Lectures on
Mean Values of The Riemann Zeta Function

By

A.Ivic

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Author
A. Ivic
S.Kovacevica, 40 Stan 41
Yu-11000, Beograd
YUGOSLAVIA

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Preface

These lectures were given at the Tata Institute of Fundamental Research in the summer of 1990. The specialized topic of mean values of the Riemann zeta-function permitted me to go into considerable depth. The central theme were the second and the fourth moment on the critical line, and recent results concerning these topic are extensively treated. In a sense this work is a continuation of my monograph [1], since except for the introductory Chapter 1, it starts where [1] ends. Most of the results in this text are unconditional, that is, they do not depend on unproved hypothesis like Riemann's (all complex zeros of $\zeta(s)$ have real parts equal to $\frac{1}{2}$) or Lindelöf's ($\zeta(\frac{1}{2}+\text{it}) \ll t^{\epsilon}$). On the other hand, in many problems concerning mean values one does not obtain the conjectured results even if one assumes some unproved hypothesis. For example, it does not seem possible to prove $E(T) \ll T^{\frac{1}{4}+\epsilon}$ even if one assumes the Riemann hypothesis. Incidentally, at the moment of writing of this text, it is not yet known whether the Riemann or Lindelof hypothesis is true or not.

Each chapter is followed by Notes, where some results not treated in the body of the text are mentioned, appropriate references: are given etc. Whenever possible, standard notation (explained at the beginning of the text) is used.

I've had the pleasure to have in my audience well-known specialists from analytic number theory such as R. Balasubramanian, K. Ramachandra, A. Sankaranarayanan, T.N. Shorey and S. Srinivasan. I am grateful to all of them for their interest, remarks and stimulating discussions. The pleasant surroundings of the Institute and the hospitality of the Tata people made my stay a happy one, inspite of heavy monsoon rains.

I wish to thank also M.N. Huxley, M. Jutila, K. Matsumoto and T. Meurman for valuable remarks concerning the present lecture notes.

Special thanks are due to K. Ramachandra, Who kindly invited me to the Tata Institute, and with whom I've been in close contact for many years.

Finally, most of all I wish to thank Y. Motohashi, whose great work on the fourth moment influenced me a lot, and who kindly permitted me to include his unpublished material in my text.

Belgrade, March 1991.

Notation

Owing to the nature of this text, absolute consistency in notation could not be attained, although whenever possible standard notation is used. The following notation will occur repeatedly in the sequel.

k, l, m, n	Natural numbers (positive integers).
$A, B, C, C_1, \dots$	Absolute positive constants (not necessarily the same at each occurrence).
s, z, w	Complex variables $\operatorname{Re} s$ and $\operatorname{Im} s$ denote the real imaginary part of s, respectively; common notation is $\sigma = \operatorname{Re}s$ and $t = \operatorname{Im}s$.
t, x, y	Real variables.
$\operatorname*{Res}_{s=s_\circ} F(s)$	The residue of $F(s)$ at the point $s = s_\circ$.
$\zeta(s)$	Riemann's zeta-function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\operatorname{Re}s > 1$ and for other values of s by analytic continuation.
$\gamma(s)$	The gamma-function, defined for $\operatorname{Re}s > 0$ by $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$, otherwise by analytic continuation.
γ	Euler's constant, defined by $r = \int_0^\infty e^{-x} \log x\, dx = 0.5772156649\dots$.
$\chi(s)$	$= \zeta(s)/\zeta(1-s) = (2\pi)^s/(2\Gamma(s)\cos\left(\frac{\pi s}{2}\right))$.

$\mu(\sigma)$	$= \limsup\limits_{t\to\infty} \dfrac{\log\left|\zeta(\sigma+\mathrm{it})\right|}{\log t}$ (σ real).
$\exp z$	$= e^z$.
$e(z)$	$= e^{2\pi iz}$.
$\log x$	$= \mathrm{Log}_e x (= \ln x)$.
$[x]$	Greatest integer not exceeding x.
$d_k(n)$	Number of ways n can be written as a product of k fixed natural numbers.
$d(n)$	$= d_2(n) =$ number of divisors of n.
$\sum\limits_{d \mid n}$	A sum over all positive divisors of n.
$\sigma_a(n)$	$= \sum\limits_{d \mid n} d^a$.
$\sum\limits_{n\le k} f(n)$	A sum taken over all natural numbers not exceeding x; the empty sum is defined to be zero.
$\sum\limits_{n\le x}{}' f(n)$	Same as above, only $'$ denotes that the last summand is halved if x^* is an integer.
$\prod\limits_j$	A product taken over all possible values of the index j; the empty sum is defined to be unity.
$I_k(T)$	$= \int_0^T \left|\zeta(\frac{1}{2}+\mathrm{it})\right|^{2k} dt (k \ge 0)$.
$E_k(T)$	The error term in the asymptotic formula when $k \ge 1$ is an integer.
$E(T)(= E_1(T))$	$\int_0^T \left|\zeta(\frac{1}{2}+\mathrm{it})\right|^2 dt - T\log\left(\frac{T}{2\pi}\right) - (2\gamma-1)T$.
$E_\sigma(T)$	$\int_0^T \left|\zeta(\sigma+it)\right|^2 dt - \zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\gamma-1)}{1-\sigma}\sin(\pi\sigma)T^{2-2\sigma}\left(\frac{1}{2}<\sigma<1\right)$.
$\operatorname{ar\,sinh} z$	$= \log(z+(z^2+1)^{\frac{1}{2}})$.
$\triangle(x)$	$= \sum\limits_{n\le x}{}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4}$.

$\triangle_k(x)$	Error term in the asymptotic formula for $\sum_{n\geq x} d_k(n)$; $\triangle_2(x) = \triangle(x)$.
$J_p(z)$, $K_p(z)$, $Y_p(z)$	Bessel functions of index p.
$G(T)$	$= \int_2^T (E(T) - \pi dt)$.
$G_\circ(T)$	$= \int_2^T (E_\sigma(t) - b(\sigma))dt \left(\frac{1}{2} < \sigma < \frac{3}{4}\right)$.
$B(\sigma)$	The constant defined by (3.3).
$S(m, n; c)$	Kloosterman sum, defined as $\sum_{1\leq d\leq c, (d,c)=1, dd'} \equiv$ (mod c)$e\left(\frac{md+nd'}{c}\right)$.
$c_r(n)$	Ramanujan sum, defined as $c_r(n) = \sum_{1\leq h\leq r, (h,r)=1} e\left(\frac{h}{r}n\right)$.
α_j, x_j, $H_j\left(\frac{1}{2}\right)$	Quantities appearing in the spectral theory of the non-Euclidean Laplace operator, defined in Section 5.3.
$f(x) \sim g(x)$ as $x \to x_0$	Means $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, with x_0 possibly infinite.
$f(x) = O(g(x))$	Means $\lvert f(x)\rvert \leq Cg(x)$ for $g(x) > 0$, $x \geq x_0$ and some absolute constant $C > 0$.
$f(x) \ll g(x)$	Means the same as $f(x) = 0(g(x))$.
$f(x) \gg g(x)$	Means the same as $g(x) = 0(f(x))$.
$f(x) \asymp g(x)$	Means that both $f(x) \ll (g(x))$ and $g(x) \ll f(x)$ hold.
(a, b)	Means the interval $a < x < b$.
$[a, b]$	Means the integral $a \leq x \leq b$.
δ, ϵ	An arbitrarily small number, not necessarily the same at each occurrence in the proof of a theorem or lemma.
$C^r[a, b]$	The class of functions having a continuous r-th derivative in $[a, b]$.

$f(x) = O(g(x))$ as $x \to x_0$	Means that $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$, with x_0 possibly infinite.
$f(x) = \Omega_+(g(x))$	Means that there exists a suitable constant $C > 0$ and a sequence x_n tending to ∞ such that $f(x) > Cg(x)$ holds for $x = x_n$.
$f(x) = \Omega_-(g(x))$	Means that there exists a suitable constant $C > 0$ and a sequence x_n tending to ∞ such that $f(x) < -Cg(x)$ holds for $x = x_n$.
$f(x) = \Omega_\pm(g(x))$	Means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ holds.
$f(x) = \Omega(g(x))$	Means that $\lvert f(x)\rvert = \Omega_+(g(x))$.

Contents

Chapter 1
Elementary Theory

1.1 Basic Properties of $\zeta(s)$

THE RIEMANN ZETA-FUNCTION $\zeta(s)$ is defined as 1

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p}(1 - p^{-s})^{-1} (\sigma = \mathrm{Re}\, s > 1), \tag{1.1}$$

where the product is over all primes p. For other values of the complex variable $s = \sigma + it$ it is defined by analytic continuation. It is regular for all values of s except $s = 1$, where it has a simple pole with residue equal to 1. Analytic continuation of $\zeta(s)$ for $\sigma > 0$ is given by

$$\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \tag{1.2}$$

since the series is (1.2) converges for Re $s > 0$.

For $x > 1$ one has

$$\begin{aligned}\sum_{n \le x} n^{-s} &= \int_{1-0}^{x} u^{-s} d[u] = [x]x^{-s} + s\int_{1}^{x} [u]u^{-s-1} du \\ &= 0(x^{1-\sigma}) + s\int_{1}^{x} ([u] - u)u^{-s-1} du + \frac{s}{s-1} - \frac{sx^{1-s}}{s-1}.\end{aligned}$$

If $\sigma > 1$ and $x \to \infty$, it follows that

$$\zeta(s) = \frac{s}{s-1} + s\int_{1}^{\infty} ([u] - u)u^{-s-1} du.$$

By using the customary notation $\psi(x) = x - [x] - 1/2$ this relation can be written as

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^\infty \psi(u)u^{-s-1}du. \tag{1.3}$$

Since $\int_y^{y+1} \psi(u)du = 0$ for any y, integration by parts shows that
2 (1.3) provides the analytic continuation of $\zeta(s)$ to the half-plane $\sigma > -1$, and in particular it follows that $\zeta(0) = -1/2$. The Laurent expansion of $\zeta(s)$ at $s = 1$ has the form

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \cdots \tag{1.4}$$

with

$$\begin{aligned}\gamma_k &= \frac{(-1)^{k+1}}{k!}\int_{1-0}^\infty x^{-1}(\log x)^k d\psi(x)\\ &= \frac{(-1)^k}{k!}\lim_{N\to\infty}\left(\sum_{n\le N}\frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1}\right),\end{aligned} \tag{1.5}$$

and in particular

$$\begin{aligned}\gamma_0 = \gamma &= \lim_{N\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log N\right) = \Gamma'(1)\\ &= -\int_0^\infty e^{-x}\log x\,dx = 0.577\ldots\end{aligned}$$

is Euler's constant. To obtain (1.4) and (1.5), write (1.3) as

$$\zeta(s) = \frac{1}{s-1} = \int_{1-0}^\infty x^{-s}d\psi(x).$$

Then

$$\begin{aligned}-\int_{1-0}^\infty x^{-s}d\psi(x) &= -\int_{1-0}^\infty x^{-1}e^{-(s-1)\log x}d\psi(x)\\ &= \int_{1-0}^\infty x^{-1}\sum_{k=0}^\infty \frac{(-1)^{k+1}}{k!}(\log x)^k(s-1)^k d\psi(x)\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left\{ \frac{(-1)^{k+1}}{k!} \int_{1-0}^{\infty} x^{-1} (\log x)^k d\psi(x) \right\} (s-1)^k.$$

The inversion of summation and integration is justified by the fact that, for $k \geq 1$, integration by parts gives

$$\int_{1-0}^{\infty} x^{-1} \log^k x d\psi(x) = \int_{1-0}^{\infty} x^{-2} \psi(x) (\log^k x - k \log^{k-1} x) dx,$$

and the last integral is absolutely convergent.

To obtain analytic continuation of $\zeta(s)$ to the whole complex plane it is most convenient to use the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \tag{1.6}$$

which is valid for all s. Namely, for $\sigma < 0$

$$-s \int_0^1 \psi(u) u^{-s-1} du = \frac{1}{s-1} + \frac{1}{2},$$

hence by (1.3)

$$\zeta(s) = -s \int_0^{\infty} \psi(u) u^{-s-1} du, \qquad (-1 < \sigma < 0). \tag{1.7}$$

By using the Fourier expansion **3**

$$\psi(x) = -\sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n\pi},$$

which is valid when x is not an integer, we obtain for $s = \sigma$, $-1 < \sigma < 0$,

$$\begin{aligned}
\zeta(s) &= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} u^{-s-1} \sin(2n\pi u) du \\
&= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{(2n\pi)^s}{n} \operatorname{Im} \left\{ \int_0^{\infty} \frac{e^{iy}}{y^{s+1}} dy \right\} \\
&= \frac{s}{\pi} \sum_{n=1}^{\infty} (2\pi)^s n^{s-1} \operatorname{Im} \left\{ i^{-s} \int_0^{-i\infty} e^{-z} z^{-s-1} dz \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s}{\pi}(2\pi)^s \zeta(1-s)\,\mathrm{Im}\left\{e^{\frac{-\pi i s}{2}} \int_0^\infty \frac{e^{-z}}{z^{s+1}} dz\right\} \\
&= \frac{s}{\pi}(2\pi)^s \zeta(1-s)\,\mathrm{Im}(e^{-\frac{1}{2}\pi i s})\Gamma(-s) \\
&= \frac{(2\pi)^s}{\pi}\zeta(1-s)\sin\left(\frac{\pi s}{2}\right)\{-s\Gamma(-s)\} \\
&= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).
\end{aligned}
$$

Here one can justify termwise integration by showing that

$$
\lim_{y\to\infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_y^\infty \sin(2n\pi u) u^{-s-1} du = 0 \qquad (-1 < \sigma < 0),
$$

which is easily established by performing an integration by parts. Thus (1.6) follows for $-1 < s = \sigma < 0$, and for other values of s it follows by analytic continuation.

One can also write (1.6) as

$$
\zeta(s) = \chi(s)\zeta(1-s), \chi(s) = (2\pi)^s \Big/ \left(2\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\right). \tag{1.8}
$$

By using Stirling's formula for the gamma-function it follows that, uniformly in σ,

$$
\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{1}{4}\pi)} \left\{1 + 0\left(\frac{1}{t}\right)\right\} (0 < \sigma \le 1, t \ge t_0 > 0) \tag{1.9}
$$

and also

$$
\chi\left(\frac{1}{2}+it\right) = \left(\frac{2\pi}{t}\right)^{-it\log(t/2\pi)+it+\frac{1}{4}i\pi} \left\{1 - \frac{i}{24t} + 0\left(\frac{1}{t^2}\right)\right\} \ (t \ge t_0 > 0). \tag{1.10}
$$

4

1.2 Elementary Mean Value Results

In general, mean value results concern the evaluation of the integral

$$
\int_1^T \left|\zeta(\sigma+it)\right|^k dt \tag{1.11}
$$

as $T \to \infty$, where σ and $k(> 0)$ are fixed. Since $\zeta(s) \gg_\sigma 1$ when $\sigma > 1$, the case $\sigma > 1$ is fairly easy to handle. By the functional equation (1.8) and (1.9) it is seen that the range $\sigma < 1/2$ can be essentially reduced to the range $\sigma \geq 1/2$. Thus the basic cases of (1.11) are $\sigma 1/2$ ("the critical line"), $1/2 < \sigma < 1$ ("the critical strip") and $\sigma = 1$. Naturally, the case when k in (1.11) is not an integer is more difficult , because $\zeta^k(s)$ may not be regular. For the time being we shall suppose that $k \geq 1$ is an integer, and we also remark that when $k = 2N$ is even the problem is somewhat less difficult, since $|\zeta(\sigma + it)|^{2N} = \zeta^N(\sigma + it)\zeta^N(\sigma - it)$.

In problems involving the evaluation of (1.11) one often encounters the general divisor function

$$d_k(n) = \sum_{n = n_1 \ldots n_k} 1,$$

which denotes the number of ways n may be written as a product of $k(\geq 2)$ fixed factors. In this notation $d(n) = d_2(n)$ denotes the number of all positive divisors of n. For Re $s > 1$

$$\zeta^k(s) = \left(\sum_{n=1}^{\infty} n^{-s}\right)^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}. \tag{1.12}$$

Note that $d_k(n)$ is a multiplicative function of n (meaning $d_k(mn) = d_k(m)d_k(n)$ for coprime m and n) and

$$d_k(p^\alpha) = \binom{\alpha + k - 1}{k - 1} = \frac{k(k+1)\ldots(\alpha + k - 1)}{\alpha!}.$$

For fixed k we have $d_k(n) \ll_\epsilon n^\epsilon$ for any $\epsilon > 0$, which follows from the stronger inequality 5

$$d_k \leq \exp(C(k) \log n / \log\log n) \qquad (n \geq 2), \tag{1.13}$$

where $C(k) > 0$ is a suitable constant. One proves (1.13) by induction on k, since $d_k(n) = \sum\limits_{\delta | n} d_{k-1}(\delta)$ and (1.13) is not difficult to establish for $k = 2$.

To obtain some basic mean-value formulas we shall use a simple so-called "approximate functional equation". This name refers to various formulas which express $\zeta(s)$ (or $\zeta^k(s)$) as a number of finite sums involving the function n^{-s}. The approximate functional equation of the simplest kind is

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}), \tag{1.14}$$

and is valid for $0 < \sigma_0 \geq \sigma \leq 2$, $x \geq |t|/\pi$, $s = \sigma + it$, where the 0-constant depends only on σ_0. We shall also need a result for the evaluation of integrals of Dirichlet polynomials, namely sums of the form $\sum\limits_{n \le N} a_n n^{it}$, where t is real and the a_n's are complex. The standard mean-value result for Dirichlet polynomials, known as the Montgomery-Vaughan theorem, is the asymptotic formula

$$\int_0^T \left| \sum_{n \le N} a_n n^{it} \right|^2 dt = T \sum_{n \le N} |a_n|^2 + O\left(\sum_{n \le N} n|a_n|^2 \right). \tag{1.15}$$

This holds for arbitrary complex numbers $a_1, \ldots, a_N$, and remains true if $N = \infty$, provided that the series on the right-hand side of (1.15) converge.

Now suppose $\frac{1}{2}T \geq t \geq T$ and choose $x = T$ in (1.14) to obtain, for $1/2 < \sigma < 1$ fixed,

$$\zeta(\sigma + it) = \sum_{n \le T} n^{-\sigma - it} + R,$$

where $R \ll T^{-\sigma}$. Since $|\zeta(s)|^2 = \zeta(s)\overline{\zeta(s)}$, we obtain

$$\int_{\frac{1}{2}T}^T |\zeta(\sigma + it)|^2 dt = \int_{\frac{1}{2}T}^T \left| \sum_{n \le T} n^{-\sigma - it} \right|^2 dt + 0(T^{1-2\sigma}) + 2\operatorname{Re}\left\{ \int_{\frac{1}{2}T}^T \sum_{n \le T} n^{-\sigma - it} \overline{R} dt \right\}.$$

6 Using (1.15) we have

$$\int_{\frac{1}{2}T}^{T} \left| \sum_{n \le T} n^{-\sigma - it} \right|^2 dt = \frac{1}{2} T \sum_{n \ge T} n^{-2\sigma} + O\left(\sum_{n \le T} n^{1-2\sigma} \right) \tag{1.16}$$
$$= \frac{1}{2} \zeta(2\sigma) T + O(T^{2-2\sigma}).$$

Trivially we have $\sum_{n \le T} n^{-\sigma - it} \ll T^{1-\sigma}, R \ll T^{-\sigma}$, hence

$$\int_{\frac{1}{2}T}^{T} \sum_{n \le T} n^{-\sigma - it} \overline{R} dt \ll T^{2-2\sigma}.$$

In the case when $\sigma = \frac{1}{2}$ the analysis is similar, only the right-hand side of (1.16) becomes

$$\frac{1}{2} T \sum_{n \le T} n^{-1} + O\left(\sum_{n \le T} 1 \right) = \frac{T}{2} \log \frac{T}{2} + O(T).$$

Thus replacing T by $T2^{-j}$ in the formulas above ($j = 1, 2, \ldots$) and adding the results we obtain

$$\int_0^T |\zeta(\sigma + it)|^2 dt = \zeta(2\sigma) T + O(T^{2-2\sigma})$$

and

$$\int_0^T |\zeta(\frac{1}{2} + it)|^2 dt = T \log T + O(T).$$

To obtain the mean square formula in the extreme case $\sigma = 1$ we use (1.14) with $s = 1 + it$, $1 \le t \le T$, $x = T$. Then we have

$$\zeta(1 + it) = \sum_{n \le T} n^{-1-it} + \frac{T^{-it}}{it} + O\left(\frac{1}{T} \right).$$

We use this formula to obtain

$$\int_1^T |\zeta(1 + it)|^2 dt = \int_1^T \left| \sum_{n \le T} n^{-1-it} \right|^2 dt - 2 \operatorname{Re} \left\{ \frac{1}{i} \int_1^T \sum_{n \ge T} n^{-1} \left(\frac{T}{n} \right)^{it} \frac{dt}{t} \right\}$$

$$+ O\left(\int_1^T \left|\sum_{n\le T} n^{-1-it}\right| \frac{dt}{T} + O(1).\right) \quad (1.17)$$

Therefore by (1.15) **7**

$$\begin{aligned}\int_1^T \left|\sum_{n\leqq T} n^{-1-it}\right|^2 dt &= (T-1)\sum_{n\le T} n^{-2} + 0\left(\sum_{n\le T} n^{-1}\right)\\ &= \zeta(2)T + O(\log T), \qquad (1.18)\end{aligned}$$

and so by the Cauchy-Schwarz inequality

$$\int_1^T \left|\sum_{n\le T} n^{-1-it}\right| \frac{dt}{T} = O(1).$$

Finally, let H be a parameter which satisfies $2 \le H \le \frac{1}{2}T$. Then

$$\begin{aligned}&\int_1^T \sum_{n\le T} n^{-1}\left(\frac{T}{n}^{it}\right)\frac{dt}{t}\\ &\quad= \sum_{n\le T(1-1/H)} n^{-1}\left\{\frac{(T/n)^{it}}{it\log(T/n)}\bigg|_1^T + \int_1^T \frac{(T/n)^{it}}{it^2\log(T/n)}dt\right\}\\ &\qquad\qquad + O\left(\sum_{T(1-1/H<n\le T)} n^{-1}\int_1^T \frac{dt}{t}\right)\\ &\quad\ll \sum_{n\le T(1-1/H)} \frac{1}{n\log(T/n)} + \log T \sum_{T(1-1/H)<n\le T} \frac{1}{n}\\ &\quad\ll \int^{T(1-1/H)} \frac{dx}{x\log(T/x)} + \frac{\log T}{H} + 1\\ &\quad= \int_{(1-1/H)^{-1}}^T \frac{du}{u\log u} + \frac{\log T}{H} + 1\\ &\quad\ll \log\log T - \log\log(1-1/H)^{-1} + \frac{\log T}{H} + 1\\ &\quad\ll \log\log T + \log H + \frac{\log T}{H} + 1 \ll \log\log T\end{aligned}$$

8 for $H = \log T$. In view of (1.17) this shows that

$$\int_1^T |\zeta(1+it)|^2 dt = \int_1^T \left|\sum_{n\le T} n^{-1-it}\right|^2 dt + O(\log\log T), \tag{1.19}$$

and in conjunction with (1.18) we obtain

$$\int_1^T |\zeta(1+it)|^2 dt = \zeta(2)T + O(\log T).$$

Hence we have proved

Theorem 1.1. *For $1/2 < \sigma < 1$ fixed we have*

$$\int_0^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + O(T^{2-2\sigma}). \tag{1.20}$$

Moreover

$$\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = T\log T + O(T) \tag{1.21}$$

and

$$\int_1^T |\zeta(1+it)|^2 dt = \zeta(2)T + O(\log T). \tag{1.22}$$

It should be remarked that the asymptotic formulas (1.20), (1.21) and (1.22) cannot be improved, that is, the error terms appearing in them are in fact of the order $T^{2-2\sigma}$, T and $\log T$, respectively. But the first two formulas in equation may be given in a much more precise form, which will be the topic of our study in Chapter 2 and Chapter 3. Note also that the argument used in the proof of Theorem 1.1yields easily, on using $d_k(n) \ll_\epsilon n^\epsilon$,

$$\int_1^T |\zeta(\sigma+it)|^{2k} dt = \left(\sum_{n=1}^{\infty} d_k^2(n) n^{-2\sigma}\right) T + O(T^{2-2\sigma}) + 0(1)$$

for $\sigma > 1$ fixed and $k \ge 1$ a fixed integer.

1.3 Bounds Over Short Intervals

We begin with a useful result which shows that pointwise estimation of 9
$\zeta^k(s)$ may be replaced by estimation of the integral of $\zeta^k(s)$ over a short interval, and the latter is in many cases easier to carry out. This is

Theorem 1.2. *Let $1/2 \le \sigma \le 1$ be fixed and let $k \ge 1$ be a fixed integer. Then for any fixed constants δ, $A > 0$*

$$|\zeta(\sigma + it)|^k \ll (\log T) \int_{-\delta}^{\delta} |\zeta(\sigma + it)|^k dv + T^{-A}. \tag{1.23}$$

Proof. Let $B, C > 0$ denote constants to be chosen later, $r = [C \log T]$, $s = \sigma + it$, $T \ge T_0$ and $X = \exp(u_1 + \cdots + u_r)$. By the residue theorem we have

$$2\pi i B^r \zeta^k(s) = \int_0^B \cdots \int_0^B \int_{|w|=\delta} \zeta^k(s + w) X^{w_w - 1} dw du_1 \ldots du_r. \tag{1.24}$$

We may clearly suppose that $0 < \delta < 1/2$, so that on the semicircle $|w| = \delta$, Re $w < 0$ we have $|e^{Bw} - 1| \le 2$ and $\zeta(s + w) \ll T^{\frac{1}{2}}$ (because from (1.14) we trivially have $\zeta(1+it) \ll \log T$, and then by the functional equation $\zeta(it) \ll T^{\frac{1}{2}} \log T$). Hence

$$\begin{aligned}
&\left| \int_0^B \cdots \int_0^B \int\limits_{|w|=\delta, \mathrm{Re}\, w<0} \zeta^k(s + w) X^w w^{-1} dw du_1 \ldots du_r \right| \\
&= \left| \int\limits_{|w|=\delta, \mathrm{Re}\, w<0} \zeta^k(s + w) \int\limits_0^B e^{wu_1} du_1 \ldots \int\limits_0^B e^{wu_r} du_r \frac{dw}{w} \right| \\
&= \left| \int\limits_{|w|=\delta, \mathrm{Re}\, w<0} \zeta^k(s + w) \left(\frac{e^{Bw} - 1}{w} \right)^r \frac{dw}{w} \right| \le \pi T^{\frac{1}{2}k} \left(\frac{2}{\delta} \right)^r.
\end{aligned}$$

On the other hand we obtain by Cauchy's theorem

$$\int\limits_{|w|=\delta, \mathrm{Re}\, w\ge 0} \zeta^k(s + w) X^w \frac{dw}{w} = \int\limits_{|w|=\delta, \mathrm{Re}\, w\ge 0} \zeta^k(s + w) \frac{X^w - X^{-w}}{w} dw$$

$$+ \int\limits_{|w|=\delta, \operatorname{Re} w \geq 0} \zeta^k(s+w) X^{-w} \frac{dw}{w}$$

$$= \int\limits_{-i\delta}^{i\delta} \zeta^k(s+w) \frac{X^w - X^{-w}}{w} dw + \int\limits_{|w|=\delta, \operatorname{Re} \geq 0} \zeta^k(s+w) X^{-w} \frac{dw}{w},$$

since $w^{-1}(X^w - X^{-w})$ is regular on the segment $[-i\delta, i\delta]$. On the semi- 10
circle $|w| = \delta$, $\operatorname{Re} w \geq 0$ we have $|X^{-w}| \leq 1$, hence the total contribution of the last integral above will be again in absolute value $\leq \pi T^{\frac{1}{2}k}(2/\delta)^r$. Thus (1.24) gives

$$|\zeta(\sigma+it)|^k \leq B^{-r} \int\limits_0^B \cdots \int\limits_0^B \left| \int\limits_{-i\delta}^{i\delta} \zeta^k(s+w) \frac{X^w - X^{-w}}{w} dw \right| du_1 \ldots du_r$$
$$+ 2\pi T^{\frac{1}{2}k} \left(\frac{2}{\delta B} \right)^r.$$

For $w = iv$, $-\delta \leq v \leq \delta$ we have

$$\left| \frac{X^w - X^{-w}}{w} \right| = 2 \log X \left| \frac{e^{iv \log X} - e^{-iv \log X}}{2iv \log X} \right| = 2 \log X \left| \frac{\sin(v \log X)}{v \log X} \right|$$
$$\leq 2 \log X = 2(u_1 + \cdots + u_r) \leq 2Br \leq \log T.$$

Taking $B = 4\delta^{-1}$, $r = [C \log T]$ with $C = C(k, \delta, A) > 0$ a sufficiently large constant, we obtain

$$|\zeta(\sigma+iT)|^k \ll B^{-r} B^r \int\limits_{-\delta}^{\delta} |\zeta(\sigma+iT+iv)|^k \log T dv + T^{\frac{1}{2}k} 2^{-r}$$
$$\ll (\log T) \int\limits_{-\delta}^{\delta} |\zeta(\sigma+iT+iv)|^k dv + T^{-A},$$

as asserted.

From Theorem 1.2 one sees immediately that the famous Lindelöf hypothesis that $\zeta(\frac{1}{2}+it) \ll_\epsilon |t|^\epsilon$ is equivalent to the statement that

$$\int_1^T |\zeta\left(\frac{1}{2}+it\right)|^k dt \ll T^{1+\epsilon} \tag{1.25}$$

11 holds for any fixed integer $k \geq 1$. That the lindelöf hypothesis implies (1.25) is obvious, and by (1.23)

$$\begin{aligned}|\zeta\left(\frac{1}{2}+iT\right)|^k &\ll \log T \int_{T-1}^{T+1} |\zeta\left(\frac{1}{2}+iu\right)|^k du + T^{-A}\\ &\ll T^{1+\epsilon}\log T \log T \ll T^{1+2\epsilon}\end{aligned}$$

if (1.25) holds. Thus for any $\epsilon_1 > 0$

$$\zeta(\frac{1}{2}+iT) \ll T^{\epsilon_1}$$

if $k = [(1+2\epsilon)/\epsilon_1] + 1$ in the last bound above. At the time of the writing of this text both the Lindelöf hypothesis and the stronger Riemann hypothesis (all complex zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$) are neither known to be true nor false. The Lindelöf hypothesis may be rephrased as $\mu(\sigma) = 0$ for $\sigma \geq 1/2$, where for any real σ one defines

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t},$$

so that $\zeta(\sigma+it) \ll t^{\mu(\sigma)+\epsilon}$ holds, but $\zeta(\sigma+it) \leq t^c$ does not hold if $c < \mu(\sigma)$. It may be shown that the function $\mu(\sigma)$ is convex downward and non-increasing. The last assertion is a consequence of $\square$

Theorem 1.3. *For* $0 \leq \sigma_1 \leq \sigma_0 \leq \sigma_1 + \frac{1}{2} \leq \frac{3}{2}$, $t \geq t_0$ *we have*

$$\zeta(\sigma_0+it) \ll 1 + \max_{|v|\leq\log\log t} |\zeta(\sigma_1+it+iv)|. \tag{1.26}$$

Proof. Let $\mathscr{D}$ be the rectangle with vertices $\sigma_1 + it \pm i \log\log t$, $\frac{11}{10} + it \pm i \log\log t$. The function

$$f(s) = \zeta(s) \exp\left(-\cos\left(\frac{\pi}{3}(s - s_0)\right)\right), s_0 = \sigma_0 + it$$

is regular in the domain bounded by $\mathscr{D}$. Therefore by the maximum modulus principle

$$|\zeta(s_0)| = e|f(s_0)| \leq e \max_{s \in \mathscr{D}} \left|\zeta(s) \exp\left(-\cos\left(\frac{\pi}{3}(s - s_0)\right)\right)\right|.$$

□

But for $w = u + iv(u, v \text{ real})$ we have 12

$$|\exp(-\cos w)| = \left|\exp\left(-\frac{1}{2}(e^{iw} + e^{-iw})\right)\right|$$
$$\left|\exp\left(-\frac{1}{2}(e^{iu}e^{-v} + e^{-iu}e^{v})\right)\right| = \exp(-\cos u \cdot ch\, v),$$

which decays like a second-order exponential as $|v| \to \infty$ if $\cos u > 0$. If $s \in \mathscr{D}$, then $|\operatorname{Re}(s - s_0)| \leq \frac{11}{10}$, hence

$$\cos\left(\frac{\pi}{3}(s - s_0)\right) \geq \cos\left(\frac{11\pi}{30}\right) : A > 0,$$

and the maximum of $|f(s)|$ on the side of $\mathscr{D}$ with $\operatorname{Re} s = \frac{11}{10}$ is $O(1)$. On the horizontal sides of $\mathscr{D}$ we have $|Im(s - s_0| = \log\log t$, and since trivially $\zeta(s) \ll t$ we have that the maximum over these sides is

$$\ll t \exp\left(-Ach\left(\frac{\pi}{3}\log\log t\right)\right) \leq t \exp\left(\frac{A}{2}e^{(\pi \log\log t)/3}\right)$$
$$= t \exp\left(-\frac{A}{2}(\log t)^{\pi/3}\right) = 0(1)$$

as $t \to \infty$. On the vertical side of $\mathscr{D}$ with $\operatorname{Re} s = \sigma_1$ the exponential factor is bounded, and (1.26) follows.

From (1.14), the functional equation and convexity it follows that

$$\zeta(\sigma+it) \ll \begin{cases} 1 & \text{for } \sigma \geq 2, \\ \log t & \text{for } 1 \leq \sigma \leq 2, \\ t^{\frac{1}{2}(1-\sigma)} \log t & \text{for } 0 \leq \sigma \leq 1, \\ t^{\frac{1}{2}-\sigma} \log t & \text{for } \sigma \leq 0. \end{cases} \tag{1.27}$$

The bound in (1.27) for $0 \leq \sigma \leq 1$ is not best possible, and the true order of $\zeta(\sigma+it)$ (or the value of $\mu(\sigma)$) is one of the deepest problems of zeta-function theory. By using the functional equation one obtains $\mu(\sigma) \leq \frac{1}{2} - \sigma + \mu(1-\sigma)$, so that the most interesting range for σ is
13 $\frac{1}{2} \leq \sigma \leq 1$. The latest bounds for $\zeta(\frac{1}{2}+it)$ are

$$\mu\left(\frac{1}{2}\right) \leq \frac{89}{560} = 0.15892\ldots, \mu\left(\frac{1}{2}\right) \leq \frac{17}{108} = 0.15740\ldots, \tag{1.28}$$
$$\mu\left(\frac{1}{2}\right) \leq 89/570 = 0.15614\ldots$$

due to N. Watt [165], M.N. Huxley and G. Kolesnik [73] and M.N. Huxley [69], respectively. These are the last in a long string of improvements obtained by the use of intricate techniques from the theory of exponential sums.

We pass now to a lower bound result for mean values over short intervals. As in the previous theorem we shall make use of the kernal $\exp(-\cos w)$, which regulates the length of the interval in our result.

Theorem 1.4. *If $k \geq 1$ is a fixed integer, $\sigma \geq 1/2$ is fixed, $12\log\log T \leq Y \leq T$, $T \geq T_0$, then uniformly in o*

$$\int_{T-Y}^{T+Y} |\zeta(\sigma+it)|^k dt \gg Y. \tag{1.29}$$

Proof. Let $\sigma_1 = \sigma + 2$, $s_1 = \sigma_1 + it$, $T - \frac{1}{2}Y \leq t \leq Y + \frac{1}{2}Y$. □

Then $\zeta(s_1) \gg 1$ and therefore

$$\int\limits_{T-\frac{1}{2}Y}^{T+\frac{1}{2}Y} |\zeta(\sigma_1 + it)|^k dt \gg Y. \tag{1.30}$$

Let now $\mathscr{E}$ be the rectangle with vertices $\sigma + iT \pm iY$, $\sigma_2 + iT \pm iY$ ($\sigma_2 = \sigma + 3$) and let X be a parameter which satisfies

$$T^{-c} X \leq T^c$$

for some $c > 0$. The residue theorem gives

$$e^{-1}\zeta^k(s_1) = \frac{1}{2\pi i} \int\limits_{\mathscr{E}} \frac{\zeta^k(w)}{w - s_1} \exp\left(-\cos\left(\frac{w - s_1}{3}\right)\right) X^{s_1 - w} dw.$$

On $\mathscr{E}$ we have $|\operatorname{Re}((w - s_1)/3)| \leq 1$, and on its horizontal sides

$$\left| Im\left(\frac{w - s_1}{3}\right)\right| \geq \frac{1}{3} \cdot \frac{Y}{2} \geq 2 \log\log T.$$

Hence if w lies on the horizontal sides of $\mathscr{E}$ we have **14**

$$\begin{aligned}\left|\exp\left(-\cos\left(\frac{w - s_1}{3}\right)\right)\right| &\geq \exp\left(-\frac{\cos 1}{2} \exp(2 \log\log T)\right)\\ &= \exp\left(-\frac{\cos 1}{2} (\log T)^2\right).\end{aligned}$$

Therefore the condition $T^{-c} \leq X \leq T^c$ ensures that, for a suitable $c_1 > 0$,

$$\begin{aligned}\zeta^k(\sigma_1 + it) \ll X^2 &\int\limits_{T-Y}^{T+Y} |\zeta(\sigma + iv)|^k \exp\left(-c_1 e^{|v-t|/3}\right) dv\\ &+ X^{-1} \int\limits_{T-Y}^{T+Y} \exp\left(-c_1 e^{|v-t|/3}\right) dv + 0(1).\end{aligned}$$

Integrating this estimate over t and using (1.30) we obtain

$$Y \ll X^2 \int\limits_{T-Y}^{T+Y} |\zeta(\sigma+iv)|^k dv \left(\int\limits_{T-\frac{1}{2}Y}^{T+\frac{1}{2}Y} \exp(-c_1 e^{|v-t|/3}) dt \right) \tag{1.31}$$

$$+ X^{-1} \int\limits_{T-Y}^{T+Y} dv \left(\int\limits_{T-\frac{1}{2}Y}^{T+\frac{1}{2}Y} \exp(-c_1 e^{|v-t|/3}) dt \right)$$

$$\ll X^2 \int\limits_{T-Y}^{T+Y} |\zeta(\sigma+iv)|^k dv + X^{-1} Y.$$

Let now

$$I := \int\limits_{T-Y}^{T+Y} |\zeta(\sigma+iv)|^k dv,$$

and choose first $X = Y^{\epsilon}$. Then (1.31) gives $I \gg Y^{1-2\epsilon}$, showing that I cannot be too small. Then we choose $X = Y^{1/3} I^{-1/3}$, so that in view of (1.28) trivially

$$T^{-k/18} \ll X \ll Y.$$

With this choice of X (1.31) reduces to $Y \ll Y^{2/3} Y^{1/3}$, and (1.29) follows.

1.4 Lower Bounds For Mean Values on The Critical Line

15 The lower bound of Theorem 1.4 is best possible when $\sigma > 1/2$ (for $Y \gg T^{2-2\sigma}$ this follows from (1.20)). However, in the most important case when $\sigma = 1/2$, this bound is poorer by a log-factor than the expected order of magnitude of the integral in question. It is conjectured that for any fixed $k \geq 0$

$$I_k(T) := \int\limits_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \sim c_k T (\log T)^{k^2} (T \to \infty) \tag{1.32}$$

for some constant $c_k(> 0)$. So far this is known to hold only for $k = 0$ (the trivial case), $k = 1$, $k = 2$ with $c_0 = 1$, $c_1 = 1$ and $c_2 = 1/(2\pi^2)$, respectively. For $k = 1$ this follows from (1.21), and for $k = 2$ this is a consequence of A.E. Ingham's classical result that

$$I_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt = \frac{T}{2\pi^2} \log^4 T + O(T \log^3 T). \tag{1.33}$$

For other values of k it is impossible at present to prove (1.32) ($k = 1/2$ would be very interesting, for example), and it seems difficult even to formulate a plausible conjectural value of c_k (this subject will be discussed more in chapter 4). The lower bound

$$I_k(T) \gg_k T(\log T)^{k^2} \tag{1.34}$$

is known to hold for all rational $k \geq 0$, and for all real $k \geq 0$ if the Riemann hypothesis is true. The following theorem proves the lower bound (1.30) (under the Lindelöf hypothesis) with the explicit constant implied by the symbol $\gg$. This is

Theorem 1.5. *Assume the Lindelöf hypothesis. If $k > 0$ is a fixed integer, then as $T \to \infty$*

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \geq (c'_k + 0(1))T(\log T)^{k^2}, \tag{1.35}$$

where

$$c'_k = \frac{1}{\Gamma(k^2 + 1)} \prod_p \left((1 - p^{-1})^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k + m)}{\Gamma(k)m!}\right)^2 p^{-m}\right) \tag{1.36}$$

Proof. Note that $c_0 = c'_0$ and $c_1 = c'_1$. The proof will give unconditionally (1.35) for $k < 2/\mu(1/2)$ (so that $\mu(1/2) = 0$, the Lindelöf hypothesis, gives the assertion of the theorem). Also, if the Riemann hypothesis is assumed then (1.35) holds for all $k \geq 0$ (and not only for integers). For this reason it is expedient to use in (1.36) the gamma-function notation. **16**

Let $a_k(s) = \sum_{n \leq Y} d_k(n) n^{-s}$, where $Y = Y(T) = o(T)$ will be chosen later. Using $|a-b|^2 = |a|^2 + |b|^2 - 2\operatorname{Re} a\bar{b}$ one has

$$0 \leq \int_1^T |\zeta^k(1/2+it) - A_k(\frac{1}{2}+it)|^2 + dt$$

$$= I_k(T) + O(1) + \int_1^T |A_k(1/2+it)|^2 dt$$

$$- 2\operatorname{Re}\left(\int_1^T \zeta^k(1/2+it) A_k(1/2-it) dt\right)$$

From (1.15) we obtain

$$\int_1^T |A_k(1/2+it)|^2 dt = (T + O(Y)) \sum_{n \leq Y} d_k^2(n) n^{-1}. \tag{1.37}$$

Consider now the rectangle with vertices $\frac{1}{2}+i$, $a+i$, $a+iT$, $\frac{1}{2}+iT$, where $a = 1 + 1/(\log T)$. Then by Cauchy's theorem we have

$$\int_1^T \left(\frac{1}{2}+it\right) A_k\left(\frac{1}{2}-it\right) dt = \frac{1}{i} \int_{a+i}^{a+iT} \zeta^k(s) A_k(1-s) ds$$

$$+ O\left(\int_{\frac{1}{2}}^{a} |\zeta(\delta+it)|^k \sum_{n \leq Y} d_k n^{\sigma-1} d\sigma\right) + O(1).$$

Now we use the Lindelöf hypothesis in the form

$$\zeta(\sigma+it) \ll t^{2\epsilon(1-\sigma)} \log t \quad \left(\frac{1}{2} \leq \sigma \leq 1, t \geq t_0\right),$$

17 which follows from $\zeta\left(\frac{1}{2}+it\right) \ll t^{\epsilon}$, $\zeta(1+it) \ll \log t$ and convexity. We also recall the elementary estimate

$$\sum_{n\le Y} d_k(n) \ll Y \log^{k-1} Y,$$

so that by partial summation

$$\begin{aligned}\int_{\frac{1}{2}}^{a} &\ll \max_{\frac{1}{2}\le\sigma\le 1} T^{2k\epsilon(1-\sigma)} Y^{\sigma} \log^{2k} T + Y \log^{2k} T\\ &\ll (T^{k\epsilon} Y^{\frac{1}{2}} + Y)\log^{2k} T \ll Y \log^{2k} T\end{aligned}$$

with the choice $\epsilon = 1/(3k)$, since obviously $k \ge 1$ may be assumed. Since

$$A_k(1-s) \ll A_k(1-a) = \sum_{n\le Y} d_k(n) n^{a-1} \ll \sum_{n\le Y} d_k(n),$$

we have by absolute convergence

$$\begin{aligned}\frac{1}{i}\int\limits_{a+i}^{a+iT} \zeta^k(s) A_k(1-s) ds &= \sum_{m=1}^{\infty} d_k(m) \sum_{n\le Y} d_k(n) n^{-1} \left\{ \frac{1}{i} \int\limits_{a+i}^{a+iT} \left(\frac{m}{n}\right)^{-s} ds \right\} \times\\ &(T-1) \sum_{n\le Y} d_k^2(n) n^{-1} + O\left(\sum_{m\ne n, n\le Y} \frac{d_k(m) d_k(n)}{|\log \frac{m}{n}| m^a n^{1-a}} \right).\end{aligned} \tag{1.38}$$

To estimate the last error term we use the elementary inequality

$$d_k(m) d_k(n) \le \frac{1}{2}(d_k^2(m) + d_k^2(n))$$

and distinguish between the cases $m \le 2Y$ and $m > 2Y$. The total contribution of the error term is then found to be

$$\begin{aligned}&\ll \sum_{m=1}^{\infty} d_k^2(m) m^{-a} \sum_{n\le Y, n\ne m} \left|\log \frac{m}{n}\right|^{-1} + \sum_{n\le Y} d_k^2(n) \sum_{m=1, m\ne n}^{\infty} m^{-a} \left|\log \frac{m}{n}\right|^{-1} \times\\ &\quad \sum_{m=1}^{\infty} d_k^2(m) m^{-a} Y \log T + \sum_{n\le Y} d_k^2(n) \log^2 T.\end{aligned}$$

From (1.37) and (1.38) it follows then

$$I_k(T) \geq T \sum_{n\leq Y} d_k^2(n)n^{-1} + O\left(\sum_{n\leq Y} d_k^2(n)\log^2 T\right)$$
$$+ O\left\{Y\left(\sum_{m\leq Y} d_k^2(m)m^{-1} + \sum_{m=1}^{\infty} d_k^2(m)m^{-a}\log T + \log^{2k} T\right)\right\}.$$

18 To finish the proof note that

$$\zeta(a) \ll \frac{1}{a-1} + 1 \qquad (a > 1)$$

for $a = 1 + 1/(\log T)$ gives

$$\sum_{m=1}^{\infty} d_k^2(m)m^{-a} \ll \zeta^{k^2}(a) \ll (\log T)^{k^2},$$

and that we have

$$\sum_{n\leq Y} d_k^2(n)n^{-1} = (c_k' + O(1))(\log Y)^{k^2} \qquad (y \to \infty) \tag{1.39}$$

with c_k' given by (1.36). Thus taking

$$Y = T\exp\left(-\frac{\log T}{\log\log T}\right)$$

we obtain the assertion of the theorem. For unconditional results we use the bound

$$\zeta(\sigma + it) \ll t^{(2\mu(1/2)+\epsilon)(1-\sigma)}\log t\ (1/2 \leq \sigma \leq 1, t \geq t_0),$$

while if the Riemann hypothesis is true we may use the bound

$$\zeta(\sigma + it) \ll \exp\left(\frac{A\log t}{\log\log t}\right) \quad \left(A > 0, \sigma \geq \frac{1}{2}, t \geq t_0\right). \tag{1.40}$$

In the last case the appropriate choice for Y is

$$Y = T\exp\left(-\frac{A_1\log T}{\log\log T}\right)$$

with $a_1 > A$. We conclude by noting that an unconditional proof of (1.35) will be given in Chapter 6. □

Notes For Chapter 1

For the elementary theory of $\zeta(s)$ the reader is referred to the first **19**
two chapters of E.C. Titchmarsh's classic [155] and to the first chapter of the author's monograph [155].

There are many ways to obtain the analytic continuation of $\zeta(s)$ outside the region $\sigma > 1$. One simple way (see T. Estermann [36]) is to write, for $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty}\left(n^{-s} - \int\limits_{n}^{n+1} u^{-s}du\right) + \frac{1}{s-1}$$

and to observe that

$$\left|n^{-s} - \int\limits_{n}^{n+1} u^{-s}du\right| = \left|s\int\limits_{n}^{n+1}\int\limits_{u}^{n} z^{-s-1}dzdu\right| \le |s|n^{-\sigma-1}.$$

Hence the second series above converges absolutely for $\sigma > 0$, and we obtain

$$\zeta(s) = \sum_{n=1}^{\infty}\left(n^{-s} - \int\limits_{n}^{n+1} u^{-s}du\right) + \frac{1}{s-1} \quad (\sigma > 0).$$

One can formalize this approach (see R. Balasubramanian and K. Ramachandra [9]) and show that

$$\sum_{a\le n<b} f(n) = \int\limits_{a}^{b} f(x)dx - \frac{1}{2}\int\limits_{0}^{1}\int\limits_{0}^{1}\sum_{a\le n<b} f'(n+u^{1/2}v)dudv$$

if $a < b$ are integers and $f(x) \in C^1[a,b]$. By repeated application of this summation formula one can obtain analytic continuation of $\zeta(s)$ to $\mathbb{C}$, and also the approximate functional equation (1.14) for $x \ge \left(\frac{1}{2}+\epsilon\right)|t|$.

The approximate functional equation (1.14) is given as Theorem 1.8 of Ivić [75] and as Theorem 4.11 is Titchmarsh [155].

The functional equation (1.6) is, together with the Euler product rep- 20
resentation (1.1), one of the fundamental features of $\zeta(s)$. There are seven proofs of (1.6) in Chapter 2 of Titchmarsh [155] (one of which is given in our text), plus another one in the Notes for Chapter 1 via the theory of Eisenstein series.

For fractional mean values, that is, the integral (1.11) when k is not necessarily a natural number, the reader is referred to Chapter 6. The formulas (6.4) - (6.7) show how (1.12) can be made meaningful for an arbitrary complex number k.

To obtain a precise form of (1.13) when $k = 2$, let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the canonical decomposition of n. Since p^α has exactly $\alpha + 1$ divisors for any prime p we have

$$d(n)n^{-\delta} = \prod_{j=1}^{r} (\alpha_j + 1) p_j^{-\alpha_j \delta},$$

where $\delta > 0$ will be suitably chosen. Now $(\alpha + 1)p^{-\alpha\delta} \le 1$ for $p \ge 2^{1/\delta}$ and

$$2^{\alpha\delta}\left(1 + \frac{1}{\delta \log 2}\right) \ge (1 + \alpha\delta \log 2)\left(1 + \frac{1}{\delta \log 2}\right) \ge 1 + \alpha$$

shows that

$$(\alpha + 1)p^{-\delta} \le 1 + \frac{1}{\delta \log 2}$$

for all primes p and $\alpha \ge 1$. Hence

$$d(n)n^{-\delta} \le \left(1 + \frac{1}{\delta \log 2}\right)^{\pi(2^{1/\delta})},$$

where $\pi(x)(\sim x/\log x$ as $x \to \infty)$ is the number of primes not exceeding x. The choice

$$\delta = \left(1 + \frac{C \log_3 n}{\log_2 n}\right) \frac{\log 2}{\log_2 n},$$

with $C > 2$, $\log_2 n = \log\log n$, $\log_3 n = \log\log\log n$ gives after a simple calculation

$$d(n) \le \exp\left\{\frac{\log 2 \log n}{\log_2 n} + o\left(\frac{\log n \log_3 n}{(\log_2 n)^2}\right)\right\}.$$

As shown by S. Ramanujan [145], this inequality holds even without $\log_3 n$ in the O-term, in which case it is actually best possible.

H.L. Montgomery and R.C. Vaughan [122] proved (1.15) from a version of the so-called Hilbert inequality. The proof of (1.15) was simplified by K. Ramachandra [139], and his proof is essentially given in Chapter 5 of A. Ivić [75]. See also the papers of S. Srinivasan [153] and E. Preissmann [134]. 21

Recently R. Balasubramanian, A. Ivić and K. Ramachandra [13] investigated the function

$$R(T) := \int_1^T |\zeta(1+it)|^2 dt - \zeta(2)T,$$

where the lower bound of integration has to be positive because of the pole of $\zeta(s)$ at $s = 1$. They proved that

$$R(T) = O(\log T),$$

which is in fact (1.22),

$$\int_1^T R(t)dt = -\pi T \log T + O(T \log\log T)$$

and

$$\int_1^T (R(t) + \log t)^2 dt = O(T \log\log T)^4.$$

From either of the last two results one can deduce that

$$R(T) = \Omega_-(\log T),$$

which justifies the claim in the text that the error term in (1.22) cannot be improved.

Theorem 1.2 is due to R. Balasubramanian and K. Ramachandra [10], [11]. It improves considerably on Lemma 7.1 of the author's work [1], which generalizes a lemma of D.R. Heath-Brown [59].

The convexity of the function $\mu(\sigma)$ follows e.g. from general results on Dirichlet series. A direct proof is given by the author [1] in Chapter 8.

Theorem 1.3 is an improved version of Lemma 1.3 of the author's work [1]. The improvement comes from the use of the kernel function $\exp(-\cos w)$, which decays like a second-order exponential.

22 This kernel function, in the alternative form $\exp(\sin^2 w)$, was introduced and systematically used by K. Ramachandra (see e.g. [137] and [141]). In Part I of [141] Ramachandra expresses the opinion that probably no function regular in a strip exists, which decays faster than a second-order exponential. This is indeed so, as was kindly pointed out to me by W.K. Hayman in a letter of August 1990. Thus Ramachandra's kernel function $\exp(\sin^2 w)$ (or $\exp(-\cos w)$) is essentially best possible. This does not imply that, for example, the range $|v| \leq \log\log t$ in (1.26) cannot be reduced, but it certainly cannot be reduced by the method of proof given in the text.

Bounds for $\mu(\sigma)$ are extensively discussed in Chapter 5 of Titchmarsh [155] and Chapter 7 of Ivić [75]. All of the latest vounds for $\mu(1/2)$, given by (1.28), are based on the powerful method introduced by E. Bombieri and H. Iwaniec [16], who proved $\mu(1/2) \leq 9/56 = 0.16071\ldots$. This method was also used by H. Iwaniec and C.J. Mozzochi [82] to prove that $\delta(x) = O(x^{7/22+\epsilon})$, where

$$\Delta(x) := \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1)$$

is the error term in the Dirichlet divisor problem. M.N. Huxley, either alone [67], [68], [69], [70], for jointly with N. Watt [165], [67], successfully investigated exponential sums via the Bombieri-Iwaniec method. Among other things, they succeeded in obtaining new exponent pairs for the estimation of exponential sums (for the definition and basic properties of exponent pairs see S.W. Graham [50] and E. Krätzel [102]).

M.N. Huxley kindly informed me that he just succeeded in proving $\mu(\frac{1}{3}) \leq 89/570 = 0.156140\ldots$. As in the previous estimates of $\mu(\frac{1}{2})$ by the Bombieri-Iwaniec method, the limit of the method, appears to be the value $3/20$. Huxley also provided me with the following summary of the salient ideas of the Bombieri-Iwaniec method, for which I am grateful.

Like the van der Corput method, the Bombieri-Iwaniec method begins by dividing the sum 23

$$S = \sum_{M \leq n \leq 2M} e(f(m))$$

of length M into short sums of the same length N. The short sums are estimated in absolute value, losing the possibility of cancellation between different short sums. The best estimate obtainable from the method must be $\Omega(MN^{-1/2})$. The first idea is to approximate $f(x)$ on each short interval by a polynomial with rational coefficients. Suppose that $f^{(r)}(x) \ll TM^{-r}$ for $r = 2, 3, 4$ and $f^{(3)}(x) \gg TM^{-3}$. Then as x runs through an interval of length N, the derivative $1/2f''(x)$ runs through an interval of length

$$\asymp NT/M^3 \asymp 1/R^2;$$

this equation defines the parameter R. A rational number a/q is chosen in the interval. Let m be the integer for which $1/2f''(m)$ is closest to a/q. Then $f(m + x)$ can be approximated by a polynomial

$$f(m) + (b + x)q^{-1}x + aq^{-1}x^2 + \mu x^3,$$

where b is an integer, x and μ are real numbers.

The sum of length N is transformed by Poisson summation first modulo q, then in the variable x, to give an exponential sum in a new variable h, whose length is proportional to q. Short sums with $q \ll R^2/N$ are called major arcs. For this the sum over h is estimated trivially. Estimating all transformed sums trivially gives the bound $O(MT^{\epsilon}(NR^2)^{-1/6})$, corresponding to the exponent pair $\left(\frac{1}{6}, \frac{2}{3}\right)$. The exponential in the transformed sum is (for q odd) essentially

$$e\left(-\frac{\overline{4a}}{q}(h^2 - 2bh) - \frac{2h^{3/2} + 3xh^{1/2}}{(27\mu q^3)^{1/2}}\right),$$

since further terms will make a negligible contribution. The expression in the exponential can be written as

$$-\underset{\sim}{x}(h) \cdot \underset{\sim}{y}(a/q),$$

where $\underset{\sim}{x}(h)$ is the vector $(h^2, h, h^{3/2}, h^{1/2})$, and $\underset{\sim}{y}(a/q)$ has entries involving $\overline{4a}$, b, q, x and μ from the approximating polynomial on the minor 24
arc ($\overline{4a}$ denotes the multiplicative inverse of $4a$ modulo q)./ Bombieri and Iwaniec interpreted these sums as analogues of those in the classical large sieve, which are sums first over integers, then over rational numbers. They devised an extremely general form of the large sieve.

The large sieve, applied to the transformed sums with q large (minor arcs) would show that the short sums have root mean square size $O(N^{1/2})$ if two spacing problems could be settled. The first spacing problem is to show that for some r the vectors

$$\underset{\sim}{x}(h_1) + \underset{\sim}{x}(h_2) + \cdots \underset{\sim}{x}(h_r)$$

corresponding to different r-tuples of integers $(h_1, \ldots, h_r)$ are usually separated by a certain distance. This is easy to show of $r = 3$. Bombieri and Iwaniec [16] gave a proof for $r = 4$, using analytic methods and ingenious induction. N. watt gave an elementary proof for $r = 4$, and a partial result for $r = 5$. Huxley and Kolesnik [73] obtained the full result for $r = 5$, combining Watt's elementary method with exponential sum techniques. The case $r = 6$ is open. For $r \geq 7$ the Dirichlet box principle shows that there must be many near-coincident sums of seven $\underset{\sim}{x}(h)$ vectors.

The second spacing problem is to show that the vectors $\underset{\sim}{y}(a/q)$ corresponding to different minor arcs are usually separated by a certain distance. Bombieri and Iwaniec [16] gave an argument for the case of Dirichlet series, showing for certain ranges of the parameters that the vectors are separated in their first and third entries. Huxley and Watt [71] and G. Kolesnik obtained arguments of the same power in the general case. The parameter ranges could be extended if one could show that vectors which coincide in their first and third entries are separated in their second and fourth entries. Huxley (to appear) considered how the entries of the vector $\underset{\sim}{y}(a/q)$ change on adjacent minor arcs, and obtained a partial result in this direction.

Bombieri and Iwaniec got the estimate $O(MT^{\epsilon}(NR)^{-1/2})$ for the originale sum S.N. Watt [165] sharpened this to $O(MT^{\epsilon}(N^{11}R^{9})^{-1/40})$, Huxley and Kolesnik [73] to $O(MT^{\epsilon}(N^3R^2)^{-1/10})$, and Huxley (to appear) to $O(MT^{\epsilon}(N^{11}R^4)^{-1/30})$. The conjectured answer to the second spacing problem could give $O(MT^{\epsilon}N^{-1/2})$ even with $r = 3$ in the first spacing problem. 25

The method was adapted by H. Iwaniec and C.J. Mozzochi [82] to show that

$$\sum_{H<h\leq 2H}\ \sum_{M<m\leq 2M} e(hf'(m)) = O(HMT^{\epsilon}(NR)^{-1/2})$$

with the approximating polynomial $(b+x)q^{-1}h+2ahq^{-1}x+3\mu hx^2$ in the notation used above. The first spacing problem is different, but easier. These sums are related to the error terms in the divisor and circle problems (see (2.4), discussion after (2.111) and Notes for Chapter 2), which Iwaniec and Mozzochi estimated as $O(x^{7/22+\epsilon})$. Huxley [67] generalized the application to other lattice-point problems in two dimensions. The improvement in the second spacing problem gives the better bound $O(HMT^{\epsilon}(HN^3R^2)^{-1/6})$, and the exponent 7/22 in the divisor and circle problems can be improved to 23/73 (Huxley, to appear). The method can be also applied to the estimation of $E(T)$. This was done by Heath-Brown and Huxley [64] (see Theorem 2.9), who obtained the exponent 7/22 in this problem also. In the case of $E(T)$ there is the extra condition $H^2 \ll NR$ which prevents one from using the improvement in the second spacing problem.

