the following. Given a mapping f verifying $f(0) = 0$, $Df(0) = 0$, $\det D^2 f(0) = 0$ and $D^2 f(0) \neq 0$, we shall construct a new mapping $f^{(1)}$

whose local zero set provides that of f through an elementary transformation. About the mapping $f^{(1)}$, the following three possibilities are *exhaustive*:

- 1) $Df^{(1)}(0) \neq 0$ so that the local zero set of $f^{(1)}$ can be found through the Implicit function theorem.
- 2) $Df^{(1)}(0) = 0$, $\det D^2 f^{(1)}(0) \neq 0$ so that the local zero set of $f^{(1)}$ can be found through the Morse lemma.
- 3) $Df^{(1)}(0) = 0$, $\det D^2 f^{(1)}(0) = 0$ and $D^2 f^{(1)}(0) \neq 0$ so that finding the local zero set of $f^{(1)}$ reduces to finding the local zero set of a new iterate $f^{(2)}$.

In so doing, we define a sequence of iterates $f^{(1)}, \dots, f^{(m)}$: The iterate $f^{(m)}$ exists under the necessary and sufficient condition that the local zero set of $f^{(m-1)}$ cannot be found through the Implicit function theorem or the Morse lemma. If the process ends after a finite number of steps, it corresponds to a *desingularization* of the initial mapping f . If it is endless, there are two possibilities, among which one of them bears a geometric characterization (see § 8).

We shall complete this section by giving a definition and establishing a preliminary property. Of course, the mapping f will henceforth verify the condition $f(0) = 0$ and

$$Df(0) = 0, \quad (1.1)$$

$$\det D^2 f(0) = 0, \quad (1.2)$$

$$D^2 f(0) \neq 0. \quad (1.3)$$

In particular, (1.2) means that the null-space of the mapping $D^2 f(0) \in \mathcal{L}(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2, \mathbb{R}))$ does not reduce to $\{0\}$. Because of (1.3), this null-space is not the whole space \mathbb{R}^2 and it is then one-dimensional.

Definition 1.1. The one-dimensional space

$$\Xi = \text{Ker } D^2 f(0),$$

will be called the characteristic of f .

The following elementary result will be used several times.

Lemma 1.1. *Under our assumptions, $D^2 f(0) \cdot (\tilde{\xi})^2 = 0$ for some $\tilde{\xi} \in \mathbb{R}^2$ if and only if $\tilde{\xi} \in \Xi$.*

Proof. It is obvious that $D^2 f(0) \cdot (\tilde{\xi})^2 = 0$, if $\tilde{\xi} \in \Xi$. Conversely, let $\tilde{\xi} \in \mathbb{R}^2$ be such that $D^2 f(0) \cdot (\tilde{\xi})^2 = 0$ and assume by contradiction that there is $\tilde{\tau} \in \mathbb{R}^2$ with $D^2 f(0) \cdot (\tilde{\xi}, \tilde{\tau}) \neq 0$. Necessarily, $\tilde{\xi}$ and $\tilde{\tau}$ are not collinear, so that $\{\tilde{\xi}, \tilde{\tau}\}$ is a basis of \mathbb{R}^2 . With respect to this basis, $D^2 f(0)$ becomes

$$\begin{bmatrix} 0 & D^2 f(0) \cdot (\tilde{\xi}, \tilde{\tau}) \\ D^2 f(0) \cdot (\tilde{\xi}, \tilde{\tau}) & D^2 f(0) \cdot (\tilde{\tau})^2 \end{bmatrix}$$

168 and its determinant is $-(D^2 f(0) \cdot (\tilde{\xi}, \tilde{\tau}))^2 < 0$, which contradicts the hypothesis that it vanishes². \square

5.2 Desingularization by Blowing-up Procedure.

We shall begin with an approach we have already used in Chapter 2. Extending (in theory) the mapping f to the whole space \mathbb{R}^2 as a C^∞ mapping, we first reduce the problem of finding the non-zero solutions of the equation $f(\tilde{x}) = 0$ with prescribed euclidian norm $\|\tilde{x}\|$ to the problem

$$\begin{cases} (t, \tilde{x}) \in (\mathbb{R} - \{0\}) \times S_1, \\ g(t, \tilde{\xi}) = 0, \end{cases} \quad (2.1)$$

where the mapping g defined on $\mathbb{R} \times \mathbb{R}^2$ with values in \mathbb{R}^2 is given by

$$g(t, \tilde{\xi}) = \begin{cases} \frac{1}{t^2} f(t\tilde{\xi}) & \text{if } t \neq 0, \\ \frac{1}{2} D^2 f(0) \cdot (\tilde{\xi})^2 & \text{if } t = 0. \end{cases} \quad (2.2)$$

The relationship of the nonzero solutions \tilde{x} of the equation $f(x) = 0$ to the solutions $(t, \tilde{\xi})$ of the equation $g(t, \tilde{\xi}) = 0$ is of course that \tilde{x} can always be written as $\tilde{x} = t\tilde{\xi}$ (in two different ways).

Remark 2.1. The mapping g in (2.2) differs from that of Chapter 2 by the multiplicative factor $1/2$ only. This modification will bring some simplifications in the formulas later.

Writing the Taylor expansion of f about the origin, we find that

$$g(t, \tilde{\xi}) = \int_0^1 (1-s) D^2 f(st\tilde{\xi}) \cdot (\tilde{\xi})^2 ds, \quad (2.3)$$

for every $(t, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^2$, which shows that the mapping g is of class C^∞ . The next result, very similar to Lemma 3.2 of Chapter 2, is based on the observation that the intersection $S_1 \cap \Xi$ reduces to two points. For the sake of convenience, we shall say that two problems are *equivalent* if all the solutions of one of them provide all the solutions of the other. We leave it to the reader to keep track of the (obvious) correspondence between the solutions of the problems under consideration.

Lemma 2.1. *Let $\tilde{\xi}_0$ be either point of the intersection $\Xi \cap S_1$ and C a given neighbourhood of $\tilde{\xi}_0$ in S_1 . Then, for $r > 0$ small enough the problems*

$$\begin{cases} 0 < |t| < r, \tilde{\xi} \in S_1, \\ g(t, \tilde{\xi}) = 0, \end{cases} \quad (2.4)$$

and

$$\begin{cases} 0 < |t| < r, \tilde{\xi} \in C, \\ g(t, \tilde{\xi}) = 0, \end{cases} \quad (2.5)$$

are equivalent.

Proof. First, note that the problem (2.5) is equivalent to

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$$\begin{cases} 0 < |t| < r, \tilde{\xi} \in C \cup (-C) \\ g(t, \tilde{\xi}) = 0, \end{cases} \quad (2.6)$$

²Recall that the condition $\det D^2 f(0) = 0$ is independent of the choice of any basis in \mathbb{R}^2 .

since each solution of (2.5) is obviously a solution of (2.6) and conversely, given any solution (t, ξ) of (2.6), either (t, ξ) or $(-t, -\xi)$ is a solution of (2.5) from the relation $g(t, \xi) = g(-t, -\xi)$. We shall prove the assertion by showing that the problems (2.4) and (2.6) *have the same solutions* for $r > 0$ small enough. Each solution of (2.6) is clearly a solution of (2.4). Conversely, assume, by contradiction, that there is no $r > 0$ such that each solution of (2.4) is a solution of (2.6). This means that there is a sequence $(t_\ell, \xi_\ell)_{\ell \geq 1}$ with $\lim_{\ell \rightarrow +\infty} t_\ell = 0$, $\xi_\ell \in S_1 \setminus C \cup (-C)$ and $g(t_\ell, \xi_\ell) = 0$. From the compactness of S_1 , we may assume that there is $\xi \in S_1$ such that ξ_ℓ converges to ξ . By the continuity of g , we deduce $g(0, \xi) = 0$. But, by definition of $g(0, \cdot)$ and Lemma 1.1, ξ must be one of the elements ξ_0 or $-\xi_0$, which is impossible, since $\xi_\ell \notin C \cup (-C)$ for every $\ell \geq 1$, by hypothesis. \square

Let us now consider a subspace T of \mathbb{R}^2 such that

$$\mathbb{R}^2 = -\Xi \oplus T. \quad (2.7)$$

Given a non-zero element $\tilde{\tau}_0 \in T$, we may write $T = \mathbb{R}\tilde{\tau}_0$. Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R} \times S_1$ be the C^∞ mapping defined by

$$\Phi(u_1, v_1) = (u_1 \|\xi_0 + v_1 \tilde{\tau}_0\|, (\xi_0 + v_1 \tilde{\tau}_0) / \|\xi_0 + v_1 \tilde{\tau}_0\|), \quad (2.8)$$

171 (where $|\cdot|$ still denotes the euclidean norm). Clearly, $\Phi(0, 0) = (0, \xi_0)$ and an elementary calculation yields

$$D\Phi(0, 0) \cdot (u_1, v_1) = (u_1, v_1 \tilde{\tau}_0 - v_1 (\xi_0 | \tilde{\tau}_0) \xi_0),$$

for $(u_1, v_1) \in \mathbb{R}^2$, where $(\cdot | \cdot)$ denotes the inner product of \mathbb{R}^2 . The above relation shows that the mapping $D\Phi(0, 0)$ is one-to-one (as an element of $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$) and hence is an isomorphism of \mathbb{R}^2 to $\mathbb{R} \times T_{\xi_0} S_1 (= \mathbb{R} \times \{\xi_0\}^\perp)$. Thus, Φ is a C^∞ diffeomorphism between a neighbourhood of the origin in \mathbb{R}^2 and a neighbourhood of the point $(0, \xi_0)$ in $\mathbb{R} \times S_1$. We can suppose that the latter contains the product $(-r, r) \times C$ of Lemma 2.1: it suffices to choose C small enough and shrink $r > 0$ accordingly.

Then, $\Phi^{-1}((-r, r) \times C)$ is a neighbourhood W_1 of the origin in \mathbb{R}^2 and each pair $(t, \xi) \in (-r, r) \times C$ has the form

$$\begin{cases} t = u_1 \|\tilde{\xi}_0 + v_1 \tilde{\tau}_0\|, \\ \tilde{\xi} = (\tilde{\xi}_0 + v_1 \tilde{\tau}_0) / \|\tilde{\xi}_0 + v_1 \tilde{\tau}_0\|, \end{cases}$$

with $(u_1, v_1) \in W_1$. As $t = 0$ if and only if $u_1 = 0$ and from lemma 2.1. our initial problem of finding the nonzero solutions \tilde{x} in the local zero set of f is equivalent to

$$\begin{cases} (u_1, v_1) \in W_1, u_1 \neq 0, \\ g(u_1 \|\tilde{\xi}_0 + v_1 \tilde{\tau}_0\|, (\tilde{\xi}_0 + v_1 \tilde{\tau}_0) / \|\tilde{\xi}_0 + v_1 \tilde{\tau}_0\|) = 0 \end{cases}$$

Coming back to the definition of g ((2.2)), this also reads

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$$\begin{cases} (u_1, v_1) \in W_1, u_1 \neq 0, \\ f(u_1(\tilde{\xi}_0 + v_1 \tilde{\tau}_0)) = 0. \end{cases}$$

Finally, multiplying by $1/u_1^2$, the problem becomes

$$\begin{cases} (u_1, v_1) \in W_1, u_1 \neq 0, \\ g(u_1, \tilde{\xi}_0 + v_1 \tilde{\tau}_0) = 0. \end{cases}$$

We shall denote by $f^{(1)}$ the C^∞ real-valued function of two variables

$$\begin{aligned} f^{(1)} : (u_1, v_1) \in \mathbb{R}^2 &\rightarrow f^{(1)}(u_1, v_1) = g(u_1, \tilde{\xi}_0 + v_1 \tilde{\tau}_0) \\ &= \begin{cases} \frac{1}{u_1^2} f(u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0) & \text{if } u_1 \neq 0, \\ \frac{1}{2} v_1^2 D^2 f(0) \cdot (\tilde{\xi}_0)^2 & \text{if } u_1 = 0. \end{cases} \end{aligned} \quad (2.9)$$

From the above comments, the nonzero solutions \tilde{x} of the equation $f(\tilde{x})$ with $\|\tilde{x}\| < r$ are of the form

$$\tilde{x} = u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0 \quad (2.10)$$

with $(u_1, v_1) \in W_1$, $u_1 \neq 0$ and $f^{(1)}(u_1, v_1) = 0$. Obviously, the characterization (2.10) remains true with the solution $\tilde{x} = 0$ since $f^{(1)}(0, 0) = 0$.

In other words, since the neighbourhood $W_1 = \Phi^{-1}((-r, r) \times C)$ can be taken arbitrarily small (by shrinking C and $r > 0$), *the problem reduces to finding the local zero set of $f^{(1)}$* , from which the local zero set of f is deduced through the transformation (2.10).

173 Remark 2.2. The reader may wonder why we have not defined the mapping $f^{(1)}$ by formula (2.9) directly. Indeed, it is clear that every element of the form (2.10) with (u_1, v_1) in the local zero set of $f^{(1)}$ is in the local zero set of f . But the problem is to show that *every* element in the local zero set of f is of the form (2.10) with (u_1, v_1) in the local zero set of $f^{(1)}$, a fact that uses the compactness of the unit circle S_1 , essential in Lemma 2.1.

COMMENT 2.1. The vector $\tilde{\xi}_0 \in \Xi$ has been chosen so that $\tilde{\xi}_0 \in S_1$ i.e. $\|\tilde{\xi}_0\| = 1$. This is without importance and $\tilde{\xi}_0$ can be taken as an arbitrary non-zero element of the characteristic Ξ (by changing the inner product of \mathbb{R}^2 for instance). From now on, we shall allow this more general choice in the definition of the mapping $f^{(1)}$ in (2.9).

All the derivatives of the mapping $f^{(1)}$ can be explicitly computed in terms of the derivatives of f at the origin. First from (2.9), for every pair $(j, \ell) \in \mathbb{N} \times \mathbb{N}$

$$(\partial^{j+\ell} f^{(1)} / \partial u_1^j \partial v_1^\ell)(0) = (\partial^{j+\ell} g / \partial t^j \partial \tilde{\xi}^\ell)(0, \tilde{\xi}_0) \cdot (\tilde{\tau}_0)^\ell.$$

Now

Lemma 2.2. *For every pair $(j, \ell) \in \mathbb{N} \times \mathbb{N}$, one has*

$$(\partial^{j+\ell} f g / \partial t^j \partial \tilde{\xi}^\ell)(0, \tilde{\xi}) = \begin{cases} 0 & \text{if } \ell > j + 2 \\ [j! / (j + 2 - \ell)!] D^{j+2} f(0) \cdot (\tilde{\xi})^{j+2-\ell} & \text{if } 0 \leq \ell \leq j + 2, \end{cases} \quad (2.11)$$

for every $\tilde{\xi} \in \mathbb{R}^2$.

174 Proof. Using the expression (2.3) of g and for every pair $(t, \tilde{\xi}) \in \mathbb{R} \times \mathbb{R}^2$,

we find

$$(\partial^j g / \partial t^j)(t, \tilde{\xi}) = \int_0^1 (1-s)s^j D^{j+2} f(st\tilde{\xi}) \cdot (\tilde{\xi})^{j+2} ds.$$

In particular, for $t = 0$,

$$(\partial^j g / \partial t^j)(0, \tilde{\xi}) = \frac{-1}{(j+1)(j+2)} D^{j+2} f(0) \cdot (\tilde{\xi})^{j+2},$$

and (2.11) follows. \square

With Lemma 2.2, we obtain for every pair $(j, \ell) \in \mathbb{N} \times \mathbb{N}$,

$$(\partial^{j+\ell} f^{(1)} / \partial u_1^j \partial v_1^\ell)(0) = \begin{cases} 0 & \text{if } \ell > j+2, \\ [j!/(j+2-\ell)!] D^{j+2} f(0) \cdot ((\tilde{\xi}_0)^{j+2-\ell}, \\ (\tilde{\tau}_0)^\ell) & \text{if } 0 \leq \ell < j+2. \end{cases} \quad (2.12)$$

In particular, taking $j = 0$ and $\ell = 2$ in formula (2.12),

$$(\partial^2 f^{(1)} / \partial v_1^2)(0) = D^2 f(0) \cdot (\tilde{\tau}_0)^2 \neq 0, \quad (2.13)$$

since $\tilde{\tau}_0 \notin \Xi$ (cf. Lemma 1.1).

5.3 Solution Through the Implicit Function Theorem : CUSP Bifurcation

Applying fromula (2.12) with $j = 1, \ell = 0$ and $j = 0, \ell = 1$ successively, we get

$$(\partial f^{(1)} / \partial u_1)(0) = \frac{1}{6} D^3 f(0) \cdot (\tilde{\xi}_0)^3 \quad (3.1)$$

and

$$(\partial f^{(1)} / \partial v_1)(0) = \frac{1}{2} D^2 f(0) \cdot (\tilde{\xi}_0, \tilde{\tau}_0).$$

But, as $\tilde{\xi}_0 \in \Xi$, $D^2 f(0) \bullet \tilde{\xi}_0 = 0$. Hence

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$$(\partial f^{(1)} / \partial v_1)(0) = 0. \quad (3.2)$$

From (3.1)-(3.2), the Implicit function theorem can be used for finding the local zero set of $f^{(1)}$ if and only if

$$D^3 f(0) \cdot (\tilde{\xi}_0)^3 \neq 0, \tag{3.3}$$

a condition which is independent of the nonzero element $\tilde{\xi}_0 \in \Xi$ and of the space T chosen for defining the mapping $f^{(1)}$. Assume then that (3.3) holds. The Implicit function theorem states that the local zero set of $f^{(1)}$ is made of exactly one curve of class C^∞ . Due to (3.2), this curve is tangent to the v_1 -axis at the origin. In other words, around the origin

$$f^{(1)}(u_1, v_1) = 0 \Leftrightarrow u_1 = u_1(v_1), u_1(0) = (du_1/dv_1)(0) = 0.$$

This result can be made more precise by showing that $(d^2u_1/dv_1^2)(0) \neq 0$: Differentiating two times the identity $f^{(1)}(u_1(v_1), v_1) = 0$ and since $(du_1/dv_1)(0) = (\partial f^{(1)}/\partial v_1)(0) = 0$, we obtain

$$(d^2u_1/dv_1^2)(0) = -\frac{(\partial^2 f^{(1)}/\partial v_1^2)(0)}{(\partial f^{(1)}/\partial u_1)(0)} = -6 \frac{D^2 f(0) \cdot (\tilde{\tau}_0)^2}{D^3 f(0) \cdot (\tilde{\xi}_0)^3} \neq 0, \tag{3.4}$$

as it follows from (2.13). In the (u_1, v_1) -plane, the graph of the function $u_1(\cdot)$ is then as in Figure 3.1

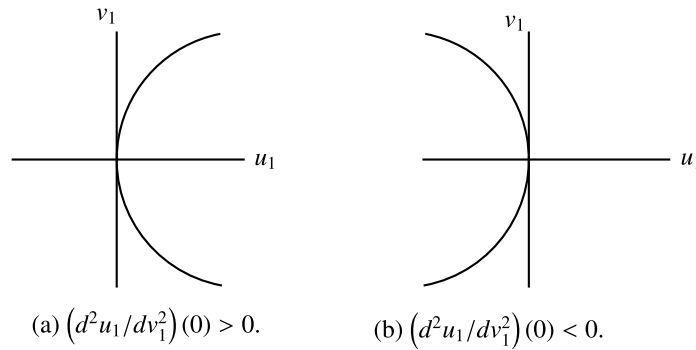


Figure 3.1: Local zero set of $f^{(1)}$

176 After changing the vector $\tilde{\xi}_0$ into $-\tilde{\xi}_0$ if necessary, we may assume (cf. (3.4))

$$(d^2u_1/dv_1^2)(0) > 0. \tag{3.5}$$

Theorem 3.1. *The vector $\tilde{\xi}_0$ being chosen so that (3.5) holds, let us set $\Xi_+ = \mathbb{R}_+ \tilde{\xi}_0$. Then, the local zero set of f is made up of two continuous half-branches emerging from the origin in the half space $\Xi_+ \oplus T$. These half-branches are of class C^∞ away from the origin and tangent to the characteristic $\Xi = \mathbb{R} \tilde{\xi}_0$ at the origin. More precisely, there exists an origin-preserving C^∞ diffeomorphism ψ in \mathbb{R} such that the local zero set of f is made up of the two half-branches ($\rho > 0$ small enough)* 177

Proof. After changing the vector $\tilde{\tau}_0$ into a scalar multiple, we may assume that $(d^2u_1/dv_1^2)(0) = 2$ (cf. (3.4)). Not changing $\tilde{\tau}_0$ introduces positive multiplicative constants in the expressions below without (of course) affecting the final result. Thus

$$u_1(v_1) = v_1^2(1 + R(v_1)), \quad (3.7)$$

where R is a C^∞ function such that $R(0) = 0$. The mapping

$$\varphi(v_1) = v_1(1 + R(v_1))^{1/2}, \quad (3.8)$$

is well defined and of class C^∞ around the origin. In addition

$$\begin{aligned} \varphi(0) &= 0, \\ \frac{d\varphi}{dv_1}(0) &= 1. \end{aligned}$$

It follows that φ is an origin-preserving C^∞ local diffeomorphism of \mathbb{R} and the relation (3.7) is simply

$$u_1(v_1) = (\varphi(v_1))^2.$$

From our choice of φ (3.8), we see that

$$\begin{aligned} \varphi(v_1) &= \sqrt{u_1(v_1)} \text{ if } v_1 > 0, \\ \varphi(v_1) &= -\sqrt{u_1(v_1)} \text{ if } v_1 < 0. \end{aligned}$$

Setting $\psi = \varphi^{-1}$, we find

$$\begin{aligned} v_1 &= \psi(\sqrt{u_1(v_1)}) \text{ if } v_1 \geq 0, \\ v_1 &= \psi(-\sqrt{u_1(v_1)}) \text{ if } v_1 \leq 0. \end{aligned}$$

□ 178

As $u_1(v_1)$ runs over some interval $[0, \rho)$ for v_1 around the origin. It is equivalent to saying, for every $u_1 \in]0, \rho[$, that the two values v_1 with $u_1 = u_1(v_1)$ are given by $v_1 = \pm \psi(\sqrt{u_1})$.

According to the general discussion before, the local zero set of f is made up of elements of the form

$$\tilde{x} = u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0,$$

with (u_1, v_1) in the local zero set of $f^{(1)}$. Here, we have then

$$\tilde{x} = \tilde{x}^{(\alpha)}(u_1) = u_1 \tilde{x}_0 + u_1 \psi((-1)^\alpha \sqrt{u_1}) \tilde{\tau}_0$$

for $u_1 \in]0, r)$. Dividing by $u_1 \neq 0$, we get

$$\lim_{u_1 \rightarrow 0} \frac{\tilde{x}^{(\alpha)}(u_1)}{u_1} = \tilde{\xi}_0 + \lim_{u_1 \rightarrow 0} \psi((-1)^\alpha \sqrt{u_1}) \tilde{\tau}_0 = \tilde{\xi}_0$$

This relation expresses that the two half-branches are tangent to the characteristic Ξ at the origin. They are distinct since $\psi(\sqrt{u_1})$ and $\psi(-\sqrt{u_1})$ have opposite signs.

From Theorem 3.1, the local zero set of f is then given by Figure 3.2 below.

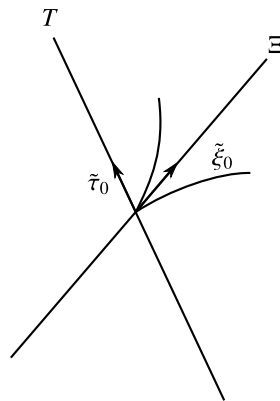


Figure 3.2: Local zero set of f .

179 Setting $\tilde{x} = (u, v)$, the simplest example of mapping f for which the

hypothesis of this section are fulfilled is $f(u, v) = u^2 \pm v^3$ (or $v^2 \pm u^2$, or, with θ any fixed real number, $(v \cos \theta - u \sin \theta)^2 \pm (u \cos \theta + v \sin \theta)^3$). This kind of bifurcation is observed in chemical reaction problems (cf. Golubitsky and Keyfitz [13]).

5.4 Solution Through the Morse Lemma : Isolated Solution and Double Limit Bifurcation.

As we saw in the previous section, a necessary and sufficient condition for the local zero set of the mapping $f^{(1)}$ (2.9) to be found through the Implicit function theorem is $D^3 f(0) \cdot (\tilde{\xi}_0)^3 \neq 0$. Here, we shall then **180** assume

$$D^3 f(0) \cdot (\tilde{\xi}_0)^3 = 0. \quad (4.1)$$

As a result, from (3.1) and (3.2)

$$(\partial f^{(1)} / \partial u_1)(0) = (\partial f^{(1)} / \partial v_1)(0) = 0. \quad (4.2)$$

Let us now examine the second derivative. Taking $j = 2\ell = 0$ and $j = 1, \ell = 1$ successively in (2.12), we find

$$(\partial^2 f^{(1)} / \partial u_1^2)(0) = \frac{1}{12} D^4 f(0) \cdot (\tilde{\xi}_0)^4, \quad (4.3)$$

$$(\partial^2 f^{(1)} / \partial u_1 \partial v_1)(0) = \frac{1}{2} D^3 f(0) \cdot ((\tilde{\xi}_0)^3, \tilde{\tau}_0) \quad (4.4)$$

and we already know that (cf. (2.13))

$$(\partial^2 f^{(1)} / \partial v_1^2)(0) = D^2 f(0) \cdot (\tilde{\tau}_0)^2 \neq 0. \quad (4.5)$$

Therefore, the structure of the local zero set of the mapping $f^{(1)}$ can be found through the Morse lemma if and only if the determinant $\det D^2 f^{(1)}(0)$ is $\neq 0$. From the above, this becomes

$$\det D^2 f^{(1)}(0) = \frac{1}{12} (D^4 f(0) \cdot (\tilde{\xi}_0)^4) (D^2 f(0) \cdot (\tilde{\tau}_0)^2) - \frac{1}{4} (D^3 f(0) \cdot (\tilde{\xi}_0)^2, \tilde{\tau}_0)^2 \neq 0. \quad (4.6)$$

Clearly, $\det D^2 f^{(1)}(0)$ being non-zero or not is independent of the choice of the non-zero elements $\widetilde{\xi}_0 \in \Xi$ and $\widetilde{\tau}_0 \in T$. It is true, but not as obvious, that our assumptions are independent of the complement T of the characteristic Ξ chosen for defining the mapping $f^{(1)}$. We prove this in

Proposition 4.1. *The condition: $Df^{(1)}(0) = 0$ and $\det D^2 f^{(1)}(0) \neq 0$ is independent of the choice of the complement T of the characteristic Ξ .*

181 *Proof.* We already know that the condition $Df^{(1)}(0) = 0$ (i.e. (4.1)) is independent of T . Assuming $Df^{(1)}(0) = 0$, we shall see that the condition $\det D^2 f^{(1)}(0) \neq 0$ is independent of T too (see Remark 4.1 below). This can be checked directly on the formula (4.6) but we prefer here to give the *reason* why this independence is true. The second derivative $D^2 f(0, \widetilde{\xi})$ of the mapping g ((2.2)) is an element of $\mathcal{L}(\mathbb{R} \times \mathbb{R}^2, \mathcal{L}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}))$ and our assertion will follow from the equivalence

$$\det D^2 f^{(1)}(0) \neq 0 \Leftrightarrow \text{Ker } D^2 g(0, \widetilde{\xi}_0) = \{0\} \times \Xi,$$

since $\text{Ker } D^2 g(0, \widetilde{\xi}_0)$ is of course independent of any choice of T . \square

Let then $(t, \widetilde{\xi}) \in \mathbb{R} \times \mathbb{R}^2$ be in the null-space $\text{Ker } D^2 g(0, \widetilde{\xi}_0)$. This means

$$\begin{cases} t(\partial^2 g / \partial t^2)(0, \widetilde{\xi}_0) + (\partial^2 g / \partial t \partial \widetilde{\xi})(0, \widetilde{\xi}) \cdot \widetilde{\xi} = 0 \in \mathbb{R}, \\ t(\partial^2 g / \partial t \partial \widetilde{\xi})(0, \widetilde{\xi}_0) + (\partial^2 g / \partial \widetilde{\xi}^2)(0, \widetilde{\xi}_0) \cdot \widetilde{\xi} = 0 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}) \end{cases} \quad (4.7)$$

From Lemma 2.2,

$$(\partial^2 g / \partial t^2)(0, \widetilde{\xi}_0) = \frac{1}{12} D^4 f(0) \cdot (\widetilde{\xi}_0)^4 \in \mathbb{R}, \quad (4.8)$$

$$(\partial^2 g / \partial t \partial \widetilde{\xi})(0, \widetilde{\xi}_0) = \frac{1}{2} D^3 f(0) \cdot (\widetilde{\xi}_0)^2 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}), \quad (4.9)$$

$$(\partial^2 g / \partial \widetilde{\xi}^2)(0, \widetilde{\xi}_0) = D^2 f(0) \in \mathcal{L}(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2, \mathbb{R})). \quad (4.10)$$

182 Since $\widetilde{\xi}_0 \in \Xi$, one has $D^2 f(0) \cdot \widetilde{\xi}_0 = 0 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$. From (4.1) and the above relations, the equations (4.7) are satisfied with $t = 0$ and $\widetilde{\xi} = \widetilde{\xi}_0$, which shows that the line $\{0\} \times \Xi$ is in the null-space $\text{Ker } D^2 g(0, \widetilde{\xi}_0)$. To prove that this line is the whole null-space and because the system

$\{\tilde{\xi}_0, \tilde{\tau}_0\}$ is a basis of \mathbb{R}^2 , it suffices to show with $\tilde{\xi} = \lambda\tilde{\tau}_0$, $\lambda \in \mathbb{R}$, that the equations (4.7) have no solution other than $t = \lambda = 0$. With (4.8)-(4.10), they become

$$\begin{cases} \frac{t}{12}D^4f(0) \cdot (\tilde{\xi}_0)^4 + \frac{\lambda}{2}D^3f(0) \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0) = 0 \in \mathbb{R}, \\ \frac{t}{2}D^3f(0) \cdot (\tilde{\xi}_0)^2 + \lambda D^2f(0) \cdot \tilde{\tau}_0 = 0 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}). \end{cases} \quad (4.11)$$

The second equation (4.11) is equivalent to two scalar equations, obtained, for instance, by expressing that the value of the left hand side vanishes at each of the two noncollinear vectors $\tilde{\xi}_0$ and $\tilde{\tau}_0$. By the same arguments as above (i.e. (4.1) and $\tilde{\xi}_0 \in \Xi$) the first scalar equation is the trivial one $0 = 0$. The second equation is

$$\frac{t}{2}D^3f(0) \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0) + \lambda D^2f(0) \cdot (\tilde{\tau}_0)^2 = 0. \quad (4.12)$$

But the system made of (4.12) and the first equation (4.11) has the unique solution $t = \lambda = 0$ if and only if $\det D^2f^{(1)}(0) \neq 0$ (cf. (4.6)), which completes the proof.

Remark 4.1. If $Df^{(1)}(0) \neq 0$ (i.e. $D^3f(0) \cdot (\tilde{\xi}_0)^3 \neq 0$), it is *not true* that the condition $\det D^2f^{(1)}(0) \neq 0$ is independent of the choice of T , as the reader can easily check on formula (4.6).

Remark 4.2. We could already make Proposition 4.1 more precise by showing that the *sign* of $\det D^2f^{(1)}(0)$ is independent of the choice of T (when $Df^{(1)}(0) = 0$) but this will be an immediate consequence of our further analysis. 183

Theorem 4.1. Assume that $Df^{(1)}(0) = 0$.

(i) If $\det D^2f^{(1)}(0) > 0$, i.e.

$$\frac{1}{3}(D^4f(0) \cdot (\tilde{\xi}_0)^4)(D^2f(0) \cdot (\tilde{\tau}_0)^2) - (D^3f(0) \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0))^2 > 0,$$

the local zero set of f reduces to the origin.

(ii) If $\det D^2f^{(1)}(0) < 0$, i.e.

$$\frac{1}{3}(D^4f(0) \cdot (\tilde{\xi}_0)^4)(D^2f(0) \cdot (\tilde{\tau}_0)^2) - (D^3f(0) \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0))^2 < 0,$$

the local zero set of f consists of exactly two distinct curves of class C^∞ , tangent to the characteristic Ξ at the origin. More precisely, there exist two real-valued functions $v_1^{(\alpha)}(u_1)$, $\alpha = 1, 2$, defined and of class C^∞ around the origin, verifying

$$\frac{dv_1^{(1)}}{du_1}(0) \neq \frac{dv_1^{(2)}}{du_1}(0), \quad (4.13)$$

such that the local zero set of f is made up of the two curves ($\rho > 0$ small enough)

$$u_1 \in (-\rho, \rho) \rightarrow \tilde{x}^{(\alpha)}(u_1) = u_1 \tilde{\xi}_0 + u_1 v_1^{(\alpha)}(u_1) \tilde{\tau}_0, \alpha = 1, 2.$$

Proof. Recall that the local zero set of f is of the form

$$\tilde{x} = u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0,$$

with (u_1, v_1) in the local zero set of $f^{(1)}$. The part (i) is then obvious since the local zero set of $f^{(1)}$ reduces to the origin.

184 Let us now prove (ii). The local zero set of $f^{(1)}$ is made up of two C^∞ curves intersecting transversally at the origin where their tangents are the two distinct lines of solutions of the equation

$$D^2 f^{(1)}(0) \cdot (u_1, v_1)^2 = 0.$$

Note that the v_1 -axis is never one of these tangents since $D^2 f(0) \cdot (\tilde{\tau}_0)^2 \neq 0$. As a result, the projection of either tangent onto the u_1 -axis and along the v_1 -axis a linear isomorphism and it follows that the two C^∞ curves can be parameterized by u_1 (cf. Figure 4.1 below)

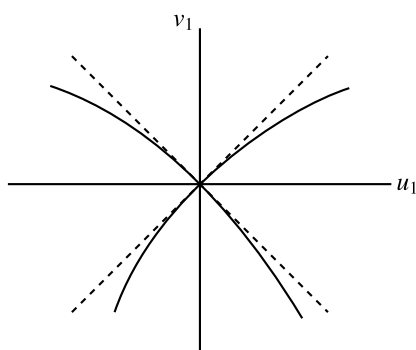


Figure 4.1:

More rigorously, each curve is of the form

$$t \rightarrow (u_1(t), v_1(t)), \quad (4.14)$$

a C^∞ function around the origin verifying $u_1(0) = v_1(0) = 0$ and

$$\frac{d}{dt}(u_1, v_1)|_{t=0} \neq 0$$

(cf. Chapter 2, Corollary 3.1). Thus, the nonzero vector $((du_1/dt)(0), (dv_1/dt)(0))$ is tangent to the curve at the origin and, as it is not collinear with the v_1 -axis, one has

$$\frac{du_1}{dt}(0) \neq 0.$$

From the Inverse mapping theorem, the relation $u_1 = u_1(t)$ is equivalent to saying that $t = t(u_1)$, a C^∞ function around the origin such that $t(0) = 0$. Therefore, the curve (4.14) is as well parametrized by

$$u_1 \rightarrow (u_1, v_1(t(u_1)))$$

as stated above. We have then proved that the local zero set of $f^{(1)}$ is made of two C^∞ curves $(u_1, v_1^{(\alpha)}(u_1))$, $\alpha = 1, 2$ and $u_1 \in (-\rho, \rho)$, $\rho > 0$ small enough. As the vectors $(1, (dv_1^{(\alpha)}/du_1)(0))$, $\alpha = 1, 2$ are tangent

to a different one of the two curves at the origin (and hence are not collinear) we deduce

$$\frac{dv_1^{(1)}}{du_1}(0) \neq \frac{dv_1^{(2)}}{du_1}(0).$$

Finally, the local zero set of f is made of the two C^∞ curves

$$u_1 \in (-\rho, \rho) \rightarrow \tilde{x}^{(\alpha)}(u_1) = u_1 \tilde{\xi}_0 + u_1 v_1^{(\alpha)}(u_1) \tilde{\tau}_0, \alpha = 1, 2.$$

Both are tangent to the characteristic Ξ at the origin since

$$\frac{d\tilde{x}^{(\alpha)}}{du_1}(0) = \tilde{\xi}_0, \alpha = 1, 2,$$

186 and they are distinct, since their second derivatives at the origin

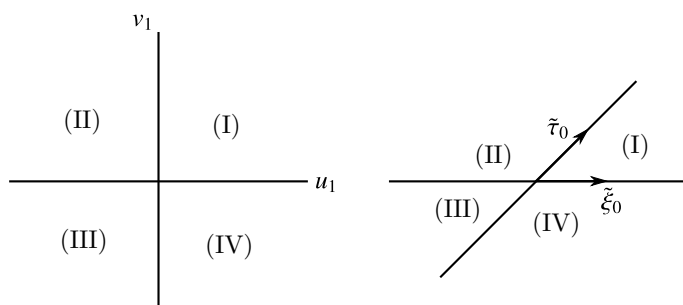
$$\frac{d^2\tilde{x}^{(\alpha)}}{du_1^2}(0) = \frac{dv_1^{(\alpha)}}{du_1}(0),$$

differ. □

Remark 4.3. We get an a posteriori proof of the fact that the sign of $\det D^2 f^{(1)}(0)$ is independent of the choice of the space T because this sign is related to the *structure of the local zero set of f* .

We shall now describe a convenient way for finding the local zero set of f from that of $f^{(1)}$. It will be repeatedly used in the applications of the next section and we shall refer to it as the “*quadrant method*”.

Identify the sum $\Xi \oplus T$ with the product $\mathbb{R}\tilde{\xi}_0 \times \mathbb{R}\tilde{\tau}_0$, the vector $\tilde{\tau}_0$ being chosen so that $\tilde{\xi}_0, \tilde{\tau}_0$ is a direct system of coordinates. In the (u_1, v_1) -plane as well as in the product $\mathbb{R}\tilde{\xi}_0 \times \mathbb{R}\tilde{\tau}_0$, we shall distinguish the usual four quadrants (I), (II), (III) and (IV).



Clearly, the mapping

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$$(u_1, v_1) \rightarrow u_1 \tilde{\xi}_0 + u_1 \tilde{\tau}_0,$$

maps the quadrants (I), \dots , (IV) of the (u_1, v_1) -plane into the quadrants (I), \dots , (IV) of $\mathbb{R}\tilde{\xi}_0 \times \mathbb{R}\tilde{\tau}_0$ according to the rule

$$\begin{array}{ll} (I) \rightarrow (I) & (III) \rightarrow (II) \\ (II) \rightarrow (III) & (IV) \rightarrow (IV) \end{array}$$

Thus, each half-branch in the local zero set of $f^{(1)}$ located in (I) (resp. (II), (III), (IV)) is transformed into a half-branch in the local zero set of f located in (I) (resp. (III), (II), (IV)). Figure 4.2 below features some possible diagrams. The simplest example of a mapping f verifying the conditions of this section is $f(u, v) = u^2 \pm v^4$ (or $v^2 \pm u^4$ or, with θ any fixed real number, $(v \cos \theta - u \sin \theta)^2 \pm (u \cos \theta + v \sin \theta)^4$).

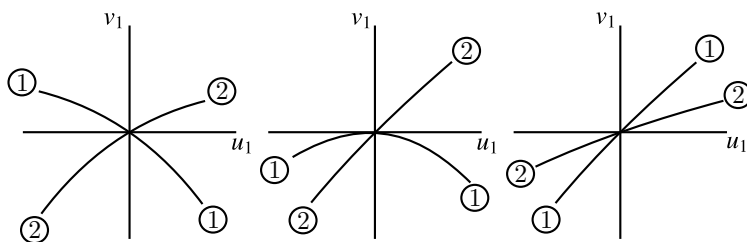


Figure 4.2: (a) local zero set of $f^{(1)}$.

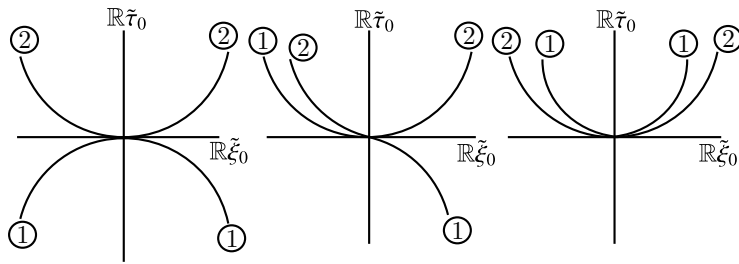


Figure 4.2: (b) local zero set of f

5.5 Iteration of the Process.

188 We shall now assume that the local zero set of the mapping $f^{(1)}$ cannot be determined through either the Implicit function theorem or the Morse lemma, that is to say

$$Df^{(1)}(0) = 0, \tag{5.1}$$

$$\det D^2 f^{(1)}(0) = 0. \tag{5.2}$$

From (2.13), we know that $(\partial^2 f^{(1)} / \partial v_1^2)(0) \neq 0$. In particular,

$$D^2 f^{(1)}(0) \neq 0, \tag{5.3}$$

189 and the problem of finding the local set of $f^{(1)}$ is the same as considered in §2 with the mapping f . The mapping $f^{(1)}$ has a characteristic Ξ_1 which is a one-dimensional subspace of the (u_1, v_1) -plane; considering any complement T_1 of Ξ_1 and choosing arbitrary nonzero elements $\tilde{\xi}_1$ and $\tilde{\tau}_1$ in Ξ_1 and T_1 respectively, we can reduce the problem to finding the local zero set of a new mapping $f^{(2)}$ obtained by blowing-up $f^{(1)}$ along Ξ_1 . The mapping $f^{(2)}$ depends on the new variables (u_2, v_2) through the relation

$$f^{(2)}(u_2, v_2) = \begin{cases} \frac{1}{u_2^2} f^{(1)}(u_2 \tilde{\xi}_1 + u_2 v_2 \tilde{\tau}_1) & \text{for } u_2 \neq 0, \\ \frac{1}{2} v_2^2 D^2 f^{(1)}(0) \cdot (\tilde{\tau}_1)^2 & \text{for } u_2 = 0. \end{cases}$$

An important observation is that the relation $(\partial^2 f^{(1)}/\partial v_1^2)(0) \neq 0$ keeps the characteristic Ξ_1 from being the v_1 -axis. This will be summarized by saying that the characteristic Ξ_1 is *not vertical*. Such a notion makes no sense as far as the characteristic Ξ is concerned, unless we *arbitrarily* fix a system (u, v) of coordinates in the original plane \mathbb{R}^2 .

If $Df^{(2)}(0) \neq 0$, the local zero set of $f^{(2)}$ can be found through the Implicit function theorem. From Theorem 3.1, we deduce the local zero set of $f^{(1)}$ and, finally, the local zero set of f . If $Df^{(2)}(0) = 0$ and $\det D^2 f^{(2)}(0) = 0$, the mapping $f^{(2)}$ has a characteristic Ξ_2 which is *not vertical* in its ambient space (u_2, v_2) and the problem reduces to finding the local zero set of a third iterate $f^{(3)}$ depending on the new variables (u_3, v_3) . More generally, we have

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Theorem 5.1. *Assume that the iterates $f^{(1)}, \dots, f^{(m)}$ are defined and $Df^{(m)}(0) \neq 0$. After changing $\tilde{\xi}_0$ into $-\tilde{\xi}_0$ if necessary and setting $\Xi_+ = \mathbb{R}_+ \Xi$, the local zero set of f is made up of two continuous half-branches emerging from the origin in the half-space $\Xi_+ \oplus T$. These half-branches are of class C^∞ away from the origin and tangent to the characteristic Ξ at the origin.*

Proof. The mappings $f^{(1)}, \dots, f^{(m)}$ are defined after the choice of non zero elements $\tilde{\xi}_0, \dots, \tilde{\xi}_{m-1}$ and $\tilde{\tau}_0, \dots, \tilde{\tau}_{m-1}$ in the characteristics $\Xi, \Xi_1, \dots, \Xi_{m-1}$ and some given complements T, T_1, \dots, T_{m-1} . In the (u_j, v_j) plane, $\tilde{\xi}_j$ and $\tilde{\tau}_j$ are of the form $\tilde{\xi}_j = (\alpha_j, \beta_j)$ and $\tilde{\tau}_j = (\lambda_j, \mu_j)$, with $\alpha_j \neq 0$ since the characteristic Ξ_j is *not vertical* for $1 \leq j \leq m-1$. \square

Clearly, the iterate $f^{(m)}$ is the first iterate of $f^{(m-1)}$. Thus, after changing $\tilde{\xi}_{m-1}$ into $-\tilde{\xi}_{m-1}$, if necessary (which does not affect the existence of the m^{th} iterate $f^{(m)}$ or the condition $Df^{(m)}(0) \neq (0)$), we know from Theorem 3.1 that the local zero set of $f^{(m-1)}$ is made up of two half-branches ($\rho > 0$ small enough, ψ an origin-preserving C^∞ local diffeomorphism of \mathbb{R})

$$0 \leq u_m < \rho - \tilde{x}_{m-1}^{(\alpha)}(u_m) = u_m \tilde{\xi}_{m-1} + u_m \psi((-1)^\alpha \sqrt{u_m}) \tilde{\tau}_{m-1}, \alpha = 1, 2.$$

In the system (u_{m-1}, v_{m-1}) , this becomes

$$u_{m-1}^{(\alpha)}(u_m) = \alpha_{m-1} u_m + \lambda_{m-1} u_m \psi((-1)^\alpha \sqrt{u_m}),$$

$$v_{m-1}^{(\alpha)}(u_m) = \beta_{m-1}u_m + \mu_{m-1}u_m\psi((-1)^\alpha\sqrt{u_m}). \quad (5.4)$$

191 The above mappings $u_{m-1}^{(\alpha)}(\cdot)$ and $v_{m-1}^{(\alpha)}(\cdot)$ are of class C^1 in $[0, \rho)$ with

$$(du_{m-1}^{(\alpha)}/du_m)(0) = \alpha_{m-1} \neq 0. \quad (5.5)$$

Now, from §2, the local zero set of $f^{(m-2)}$ is made of elements of the form

$$\tilde{x}_{m-2} = u_{m-1}\tilde{\xi}_{m-2} + u_{m-1}v_{m-1}\tilde{\tau}_{m-2},$$

with (u_{m-1}, v_{m-1}) in the local zero set of $f^{(m-1)}$. From (5.4), $\tilde{x}_{m-2} = \tilde{x}_{m-2}^{(\alpha)}(u_m)$, $\alpha = 1, 2$, with

$$\tilde{x}_{m-2}(u_m) = u_{m-1}^{(\alpha)}(u_m)\tilde{\xi}_{m-2} + u_{m-1}^{(\alpha)}(u_m)v_{m-1}^{(\alpha)}(u_m)\tilde{\tau}_{m-2},$$

which, in the system (u_{m-2}, v_{m-2}) , expresses as

$$\begin{aligned} u_{m-2}^{(\alpha)}(u_m) &= \alpha_{m-2}u_{m-1}^{(\alpha)}(u_m) + \lambda_{m-2}u_{m-1}^{(\alpha)}(u_m)v_{m-1}^{(\alpha)}(u_m), \\ v_{m-2}^{(\alpha)}(u_m) &= \beta_{m-2}u_{m-1}^{(\alpha)}(u_m) + \mu_{m-2}u_{m-1}^{(\alpha)}(u_m)v_{m-1}^{(\alpha)}(u_m). \end{aligned}$$

The mappings $u_{m-2}^{(\alpha)}(\cdot)v_{m-2}^{(\alpha)}(\cdot)$ are of class C^1 in $(0, \rho)$ and, from (5.5)

$$(du_{m-2}^{(\alpha)}/du_m)(0) = \alpha_{m-2}\alpha_{m-1} \neq 0, \alpha = 1, 2,$$

since $\alpha_{m-2} \neq 0$. Iterating the process, we find that the local zero set of f is of the form

$$\tilde{x}^{(\alpha)}(u_m) = u_1^{(\alpha)}(u_m)\tilde{\xi}_0 + u_1^{(\alpha)}(u_m)v_1^{(\alpha)}(u_m)\tilde{\tau}_0, \alpha = 1, 2,$$

where the functions $u_1^{(\alpha)}(\cdot)$ and $v_1^{(\alpha)}(\cdot)$ are of class C^1 in $[0, \rho)$, vanish at the origin and

$$(du_1^{(\alpha)}/du_m)(0) = \alpha_1 \cdots \alpha_{m-1} \neq 0.$$

192 After shrinking $\rho > 0$ if necessary, the sign of $u_1^{(\alpha)}(u_m)$ is that of the product $\alpha_1 \cdots \alpha_{m-1}$ for $0 < u_m < \rho$ and both values $\alpha = 1$ and $\alpha = 2$. Hence our assertion, since all the mappings we have considered are C^∞ away from the origin.

When Theorem 5.1 applies, the partial determination of the local zero set of f from that of $f^{(m)}$ follows by successive applications of the “quadrant method” described in §4. Note, however, for $1 \leq j \leq m - 1$, that the quadrants (I), \dots , (IV) used for finding the local zero set of $f^{(j)}$ from the local zero set of $f^{(j+1)}$ are (in general) *different* from the quadrants (I), \dots , (IV) used for finding the local zero set of $f^{(j-1)}$ from that of $f^{(j)}$.

Figure 5.1 (a), (b), (c) below gives an example of the local zero set of $f^{(2)}$, $f^{(1)}$ and f respectively when Theorem 5.1 applies with $m = 2$. A mapping whose local zero set looks like that in Figure 5.1(c) (and for which the process stops at $m = 2$ of course) is

$$f(u, v) = (v - u^2)^2 + u^5.$$

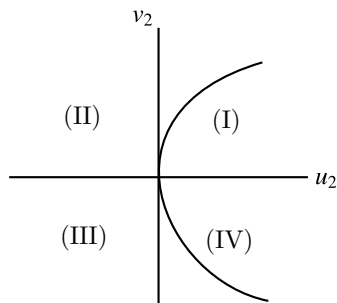


Figure 5.1: (a) local zero set of $f^{(2)}$.

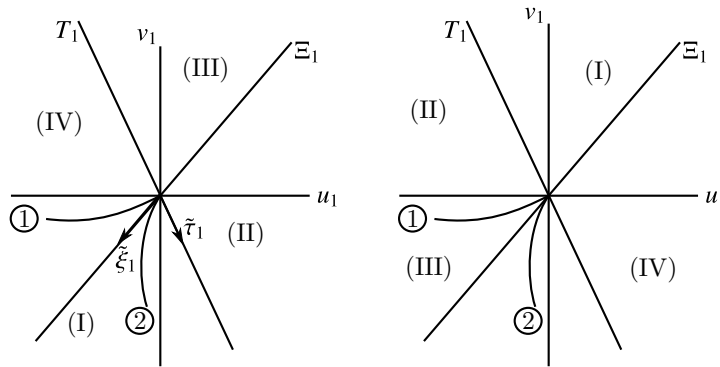


Figure 5.1: (b) local zero set of $f^{(1)}$

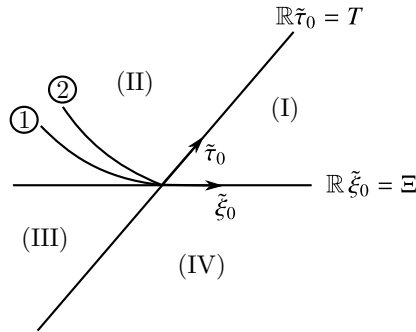


Figure 5.1: (c) local zero set of f

- 193 **Remark 5.1.** In theory at least, applying Theorem 5.1 does not require the computation of $f^{(1)}, \dots, f^{(n)}$ and reduces to checking some conditions on the partial derivatives of f . For a general m , this follows
- 194 from the proof of the *intrinsic character of the process*, namely, that whether or not it stops at some rank m with $Df^{(m)}(0) \neq 0$ is independent of the choice of the complements T, T_1, \dots, T_{m-1} and of the elements $\tilde{\xi}_0, \dots, \tilde{\xi}_{m-1}$ and $\tilde{\tau}_0, \dots, \tilde{\tau}_{m-1}$ for defining the iterates $f^{(1)}, \dots, f^{(m)}$ ³.

³Observe that modifying T, \dots, T_{m-2} affects the characteristics Ξ_1, \dots, Ξ_{m-1} as well.

For $m = 1$, this was shown in § 3 (and the result is obvious). The proof of this assertion for any m is very technical and will not be given in these notes (cf. Rabier [32])⁴. We shall only give the example of the case when $m = 2$; The iterate $f^{(2)}$ is defined under the conditions (5.1) and (5.2), namely (cf. §§3 and 4)

$$D^3 f(0) \cdot (\tilde{\xi}_0)^3 = 0, \quad (5.6)$$

$$\frac{1}{3}(D^4 f(0) \cdot (\tilde{\xi}_0)^4)(D^2 f(0) \cdot (\tilde{\tau}_0)^2) - (D^3 f(0) \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0))^2 = 0. \quad (5.7)$$

Now, a nonzero element $\tilde{\xi}_1$ of the characterisic Ξ_1 is

$$\tilde{\xi}_1 = (D^2 f(0) \cdot (\tilde{\tau}_0)^2), -\frac{1}{2}D^3 f(0) \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0).$$

From §3, the condition $Df^{(2)}(0) \neq 0$ is equivalent to

$$D^3 f^{(1)}(0) \cdot (\tilde{\xi}_1)^3 \neq 0.$$

Using (5.7) and formula (2.12), this can be rewritten as

$$\begin{aligned} \frac{4}{5}(D^5 F(0) \cdot (\tilde{\xi}_0)^5)(D^2 f(0) \cdot (\tilde{\tau}_0)^2) - 8(D^4 f(0) \cdot ((\tilde{\xi}_0)^3, \tilde{\tau}_0))(D^3 f(0) \\ \cdot ((\tilde{\xi}_0)^2, \tilde{\tau}_0)) + 4D^3 f(0) \cdot (\tilde{\xi}_0, (\tilde{\tau}_0)^2)(D^4 f(0) \cdot (\tilde{\xi}_0)^4) \neq 0. \end{aligned} \quad (5.8)$$

When (5.6) and (5.7) hold (which is independent of the choice of $\tilde{\xi}_0 \in \Xi$ and $\tilde{\tau}_0$ from Proposition 4.1), it is easy to see that (5.8) also is independent of $\tilde{\xi}_0 \in \Xi$ and $\tilde{\tau}_0$. However, the reader can already guess that proving the intrinsic character of the process by using formulas such as (5.6)-(5.8) is impossible in the general case. 195

Assume now that $Df^{(2)}(0) = 0$. Then, the local zero set of $f^{(2)}$ can be found through the Morse lemma and Theorem 4.1 provides the structure of the local zero set of $f^{(1)}$, from which the structure of the local zero set of f is easily derived. More generally, we have

⁴However, § 6 gives a partial result in this direction.

Theorem 5.2. *Assume that the iterates $f^{(1)}, \dots, f^{(m)}$ are defined and $Df^{(m)}(0) = 0$, $\det D^2 f^{(m)}(0) \neq 0$. Then, the local zero set of f reduces to the origin if $\det D^2 f^{(m)}(0) > 0$ and is made up of exactly two distinct curves of class C^∞ tangent to the characteristic Ξ at the origin if $\det D^2 f^{(m)}(0) < 0$.*

Proof. It is possible to parallel the proof of Theorem 5.1. However, we are going to give a more “geometrical” one. We limit ourselves to the case $m = 2$, the general situation being identical by repeating the same arguments. From §2 we know that the local zero set of f is of the form

$$\tilde{x} = u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0,$$

196 with (u_1, v_1) in the local zero set of $f^{(1)}$. The assertion is then obvious if $\det D^2 f^{(2)}(0) > 0$ since the local zero set of $f^{(1)}$ reduces to the origin (cf. Theorem 4.1). Assume then that $\det D^2 f^{(2)}(0) < 0$ so that the local zero set of $f^{(1)}$ is made of two C^∞ curves tangent to the characteristic Ξ_1 at the origin as Theorem 4.1 states. In addition, from the proof of Theorem 4.1, these curves coincide with the graphs of two C^∞ functions defined around the origin in Ξ_1 with values in T_1 . The important point is that this property is not affected, if T_1 is replaced by any other complement of T_1 (express the general form of a linear change of coordinates leaving the characteristic Ξ_1 globally invariant and use the Implicit function theorem). In particular, as the characteristic Ξ_1 is not vertical, this complement can be taken as the v_1 -axis $\mathbb{R}(0, 1)$. Thus, the local zero set of $f^{(1)}$ coincides with the graphs of two C^∞ functions defined around the origin Ξ_1 with values in $\mathbb{R}(0, 1)$. Because the characteristic Ξ_1 is not vertical again, its projection onto the u_1 -axis is a linear isomorphism. It follows that the local zero set of $f^{(1)}$ coincides with the graphs of two C^∞ functions $v_1^{(\alpha)}(u_1)$, $\alpha = 1, 2$, defined for $|u_1| < \rho$ ($\rho > 0$ small enough) verifying $v_1^{(\alpha)}(0) = 0$. Hence, the local zero set of f is made up of the two curves

$$u_1 \in (-\rho, \rho) \rightarrow \tilde{x}^{(\alpha)}(u_1) = u_1 \tilde{\xi}_0 + u_1 v_1^{(\alpha)}(u_1) \tilde{\tau}_0, \alpha = 1, 2.$$

Theses two curves are tangent to the characteristic Ξ at the origin and they are distinct, since they coincide with the graphs of the two

197 distinct functions $u_1 v_1^{(\alpha)}(u_1)$ in $\mathbb{R}_{\tilde{\xi}_0} \times \mathbb{R}_{\tilde{\tau}_0}$. Finally, they are of class C^∞ , since the functions $v_1^{(\alpha)}$ are and the proof is complete. \square

For practically finding the local zero set of f from the local zero set of $f^{(m)}$, the quadrant method can be applied again. We now show it on the example when

$$f(u, v) = u^2 + 2uv^2 + u^3 + 2u^2v + v^4 + 2uv^3 + uv^4 + v^5 + v^6.$$

Clearly, $f(0) = 0, Df(0) = 0, D^2f(0) \neq 0$ and $\det D^2f(0) = 0$. Here, the characteristic Ξ is the space $\Xi = \mathbb{R}(0, 1)$. It is convenient to take $T = \mathbb{R}(1, 0)$ (however, the process being intrinsic, all the possible choices are equivalent). If so, with $\tilde{\xi}_0 = (0, 1)$ and $\tilde{\tau}_0 = (1, 0)$, we find

$$\begin{aligned} f^{(1)}(u_1, v_1) &= \frac{1}{u_1^2} f(u_1(0, 1) + u_1 v_1(1, 0)) = \frac{1}{u_1^2} f(u_1 v_1, u_1) = \\ &= v_1^2 + 2u_1 v_1 + u_1 v_1^3 + u_1 v_1^2 + u_1^2 + 2u_1^2 v_1 + u_1^3 v_1 + u_1^3 + u_1^4. \end{aligned}$$

We see that $Df^{(1)}(0) = 0$ while the quadratic part of $f^{(1)}$ is $(u_1 + v_1)^2$. Thus, the characteristic Ξ_1 is $\Xi_1 = \mathbb{R}(1, -1)$ and we choose $T_1 = \mathbb{R}(0, 1)$. Taking $\tilde{\xi}_1 = (1, -1)$ and $\tilde{\tau}_1 = (0, 1)$, we find

$$\begin{aligned} f^{(2)}(u_2, v_2) &= \frac{1}{u_2^2} f^{(1)}(u_2(1, -1) + u_2 v_2(0, 1)) = \frac{1}{u_2^2} f^{(1)}(u_2, u_2(v_2 - 1)) \\ &= v_2^2 - u_2^2 + 4u_2^2 v_2 + u_2 v_2^2 + u_2^2 v_2^3 - 3u_2^2 v_2^2. \end{aligned}$$

The mapping $f^{(2)}$ verifies $Df^{(2)}(0) = 0$ and $\det D^2f^{(2)}(0) = -4 < 0$. The two lines in the local zero set of $D^2f^{(2)}(0) \cdot (u_2, v_2)^2$ are the lines **198** $u_2 = v_2$ and $u_2 = -v_2$. Figure 5.2 shows how the quadrant method allows to find the two curves in the local zero set of f .