Theorem 1.4 improves Theorem 9.6 of Ivić [75]. The method of proof is the same, only again the use of the kernel function $\exp(-\cos w)$ improves the range for Y from $\log^{1+\epsilon} T \leq Y \leq T$ to $12\log\log T \leq Y \leq T$. A result analogous to Theorem 1.4 holds for a large class of Dirichlet series, since the proof uses very little from the properties of $\zeta(s)$.

The lower bound (1.34) when k is an integer was proved first by K. Ramachandra [138], who obtained many significant results on lower bounds for power moments. For more about his results, see the Notes for Chapter 6.

Theorem 1.5 is due to J.B. Conrey and A. Ghosh [23]. Their research is continued in [25] and [26]. I have included the proof of Theorem 1.5 because of its simplicity, since a sharper result (Theorem 6.5), due to Balasubramanian and Ramachandra, will be proved later. 26

The asymptotic formula (1.39) follows by standard methods of analytic number theory. Namely,

$$c'_k = \lim_{s\to 1+0} (s-1)^{k^2} F_k(s)/(k^2)!,$$

where (see also Section 4.4) for $\operatorname{Re} s > 1$

$$F_k(s) = \sum_{n=1}^{\infty} d_k^2(n) n^{-s} = \zeta^{k^2}(s) \prod_p ((1-p^{-s})^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{m!\Gamma(k)}\right)^2 p^{-ms}.$$

Since $\lim_{s\to 1}(s-1)\zeta(s) = 1$, one obtains from the above representation the value of c'_k given by (1.36).

In what concerns unconditional bounds for $\zeta(s)$, the best known bound in the vicinity of $\sigma = 1$ is of the form

$$\zeta(\sigma+it) \ll |t|^{A(1-\sigma)^{3/2}} \log^{2/3}|t| \quad (t \ge 2, \frac{1}{2} \le \sigma \le 1). \tag{1.41}$$

This was proved (with $C = 100$) by H.-E. Richert [148] by using Vinogradoc's method. In Chapter 6 of [157] the author proved that the estimate

$$\sum_{N<N\le N'\le 2N} n^{it} \ll N \exp\left(\frac{D\log^3 N}{\log^2 t}\right) \quad (1 \ll N \le t), \tag{1.42}$$

which also follows from Vinogradov's method, implies (1.41) for $1-\eta \le \sigma \le 1$ (the relevant range for σ) with $C = \frac{2}{3}(3D)^{-1/2}$. The estimate (1.42) was shown to hold with $D = 10^{-5}$, giving $C = 122$. Recently E.I. Panteleeva [133] improved the value of D to $D = \frac{1}{2976}$. Since $\frac{2}{3}\left(\frac{2976}{3}\right)^{1/2} = 20.99735\ldots$, this proves (1.41) with $C = 21$, a fact which was also independently obtained by K.M. Bartz [14].

On the other hand, the best conditional bound for $\zeta\left(\frac{1}{2}+it\right)$ under the Riemann hypothesis is (1.40), a proof of which is to be found in Titchmarsh [155] or Ivić [75] (thus by (1.40)) it is seen that the Riemann hypothesis implies the Lindelöf hypothesis). An explicit value of the constant A appearing in (1.40), namely $A = 0.46657\ldots$, was found recently by K. Ramachandra and A. Sankaranarayanan [144].

Chapter 2
The Error Terms In Mean Square Formulas

2.1 The Mean Square Formulas

THE EVALUATION OF the mean square integral $\int_0^T |\zeta(\sigma + it)|^2 dt$ is 27
one of the central problems in zeta-function theory. In view of the functional equation $\zeta(s) = \chi(s)\zeta(1 - s)$ is turns out that the relevant range for σ is the so called "critical strip" $1/2 \leq \sigma \leq 1$. Of particular interest is the case $\sigma = 1/2$ (i.e. the so-called "critical line"). This problem was considered in Chapter 1 (Theorem 1.1). It is possible to push the analysis much further, and to obtain fairly precise formulas for the integrals in equation. These in turn yield relevant information about $|\zeta(\sigma + it)|$ and $|\zeta(1/2 + it)|$. With this in mind we define

$$E(T) := \int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt - T\log\left(\frac{T}{2\pi}\right) - (2\gamma - 1)T, \tag{2.1}$$

where γ is Euler's constant, and for $\frac{1}{2} < \sigma < 1$ fixed

$$E_\sigma(T) := \int\limits_0^T |\zeta(\sigma+it)|^2 dt - \zeta(2\sigma)T - \frac{\zeta(2\sigma - 1)\Gamma(2\sigma - 1)}{1 - \sigma}\sin(\pi\sigma)T^{2-2\sigma}. \tag{2.2}$$

The functions $E(T)$ and $E_\sigma(T)$ are obviously related, and in fact we have

$$\lim_{\sigma\to\frac{1}{2}+0} E_\sigma(T) = E(T). \tag{2.3}$$

To show this we use the Laurent expansions

$$\begin{aligned}
\Gamma(s) &= \frac{1}{s} - \gamma + a_1 s + a_2 s^2 + \cdots && \text{(near } s = 0),\\
\zeta(s) &= \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \cdots && \text{(near } s = 1),\\
\zeta(s) &= \zeta(0) + \zeta'(0)s + b_2 s^2 + \cdots &&\\
&= -\frac{1}{2} - \frac{1}{2}\log(2\pi)s + b_2 s^2 + \cdots && \text{(near } s = 0),\\
\frac{1}{1-s} &= 2 - 2(1-2s) + 2(1-2s)^2 + \cdots && \text{(near } s = \tfrac{1}{2}).
\end{aligned}$$

28 Then for $\sigma \to \frac{1}{2} + 0$ and T fixed we have

$$\begin{aligned}
&\zeta(2\sigma)T + \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)}{1-\sigma}\cos(\pi(\sigma-\frac{1}{2}))T^{2-2\sigma} = \frac{T}{2\sigma-1}\\
&+ \gamma T + O((2\sigma-1)) + \left\{2 - 2(1-2\sigma) + O((2\sigma-1)^2)\right\}\times\\
&\left\{-\frac{1}{2} - \frac{1}{2}\log(2\pi)(2\sigma-1) + O((2\sigma-1)^2)\right\}\times\\
&\left\{\frac{1}{2\sigma-1} - \gamma + a_1(2\sigma-1) + O((2\sigma-1)^2)\right\}\times\\
&\left\{T + (1-2\sigma)T\log T + O((2\sigma-1)^2)\right\}\times\\
&= T\log T + (2\gamma - 1 - \log(2\pi))T + O((2\sigma-1)),
\end{aligned}$$

and this proves (2.3).

An explicit formula for $E(T)$ was discovered in 1949 by F.V. Atkinson, and its analogue for $E_\sigma(T)(\frac{1}{2} < \sigma < \frac{3}{4})$ in 1989 by K. Matsumoto. Atkinson's formula shows certain analogies with the classical Voronoi formula for

$$\Delta(x) := \sum_{n\le x}{}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4}, \tag{2.4}$$

the error term in the Dirichlet divisor problem (see (2.25)), where $\sum_{n\le x}'$ means that the last term in the sum is to be halved if x is an integer. In fact, the analogy between $E(T)$ and $\Delta(x)$ was one of the primary motivations of Atkinson's work on $E(T)$, and his formula will be given below as Theorem 2.1, It has inspired much recent research on the zeta-function. One of its remarkable aspects is that it provides various results on $E(T)$ and related topics, some of which will be discussed in Chapter 3. Another important thing is that Atkinso's formula may be generalized in several ways. Instead of the mean square of $|\zeta(\frac{1}{2}+it)|$ one may consider the mean square of $|L(\frac{1}{2}+it,\chi)|$. The possibility of considering the mean square of a Dirichlet polynomial together with $|\zeta(\frac{1}{2}+it)|$ will be treated in Section 2.8. The approach to the fourth power moment, found recently by Y. Motohashi and expounded in Chapter 5, is based on a generalization of Atkinson's approach. As already mentioned, it **29**
is also possible to obtain the analogue of $E(T)$ for $E_\sigma(T)$ in the range $\frac{1}{2}<\sigma<\frac{3}{4}$ ($\sigma=\frac{3}{4}$ appears to be the limit of the present method). The formula for $E_\sigma(T)$ will be given as Theorem 2.2, and its proof will be given parallel to the proof of Theorem 2.1. A discussion of the mean square formula for $E(T)$, which is analogous to the corresponding problem for $\Delta(x)$, is presented in Section 2.6. Upper bound results on $E(T)$ and $E_\sigma(T)$ are contained in Section 2.7, and some aspects of $E(T)$ are discussed also in Chapter 4, where a general approach to even moments of $|\zeta(\frac{1}{2}+it)|$ is given.

We present now the formulas for $E(T)$ and $E_\sigma(T)$.

Theorem 2.1. *Let* $0<A<A'$ *be any two fixed constants such that* $AT<N<A'T$, $N'=N'(T)=T/(2\pi)+N/2-(N^2/4+NT/(2\pi))^{\frac{1}{2}}$, *and let*

$$f(T,n):=2T\,ar\sinh\sqrt{\frac{\pi n}{2T}}+(2\pi nT+\pi^2n^2)^{1/2}-\frac{\pi}{4},$$
$$g(T,n):=T\log\left(\frac{T}{2\pi n}\right)-T+\frac{\pi}{4},\ ar\sinh x:=\log(x+\sqrt{x^2+1}).$$

Then

$$E(T) = 2^{-\frac{1}{2}} \sum_{n\le N} (-1)^n d(n) n^{-\frac{1}{2}} \left(ar\sinh \sqrt{\frac{\pi n}{2T}} \right)^{-1} \left(\frac{T}{2\pi n} + \frac{1}{4} \right)^{\frac{1}{4}} \times$$
$$\cos(f(T,n)) - 2 \sum_{n\le N'} d(n) n^{-\frac{1}{2}} \left(\log \frac{T}{2\pi n} \right)^{-1} \times$$
$$\cos(g(T,n)) + O(\log^2 T). \tag{2.5}$$

Theorem 2.2. *Let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let σ be a fixed number satisfying $\frac{1}{2} < \sigma < \frac{3}{4}$. If $\sigma_a(n) = \sum_{d|n} d^a$, then with the notation introduced in Theorem 2.1 we have*

$$E_\sigma(T) = 2^{\sigma-1} \left(\frac{\pi}{T} \right)^{\sigma-\frac{1}{2}} \sum_{n\le N} (-1)^n \sigma_{1-2\sigma}(n) n^{\sigma-1} \left(ar\sinh \sqrt{\frac{\pi n}{2T}} \right)^{-1} \times$$
$$\left(\frac{T}{2\pi n} + \frac{1}{4} \right)^{-\frac{1}{4}} \cos((f(T,n)) - 2 \left(\frac{2\pi}{T} \right)^{\sigma-\frac{1}{2}} \sum_{n\le N}, \sigma_{1-2\sigma}(n) n^{\sigma-1} \times \tag{2.6}$$
$$\left(\log \frac{T}{2\pi n} \right)^{-1} \cos(g(T,n)) + O(\log T).$$

30 Note that, as $\sigma \to \frac{1}{2} + 0$, the sums in Theorem 2.2 become the sums in Theorem 2.1. In the next sections we shall give a proof of Theorem 2.1 and Theorem 2.2. It will transpire from the proof why the restriction $\frac{1}{2} < \sigma < \frac{3}{4}$ is a natural one in Theorem 2.2.

2.2 The Beginning of Proof

We start with the initial stages of proof of Theorem 2.1 and Theorem 2.2. Atkinson's basic idea was to use the obvious identity, valid for $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$,

$$\zeta(u)\zeta(v) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-u} n^{-v} = \zeta(u+v) + f(u,v) + f(v,u), \tag{2.7}$$

where

$$f(u,v) := \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r^{-u}(r+s)^{-v}. \tag{2.8}$$

What is needed is the analytic continuation of $f(u,v)$ to a region containing the points $u = \frac{1}{2}+it$, $v = \frac{1}{2}-it$, $v = \frac{1}{2}-it$, so that eventually the integration of (2.7) will lead to (2.5). To carry out this plane we show first that $f(u,v)$ is a meromorphic function of u and v for $\operatorname{Re}(u+v) > 0$. Taking $\operatorname{Re} v > 1$ and writing

$$\psi(x) = x - [x] - \frac{1}{2}, \psi_1(x) = \int_1^x \psi(y)dy,$$

so that $\psi_1(x) \ll 1$ uniformly in x, it follows on integrating by parts that

$$\begin{aligned}
\sum_{s=1}^{\infty}(r+s)^{-v} &= \int_{1-0}^{\infty}(r+y)^{-v}d[y] = \int_r^{\infty}([x]-r)x^{-v-1}dx \\
&= r^{1-v}(v-1)^{-1} - \frac{1}{2}r^{-v} - v\int_r^{\infty}\psi(x)x^{-v-1}dx \\
&= r^{1-v}(v-1)^{-1} - \frac{1}{2}r^{-v} - v(v+1)\int_r^{\infty}\psi_1(x)x^{-v-2}dx \\
&= r^{1-v}(v-1)^{-1} - \frac{1}{2}r^{-v} + O(|v|^2 r^{-\operatorname{Re} v-1}).
\end{aligned}$$

Hence

$$f(u,v) = (v-1)^{-1}\sum_{r=1}^{\infty} r^{1-u-v} - \frac{1}{2}\sum_{r=1}^{\infty} r^{-u-v} + O\left(|v|^2\sum_{r=1}^{\infty} r^{-\operatorname{Re} u - \operatorname{Re} v - 1}\right),$$

and therefore **31**

$$f(u,v) - (v-1)^{-1}\zeta(u+v-1) + \frac{1}{2}\zeta(u+v)$$

is regular for $\operatorname{Re}(u+v) > 0$. Thus (2.7) holds by analytic continuation when u and v both lie in the critical strip, apart from the poles at $v = 1$, $u+v = 1$ and $u+v = 2$.

We consider next the case $-1 < \operatorname{Re} u < 0$, $\operatorname{Re}(u+v) > 2$. We use the well-known Poisson summation formula: If a, b are integers such that $a < b$ and $f(x)$ has bounded first derivative on $[a, b]$, then

$$\sum_{a\le n\le b}{}' f(n) = \int\limits_a^b f(x)dx + 2\sum_{n=1}^{\infty}\int\limits_a^b f(x)\cos(2\pi nx)dx, \tag{2.9}$$

where $\sum'$ means that the first and the last term in the sum is to be halved. By using (2.9) with $a = 0$, $b = \infty$ it follows that

$$\sum_{r=1}^{\infty} r^{-u}(r+s)^{-v} = \int\limits_0^{\infty} x^{-u}(x+s)^{-v}dx + 2\sum_{m=1}^{\infty}\int\limits_0^{\infty} x^{-u}(x+s)^{-v}\times$$

$$\cos(2\pi mx)dx = s^{1-u-v}\left(\int\limits_0^{\infty} y^{-u}(1+y)^{-v}dy + 2\times\right.$$

$$\left.\sum_{m=1}^{\infty}\int\limits_0^{\infty} y^{-u}(1+y)^{-v}\cos(2\pi mys)dy\right)$$

after the change of variable $x = sy$. Recalling the beta-integral formula

$$B(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(\operatorname{Re} a > 0, \operatorname{Re} b > 0), \tag{2.10}$$

we have with $1 + y = 1/z$

$$\int\limits_o^{\infty} y^{-u}(1+y)^{-v}dy = \int\limits_0^1 (1-z)^{-u}z^{u+v-2}dz = \frac{\Gamma(u+v-1)\Gamma(1-u)}{\Gamma(v)}. \tag{2.11}$$

Since $\operatorname{Re}(u+v) > 2$, summation over s gives

$$g(u,v) := f(u,v) - \Gamma(u+v-1)\Gamma(1-u)\Gamma^{-1}(v)\zeta(u+v-1)$$

$$= 2\sum_{s=1}^{\infty} s^{1-u-v}\sum_{m=1}^{\infty}\int\limits_0^{\infty} y^{-u}(1+y)^{-v}\cos(2\pi msy)dy. \tag{2.12}$$

32 To investigate the convergence of the last expression we note that for $\mathrm{Re}\, u < 1$, $\mathrm{Re}(u+v) > 0$, $m \geq 1$,

$$2\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi ny)dy = \int_0^\infty y^{-u}(1+y)^{-v}(e(ny)+e(-ny))dy \tag{2.13}$$

$$= \int_0^{i\infty} y^{-u}(1+y)^{-v}e^{ny}dy + \int_0^{-i\infty} y^{-u}(1+y)^{-v}e(-ny)dy$$

$$= n^{u-1}\int_0^{i\infty} y^{-u}\left(1+\frac{y}{n}\right)^{-v} e(y)dy + n^{u-1}\int_0^{-i\infty} y^{-u}\left(1+\frac{y}{n}\right)^{-v} e(-y)dy$$

$$\ll n^{\mathrm{Re}\, u-1}|u-1|^{-1}$$

uniformly for bounded u and v, which follows after an integration by parts. Therefore the double series in (2.12) is absolutely convergent for $\mathrm{Re}\, u < 0$, $\mathrm{Re}\, v > 1$, $\mathrm{Re}(u+v) > 0$, by comparison with

$$\sum_{s=1}^\infty |s^{-v}| \sum_{m=1}^\infty |m^{u-1}|.$$

Hence (2.12) holds throughout this region, and grouping together the terms with $ms = n$ we have

$$g(u,v) = 2\sum_{n=1}^\infty \sigma_{1-u-v}(n)\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi ny)dy,$$

where as before $\sigma_a(n) = \sum_{d|n} d^a$, so that $\sigma_0(n) = \sum_{d|n} 1 = d(n)$ is the number of divisors of n.

Therefore if $g(u,v)$ is the analytic continuation of the function given by (2.12), then for $0 < \mathrm{Re}\, u < 1$, $0 < \mathrm{Re}\, v < 1$, $u+v \neq 1$, we have

$$\zeta(u)\zeta(v) = \zeta(u+v) + \zeta(u+v-1)\Gamma(u+v-1) \left(\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right) + g(u,v) + g(v,u). \tag{2.14}$$

So far our discussion was general, but at this point we shall distin-
33 guish between the cases $\sigma = \frac{1}{2}$ and $\sigma > \frac{1}{2}$, which eventually lead to the expressions for $E(T)$ and $E_\sigma(T)$, given by Theorem 2.1 and Theorem 2.2, respectively.

a) The Case $\sigma = 1/2$.

We are interested in (2.14) in the exceptional case $u + v = 1$. We shall use the continuity of $g(u, v)$ and set $u + v = 1 + \delta$, $0 < |\delta| < 1/2$, with the aim of letting $\delta \to 0$. Then the terms on the right-hand side of (2.14) not containing g become

$$\begin{aligned}
&\zeta(1+\delta) + \zeta(\delta)\Gamma(\delta)\left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)} + \frac{\Gamma(u-\delta)}{\Gamma(u)}\right) \\
&= \zeta(1+\delta) + \zeta(1-\delta)\frac{(2\pi)^\delta}{2\cos(\frac{1}{2}\pi\delta)}\left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)} + \frac{\Gamma(u-\delta)}{\Gamma(u)}\right) \\
&= \delta^{-1} + \gamma + (\gamma - \delta^{-1})\left(\frac{1}{2} + \frac{\delta}{2}\log 2\pi\right)\left(1 - \frac{\Gamma'(1-u)}{\Gamma(1-u)}\delta + 1 - \frac{\Gamma'(u)}{\Gamma(u)}\delta\right) \\
&\qquad + O(|\delta|) \\
&= \frac{1}{2}\left(\frac{\Gamma'(1-u)}{\Gamma(1-u)} + \frac{\Gamma'(u)}{\Gamma(u)}\right) + 2\gamma - \log(2\pi) + O(|\delta|),
\end{aligned}$$

where we used the functional equation for $\zeta(s)$. Hence letting $\delta \to 0$ we have, for $0 < \operatorname{Re} u < 1$,

$$\zeta(u)\zeta(1-u) = \frac{1}{2}\left(\frac{\Gamma'(1-u)}{\Gamma(1-u)} + \frac{\Gamma'(u)}{\gamma(u)}\right) + 2\gamma - \log(2\pi) + g(u, 1-u) + g(1-u, u), \tag{2.15}$$

where reasoning as in (2.13) we have, for $\operatorname{Re} u < 0$,

$$g(u, 1-u) = 2\sum_{n=1}^\infty d(n)\int_0^\infty y^{-u}(1+y)^{u-1}\cos(2\pi ny)dy. \tag{2.16}$$

What is needed now is the analytic continuation of $g(u, 1-u)$, valid for $\operatorname{Re} u = 1/2$. This will be obtained in the next section by the use of the Voronoi formula. Right now, assuming that we have such a continuation, we may write an expression for $E(T)$ that will be used in later evaluations. We set $u = \frac{1}{2} + it$ and note that $\zeta(u)\zeta(1-u) = |\zeta(\frac{1}{2}+it)|^2$. Integration of (2.15) gives then

$$2i\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} \zeta(u)\zeta(1-u)du$$

$$\frac{1}{2}(-\log\Gamma(1-u)+\log\Gamma(u))\Bigg|_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} + 2iT(2\gamma - \log 2\pi)$$

$$+\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} (g(u,1-u)+g(1-u,u))du$$

$$\log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)} + 2iT(2\gamma-\log 2\pi) + 2\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g(u,1-u)du.$$

To simplify this expression we use Striling's formula in the form **34**

$$\log\Gamma(s+b) = \left(s+b-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log(2\pi) + O(|s|^{-1}), \quad (2.17)$$

which is valid for b a constant and $|\arg s| \leq \pi - \delta(\delta > 0)$, if $s = 0$ and the neighbourhoods of the poles of $\Gamma(s+b)$ are excluded. We obtain

$$\int_0^T |\zeta\left(\frac{1}{2}+iT\right)|^2 dt = T\log\left(\frac{T}{2\pi}\right) + (2\gamma-1)T - i$$

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g(u,1-u)du + O(1), \quad (2.18)$$

or

$$E(T) = -\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g(u,1-u)du + O(1). \quad (2.19)$$

b) The Case $1/2 < \sigma < 3/4$**.**

We start again from (2.14), setting $u = \sigma + it$, $v = 2\sigma - u = \sigma - it$ and integrating the resulting expression over t. It follows that

$$\int\limits_0^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + 2\zeta(2\sigma-1)\Gamma(2\sigma-1)\,\mathrm{Re}\times$$

$$\left\{\int\limits_0^T \frac{\Gamma(1+\sigma+it)}{\Gamma(\sigma+it)}dt\right\} - i\int\limits_{\sigma-iT}^{\sigma+iT} g(u, 2\sigma-u)du. \tag{2.20}$$

To evaluate the first integral on the right-hand side of (2.20) we use Stirling's formula in the form

$$\Gamma(s) = \sqrt{2\pi}t^{\sigma-\frac{1}{2}}\exp\left\{-\frac{\pi}{2}t + i\left(t\log t - t + \frac{\pi}{2}\left(\sigma - \frac{1}{2}\right)\right)\right\}\cdot\left(1 + O\left(\frac{1}{t}\right)\right) \tag{2.21}$$

for $0 \le \sigma \le 1$, $t \ge t_0 > 0$. Then for $t \ge t_0$

$$\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)} = t^{1-2\sigma}\exp\left(\frac{i\pi}{2}(1-2\sigma)\right)\cdot\left(1 + O\left(\frac{1}{t}\right)\right),$$

$$\int\limits_0^T \frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}dt = \int\limits_{t_0}^T +O(1) = \exp\left(\frac{i\pi}{2}(1-2\sigma)\right)\frac{T^{2-2\sigma}}{2-2\sigma} + O(1),$$

and

$$2\zeta(2\sigma-1)\Gamma(2\delta-1)\,\mathrm{Re}\left\{\int\limits_0^T \frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}dt\right\}$$

$$= \frac{\zeta(2\sigma-1)\gamma(2\sigma-1)\cos(\frac{1}{2}\pi(1-2\sigma))}{1-\sigma}T^{2-2\sigma} + O(1).$$

35 Inserting the last expression in (2.20) we obtain

$$\int\limits_0^T |\zeta(\sigma+it)|^2 dt = \zeta(2\sigma)T + \frac{\zeta(2\sigma)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}T^{2-2\sigma} + E_\sigma(T)$$

with

$$E_\sigma(T) = -i \int\limits_{\sigma-iT}^{\sigma+iT} g(u, 2\sigma - u)du + O(1). \tag{2.22}$$

2.3 Transformation of the Expressions for the Error Terms

We shall transform the integrals in (2.19) and (2.22), providing incidentally analytic continuation for the function g which appears in them. First we shall deal with the case of $E(T)$ in (2.19), using the Voronoi formula for $\Delta(x)$ (see (2.24)) to transform the integral in (2.16). This is

$$\Delta(x) = -\frac{2}{\pi}\sqrt{x}\sum_{n=1}^{\infty} d(n)n^{-\frac{1}{2}}\left(K_1(4\pi\sqrt{4x}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})\right), \tag{2.23}$$

where K_1, Y_1 are standard notation for the Bessel functions. It is known from Analysis that there exist asymptotic formulas for the Bessel functions with any desired degree of accuracy. In particular, one has

$$\begin{aligned}\Delta(x) = {} & \frac{x^{\frac{1}{4}}}{\pi 2^{\frac{1}{2}}}\sum_{n=1}^{\infty} d(n)n^{-\frac{3}{4}}\cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) \\ & - \frac{x^{-\frac{1}{4}}}{32\pi 2^{\frac{1}{2}}}\sum_{n=1}^{\infty} d(n)n^{-5/4}\sin\left(4\pi\sqrt{4x} - \frac{\pi}{4}\right) \\ & + \frac{15x^{-\frac{1}{4}}}{2^{11}\pi 2^{\frac{1}{2}}}\sum_{n=1}^{\infty} d(n)n^{-7/4}\cos\left(4x\sqrt{nx} - \frac{\pi}{4}\right) + O(x^{-5/4}).\end{aligned} \tag{2.24}$$

The series in (2.23) is boundedly convergent when x lies in any fixed closed subinterval of $(0,\infty)$, and it is uniformly convergent when the interval is free of integers. Instead of (2.23) or (2.24) one often uses a truncated expression for $\Delta(x)$, namely **36**

$$\Delta(x) = (\pi\sqrt{2})^{-1}x^{\frac{1}{4}}\sum_{n\le N} d(n)n^{-\frac{1}{4}}\cos\left(4\sqrt{nx} - \frac{\pi}{4}\right)$$

$$O(x^{\epsilon}) + O(x^{\frac{1}{2}+\epsilon}N^{-\frac{1}{2}}) \tag{2.25}$$

when $1 \ll N \ll x^A$ for any fixed $A > 0$.

Let now N be a large positive integer, $X = N + 1/2$, and

$$h(u,x) := 2\int_0^{\infty} y^{-u}(1+y)^{u-1}\cos(2\pi xy)dy. \tag{2.26}$$

With $D(x) = \sum_{n\le x}{}' d(n)$ we have

$$\begin{aligned}
\sum_{n>N} d(n)h(u,n) &= \int_X^{\infty} h(u,x)dD(x)\\
&= \int_X^{\infty}(\log x + 2\gamma)h(u,x)dx + \int_X^{\infty} h(u,x)d\Delta(x)\\
&= -\Delta(X)h(u,x) + \int_X^{\infty}(\log x + 2\gamma)h(u,x)dx\\
&\quad - \int_X^{\infty}\Delta(x)\frac{\partial h(u,x)}{\partial x}dx.
\end{aligned}$$

Hence (2.16) becomes

$$\begin{aligned}
g(u, 1-u) &= \sum_{n\le N} d(n)h(u,n) - \Delta(X)h(u,X)\\
&\quad + \int_X^{\infty}(\log x + 2\gamma)h(u,x)dx - \int_X^{\infty}\Delta(x)\frac{\partial h(u,x)}{\partial x}dx\\
&= g_1(u) - g_2(u) + g_3(u) - g_4(u),
\end{aligned}$$

say. Here $g_1(u)$ and $g_2(u)$ are analytic functions of u in the region $\operatorname{Re} u < 1$, since the right-hand side of (2.26) is analytic in this region. Next we have

$$g_3(u) = \int\limits_X^\infty (\log x + 2\gamma)\times$$
$$\left\{\int\limits_0^{i\infty} y^{-u}(1+y)^{u-1}e(xy)dy + \int\limits_0^{-i\infty} y^{-u}(1+y)^{u-1}e(-xy)dy\right\}dx. \tag{2.27}$$

The last integral may be taken over $[0, \infty)$ and the variable changed from y to y/X. The other two integrals in (2.27) are treated similarly, and the results may be combined to produce 37

$$g_3(u) = -\pi^{-1}(\log X + 2\gamma)\int\limits_0^\infty y^{-u-1}(1+y)^{u-1}\sin(2\pi Xy)dy$$
$$+ (\pi u)^{-1}\int\limits_0^\infty y^{-u-1}(1+y)^{u}\sin(2\pi Xy)dy. \tag{2.28}$$

To treat $g_4(u)$, write first

$$h(u, x) = \int\limits_0^{i\infty} y^{-u}(1+y)^{u-1}e(xy)dy + \int\limits_0^{-i\infty} y^{-u}(1+y)^{u-1}e(-xy)dy.$$

Then

$$\begin{aligned}\frac{\partial h(u,x)}{\partial x} &= 2\pi i\int\limits_0^{i\infty} y^{1-u}(1+y)^{u-1}e(xy)dy - 2\pi i\int\limits_0^{-i\infty} y^{1-u}(1+y)^{u-1}e(-xy)dy\\ &= 2\pi i x^{u-2}\left(\int\limits_0^{i\infty} y^{1-u}\left(1+\frac{y}{x}\right)^{u-1}e(y)dy - \int\limits_0^{-i\infty} y^{1-u}\left(1+\frac{y}{x}\right)^{u-1}e(-y)dy\right)\\ &\ll x^{\operatorname{Re} u-2}\end{aligned}$$

for $\operatorname{Re} u \le 1$ and bounded u. From (2.25) with $N = x^{1/3}$ one has by trivial estimation $\Delta(x) \ll x^{1/3+\epsilon}$, which suffices to show that $g_4(u)$ is an analytic function of u when $\operatorname{Re} u < 2/3$. Therefore from (2.19) and the expressions for $g_n(u)$ we obtain

$$E(T) = I_1 - I_2 + I_3 - I_4 + O(1), \tag{2.29}$$

where for $n = 1, 2, 3, 4$

$$I_n = -i \int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g_n(u). \tag{2.30}$$

Hence

$$I_1 = 4 \sum_{n \le N} d(n) \int\limits_0^\infty \frac{\sin(T \log(1 + 1/y)) \cos(2\pi ny)}{y^{\frac{1}{2}}(1 + y)^{\frac{1}{2}} \log(1 + 1/y)} dy, \tag{2.31}$$

38

$$I_2 = 4\Delta(X) \int\limits_0^\infty \frac{\sin(T \log(1 + 1/y)) \cos(2\pi Xy)}{y^{\frac{1}{2}}(1 + y)^{\frac{1}{2}} \log(1 + 1/y)} dy, \tag{2.32}$$

$$\begin{aligned} I_3 &= -\frac{2}{\pi}(\log X + 2\gamma) \int\limits_0^\infty \frac{\sin(T \log(1 + 1/y)) \sin(2\pi Xy)}{y^{\frac{3}{2}}(1 + y)^{\frac{1}{2}} \log(1 + 1/y)} dy, \\ &+ (\pi i)^{-1} \int\limits_0^\infty y^{-1} \sin(2\pi Xy) dy \int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} (1 + y^{-1})^u u^{-1} du, \end{aligned} \tag{2.33}$$

and lastly

$$I_4 = -i \int\limits_X^\infty \Delta(x) dx \int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\partial h(u, x)}{\partial x} du, \tag{2.34}$$

where N is a positive integer, $X = N + 1/2$, and as in the formulation of Theorem 2.1 we get $AT < N < A'T$. A more explicit formula for I_4

may be derived as follows. Using (2.26) we have

$$\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\partial h(u,x)}{\partial x} du = 4i\frac{\partial}{\partial x}\left\{\int\limits_0^\infty \frac{\sin(T\log(1+1/y))\cos(2\pi xy)}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}}\log(1+1/y)}dy\right\}$$
$$= 4i\frac{\partial}{\partial x}\left\{\int\limits_0^\infty \frac{\sin(T\log(x+y)/y)\cos(2\pi y)}{y^{\frac{1}{2}}(x+y)^{\frac{1}{2}}\log(x+y)/y}dy\right\}$$
$$= 4i\int\limits_0^\infty \frac{\cos(2\pi y)}{y^{\frac{1}{2}}(x+y)^{3/2}\log(x+y)/y}$$
$$\left\{T\cos(T\log(x+y)/y) - \sin(T\log(x+y)/y)\left(\frac{1}{2}\right.\right.$$
$$\left.\left. + \log^{-1}\left(\frac{x+y}{y}\right)\right)\right\} dy$$

Hence replacing y by xy we obtain

$$I_4 = 4\int\limits_x^\infty \frac{\Delta(x)}{x}dx\int\limits_0^\infty \frac{\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}\log(1+1/y)}\times$$
$$\left\{T\cos\left(T\log\frac{1+y}{y}\right) - \sin\left(T\log\frac{1+y}{y}\right)\left(\frac{1}{2}+\log^{-1}\left(\frac{1+y}{y}\right)\right)\right\} dy. \tag{2.35}$$

We pass now to the discussion of $E_\sigma(T)$, namely the case $1/2 < \sigma < 3/4$. Let

$$D_{1-2\sigma}(x) = \sum_{n\le x}{}' \sigma_{1-2\sigma}(n), \tag{2.36}$$

where, as in the definition of $\Delta(x)$, $\sum'$ means that the last term in the sum is to be halved if x is an integer, and the error term $\Delta_{1-2\sigma}(x)$ is defined by the formula **39**

$$D_{1-2\sigma}(x) = \zeta(2\sigma)x + (2-2\sigma)^{-1}\zeta(2-2\sigma)x^{2-2\sigma}$$

$$-\frac{1}{2}\zeta(2\sigma-1)+\Delta_{1-2\sigma}(x). \qquad (2.37)$$

The analytic continuation of $g(u, 2\sigma-u)$ in (2.22) can be obtained by the use of the analogue of Voronoi's classical formula for the function $\Delta_{1-2\sigma}(x)$. In the usual notation of Bessel functions one has

$$\Delta_{1-2\sigma}(X)=-x^{1-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-1}\Big\{\cos(\sigma\pi)J_{2-2\sigma}(4\pi\sqrt{nx}) \qquad (2.38)$$
$$+\sin(\sigma\pi)\Big(Y_{2-2\sigma}(4\pi\sqrt{nx})+\frac{2}{\pi}K_{2-2\sigma}(4\pi\sqrt{nx})\Big)\Big\},$$

and it may be noted that the right-hand side of (2.38) becomes the right-hand side of (2.23) when $\sigma\to\frac{1}{2}+0$. Actually we have the generating Dirichlet series

$$\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{-s}=\zeta(s)\zeta(2\sigma-1+s) \qquad (\operatorname{Re} s>1),$$
$$\sum_{n=1}^{\infty}d(n)n^{-s}=\zeta^2(s) \qquad (\operatorname{Re} s>1),$$

and

$$\lim_{\sigma\to\frac{1}{2}+0}\sigma_{1-2\sigma}(n)=d(n).$$

The series in (2.38) is boundedly convergent when x lies in any fixed closed subinterval of $(0,\infty)$, provided that $1/2<\sigma<3/4$, which is a condition whose significance in the analysis of $E_\sigma(T)$ now becomes apparent. Using the asymptotic formulas for the Bessel functions we obtain from (2.38)

$$\Delta_{1-2\sigma}(x)=O(x^{-\sigma-\frac{1}{4}}+(\sqrt{2}\pi)^{-1}x^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-5/4}\Big\{\cos\Big(4\pi\sqrt{nx}$$
$$-\frac{\pi}{4}\Big)-(32\pi\sqrt{nx})^{-1}(16(1-\sigma)^2-1)\sin\Big(4\pi\sqrt{nx}-\frac{\pi}{4}\Big)\Big\}. \qquad (2.39)$$

The analogue of the truncated Voronoi formula (2.25) for $\Delta_{1-2\sigma}(x)$
40 is, for $1 \ll N \ll x^A$, $1/2 < \sigma < 3/4$,

$$\Delta_{1-2\sigma}(x) = (\sqrt{2}\pi)^{-1} x^{3/4-\sigma} \sum_{n \leq N} \sigma_{1-2\sigma}(n) n^{\sigma-5/4} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right)$$
$$+ O(x^{1/2-\sigma} N^{\sigma-1/2+\epsilon}) + O(x^{\frac{1}{2}+\epsilon} N^{-\frac{1}{2}}). \tag{2.40}$$

Taking in (2.40) $N = x^{(4\sigma-1)/(4\sigma+1)}$ we obtain by trivial estimation

$$\Delta_{1-2\sigma(x)} \ll x^{1/(4\sigma+1)+\epsilon}. \tag{2.41}$$

One can prove (2.40) much in the same way as one proves (2.35). We shall now briefly sketch the proof. By the Perron inversion formula

$$\sum_{n \leq x}{}' \sigma_{1-2\sigma}(n) = \frac{1}{2\pi i} \int\limits_{1+\epsilon-iT}^{1+\epsilon+iT} L(w) \frac{x^w}{w} dw + O(x^\epsilon) + O(x^{1+\epsilon} T^{-1}), \tag{2.42}$$

where T is a parameter, $w = u + iv (v \geq v_0)$, and

$$L(w) = \zeta(w)\zeta(2\sigma - 1 + w) = \psi(w) L(1 - w),$$
$$\psi(w) = \chi(w)\chi(2\sigma - 1 + w) = \left(\frac{v}{2\pi}\right)^{2-2\sigma-2u}$$
$$\exp\left\{-2iv \log\left(\frac{v}{2\pi}\right) + 2iv + \frac{i\pi}{2}\right\}.\left(1 + O\left(\frac{1}{v}\right)\right)$$

by (1.8) and (1.9) of Chapter 1. The segment of integration in (2.42) is replaced by the segment $[-\delta - iT, -\delta + iT]$ ($\delta > 0$ arbitrarily small, but fixed) with an error which is

$$\ll \quad (xT)^\epsilon (xT^{-1} + T^{-1-2\delta+2\delta} x^{-\delta}).$$

Let N be an integer and $T^2/(4\pi^2 x) = N + 1/2$. In view of the poles of the integrand in (2.42) at $w = 1$, $w = 2 - 2\delta$ and $w = 0$, it follows that

$$\Delta_{1-2\sigma}(x) = \frac{1}{2\pi i} \int\limits_{-\delta-iT}^{-\delta+iT} \psi(w) L(1 - w) \frac{x^w}{w} dw + O \times$$

$$\left\{(xT)^{\epsilon}(1 + xT^{-1} + T^{1-2\sigma+2\delta}x^{-\delta})\right\}$$

$$= \sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)\left(\frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT}\psi(w)n^{w-1}\frac{x^w}{w}dw\right) + O\left\{(xT)^{\epsilon}(1 + xT^{-1} + T^{1-2\sigma+2\delta}x^{-\delta})\right\}$$

41 because of the absolute convergence of the above series. Using the asymptotic formula for $\psi(w)$ it is seen that the terms in the above series for $n > N$ contribute $\ll N^{\epsilon}T^{1-2\sigma}$. In the remaining terms we replace the segment $[-\delta - iT, -\delta + iT]$ by $[-\delta - i\infty, -\delta + i\infty]$ with an admissible error. By using

$$\psi(w) = 2^{2w+2\sigma-1}\pi^{2w+2\sigma-3}\sin\left(\frac{\pi w}{2}\right)\sin\left(\frac{\pi w}{2} + \frac{\pi(2\sigma-1)}{2}\right) \Gamma(1-w)\Gamma(2-2\sigma-w)$$

and properties of Mellin transforms for the Bessel functions we arrive at

$$\Delta_{1-2\sigma}(x) = O(x^{1/2-\sigma}N^{\sigma-1/2+\epsilon}) + O(x^{1/2+\epsilon}N^{-\frac{1}{2}}) - x^{1-\sigma}\sum_{n\le N}\sigma_{1-2\sigma}(n)n^{\sigma-1} \left\{\cos(\sigma\pi)J_{2-2\sigma}(4\pi\sqrt{nx}) + \sin(\sigma\pi)\left(Y_{2-2\sigma}(4\pi\sqrt{nx}) + \frac{2}{\pi}K_{2-2\sigma}(4\pi\sqrt{nx})\right)\right\}.$$

Finally with the asymptotic formulas

$$\begin{aligned}
J_{\nu}(x) &= \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) + O_{\nu}(x^{-3/2}),\\
Y_{\nu}(x) &= \left(\frac{2}{\pi x}\right)^{1/2}\sin\left(x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) + O_{\nu}(x^{-3/2}),\\
K_{\nu}(x) &\ll_{\nu} \quad x^{-1/2}e^{-x},
\end{aligned}$$

the above expression for $\Delta_{1-2\sigma}(x)$ easily reduces to (2.40). The derivation of (2.38) is analogous to the derivation of the Voronoi formula (2.23) for $\Delta(x)$.

Having finished this technical preparation we pass to the evaluation of the integral in (2.22). We shall write

$$g(u, 2\sigma - u) = \sum_{n=1}^{\infty} \sigma_{1-2\sigma}(n) h(u, n),$$

$$h(u, x) = 2 \int_0^{\infty} y^{-u}(1 + y)^{u-2\sigma} \cos(2\pi xy) dy,$$

as no confusion with (2.26) will arise. As in the case of $g(u, 1 - u)$ we obtain analogously ($X = N + 1/2$) 42

$$\begin{aligned} g(u, 2\sigma - u) &= \sum_{n \le N} \sigma_{1-2\sigma}(n) h(u, n) - \Delta_{1-2\sigma}(X) h(u, X) \\ &\quad + \int_X^{\infty} (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) h(u, x) dx \\ &\quad - \int_X^{\infty} \Delta_{1-2\sigma}(x) \frac{\partial}{\partial x} h(u, x) dx \\ &= g_1(u) - g_2(u) + g_3(u) - g_4(u), \end{aligned}$$

say. Here again a slight abuse of notation was made to keep the analogy with $E(T)$, since $g_n(u) = g_n(\sigma, u)$. We wish to show that $g(u, 2\sigma - u)$ can be analytically continued to the line $\operatorname{Re} u = \sigma$, $1/2 < \sigma < 3/4$. Since the integral $h(u, x)$ is absolutely convergent for $\operatorname{Re} u < 1$, the functions $g_1(u)$ and $g_2(u)$ clearly possess analytic continuation to the line $\operatorname{Re} u = \sigma$. For $g_3(u)$ we have, assuming first $\operatorname{Re} u < 0$,

$$\begin{aligned} g_3(u) &= \int_X^{\infty} (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) h(u, x) dx \\ &= \int_X^{\infty} (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) \int_0^{i\infty} y^{-u}(1 + y)^{u-2\sigma} e(xy) dy dx \\ &\quad + \int_X^{\infty} (\zeta(2\sigma) + \zeta(2 - 2\sigma) x^{1-2\sigma}) \int_0^{-i\infty} y^{-u}(1 + y)^{u-2\sigma} e(-xy) dy dx \end{aligned}$$

$$= I_1 + I_2,$$

say. Using Fubini's theorem and integrating by parts we obtain

$$\begin{aligned}
I_1 &= \int_0^{i\infty} y^{-u}(1+y)^{u-2\sigma}\left\{\int_X^\infty (\zeta(2\sigma)+\zeta(2-2\sigma)x^{1-2\sigma})e(xy)dx\right\}dy \\
&= -\int_0^{i\infty} y^{-u}(1+y)^{u-2\sigma}\left\{(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})\frac{e(Xy)}{2\pi iy}\right\}dy \\
&\quad - \int_X^\infty \zeta(2-2\sigma)(1-2\sigma)x^{-2\sigma}\int_0^{i\infty} y^{-u}(1+y)^{u-2\sigma}\frac{e(xy)}{2\pi iy}dydx \\
&= -\frac{1}{2\pi i}\int_0^{i\infty}\left(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma}\right)y^{-1-u}(1+y)^{u-2\sigma}e(Xy)dy \\
&\quad - \frac{(1-2\sigma)\zeta(2-2\sigma)}{2\pi i}\int_X^\infty x^{-2\sigma}\int_0^{i\infty} y^{-1-u}(1+y)^{u-2\sigma}e(xy)dy\,dx.
\end{aligned}$$

43 We denote the last double integral by J and make a change of variable $xy = w$. Then we obtain

$$\begin{aligned}
J &= \int_X^\infty x^{-2\sigma+1+u-u+2\sigma-1}\int_0^{i\infty} w^{-1-u}(x+w)^{u-2\sigma}e(w)dw\,dx \\
&= \int_0^{i\infty} w^{-1-u}e(w)\int_X^\infty (x+w)^{u-2\sigma}dx\,dw \\
&= -\int_0^{i\infty} w^{-1-u}e(w)(X+w)^{u-2\sigma+1}(u-2\sigma+1)^{-1}dw.
\end{aligned}$$

Change of variable $y = w/X$ gives now

$$J = -\frac{X^{1-2\sigma}}{u-2\sigma+1}\int_0^{i\infty} y^{-1-u}(1+y)^{u-2\sigma+1}e(Xy)dy,$$

and inserting this in the expression for I_1 we obtain

$$I_1 = -\frac{1}{2\pi i}\int_0^{i\infty}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})y^{-1-u}(1+y)^{u-2\sigma}e(Xy)dy$$
$$+\frac{(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}}{2\pi i(u+1-2\sigma)}\int_0^{i\infty}y^{-1-u}(1+y)^{u+1-2\sigma}e(Xy)dy.$$

Similarly we obtain 44

$$I_2 = \frac{1}{2\pi i}\int_o^{i\infty}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})y^{-1-u}(1+y)^{u-2\sigma}e(-Xy)dy$$
$$-\frac{(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}}{2\pi i(u+1-2\sigma)}\int_0^{-i\infty}y^{-1-u}(1+y)^{u+1-2\sigma}e(-Xy)dy.$$

Since $g_3(u) = I_1 + I_2$, we finally obtain the desired analytic continuation of $g_3(u)$ to the line $\operatorname{Re} u = \sigma$ in the form (after changing the lines of integration to $[0,\infty)$)

$$g_3(u) = \frac{1}{\pi}\int_0^{\infty}(\zeta(2\delta)+\zeta(2-2\sigma)X^{1-2\sigma})y^{-1-u}(1+y)^{u-2\sigma}\sin(2\pi Xy)dy$$
$$+\frac{(1-2\sigma)\zeta(2-2\sigma)}{\pi(u+1-2\sigma)}X^{1-2\sigma}\int_0^{\infty}y^{-1-u}(1+y)^{u+1-2\sigma}\sin(2\pi Xy)dy. \tag{2.43}$$

To show that

$$g_4(u) = \int_X^{\infty}\Delta_{1-2\sigma}(x)\frac{\partial}{\partial x}h(u,x)dx$$

converges for $\operatorname{Re} u \geq \sigma$ we use (2.41) and

$$\frac{\partial}{\partial x}h(u,x) \ll x^{\operatorname{Re} u-2},$$

which is obtained as in the corresponding estimate for (2.26). Thus the integral for $g_4(u)$ converges absolutely for $\mathrm{Re}\, u < 1 - 1/(1+4\sigma)$. But if $\sigma < 3/4$, then $\sigma < 1 - 1/(1+4\sigma)$, which means that the integral representing $g_4(u)$ converges for $\mathrm{Re}\, u = \sigma$. Now we can integrate the expression for $g(u, 2\sigma - u)$ in (2.22) to obtain

$$E_\sigma(T) = -i(G_1 - G_2 + G_3 - G_4) + O(1), G_j = \int\limits_{\sigma-iT}^{\sigma+iT} g_j(u)du \tag{2.44}$$

for $1 \leq j \leq 4$ with

$$\begin{aligned} G_1 = 4i \sum_{n \leq N} \sigma_{1-2\sigma}(n) \int_0^\infty y^{-\sigma} \log^{-1}\left(1 + \frac{1}{y}\right) \times \\ \cos(2\pi n y) \sin\left(T \log\left(1 + \frac{1}{y}\right)\right) dy, \qquad (2.45) \end{aligned}$$

$$\begin{aligned} G_2 = 4i\Delta_{1-2\sigma}(X) \int_0^\infty y^{-\sigma}(1+y)^{-\sigma} \log^{-1}\left(1 + \frac{1}{y}\right) \times \\ \cos(2\pi X y) \sin\left(T \log\left(1 + \frac{1}{y}\right)\right) dy, \quad (2.46) \end{aligned}$$

$$\begin{aligned} G_3 = \frac{-2i}{\pi}(\zeta(2\sigma) + \zeta(2-2\sigma)X^{1-2\sigma}) \int\limits_0^\infty y^{-\sigma-1}(1+y)^{-\sigma} \log^{-1}\left(1 + \frac{1}{y}\right) \times \\ \sin(2\pi X y) \sin\left(T \log\left(1 + \frac{1}{y}\right)\right) dy + \frac{1-2\sigma}{\pi} \zeta(2-2\sigma) X^{1-2\sigma} \quad (2.47) \\ \int\limits_0^\infty y^{-1}(1+y)^{1-2\sigma} \sin(2\pi X y) dy \int\limits_{\sigma-iT}^{\sigma+iT} (u+1-2\sigma)^{-1}\left(1 + \frac{1}{y}\right)^u du. \end{aligned}$$

45 To obtain a suitable expression for G_4 note that

$$\begin{aligned} \int\limits_{\sigma-iT}^{\sigma+iT} \frac{\partial}{\partial x} h(u,x) du &= 2\frac{\partial}{\partial x}\left\{\int\limits_0^\infty \int\limits_{\sigma-iT}^{\sigma+iT} y^{-u}(1+y)^{u-2\sigma} \cos(2\pi x y) du\, dy\right\} \\ &= 4i \frac{\partial}{\partial x}\left\{\int\limits_0^\infty y^{-\sigma}(1+y)^{-\sigma} \log^{-1}\left(1 + \frac{1}{y}\right) \times \right. \end{aligned}$$

$$\sin\left(T\log\left(1+\frac{1}{y}\right)\right)\cos(2\pi xy)dy\Biggr\}$$
$$= 4i\frac{\partial}{\partial x}\left\{\int_0^\infty x^{2\sigma-1}y^{-\sigma}(x+y)^{-\sigma}\log^{-1}\left(\frac{x+y}{y}\right)\times\right.$$
$$\left.\sin\left(T\log\left(\frac{x+y}{y}\right)\right)\cos(2\pi y)dy\right\},$$

similarly as in the derivation of (2.35) from (2.34). Hence differentiating the last expression under the integral sign we arrive at

$$G_4 = 4i\int_X^\infty x^{-1}\Delta_{1-2\sigma}(x)dx\int_0^\infty y^{-\sigma}(1+y)^{-\sigma-1}\log^{-1}\left(1+\frac{1}{y}\right)\times$$
$$\cos(2\pi xy)\left(T\cos\left(T\log\left(1+\frac{1}{y}\right)\right)+\sin\left(T\log\left(1+\frac{1}{y}\right)\right)\times\right.$$
$$\left.\left\{(2\sigma-1)(1+y)-\sigma-\log^{-1}\left(1+\frac{1}{1}y\right)\right\}\right)dy. \tag{2.48}$$

Note that this expression corresponds exactly to (2.35) when $\sigma \to \frac{1}{2}+0$.

2.4 Evaluation of Some Exponential Integrals

We shall first state and prove an elementary result, often used in the estimation of exponential integrals. This is 46

Lemma 2.1. *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic and $F'(X) \geq m > 0$ or $F'(x) \leq -m < 0$ for $a \leq x \leq b$. Then*

$$\left|\int_a^b e^{iF(x)}dx\right| \leq \frac{4}{m}. \tag{2.49}$$

If in addition $G(x)$ is a positive, monotonic function for $a \leq x \leq b$ such that $|G(x)| \leq G$, $G(x) \in C^1[a,b]$, then

$$\left|\int_a^b G(x)e^{iF(x)}dx\right| \leq \frac{4G}{m}. \tag{2.50}$$

Proof. The merit of the above estimates is that they do not depend on the length of the interval of integration. Recall that by the second mean value theorem for integrals

$$\int_a^b f(x)g(x)dx = \begin{cases} f(b)\int_c^b g(x)dx & \text{if } f(x) \ge 0, f'(x) \ge 0, \\ f(a)\int_a^c g(x)dx & \text{if } f(x) \ge 0, f'(x) \le 0, \end{cases} \tag{2.51}$$

where $a < c < b$ and $f'(x)$, $g(x) \in C[a, b]$. We write

$$e^{iF(x)} = \cos F(x) + i \sin F(x),$$

$$\int_a^b \cos F(x)dx = \int_a^b (F'(x))^{-1} d \sin F(x),$$

and use (2.51), since $(F')^{-1}$ as the reciprocal of F' is monotonic in $[a, b]$. It is seen that

$$\left| \int_a^b \cos F(x)dx \right| \le \frac{2}{m}.$$

□

47 and the same bound holds for the integral with $\sin F(x)$. Hence (2.49) follows. To obtain (2.50) write again $e^{iF(x)} = \cos F(x) + i \sin F(x)$, and to each resulting integral apply (2.51), since $G(x)$ is monotonic. Using then (2.49), (2.50) follows.

From the formulas of Section 2.3 it is seen that the proof of Theorem 2.1 and Theorem 2.2 is reduced to the evaluation of certain integrals of the form

$$I = \int_a^b \varphi(x)e(f(x) + kx)dx,$$

where $\varphi(x)$, $f(x)$ are continuous, real-valued functions on $[a, b]$, and k is real. The evaluation of this type of integrals, which properly represents a theory of its own, is most often carried out by the so-called “saddle point" method, or the method of "stationary phase". It consists of considering I as the complex integral

$$I = \int\limits_a^b \varphi(z)e(f(z) + kz)dz,$$

where naturally one imposes some conditions on $\varphi(z)$ and $f(z)$ as functions of the complex variable z. One then replaces the segment of integration $[a, b]$ by a suitable contour in the z-plane. If $(f(z) + kz)'$ has a (unique) zero in $[a, b]$ ("saddle point"), then in many cases arising in practice the main contribution to I comes from a neighbourhood of the saddle point x_0. There are several results in the literature which give an evaluation of I under different hypotheses. The one we shall use, due to F.V. Atkinson, will be stated without proof as

Theorem 2.3. *Let $f(z)$, $\varphi(z)$ be two functions of the complex variable z, and $[a, b]$ a real interval such that:*

1. *For $a \leq x \leq b$ the function $f(x)$ is real and $f''(x) > 0$.*

2. *For a certain positive differentiable function $\mu(x)$, defined on $a \leq x \leq b$, $f(z)$ and $\varphi(z)$ are analytic for $a \leq x \leq b$, $|z - x| \leq \mu(x)$.*

3. *There exist positive functions $F(x)$, $\Phi(x)$ defined on $[a, b]$ such that for $a \leq x \leq b$, $|z - x| \leq \mu(x)$ we have*

$$\varphi(z) \ll \Phi(x), f'(z) \ll F(x)\mu^{-1}(x), |f''(z)|^{-1} \ll \mu^2(x)F^{-1}(x),$$

and the $\ll$-constants are absolute. 48

Let k be any real number, and if $f'(x) + k$ has a zero in $[a, b]$ denote it by x_0. Let the values of $f(x)$, $\varphi(x)$, and so on, at a, x_0 and b characterised by the suffixes $a, 0$ and b, respectively. Then

$$\int\limits_a^b \varphi(x)e(f(x) + kx)dx = \varphi_0(f_0'')^{-\frac{1}{2}} e\left(f_0 + kx_0 + \frac{1}{8}\right) + O(\Phi_0\mu_o F_0^{-3/2})$$
$$+ O\left(\int\limits_a^b \Phi(x)\exp\{-C|k|\mu(x) - CF(x)\}(dx + |d\mu(x)|)\right)$$

$$+ O\left(\Phi_a\left(|f'_a + k| + f''^{\frac{1}{2}}_a\right)^{-1}\right)$$
$$+ O\left(\Phi_b\left(|f'_b + k| + f''^{\frac{1}{2}}_b\right)^{-1}\right). \qquad (2.52)$$

If $f'(x) + k$ has no zeros in $[a, b]$, then the terms involving x_0 are to be omitted.

a simplified version of the above result may be obtained if the conditions 2. and 3. of Theorem 2.3 are replaced by

2′. There exists $\mu > 0$ such that $f(z)$ and $\varphi(z)$ are analytic in the region

$$D = D(\mu) = \{z : |z - x| < \mu \text{ for some } x \in [a, b]\}.$$

3′. There exist $F, \Phi > 0$ such that for $z \in D$

$$\varphi(z) \ll \Phi, f'(z) \ll F\mu^{-1}, |f''(z)|^{-1} \ll \mu^2 F^{-1}.$$

Then in the notation of Theorem 2.3 one has

$$\int\limits_a^b \varphi(x)e(f(x) + kx)dx = \varphi_0(f''_0)^{-\frac{1}{2}} e\left(f_0 + kx_0 + \frac{1}{8}\right)$$
$$+ O\left(\frac{\Phi\mu}{F\mu^{-1}\Delta + F^{\frac{1}{2}}}\right), \qquad (2.53)$$

where

$$\Delta = \min(|a - x_0|, |b - x_0|).$$

If $f'(x) + k$ has no zeros in $[a, b]$, then the terms involving x_0 are to be omitted.

In Atkinson's Theorem 2.3 we would have two additional error
49 terms, which would be

$$O(\Phi\mu F^{-3/2}) + O(\Phi(b - a)\exp(-A(|k|\mu + F))),$$

where $A > 0$ is an absolute constant. However, these error terms are negligible if $F \gg 1$, since $b - a \ll \mu$. Indeed,

$$(b - a)F\mu^{-2} \ll \int\limits_a^b f''(x)dx = f'(b) - f'(a) \ll F\mu^{-1},$$

whence $b - a \ll \mu$. If $F \ll 1$, then (2.53) is verified directly: in any case the left-hand side of (2.53) is $O(\mu\Phi)$ and the first term on the right-hand side is $O(\mu\Phi F^{-\frac{1}{2}})$. Both of these are absorbed in the error term of (2.53).

Taking $\varphi(x) = x^{-\alpha}(1+x)^{-\beta}\left(\log\frac{1+x}{x}\right)^{-\gamma}$, $f(x) = \frac{T}{2\pi}\log\frac{1+x}{x}$, $\phi(x) = x^{\alpha}(1+x)^{\gamma-\beta}$, $F(x) = T/(1+x)$, $\mu(x) = x/2$, we obtain after some calculations from Theorem 2.3 the following

Lemma 2.2. *Let $\alpha, \beta, \gamma, a, b, k, T$ be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < \frac{1}{2}$, $a < T/(8\pi k)$, $b \geq T$, $k \geq 1$ and $T \geq 1$. Then*

$$\int_a^b y^{-\alpha}(1+y)^{-\beta}\left(\log\frac{1+y}{y}\right)^{-\gamma}\exp\left(iT\log\frac{1+y}{y} + 2\pi kiy\right)dy = (2k\pi^{\frac{1}{2}})^{-1}\times$$
$$T^{\frac{1}{2}}V^{-\gamma}U^{-\frac{1}{2}}\left(U - \frac{1}{2}\right)^{-\alpha}\left(U + \frac{1}{2}\right)^{-\beta}\exp\left(iTV + 2\pi ikU - \pi ik + \frac{\pi i}{4}\right)$$
$$+ O(a^{1-\alpha}T^{-1}) + O(b^{\gamma-\alpha-\beta}k^{-1}) + R(T,k) \tag{2.54}$$

uniformly for $|\alpha - 1| > \epsilon$, where

$$U = \left(\frac{T}{2\pi k} + \frac{1}{4}\right)^{\frac{1}{2}}, V = 2ar\sinh\left(\frac{\pi k}{2T}\right)^{\frac{1}{2}},$$
$$R(T,k) \ll T^{(\gamma-\alpha-\beta)/2-1/4}k^{-(\gamma-\alpha-\beta)/2-5/4} \qquad \textit{for } 1 \leq k \leq T,$$
$$T(T,k) \ll T^{-\frac{1}{2}-\alpha}k^{\alpha-1} \qquad \textit{for } k \geq T.$$

A similar result holds for the corresponding integral with $-k$ in place of k, except that in that case the main term on the right-hand side of (2.54) is to be omitted.

For the next lemma we apply Theorem 2.3 with a, b as the limits of integration, where $b > T$, and **50**

$$\varphi(x) = x^{-\alpha}\left(ar\sinh\left(x\sqrt{\frac{\pi}{2T}}\right)\right)^{-1}\left(\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{2}\right)^{-1}\left(\frac{1}{4} + \frac{T}{2\pi x^2}\right)^{-\frac{1}{4}},$$

$$f(x) = \frac{1}{2}x^2 - \left(\frac{Tx^2}{2\pi} + \frac{x^4}{4}\right)^{\frac{1}{2}} - \frac{T}{\pi}ar\sinh\left(\sqrt{\frac{\pi}{2T}}\right).$$

$\mu(x) = \dfrac{1}{2}x, \Phi(x) = x^{-\alpha}, F(x) = T.$

Then we obtain

Lemma 2.3. *For* $AT^{\frac{1}{2}} < a < A'T^{\frac{1}{2}}$, $0 < A < A'$, $\alpha > 0$,

$$\int\limits_a^b \frac{\exp i\left\{4\pi x\sqrt{n} - 2Tar\sinh(x\sqrt{\pi/2T}) - (2\pi x^2 T + \pi^2 x^4)^{\frac{1}{2}} + x^2\right\}}{x^\alpha ar\sinh(x\sqrt{\pi 2T})\left(\left(\frac{1}{2} + \left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{\frac{1}{2}}\right)\left(\frac{1}{4} + \frac{T}{2\pi x^2}\right)^{\frac{1}{4}}\right)} dx = 4\pi T^{-1}$$
$$n^{\frac{1}{2}(\alpha-1)}\left(\log\frac{T}{2\pi n}\right)^{-1}\left(\frac{T}{2\pi} - n\right)^{3/2-\alpha}\exp\left\{i\left(T - T\log\left(\frac{T}{2\pi n}\right) - 2\pi n + \frac{\pi}{4}\right)\right\}$$
$$+ O(T^{-\frac{1}{2}\alpha}\min(1, |2\sqrt{n} + a - (a^2 + 2T/n)^{\frac{1}{2}}|^{-1})) + O\left(n^{\frac{1}{2}(\alpha-1)}\left(\frac{T}{2\pi} - n\right)^{1-\alpha} T^{-3/2}\right), \tag{2.55}$$

provided that $n \geq 1$, $n < T/(2\pi)$, $(T/(2\pi) - n)^2 > na^2$. *If the last two conditions on n are not satisfied, or if* $\sqrt{n}$ *is replaced by* $-\sqrt{n}$, *then the main term and the last error term on the right-hand side of* (2.55) *are to be omitted.*

2.5 Completion of the Proof of the Mean Square Formulas

Having at our disposal Lemmas 2.2 and 2.3 we proceed to evaluate $I_n (1 \leq n \leq 4)$, as given by (2.31)- (2.34). We consider first I_1, taking in Lemma 2.2 $0 < \alpha < 1$, $\alpha + \beta > \gamma$, so that we may let $a \to 0$, $b \to \infty$. Hence, if $\frac{1}{2} < \alpha < \frac{3}{4}$, $1 \leq k < AT$, we obtain

$$\int\limits_0^\infty \frac{\sin(T\log(1 + 1/y))\cos(2\pi ky)}{y^\alpha(1 + y)^{\frac{1}{2}}\log(1 + 1/y)} dy = (4k)^{-1}\times$$
$$\left(\frac{T}{\pi}\right)^{\frac{1}{2}} \frac{\sin(TV + 2\pi kU - \pi k + \pi/4)}{VU^{\frac{1}{2}}\left(U - \frac{1}{2}\right)^\alpha\left(U + \frac{1}{2}\right)^{\frac{1}{2}}} + O(T^{-\alpha/2}k^{(\alpha-3)/2}). \tag{2.56}$$

51 Since this formula holds uniformly in α, we may put $\alpha = 1/2$. Tak-

ing into account that $\sin(x - \pi k) = (-1)^k \sin x$ we obtain, after substituting (2.56) with $\alpha = 1/2$ into (2.31),

$$I_1 = O(T^{-1/4}) + 2^{-\frac{1}{2}} \sum_{n \le N} (-1)^n d(n) n^{-\frac{1}{2}} \times$$

$$\left\{ \frac{\sin(2Tar\sinh\sqrt{\pi n/2T} + \sqrt{2\pi nT + \pi^2 n^2} + \pi/4)}{\left(ar\sinh\sqrt{\pi n/2T})(T/2\pi n) + \frac{1}{4}\right)^{\frac{1}{4}}} \right\}, \tag{2.57}$$

by taking $AT < N < A'T$. Similarly, using $\Delta(x) \ll x^{1/3+\epsilon}$ we obtain from (2.32)

$$I_2 \ll |\Delta(X)| X^{-1/2} \ll T^{\epsilon - 1/6}. \tag{2.58}$$

To deal with I_3 we write (2.33) in the form

$$I_3 = -\frac{2}{\pi}(\log X + 2\gamma) I_{31} + (\pi i)^{-1} I_{32}, \tag{2.59}$$

and consider first I_{31}. We have

$$\int\limits_0^\infty \frac{\sin(T\log(1+1/y))\sin(2\pi Xy)}{y^{3/2}(1+y)^{1/2}\log(1+1/y)} dy = \int\limits_0^{(2X)^{-1}} + \int\limits_{(2X)^{-1}}^\infty \ll T^{-1/2}.$$

Here we estimated the integral from 0 to $(2X)^{-1}$ by the second mean value theorem for integrals (see (2.51)). Namely setting

$$f(x) = x^{1/2}(1+x)^{1/2}/\log(1+1/x)$$

we find that $f'(x) > 0$, hence the integral in question equals

$$2\pi X \int\limits_0^\xi \frac{\sin(T\log(1+1/y))}{y(1+y)} f(y) dy = 2\pi X f(\xi) \times$$

$$\int\limits_\eta^\xi \frac{\sin(T\log(1+1/y))}{y(1+y)} dy$$

$$= 2\pi X \xi^{\frac{1}{2}} (1+\xi)^{\frac{1}{2}} (\log(1+1/\xi))^{-1} \times$$

$$\left\{T^{-1}\cos(T\log(1+1/y))\right\}\Big|_{\eta}^{\xi} \ll T^{-\frac{1}{2}},$$

where $0 \le \eta \le \xi \le (2X)^{-1}$. The remaining integral is estimated by Lemma 2.2 by treating the main terms on the right-hand side of (2.54) as an error term. 52

Take next I_{32} and write

$$\begin{aligned} I_{32} &= \int_0^\infty y^{-1}\sin(2\pi Xy)dy \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} \\ &= \int_0^1 \cdots dy + \int_1^\infty \cdots dy = I'_{32} + I''_{32}, \end{aligned}$$

say. Note that

$$\begin{aligned} \int_0^1 y^{-1}\sin(2\pi Xy)dy &= \int_0^\infty y^{-1}\sin(2\pi Xy)dy - \int_1^\infty y^{-1}\sin(2\pi Xy)dy \\ &= \int_0^\infty v^{-1}\sin v\,dv + \frac{y^{-1}\cos(2\pi Xy)}{2\pi X}\Big|_1^\infty \\ &\quad + \int_1^\infty \frac{\cos(2\pi Xy)}{2\pi Xy^2}dy = \frac{\pi}{2} + O\left(\frac{1}{X}\right). \end{aligned} \tag{2.60}$$

In I'_{32} we have $0 < y \le 1$, hence by the residue theorem

$$\begin{aligned} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} &= 2\pi i - \left[\int_{\frac{1}{2}+iT}^{\infty+iT} + \int_{-\infty-iT}^{\frac{1}{2}-iT}\right]\left(\frac{1+y}{y}\right)^u \frac{du}{u} \\ &= 2\pi i + O(T^{-1}y^{-1/2}), \end{aligned}$$

since

$$\int_{\frac{1}{2}\pm iT}^{-\infty\pm iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} \ll T^{-1} \int_{-\infty}^{\frac{1}{2}} \left(\frac{1+y}{y}\right)^t dt \ll T^{-1}y^{-1/2}.$$

Hence

$$I'_{32} = 2\pi i \int_0^1 y^{-1}\sin(2\pi Xy)dy + O(T^{-1}\int_0^1 |\sin(2\pi Xy)|y^{-3/2}dy)$$

$$= 2\pi i\left(\frac{\pi}{2}\right) + O(X^{-1}) + O\left(T^{-1}\int_0^{X^{-1}} Xy^{-\frac{1}{2}}dy\right) + O\left(T^{-1}\int_{X^{-1}}^{\infty} y^{-3/2}dy\right) = \pi^2 i + O(T^{-\frac{1}{2}}).$$

Next, an integration by parts gives

$$I''_{32} = \int_1^\infty y^{-1}\sin(2\pi xy)dy \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{du}{u} = \left\{-\frac{\cos(2\pi Xy)}{2\pi Xy}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u \frac{du}{u}\right\}\Bigg|_1^\infty$$

$$-\int_1^\infty \frac{\cos(2\pi Xy)}{2\pi Xy^2}dy \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u \frac{du}{u}$$

$$-\int_1^\infty \frac{\cos(2\pi Xy)}{2\pi Xy}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^{u-1} y^{-2}du \ll T^{-1}\log T,$$

since for $y \geq 1$ **53**

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u \frac{du}{u} \ll \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left|\frac{du}{u}\right| \ll \log T,$$

so that finally

$$I_3 = \pi + O(T^{-1/2}\log T). \tag{2.61}$$

It remains yet to evaluate I_4, as given by (2.35), which will yield the terms $-2\sum_{n\leq N'}\cdots$ in (2.5) in the final formula for $E(T)$. We estimate first the inner integrals in (2.35), making $a \to 0$, $b \to \infty$ in Lemma 2.2. We have then in the notation of Lemma 2.2, for $k = x > AT$,

$$\int\limits_0^\infty \frac{\cos(T\log(1+1/y))\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}\log(1+1/y)}dy$$
$$= (4x)^{-1}\left(\frac{T}{\pi}\right)^{\frac{1}{2}} \frac{\cos(TV + 2\pi xU - \pi x + \frac{1}{4}\pi)}{VU^{1/2}\left(U-\frac{1}{2}\right)^{1/2}\left(U+\frac{1}{2}\right)^{3/2}} + O\left(Tx^{-1/2}\right),$$

and similarly for $r = 1, 2$,

$$\int\limits_0^\infty \frac{\sin(T\log(1+1/y))\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}(\log(1+1/y))^r}dy$$
$$= O\left(T^{1/2}\left(U-\frac{1}{2}\right)^{-1/2}x^{-1}\right) + O\left(T^{-1}x^{-1/2}\right) = O(x^{-1/2}).$$

Thus we have

$$I_4 = \int\limits_X^\infty \frac{\Delta(x)}{x}\times$$
$$\left\{\frac{T\cos(2Tar\sinh\sqrt{\pi x/2T} + (2\pi xT + \pi^2x^2)^{\frac{1}{2}} - \pi x + \pi/4}{(\sqrt{2x}ar\sinh\sqrt{\pi x/2T})\left(\left(\frac{T}{2\pi x}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)\left(\frac{T}{2\pi x}+\frac{1}{4}\right)^{1/4}} + O(x^{-1/2})\right\}dx.$$

54 Now from (2.25) it follows without difficulty that $\Delta(x)$ is $\approx x^{1/4}$ in mean square (see Theorem 2.5 for a sharp result). Hence by the Cauchy-Schwarz inequality and integration by parts we obtain

$$\int\limits_X^\infty x^{-1}\Delta(x)O(x^{-1/2})dx \ll \left(\int_X^\infty x^{-4/3}dx\right)^{1/2}\left(\int\limits_X^\infty x^{-5/3}\Delta^2(x)dx\right)^{1/2} \tag{2.62}$$

$$\ll X^{-1/6}\left\{(x^{-5/3}\int_0^x \Delta^2(t)dt)\Big|_X^\infty + \frac{5}{3}\int_X^\infty\left(\int_0^x \Delta^2(t)dt\right)x^{-8/3}dx\right\}^{1/2}$$
$$\ll X^{-1/6}X^{3/4-5/6} = X^{-1/4}.$$

Changing the variable x to $x^{1/2}$ in the integral for I_4 and using (2.24) we obtain

$$I_4 = \frac{T}{\pi}\sum_{n=1}^\infty d(n)n^{-3/4}\times \tag{2.63}$$
$$\int_{X^{1/2}}^\infty \frac{\cos\left\{2T ar\sinh(x\sqrt{\pi/2T}) + (2\pi^2x^2T + \pi^2x^4)^{1/2} - \pi x^2 + \pi/4\right\}}{x^{3/2}\left(ar\sinh(x\sqrt{\pi 2/T})\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{1/2}+\frac{1}{2}\right)\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{1/4}}\times$$
$$\left\{\cos(4\pi x\sqrt{n} - \frac{1}{4}\pi) - 3(32\pi x\sqrt{n})^{-1}\sin(4\pi x\sqrt{n} - \frac{1}{4}\pi)\right\}dx + O(T^{-\frac{1}{4}})$$
$$= \frac{T}{\pi}\sum_{n=1}^\infty d(n)n^{-3/4}J_n + O(T^{-1/4}),$$

say. Strictly speaking, we should write the integral for I_4 ad

$$\lim_{b\to\infty}\int_X^{b^2} x^{-1}\Delta(x)(\ldots)dx$$

and then apply this procedure, since the series for $\Delta(x)$ is boundedly convergent in finite intervals only. But it is not difficult to see that working with $\lim\limits_{b\to\infty}$ leads to the same final result.

Now it is clear why we formulated Lemma 2.3. It is needed to evaluate the integral J_n in (2.63). Indeed, if $\left(\frac{T}{2\pi} - n\right)^2 > nX$, $n < T/(2\pi)$, that is to say if

$$n < \frac{T}{2\pi} + \frac{X}{2} - \left(\frac{X^2}{4} + \frac{XT}{2\pi}\right)^{1/2} = Z, \tag{2.64}$$

then an application of Lemma 2.3 gives, with $\alpha = 3/2$, $\alpha = 5/2$, **55**

$$
\begin{aligned}
I_4 &= 2\sum_{n<Z} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos\left(T\log\left(\frac{T}{2\pi n}\right)-T+\frac{\pi}{4}\right)\\
&+O\left(\sum_{n<Z} d(n)n^{-1/2}(T-2\pi n)^{-1}\right)+O\left(T^{-1/2}\sum_{n<Z} d(n)n^{-1/2}\sum_{n<Z}(T-2\pi n)^{-1/2}\right)\\
&+O\left(T^{1/4}\sum_{n=1}^{\infty} d(n)n^{-3/4}\min(1,|2\sqrt{n}+\sqrt{X}-(X+T/2\pi))^{1/2}|^{-1}\right)+O(T^{-1/4})\\
&= I_{41}+O(I_{42})+O(I_{43})+O(I_{44})+O(T^{-1/4}),
\end{aligned}
$$

say, Here I_{41} contributes the second main term on the right-hand side of (2.5), while the contribution of the other error terms (I_{42}comes from applying Lemma 2.3 to estimate the sine terms in (2.63) with $\alpha = 5/2$) is $O(\log^2 T)$. To see this, observe that in view of $AT < X < A'T$ we have $Z \ll T$, $T/(2\pi) - Z \gg T$. Hence

$$
\begin{aligned}
I_{42} &\ll T^{-1}\sum_{n\le Z} d(n)n^{-1/2} \ll T^{-1/2}\log T,\\
I_{43} &\ll T^{-1/2}T^{-1/2}\sum_{n\le Z} d(n)n^{-1/2} \ll T^{-1/2}\log T,
\end{aligned}
$$

and it remains yet to deal with I_{44}. Since

$$
\left(\frac{1}{2}\sqrt{X+\frac{2T}{\pi}}-\frac{1}{2}\sqrt{X}\right)^2 = \frac{X}{2}+\frac{T}{2\pi}-\sqrt{\frac{X^2}{4}+\frac{XT}{2\pi}} = Z,
$$

we have

$$
\begin{aligned}
I_{44} &\ll T^{1/4}\sum_{n=1}^{\infty} d(n)n^{-3/4}\min(1,|n^{1/2}-Z^{1/2}|^{-1})\\
&= T^{1/4}\left(\sum_{n\le 1/2Z}+\sum_{\frac{1}{2}Z<n\le Z-Z^{1/2}}+\sum_{Z-Z^{1/2}<n\le Z+Z^{1/2}}+\sum_{Z+Z^{1/2}<n\le 2Z}+\sum_{n>2Z}\right)\\
&= T^{1/4}(S_1+S_2+S_3+S_4+S_5),
\end{aligned}
$$

say. Using partial summation and the crude formula $\sum_{n\le x} d(n) \sim x\log x$, we obtain

$$
S_1 = \sum_{n\le\frac{1}{2}Z} d(n)n^{-3/4}(Z^{1/2}-n^{1/2})^{-1} \ll Z^{-\frac{1}{2}}\sum_{n\le\frac{1}{2}Z} d(n)n^{-3/4} \ll T^{-1/4}\log T,
$$

$$S_2 = \sum_{\frac{1}{2}Z<n\le Z-Z^{1/2}} d(n)n^{-3/4}(Z^{1/2}-n^{1/2})^{-1} \ll Z^{-1/4} \sum_{\frac{1}{2}Z<n\le Z-Z^{1/2}} d(n)(z-n)^{-1}$$
$$\ll T^{-1/4} \sum_{Z^{1/2}\le k\le \frac{1}{2}Z} d([Z]-k)k^{-1} \ll T^{-1/4}(Z(\log Z)Z^{-1} + \int\limits_{Z^{1/2}}^{Z} t\log t\frac{dt}{t^2})$$
$$\ll T^{-1/4}\log^2 T,$$
$$S_3 = \sum_{Z-Z^{1/2}<n\le Z+Z^{1/2}} d(n)n^{-3/4} \ll T^{-1/4\log T},$$

while 56

$$S_4 \ll T^{-1/4}\log^2 T$$

follows analogously as the estimate for S_2. Finally

$$S_5 \ll \sum_{n>2Z} d(n)n^{-3/4}(n^{1/2}-Z^{1/2})^{-1} \ll \sum_{n>2Z} d(n)n^{-5/4} \ll T^{-1/4}\log T.$$

Therefore we obtain

$$I_4 = 2\sum_{n\le Z} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos\left(T\log\left(\frac{T}{2\pi n}\right)-T+\frac{\pi}{4}\right)+O(\log^2 T). \tag{2.65}$$

It remains to note that in (2.61) the limit of summation Z may be replaced by

$$N' = N'(T) = \frac{T}{2\pi}+\frac{N}{2}-\left(\frac{N^2}{4}+\frac{NT}{2\pi}\right)^{1/2},$$

as in the formulation of Theorem 2.1, with a total error which is $O(\log^2 T)$. Theorem 2.1 follows now from (2.29), (2.57), (2.58), (2.61) and (2.65) if N is an integer. If N is not an integer, then in (2.5) we replace N by $[N]$ again with an error which is $O(\log^2 T)$.

It remains yet to indicate how the expression (2.44) for $E_\sigma(T)$, with 57
$G_j (1 \le j \le 4)$ given by (2.45) - (2.48), is transformed into the one given by Theorem 2.2. Firstly, applying Lemma 2.2 to (2.45), we obtain (analogously as for I_1 given by (2.31))

$$G_1 = 2^{\sigma-1} i \left(\frac{\pi}{T}\right)^{\sigma-1/2} \sum_{n\le N} (-1)^n n^{\sigma-1} \sigma_{1-2\sigma}(n) \left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-1}$$
$$\left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4} \cos f(T,n) + O(T^{1/4-\sigma}).$$

Secondly, the estimate

$$G_2 \ll T^{(1-4\sigma)/(2+8\sigma)+\epsilon}$$

follows from (2.46), (2.41) and Lemma 2.2. Thirdly, starting from (2.47) and using a technique similar to the one used in obtaining (2.61), we arrive at

$$G_3 = i\pi(1-2\sigma)(2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{\Gamma(2\sigma)\sin(\sigma\pi)} + O(T^{\sigma-1}). \tag{2.66}$$

Note that in (2.66) the main term tends to π as $\sigma \to \frac{1}{2}+0$ in analogy with (2.61). We also remark that in proving (2.66), instead of (2.60), we use the formula

$$\int\limits_0^\infty w^{-2\sigma} \sin w dw = \frac{\pi}{2\Gamma(2\sigma)\sin(\pi\sigma)}, \tag{2.67}$$

which follows with $p = 1-2\sigma$ from the well-known formula

$$\int\limits_0^\infty t^{p-1} \sin t\, dt = \Gamma(p) \sin\left(\frac{p\pi}{2}\right) \qquad (-1 < p < 0)$$

and the relation $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$.

To evaluate G_4 we apply Lemma 2.2 to the inner integral in (2.48) to obtain

$$G_4 = O\left(\int_X^\infty |\Delta_{1-2\sigma}(x)| x^{\sigma-2} T^{1/2-\sigma} dx\right)$$
$$+ 2^{\sigma-1} i\pi^{\sigma-1/2} \int\limits_X^\infty \Delta_{1-1\sigma}(x) T^{3/2-\sigma} x^{\sigma-2} \times$$

$$\left(ar\sinh\sqrt{\frac{\pi x}{2T}}\right)\left(\frac{T}{2\pi x}+\frac{1}{4}\right)^{-1/4}\times$$
$$\left(\left(\frac{T}{2\pi x}+\frac{1}{4}\right)^{1/2}+\frac{1}{2}\right)^{-1}\cos(f(T,x))dx, \tag{2.68}$$

where $f(T,x)$ is as in the formulation of Theorem 2.1. From (2.40) it follows that $\Delta_{1-2\sigma}(x)$ is $\approx\ x^{3/4-\sigma}$ in mean square, hence as in the corresponding estimate in (2.62) for I_4 it follows that the contribution of the error term in (2.68) is $O(T^{1/4-\sigma})$. **58**

To evaluate the main term in (2.68) we proceed as in the evaluation of I_4 (see (2.63)), only instead of $\Delta(x)$ we use the formula (2.38) for $\Delta_{1-2\sigma}(x)$. To be rigorous, one should treat $\int_X^\infty$ as $\lim\limits_{b\to\infty}\int\limits_X^{b^2}$, since $\Delta_{1-2\sigma}(x)$ is boundedly convergent only in finite intervals. However, in both cases the final result will be the same, namely

$$G_4=2i\left(\frac{2\pi}{T}\right)^{\sigma-1/2}\sum_{n\le N'}\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos(g(T,n))+O(\log T).$$

The error term $O(\log T)$, which is better than the error term $O(\log^2 T)$ appearing in the expression (2.65) for I_4, comes from the fact that

$$\sum_{n\le x}\sigma_{1-2\sigma}(n)\sim\zeta(2\sigma)x \qquad \left(\sigma>\frac{1}{2},x\to\infty\right),$$

whereas

$$\sum_{n\le x}d(n)\sim x\log x \qquad (x\to\infty),$$

and this accounts for the saving of a log-factor. This completes our discussion of the proof of Theorem 2.2.

2.6 The Mean Square Formula For $E(T)$

From the expressions for $E(T)$ and $E_\sigma(T)$, given by Theorem 2.1 and Theorem 2.2, one can obtain by squaring and termwise integration the asymptotic formula

$$\int_2^T E^2(t)dt = cT^{3/2} + O(T^{5/4}\log^2 T) \tag{2.69}$$

with

$$c = \frac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^\infty d^2(n)n^{-3/2} = \frac{2}{3}(2\pi)^{-1/2}\frac{\zeta^4(3/2)}{\zeta(3)} = 10.3047\ldots. \tag{2.70}$$

59 Similarly for σ fixed satisfying $1/2 < \sigma < 3/4$ we obtain

$$\int_2^T E^2(t)dt = c(\sigma)T^{5/2-2\sigma} + O(T^{7/4-\sigma}\log T) \tag{2.71}$$

with

$$c(\sigma) = \frac{2}{5-4\sigma}(2\pi)^{2\sigma-3/2}\sum_{n=1}^\infty \sigma^2_{1-2\sigma}(n)n^{2\sigma-5/2} \tag{2.72}$$

From (2.69) and (2.71) we can get weak omega results for $E(T)$ and $E_\sigma(T)$ (recall that $f(x) = \Omega(g(x))$ means that $\lim\limits_{x\to\infty} f(x)/g(x) = 0$ does not hold). For example, (2.69) implies

$$E(T) = \Omega(T^{1/4}), \tag{2.73}$$

and (2.71) gives

$$E_\sigma(T) = \Omega(T^{3/4-\sigma}). \tag{2.74}$$

There are reasons to believe that (2.73) and (2.74) are fairly close to the actual order of magnitude of $E(T)$ and $E_\sigma(T)$ (for $1/2 < \sigma < 3/4$), as one may conjecture that $E(T) \ll T^{1/4+\epsilon}$ and $E_\sigma(T) \ll T^{3/4-\sigma+\epsilon}$ (for $1/2 < \sigma < 3/4$). The proofs of both of these bounds are out of reach at present; the conjectural bound for $E(T)$ implies by Theorem 2.2 that $\zeta(1/2+it) \ll t^{1/8+\epsilon}$. The main aim of this section is to prove a sharpening of (2.69). We remark that by the same method a sharpening of (2.71) may be also obtained, but we shall treat only the more interesting case of $E(T)$. In Chapter 3 we shall deal with the omega results, and we shall improve (2.74) a little and (2.73) substantially, obtaining $\Omega_\pm$−results for $E(T)$ which correspond to the best known $\Omega_\pm$−results for $\Delta(x)$. Our main result is

Theorem 2.4. *With c given by* (2.70) *we have*

$$\int\limits_2^T E^2(t)dt = cT^{3/2} + O(T\log^5 T). \tag{2.75}$$

This result corresponds to the mean square result for

$$\Delta(x) = \sum_{n\le x}{}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4},$$

which also shows the analogy between $E(T)$ and $\Delta(x)$. The mean square 60
result for $\Delta(x)$ is contained in

Theorem 2.5.

$$\int\limits_2^X \Delta^2(x)dx = \frac{\zeta^4(3/2)}{6\pi^2\zeta(3)}X^{3/2} + O(X\log^5 X). \tag{2.76}$$

We note that from (2.75) we can obtain an order estimate for $E(T)$, which is much better than $E(T) \ll T^{1/2}\log T$, a result that is a trivial consequence of Atkinson's formula (2.5). Namely, from the definition

$$E(T) = \int\limits_0^T |\zeta(1/2+it)|^2 dt - T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right)$$

we find that, for $0 \le x \le T$,

$$E(T+x) - E(T) \ge -2Cx\log T \tag{2.77}$$

for $T \geq T_0$ and some absolute $C > 0$. Hence

$$\int\limits_T^{T+x} E(t)dt = xE(T) + \int_0^x (E(T+u) - E(T))du$$
$$\geq xE(T) - 2C\log T \int_0^x u\,du = xE(T) - Cx^2 \log T.$$

Therefore we obtain

$$E(T) \leq x^{-1} \int\limits_T^{T+x} E(t)dt + Cx\log T \ (0 < x \leq T, T \geq T_0), \tag{2.78}$$

and analogously

$$E(T) \geq x^{-1} \int\limits_{T-x}^{T} E(t)dt - Cx\log T \ (0 < x \leq T, T \geq T_0). \tag{2.79}$$

Combining (2.78) and (2.79), using the Cauchy-Schwarz inequality and Theorem 2.4 we obtain

$$\begin{aligned}
|E(T)| &\leq x^{-1} \int\limits_{T-x}^{T+x} |E(t)|dt + 2Cx\log T \\
&\leq x^{-1/2} \left(\int\limits_{T-x}^{T+x} E^2(t)dt \right)^{1/2} + 2Cx\log T \\
&= x^{-1/2}(c(T+x)^{3/2} - c(T-x)^{3/2} + O(T\log^5 T))^{1/2} + 2Cx\log T \\
&\ll T^{1/4} + T^{1/2}x^{-1/2}\log^{5/2} T + x\log T \ll T^{1/3}\log^2 T
\end{aligned}$$

61 with the choice $x = T^{1/3}\log T$. Therefore the bound

$$E(T) \ll T^{1/3}\log^2 T \tag{2.80}$$

is a simple corollary of Theorem 2.4. the foregoing argument gives in fact a little more. Namely, if we define $F(T)$ by

$$\int\limits_2^T E^2(t)dt = cT^{3/2} + F(T), \tag{2.81}$$

and η is the infimum of numbers d such that $F(T) \ll T^d$, then

$$E(T) \ll T^{\eta/3+\epsilon}. \tag{2.82}$$

Since $E(T) = \Omega(T^{1/4})$ by (2.73), this means that we must have $3/4 \leq \eta \leq 1$, the upper bound being true by Theorem 2.4. It would be interesting to determine η, although this problem looks difficult. My (optimistic) conjecture is that $\eta \geq 3/4$. An Ω-result for $F(T)$, which is sharper than just $\eta \geq 3/4$, is given by Theorem 3.8. We also remark here that the bound in (2.80) was obtained without essentially taking into account the structure of the exponential sums (with the divisor function $d(n)$) that appear in Atkinson's formula for $E(T)$. The use of the exponential sum techniques improves (2.80). A brief discussion of this topic will be given in Section 2.7.

Before we pass to the proof of Theorem 2.4 and Theorem 2.5, we remark that Theorem 2.4 was recently proved independently by Y. Motohashi and T. Meurman. We shall only outline the salient points of Motohashi's proof, as it is based on his deep work on the approximate functional equation for $\zeta^2(s)$, which represents the analogue of the classical Riemann-Siegel formula for $\zeta(s)$. We state Motohashi's formula **62**
for $E(T)$ as

Theorem 2.6. *Let $\delta > 0$ be a small constant and $\delta \leq \alpha < \beta \leq 1 - \delta$. Define*

$$\lambda(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \alpha, \\ (\beta - x)/(\beta - \alpha) & \text{if } \alpha \leq x \leq \beta, \\ 0 & \text{if } \beta \leq x \leq 1, \end{cases}$$

$\omega(n) = \lambda\left(\frac{2\pi n}{T}\right), \overline{\omega}(n) = 1 - \lambda\left(\exp\left(-2ar\sinh\sqrt{\frac{\pi n}{2T}}\right)\right)$. *Then we have, with an absolute constant c_0 and $T(\alpha) = \frac{T}{2\pi\alpha}(1-\alpha)^2$,*

$$E(T) = 2^{-1/2} \sum_{n \leq T(\alpha)} (-1)^n \overline{\omega}(n) d(n) n^{-1/2} \left(ar\sinh\sqrt{\frac{\pi n}{2}}\right)^{-1} \left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4}$$
$$\times \cos(f(T,n)) - \sum_{n \leq \beta T/(2\pi)} \omega(n) d(n) n^{-1/2} \left(\log\frac{T}{2\pi n}\right)^{-1} \cos(g(T,n))$$

$$+ c_0 + O(T^{-1/4}) + O((\beta-\alpha)^{-1}T^{-1/2}\log T)$$
$$+ O\Big((\beta-\alpha)^{-1/2}T^{-1/2}\log^{5/2}T\Big). \tag{2.83}$$

In this formula $f(T,n)$ and $g(T,n)$ are as in Theorem 2.1. If we consider the special case when $\beta = \alpha + T^{-1/2}$, replace ω and $\overline{\omega}$ by 1 with total error $O(\log T)$, we obtain Atkinson's formula (2.5) with $N = T(\alpha)$ and $\delta \le \alpha \le 1-\delta$, only with the error term $O(\log T)$, which is better than $O(\log^2 T)$ in Atkinson's formula. This saving of a log-factor comes essentially from the smoothing technique inherent in Motohashi's approach. The asymptotic formula (2.83) is strong enough to produce, by standard termwise integration technique, the asymptotic formula (2.75) with the good error term $O(T\log^5 T)$.

The proof of (2.83) is based on Motohashi's expression for

$$R_2(s,x) := \zeta^2(s) - {\sum_{n\ge x}}' d(n)n^{-s} - \chi^2(s){\sum_{n\le y}}' d(n)n^{s-1}, \tag{2.84}$$

where $xy = (t/(2\pi))^2$. In the most important case when $x = y = t/(2\pi)$, his asymptotic formula reduces to

$$\chi(1-s)R_2\left(s,\frac{t}{2\pi}\right) = -2\left(\frac{t}{\pi}\right)^{-1/2}\Delta\left(\frac{t}{2\pi}\right) + O\left(t^{-1/4}\right), \tag{2.85}$$

63 which shows the explicit connection between the divisor problem and the approximate functional equation for $\zeta^2(s)$. Using the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$ and (2.84) one can write

$$|\zeta(1/2+it)|^2 = 2\,\mathrm{Re}\left\{\chi(1/2-it){\sum_{n\le t/2\pi}}' d(n)n^{-1/2-it}\right\}$$
$$+\chi(1/2-it)R_2\left(1/2+it,\frac{t}{2\pi}\right).$$

Integration gives

$$\int_0^T |\zeta(1/2+it)|^2 dt = I_1(T) + I_2(T),$$

say, where

$$I_1(T) = 2\,\mathrm{Re}\left\{\sum_{n\le T/2\pi} d(n)n^{-1/2}\int\limits_{2\pi n}^{T}\chi(1/2-it)n^{-it}dt\right\},$$

$$I_2(T) = \int\limits_{0}^{T}\chi(1/2-it)R_2\left(1/2+it,\frac{t}{2\pi}\right)dt.$$

Quite a delicate analysis is required to evaluate $I_1(T)$, which necessitates the use of the saddle-point method to evaluate certain exponential integrals. The perturbation device

$$I_1(T) = \frac{1}{\beta-\alpha}\int_\alpha^\beta I_1(T)d\xi$$

induces a lot of cancellations, when the sum over n in the expression for $I_1(T)$ is split into two parts according to whether $n \le \xi T/(2\pi)$ or $\xi T/(2\pi) < n \le T/(2\pi)$ $(\delta \le \xi \le 1-\delta)$. The integral $I_2(T)$ is evaluated with the aid of the ζ^2-analogue of the Riemann-Siegel formula.

In Meurman's proof of Theorem 2.4, which we proceed to present now, one first derives in fact a special case of Motohashi's Theorem 2.6, which essentially corresponds to the case $\beta-\alpha \approx T^{-1/4}$. To obtain this we return to the proof of Atkinson's formula (2.5). Analyzing the proof, it is seen that one gets

$$\begin{aligned}E(T) = \left(\frac{2T}{\pi}\right)^{-\frac{3}{4}}&\sum_{n\le N}(-)^n d(n)n^{-1/4}e(T,n)\cos(f(T,n))\\&-\frac{T}{\pi}\sum_{n=1}^{\infty}d(n)n^{-3/4}J_n(T,\sqrt{N})+\pi+O(|\Delta(N)|N^{-1/2})\\&\qquad+O\left(\int_N^\infty X^{-3/2}|\Delta(x)|dx\right)+O(T^{-1/4}),\end{aligned}$$

where $T \ll N \ll T$, $f(T,n)$ is as in Theorem 2.1, **64**

$$e(T,n) = \left(1+\frac{\pi n}{2T}\right)^{-1/2}\left(\sqrt{\frac{2T}{\pi n}}ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-1},$$

$$J_n(T,Y) = \int_Y^\infty g_{3/2}(x)\cos\left(f(T,x^2) - \pi x^2 + \frac{\pi}{2}\right)$$
$$\left\{\cos\left(4\pi x\sqrt{n} - \frac{\pi}{4}\right) - Cx^{-1}n^{-1/2}\sin\left(4\pi x\sqrt{n} - \frac{\pi}{4}\right)\right\}dx,$$
$$g_\alpha(x) = \left\{x^\alpha ar\sinh\left(x\sqrt{\frac{\pi}{2T}}\right)\left(\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{1/2} + \frac{1}{2}\right)\left(\frac{T}{2\pi x^2} + \frac{1}{4}\right)^{1/4}\right\}^{-1}$$

and C is a constant. Now we average the expression for $E(T)$ by taking $N = (a+u)^2$, integrating over $0 \le u \le U$ and dividing by U. Choosing $T^{1/4} \ll U \ll T^{1/4}$ we have $T^{1/2} \ll a \ll T^{1/2}$. Since $\Delta(x)$ is $\approx x^{1/4}$ in mean square we easily obtain

$$\int_N^\infty x^{-3/2}|\Delta(x)|dx \ll T^{-1/4}$$

without the averaging process. But by using Theorem 2.5 (whose proof is independent of Theorem 2.4) we obtain

$$U^{-1}\int_0^U |\Delta(N)|N^{-1/2}du \ll T^{-1/2}U^{-1}\int_0^U |\Delta((a+u)^2)|du$$
$$\ll T^{-1/2}U^{-1/2}\left(\int_{a^2}^{(a+U)^2} \Delta^2(v)v^{-1/2}dv\right)^{1/2} \ll T^{-3/4}U^{-1/2}\left(\int_{a^2}^{a^2+4aU} \Delta^2(v)dv\right)^{1/2}$$
$$\ll T^{-3/4}U^{-1/2}(Ua^2 + a^2\log^5 T)^{1/2} \ll T^{-1/4}.$$

Thus we obtain

$$E(T) = \sum\nolimits_1^*(T) - \pi^{-1}T\sum_{n=1}^\infty d(n)n^{-3/4}K_n + \pi + O(T^{-1/4}) \tag{2.86}$$

where

$$\sum\nolimits_1^*(T) = \left(\frac{2T}{\pi}\right)^{1/4} \sum_{n \le (a+U)^2} \eta(n)(-1)^n d(n)n^{-3/4}e^{T,n}\cos(f(T,n)),$$

$$\eta(n) = 1 - \max(0, U^{-1}(n^{1/2} - a)),$$

$$K_n = \frac{1}{2}\operatorname{Im}(K^+_{n,3/2}) + O(|K^-_{n,3/2}|+n^{-1/2}|K^+_{n,5/2}|+n^{-1/2}|K^-_{n,5/2}|),$$

$$K^\pm_{n,\alpha} = U^{-1}\int\limits_0^U\int\limits_{a+u}^\infty g_\alpha(x)\exp\left\{i(\pm 4\pi x\sqrt{n} - f(T,x^2) + \pi x^2 - \frac{\pi}{4})\right\}dx\,du.$$

The important feature of (2.86) is that the error term is $O(T^{-1/4})$. we define now **65**

$$Z(u) = Z(T,u) = \frac{T}{2\pi} + \frac{1}{2}(a+u)^2 - \left\{\frac{T}{2\pi}(a+u)^2 + \frac{1}{4}(a+u)^4\right\}^{1/2},$$

$$\xi(T,n) = \max\left\{\min(1, U^{-1}(n^{-1/2}\left(\frac{T}{2\pi} - n\right) - a)), 0\right\},$$

and note that $\xi(T,n) = 0$ for $n \geq Z(0)$. We can evaluate $K^\pm_{n,\alpha}$ by using the saddle-point method as in Theorem 2.3 (see also M. Jutila [95]). One obtains, for $T \geq 2$, $\alpha > 0$, $T^{1/2} \ll a \ll T^{1/2}$, $1 \leq U \leq a$ and $n \geq 1$

$$\begin{aligned} K^\pm_{n,\alpha} &= \delta^\pm\xi(T,n)\frac{4\pi}{T}n^{\frac{1}{2}(\alpha-1)}\left(\log\frac{T}{2\pi n}\right)^{-1}\left(\frac{T}{2\pi} - n\right)^{\frac{3}{2}-\alpha}\exp\left\{i\left(-g(T,n) + \frac{\pi}{2}\right)\right\} \\ &\quad + O\left(U^{-1}T^{-1/2}\sum_{u=0,U}\min\left\{1, (\sqrt{n} - \sqrt{Z(u)})^{-2}\right\}\right) + O(\delta^\pm R(n)T^{-\alpha}n^{\frac{1}{2}(\alpha-1)}), \end{aligned} \tag{2.87}$$

where $g(T,n)$ is as in (2.5), $\delta^+ = 1$, $\delta^- = 0$, $R(n) = T^{-1/2}$ for $n < Z(u)$, $R(n) = 1$ for $z(u) \leq n < Z(0)$, and $R(n) = 0$ for $n \geq Z(0)$. Using (2.87) it follows that

$$-\pi^{-1}T\sum_{n=1}^\infty d(n)n^{-3/4}K_n = \sum\nolimits_2^*(T) + O(R_1) + O(R_2) + O(R_3),$$

where

$$\sum\nolimits_2^*(T) = -2\sum_{n\geq Z(0)}\xi(T,n)d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos(g(T,n)), \tag{2.88}$$

$$R_1 = U^{-1}T^{-1/2}\sum_{u=0,U}\sum_{n\geq 1}d(n)n^{-3/4}\min(1, (\sqrt{n} - \sqrt{Z(u)})^{-2}),$$

$$R_2 = T^{-1}\sum_{n\le Z(0)} d(n)n^{-1/2}, R_3 = T^{-1/2}\sum_{Z(U)\le n\le Z(0)} d(n)n^{-1/2}.$$

66 Trivially

$$R_2 \ll T^{-1/2}\log T,$$

and since $dZ(u)/du \ll T^{1/2}$ we have

$$R_3 \ll T^{-1}(Z(0)-Z(U))\log T \ll UT^{-1/2}\log T \ll T^{-1/4}\log T.$$

In R_1 the values of n in $(1/2Z(u), 2Z(u)]$ contribute

$$\begin{aligned}
&\ll U^{-1}T^{-\frac{1}{2}}\sum_{\frac{1}{2}Z(u)<n\le 2Z(u)} d(n)\min(1, Z(u)(n-Z(u))^{-2}\\
&\ll U^{-1}T^{-\frac{1}{2}}\times\\
&\left(\sum_{Z(u)-\sqrt{z(u)}<n<Z(u)+\sqrt{z(u)}} d(n)+\sum_{|n-Z(u)|>\sqrt{Z(u)},1/2Z(u)<n\le 2Z(u)}\frac{Z(u)d(n)}{(n-Z(u))^2}\right)\\
&\ll U^{-1}T^{-1/2}T^{1/2}\log T \ll T^{-1/4}\log T.
\end{aligned}$$

The remaining n's contribute to R_1 only $\ll U^{-1}T^{-1/2+\epsilon} \ll T^{-3/4+\epsilon}$. Therefore from (2.86), (2.87) and the above estimates we obtain, for $T\ge 2, T^{1/4}\ll U\ll T^{1/4}, T^{1/2}\ll a\ll T^{1/2}$,

$$E(T) = \sum\nolimits_1^*(T) + \sum\nolimits_2^*(T) + \quad +O(T^{-1/4}\log T). \tag{2.89}$$

We note that by using multiple averaging (i.e. not only over one variable u, but over several) one can remove the log-factor from the error term in (2.89), although this is not needed for the proof of Theorem 2.4. It is the variant of Atkinson's formula, given by (2.89), that is suitable for the proof of Theorem 2.4. We use it with $a = T^{1/2} - U$, $U = T^{1/4}$, and combine with the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\int_T^{2T} E^2(t)dt &= I_{11} + 2I_{12} + I_{22} + 2\pi I_1\\
&\quad + O(T^{1/4}I_{11}\log T + T^{1/2}I_{22}^{1/2} + T),
\end{aligned} \tag{2.90}$$

67 where

$$I_{jk} = \int\limits_{T}^{2T} \sum\nolimits_j^*(t) \sum\nolimits_k^*(t)dt \quad (j, k = 1 \text{ or } 2),$$

$$I_1 = \int\limits_{T}^{2T} \sum\nolimits_1^*(t)dt \ll T^{3/4}$$

by Lemma 2.1. From now on the proof is essentially the same as the one that will be given below by Theorem 2.5. Termwise integration gives (the factors $\eta(n)$ and $\xi(t, n)$ will cause no trouble)

$$I_{11} = c(2T)^{3/2} - cT^{3/2} + O(T \log^5 T), \tag{2.91}$$

$$I_{22} = O(T \log^4 T), \tag{2.92}$$

so that the main contribution to (2.75) comes from (2.91), which will follow then on replacing T in (2.90) by $T2^{-j}(j = 1, 2, \ldots)$ and adding all the resulting expressions. Thus the proof will be finished if the contribution of I_{12} in (2.90) is shown to be negligible. To see this let

$$\sum\nolimits_1^*(t) = \sum\nolimits_1'(t) + \sum\nolimits_2''(t),$$

where in $\Sigma'(t)$ we have $n \leq T/A$, and in $\Sigma''(t)$ we have $T/A < n \leq T$, $A > 0$ being a large constant. Then by termwise integration we obtain, as in the proof of (2.91),

$$\begin{aligned} \int\limits_{T}^{2T} \left(\sum\nolimits''(t)\right)^2 &\ll T^{3/2} \sum_{n>T/A} d^2(n) n^{-\frac{3}{2}} \sum_{n>T/A} d^2(n) n^{-3/2} + T \log^4 T \\ &\ll T \log^4 T. \end{aligned} \tag{2.93}$$

Set $I_{12} = I' + I''$, where

$$I' = \int\limits_{2}^{2T} \sum\nolimits'(t) \sum\nolimits_2^*(t)dt, I'' = \int\limits_{T}^{2T} \sum\nolimits''(t) \sum\nolimits_2^*(t)dt.$$

Thus by the Cauchy-Schwarz inequality, (2.92) and (2.93) we infer that $I'' \ll \log^4 T$. Consider now I'. We have

$$I' \ll \sum_{n \le T/A} \sum_{m \le Z(2T,0)} d(n)d(m)n^{-3/4}m^{-1/2}\left(|J^{+}_{n,m}| + |J^{-}_{n,m}|\right).$$

68 where

$$J^{\pm}_{n,m} = \int\limits_{H} \xi(t,m)t^{1/4}e(t,n)\left(\log\frac{t}{2\pi m}\right)^{-1}\exp\{i(f(t,n) \pm g(t,m))\}dt$$

and H is a subinterval of $[T, 2T]$ such that $m \le Z(t,0)$ for $t \in H$. Now

$$f'(t,n) = 2ar\sinh\sqrt{\frac{\pi n}{2t}},\, g'(t,m) = \log\left(\frac{t}{2\pi m}\right),$$

and for $n \le T/A$ we have $|f'(t,n) \pm g'(t,m)| \gg 1$ in H, which was the reason for considering separately $\Sigma'(t)$ and $\Sigma''(t)$. Hence by Lemma 2.1 we obtain $J^{\pm}_{n,m} \ll T^{1/4}$, and consequently $I' \ll T\log^2 T$, $I_{12} \ll T\log^4 T$, and Theorem 2.4 follows.

It remains to prove Theorem 2.5, whose method of proof incidentally establishes (2.91). Suppose $x \ge 1$ is not an integer and let

$$\delta_M(x) := (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n \le M} d(n)n^{-3/4}\cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right).$$

From the Voronoi formula (2.24) one has

$$\begin{aligned}\Delta(x) &= \delta_M(x) + O(x^{1/4}(|S_1| + |S_2|) + x^{-1/4}), \qquad (2.94)\\ S_1 &= \sum_{n>M}(d(n) - \log n - 2\gamma)n^{-3/4}\exp(4\pi i\sqrt{nx}),\\ S_2 &= \sum_{n>M}(\log n + 2\gamma)n^{-3/4}\exp(4\pi i\sqrt{nx}).\end{aligned}$$

For $\xi > M \ge 2x$ we have

$$G(\xi) := \sum_{M<n\le\xi}\exp(4\pi i\sqrt{nx}) = \int\limits_{M}^{\xi}\exp(4\pi i\sqrt{tx})dt + O(1) = O((\xi/x)^{1/2}), \qquad (2.95)$$

where we used Lemma 2.1 and the elementary relation

$$\sum_{a<n\le b} e(f(n)) = \int_a^b e(f(x))dx + O_\delta(1), \tag{2.96}$$

provided that $f(x) \in C^2[a,b]$, $f'(x)$ is monotonic on $[a,b]$ and $|f'(x)| \le \delta < 1$. Then by partial summation it follows that

$$\begin{aligned}
S_2 &= \lim_{A\to\infty} \sum_{M<n\le A} (\log n + 2\gamma) n^{-3/4} \exp(4\pi i\sqrt{nx}) \\
&= \lim_{A\to\infty} \left\{ (\log t + 2\gamma)t^{-3/4} G(t)\Big|_M^A - \int_M^\infty ((\log t + 2\gamma)t^{-3/4})' G(t)dt \right\} \\
&\ll x^{-1/2} M^{-1/4} \log M \ll x^{-1/2}.
\end{aligned}$$

For S_1, note that for any $t > 1$ **69**

$$\sum_{n\le t} (d(n) - \log n - 2\gamma) = \Delta(t) + O(t^\epsilon),$$

since the definition of $\Delta(t)$ contains $\sum_{n\le t}{}'$ and $d(n) \ll n^\epsilon$. Partial summation and the weak estimate $\Delta(x) \ll x^{1/3+\epsilon}$ give

$$\begin{aligned}
S_1 = \lim_{A\to\infty} &\left\{ (\Delta(t) + O(t^\epsilon)) t^{-3/4} \exp(4\pi i\sqrt{tx})\Big|_M^A \right. \\
&\left. - \int_M^A (\Delta(t) + O(t^\epsilon))(t^{-3/4}\exp(4\pi i\sqrt{tx}))' dt \right\} \\
\ll M^{\epsilon-5/12} &+ x^{1/2} \left| \int_M^\infty \Delta(t) t^{5/4} \exp(4\pi i\sqrt{tx}) dt \right| + \int_M^\infty t^{\epsilon-5/4} x^{1/2} dt \\
\ll x^{1/2} &\left| \int_M^\infty \Delta(t) t^{-5/4} \exp(4\pi i\sqrt{tx}) dt \right| + x^{1/2} M^{\epsilon-1/4}
\end{aligned}$$

$$= x^{1/2}|I| + x^{1/2}M^{\epsilon-1/4} = x^{1/2}\sum_{k=0}^{\infty} I(2^k M) + x^{1/2}M^{\epsilon-1/4},$$

where we have set

$$I(Y) := \int\limits_{Y}^{2Y} \Delta(y)y^{-5/4}\exp(4\pi i\sqrt{xy})dy.$$

To estimate $I(Y)$ we use

$$\Delta(y) = (\pi 2^{1/2})^{-1}y^{1/4}\sum_{n\le Y} d(n)n^{-3/4}\cos\left(4\pi\sqrt{ny}-\frac{\pi}{4}\right) + O(Y^\epsilon)\ (Y \le y \le 2Y).$$

This gives

$$I(Y) \ll \sum_{n\le Y} d(n)n^{-3/4}\left(|I_n^+(Y)| + |I_n^-(Y)|\right) + Y^{\epsilon-1/4},$$

70 where using Lemma 2.1 we have

$$I_n^{\pm}(Y) = \int\limits_{Y}^{2Y} y^{-1}\exp(4\pi i(\sqrt{x}+\sqrt{n})\sqrt{y})dy \ll \frac{Y^{-1/2}}{|\sqrt{x}\pm\sqrt{n}|}$$

The contribution of I_n^+ is trivially $O(Y^{-1/2})$, and that of I_n^- is (since by assumption x is not an integer)

$$\ll \sum_{n\le Y} d(n)n^{-3/4}Y^{-1/2}|x^{1/2}-n^{1/2}|^{-1} \ll Y^{-1/2} + Y^{-1/2}x^{1/2}\sum_{\frac{1}{2}x<n\le 2x}\frac{d(n)n^{-3/4}}{|x-n|}$$
$$= Y^{-1/2} + 0(Y^{-1/2}x^{-1/4+\epsilon}\parallel x\parallel^{-1}) + O(Y^{-1/2})\sum_{|r|\le x-1, r\ne 0} r^{-1}d([x]+r)x^{-1/4}$$
$$\ll Y^{-1/2} + x^{-1/4+\epsilon}\parallel x\parallel^{-1} Y^{-1/2},$$

where $\parallel x\parallel$ is the distance for x to the nearest integer, and where we set $n = [x] + r$. Thus

$$I(Y) \ll Y^{-1/2}x^{-1/4+\epsilon}\parallel x\parallel^{-1} + Y^{\epsilon-1/4},$$

$$S_1 \ll M^{-1/2} x^{1/4+\epsilon} \parallel x \parallel^{-1} + x^{1/2} M^{\epsilon-1/4}.$$

But $\epsilon > 0$ sufficiently small

$$M^{-1/4+\epsilon} x^{1/2} \leq x^{-1/2} \qquad \text{for } M \geq x^{4+20\epsilon},$$
$$M^{-1/2} x^{1/4+\epsilon} \parallel x \parallel^{-1} \leq x^{-1/2} \qquad \text{for } M \leq x^{3/2+2\epsilon} \parallel x \parallel^{-2}.$$

The truncated Voronor formula (2.25) gives $\Delta(x) = \delta_M(x) + O(x^\epsilon)$ if $x \geq 1$, $M \gg x$, so that we obtain

$$\Delta(x) = \delta_M(x) + R(x), R(x) \ll \begin{cases} x^{-1/4} & \text{if } M \geq \max(x^{4+20\epsilon}, x^{3/2+2\epsilon} \parallel x \parallel^{-2}), \\ x^\epsilon & \text{otherwise.} \end{cases}$$

If $M \geq x^5 \parallel x \parallel^{-2}$, then $M \geq 4x^5 \geq x^{4+20\epsilon}$, $M \geq x^{3/2+2\epsilon} \parallel x \parallel^{-2}$ for $o < \epsilon < 1/20$, hence $R(x) \ll x^{-1/4}$ for $M \geq x^5 \parallel x \parallel^{-2}$, and $R(x) \ll x^\epsilon$ 71
otherwise. It is this estimate for $R(x)$, which shows that it is $\ll x^{-1/4}$ "most of the time", that makes the proof of Theorem 2.5 possible. So we take now $M = X^9$, $X \leq x \leq 2X$, obtaining

$$\Delta(x) = \begin{cases} \delta_M(x) + O(X)^{-1/4} & \text{if } \parallel x \parallel \gg X^{-2}, \\ \delta_M(x) + O(X^\epsilon) & \text{if } \parallel x \parallel \ll X^{-2}. \end{cases} \tag{2.97}$$

To prove Theorem 2.5 it will be sufficient to show

$$D := \int\limits_X^{2X} \delta_M^2(x)dx = \frac{1}{6\pi^2} \left(\sum_{n=1}^{\infty} d^2(n) n^{-3/2} \right) ((2X)^{3/2} - X^{3/2}) + O(X \log^5 X), \tag{2.98}$$

since using (2.97) we have

$$\int\limits_X^{2X} \Delta^2(x)dx = D + O\left(\int\limits_{X, \parallel x \parallel \ll X^{-2}}^{2X} X^{2\epsilon} dx \right) + O(X^{1/2} + X^{1/4} D^{1/2}),$$

by applying the Cauchy-Schwarz inequality. Thus if (2.98) holds we obtain

$$\int\limits_X^{2X} \Delta^2(x)dx = D + O(X) = d((2X)^{3/2} - X^{3/2}) + O(X \log^5 X)$$

with
$$d = \frac{1}{6\pi^2}\left(\sum_{n=1}^{\infty} d^2(n)n^{-3/2}\right) = \frac{\zeta^4(3/2)}{6\pi^2\zeta(3)}.$$

To prove (2.98) we use $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$. It follows from the definition of $\delta_M(x)$ that

$$\begin{aligned} D &= \frac{1}{4\pi^2}\sum_{m,n\le M} d(m)d(n)(mn)^{-3/4}\int\limits_X^{2X} x^{1/2}\times \\ &\quad \left\{\cos(4\pi(m^{1/2}-n^{1/2})x^{1/2}) + \sin(4\pi(m^{1/2}+n^{1/2})x^{1/2})\right\}dx \\ &= \frac{1}{6\pi^2}\sum_{m\le M} d^2(m)m^{-3/2}((2X)^{3/2}-X^{3/2}) \\ &\quad + O\left(X\sum_{m,n\le M, m\ne n}\frac{d(m)d(n)}{(mn)^{1/4}|m^{1/2}-n^{1/2}|}\right) \end{aligned} \tag{2.99}$$

after an integration by parts. Since $M = X^9$, we have

$$\begin{aligned} \sum_{m\le M} d^2(m)m^{-3/2} &= \sum_{m=1}^{\infty} d^2(m)m^{-3/2} + O(M^{-1/2}\log^4 M) \\ &= \frac{\zeta^4(3/2)}{\zeta(3)} + O(X^{-4}). \end{aligned}$$

72 The double sum in the O-term in (2.99) is

$$\ll \sum_{n<m\le M}\frac{d(m)d(n)}{n^{3/4}m^{1/4}(m-n)} \ll \sum_{r\le M, n\le M, n+r\le M}\frac{d(n)d(n+r)}{n^{3/4}r(n+r)^{1/4}}$$

if we write $m = n + r$. In the portion of the last sum with $n \le M$, $\frac{1}{2}n < r \le M$ we have $n + r \ge \frac{3}{2}n$ and $\frac{1}{r} \le \frac{3}{n+r}$. Hence

$$\begin{aligned} \sum_{\frac{1}{2}n<r\le M, n+r\le M}\frac{d(n)d(n+r)}{n^{3/4}r(n+r)^{1/4}} &\le 3\sum_{n\le M}\frac{d(n)}{n^{3/4}}\sum_{n+r\le M, r>1/2n}\frac{d(n+r)}{(n+r)^{5/4}} \\ &\ll \sum_{n\le M} d(n)n^{-3/4}\sum_{m\ge 3n/2} d(m)m^{-5/4} \ll \sum_{n\le M} d(n)n^{-1}\log n \ll \log^2 M. \end{aligned}$$

Also

$$\sum_{r\le\frac{1}{2}n, n<M, n+r\le M} \frac{d(n)d(n+r)}{n^{3/4}r(n+r)^{1/4}}$$
$$\le \sum_{r\le\frac{1}{2}M} r^{-1} \sum_{2r\le n\le M} \frac{d(n)d(n+r)}{n^{3/4}(n+r)^{1/4}}$$
$$\le 2\sum_{r\le\frac{1}{2}M} r^{-1}\left(\sum_{2r\le n\le M}\frac{d(n)}{n^{1/2}}\cdot\frac{d(n+r)}{(n+r)^{1/2}}\right)$$
$$\le 2\sum_{r\le\frac{1}{2}M} r^{-1}\left(\sum_{n\le M} d^2(n)n^{-1}\right)^{1/2}\left(\sum_{m\le 2M} d^2(m)m^{-1}\right)^{1/2}$$
$$\ll \log^5 M.$$

In view of $M = X^9$ this implies

$$\sum_{r\le M, n\le M, n+r\le M} \frac{d(n)d(n+r)}{n^{3/4}r(n+r)^{1/4}} \ll \log^5 X$$

and therefore establishes (2.98).