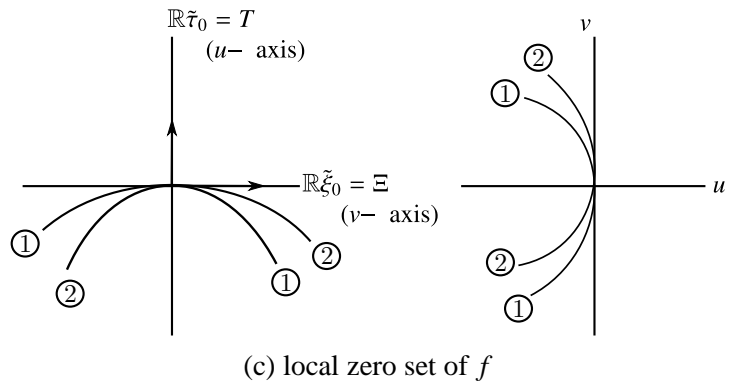
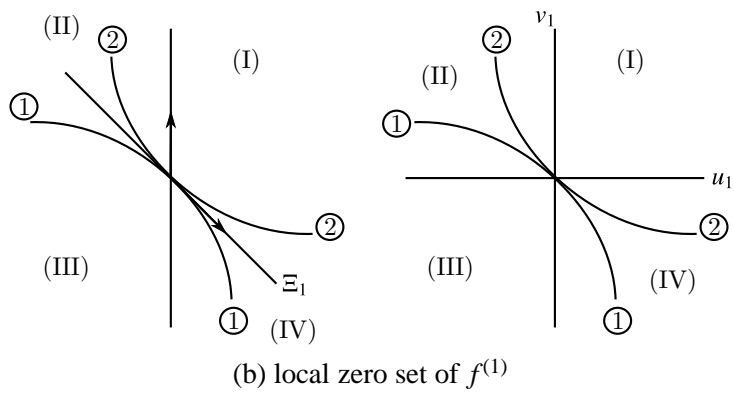
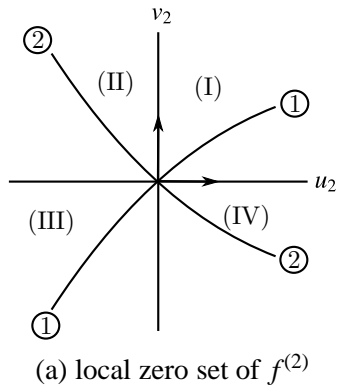


Figure 5.2:

5.6 Partial Results on the Intrinsic Character of the Process

A natural question we have already mentioned is to know whether the process described in §5 is intrinsic or depends on the successive spaces T, T_1, \dots, T_{m-1} and elements $\tilde{\xi}_0, \tilde{\xi}_1, \dots, \tilde{\xi}_{m-1}$ and $\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_{m-1}$ chosen for defining the iterates $f^{(1)}, \dots, f^{(m)}$. In § 3, we saw that the condition $Df^{(1)}(0) \neq 0$ was independent of the choice of T and $\tilde{\xi}_0, \tilde{\tau}_0$ and we also proved a similar result as concerns the conditions $Df^{(1)}(0) = 0$ and $\det D^2 f^{(1)}(0) \neq 0$ together (cf. Proposition 4.1). The existence of a second iterate $f^{(2)}$ (i.e. the condition $Df^{(1)}(0) = 0, \det D^2 f^{(1)}(0) = 0$) is then independent of T and $\tilde{\xi}_0, \tilde{\tau}_0$. Given such a space T and elements $\tilde{\xi}_0 \in \Xi, \tilde{\tau}_0 \in T$ from which $f^{(1)}$ is defined, the existence of a third iterate $f^{(3)}$ is then independent of the choice of the complement T_1 of the characteristic Ξ_1 and of the non-zero elements $\tilde{\xi}_1 \in \Xi_1, \tilde{\tau}_1 \in T_1$. But whether or not its existence is independent of the initial choice of T and $\tilde{\xi}_0, \tilde{\tau}_0$ is not ensured by our previous results. However, we can reduce the question as follows. Since the characteristic Ξ_1 is not vertical and since the result is known to be independent of T_1 , it suffices to prove that the existence of $f^{(3)}$ is independent of $T, \tilde{\xi}_0, \tilde{\tau}_0$ when T_1 is taken as the v_1 -axis. Suppose then that this step has been solved successfully. We deduce that the existence of a fourth iterate $f^{(4)}$ is independent of the choice of T_1, T_2 and of the non-zero elements $\tilde{\xi}_1, \tilde{\xi}_2$ and $\tilde{\tau}_1, \tilde{\tau}_2$ in the characteristics Ξ_1, Ξ_2 and their complements T_1, T_2 respectively. Again, the non-dependence on the initial choice of T and $\tilde{\xi}_0, \tilde{\tau}_0$ is not a consequence of the results we have established so far, but the problem can be reduced to the case when T_1 is taken as the v_1 -axis and T_2 is taken as the v_2 -axis. More generally, we see that proving that the process of §5 is intrinsic reduces to proving for every m that it is independent of T and $\tilde{\xi}_0, \tilde{\tau}_0$ when the space T_1, \dots, T_{m-1} are taken as the v_1, \dots, v_{m-1} -axes.

Therefore, we shall henceforth assume that the sequence $f^{(1)}, \dots, f^{(n)}$ is defined when T_j is taken as the v_j -axis, $1 \leq j \leq m - 1$. Equivalently, we shall say that each space $T_j, 1 \leq j \leq m - 1$ is vertical. A proof of the independence of T based on some generalization of Proposition 4.1 is not available. Indeed, Proposition 4.1 allows to prove the first in-

dependence result and is based on “linear” arguments while the question of the independence of T in the general case is a *higher order (hence) nonlinear* problem. By showing that the two half-branches/curves (following that either Theorem 5.1 or Theorem 5.2 applies) in the local zero set of f have a contact of order m exactly at the origin, it is possible to establish that the process is intrinsic if, for choice of T , $Df^{(m)}(0) \neq 0$ or $Df^{(m)}(0) = 0$ and $\det D^2 f^{(m)}(0) < 0$ (thus, in such a case, m and either condition $Df^{(m)}(0) \neq 0$ or $Df^{(m)}(0) = 0$ and $\det D^2 f^{(m)}(0) < 0$ is independent of T). If $Df^{(m)}(0) = 0$ and $\det D^2 f^{(m)}(0) > 0$, the local zero set of f reduces to the origin and no useful information of an invariant geometrical character can be derived from the structure of the local zero set of f . Nevertheless, this situation also can be shown to be intrinsic. The details of the above assertions can be found in Rabier [32]. Here, it will be sufficient for our purposes to prove that the process is intrinsic in a particular case that we shall now describe.

We begin with a definition. We shall say that the characteristic Ξ_j is *horizontal* if it coincides with u_j -axis in its ambient space (u_j, v_j) . Similarly, *after the choice of a system* of coordinates (u, v) in the original plane \mathbb{R}^2 , we shall say that the characteristic Ξ is horizontal if it coincides with the u -axis. The v -axis will be referred to as the vertical axis.

Remark 6.1. Note that we can always make a choice of the system (u, v) so that the characteristic Ξ is horizontal but we have no freedom of making the characteristics Ξ_1, \dots, Ξ_{m-1} horizontal. Conversely, the characteristics Ξ_1, \dots, Ξ_{m-1} being horizontal is independent of the system (u, v) of coordinates in the original plane \mathbb{R}^2 , since no particular system is involved in their definition.

Theorem 6.1. *Let (u, v) be a system of coordinates in which the characteristic Ξ is not vertical. Then, the characteristics $\Xi, \Xi_1, \dots, \Xi_{m-1}$ are horizontal if and only if*

$$(\partial^j f / \partial u^i)(0) = 0, 0 \leq j \leq 2m, \quad (6.1)$$

$$(\partial^{j+1} f / \partial u^j \partial v)(0) = 0, 0 \leq j \leq m. \quad (6.2)$$

Proof. Note that the relations (6.1) and (6.2) are independent of the change of coordinates leaving the u -axis invariant. Indeed, such a change of coordinates is of the form

$$\begin{aligned} U &= au + bv, \\ V &= cv, \end{aligned}$$

with $ac \neq 0$. Thus, at the origin

$$\partial^j / \partial u^j = a^j \partial^j / \partial U^j, \partial^{j+1} / \partial u^j \partial u = a^j b \partial^{j+1} / \partial U^{j+1} + a^j c \partial^{j+1} / \partial U^j \partial V, \tag{6.3}$$

so that the relations (6.1) and (6.2) hold in the system (U, V) if and only if they hold in the system (u, v) .

As the characteristic Ξ is not vertical, it is therefore not restrictive to suppose that T is the v -axis (by changing the latter, not T). On the other hand, the equivalence is true with $m = 1$ as is immediately checked. Assume then $m > 1$ and the equivalence holds. We shall prove that the characteristics Ξ, Ξ_1, \dots, Ξ_m exist and are horizontal if and only if (6.1) and (6.2) hold with $m + 1$ replacing m . Let then the characteristics $\Xi, \Xi_1, \dots, \Xi_{m-1}$ exist and be horizontal. In particular, this is true with $\Xi, \Xi_1, \dots, \Xi_{m-1}$, which is equivalent to (6.1) - (6.2) by hypothesis. Now, saying that Ξ_m exists and is horizontal means that

$$Df^{(m)}(0) = D^2 f^{(m)}(0) \cdot (1, 0) = 0 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}),$$

a condition expressed by the four scalar equations

$$\begin{aligned} (\partial f^{(m)} / \partial u_m)(0) &= (\partial f^{(m)} / \partial v_m)(0) \\ &= (\partial^2 f^{(m)} / \partial u_m \partial v_m)(0) \\ &= (\partial^2 f^{(m)} / \partial u_m^2)(0) = 0. \end{aligned}$$

Applying formula (2.12) with $f^{(m)}, f^{(m-1)}, \tilde{\xi}_{m-1}$ and $\tilde{\tau}_{m-1}$ replacing $f^{(1)}, f, \tilde{\xi}_0$ and $\tilde{\tau}_0$ respectively, where $\tilde{\xi}_{m-1}$ and $\tilde{\tau}_{m-1}$ are two given nonzero elements of Ξ_{m-1} and T_{m-1} ($= v_{m-1}$ -axis), the equation $(\partial f^{(m)} / \partial v_m)(0) = 0$ is automatically satisfied. The remaining three ones are

$$D^3 f^{(m-1)}(0) \cdot (\tilde{\xi}_{m-1})^3 = D^4 f^{(m-1)}(0) \cdot (\tilde{\xi}_{m-1})^4$$

$$= D^3 f^{(m-1)}(0) \cdot ((\tilde{\xi}_{m-1})^2, \tilde{\tau}_{m-1}) = 0$$

By hypothesis, $\tilde{\xi}_{m-1}$ and $\tilde{\tau}_{m-1}$ are collinear with the u_{m-1} - and v_{m-1} -axes respectively. Therefore, we get the equivalent relation

$$\begin{aligned} (\partial^3 f^{(m-1)} / \partial u_{m-1}^3)(0) &= (\partial^4 f^{(m-1)} / \partial u_{m-1}^4)(0) \\ &= (\partial^3 f^{(m-1)} / \partial u_{m-1}^2 \partial v_{m-1})(0) = 0. \end{aligned}$$

Again, with the same arguments, each of these equations can be expressed in terms of $f^{(m-2)}$ and the variables u_{m-2} and v_{m-2} . Iterating the process and since Ξ and T coincide with the u - and v -axis respectively, we finally find

$$(\partial^{2m+1} f / \partial u^{2m+1})(0) = (\partial^{2m+2} f / \partial u^{2m+2})(0) = (\partial^{m+2} f / \partial u^{m+1} \partial v)(0) = 0,$$

to be necessary and sufficient condition for the characteristic Ξ_m to exist and be horizontal, under the equivalent assumptions that $\Xi, \Xi_1, \dots, \Xi_{m-1}$ exist and are horizontal or that (6.1)-(6.2) hold. The equivalence at rank $m + 1$ follows. \square

Remark 6.2. Theorem 6.1 is clearly independent of the choice of T . By the arguments of the beginning of this section, the fact that Ξ_1, \dots, Ξ_{m-1} are horizontal is then independent of any choice T, T_1, \dots, T_{m-2} for defining the iterates $f^{(1)}, \dots, f^{(m-1)}$.

204 Remark 6.3. (Intrinsic character of the process when the characteristics Ξ_1, \dots, Ξ_{m-1} are horizontal): From Theorem 6.1 and by successive applications of formula (2.12), it is easily seen in a system (u, v) of coordinates in which the characteristic Ξ is horizontal and its complement T vertical (so that $\tilde{\xi}_0 = (\alpha_0, 0)$, $\tilde{\tau}_0 = (0, \mu_0)$) and when the characteristics Ξ_1, \dots, Ξ_{m-1} are horizontal, that, setting $\tilde{\xi}_j = (\alpha_j, 0)$ and $\tilde{\tau}_j = (0, \mu_j)$

$$\begin{aligned} (\partial f^{(m)} / \partial u_m)(0) &= [1/(2m + 1)!] \alpha_{m-1}^3 \cdots \alpha_0^{2m+1} \\ &\quad (\partial^{2m+1} f / \partial \mu^{2m+1})(0), \end{aligned} \tag{6.4}$$

$$\begin{aligned} (\partial^2 f^{(m)} / \partial u_m \partial v_m)(0) &= [1/(m + 1)!] \mu_{m-1} \cdots \mu_0 \alpha_{m-1}^2 \cdots \alpha_0^{m+1} \\ &\quad (\partial^{m+2} f / \partial u^{m+1} \partial v)(0), \end{aligned} \tag{6.5}$$

$$(\partial^2 f^{(m)} / \partial u_m^2)(0) = [2/(2m + 2)!] \alpha_{m-1}^4 \cdots \alpha_0^{2m+2} (\partial^{2m+2} f / \partial u^{2m+2})(0), \tag{6.6}$$

$$(\partial^2 f^{(m)} / \partial v_m^2)(0) = \mu_{m-1}^2 \cdots \mu_0^2 (\partial^2 f / \partial v^2)(0). \tag{6.7}$$

Together with the relation $(\partial f^{(m)} / \partial v_m)(0) = 0$ (cf. Theorem 6.1), these relations show that whether or not the characteristic Ξ_m exist (but is not necessarily horizontal) is independent of T . Indeed, the conditions $Df^{(m)}(0) \neq 0$ or $Df^{(m)}(0) = 0$ and $\det D^2 f^{(m)}(0) \neq 0$ are independent of the change of coordinates in the (u, v) -plane leaving the u -axis invariant because, in such a change $U = au + bv, V = cv (ac \neq 0)$, the right hand sides of (6.4)-(6.7), are multiplied by $a^{-2m-1}, a^{-m-1}c^{-1}, a^{-2m-2}$ and c^{-2} respectively (cf. (6.3)). Using the arguments at the beginning of this section and Remark 6.2, we conclude that the assumption: “ Ξ_1, \dots, Ξ_{m-1} exist and are horizontal and Ξ_m does not exist” is independent of any choice of T, T_1, \dots, T_{m-1} (and, of course, of $\tilde{\xi}_0, \tilde{\xi}_1, \dots, \tilde{\xi}_{m-1}$ and $\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_{m-1}$) for defining $f^{(1)}, \dots, f^{(m)}$.

In other words, the rank m at which the process stops (if at all) is intrinsically linked to f . 205

Corollary 6.1. *Let f verify the condition*

$$f(u, 0) = 0 \tag{6.8}$$

for u around the origin. Then, the characteristic Ξ is horizontal and Ξ_1, \dots, Ξ_{m-1} exist if and only if

$$(\partial^{j+1} f / \partial u^j \partial v)(0) = 0, 0 \leq j \leq m. \tag{6.9}$$

If so, they are horizontal. Finally, if there is a largest integer $m (\geq 1$ necessarily) such that (6.9) holds, the process §5 ends at the step m exactly and the local zero set of f is the union of the u -axis and one distinct C^∞ curve tangent to the u -axis at the origin.

Proof. Under the assumption (6.8), the characteristic Ξ is horizontal. Indeed, one has $(\partial^2 f / \partial u^2)(0) = 0$ and, since $\det D^2 f(0) = 0$, we deduce $(\partial^2 f / \partial u \partial v)(0) = 0$. Thus, $D^2 f(0) \cdot (1, 0) = 0 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$, a relation which expresses that Ξ is horizontal. Taking $\tilde{\xi}_0 = (1, 0)$ and $\tilde{\tau}_0 = (0, 1)$ (we

leave it to the reader to check that any other choice leads to the same result) the iterate $f^{(1)}$ is

$$f^{(1)}(u_1, v_1) = \begin{cases} \frac{1}{u_1^2} f(u_1, u_1 v_1) & \text{for } u_1 \neq 0, \\ \frac{1}{2} v_1^2 (\partial^2 f / \partial v^2)(0) & \text{for } u_1 = 0. \end{cases}$$

Clearly, from (6.8), $f^{(1)}(u_1, 0) = 0$ for u_1 around the origin. Hence, if Ξ_1 exists, it must be horizontal for the same reason as Ξ is. More generally, each iterate $f^{(j)}$ verifies $f^{(j)}(u_j, 0) = 0$ for u_j around the origin and Ξ_j must be horizontal if it exists. Theorem 6.1 is then available and states that assuming that Ξ_1, \dots, Ξ_{m-1} exist is equivalent to assuming that the relations (6.1) and (6.2) hold. But (6.1) is automatically satisfied because of (6.8) and the first part of our assertion follows.

Let now m be the largest integer such that (6.9) holds (if at all). Then

$$\begin{aligned} (\partial^{j+1} f / \partial u^j \partial v)(0) &= 0, 0 \leq j \leq m, \\ (\partial^{m+2} f / \partial u^{m+1} \partial v)(0) &\neq 0, \end{aligned}$$

and the characteristics Ξ_1, \dots, Ξ_{m-1} exist ($m \geq 1$) while Ξ_m does not. This means either $Df^{(m)}(0) \neq 0$ or $Df^{(m)}(0) = 0$ and $\det D^2 f^{(m)}(0) \neq 0$. From Theorem 5.1 and 5.2, it is immediately checked that the only case compatible with the fact that the trivial branch $(u, 0)$ for $|u|$ small enough is in the local zero set of f is when $Df^{(m)}(0) = 0$, $\det D^2 f^{(m)}(0) < 0$ and the conclusion is given by Theorem 5.2. \square

Remark 6.4. The second part of Corollary 6.1 can also be derived from formulas (6.4)-(6.7) and Theorem 5.2.

Remark 6.5. The mapping f verifying the condition $f(u, 0) = 0$ for u around the origin, it can of course happen that all the derivatives $(\partial^{j+1} f / \partial u^j \partial v)(0)$ vanish. As all the derivatives $(\partial^j f / \partial u^j)(0)$ also vanish because of the special form of f , we deduce, when f is *real-analytic*, that f can be written as $f(u, v) = v^2 h(u, v)$. Since $D^2 f(0) \neq 0$ by hypothesis, one has $h(0) \neq 0$ and the local zero set of f reduces to the trivial branch. If f is not real-analytic, the result is false as is easily seen by taking the example of $f(u, v) = v^2 - v e^{-1/(u^2+v^2)}$. This observation will be generalized in §8.

5.7 An Analytic Proof of Krasnoselskii's Theorem in a Non-Classical Particular Case.

In this section, we consider again the problem of finding the local zero set of a mapping of the form

$$G(\mu, x) = (I - (\lambda_0 + \mu)L)x + \Gamma(\mu, x), \quad (7.1)$$

defined on a neighbourhood of the origin in the product $\mathbb{R} \times X$ and taking its values in the real Banach space X . As in Chapter 1, the operator $L \in \mathcal{L}(X)$ is supposed to be compact and the nonlinear operator Γ verifies

$$\Gamma(\mu, 0) = 0, \quad (7.2)$$

$$D_x \Gamma(\mu, 0) = 0, \quad (7.3)$$

for $|\mu|$ small enough. For the sake of simplicity, we shall assume that Γ is of class C^∞ , although this hypothesis of regularity can be weakened without affecting the final results.

Recall the notation introduced in Chapter 1

$$X_1 = \text{Ker}(I - \lambda_0 L), \quad (7.4)$$

$$Y_2 = \text{Range}(I - \lambda_0 L) \quad (7.5)$$

while X_2 and Y_1 are two topological complements of X_1 and Y_2 respectively. Denoting by Q_1 and Q_2 the projection operators onto the spaces Y_1 and Y_2 , the Lyapunov-Schmidt reduction leads to the reduced equation (cf. Chapter 1, §3) 208

$$f(\mu, x) = -\frac{\mu}{\lambda_0} Q_1 x - \frac{\mu}{\lambda_0} Q_1 \varphi(\mu, x) + Q_1 \Gamma(\mu, x + \varphi(\mu, x)) = 0, \quad (7.6)$$

for (μ, x) around the origin of $\mathbb{R} \times X_1$, where the mapping φ (here of class C^∞) is characterized by

$$-\frac{\mu}{\lambda_0} Q_2 x + (I - \lambda_0 L)\varphi(\mu, x) - \mu Q_2 L\varphi(\mu, x) + Q_2 \Gamma(\mu, x + \varphi(\mu, x)) = 0, \quad (7.7)$$

and verifies the conditions

$$\varphi(\mu, 0) = 0 \quad (7.8)$$

for $|\mu|$ small enough and

$$D\varphi(0) = 0. \quad (7.9)$$

When the algebraic and geometric multiplicities of λ_0 coincide, namely

$$X = X_1 \oplus Y_2 (= \text{Ker}(I - \lambda_0 L) \oplus \text{Range}(I - \lambda_0 L)), \quad (7.10)$$

the problem was studied in Chapter 1 if $\dim \text{Ker}(I - \lambda_0 L) = 1$ and in Chapter 3 if $\dim \text{Ker}(I - \lambda_0 L) = n \geq 2$. In what follows, we shall assume

$$\dim \text{Ker}(I - \lambda_0 L) = 1, \quad (7.11)$$

209 but drop the condition (7.10). This means that we consider the case

$$X_1 \subset Y_2 (\text{i.e. } \text{Ker}(I - \lambda_0 L) \subset \text{Range}(I - \lambda_0 L)). \quad (7.12)$$

Thus, for $x \in X_1$, $Q_1 x = 0$ and $Q_2 x = x$ so that the reduced mapping (7.6) becomes

$$f(\mu, x) = -\frac{\mu}{\lambda_0} Q_1 \varphi(\mu, x) + Q_1 \Gamma(\mu, x + \varphi(\mu, x)), \quad (7.13)$$

while the characterization (7.7) of the mapping φ can be rewritten as

$$-\frac{\mu}{\lambda_0} x + (I - \lambda_0 l) \varphi(\mu, x) - \mu Q_2 L \varphi(\mu, x) + Q_2 \Gamma(\mu, x + \varphi(\mu, x)) = 0. \quad (7.14)$$

The *generalized null-space* of the operator $(I - \lambda_0 L)$ will play a key role. Recall that it is defined as

$$\text{Ker}(I - \lambda_0 L)^\gamma,$$

where γ is the smallest positive integer such that ⁵

$$\text{Ker}(I - \lambda_0 L)^{\gamma'} = \text{Ker}(I - \lambda_0 L)^\gamma,$$

for every $\gamma' \geq \gamma$. Because of (7.12), one has $\gamma \geq 2$ and, by definition the algebraic multiplicity of λ_0 is the integer

$$\dim \text{Ker}(I - \lambda_0 L)^\gamma.$$

⁵The existence of γ is known from the spectral theory of compact operators as recalled in Chapter 1.

Remark 7.1. As $\dim \text{Ker}(I - \lambda_0 L) = 1$ by hypothesis, one has

$$\dim \text{Ker}(I - \lambda_0 L)^\gamma = \gamma. \quad (7.15)$$

Indeed, by a simple induction argument

$$\dim \text{Ker}(I - \lambda_0 L)^j \leq j, \quad (7.16)$$

for every $j \in \mathbb{N}$ and

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$$\dim \text{ker}(I - \lambda_0 L)^{j-1} < \dim \text{Ker}(I - \lambda_0 L)^j,$$

for $j \leq \gamma$, by definition of γ . Such a simple relation as (7.15) between γ and $\dim \text{Ker}(I - \lambda_0 L)^\gamma$ is no longer available when $\dim \text{Ker}(I - \lambda_0 L) \geq 2$.

Applying Krasnoselskii's theorem (Theorem 1.2) of Chapter 1 it follows that existence of nontrivial solutions of the equation $G(\mu, x) = 0$ arbitrarily close to the origin of $\mathbb{R} \times X$ is ensured when γ is *odd*. We shall complement this result in a particular case by showing that bifurcation occurs regardless of the parity of γ and obtain a precise description of the local zero set of G . Before that, we need to establish some preliminary properties of the reduced mapping f in (7.13). First, let e_0 denote a given eigenvector of L associated with the characteristic value λ_0 so that every $x \in X_1 = \text{Ker}(I - \lambda_0 L)$ can be written in the form $x = \epsilon e_0$, $\epsilon \in \mathbb{R}$. With an obvious abuse of notation, the reduced mapping f in (7.13) identifies with

$$f(\mu, \epsilon) = -\frac{\mu}{\lambda_0} Q_1 \varphi(\mu, \epsilon) + Q_1 \Gamma(\mu, \epsilon e_0 + \varphi(\mu, \epsilon)), \quad (7.17)$$

where $\varphi(\mu, \epsilon)$ is characterized by (cf. (7.14))

$$-\frac{\mu \epsilon}{\lambda_0} e_0 + (I - \lambda_0 L) \varphi(\mu, \epsilon) - \mu Q_2 L \varphi(\mu, \epsilon) + Q_2 \gamma(\mu, \epsilon e_0 + \varphi(\mu, \epsilon)) = 0 \quad (7.18)$$

Differentiating (7.17) with respect to ϵ , we find

$$\frac{\partial f}{\partial \epsilon}(\mu, \epsilon) = -\frac{\mu}{\lambda_0} Q_1 \frac{\partial \varphi}{\partial \epsilon}(\mu, \epsilon) + Q_1 D_x \Gamma(\mu, \epsilon e_0 + \varphi(\mu, \epsilon)) \cdot \left(e_0 + \frac{\partial \varphi}{\partial \epsilon}(\mu, \epsilon) \right). \quad (7.19)$$

For $\epsilon = 0$ and from (7.3) and (7.8), this expression is simply 211

$$\frac{\partial f}{\partial \epsilon}(\mu, 0) = -\frac{\mu}{\lambda_0} Q_1 \frac{\partial \varphi}{\partial \epsilon}(\mu, 0). \quad (7.20)$$

In particular

$$\frac{\partial f}{\partial \epsilon}(0) = 0. \quad (7.21)$$

Next, differentiating (7.19) with respect to ϵ at the origin, we get

$$\frac{\partial^2 f}{\partial \epsilon^2}(0) = Q_1 D_x^2 \Gamma(0) \cdot (e_0)^2. \quad (7.22)$$

Finally, differentiating (7.20) with respect to μ for any order j at the origin, it is easy to see that

$$\frac{\partial^{j+1} f}{\partial \mu^j \partial \epsilon}(0) = -\frac{j!}{\lambda_0} Q_1 \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0). \quad (7.23)$$

Theorem 7.1. *Assume that*

$$Q_1 D_x^2 \Gamma(0) \cdot (e_0)^2 \neq 0 \quad (7.24)$$

Then, the local zero set of the mapping G (7.1) is the union of the trivial branch and exactly one distinct C^∞ curve tangent to the trivial branch at the origin.

Proof. It suffices to prove an analogous result for the reduced mapping f . As $f(\mu, 0) = 0$, one has

$$\frac{\partial f}{\partial \mu}(0) = \frac{\partial^2 f}{\partial \mu^2}(0) = 0. \quad (7.25)$$

Meanwhile, for $j = 1$ in (7.23) and due to (7.9),

$$\frac{\partial^2 f}{\partial \mu \partial \epsilon}(0) = 0. \quad (7.26)$$

□

212 Thus, the relations (7.21)-(7.26) show that $Df(0) = 0$, $\det D^2f(0) = 0$ and $D^2f(0) \neq 0$. According to Corollary 6.1, it suffices to prove that there exists an index j (necessarily ≥ 2) such that $(\partial^{j+1}f/\partial\mu^j\partial\epsilon)(0) \neq 0$. From (7.23), this is equivalent to showing that

$$Q_1 \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0) \neq 0.$$

for some $j \geq 2$. To do this, we go back to the characterization of φ (7.18). Differentiating this identity with respect to ϵ and setting $\epsilon = 0$, it follows from (7.3) that

$$-\frac{\mu}{\lambda_0}e_0 + (I - \lambda_0L)\frac{\partial \varphi}{\partial \epsilon} - \mu Q_2L\left(\frac{\partial \varphi}{\partial \epsilon}\right)(\mu, 0) = 0.$$

Now, differentiating with respect to μ yields

$$-\frac{1}{\lambda_0}e_0 + (I - \lambda_0L)\frac{\partial^2 \varphi}{\partial \mu \partial \epsilon}(\mu, 0) - \mu Q_2L\frac{\partial^2 \varphi}{\partial \mu \partial \epsilon}(\mu, 0) - Q_2L\frac{\partial \varphi}{\partial \epsilon}(\mu, 0) = 0, \quad (7.27)$$

which, because of (7.9), provides

$$(I - \lambda_0L)\frac{\partial^2 \varphi}{\partial \mu \partial \epsilon}(0) = \frac{1}{\lambda_0}e_0. \quad (7.28)$$

Assume first that $\gamma = 2$. If $Q_1\frac{\partial^2 \varphi}{\partial \mu \partial \epsilon}(0) = 0$, one has $\frac{\partial^2 \varphi}{\partial \mu \partial \epsilon}(0) \in Y_2 = \text{Range}(I - \lambda_0L)$. Thus, there exists $\xi \in X$ such that

$$(I - \lambda_0L)\xi = \frac{\partial^2 \varphi}{\partial \mu \partial \epsilon}(0). \quad (7.29)$$

Hence, from (7.28)

$$(I - \lambda_0L)^2\xi = \frac{1}{\lambda_0}, \quad (7.30)$$

and consequently $(I - \lambda_0L)^3\xi = 0$. But, since $\gamma = 2$,

$$\text{Ker}(I - \lambda_0L)^3 = \text{Ker}(I - \lambda_0L)^2.$$

We deduce that $(I - \lambda_0 L)^2 \xi = 0$, which contradicts (7.30). Our as- 213
 ssertion is then proved when $\gamma = 2$. When $\gamma \geq 3$, we shall use the same
 method but we need some preliminary observations. First, by differen-
 tiating (7.27) at any order $j - 1$, $j \geq 2$, at the origin, we get

$$(I - \lambda_0 L) \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0) = (j - 1) Q_2 L \frac{\partial^{j-1} \varphi}{\partial \mu^{j-2} \partial \epsilon}(0). \quad (7.31)$$

If there is $3 \leq j \leq \gamma$ such that $Q_1 \frac{\partial^{j-1} \varphi}{\partial \mu^{j-2} \partial \epsilon}(0) \neq 0$, the problem is
 solved. Assume then

$$Q_1 \frac{\partial^{j-1} \varphi}{\partial \mu^{j-2} \partial \epsilon}(0) = 0, \quad 3 \leq j \leq \gamma,$$

or equivalently,

$$\frac{\partial^{j-1} \varphi}{\partial \mu^{j-2} \partial \epsilon}(0) \in Y_2 = \text{Range}(I - \lambda_0 L), \quad 3 \leq j \leq \gamma$$

Clearly, the space $\text{Range}(I - \lambda_0 L)$ is stable under L and, for the
 indices $3 \leq j \leq \gamma$, (7.31) reads

$$(I - \lambda_0 L) \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0) = (j - 1) L \frac{\partial^{j-1} \varphi}{\partial \mu^{j-2} \partial \epsilon}(0). \quad (7.32)$$

In particular, for $j = \gamma$ and applying $(I - \lambda_0 L)$ to both sides

$$(I - \lambda_0 L)^2 \frac{\partial^\gamma \varphi}{\partial \mu^{\gamma-1} \partial \epsilon}(0) = (\gamma - 1) L (I - \lambda_0 L) \frac{\partial^{\gamma-1} \varphi}{\partial \mu^{\gamma-2} \partial \epsilon}(0).$$

If $\gamma = 3$, it follows from (7.28) that

$$(I - \lambda_0 L)^2 \frac{\partial^3 \varphi}{\partial \mu^2 \partial \epsilon}(0) = \frac{2}{\lambda_0^2} e_0,$$

214 while, if $\gamma \geq 4$, (7.32) provides

$$(I - \lambda_0 L)^2 \frac{\partial^\gamma \varphi}{\partial \mu^{\gamma-1} \partial \epsilon}(0) = (\gamma - 1)(\gamma - 2) L^2 \frac{\partial^{\gamma-2} \varphi}{\partial \mu^{\gamma-3} \partial \epsilon}(0).$$

Iterating the process, it is clear in any case that

$$(I - \lambda_0 L)^{\gamma-1} \frac{\partial^\gamma \varphi}{\partial \mu^{\gamma-1} \partial \epsilon}(0) = \frac{(\gamma-1)!}{\lambda_0^{\gamma-1}} e_0. \quad (7.33)$$

As $Q_1 \frac{\partial^\gamma \varphi}{\partial \mu^{\gamma-1} \partial \epsilon}(0) = 0$, there is $\xi \in X$ such that

$$(I - \lambda_0 L)\xi = \frac{\partial^\gamma \varphi}{\partial \mu^{\gamma-1} \partial \epsilon}(0).$$

From (7.33)

$$(I - \lambda_0 L)^\gamma \xi = \frac{(\gamma-1)!}{\lambda_0^{\gamma-1}} e_0. \quad (7.34)$$

Hence, $(I - \lambda_0 L)^{\gamma+1} \xi = 0$. But, by the definition of γ , one has $\text{Ker}(I - \lambda_0 L)^{\gamma+1} = \text{Ker}(I - \lambda_0 L)^\gamma$. It follows that $(I - \lambda_0 L)^\gamma \xi = 0$, which contradicts (7.34).

Remark 7.2. The theorem can be complemented by showing that

$$Q_1 \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0) = 0, 2 \leq j \leq \gamma - 1.$$

Indeed, let us denote by $p(\geq 2)$ the first integer such that

$$Q_1(\partial^p \varphi / \partial \mu^{p-1} \partial \epsilon)(0) \neq 0.$$

From the proof of Theorem 7.1, we have $p \leq \gamma$. With the arguments we used for proving (7.33), we find

$$(I - \lambda_0 L)^{j-1} \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0) = \frac{(j-1)!}{\lambda_0^{j-1}} e_0, 2 \leq j \leq p.$$

Setting

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$$e_{j-1} = \frac{\partial^j \varphi}{\partial \mu^{j-1} \partial \epsilon}(0), 2 \leq j \leq p, \quad (7.35)$$

this relation yields

$$(I - \lambda_0 L)^j e_j = \frac{j!}{\lambda_0^j} e_0, 1 \leq j \leq p - 1.$$

Clearly, for $1 \leq j \leq p-1$, the vectors e_0, \dots, e_j belong to the space $\text{Ker}(I - \lambda_0 L)^{j+1}$ and they are linearly independent: indeed, assuming

$$\sum_{\ell=0}^j \alpha_\ell e_\ell = 0,$$

and applying $(I - \lambda_0 L)^j, \dots, (I - \lambda_0 L)$ successively, it follows that $\alpha_j = \dots = \alpha_0 = 0$. Owing to (7.16), we deduce that the vectors e_0, \dots, e_j form a *basis* of $\text{Ker}(I - \lambda_0 L)^{j+1}$ for $1 \leq j \leq p-1$. As a result,

$$\text{Ker}(I - \lambda_0 L)^{p+1} = \text{Ker}(I - \lambda_0 L)^p. \quad (7.36)$$

To see this, consider an element $\xi \in \text{Ker}(I - \lambda_0 L)^{p+1}$. Then, $(I - \lambda_0 L)\xi \in \text{Ker}(I - \lambda_0 L)^p$, namely

$$(I - \lambda_0 L)\xi = \sum_{\ell=0}^{p-1} \alpha_\ell e_\ell.$$

By definition of p , one has $e_{p-1} \notin Y_2 = \text{Range}(I - \lambda_0 L)$ (cf. (7.35)) while $e_\ell \in \text{Range}(I - \lambda_0 L)$, $0 \leq \ell \leq p-1$. As $(I - \lambda_0 L)\xi \in \text{Range}(I - \lambda_0 L)$, the coefficient α_{p-1} must vanish. But, if so,

$$(I - \lambda_0 L)\xi = \sum_{\ell=0}^{p-2} \alpha_\ell e_\ell \in \text{Ker}(I - \lambda_0 L)^{p-1}.$$

216 Hence, $\xi \in \text{Ker}(I - \lambda_0 L)^p$ and (7.36) follows. We conclude that $p \geq \gamma$ and therefore $p = \gamma$.

This conclusion is important because, according to our discussion of § 6, it means that the process of desingularization of the reduced mapping f ends at the step $\gamma - 1$ exactly (i.e. the iterates $f^{(1)}, \dots, f^{(\gamma-1)}$ are defined and the local zero set of $f^{(\gamma-1)}$ is found through the Morse lemma).

Remark 7.3. Theorem 7.1 holds as soon as we can apply the Morse lemma to the iterate $f^{(\gamma-1)}$. According to the results of Chapter 2, this requires $f^{(\gamma-1)} \in C^2$, which is the case if $f \in C^{2\gamma}$ and hence if $\Gamma \in C^{2\gamma}$. Also, the condition (7.3) can be weakened and replaced by

$$D_\mu^j D_x \Gamma(0) = 0, 1 \leq j \leq \gamma.$$

Remark 7.4. As mentioned in §6, since the desingularization process stops at rank $\gamma - 1$ exactly (cf. Remark 7.2), the two curves in the local zero set of f (so those in the zero set of G) have a contact of order $\gamma - 1$ exactly at the origin. Following the parity of γ and apart from the symmetry with respect to the μ -axis, the local zero set of f is given by Figure 7.1 below. This can be easily confirmed by using the “quadrant method” described in § 4.

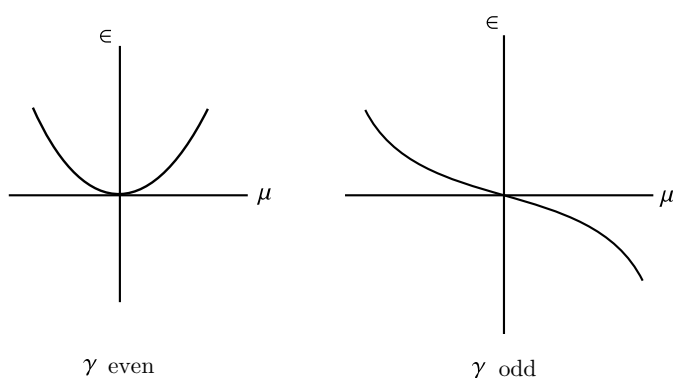


Figure 7.1: Structure of the local zero set of f (7.17).

We shall complete this section with an example: Let us take $X = \mathbb{R}^2$ 217
 and $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so that $\lambda_0 = 1$ and $X_1 = Y_2 = \mathbb{R}(1, 0)$ and we can choose
 $e_0 = (1, 0)$ and $X_2 = Y_1 = \mathbb{R}(0, 1)$. With

$$\Gamma(\mu, x) = \Gamma(x) = \Gamma(\epsilon, w) = (w^2, \epsilon^2),$$

it is immediate that the assumptions (7.2), (7.3) and (7.24) are satisfied and $\gamma = 2$. The equation $G(\mu, x) = G(\mu, \epsilon, w) = 0$ is

$$\begin{aligned} -\mu\epsilon - (1 + \mu)w + w^2 &= 0, \\ -\mu w + \epsilon^2 &= 0. \end{aligned} \tag{7.37}$$

The first equation yields

$$\begin{aligned}
 w = \varphi(\mu, \epsilon) &= \frac{(1 + \mu) - \sqrt{((1 + \mu)^2 + 4\mu\epsilon)}}{2} \\
 &= -\frac{2\mu\epsilon}{(1 + \mu) + \sqrt{((1 + \mu)^2 + 4\mu\epsilon)}},
 \end{aligned}$$

so that

$$\varphi(\mu, \epsilon) \sim -\mu\epsilon,$$

around the origin. By referring to the second equation (7.37), we find the reduced mapping $f(\mu, \epsilon) \sim -\mu^2\epsilon + \epsilon^2$. The solutions of the equation $-\mu^2\epsilon + \epsilon^2 = 0$ are $(\mu, 0)$ and (μ, μ^2) and the local zero set of f is as in Figure 7.1 with γ even. Note that the “natural” choice

$$\Gamma(\epsilon, v) = (\epsilon^2, v^2)$$

cannot be treated here because the condition (7.24) fails (and hence $D^2f(0) = 0$). Actually, the associated reduced equation is

$$-\mu\varphi(\mu, \epsilon) + \varphi^2(\mu, \epsilon) = 0, \quad (7.38)$$

with

$$\varphi(\mu, \epsilon) = \frac{\epsilon(\epsilon - \mu)}{1 + \mu}$$

It is easily checked that the local zero set of the reduced mapping is made up of the *three transversal curves* $(\mu, 0)$, (μ, μ) and $(2\epsilon / [1 + \epsilon + \sqrt{((1 + \epsilon)^2 + 4\epsilon^2)}], \epsilon) \sim (\epsilon^2, \epsilon)$. Observe in this case that Theorem 4.1 of Chapter 2 applies to the reduced mapping (7.38) because its first nonzero derivative at the origin is of order 3 and the associated polynomial mapping verifies the condition ($\mathbb{R} - N.D.$). This is not in contradiction with the analysis of Chapter 3, because the condition (1.7) (as well as its generalized form (1.26)) of Chapter 3 *fails*.

5.8 The Case of an Infinite Process.

To be complete, we analyze now the case when the process of §5 is endless. We begin with the following definition

Definition 8.1. We shall say that the mapping f is m -degenerate if the process of §5 stops after m steps exactly. When f is m -degenerate for some m , f will be referred to as finitely degenerate. Finally, if the process of § 5 is endless, we shall say that f is indefinitely degenerate (in short ∞ -degenerate).

Remark 8.1. This definition assumes the intrinsic character of the process of §5, proved in [32], that we shall then admit, for Definition 8.1 to make sense.

With this vocabulary, the study we made in §5 was devoted to the determination of the local zero set of a finitely degenerate mapping. When f is ∞ -degenerate, the structure of its local zero set can still be determined in a general framework. To see this, we need

Proposition 8.1. Assume that the mapping f is of the form

$$f(\bar{x}) = (\rho(\bar{x}))^2 h(\bar{x}), \quad (8.1)$$

where ρ and h are two C^∞ real-valued mappings and $\rho(0) = 0$. Then, f verifies the condition $Df(0) = 0$, $\det D^2 f(0) = 0$, $D^2 f(0) \neq 0$ if and only if $h(0) \neq 0$ and $D\rho(0) \neq 0$. If so, f is ∞ -degenerate. 220

Proof. By differentiating (8.1) and since $\rho(0) = 0$, we find $Df(0) = 0$ and $D^2 f(0) \cdot (\tilde{\xi}, \tilde{\zeta}) = 2h(0)(D\rho(0) \cdot \tilde{\xi})(D\rho(0) \cdot \tilde{\zeta})$. Hence, $D^2 f(0) \neq 0$ if and only if $h(0) \neq 0$ and $D\rho(0) \neq 0$. If so, the characteristic Ξ exists with

$$\Xi = \text{Ker } D\rho(0) \quad (8.2)$$

Let T be a complement of Ξ in \mathbb{R}^2 and take two nonzero elements $\tilde{\xi}_0 \in \Xi$ and $\tilde{\tau}_0 \in T$: The mapping $f^{(1)}$ is defined by

$$f^{(1)}(u_1, v_1) = \begin{cases} \frac{1}{u_1^2} (\rho(u - 1\tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0))^2 h(u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0) & \text{for } u_1 \neq 0 \\ h(0)(D\rho(0) \cdot \tilde{\tau}_0)^2 v_1^2 & \text{for } u_1 = 0. \end{cases}$$

Let us introduce the two C^∞ real - valued mappings

$$\rho^{(1)}(u_1, v_1) = \int_0^1 D\rho(su_1 \tilde{\xi}_0 + su_1 v_1 \tilde{\tau}_0) \cdot (\tilde{\xi}_0 + v_1 \tilde{\tau}_0) ds,$$

$$h^{(1)}(u_1, v_1) = h(u_1 \tilde{\xi}_0 + u_1 v_1 \tilde{\tau}_0).$$

One has $\rho^{(1)}(0) = 0, h^{(1)}(0) = 0$ and the derivative $D\rho^{(1)}(0)$ is the nonzero linear mapping

$$(u_1, v_1) \in \mathbb{R}^2 \rightarrow \frac{1}{2}(D^2\rho(0) \cdot (\tilde{\xi}_0)^2)u_1 + (D\rho(0) \cdot \tilde{\tau}_0)v_1.$$

From the relation $\rho(\tilde{x}) = \int_0^1 D\rho(s\tilde{x}) \cdot \tilde{x} ds$, it is immediate that $f^{(1)} = \rho^{(1)}h^{(1)}$ with $\rho^{(1)}$ and $h^{(1)}$ verifying the same hypotheses as ρ and h . Hence the existence of a second iterate $f^{(2)}$ and, more generally, of $f^{(m)}$, $m \geq 1$, so that f is ∞ -degenerate. \square

The above result motivates the following definition.

221 Definition 8.2. We shall say that the ∞ -degenerate mapping f is reducible if we can write

$$f(\tilde{x}) = (\rho(\tilde{x}))^2 h(\tilde{x}) \quad (8.3)$$

where ρ and h are two C^∞ real-valued mappings such that $\rho(0) = 0, D\rho(0) \neq 0$ and $h(0) \neq 0$.

When f is a reducible ∞ -degenerate mapping, its local zero set is that of the mapping ρ . As $D\rho(0) \neq 0$, the Implicit function theorem shows that it consists of exactly one C^∞ curve, tangent to the characteristic Ξ at the origin (cf. (8.2)). In any system (u, v) of coordinates such that Ξ does not coincide with the v -axis (i.e. in which Ξ is not vertical), this curve is the graph of a C^∞ function $v = v(u)$. In such a system of coordinates, we can always take $\rho(u, v) = v - v(u)$: Indeed, from the identity $\rho(u, v(u)) = 0$ we get

$$\rho(u, v) = \left(\int_0^1 (\partial\rho/\partial v)(u, v(u) + s(v - v(u))) ds \right) (v - v(u))$$

and it suffices to replace h by the mapping

$$\left[\int_0^1 (\partial\rho/\partial v)(u, v(u) + s(v - v(u))) ds \right]^2 h(u, v),$$

(which does not vanish at the origin).

We shall see how the mapping v is related to the characteristics $\Xi, \Xi_1, \dots, \Xi_m, \dots$. With this aim, there is a choice of elements $\tilde{\xi}_j$ and $\tilde{\tau}_j, j \geq 0$, that is especially convenient for defining the iterates $f^{(j)}$: Let γ_0 be the slope of the characteristic Ξ in the system (u, v) (recall that γ_0 is well defined as a real number since Ξ is not vertical) and set $\tilde{\xi}_0 = (1, \gamma_0)$. Taking T as the v -axis, we choose $\tilde{\tau}_0 = (0, 1)$. The first iterate $f^{(1)}$ is entirely determined from $\tilde{\xi}_0$ and $\tilde{\tau}_0$ and its characteristic Ξ_1 is not vertical in the (u_1, v_1) -plane. Denoting by γ_1 the slope of Ξ_1 , we set $\tilde{\xi}_1 = (1, \gamma_1)$. Taking T_1 as the v_1 -axis, we choose $\tilde{\tau}_1 = (0, 1)$ which determines the second iterate $f^{(2)}$. In general, the choice of the $\tilde{\xi}_j$'s and $\tilde{\tau}_j$'s is then as follows

$$\tilde{\xi}_j = (1, \gamma_j), j \geq 0, \quad (8.4)$$

where γ_j denotes the slope of the characteristic Ξ_j in the (u_j, v_j) plane and

$$\tilde{\tau}_j = (0, 1), j \geq 0. \quad (8.5)$$

Remark 8.2. Note for every $j \geq 1$ that the characteristic Ξ_j depends on $\tilde{\xi}_\ell$ and $\tilde{\tau}_\ell, 0 \leq \ell \leq j - 1$. Thus, by modifying one of the $\tilde{\xi}_j$'s (or $\tilde{\tau}_j$'s) we affect the definition of all the characteristics of order $> j$. This is to say that our further results hold with the choice (8.4)-(8.5) and *with this choice only*. If any other choice of elements $\tilde{\xi}_j$ and $\tilde{\tau}_j$ is made, the relationship to the characteristics $\Xi, \Xi_1, \dots, \Xi_m, \dots$ is *different*.

Before the first important theorem of this section, we need to establish

Lemma 8.1. *Let $m \geq 1$ be a given integer. We set*

$$\hat{f}(u, v) = f \left(u, v + \sum_{j=0}^{m-1} \gamma_j u^{j+1} \right). \quad (8.6)$$

Then, the characteristics $\hat{\Xi}, \hat{\Xi}_1, \dots, \hat{\Xi}_{m-1}$ of $\hat{f}, \hat{f}^{(1)}, \dots, \hat{f}^{(m-1)}$ exist and are horizontal.

Proof. It is clear that $D\hat{f}(0) = 0$. Now, we have

$$\begin{aligned}\frac{\partial^2 \hat{f}}{\partial u^2}(0) &= \frac{\partial^2 f}{\partial u^2}(0) + 2\frac{\partial^2 f}{\partial u \partial v}(0)\gamma_0 + \frac{\partial^2 f}{\partial v^2}(0)\gamma_0^2 = \\ &= D^2 f(0) \cdot (1, \gamma_0)^2 = D^2 f(0) \cdot (\tilde{\xi}_0)^2 = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \hat{f}}{\partial u \partial v}(0) &= \frac{\partial^2 f}{\partial u \partial v}(0) + \frac{\partial^2 f}{\partial v^2}(0)\gamma_0 \\ &= D^2 f(0) \cdot ((1, \gamma_0), (0, 1)) \\ &= D^2 f(0) \cdot (\tilde{\xi}_0, \tilde{\tau}_0) = 0\end{aligned}$$

Finally, since

$$\frac{\partial^2 \hat{f}}{\partial v^2}(0) = \frac{\partial^2 f}{\partial v^2}(0) = D^2 f(0) \cdot (0, 1)^2 = D^2 f(0) \cdot (\tilde{\tau}_0)^2 \neq 0, \quad (8.7)$$

the above relations show that the characteristic $\hat{\Xi}$ of \hat{f} exists and is horizontal.

Taking $\tilde{\xi}_0 = (1, 0)$ and $\tilde{\tau}_0 = (0, 1)$, the first iterate $\hat{f}^{(1)}$ is defined by

$$\hat{f}^{(1)}(u_1, v_1) = \begin{cases} \frac{1}{u_1^2} \hat{f}(u_1, u_1 v_1) & \text{if } u_1 \neq 0, \\ \frac{1}{2} \frac{\partial^2 \hat{f}}{\partial v^2}(0) v_1^2 & \text{if } u_1 = 0. \end{cases} \quad (8.8)$$

By definition, one has

$$\hat{f}^{(1)}(u_1, v_1) = \begin{cases} \frac{1}{u_1^2} f(u_1, \gamma_0 u_1 + u_1 v_1) & \text{if } u_1 \neq 0, \\ \frac{1}{2} \frac{\partial^2 f}{\partial v^2}(0) v_1^2 & \text{if } u_1 = 0. \end{cases} \quad (8.9)$$

224 Thus, from (8.7)-(8.9) we deduce

$$\hat{f}^{(1)}(u_1, v_1) = f^{(1)}\left(u_1, \sum_{j=1}^{m-1} \gamma_j u_1^j\right). \quad (8.10)$$

Setting

$$\gamma_j^{(1)} = \gamma_{j+1}, 0 \leq j \leq m-2,$$

we see that $\gamma_j^{(1)}, 0 \leq j \leq m-2$ play with $f^{(1)}$ the same role as $\gamma_j, 0 \leq j \leq m-1$, play with f . Therefore, it follows from (8.10) that the iterate $\hat{f}^{(1)}$ is defined from $f^{(1)}$ as \hat{f} is defined from f . The same observation can be repeated $m-1$ times and our assertion follows. \square

Theorem 8.1. *Let f be a reducible ∞ -degenerate mapping. The iterate $f^{(1)}, \dots, f^{(m)}, \dots$ being defined from $\tilde{\xi}_j$ and $\tilde{\tau}_j, j \geq 0$ as in (8.4)-(8.5), the function $v(u)$ whose graph is the local zero set of f verifies*

$$\frac{d^j v}{du^j}(0) = j! \gamma_{j-1} \text{ for every } j \geq 1. \quad (8.11)$$

Conversely, if f is ∞ -degenerate and real-analytic and the disk of convergence of the power series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ is not $\{0\}$, the mapping f is reducible and its local zero set coincides with the graph of the analytic function

$$v(u) = \sum_{j=0}^{\infty} \gamma_j u^{j+1}. \quad (8.12)$$

Proof. Let $m \geq 1$ be a given integer. Set

$$\hat{v}(u) = v(u) - \sum_{j=0}^{m-1} \gamma_j u^{j+1}. \quad (8.13)$$

With the characterization $f(u, v(u)) = 0$, we find

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$$\hat{f}(u, \hat{v}(u)) = 0, \quad (8.14)$$

where \hat{f} is the mapping (8.11). From Lemma 8.1, the characteristics $\hat{\xi}, \hat{\xi}_1, \hat{\xi}_{m-1}, \dots$ are horizontal. Thus, using Theorem 6.1,

$$\begin{aligned} (\partial^j \hat{f} / \partial u^j)(0) &= 0, 0 \leq j \leq 2m, \\ (\partial^{j+1} \hat{f} / \partial u^j \partial v)(0) &= 0, 0 \leq j \leq m. \end{aligned}$$

By implicit differentiation in (8.14), it easily follows that $(d^j \hat{v} / du^j)(0) = 0, 0 \leq j \leq m$. By definition of \hat{v} (cf. (8.13)), this is equivalent to

$$\frac{d^j v}{du^j}(0) = j! \gamma_{j-1}, 0 \leq j \leq m.$$

But this relation holds for any $m \geq 1$, which proves (8.11).

Now, assume that f is ∞ -degenerate and real-analytic and the disk of convergence of the power series $\sum_{j=0}^{\infty} j^{u^{j+1}}$ is not $\{0\}$. The mapping

$$\hat{f}(u, v) = f \left(u, v + \sum_{j=0}^{\infty} \gamma_j u^{j+1} \right), \quad (8.15)$$

is real-analytic and, by the same arguments as in Lemma 8.1, we see that \hat{f} is ∞ -degenerate and all the characteristics $\hat{\Xi}, \hat{\Xi}_1, \dots, \hat{\Xi}_m, \dots$ are horizontal. From Theorem 6.1 again,

$$(\partial^j \hat{f} / \partial u^j)(0) = 0 \text{ for every } j \geq 0,$$

$$(\partial^{j+1} \hat{f} / \partial u^j \partial v)(0) \text{ for every } j \geq 0.$$

As \hat{f} is real-analytic, we deduce that

$$\hat{f}(u, v) = v^2 \hat{h}(u, v),$$

226 where \hat{h} is real-analytic and verifies $\hat{h}(0) \neq 0$ (for otherwise $D^2 \hat{f}(0)$ would vanish). According to (8.15), we conclude

$$f(u, v) = \left(v - \sum_{j=0}^{\infty} \gamma_j u^{j+1} \right)^2 \hat{h} \left(u, v - \sum_{j=0}^{\infty} \gamma_j u^{j+1} \right),$$

a relation from which it is obvious that the local zero set of f is the graph of the mapping v (8.12). \square

Remark 8.3. When f is ∞ -degenerate but not real-analytic, it may happen that the disk of convergence of the power series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ is not $\{0\}$

but the graph of the function $v(u) = \sum_{j=0}^{\infty} \gamma_j u^{j+1}$ is *not* in the local zero set of f . This is the case with

$$f(u, v) = v^2 + e^{-1/u^2},$$

whose characteristics $\Xi, \Xi_1, \dots, \Xi_1, \dots$ are horizontal (i.e. $\gamma_j = 0$ for $j \geq 0$) but whose local zero set reduces to the origin. Of course, such a mapping is not reducible.

From now on, we assume that f is real-analytic and ∞ -degenerate. Let (u, v) be a system of coordinates in which the characteristic Ξ is not vertical. The next result (Theorem 8.2) will show that the assumption that the disk of convergence of the series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ does not reduce to $\{0\}$ in Theorem 8.1 is *vacuous*.

For (u, v) around the origin, we can write

$$f(u, v) = \sum_{k, \ell \geq 0} a_{k\ell} u^k v^\ell, \quad (8.16)$$

where $a_{00} = a_{10} = a_{01} = 0$, $a_{02}a_{20} - a_{11}^2 = 0$. Similarly, for every $m \geq 1$, the iterate $f^{(m)}$ is real-analytic (which can be easily checked) and we shall set 227

$$f^{(m)}(u_m, v_m) = \sum_{k, \ell \geq 0} a_{k\ell}^{(m)} u_m^k v_m^\ell, \quad (8.17)$$

where $a_{00}^{(m)} = a_{10}^{(m)} = a_{01}^{(m)} = 0$, $a_{02}^{(m)}a_{20}^{(m)} - (a_{11}^{(m)})^2 = 0$. As Ξ is not vertical and none of the characteristics $\Xi_1, \dots, \Xi_m, \dots$ is vertical either, one has $a_{02}^{(m)} \neq 0$. Hence, for $m \geq 0$

$$\gamma_m = -\frac{a_{11}^{(m)}}{2a_{02}^{(m)}}.$$

Now, due to the choice of $\tilde{\xi}_{m-1}$ and $\tilde{\tau}_{m-1}$ (8.4)-(8.5)

$$f^{(m)}(u_m, v_m) = \begin{cases} \frac{1}{u_m^2} f^{(m-1)}(u_m, \gamma_{m-1} u_m + u_m v_m) & \text{for } u_m \neq 0, \\ \frac{1}{2} v_m^2 (\partial^2 f^{(m-1)} / \partial v_{m-1}^2)(0) = a_{02}^{m-1} v_m^2 & \text{for } u_m = 0. \end{cases} \quad (8.18)$$

In particular,

$$a_{02}^{(m)} = \frac{1}{2} \frac{\partial^2 f^{(m)}}{\partial v_m^2}(0) = \frac{1}{2} \frac{\partial^2 f^{(m-1)}}{\partial v_{m-1}^2}(0) = a_{02}^{(m-1)}.$$

Thus

$$a_{02}^{(m)} = a_{02} \text{ for every } m \geq 0,$$

so that

$$\gamma_m = -\frac{a_{11}^{(m)}}{2a_{02}} \text{ for every } m \geq 0. \quad (8.19)$$

228 It is easy to find a relationship between the coefficients $a_{k\ell}^{(m)}$, $a_{k\ell}^{(m-1)}$ and γ_{m-1} . To do this, we can formula (2.12) or identify the coefficients in (8.18). We get

$$a_{k\ell}^{(m)} = \begin{cases} \sum_{j=\ell}^{k+2} \binom{j}{\ell} a_{k+2-j,j}^{(m-1)} \gamma_{m-1}^{j-1} & \text{for } 0 \leq \ell \leq k+2 \\ 0 & \text{for } \ell > k+2. \end{cases} \quad (8.20)$$

Lemma 8.2. (Weierstrass preparation theorem for functions of two variables): Let $f(u, v)$ be an analytic function of the two complex variables (u, v) with values in \mathbb{C} . Assume that there is an integer $\ell \geq 1$ such that

$$D^j f(0) = 0, 0 \leq j \leq \ell - 1,$$

and

$$\frac{\partial^\ell f}{\partial v^\ell}(0) \neq 0.$$

Then, there exist analytic functions $\theta_0(u), \dots, \theta_{\ell-1}(u)$ and $h(u, v)$ verifying

$$\begin{aligned} D^i \theta_j(0) &= 0, 0 \leq i \leq \ell - j - 1, \\ h(0) &\neq 0, \end{aligned}$$

such that

$$f(u, v) = (v^\ell + \sum_{j=0}^{\ell-1} \theta_j(u) v^j) h(u, v).$$

Moreover, the functions $\theta_0, \dots, \theta_{\ell-1}$ and h are unique.

Proof. See e.g. Bers [3], Golubitsky and Guillemin [12]. □

Theorem 8.2. Every real-analytic mapping f such that $f(0) = 0$, $Df(0) = 0$, $D^2 f(0) \neq 0$ which is ∞ -degenerate is reducible and its local zero set is made up of exactly one analytic curve.

229 *Proof.* From Theorem 8.1, it suffices to show that the disk of convergence of the power series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ does not reduce to $\{0\}$. Extending f to the complex values of u and v and by Lemma 8.2 with $\ell = 2$ (recall that $(\partial^2 f / \partial v^2)(0) \neq 0$ from the choice of the system (u, v) of coordinates), we find

$$f = ph, \tag{8.21}$$

where

$$p(u, v) = v^2 + \theta_1(u)v + \theta_0(u),$$

is the Weierstrass polynomial of f . By the uniqueness of the decomposition (8.21) and from $f = \bar{f} = \bar{p}\bar{h}$, we deduce $\theta_1 = \bar{\theta}_1$, $\theta_0 = \bar{\theta}_0$ and $h = \bar{h}$ so that the power series expansions of the functions θ_0, θ_1 and h at the origin have real coefficients. They are then real-analytic functions of the variables u and v .

As $p(0) = 0$ and $h(0) \neq 0$, the assumption $Df(0) = 0$ is equivalent to $Dp(0) = 0$. Hence, $D^2 f(0) = h(0)D^2 p(0)$ so that $D^2 p(0) \neq 0$ and $\det D^2 p(0) = 0$ since the same relation holds with f . In addition, the characteristic Ξ of f , and then its slope γ_0 , is none other than that of p . Denoting by $p^{(1)}$ the first iterate of p defined through the choice $\xi_0 = (1, \gamma_0)$ and $\bar{\tau}_0 = (0, 1)$ and setting

$$h^{(1)}(u_1, v_1) = h(u_1, \gamma_0 u_1 + u_1 v_1),$$

one has $f^{(1)} = p^{(1)}h^{(1)}$, with $p^{(1)}$ and $h^{(1)}$ real-analytic and $h^{(1)}(0) \neq 0$. By the same arguments as above, $p^{(1)}$ has a characteristic which is none other than the characteristic Ξ_1 of $f^{(1)}$ so that the slope γ_1 can be determined through $p^{(1)}$ as well as through $f^{(1)}$ and so on. To sum up, the Weierstrass polynomial p of f inherits the property of ∞ -degeneracy and the numbers $\gamma_j, j \geq 0$, are nothing but the slopes of the characteristics of $p, p^{(1)}, \dots$ so that all amounts to proving that the disk of convergence of the power series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ does not reduce to $\{0\}$ when $f(u, v) = p(u, v) = v^2 + \theta_1(u)v + \theta_0(u)$. If so, and with the notation (8.16) we have

$$\theta_1(u) = \sum_{k=1}^{\infty} a_{k1} u^k, \tag{8.22}$$

$$\theta_0(u) = \sum_{k=2}^{\infty} a_{k0} u^k, \quad (8.23)$$

with $a_{02} = 1$ and all the other coefficients $a_{k\ell} = 0$. In particular, for $\ell \geq 3$ and an induction based on formula (8.20) shows that this remains true at any rank $m \geq 0$:

$$a_{k\ell}^{(m)} = 0 \text{ for every } k \geq 0, \text{ every } \ell \geq 3 \text{ and every } m \geq 0. \quad (8.24)$$

With this observation, (8.20) yields in particular

$$a_{k2}^{(m)} = a_{k2}^{(m-1)} = a_{k2} \text{ for every } k \geq 0 \text{ and every } m \geq 1.$$

But $a_{k2} = 0$ for $k \geq 1$ and hence

$$a_{k2}^{(m)} = a_{k2} = 0 \text{ for every } k \geq 1 \text{ and every } m \geq 0. \quad (8.25)$$

Finally, with (8.20) again and (8.24), we find

$$a_{k1}^{(m)} = a_{k+1,1}^{(m-1)} + 2a_{k2}^{(m-1)} \gamma_{m-1},$$

231 for $k \geq 0$ and $m \geq 1$. Owing to (8.25), this is only

$$a_{k1}^{(m)} = a_{k+1,1}^{(m-1)},$$

for $k \geq 1$. Therefore

$$a_{k1}^{(m)} = a_{k+m,1} \text{ for every } k \geq 1 \text{ and every } m \geq 0. \quad (8.26)$$

(8.19) can be rewritten in the simple form

$$\gamma_m = -\frac{1}{2} a_{m+1,1} \text{ for every } m \geq 0.$$

As a result

$$\sum_{j=0}^{\infty} \gamma_j u^{j+1} = -\frac{1}{2} \theta_1(u), \quad (8.27)$$

which proves that the disk of convergence of the power series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ does not reduce to $\{0\}$, □

Remark 8.4. The reader can check that proving that

$$\overline{\lim}_{m \rightarrow \infty} |\gamma_m|^{1/(m+1)} < +\infty$$

without using the Weirstrass preparation theorem and trying to express γ_m in terms of the coefficients $(a_{k\ell})$ through successive applications of formula (8.20) is inextricable.