Finally we discuss how to improve slightly on the error terms in Theorem 2.4 and Theorem 2.5 and obtain

$$\int_2^T E^2(t)dt = cT^{3/2} + O(T\log^4 T), \tag{2.100}$$

and analogously the error term in (2.76) may be replaced by $O(X\log^4 X)$. **73**
Instead of the fairly straightforward estimation of the O-term in (2.99) one has to use the inequality

$$\sum_{r,s\le R; r\ne s} \frac{a_r\bar{a}_s\exp(ib_r - ib_s)}{(rs)^{1/4}(r^{1/2}-s^{1/2})} \ll \sum_{r\le R}|a_r|^2, \tag{2.101}$$

where the a_r's are arbitrary complex numbers, and the b_r's are arbitrary real numbers. This follows from a variant of the so-called Hilbert's inequality: Suppose that $\lambda_1, \lambda_2, \ldots \lambda_R$ are distinct reals and $\delta_r = \min\limits_{s} \ _{+} |\lambda_r -$

$\lambda_s|$, where $\min_+ f$ denotes the least positive value of f. Then

$$\left|\sum_{r,s\le R;r\ne s}\frac{u_r\bar{u}_s}{\lambda_r-\lambda_s}\right|\le\frac{3\pi}{2}\sum_{r\le R}|u_r|^2\delta_r^{-1}. \tag{2.102}$$

By taking in (2.102) $\lambda_r = r^{1/2}$, $u_r = a_r r^{-1/4}\exp(ib_r)$ and noting that in this case

$$\delta_r = \min_{s\ +}|r^{1/2}-s^{1/2}| = \min_{s\ +}\left|\frac{r-s}{r^{1/2}+s^{1/2}}\right| \ll r^{-1/2},$$

One obtains (2.101). This possibility was noted recently by E. Preissmann [135], who considered the mean square of $\Delta(x)$, using a classical differencing technique of E. Landau based on the Voronoi formula for the integral of $\Delta(x)$. This seems more appropriate in this context than the use of the method of proof of Theorem 2.5, where the choice $M = X^9$ is large for the application of (2.101). But for (2.100) this can be done directly, and it will be sufficient that the non-diagonal terms in the mean square for $\Sigma_1^*(t)$ are $\ll T\log^4 T$. The non-trivial contribution to the sum in question comes from

$$\sum_{m\ne n\le T} d(m)d(n)\eta(m)\eta(n)(mn)^{-3/4}\int_T^{2T} e(t,m)e(t,n)\cos(f(t,m)-f(t,n))dt$$
$$=\sum_{m\ne n\le T} d(m)d(n)\eta(m)\eta(n)(mn)^{-3/4}\times$$
$$\int_T^{2T}\frac{e(t,m)e(t,n)}{2\left(ar\sinh\left(\frac{\pi n}{2T}\right)^{1/2}-ar\sinh\left(\frac{\pi m}{2T}\right)^{1/2}\right)}\frac{d}{dt}\left\{\sin(f(t,n)-f(t,m))\right\}dt$$
$$=\frac{1}{2}\sum_{m\ne n\le T} d(m)d(n)\eta(m)\eta(n)(mn)^{-1/4}\times$$
$$\left\{\frac{e(t,m)e(t,n)}{ar\sinh\left(\frac{\pi n}{2T}\right)^{1/2}-ar\sinh\left(\frac{\pi m}{2T}\right)^{1/2}}(\sin(f(t,n)-f(t,m)))\right\}\Bigg|_T^{2T}$$
$$-\frac{1}{2}\sum_{m\ne n\le T} d(m)d(n)\eta(m)\eta(n)(mn)^{-3/4}\int_T^{2T}\sin(f(t,n)-f(t,m))\frac{d}{dt}\times$$

$$\left\{\frac{e(t,m)e(t,n)}{ar\sinh\left(\frac{\pi n}{2T}\right)^{1/2} - ar\sinh\left(\frac{\pi m}{2T}\right)^{1/2}}\right\} dt.$$

The integrated terms are first simplified by Taylor's formula and then estimated by (2.101), giving sums like 74

$$T \sum_{m\neq n\leq T} \frac{d(m)d(n)\exp(if\,(T,n) - if\,(t,m))}{(m^{1/2} - n^{1/2})(mn)^{1/4}} \ll T\log^4 T.$$

The integral $\int\limits_T^{2T}$ in the above sum is estimated by Lemma 2.1 and the ensuing sums are treated analogously as in the proof of Theorem 2.5. Their total contribution will be again $\ll \log^4 T$, and (2.100) follows.

2.7 The Upper Bounds for the Error Terms

We have seen in Section 2.6 how by an averaging technique Theorem 2.4 can be used to yield the upper bound $E(T) \ll T^{1/3}\log^2 T$. Now we shall use another, similar averaging technique to obtain an upper bound for $E(T)$ which is in many ways analogous to the truncated formula (2.25) for $\Delta(x)$. This bound will have the advantage that exponential sum techniques can be applied to it, thereby providing the possibility to improve on $E(T) \ll T^{1/3}\log^2 T$. We shall also give upper bounds for $E_\sigma(T)$, which will be valid in the whole range $1/2 < \sigma < 1$, and not just for $1/2 < \sigma < 3/4$, which is the range for which Theorem 2.2 holds.

We shall consider first $E(T)$. In case $E(T) > 0$ we start from the inequality

$$E(T) \leq H^{-N}\int\limits_0^H \cdots \int\limits_0^H E(T + u_1 + \cdots + u_N)du_1\cdots du_N + CH\log T, \tag{2.103}$$

whose proof is analogous to the proof of (2.78). A similar lower bound 75
inequality, analogous to (2.79), also holds and may be used to estimate $E(T)$ when $E(T) < 0$. We suppose $T \geq T_0$, $T^\epsilon \ll H \ll T^{1/2}$, and take

N to be a large, but fixed integer. In Atkinson's formula (2.5), which we write as

$$E(T) = \sum\nolimits_1(T) + \sum\nolimits_2(T) + O(\log^2 T),$$

take $N = T$, $N' = \alpha T$, $\alpha = \frac{1}{2\pi} + \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{2\pi}\right)^{1/2}$. First we shall show that the contribution of

$$\sum\nolimits_2(T) = -2\sum_{n\le N'} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos(g(T,n))$$

is negligible. We have

$$g(T,n) = T\log\left(\frac{T}{2\pi n}\right) - T + \frac{\pi}{4}, \frac{\partial g(T,n)}{\partial T} = \log\left(\frac{T}{2\pi n}\right) \gg 1$$

for $n \le N$. Hence

$$\begin{aligned}
&H^{-N}\int\limits_0^H\cdots\int\limits_0^H \sum\nolimits_2(T+u_1+\cdots+u_N)du_1\cdots du_N\\
&= -2H^{-n}\sum_{n\le N'} d(n)n^{-1/2}\int\limits_0^H\cdots\int\limits_0^H \log^{-2}\left(\frac{T+u_1+\cdots+u_n}{2\pi n}\right)\\
&\qquad \frac{\partial \sin g(T+u_1+\cdots+u_{N'}n)}{\partial u_1}du_1\cdots du_N\\
&= -2H^{-N}\sum_{n\le N'} d(n)n^{-1/2}\left\{\int\limits_0^H\cdots\int\limits_0^H\left(\frac{\sin g(T+H+u_2+\cdots u_n,n)}{\log^2\left(\frac{T+H+u_2+\cdots+u_N}{2\pi n}\right)}\right.\right.\\
&\qquad \left.-\frac{\sin g(T+u_2+\cdots+u_n,n)}{\log^2\left(\frac{T+u_2+\cdots+u_N}{2\pi n}\right)}\right)du_2\cdots du_N\\
&\qquad +2\int\limits_0^H\cdots\int\limits_0^H \log^{-3}\left(\frac{T+u_1+\cdots+u_N}{2\pi n}\right)\\
&\qquad \left.\frac{\sin g(T+u_1+\cdots+u_N,n)}{T+u_1+\cdots+u_N}du_1\cdots du_N\right\}
\end{aligned}$$

$$= -2H^{-N} \sum_{n \le N'} d(n) n^{-1/2} \int_0^H \cdots \int_0^H \left(\frac{\sin g(T+H+u_2+\cdots+u_N, n)}{\log^2\left(\frac{T+H+u_2+\cdots+u_N}{2\pi n}\right)} \right.$$
$$\left. - \frac{\sin g(T+u_2+\cdots+u_N, n)}{\log^2\left(\frac{T+u_2+\cdots+u_N}{2\pi n}\right)} \right) du_2 \cdots du_N$$
$$+ O(T^{-1/2}\log T) = \ldots = O(H(^{-N} T^{1/2} \log T)$$
$$+ O(T^{-1/2}\log T)$$

after N integrations by parts. Thus for N sufficiently large the contribution of Σ_2 is negligible, namely it is asorbed in the term $CH\log T$ in (2.103). 76

Further we have

$$\sum\nolimits_1 (T) = 2^{-\frac{1}{2}} \sum_{n \le T} (-1)^n d(n) n^{-1/2} \left(ar\sinh \sqrt{\frac{\pi n}{2T}} \right)^{-1} \left(\frac{T}{2\pi n} + \frac{1}{4} \right)^{-1/4} \cos(f(T,n)),$$

where

$$\frac{\partial f(T,n)}{\partial T} = 2ar\sinh \sqrt{\frac{\pi n}{2T}}.$$

It will be shown now that the contribution of n for which $T^{1+\epsilon}H^{-2} < n \le T$ ($\epsilon > 0$ arbitrarily small and fixed) in the integral of $\Sigma_1(T+u_1+\cdots+u_N)$ in (2.103) is negligible. To see this set

$$F_1(T,n) := 2^{-3/2} \left(ar\sinh \sqrt{\frac{\pi n}{2T}} \right)^{-2} \left(\frac{T}{2\pi n} + \frac{1}{4} \right)^{-1/4}.$$

On integrating by parts it is seen that the integral in question is equal to

$$H^{-N} \sum_{T^{1+\epsilon}H^{-2} < n \le T} (-1)^n d(n) n^{-1/2} \int_0^H \cdots \int_0^H F_1(T+u_1+\cdots+u_N, n) \times$$
$$\frac{\partial \sin f(T+u_1+\cdots+u_N, n)}{\partial u_1} du_1 \cdots du_N$$
$$= H^{-N} \sum_{T^{1+\epsilon}H^{-2} < n \le T} (-1)^n d(n) n^{-1/2} \left\{ \int_0^H \cdots \int_0^H (F_1(T+H+u_2+\cdots+u_N, n) \right.$$

$$\sin f(T+H+u_2+\cdots+u_N,n) - F_1(T+u_2+\cdots+u_N,n)$$

$$\sin f(T+u_2+\cdots+u_N,n))du_2\cdots du_N \int\limits_0^H \cdots \int_0^H$$

$$\left.\frac{\partial F_1(T+u_1+\cdots+u_N,n)}{\partial u_1}\sin f(T+u_1+\cdots+u_N,n)du_1\cdots du_n\right\}.$$

77 But we have

$$\frac{\partial F_1(T,n)}{\partial T} = -\frac{1}{8\sqrt{8}\pi n}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-5/4}\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-2} + 2^{-3/2}\times$$
$$\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}(ar\ \sinh)\left(\sqrt{\frac{\pi n}{2T}}\right)^{-3}\left(1+\frac{\pi n}{2T}\right)^{-1/2}\left(\frac{\pi n}{2}\right)^{1/2}T^{-3/2}$$
$$\ll T^{-1/4}n^{-3/4},$$

and since $\sum\limits_{n\geq 1} d(n)n^{55/4}$ converges, trivial estimation shows that the contribution of the N-fold integral is negligible. Now set

$$F_2(T,n) := 2^{-1}\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-1}F_1(T,n)$$
$$= 2^{-5/2}\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-3}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}.$$

Then we have (the other $(N-1)$-fold integral is treated analogously:)

$$H^{-N}\sum_{T^{1+\epsilon}H^{-2}<n\leq T}(-1)^n d(n)n^{-1/2}\int_0^H\cdots\int_0^H(-F_1(T+u_2+\cdots+u_N,n)\times$$
$$\sin f(T+u_2+\cdots+u_N,n))du_2\cdots du_N$$
$$= H^{-N}\sum_{T^{1+\epsilon}H^{-2}<n\leq T}(-1)^n d(n)n^{-1/2}\int\limits_0^H\cdots\int\limits_0^H F_2(T+u_2+\cdots+u_N,n)\times$$
$$\frac{\partial\cos f(T+u2+\cdots+u_N,n)}{\partial u_2}du_2\cdots du_N$$

$$= H^{-N} \sum_{T^{1+\epsilon}H^{-2}<n\leq T} (-1)^n d(n) n^{-1/2} \left\{ \int_0^H \cdots \int_0^H (F_2(T+H+u_3+\cdots+u_N, n)\times \right.$$
$$\cos f(T+H+u_3+\cdots+u_N, n) - F_2(T+u_3+\cdots+u_N.n)\times$$
$$\cos f(T+u_3+\cdots+u_N, n))du_3\cdots du_N - \int_0^H \cdots \int_0^H \times$$
$$\left. \frac{\partial F_2(T+u_2+\cdots+u_N, n)}{\partial u_2} \cos f(T+u_2+\cdots+u_N, n) du_2\cdots du_N \right\}$$

But

$$\frac{\partial F_2(T,n)}{\partial T} \ll T^{1/4} n^{-5/4}.$$

Trivial estimation shows that the total contribution of the $(N-1)$-fold integral is 78

$$\ll H^{N-1}H^{-N} \sum_{n>T^{1+\epsilon}H^{-2}} d(n) n^{-1/2} T^{1/4} n^{-5/4}$$
$$\ll H^{-1}T^{1/4}(T^{1+\epsilon}H^{-2})^{-3/4}\log T \ll 1.$$

Thus defining

$$F_k(T,n) := \left(2^{-1} ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{1-k} F_1(T,n)$$

we continue integrating, until all N integrations are completed. We estimate the resulting expression trivially, obtaining a contribution of all integrated terms as

$$H^{-N} \sum_{n>T^{1+\epsilon}H^{-2}} d(n) n^{-1/2} \left(\frac{n}{T}\right)^{\frac{1}{2}(1-N)} F_1(T,n) \ll H^{-N} T^{\frac{1}{2}N+\frac{1}{4}}\times$$
$$\sum_{n>T^{1+\epsilon}H^{-2}} d(n) n^{-\frac{3}{4}-\frac{1}{2}N}\times$$
$$H^{-N}T^{\frac{1}{2}N+\frac{1}{4}}(T^{1+\epsilon}H^{-2})^{\frac{1}{4}-\frac{1}{2}N}\log T = T^{\frac{1}{2}+\frac{1}{4}\epsilon-\frac{1}{2}\epsilon N} H^{1/2}\log T \ll 1$$

if $N = N(\epsilon)$ is sufficiently large. The contribution of the remaining n's in Σ_1 for which $n \leq T^{1+\epsilon}H^{-2}$ is estimated trivially, by taking the supremum of the expression in question. Therefore we obtain

Theorem 2.7. *For $T^\epsilon \ll H \ll T^{1/2}$, $0 < \epsilon < 1/2$ fixed, we have*

$$E(T) \ll H \log T + \sup_{\frac{1}{2}T < \tau \leq 2T} \Bigg| \sum_{n \leq T^{1+\epsilon}H^{-2}} (-1)^n d(n) n^{n^{-\frac{1}{2}}}$$
$$\left(ar\sinh\sqrt{\frac{\pi n}{2\tau}}\right)^{-1} \left(\frac{\tau}{2\pi n} + \frac{1}{4}\right)^{-1/4} \cos(f(\tau, n))\Bigg|. \tag{2.104}$$

Estimating the sum over n in (2.104) trivially we obtain

$$E(T) \ll H \log T + T^{1/4} \sum_{n \leq T^{1+\epsilon}H^{-2}} d(n) n^{-1/4} \ll H \log T + T^{1/2+\epsilon} H^{-1/2}.$$

79 Choosing $H = T^{1/3}$ we obtain

$$E(T) = O(T^{1/3+\epsilon}),$$

which is essentially (2.80). A similar analysis may be made for $E_\sigma(T)$, if one uses (2.6) and follows the foregoing argument. In this way one obtains

Theorem 2.8. *For $T^\epsilon \ll H \ll T^{1/2}$, $0 < \epsilon < 1/2$ and $1/2 < \sigma < 3/4$ fixed, we have*

$$E_\sigma(T) \ll H + T^{1/2-\sigma} \sup_{\frac{1}{2}T \leq \tau \leq 2T} \Bigg| \sum_{n \leq T^{1+\epsilon}H^{-2}} (-1)^n \sigma_{1-2\sigma}(n) n^{\sigma-1} \times$$
$$\left(ar\sinh\sqrt{\frac{\pi n}{2\tau}}\right)^{-1} \left(\frac{\tau}{2\pi n} + \frac{1}{4}\right)^{-1/4} \cos f(\tau, n)\Bigg|. \tag{2.105}$$

Choosing $H = T^{1/(1+4\sigma)}$ and estimating trivially the sum in (2.105) one obtains

$$E_\sigma(T) \ll T^{1/(1/4\sigma)+\epsilon} \quad (1/2 < \sigma < 3/4), \tag{2.106}$$

and using the technique similar to the one employed in deriving (2.80), it can be seen that "ϵ" in (2.106) can be replaced by a suitable log-power. The point of having upper bound such as (2.104) and (2.105) is that both (2.80) and (2.106) can be improved by the use of exponential sum techniques on writing $d(n) = \sum_{kl=n} 1$ and $\sigma_{1-2\sigma}(n) = \sum_{kl=n} k^{1-2\sigma}$, respectively. In this way one obtains a double sum which is suitable for further transformations and estimations. We shall consider first the estimation of $E(T)$ in more detail and treat $E_\sigma(T)$ later on. However, right now we remark again that (2.104) corresponds to the truncated formula (2.25) for $\Delta(x)$, namely

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le N} d(n) n^{-3/4} \cos\left(4\pi\sqrt{\pi x} - \frac{\pi}{4}\right) + O\left(x^{1/2+\epsilon} N^{-1/2}\right), \tag{2.107}$$

where $1 \ll N \ll x$. In this analogy N corresponds to $T^{1+\epsilon}H^{-2}$, x to T, 80
$4\pi\sqrt{nx} - \frac{\pi}{4}$ to $f(t, n)$, since

$$f(t, n) = -\frac{\pi}{4} + 4\pi\sqrt{\frac{nt}{2\pi}} + O(n^{3/2}t^{-1/2}) \ (n = o(t)).$$

Hence we may consider for simplicity the estimation of $\Delta(x)$, since the estimation of $E(T)$ via (2.106) will be completely analogous. Removing by partial summation $n^{-3/4}$ from the sum in (2.107), we have by the hyperbola method

$$\sum_{n \le N} d(n) e(2\sqrt{nx}) = 2 \sum_{m \le \sqrt{N}} \sum_{n \le N/m} e(2\sqrt{mnx}) - \sum_{m \le \sqrt{N}} \sum_{n \le \sqrt{N}} e(2\sqrt{mnx}).$$

If $(x.\lambda)$ is a one-dimensional exponent pair, then for $1 \ll M \ll x$

$$\sum_{M < n \le M' \le 2M} e(2\sqrt{mnx}) \ll (mxM^{-1})^{\frac{1}{2}x} M^{\lambda} = (mx)^{\frac{1}{2}x} M^{\lambda - \frac{1}{2}x}.$$

Hence

$$\sum_{n \le N} d(n) e^{(2\sqrt{nx})} \ll \sum_{m \le \sqrt{N}} \left\{ (mx)^{\frac{1}{2}x} (N/m)^{\lambda - \frac{1}{2}x} + (mx)^{\frac{1}{2}x} N^{\frac{1}{2}\lambda - \frac{1}{4}x} \right\}$$

$$\ll x^{\frac{1}{2}x} (N^{\lambda - \frac{1}{2}x} N^{\frac{1}{2}x - \frac{1}{2}\lambda + \frac{1}{2}} + N^{\frac{1}{2}\lambda - \frac{1}{2}x} N^{\frac{1}{4}x + \frac{1}{2}}) \ll x^{\frac{1}{2}x} N^{\frac{1}{2}(1+\lambda)}.$$

Partial summation gives

$$\sum_{n \le N} d(n) n^{-3/4} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) \ll x^{1/2x} N^{1/2\lambda - 1/4x},$$

and from (2.107) it follows that

$$\Delta(x) \ll x^{1/4+1/2x} N^{1/2\lambda-1/4} + x^{1/2+\epsilon} N^{-1/2} \ll x^{(x+\lambda)/(2\lambda+1)+\epsilon} \tag{2.108}$$

with the choice

$$N = x^{(1-2x)/(2\lambda+1)}.$$

The same approach can be used to estimate $E(T)$, by considering separately even and odd n to get rid of the factor $(-1)^n$ in (2.104). We therefore obtain the bound

$$E(T) \ll T^{(x+\lambda)/(2\lambda+1)+\epsilon}. \tag{2.109}$$

81 But if (x_0, λ_0) is exponent pair, then so is also

$$(x, \lambda) = B(x_0, \lambda_0) = \left(\lambda_0 - \frac{1}{2}, x_0 + \frac{1}{2}\right).$$

This is the so-called B-process (Poisson step) in the theory of exponent pairs. Then

$$\frac{x+\lambda}{2\lambda+1} = \frac{x_0+\lambda_0}{2(x_0+1)},$$

hence we also have

$$E(T) \ll T^{(x+\lambda)/(2x+2)+\epsilon}, \tag{2.110}$$

where (x, λ) is any exponent pair obtained after applying the B-process. In (2.109) the exponent of T is less than $1/3$ if $3x + \lambda < 1$, i.e. in this case (2.109) improves (2.80). We remark that we shall also consider the estimation of $E(T)$ in Section 4.5. There we shall obtain (2.110) in another way, first from a general approach to even moments, and then from an explicit formula of R. Balasubramanian on $E(T)$. The latter gives an estimate which is slightly sharper than (2.110), namely

$$E(T) \ll T^{(x+\lambda)/(2x+2)}(\log T)^2.$$

Nevertheless, it is worth noting that upper bounds for $E(T)$ such as (2.109) or (2.110), which involve the explicit use of one-dimensional exponent pairs, are not likely to produce the sharpest known upper bound for $E(T)$, which we state without proof as

Theorem 2.9.
$$E(T) \ll T^{7/22+\epsilon}. \tag{2.111}$$

This is a recent result of D.R. Heath-Brown ad M.N. Huxley [66], which is analogous to the well-known bound $\Delta(x) \ll x^{7/22+\epsilon}$ of H. Iwaniec and C.J. Mozzochi [82] (in both cases "ϵ" can be replaced by a suitable log-power). Thus the analogy between $E(T)$ and $\Delta(x)$ holds in this case in the sense that at best known exponents in both problems are $7/22 = 0.3181818\cdots$, and as we shall see in Chapter 3, analogous $\Omega_{\pm}$-results also hold in both problems.

The proof of Theorem 2.9 is quite involved, and its presentation would lead us too much astray. It rests on the application of the Bombieri-Iwaniec method for the estimation of exponential sums (see Notes for Chapter 1), applied to the sum **82**

$$S = \sum_{h=H}^{2H-1} \sum_{m=M+2H-1}^{2M-2H} e\left(TF\left(\frac{m+h}{M}\right) - TF\left(\frac{m-h}{M}\right)\right).$$

As is common in such problems, $F(x)$ is a real-valued function with sufficiently many derivatives which satisfy certain non-vanishing conditions for $1 \le x \le 2$. Heath-Brown and Huxley succeeded in treating S by the ideas of Iwaniec-Mozzochi. They obtained a general mean value bound which gives

$$\int_T^{T+\Delta} |\zeta(\frac{1}{2}+it)|^2 dt \ll \Delta \log T \quad (T^{7/22} \log^{45/22} T \ll \Delta \le T).$$

This bound implies

$$E(T+\Delta) - E(T) \ll \Delta \log T + T^{7/22} \log^{67/22} T, \tag{2.112}$$

and they obtain (2.111) in an even slightly sharper form, namely

$$E(T) \ll T^{7/22} \log^{111/22} T,$$

by the use of an averaging technique, similar to the one used in Section 5.6 for the fourth moment of $\zeta(\frac{1}{2}+it)$.

For the rest of this section we shall suppose that $1/2 < \sigma < 1$ is fixed, and we shall consider the estimation of

$$E_\sigma(T) = \int\limits_0^T |\zeta(\sigma+it)|^2 dt - \zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)}{1-\sigma}\sin(\pi\sigma)T^{2-2\sigma}.$$

To this end we shall introduce also the function

$$I(T,\sigma;\Delta) := (\Delta\sqrt{\pi})^{-1}\int\limits_{-\infty}^{\infty}|\zeta(\sigma+iT+it)|^2 e^{-t^2/\Delta^2}dt \tag{2.113}$$

and

$$E(T,\sigma;\Delta) := I(T,\sigma;\Delta) - \zeta(2\sigma) - 2\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)T^{1-2\sigma}, \tag{2.114}$$

where $0 < \Delta \leq T/\log T$. Since

$$\lim_{\Delta\to o+} I(T,\sigma;\Delta) = |\zeta(\sigma+iT)|^2,$$

83 it follows that $E(T,\sigma;\Delta)$ may be thought of as a sort of a derivative of $E_\sigma(T)$. The connection between $E(T,\sigma;\Delta)$ and $E_\sigma(T)$ will be made in the following lemmas.

Lemma 2.4. *For $1/2 < \sigma < 1$ fixed, $T^\epsilon \leq \Delta \leq T^{1-\epsilon}$ and $L = 100\sqrt{\log T}$ we have uniformly*

$$\int\limits_T^{2T}|\zeta(\sigma+it)|^2 dt \leq \int\limits_{T-L\Delta}^{2T+L\Delta} I(t,\sigma;\Delta)dt + O(1)$$

and

$$\int\limits_T^{2T}|\zeta(\sigma+it)|^2 dt \geq \int\limits_{T+L\Delta}^{2T-L\Delta} I(t,\sigma;\Delta)dt + O(1).$$

Proof of Lemma 2.4. This lemma is similar to Lemma 5.1, and the proof is similar. We have

$$\int_{T-L\Delta}^{2T+L\Delta} I(t,\sigma;\Delta)dt = \int_{-\infty}^{\infty} |\zeta(\sigma+iu)|^2 \left((\Delta\sqrt{\pi})^{-1}\int_{T-L\Delta}^{2T+L\Delta} e^{-(t-u)^2/\Delta^2}dt\right)du$$
$$\geq \int_{T}^{2T} |\zeta(\sigma+iu)|^2 \left((\Delta\sqrt{\pi})^{-1}\int_{T-L\Delta}^{2T+L\Delta} e^{-(t-u)^2.\Delta^2}dt\right)du.$$

But for $T \leq u \leq 2T$ we have, on setting $t-u=\Delta v$,

$$(\Delta\sqrt{\pi})^{-1}\int_{T-L\Delta}^{2T+L\Delta} e^{-(t-u)^2/\Delta^2}dt = \pi^{-1/2}\int_{(T-u)/\Delta-L}^{(2T-u)/\Delta+L} e^{-v^2}dv$$
$$= \pi^{-1/2}\int_{-\infty}^{\infty} e^{-v^2}dv + O\left(\int_{100\sqrt{\log T}}^{\infty} e^{-v^2}dv + \int_{-\infty}^{-100\sqrt{\log T}} e^{-v^2}dv\right)$$
$$= 1 + O(T^{-10}).$$

Therefore

$$(1+O(T^{-10}))\int_{T}^{2T} |\zeta(\sigma+it)|^2 dt \leq \int_{T-L\Delta}^{2T+L\Delta} I(t,\sigma;\Delta)dt,$$

which proves the upper bound inequality in the lemma. The lower bound 84
is proved analogously.

Lemma 2.5. *For $1/2<\sigma<1$ fixed, $T^\epsilon \leq \Delta \leq T^{1-\epsilon}$ and $L = 100\sqrt{\log T}$ we have uniformly*

$$|E_\sigma(2T)-E_\sigma(T)| \leq \left|\int_{T-L\Delta}^{2T+L\Delta} E(t,\sigma;\Delta)dt\right| + \left|\int_{T+L\Delta}^{2T-L\Delta} E(t,\sigma;\Delta)dt\right| + O(L\Delta).$$

Proof of Lemma 2.5. This follows from Lemma 2.4 when we note that the integrals appearing in Lemma 2.4 on the left-hand sides are equal to

$$E_\sigma(2T) - E_\sigma(T) + \left\{\zeta(2\sigma) + \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)t^{2-2\sigma}}{1-\sigma}\right\}\Bigg|_T^{2T},$$

and then use (2.114) and simplify.

Lemma 2.6. *For* $1/2 < \sigma < 1$ *fixed,* $T^\epsilon \leq \Delta \leq T^{1-\epsilon}$ *we have uniformly*

$$\int\limits_0^T E(t,\sigma;\Delta)dt = O(\log T) + 2^{\sigma-1}\left(\frac{T}{\pi}\right)^{1/2-\sigma}\sum_{n=1}^\infty (-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\times$$
$$\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-1}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}\times$$
$$\exp\left(-\left(\Delta ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^2\right)\cos(f(T,n)),$$

where $f(T,n)$ *is in Theorem 2.1.*

Proof of Lemma 2.6. Follows from the method of Y. Motohashi [130], which is analogous to the method that he used in proving Theorem 5.1, the fundamental result on the fourth moment. However, the case of the fourth moment is much more difficult than the case of the mean square, hence only the salient points of the proof of lemma 2.6 will be given.

Consider, for $\operatorname{Re} u, \operatorname{Re} v > 1$, $0 < \Delta \leq T/\log T$,

$$I_o(u,v;\Delta) := (\Delta\sqrt{\pi})^{-1}\int\limits_{-\infty}^{\infty}\zeta(u+it)\zeta(v-it)e^{-t^2/\Delta^2}dt.$$

85 Then $I(u,v;\Delta)$ can be continued meromorphically to $\mathbb{C}^2$, and by shifting appropriately the line of integration one has, for $\operatorname{Re} u, \operatorname{Re} v < 1$,

$$I_0(u,v;\Delta) = (\Delta\sqrt{\pi})^{-1}\int\limits_{-\infty}^{\infty}\zeta(u+it)\zeta(v-it)e^{-t^2/\Delta^2}dt + \frac{2\sqrt{\pi}}{\Delta}\zeta(u+v-1)$$

$$\left\{\exp\left(\left(\frac{u-1}{\Delta}\right)^2\right) + \exp\left(\left(\frac{v-1}{\Delta}\right)^2\right)\right\}.$$

On the other hand, in the region of absolute convergence we have

$$I_0(u, v; \Delta) = \zeta(u+v) + I^{(1)}(u, v; \Delta) + I^{(1)}(v, u; \Delta),$$

where

$$I^{(1)}(u, v; \Delta) := \sum_{m,n=1}^{\infty} m^{-u}(m+n)^{-v} \exp\left(-\frac{\Delta^2}{4}\log^2\left(1+\frac{n}{m}\right)\right).$$

For $\operatorname{Re} v > \operatorname{Re} s > 0$ we define

$$M(s, v; \Delta) := (\Delta\sqrt{\pi})^{-1}\Gamma(s)\int_{-\infty}^{\infty} \frac{\Gamma(v+it-s)}{\Gamma(v+it)} e^{-t^2/\Delta^2} dt,$$

so that for $x > 0$, $a > 0$ we have

$$(1+x)^{-v}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} M(s, v; \Delta) x^{-s} ds.$$

This gives, for $\operatorname{Re}(u+v) > a+1 > 2$,

$$I^{(1)}(u, v; \Delta) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \zeta(s)\zeta(u+v-s)M(s, v; \Delta) ds.$$

If (u, v) lies in the region $b > \operatorname{Re}(u+v) > a+1 > 2$, then we move the line of integration to $\operatorname{Re} s = b$. In this way we obtain

$$I^{(1)}(u, v; \Delta) = \zeta(u+v-1)M(u+v-1, v; \Delta) + \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \zeta(s)\zeta(u+v-s)M(s, v; \Delta) ds. \tag{2.115}$$

From (2.115) we have a meromorphic continuation of $I^{(1)}$ to the region $b > \mathrm{Re}(u+v)$. Then we use the functional equation for $\zeta(s)$ and obtain from (2.115), for any $\sigma < 1$ and $b > 2$,

$$\begin{aligned}(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}|\zeta(\sigma+iT+it)|^2 e^{-t^2/\Delta^2}dt &= \zeta(2\sigma)+\zeta(2\sigma-1)\times\\ &\{M(2\sigma-1,\sigma+iT;\Delta)+M(2\sigma-1,\sigma-iT;\Delta)\}\\ &-2i(2\pi)^{2\sigma-2}\sum_{n=1}^{\infty}\sigma_{2\sigma-1}(n)\int_{b-i\infty}^{b+i\infty}(2\pi n)^{-s}\sin\left(\frac{2\sigma\pi-\pi s}{2}\right)\times\\ &\Gamma(s+1-2\sigma)\{M(s,\sigma+iT\Delta)+M(s\sigma-iT;\Delta)\}\,ds\\ &-4\pi(\Delta\sqrt{\pi})^{-1}\zeta(2\sigma-1)\,\mathrm{Re}\left\{\exp\left(\frac{\sigma-1+iT}{\Delta}\right)^2\right\}. \qquad (2.116)\end{aligned}$$

86 In (2.116) the sine is written by means of the exponential function, and the resulting integrals are simplified by changing the order of integration. In this way we obtain from (2.116), for $T^{\epsilon} \le \Delta \le T^{1-\epsilon}$, $T \le t \le 2T$, and I defined by (2.113),

$$\begin{aligned}I(t,\sigma;\Delta) &= \zeta(2\sigma)+2\zeta(2\sigma-1)\Gamma(2\sigma-1)(\Delta\sqrt{\pi})^{-1}\\ &\mathrm{Re}\left\{\int_{-\infty}^{\infty}\frac{\Gamma(1-\sigma+it+iu)}{\Gamma(\sigma+it+iu)}e^{-u^2/\Delta^2}du\right\}\\ &+4\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)\int_0^{\infty}y^{-\sigma}(1+y)^{-\sigma}\cos(2\pi ny)\times\\ &\cos\left(t\log\left(1+\frac{1}{y}\right)\right)\exp\left(-\frac{\Delta^2}{4}\log^2\left(1+\frac{1}{y}\right)\right)dy\\ &+O\left(\exp\left(-\frac{1}{2}\log^2 T\right)\right) \qquad (2.117)\end{aligned}$$

By Stirling's formula we find that, for $T \le t \le 2T$,

$$(\Delta\sqrt{\pi})^{-1}\,\mathrm{Re}\left\{\int\limits_{-\infty}^{\infty}\frac{\Gamma(1-\sigma+it+iu)}{\Gamma(\sigma+it+iu)}e^{-u^2/\Delta^2}du\right\}t^{1-2\sigma}\sin(\pi\sigma)+O\left(\frac{1}{T}\right). \tag{2.118}$$

On inserting (2.118) in (2.117) and integrating it follows that

$$\int\limits_{T}^{2T}E(t,\sigma;\Delta)dt = O(1)+4\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(\sigma) \tag{2.119}$$

$$\left\{\int\limits_{0}^{\infty}\frac{\cos(2\pi ny)\sin(t\log(1+1/y))}{y^{\sigma}(1+y)^{\sigma}\log(1+1/y)}\exp\left(-\frac{\Delta^2}{4}\log^2(1+1/y)\right)dy\right\}\Bigg|_{T}^{2T}.$$

It remains to evaluate the exponential integrals in (2.119). This can **87**
be done directly by the saddle-point, as was done by Y. Motohashi [130], or one can use Atkinson's Theorem 2.3. In fact, save for the factor $\exp\left(-\frac{\Delta^2}{4}\cdots\right)$, the integrals in question are of the same type as those considered by Lemma 2.2. By either method one obtains Lemma2.6, replacing T by $T2^{-j}(j=1,2,\ldots)$ and adding all the results.

As we have

$$ar\sinh x = x-\frac{x^3}{6}+O(|x|^5)\quad(|x|\le 1),$$

it follows that we may truncate the series in Lemma 2.6 at $n = 100T\Delta^{-2}\log T$ with an error which is $O(1)$, provided that $\Delta \ll T^{1/2}$. Hence combining Lemma 2.5 and Lemma 2.6 we obtain

Theorem 2.10. *For $1/2<\sigma<1$ fixed, $T^{\epsilon}\le\Delta\le T^{1/2}$ and $f(T,n)$ as in Theorem 2.1, we have uniformly*

$$E_{\sigma}(2T)-E_{\sigma}(T) \ll \Delta\sqrt{\log T}+\sup_{\frac{2T}{3}\le\tau\le 3T}\tau^{1/2-\sigma}\times$$

$$\Bigg|\sum_{n\le 100T\Delta^{-2}\log T}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(ar\sinh\sqrt{\frac{\pi n}{2\tau}}\right)^{-1}\times$$

$$\left(\frac{\tau}{2\pi n} + \frac{1}{4}\right)^{-1/4} \exp\left(-\left(\Delta ar \sinh \sqrt{\frac{\pi n}{2\tau}}\right)^2\right) \cos(f(\tau, n))\Bigg|. \quad (2.120)$$

This result may be compared to Theorem 2.9. The parameter H in (2.105) corresponds to Δ in (2.120). The latter has two distinct advantages: the range for σ is the whole interval $1/2 < \sigma < 1$, and in the range for n there is no T^{ϵ}, but only $\log T$ present.

To obtain bounds for $E_\sigma(T)$ from Theorem 2.10, we use first the fact that, for $\sigma > 1/2$ and $x \to \infty$,

$$\sum_{n \le x} \sigma_{1-2\sigma}(n) \sim \zeta(2\sigma)x.$$

Hence by partial summation and trivial estimation one obtains from Theorem 2.10

$$E_\sigma(2T) - E_\sigma(T) \ll T^{3/4-\sigma} \sum_{n \le 100T\Delta^{-2} \log T} \sigma_{1-2\sigma}(n) n^{\sigma-5/4}$$
$$+ \Delta(\log T)^{1/2} \ll T^{1/2}\Delta^{-2\sigma+1/2}(\log T)^{\sigma-1/2} + \Delta(\log T)^{1/2}.$$

88 Choosing

$$\Delta = T^{1/(1+4\sigma)}(\log T)^{(4\sigma-3)/(8\sigma+2)}$$

we obtain

$$E_\sigma(T) \ll T^{1/(1+4\sigma)}(\log T)^{(4\sigma-1)/(4\sigma+1)} \quad (1/2 < \sigma < 1). \qquad (2.121)$$

This bound obviously sharpens and extends (2.106). Note that, for

$$\sigma > \frac{3 + \sqrt{17}}{8} = 0.890388\ldots, \qquad (2.122)$$

we have

$$\frac{1}{1 + 4\sigma} > 2 - 2\sigma,$$

so that in this range the bound (2.121) for $E_\sigma(T)$ becomes larger than the second main term (see (2.2)) in the asymptotic formula for

$$\int_0^T |\zeta(\sigma + it)|^2 dt.$$

Now we shall use the theory of exponent pairs to estimate $E_\sigma(T)$, which will yield non-trivial bounds in the whole range $1/2 < \sigma < 1$, and in particular when σ satisfies (2.122). To this end first note that the factor $(-1)^n$ appearing in the sum in (2.120) is harmless, since one may consider separately subsums over even and odd integers. Thus, with a slight abuse of notation, this factor may be discarded. The functions $ar\sinh(\pi n/2T)^{1/2}$ and $\left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4}$ may be approximated by the first few terms coming from the Taylor expansion. The factor $\exp(-(\Delta\ldots)^2)$ lies between 0 and 1, and since it is monotonically decreasing as a function of n, it may be removed from the sum by partial summation. Thus the problem is reduced to the estimation

$$S(T,N) := T^{3/4-\sigma}\left|\sum_{n\le N}\sigma_{1-2\sigma}(n)n^{\sigma-5/4}\exp(if\ (T,n))\right|,$$

where $N \ll T\Delta^{-2}\log T$. We estimate first

$$\sum(T,N) := \sum_{n\le N}\sigma_{1-2\sigma}(n)\exp(if\ (T,n)).$$

Write **89**

$$\begin{aligned}\sum(T,N) &= \sum_{km\le N}k^{1-2\sigma}\exp(\text{ if }(T,km))\\ &= 2\sum_{k\le N^{1/2}}k^{1-2\sigma}\sum_{m\le N/k}\exp(if\ (T,km))\\ &\quad-\sum_{k\le N^{1/2}}k^{1-2\sigma}\sum_{m\le N^{1/2}}\exp(if\ (T,km)).\end{aligned}$$

The sums over m are split into subsums of the form

$$S_M := \sum_{M<m\le M'\le 2M}\exp(if\ (T,km)) \qquad (1\le M\le N).$$

Note that

$$\frac{\partial f(T,km)}{\partial m} \sim \left(\frac{2\pi kT}{m}\right)^{\frac{1}{2}} > 1,$$

and that higher derivatives with respect to m may be also easily evaluated. Their form is such that S_M may be estimated by the theory of exponent pairs as

$$S_M \ll (Tk)^{\frac{1}{2}x} M^{\lambda - \frac{1}{2}x},$$

and consequently

$$\begin{aligned}
\sum(T,N) &\ll \sum_{k \le N^{\frac{1}{2}}} k^{1-2\sigma} (Tk)^{\frac{1}{2}} (N/k)^{\lambda - \frac{1}{2}x} + \sum_{k \le N^{\frac{1}{2}}} k^{1-2\sigma} (Tk)^{\frac{1}{2}x} N^{\frac{1}{2}\lambda - \frac{1}{4}x} \\
&\ll T^{\frac{1}{2}x} N^{\lambda - \frac{1}{2}x} \sum_{k \le N^{\frac{1}{2}}} k^{1-2\sigma+x-\lambda} + T^{\frac{1}{2}x} N^{\frac{1}{2}\lambda - \frac{1}{4}x} \sum_{k \le N^{\frac{1}{2}}} k^{1-2\sigma+\frac{1}{2}x} \\
&\ll T^{\frac{1}{2}x} N^{1-\sigma+\frac{1}{2}\lambda}
\end{aligned}$$

if

$$\lambda - x < 2 - 2\sigma, \tag{2.123}$$

and if equality holds in (2.123) we get an extra log-factor. Partial summation gives then

$$S(T,N) \ll T^{\frac{3}{4}-\sigma+\frac{1}{2}x} N^{\frac{1}{2}\lambda - \frac{1}{4}} \ll T^{\frac{3}{4}-\sigma+\frac{1}{2}x} (T\Delta^{-2} \log T)^{\frac{1}{2}\lambda - \frac{1}{4}},$$

and consequently we obtain

$$E_\sigma(2T) - E_\sigma(T) \ll T^{1/2(x+\lambda+1)-\sigma} \Delta^{1/2-\lambda} (\log T)^{1/2\lambda - 1/4} + \Delta (\log T)^{1/2}. \tag{2.124}$$

90 If we choose

$$\Delta = T^{\frac{(1-2\sigma)+\chi+\lambda}{2\lambda+1}} (\log T)^{\frac{2\lambda-3}{4\lambda+2}}$$

to equalize the terms on the right-hand side of (2.124), we obtain the following

Theorem 2.11. *If (χ, λ) is an exponent pair and $1/2 < \sigma < 1$ is a fixed number such that $\lambda - x < 2 - 2\sigma$, then*

$$E_\sigma(T) \ll T^{(1-2\sigma+\chi+\lambda)/(2\lambda+1)} (\log T)^{(2\lambda-1)/(2\lambda+1)}. \tag{2.125}$$

By specialising (χ, λ) one can get several interesting estimates from (2.125). For example, by taking $(\chi, \lambda) = (1/2, 1/2)$ it is seen that (2.125) holds for $1/2 < \sigma < 1$, hence we have

Corollary 1. For $1/2 < \sigma < 1$ fixed we have

$$E_\sigma(T) \ll T^{1-\sigma}. \tag{2.126}$$

Note that (2.126) provides a bound which is always of a lower order of magnitude than the main terms, and it improves (2.121) for $3/4 < \sigma < 1$.

Another possibility is to choose (x, λ) such that $\lambda = x + \frac{1}{2}$. Then (2.123) reduces to $\sigma < 3/4$, and the estimate given by (2.125) is optimal for χ minimal. Taking $(\chi, \lambda) = \left(\frac{9}{56} + \epsilon, \frac{37}{56} + \epsilon\right)$, which is a recent exponent pair of Hauley-Watt, we see that even the case $\sigma = \frac{3}{4}$ may be treated, and we obtain

Corollary 2. For $1/2 < \sigma \leq 3/4$ fixed we have

$$E_\sigma(T) \ll T^{(51-56\sigma)/65+\epsilon}, \tag{2.127}$$

and in particular

$$E_{3/4}(T) \ll T^{9/65+\epsilon}. \tag{2.128}$$

Note that (2.121) gives only $E_{3/4}(T) \ll T^{1/4}(\log T)^{1/2}$, which is much weaker than (2.128). There are possibilities to choose various 91
other exponent pairs which, depending on the condition (2.123), will provide various estimates for $E_\sigma(T)$. We list here only one more specific example. Namely, with $(\chi, \lambda) = \left(\frac{11}{30}, \frac{16}{30}\right)$ we see that (2.123) holds for $\sigma \leq 11/12$, and (2.125) yields

Corollary 3. For $\frac{1}{2} < \sigma < \frac{11}{30}$ fixed we have

$$E_\sigma(T) \ll T^{(57-60\sigma)/62}(\log T)^{1/2}$$

and we also have

$$E_{11/12}(T) \ll T^{1/31}(\log T)^{3/2}.$$

Of course, there are possibilities to use the techniques of two-dimensional exponential sums to estimate $\Sigma(T, N)$. These techniques may, at least in some ranges of σ, lead to further small improvements of our results. Also it may be remarked that (2.125) in the limiting case when $\sigma \to 1/2 + 0$ reduces to the bound (2.110) for $E(T)$.

2.8 Asymptotic Mean Square of the Product of the Zeta Function and a Dirichlet Polynomial

The title of this section refers to the asymptotic formula for the integral

$$I(T,A) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|A\left(\frac{1}{2}+it\right)\right|^2 dt,$$

where

$$A(s) = \sum_{m\le M} a(m)m^{-s}$$

is a Dirichlet polynomial. Problems involving the application of $I(T,A)$ with various (complex) coefficients $a(m)$ frequently occur in analytic number theory. Thus it is desirable to have an asymptotic formula for $I(T,A)$, or at least a good upper bound. It is natural to expect that for relatively small M an asymptotic formula for $I(T,A)$ may be derived,
92 and we know that in the trivial case $M = 1$ such a formula exists. R. Balasubramanian et al. [6] established an asymptotic formula in the case when $a(m) \ll_\epsilon m^\epsilon$ and $\log M \ll \log T$. They proved that

$$\begin{aligned} I(T,A) = T \sum_{h,k\le M} \frac{a(h)\overline{a(k)}}{hk}(h,k)\left(\log\frac{T(h,k)^2}{2\pi hk} + 2\gamma - 1\right) \\ + O_\epsilon(T^\epsilon M^2) + O_B(T\log^{-B} T) \end{aligned} \tag{2.129}$$

for any $B > 0$, so that one gets a true asymptotic formula from (2.129) in the range $1 \le M \ll T^{1/2-\epsilon}$. In some special cases, or in the case when some additional conjectures are assumed (e.g. like Hooley's Conjecture R^* for Kloosterman sums), the error term in (2.129) can be improved. The asymptotic formula for $I(T,A)$ can be recovered (by an argument similar to the one used in Sections 5.6 and 5.7) from the asymptotic formula for

$$g(u) := \left(\Delta\sqrt{\pi}\right)^{-1} \int_{-\infty}^{\infty} |\zeta(1/2+it)|^2 |A(1/2+it)|^2 e^{(-t-u)^2/\Delta^2} dt,$$

where $T \leq u \leq 2T$, $\exp(5\sqrt{\log T}) \leq \Delta \leq T/\log T$. The evaluation of $g(u)$ is rather delicate. Putting $s_0 = 1/2 + iu$ one can write

$$g(u) = \left(i\Delta\sqrt{\pi}\right)^{-1} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{(s-s_0)^2/\Delta^2} \zeta^2(s)\chi(1-s)A(s)\overline{A(1-s)}ds$$

and move the line of integration to $\operatorname{Re} s = 1 + \eta (\eta > 0)$. This procedure is difficult, as it involves the evaluation of certain complicated integrals containing the exponential function.

Another approach to the evaluation of $I(T, A)$ is given by Y. Motohashi [3, Part V]. He observed that the argument of Atkinson which leads to the proof of Theorem 2.1 may be generalized to produce the asymptotic formula for $I(T, A)$. His result is

Theorem 2.12. *If* $A(S) = \sum_{m \leq M} a(m)m^{-s}$ *with* $a(m) \ll_\epsilon m^\epsilon$ *and* $\log M \ll$ *logT, then*

$$I(T, A) = T \sum_{h,k \leq M} \frac{a(h)\overline{a(k)}}{hk}(h, k)\left(\log \frac{T(h, k)^2}{2\pi hk} + 2\gamma - 1\right) + O_\epsilon\left(T^{\frac{1}{3}+\epsilon}M^{4/3}\right). \tag{2.130}$$

Proof. Only an outline of the proof will be given, since full details require considerable technicalities. Unlike (2.129), from (2.130) one can obtain the estimate $E(T) \ll_\epsilon T^{1/3+\epsilon}$ simply by taking $m = 1$. We have, for $\operatorname{Re} u > 1$, $\operatorname{Re} v > 1$, 93

$$\zeta(u)\zeta(v)A(u)\overline{A(\overline{v})} = \zeta(u + v) \sum_{k,l \leq M} a(k)\overline{a(l)}[k, l]^{-u-v}$$
$$M(u, v) + \overline{M(\overline{v}, \overline{u})},$$

where

$$M(u, v) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\sum_{k|m} a(k)\right) \left(\sum_{l|(m+n)} \overline{a(l)}m^{-u}(m + n)\right)^{-v}.$$

□

This relation is an easy generalization of (2.7) and (2.8) in the initial stage of the proof of Atkinson's formula. To obtain analytic continuation of $M(u, v)$ to the region $\operatorname{Re} u < 1$ we use the formula

$$r^{-s} = \frac{1}{\Gamma(s)} \int\limits_0^\infty y^{s-1} e^{-ry} dy \qquad (r > 0, \operatorname{Re} s > 0)$$

to replace summation over m and n by a double integral over $(0, \infty) \times (0, \infty)$. Then we replace this integral by integrals over the contour $\mathscr{C}$. This is the loop contour which starts at infinity, proceeds along the positive real axis to $\delta (0 < \delta < 1/2)$, describes a circle of radius δ counterclockwise around the origin and returns to infinity along the positive axis. This procedure leads to

$$M(u, v) = \Gamma(u + v + 1) \frac{\Gamma(1 - u)}{\Gamma(v)} \zeta(u + v - 1) \sum_{k,l} \frac{(k, l)^{1-u-v}}{[k, l]} a(k)\overline{a(l)} + g(u, v; A),$$

where

$$g(u, v; A) := \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi iu} - 1)(e^{2\pi iv} - 1)} \sum_{k,l} \frac{a(k)\overline{a(l)}}{l} \sum_{f=1}^{l}$$
$$\int\limits_{\mathscr{C}} \frac{y^{v-1}}{e^{y-2\pi if/l} - 1} \int\limits_{\mathscr{C}} x^{u-1} \left(\frac{1}{e^{kx+ky-2\pi ifk/l} - 1} - \frac{\delta(f)}{kx + ky} \right) dx\, dy.$$

94 Here $\delta(f) = 1$ if $l \mid kf$ and zero otherwise, and the double integral is absolutely convergent for $\operatorname{Re} u < 1$. Collecting the above estimates it follows that, for $0 < \operatorname{Re} u < 1$ and $u + v \to 1$, we have

$$\zeta(u)\zeta(1 - u)A(u)\overline{A(1 - \overline{u})} = \sum_{k,l} \frac{a(k\overline{a(l)})}{[k, l]}$$
$$\left\{ \frac{1}{2} \left(\frac{\Gamma'(u)}{\Gamma(u)} + \frac{\Gamma'(1 - u)}{\Gamma(1 - u)} \right) + \log \frac{(k, l)^2}{kl} + 2\gamma - 2\log(2\pi) \right\}$$
$$+ g(u, l - u; A) + g(1 - u, u; \overline{A}).$$

Setting $k^* = k/(k,l)$, $l^* = 1/(k,l)$, $\overline{k^*k} \equiv 1 \mod l^*$, one transforms g into

$$g(u, l-u; A) = \sum_{k,l} \frac{a(k)\overline{a(l)}}{[k,l]} \tag{2.131}$$

$$\sum_{n\neq 0} d(|n|) \exp(2\pi i \overline{k^*}/l^*) \int_0^\infty \exp((2\pi i n y)/(k^* l^*)) y^{-u}(1+y)^{u-1} dy.$$

Further transformations of (2.130) may be carried out by the analogue of the Voronoi formula for the function

$$\Delta(x, \overline{k^*}/l^*) : = \sum_{n\leq x} d(n) \exp\left(2\pi i \frac{\overline{k^*}}{l^*} n\right)$$
$$- \frac{x}{l^*}(\log x + 2\gamma - 1 - 2\log l^*) - D\left(0, \frac{\overline{k^*}}{l^*}\right), \tag{2.132}$$

where

$$D\left(s, \frac{\overline{k^*}}{l^*}\right) := \sum_{n=1}^\infty d(n) \exp\left(2\pi i \frac{\overline{k^*}}{l^*} n\right) n^{-s} \qquad (\mathrm{Re}\, s > 1).$$

Formulas for (2.132) are given by M. Jutila [95], and their use makes it possible to estimate the error term in (2.130). If the function in question is denoted by $E(T, A)$, then the result of Theorem 2.12 follows by using an averaging technique, either the analogue of (2.13), or by considering

$$\int_{-\infty}^{\infty} E(T) + t, A) e^{-(t/G)^2} dt,$$

the suitable choice for G being $G = T^{1/3+\epsilon} M^{4/3}$. Note that the error term in (2.130) does not exceed the error term in (2.129). 95

Notes For Chapter 2

The definition of $E_\sigma(T)$ in (2.2) differs from K. Matsumoto's definition in [115] by a factor of 2. This is because Matsumoto considers $\int\limits_{-T}^{T}|\zeta(\sigma+it)|^2dt$, but I thought it more appropriate to define $E_\sigma(T)$ in such a way that the limiting formula (2.3) holds. I have followed Matsumoto in the use of the function $\sigma_{1-2\sigma}(n)$ in (2.6) and in the sequel. This notation is perhaps a little awkward because of the double appearance of "sigma", but I hope this will cause no confusion.

In view of (2.4) and the fact that $G(T) = O(T^{3/4})$ (see (3.1) and Lemma 3.2) perhaps it would be more consistent to define

$$E(T) = \int\limits_0^T |\zeta(1/2+it)|^2dt - T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right) - \pi,$$

which would be stressing the analogy between $E(T)$ and $\Delta(x)$. However, I prefer to use the standard notation introduced in the fundamental paper of F.V. Atkinson [4] (see also his work [5] which contains the ideas used in [4]). Moreover, one defines often in the literature $\Delta(x)$ not by (2.4) but as

$$\Delta(x) = \sum_{n\le x} d(n) \ - \ x(\log x + 2\gamma - 1),$$

which is then more in true with the definition of $E(T)$ is extensively studied by M. Jutila [87], [90] and [91].

Generalizations of Atkinson's method to L-functions were made independently by Y. Motohashi in the series of papers [125], and by T.
96 Meurman [117], [120]. For example the latter notes that, for $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$ one has

$$\sum_{\chi \pmod q} L(u,\chi)L(v,\overline{\chi}) = \varphi(q)(L(u+v,\chi_0) + f_q(u,v) + f_q(v,u)),$$

where χ_0 is the principal character modulo q and

$$f_q(u,v) = \sum_{r=1,(r,1)=1}^{\infty}\sum_{s=1}^{\infty} r^{-u}(r+qs)^{-v},$$

the double series being absolutely convergent for $\operatorname{Re}(u+v) > 2$, $\operatorname{Re} v > 1$. Writing

$$f_q(u,v) = \sum_{k|q} \mu(k) \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} (kr+qs)^{-v},$$

one can apply Poisson's summation formula to the sum over r if $\operatorname{Re} u < -1$, $\operatorname{Re}(u+v) > 2$, and then carry on the analysis in the manner of Atkinson. In this fashion Meurman eventually obtains an asymptotic formula for

$$\begin{aligned} E(q.T) := &\sum_{\chi \pmod q} \int_0^T |L(1/2+it,\chi)|^2 dt \\ &- \frac{\varphi^2(q)}{q} T \left(\log \frac{qT}{2\pi} + \sum_{p|q} \frac{\log p}{p-1} + 2\gamma - 1 \right), \end{aligned}$$

which generalizes Atkinson's formula for $E(T) \equiv E(1,T)$. Y. Motohashi also obtains several interesting results concerning applications of Atkinson's method to L-functions. Thus in Part I of [125] he proves that, if t is a fixed real and q is a prime, then

$$\begin{aligned} (q-1)^{-1} \sum_{\chi \pmod q} |L(1/2+it,\chi)|^2 &= \log \frac{q}{2\pi} + 2\gamma + \operatorname{Re} \frac{\Gamma'}{\Gamma}(1/2+it) \\ + 2q^{-1/2} |\zeta(1/2+it)|^2 \cos(t \log q) &- q^{-1} |\zeta(1/2+it)|^2 + O(q^{-3/2}), \end{aligned}$$

a result that suggests some peculiar relation between the zeros of $\zeta(s)$ and the values of L-functions. Part V of Motohashi [125] is discussed in Section 2.8.

Theorem 2.1 is due to F. V. Atkinson [4], and Theorem 2.2 was proved by K. Matsumoto [115].

In deriving (2.28) we started from $\operatorname{Re} u < 0$, but (2.28) in fact can be seen to hold for $\operatorname{Re} u < 1$.

The classical formula of G.F. Voronoi [162], [163] for $\Delta(x)$ is dis- **97**
cussed also in Chapter 3 of Ivić [75] and by M. Jutila [95]. The asymptotic formula for (2.38) for $\Delta_{1-2\sigma}(x)$ is due to A. Oppenheim [132]. The

proof of the truncated formula (2.40) for $\Delta_{1-2\sigma}(x)$ is analogous to the proof of the formula for $\Delta(x)$ given in Chapter 12 of E.C. Titchmarsh [155]. for properties of Bessel functions the reader is referred to the monograph of G.N. Watson[164].

A more general version of the second mean value theorem for integrals than (2.51) is as follows. Suppose $f(x)$ is monotonic and $g(x)$ integrable on $[a, b]$, $a < b$. Then there exists $a \leq \xi \leq b$ such that

$$\int\limits_a^b f(x)g(x)dx = f(a)\int\limits_a^\xi g(x)dx + f(b)\int\limits_\xi^b g(x)dx.$$

Namely, let $G(x) = \int\limits_a^x g(t)dt$. Integration by parts gives

$$\int\limits_a^b f(x)g(x)dx = \int\limits_a^b f(x)dG(x) = G(b)f(b) - \int\limits_a^b G(x)df(x).$$

Suppose now $f(x)$ is increasing. Then $df(x)$ is a positive Stieltjes measure, and in view of continuity of $G(x)$ the last integral above equals

$$G(\xi)(f(b) - f(a)) \qquad (a \leq \xi \leq b),$$

so that after rearrangement we obtain the result.