Remark 8.5. According to (8.27) and Theorem 8.1, one must also have

$$\theta_0(u) = \left(\sum_{j=0}^{\infty} \gamma_j u^{j+1} \right)^2,$$

which a direct calculation confirms.

Remark 8.6. Even when f is not real-analytic, it may happen that the disk of convergence of the power series $\sum_{j=0}^{\infty} \gamma_j u^{j+1}$ is not $\{0\}$ and the

graph of the function $v(u) = \sum_{j=0}^{\infty} \gamma_j u^{j+1}$ is in the local zero set of f . For instance, with

$$f(u, v) = v^2 + ve^{-1/u^2},$$

one has $\gamma_j = 0$ for every $j \geq 0$ and $v = 0$ is in the local zero set of f . However, this is not always the case as already observed in Remark 8.3

5.9 Concluding Remarks on Further Possible Developments.

Formally, the method we have described can be generalized to C^∞ mappings from \mathbb{R}^{n+1} into \mathbb{R}^n (again, the regularity of f can be weakened without really affecting most of the results) for which there is an integer $k \geq 2$ such that

$$D^j f(0) = 0, 0 \leq j \leq k - 1, \tag{9.1}$$

and the zero set of the mapping

$$\widetilde{\xi} \in \mathbb{R}^{n+1} \rightarrow D^k f(0) \cdot (\widetilde{\xi})^k, \tag{9.2}$$

consists of a finite number of lines through the origin. In particular, when $n = 1$, this assumption is fulfilled as soon as $D^k f(0) \neq 0$ (cf. Chapter 2, §2). We can proceed as follows. Let $\tilde{\xi}_0 \in \mathbb{R}^{n+1}$ be a non-zero vector generating one of the lines in the zero set of the mapping (9.2). As the analysis we made in Chapter 2 deals with each direction *separately*, we deduce the existence of one and only curve in the local zero set of f which is tangent to the line $\mathbb{R}\tilde{\xi}_0$ at the origin if the mapping

$$D^k f(0) \cdot ((\tilde{\xi})^{k-1}, \cdot) \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n)$$

233 is *onto*. If it is not, a first iterate (depending on $\tilde{\xi}_0$) is defined by setting

$$f^{(1)}(u_1, \tilde{v}_1) = \begin{cases} \frac{1}{u_1^k} f(u_1 \tilde{\xi}_0 + u_1 \tilde{v}_1) & \text{if } u_1 \neq 0, \\ \frac{1}{k!} D^k f(0) \cdot (\tilde{\xi}_0 + \tilde{v}_1)^k = \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} D^j f(0) \cdot (\tilde{\xi}_0)^{k-j} (\tilde{v}_1)^j, & \text{if } u_1 = 0, \end{cases} \quad (9.3)$$

where $u_1 \in \mathbb{R}$ and \tilde{v}_1 belongs to some given n -dimensional complement T of $\mathbb{R}\tilde{\xi}_0$ (in the case treated in this chapter, T is one-dimensional and $\tilde{v}_1 = v_1 \tilde{\tau}_0$ with $v_1 \in \mathbb{R}$ and $\tilde{\tau}_0$ is a fixed nonzero element of T)

The mapping $f^{(1)}$ (9.3) is easily seen to be class C^∞ . If $Df^{(1)}(0) \in \mathcal{L}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is *onto*, the local zero set of $f^{(1)}$ is found through the Implicit function theorem and reduces to one curve: The part of the local zero set of f which is “along the space $\mathbb{R}\tilde{\xi}_0$ ” (we hope this expression self-explanatory) is obtained by

$$\tilde{x} = u_1 \tilde{\xi}_0 + u_1 \tilde{v}_1, \quad (9.4)$$

with (u_1, \tilde{v}_1) in the local zero set of $f^{(1)}$.

Remark 9.1. It is easy to check that $Df^{(1)}(0)$ is onto in the following two situations

- 1) $D^k f(0) \cdot ((\tilde{\xi}_0)^{k-1}, \cdot) \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n)$ is onto: This is precisely the case when the problem is solved through the method of Chapter 2 and defining $f^{(1)}$ is unnecessary.

2) Rank $D^k f(0) \cdot ((\tilde{\xi}_0)^{k-1}, \cdot) = n - 1$ and

$$D^{k+1} f(0) \cdot (\tilde{\xi}_0)^{k+1} \notin \text{Range} D^k f(0) \cdot ((\tilde{\xi}_0)^{k-1}, \cdot).$$

When $n = 1$ and $k = 2$, this is exactly the situation studied in § 3. 234

More generally, if there exists an integer $k_1 \geq 1$ such that

$$D^j f^{(1)}(0) = 0, 0 \leq j \leq k_1 - 1, \quad (9.5)$$

and the mapping

$$(u_1, \tilde{v}_1) \in \mathbb{R} \times \mathbb{R}^n \rightarrow D^{k_1} f^{(1)}(0) \cdot (u_1, \tilde{v}_1)^{k_1} \in \mathbb{R}^n, \quad (9.6)$$

verifies the condition ($\mathbb{R} - N.D.$), the local zero set of $f^{(1)}$ is found through Corollary 3.1 of Chapter 2. Again, the local zero set of f “along the space $\mathbb{R}\tilde{\xi}_0$ ” is obtained through the transformation (9.4).

How every, in a number of cases, it can be expected that $Df^{(1)}(0) \neq 0$ while $Df^{(1)}(0)$ is not onto either (for instance, if $1 \leq \text{rank } D^k f(0) \cdot ((\tilde{\xi}_0)^{k-1}, \cdot) \leq n - 2$; cf. Remark 9.1). That is when Theorem 4.2 - that has not been used in these lectures - has a role to play since it does not require the first derivative to vanish at the origin.

Note that, contrary to what happens in the simplest case $n = 1$ and $k = 2$, there is not necessarily a unique direction $\tilde{\xi}_0$ along which $D^k f(0) \cdot ((\tilde{\xi}_0)^{k-1}, \cdot)$ is not onto and a first iterated $f^{(1)}$ must be defined for *each* such direction.

Suppose now that (9.5) holds but the mapping (9.6) does not verify the condition ($\mathbb{R} - N.D.$): If however its zero set is made up of a finite number of lines through the origin, we can reduce the study of the local zero set of $f^{(1)}$ to the study of the local zero set of new iterates $f^{(2)}$. The same idea can be applied - with some modifications - to the more general case when $Df^{(1)}(0) \neq 0$ but is not onto and Theorem 4.2 fails to hold. Formally, the blowing-up process can be repeated under the assumption that the zero sets of the first nonzero derivative (of order > 1 actually) at the origin of each necessary iterate consists of a finite number of lines. This is a serious obstacle to the development of a general theory since it seems hard to find reasonable assumptions on f ensuring that it will *a* 235

priori be so, but the method can be tried with each particular example. Nevertheless, the above hypothesis that the zero set of the first nonzero derivative at the origin of each necessary iterate is made up of a finite number of lines is *automatic* if $n = 1$. It is then conceivable that this case could be studied in a somewhat general frame work and much progress has already been made in this direction.

Remark 9.2. More generally, for any n and when rank

$$D^k f(0) \cdot ((\tilde{\xi}_0)^{k-1}, \cdot) = n - 1$$

(in the notation of Remark 9.1), the behaviour of the iterate $f^{(1)}$ and hence of all the necessary other ones - is “as if n were 1” so that this case also can be expected to bear some general analysis.

When $n = 1$, the general case when k is arbitrary must be handled rather differently by using results and methods of algebraic geometry. **236** The relationship to *singularity theory* has also become clear although some problems remain open and the related results will be found in forthcoming publications (In the present situation when $n = 1$ and $k = 2$, there is a simple and explicit relation between the number m of iterates necessary for desingularizing f and the codimension of f but this is no longer true if $k \geq 3$).

Singularity theory seems to play an increasing role in the study of *perturbed bifurcation* since the publication of a famous paper by Golubitsky and Schaeffer ([5]).

Appendix 1

Practical Verification of the Conditions ($\mathbb{R} - N.D.$) and ($\mathbb{C} - N.D.$) when $n = 2$ and Remarks on the General Case.

Let $q = (q_\alpha)_{\alpha=1,2}$ be a homogeneous polynomial mapping of degree k on \mathbb{R}^3 with values in \mathbb{R}^2 . Given a system of coordinates $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ of \mathbb{R}^3 , we set for $\tilde{\xi} \in \mathbb{R}^3$

$$\tilde{\xi} = \xi_1 \tilde{e}_1 + \xi_2 \tilde{e}_2 + \xi_3 \tilde{e}_3, \quad (\text{A.1})$$

where $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and define

$$\tilde{\xi}' = \xi_1 \tilde{e}_1 + \xi_2 \tilde{e}_2, \quad (\text{A.2})$$

so that

$$\tilde{\xi} = \tilde{\xi}' + \xi_3 \tilde{e}_3. \quad (\text{A.3})$$

With these notations, each polynomial q_α is expressed as

$$q_\alpha(\tilde{\xi}) = \sum_{s=0}^k a_{\alpha,s}(\tilde{\xi}') \xi_3^s, \quad \alpha = 1, 2 \quad (\text{A.4})$$

where $a_{\alpha,s}$ is a homogeneous polynomial of degree $k - s$. Now, note that either the condition $(\mathbb{R} - N.D.)$ fails of \tilde{e}_3 can be chosen so that $q_1(\tilde{e}_3) \neq 0$ and $q_2(\tilde{e}_3) \neq 0$. Indeed, this is possible unless $q_1q_2 = 0$, namely $q_1 \equiv 0$ or $q_2 \equiv 0$. But, in this case, the condition $(\mathbb{R} - N.D.)$ is not satisfied by q .

With this choice of \tilde{e}_3 , one has

$$a_{\alpha,k} = q_\alpha(\tilde{e}_3) \neq 0, \alpha = 1, 2. \quad (\text{A.5})$$

238 Then, $q(\tilde{\xi}) = 0$ if and only if the two polynomials in ξ_3 of degree exactly k

$$\xi_3 \rightarrow q_\alpha(\tilde{\xi}' + \xi_3\tilde{e}_3), \alpha = 1, 2, \quad (\text{A.6})$$

have a common real root for the prescribed value $\tilde{\xi}'$. As the leading coefficients $a_{1,k}$ and $a_{2,k}$ are nonzero, the resultant $\mathbb{R}(\tilde{\xi}')$ of these two polynomials vanishes for this values of $\tilde{\xi}'$.

We examine the converse of this result. First, the resultant \mathbb{R} can identically vanish: It is well - known that this happens if and only if q_1 and q_2 have a nonconstant (homogeneous) common factor. Let us then write

$$q_1 = rp_1, q_2 = rp_2$$

with p_1 and p_2 relatively prime, $\deg r \geq 1$. If r vanishes at some point $\tilde{\xi} \in \mathbb{R}^3 - \{0\}$, one has

$$q_\alpha(\tilde{\xi}) = 0, \alpha = 1, 2,$$

and

$$Dq_\alpha(\tilde{\xi}) = P_\alpha(\tilde{\xi})Dr(\tilde{\xi}), \alpha = 1, 2.$$

Thus, $\text{Ker } Dq(\tilde{\xi}) \supset \text{Ker } Dr(\tilde{\xi})$ whose dimension is ≥ 2 and q does not satisfy the condition $(\mathbb{R} - N.D.)$.

If r does not vanish in $\mathbb{R}^3 - \{0\}$ (note that r does vanish in $\mathbb{C}^3 - \{0\}$ from Hilbert's zero theorem), there is no line in the zero set of q in \mathbb{R}^3 which is in the zero set of r and we can replace q_1 and q_2 by p_1 and p_2 . From now, on, we can then assume that q_1 and q_2 are relatively prime, so that

$$\mathcal{R} \neq 0.$$

239 It can be shown by simple arguments that \mathcal{R} is a homogeneous poly-

nomial of degree k^2 (see e.g. [16, 18]). Its zero set in $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ then consists of a finite number of lines ($\leq k^2$) through the origin and the same result is true in the space $\mathbb{C}\tilde{e}_1 \oplus \mathbb{C}\tilde{e}_2$ (of course, in this case, the lines in the zero set of \mathcal{R} are *complex* ones). For the sake of convenience, we shall denote by \mathbb{K} either field \mathbb{R} or \mathbb{C} . Given a point $\tilde{\xi}' \in \mathbb{K}\tilde{e}_1 \oplus \mathbb{K}\tilde{e}_2$ such that $\mathcal{R}(\tilde{\xi}') = 0$, there is a finite number (≥ 1) of values $\xi_3 \in \mathbb{C}$ such that $q(\tilde{\xi}' + \xi_3\tilde{e}_3) = 0$: These values are the common roots in \mathbb{C} of the two polynomials (A.6) (hence their number is finite). It follows that, above each \mathbb{K} -line in the zero set of \mathcal{R} in $\mathbb{K}\tilde{e}_1 \oplus \mathbb{K}\tilde{e}_2$, there is a finite number of \mathbb{K} -lines in the zero set of q . However, when $\mathbb{K} = \mathbb{R}$, there lines may not lie in \mathbb{R}^3 since, for a given $\tilde{\xi}' \in \mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ in the zero set of \mathcal{R} , some corresponding values ξ_3 such that $q(\tilde{\xi}' + \xi_3\tilde{e}_3) = 0$ may belong to $\mathbb{C} \setminus \mathbb{R}$. But this happens in “almost” no system $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$. To see this, let us consider an arbitrary change of \tilde{e}_3 into \tilde{e}_3^* such that $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3^*)$ is a basis of \mathbb{R}^3 . The coordinates $(\xi_1^*, \xi_2^*, \xi_3^*)$ of any point $\tilde{\xi} \in \mathbb{R}^3$ in this basis are given by

$$\begin{bmatrix} \xi_1^* \\ \xi_2^* \\ \xi_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \quad (\text{A.7})$$

where (ξ_1, ξ_2, ξ_3) are the coordinates of $\tilde{\xi}$ in the basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ and the coefficients a, b and c are real with $c \neq 0$. Of course, $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ and $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3^*)$ are also bases in \mathbb{C}^3 and the change of coordinates is still given by (A.7). 240

Let the \mathbb{C} -lines in the zero set of q in \mathbb{C}^3 be generated by the nonzero vectors $\tilde{\xi}_1, \dots, \tilde{\xi}_\nu$ with $\nu \leq k^2$ from Bezout's theorem. Denote by $(\xi_{j1}, \xi_{j2}, \xi_{j3})$ and $(\xi_{j1}^*, \xi_{j2}^*, \xi_{j3}^*)$ the components of $\tilde{\xi}_j$, $1 \leq j \leq \nu$, in the bases $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ and $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3^*)$ respectively. For a fixed index $1 \leq j \leq \nu$, the first two components ξ_{j1}^* and ξ_{j2}^* will have the same argument if and only if (agreeing that 0 has the same argument as any complex number)

$$\begin{aligned} & (\operatorname{Re}\xi_{j1}\operatorname{Im}\xi_{j2} - \operatorname{Re}\xi_{j2}\operatorname{Im}\xi_{j1}) + a(\operatorname{Re}\xi_{j3}\operatorname{Im}\xi_{j2} - \operatorname{Re}\xi_{j2}\operatorname{Im}\xi_{j3}) + \\ & + b(\operatorname{Re}\xi_{j1}\operatorname{Im}\xi_{j3} - \operatorname{Re}\xi_{j3}\operatorname{Im}\xi_{j1}) = 0 \end{aligned}$$

In other words, if ξ_{j1}, ξ_{j2} and ξ_{j3} do not have the same argument, the above equality holds for a and b in a one-dimensional affine manifold

of \mathbb{R}^2 while, from (A.7), if ξ_{j1}, ξ_{j2} and ξ_{j3} have the same argument, the three components ξ_{j1}^*, ξ_{j2}^* and ξ_{j3}^* also have the same argument. To sum up, for a and b outside the union of a finite number $\leq \nu$ of one-dimensional affine manifolds in \mathbb{R}^2 , the following property holds: For any $1 \leq j \leq \nu$, the two components ξ_{j1}^* and ξ_{j2}^* have the same argument if and only if ξ_{j1}^*, ξ_{j2}^* and ξ_{j3}^* have the same argument.

Now, let

$$\tilde{\xi} = \xi_1^* \tilde{e}_1 + \xi_2^* \tilde{e}_2 + \xi_3^* \tilde{e}_3,$$

241 be a nonzero element in the zero set of q in \mathbb{C}^3 with $\xi_1^*, \xi_2^* \in \mathbb{R}$. One has

$$\tilde{\xi} = \lambda \tilde{\xi}_j,$$

for some $\lambda \in \mathbb{C} - \{0\}$ and some $1 \leq j \leq \nu$. Thus $\xi_1^* = \lambda \xi_{j1}^*$, $\xi_2^* = \lambda \xi_{j2}^*$ and $\xi_3^* = \lambda \xi_{j3}^*$. As ξ_1^* and ξ_2^* are real by hypothesis, ξ_{j1}^* and ξ_{j2}^* have the same argument (namely, $-\text{Arg } \lambda$). Hence $\text{Arg } \xi_{j3}^* = -\text{Arg } \lambda$ too and it follows that ξ_3^* is real.

Thus, by simply modifying \tilde{e}_3 , one can assume that above each \mathbb{R} -line in the zero set of \mathcal{R} in $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$, all the corresponding \mathbb{R} -lines in the zero set of q in \mathbb{C}^3 lie in \mathbb{R}^3 (an equivalent way of saying that for every $\tilde{\xi}' \in \mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ such that $\mathcal{R}(\tilde{\xi}') = 0$, all the solutions $\xi_3 \in \mathbb{C}$ of the equation $q(\tilde{\xi}' + \xi_3 \tilde{e}_3) = 0$ are *real*). From the above proof, this property is unchanged by an arbitrarily small change of the vector \tilde{e}_3 . Of course, above a given \mathbb{R} -line in the zero set of \mathcal{R} in $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ may lie several \mathbb{R} -lines in the zero set of q in \mathbb{R}^3 . Again, this happens in exceptional cases only. Indeed, the \mathbb{R} -lines in the zero set of \mathcal{R} are the projections along \tilde{e}_3 of the lines in the zero set of q . By slightly changing \tilde{e}_3 and since there are only finitely many \mathbb{R} -lines in the zero set of q in \mathbb{R}^3 , we can manage so that the lines in the zero set of q and the lines in the zero set of \mathcal{R} are in one-to-one correspondence. We leave it to the reader to give a rigorous proof of this result, exemplified on Figure A.1 below: Given a basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ of \mathbb{R}^3 and a finite number of lines in \mathbb{R}^3 which project onto the same line of the plane $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ along \tilde{e}_3 , any change of \tilde{e}_3 into a non-collinear vector is so that the new projection onto $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ along \tilde{e}_3 transforms these lines into the same number of *distinct* lines of the plane $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$.

Remark A.1: *Of course, each change of the vector \tilde{e}_3 modifies the polynomial \mathcal{R} , which is the reason why the above properties can be established.*

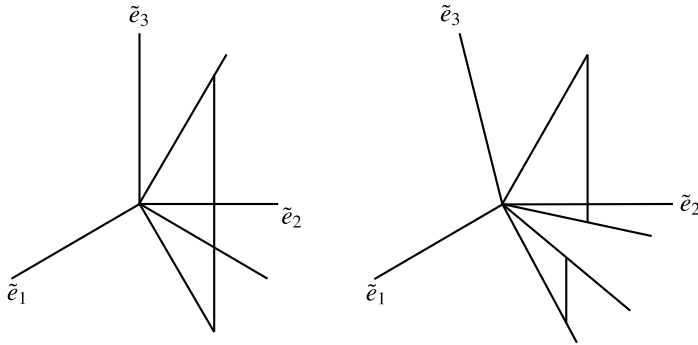


Figure A.1:

The basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ being now fixed so that the previous properties hold, we are in position to give a characterization of the condition $(\mathbb{R} - N.D.)$. First, as $\mathcal{R} \neq 0$ and by a suitable choice of \tilde{e}_2 *without modifying* $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ (so that the definition of \mathcal{R} is not affected) we may assume $\mathcal{R}(\tilde{e}_2) \neq 0$. As \mathcal{R} is homogeneous of degree k^2 , $\tilde{\xi}' = 0$ is always in the zero set of \mathcal{R} . From (A.5), this value corresponds with the value $\tilde{\xi} = 0$ in the zero set of q , a solution in which we have no interest. All then amounts to finding the nonzero solutions of $\mathcal{R}(\tilde{\xi}') = 0$. By our choice of \tilde{e}_2 , none of them is of the form $\xi_2\tilde{e}_2$ (i.e. $\xi_1 = 0$) and, after dividing by $\xi_1^{k^2} \neq 0$ 243

$$\mathcal{R}(\tilde{\xi}') = 0 \leftrightarrow \mathcal{R}\left(\tilde{e}_1 + \frac{\xi_2}{\xi_1}\tilde{e}_2\right) = 0$$

Setting $\tau = \xi_2/\xi_1$, each real root of the polynomial

$$a(\tau) = \mathcal{R}(\tilde{e}_1 + \tau\tilde{e}_2) = 0, \tag{A.8}$$

corresponds with one and only one real line in the zero set of \mathcal{R} in $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$ and hence with *one and only one line in the zero set of q in \mathbb{R}^3* . Then, the condition $(\mathbb{R} - N.D.)$ is equivalent to assuming that *each*

real root of polynomial $a(\tau)$ is simple. We only sketch the proof of this result (which uses the notion of multiplicity of intersection). For the sake of brevity, we shall refer to the real zero set of q (resp. \mathcal{R}) as being the zero set of q (resp. \mathcal{R}) in \mathbb{R}^3 (resp. $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$). Similar definitions will be used for the complex zero sets of q and \mathcal{R} .

A real line $L_{\mathbb{R}}$ in the real zero set of q (resp. \mathcal{R}) is than the intersection of one and only one complex line $L_{\mathbb{C}}$ in the complex zero set of q (resp. \mathcal{R}) with \mathbb{R}^3 (resp. $\mathbb{R}\tilde{e}_1 \oplus \mathbb{R}\tilde{e}_2$). Actually, if

$$L_{\mathbb{R}} = \mathbb{R}\tilde{\xi}_0(\text{resp. } \mathbb{R}\tilde{\xi}'_0),$$

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$$L_{\mathbb{C}} = \mathbb{C}\tilde{\xi}_0(\text{resp. } \mathbb{C}\tilde{\xi}'_0).$$

The line $L_{\mathbb{C}}$ will be called the complex extension of $L_{\mathbb{R}}$. From our previous results, above the extension $L_{\mathbb{C}}$ of a given line $L_{\mathbb{R}}$ in the real zero set of \mathcal{R} lines exactly one \mathbb{C} -line in the complex zero set of q (whose intersection with \mathbb{R}^3 is nothing but this one \mathbb{R} -line in the real zero set of q above $L_{\mathbb{R}}$).

To each \mathbb{C} -line $L_{\mathbb{C}}$ in the complex zero set of q is associated a *multiplicity*, called the *multiplicity of intersection* of the surface $q_1(\xi) = 0$ and $q_2(\tilde{\xi}) = 0$ along $L_{\mathbb{C}}$. The following result is true in general (cf. [11]): The multiplicity of the root $\tau \in \mathbb{C}$ of the polynomial $a(\tau)$ (A.8) equals the number of \mathbb{C} -lines in the complex zero set of q which lie above the \mathbb{C} -line

$$\{\xi_1\tilde{e}_1 + \tau\xi_1\tilde{e}_2, \xi_1 \in \mathbb{C}\},$$

in the complex zero set of \mathcal{R} , *counted with multiplicity*. On the other hand, from the condition ($\mathbb{R} - N.D.$), the multiplicity of the complex extension of any \mathbb{R} -line in the real zero set of q happens to be *one*, which, from our choice of the basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$, proves the equivalence of the condition ($\mathbb{R} - N.D.$) with the fact that each real root of the polynomial $a(\tau)$ (A.8) is simple.

245 The condition ($\mathbb{C} - N.D.$) also bears a similar characterization. Here, the basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ must be taken so that the projection onto $\mathbb{C}\tilde{e}_1 \oplus \mathbb{C}\tilde{e}_2$ of all the \mathbb{C} -lines in the complex zero set of q are distinct. If so, the same

arguments of multiplicity show that the condition ($\mathbb{C} - N.D.$) is equivalent¹ to assuming that *each root of the polynomial $a(\tau)$ (A.8) simple* (as a result, it has exactly k^2 distinct roots, each of them providing a different \mathbb{C} -line in the complex zero set of q). It follows that the condition ($\mathbb{C} - N.D.$) amounts to assuming that the *discriminant* of $a(\tau)$ (A.8) is $\neq 0$: It is a $(2k^2 - 1) \times (2k^2 - 1)$ determinant² whose coefficients are completely determined by the polynomials q_1 and q_2 and is then particularly simple to work out in practice.

In both cases, besides a criterion for checking the non-degeneracy condition, we have got a practical means of calculation of (approximations of) the \mathbb{R} -lines in the real zero set of q by calculating (approximations of) the real roots of the polynomial $a(\tau)$, an important feature for developing the algorithmic process of Chapter 4.

REMARKS ON THE GENERAL CASE

When n is arbitrary, the problem of practical verification of the condition ($\mathbb{R} - N.D.$) and calculation of (approximations of) the \mathbb{R} -lines in the real zero set of q is imperfectly solved. However, the practical verification of the condition ($\mathbb{C} - N.D.$) remains possible. Observe that it will *fail* if and only if the $(n + 1)$ systems of $(n + 1)$ homogeneous equations in $(n + 1)$ variables 246

$$\begin{cases} q(\tilde{\xi}) = 0 \in \mathbb{C}^n, \\ \Delta_j(\xi) = 0 \in \mathbb{C}, \end{cases} \quad (\text{A.9})$$

where Δ_j is one of the $(n + 1) \times n$ minors of the derivative $Dq(\tilde{\xi})$, are satisfied by some nonzero $\tilde{\xi} \in \mathbb{C}^{n+1}$. Each minor Δ_j is a polynomial whose coefficients are polynomials in the coefficients of q . It so happens that a system of algebraic relations between the coefficients of q can be found so that the system (A.9) is solvable if and only if these relations hold (cf. Hodge and Pedoe [16], where it is also shown that *one* algebraic relation is sufficient). Hence the existence of a finite system of

¹As every nonconstant homogeneous polynomial vanishes at nonzero points in \mathbb{C}^3 , it is necessary that q_1 and q_2 be relatively prims for the condition ($\mathbb{C} - N.D.$) to hold.

²A note by H. Gerber (Amer.Math. Monthly, 91(1984), pp.644-646) on reducing the size of determinants for calculating resultants may be applicable here.

algebraic relations between the coefficients of q which will be satisfied Marsden and Schechter have suggested the use of Seidenberg's algorithm ([36]) to find such a system of algebraic relations characterizing the condition ($\mathbb{R} - N.D.$) but simplifications would be desirable to get an actual practical method of verification. Also, this approach does not yield any means of calculation of (approximations of) the \mathbb{R} -lines in the zero set of q .

Appendix 2

Complements of Chapter IV

In this appendix, we give complete proofs for several results we used in Chapter 4, §§5 and 6 and which were omitted there for the sake of brevity. The notation is that of Chapter 4 and our exposition follows E1 Hajji [10]. 247

To begin with, we make precise a few definitions and elementary properties. Let E be a real Banach space with norm $\|\cdot\|$ and Q a compact space. The Banach space $C^0(Q, E)$ will be equipped with the norm $|\cdot|_{\infty, Q}$ defined by

$$\psi \in C^0(Q, E) \rightarrow |\psi|_{\infty, Q} = \sup_{x \in Q} \|\psi(x)\|.$$

Now, if Ω denotes an open subset of some other real Banach space and $m \geq 0$ a given integer, we call $C^m(\overline{\Omega}, E)$ the space of mappings m times differentiable in Ω whose derivatives of order $\leq m$ can be continuously extended to the closure $\overline{\Omega}$. Note that if the ambient space of Ω is infinite-dimensional, $C^m(\overline{\Omega}, E)$ is not a normed space, even when Ω is bounded. On the contrary, if Ω is bounded and its ambient space is finite-dimensional, $C^m(\overline{\Omega}, E)$ is a Banach space with the norm

$$\psi \in C^m(\overline{\Omega}, E) \rightarrow \sum_{i=0}^m \sup_{x \in \overline{\Omega}} \|D^i \psi(x)\| = \sum_{i=0}^m |D^i \psi|_{\infty, \overline{\Omega}}.$$

In other words, convergence in the space $C^m(\overline{\Omega}, E)$ is equivalent to convergence of the derivatives of order $\leq m$ in the uniform norm.

248 Our first task is to prove, for $r > 0$ and $\delta > 0$ small enough, that the sequence (g_ℓ) (resp. $(D_{\tilde{\xi}}g_\ell)$) tends to g (resp. $D_{\tilde{\xi}}g$) in the space $C^0([-r, r] \times \Delta, \mathbb{R}^n)$ (resp. $C^0([-r, r] \times \Delta, \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^n))$). Since the results we are looking for are local, it is not restrictive to assume that \mathcal{O} (in IV.5) is convex and $r > 0$ is chosen so that $t\tilde{\xi} \in \mathcal{O}$ for every $t \in [-r, r]$ and every $\tilde{\xi} \in \Delta$.