Theorem 2.3, Lemmas 2.2 and 2.3 are all from F.V. Atkinson [4]. Proofs of these results may be also found in Chapter 2 of Ivić [75], and results on exponential integrals in the monographs of M. Jutila [95] and E. Krätzel [102]. For this reason and because exponential integrals are not the main topic of this text, I have not given the proofs of these results.

The discussion concerning the simplified version of Atkinson's Theorem 2.3 (with the conditions 2.' and 3.') is due to T. Meurman [116].

98 The mean square formula (2.69) is due to D.R. Heath-Brown [60]. Theorem 2.4 was obtained independently by T. Meurman [118] and Y. Motohashi [126], and the latter work contains a comprehensive account

on his important work on the analogue of the Riemann-Siegel formula for $\zeta^2(s)$. Theorem 2.5 is due to K.-C. Tong [156]. The asymptotic formula (2.71), in the range $1/2 < \sigma < 3/4$, is due to K. Matsumoto [115]. He kindly informed me that, jointly with T. Meurman, he succeeded in improving the error term in (2.71) to $O(T\log^4 T)$.

Concerning the use of (2.101) for the proof of (2.100) (and the sharpening of (2.76) by a log-factor), it may be remarked that E. Preissmann in correspondence informed me that he also obtained a proof of (2.100). In [135] he actually treats in detail the circle problem (i.e., the function $P(x) = \sum_{n\le x} r(n) - \pi x$, where $r(n)$ is the number of representation of n as a sum of two integer squares) by the classical method of E. Landau [108] and (2.101), getting

$$\int_1^X P^2(x)dx = CX^{3/2} + O(X\log^2 X), C = \frac{1}{2\pi^2}\sum_{n=1}^{\infty} r^2(n)n^{-\frac{3}{2}}.$$

The divisor problem is closely related to the circle problem (see Chapter 13 of Ivić [75]), and similar methods may be applied to both. However, the above result for $P(x)$ is not new, since it was proved long ago by I. Kátai [100]. Kátai used the estimate

$$\sum_{n\le x} r(n)r(n+k) \ll x\sum_{d|k}\frac{1}{d} \quad (\text{uniformly for } 1\le k\le x^{1/3}),$$

which he proved by ingenious elementary arguments.

Y. Motohashi remarked that alternatively one can prove

$$\sum_{r\le M, n\le M, n+r\le M}\frac{d(n)d(n+r)}{rn^{3/4}(n+r)^{1/4}} \ll \log^4 x$$

by noting that, for $r \ll x$,

$$\sum_{n\le x} d(n)d(n+r) \ll \sigma_{-1}(r)x\log^2 x,$$

which follows e.g. from a theorem of P. Shiu [151] on multiplicative functions. **99**

Theorem 2.7 corresponds to Theorem 15.5 of A. Ivić [75], which is a result of M. Jutila [87]. Theorem 2.8 is given by K. Matsumoto [115].

R. Balasubramanian's formula [6] for $E(T)$ will be discussed in Chapter 4. There it will be shown how a smoothing technique, combined with the method that gives (2.110), leads to the estimate

$$E(T) \ll T^{(x+\lambda)/(2x+2)} \log^2 T.$$

This result is superseded by a bound of D.R. Heath-Brown and M.N. Huxley [64], contained in Theorem 2.9.

In [130] Y. Motohashi uses the argument briefly described in the proof of Lemma 2.6 to prove, for $1/2 < \sigma < 1$ fixed and $T^\epsilon \leq \Delta \leq T^{1-\epsilon}$,

$$\begin{aligned} E(T,\sigma;\Delta) = O\left(\Delta^2 T^{-1-2\sigma}\right) + O\left(\Delta^{1/2-2\sigma} T^{-1/2\log^5 T}\right) + \\ + 2^\sigma \left(\frac{T}{\pi}\right)^{1/2-\sigma} \sum_{n=1}^{\infty} (-1)^{n-1} \sigma_{1-2\sigma}(n) n^{\sigma-1} \left(\frac{T}{2\pi n} + \frac{1}{4}\right)^{-1/4} \\ \sin f(T,n) \exp\left(-\left(\Delta\, ar \sinh \sqrt{\frac{\pi n}{2T}}\right)^2\right) \end{aligned}$$

uniformly in Δ. Motohashi's idea to work with the smoothed integral (2.113) is used in his fundamental works [3, Part VI], [128] on the fourth power moment, which is discussed in Chapter 5. There is the reader will find more on the properties of the function $M(s,v;\Delta)$, which is crucial in establishing (2.116). For details on (2.118), see (3.19), where a similar expression is also evaluated by Stirling's formula.

Proofs of Theorem 2.10 and Theorem 2.11 have not been published before.

As was already remarked in Notes for Chapter 1, M.N. Huxley and N. Watt [71] discovered new exponent pairs. This means that these exponent pairs, one of which is the pair $\left(\frac{9}{56}+\epsilon, \frac{37}{56}+\epsilon\right)$ used in the text, cannot be obtained by a finite number of applications of the so-called $A-$, $B-$processes and convexity to the trivial exponent pair $(x,\lambda) = (0,1)$.

100 For C. Hooley's Conjecture R^*, which is important for the asymptotic formula (2.129) of R. Balasubramanian et. al. [6], see C. Hooley [65].

The Dirichlet series

$$D\left(s, \frac{k}{\ell}\right) = \sum_{n=1}^{\infty} d(n) \exp\left(2\pi i \frac{k}{\ell} n\right) n^{-s} \quad (\operatorname{Re} s > 1),$$

which appears in (2.132) is sometimes called the Estermann zeta - function. This is in honour of T. Estermann, who in [33] studied analytic properties of this function. It will appear again in Chapter 5 in connection with the fourth power moment. For its properties one can also see M. Jutila [95].

Chapter 3
The Integrals of the Error Terms in Mean Square Formulas

3.1 The Formulas for the Integrated Error Terms

This Chapter is in a certain sense a continuation of Chapter 2, where 101
we established explicit formulas for the functions $E(T)$, $E_\sigma(T)$ defined by (2.1) and (2.2), respectively. These functions, which represent the error terms in mean square formulas, contain much information about $\zeta(s)$ on the critical line $\operatorname{Re} s = 1/2$ and $\operatorname{Re} s = \sigma$. This topic was in part discussed at the end of Chapter 2, and further results may be derived from the asymptotic formulas for

$$G(T) := \int_2^T (E(t) - \pi) dt \tag{3.1}$$

and

$$G_\sigma(T) := \int_2^T (E_\sigma(t) - B(\sigma)) dt \quad \left(\frac{1}{2} < \sigma < \frac{3}{4}\right), \tag{3.2}$$

where

$$B(\sigma) := \zeta(2\sigma-1)\Gamma(2\sigma-1)\int\limits_0^\infty \left\{\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} - 2u^{1-2\sigma}\sin(\pi\sigma)\right\} du \tag{3.3}$$
$$+ \frac{\pi(1-2\sigma)\zeta(2-2\sigma)(2\pi)^{2\sigma-1}}{\Gamma(2\sigma)\sin(\pi\sigma)}.$$

Throughout this chapter it will be assumed that σ is fixed and is restricted to the range $1/2 < \sigma < 3/4$. It may be shown that

$$\lim_{\sigma\to 1/2+0} B(\sigma) = \pi, \tag{3.4}$$

so that (3.1) may be thought of as the limiting case of (3.2), which in view of continuity one certainly expects to be true. The lower limit of integration in (3.1) and (3.2) is unimportant, especially in applications.
102 It could be, of course, taken as zero, but a strictly positive lower limit enables one to use asymptotic formulas such as Stirling's for the gamma-function. Our main results are contained in the following theorems.

Theorem 3.1. *Let* $0 < A < A'$ *be any two fixed constants such that* $AT < N < A'T$, $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$, *and let*

$$f(T,n) = 2T ar\sinh\sqrt{\frac{\pi n}{2T}} + \left(2\pi nT + \pi^2 n^2\right)^{1/2} - \frac{\pi}{4},$$
$$g(T,n) = T\log\left(\frac{T}{2\pi n}\right) - T + \frac{\pi}{4},\ ar\sinh x = \log\left(x + \sqrt{x^2+1}\right).$$

Then if $G(T)$ *is defined by* (3.1) *we have*

$$G(T) = 2^{-3/2}\sum_{n\le N}(-1)^n d(n) n^{-1/2}\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-2}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4} \tag{3.5}$$
$$\sin(f(T,n)) - 2 = \sum_{n\le N'} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin(g(T,n)) + O(T^{1/4}).$$

Theorem 3.2. *Let* $1/2 < \sigma < 3/4$ *be fixed. Then with the above notation and* $G_0 - (T)$ *defined by* (3.2) *we have*

$$G_\sigma(T) = 2^{\sigma-2}\left(\frac{\pi}{T}\right)^{\sigma-1/2}\sum_{n\le N}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1} \tag{3.6}$$
$$\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-2}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}\sin(f(T,n))$$
$$-2\left(\frac{2\pi}{T}\right)^{\sigma-1/2}\sum_{n\le N'}\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\frac{T}{2\pi n}\right)^{-2}\sin(g(T,n)) + O(T^{3/4-\sigma}).$$

We remark first that, when $\sigma \to 1/2+0$, the expression on the right-hand side of (3.6) reduces to the corresponding expression in (3.5). Since

$$\frac{\partial f(T,n)}{\partial T} = 2ar\sinh\sqrt{\frac{\pi n}{2T}},\ \frac{\partial g(T,n)}{\partial T} = \log\left(\frac{T}{2\pi n}\right),$$

it is seen that formal differentiation of the sine terms in (3.5) and (3.6) yields the sums in (2.5) and (2.6), respectively. This is natural to expect, 103
because

$$\frac{dG(T)}{dT} = E(T), \quad \frac{dG_\sigma(T)}{dT} = E_\sigma(T).$$

The formulas (3.5) and (3.6) are of intrinsic interest, and in Section (3.2) we shall use them to obtain omega-results for $E(T)$, $E_\sigma(T)$ and $G_\sigma(T)$. Theorem 3.4 brings forth the sharpest $\Omega_\pm$- results for $E(T)$ which correspond to the sharpest known results for $\Delta(x)$. Mean square estimates for $G(T)$ and $G_\sigma(T)$ are also of interest, and they will be discussed in Section 3.4 .

We shall begin now with the proof of Theorem 3.1 and Theorem 3.2. Details will be given only for Theorem 3.1, and then it will be indicated what changes are to be made in proving Theorem 3.2. One applies (3.5) most often in the case when $N = T$, namely

$$G(T) = 2^{-3/2}\sum_{n\le T}(-1)^n d(n)n^{-1/2}\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-2}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4} \tag{3.7}$$
$$\sin(f(T,n)) - 2\sum_{n\le c_0T}d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin(g(T,n)) + O(T^{1/2}),$$

where

$$c_0 = \frac{1}{2\pi} + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{2\pi}}.$$

In proving either (3.5) or (3.7) one would naturally wish to use Atkinson's formula for $E(T)$, but unfortunately it contains the error term $O(\log^2 T)$ which is much too large for this purpose. We shall actually prove (3.7), and then indicate how it can be used to yield the slightly more general result contained in (3.5). We start from the formulas (2.18) and (2.19) with T replaced by t and integrate. We obtain

$$\begin{aligned}\int\limits_T^{2T} E(t)dt &= \int\limits_T^{2T}\int\limits_{-t}^{t} g(1/2+i\sigma, 1/2-i\sigma)d\sigma\, dt + O(1)\\ &= \int\limits_T^{2T}(I_1(t) - I_2(t) + I_3(t) - I_4(t))dt + O(1),\end{aligned}$$

104 where $I_n = I_n(t)$ is as in (2.30) - (2.34), only with T replaced by t and $N = T$. To prove (3.7) i twill suffice to prove

$$\int\limits_T^{2T} E(t)dt = \pi T + H(2T) - H(T) + K(2T) - K(T) + O(T^{1/4}), \quad (3.8)$$

where

$$H(x) : = 2^{-3/2}\sum_{n\le x}(-1)^n d(n)n^{-\frac{1}{2}}\left(\frac{x}{2\pi n} + \frac{1}{4}\right)^{-\frac{1}{4}} \quad (3.9)$$

$$\left(ar\sinh\sqrt{\frac{\pi n}{2x}}\right)^{-2}\sin(f(x,n)),$$

$$H(x) : = -2\sum_{n\le c_0 x} d(n)n^{-1/2}\left(\log\frac{x}{2\pi n}\right)^{-2}\sin(g(x,n)), \quad (3.10)$$

and then to replace T by $T2^{-j}$ and sum over $j = 1, 2, \ldots$ The main term πT in (3.8) comes from $I_3(t)$, while the sums defined by H will appear in the evaluation of $\int\limits_T^{2T} I_1(t)dt$. The integral I_1 was evaluated in Section 2.5 with an error term $O(T^{-1/2})$ which, when integrated, is too large for our purposes. To avoid this difficulty we take advantage of the extra averaging over t via the following lemma.

Lemma 3.1. *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < 1/2$, $a < T/(8\pi k)$, $b \geq T$, $k \geq 1$, $T \geq 1$,*

$$U(t) = \left(\frac{t}{2\pi k} + \frac{1}{4}\right)^{1/2}, V(t) = 2ar\sinh\sqrt{\frac{\pi k}{2t}},$$

$$L(t) = \frac{1}{i}(2k\sqrt{\pi})^{-1} t^{1/2} V^{-\gamma-1}(t) U^{-1/2}(t)\left(U(t) - \frac{1}{2}\right)^{-\alpha}\left(U(t) + \frac{1}{2}\right)^{-\beta}$$
$$\exp\left\{itV(t) + 2\pi kiU(t) - \pi ik + \frac{\pi i}{4}\right\},$$

and

$$J(T) = \int\limits_T^{2T}\int\limits_a^b y^{-\alpha}(1+y)^{-\beta}\left(\log\left(1+\frac{1}{y}\right)\right)^{-\gamma}$$
$$\exp\left\{it\log\left(1+\frac{1}{y}\right) + 2\pi iky\right\} dy\, dt. \tag{3.11}$$

Then uniformly for $|\alpha - 1| \geq \epsilon$, $1 \leq k \leq T + 1$, we have

$$J(T) = L(2T) - L(T) + O(a^{1-\alpha}) + O(Tk^{-1}b^{\gamma-\alpha-\beta})$$
$$+ O\left((T/k)^{1/2(\gamma+1-\alpha-\beta)} T^{-1/4} k^{-5/4}\right). \tag{3.12}$$

A similar result. without $L(2T) - L(T)$, holds for the corresponding **105**
integral with $-k$ in place of k.

This result is a modified version of Lemma 2.2, and likewise also follows from Atkinson's Theorem 2.3. Therein one takes

$$f(z) = \frac{t}{2\pi}\log\frac{1+z}{z}, \quad \Phi(x) = x^{-\alpha}(1+x)^{-\beta}, \quad F(x) = \frac{t}{1+x}, \mu(x) = \frac{x}{2},$$

follows the proof of Atkinson's theorem for the inner integral in (3.11), and then one integrates over t. The contribution of the integrals I_{31} and I_{32} (see p. 63 of Ivić [75]) is contained in the O-term in (3.12), since in our case we find that

$$I_{31} + I_{32} \ll \begin{cases} e^{-ck} & \text{if } k \geq \log^2 T, \\ e^{-(tk)^{1/2}} & \text{if } k \leq \log^2 T. \end{cases}$$

Likewise, the error terms

$$\Phi_a \left(|f'_a + k| + f''^{1/2}_a \right)^{-1}, \Phi_b \left(|f'_b + k| + f''^{1/2}_b \right)^{-1}$$

give after integration the terms $O(a^{1-\alpha})$ and $O(Tk^{-1}b^{\gamma-\alpha-\beta})$ which are present in (3.12). The main contribution comes from I_{32}, only now one has to integrate over t for $T \leq t \leq 2T$. This leads to the same type of integral (the factor $1/i$ is unimportant) at T and $2T$ respectively. The only change is that $\gamma + 1$ appears instead of γ, because of the extra factor $\log(1 + 1/y)$ in the denominator. Hence the main terms will be $L(2T) - L(T)$, and as in Theorem 2.3 the error term is $\Phi_0 \mu_0 F_0^{-3/2}$ with again $\gamma + 1$ replacing γ. This gives the last O-term in (3.12) (see the analogous computation on p 453 of Ivić [75]), and completes the proof of Lemma (3.1).

Now we write

$$\begin{aligned} \int\limits_T^{2T} I_1(t)dt &= 4 \sum_{n \leq T} d(n) \lim_{\alpha \to 1/2+0} \lim_{b \to \infty} \int\limits_T^{2T} \int\limits_0^b \frac{\sin(t \log(1 + 1/y)) \cos(2\pi ny)}{y^\alpha (1 + y)^{1/2} \log(1 + 1/y)} dy\, dt \\ &= 2 \sum_{n \leq T} d(n) \lim_{\alpha \to 1/2+0} \lim_{b \to \infty} \operatorname{Im} \\ &\qquad \left\{ \int\limits_T^{2T} \int\limits_0^b \frac{\exp(it \log(1 + 1/y) + 2\pi iny)}{y^\alpha (1 + y)^{1/2} \log(1 + 1/y)} dy\, dt \right\} + O(T^{1/4}). \end{aligned}$$

106 The first equality above holds because the integral defining $I_1(t)$ converges uniformly at ∞ and 0 for $1/2 \leq \alpha \leq 1 - \epsilon$. The second equality comes from using the case of Lemma 3.1 "$-k$ inplace of k" for the other two integrals coming from $\sin(\ldots)$ and $\cos(\ldots)$ in $I_1(t)$. We evaluate the

double integral above by applying Lemma 3.1 with $\beta = 1/2$, $\gamma = 1$, $a \to 0$. Then we let $b \to \infty$ and $\alpha \to 1/2 + 0$. we obtain

$$\int\limits_T^{2T} I_1(t)dt = H(2T) - H(T) - 2^{-3/2} \sum_{T \le n \le 2T} (-1)^n d(n) n^{-1/2}$$
$$\left(\frac{2T}{2\pi n} + \frac{1}{4}\right)^{-1/4} \left(ar\sinh\sqrt{\frac{\pi n}{4T}}\right)^{-2} \sin(f(2T, n)) + O(T^{1/4}) \tag{3.13}$$

where $H(x)$ is given by (3.9).

Hence forth we set for brevity $X = [T] + \frac{1}{2}$. Note that the contribution of the integral

$$I_2(t) = 4\Delta(X) \int\limits_0^\infty \frac{\sin(t\log(1 + 1/y))\cos(2\pi Xy)}{y^{1/2}(1 + y)^{1/2}\log(1 + 1/y)} dy$$

to $\int\limits_T^{2T} E(t)dt$ is estimated again by Lemma 3.1. Using the weak estimate $\Delta(X) \ll X^{1/3+\epsilon}$ it follows at once that

$$\int\limits_T^{2T} I_2(T)dt \ll T^{\epsilon - 1/6}.$$

We now turn to

$$\begin{aligned} I_3(t) &= -\frac{2}{\pi}(\log X + 2\gamma) \int\limits_0^\infty \frac{\sin(t\log(1 + 1/y))\sin(2\pi Xy)}{y^{3/2}(1 + y)^{1/2}\log\left(1 + \frac{1}{y}\right)} dy \\ &\quad + \frac{1}{\pi i} \int\limits_0^\infty \frac{\sin(2\pi Xy)}{y} dy \int\limits_{1/2 - it}^{1/2 + it} \left(1 + \frac{1}{y}\right)^u u^{-1} du \\ &= -\frac{2}{\pi}(\log X + 2\gamma) I_{31}(t) + \frac{1}{\pi i} I_{32}(t), \end{aligned}$$

say. We have first

$$\int_T^{2T} I_{31}(t)dt = \int_T^{2T}\int_0^{3T}\cdots + \int_T^{2T}\int_{3T}^{\infty}\cdots$$
$$= \int_0^{3T} \frac{\{\cos(T\log(1+1/y)) - \cos(2T\log(1+1/y))\}\sin(2\pi Xy)}{y^{3/2}(1+y)^{1/2}\log^2(1+1/y)}dy + O(T^{-1})$$

on estimating $\int_{3T}^{\infty}\cdots$ as $O(T^{-2})$ by writing the sine terms in $I_{31}(t)$ as exponentials, and applying Lemma 2.1. The remaining integral above is written as

$$\int_0^{3T} = \int_0^{(2X)^{-1}} + \int_{(2X)^{-1}}^{3T} = I' + I'',$$

say. By applying twice the second mean value theorem for integrals it is seen that the part of I' containing $\cos(T\log(1+1/y))$ equals, for some $0 < \eta \le \xi \le (2X)^{-1}$,

$$2\pi X \int_0^{\xi} \frac{\cos(T\log(1+1/y))}{y(1+y)} \cdot \frac{y^{1/2}(1+y)^{1/2}}{\log^2(1+1/y)}dy$$
$$= 2\pi x \frac{\xi^{1/2}(1+\xi)^{1/2}}{\log^2(1+1/\xi)} \int_{\eta}^{\xi} \frac{\cos(T\log(1+1/y))}{y(1+y)}dy$$
$$- 2\pi X \frac{\xi^{1/2}(1+\xi^{1/2})}{\log^2(1+1/\xi)} \left\{-\frac{1}{T}\sin(T\log(1+1/y))\right\}\Bigg|_{\eta}^{\xi} \ll T^{-1/2},$$

since $y^{-1}\sin(2\pi Xy)$ is a monotonically decreasing function of y in $[0,(2X)^{-1}]$, and $y^{1/2}(1+y)^{1/2}\log^{-2}(1+1/y)$ is monotonically increasing.
108 The same reasoning applies to the integral with $\cos(2T\log(1+1/y))$, and $I'' \ll T^{-1/2}$ follows on applying Lemma 2.1. Hence

$$\int_T^{2T} I_{31}(t)dt \ll T^{-1/2}.$$

Next take $I_{32}(t)$ and write

$$I_{32}(t) = \int\limits_0^1 \frac{\sin(2\pi Xy)}{y} dy \int\limits_{1/2-it}^{1/2+it} (1-1/y)^u u^{-1} du$$
$$+ \int_1^\infty \frac{\sin(2\pi Xy)}{y} \int\limits_{1/2-it}^{1/2+it} (1+1/y)^u u^{-1} du = I'_{32}(t) + I''_{32}(t),$$

say. As in the corresponding estimation in the proof of Theorem 2.1 one has $I''_{32}(t) \ll t^{-1} \log t$, which gives

$$\int\limits_T^{2T} I''_{32}(t) dt \ll \log T.$$

In $I''_{32}(t)$ we have $o < y \leq 1$, hence by the residue theorem

$$\int\limits_{1/2-it}^{1/2+it} (1+1/y)^u u^{-1} du = 2\pi i - \left(\int\limits_{1/2+it}^{-\infty+it} + \int\limits_{-\infty-it}^{1/2-it} \right) (1+1/y)^u u^{-1} du.$$

If we use

$$\int\limits_0^1 \frac{\sin(2\pi Xy)}{y} dy = \int\limits_0^{2\pi X} \frac{\sin z}{z} dz = \int\limits_0^\infty - \int_{2\pi X}^\infty = \frac{\pi}{2} + O(T^{-1})$$

and integrate, we obtain

$$\int\limits_T^{2T} I'_{32}(t) dt = \pi^2 i T - \int\limits_T^{2T} \int\limits_0^1 \frac{\sin(2\pi Xy)}{y} \int\limits_{1/2+it}^{-\infty+it} (1+1/y)^u u^{-1} du\, dy\, dt$$
$$- \int_T^{2T} \int\limits_0^1 \frac{\sin(2\pi Xy)}{y} \int\limits_{-\infty-it}^{1/2-it} (1+1/y)^u u^{-1} du\, dy\, dt + O(1).$$

Both triple integrals are estimated similarly, each being $\ll T^{-1/2}$. Namely, changing the order of integration and integrating by parts we have 109

$$\int_T^{2T}\int_{-\infty+it}^{1/2+it}(1+1/y)^u u^{-1}du\,dt = \int_{-\infty}^{1/2}(1+1/y)^{\sigma}\left\{\int_T^{2T}(1+1/y)^{it}\frac{dt}{\sigma+it}\right\}d\sigma$$

$$= \int_{-\infty}^{1/2}\left(1+\frac{1}{y}\right)^{\sigma}\left\{\frac{(1+1/y)^{2iT}}{i(\sigma+2iT)\log(1/1/y)} - \frac{(1+1/y)^{iT}}{i(\sigma+iT)\log(1+1/y)}+\right.$$

$$\left.+\int_T^{2T}\frac{(1+1/y)^{it}}{(\sigma+it)^2\log(1+1/y)}dt\right\}d\sigma \ll T^{-1}\int_{-\infty}^{1/2}\frac{(1+1/y)^{\sigma}d\sigma}{\log(1+1/y)} \ll T^{-1}y^{-1/2}$$

for $0 < y \le 1$, and

$$T^{-1}\int_0^1 |\sin(2\pi Xy)|y^{-3/2}dy \ll T^{-1}\int_0^{X^{-1}} Xy^{-1/2}dy + T^{-1}\int_{X^{-1}}^{\infty} y^{-3/2}dy \ll T^{-1/2}.$$

Therefore combining the preceding estimates we have

$$\int_T^{2T} I_3(t)dt = \pi T + O(\log T).$$

Finally, it remains to deal with the contribution of the integral

$$I_4(t) = -i\int_X^{\infty}\Delta(x)\left(\int_{1/2-it}^{1/2+it}\frac{\partial h(u,x)}{\partial x}du\right)dx$$

to $\int_T^{2T} E(t)$, where as in the proof of Theorem 2.1

$$h(u,x) = 2\int_0^{\infty} y^{-u}(1+y)^{u-1}\cos(2\pi xy)dy$$

$$= 2\int_0^\infty w^{-u}(x+w)^{u-1}\cos(2\pi w)dw.$$

It is easy to see that this integral h is uniformly convergent, and so we can differentiate under the integral sign to get (after changing variables again)

$$\frac{\partial}{\partial x}h(u,x) = \frac{2}{x}(u-1)\int_0^\infty y^{-u}(1+y)^{u-2}\cos(2\pi xy)dy.$$

This integral is absolutely convergent at both endpoints, so we insert 110
it in the definition of $I_4(t)$ to obtain

$$-\int_T^{2T} I_4(t)dt = 2i\int_X^\infty \Delta(x)x^{-1}dx\int_0^\infty y^{-1/2}(1+y)^{-3/2}$$
$$\cos(2\pi xy)dy\int_T^{2T}\int_{1/2-it}^{1/2+it}(u-1)\left(1+\frac{1}{y}\right)^{u-1/2}du\,dt.$$

We can now evaluate explicitly the integrals with respect to u and t. We shall see from subsequent estimates that what remains provides absolute convergence for the integral in x, so that this procedure is justified. We have

$$-\int_T^{2T} I_4(t)dt = 4\int_X^\infty \Delta(x)(x)^{-1}(I(x,T)+\Gamma(x,T)dx, \tag{3.14}$$

where

$$I(x,z) := \int_0^\infty \frac{-z\sin(z\log(1+1/y))\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}\log^2(1+1/y)}dy,$$

$$r(x,T) := \int_0^\infty \frac{\{\cos(T\log(1+1/y)) - \cos(2T\log(1+1/y))\}\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}\log^2(1+1/y)}$$

$$\left\{\frac{1}{2} + \frac{2}{\log(1+1/y)}\right\} dy.$$

Split now

$$r(x,T) = \int_0^\infty = \int_0^{3T} + \int_{3T}^\infty = I' + I'',$$

say. In I'' we write

$$\cos(T\log(1+1/y)) - \cos(2T\log(1/y))$$
$$= 2\sin\left(\frac{3T}{2}\log(1+1/y)\right)\sin\left(\frac{T}{2}\log(1+1/y)\right)$$

and use the second mean value theorem for integrals. Thus we have, for some $c > 3T$,

$$I'' = \frac{3T^2}{2}\left\{\frac{\sin\left(\frac{3T}{2}\log\left(1+\frac{1}{3T}\right)\right)}{\frac{3T}{2}\log\left(1+\frac{1}{3T}\right)} \cdot \frac{\sin\left(\frac{T}{2}\log\left(1+\frac{1}{3T}\right)\right)}{\frac{T}{2}\log\left(1+\frac{1}{3T}\right)}\right\} \times$$
$$\times \int_{3T}^{c} \frac{\cos(2\pi xy)}{y^{1/2}(1+y)^{3/2}}\left\{\frac{1}{2} + \frac{2}{\log(1+1/y)}\right\} dy \ll Tx^{-1},$$

111 since the first expression in curly brackets is $O(1)$, and the above integral is $O(T^{-1}x^{-1})$ on applying Lemma 2.1. Hence using Theorem 2.5 the Cauchy-Schwarz inequality we obtain

$$4\int_X^\infty \Delta(x)x^{-1}I''dx \ll \left(\int_X^\infty \Delta^2(x)x^{-2}dx\right)^{1/2}\left(\int_X^\infty x^{-2}dx\right)^{1/2} \ll T^{1/4}. \tag{3.15}$$

To evaluate I' we use Lemma 2.2 (treating the main terms as an error term) to get the analogue of (3.15) for I'. the integral $I(x,2T) - I(x,T)$ is also evaluated by Lemma 2.2 with $\alpha \to 1/2 + 0, \beta = 3/2, \gamma = 2$. The error terms will be $\ll T^{1/4}$ as in (3.15). The main terms will be

$$\left\{-z(4x)^{-1}\left(\frac{z}{\pi}\right)^{1/2} V^{-2}U^{-1/2}\left(U-\frac{1}{2}\right)^{-1/2}\left(U+\frac{1}{2}\right)^{-3/2}\sin\left(zV + 2\pi xU - \pi x + \frac{\pi}{x}\right)\right\}\Bigg|_T^{2T},$$

where

$$U = \left(\frac{z}{2\pi x} + \frac{1}{4}\right)^{1/2}, \quad V = 2ar\sinh\sqrt{\frac{\pi x}{2z}}.$$

Thus (3.13) becomes

$$-\int_T^{2T} I_4(t)dt = O(T^{1/4}) - \int_Z^\infty \Delta(x)x^{-3/2}$$

$$\left\{\sqrt{2}zV^{-2}U^{-1/2}\left(U+\frac{1}{2}\right)^{-1}\sin\left(zV + 2\pi xU - \pi x + \frac{\pi}{x}\right)\Bigg|_T^{2T}\right\}dx. \tag{3.16}$$

The last integral bears resemblance to the integral for I_4 in Section 2.5. The difference is that instead of V^{-1} we have V^{-2} and sine (at T and $2T$) instead of cosine in (3.16). This difference is not important at all, and after using the Voronoi series expansion for $\Delta(x)$ and changing the variable x to x^2 the above integral may be evaluated by Lemma 2.3. The modification is that, as on p. 454 of Ivić [75], we have $V = 2ar\sinh(x_0(\pi/(2T))^{1/2}) = \log\left(\frac{T}{2\pi n}\right)$; hence if we replace $ar\sinh\left(x\sqrt{\frac{\pi}{2T}}\right)$ by its square in Lemma 2.3 we obtain in the main term the additional factor $2\left(\log\left(\frac{T}{2\pi n}\right)\right)^{-1}$, the error terms remain unchanged. With this remark one can proceed exactly as was done in the evaluation of I_4 in the proof of Atkinson's formula, and the details for this reason may be omitted. We obtain 112

$$-\int_T^{2T} I_4(t)dt = -2\sum_{n<Z} d(n)n^{-1/2}\left(\log\frac{z}{2\pi n}\right)^{-2}\sin(g(z,n))\Bigg|_T^{2T} + O(T^{1/4})$$

$$= K(2T) - K(T) - 2\sum_{N_2'\le n\le N_1'} d(n)n^{-1/2}\left(\log\frac{2T}{2\pi n}\right)^{-2}$$

$$\sin(g(2T,n)) + O(T^{1/4}), \tag{3.17}$$

where as in the proof of Atkinson's formula

$$Z = N'(z,X) = \frac{z}{2\pi} + \frac{X}{2} - \left(\frac{X^2}{4} + \frac{Xz}{2\pi}\right)^{1/2},$$

$N'_j = N'(2T, jT)$, and $K(x)$ is given by (3.10).

Thus except for the extra sums in (3.17) and the expression for $\int_T^{2T} I_1(t)dt$ in (3.13) we are near the end of the proof of (3.8). But the sums in question may be transformed into one another (plus a small error term) by the method of M. Jutila [89]. Indeed, from Jutila's work we obtain (analogously to eq (15.45) of Ivić [75])

$$
\begin{aligned}
&-2\sum_{N'_2\le n\le N'_1} d(n)n^{-1/2}\left(\log\frac{2T}{2\pi n}\right)^{-2}\sin(g(2T,n))\\
&= 2^{-3/2}\sum_{T\le n\le 2T}(-1)^n d(n)n^{-1/2}\left(\frac{2T}{2\pi n}+\frac{1}{4}\right)^{-1/4}\\
&\left(ar\sinh\sqrt{\frac{\pi n}{4T}}\right)^{-2}\sin(f(2T,n)) + O(\log^2 T),
\end{aligned}
$$

113 the difference from (15.45) of Ivić [75] being in $(\log\ldots)^{-2}$ and $(ar\sinh\ldots)^{-2}$, and in $2T$ instead of T. Hence collecting all the expressions for $\int_T^{2T} I_n(t)dt$ $(1 \le n \le 4)$ we obtain (3.8). But applying the same procedure (i.e. (15.45) of Ivić [75]) we obtain without difficulty Theorem 3.1 from (3.8).

We pass now to the proof of Theorem 3.2, basing our discussion on the method of proof of Theorem 2.2, and supposing $1/2 < \sigma < 3/4$ throughout the proof. In the notation of (2.20) we have (with t in place of T)

$$
\begin{aligned}
&\int_0^t |\zeta(\sigma+it)|^2 du = \zeta(2\sigma)t + \zeta(2\sigma-1)\Gamma(2\sigma-1)\\
&\int^t\left(\frac{\Gamma(1-\sigma-iu)}{\gamma(\sigma-iu)}+\frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)}\right)du - i\int_{\sigma-it}^{\sigma+it} g(u,2\sigma-u)du, \qquad (3.18)
\end{aligned}
$$

where $g(u, v)$ is the analytic continuation of the function which is for

$\operatorname{Re} u < 0$, $\operatorname{Re}(u+v) > 2$ given by

$$g(u,v) = 2\sum_{n=1}^{\infty} \sigma_{1-u-v}(n) \int\limits_0^{\infty} y^{-u}(1+y)^{-v}\cos(2\pi ny)dy.$$

Now we use Stirling's formula for the gamma-function in the from $(s) = \sigma + it$, $0 \leq \sigma \leq 1$, $t \geq e$

$$\Gamma(s) = \sqrt{2\pi} t^{\sigma - 1/2} k(\sigma, t) \exp\left\{-\frac{\pi}{2}t + i\left(t\log t - t + \frac{\pi}{4}\left(\sigma - \frac{1}{2}\right)\right)\right\}$$

with

$$k(\sigma,t) = 1 + c_1(\sigma)t^{-1} + \cdots + c_N(\sigma)t^{-N} + O_N(t^{-N-1})$$

for any fixed integer $N \geq 1$, where $c_1(\sigma) = \frac{1}{2}i(\sigma - \sigma^2 - 1/6)$. Therefore $c_1(\sigma) = c_1(1-\sigma)$, and for $u \geq e$ we obtain

$$\begin{aligned}
\frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} &= u^{1-2\sigma}\exp\left(\frac{i\pi}{2}(1-2\sigma)\right)\frac{k(1-\sigma,u)}{k(\sigma,u)} \\
&= u^{1-2\sigma}\exp\left(\frac{i\pi}{2}(1-2\sigma)\right)\cdot(1+m(\sigma,u)), \qquad (3.19) \\
m(\sigma,u) &= d_2(\sigma)u^{-2} + \cdots + d_N(\sigma)u^{-N} + O_N(u^{-N-1}).
\end{aligned}$$

Thus **114**

$$\begin{aligned}
\int_0^t \left(\frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} + \frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)}\right)du &= \int_0^t 2u^{1-2\sigma}\cos(\pi\sigma - 1/2\pi)du \\
+ \int\limits_0^t \left\{\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} - 2u^{1-2\sigma}\cos(\pi\sigma - 1/2\pi)\right\}du
\end{aligned}$$

$$= \frac{t^{2-2\sigma}}{1-\sigma}\sin(\pi\sigma) + \int\limits_0^{\infty}\left\{\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)}\right\}du + O(t^{-2\sigma}),$$

where (3.19) was used. Then we have

$$\zeta(2\sigma-1)\Gamma(2\sigma-1)\int\limits_T^{2T}\int_0^t \left(\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)}\right)du\,dt$$

$$= O(T^{1-2\sigma}) + \int\limits_T^{2T} \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)}{1-\sigma}\sin(\pi\sigma)t^{2-2\sigma}dt + 2\zeta(2\sigma-1)$$

$$\Gamma(2\sigma-1)\int\limits_0^\infty \left\{\operatorname{Re}\frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} - u^{1-2\sigma}\sin(\pi\sigma)\right\}du.$$

Taking into accout (3.18) and the definition of $E_\sigma(T)$ it follows that

$$\int\limits_T^{2T} E_\sigma(t)dt = A(\sigma)T - i\int_T^{2T}\int\limits_{\sigma-it}^{\sigma+it} g(u, 2\sigma-u)du\,dt + O(T^{1-2\sigma}), \quad (3.20)$$

where

$$A(\sigma) = \zeta(2\sigma-1)\Gamma(2\sigma-1)\int\limits_0^\infty \left\{\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \right.$$

$$\left.\frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} - 2u^{1-2\sigma}\sin(\pi u)\right\}du. \quad (3.21)$$

For $E(T)$ we had an analogous formula, only without a term corresponding to $A(\sigma)T$. Therefore it seems natural to expect that

$$\lim_{\sigma\to 1/2+0} A(\sigma) = 0, \quad (3.22)$$

which will indirectly establish (3.4). We write

$$A(\sigma) = \lim_{V\to\infty}\int\limits_0^V \left\{\zeta(2\sigma-1)\Gamma(2\sigma-1)\right. \quad (3.23)$$

$$\left.\left(\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} - 2u^{1-2\sigma}\cos(\pi\sigma - 1/2\pi)\right)\right\}du.$$

115 For a fixed u and $\sigma \to 1/2+0$ the expression in curly brackets equals

$$\left(-\frac{1}{2} - \frac{1}{2}\log(2\pi)(2\sigma-1) + O((2\sigma-1)^2)\right)\left(\frac{1}{2\sigma-1} - \gamma + O((2\sigma-1))\right)\times$$

$$\times\left\{2-(2\sigma-1)\left(\frac{\Gamma'}{\Gamma}(1/2-iu)+\frac{\Gamma'}{\Gamma}(1/2+iu)\right)-2(1+(1-2\sigma)\log u+O((2\sigma-1)^2))\right\},$$

which tends to

$$\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-iu\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+iu\right)-2\log u\right).$$

Because of uniform convergence the integral in (3.23) is

$$\begin{aligned}&\frac{1}{2}\int_0^V\left(\frac{\Gamma'}{\Gamma}(1/2-iu)+\frac{\Gamma'}{\Gamma}(1/2-iu)-2\log u\right)du \qquad (3.24)\\ &=\frac{1}{2i}\log\frac{\Gamma(1/2+iV)}{\Gamma(1/2-iV)}-V\log V+V.\end{aligned}$$

But for $V\geq V_0>0$ Stirling's formula gives

$$\begin{aligned}\Gamma(1/2+iV)&=\sqrt{2\pi}\exp(-1/2\pi V+i(V\log V-V))\cdot(1+O(1/V)),\\ \Gamma(1/2-iV)&=\sqrt{2\pi}\exp(-1/2\pi.V+i(-V\log V+V))\cdot(1+O(1/v)).\end{aligned}$$

Therefore

$$\log\frac{\Gamma(1/2+iV)}{\Gamma(1/2-iV)}=2i(V\log V-V)+O(1/V). \qquad (3.25)$$

Inserting (3.25) in (3.24) and taking the limit in (3.23) we obtain (3.22).

Hence, analogously to the proof of Theorem 3.1, we obtain

$$\begin{aligned}\int_T^{2T}(E_\sigma(t)-A(\sigma))dt&=-i\int_T^{2T}\int_{\sigma-it}^{\sigma+it}g(u,2\sigma-u)du\,dt+O(1)\\ &=-i\int_T^{2T}(G_1-G_2+G_3-G_4)dt+O(1), \qquad (3.26)\end{aligned}$$

where G_n for $1\leq n\leq 4$ is given by (2.45)- (2.48). 116

As in the evaluation of $\int_T^{2T} I_1(t)dt$ in the proof of Theorem 3.2 it will be seen that in the final result the contribution of G_1 will be $2^{\sigma-2}\left(\frac{\pi}{T}\right)^{\sigma-1/2} \sum_{n\le N} \ldots$ in (3.6) plus $O(T^{3/4} - \sigma)$, which is the largest O-term appearing in the estimation of various integrals. Using (2.41), namely

$$\Delta_{1-2\sigma}(x) \ll x^{1/(4\sigma+1)+\epsilon},$$

it follows for sufficiently small ϵ that

$$\int_T^{2T} G_2 dt \ll T^{1/4\sigma+!} - 1/2\epsilon \ll 1.$$

The contribution of G_3 is, however, more involved. We have

$$-i\int_T^{2T} G_3 dt = \frac{\pi(1-2\sigma)\zeta(2-2\sigma)(2\pi)^{2\sigma-1}}{\Gamma(2\sigma)\sin(\pi\sigma)} T + O(\log T). \tag{3.27}$$

Since it will be indicated that he contribution of G_4 will be essentially $-2\left(\frac{2\pi}{T}\right)^{\sigma-1/2} \sum_{n\le N'} \ldots$ in (3.6), it follows from (3.27) that $B(\sigma)$ in (3.2) is indeed given by (3.3). In view of (3.21) we may write

$$B(\sigma) = A(\sigma) + \frac{\pi(1-2\sigma)\zeta(2-2\sigma)(2\pi)^{2\sigma-1}}{\Gamma(2\sigma)\sin(\pi\sigma)},$$

hence taking the limit as $\sigma \to 1/2 + 0$ and using (3.22) we obtain (3.4). To obtain (3.27) we split ($X = N + 1/2$) the contribution of the first integral appearing in the definition of G_3 as

$$\int_T^{2T}\int_0^{3T} \cdots + \int_T^{2T}\int_{3T}^{\infty} \cdots = -\left(\frac{2i}{\pi}\right)\left\{\zeta(2\sigma) + \zeta(2-2\sigma)x^{1-2\sigma}\right\}$$

$$\int_0^{3T} \frac{\sin(2\pi Xy)\left\{\cos(T\log(1+1/y)) - \cos(2T\log(1+1/y))\right\}}{y^{\sigma+1}(1+y)^{\sigma}\log^2(1+1/y)} dy + O(T^{-2\sigma}),$$

117 analogously as in the treatment of $I_3(t)$ in the proof of Theorem (3.1). By the same technique the remaining integral is found to be

$$\int\limits_0^{3T} = \int\limits_0^{(2x)^{-1}} + \int\limits_{(2X)^{-1}}^{3T} = I' + II'' \ll T^{1-2\sigma} \ll 1.$$

The main term in (3.27) comes from the second integral in the expression for G_3. Integration yields (with $-i$ as a factor)

$$-\frac{i(1-2\sigma)}{\pi}\zeta(2-2\sigma)X^{1-2\sigma}\left\{\int\limits_T^{2T}\int\limits_0^{1}\frac{\sin(2\pi Xy)dy}{y(1+y)^{2\sigma-1}}\int\limits_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}\left(1+\frac{1}{y}\right)^u du dt\right.$$
$$\left.+\int\limits_T^{2T}\int_1^{\infty}\frac{\sin(2\pi Xy)dy}{y(1+y)^{2\sigma-1}}\int\limits_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}\left(1+\frac{1}{y}\right)^u du\, dt\right\}.$$

The total contribution of the second triple integral above is $\ll \log T$, after integration by parts, analogously as in the corresponding part of the proof of Theorem 2.1. To evaluate the first integral note that the theorem of residues gives, for $0 < y \leq 1$,

$$\int\limits_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}\left(1+\frac{1}{y}\right)^u du$$
$$= 2\pi i\left(1+\frac{1}{y}\right)^{2\sigma-1} - \left(\int\limits_{\sigma+it}^{-\infty+it} + \int\limits_{-\infty-it}^{\sigma-it}\right)\left(1+\frac{1}{y}\right)^u \frac{du}{u+1-2\sigma}$$
$$= 2\pi i\left(1+\frac{1}{y}\right)^{2\sigma-1} + J' + J'',$$

say. Then 118

$$-\frac{i(1-2\sigma)}{\pi}\zeta(2-2\sigma)X^{1-2\sigma}\int\limits_0^1\frac{\sin(2\pi Xy)}{y(1+y)^{2\sigma-1}}\cdots 2\pi i\left(1+\frac{1}{y}\right)^{2\sigma-1}dy$$
$$= 2(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}\int\limits_0^1 y^{-2\sigma}\sin(2\pi Xy)dy$$

$$= 2(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}\int_0^\infty y^{-2\sigma}\sin(2\pi Xy)dy + O(T^{-2\sigma})$$

$$= 2(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}\int_0^\infty X^{2\sigma}(2\pi)^{2\sigma}w^{-2\sigma}\sin w\cdot\frac{dw}{2\pi X} + O(T^{-2\sigma})$$

$$= 2(2\pi)^{2\sigma-1}(1-2\sigma)\zeta(2-2\sigma)\int_0^\infty w^{-2\sigma}\sin W\,dw + O(T^{-2\sigma})$$

$$= \frac{\pi(1-2\sigma)\zeta(2-2\sigma)(2\pi)^{2\sigma-1}}{\Gamma(2\sigma)\sin(\pi\sigma)} + O(T^{-2\sigma}),$$

where, similarly as in the proof of Theorem (2.2), we used

$$\int_0^\infty w^{-2\sigma}\sin w\,dw = \frac{\pi}{2\Gamma(2\sigma)\sin(\pi\sigma)}.$$

The contribution of J'' (similarly for J') is

$$\int_T^{2T}\int_{-\infty-it}^{\sigma-it}\left(1+\frac{1}{y}\right)^u\frac{du\,dt}{u+1-2\sigma}$$

$$= \int_{-\infty}^{\sigma}\left(1+\frac{1}{y}\right)^v dv\left\{\int_T^{2T}\left(1+\frac{1}{y}\right)^{-it}\frac{dt}{v-it+1-2\sigma}\right\}$$

$$= \int_{-\infty}^{\sigma}\frac{(1+1/y)^v}{T\log(1+1/y)}dv \ll T^{-1}y^{-\sigma},$$

on integrating the middle integral by parts. Then

$$T^{-1}\int_0^1\frac{|\sin(2\pi Xy)|}{y^{\sigma+1}(1+y)^{2\sigma-1}}dy$$

$$= T^{-1}\left(\int_0^{X^{-1}}\frac{|\sin(2\pi Xy)|}{y^{\sigma+1}(1+y)^{2\sigma-1}}dy + \int_{X^{-1}}^1\frac{|\sin(2\pi Xy)|}{y^{\sigma+1}(1+y)^{2\sigma-1}}dy\right)$$

$$\ll T^{-1}\left(\int\limits_0^{X^{-1}} Xy^{-\sigma}dy + \int\limits_{X^{-1}}^{\infty} y^{-\sigma-1}dy\right) \ll T^{\sigma-1} \ll \log T,$$

which proves (3.27). **119**

Further with

$$c = c(y,\sigma) = (2\sigma-1)(1+y) - \sigma - \log^{-1}\left(1+\frac{1}{y}\right) \asymp y(y \to \infty)$$

we have

$$\int\limits_T^{2T} G_4 dt = 4i\int\limits_X^{\infty} x^{-1}\Delta_{1-2\sigma}(x)dx \int\limits_0^{\infty} y^{-\sigma}(1+y)^{-\sigma-1}\log^{-1}\left(1+\frac{1}{y}\right)\cos(2\pi xy)$$
$$\int\limits_T^{2T}\left\{t\cos\left(t\log\left(1+\frac{1}{y}\right)\right) + c\sin\left(t\log\left(1+\frac{1}{y}\right)\right)\right\}dy\,dt$$
$$= 4i\int\limits_X^{\infty} x^{-1}\Delta_{1-2\sigma}(x)dx(I_\sigma(x,2T) - I_\sigma(x,T) + \Gamma_\sigma(x,T)),$$

where, analogously as in the treatment of $\int_T^{2T} I_4(t)dt$ in Theorem 3.1,

$$I_\sigma(x,z) = \int\limits_0^{\infty} \frac{z\sin(z\log(1+1/y))\cos(2\pi xy)}{y^\sigma(1+y)^{1+\sigma}\log^2(1+1/y)}dy,$$
$$r_\sigma(x,T) = \int\limits_0^{\infty} \frac{\{\cos(2T\log(1+1/y)) - \cos(T\log(1+1/y))\}\cos(2\pi xy)(-c(y,\sigma))}{y^{\sigma(1+y)^{1+\sigma}\log^2(1+1/y)}}dy.$$

Then analogously as in the proof of Theorem 3.1 we write

$$r_\sigma(x,T) = \int\limits_0^{3T} + \int\limits_{3T}^{\infty} = I' + I''$$

and show that, for some $c' > 3T$, **120**

$$
\begin{aligned}
I'' &\ll T^2 \left| \int\limits_{3T}^{c'} \frac{\cos(2\pi xy)}{y^\sigma (1+y)^{1+\sigma}} \left\{ \sigma + \frac{2}{\log\left(1+\frac{1}{y}\right)} - (2\sigma-1)(1+y) \right\} dy \right| \\
&\ll T^2 T T^{-\sigma-\sigma-1} x^{-1} = T^{2-2\sigma} x^{-1}.
\end{aligned}
$$

Since in mean square $\Delta_{1-2\sigma}(x)$ is of the order $x^{3/4} - \sigma$, we find that

$$
\begin{aligned}
\int\limits_X^\infty I'' x^{-1} \Delta_{1-2\sigma}(x) dx &\ll T^{2-2\sigma} \left(\int\limits_X^\infty \Delta^2_{1-2\sigma}(x) x^{-2} dx \right)^{1/2} \left(\int\limits_X^\infty x^{-2} dx \right)^{1/2} \\
&\ll T^{2-2\sigma} (T^{-1} T^{3/2-2\sigma})^{1/2} T^{-1/2} = T^{7/4-3\sigma} \leq T^{3/4-\sigma}
\end{aligned}
$$

for $\sigma \geq 1/2$. Using Lemma 2.1 we also obtain $I' \ll T^{3/4} - \sigma$.

The integral $I_\sigma(x, 2T) - I_\sigma(x, T)$ is evaluated by using Lemma 2.3. The remainder terms will be $\ll T^{2-2\sigma}(T^{-1/4} x^{-5/4}) + T^{-1} x^{-1/2})$, and their total contribution will be (as in the previous case) $\ll T^{3/4-\sigma}$. Thus we shall obtain

$$
\begin{aligned}
i \int\limits_T^{2T} G_4 dt = O(T^{3/4-\sigma}) - 2^\sigma \pi^{\sigma-1/2} \int\limits_X^\infty \Delta_{1-2\sigma}(x) x^{\sigma-2} \\
\left\{ z^{3/2-\sigma} V^{-2} U^{-1/2} \left(U + \frac{1}{2} \right)^{-1} \sin\left(zV + 2\pi x U - \pi x + \frac{\pi}{4} \right) \Bigg|_{z=T}^{2T} \right\} dx.
\end{aligned}
$$

From this point on the proof is very similar to the corresponding part of the proof of Theorem 3.1. Instead of the Voronoi formula (2.23) for $\Delta(x)$ we use the analogue (2.38) for $\Delta_{1-2\sigma}(x)$. The main terms will be the ones appearing in (3.6), and all the error terms will be $\ll T^{3/4-\sigma}$. Finally one may derive the transformation formulae for Dirichlet polynomials for $\sigma_{1-2\sigma}(n)$ by the same technique used by M. Jutila [89] in deriving
121 the transformation formulae for Dirichlet polynomials containing $d(n)$. In this way the proof of Theorem 3.2 is completed.

3.2 Omega-Results

One of the principal uses of Theorem 3.1 and Theorem 3.2 is to provide omega-results for $E(T)$, $E_\sigma(T)$ and $G_\sigma(T)$. We recall the common notation: $f(x) = \Omega(g(x))$ as $x \to \infty$ means that $f(x) = o(g(x))$ does not hold, $f(x) = \Omega_+(g(x))$ means that there exists $C > 0$ and a sequence x_n tending to infinity such that $f(x_n) > Cg(x_n)$. Analogously, $f(x) = \Omega_-(g(x))$ means that $f(y_n) < -Cg(y_n)$ for a sequence y_n tending to infinity, while $f(x) = \Omega_\pm(g(x))$ means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold. To obtain omega-results we shall work not directly with Theorem 3.1, but with a weaker version of it, contained in

Lemma 3.2.

$$G(T) = 2^{-1/4}\pi^{-3/4}T^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-5/4} \sin(\sqrt{8\pi nT} - \frac{\pi}{4}) + O(T^{2/3}\log T), \tag{3.28}$$

$$G_\sigma(T) = 2^{\sigma-3/4}\pi^{\sigma-5/4}T^{5/4-\sigma}\sum_{n=1}^{\infty}(-1)^n \sigma_{1-2\sigma}(n)n^{\sigma-7/4} \sin\left(\sqrt{8\pi nT} - \frac{\pi}{4}\right) + O\left(T^{1-\frac{2}{3}\sigma}\log T\right). \tag{3.29}$$

Proof. Both proofs are analogous, so we shall only sketch the proof of (3.29), using Theorem 3.2, Trivially we have

$$T^{1/2-\sigma}\sum_{n\le N'}\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\log\frac{T}{2\pi n}\right)^{-2}\sin(g(T,n)) \ll T^{1/2-\sigma}T^{\sigma}\log T = T^{1/2}\log T,$$

and $\frac{1}{2} < 1 - \frac{2}{3}\sigma$ for $\sigma < \frac{3}{4}$. Also $\frac{3}{4} - \sigma < 1 - \frac{2}{3}\sigma$ and

$$T^{1/2-\sigma}\sum_{T^{1/3}<n\le N}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(ar\sinh\sqrt{\frac{\pi n}{2T}}\right)^{-2}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4} \sin(f(T,n))$$

$$\ll T^{1/2-\sigma} \sum_{T^{1/3}<n\leq N} d(n)n^{\sigma-1}\left(\frac{n}{T}\right)^{-1}\left(\frac{T}{n}\right)^{-1/4} \ll T^{5/4-\sigma}\sum_{n>T^{1/3}} d(n)n^{\sigma-7/4}$$
$$\ll T^{5/4-\sigma}T^{(\sigma-3/4)/3}\log T = T^{1-\frac{2}{3}\sigma}\log T.$$

$\square$

122 Further we have

$$2^{\sigma-2}\pi^{\sigma-1/2}T^{1/2-\sigma}\sum_{n\leq T^{1/3}}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(ar\sinh\sqrt{\frac{\pi}{n}2T}\right)^{-2}\left(\frac{T}{2\pi n}+\frac{1}{4}\right)^{-1/4}\sin(f(T,n))$$
$$= 2^{\sigma-2}\pi^{\sigma-1/2}T^{1/2-\sigma}\sum_{n\leq T^{1/3}}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\left(\frac{\pi n}{2T}\right)^{-1}+O\left(\left(\frac{n}{T}\right)^{1/2}\right)\right)\left(\frac{T}{2\pi n}\right)^{-1/4}$$
$$\left(1+O\left(\frac{n}{T}\right)\right)\sin(f(T,n))$$
$$= 2^{\sigma-3/4}\pi^{\sigma-5/4}T^{5/4-\sigma}\sum_{n\leq T^{1/3}}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}n^{-1+1/4}\sin(f(T,n))$$
$$+O\left(T^{1/2-\sigma}\sum_{n\leq T^{1/3}}d(n)n^{\sigma-1}\left(\frac{n}{T}\right)^{3/4}\right)+O\left(T^{1/2-\sigma}\sum_{n\leq T^{1/3}}d(n)n^{\sigma-1}\left(\frac{n}{T}\right)^{1/4}\right)$$
$$= 2^{\sigma-3/4}\pi^{\sigma-5/4}T^{5/4-\sigma}\sum_{n\leq T^{1/3}}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\sin(f(T,n))+O\left(T^{1-\frac{2}{3}\sigma}\log T\right).$$

Finally, for $1\leq n\ll T^{1/3}$, we have by Taylor's formula

$$f(T,n)=(8\pi nT)^{1/2}-\frac{\pi}{4}+O\left(n^{3/2}T^{-1/2}\right),$$

and the total contribution of the error term above will be

$$\ll T^{5/4-\sigma}\sum_{n\leq T^{1/3}}d(n)n^{\sigma-7/4}n^{3/2}T^{-1/2} = T^{3/4-\sigma}\sum_{n\leq T^{1/3}}d(n)n^{\sigma-1/4}$$
$$\ll T^{3/4-\sigma}\log T\cdot T^{(\sigma+\frac{3}{4})/3} = T^{1-\frac{3}{2}\sigma}\log T.$$

123 Therefore if we write

$$\sum_{n\leq T^{1/3}} = \sum_{n=1}^{\infty} - \sum_{n>T^{1/3}}$$

and estimate the last sum above trivially, we obtain (3.29).

Observe now that $G_\sigma(t)$ is a continuous function of t which may be written as

$$G_\sigma(t) = 2^{\sigma-3/4}\pi^{\sigma-5/4}t^{5/4-\sigma}g_\sigma(t) + O\left(t^{1-\frac{2}{3}\sigma}\log t\right), \tag{3.30}$$

where

$$g_\sigma(t) = \sum_{n=1}^{\infty} h(n)\sin\left(\sqrt{8\pi nt} - \frac{\pi}{4}\right), h(n) = h_\sigma(n) = (-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-\frac{7}{4}}, \tag{3.31}$$

and the series in (3.31) is absolutely convergent for $\sigma < 3/4$, which is of crucial importance. Namely, we shall first deduce our omega-results from the following

Lemma 3.3. *If $g_\sigma(t)$ is defined by* (3.31)*, then there exists a constant $C > 0$ such that, uniformly for $1 \ll G \leq T$,*

$$\int_T^{T+G} g_\sigma^2(t)dt = CG + O(T^{1/2}). \tag{3.32}$$

Proof. By absolute convergence the series in (3.31) may be squared and integrated termwise. Therefore the left-hand side of (3.32) equals

$$\sum_{n=1}^{\infty} h^2(n)\int_T^{T+G}\sin^2\left(\sqrt{8\pi nt} - \frac{\pi}{4}\right)dt \tag{3.33}$$
$$+ O\left\{\sum_{m,n=1;m\neq n}^{\infty}|h(m)h(n)|\left|\int_T^{T+G}\exp\left(i\sqrt{8\pi nt}(\sqrt{m}\pm\sqrt{n})\right)dt\right|\right\}$$

□

The first integral in (3.33) is **124**

$$\frac{1}{2}\int_T^{T+G}\left(1-\cos\left(\sqrt{32\pi nt}-\frac{\pi}{2}\right)\right)dt = \frac{1}{2}G + O\left(T^{1/2}n^{-1/2}\right)$$

uniformly in G on applying Lemma 2.1. Therefore

$$\sum_{n=1}^{\infty} h^2(n) \int\limits_{T}^{T+G} \sin^2\left(\sqrt{8\pi nt} - \frac{\pi}{4}\right) = \frac{1}{2} G \sum_{n=1}^{\infty} h^2(n) + O\left(T^{1/2} \sum_{n=1}^{\infty} h^2(n) n^{-1/2}\right)$$
$$= \frac{1}{2} G \sum_{n=1}^{\infty} \sigma^2_{1-2\sigma}(n) n^{2\sigma-7/2} + O(T^{1/2})$$

uniformly in G, and the last series above is absolutely convergent and positive.