Due to the definition of k , the Taylor formula shows for $0 < |t| \leq r$ and $\tilde{\xi} \in \Delta$ that

$$g_\ell(t, \tilde{\xi}) - g(t, \tilde{\xi}) = k \int_0^1 (1-s)^{k-1} D^k(f_\ell - f)(st\tilde{\xi}) \cdot (\tilde{\xi})^k ds.$$

From the definitions, this remains true for $t = 0$ and, as $\|\tilde{\xi}\| \leq 1 + \delta/2$ we get

$$|g_\ell - g|_{\infty, [-r, r] \times \Delta} \leq \left(1 + \frac{\delta}{2}\right)^k |D^k(f_\ell - f)|_{\infty, \overline{\mathcal{O}}}. \quad (\text{A2.1})$$

Also, for $t \neq 0$, we have

$$D_{\tilde{\xi}}(g_\ell - g)(t, \tilde{\xi}) = \frac{k!}{t^{k-1}} D(f_\ell - f)(t\tilde{\xi}), \quad (\text{A2.2})$$

while the relation $g_\ell(0, \cdot) = g(0, \cdot) = q$ for every $\ell \geq 0$ yields

$$D_{\tilde{\xi}}(g_\ell - g)(0, \tilde{\xi}) = 0. \quad (\text{A2.3})$$

Thus, if $k = 1$

$$|D_{\tilde{\xi}}(g_\ell - g)|_{\infty, [-r, r] \times \Delta} \leq k \left(1 + \frac{\delta}{2}\right)^{k-1} |D^k(f_\ell - f)|_{\infty, \overline{\mathcal{O}}}. \quad (\text{A2.4})$$

249 In order to show that the above formula remains true also when $k \geq 2$, it suffices to apply the Taylor formula to the term $D(f_\ell - f)$ and to substitute it in (A2.2); we then find, for $0 < |t| \leq r$ and $\tilde{\xi} \in \Delta$, that

$$D_{\tilde{\xi}}(g_\ell - g)(t, \tilde{\xi}) = k(k-1) \int_0^1 (1-s)^{k-2} D^k(f_\ell - f)(st\tilde{\xi}) \cdot (\tilde{\xi})^{k-1} ds,$$

and this relation remains valid for $t = 0$, which establishes (A2.4) immediately. The assertion follows from (A2.1) and (A2.4) and the convergence of the sequence (f_ℓ) to f in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$.

It is then easy to deduce that the sequence (M_ℓ) (respectively $(D_{\tilde{\xi}} M_\ell)$) tends to M (respectively $D_{\tilde{\xi}} M$) uniformly in the set $[-r, r] \times C \times \Delta$. More precisely, introducing

$$w(\delta) = \sup_{\tilde{\zeta}_0 \in C} \|A(\tilde{\zeta}_0) - A(\tilde{\xi}_0)\| = \|A - A(\tilde{\xi}_0)\|_{\infty, C}, \quad (\text{A2.5})$$

one has

$$\|M_\ell - M\|_{\infty, [-r, r] \times C \times \Delta} \leq (\|A(\tilde{\xi}_0)\| + w(\delta)) |g_\ell - g|_{\infty, [-r, r] \times \Delta} \quad (\text{A2.6})$$

Besides, the same arguments leads to the inequality

$$\|M_\ell(t, \cdot, \cdot) - M(t, \cdot, \cdot)\|_{\infty, C \times \Delta} < (\|A(\tilde{\xi}_0)\| + w(\delta)) |g_\ell(t, \cdot) - g(t, \cdot)|_{\infty, \Delta}, \quad (\text{A2.6}')$$

for every $t \in [-r, r]$.

Now, for $(t, \tilde{\zeta}_0, \tilde{\xi}) \in [-r, r] \times C \times \Delta$

$$\|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - N(t, \tilde{\zeta}_0, \tilde{\xi})\| \leq 2 \frac{\|M_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - M(t, \tilde{\zeta}_0, \tilde{\xi})\|}{\|M(t, \tilde{\zeta}_0, \tilde{\xi})\|}$$

Hence, from Lemma 2.1 of Chapter 4

$$\|N_\ell - N\|_{\infty, [-r, r] \times C \times \Delta} \leq \frac{2}{1 - \delta} \|M_\ell - M\|_{\infty, [-r, r] \times C \times \Delta}, \quad (\text{A2.7})$$

from which it follows that the sequence (N_ℓ) tends to N uniformly in the set $[-r, r] \times C \times \Delta$. The same method shows that **250**

$$\|N_\ell(t, \cdot, \cdot) - N(t, \cdot, \cdot)\|_{\infty, C \times \Delta} < \frac{2}{1 - \delta} \|M_\ell(t, \cdot, \cdot) - M(t, \cdot, \cdot)\|_{\infty, C \times \Delta}, \quad (\text{A2.7}')$$

for every $t \in [-r, r]$.

The proof of Theorem 5.1 of Chapter 4 is based on the following estimate.

Lemma A2.1: For every triple $(t, \tilde{\zeta}_0, \tilde{\xi}) \in [-r, r] \times C \times \Delta$ and every $\ell \in \mathbb{N}$, one has

$$\|D_{\tilde{\xi}} N_\ell(t, \tilde{\zeta}_0, \tilde{\xi})\| \leq \frac{3\delta}{(1 - \delta^3)} + \frac{2\|D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0)\|}{(1 - \delta)} w(\delta) +$$

$$+ 2 \left[\frac{\|A(\tilde{\xi}_0)\| + w(\delta)}{(1-\delta)} \right] \left[\frac{3}{(1-\delta)^2} \|g_\ell(t, \tilde{\xi})\| + \|D_{\tilde{\xi}} g_\ell(t, \tilde{\xi}) - D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0)\| \right]. \quad (\text{A2.8})$$

Proof. In this proof, it will be convenient to use the following notation: M_ℓ^* (resp. Λ_ℓ^*) will denote the value $M(t, \tilde{\zeta}_0, \tilde{\xi})$ (resp. $D_{\tilde{\xi}} M_\ell(t, \tilde{\zeta}_0, \tilde{\xi})$) at some arbitrary point $(t, \tilde{\zeta}_0, \tilde{\xi}) \in [-r, r] \times C \times \Delta$ and M_ℓ^0 (resp. Λ_ℓ^0) the particular value $M_\ell(0, \tilde{\xi}_0, \tilde{\xi}_0)$ (resp. $D_{\tilde{\xi}} m_\ell(0, \tilde{\xi}_0, \tilde{\xi}_0)$). Thus, by the definition of M_ℓ

$$M_\ell^0 = \tilde{\xi}_0. \quad (\text{A2.9})$$

On the other hand, the calculation of $D_{\tilde{\xi}} M(0, \tilde{\xi}_0, \tilde{\xi}_0)$ made in Chapter 4, §3 can be repeated for finding $D_{\tilde{\xi}} M_\ell(0, \tilde{\xi}_0, \tilde{\xi}_0) = \Lambda_\ell^0$ so that

$$\Lambda_\ell^0 \tilde{h} = (\tilde{\xi}_0 \tilde{h}) \tilde{\xi}_0, \quad (\text{A2.10})$$

for every $\tilde{h} \in \mathbb{R}^{n+1}$. As a result

$$\|M_\ell^0\| = 1, \quad (\text{A2.11})$$

$$\|\Lambda_\ell^0\| = 1. \quad (\text{A2.12})$$

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With the notation introduced above, an elementary calculation gives

$$D_{\tilde{\xi}} N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h} = \frac{\Lambda_\ell^* \tilde{h}}{\|M_\ell^*\|} - \frac{(\Lambda_\ell^* \tilde{h} | M_\ell^*)}{\|M_\ell^*\|^3} M_\ell^*, \quad (\text{A2.13})$$

for every $\tilde{h} \in \mathbb{R}^{n+1}$. In particular, choosing $(t, \tilde{\zeta}_0, \tilde{\xi}) = (0, \tilde{\xi}_0, \tilde{\xi}_0)$, we get

$$D_{\tilde{\xi}} N_\ell(0, \tilde{\xi}_0, \tilde{\xi}_0) = \Lambda_\ell^0 \tilde{h} - (\Lambda_\ell^0 \tilde{h} | M_\ell^0) M_\ell^0 \quad (\text{A2.14})$$

Next, from (A2.9)-(A2.10), it follows that $D_{\tilde{\xi}} N_\ell(0, \tilde{\xi}_0, \tilde{\xi}_0) = 0$. Therefore the relation (A2.13) is unchanged by subtracting (A2.14) from it, which means that

$$D_{\tilde{\xi}} N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h} = \left(\frac{\Lambda_\ell^* \tilde{h}}{\|M_\ell^*\|} - \Lambda_\ell^0 \tilde{h} \right) - \left(\frac{(\Lambda_\ell^* \tilde{h} | M_\ell^*)}{\|M_\ell^*\|^3} M_\ell^* - (\Lambda_\ell^0 \tilde{h} | M_\ell^0) M_\ell^0 \right)$$

for every $\tilde{h} \in \mathbb{R}^{n+1}$. Hence the inequality

$$\|D_{\tilde{\xi}} N_{\ell}(t, \tilde{\zeta}_0, \tilde{\xi}) \cdot \tilde{h}\| \leq \left\| \frac{\Lambda_{\ell}^* \tilde{h}}{\|M_{\ell}^*\|} - \Lambda_{\ell}^0 \tilde{h} \right\| + \left\| \frac{(\Lambda_{\ell}^* \tilde{h} | M_{\ell}^*)}{\|M_{\ell}^*\|^3} M_{\ell}^* - (\Lambda_{\ell}^0 \tilde{h} | M_{\ell}^0) M_{\ell}^0 \right\|. \quad (\text{A2.15})$$

As a first step, we establish an estimate for the term $\left\| \frac{\Lambda_{\ell}^* \tilde{h}}{\|M_{\ell}^*\|} - \Lambda_{\ell}^0 \tilde{h} \right\|$. One has

$$\frac{\Lambda_{\ell}^* \tilde{h}}{\|M_{\ell}^*\|} - \Lambda_{\ell}^0 \tilde{h} = \frac{1}{\|M_{\ell}^*\|} ((\Lambda_{\ell}^* \tilde{h}) + (1 - \|M_{\ell}^*\|) \Lambda_{\ell}^0 \tilde{h}).$$

From (A2.11), we first find a majorisation by

$$\frac{1}{\|M_{\ell}^*\|} \left(\|\Lambda_{\ell}^* - \Lambda_{\ell}^0\| + |1 - \|M_{\ell}^*\|| \right) \|\tilde{h}\|.$$

Now, since $|1 - \|M_{\ell}^*\|| \leq \|M_{\ell}^* - M_{\ell}^0\|$ (cf. (A2.11))

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$$\left\| \frac{\Lambda_{\ell}^* \tilde{h}}{\|M_{\ell}^*\|} - \Lambda_{\ell}^0 \tilde{h} \right\| \leq \frac{1}{\|M_{\ell}^*\|} (\|\Lambda_{\ell}^* - \Lambda_{\ell}^0\| + \|M_{\ell}^* - M_{\ell}^0\|) \|\tilde{h}\|. \quad (\text{A2.16})$$

Next, we look for an appropriate majorisation of the second term appearing in the right hand side of (A2.15), namely

$$\left\| \frac{(\Lambda_{\ell}^* \tilde{h} | M_{\ell}^*)}{\|M_{\ell}^*\|^3} M_{\ell}^* - (\Lambda_{\ell}^0 \tilde{h} | M_{\ell}^0) M_{\ell}^0 \right\|.$$

Here, we use the identity

$$\begin{aligned} \frac{(\Lambda_{\ell}^* \tilde{h} | M_{\ell}^*)}{\|M_{\ell}^*\|} M_{\ell}^* - (\Lambda_{\ell}^0 \tilde{h} | M_{\ell}^0) M_{\ell}^0 &= \frac{1}{\|M_{\ell}^*\|^3} \{ (\Lambda_{\ell}^* - \Lambda_{\ell}^0) \tilde{h} | M_{\ell}^* \} M_{\ell}^* + \\ &+ (\Lambda_{\ell}^0 \tilde{h} | M_{\ell}^*) (M_{\ell}^* - M_{\ell}^0) + (\Lambda_{\ell}^0 \tilde{h}, M_{\ell}^* - M_{\ell}^0) M_{\ell}^0 + \\ &+ (1 - \|M_{\ell}^*\|^3) (\Lambda_{\ell}^0 \tilde{h}, M_{\ell}^0) M_{\ell}^0. \end{aligned}$$

As $|1 - \|M_{\ell}^*\|^3| = |1 - \|M_{\ell}^*\|| (1 + \|M_{\ell}^*\| + \|M_{\ell}^*\|^2)$ and using the inequality $|1 - \|M_{\ell}^*\|| \leq \|M_{\ell}^* - M_{\ell}^0\|$, we obtain

$$\left\| \frac{(\Lambda_{\ell}^* \tilde{h} | M_{\ell}^*)}{\|M_{\ell}^*\|} M_{\ell}^* - (\Lambda_{\ell}^0 \tilde{h} | M_{\ell}^0) M_{\ell}^0 \right\| \leq \quad (\text{A2.17})$$

$$\leq \left\{ \frac{1}{\|M_\ell^*\|} \|\Lambda_\ell^* - \Lambda_\ell^0\| + \left(\frac{1}{\|M_\ell^*\|} + \frac{2}{\|M_\ell^*\|^2} + \frac{2}{\|M_\ell^*\|^3} \right) \|M_\ell^* - M_\ell^0\| \right\} \|\tilde{h}\|.$$

Using (A2.16) and (A2.17) in (A2.15), we arrive at

$$\|D_{\tilde{\xi}} N_\ell(t, \tilde{\zeta}_0, \tilde{\xi})\| \leq \frac{2}{\|M_\ell^*\|} \left\{ \|\Lambda_\ell^* - \Lambda_\ell^0\| + \left(1 + \frac{1}{\|M_\ell^*\|} + \frac{1}{\|M_\ell^*\|^2} \right) \|M_\ell^* - M_\ell^0\| \right\}.$$

253 In view of Lemma 2.1 of Chapter 4 and from $0 < 1 - \delta < 1$, this inequality takes the simpler form

$$\|D_{\tilde{\xi}} N_\ell(t, \tilde{\zeta}_0, \tilde{\xi})\| \leq \frac{2}{1 - \delta} \left(\|\Lambda_\ell^* - \Lambda_\ell^0\| + \frac{3}{(1 - \delta)^2} \|M_\ell^* - M_\ell^0\| \right). \quad (\text{A2.18})$$

At this stage, proving our assertion reduces to finding a suitable estimate for the two terms $\|\Lambda_\ell^* - \Lambda_\ell^0\|$ and $\|M_\ell^* - M_\ell^0\|$. We begin with a majorisation of $\|M_\ell^* - M_\ell^0\|$. By (A2.9) and the definition of the mapping M_ℓ ,

$$\begin{aligned} \|M_\ell^* - M_\ell^0\| &\leq \|\tilde{\xi} - \tilde{\xi}_0\| + \|A(\tilde{\zeta}_0) \cdot g_\ell(t, \tilde{\xi})\| \\ &\leq \frac{\delta}{2} + \|A(\tilde{\zeta}_0)\| \|g_\ell(t, \tilde{\xi})\|. \end{aligned}$$

As $\|A(\tilde{\zeta}_0)\| \leq \|A(\tilde{\xi}_0)\| + w(\delta)$ by the definition of $w(\delta)$ (of. (A2.5))

$$\|M_\ell^* - M_\ell^0\| \leq \frac{\delta}{2} + \left(\|A(\tilde{\xi}_0)\| + w(\delta) \right) \|g_\ell(t, \tilde{\xi})\|. \quad (\text{A2.19})$$

Finally, it is immediate that

$$\Lambda_\ell^* - \Lambda_\ell^0 \tilde{h} = A(\tilde{\zeta}_0) D_{\tilde{\xi}} g_\ell(t, \tilde{\xi}) \cdot \tilde{h} - A(\tilde{\xi}_0) D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0) \cdot \tilde{h}.$$

Hence

$$\begin{aligned} \|\Lambda_\ell^* - \Lambda_\ell^0\| &< \|A(\tilde{\zeta}_0) - A(\tilde{\xi}_0)\| \|D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0)\| + \\ &\quad + \|A(\tilde{\zeta}_0)\| \|D_{\tilde{\xi}} g_\ell(t, \tilde{\xi}) - D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0)\|. \end{aligned}$$

From the inequality $\|A(\tilde{\zeta}_0)\| < \|A(\tilde{\xi}_0)\| + w(\delta)$ again and by the definition of $w(\delta)$, we find then

$$\|\Lambda_\ell^* - \Lambda_\ell^0\| < w(\delta) \|D_{\tilde{\xi}} g_\ell(0)\| +$$

$$+ (\|A(\tilde{\xi}_0)\| + w(\delta)) \|D_{\tilde{\xi}} g_\ell(t, \tilde{\xi}) - D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0)\|. \quad (\text{A2.20})$$

The combination of inequalities (A2.18) - (A2.20) yields the desired estimate (A2.8). \square

Proof of Theorem 5.1: As a first step, we prove that the mapping $N_\ell(t, \tilde{\xi}_0, \cdot)$ is Lipschitz continuous with constant γ . The fact that $N(t, \tilde{\xi}_0, \cdot)$ maps the ball Δ into itself will be shown afterwards. The starting point is the estimate (A2.8) of Lemma A2.1: From the relation $D_{\tilde{\xi}} g_\ell(0, \tilde{\xi}_0) = D_{\tilde{\xi}} g(0, \tilde{\xi}_0) (= Dq(\tilde{\xi}_0))$, Chapter 2, §3) and writing $g_\ell = g_\ell - g + g$, it is easily seen that

$$\begin{aligned} \|D_{\tilde{\xi}} N_\ell(t, \tilde{\xi}_0, \tilde{\xi})\| &\leq \frac{3\delta}{(1-\delta)^3} + \frac{2\|D_{\tilde{\xi}} g(0, \tilde{\xi}_0)\|}{1-\delta} w(\delta) + \frac{2(\|A(\tilde{\xi}_0)\| + w(\delta))}{1-\delta} \\ &\quad \left[\frac{3}{(1-\delta)^2} \|g(t, \tilde{\xi})\| + \|D_{\tilde{\xi}} g(t, \tilde{\xi}) - D_{\tilde{\xi}} g(0, \tilde{\xi}_0)\| \right] + \\ &\quad + \frac{2(\|A(\tilde{\xi}_0)\| + w(\delta))}{1-\delta} \\ &\quad \left[\frac{3}{(1-\delta)^2} \|(g_\ell - g)(t, \tilde{\xi})\| + \|D_{\tilde{\xi}}(g_\ell - g)(t, \tilde{\xi})\| \right]. \end{aligned}$$

Given any constant $\gamma > 0$, the sum of the first three terms of the above inequality can clearly be made $\leq \gamma/2$ provided $r > 0$ and δ with $0 < \delta < 1$ are taken small enough. Fixing then r , and δ (for the time being), the last term is uniformly bounded by

$$\frac{2(\|A(\tilde{\xi}_0)\| + w(\delta))}{1-\delta} \left[\frac{3}{(1-\delta)^2} |g_\ell - g|_{\infty, [-r, r] \times \Delta} + |D_{\tilde{\xi}}(g_\ell - g)|_{\infty, [-r, r] \times \Delta} \right].$$

For ℓ large enough, say $\ell \geq \ell_0$, it can then be made $\leq \gamma/2$ as well. By the mean value theorem, the mapping $N(t, \tilde{\xi}_0, \cdot)$ is then Lipschitz continuous with constant γ in the ball Δ for every pair $(t, \tilde{\xi}_0) \in [-r, r] \times C$ and every $\ell \geq \ell_0$. This property is not affected by shrinking $r > 0$ and $0 < \delta < 1$ arbitrarily. Applying then Theorem 3.1 to each mapping $N_\ell, 0 \leq \ell \leq \ell_0 - 1$ (thus a finite number of times), it is not restrictive to assume that $r > 0$ and δ with $0 < \delta < 1$ are such that the mapping

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$N_\ell(t, \tilde{\zeta}_0, \cdot)$ is a contraction with constant γ in the ball Δ for every $\ell \in \mathbb{N}$. Again, this property remains true if $r > 0$ is arbitrarily diminished. Let us then fix δ as above. We shall show that $N_\ell(t, \tilde{\zeta}_0, \cdot)$ maps the ball Δ into itself for every $\ell \in \mathbb{N}$ after modifying $r > 0$ if necessary.

For every triple $(t, \tilde{\zeta}_0, \tilde{\xi}) \in [-r, r] \times C \times \Delta$ and since $N(0, \tilde{\zeta}_0, \tilde{\xi}_0) = \tilde{\xi}_0$ (cf. relation (2.10) of Chapter 4), one has

$$\begin{aligned} \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - \tilde{\xi}_0\| &\leq \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}_0)\| + \\ &+ \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N(t, \tilde{\zeta}_0, \tilde{\xi}_0)\| + \|N(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N(0, \tilde{\zeta}_0, \tilde{\xi}_0)\|. \end{aligned}$$

The first term in the right hand side is bounded by $\gamma\|\tilde{\xi} - \tilde{\xi}_0\| \leq \gamma\delta/2$. As $\gamma < 1$ and further the mapping $N(\cdot, \cdot, \tilde{\xi}_0)$ is uniformly continuous on the compact set $[-r, r] \times C$, we may assume that $r > 0$ is small enough for the inequality

$$\|N(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N(0, \tilde{\zeta}_0, \tilde{\xi}_0)\| \leq (1 - \gamma)\frac{\delta}{4},$$

to hold. The term $\|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N(t, \tilde{\zeta}_0, \tilde{\xi}_0)\|$ is bounded by

$$\|N_\ell - N|_{[-r, r] \times C \times \Delta}$$

and can therefore be made $\leq (1 - \gamma)\delta/4$ for ℓ large enough, say $\ell \geq \ell_0$. Thus

$$\|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - \tilde{\xi}_0\| \leq \frac{\delta}{2},$$

for every triple $(t, \tilde{\zeta}_0, \tilde{\xi}) \in [-r, r] \times C \times \Delta$ and every $\ell \geq \ell_0$. This property not being affected by diminishing $r > 0$ arbitrarily (without (modifying δ , of course) and observing that $N_\ell(0, \tilde{\zeta}_0, \tilde{\xi}_0) = \tilde{\xi}_0$ for every $\tilde{\zeta}_0 \in C$, let us write for $0 \leq \ell \leq \ell_0 - 1$

$$\begin{aligned} \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - \tilde{\xi}_0\| &\leq \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}_0)\| + \\ &+ \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N_\ell(0, \tilde{\zeta}_0, \tilde{\xi}_0)\|. \end{aligned}$$

The first term in the right hand side is bounded by $\gamma\|\tilde{\xi} - \tilde{\xi}_0\| \leq \gamma\delta/2$. Owing to the uniform continuity of the mapping $N_\ell(\cdot, \cdot, \tilde{\xi}_0)$ on the

compact set $[-r, r] \times C$, we may assume that $r > 0$ is small enough for the inequality

$$\|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}_0) - N_\ell(0, \tilde{\zeta}_0, \tilde{\xi}_0)\| \leq (1 - \gamma)\delta/2,$$

to hold for $0 \leq \ell \leq \ell_0 - 1$ and, for these indices as well, we get

$$\|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - \tilde{\xi}_0\| \leq \frac{\delta}{2},$$

which completes the proof.

In chapter 4, Theorem 5.1 has been used for proving the convergence of the sequence $\tilde{x}_\ell = t\tilde{\zeta}_\ell$ (where $\tilde{\zeta}_{\ell+1} = N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell)$) to $\tilde{x} = t\tilde{\xi}$ (where $\tilde{\xi}$ is the unique fixed point of the mapping $N(t, \tilde{\zeta}_0, \cdot)$ in the ball Δ). We shall now give an estimate of the rate of convergence of the sequence (\tilde{x}_ℓ) to \tilde{x} under suitable assumptions on the rate of convergence of the sequence (f_ℓ) to f in appropriate spaces.

Theorem A2.1: (i) *Assume that the sequence (f_ℓ) tends to f geometrically in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$. Then, there are constant $0 < \gamma' < 1$ and $K > 0$ such that, for every $t \in [-r, r]$*

$$\|\tilde{x}_\ell - \tilde{x}\| \leq K|t|\gamma'^\ell, \quad (\text{A2.21})$$

for every $\ell \geq 0$.

(ii) *Assume only that the sequence (f_ℓ) tends to f in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$, the convergence being geometrical in the space $C^{k-1}(\overline{\mathcal{O}}, \mathbb{R}^n)$. Then, there are constant $0 < \gamma' < 1$ and $K > 0$ such that, for every $t \in [-r, r]$*

$$\|\tilde{x}_\ell - \tilde{x}\| \leq K\gamma'^\ell. \quad (\text{A2.22})$$

Proof. From the relation

$$\begin{aligned} \tilde{\zeta}_{\ell+1} - \tilde{\xi} &= N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell) - N(t, \tilde{\zeta}_0, \tilde{\xi}) \\ &= N_\ell(t, \tilde{\zeta}_0, \tilde{\zeta}_\ell) - N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) + N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - N(t, \tilde{\zeta}_0, \tilde{\xi}), \end{aligned}$$

and from Theorem 5.1 of Chapter 4, we get

$$\|\tilde{\zeta}_{\ell+1} - \tilde{\xi}\| < \gamma\|\tilde{\zeta}_\ell - \tilde{\xi}\| + \|N_\ell(t, \tilde{\zeta}_0, \tilde{\xi}) - N(t, \tilde{\zeta}_0, \tilde{\xi})\|.$$

Hence

$$\|\widetilde{\zeta}_{\ell+1} - \widetilde{\xi}\| \leq \gamma \|\widetilde{\zeta}_\ell - \widetilde{\xi}\| + |N_\ell(t, \cdot, \cdot) - N(t, \cdot, \cdot)|_{\infty, C \times \Delta}, \quad (\text{A2.23})$$

for every $t \in [-r, r]$. To prove (i), observe that $|N_\ell(t, \cdot, \cdot) - N(t, \cdot, \cdot)|_{\infty, C \times \Delta} \leq |N - N_\ell|_{\infty, [-r, r] \times C \times \Delta}$. From (A2.6) - (A2.7) and (A2.1)

$$\begin{aligned} \|\widetilde{\zeta}_{\ell+1} - \widetilde{\xi}\| &\leq \gamma \|\widetilde{\zeta}_\ell - \widetilde{\xi}\| + \frac{2(\|A(\widetilde{\xi}_0)\| + w(\delta))}{1 - \delta} |g_\ell - g|_{\infty, [-r, r] \times \Delta} \\ &\leq \gamma \|\widetilde{\zeta}_\ell - \widetilde{\xi}\| + \frac{2(\|A(\widetilde{\xi}_0)\| + w(\delta))}{1 - \delta} \left(1 + \frac{\delta}{2}\right)^k |D^k(f_\ell - f)|_{\infty, \overline{\mathcal{O}}}. \end{aligned}$$

As the sequence (f_ℓ) tends to f geometrically in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$ by hypothesis and after increasing $0 < \gamma < 1$ if necessary, there is a constant $C > 0$ such that

$$\|\widetilde{\zeta}_{\ell+1} - \widetilde{\xi}\| \leq \gamma \|\widetilde{\zeta}_\ell - \widetilde{\xi}\| + C\gamma^\ell.$$

258 By a simple induction argument, we find

$$\|\widetilde{\zeta}_{\ell+1} - \widetilde{\xi}\| \leq \gamma^{\ell+1} \|\widetilde{\zeta}_0 - \widetilde{\xi}\| + (\ell + 1)C\gamma^\ell.$$

Thus

$$\|\widetilde{\zeta}_\ell - \widetilde{\xi}\| \leq \gamma^\ell \left(\|\widetilde{\zeta}_0 - \widetilde{\xi}\| + \frac{\ell C}{\gamma} \right),$$

for every $\ell \geq 0$. As $\widetilde{\zeta}_0 \in C \subset \Delta$ and $\widetilde{\xi} \in C \subset \Delta$ and the ball Δ has diameter $\delta < 1$

$$\|\widetilde{\zeta}_\ell - \widetilde{\xi}\| \leq \gamma^\ell \left(1 + \frac{\ell C}{\gamma} \right). \quad (\text{A2.24})$$

Multiplying by $|t|$, we see that

$$\|\widetilde{x}_\ell - \widetilde{x}\| \leq |t| \gamma^\ell \left(1 + \frac{\ell C}{\gamma} \right).$$

Choosing $\gamma < \gamma' < 1$, inequality (A2.21) follows with

$$K = \sup_{\ell \geq 0} \left(\frac{\gamma}{\gamma'} \right)^\ell \left(1 + \frac{\ell C}{\gamma} \right) < +\infty.$$

To prove (ii), we use the same method, replacing the relations (A2.6) and (A2.7) by (A2.6)' and (A2.7)'. With (A2.23), we get

$$\|\tilde{\zeta}_{\ell+1} - \tilde{\xi}\| \leq \gamma \|\tilde{\zeta}_\ell - \tilde{\xi}\| + \frac{2(\|A(\tilde{\xi}_0)\| + w(\delta))}{1 - \delta} |g_\ell(t, \cdot) - g(t, \cdot)|_{\infty, \Delta}.$$

Now, note that

$$|g_\ell(t, \cdot) - g(t, \cdot)|_{\infty, \Delta} \leq \frac{k}{|t|} \left(1 + \frac{\delta}{2}\right)^{k-1} |D^{k-1}(f_\ell - f)|_{\infty, \bar{\mathcal{O}}}.$$

Indeed, this is obvious if $k = 1$. If $k \geq 2$, the inequality follows from the relation

$$g_\ell(t, \tilde{\zeta}) - g(t, \tilde{\zeta}) = \frac{k(k-1)}{t} \int_0^1 (1-s)^{k-1} D^{k-1}(f_\ell - f)(st\tilde{\zeta}) \cdot (\tilde{\zeta})^{k-1} ds.$$

Arguing as before, we find the analogue of (A2.24), namely

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$$\|\tilde{\zeta}_\ell - \tilde{\xi}\| \leq \gamma^\ell \left(1 + \frac{\ell C}{\gamma |t|}\right),$$

for every $\ell \geq 0$. Multiplying by $|t| > 0$, we obtain

$$\|\bar{x}_\ell - \bar{x}\| \leq \gamma^\ell \left(|t| + \frac{\ell C}{\gamma}\right) \leq \gamma^\ell \left(r + \frac{\ell C}{\gamma}\right).$$

Choosing $\gamma < \gamma' < 1$, the inequality (A2.22) follows with

$$K = \sup_{\ell \geq 0} \left(\frac{\gamma}{\gamma'}\right)^\ell \left(r + \frac{\ell C}{\gamma}\right) < +\infty,$$

and remains valide for $t = 0$. □

We shall now prove the results about convergence in the spaces of type C^k that we used in §6 of Chapter 4. Recall that given a real Banach space \tilde{Z} and a mapping $\Phi(= \Phi(\bar{x}, \bar{z})) \in C^k(\bar{\mathcal{O}} \times \bar{B}(0, \rho), \tilde{Z})$, $k \geq 1$, where $\bar{B}(0, \rho)$ denotes the closed ball with radius $\rho > 0$ centered at the origin of \tilde{Z} , verifying $\Phi(0) = 0$ and $D_{\bar{z}}\Phi(0) = 0$, it is possible to shrink $\rho > 0$ and

the neighbourhood \mathcal{O} so that, given any arbitrary constant $0 < \beta < 1$, one has

$$\|D_{\bar{z}}\Phi(\bar{x}, \bar{z})\| \leq \beta \quad (\text{A2.25})$$

for every $(\bar{x}, \bar{z}) \in \overline{\mathcal{O}} \times \overline{B}(0, \rho)$ and the mapping $\Phi(\bar{x}, \cdot)$ is a contraction with constant β from $\overline{B}(0, \rho)$ to itself for every $\bar{x} \in \overline{\mathcal{O}}$. If so, for $\bar{x} \in \overline{\mathcal{O}}$, the sequence

$$\begin{cases} \tilde{\varphi}_0(\bar{x}) = 0, \\ \tilde{\varphi}_{\ell+1}(\bar{x}) = \Phi(\bar{x}, \tilde{\varphi}_\ell(\bar{x})), \ell \geq 0, \end{cases}$$

is well defined, each mapping $\tilde{\varphi}_\ell$ being in the space $C^k(\overline{\mathcal{O}}, \overline{Z})$ with values in $\overline{B}(0, \rho)$. We already know that this sequence converges pointwise to the mapping $\tilde{\varphi} \in C^k(\overline{\mathcal{O}}, \overline{Z})$ characterized by

$$\tilde{\varphi}(\bar{x}) = \Phi(\bar{x}, \tilde{\varphi}(\bar{x})),$$

for every $\bar{x} \in \overline{\mathcal{O}}$. In §6 of Chapter 4, we used the fact that the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^k(\overline{\mathcal{O}}, \overline{Z})$, the convergence being geometrical in the space $C^{k-1}(\overline{\mathcal{O}}, \overline{Z})$. Proving this assertion will take us a certain amount of time. To begin with, we show that

Lemma A2.2: *The sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^0(\overline{\mathcal{O}}, \overline{Z})$.*

Proof. Let $\ell \geq 1$ be fixed. For every $\bar{x} \in \overline{\mathcal{O}}$

$$\begin{aligned} \tilde{\varphi}_{\ell+1}(\bar{x}) - \tilde{\varphi}_\ell(\bar{x}) &= \Phi(\bar{x}, \tilde{\varphi}_\ell(\bar{x})) - \Phi(\bar{x}, \tilde{\varphi}_{\ell-1}(\bar{x})) \\ &= \int_0^1 D_{\bar{z}}\Phi(\bar{x}, \tilde{\varphi}_{\ell-1}(\bar{x}) + s(\tilde{\varphi}_\ell(\bar{x}) - \tilde{\varphi}_{\ell-1}(\bar{x}))) \cdot \\ &\quad (\tilde{\varphi}_\ell(\bar{x}) - \tilde{\varphi}_{\ell-1}(\bar{x})) ds. \end{aligned}$$

Due to (A2.25) we obtain

$$\|\tilde{\varphi}_{\ell+1}(\bar{x}) - \tilde{\varphi}_\ell(\bar{x})\| \leq \beta \|\tilde{\varphi}_\ell(\bar{x}) - \tilde{\varphi}_{\ell-1}(\bar{x})\|,$$

for every $\bar{x} \in \overline{\mathcal{O}}$. Thus

$$|\tilde{\varphi}_{\ell+1} - \tilde{\varphi}_\ell|_{\infty, \overline{\mathcal{O}}} \leq \beta |\tilde{\varphi}_\ell - \tilde{\varphi}_{\ell-1}|_{\infty, \overline{\mathcal{O}}}.$$

Hence, the sequence $(\tilde{\varphi}_\ell)$ tends (geometrically) to a limit in the space $C^0(\tilde{\mathcal{O}}, \tilde{\mathcal{Z}})$. Of course, this limit must be the pointwise limit $\tilde{\varphi}$ and the proof is complete.