Also using Lemma 2.1 we obtain that the O-term (3.33) is, uniformly in G,

$$\ll T^{1/2} \sum_{m,n=1;1\le n<m}^{\infty} m^{\sigma-7/4+\epsilon} n^{\sigma-7/4+\epsilon} (m^{1/2} - n^{1/2})^{-1/2}$$
$$\ll T^{1/2} \sum_{n=1}^{\infty} n^{\sigma-7/4+\epsilon} \sum_{n<m\le 2n} \frac{m^{\sigma-7/4+\epsilon+1/2}}{m-n}$$
$$+ T^{1/2} \sum_{n=1}^{\infty} n^{\sigma-7/4+\epsilon} \sum_{m>2n} \frac{m^{\sigma-7/4+\epsilon+1/2}}{m-n}$$
$$\ll T^{1/2} \sum_{n=1}^{\infty} n^{2\sigma-3+2\epsilon} \log(n+1) + T^{1/2} \sum_{n=1}^{\infty} n^{\sigma-7/4+\epsilon} \sum_{m>2n} m^{\sigma+\epsilon-9/4}$$
$$\ll T^{1/2} \sum_{n=1}^{\infty} n^{2\sigma-3+2\epsilon} \log(n+1) + T^{1/2} \sum_{n=1}^{\infty} n^{2\sigma-3+2\epsilon} \ll T^{1/2}$$

if $\epsilon > 0$ is sufficiently small, since $\sigma < 3/4$.

It follows from Lemma 3.2 that there exist two constants $B, D > 0$ and a point $t_0 \in [T, T + DT^{1/2}]$ such that $|g_\sigma(t_0)| > B$ whenever $T \ge$
125 T_0. However, it is not clear whether $g_\sigma(t_0)$ is positive or negative. The following lemma shows that both positive and negative values of $g_\sigma(t)$ may occur.

Lemma 3.4. *If $g_\sigma(t)$ is given by* (3.31)*, then there exist two constants $B, D > 0$ such that for $T \ge T_0$ every interval $[T, T + DT^{1/2}]$ contains two points t_1, t_2 for which*

$$g_\sigma(t_1) > B, g_\sigma(t_2) < -B. \tag{3.34}$$

Proof. Both inequalities in (3.34) are proved analogously, so the details will be given only for the second one. Suppose that $g_\sigma(t) > -\epsilon$ for any given $\epsilon > 0$, and $t \in [T, T + DT^{1/2}]$ for $T \geq T_0(\epsilon)$ and arbitrary $D > 0$. If $C_1, C_2, \ldots$ denote absolute, positive constants, then for D sufficiently large and $G = DT^{1/2}$ we have from (3.32)

$$C_1 G \leq \int\limits_T^{T+G} g_\sigma^2(t)dt = \sum_k \int\limits_{I_k} g_\sigma^2(t)dt + \sum_\ell \int\limits_{J_\ell} g_\sigma^2(t)dt,$$

where I_k's denote subintervals of $[T, T + G]$ in which $g_\sigma(t) > 0$, and the J_ℓ's subintervals in which $g_\sigma(t) < 0$. In each J_ℓ we have $g_\sigma^2(t) < \epsilon$, and since

$$|g_\sigma(t)| \leq \sum_{n=1}^\infty |h(n)| = \sum_{n=1}^\infty \sigma_{1-2\sigma}(n) n^{\sigma-7/4} = C_2,$$

we have

$$\begin{aligned}
C_G &\leq C_2 \sum_k \int\limits_{I_k} g_\sigma(t)dt + G\epsilon^2 \\
&= C_2 \int\limits_T^{T+G} g_\sigma(t)dt + C_2 \sum_\ell \int\limits_{J_\ell} (-g_\sigma(t))dt + G\epsilon^2 \\
&\leq C_2 \int\limits_T^{T+G} g_\sigma(t)dt + C_2 G\epsilon + G\epsilon^2.
\end{aligned}$$

But using (3.31) and Lemma 2.1 it follows that

$$\begin{aligned}
\int_T^{T+G} g_\sigma(t)dt &= \sum_{n=1}^\infty h(n) \int\limits_T^{T+G} \sin\left(\sqrt{8\pi nt} - \frac{\pi}{4}\right) dt \\
&\ll T^{1/2} \sum_{n=1}^\infty |h(n)| n^{-1/2} = C_3 T^{1/2},
\end{aligned}$$

hence **126**

$$C_1G \le C_4T^{1/2} + C_2G\epsilon + G\epsilon^2. \tag{3.35}$$

If we take $G = DT^{1/2}$, $D > C_4/C_1$ and ϵ sufficiently small, then (3.35) gives a contradiction which proves the second inequality in (3.34), and the first one is proved similarly. □

Now we turn to (3.30) and observe that $\frac{5}{4}-\sigma > 1-\frac{2}{3}\sigma$ for $\frac{1}{2} \le \sigma < \frac{3}{4}$. Therefore, by continuity, Lemma 3.4 yields the following proposition (B, D, t_1, t_2 are not necessarily the same as in Lemma 3.4): There exist two positive constants B and D such that for $T \ge T_0$ every interval $[T, T + DT^{1/2}]$ contains two points t_1, t_2 for which $G_\sigma(t_1) > Bt_1^{5/4-\sigma}$, $G_\sigma(t_2) < -Bt_2^{5/4-\sigma}$. By continuity, this also implies that there is a zero t_3 of $G_\sigma(t)$ in $[T, T + DT^{1/2}]$.

Next consider for $H > 0$

$$G_\sigma(T + H) - G_\sigma(T) = \int\limits_T^{T+H} (E_\sigma(t) - B(\sigma))dt. \tag{3.36}$$

Let T be a zero of $G_\sigma(T)$, and let H be chosen in such a way that $G_\sigma(T + H) > B(T + H)^{5/4-\sigma}$. By the preceding discussion we may take $H \le FT^{1/2}$ with suitable $F > 0$. then (3.36) gives

$$B(T + H)^{5/4-\sigma} < \int\limits_T^{T+H} (E_\sigma(t) - B(\sigma))dt = H(E_\sigma(t_4) - B(\sigma))$$

for some $t_4 \in [T, T + H]$ by the mean value theorem for integrals. Therefore

$$E_\sigma(t_4) > Ct_4^{3/4-\sigma}$$

127 with a suitable $C > 0$, and similarly we obtain

$$E_\sigma(t_5) < -Ct_5^{3/4-\sigma}$$

with $t_4, t_5 \in [T, T + FT^{1/2}]$. the foregoing analysis may be repeated, by virtue of (3.28), with $G(T)$ in place of $G_\sigma(T)$ (in (3.31) we shall have $h(n) = (-1)^n d(n)n^{-5/4}$). In this way we are led to our first omega-result, which is

Theorem 3.3. *There exist constants $B, D > 0$ such that for $T \geq T_0$ every interval $[T, T + DT^{1/2}]$ contains points $t_1, t_2, t_3, t_4 \tau_1, \tau_2, \tau_3, \tau_4$ such that*

$$E(t_1) > Bt_1^{1/4}, E(t_2) < -Bt_2^{1/4}, G(t_3) > Bt_3^{3/4}, G(t_4) < -Bt_4^{3/4}, \quad (3.37)$$

$$E_\sigma(\tau_1) > B\tau_1^{3/4-\sigma}, E_\sigma(\tau_2) < -B\tau_2^{3/4-\sigma}, \quad (3.38)$$

$$G_\sigma(\tau_3) > B\tau_3^{5/4-\sigma}, G_\sigma(\tau_4) < -B\tau_4^{5/4-\sigma}.$$

Since $G(T) = O(T^{3/4})$, $G_\sigma(T) = O(T^{5/4-\sigma})$ by (3.28) and (3.29), this means that we have proved

$$G(T) = O(T^{3/4}), G(T) = \Omega_\pm(T^{3/4}), G_\sigma(T) =)(T^{5/4-\sigma}),$$
$$G_\sigma(T) = \Omega_\pm(T^{5/4-\sigma}) \quad (3.39)$$

and also

$$E(T) = \Omega_\pm(T^{1/4}), E_\sigma(T) = \Omega_\pm(T^{3/4-\sigma}). \quad (3.40)$$

Thus (3.39) shows that, up to the values of the numerical constants involved, we have determined the true order of magnitude of $G(T)$ and $G_\sigma(T)$. Moreover, not only does Theorem 3.3 provide $\Omega_\pm$-results for the values where these functions attain large positive and small negative values, respectively. As mentioned in Section 2.6, the mean square formulas (2.69) and (2.71) provide weak omega-results, namely

$$E(T) = \Omega(T^{1/4}), E_\sigma(T) = \Omega(T^{3/4-\sigma}),$$

which are superseded by (3.40). One can, of course, try to go further and 128
sharpen (3.40) by using special properties (arithmetical structure) of the functions $d(n)$ and $\sigma_{1-2\sigma}(n)$ appearing in (3.28) and (3.29), respectively. As already mentioned in Chapter 2, there are several deep analogies between $E(T)$ and $\Delta(x)$. Even the formula (3.28) has its counterpart in the theory of $\Delta(x)$, namely

$$\int_2^T \Delta(t)dt = \frac{T}{4} + \frac{T^{3/4}}{2\sqrt{2}\pi^2} \sum_{n=1}^{\infty} d(n) n^{-5/4} \sin\left(4\pi\sqrt{nT} - \frac{\pi}{4}\right) + O(1),$$

which is a classical formula of G.F. Voronoi. For $\Delta(x)$ the best known omega-results are

$$\Delta(T) = \Omega_+ \left\{ (T \log T)^{1/4} (\log\log T)^{1/4(3+\log 4)} e^{-C\sqrt{\log\log\log T}} \right\} \tag{3.41}$$

and

$$\Delta(T) = \Omega_- \left\{ T^{1/4} \exp\left(\frac{D(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}} \right) \right\}, \tag{3.42}$$

due to J.L. Hafner [51] and K. Corrádi - I. Kátai [28], where $C, D > 0$ are absolute constants. The corresponding problems involving $\Delta_{1-2\sigma}(x)$ may be also considered, but it seems appropriate to make the following remark here. The arithmetical function $\sigma_{1-2\sigma}(n)$, which by (2.37) and (2.41) has the mean value $\zeta(2\sigma)$, is much more regularly distributed than $d(n)$, whose average order is $\log n$. For this reason sharp omega-results for $\sigma_{1-2\sigma}(n)$ are harder to obtain than sharp omega-results for $d(n)$, since for the latter one somehow tries to exploit the irregularities of distribution of $d(n)$. Observe that in (3.28) there is the factor $(-1)^n$, which is not present in the above Voronoi formula for $\int_2^T \Delta(t)dt$. It was thought
129 by many that the oscillating factor $(-1)^n$, present already in Atkinson's formula (2.5) for $E(T)$, would hinder the possibility of obtaining sharp $\Omega_\pm$-results for $E(T)$ analogous to (3.41) and (3.42). The theorem that follows shows that this is not the case, and that (3.28) is in fact strong enough to render (when suitable techniques are applied to it) the analogues of (3.41) and (3.42). In the casee of $E_\sigma(T)$ we would have to cope with the regularity of distribution of $\sigma_{1-2\sigma}(n)$, and the presence of the oscillating factor $(-1)^n$. For this reason we shall content ourselves only with sharp omega-results for $E(T)$, contained in

Theorem 3.4. *There exist absolute constants $C, D > 0$ such that*

$$E(T) = \Omega_+ \left\{ (T \log T)^{1/4} (\log\log T)^{1/4}(3 + \log 4) e^{-C\sqrt{\log\log\log T}} \right\} \tag{3.43}$$

and

$$E(T) = \Omega_- \left\{ T^{1/4} \exp\left(\frac{D(\log\log T)^{1/4}}{(\log\log\log T)^{3/4}} \right) \right\}. \tag{3.44}$$

These formulas are the exact analogues of (3.41) and (3.42). Since the problem of $\Omega_\pm$-results for $\Delta(x)$ is certainly not more difficult than the corresponding problem for $E(T)$ (the Voronoi formula for $\Delta(x)$ is simpler and sharper than Atkinson's formula for $E(T)$ or any of its variants), it is hard to imagine improvements of (3.43) and (3.44) which would come from methods not capable of improving (3.41) and (3.42). On the other hand, although (3.43) and (3.44) are such sharper than just $E(T) = \Omega_\pm(T^{1/4})$, which follows from Theorem 3.3, one does not obtain in the proof of Theorem 3.4 the localization of points where large positive and small negative values of $E(T)$ are taken (or to be precise, the localization will be very poor). Theorem 3.3 provides good localization, and thus the omega-results furnished by Theorem 3.3 and Theorem 3.4 both have their merits and are in a certain sense complementary to one another.

Proof of Theorem 3.4. First we prove the Ω_+-result (3.43). Let 130

$$E^*(t) := \int\limits_{-1}^{1} E_0(t+u)k_M(u)du, E_0(t) := (2t)^{-1/2}e(2\pi t^2), \tag{3.45}$$

where E_0 is introduced because it is more convenient to work without square roots in Atkinson's formula. Further let

$$k_n(u) = K_{1/2\lambda_n}(u) := \frac{\lambda_n}{2\pi}\left(\frac{\sin(1/2\lambda_n u)}{1/2\lambda_n u}\right)^2 \quad (\lambda_n = 4\pi\sqrt{n}) \tag{3.46}$$

be the Fejér kernel of index $1/2\lambda_n$, and M is a large positive integer. Because

$$k_M(u) > 0, 0 < \int\limits_{-1}^{1} k_M(u)du < 1,$$

(3.43) is a consequence of the following assertion: there exist absolute, positive constants A and C such that

$$E^*(t) > A(\log t)^{1/4}(\log\log t)^{1/4(3+\log 4)}e^{-C\sqrt{\log\log\log t}} \tag{3.47}$$

for some arbitrarily large values of t. To prove (3.47) we shall show that, uniformly for $1 \leq M \leq t^{1/2}$,

$$E^*(t) = \sum_{n\leq M} (-1)^n d)n)n^{-3/4} \left(1 - \left(\frac{n}{M}\right)^{1/2}\right) \cos\left(4\pi\sqrt{n}t - \frac{\pi}{4}\right) + o(1) \tag{3.48}$$

and then deduce (3.47) from (3.48). To this end let

$$f(t) := 2^{-1/2} t^{3/2} \int_2^t (E(2\pi y^2) - \pi) y\, dy,$$

so that by (3.28) we obtain

$$f(t) = \frac{1}{4\pi} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-5/4} \sin(\lambda_n t - \frac{\pi}{4}) + o(1), \tag{3.49}$$

and by direct computation we find that

$$E_0(t) = \frac{d}{dt} f(t) + o(t^{-1/2}). \tag{3.50}$$

131 Using (3.50) in (3.45), integrating by parts, and then using (3.49) we obtain, uniformly for $1 \leq M \leq t^{1/2}$,

$$\begin{aligned} E^*(t) &= -\int_{-1}^{1} f(t+u) k'_M(u) du + o(1) \\ &= -\frac{1}{4\pi} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-5/4} \operatorname{Im} \left\{ e^{i(\lambda_n t - \frac{\pi}{4})} \int_{-1}^{1} e^{i\lambda_n u} k'_M(u) du \right\} + o(1). \end{aligned}$$

The last integral is readily evaluated as

$$\int_{-1}^{1} e^{i\lambda_n u} k'_M(u) du = \begin{cases} -i\lambda_n \left(1 - \frac{\lambda_n}{\lambda_n}\right) + o(1) & \text{if } n \leq M, \\ o(1) & \text{if } n > M, \end{cases}$$

and (3.48) follows.

To take advantage of (3.48) we need some facts about sums involving the divisor function, which reflect the irregularities of its distribution. This is

Lemma 3.5. *For each positive constant C and positive integer $K \geq 2$, there is a set $P_C \subseteq \{1, 2, \ldots K\}$ such that uniformly*

$$\sum_{n \notin P_C, n \leq K} d(n) n^{-3/4} \ll C^{-2} K^{1/4} \log K,$$

and if $|P_C|$ denotes the cardinality of P_C, then

$$|P_C| \ll K (\log K)^{1 - \log 4} \exp\left(C \sqrt{\log \log K}\right).$$

Proof. First we show that

$$\sum_{n \leq x} d(n)(\omega(n) - 2 \log \log x)^2 \ll x \log x \log \log x, \tag{3.51}$$

where as usual $\omega(n)$ denotes the number of distinct prime factors of n. **132**
To obtain (3.51) note that if p, q are primes, then

$$d(np) = 2d(n) - d\left(\frac{n}{p}\right)$$

and

$$d(npq) = 4d(n) - 2d\left(\frac{n}{p}\right) - 2d\left(\frac{n}{q}\right) + d\left(\frac{n}{pq}\right),$$

where we put $d(x) = 0$ if x is not an integer. Then, for distinct primes p, q we obtain

$$\begin{aligned}
\sum_{n \leq x} d(n) \omega^2(n) = &\sum_{pq \leq x, p \neq q} \\
&\left\{4 \sum n \leq x/pq d(n) - 2 \sum_{n \leq x/p^2 q} d(n) - 2 \sum_{n \leq x/p\, q^2} d(n) \sum_{n \leq xp^2 q^2} d(n)\right\} \\
&\qquad + \sum_{p \leq x} \left\{2 \sum_{n \leq x/p} d(n) - \sum_{n \leq x/p^2} d(n)\right\} \\
&= 4 \sum_{pq \leq x} \frac{x}{pq} \log \frac{x}{pq} + o(x \log x \log \log x) \\
&= 4x \log x (\log \log x)^2 + o(x \log x \log \log x).
\end{aligned}$$

$\square$

In a similar fashion it may be shown that

$$\sum_{n\le x} d(n)\omega(n) = 2x\log x\log\ \log x + o(x\log x),$$

and (3.51) follows. Let now

$$P_C = \left\{n \le K : \omega(n) \ge 2\log logK - C\sqrt{\log\log K}\right\}.$$

By using $d(n) \ge 2^{\omega(n)}$ it follows that

$$|P_C| 2^{2\log\log K - C\sqrt{\log\log K}} \le \sum_{n\in P_C} 2^{\omega(n)} \le \sum_{n\le K} d(n) \ll K\log K,$$

as asserted. Also, using (3.51) and partial summation we find that

$$\begin{aligned}\sum_{n\notin P_C} d(n)n^{-3/4} &\le (C^2\log\log K)^{-1}\sum_{n\notin P_C} d(n)n^{-3/4}(\omega(n) - 2\log\log K)^2\\ &\le (C^2\log\log K)^{-1}\sum_{n\le K} d(n)n^{-3/4}(\omega(n) - 2\log\log K)^2\\ &\ll C^{-2}K^{1/4}\log K.\end{aligned}$$

133 Now let $K = [M/2]$ and let P_C be as in Lemma 3.5 for this K and some C to be chosen later. By Dirichlet's approximation theorem there exists t satisfying

$$M^2 \le t \le M^2 64^{|P_C|}$$

and such that for each m in P_C and $n = 2m$ we have $|t\sqrt{n} - x_n| \le \frac{1}{64}$ for some integers x_n. For these n and this t it follows that

$$\cos\left(4\pi t\sqrt{n} - \frac{\pi}{4}\right) \ge \cos\left(\frac{\pi}{16} + \frac{\pi}{4}\right) > \frac{1}{2}.$$

Note that each pair M, t constructed in this way satisfies $1 \le M \le t^{1/2}$. For this pair (3.48) gives

$$E^*(t) = \left\{\frac{1}{2}\sum_{\substack{n\le M, n=2m\\ m\in P_C}} - \sum_{\substack{n\notin M, n=2m\\ m\neq P_C}} - \sum_{n\le M, n=2m+1}\right\} d(n)n^{-3/4}\left(1 - \left(\frac{n}{m}\right)^{1/2}\right) + o(1)$$

$$= \left\{ \frac{1}{2} \sum_{n \le M} - \frac{3}{2} \sum_{\substack{n \le M, n=2m \\ m \notin P_C}} - \frac{3}{2} \sum_{n \le M, n=2m+1} \right\} d(n) n^{-3/4} \left(1 - \left(\frac{n}{m} \right)^{1/2} \right) + o(1).$$

But we have elementarily

$$\sum_{n \le x, n=2m} d(n) = \frac{3}{4} x \log x + O(x), \quad \sum_{n \le x, n=2m+1} d(n) = \frac{1}{4} x \log x + O(x),$$

hence by partial summation

$$\sum_{n \le M} d(n) n^{-3/4} \left(1 - \left(\frac{n}{M} \right)^{1/2} \right) = \frac{8}{3} M^{1/4} \log M + O(M^{1/4})$$

and

$$\sum_{n \le M, n=2m+1} d(n) n^{-3/4} \left(1 - \left(\frac{n}{M} \right)^{1/2} \right) = \frac{2}{3} M^{1/4} \log M + O(M^{1/4}).$$

With these formulas and Lemma (3.5) we obtain for this pair t, M **134**
and C sufficiently large that

$$E^*(t) \ge \left(\frac{1}{3} + O(C^{-2}) \right) M^{1/4} \log M \ge \frac{1}{4} M^{1/4} \log M. \tag{3.52}$$

Note that from $t \le M^2 64^{|P_C|}$ and the second part of Lemma (3.5) we obtain

$$M \gg \log t (\log\log t)^{\log 4 - 1} \exp\left(-C \sqrt{\log\log\log t} \right) \tag{3.53}$$

for some (perhaps different) constant $C > 0$. Combining (3.52) and (3.53) it is seen that (3.47) holds, and so (3.43) is proved.

We proceed now to the proof of the Ω-result (3.44), using again (3.28) as our starting point. First we are going to prove a weaker Ω-result, namely

$$\liminf_{T \to \infty} E(T) T^{-1/4} = -\infty. \tag{3.54}$$

This result will be then used in deriving the stronger Ω_--result given by (3.44). To prove (3.54) it suffices to show that

$$\liminf_{T\to\infty} E^*(T) = -\infty, \tag{3.55}$$

where $E^*(T)$ is defined by (3.45). Write each $n \le M$ in (3.48) as $n = \nu^2 q$, where q is the largest squarefree divisor of n. By Kronecker's approximation theorem there exist arbitrarily large T such that

$$T\sqrt{q} = \begin{cases} m_q + \delta_q & \text{if } q \text{ is odd,} \\ \frac{1}{4} + n_q + \delta_q & \text{if } q \text{ is even,} \end{cases}$$

135 with some integers m_q and $|\delta_q| < \delta$ for any given $\delta > 0$. With these T we conclude that

$$(-1)^n \cos\left(4\pi T\sqrt{n} - \frac{\pi}{4}\right) = -\epsilon_n \cos\left(\frac{\pi}{4}\right) + O(\sqrt{n}\delta),$$

where

$$\epsilon_n = \begin{cases} -1 & \text{if } n \equiv 0 \pmod 4). \\ 1 & \text{if } n \not\equiv 0 \pmod 4). \end{cases}$$

We deduce from (3.48) that

$$\liminf_{T\to\infty} E^*(T) \le -\cos\left(\frac{\pi}{4}\right)\sum_{n\le M} \epsilon_n d(n) n^{-3/4}\left(1 - \left(\frac{n}{M}\right)^{1/2}\right) + O(\delta M^{3/4}\log M). \tag{3.56}$$

On letting $\delta \to 0$ we obtain (3.55), since the sum in (3.56) can be shown elementarily to be unbounded as $M \to \infty$.

Now we pass to the actual proof of (3.44) by using a variant of a technique of K.S. Gangadharan [40]. Let P_x be the set of odd primes $\le x$, and Q_x the set of squarefree numbers composed of primes from P_x. Let $|P_x|$ be the cardinality of P_x and $M = 2^{|P_x|}$ the cardinality of Q_x. Then we have

$$\frac{x}{\log x} \ll |P_x| \ll \frac{x}{\log x}, \quad M \ll \exp\left(\frac{cx}{\log x}\right) \tag{3.57}$$

for some $c > 0$, and also that all elements in Q_x do not exceed e^{2x}.

Now let S_x be the set of numbers defined by

$$S_x = \left\{ \mu = \sum_{q \in Q_x} r_q \sqrt{q} : r_q \in \{-1, 0, 1\}; \sum r_q^2 \geq 2 \right\},$$

and finally

$$\tilde{\eta}(x) = \inf \left\{ | \sqrt{m} + \mu | : m = 1, 2, \ldots; \mu \in S_x \right\}.$$

Taking $m = [\sum \sqrt{q}]^2$ it is seen that $| \sqrt{m} - \sum \sqrt{q}| < 1$, hence $\tilde{\eta}(x) < 1$. Also there are only finitely many distinct values of $| \sqrt{m} + \mu|$ in (0,1). **136**
Then one has (see Gangadharan [40]):

Lemma 3.6. *If* $q(x) = -\log \tilde{\eta}(x)$, *then for some* $c > 0$

$$x \leq q(x) \ll \exp\left(\frac{cx}{\log x} \right).$$

Similarly as in the proof of (3.43) we avoid square roots by introducing the functions

$$\tilde{E}(t) = \sqrt{2\pi} \left\{ E\left(\frac{t^2}{8\pi} \right) - \pi \right\}, E_+(T) = \int_0^T t \tilde{E}(t) dt. \tag{3.58}$$

From (3.28) we have then

$$E_+(T) = T^{3/2} \sum_{n=1}^{\infty} (-1)^n d(n) n^{-5/4} \sin\left(T\sqrt{n} - \frac{\pi}{4} \right) + O(T^{4/3} \log T). \tag{3.59}$$

If we could differentiate this series (and the O-term) we could deal with $E(T)$ directly. This is not possible, but we can use integration by parts in subsequent integrals that appear to take advantage of (3.59).

We let

$$P(x) = \exp\left(\frac{\alpha x}{\log x} \right)$$

be such that

$$P(x) \geq \max(q(x), M^2) \tag{3.60}$$

and define, for a fixed x,

$$\gamma_x = \sup_{u>0} \left\{ \frac{-\sqrt{2\pi}E(u^2/(8\pi))}{u^{1/2+1/P(x)}} \right\}. \tag{3.61}$$

Now for $T \to 0+$, $E(T) \sim -T\log T$, so that the expression in brackets in (3.61) is bounded for small u. If this expression is not bounded for all u then more than (3.44) would be true. Also, by our earlier Ω_{-}-result (3.54) there exists a $u > 0$ for which this expression is positive. Hence we can conclude that $0 < \gamma_x < \infty$, or, in other words,

$$\gamma_x u^{1/2+1/P(x)} + A + \tilde{E}(u) \geq 0$$

137 for all $u > 0$, where $A = \sqrt{2}\pi^{3/2}$.

Our next step is to describe the part of the kernel function we use to isolate certain terms of the "series" for $E(u)$, and to point them in an appropriate direction. Let

$$V(z) = 2\cos^2\frac{z}{2} == \frac{e^{iz}+e^{-iz}}{2} + 1$$

and set

$$T_x(u) = \prod_{q\in Q_x} V\left(u\sqrt{q} - \frac{5\pi}{4}\right).$$

Note that $T_x(u) \geq 0$ for all u. Finally, put $\sigma_x = \exp(-2P(x))$ and

$$J_x := \sigma_x^{5/2} \int_0^\infty \left(\gamma_x u^{1/2+1/P(x)} + A + \tilde{E}(u)\right) u e^{-\sigma_x u} T_x(u) du. \tag{3.62}$$

From the remarks above we see immediately that $J_x \geq 0$. In the next two lemmas we provide the tools for an asymptotic expansion for J_x. In the first we cover the first two terms of J_x.

Lemma 3.7. *For $\frac{1}{2} < \theta < 2$ and $x \to \infty$ we have*

$$\int_0^\infty u^\theta e^{-\sigma_x u} T_x(u) du = \sigma_x^{-1-\theta}\Gamma(1+\theta) + o(\sigma_x^{-5/2}).$$

Proof. We expand the trigonometric polynomial $T_x(u)$ into exponential polynomials as
$$T_x(u) = T_0 + T_1 + \overline{T}_1 + T_2,$$
where
$$T_0 = 1, T_1 = \frac{1}{2}e^{1/4(5\pi)} \sum_{q \in QW_x} e^{-iu\sqrt{q}}, T_2 = \sum_{\mu \in S_x} h_\mu e^{-iu\mu},$$
$\overline{T}_1$ is the complex conjugate of T_1, and h_μ are constants bounded by $1/4$ in absolute value. □

Note that T_0 contributes to the integral exactly the first term, so that we have to consider the other parts of T_x. The part T_1 contributes exactly
$$\frac{1}{2}e^{1/4(5\pi)i}\Gamma(1+\theta) \sum_{q \in Q_x} (\sigma_x + i\sqrt{q})^{-1-\theta} \ll \sum_{q \in Q_x} q^{-1/2(1+\theta)} \ll M = o\left(\sigma_x^{-5/2}\right)$$
since $\theta + 1 > 0$ and (3.60) holds. The contribution of $\overline{T}_1$ is likewise 138
$o(\sigma_x^{-5/2})$, and T_2 provides the term
$$\begin{aligned}\Gamma(1+\theta)\sum_\mu h_\mu(\sigma_x + iu)^{-1-\theta}. &\ll 3^M \left(\inf_{\mu \in S_x} |\mu|\right)^{-1-\theta} \ll 3^M \tilde{\eta}(x)^{-1-\theta}\\ &= \exp\left\{c\sqrt{P(x)} + P(x)(1+\theta)\right\} = o\left(\sigma_x^{-5/2}\right),\end{aligned}$$
again by (3.60) and the fact that $1 + \theta < 3$.

In the next lemma we cover the contribution to J_x from $\tilde{E}(U)$. It is here that we appeal to the identity (3.59) for $E_+(T)$.

Lemma 3.8. *For $x \to \infty$ we have*
$$\int_0^\infty \tilde{E}(u)ue^{-\sigma_x u}T_x(u)du = -\frac{1}{2}\Gamma\left(\frac{5}{2}\right)\sigma_x^{-5/2} \sum_{q \in Q_x} d(q)q^{-3/4} + o(\sigma_x^{-5/2}).$$

Proof. Or first step is to integrate by parts to introduce $E_+(T)$ in the integral so that we can use (3.59). Thus our integral can be written as
$$E_+(u)e^{-\sigma_x u}T_x(u)\Big|_0^\infty - \int_0^\infty E_+(u)\frac{d}{du}\left(e^{-\sigma_x u}T_x(u)\right)du.$$

Now since

$$E_+(u) = \begin{cases} O(u^2) & \text{if } O \le u \le 10, \\ O(u^{3/2}) & \text{if } u \ge 10, \end{cases}$$

the integrated terms vanish. In the remaining integral we wish to replace $E_+(u)$ by (3.59). However, we must be careful how we deal with the error term. Write the integral in question as

$$-\int_0^\infty h(u)u^{3/2}\frac{d}{du}\left(e^{-\sigma_x u}T_x(u)\right)du + O\left(\int_{10}^\infty u^{4/3}\log u\left|\frac{d}{du}(\ldots)\right|du\right)$$
$$+\int_0^{10} h(u)u^{3/2}\frac{d}{du}(\ldots)du + O\left(\int_0^{10} u^2\left|\frac{d}{du}(\ldots)\right|du\right)$$
$$= I_1 + O(I_2) + I_3 + O(I_4),$$

139 say, where $h(u)$ is defined by

$$h(u) = \sum_{n=1}^\infty (-1)^n d(n)n^{-5/4}\sin\left(u\sqrt{n} - \frac{\pi}{4}\right).$$

□

The integral I_3 is bounded by

$$I_3' = \int_0^{10} u^{3/2}\left|\frac{d}{du}(\ldots)\right|du,$$

and this dominates the last integral I_4. Hence, we should estimate I_3' and I_2 and calculate I_1.

For the two integral estimates, we need a bound on the expression in absolute values. For this we note that from the definition and from the decomposition used in the proof of Lemma 3.7 we have

$$T_x(u) \ll 2^M, T_x'(u) \ll 3^M Me^{cx},$$

so that
$$\frac{d}{du}\left(e^{-\sigma_x u}T_x(u)\right) \ll \quad e^{-\sigma_x u}4^M.$$
In I_3' this contributes at most
$$4^M \int\limits_0^{10} u^{-1/2} du \ll ec\sqrt{P(x)} = O(\sigma_x^{-5/2}).$$
In I_2 the estimate becomes
$$4^M \int\limits_{10}^{\infty} u^{4/3} e^{-\sigma_x u} \log u \, du \ll e^{\sqrt{P(x)}} \sigma_x^{-7/3-\epsilon} = O(\sigma_x^{-5/2}).$$
For I_1 we expand the expression $\frac{d}{du}(\ldots)$ as
$$u^{-3/2}\frac{d}{du}(u^{3/2}e^{-\sigma_x u}T_x(u)) - \frac{3}{2}u^{-1}e^{-\sigma_x u}T_x(u).$$
The last term contributes to I_1 at most (since $h(u)$ is bounded)
$$2^M \int\limits_0^{\infty} u^{1/2} e^{-\sigma_x u} du \ll 2^M \sigma_x^{-3/2} = O(\sigma_x^{-5/2}).$$
Finally, we are left to deal with the following: **140**
$$-\int\limits_0^{\infty} h(u)\frac{d}{du}\left(u^{3/2}e^{-\sigma_x u}T_x(u)\right) du.$$

We replace $h(u)$ by its series definition and integrate term by term. This is legitimate because of absolute and uniform convergence. We obtain
$$-\sum_{n=1}^{\infty}(-1)^n d(n) n^{-5/4}\,\mathrm{Im}(e^{-1/4\pi i} I(n)), \tag{3.63}$$
where
$$I(n) := \int\limits_0^{\infty} e^{iu\sqrt{n}}\frac{d}{du}(u^{3/2}e^{-\sigma_x u}T_x(u))du.$$

In this integral we can reintegrate by parts and expand $T_x(u)$ as we did in the proof of Lemma 3.7 to obtain

$$\begin{aligned} I(n) &= i\sqrt{n}\int\limits_0^\infty e^{iu\sqrt{n}}u^{3/2}e^{-\sigma_x u}(T_0(u)+T_1(u)+\overline{T_1(u)}+T_2(u))du \\ &= I_0(n)+I_1(n)+I_1^*(n)+I_2(n), \end{aligned}$$

say. The only significant contribution will come from $I_1(n)$, as we shall see. First we have

$$I_0(n) \ll \sqrt{n}|\sigma_x - i\sqrt{n}|^{-5/2} \ll n^{-3/4}.$$

Second,

$$I^*(n) \ll \sqrt{n}\sum_{q\in Q_x}|\sigma_x - i(\sqrt{n}+\sqrt{q})|^{-5/2} \ll n^{-3/4}M.$$

Third,

$$\begin{aligned} I_2(n) &\ll \sum_{\mu\in S_x}|\sigma_x - i(\sqrt{n}-\mu)|^{-5/2} \\ &\ll \begin{cases} 3^M n^{-3/4} & \text{if } n > 2\max\{|\mu| : \mu\in S_x\}, \\ 3^M\tilde{\eta}(x)^{-5/2}\sqrt{n} & \text{if } n \le 2\max\{|\mu| : \mu\in S_x\}. \end{cases} \end{aligned}$$

141 This $\max\{|\mu|\ldots\}$ is bounded by Me^{cx}. Hence all of these contribute to our series (3.63) no more than

$$3^M\tilde{\eta}(x)^{-5/2}(Me^{cx})^{1/4+\epsilon} = O(\sigma_x^{-5/2}).$$

as required. There remains only the contribution of $I_1(n)$. We need to distinguish two cases. If $n \neq q$ for all $q \in Q_x$, then we obtain a bound exactly as above for $I_2(n)$, but with M replacing the factor 3^M which comes from the number of terms in the sum. Now suppose $n = q$ for some q in Q_x. The term in the sum defining $T_1(u)$ corresponding to this q along contribute exactly

$$\frac{1}{2}i\,e^{5\pi i/4}\Gamma\left(\frac{5}{2}\right)\sqrt{q}\sigma_x^{-5/2}.$$

The other terms contribute as in the case $n \neq q$. Combining all these contributions to (3.63) we see that the lemma is proved. It should be noted that each q in Q_x is odd so that the factor $(-1)^q$ in (3.63) is always negative for the significant terms.

We can now complete the proof of (3.44). For J_x in (3.62) we first have $J_x \geq 0$. thus by Lemma 3.7 and Lemma 3.8 we also have, as $x \to \infty$,

$$J_x = \gamma_x \sigma_x^{-1/P(x)} \Gamma\left(\frac{5}{2} + \frac{1}{P(x)}\right) - \frac{1}{2}\Gamma\left(\frac{5}{2}\right) \prod_{q \in Q_x} d(q) q^{-3/4} + o(\gamma_x) + o(1).$$

Hence if x is sufficiently large we deduce that

$$\begin{aligned} \gamma_x &\gg \sum_{q \in Q_x} d(q) q^{-3/4} \gg \prod_{2<p\leq x} (1 + 2p^{-3/4}) \\ &= \exp\left(\sum_{2<p\leq x} \log(1 + 2p^{-3/4})\right) \gg \exp\left(\frac{cx^{1/4}}{\log x}\right). \end{aligned}$$

In other words, for each sufficiently large x there exists a u_x such that for some absolute constant $A > 0$

$$-E(u_x^2) u_x^{-1/2} \geq A \exp\left(\frac{\log u_x}{P(x)} + \frac{cx^{1/4}}{\log x}\right). \tag{3.64}$$

This implies first that u_x tends to infinity with x. If the second term in the exponential dominates, then it is easy to see on taking logarithms **142**
and recalling the definition of $P(x)$ that

$$\log \log u_x \ll \frac{x}{\log x}.$$

Hence

$$x \gg \log \log u_x \log \log \log u_x,$$

and since the function $x^{1/4}/\log x$ is increasing for $x \geq x_0$, we obtain (3.44) from (3.64). If the first term in the exponential in (3.64) dominates, then we may assume

$$\frac{(\log \log u_x)^{1/4}}{(\log \log \log x)^{3/4}} \gg \frac{\log u_x}{P(x)},$$

since otherwise the Ω_--result holds again. But the last condition gives again

$$\log\log u_x \ll \frac{x}{\log x},$$

so that (3.44) holds in this case as well.

3.3 Mean Square Formulas

The explicit formulas for $G(T)$ and $G_\sigma(T)$, contained in Theorem 3.1 and Theorem 3.2, enable us to obtain mean square formulas for these functions. The results are given by

Theorem 3.5.

$$\int\limits_2^T G^2(t)dt = BT^{5/2} + O(T^2), B = \frac{\zeta^4(5/2)}{5\pi\sqrt{2\pi}\zeta(5)} = 0.079320\ldots \quad (3.65)$$

and for $1/2 < \sigma < 3/4$ *fixed*

$$\int\limits_2^T G_\sigma^2(t)dt = C(\sigma)T^{7/2-2\sigma} + O(T^{3-2\sigma}) \quad (3.66)$$

] with

$$C(\sigma) = \frac{4^{\sigma-1}\pi^{2\sigma-3}\sqrt{2\pi}}{7-4\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}^2(n)n^{2\sigma-7/2}. \quad (3.67)$$

143 *Proof.* Note that here, as on some previous occasions, the asymptotic formula (3.66) for G_σ reduces to the asymptotic formula for G as $\sigma \to 1/2+0$. The proofs of both (3.65) and (3.66) are analogous. The proof of (3.65) is given by Hafner-Ivić [54], and here only (3.66) will be proved. We use Theorem 3.2 to write

$$\begin{aligned}
G_\sigma(t) &= \left\{2^{\sigma-2}\left(\frac{\pi}{t}\right)^{\sigma-1/2}\sum_{n\le t}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(ar\sinh\sqrt{\frac{\pi n}{2t}}\right)^{-2}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/4}\sin f(t,n)\right\}\\
&+\left\{-2\left(\frac{2\pi}{t}\right)^{\sigma-\frac{1}{2}}\sum_{n\le c_0t}\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\log\frac{t}{2pin}\right)^{-2}\sin g(t,n) + O(T^{3/4-\sigma})\right\}\\
&= \sum\nolimits_1(t) + \sum\nolimits_2(t),
\end{aligned}$$

say, and assume $T \leq t \leq 2T$. Then we have

$$\int\limits_T^{2T} G_\sigma^2(t)dt = \int\limits_T^{2T} \sum\nolimits_1^2(t)dt + \int\limits_T^{2T} \sum\nolimits_2^2(t)dt + 2\int\limits_T^{2T} \sum\nolimits_1(t) \sum\nolimits_2(t)dt,$$

so that in view of the Cauchy-Schwarz inequality (3.66) follows from

$$\int\limits_T^{2T} \sum\nolimits_2^2(t)dt \ll T^{5/2-2\sigma} \tag{3.68}$$

and

$$\int\limits_T^{2T} \sum\nolimits_1^2(t)dt = C(\sigma)(2T)^{7/2-2\sigma} - C(\sigma)T^{7/2-2\sigma} + O(T^{3-2\sigma}) \tag{3.69}$$

on replacing T by $T2^{-j}$ and summing over $j = 1, 2, \ldots$. the bound given by (3.68) follows easily by squaring and integrating termwise, since the sum in $\sum_2(t)$ is essentially a Dirichlet polynomial of length $\ll T$, so its contribution to the left-hand side of (3.68) will be $\ll T^{1+\epsilon}$, and the error term $O(T^{3/4-\sigma})$ makes a contribution which is $\ll T^{5/2-2\sigma}$. By grouping together terms with $m = n$ and $m \neq n$ it is seen that the left-hand side of (3.69) equals

$$4^{\sigma-2}\pi^{2\sigma-1}\int\limits_T^{2T} t^{1-2\sigma}\sum_{n\leq t}\sigma_{1-2\sigma}^2(n)n^{2\sigma-2}\left(ar\sinh\sqrt{\frac{\pi n}{2t}}\right)^{-4}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/2}$$

$$\sin^2 f(t,n)dt + 4^{\sigma-2}\pi^{2\sigma-1}\int\limits_T^{2T} t^{1-2\sigma}\Bigg\{\sum_{m\neq n\leq t}(-1)^{m+n}\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)(mn)^{\sigma-1}$$

$$\times\left(\frac{t}{2\pi m}+\frac{1}{4}\right)^{-1/4}\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/4}\left(ar\sinh\sqrt{\frac{\pi n}{2t}}\right)^{-2}\left(ar\sinh\sqrt{\frac{\pi n}{2t}}\right)^{-2}$$

$$\sin f(t,m)\sin f(t,n)\Bigg\}dt = \sum\nolimits' + \sum\nolimits'',$$

say, and the main terms in (3.69) will come from $\sum'$. We have 144

$$\sum{}' = S_1 + S_2,$$

where

$$S_1 = 4^{\sigma-2}\pi^{2\sigma-1} \sum_{n \le T^{1/2}} \sigma^2_{1-2\sigma}(n) n^{2\sigma-2} \int\limits_T^{2T} t^{1-2\sigma} \left(\frac{t}{2\pi n} + \frac{1}{4}\right)^{-1/2}$$
$$\left(ar\sinh\sqrt{\frac{\pi n}{2t}}\right)^{-4} \sin^2 f(t,n) dt,$$
$$S_2 \ll \sum_{T^{1/2} < n \le T} d^2(n) n^{2\sigma-2} T^{2-2\sigma} \left(\frac{T}{n}\right)^{-1/2} \left(\frac{T}{n}\right)^2$$
$$= T^{7/2-2\sigma} \sum_{T^{1/2} < n \le T} d^2(n) n^{2\sigma-7/2} \ll T^{\sigma-5/4} \log^3 T \cdot T^{7/2-2\sigma}$$
$$= T^{\frac{9}{4}-\sigma} \log^3 T$$

so that

$$S_2 \ll T^{3-2\sigma},$$

since $9/4 - \sigma < 3 - 2\sigma$ for $\sigma < 3/4$.

Simplifying S_1 by Taylor's formula and using Lemma 2.1 it follows that

$$S_4 = 4^{\sigma-2}\pi^{2\sigma-1} \sum_{n \le T^{1/2}} \sigma^2_{1-2\sigma}(n) n^{2\sigma-2} \int\limits_T^{2T} t^{1-2\sigma} t^{-1/2} (2\pi)^{1/2} n^{1/2} \pi^{-2} n^{-2} 4t^2 \cdot$$
$$\cdot \frac{1 - \cos 2f(t,n)}{2} \left(1 + O\left(\left(\frac{n}{T}\right)\right)\right)^{1/2} dt$$
$$= \sqrt{2\pi}\frac{4^{\sigma-1}}{2}\pi^{2\sigma-3} \sum_{n \le T^{1/2}} \sigma^2_{1-2\sigma}(n) n^{2\sigma-7/2} \int\limits_T^{2T} t^{5/2-2\sigma} (1 - \cos 2f(t,n)) dt$$
$$+ O\left(\sum_{n \le T^{1/2}} \sigma^2_{1-2\sigma}(n) n^{2\sigma-5/2} \int\limits_T^{2T} t^{2-2\sigma} dt\right)$$
$$= \frac{1}{2} 4^{\sigma-1} \pi^{2\sigma-3} (2\pi)^{1/2} \sum_{n \le T^{1/2}} \sigma^2_{1-2\sigma}(n) n^{2\sigma-7/2} \int\limits_T^{2T} t^{5/2-2\sigma} dt + O(T^{3-2\sigma})$$

$$= C(\sigma)(2T)^{7/2-2\sigma} - C(\sigma)T^{7/2-2\sigma} + O(T^{7/2-2\sigma}T^{\sigma-5/4}\log^4 T + T^{3-2\sigma})$$

on writing 145

$$\sum_{n\le T^{1/2}} = \sum_{n=1}^{\infty} - \sum_{n>T^{1/2}}$$

and estimating the tails of the series trivially. It remains yet to consider $\sum''$. By symmetry we have

$$\sum{}'' \ll \sum_{n<m\le 2T}\Bigg|\int\limits_{\max(m,T)}^{2T} \sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)(mn)^{\sigma-1} \times \left(\frac{t}{2\pi m}+\frac{1}{4}\right)^{-1/4}$$
$$\times\left(\frac{t}{2\pi n}+\frac{1}{4}\right)^{-1/4}\left(ar\sinh\sqrt{\frac{\pi m}{2t}}\right)^{-2}\left(ar\sinh\sqrt{\frac{\pi n}{2t}}\right)^{-2}\sin f(t,m)\sin f(t,n)dt\Bigg|.$$

The sine terms yield exponentials of the form $\exp\{if(t,m)\pm if(t,n)\}$, and the contribution of the terms with the plus sign is easily seen to be $\ll T^{3-2\sigma}$ by Lemma 2.1. For the remaining terms with the minus sign put

$$F(t) := f(t,m) - f((t,n)$$

for any fixed $n < m \le 2T$, so that by the mean value theorem

$$F'(t) = 2ar\sinh\sqrt{\frac{\pi n}{2t}} - 2ar\sinh\sqrt{\frac{\pi n}{2t}} \asymp T^{-1/2}(m^{1/2}-n^{1/2}).$$

Again by Lemma 2.1

$$\sum{}'' \ll T^{3-2\sigma}\sum_{n<m\le 2T}(mn)^{\epsilon-5/4+\sigma-1/2}(m^{1/2}-n^{1/2})^{-1}$$
$$= T^{3-2\sigma}\left(\sum_{n\le 1/2m} + \sum_{n>1/2m}\right) = T^{3-2\sigma}(S_3+S_4),$$

say. Since $1/2 < \sigma < 3/4$ then trivially $S_3 \ll 1$ if ϵ is sufficiently small, and also 146

$$S_4 \ll \sum_{m\le 2T} m^{2\epsilon+2\sigma-7/2}\sum_{1/2m<n<m}\frac{m^{1/2}}{m-n} \ll \sum_{m\le 2T} m^{2\epsilon+2\sigma-3}\log m \ll 1.$$

This proves (3.69), and completes the proof of (3.66). □

3.4 The Zeros of $E(T) - \pi$

From Theorem 3.3 it follows by continuity that every interval $[T, T + DT^{1/2}]$ for $T \geq T_0$ and suitable $D > 0$ contains a zero of $E(t) - \pi$. The same is true, of course, for the function $E(t)$ itself, but it seems more appropriate to consider the zeros of $E(t) - \pi$, since by Theorem 3.1 $E(t)$ has the mean value π. Naturally, one can make the analogous conclusion also for $E_\sigma(t) - B(\sigma)$, but the numerical calculations of zeros of $E_\sigma(t) - E(\sigma)$ would be more tedious. Also the function $E(T)$ seems more important, as the results concerning it embody usually much information about the behaviour of $\zeta(s)$ opn the critical line $\sigma = 1/2$, which is one of the main topics of zeta-function theory.

In [1] the author and H.J.J. te Riele investigated the zeros of $E(T) - \pi$ both from theoretical and numerical viewpoint. From many numerical data obtained in that work we just present here a table with the first 100 zeros of $E(T) - \pi$. Hence forth t_n will denote the n^{th} distinct zero of $E(T) - \pi$. All the zeros not exceeding 500 000 were found; all were simple and $t_n = 499993.656034$ for $n = 42010$ was the largest one. The interested reader will find other data, as well as the techniques used in
147 computations, in the aforementioned work of Iveć-te Riele.

n	t_n	n	t_n	n	t_n	n	t_n
1	1.199593	26	99.048912	51	190.809257	76	318.788055
2	4.757482	27	99.900646	52	192.450016	77	319.913514
3	9.117570	28	101.331134	53	199.646158	78	321.209365
4	13.545429	29	109.007151	54	211.864426	79	326.203904
5	17.685444	30	116.158343	55	217.647450	80	330.978187
6	22.098708	31	117.477368	56	224.290283	81	335.589281
7	27.736900	32	119.182848	57	226.323460	82	339.871410
8	31.884578	33	119.182848	58	229.548079	83	343.370082
9	35.337567	34	121.514013	59	235.172515	84	349.890794
10	40.500321	35	126.086873	60	239.172515	85	354.639224
11	45.610584	36	130.461139	61	245.494672	86	358.371624
12	50.514621	37	136.453527	62	256.571746	87	371.554495
13	51.658642	38	141.371299	63	362.343301	88	384.873869

n	t_n	n	t_n	n	t_n	n	t_n
14	52.295421	39	144.418515	64	267.822499	89	390.001409
15	54.295421	40	149.688528	65	280.805140	90	396.118200
16	56.819660	41	154.448617	66	289.701637	91	399.102390
17	63.010778	42	159.295786	67	290.222188	92	402.212210
18	69.178386	43	160.333263	68	294.912620	93	406.737516
19	73.799939	44	160.636660	69	297.288651	94	408.735190
20	76.909522	45	171.712482	70	297.883251	95	417.047725
21	81.138399	46	179.509721	71	298.880777	96	430.962383
22	85.065530	47	181.205224	72	299.919407	97	434.927645
23	90.665198	48	182.410680	73	308.652004	98	439.425963
24	95.958639	49	182.899197	74	314.683833	99	445.648250
25	97.460878	50	185.733682	75	316.505614	100	448.037348

As already mentioned, from Theorem 3.3 it follows that every interval $[T, T + DT^{1/2}]$ contains a zero of $E(t) - \pi$, hence

$$T_{n+1} - t_n \ll t_n^{1/2}. \tag{3.70}$$

On the other hand, the gaps between the consecutive t_n's may be sometimes quite large. This follows from the inequality

$$\max_{t_n \le t \le t_{n+1}} |E(t) - \pi| \ll (t_{n+1} - t_n) \log t_n, \tag{3.71}$$

so if we define

$$\chi = \inf\{c > 0 : t_{n+1} - t_n \ll t_n^c\}, \alpha = \inf\{c \ge 0 : E(t) \ll t^c\},$$

then (3.70) gives $x \le 1/2$ and (3.71) gives **148**

$$x \ge \alpha. \tag{3.72}$$

Since we know from Ω-results on $E(T)$ that $\alpha \ge 1/4$, it follows that $x \ge 1/4$. There is some numerical evidence which supports our conjecture that $x = 1/4$, which if true would be very strong, and is certainly out of reach at present. To prove (3.71) let

$$|E(\bar{t}) - \pi| = \max_{t_n \le t \le t_{n+1}} |E(t) - \pi|.$$

Suppose $E(\bar{t}) - \pi > 0$ (the other case is analogous) and use (2.77). Then for some $C > 0$, $n \geq n_0$ and $0 \leq H \leq \frac{1}{2}t_n$,

$$E(\bar{t} + H) - \pi \geq E(\bar{t}) - \pi - CH \log t_n > 0$$

holds if $0 \leq H \leq (E(\bar{t}) - \pi)/(2C \log t_n)$. Thus $E(t) - \pi$ has no zeros in $[\bar{t}, \bar{t} + H]$ with $H = (E(\bar{t})/2C \log t_n)$. Consequently

$$(E(\bar{t}) - \pi)/(2C \log t_n) = H \leq t_{n+1} - t_n,$$

and (3.71) follows.

Another important problem is the estimation of t_n as a function of n. Alternatively, one may consider the estimation of the counting function

$$K(T) := \sum_{t_n \leq T} 1.$$

Since $[T, T + DT^{1/2}]$ contains a t_n for $T \geq T_0$, it follows that $K(T) \gg T^{1/2}$. Setting $T = t_n$ we have $n = K(t_n) \gg t_n^{1/2}$, giving

$$t_n \ll n^2. \tag{3.73}$$

This upper bound appears to be crude, and we proceed to deduce a lower bound for t_n, which is presumably closer to the true order of
149 magnitude of t_n. Note that $K(T) \ll M(T)$, where $M(T)$ denotes the number of zeros in $[0, T]$ of the function

$$E'(t) = \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 - \log\left(\frac{t}{2\pi}\right) - 2\gamma = Z^2(t) - \log\left(\frac{t}{2\pi}\right) - 2\gamma.$$

Here, as usual, we denote by $Z(t)$ the real-valued function

$$z(t) = \chi^{-1/2}(\frac{1}{2} + it)\zeta(\frac{1}{2} + it),$$

where

$$\chi(s) = \frac{\zeta(s)}{\zeta(1 - s)} = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s).$$

Thus $|Z(t)| = |\zeta(^1/2 + it)|$, and the real zeros of $Z(t)$ are precisely the ordinates of the zeros of $\zeta(s)$ on $\mathrm{Re}\, s = 1/2$. But

$$M(T) = M_1(T) + M_2(T),$$

where $M_1(T)$ and $M_2(T)$ denote the number of zeros of

$$Z(t) - \left(\log \frac{t}{2\pi} + 2\gamma\right)^{1/2}, \quad Z(t) + \left(\log \frac{t}{2\pi} + 2\gamma\right)^{1/2}$$

in $[0, T]$, respectively. Note that $M_j(T) \ll L_j(T)$, where $L_j(T)$ is the number of zeros of

$$Z'(t) + \frac{(-1)^j}{2t\sqrt{\log(t/2\pi)}}$$

in $[0, T]$. It was shown by R.J. Anderson [1] that the number of zeros of $Z'(t)$ in $[0, T]$ is asymptotic to $\frac{T}{2\pi} \log T$, and by the same method it follows that $L_j(T) = O(T \log T)$. Hence $K(T) \ll T \log T$, and taking $T = t_n$ we obtain

$$t_n \gg n/\log n. \tag{3.74}$$

In the range for n that was investigated numerically by Ivić- te Riele [75], t_n behaves approximately like $n \log n$, but it appears quite difficult to prove this.

Another inequality involving the t_n's may be obtained as follows. Observe that $(T(t) - \pi)$' must vanish at least once in (t_n, t_{n+1}). Hence this interval contains a point t_0 such that

$$0 = E'(t_0) = \left|\zeta\left(\frac{1}{2} + it_0\right)\right|^2 - \left(\log \frac{t_0}{2\pi} + 2\gamma\right).$$

Therefore it follows that **150**

$$\max_{t_n \le t \le t_{n+1}} \left|\zeta\left(\frac{1}{2} + it\right)\right| \ge \left(\log \frac{t_n}{2\pi} + 2\gamma\right)^{1/2}. \tag{3.75}$$

This inequality shows that the maximum of $|\zeta(\frac{1}{2} + it)|$ between consecutive zeros of $E(T) - \pi$ cannot be too small, even if the gap between such zeros is small. On the other hand, the maximum of $|\zeta(\frac{1}{2}+it)|$ can be larger over long intervals, since R. Balasubramanian [7] proved

$$\max_{T\le t\le T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right| \ge \exp\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right) \tag{3.76}$$

in the range $100\log\log T \le H \le T$. Using (3.71) we may investigate sums of powers of consecutive gaps $t_{n+1}-t_n$. Namely from Theorem 2.4 we have, as $T\to\infty$,

$$C_1T^{3/2} \sim \int\limits_T^{2T} E^2(t)dt \sim \sum_{\substack{T<t_n\le 2T\\ T^{1/4}\log^{-2}T\le T_{n+1}-t_n}} \int\limits_{t_n}^{t_{n+1}} E^2(t)dt. \tag{3.77}$$

The contribution of gaps less than $T^{1/4}\log^{-2}T$ is negligible by (3.71) and trivial estimation. From (3.77) we infer by using (3.71) that

$$\begin{aligned} T^{3/2} &\ll \sum_{T<t_n\le 2T, t_{n+1}-t_n\ge T^{1/4}\log^{-2}T} (t_{n+1}-t_n)\left(\max_{t\in[t_n,t_{n+1}]}|E(t)-\pi|^2+1\right)\\ &\ll \log^2 T \sum_{T<t_n\le 2T, t_{n+1}-t_n\ge T^{1/4}\log^{-2}T} (t_{n+1}-t_n)^3+T. \end{aligned}$$

Replacing T by $T2^{-j}$ and summing over $j\ge 1$ this gives

$$T^{3/2}\log^{-2}T \ll \sum_{t_n\le T}(t_{n+1}-t_n)^3. \tag{3.78}$$

In general, for any fixed $\alpha\ge 1$ and any given $\epsilon>0$

$$T^{1/4(3+\alpha-\epsilon)} \ll_{\epsilon,\alpha} \sum_{t_n\le T}(t_{n+1}-t_n)^{\alpha}. \tag{3.79}$$

151 This follows along the same lines as (3.78), on using

$$T^{1+1/4a-\epsilon} \ll_{a,\epsilon} \int\limits_2^T |E(t)|^a dt \quad (a\ge 0, \epsilon>0) \tag{3.80}$$

with $a = \alpha - 1$. The bound (3.80) for $a > 2$ (without "ϵ") follows easily from Theorem 2.4 and Hölder's inequality, and for $o < a < 2$ it follows from

$$T^{3/2} \ll \int_2^T E^{1/2a}(t)E^{2-1/2a}(t)dt \leq \left(\int_2^T |E(t)|^2 dt\right)^{1/2} \left(\int_2^T |E(t)|^{4-a} dt\right)^{1/2}$$

on using

$$\int_0^T |E(t)|^A dt \ll T^{1+1/4A+\epsilon} \quad \left(0 \leq A \leq \frac{35}{4}\right), \tag{3.81}$$

a proof of which is given by the author in Chapter 15 of [1]. It maybe conjectured that the lower bound in (3.79) is close to the truth, that is, we expect that

$$\sum_{t_n \leq T} (t_{n+1} - t_n)^\alpha = T^{1/4(3+\alpha+o(1))} \quad (\alpha \geq 1, T \to \infty), \tag{3.82}$$

but unfortunately at present we cannot prove this for any specific $\alpha > 1$ (for $\alpha = 1$ this is trivial).