For $j \geq 0$, let us denote by $\mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}})$ the space of j -linear mappings from \mathbb{R}^{n+1} into $\tilde{\mathcal{Z}}$ with the usual abuse of notation $\mathcal{L}_0(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}) = \tilde{\mathcal{Z}}$. Also, recall the canonical isomorphism

$$\mathcal{L}_{i+1}(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}) \simeq \mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}})), \quad (\text{A2.26})$$

which will be repeatedly used in sequel. □ 261

We shall denote by λ_i the generic element of the space $\mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}})$. Due to the identification $\mathcal{L}_0(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}) = \tilde{\mathcal{Z}}$, this means, in particular, that we shall identify the element $\tilde{z} \in \tilde{\mathcal{Z}}$ with λ_0 so that the assumption $D_{\tilde{x}}\Phi(0) = 0$ will be rewritten as

$$D_{\lambda_0}\Phi(0) = 0. \quad (\text{A2.27})$$

Now, setting $\Phi = \Phi_0$, introduce the mappings

$$\Phi_j : \tilde{\mathcal{O}} \times \bar{B}(0, \rho) \times \prod_{i=1}^j \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}) \rightarrow \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}), \quad 1 \leq j \leq k,$$

by

$$\begin{aligned} \Phi_j(\tilde{x}, \lambda_0, \dots, \lambda_j) &= \frac{\partial \Phi_{j-1}}{\partial x}(\tilde{x}, \lambda_0, \dots, \lambda_{j-1}) + \\ &+ \sum_{i=0}^{j-1} \frac{\partial \Phi_{j-1}}{\partial \lambda_i}(\tilde{x}, \lambda_0, \dots, \lambda_{j-1}) \lambda_{i+1}, \end{aligned} \quad (\text{A2.28})$$

where the term $\frac{\partial \Phi_{j-1}}{\partial \lambda_i}(\tilde{x}, \lambda_0, \dots, \lambda_{j-1}) \lambda_{i+1}$ is the product of the linear mappings

$$\frac{\partial \Phi_{j-1}}{\partial \lambda_i}(\tilde{x}, \lambda_0, \dots, \lambda_{j-1}) \in \mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}), \mathcal{L}_{j-1}(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}))$$

and

$$\lambda_{i+1} \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}})).$$

Remark A2.1: Observe, in particular, that the only term involving λ_j in the definition of Φ_j , $j \geq 1$, is the term $\frac{\partial \Phi_{j-1}}{\partial \lambda_{j-1}}(\tilde{Z}, \lambda_0, \dots, \lambda_{j-1})\lambda_j$ it follows that $\Phi_j(\tilde{x}, \lambda_0, \dots, \lambda_j)$ is linear with respect to λ_j when $j \geq 1$.

The importance of the mappings Φ_j , $0 \leq j \leq k$ lies in the fact that

$$D^j \tilde{\varphi}(\tilde{x}) = \Phi_j(\tilde{x}, \tilde{\varphi}(\tilde{x}), D\tilde{\varphi}(\tilde{x}), \dots, D^j \tilde{\varphi}(\tilde{x})), 0 \leq j \leq k \quad (\text{A2.29})$$

262 for every $\tilde{x} \in \overline{\mathcal{O}}$ and, for every $\ell \geq 0$

$$D^j \tilde{\varphi}_{\ell+1}(\tilde{x}) = \Phi_j(\tilde{x}, \tilde{\varphi}_\ell(\tilde{x}), D\tilde{\varphi}_\ell(\tilde{x}), \dots, D^j \tilde{\varphi}_\ell(\tilde{x})), 0 \leq j \leq k, \quad (\text{A2.30})$$

for every $x \in \overline{\mathcal{O}}$. These properties can be immediately checked by induction. Also, it is clear that the mapping Φ_j is of class C^{k-j} and, for $1 \leq j \leq k$, Φ_j is of class C^∞ with respect to $(\lambda_1, \dots, \lambda_j)$. We now establish two simple preliminary lemmas.

Lemma A2.3: Given any index $0 \leq j \leq k$, one has

$$\frac{\partial \Phi_j}{\partial \lambda_j}(0, 0, \lambda_1, \dots, \lambda_j) = 0, \quad (\text{A2.31})$$

for every $(\lambda_1, \dots, \lambda_j) \in \prod_{i=1}^j \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z})$.

Proof. For $j = 0$, the result is nothing but (A2.27). For $j \geq 1$, it follows from Remark A2.1 that for every $\mu_j \in \mathcal{L}(\mathbb{R}^{n+1}, \tilde{Z})$

$$\frac{\partial \Phi_j}{\partial \lambda_j}(\tilde{x}, \lambda_0, \dots, \lambda_j) \cdot \mu_j = \frac{\partial \Phi_{j-1}}{\partial \lambda_{j-1}}(\tilde{x}, \lambda_0, \dots, \lambda_{j-1})\mu_j.$$

Choosing $\tilde{x} = 0, \lambda_0 = 0$, the above relation yields the desired result by an induction argument. \square

Lemma A2.4: Let the index $0 \leq j \leq k$ be fixed. If the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^j(\overline{\mathcal{O}}, \tilde{Z})$ (which is already known for $j = 0$), the set

$$\Lambda_j = \bigcup_{\ell \geq 0} D^j \tilde{\varphi}_\ell(\overline{\mathcal{O}}) \subset \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z}), \quad (\text{A2.32})$$

is compact and one has

$$D^j \tilde{\varphi}(\overline{\mathcal{O}}) \subset \Lambda_j. \quad (\text{A2.33})$$

263 *Proof.* A sequence in the set $\bigcup_{\ell \geq 0} D^j \tilde{\varphi}_\ell(\overline{\mathcal{O}})$ is of the form $(\lambda_j^{(p)})_{p > 0}$ where for each $p \in \mathbb{N}$ there is an index $\ell = \ell(p)$ and an element $\tilde{x}_p \in \overline{\mathcal{O}}$ such that $\lambda_j^{(p)} = D^j \tilde{\varphi}_{\ell(p)}(\tilde{x}_p)$. If, for infinitely many indices p , namely for a subsequence (p_m) , the index $\ell(p_m)$ equals some fixed value ℓ , one has

$$\lambda_j^{(p_m)} = D^j \tilde{\varphi}_\ell(\tilde{x}_{p_m}).$$

As the set $\overline{\mathcal{O}}$ is compact and after extracting a subsequence, we may assume that there is $\tilde{x} \in \overline{\mathcal{O}}$ such that the sequence (\tilde{x}_{p_m}) tends to \tilde{x} . By continuity of the mapping $D^j \tilde{\varphi}_\ell$, we deduce that the sequence $\lambda_j^{(p_m)}$ tends to $D^j \tilde{\varphi}_\ell(\tilde{x})$ as m tends to $+\infty$.

Now, assume that the mapping $p \rightarrow \ell(p)$ takes only finitely many times any given value ℓ . Then, there is a subsequence $\ell(p_m)$ which is strictly increasing and thus tends to $+\infty$ as m tends to $+\infty$. For every m , let us write

$$D^j \tilde{\varphi}_{\ell(p_m)}(\tilde{x}_{p_m}) = D^j \tilde{\varphi}_{\ell(p_m)}(\tilde{x}_{p_m}) - D^j \tilde{\varphi}(\tilde{x}_{p_m}) + D^j \tilde{\varphi}(\tilde{x}_{p_m}).$$

Again, in view of the compactness of the set $\overline{\mathcal{O}}$, we may assume that the sequence (\tilde{x}_{p_m}) tends to $\tilde{x} \in \overline{\mathcal{O}}$. As the sequence $(D^j \tilde{\varphi}_\ell)_{\ell \geq 0}$ tends to $D^j \tilde{\varphi}$ in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}}))$, by hypothesis, the same property holds for the subsequence $(D^j \tilde{\varphi}_{\ell(p_m)})_{m \geq 0}$. Therefore, in the space $\mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}})$

$$\lim_{m \rightarrow +\infty} [D^j \tilde{\varphi}_{\ell(p_m)}(\tilde{x}_{p_m}) - D^j \tilde{\varphi}(\tilde{x}_{p_m})] = 0$$

On the other hand, by the continuity of the mapping $D^j \tilde{\varphi}$, we get

$$\lim_{m \rightarrow +\infty} D^j \tilde{\varphi}(\tilde{x}_{p_m}) = D^j \tilde{\varphi}(\tilde{x}).$$

As a result, the sequence $(\lambda_j^{(p_m)})_{m \geq 0}$ tends to $D^j \tilde{\varphi}(\tilde{x})$.

To sum up, we have shown that every sequence $(\lambda_j^{(p)})_{p \geq 0}$ of the set $\bigcup_{\ell \geq 0} D^j \tilde{\varphi}_\ell(\overline{\mathcal{O}})$ has a cluster point in the space $\mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{\mathcal{Z}})$ and hence is relatively compact, which proves (A2.32). The relation (A2.33) is now obvious since $D^j \tilde{\varphi}(\tilde{x})$ is the limit of the sequence $(D^j \tilde{\varphi}_\ell(\tilde{x}))_{\ell \geq 0}$ for every $\tilde{x} \in \overline{\mathcal{O}}$. \square

In our new notation, the condition (A2.25) can be rewritten as

$$\left\| \frac{\partial \Phi_0}{\partial \lambda_0}(\bar{x}, \lambda_0) \right\| \leq \beta, \quad (\text{A2.34})$$

for every $(\bar{x}, \lambda_0) \in \bar{\mathcal{O}} \times \bar{B}(0, \rho)$. It will be essential in the sequel to have an appropriate generalization of it, which we establish in the following lemma.

Lemma A2.5: *Let the index $0 \leq j \leq k$ be fixed. If the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^j(\bar{\mathcal{O}}, \bar{Z})$ (which is already known for $j = 0$) there is no loss of generality in assuming that*

$$\left\| \frac{\partial \Phi_j}{\partial \lambda_j}(\bar{x}, \lambda_0, \dots, \lambda_j) \right\| \leq \beta, \quad (\text{A2.35})$$

for every $(\bar{x}, \lambda_0, \dots, \lambda_j) \in \bar{\mathcal{O}} \times \Lambda_0 \times \dots \times \Lambda_j$.¹

Proof. First from the previous lemma, the set $\bar{\mathcal{O}} \times \Lambda_0 \times \dots \times \Lambda_j$ is compact and the restriction of the continuous mapping $(\partial \Phi_j / \partial \lambda_j)$ to $\bar{\mathcal{O}} \times \Lambda_0 \times \dots \times \Lambda_j$ is then *uniformly continuous*. Next, since the mappings $\tilde{\varphi}$ and $\tilde{\varphi}_\ell$, $\ell \geq 0$, take their values in the ball $\bar{B}(0, \rho)$, one has $\Lambda_0 \subset \bar{B}(0, \rho)$. Due to Lemma A2.3, and the uniform continuity of the mapping $(\partial \Phi_j / \partial \lambda_j)$, there is a neighbourhood $\mathcal{O}' \subset \mathcal{O}$ of the origin in \mathbb{R}^{n+1} and $0 < \rho' \leq \rho$ such that (A2.35) holds for $(\bar{x}, \lambda_0, \dots, \lambda_j) \in \bar{\mathcal{O}}' \times [\Lambda_0 \cap \bar{B}(0, \rho')] \times \Lambda_1 \times \dots \times \Lambda_j$. Let then ρ' be fixed. Arguing as in §6 of Chapter 4, we can shrink the neighbourhood $\bar{\mathcal{O}}'$ so that the mappings $\tilde{\varphi}_\ell$, $\ell \geq 0$, take their values in the ball $\bar{B}(0, \rho')$ for $x \in \bar{\mathcal{O}}'$. Of course, the sequence $(\tilde{\varphi}_\ell)$ still tends to $\tilde{\varphi}$ in the space $C^j(\bar{\mathcal{O}}, \bar{Z})$ and the sets $\Lambda'_i = \bigcup_{\ell \geq 0} D^i \tilde{\varphi}_\ell(\bar{\mathcal{O}}')$, $0 \leq i \leq j$, are compact (Lemma A2.4) with $\Lambda'_i \subset \Lambda_i$. Also, $\Lambda'_0 \subset \bar{B}(0, \rho')$ so that inequality (A2.35) is a fortiori valid with $(\bar{x}, \lambda_0, \dots, \lambda_k) \in \bar{\mathcal{O}}' \times \Lambda'_0 \times \dots \times \Lambda'_j$. In other words, none of the properties we have proved is affected if we restrict ourselves to the neighbourhood \mathcal{O}' instead of \mathcal{O} and, in addition, inequality (A2.35) holds. \square

¹For $j = 0$, (A2.35) is nothing but (A2.34) for the elements $(\bar{x}, \lambda_0) \in \bar{\mathcal{O}} \times \Lambda_0$, thus a weak form of (A2.34).

For $0 \leq j \leq k-1$ and $\tilde{x} \in \overline{\mathcal{O}}$, we shall set

$$b^j(\tilde{x}) = \frac{\partial \Phi_j}{\partial \tilde{x}}(\tilde{x}, \tilde{\varphi}(\tilde{x}), D\tilde{\varphi}(\tilde{x}), \dots, D^j \tilde{\varphi}(\tilde{x})), \quad (\text{A2.36})$$

$$b_\ell^j(\tilde{x}) = \frac{\partial \Phi_j}{\partial \tilde{x}}(\tilde{x}, \tilde{\varphi}_\ell(\tilde{x}), D\tilde{\varphi}_\ell(\tilde{x}), \dots, D^j \tilde{\varphi}_\ell(\tilde{x})) \text{ for } \ell \geq 0. \quad (\text{A2.37})$$

Similarly, for $0 \leq j \leq k-1$, $0 \leq i \leq j$ and $\tilde{x} \in \overline{\mathcal{O}}$, set

$$B_i^j(\tilde{x}) = \frac{\partial \Phi_j}{\partial \lambda_i}(\tilde{x}, \tilde{\varphi}(\tilde{x}), D\tilde{\varphi}(\tilde{x}), \dots, D^j \tilde{\varphi}(\tilde{x})), \quad (\text{A2.38})$$

$$B_{i\ell}^j(\tilde{x}) = \frac{\partial \Phi_j}{\partial \lambda_i}(\tilde{x}, \tilde{\varphi}_\ell(\tilde{x}), D\tilde{\varphi}_\ell(\tilde{x}), \dots, D^j \tilde{\varphi}_\ell(\tilde{x})) \text{ for } \ell \geq 0. \quad (\text{A2.39})$$

For every $\tilde{x} \in \overline{\mathcal{O}}$, $b^j(\tilde{x})$ (resp. $b_\ell^j(\tilde{x})$) is an element of the space $\mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z}))$ and $B_i^j(\tilde{x})$ (resp. $B_{i\ell}^j(\tilde{x})$) is an element of the space $\mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z}), \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z}))$. 266

Lemma A2.6: (i) Assume that $k \geq 2$ and $0 \leq j \leq k-2$. Then, if the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ geometrically in the space $C^j(\overline{\mathcal{O}}, \tilde{Z})$, the sequence $(b_\ell^j)_{\ell \geq 0}$ tends to b^j geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z})))$ and the sequence $(B_{i\ell}^j)_{\ell \geq 0}$ tends to B_i^j geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z}), \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z})))$.

(ii) Assume only $k \geq 1$ and $0 \leq j \leq k-1$ and that the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^j(\overline{\mathcal{O}}, \tilde{Z})$. Then, the sequence $(b_\ell^j)_{\ell \geq 0}$ tends to b^j in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z})))$ and the sequence $(B_{i\ell}^j)_{\ell > 0}$ tends to B_i^j in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z}), \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z})))$.

Proof. First, as the mapping Φ_j is of class C^{k-j} , the mappings b^j , b_ℓ^j , $B_{i\ell}^j$ and B_i^j are of class C^1 in the case (i) and of class C^0 in the case (ii). Throughout this proof and for the sake of convenience, we shall set for $\tilde{x} \in \overline{\mathcal{O}}$

$$V^j(\tilde{x}) = (\tilde{\varphi}(\tilde{x}), D\tilde{\varphi}(\tilde{x}), \dots, D^j \tilde{\varphi}(\tilde{x})) \quad (\text{A2.40})$$

and

$$V_\ell^j(\tilde{x}) = (\tilde{\varphi}_\ell(\tilde{x}), D\tilde{\varphi}_\ell(\tilde{x}), \dots, D^j \tilde{\varphi}_\ell(\tilde{x})). \quad (\text{A2.41})$$

This allows to write the relations (A2.36) - (A2.39) in the form

$$b^j(\bar{x}) = \frac{\partial \varphi_j}{\partial \bar{x}}(\bar{x}, V^j(\bar{x})), \quad (\text{A2.42})$$

$$b_\ell^j(\bar{x}) = \frac{\partial \Phi_j}{\partial \bar{x}}(\bar{x}, V_\ell^j(\bar{x})), \quad (\text{A2.43})$$

$$B_i^j(\bar{x}) = \frac{\partial \Phi_j}{\partial \lambda_i}(\bar{x}, V^j(\bar{x})), \quad (\text{A2.44})$$

$$B_{i\ell}^j(\bar{x}) = \frac{\partial \Phi_j}{\partial \lambda_i}(\bar{x}, V - \ell^j(\bar{x})). \quad (\text{A2.45})$$

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Before proving (i), note that, for $\bar{x} \in \bar{\mathcal{O}}$ and $0 \leq s \leq 1$, the combinations

$$V^j(\bar{x}) + s(V_\ell^j(\bar{x}) - V^j(\bar{x})),$$

belong to the product $\Lambda_0^c \times \cdots \times \Lambda_j^c$ of the closed convex hulls² Λ_i^c of the compact sets Λ_i (cf. Lemma A2.4). As the closed convex hull of a compact set is by a classical result from topology, the product $\Lambda_0^c \times \cdots \times \Lambda_j^c$ is compact. On the other hand, since the mappings $\bar{\varphi}$ and $\bar{\varphi}_\ell$ for $\ell \geq 0$ take their values in the ball $\bar{B}(0, \rho)$, we deduce that Λ_0 and hence Λ_0^c is contained in $\bar{B}(0, \rho)$. Therefore, for every $0 \leq i \leq j \leq k-2$, an expression such as

$$\frac{\partial^2 \Phi_j}{\partial \bar{x} \partial \lambda_i}(\bar{x}, V^j(\bar{x}) + s(V_\ell^j(\bar{x}) - V^j(\bar{x}))),$$

is well defined for $\bar{x} \in \bar{\mathcal{O}}$ and $0 \leq s \leq 1$. For $\bar{x} \in \bar{\mathcal{O}}$ and due to the Taylor formula (which can be used in view of the above observation) we get

$$b_\ell^j(\bar{x}) - b^j(\bar{x}) = \sum_{i=0}^j \int_0^1 \frac{\partial^2 \Phi_j}{\partial \bar{x} \partial \lambda_i}(\bar{x}, V^j(\bar{x}) + s(V_\ell^j(\bar{x}) - V^j(\bar{x}))) \cdot (D^i \bar{\varphi}_\ell(\bar{x}) - D^i \bar{\varphi}(\bar{x})) ds.$$

268 From the compactness of the set $\bar{\mathcal{O}} \times \Lambda_0^c \times \cdots \times \Lambda_j^c$, the quantities $\left\| \frac{\partial^2 \Phi_j}{\partial \bar{x} \partial \lambda_i}(\bar{x}, \lambda_0, \dots, \lambda_j) \right\|$ are bounded by a constant $K > 0$ for $(\bar{x}, \lambda_0, \dots,$

²Recall that the closed convex hull of a set is defined as the closure of its convex hull.

$\lambda_j) \in \overline{\mathcal{O}} \times \Lambda_0^c \times \cdots \times \Lambda_j^c$ and $0 \leq i \leq j$. Thus

$$\|b_\ell^j(\bar{x}) - b^j(\bar{x})\| \leq K \sum_{i=0}^j \|D^i \tilde{\varphi}_\ell(\bar{x}) - D^i \tilde{\varphi}(\bar{x})\|.$$

As the sequence $(D^i \varphi_\ell)_{\ell \geq 0}$ tends to $D^i \tilde{\varphi}$ geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z}))$ for every $0 \leq i \leq j$ by hypothesis, there exist q with $0 < q < 1$ and a constant $C > 0$ such that

$$\|D^i \tilde{\varphi}_\ell(\bar{x}) - D^i \tilde{\varphi}(\bar{x})\| \leq Cq^\ell,$$

for every $0 \leq i \leq j$ and every $\bar{x} \in \overline{\mathcal{O}}$. Hence

$$\|b_\ell^j(\bar{x}) - b^j(\bar{x})\| \leq (j+1)KCq^\ell,$$

for every $\bar{x} \in \overline{\mathcal{O}}$ and then

$$\sup_{\bar{x} \in \overline{\mathcal{O}}} \|b_\ell^j(\bar{x}) - b^j(\bar{x})\| \leq (j+1)KCq^\ell.$$

The above inequality proves that the sequence $(b_\ell^j)_{\ell \geq 0}$ tends to b^j geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{B})))$. Similar arguments can be used for showing that the sequence $(B_{i\ell}^j)_{\ell \geq 0}$ tends to B_i^j geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z}), \mathcal{L}_j(\mathbb{R}^{n+1}, \tilde{Z})))$, $0 \leq i \leq j$.

Now, we prove part (ii) of the statement. Here, the Taylor formula is not available because of the lack of regularity for $j = k - 1$. Actually, as we no longer interested in the rate of convergence, the results can be easily deduced from the *uniform continuity* of the continuous mapping $(\partial\Phi_j/\partial\bar{x})$ and $(\partial\Phi_j/\partial\lambda_i)$, $0 \leq i \leq j$ on the compact set $\overline{\mathcal{O}} \times \Lambda_0 \times \cdots \times \Lambda_j$ together with the *uniform convergence* (i.e. in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_i(\mathbb{R}^{n+1}, \tilde{Z}))$ of the sequence $(D^i \tilde{\varphi}_\ell)_{\ell \geq 0}$ to the mapping $D^i \tilde{\varphi}$ for $0 \leq i \leq j$ from which the uniform convergence of the sequence $(V_\ell^j)_{\ell \geq 0}$ to the mapping V^j is derived.) \square 269

§ We are now in a position to prove an important part of the results announced before.

Theorem A2.2 : *After shrinking the neighbourhood \mathcal{O} is necessary, the sequence $(\widetilde{\varphi}_\ell)$ tends to the mapping $\widetilde{\varphi}$ geometrically in the space $C^{k-1}(\overline{\mathcal{O}}, \widetilde{\mathcal{Z}})$.*

Proof. Equivalently, we have to prove for $0 \leq j \leq k-1$ the sequence $(D^j \widetilde{\varphi}_\ell)_{\ell \geq 0}$ tends to $D^j \widetilde{\varphi}$ geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_j(\mathbb{R}^{n+1}, \widetilde{\mathcal{Z}}))$. This result has already been proved for $j = 0$ in Lemma A2.2 and we can thus assume $j \geq 1$ and $k \geq 2$. We proceed by induction and therefore assume that the sequence $(D^i \widetilde{\varphi}_\ell)_{\ell \geq 0}$ tends to $D^i \widetilde{\varphi}$ geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_i(\mathbb{R}^{n+1}, \widetilde{\mathcal{Z}}))$ for $0 \leq i \leq j-1$.

Let $\ell \geq 0$ be fixed. From (A2.29) and (A2.30), one has for $\widetilde{x} \in \overline{\mathcal{O}}$

$$\begin{aligned} D^j \widetilde{\varphi}_{\ell+1}(\widetilde{x}) - D^j \widetilde{\varphi}_\ell(\widetilde{x}) &= \Phi_j(\widetilde{x}, \widetilde{\varphi}_\ell(\widetilde{x}), D\widetilde{\varphi}_\ell(\widetilde{x}), \dots, D^j \widetilde{\varphi}_\ell(\widetilde{x})) \\ &\quad - \Phi_j(\widetilde{x}, \widetilde{\varphi}(\widetilde{x}), D\widetilde{\varphi}(\widetilde{x}), \dots, D^j \widetilde{\varphi}(\widetilde{x})). \end{aligned}$$

By definition of the mapping Φ_j (cf. (A2.48)) and in the notations (A2.42) - (A2.45), this identity can be rewritten as

$$\begin{aligned} D^j \widetilde{\varphi}_{\ell+1}(\widetilde{x}) - D^j \widetilde{\varphi}_\ell(\widetilde{x}) &= b_\ell^{j-1}(\widetilde{x}) - b^{j-1}(\widetilde{x}) \\ &\quad + \sum_{i=0}^{j-1} [B_{i\ell}^{j-1}(\widetilde{x}) D^{i+1} \widetilde{\varphi}_\ell(\widetilde{x}) - B_i^{j-1}(\widetilde{x}) D^{i+1} \widetilde{\varphi}(\widetilde{x})]. \end{aligned}$$

Recall in this formula that the element $D^{i+1} \widetilde{\varphi}_\ell(\widetilde{x})$ (resp. $D^{i+1} \widetilde{\varphi}(\widetilde{x})$) is considered as an element of $\mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_i(\mathbb{R}^{n+1}, \widetilde{\mathcal{Z}}))$ and the operation $B_{i\ell}^{j-1}(\widetilde{x}) D^{i+1} \widetilde{\varphi}_\ell(\widetilde{x})$ (resp. $B_i^{j-1}(\widetilde{x}) D^{i+1} \widetilde{\varphi}(\widetilde{x})$) denotes composition of linear mappings. Thus, we can as well write

$$\begin{aligned} D^j \widetilde{\varphi}_{\ell+1}(\widetilde{x}) - D^j \widetilde{\varphi}_\ell(\widetilde{x}) &= b_\ell^{j-1}(\widetilde{x}) - b^{j-1}(\widetilde{x}) \\ &\quad + \sum_{i=0}^{j-1} (B_{i\ell}^{j-1}(\widetilde{x}) - B_i^{j-1}(\widetilde{x})) D^{i+1} \widetilde{\varphi}(\widetilde{x}) \\ &\quad + \sum_{i=0}^{j-1} B_{i\ell}^{j-1}(\widetilde{x}) (D^{i+1} \widetilde{\varphi}_\ell(\widetilde{x}) - D^{i+1} \widetilde{\varphi}(\widetilde{x})). \end{aligned}$$

We deduce the inequality

$$\|D^j \widetilde{\varphi}_{\ell+1}(\widetilde{x}) - D^j \widetilde{\varphi}_\ell(\widetilde{x})\| \leq \|b_\ell^{j-1}(\widetilde{x}) - b^{j-1}(\widetilde{x})\|$$

$$\begin{aligned}
& + \sum_{i=0}^{j-1} \|B_{i\ell}^{j-1}(\bar{x}) - B_i^{j-1}(\bar{x})\| \|D^{i+1}\tilde{\varphi}(\bar{x})\| \quad (\text{A2.46}) \\
& + \sum_{i=0}^{j-1} \|B_{i\ell}^{j-1}(\bar{x})\| \|D^{i+1}\tilde{\varphi}_\ell(\bar{x}) - D^{i+1}\tilde{\varphi}(\bar{x})\|.
\end{aligned}$$

It is essential to pay special attention to the term corresponding with $i = j - 1$ in the last sum, namely $\|B_{j-1,\ell}^{j-1}(\bar{x})\| \|D^j\tilde{\varphi}_\ell(\bar{x}) - D^j\tilde{\varphi}(\bar{x})\|$. From (A2.39)

$$B_{j-1,\ell}^{j-1}(\bar{x}) = \frac{\partial \Phi_{j-1}}{\partial \lambda_{j-1}}(\bar{x}, \tilde{\varphi}_\ell(\bar{x}), D\tilde{\varphi}_\ell(\bar{x}), \dots, D^{j-1}\tilde{\varphi}_\ell(\bar{x})).$$