If u_n denotes the n^{th} zero of $G(T)$, then by Theorem 3.3 there exist infinitely many u_n's the sequence $\{u_n\}_{n=1}^\infty$ is unbounded, and moreover

$$u_{n+1} - u_n \ll u_n^{1/2}.$$

This bound is actually close to being best possible, since we can prove without difficulty that

$$\beta := \limsup_{n \to \infty} \frac{\log(u_{n+1} - u_n)}{\log u_n} = \frac{1}{2}.$$

This should be contrasted with what we have succeeded in proving for the sequence $\{t_n\}_{n=1}^\infty$, namely only

$$\frac{1}{4} \leq \limsup_{n \to \infty} \frac{\log(t_{n+1} - t_n)}{\log t_n} \leq \frac{1}{2}.$$

To prove that $\beta = \frac{1}{2}$ it remains to show that $\beta < \frac{1}{2}$ cannot hold. We 152 have

$$G(T+H) - G(T) = \int_T^{T+H} (E(t) - \pi)dt,$$

and we choose T (this is possible by Theorem 3.3) such that $G(T) < -BT^{3/4}$, and H such that $G(T+H) = 0$. Then we have $O < H \ll T^{\beta+\epsilon}$, and using the Cauchy-Schwarz inequality it follows that

$$T^{3/2} \ll G^2(T) \ll H \int_T^{T+H} E^2(t)dt + H^2 \ll H^2 T^{1/2} + HT \log^5 T$$

by appealing to Theorem 2.4. Hence

$$T^{3/2} \ll T^{1/2+2\beta+2\epsilon} + T^{1+\beta+\epsilon} \log^5 T,$$

which is impossible if $\beta < \frac{1}{2}$ and $\epsilon > 0$ is sufficiently small. This proves that $\beta = \frac{1}{2}$. The reason that one can obtain a sharper result for the sequence $\{u_n\}_{n=1}^{\infty}$ than for the sequence $\{t_n\}_{n=1}^{\infty}$ is essentially that $G(T)$ is the integral of $E(T) - \pi$, and possesses the representation (3.28) involving an absolutely convergent series which is easily manageable. No expression of this type seems to exist for $E(T)$.

3.5 Some Other Results

In this section we shall investigate some problems concerning $E(T)$ that were not treated before. In particular, we shall consider integrals of the type

$$I = I(T, H) = \int_T^{T+H} f(E(t))|\zeta(1/2 + it)|^2 dt, \tag{3.83}$$

where $T \geq T_0$, $0 \leq H \leq T$, and $f(t)$ is a given function which is
153 continuous in $[T, T+H]$. The method for evaluating the integral in (3.83) is very simple. Namely, if $F' = f$, then from the definition of

$E(T)$ it follows that

$$I = \int_T^{T+H} f(E(t))\left(E'(t) + \log\frac{t}{2\pi} + 2\gamma\right)dt \tag{3.84}$$
$$= F(E(T+H)) - F(E(T)) + \int_T^{T+H} f(E(t))\left(\log\frac{t}{2\pi} + 2\gamma\right)dt.$$

Therefore the problem is reduced to a simpler one, namely to the evaluation of the integral where $|\zeta|^2$ is replaced by $\log(t/2\pi) + 2\gamma$. If T and $T + H$ are points at which $E(T) = E(T + H)$, then (3.84) simplifies even further. As the first application we prove

Theorem 3.6. *With* $c = \frac{2}{3}(2\pi)^{-1/2}\zeta^4\left(\frac{3}{2}\right)/\zeta(3)$ *we have*

$$\int_0^T E^2(t)|\zeta(1/2+it)|^2 dt = c\left(\log\frac{T}{2\pi} + 2\gamma - \frac{2}{3}\right)T^{3/2} + O(T\log^5 T) \tag{3.85}$$

and

$$\int_0^T E^4(t)|\zeta(1/2+it)|^2 dt \ll T^{3+\epsilon}, \tag{3.86}$$

$$\int_0^T E^6(t)|\zeta(1/2+it)|^2 dt \ll T^{5/2+\epsilon}, \tag{3.87}$$

$$\int_0^T E^8(t)|\zeta(1/2+it)|^2 dt \ll T^{3+\epsilon}. \tag{3.88}$$

Proof. To prove (3.85) we apply (3.84) with $H = T$, $f(t) = t^2$, $F(t) = t^3$. Using the weak bound $E(t) \ll t^{1/3}$ and the mean square formula (2.100) with the error term $O(T\log^4 T)$, it follows that

$$\int_T^{2T} E^2(t)|\zeta(1/2+it)|^2 dt = O(T) + \int_T^{2T} E^2(t)\left(\log\frac{t}{2\pi} + 2\gamma\right)dt$$

$$= \left(\int_0^t E^2(u)du\right)\left(\log\frac{t}{2\pi} + 2\gamma\right)\Bigg|_T^{2T} - \int_T^{2T}\left(\int_0^T E^2(u)du\right)\frac{dt}{t} + o(T)$$

$$= ct^{3/2}\left(\log\frac{t}{2\pi} + 2\gamma - \frac{2}{3}\right)\Bigg|_T^{2T} + O(T\log^5 T).$$

$\square$

154 Replacing T by $T2^{-j}$ and summing over $j = 1, 2, \ldots$ we obtain (3.85). the remaining estimates (3.86)-(3.88) are obtained analogously by using (3.81). The upper bounds in (3.86)-(3.88) are close to being best possible, since

$$\int_0^T |E(t)|^A|\zeta(1/2+it)|^2 dt \gg \begin{cases} T^{1+1/4A-\epsilon} & (0 \le A < 2), \\ T^{1+1/4A}\log T & (A \ge 2), \end{cases} \tag{3.89}$$

for any fixed $A \ge 0$ and $\epsilon > 0$. For $A \ge 2$ this follows easily from (3.85) and Hölder's inequality for integrals, since

$$\int_0^T E^2(t)|\zeta(1/2+it)|^2 dt = \int_0^T E^2(t)|\zeta|^{4/A}|\zeta|^{2(1-2/A)}dt$$

$$\le \left(\int_0^T |E(t)|^A|\zeta(1/2+it)|^2 dt\right)^{2/A}\left(\int_0^T |\zeta(1/2+it)|^2 dt\right)^{1-\frac{2}{A}}.$$

For $O \le A \le 2$ we use again (3.85) and Hölder's inequality to obtain

$$T^{3/2}\log T \ll \int_0^T E^{1/2A}(t)|\zeta(1/2+it)|E^{2-1/2A}(t)|\zeta(1/2+it)|dt$$

$$\le \left(\int_0^T |E(t)|^A|\zeta(1/2+it)|^2 dt\right)^{1/2}\left(\int_0^T |E(t)|^{8-2A}dt\right)^{1/4}\left(\int_0^T |\zeta(1/2+it)|^4 dt\right)^{1/4}.$$

Using (3.81) (with A replaced by $8-2A$) and the weak bound

$$\int_0^T |\zeta(1/2+it)|^4 dt \ll T^{1+\epsilon}$$

we obtain the first part of (3.89).

Perhaps the most interesting application of our method is the evaluation of the integral

$$\int_0^T E(t)|\zeta(1/2+it)|^2 dt. \tag{3.90}$$

The function $E(t)$ has the mean value π in view of Theorem 3.1, **155**
while $|\zeta(1/2+it)|^2$ has the average value $\log t$. Therefore the integral in (3.90) represents in a certain sense the way that the fluctuations of these important functions are being superimposed. We shall prove the following

Theorem 3.7. *Let $U(T)$ be defined by*

$$\int_0^T E(t)|\zeta(1/2+it)|^2 dt = \pi T\left(\log\frac{T}{2\pi}+2\gamma-1\right)+U(T), \tag{3.91}$$

and let $V(T)$ be defined by

$$\int_0^T U(t)dt = \frac{\zeta^4(3/2)}{3\sqrt{2\pi}\zeta(3)}T^{3/2}+V(T). \tag{3.92}$$

Then

$$U(T)=O(T^{3/4}\log T), U(T)=\Omega_\pm(T^{3/4}\log T), \tag{3.93}$$

$$V(T)=O(T^{5/4}\log T), V(T)=\Omega_\pm(T^{5/4}\log T) \tag{3.94}$$

and

$$\int_2^T U^2(t)dt = T^{5/2}P_2(\log T)+O(T^{9/4+\epsilon}), \tag{3.95}$$

where $P_2(x)$ is a suitable quadratic function in x.

Proof. We begin the proof by noting that (3.84) gives

$$\int_2^T E(t)|\zeta(1/2+it)|^2 dt = \frac{1}{2}E^2(T) + \int_2^T E(t)\left(\log\frac{t}{2\pi} + 2\gamma\right) + O(1).$$

In view of the definition (3.1) of $G(T)$ the integral on the right-hand side of this equality becomes

$$\begin{aligned}
&\int_2^T \pi\left(\log\frac{t}{2\pi} + 2\gamma\right)dt + \int_2^T \left(\log\frac{t}{2\pi} + 2\gamma\right)dG(t)\\
&= \pi T\left(\log\frac{T}{2\pi} + 2\gamma\right) - \pi\int_2^T dt + O(1) + G(T)\left(\log\frac{T}{2\pi} + 2\gamma\right) - \int_2^T G(t)\frac{dt}{t}.
\end{aligned}$$

156 Hence

$$U(T) = \frac{1}{2}E^2(T) + G(T)\left(\log\frac{T}{2\pi} + 2\gamma\right) - \int_2^T G(t)\frac{dt}{t} + O(1).$$

Using Theorem 3.1 and Lemma 2.1 it follows that

$$\int_2^T G(t)\frac{dt}{t} = O(T^{1/4}). \tag{3.96}$$

This gives at once

$$U(T) = \frac{1}{2}E^2(T) + G(T)\left(\log\frac{T}{2\pi} + 2\gamma\right) + O(T^{1/4}). \tag{3.97}$$

Since $E(T) \ll T^{1/3}$, then using (3.28) and $G(T) = \Omega_\pm(T^{3/4})$ we obtain (3.93). Therefore the order of magnitude of $U(T)$ is precisely determined, and we pass on to the proof of (3.94) and (3.95). From (3.97) we have

$$\int_{1/2T}^T U^2(t)dt = \int_{1/2T}^T G^2(t)\left(\log\frac{t}{2\pi} + 2\gamma\right)^2 dt + O\left(\int_{1/2T}^T (E^4(t) + T^{1/2})dt\right)$$

$$+ O\left(\int\limits_{1/2T}^{T} |G(t)|(E^2(t) + T^{1/4}) \log T dt\right).$$

Using (3.81) with $A = 4$ it is seen that the contribution of the first O-term above is $T^{2+\epsilon}$. To estimate the second O-term we use (3.65) and the Cauchy-Schwarz inequality. We obtain a contribution which is

$$\ll \log T \left\{\int\limits_{1/2T}^{T} G^2(t)dt \left(\int\limits_{1/2T}^{T} E^4(t)dt + T^{3/2}\right)\right\}^{1/2} \ll T^{9/4+\epsilon}$$

Integration by parts and (3.65) yield ($B = \zeta^4(5/2)/(5\pi\sqrt{2\pi}\zeta(5))$)

$$\begin{aligned}\int\limits_{1/2T}^{T} G^2(t)\left(\log\frac{t}{2\pi} + 2\gamma\right)^2 dt &= (Bt^{5/2} + O(t^{2+\epsilon}))\left(\log\frac{t}{2\pi} + 2\gamma\right)^2\Bigg|_{1/2T}^{T} \\ -2\int 1/2T^T(Bt^{3/2} + O(t^{1+\epsilon}))\left(\log\frac{t}{2\pi} + 2\gamma\right)dt &= t^{5/2}P_2(\log t)\Bigg|_{1/2T}^{T} \\ &+ O(T^{2+\epsilon}),\end{aligned}$$

where $P_2(x) = a_0x^2 + a_1x + a_2$ with a_0, a_1, a_2 effectively computable. 157
This means that we have shown that

$$\int\limits_{1/2T}^{T} U^2(t)dt = t^{5/2}P_2(\log t)\Bigg|_{1/2T}^{T} + O(T^{9/4+\epsilon}),$$

so that replacing T by $T2^{-j}$ and summing over $j = 0, 1, 2, \ldots$ we obtain (3.95). Probably the error term in (3.95) could be improved to $O(T^{2+\epsilon})$ (in analogy with (3.65)), but this appears to be difficult.

Now we shall establish (3.92) with the O-result (3.94). Integrating (3.97) with the aid of Theorem 2.4 we have

$$\int\limits_{1/2T}^{T} U(t)dt = \frac{1}{2}c\left(T^{3/2} - \left(\frac{1}{2}\right)^{3/2}\right) + O(T^{5/4}) + \int\limits_{1/2T}^{T} G(t)\left(\log\frac{t}{2\pi} + 2\right)dt. \quad (3.98)$$

Using Theorem 3.1 (with $N = T$) it is seen that the contribution of the sum $\sum\limits_{n\leq N'}$ to the last integral is $O(T)$, while the contribution of the error term $O(T^{1/4})$ is trivially $O(T^{5/4}\log T)$. The contribution of the sum $\sum\limits_{n\leq N}$ is, after simplification by Taylor's formula,

$$\int\limits_{1/2T}^{T} 2^{-1/4}\pi^{-3/4}t^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-5/4}\sin\left(\sqrt{8\pi nt}-\frac{\pi}{4}\right)\left(\log\frac{t}{2\pi}+2\gamma\right)dt + O(T^{5/4})$$

$$= 2^{-3/4}\pi^{-5/4}\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}\int\limits_{1/2T}^{T} t^{5/4}\left(\log\frac{t}{2\pi}+2\gamma\right)\frac{d}{dt}\left\{\cos(\sqrt{8\pi nt}-\frac{\pi}{4})\right\} + O(T^{5/4})$$

$$= 2^{-3/4}\pi^{-5/4}\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}\left\{t^{5/4}\left(\log\frac{t}{2\pi}+2\gamma\right)\cos\left(\sqrt{8\pi nt}-\frac{\pi}{4}\right)\right\}\Bigg|_{1/2T}^{T}$$

$$- 2^{-3/4}\pi^{-5/4}\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}\int\limits_{1/2T}^{T}\left(\frac{5}{4}t^{\frac{1}{4}}\left(\log\frac{t}{2\pi}+2\gamma\right)+t^{\frac{1}{4}}\right)$$

$$\cos\left(\sqrt{8\pi nt}-\frac{t}{4}\right)dt + O(T^{5/4})$$

$$= 2^{-3/4}\pi^{5/4}\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}\left\{t^{5/4}\left(\log\frac{t}{2\pi}+2\gamma\right)\cos\left(\sqrt{8\pi nt}\right)\right\}\Bigg|_{1/2T}^{T} + O(T^{5/4})$$

158 if we use Lemma 2.1 to estimate the last integral above. Inserting this expression in (3.98) we obtain (3.92) with

$$V(T) = W(T) + 2^{-3/4}\pi^{-5/4}T^{5/4}\left(\log\frac{T}{2\pi}+2\gamma\right)$$
$$\sum_{n=1}^{\infty}(-1)^{n-1}d(n)n^{-7/4}\cos\left(\sqrt{8\pi nT}-\frac{\pi}{4}\right), \tag{3.99}$$

where

$$W(T) = O(T^{5/4}\log T). \tag{3.100}$$

This proves the O-result of (3.94). By the method of Lemma 3.3 and Lemma 3.4 it is easily seen that the series in (3.99) is $\Omega_{\pm}(1)$, so that the omega-result of (3.94) follows from a sharpening of (3.100), namely

$$W(T) = O(T^{5/4}). \tag{3.101}$$

To see that (3.101) holds we have to go back to (3.98) and recall that it is the error term $O(T^{1/4})$ which, after integration in (3.98), leads to (3.100) since actually all other error terms will be $O(T^{5/4})$. Observe that in (3.98) we have in fact a double integration, since $G(T)$ itself is an integral (the log-factor is unimportant). By analyzing the proof of Lemma 3.1 it will be seen that instead of the error term $\Phi_0\mu_0 F_0^{-3/2}$ with $\gamma+1$ replacing γ we shall have $\gamma+2$ replacing γ in view of double integration. This produces in our case the error term $O((T/k)^{1/2(\gamma+2-\alpha-\beta)}T^{-1/4}k^{-5/4})$, which for $\beta=\frac{1}{2}, \gamma=1, \alpha\to\frac{1}{2}+0$ will eventually give $O(T^{3/4}\log T)$ and therefore (3.101) holds. This completes the proof of Theorem 3.7. $\square$

We shall conclude this chapter by establishing some further analogues between the functions $E(T)$ and $\Delta(x)$. Namely, we shall use the Ω_+-results (3.43) and (3.41) to derive omega-results in the mean square for the functions in question. Thus we write

$$\int\limits_2^T E^2(t)dt = cT^{3/2} + F(T)\ \left(c = \frac{2}{3}(2\pi)^{-1/2}\frac{\zeta^4(3/2)}{\zeta(3)}\right), \tag{3.102}$$

$$\int\limits_2^T \Delta^2(t)dt = dT^{3/2} + H(T)\ \left(d = \frac{\zeta^4(3/2)}{6\pi^2\zeta(3)}\right). \tag{3.103}$$

Upper bound results $F(T) \ll T\log^5 T$ and $H(T) \ll T\log^5 T$ were **159**
given by Theorem 2.4 and Theorem 2.5, respectively. Moreover, it was pointed out that in both bounds $\log^5 T$ may be replaced by $\log^4 T$. Now we shall prove

Theorem 3.8. *If $F(T)$ and $H(T)$ are defined by (3.102) and (3.103), respectively, then with suitable constants $B_1, B_2 > 0$*

$$F(T) = \Omega\left\{T^{3/4}(\log T)^{-1/4}(\log\log T)^{3/4(3+\log 4)}e^{-B_1\sqrt{\log\log\log T}}\right\} \tag{3.104}$$

and

$$H(T) = \Omega\left\{T^{3/4}(\log T)^{-1/4}(\log\log T)^{3/4(3+\log 4)}e^{-B_2\sqrt{\log\log\log T}}\right\}. \tag{3.105}$$

Proof. In Chapter 2 we proved the inequalities

$$E(T) \le H^{-1} \int\limits_{T}^{T+H} E(t)dt + CH \log T \ (O < H \le T, T \ge T_0, C > 0)$$

and

$$E(T) \ge H^{-1} \int\limits_{T-H}^{T} E(t)dt - CH \log T.$$

Thus by the Cauchy-Schwarz inequality we have

$$E^2(T) \le 2H^{-1} \int\limits_{T}^{T+H} E^2(t)dt + 2C^2H^2 \log^2 T \quad (E(T) > 0)$$

and $$E^2(T) \le 2H^{-1} \int\limits_{T-H}^{T} E^2(t)dt + 2C^2H^2 \log^2 T \quad (E(T) < 0)$$

so that in any case for $T \ge T_0$ and a suitable $C_1 > 0$ we obtain

$$E^2(T) \le 2H^{-1} \int\limits_{T-H}^{T+H} E^2(t)dt + C_1H^2 \log^2 T. \tag{3.106}$$

In (3.106) we take $T = T_n$, the sequence of points $T_n \to \infty$ for which
160 the Ω_+-result (3.43) is attained. If $C_1, C_2, \ldots$ denote absolute, positive constants, then we obtain

$$\begin{aligned}
&C_2H(T \log T)^{1/2}(\log\log T)^{1/2(3+\log 4)} \exp\left(-C_3 \sqrt{\log\log\log T}\right) \\
&\le 2c(T+H)^{3/2} - 2c(T-H)^{3/2} + 2F(T+H) + 2F(T-H) + C_1H^3 \log^2 T \\
&\le C_3HT^{1/2} + 2F(T+H) - 2F(T-H) + C_1H^3 \log^2 T.
\end{aligned}$$

Suppose (3.104) does not hold. Then

$$\begin{aligned}
&F(T) \ll K(T, B_1) := \\
&\qquad T^{3/4}(\log T)^{-1/4}(\log\log T)^{3/4(3+\log 4)} \exp\left(-B_1 \sqrt{\log\log\log T}\right)
\end{aligned}$$

for any given $B_1 > 0$, where $K(T, B_1)$ is an increasing function of T for $T \geq T_0(B_1)$. Now let

$$H = H(T, B_1) = (K(T, B_1)\log^{-2} T)^{1/3}.$$

Then we obtain

$$\begin{aligned} &C_4 H(T\log T)^{1/2}(\log\log T)^{1/2(3+\log 4)}\exp\left(-C_3\sqrt{\log\log\log T}\right)\\ &\leq C_5 T^{3/4}(\log T)^{-1/4}(\log\log T)^{3/4(3+\log 4)}\exp\left(-B_1\sqrt{\log\log\log T}\right), \end{aligned}$$

or after simplification

$$C_4\exp\left(-\left(\frac{1}{3}B_1 + C_3\right)\sqrt{\log\log\log T}\right) \leq C_5\exp\left(-B_1\sqrt{\log\log\log T}\right).$$

Taking $B_1 = 3C_3$ the last inequality gives a contradiction if $T \geq T(C_3)$, which proves (3.104). One proves (3.105) analogously, only the proof is even slightly easier, since in this case one has

$$\Delta(x) = H^{-1}\int_x^{x+H}\Delta(t)dt + O(H\log x)\ (x^\epsilon \leq H \leq x). \tag{3.107}$$

Namely

$$\begin{aligned} \Delta(x) - H^{-1}\int_x^{x+H}\Delta(t)dt &= H^{-1}\int_x^{x+H}(\Delta(x) - \Delta(t))dt\\ &\ll H\log x + H^{-1}\int_x^{x+H}\left(\sum_{x<n\leq x+H} d(n)\right)dt \ll H\log x, \end{aligned}$$

since **161**

$$\sum_{x<n\leq x+H} d(n) \ll H\log x \quad (x^\epsilon \leq H \leq x). \tag{3.108}$$

□

Notes For Chapter 3

Theorem 3.2 is new, while Theorem 3.1 is to be found in J.L. Hafner - A. Ivić [54]. The motivation for the study of $G(T)$ is the fact, already exploited by Hafner [53], that it is not essential to have a functional equation for the generating Dirichlet series connected with a number-theoretic error term, but a Voronoi-type representation for the error term in question. By this we mean representations such as

$$\Delta(T) = \pi^{-1}2^{-1/2}T^{1/4}\sum_{n\le T} d(n)n^{-3/4}\cos\left(4\pi\sqrt{4\pi}-\frac{\pi}{4}\right) + O(T^{\epsilon}),$$

which follows from Voronoi's classical formula for $\Delta(T)$ (see [162], [163]), and is considerably simpler than Atkinson's formula for $E(T)$. Indeed, for $E(T)$ it is not clear what the corresponding Dirichlet series should look like, while for $\Delta(T)$ it is obviously $\zeta^2(s)$.

Lemmas 3.3 and 3.4 are from A. Ivić [78], where a more general result is proved. This enables one to prove analogous results for error terms corresponding to Dirichlet series with a functional equation involving multiple gamma-factors. These error terms, studied extensively in the fundamental papers of K. Chandrasekharan and R. Narasimhan [20], [21], [22], involve functional equations of the form

$$\Delta(s)\varphi(s) = \Delta(r-s)\Psi(r-s),$$

where

$$\varphi(s) = \sum_{n=1}^{\infty} f(n)\lambda_n^{-s}, \quad \Psi(s) = \sum_{n=1}^{\infty} g(n)\mu_n^{-s}, \quad \Delta(s) = \prod_{\nu=1}^{N}\Gamma(\alpha_\nu s+\beta_\nu),$$

α_ν's are real, and some natural conditions on φ and Ψ are imposed. The class of functions possessing a functional equation of the above type is large, and much work has been done on this subject. For this the
162 reader is referred to J.L. Hafner [52], where additional references may be found.

For the method of transforming Dirichlet polynomials with the divisor function see M. Jutila [89], [92] and his monograph [95]. Further applications of this theory are to be found in M. Jutila [96], [97], [98], [99], where among other things, he reproves H. Iwaniec's result [82]

$$\int_T^{T+G} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll T^\epsilon G \left(T^{2/3} \le G \le T\right)$$

without the use of spectral theory and Kloosterman sums. The above result will be proved in this text in Chapter 4 and Chapter 5, where the fourth moment will be extensively discussed.

The first part of Theorem 3.3 is contained in A. Ivić [78], while (3.38) is new.

The first Ω-result for $E(T)$, namely $E(T) = \Omega(T^{1/4})$, was obtained by A. Good [46], and it of course follows also from the mean value result of D.R. Heath-Brown [60], given by (2.69). Good's method is based on the use of a smoothed approximate functional equation for $\zeta(s)$, obtained by the method of Good [44]. This approximate functional equation is similar in nature to the one that will be used in Chapter 4 for the investigation of higher power moments of $|\zeta(\frac{1}{2}+it)|$. In [45] Good obtained an explicit expression for $E(T)$, similar to the one obtained by R. Balasubramanian [6]. The use of the smoothing device, a common tool nowadays in analytic number theory, makes Good's proof less complicated (his error term is only $0(1)$, while Balasubramanian had $0(\log^2 T)$). However, Good's final formula contains expressions involving the smoothing functions that are not of a simple nature. Good's method is powerful enough to enable him in [46] to derive an asymptotic formula for

$$\int_0^X (E(T+H)-E(T))^2 dT \qquad (H \ll X^{1/2}),$$

and then to deduce from his result that $E(T) = \Omega(T^{1/4})$.

Theorem 3.4 is from Hafner -Ivić [54].

Concerning (3.43) and Lemma (3.5) (the latter is due to J. L. Hafner [51]), it should be remarked that $(3+\log 4)/4$ is the best possible exponent of $\log\log T$ in (3.43) that the method allows. This was observed by 163

S. Srinivasan, who remarked that in the proof one wants to have

$$\sum_{n\le x}{}' d(n) \ge (1-\delta)x\log x,\ \sum_{n\le x}{}' 1 \le x/y$$

with y as large as possible, where $\sum'$ denotes summation over a subset of natural numbers. If $\alpha > 1$, then by Hölder's inequality it follows that

$$\begin{aligned} x\log x &\ll \sum_{n\le x}{}' d(n) \le \left(\sum_{n\le x}{}' 1\right)^{1-1/\alpha}\left(\sum_{n\le x} d^\alpha(n)\right)^{1/\alpha} \\ &\le \left(\frac{x}{y}\right)^{1-1/\alpha}\left(x(\log x)^{2^\alpha-1}\right)^{1/\alpha}, \end{aligned}$$

$$\text{hence}\qquad y \ll (\log x)^{C(\alpha)},$$

$$\text{where}\qquad C(\alpha) = \frac{2^\alpha-1-\alpha}{\alpha-1} \to \log 4 - 1$$

as $\alpha \to 1+O$. Thus it is only the small factors like $\exp(C\sqrt{\log\log K})$ in Hafner's Lemma 3.5 that can be possibly improved by this method.

In the proof of Lemma 3.5 one encounters sums (see (3.51)) which are a special case of the sum

$$\sum_{n\le x} d_k(n)\omega^m(n),$$

where m, k are fixed natural numbers. An analytic method is presented in A. Ivić [76], which enables one to evaluate asymptotically sums of the form $\sum_{n\le x} f(n)g(n)$, where $f(n)$ is a suitable "small" multiplicative, and $g(n)$ a "small" additive function. In particular, it is proved that, if $m, N \ge 1$ and $k \ge 2$ are fixed integers, then there exist polynomials $P_{k,m,j}(t)$ $(j = 1,\ldots,N)$ of degree m in t with computable coefficients such that

$$\begin{aligned} \sum_{n\le x} d_k(n)\omega^m(n) = x\sum_{j=1}^{N} &P_{k,m,j}(\log\log x)\log^{k-j} x \\ &+ O\left(x(\log x)^{k-N-1}(\log\log x)^m\right). \end{aligned}$$

164 For a formulation and proof of Dirichlet's and Kronecker's approximation theorems the reader is referred to Chapter 9 of Ivić [75].

R. Balasubramanian's paper [7] in which he proves (3.76) is a continuation of Balasubramanian - Ramachandra [8], where a weaker version of this result was proved ($\frac{3}{4}$ was replaced by a smaller constant). It turns out that the limit of the value that the method can give is only slightly larger than $\frac{3}{4}$, so that any substantial improvement of (3.76) will require new ideas.

One can obtain a more precise result than $\chi \geq \frac{1}{4}$. Namely, from the Ω_+-result (3.43) it follows that

$$t_{n+1} - t_n > Bt_n^{1/4}(\log t_n)^{3/4}(\log\log t_n)^{1/4(3+\log 4)} \exp\left(-C\sqrt{\log\log\log t_n}\right)$$

for infinitely many n with suitable $B, C > O$.

The conjecture (3.82) was made by Ivić- te Riele [75]. the numerical data are so far insufficient to make any reasonable guess about the "$O(1)$" in the exponent on the right-hand side of (3.82).

The proof that

$$\limsup_{n\to\infty} \frac{\log(u_{n+1} - u_n)}{\log u_n} = \frac{1}{2}$$

is given by the author in [78], while Theorem 3.6 and Theorem 3.7 are from [76]. Thus the omega-result (3.106) answers the question posed in Ch. 15.4 of Ivić [75], and a weaker result of the same type was announced in Ch. VII of E.C. Titchmarsh [155]. However, the argument given therein is not quite correct (only $E(T) = \Omega(T^{1/4})$) is not enough to give $F(T) = \Omega(T^{3/4}(\log T)^{-1})$. This oversight was pointed out to me by D.R. Heath-Brown. Both Heath-Brown and T. Meurman independently pointed out in correspondence how (3.104) is possible if (3.43) is known.

The bound (3.107) is a special case of a useful general result of P. Shiu [151] on multiplicative functions.

Chapter 4
A General Study of Even Moments

4.1 The Error Term for the $2k^{\text{th}}$ Moment

In This Chapter we are going to study the asymptotic evaluation of the 165
integral

$$I_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \tag{4.1}$$

when $k \geq 1$ is a fixed integer. This is a problem that occupies a central position in zeta-function theory. Upper bounds for $I_k(T)$ have numerous applications, for example in zero-density theorems for $\zeta(s)$ and various divisor problems. In Chapter 2 and Chapter 3 we studied extensively the function $E(T)$, defined by

$$I_1(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right) + E(T),$$

and in Chapter 5 we shall investigate the asymptotic formula for $I_2(T)$. There we shall present the recent work of Y. Motohashi, who succeeded in obtaining a sharp asymptotic formula for a weighted integral connected with $I_2(T)$. A classical result of A.E. Ingham states that

$$I_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt = \frac{T}{2\pi^2}\log^4 T + O(T\log^3 T). \tag{4.2}$$

Ingham's proof of (4.2) was difficult, and (4.2) remained the best result of its kind for more than half a century. It should be remarked that precise asymptotic formulas for

$$\int_0^\infty e^{-\delta t}\left|\left(\frac{1}{2} + it\right)\right|^2 dt, \quad \int_0^T e^{-\delta t}\left|\left(\frac{1}{2} + it\right)\right|^4 dt \quad (\delta \to O+)$$

166 were obtained by H. Kober [101] and F.V. Atkinson [3], respectively. So far no one has succeeded in deriving from these formulas correspondingly sharp results for $I_1(T)$. In 1979 D.R. Heath Brown [61] substantially improved (4.2) by showing that

$$I_2(T) = \int_O^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt = T\sum_{j=0}^{4} a_j \log^j T + E_2(T) \tag{4.3}$$

with

$$a_4 = 1/(2\pi^2), a_3 = a\left(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2}\pi^{-2}\right) \tag{4.4}$$

and

$$E_2(T) \ll T^{7/8+\epsilon}. \tag{4.5}$$

The remaining a_j's in (4.3) can be also written down explicitly, but are not of such a simple form as a_4 and a_3. Heath-Brown's method of proof, which will be briefly discussed in Section 4.7, rests on evaluating a certain weighted integral rather than evaluating $I_2(T)$ directly. This type of technique, used also in Chapter 5, is becoming prominent in analytic number theory. It usually gives good upper bound estimates for the error terms in question. Its disadvantage is that it rarely produces an explicit expression (such as Atkinson's formula for $E(T)$ does) for the error term. Recently N.I. Zavorotnyi improved (4.5) by showing that

$$E_2(T) \ll T^{2/3+\epsilon}. \tag{4.6}$$

The proof makes heavy use of spectral theory of automorphic functions and N.V. Kuznetsov's "trace formula". Earlier H. Iwaniec [82] proved

$$\int_T^{T+G} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll GT^\epsilon \quad \left(T^{2/3} \le G \le T\right)$$

by a related technique involving results on sums of Kloosterman sums. Iwaniec's result was reproved by M. Jutila [96], [97], who used a more classical approach, based on transformation formulas involving Dirichlet polynomials with the divisor function $d(n)$. The upper bound in (4.6) will be improved in Chapter 5, where we shall show that T^ϵ may be replaced by a suitable log-power. Any improvement of the exponent $2/3$ **167**
necessitates non-trivial estimates for exponential sums with the quantities $\alpha_j H_j^3(\frac{1}{2})$ from the theory of automorphic L-functions.

When $k \ge 3$ no asymptotic formulas for $I_k(T)$ are known at present. Even upper bounds of the form

$$I_k(T) \ll_{k,\epsilon} T^{1+\epsilon}$$

would be of great interest, with many applications. The best known unconditional bound for $2 \le k \le 6$ is, apart from T^ϵ which can be replaced by log-factors,

$$I_k(T) \ll_\epsilon T^{1+\epsilon+\frac{1}{4}(k-2)}. \tag{4.7}$$

Concerning lower bounds for $I_k(T)$ we already mentioned (see (1.34)) that

$$I_k(T) \ll_k T(\log T)^{k^2}, \tag{4.8}$$

which is a result of K. Ramachandra. Presumably this lower bound is closer to the true order of magnitude than the upper bound in (4.7). Therefore it seems to make sense to define, for any fixed integer $k \ge 1$,

$$E_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt - TP_{k^2}(\log T), \tag{4.9}$$

where for some suitable constants $a_{j,k}$

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k} y^j. \tag{4.10}$$

Thus $E_1(T) = E(T)$, $P_1(y) = y + 2\gamma - 1 - \log(2\pi)$, $P_4(y)$ is given by (4.3) and (4.4). However, in the case of general k one can only hope that $E_k(T) = o(T)$ as $T \to \infty$ will be proved in the foreseeable future for any $k \geq 3$. Another problem is what are the values (explicit expressions) for the constants $a_{j,k}$ in (4.10). In fact, heretofore it has not been easy to define properly (even on heuristic grounds) the value of

$$c(k) = a_{k^2,k} = \lim_{T\to\infty} \left(T \log^{k^2} T\right)^{-1} \int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt, \tag{4.11}$$

168 provided that the existence of the limit is assumed. Even assuming unproved hypotheses, such as the Riemann hypothesis or the Lindelöf hypothesis, it does not seem easy to define $c(k)$. I believe that, if $c(k)$ exists, then for all integers $k \geq 1$

$$c(k) = 2\left(\frac{k}{2}\right)^{k^2} \frac{1}{\Gamma(k^2+1)} \prod_p \left\{\left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{j=0}^{\infty} \left(\frac{\Gamma(k+j)}{j!\Gamma(k)}\right)^2 p^{-j}\right)\right\}. \tag{4.12}$$

This formula gives $c(1) = 1$, $c(2) = 1/(2\pi^2)$, which are the correct values. A conditional lower bound for $I_k(T)$ is given by Theorem 1.5. Note that we have $c(k) = 2\left(\frac{1}{2}k\right)^{k^2} c'_k$, where c'_k is the constant defined by (1.36).

We proceed now to establish some general properties of $E_k(T)$, defined by (4.9). We have $E_k \in C^\infty(O, \infty)$ with

$$\begin{aligned} E'_k(T) &= \left|\zeta\left(\frac{1}{2} + iT\right)\right|^{2k} - \left(P_{k^2}(\log T) + P'_{k^2}(\log T)\right) \\ &= Z^{2k}(T) - \left(P_{k^2}(\log T) + P'_{k^2}(\log T)\right), \end{aligned}$$

where as usual

$$Z(t) = \chi^{-1/2}\left(\frac{1}{2} + it\right)\zeta\left(\frac{1}{2} + it\right),$$
$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right).$$

For $r \geq 2$ we obtain

$$E_k^{(r)}(T) = (Z^{2k}(T))^{(r-1)} + O_{r,\epsilon}(T^{\epsilon-1}),$$

where the $(r-1)^{\text{st}}$ derivative can be easily found by an application of Leibniz's rule and the Riemann-Siegel formula for $Z(T)$.

One can use $E_k(T)$ to obtain bounds for $\zeta\left(\frac{1}{2} + it\right)$. Using Theorem 1.2 with $\delta = 1$ we obtain

$$\begin{aligned}\left|\zeta\left(\frac{1}{2} + iT\right)\right|^{2k} &\ll \log T\left(1 + \int\limits_{T-1}^{T+1}\left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt\right)\\ &= \log T\left\{1 + (tP_{k^2}(\log t))\Big|_{T-1}^{T+1} + E_k(T+1) - E_k(T-1)\right\}.\end{aligned}$$

Therefore we obtain 169

Lemma 4.1. *For $k \geq 1$ a fixed integer we have*

$$\zeta\left(\frac{1}{2} + iT\right) \ll (\log T)^{(k^2+1)/(2k)} + \left(\log T \max_{t\in[T-1,T+1]}|E_k(t)|\right)^{1/2k}. \quad (4.13)$$

Note that Lemma 4.1 in conjunction with (4.6) gives

$$\zeta\left(\frac{1}{2} + iT\right) \ll T^{1/6+\epsilon}, \quad (4.14)$$

which is essentially the classical bound of Hardy-Littlewood.

The function $E_k(T)$ can grow fast (near the points where $\zeta\left(\frac{1}{2} + iT\right)$ is large), but it can decrease only relatively slowly. Namely, for $0 \leq x \leq T$ one obtains, by integrating the expression for $E_k'(t)$,

$$O \le \int\limits_T^{T+x} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = \int\limits_T^{T+x} \left(P_{k^2}(\log t) + P'_{k^2}(\log t)\right) dt$$
$$+ E_k(T+x) - E_k(T) \le C_k x \log^{k^2} T + E_k(T+x) - E_k(T)$$

for a suitable constant $C_k > 0$. Thus

$$E_k(T) \le E_k(T+x) + C_k x \log^{k^2} T,$$

and integrating this inequality over x from 0 to $H (0 < H \le T)$ we obtain

$$E_k(T) \le H^{-1} \int\limits_T^{T+H} E_k(t) dt + C_k H \log^{k^2} T \quad (0 < H \le T).$$

Similarly we obtain a lower bound for $E_k(T)$. The above discussion is contained in

Lemma 4.2. *For a suitable $C_k > 0$ and any fixed integer $k \ge 1$*

$$E_k(T) \le E_k(T+x) + C_k x \log^{k^2} T \qquad (0 \le x \le T), \quad (4.15)$$
$$E_k(T) \ge E_k(T-x) - C_k x \log^{k^2} T \qquad (0 \le x \le T), \quad (4.16)$$
$$E_k(T) \le H^{-1} \int\limits_T^{T+H} E_k(t) dt + C_k H \log^{k^2} T \quad (0 < H \le T), \quad (4.17)$$
$$E_k(T) \ge H^{-1} \int\limits_{T+H}^{T} E_k(t) dt - C_k H \log^{k^2} T \quad (0 < H \le T). \quad (4.18)$$

170 The Lindelöf hypothesis that $\zeta\left(\frac{1}{2}+it\right) \ll t^\epsilon$ has many equivalent formulations. A simple connection between the Lindelöf hypothesis and the differences $E_k(T+H) - E_k(T)$ is contained in

Theorem 4.1. *Let $k > 1$ be a fixed integer. The Lindelöf hypothesis is equivalent to the statement that*

$$E_k(T+H) - E_k(T) = o(|H| T^\epsilon) \quad (0 < |H| \le T) \qquad (4.19)$$

or

$$E_k(T) = H^{-1} \int\limits_T^{T+H} E_k(t) dt + o(|H| T^\epsilon) \quad (0 < |H| \le T). \qquad (4.20)$$

Proof. Let the Lindelöf hypothesis be true. Then $\zeta(\frac{1}{2}+it) \ll t^{\epsilon_1}$ for any $\epsilon_1 > 0$ and consequently

$$E_k(T+H) - E_k(T) = \int\limits_T^{T+H} \left|\left(\frac{1}{2}+it\right)\right|^{2k} dt - \int\limits_T^{T+H} \left(P_{k^2}(\log t) + P'_{k^2}(\log t)\right) dt$$
$$\ll |H| \left(T^{2\epsilon_1 k} + \log^{k^2} T\right) \ll |H| T^{\epsilon}$$

for $\epsilon = 2\epsilon_1 k$. Conversely, if (4.19) holds, then by following the proof of Lemma 4.1 we have

$$\left|\zeta\left(\frac{1}{2}+iT\right)\right|^{2k} \ll \log T \left\{1 + tP_{k^2}(\log t)\Big|_{T-1}^{T+1} + e_k(T-1) - E_k(T-1)\right\}$$
$$\ll (\log T)^{k^2+1} + T^{\epsilon} \log T \ll T^{2\epsilon},$$

which gives $\zeta(\frac{1}{2}+iT) \ll_\epsilon T^{\epsilon}$. Further we have 171

$$E_k(T) - H^{-1} \int\limits_T^{T+H} E_k(t) dt = H^{-1} \int\limits_T^{T+H} (E_k(T) - E_k(t))\, dt,$$

so that (4.19) implies (4.20). Finally, if (4.20) holds then we use it once with H, and once with T replaced by $T+H$ and H by $-H$ to get (4.19). □

Concerning the true order of magnitude of $E_k(T)I$ conjecture that for $k \geq 1$ fixed and any $\epsilon > 0$

$$E_k(T) = O_\epsilon(T^{1/4k+\epsilon}) \tag{4.21}$$

and for $1 \leq k \leq 4$

$$E_k(T) = \Omega(T^{1/4k}), \tag{4.22}$$

the last result being true for $k = 1$ and $k = 2$ (see Theorem 5.7). The reason I am imposing the restriction $1 \leq k \leq 4$ in (4.22) is that if (4.22) holds for $k = 5$, then this would *disprove* the Lindelöf hypothesis which trivially implies $E_k(T) \ll_\epsilon T^{1+\epsilon}$. Anyway, the result $E_2(T) = \Omega(T^{1/2})$ of Theorem 5.7 clearly shows that it is reasonable to expect that $E_k(T)$ gets bigger as k increases, and that in some sense the eighth moment will be a "turning point" in the theory of $I_k(T)$.

4.2 The Approximate Functional Equation

We pass now to the derivation of an approximate functional equation for $\zeta^k(s)$ that will prove to be very useful in the evaluation of $I_k(T)$. There are many approximate functional equations in the literature. For example, one has

$$\zeta(1/2+it) = \sum_{n\le(t/2\pi)^{1/2}} n^{-1/2-it} + \chi\left(\frac{1}{2}+it\right) \sum_{n\le(t/2\pi)^{1/2}} n^{-1/2+it} + O(t^{-1/4}), \quad (4.23)$$

which is weakened form of the classical Riemann-Siegel formula, and Motohashi's result

$$\zeta^2(1/2+it) = \sum_{n\le t/2\pi} d(n)n^{-1/2-it} + \chi^2\left(\frac{1}{2}+it\right) \sum_{n\le t/2\pi} d(n)n^{-1/2+it} + O(t^{-1/6}).$$

172 If we multiply these equations by $\chi^{-1/2}\left(\frac{1}{2}+it\right)$ and $\chi^{-1}\left(\frac{1}{2}+it\right)$, respectively, we see that they have s symmetric form, in the sense that one main term in each of them is the conjugate of the other main term. However, the error term in (4.23) is best possible, and the other error term is also too large for the evaluation of the general $I_k(T)$. What we seek is a smoothed approximate functional equation, which is symmetric like (4.23), but contains a fairly small error term. Such an expression can be squared and integrated termwise, with the aim of evaluating the integral $I_k(T)$. Before we formulate our main result, we shall prove a lemma which guarantees the existence of the type of smoothing function that we need. This is

Lemma 4.3. *Let $b > 1$ be a fixed constant. There exists a real-valued function $\rho(x)$ such that*

(i) $\rho(x) \in C^\infty(0,\infty)$,

(ii) $\rho(x) + \rho\left(\frac{1}{x}\right) = 1$ *for* $x > 0$,

(iii) $\rho(x) = 0$ *for* $x \ge b$.

Proof. Let us define for $\alpha > \beta > 0$

$$\varphi(t) = \exp\left((t^2 - \beta^2)^{-1}\right)\left(\int\limits_{-\beta}^{\beta} \exp\left((u^2 - \beta^2)^{-1}\right) du\right)^{-1}$$

if $|t| < \beta$, and put $\varphi(t) = 0$ if $|t| \geq \beta$, and let

$$f(x) := \int\limits_{x-\alpha}^{x+\alpha} \varphi(t)dt = \int\limits_{-\infty}^{x} (\varphi(t+\alpha) - \varphi(t-\alpha))\, dt.$$

Then $\varphi(t) \in C(-\infty, \infty)$, $\varphi(t) \geq 0$ for all t, and from the definition of φ and f it follows that $f(x) \in C^\infty(-\infty, \infty)$, $f(x) \geq 0$ for all x and

$$f(x) = \begin{cases} 0 & \text{if } |x| \geq \alpha + \beta, \\ 1 & \text{if } |x| < \alpha - \beta. \end{cases}$$

Moreover if $\varphi(t)$ is even, then $f(x)$ is also even. Now choose $\alpha = \frac{1}{2}(1+b), \beta = \frac{1}{2}(b-1)$. then **173**

$$f(x) = \begin{cases} 0 & \text{if } x \geq b, \\ 1 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Set

$$\rho(x) := \frac{1}{2}\left(1 + f(x) - f\left(\frac{1}{x}\right)\right). \tag{4.24}$$

The property $\rho(x) \in C^\infty(O, \infty)$, which is i) of Lemma (4.3), is obvious.

Next

$$\rho(x) + \rho\left(\frac{1}{x}\right) = \frac{1}{2}\left(1 + f(x) - f\left(\frac{1}{x}\right)\right) + \frac{1}{2}\left(1 + f\left(\frac{1}{x}\right) - f(x)\right) = 1,$$

which establishes ii) of the lemma. Lastly, if $x \geq b$,

$$\rho(x) = \frac{1}{2}\left(1 - f\left(\frac{1}{x}\right)\right) = 0$$

since $1/x \leq 1/b < 1$ and $f(x) = 1$ for $0 \leq x \leq 1$. Thus $\rho(x)$, defined by (4.24), satisfies the desired requirements i) - iii), and the lemma is proved. □

The main result in this section is

Theorem 4.2. *For* $1/2 \leq \sigma < 1$ *fixed* $1 \ll x, y \ll t^k$, $s = \sigma + it$, $xy = \left(\frac{t}{2\pi}\right)^k$, $t \geq t_0$ *and* $k \geq 1$ *a fixed integer we have*

$$\begin{aligned}\zeta^k(s) &= \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} + \chi^k(s) \sum_{n=1}^{\infty} \rho\left(\frac{n}{y}\right) d_k(n) n^{s-1} \\ &\quad + O\left(t^{\frac{k(1-\sigma)}{3}-1}\right) + O\left(t^{k(1/2-\sigma)-2} y^{\sigma} \log^{k-1} t\right),\end{aligned} \tag{4.25}$$

where $\rho(x)$ *is a function satisfying the conditions of Lemma* (4.3). *furthermore, if* $c > 1$ *is a fixed constant, then*

$$\begin{aligned}\zeta^k(s) &= \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} + \sum_{n=1}^{\infty} \rho\left(\frac{x}{n}\right) \rho\left(\frac{n}{cx}\right) d_k(n) n^{-s} \\ &\quad + \chi^k(s) \sum_{n=1}^{\infty} \rho\left(\frac{n}{y}\right) \rho\left(\frac{nc}{y}\right) d_k(n) n^{s-1} + O\left(t^{\frac{k(1-\sigma)}{3}-1}\right) \\ &\quad + O\left(t^{k(1/2-\sigma)^{-2}} y^{\sigma} \log^{k-1} t\right).\end{aligned} \tag{4.26}$$

174 *Proof.* Note that in the most important special case when $s = \frac{1}{2} + it$, $x = y = \left(\frac{t}{2\pi}\right)^{1/2k}$ it follows that

$$\begin{aligned}\zeta^k(s) &= \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} + \chi^k(s) \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{s-1} \\ &\quad + O\left(t^{k/6-1} + t^{1/4k-2} \log^{k-1} t\right),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\zeta^k(s) &= \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} + \sum_{n=1}^{\infty} \rho\left(\frac{x}{n}\right) \rho\left(\frac{n}{cx}\right) d_k(n) n^{-s} + \chi^k(s) \\ &\quad \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) \rho\left(\frac{nc}{x}\right) d_k(n) n^{s-1} + O\left(t^{k/6-1} + t^{1/4k-2} \log^{k-1} t\right).\end{aligned} \tag{4.28}$$

Thus (4.27) for $k = 1$ is a smoothed variant of (4.23) (since all the series are in fact finite sums), only it contains an error term for $k = 1$ which is only $O(t^{5/6})$.

Our method of proof is based on the use of Mellin transforms. Namely, let

$$R(s) := \int_0^\infty \rho(x) x^{s-1}\,dx \tag{4.29}$$

denote the Mellin transform of $\rho(x)$. This is a regular function for $\operatorname{Re} s > 0$, but it has analytic continuation to the whole complex plane. Its only singularity is a simple pole at $s = 0$ with residue 1. For $\operatorname{Re} s > 0$

$$R(s) = \int_0^b \rho(x) x^{s-1}\,dx = \frac{x^2}{s}\rho(x)\Big|_0^b - \int_0^b \frac{x^s}{s}\rho'(x)\,dx = \frac{1}{s}\int_0^\infty \rho'(x) x^s\,dx,$$

and the last integral is an analytic function for $\operatorname{Re} s > -1$. Since

$$-\int_0^b \rho'(x)\,dx = 1,$$

the residue of $R(s)$ at $s = 0$ equals 1. In general, repeated integration by parts gives, for $N \geq 0$ an integer

$$R(s) = \frac{(-1)^{N+1}}{s(s+1)\ldots(s+N-1)(s+N)} \int_0^b \rho^{(N+1)}(x) x^{s+N}\,dx. \tag{4.30}$$

Taking N sufficiently large it is seen that (4.30) provides analytic **175**
continuation of $R(s)$ to the whole complex plane. For any N, and σ in a fixed strip ($s = \sigma + it$)

$$R(s) \ll_N |t|^{-N} \quad (|t| \to \infty). \tag{4.31}$$

In fact, (4.30) shows that $R(s)$ is regular at all points except $s = 0$, since for $N \geq 1$

$$\int_0^b \rho^{(N+1)}(x)\,dx = \rho^{(N)}(b) - \rho^{(N)}(0) = 0.$$

For $-1 < \operatorname{Re} s < 1$ and $s \neq 0$ we have

$$R(s) = -\frac{1}{s}\int_0^b \rho'(x)x^s dx = -\frac{1}{s}\int_0^\infty \rho'(x)x^s dx = -\frac{1}{s}\int_0^\infty \rho'\left(\frac{1}{t}\right)t^{-s-2}dt$$

after change of variable $x = 1/t$. But

$$\rho(t) + \rho\left(\frac{1}{t}\right) = 1, \quad \rho'(t) - t^{-2}\rho'\left(\frac{1}{t}\right) = 0, \quad \rho'\left(\frac{1}{t}\right) = t^2\rho'(t).$$

Hence

$$R(-s) = \frac{1}{s}\int_O^\infty \rho'\left(\frac{1}{t}\right)t^{s-2}dt = \frac{1}{s}\int_0^\infty \rho'(t)t^s dt = -R(s)$$

for $-1 < \operatorname{Re} s < 1$, and then by analytic continuation for all s

$$R(-s) = -R(s) \tag{4.32}$$

Conversely, from (4.32) by the inverse Mellin transform formula we can deduce the functional equation $\rho(x) + \rho(1/x) = 1$. The fact that $R(s)$ is an odd function plays an important role in the proof of Theorem 4.2. □

Having established the necessary analytic properties of $R(s)$ we pass to the proof, supposing $s = \sigma + it$, $1/2 \leq \sigma < 1$, $t \geq t_0$, $d = \operatorname{Re} z > 1 - \sigma$. Then for any integer $n \geq 1$ we have from (4.29), by the inversion formula for Mellin transforms

$$\rho\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} R(z)\left(\frac{x}{n}\right)^z dz.$$

By the absolute convergence of the series for $\zeta^k(s)$ this gives

$$\sum_{n=1}^\infty \rho\left(\frac{n}{x}\right)d_k(n)n^{-s} = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} R(z)x^z\zeta^k(s+z)dz.$$

176 We shift the line of integration to $\operatorname{Re} z = -d$, passing the poles of the integrand at $z = 1 - s$ and $z = 0$. Using (4.31) it is seen that the first residue is $O(t^{-A})$ for any fixed $A > 0$. Hence by the residue theorem and the functional equation $\zeta(w) = \chi(w)\zeta(1-w)$ we have

$$\sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} = \zeta^k(s) + O(t^{-A}) + \frac{1}{2\pi i} \int\limits_{-d+i\infty}^{-d-i\infty} R(z) x^z \zeta^k(s+z)\,dz$$

$$= \zeta^k(s) + O(t^{-A}) + \frac{1}{2\pi i} \int\limits_{-d+i\infty}^{-d-i\infty} R(z) x^z \chi^k(s+z)\zeta^k(1-s-z)\,dz$$

$$= \zeta^k(s) + O(t^{-A}) - \frac{1}{2\pi i} \int\limits_{d+i\infty}^{d-i\infty} R(-w) x^{-w} \chi^k(s-w)\zeta^k(1-s+w)\,dw$$

$$= \zeta^k(s) + O(t^{-A}) - \frac{1}{2\pi i} \int\limits_{d-i\infty}^{d+i\infty} R(w) y^w T^{-w} \chi^k(s-w)\zeta^k(1-s+w)\,dw.$$

Here we supposed $1 \ll x, y \ll t^k$ and $xy = T$, where

$$\log T = -k\frac{\chi'(1/2+it)}{\chi(1/2+it)}. \tag{4.33}$$

Since

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = \pi^{s-\frac{1}{2}} \frac{\Gamma(1/2 - 1/2s)}{\Gamma(1/2s)}, \tag{4.34}$$

logarithmic differentiation and Stirling's formula for the gamma function give

$$\frac{\chi'(1/2+it)}{\chi(1/2+it)} = -\log t + \log(2\pi) + O(t^{-2}). \tag{4.35}$$

Hence

$$T = \left(\frac{t}{2\pi}\right)^k (1 + O(t^{-2})). \tag{4.36}$$

In fact, (4.34) gives

$$\frac{\chi'(s)}{\chi(s)} = \log \pi - \frac{\Gamma'(\frac{1}{2} - \frac{1}{2}s)}{2\Gamma(\frac{1}{2} - \frac{1}{2}s)} - \frac{\Gamma'(\frac{1}{2}s)}{2\Gamma(\frac{1}{2}s)},$$

and using

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\left(\frac{1}{t^2}\right),$$

177 which is just a first approximation to a full asymptotic expansion of Stirling, it follows that

$$\frac{\chi'(s)}{\chi(s)} = \log(2\pi) + \frac{1}{s(1-s)} - \frac{1}{2}\log s(1-s) + O\left(\frac{1}{2}\right).$$

If $s = \frac{1}{2} + it$, then

$$\frac{1}{2}\log s(1-s) = \frac{1}{2}\log\left(\frac{1}{4} + t^2\right) = \log t + O\left(\frac{1}{t^2}\right)$$

and (4.35) follows. For $s = \sigma + it$ we obtain similarly

$$\frac{\chi'(\sigma+it)}{\chi(\sigma+it)} = -\log t + \log(2\pi) + O\left(\frac{1}{t}\right). \tag{4.37}$$

We shall derive first the approximate functional equation with x, y satisfying $xy = T$, and then replace this by the condition $xy = \left(\frac{t}{2\pi}\right)^k$, estimating the error term trivially. This is the method due originally to Hardy and Littlewood, who used it to derive the classical approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$. Therefore so far we have shown that

$$\begin{aligned}\zeta^k(s) &= \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} + O(t^{-A}) \\ &\quad + \frac{1}{2\pi i}\int\limits_{d-i\infty}^{d+i\infty} R(w)\left(\frac{y}{T}\right)^w \chi^k(s-w)\zeta^k(1-s+w)\,dw. \end{aligned} \tag{4.38}$$

We choose d to satisfy $d > \sigma$, so that the series for ζ^k in (4.38) is absolutely convergent and may be integrated termwise. Since $R(w) \ll |v|^{-A}$ for any fixed $A > 0$ if $v = \operatorname{Im} w$, it is seen that the portion of the integral in (4.38) for which $|v| \ge t^\epsilon$ makes a negligible contribution. Suppose now N is a (large) fixed integer. For $|v| \le t^\epsilon$

$$T^{-w}\chi^k(s-w) =$$
$$\exp\left\{kw\frac{\chi'(1/2+it)}{\chi(1/2+it)} + k\log\chi(s) - kw\frac{\chi'(s)}{\chi(s)} + k\sum_{j=2}^{N}\frac{(-w)^j}{j!}\frac{d^j}{ds^j}(\log\chi(s))\right\}$$
$$(1+O(t^{\epsilon-N}))$$

because $\frac{d^j}{ds^j}(\log\chi(s)) \ll_j t^{-j+1}$ for $j \geq 2$. Using (4.35) and (4.37) we **178**
infer that

$$\exp\{kw\ldots\} = \chi^k(s)(1+G(w,s))$$

with $G(w,s) \ll t^{\epsilon-1}$ for $|v| \leq t^\epsilon$. For $\delta > 0$ and N sufficiently large we obtain

$$\frac{1}{2\pi i}\int\limits_{d-i\infty}^{d+i\infty} R(w)y^w T^{-w}\chi^k(s-w)\zeta^k(1-s+w)dw$$
$$= O(t^{-A}) + \frac{1}{2\pi i}\int\limits_{d-i\infty}^{d+i\infty} R(w)y^w\chi^k(s)\zeta^k(1-s+w)dw$$
$$+ \frac{1}{2\pi i}\int\limits_{\delta-i\infty}^{\delta_i\infty} R(w)y^w\chi^k(s)\zeta^k(1-s+w)G(w,s)dw.$$

We have, since $d > \sigma$,

$$\frac{1}{2\pi i}\int\limits_{d-i\infty}^{d+i\infty} R(w)y^w\chi^k(s)\zeta^k(1-s+w)dw$$
$$= \chi^k(s)\sum_{n=1}^{\infty} d_k(n)n^{s-1}\left\{\frac{1}{2\pi i}\int\limits_{d-i\infty}^{d+i\infty} R(w)\left(\frac{n}{y}\right)^{-w}dw\right\}$$
$$= \chi^k(s)\sum_{n=1}^{\infty} d_k(n)\rho\left(\frac{y}{n}\right)n^{s-1}.$$

For $\delta > 0$ sufficiently small

$$\frac{1}{2\pi i}\int\limits_{\delta-i\infty}^{\delta+i\infty} R(w)y^w\chi^k(s)\zeta^k(1-s+w)G(w,s)dw \tag{4.39}$$

$$\ll t^{-A} + \left| \int\limits_{|\operatorname{Im} w| \le t^\epsilon, \operatorname{Re} w = \delta} R(w) y^w \zeta^k(s-w)(\chi^k(s)\chi^k(1-s+w)) G(w,s)dw \right|$$

$$\ll t^{-A} + t^{2\epsilon-1} \int\limits_{-t^\epsilon}^{t^\epsilon} |\zeta(\sigma + it - \delta + iv)|^k dv \ll t^{-A} + t^{k\mu(\sigma)+\epsilon_1 - 1}$$

where $\epsilon_1 \to 0$ as $\epsilon \to 0$, and where as in Chapter 1 we set

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t}.$$

179 For our purposes the standard bound

$$\mu(\sigma) < \frac{1-\sigma}{3} \quad \left(\frac{1}{2} \le \sigma < 1\right) \tag{4.40}$$

will suffice. This comes from $\mu\left(\frac{1}{2}\right) < \frac{1}{6}$, $\mu(1) = 0$ and convexity, and the use of the recent bounds for $\mu(1/2)$ (see (1.28)) would lead to small improvements. Thus, if $\epsilon = \epsilon(k)$ is sufficiently small

$$\zeta^k(s) = \sum_{n=1}^{\infty} \rho\left(\frac{n}{x}\right) d_k(n) n^{-s} + \chi^k(s) \sum_{n=1}^{\infty} \rho\left(\frac{n}{y}\right) d_k(n) n^{s-1} + O\left(t^{\frac{k(1-\sigma)}{3}-1}\right) \tag{4.41}$$

It is the use of (4.40) that required the range $\frac{1}{2} \le \sigma < 1$. We could have considered also the range $0 < \sigma \le \frac{1}{2}$. The analysis is, of course, quite similar but instead of (4.40) we would use

$$\mu(\sigma) \le \frac{1}{2} - \frac{2\sigma}{3} \quad (0 \le \sigma \le \frac{1}{2}).$$

This means that we would have obtained (4.41) with $O(t^{k(1-\sigma)/3-1})$ replaced by $O\left(t^{k\left(\frac{1}{2}-\frac{2\sigma}{3}\right)+\epsilon-1}\right)$. As the final step of the proof of (4.25) we replace $y = Tx^{-1}$ by Y, where $Y = x^{-1}\left(\frac{t}{2\pi}\right)^k$. Then, for $n \ll y$, we have

$$Y - y = O(t^{k-2}x^{-1}),$$
$$\rho\left(\frac{n}{y}\right) - \rho\left(\frac{n}{Y}\right) \ll \frac{|Y-y|n}{y^2} \ll t^{k-2}x^{-1}y^{-1} \ll t^{-2}.$$

Since $\rho\left(\frac{n}{y}\right) = 0$ for $n \geq by$ this means that if in (4.41) we replace y by Y the total error is

$$\ll t^{k(1/2-\sigma)} \sum_{n \leq by} t^{-2} d_k(n) n^{\sigma-1} \ll t^{t(1/2-\sigma)-2} Y^{\sigma} \log^{k-1} t.$$

Writing then y for Y we obtain (4.25).