As the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^{j-1}(\bar{\mathcal{O}}, \bar{\mathcal{Z}})$ by hypothesis, we can apply Lemma A2.5 with $j - 1$ (which amount to shrinking the neighbourhood $\bar{\mathcal{O}}$ if necessary) and hence

$$\|B_{j-1,\ell}^{j-1}(\bar{x})\| \leq \beta,$$

for every $\bar{x} \in \bar{\mathcal{O}}$. Inequality (A2.46) becomes

$$\begin{aligned}
\|D^j\tilde{\varphi}_{\ell+1}(\bar{x}) - D^j\tilde{\varphi}(\bar{x})\| & < \|b_\ell^{j-1}(\bar{x}) - b^{j-1}(\bar{x})\| \\
& + \sum_{i=0}^{j-1} \|B_{i\ell}^{j-1}(\bar{x}) - B_i^{j-1}(\bar{x}) - B_i^{j-1}(\bar{x})\| \|D^{i+1}\tilde{\varphi}\| \\
& + \sum_{i=0}^{j-2} \|B_{i\ell}^{j-1}(\bar{x})\| \|D^{i+1}\tilde{\varphi}_\ell(\bar{x}) - D^{i+1}\tilde{\varphi}(\bar{x})\| + \\
& + \beta \|D^j\tilde{\varphi}_\ell - D^j\tilde{\varphi}(\bar{x})\|. \quad (\text{A2.47})
\end{aligned}$$

Now, since $j \leq k - 1$, we can apply Lemma A2.6 (i) with $j - 1$: **271**
There are constant q with $0 < q < 1$ and $C > 0$ such that

$$\begin{aligned}
\|b_\ell^{j-1}(\bar{x}) - b^{j-1}(\bar{x})\| & \leq Cq^\ell, \\
\|B_{i\ell}^{j-1}(\bar{x}) - B_i^{j-1}(\bar{x})\| & \leq Cq^\ell, 0 \leq i \leq j - 1,
\end{aligned}$$

for every $\ell \geq 0$ and every $\tilde{x} \in \overline{\mathcal{O}}$. As $\tilde{\varphi} \in C^k(\overline{\mathcal{O}}, \tilde{Z}) \subset C^j(\overline{\mathcal{O}}, \tilde{Z})$, there is a constant $K > 0$ such that

$$\|D^{i+1}\tilde{\varphi}(\tilde{x})\| \leq K,$$

for every $\tilde{x} \in \overline{\mathcal{O}}$ and $0 \leq i \leq j-1$. Together with (A2.47), these observations give

$$\begin{aligned} \|D^j\tilde{\varphi}_{\ell+1}(\tilde{x}) - D^j\tilde{\varphi}(\tilde{x})\| &< (jK+1)Cq^\ell \\ &+ \sum_{i=0}^{j-2} \|B_{i\ell}^{j-1}(\tilde{x})\| \|D^{i+1}\tilde{\varphi}_\ell(\tilde{x}) - D^{i+1}\tilde{\varphi}(\tilde{x})\| \\ &+ \beta \|D^j\tilde{\varphi}_\ell(\tilde{x}) - D^j\tilde{\varphi}(\tilde{x})\|. \end{aligned} \tag{A2.48}$$

From the convergence of the sequence $(\tilde{\varphi}_\ell)$ to $\tilde{\varphi}$ geometrically in the space $C^{j-1}(\overline{\mathcal{O}}, \tilde{Z})$, it is not restrictive to assume that q with $0 < q < 1$ and $C > 0$ are such that

$$\|D^{i+1}\tilde{\varphi}_\ell(\tilde{x}) - D^{i+1}\tilde{\varphi}(\tilde{x})\| \leq Cq^\ell,$$

for every $\tilde{x} \in \overline{\mathcal{O}}$, $0 \leq i \leq j-2$. Besides, we can suppose the constant K large enough for the inequality

$$\|B_{i\ell}^{j-1}(\tilde{x})\| \leq K,$$

to hold for every $\tilde{x} \in \overline{\mathcal{O}}$, $0 \leq i \leq j-2$ and $\ell \geq 0$. Indeed, this follows from the definition (A2.39), namely

$$B_{i\ell}^{j-1}(\tilde{x}) = \frac{\partial \Phi_{j-1}}{\partial \lambda_i}(\tilde{x}, \tilde{\varphi}_\ell(\tilde{x}), D\tilde{\varphi}_\ell(\tilde{x}), \dots, D^{j-1}\tilde{\varphi}_\ell(\tilde{x}))$$

272 and the compactness of the set $\overline{\mathcal{O}} \times \Lambda_0 \times \dots \times \Lambda_{j-1}$ (Lemma A2.4 with $j-1$). Inequality (A2.48) then takes the simpler form

$$\|D^j\tilde{\varphi}_{\ell+1}(\tilde{x}) - D^j\tilde{\varphi}(\tilde{x})\| \leq [(2j-1)K+1]Cq^\ell + \beta \|D^j\tilde{\varphi}_\ell(\tilde{x}) - D^j\tilde{\varphi}(\tilde{x})\|.$$

Replacing q by $\max(q, \beta) < 1$ and setting

$$K_0 = \frac{1}{q} [(2j-1)K+1]C,$$

the above inequality becomes

$$\|D^j \widetilde{\varphi}_{\ell+1}(\bar{x}) - D^j \widetilde{\varphi}(\bar{x})\| < K_0 q^{\ell+1} + q \|D^j \widetilde{\varphi}_\ell(\bar{x}) - D^j \widetilde{\varphi}(\bar{x})\|$$

and yields

$$|D^j(\widetilde{\varphi}_{\ell+1} - \widetilde{\varphi})|_{\infty, \overline{\mathcal{O}}} \leq K_0 q^{\ell+1} + q |D^j(\widetilde{\varphi}_\ell - \widetilde{\varphi})|_{\infty, \overline{\mathcal{O}}}. \quad (\text{A2.49})$$

Let us define the sequence $(a_\ell)_{\ell \geq 0}$ by

$$\begin{cases} a_0 &= |D^j \widetilde{\varphi}|_{\infty, \overline{\mathcal{O}}}, \\ a_{\ell+1} &= K_0 q^{\ell+1} + q a_\ell, \ell \geq 0 \end{cases}$$

It is immediately checked that

$$|D^j(\widetilde{\varphi}_\ell - \widetilde{\varphi})|_{\infty, \overline{\mathcal{O}}} \leq a_\ell, \quad (\text{A2.50})$$

for every $\ell \geq 0$ (recall that $\widetilde{\varphi}_0 = 0$). On the other hand, the sequence (a_ℓ) is equivalently defined by

$$a_\ell = (K_0 \ell + a_0) q^\ell, \ell \geq 0.$$

Choosing $q < q' < 1$, this is the same as

$$a_\ell = (K_0 \ell + a_0) \left(\frac{q}{q'}\right) (q')^\ell, \ell \geq 0.$$

As $0 < q/q' < 1$, the term $(K_0 \ell + a_0)(q/q')^\ell$ tends to 0 and hence is uniformly bounded by a constant $K_1 > 0$. Thus, 273

$$a_\ell \leq K_1 (q')^\ell, \ell \geq 0$$

and our assertion follows from (A2.50). □

Replacing k by $k + 1$ in Theorem A2.2, we get

Corollary A2.1: *Assume that the mapping Φ is of class C^{k+1} . Then, after shrinking the neighbourhood \mathcal{O} is necessary, the sequence $(\widetilde{\varphi}_\ell)$ tends to the mapping $\widetilde{\varphi}$ geometrically in the space $C^k(\overline{\mathcal{O}}, \widetilde{Z})$.*

Our next theorem shows that even when the mapping Φ is only of class C^k , the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^k(\overline{\mathcal{O}}, \tilde{Z})$ but the rate of convergence is not known in general (compare with Corollary A2.1 above).

Theorem A2.3: *After shrinking the neighbourhood \mathcal{O} if necessary, the sequence $(\tilde{\varphi}_\ell)$ tends to the mapping $\tilde{\varphi}$ in the space $C^k(\overline{\mathcal{O}}, \tilde{Z})$.*

Proof. From Theorem A2.2, we know that the sequence $(\tilde{\varphi}_\ell)$ tends to $\tilde{\varphi}$ in the space $C^{k-1}(\overline{\mathcal{O}}, \tilde{Z})$ and it remains to prove the convergence of the sequence $(D^k \tilde{\varphi}_\ell)_{\ell \geq 0}$ to $D^k \tilde{\varphi}$ in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_k(\mathbb{R}^{n+1}, \tilde{Z}))$. To do this, we can repeat a part of the proof of Theorem A2.2. Replacing j by k , the observations we made up to and including formula (A2.47) are still valid and we then find

$$\begin{aligned} \|D^k \tilde{\varphi}_{\ell+1}(\tilde{x}) - D^k \tilde{\varphi}(\tilde{x})\| &\leq \|b_\ell^{k-1}(\tilde{x}) - b^{k-1}(\tilde{x})\| \\ &+ \sum_{i=0}^{k-1} \|B_{i\ell}^{k-1}(\tilde{x}) - B_i^{k-1}(\tilde{x})\| \|D^{i+1} \tilde{\varphi}(\tilde{x})\| \quad (\text{A2.51}) \\ &+ \sum_{i=0}^{k-2} \|B_{i\ell}^{k-1}(\tilde{x})\| \|D^{i+1} \tilde{\varphi}_\ell(\tilde{x}) - D^{i+1} \tilde{\varphi}(\tilde{x})\| \\ &+ \beta \|D^k \tilde{\varphi}_\ell(\tilde{x}) - D^k \tilde{\varphi}(\tilde{x})\|, \end{aligned}$$

274 for every $\tilde{x} \in \overline{\mathcal{O}}$ and every $\ell \geq 0$. Here, we can no longer use Lemma A2.6 (i) for finding an estimate of the terms $\|b_\ell^{k-1}(\tilde{x}) - b^{k-1}(\tilde{x})\|$ or $\|B_{i\ell}^{k-1}(\tilde{x}) - B_i^{k-1}(\tilde{x})\|$ but Lemma A2.6 (ii) applies with $j = k - 1$: Given $\epsilon > 0$ and ℓ large enough, say $\ell \geq \ell_0$, one has

$$\begin{aligned} \|b_\ell^{k-1}(\tilde{x}) - b^{k-1}(\tilde{x})\| &\leq \epsilon, \\ \|B_{i\ell}^{k-1}(\tilde{x}) - B_i^{k-1}(\tilde{x})\| &\leq \epsilon, \quad 0 \leq i \leq k-1, \end{aligned}$$

for every $\tilde{x} \in \overline{\mathcal{O}}$. By the same arguments as in Theorem A2.2 and from the regularity $\tilde{\varphi} \in C^k(\overline{\mathcal{O}}, \tilde{Z})$, we get the existence of a constant $K > 0$ such that

$$\|D^{i+1} \tilde{\varphi}(\tilde{x})\| \leq K,$$

for every $\tilde{x} \in \overline{\mathcal{O}}$, $0 \leq i \leq k-1$ and

$$\|B_{i\ell}^{k-1}(\tilde{x})\| \leq K,$$

for every $\tilde{x} \in \overline{\mathcal{O}}$, $0 \leq i \leq k-2$ and $\ell \geq 0$ and ℓ_0 can be taken large enough for the inequality

$$\|D^{i+1}\tilde{\varphi}_\ell(\tilde{x}) - D^{i+1}\tilde{\varphi}(\tilde{x})\| \leq \epsilon$$

to hold for every $\tilde{x} \in \overline{\mathcal{O}}$, $0 \leq i \leq k-2$. Together with (A2.51), we arrive at

$$\|D^k\tilde{\varphi}_{\ell+1}(\tilde{x}) - D^k\tilde{\varphi}(\tilde{x})\| \leq [(2k-1)K+1]\epsilon + \beta\|D^k\tilde{\varphi}_\ell(\tilde{x}) - D^k\tilde{\varphi}(\tilde{x})\|.$$

Setting

$$K_0 = (2k-1)K+1,$$

the above inequality becomes

$$\|D^k\tilde{\varphi}_{\ell+1}(\tilde{x}) - D^k\tilde{\varphi}(\tilde{x})\| \leq K_0\epsilon\beta\|D^k\tilde{\varphi}_\ell(\tilde{x}) - D^k\tilde{\varphi}(\tilde{x})\|$$

and yields

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$$|D^k(\tilde{\varphi}_{\ell+1} - \tilde{\varphi})|_{\infty, \overline{\mathcal{O}}} < K_0\epsilon + \beta|D^k(\tilde{\varphi}_\ell - \tilde{\varphi})|_{\infty, \overline{\mathcal{O}}}.$$

Let us define the sequence $(a_\ell)_{\ell \geq \ell_0}$ by

$$\begin{cases} a_{\ell_0} = |D^k(\tilde{\varphi}_{\ell_0} - \tilde{\varphi})|_{\infty, \overline{\mathcal{O}}}, \\ a_{\ell+1} = K_0\epsilon + \beta a_\ell, \ell \geq \ell_0. \end{cases}$$

It is immediately checked that

$$|D^k(\tilde{\varphi}_\ell - \tilde{\varphi})|_{\infty, \overline{\mathcal{O}}} \leq a_\ell, \quad (\text{A2.52})$$

for every $\ell \geq \ell_0$. On the other hand, the sequence $(a_\ell)_{\ell \geq \ell_0}$ is equivalently defined by

$$a_\ell = K_0 \frac{1 - \beta^{\ell - \ell_0}}{1 - \beta} \epsilon + \beta^{\ell - \ell_0} a_{\ell_0}, \ell \geq \ell_0 + 1.$$

For $\ell \geq \ell_0$ large enough, say $\ell \geq \ell_1$, one has $\beta^{\ell-\ell_0} a_{\ell_0} \leq \epsilon$.

Thus

$$a_\ell \leq \left[\frac{K_0}{1-\beta} + 1 \right] \epsilon,$$

for $\ell \geq \ell_1$ and our assertion follows from (A2.52). \square

Here is now a result of a different kind that we shall use later on.

Lemma A2.7: *For every $\ell \geq 0$, one has*

$$D^i \tilde{\varphi}_{\ell+k}(0) = D^i \tilde{\varphi}(0), \quad (\text{A2.53})$$

for $0 \leq i \leq k$.

276 *Proof.* Again, we shall proceed by induction and show for $0 \leq j \leq k$ and every $\ell \geq 0$

$$D^i \tilde{\varphi}_{\ell+j}(0) = D^i \tilde{\varphi}(0), \quad (\text{A2.54})$$

for $0 \leq i \leq j$. This result is true for $j = 0$ since $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}_\ell(0) = 0$ as it follows from the definition. Assume then that $j \geq 1$ and the result is true up to rank $j - 1$, namely, that

$$D^i \tilde{\varphi}_{\ell+j-1}(0) = D^i \tilde{\varphi}(0), \quad (\text{A2.55})$$

for $0 \leq i \leq j - 1$. From the relations (A2.29) and (A2.30)

$$D^i \tilde{\varphi}_{\ell+j}(0) = \Phi_i(0, \tilde{\varphi}_{\ell+j-1}(0), \dots, D^i \tilde{\varphi}_{\ell+j-1}(0)), \quad (\text{A2.56})$$

$$D^i \tilde{\varphi}(0) = \Phi_i(0, \tilde{\varphi}(0), \dots, D^i \tilde{\varphi}(0)), \quad (\text{A2.57})$$

for $0 \leq i \leq k$ and hence for $0 \leq i \leq j$. Clearly, from (A2.55), the relation (A2.54) holds for $0 \leq i \leq j - 1$ but we must show that it also holds for $i = j$.

The fact that $\tilde{\varphi}(0) = \tilde{\varphi}_{\ell+j-1}(0) = 0$ is essential here because taking $i = j$ in (A2.56) and (A2.57) and coming back to the definition (cf. (A2.28)), we get

$$D^j \tilde{\varphi}_{\ell+j}(0) = \frac{\partial \Phi_{j-1}}{\partial \tilde{x}}(0, 0, D \tilde{\varphi}_{\ell+j-1}(0), \dots, D^{j-1} \tilde{\varphi}_{\ell+j-1}(0)) + \quad (\text{A2.58})$$

$$+ \sum_{i=0}^{j-1} \frac{\partial \Phi_{j-1}}{\partial \lambda_i}(0, 0, D\tilde{\varphi}_{\ell+j-1}(0), \dots, D^{j-1}\tilde{\varphi}_{\ell+j-1}(0))D^{i+1}\tilde{\varphi}_{\ell+j-1}(0)$$

and

$$\begin{aligned} D^j\tilde{\varphi}(0) &= \frac{\partial \Phi_{j-1}}{\partial \bar{x}}(0, 0, D\tilde{\varphi}(0), \dots, D^{j-1}\tilde{\varphi}(0)) + \\ &+ \sum_{i=0}^{j-1} \frac{\partial \Phi_{j-1}}{\partial \lambda_i}(0, 0, D\tilde{\varphi}(0), \dots, D^{j-1}\tilde{\varphi}(0))D^{i+1}\tilde{\varphi}(0). \end{aligned} \quad (\text{A2.59})$$

But, in view of Lemma A2.3, the term corresponding with $i = j - 1$ vanishes in both expressions (A2.58) and (A2.59). It follows that $D^j\tilde{\varphi}_{\ell+j}(0)$ and $D^j\tilde{\varphi}(0)$ are given by the same formula involving the derivatives of order $\leq j - 1$ of the mappings $\tilde{\varphi}_{\ell+j-1}$ and $\tilde{\varphi}$ respectively. As these derivatives coincide by the hypothesis of induction, relation (A2.54) at rank j is established and the proof is complete. \square

let us now consider the mapping $F \in C^k(\overline{\mathcal{O}} \times \overline{B}(0, \rho), \mathbb{R}^n)$ introduced in §6 of Chapter 4 verifying $F(0) = 0$. So as to fully prove the results we used in §6 of Chapter 4, we still have to show that the sequence of mapping

$$\tilde{x} \in \overline{\mathcal{O}} \rightarrow f_\ell(\tilde{x}) = F(\tilde{x}, \tilde{\varphi}_{\ell+k}(\tilde{x})),$$

of class C^k from $\overline{\mathcal{O}}$ to \mathbb{R}^n tends to the mapping

$$\tilde{x} \in \overline{\mathcal{O}} \rightarrow f(\tilde{x}) = F(\tilde{x}, \tilde{\varphi}(\tilde{x})),$$

in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$, the convergence being geometrical in the space $C^{k-1}(\overline{\mathcal{O}}, \mathbb{R}^n)$ (and also geometrical in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$ if both mappings Φ and F are of class C^{k+1}). Besides, we used the relation

$$D^j f_\ell(0) = D^j f(0), 0 \leq j \leq k,$$

which is now immediate from Lemma A2.7 because, for a given index $0 \leq j \leq k$, the derivatives of order j of the mappings f_ℓ and f are given by the same formula involving the derivatives of order $\leq j$ of the mappings F and $\tilde{\varphi}_{\ell+k}$ on the one hand those of F and $\tilde{\varphi}$ on the other hand.

Still denoting by λ_0 the generic element of the space \widetilde{Z} and setting $F_0 = F$, define for $1 \leq j \leq k$ the mappings

$$F_j : \overline{\mathcal{O}} \times \overline{B}(0, \rho) \times \prod_{i=1}^j \mathcal{L}_i(\mathbb{R}^{n+1}, \widetilde{Z}) \rightarrow \mathcal{L}_j(\mathbb{R}^{n+1}, \mathbb{R}^n),$$

by

$$\begin{aligned} F_j(\overline{x}, \lambda_0, \dots, \lambda_j) &= \frac{\partial F_{j-1}}{\partial \overline{x}}(\overline{x}, \lambda_0, \dots, \lambda_{j-1}) + \\ &+ \sum_{i=0}^{j-1} \frac{\partial F_{j-1}}{\partial \lambda_i}(\overline{x}, \lambda_0, \dots, \lambda_{j-1}) \lambda_{i+1}. \end{aligned} \quad (\text{A2.60})$$

Of course, the mappings F_j play with F the role that the mappings Φ_j play with Φ and have similar properties. For instance, the mapping F_j is of class C^{k-j} and for $\overline{x} \in \overline{\mathcal{O}}$,

$$D^j f_\ell(\overline{x}) = F_j(\overline{x}, \widetilde{\varphi}_{\ell+k}(\overline{x}), D\widetilde{\varphi}_{\ell+k}(\overline{x}), \dots, D^j \widetilde{\varphi}_{\ell+k}(\overline{x})) \quad (\text{A2.61})$$

for every $\ell \geq 0$, while

$$D^j f(\overline{x}) = F_j(\overline{x}, \widetilde{\varphi}(\overline{x}), D\widetilde{\varphi}(\overline{x}), \dots, D^j \widetilde{\varphi}(\overline{x})). \quad (\text{A2.62})$$

For $0 \leq j \leq k-1$ and $\overline{x} \in \overline{\mathcal{O}}$, we shall set

$$c^j(\overline{x}) = \frac{\partial F_j}{\partial \overline{x}}(\overline{x}, \widetilde{\varphi}(\overline{x}), D\widetilde{\varphi}(\overline{x}), \dots, D^j \widetilde{\varphi}(\overline{x})), \quad (\text{A2.63})$$

$$c_\ell^j(\overline{x}) = \frac{\partial F_j}{\partial \overline{x}}(\overline{x}, \widetilde{\varphi}_{\ell+k}(\overline{x}), D\widetilde{\varphi}_{\ell+k}(\overline{x}), \dots, D^j \widetilde{\varphi}_{\ell+k}(\overline{x})) \text{ for } \ell \geq 0. \quad (\text{A2.64})$$

Similarly, for $0 \leq j \leq k-1$, $0 \leq i \leq j$ and $\overline{x} \in \overline{\mathcal{O}}$, set

$$C_i^j(\overline{x}) = \frac{\partial F_j}{\partial \lambda_i}(\overline{x}, \widetilde{\varphi}(\overline{x}), D\widetilde{\varphi}(\overline{x}), \dots, D^i \widetilde{\varphi}(\overline{x})), \quad (\text{A2.65})$$

$$C_{i\ell}^j(\overline{x}) = \frac{\partial F_j}{\partial \lambda_j}(\overline{x}, \widetilde{\varphi}_{\ell+k}(\overline{x}), D\widetilde{\varphi}_{\ell+k}(\overline{x}), \dots, D^i \widetilde{\varphi}_{\ell+k}(\overline{x})) \text{ for } \ell \geq 0 \quad (\text{A2.66})$$

In what follows, we assume that the neighbourhood \mathcal{O} has been chosen so that Theorem A2.2 and A2.3 apply

279 Lemma A2.8: (i) Assume $k \geq 2$ and $0 \leq j \leq k - 2$. Then, the sequence $(c_\ell^j)_{\ell \geq 0}$ tends to c^j geometrically in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_j(\mathbb{R}^{n+1}, \mathbb{R}^n)))$ and the sequence $(C_{i\ell}^j)_{\ell \geq 0}$ tends to C_i^j in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \mathbb{R}^n), \mathcal{L}_j(\mathbb{R}^{n+1}, \mathbb{R}^n)))$.

(ii) Assume only $k \geq 1$ and $0 \leq j \leq k - 1$. Then, the sequence $(c_\ell^j)_{\ell \geq 0}$ tends to c^j in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathbb{R}^{n+1}, \mathcal{L}_j(\mathbb{R}^{n+1}, \mathbb{R}^n)))$ and the sequence $(C_{i\ell}^j)_{\ell \geq 0}$ tends to C_i^j in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}(\mathcal{L}_i(\mathbb{R}^{n+1}, \mathbb{R}^n), \mathcal{L}_i(\mathbb{R}^{n+1}, \mathbb{R}^n)))$.

Proof. The proof of this lemma parallels that of Lemma A2.6. Note only that the (geometrical) convergence of the sequence $(\tilde{\varphi}_{\ell+k})$ to $\tilde{\varphi}$ in the space $C^j(\overline{\mathcal{O}}, \tilde{Z})$ need not be listed in the assumption since it is known from Theorem A2.2 \square

Theorem A2.4: The sequence (f_ℓ) tends to f in the space $C^k(\overline{\mathcal{O}}, \mathbb{R}^n)$ and the convergence is geometrical in the space $C^{k-1}(\overline{\mathcal{O}}, \mathbb{R}^n)$.

Proof. Once again, it is enough to prove equivalently that the sequence $(D^j f_\ell)_{\ell \geq 0}$ tends to $D^j f$ in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_j(\mathbb{R}^{n+1}, \mathbb{R}^n))$ for $0 \leq j \leq k$, the convergence being geometrical for $0 \leq j \leq k - 1$. We shall distinguish the case $j = 0$ and $j \geq 1$.

When $j = 0$, we must prove that the sequence (f_ℓ) tends to f geometrically in the space $C^0(\overline{\mathcal{O}}, \mathbb{R}^n)$. By definition and with the Taylor formula (since F is at least C^1), one has for $\tilde{x} \in \overline{\mathcal{O}}$ and $\ell \geq 0$

$$\begin{aligned} f_\ell(\tilde{x}) - f(\tilde{x}) &= F(\tilde{x}, \tilde{\varphi}_{\ell+k}(\tilde{x})) - F(\tilde{x}, \tilde{\varphi}(\tilde{x})) \\ &= \int_0^1 \frac{\partial F}{\partial \lambda_0}(\tilde{x}, \tilde{\varphi}(\tilde{x}) + s(\tilde{\varphi}_{\ell+k}(\tilde{x}) - \tilde{\varphi}(\tilde{x}))) \cdot (\tilde{\varphi}_{\ell+k}(\tilde{x}) - \tilde{\varphi}(\tilde{x})) ds. \end{aligned}$$

By the continuity of the mapping $(\partial F / \partial \lambda_0)$ on the compact set $\overline{\mathcal{O}} \times \Lambda_0^c$, where $\Lambda_0^c \subset \overline{B}(0, \rho)$ denotes the closed convex hull of the compact set Λ_0 (cf. Lemma A2.4), there is a constant $K > 0$ such that

$$\left\| \frac{\partial F}{\partial \lambda_0}(\tilde{x}, \tilde{\varphi}(\tilde{x}) + s(\tilde{\varphi}_{\ell+k}(\tilde{x}) - \tilde{\varphi}(\tilde{x}))) \right\| \leq K,$$

for every $\tilde{x} \in \overline{\mathcal{O}}$ and $\ell \geq 0$. Hence

$$|f_\ell - f|_{\infty, \overline{\mathcal{O}}} \leq K |\tilde{\varphi}_{\ell+k} - \tilde{\varphi}|_{\infty, \overline{\mathcal{O}}}$$

and the geometrical rate off convergence of the sequence (f_ℓ) to f in the space $C^0(\bar{\mathcal{O}}, \mathbb{R}^n)$ follows from Theorem A2.2.

When $1 \leq j \leq k$ and with (A2.60) - (A2.62) and the notation (A2.63)-(A2.66), the method we used in Theorem A2.2 and A2.3 leads to the inequality

$$\begin{aligned} \|D^j f_\ell(\bar{x}) - D^j f(\bar{x})\| &\leq \|c_\ell^{j-1}(\bar{x}) - C^{j-1}(\bar{x})\| \\ &+ \left\| \sum_{i=0}^{j-1} \|c_{i\ell}^{j-1}(\bar{x}) - C_i^{j-1}(\bar{x})\| \|D^{i+1}\tilde{\varphi}(\bar{x})\| \right. \\ &\left. + \sum_{i=0}^{j-1} \|C_{i\ell}^{j-1}(\bar{x})\| \|D^{i+1}\tilde{\varphi}_{\ell+k}(\bar{x}) - D^{i+1}\tilde{\varphi}(\bar{x})\|, \right. \end{aligned}$$

for every $\bar{x} \in \bar{\mathcal{O}}$. Due to the compactness of the set $\bar{\mathcal{O}} \times \Lambda_0 \times \cdots \times \Lambda_{j-1}$ (Lemma A2.4) and the regularity $\tilde{\varphi} \in C^k(\bar{\mathcal{O}}, \bar{\mathcal{Z}})$, there is a constant $K > 0$ such that

$$\begin{aligned} \|D^{i+1}\tilde{\varphi}(\bar{x})\| &\leq K, \\ \|C - 1\ell^{j-1}(\bar{x})\| &\leq K, \end{aligned}$$

for $\bar{x} \in \bar{\mathcal{O}}, 0 \leq i \leq j-1$ and $\ell \geq 0$. Thus, we first obtain

$$\begin{aligned} \|D^j f_\ell(\bar{x}) - D^j f(\bar{x})\| &< \|c_\ell^{j-1}(\bar{x}) - c^{j-1}(\bar{x})\| \\ &+ K \left[\sum_{i=0}^{j-1} \|C_{i\ell}^{j-1}(\bar{x}) - C_i^{j-1}(\bar{x})\| + \|D^{i+1}\tilde{\varphi}_{\ell+k}(\bar{x}) - D^{i+1}\tilde{\varphi}(\bar{x})\| \right] \end{aligned}$$

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$$\begin{aligned} |D^j(f_\ell - f)|_{\infty, \bar{\mathcal{O}}} &\leq |c_\ell^j - c^{j-1}|_{\infty, \bar{\mathcal{O}}} \\ &+ K \left[\sum_{i=0}^{j-1} |C_{i\ell}^{j-1} - C_i^{j-1}|_{\infty, \bar{\mathcal{O}}} + D^{i+1}(\tilde{\varphi}_{\ell+k} - \tilde{\varphi})|_{\infty, \bar{\mathcal{O}}} \right]. \end{aligned} \tag{A2.67}$$

For $k \geq 2$ and $1 \leq j \leq k-1$ (i.e. $0 \leq j-1 \leq k-2$). this inequality yields the geometrical convergence of the sequence $(D^j f_\ell)_{\ell \geq 0}$ to $D^j f$ in

the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_j(\mathbb{R}^{n+1}, \mathbb{R}^n))$ by applying Theorem A2.2 and Lemma A2.8 (i). For $k \geq 1$ and $j = k$, convergence of the sequence $(D^k f_\ell)_{\ell \geq 0}$ to $D^k f$ (with no rate of convergence in the space $C^0(\overline{\mathcal{O}}, \mathcal{L}_k(\mathbb{R}^{n+1}, \mathbb{R}^n))$) also follows from (A2.67) by applying Theorem A2.3 and Lemma A2.8 (ii). \square

Corollary A2.2: *Assume that both mappings F and Φ are of class C^{k+1} . Then, the sequence (f_ℓ) tends to the mapping f geometrically in the space $C^k(\mathcal{O}, \mathbb{R}^n)$.*

Proof. When the mapping F is of class C^{k-1} , the mappings F_j $0 \leq j \leq k$, are of class C^{k+1-j} and hence the mappings c^j, c_ℓ^j, C_i^j and $C_{i\ell}^j$ are of class C^1 at least. This allows us to prove Lemma A2.8 (i) for the indices $0 \leq j \leq k-1$ (instead of $0 \leq j \leq k-2$). On the other hand, if the mapping Φ is of class C^{k+1} too, the sequence $(\tilde{\varphi}_\ell)$ tends to the mapping $\tilde{\varphi}$ geometrically in the space $C^k(\overline{\mathcal{O}}, \tilde{\mathcal{Z}})$ (Corollary A2.1) and so does the sequence $(\tilde{\varphi}_{\ell+k})$. The result follows now from inequality (A2.67) for $1 \leq j \leq k$ (i.e. $0 \leq j-1 \leq k-1$), the geometrical convergence of the sequence (f_ℓ) to f in the space $C^0(\overline{\mathcal{O}}, \mathbb{R}^n)$ having been already proved in Theorem A2.3. \square

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