To prove (4.26) we proved analogously, taking $\alpha, \beta > 0, \beta > 1-\sigma+\alpha$. Then for $c > 1$ fixed and any fixed $A > 0$

$$\sum_{n=1}^{\infty} d_k(n)\rho\left(\frac{x}{n}\right)\rho\left(\frac{n}{cx}\right)n^{-s} = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} R(w)R(z)c^z x^{z-w}\zeta^k(s+z-w)dzdw$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} R(w)x^{-w}\left\{\zeta^k(s-w) + O(t^{-A}) + \frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} x^z c^z R(z)\zeta^k(s+z-w)dz\right\}dw$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} R(w)x^{-w}\chi^k(s-w)\zeta^k(1-s+w)dw + O(t^{-A})$$

$$-\frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta+i\infty}^{\beta-i\infty} R(w)R(-z)x^{-w-z}c^{-z}\chi^k(s-z-w)\zeta^k(1-s+z+w)dzdw,$$

where we used the residue theorem and the functional equation for $\zeta(s)$. 180
Now we treat the above integrals as in the previous case, using $R(-z) = -R(z)$, choosing suitably α and β and integrating termwise the series for ζ^k. We also replace $\chi^k(s-w)$ and $\chi^k(s-z-w)$ by $\chi^k(s)$ plus an error term, as was done before. In this way we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} d_k(n)\rho\left(\frac{x}{n}\right)\rho\left(\frac{n}{cx}\right)n^{-s} \qquad (4.42)\\
&= \chi^k(s)\sum_{n=1}^{\infty} d_k(n)\rho\left(\frac{n}{y}\right)n^{s-1} - \chi^k(s)\sum_{n=1}^{\infty} d_k(n)\rho\left(\frac{n}{y}\right)\rho\left(\frac{nc}{y}\right)n^{s-1}\\
&\quad + O\left(t^{\frac{k(1-\sigma)}{3}-1}\right) + O\left(t^{k(\frac{1}{2}-\sigma)-2}y^{\sigma}\log^{k-1} t\right)
\end{aligned}$$

if $1 \ll x, y \ll t^k$, $xy = \left(\frac{t}{2\pi}\right)^k$, $s = \sigma + it$, $\frac{1}{2} \leq \sigma < 1$. Combining (4.42) with (4.25) we obtain (4.26), so that Theorem 4.2 is completely proved.

4.3 Evaluation of $I_k(T)$

In this section we proved to evaluate asymptotically $I_k(T)$, defined by (4.1), when $k \geq 1$ is a fixed integer. The approach is in principle general,
181 but the error terms that will be obtained can be $O(T^{1+\epsilon})$ only for $k \leq 4$. Thus a weakened form of the eighth moment is in fact the limit of the method. Of course, we still seem to be far from proving $I_4(T) \ll T^{1+\epsilon}$ (the proof of N.V. Kuznetsov [107] is not complete). On the other hand our method, which is a variation of the method used by N. Zavorotnyi to evaluate $I_2(T)$, leaves hope for evaluating successfully $I_3(T)$. To follow Zavorotnyi more closely we make a formal change of notation in Theorem 4.2 by setting

$$\nu(x) := \rho\left(\frac{1}{x}\right),$$

where ρ is the function explicitly constructed in Lemma 4.3. Then $\nu(x) \in C^\infty(O, \infty)$, $\nu(x) + \nu\left(\frac{1}{x}\right) = 1$, $\nu(x) = 0$ for $x \leq 1/b$, $\nu(x) = 1$ for $x \geq b$ for a fixed $b > 1$, $\nu(x)$ is monotonically increasing in (b^{-1}, b) and ν', ν'' are piecewise monotonic. In Theorem 4.2 we set $s = \frac{1}{2} + it$, $x = y = \left(\frac{t}{2\pi}\right)^k$, obtaining

$$\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n)\nu\left(\frac{x}{n}\right)n^{-s} + \sum_{n=1}^{\infty} d_k(n)\nu\left(\frac{x}{n}\right)n^{s-1} + O(R_k(t)), \tag{4.43}$$

$$\begin{aligned}\zeta^k(s) = O(R_k(t)) &+ \sum_{n-1}^{\infty} d_k(n)\nu\left(\frac{x}{n}\right)n^{-s} \\ &+ \sum_{n=1}^{\infty} d_k(n)\nu\left(\frac{n}{k}\right)\nu\left(\frac{cx}{n}\right)n^{-s} + \chi^k(s)\sum_{n=1}^{\infty} d_k(n)\nu\left(\frac{x}{n}\right)\nu\left(\frac{x}{cn}\right)n^{s-1},\end{aligned} \tag{4.44}$$

where

$$R_k(t) = t^{\frac{k}{6}-1} + t^{\frac{k}{4}-2}\log^{k-1} t. \tag{4.45}$$

The case $k = 2$ of these equations was used by Zavorotnyi, who had however the weaker error term $R_2(t) = t^{-1/2}\log t$, which was still sufficiently small for the evaluation of $I_2(T)$.

As is common in analytic number theory, it is often not easy to evaluate directly an integral, but it is more expedient to consider a weighted

(or smoothed) version of it. This approach was used by Heath-Brown, Zavorotnyi and Motohashi (see Chapter 5) in their evaluation of $I_2(T)$, so it is natural that we shall proceed here in a similar vein. The smooth- **182**
ing functions used in the evaluation of $I_k(T)$ are introduced similarly as the function ρ in Lemma 4.3. Let, for $|t| < 1$,

$$\beta(t) := \exp\left(\frac{1}{t^2-1}\right)\cdot\left(\int_{-1}^{1}\exp\left(\frac{1}{u^2-1}\right)\right)^{-1}$$

and $\beta(t) = 0$ for $|t| \geq 1$. Define then

$$\alpha(t) := \int_{t-1}^{t+1}\beta(u)du.$$

By construction $\int_{-\infty}^{\infty}\beta(u)du = 1$, and it follows that $\alpha(t) = \alpha(-t)$, $\alpha(t) \in C^{\infty}(0,\infty)$, $\alpha(t) = 1$ for $|t| \leq 1$, $\alpha(t) = 0$ for $|t| \geq 2$, and $\alpha(t)$ is decreasing for $1 < t < 2$. Henceforth assume T is large and let the parameter T_0 satisfy $T^{\epsilon} \ll T_0 \ll T^{1-\epsilon}$, and define

$$\overline{f}(t) = \alpha\left(\frac{2|t-\frac{1}{2}(3T)| + 2T_0 - T}{2T_0}\right), \quad \underline{f}(t) = \alpha\left(\frac{2|t-\frac{1}{2}(3T)| + 4T_0 - T}{2T_0}\right).$$

The functions $\overline{f}$ and $\underline{f}$ are constructed in such a way that, for any t,

$$0 \leq \underline{f}(t) \leq X_T(t) \leq \overline{f}(t)$$

if $\chi_T(t)$ denotes the characteristic function of the interval $[T, 2T]$. Thus we have

$$I_{k,\underline{f}}(T) \leq I_k(2T) - I_k(T) \leq I_{k,\overline{f}}(T), \tag{4.46}$$

where

$$I_{k,\underline{f}}(T) = \int_0^{\infty}\underline{f}(t)|\zeta(1/2+it)|^{2k}dt, I_{k,\overline{f}}(T) = \int_0^{T}\overline{f}(t)|\zeta(1/2+it)|^{2k}dt. \tag{4.47}$$

Therefore if we can prove

$$I_{k,f}(T) = tP_{k^2}(\log t)\Big|_T^{2T} + O(T^{\gamma_k+\epsilon}) \tag{4.48}$$

with a suitable $0 < \gamma_k \leq 1$, where henceforth f will denote either $\underline{f}$, then in view of (4.46) we obtain

$$I_k(T) = \int_0^T |\zeta(1/2+it)|^{2k} dt = TP_{k^2}(\log T) + O(T^{\gamma+k+\epsilon}) \tag{4.49}$$

183 on replacing T by $T2^{-j}$ and summing over $j \geq 1$. We can reinterpret our current knowledge on $I_k(T)$ by saying that

$$\frac{1}{4} \leq \gamma_1 \leq \frac{7}{22}, \quad \frac{1}{2} \leq \gamma_2 \leq \frac{2}{3}, \quad \gamma_k \leq 1 + \frac{k-2}{4} \ \text{ for } \ 3 \leq k \leq 6.$$

The problem of evaluating $I_k(T)$ is therefore reduced to the (somewhat less difficult) problem of the evaluation of $I_{k,f}(T)$, where f is either $\underline{f}$ or $\overline{f}$. For any integer $g \geq 1$

$$\overline{f}^{(r)}(t) = sgn^r\left(t - \frac{3}{2}T\right) \cdot \alpha^{(r)}\left(\frac{2|t - \frac{1}{2}(3T)| + 2T_0 - T}{2T_0}\right) T_0^{-r},$$

and an analogous formula also holds for $\underline{f}^{(r)}(t)$. This gives

$$f^{(r)}(t) \ll_r T_0^{-r} \qquad (r = 1, 2, \ldots), \tag{4.50}$$

which is an important property of smoothing functions. It provides the means for the regulation of the length of exponential sums (Dirichlet series) that will appear in the sequel. A general disadvantage of this type of approach is that it is difficult to obtain an explicit formula for $E_k(T)$ when $k \geq 2$. Hopes for obtaining analogues of Atkinson's formula for $E_1(T) = E(T)$ in the case of general $E_k(T)$, may be unrealistic. Namely, Theorem 5.1 shows that the explicit formula for the integral in question does not contain any divisor function, but instead it contains quantities from the spectral theory of automorphic functions.

Let now

$$\sum(t) := \sum_{m=1}^{\infty} \nu\left(\frac{x}{m}\right) d_k(m) m^{-1/2-it} \cdot \left(x = \left(\frac{t}{2\pi}\right)^{1/2k}\right). \tag{4.51}$$

Then (4.43) yields

$$\zeta^k(1/2+it) = \sum(t) + \chi^k(1/2+it)\overline{\sum}(t) + O(R_k(t)),$$

hence taking conjugates one obtains

$$\zeta^k(1/2-it) = \overline{\sum}(t)\chi^k(1/2-it)\sum(t) + O(R_k(t)).$$

From the functional equation one obtains $\chi(s)\chi(1-s) = 1$, hence multiplying the above expressions we obtain **184**

$$|\zeta(1/2+it)|^{2k} = 2|\sum(t)|^2 + 2\,\mathrm{Re}\left(\chi^k(1/2+it)\overline{\sum}^2(t)\right) + O\left(R_k(t)|\sum(t)|\right) + O(R_k^2(t)).$$

Now we multiply the last relation by $f(t)$ and integrate over t from $T/2$ to $(5T)/2$. Since the support of f is contained in $[T/2, (5T)/2]$ we have

$$\frac{1}{2}I_{k,f}(T) = \int_0^\infty f(t)|\sum(t)|^2 dt + \mathrm{Re}\left(\int_0^\infty f(t)\chi^k(1/2+it)\overline{\sum}^2(t)dt\right) + O\left(\int_{1/2T}^{1/2(5T)} R_k(t)|\sum(t)|dt\right) + \left(\int_{1/2T}^{1/2(5T)} R_k^2(t)dt\right).$$

We can proceed now by using

$$R_k(t) = t^{\frac{k}{6}-1} + t^{\frac{k}{4}-2}\log^{k-1} t \tag{4.52}$$

and applying the Cauchy-Schwarz inequality to the first O-term. We shall restrict k to $1 \leq k \leq 4$, although we could easily carry out the subsequent analysis for general k. The reason for this is that in the sequel the error terms for $k \geq 5$ will be greater than $T^{1+\eta_k}$ for some $\eta_k > 0$, hence there is no hope of getting an asymptotic formula for $I_k(T)$ by this method. However, instead of using the pointwise estimate (4.52), a sharper result will be obtained by using (4.39) with $\sigma = 1/2$, namely

$$R_k(t) \ll t^{1/4k-2} \log^{k-1} t + t^{2\epsilon-1} \int\limits_{-t^\epsilon}^{t^\epsilon} |\zeta(1/2 + it - \delta + iv)|^k dv. \tag{4.53}$$

we have

$$\int\limits_{1/2T}^{1/2(5T)} |\zeta(1/2 + it)|^k dt \ll T \log^4 T \quad (1 \leq k \leq 4),$$

and by standard methods (Perron's inversion formula) it follows that

$$\sum(t) \ll T^{k\mu(1/2)+\epsilon}$$

185 Thus in view of $\mu(1/2) < 1/6$ we obtain

$$\frac{1}{2} I_{k,f}(T) = \int\limits_0^\infty f(t) |\sum(t)|^2 dt \tag{4.54}$$
$$+ \operatorname{Re}\left\{ \int\limits_0^\infty f(t) \chi^k (1/2 + it) \overline{\sum}^2 (t) dt \right\} + O(T^{k/6}),$$

which shows that the error term coming from the error term in our approximate functional equation for $\zeta^k(s)$ is fairly small. However, in the sequel we shall encounter larger error terms which hinder the possibility of obtaining a good asymptotic evaluation of $I_k(T)$ when $k \geq 5$. In (4.54) write

$$\chi^k \cdot \overline{\sum}^2 = \left(\chi^k \overline{\sum}\right) \cdot \overline{\sum}$$

and evaluate the factor in brackets by subtracting (4.44) from (4.43). We obtain

$$\int\limits_0^\infty f(t) \chi^k (1/2 + it) \overline{\sum}^2 (t) dt = \sum_{m,n=1}^\infty d_k(m) d_k(n) (mn)^{-1/2}$$

$$\int_0^\infty f(t)\nu\left(\frac{x}{m}\right)\nu\left(\frac{n}{x}\right)\nu\left(\frac{cx}{n}\right)\left(\frac{m}{n}\right)^{it} dt + O(T^{k/6}) + \sum_{m,n=1}^\infty d_k(m)d_k(n)(mn)^{-1/2}$$

$$\int_0^\infty f(t)e^{ki(t\log\frac{2\pi}{t}+t+\frac{1}{4}\pi)}(1+g_k(t))\nu\left(\frac{x}{m}\right)\nu\left(\frac{x}{n}\right)\nu\left(\frac{x}{cn}\right)(mn)^{it}dt,$$

where $g_k(t) \ll 1/t$. The last bound is a consequence of the asymptotic expansion

$$\chi(1/2+it) = e^{i\left(t\log\frac{2\pi}{t}+t+\frac{1}{4}\pi\right)}\left(1-\frac{i}{24t}+\frac{A_2}{t^2}+\ldots+O\left(\frac{1}{t^N}\right)\right),$$

which follows, for any fixed integer $N \geq 2$, from Stirling's formula for the gamma-function. Therefore the method used to estimate

$$\sum_{m,n=1}^\infty d_k(m)d_k(n)(mn)^{-1/2}\int_0^\infty f(t)e^{ki\left(t\log\frac{2\pi}{t}+t+\frac{1}{4}\pi\right)} \nu\left(\frac{x}{m}\right)\nu\left(\frac{x}{n}\right)\nu\left(\frac{x}{cn}\right)(mn)^{it}dt \tag{4.55}$$

will give a bound of a lower order of magnitude (by a factor or T) for the corresponding integral containing $g_k(t)$. In the integral in (4.55), which will be denoted by $I(m,n)$, we make the substitution $t = 2\pi u(mn)^{1/k}$. Since $x = (t/2\pi)^{1/2k}$ we have **186**

$$\begin{aligned} I(m,n) :&= 2\pi e^{\frac{ki\pi}{4}}(mn)^{\frac{4}{k}}\int_0^\infty f(2\pi u(mn)^{\frac{4}{k}})\nu\left(u^{\frac{k}{2}}\sqrt{\frac{n}{m}}\right) \\ &\qquad \nu\left(u^{\frac{k}{2}}C^{-1}\sqrt{\frac{m}{n}}\right)e^{2\pi ki(mn)^{1/k}(-u\log u+\mu)}du \\ &= 2\pi e^{1/4ki\pi}(mn)^{1/k}\int_A^B H(t;m,n)e^{2\pi ki(mn)(-t\log t+t)}dt \end{aligned}$$

with

$$H(t;m,n) := f(2\pi t(mn)^{1/k})\nu\left(t^{1/2k}\left(\frac{n}{m}\right)^{1/2}\right)\nu\left(t^{1/2k}\left(\frac{m}{n}\right)^{1/2}\right)\nu\left(t^{1/2k}c^{-1}\left(\frac{m}{n}\right)^{1/2}\right),$$

$$A := \max\left(\frac{T \mp T_0}{2\pi(mn)^{1/k}}, \left(\frac{m}{n}\right)^{1/k} b^{-2/k}, \left(\frac{c}{b}\right)^{2/k} \left(\frac{n}{m}\right)^{1/k}\right) \leq t \leq B := \frac{2T \pm T_0}{2\pi(mn)^{1/k}},$$

where upper signs refer to $\overline{f}$, and lower signs to $\underline{f}$. If

$$F(t) = -t\log t + t,$$

then

$$|F'(t)| = \log t \geq \log(1+\delta) \gg \delta$$

for a given $O > \delta < 1$ when $\left(\frac{m}{n}\right)^{1/k} b^{-2/k} \geq 1 + \delta$. On the other hand, if $\left(\frac{m}{n}\right) b^{-2/k} < 1 + \delta$, then

$$\left(\frac{c}{b}\right)^{2/k} \left(\frac{n}{m}\right)^{1/k} > \left(\frac{c}{b}\right)^{2/k_b - 2/k} (1+\delta)^{-1} \geq 1 + \delta$$

for

$$c = b^2(1+2\delta)^k. \tag{4.56}$$

Thus $|F'(t)| = \log t \gg \delta$ for $A \leq t \leq B$, provided that (4.56) holds. Since $H(A; m, n) = H(B; m, n) = 0$, integration by parts yields

$$\begin{aligned}
&\int\limits_A^B H(t; m, n) e\left(k(mn)^{1/k} F(t)\right) dt \\
&= -\frac{(mn)^{-1/k}}{2\pi ki} \int\limits_A^B \frac{H(t; m, n)}{\log t} \frac{d}{dt}\left\{e\left(k(mn)^{1/k} F(t)\right)\right\} \\
&= \frac{(mn)^{-1/k}}{2\pi ki} \int\limits_A^B \left(\frac{H'(t; m, n)}{\log t} - \frac{H(t; m, n)}{t\log^2 t}\right) e\left(k(mn)^{1/k} f(t)\right) dt.
\end{aligned}$$

Now we use the standard estimate (see Lemma 2.1) for exponential
187 integrals to show that the portion of the last integral containing $H(t; m, n)$ is $\ll (mn)^{-2/k_T - 1}$. Consider next, in obvious notation,

$$\begin{aligned}
H'(t; m, n) = 2\pi(mn)^{1/k} f'\left(2\pi(mn)^{1/k} t\right) \nu(\cdot)\nu(\cdot)\nu(\cdot) \\
+ \sum f(\cdot) \frac{d\nu(\cdot)}{dt} \nu(\cdot)\nu(\cdot).
\end{aligned} \tag{4.57}$$

One has $\nu'(x) = 0$ for $x \leq 1/b$ or $x \geq b$, so that the terms coming from $\sum f(\cdot)\frac{d\nu(\cdot)}{dt}\ldots$ in (4.57) vanish, because of the conditions

$$\frac{1}{b} \leq t^{1/2k}\left(\frac{m}{n}\right)^{1/2} \leq b \ \text{ or } \ \frac{1}{b} \leq t^{1/2k}\left(\frac{n}{m}\right)^{1/2} \leq b,$$

which cannot hold in view of $t \asymp T$, $1 \leq m, n \ll T^{1/2k}$. Taking into account that $f'(t) \ll 1/T_0$ and using again Lemma (2.1) it follows that the expression in (4.55) equals

$$\sum_{m,n=1}^{\infty} I(m,n)d_k(n)(mn)^{-1/2} \ll T_0^{-1} \sum_{m \ll T^{1/2k}} \sum_{n \ll T^{1/2k}} d_k(m)d_k(n)(mn)^{-1/2}$$
$$\ll T^{1/2k}T_0^{-1}(\log T)^{2k-2}.$$

This gives

$$\frac{1}{2}I_{k,f}(T) = \sum_{\substack{m,n=1\\ 1-\delta \leq \frac{m}{n} \leq 1+\delta}} d_k(m)d_k(n)(mn)^{-1/2}\,\mathrm{Re}\left\{\int_O^\infty f(t)\nu\left(\frac{x}{m}\right)\nu\left(\frac{x}{n}\right)^{it} dt\right\} \tag{4.58}$$

$$+ O(T^{k/6}) + O\left(T^{1/2k+\epsilon}T_0^{-1}\right) + \sum_{\substack{m,n=1\\ 1-\delta \leq \frac{m}{n} \leq 1+\delta}}^{\infty} d_k(m)d_k(n)(mn)^{-1/2}$$

$$\mathrm{Re}\left\{\int_0^\infty f(t)\nu\left(\frac{x}{m}\right)\nu\left(\frac{n}{x}\right)\nu\left(\frac{m}{n}\right)^{it} dt\right\} + \sum_{\substack{m,n=1\\ \frac{m}{n}<1-\delta, \frac{m}{n}>1+\delta}}^{\infty} d_k(m)d_k(n)(mn)^{-1/2}$$

$$\left[\mathrm{Re}\left\{\int_0^\infty f(t)\left(\nu\left(\frac{x}{m}\right)\nu\left(\frac{x}{n}\right) + \nu\left(\frac{x}{m}\right)\nu\left(\frac{n}{x}\right)\nu\left(\frac{cx}{n}\right)\right)\left(\frac{m}{n}\right)^{it} dt\right\}\right].$$

We recall that we have at out disposal the parameters T_0 and δ such that $T^\epsilon \ll T_0 \ll T^{1-\epsilon}$ and $0 < \delta < 1$. In the last sum in (4.57) we have $\log(m/n) \gg \delta$. We integrate the integral in that sum by parts, setting 188

$$u = f(t), dv = \left(\nu\left(\frac{x}{m}\right)\nu\left(\frac{x}{n}\right) + \nu\left(\frac{x}{m}\right)\nu\left(\frac{n}{x}\right)\nu\left(\frac{cx}{n}\right)\right)\left(\frac{m}{n}\right)^{it} dt,$$

and after integration we use the second mean value theorem for integrals to remove the ν-function. Since $f'(t) \ll 1/T_0$, on applying again Lemma 2.1 it is seen that the last sum in (4.58) is $O(T^{1/2k+\epsilon}T_0^{-1})$.

It remains to simplify the remaining two sums in (4.58), and it is at this point that we shall make use of the parameter δ. Namely, it will be shown that for δ sufficiently small $\nu\left(\frac{cx}{n}\right) = 1$ if (4.56) holds. This happens for $\frac{cx}{n} \geq b$, or $t \geq 2\pi(nb/c)^{2/k}$. The integral in question is non-zero for $x/m \geq 1/b$, giving $t \geq 2\pi\left(\frac{m}{b}\right)^{2/k}$. Since in both sums $m \geq (1-\delta)n$, we have

$$t \geq 2\pi\left(\frac{m}{b}\right)^{2/k} \geq 2\pi\left(\frac{n}{b}\right)^{2/k}(1-\delta)^{2/k} \geq 2\pi\left(\frac{nb}{c}\right)^{2/k}$$

for

$$(1-\delta)c \geq b^2.$$

In view of (4.56) this condition reduces to showing that

$$(1+2\delta)^k(1-\delta) \geq 1.$$

The last inequality is certainly true for $0 < \delta < \frac{1}{2}$, since $(1+2\delta)^k \geq 1+2\delta$. Therefore, in the second integral in (4.58), $\nu\left(\frac{cx}{n}\right)$ may be omitted for $0 < \delta < \frac{1}{2}$. Using $\nu\left(\frac{x}{n}\right) + \nu\left(\frac{n}{x}\right) = 1$ we finally obtain

Theorem 4.3. *for* $1 < k \leq 4$ *an integer,* $0 < \delta < \frac{1}{2}$, $T^\epsilon \ll T_0 \ll T^{1-\epsilon}$ *and* $x = \left(\frac{t}{2\pi}\right)^{1/2k}$ *we have*

$$\frac{1}{2}I_{k,f}(T) = \sum_{n=1}^{\infty} d_k^2(n)n^{-1}\operatorname{Re}\left\{\int_0^\infty f(t)\nu\left(\frac{x}{n}\right)dt\right\} + O(T^{k/6}) + O\left(T^{1/2k+\epsilon}T_0^{-1}\right) \tag{4.59}$$

$$+ \sum_{\substack{m,n=1\\ m\neq n, 1-\delta\leq\frac{m}{n}\leq 1+\delta}} d_k(m)d_k(n)(mn)^{-1/2}\operatorname{Re}\left\{\int_0^\infty f(t)\nu\left(\frac{x}{m}\right)\left(\frac{m}{n}\right)^{it}dt\right\}.$$

189 The above formulas may be considered as the starting point for

the evaluation of $I_k(T)$. Two sums will emerge, of which the first is $TR_{k^2}(\log T)$ plus an error term. This will be shown in the next section, where $R_{k^2}(y)$ is a polynomial of degree k^2 in y whose first two coefficients do not depend on ν, but the others do. The other (double) sum, presumably of order lower by a factor of $\log^2 T$ than the first one, is much more complicated. This sum represents the major difficulty in the evaluation of $I_k(T)$, and so far can be evaluated asymptotically only for $k = 1$ and $k = 2$. Since the final (expected) formula (4.9) (with $E_k(T) = o(T)$ as $T \to \infty$) does not contain in its main term the ν-function, it is natural to expect that the coefficients in both sums containing the ν-function will eventually cancel each other. This is because the ν-function does not pertain intrinsically to the problem of evaluating asymptotically $I_k(T)$, and can be in fact chosen with a reasonable degree of arbitrariness.

4.4 Evaluation of the First Main Term

The title of this section refers to the evaluation of the sum

$$S_k(T) := \sum_{n=1}^{\infty} d_k^2(n) n^{-1} \operatorname{Re}\left\{\int_0^{\infty} f(t)\nu\left(\frac{x}{n}\right) dt\right\} \tag{4.60}$$

which figures in (4.59). This sum can be precisely evaluated. We shall show that, for $1 \le k \le 4$,

$$S_k(T) = (tQ_{k^2}(\log t))\Big|_T^{2T} + (tH_{k,\nu}(\log t))\Big|_T^{2T} + O(T^{\epsilon}T_0) + O(T^{\eta_k+\epsilon}), \tag{4.61}$$

where $\eta_1 = 0$, $\eta_k = \frac{1}{2}$ for $2 \le k \le 4$. In (4.61) $H_{k,\nu}(y)$ is a polynomial **190**
of degree $k^2 - 2$ in y whose coefficients depend on k and the function ν, while $Q_{k^2}(y)$ is a polynomial of degree k^2 in y whose coefficients depend only on k, and may be explicitly evaluated. Before this is done, observe that the error terms $O(T^{\epsilon}T_0)$ and $O(T^{1/2k+\epsilon}T_0^{-1})$ in (4.59) and (4.61) set the limit to the upper bound on $E_k(T)$ as

$$E_k(T) = O\left(T^{1/4k+\epsilon}\right)$$

on choosing $T_0 = T^{1/4k}$, $1 \leq k \leq 4$. This is the conjectural bound mentioned in (4.21), and the above discussion provides some heuristic reasons for its validity. Coupled with (4.22), it means that heuristically we have a fairly good idea of what the true order of magnitude of $E_k(T)$ should be for $1 \leq k \leq 4$. We are certainly at present far from proving the above upper bound for any k, which is quite strong, since by Lemma 4.1 it implies $\mu(1/2) \leq 1/8$.

To prove (4.61) we define first, for $\mathrm{Re}\, s > 1$ and $k \geq 1$ a fixed integer

$$F_k(s) := \sum_{n=1}^{\infty} d_k^2(n) n^{-s}.$$

Since

$$d_k(p^\alpha) = \frac{k(k+1)\dots(\alpha+k-1)}{\alpha!} = \frac{\Gamma(k+\alpha)}{\alpha!\Gamma(k)},$$

we have

$$F_1(s) = \zeta(s), F_2(s) = \frac{\zeta^4(s)}{\zeta(2s)},$$

$$\begin{aligned} F_k(s) &= \zeta^{k^2}(s) \prod_p (1-p^{-s})^{k^2} \left(1 + k^2 p^{-s} + \frac{k^2(k+1)^2}{4} p^{-2s} + \dots\right) \\ &= \zeta^{k^2}(s) \prod_p \left(1 - \frac{k^2(k-1)^2}{4} p^{-2s}\right) + O_k(p^{3\sigma}). \end{aligned}$$

The Dirichlet series defined by the last product converges absolutely for $\mathrm{Re}\, s \geq \frac{1}{2} + \epsilon$ and any fixed $\epsilon > 0$ (the O -term vanishes for $k = 2$), so that the last formula provides analytic continuation of $F_k(s)$ to $\mathrm{Re}\, s \geq \frac{1}{2} + \epsilon$.

191 Further we introduce the function

$$P(s) = \int_0^\infty v(x) x^{-s} dx = \frac{1}{s-1} \int_0^\infty v'(x) x^{1-s} dx \ll_A |\mathrm{Im}\, s|^{-A}$$

for any fixed $A > 0$ if Re s lies in any fixed strip, the last bound being obtained by successive integrations by parts. $P(s)$ is regular in the complex plane, and has only a simple pole at $s = 1$ with residue $1 \left(= \int\limits_0^\infty f(x)x^{s-1}dx \right)$ is the Mellin transform of f, then $\hat{f}$ is regular (since the support of f does not contain zero). By the Mellin inversion formula

$$f(x) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \hat{f}(s)x^{-s}ds \qquad (c > 0).$$

In (4.60) we make the substitution $t = 2\pi(nu)^{2/k}$, $dt = \frac{4\pi}{k} n^{2/k} u^{2/k-1}$. Then for $c > 1$

$$\begin{aligned} S_k(T) &= \frac{4\pi}{k} \sum_{n=1}^\infty d_k^2(n) n^{-(1-\frac{2}{k})} \operatorname{Re} \left\{ \int\limits_0^\infty f\left(2\pi(nu)^{2/k}\right) v(u) u^{\frac{2}{k}-1} du \right\} \qquad (4.62) \\ &= \frac{4\pi}{k} \sum_{n=1}^\infty d_k^2(n) n^{-\left(1-\frac{2}{k}\right)} \operatorname{Re} \left\{ \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \int\limits_0^\infty \hat{f}(s) \left(2\pi(nx)^{2/k}\right)^{-s} x^{2/x-1} v(x) dx ds \right\} \\ &= \operatorname{Re} \left\{ \frac{4\pi}{k} \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \hat{f}(s)(2\pi)^{-s} \left(\int_0^\infty v(x) x^{2/k(1-s)-1} dx \right) \left(\sum_{n=1}^\infty d_k^2(n) n^{-\left(1+\frac{2}{k}(s-1)\right)} \right) ds \right\} \\ &= \operatorname{Re} \left\{ \frac{2}{k} \cdot \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \hat{f}(s)(2\pi)^{1-s} F_k\left(1 + \frac{2}{k}(s-1)\right) P\left(1 + \frac{2}{k}(s-1)\right) ds \right\}, \end{aligned}$$

where the interchange of summation and integration is justified by absolute convergence. We have

$$\frac{d^r}{ds^r} \hat{f}(s) = \int\limits_0^\infty f(x)x^{s-1} \log^r x \, dx \quad (r = 0, 1, 2, \ldots),$$

$$\hat{f}(s) = \int\limits_{T/2}^{5T/2} f(x)x^{s-1}dx \ll \int\limits_{T/2}^{5T/2} x^{\sigma-1}dx \ll T^\sigma. \qquad (4.63)$$

Moreover, if $r \geq 1$ is an integer and Re s lies in a fixed strip, then **192**
integration by parts and (4.60) give

$$\hat{f}(s) = \frac{(-1)^r}{s(s+1)\ldots(s+r-1)} \int\limits_0^\infty f^{(r)}(x)x^{s+r-1}dx \ll |\operatorname{Im} s|^{-r} T_0^{-r} T^{\sigma+r}.$$

In the last integral in (4.62) we move the line of integration to $\operatorname{Re} s = \epsilon$ if $k = 1$, and to $\operatorname{Re} s = \frac{1}{2} + \epsilon$ if $k \geq 2$. We encounter the pole of the integrand of order $k^2 + 1$ at $s = 1$. By the residue theorem we have

$$S_k(T) = \frac{2}{k} \operatorname{Re} \left\{ \operatorname*{Res}_{s=1} \hat{f}(s)(2\pi)^{1-s} F_k\left(1 + \frac{2}{k}(s-1)\right) P\left(1 + \frac{2}{k}(s-1)\right) \right\} + O(T^{\eta_k+\epsilon})$$

with $\eta_1 = 0$ for $k = 1$ and $\eta_k = \frac{1}{2}$ for $2 \leq k \leq 4$. Here we used the properties of $F_k(s)$, (4.63) and $P(s) \ll_A |\operatorname{Im} s|^{-A}$. To evaluate the residue at $s = 1$, we shall use the following power series expansions near $s = 1$:

$$(2\pi)^{1-s} = 1 + a_1(s-1) + a_2(s-1)^2 + \cdots \left(a_j = \frac{(-1)^j (\log 2\pi)^j}{j!}\right),$$

$$F_k(s) = \frac{d_{-k^2}(k)}{(s-1)^{k^2}} + \cdots + \frac{d_{-1}(k)}{s-1} + d_0(k) + d_1(k)(s-1) + \cdots,$$

$$\hat{f}(s) = \sum_{j=0}^{\infty} c_j (s-1)^j, \tag{4.64}$$

$$c_j = \frac{1}{j!}\hat{f}^{(j)}(1) = \frac{1}{j!}\int_0^\infty f(x)\log^j x dx = \frac{1}{j!}\left(\int_T^{2T} \log^j x dx + O(T_0 \log^j T)\right)$$

$$= \frac{1}{j!}\left(t\log^j t + t\sum_{\ell=1}^{j}(-1)^\ell j(j-1)\dots(j-\ell+1)\log^{j-\ell} t\right)\Bigg|_T^{2T} + O(T_0\log^j T),$$

$$P(s) = \frac{1}{s-1} + \sum_{j=0}^{\infty} h_j(v)(s-1)^j,$$

193 with $h_{2j}(v) = 0$ for $j = 0, 1, 2, \ldots$. The last assertion follows because

$$h_j(v) = \frac{(-1)^{j+1}}{(j+1)!}\int_0^\infty v'(x)\log^{j+1} x\, dx,$$

and using $v(x) + v\left(\frac{1}{x}\right) = 1$ we have $v'(x) - x^{-2}v'\left(\frac{1}{x}\right) = 0$, which gives

$$\int_0^\infty v'(x)\log^{2n-1} x\, dx = \int_0^1 v'(x)\log^{2n-1} x\, dx + \int_1^\infty v'(x)\log^{2n-1} x\, dx$$

$$= \int\limits_0^1 v'(x)\log^{2n-1} x\,dx - \int\limits_0^1 v'\left(\frac{1}{x}\right)x^{-2}\log^{2n-1} x\,dx = 0$$

for $n = 1, 2, \ldots$. Near $s = 1$ we thus have the expansion

$$\frac{2}{k}\hat{f}(s)(2\pi)^{1-s}F_k\left(1+\frac{2}{k}(s-1)\right)P\left(1+\frac{2}{k}(s-1)\right)$$

$$= \frac{2}{k}\left(1 + a_1(s-1) + a_2(s-1)^2 + \cdots\right)\left(c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots\right)$$

$$\times\left(\frac{1}{\frac{2}{k}(s-1)} + \frac{2}{k}h_1(v)(s-1) + \left(\frac{2}{k}\right)^3 h_3(v)(s-1)^3 + \cdots\right)$$

$$\times\left(\frac{d_{-k^2}(k)}{\left(\frac{2}{k}\right)^{k^2}(s-1)^{k^2}} + \cdots + \frac{d_{-1}(k)}{\frac{2}{k}(s-1)} + d_0(k) + d_1(k)\frac{2}{k}(s-1) + \cdots\right)$$

$$= \Big\{c_0 t(c_1 + c_0a_1)(s-1) + (c_2 + c_1a_1 + c_0a_2)(s-1)^2 + \cdots$$
$$+ (c_{k2} + c_{k^2-1}a_1 + \cdots + c_0a_{k2})(s-1)^{k^2} + \cdots\Big\}$$

$$\times\left\{\frac{d_{-k^2}(k)}{\left(\frac{2}{k}\right)^{k^2}(s-1)^{k^2+1}} + \frac{d_{-(k^2-1)}(k)}{\left(\frac{2}{k}\right)^{k^2-1}(s-1)^{k^2}} + \sum_{j=-(k^2-1)}^{\infty} e_j(s-1)^j\right\}$$

with suitable coefficients $e_j = e_j(k, v)$ depending on k and the function v. The residue in the formula for $S_k(T)$ is the coefficient of $(s-1)^{-1}$ in the above power series expansion. It will be a linear combination of the c_j's with j running from 0 to k^2. The e_j's (which depend on v) will appear in the expression for the residue as the coefficients of the c_j's for $j \le k^2 - 2$. In view of (4.64) we obtain then the main in (4.61). If $Q_{k^2}(y)$ in (4.61) is written as 194

$$Q_{k^2}(y) = \sum_{j=0}^{k^2} A_j(k)y^{k^2-j},$$

then the above power series expansion enables us to calculate explicitly $A_0(k)$ and $A_1(k)$. We have

$$A_0(k) = (k^2!)\left(\frac{2}{k}\right)^{-k^2} d_{-k^2}(k) = (k^2!)^{-1}\left(\frac{k}{2}\right)^{k^2}\lim_{s\to 1+0}(s-1)^{k^2}F_k(s)$$

$$= (k^2!)^{-1}\left(\frac{k}{2}\right)^{k^2}\lim_{s\to 1+0}\left(\zeta(s)^{-k^2}\sum_{n=1}^{\infty} d_k^2(n)n^{-s}\right)$$
$$= \Gamma^{-1}(k^2+1)\left(\frac{k}{2}\right)^{k^2}\prod_p\left\{\left(1-\frac{1}{p}\right)^{k^2}\left(\sum_{j=0}^{\infty}\left(\frac{\Gamma(j+k)}{j!\Gamma(k)}\right)p^{-j}\right)\right\}.$$

Note that $A_0(k)$ is the coefficient of $(\log T)^{k^2}$ in the asymptotic formula for $\sum\limits_{n\le(T/2\pi)^{\frac{1}{2}k}} d_k^2(n)n^{-1}$. I conjecture that for $k=3,4$

$$I_k(T) = 2S_k(T) + O\left(T\log^{k^2-2} T\right) \tag{4.65}$$

holds, and (4.65) is in fact true for $k=1,2$ (when, of course, much more is known). Hence the expression for $A_0(k)$ leads to the conjectural value (4.12) for $c(k)$, namely

$$C(k) = \lim_{T\to\infty}\left(T\log^{k^2} T\right)^{-1}\int_0^T |\zeta(1/2+it)|^{2k}dt$$
$$= 2\left(\frac{k}{2}\right)^{k^2}\frac{1}{\Gamma(k^2+1)}\prod_p\left\{\left(1-\frac{1}{p}\right)^{k^2}\left(\sum_{j=0}^{\infty}\left(\frac{\Gamma(k+j)^2}{j!\Gamma(k)}\right)^2 p^{-j}\right)\right\}.$$

Other coefficients $A_j(k)$ can be also calculated, although the calculations turn out to be quite tedious as j increases. I shall only evaluate here $A_1(k)$ explicitly. It is the coefficient of $t\log^{k^2-1} t\Big|_T^{2T}$ in

$$\left(\frac{k}{2}\right)^{k^2}(c_{k^2}+c_{k^2-1}a_1)\,d_{-k^2}(k)+\left(\frac{k}{2}\right)^{k^2-1}c_{k^2-1}d_{-(k^2-1)}(k).$$

195 Thus in view of (4.64) we obtain

$$A_1(k) = \left(\frac{k}{2}\right)^{k^2}\left(-\frac{k^2}{(k^2)!}-\frac{\log(2\pi)}{(k^2-1)!}\right)d_{-k^2}(k)+\left(\frac{k}{2}\right)^{k^2-1}\frac{1}{(k^2-1)}d_{-(k^2-1)}(k) \tag{4.66}$$

$$= \left(\frac{k}{2}\right)^{k^2-1} \cdot \frac{1}{(k^2-1)!}\left\{-\frac{k}{2}d_{-k^2}(k)(1+\log(2\pi)) + d_{-(k^2-1)}(k)\right\}.$$

For $k = 2$

$$d_{-4}(2) = \lim_{s\to 1+0}(s-1)^4\frac{\zeta^4(s)}{\zeta(2)} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Thus

$$c(2) = \lim_{T\to\infty}(T\log^4 T)^{-1}\int_0^T |\zeta(1/2+it)|^4 dt = 2\cdot\frac{1}{4!}\cdot 6\pi^{-2} = \frac{1}{2\pi^2}.$$

Further, near $s = 1$,

$$\begin{aligned} F_2(s) = \frac{\zeta^4(s)}{\zeta(2s)} &= \left(\frac{1}{s-1} + \gamma + \gamma_1(s-1) + \cdots\right)^4 \\ &\quad \left(\frac{1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}(s-1) + \ldots\right) \\ &= d_{-4}(2)(s-1)^{-4} + d_{-3}(2)(s-1)^{-3} + \cdots \end{aligned}$$

with

$$d_{-3}(2) = \frac{4\gamma}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}.$$

Hence (4.66) for $k = 2$ yields

$$\begin{aligned} A_1(2) &= \frac{1}{6}\left(-\frac{(1+\log(2\pi))}{\zeta(2)} + \frac{4\gamma - 12\zeta'(2)\pi^{-2}}{\zeta(2)}\right) \\ &= \pi^{-2}\left(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2}\right), \end{aligned}$$

which coincides with (4.4), since for a_3 given by (4.3) we have $a_3 = 2A_1(2)$.

To find a more explicit form of the expression for $A_1(k)$, as given by (4.66), we proceed as follows. Write

$$F_k(s) = \zeta^{k^2}(s)G_k(s) \qquad (\mathrm{Re}\, s > 1),$$

where

$$G_k(s) = \prod_p \left\{ (1-p^{-s})^{k^2} \left(\sum_{j=0}^{\infty} \left(\frac{\Gamma(k+j)}{j!\Gamma(k)} \right)^2 p^{-js} \right) \right\} \left(\operatorname{Re} s > \frac{1}{2} \right). \quad (4.67)$$

196 Near $s = 1$ we have the Laurent expansion

$$\begin{aligned} F_k(s) &= d_{-k^2}(k)(s-1)^{-k^2} + d_{-(k^2-1)}(k)(s-1)^{-(k^2-1)} + \cdots \\ &= \left(\frac{1}{s-1} + \gamma + \gamma_1(s-1) + \cdots \right)^{k^2} \left(G_k(1) + G_k'(1)(s-1) + \cdots \right). \end{aligned}$$

This yields

$$d_{-k^2}(k) = G_k(1) = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^{k^2} \left(\sum_{j=0}^{\infty} \left(\frac{\Gamma(k+j)}{j!\Gamma(k)} \right)^2 p^{-j} \right) \right\}.$$

$$d_{k^2-1}(k) = \gamma k^2 G_k(1) + G_k'(1).$$

Hence we obtain

Theorem 4.4. *If* $Q_{k^2}(y) = \sum_{j=0}^{k^2} A_j(k) y^{k^2-j}$ *is the polynomial in* (4.61)*, we obtain*

$$\begin{aligned} A_0(k) &= \frac{1}{(k^2)!} \left(\frac{k}{2} \right)^{k^2} G_k(1), \\ A_1(k) &= \frac{1}{(k^2-1)!} \left(\frac{k}{2} \right)^{k^2-1} \left\{ \left(\gamma k^2 - \frac{k}{2}(1+\log(2\pi)) \right) G_k(1) + G_k'(1) \right\}, \end{aligned}$$

where $G_k(s)$ *is given by* (4.67).

Note that the conjecture (4.65) may be formulated more explicitly as ($k = 3, 4$)

$$\int_0^T |\zeta(1/2+it)|^{2k} dt = T \left(2A_0(k) \log^{k^2} T + 2A_1(k) \log^{k^2-1} T + O(\log^{k^2-2} T) \right).$$

A. Odlyzko has kindly calculated the values

$$2A_o(3) = 1.04502424 \times 10^{-5}, 2A_0(4) = 1.34488711 \times 10^{-12}.$$

4.5 The Mean Square Formula

The mean square formula

$$I_1(T) = \int\limits_0^T |\zeta(1/2+it)|^2 dt = T\left(\log\frac{T}{2\pi} + 2\gamma - 1\right) + E(T)$$

and results concerning $E(T)$ were discussed extensively in Chapter 2 and Chapter 3. In this section we shall show how bounds for $E(T)$ may be obtained from Theorem 4.3, which for $k = 1$ gives **197**

$$\frac{1}{2}I_{1,f}(T) = \sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Re}\left(\int\limits_0^{\infty} f(t)v\left(\frac{x}{n}\right)dt\right) + o(T^{1/6}) + o\left(T^{1/2+\epsilon}T_0^{-1}\right)$$
$$+ \sum_{\substack{m,n=1; m\neq n\\ 1-\delta\le m/n\le 1+\delta}} (mn)^{-1/2}\operatorname{Re}\left\{\int\limits_0^{\infty} f(t)v\left(\frac{x}{m}\right)\left(\frac{m}{n}\right)^{it} dt\right\},$$

where $x = \left(\frac{t}{2\pi}\right)^{1/2}$, and $0 < \delta < \frac{1}{2}$. From (4.60) and (4.62) we have

$$S_1(T) = \sum_{n=1}^{T}\frac{1}{n}\operatorname{Re}\left(\int_0^{\infty} f(t)v\left(\frac{x}{n}\right)dt\right) \tag{4.68}$$
$$= 2\operatorname{Re}\left\{\operatorname*{Res}_{s=1}\hat{f}(s)(2\pi)^{1-s}\zeta(1+2(s-1))P(1+2(s-1))\right\} + O(T^{\epsilon}).$$

Near $s = 1$

$$2\hat{f}(s)(2\pi)^{1-s}\zeta(1+2(s-1))P(1+2(s-1))$$
$$= 2\left(c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots\right)(1 + a_1(s-1) + a_2(s-1)^2 + \cdots$$
$$\times\left(\frac{1}{2(s-1)} + \gamma + 2\gamma_1(s-1) + \cdots\right)\left(\frac{1}{2(s-1)} + 2h_1(s-1) + \cdots\right).$$

The residue at $s = 1$ therefore equals

$$\frac{1}{2}\left(c_90a_1 + c_1 + 2\gamma c_0\right).$$

But

$$a_1 = -\log(2\pi), c_0 = t\Big|_T^{2T} + O(T_0),$$

$$c_1 = \int\limits_T^{2T} \log t dt + O(T_0 \log T) = (t \log t - t)\Big|_T^{2T} + O(T_0 \log T).$$

Hence (4.68) becomes

$$S_1(T) = \frac{1}{2}\left(t \log \frac{t}{2\pi} + (2\gamma - 1)t\right)\Bigg|_T^{2T} + O(T^\epsilon) + O(T_0 \log T). \quad (4.69)$$

198 In view of (4.61) it is seen that the above expression (with a suitable choice for T_0) will indeed furnish the main term in the asymptotic formula for $I_1(T)$. Therefore it remains to estimate the double sum

$$\begin{aligned}\sum(T, \delta) : &= \sum_{\substack{m,n=1; m\neq n\\ 1-\delta\leq \frac{m}{n}\leq 1+\delta}} (mn)^{-1/2} \int\limits_0^\infty f(t) \nu\left(\frac{x}{m}\right)\left(\frac{m}{n}\right)^{it} dt \qquad (4.70)\\ &= \sum_{\substack{m,n=1; m\neq n\\ 1-\delta\leq \frac{m}{n}\leq 1+\delta}} (mn)^{-1/2} J(T; m, n),\end{aligned}$$

where both m, n take at take at most $2\sqrt{T}$ values, and

$$J(T; m, n) := \int\limits_0^\infty f(t) \nu\left(\frac{x}{m}\right)\left(\frac{m}{n}\right)^{it} dt. \quad (4.71)$$

Integrating by parts the integral in (4.71) we obtain

$$J(T; m, n) = -\int\limits_0^\infty f'(t)\left(\int\limits_0^\infty \nu\left(m^{-1}\sqrt{\frac{u_1}{2\pi}}\right)\left(\frac{m}{n}\right)^{iu_1} du_1\right) dt = \ldots =$$

$$= (-1)^R \int\limits_0^\infty f^{(R)}(t)\left(\underbrace{\int\limits_0^t \cdots}_{R} \int\limits_0^{u_1} \nu\left(m^{-1}\sqrt{\frac{u_1}{2\pi}}\right)\left(\frac{m}{n}\right)^{iu_1} du_1 \cdots du_R\right) dt.$$

We evaluate the R-fold integral by using $\int c^{it}dt = c^{it}/(i\log c)$, after removing the ν-function from the innermost integral by the second mean value theorem for integrals. Hence this integral is $\ll_R |\log \frac{m}{n}|^{-R}$. Moreover, since $f^{(R)}(t) \ll_R T_0^{-R}$, we obtain uniformly in m and n

$$J(T;m,n) \quad \ll_R \quad TT_0^R\Big|\log \frac{m}{n}\Big|^{-R}.$$

To use this bound for J we write the sum in (4.70) ad

$$\begin{aligned}\sum(T,\delta) &= \sum_{\substack{m,n=1;m\neq n\\ 1-\delta\le\frac{m}{n}\le 1+\delta,|m-n|>T^{1/2+\epsilon}T_0^{-1}}}^{\infty} + \sum_{\substack{m,n=1;m\neq n\\ 1-\delta\le\frac{m}{n}\le 1+\delta,|m-n|\le T^{1/2+\epsilon}T_0^{-1}}}^{\infty}\\ &= \sum\nolimits_1 + \sum\nolimits_2,\end{aligned}$$

say, where $\epsilon > 0$ is an arbitrarily small, but fixed number. We obtain **199**

$$\sum\nolimits_1 \ll_R \sum_{\substack{m,n\le 2\sqrt{T};m\neq n\\ 1-\delta\le\frac{m}{n}\le 1+\delta;|m-n|\ge T^{1/2+\epsilon}T_0^{-1}}} (mn)^{-1/2}|\log\frac{m}{n}|^{-R}TT_0^{-R}.$$

Since $1-\delta \le \frac{m}{n} \le 1+\delta$ and $0 < \delta < \frac{1}{2}$, we have

$$\log\frac{m}{n} = \log\left(1+\frac{m-n}{n}\right) \gg_\delta \frac{|m-n|}{n} \ge T^{1/2+\epsilon}T_0^{-1}n^{-1}.$$

Therefore

$$\begin{aligned}\sum\nolimits_1 &\ll_R \sum_{m,n\le 2T^{1/2}} TT_0^{-R}(mn)^{-1/2}n^RT^{-1/2R-\epsilon R}T_0^R\\ &\ll_R T^{1-\epsilon R}\sum_{m\le 2T^{1/2}}\sum_{n\le T^{1/2}} n^{-1/2} \ll T^{3/2-\epsilon R} \ll 1\end{aligned}$$

if we choose $R = [3/(2\epsilon)] + 1$. In other words, the smoothing function f, with its salient property that $f^{(R)}(t) \ll_R T_0^{-R}$, has the effect of making the contribution of $\sum_1$ negligible What remains is the "shortened" sum $\sum_3$, which becomes, on writing $m - n = r$,

$$\sum\nolimits_2 = \sum_{\substack{0<|r|\le T^{1/2+\epsilon}T_0^{-1}\\ r\ge 2\sqrt{T},-\delta n\le r\le\delta n}} n^{-1/2}(n+r)^{-1/2}\int\limits_0^\infty f(t)\nu\left(\frac{x}{n+r}\right)\left(1+\frac{r}{n}\right)^{it} dt.$$

We suppose $T^\epsilon \ll T_0 \ll T^{1/2}$, since we do not wish the sum over r to be empty. To estimate $\sum_2$ write

$$\int\limits_0^\infty f(t)\nu\left(\frac{x}{n+r}\right)\left(1+\frac{r}{n}\right)^{it} dt = \int\limits_{T+T_0}^{2T-T_0} f(t)\nu\left(\frac{x}{n+r}\right)\left(\frac{r}{n}\right)^{it} dt + I_2$$
$$= I_1 + I_2,$$

say, where in I_2 the intervals of integration are $\ll T_0$ in length. In I_1 we have $f(t) = 1$, and the ν-function is removed by the second mean value theorem for integrals. We use

$$\int\limits_A^B \left(\frac{m}{n}\right)^{it} dt = \frac{(m/n)^{iB} - (m/n)^{iA}}{i\log(m/n)}$$

200 and perform summation over n. Since $m = n + r$ we encounter sums of the type

$$\sum_{n\le 2T^{1/2}} n^{-1}\left(\log\left(1+\frac{r}{n}\right)\right)^{-1} \exp\left(i\tau\log\left(1+\frac{r}{n}\right)\right) \tag{4.72}$$

with $\tau \asymp T$. If (χ, λ) is an exponent pair, then for $N \ll T^{1/2}$

$$\sum_{N<n\le N'\le 2N} \exp\left(i\tau\log\left(1+\frac{r}{n}\right)\right) \ll \left(\frac{Tr}{N^2}\right)^{\chi} N^\lambda, \tag{4.73}$$

and since $\log\left(1+\frac{r}{n}\right) \sim \frac{r}{n}$ it follows by partial summation that the sum in (4.72) is

$$\ll r^{\chi-1}T^xT^{1/2}(\lambda - 2x) = r^{x-1}T^{1/2\lambda}.$$

Summation over r shows then that the contribution of I_1 is

$$\ll \sum_{1\le r\le T^{1/2+\epsilon}T_0^{-1}} r^{x-1}T^{1/2\lambda} \ll T^{\epsilon+1/2}(\chi+\lambda)T_0^{-x}.$$

The contribution of I_2 is obtained not by integration, but by estimation, first over n using again (4.73), then over r under the integral sign. Trivial estimation of the integral as $O(T_0)$ times the maximum of the sums shows that this contribution will be $\ll T^{\epsilon+1/2(\chi+\lambda)T_0^{-x}}$, just as in the previous case. Combining the preceding estimates we have, for $T^\epsilon \ll T_0 \ll T^{1/2}$,

$$E(T) \ll T^\epsilon \left(T_0 + T^{1/2}T_0^{-1} + T^{1/2(x+\lambda)}T_0^{-x}\right).$$

We choose now $T_0 = T^{(x+\lambda)/(2x+2)}$. Since $1/4 \leq (x+\lambda)/(2x+2) \leq 1/2$ for any (x, λ), we have $T^{1/4} \leq T_0 \leq T^{1/2}$ and therefore

$$E(T) \ll \frac{x+\lambda}{T^{2(x+1)}} + \epsilon. \tag{4.74}$$

This is the same upper bound that was obtained in Section 2.7 from Atkinson's formula for $E(T)$ by an averaging process. However, we already remarked that it seems difficult to obtain an explicit formula **201** (such as Atkinson's) for $E(T)$ from Theorem 4.3.

There is yet another approach to the estimation of $E(T)$, which gives essentially the same bound as (4.74). It involves the use of an explicit formula for $E(T)$, due to R. Balasubramanian [6]. He integrated the classical Riemann-Siegel formula and obtained

$$E(T) = 2\sum_{n\leq K}\sum_{m\leq K, m\neq n}\left\{\frac{\sin(T\log\frac{m}{n})}{(mn)^{1/2}\log\frac{m}{n}} + \frac{\sin(2\theta - T\log mn)}{(mn)^{1/2}(2\theta' - \log mn)}\right\} + O(\log^2 T), \tag{4.75}$$

where $K = K(T) = \left(\frac{T}{2\pi}\right)^{1/2}$, $\theta = \theta(T) = \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8}$. As already seen in Chapter 2, from the definition of $E(T)$ it follows that

$$E(T) \leq E(T+u) + C_1 u \log T \quad (C_1 > 0, 0 \leq u \leq T).$$

Set $u = t + GL$, multiply by $exp(-t^2G^{-2})$ and integrate over t from $-GL$ to GL, where $L = \log T$. We obtain

$$\int_{-GL}^{GL} E(T)e^{-(t/G)^2}dt \leq \int_{-GL}^{GL} E(T+t+GL)e^{-(t/G)^2}dt + CG^2L^2,$$

whence for $1 \ll G \ll T^{1/2}$ and a suitable $C > 0$

$$E(T) \leq \left(\sqrt{\pi}G\right)^{-1}\int_{-GL}^{GL} E(T+t+GL)e^{-(t/G)^2}dt + CGL^2, \tag{4.76}$$

and similarly

$$E(T) \geq \left(\sqrt{\pi}G\right)^{-1} \int\limits_{-GL}^{GL} E(T+t-GL)e^{-(t/G)^2}dt - CGL^2. \tag{4.77}$$

If $E(T) > 0$ we use (4.76), otherwise (4.77), and since the analysis is similar in both cases we may concentrate on (4.76). The parameter G may be restricted to the range $T^{1/4} \leq G \leq T^{1/3}$ since $E(T) = \Omega(T^{1/4})$ and $E(T) = O(T^{1/3})$. We insert (4.75) in the integral in (4.76), with T replaced by $T + t + GL$. The first step is to replace $K(T + t + GL)$ by $K(T)$. In doing this we make an error which is $0(GL)$. Then we integrate Balasubramanian's formula by using

$$\int\limits_{-\infty}^{\infty} e^{-Bx^2}\sin(Ax+C)dx = \sqrt{\frac{\pi}{B}}e^{\frac{A^2}{4B}}\sin C \quad (\operatorname{Re} B > 0). \tag{4.78}$$

202 We set $\tau = T + GL$ and use

$$\begin{aligned}
&\sin(2\theta(\tau+t) - (\tau+t)\log mn) \\
&= \sin\left(\tau\log\frac{\tau}{2\pi mn} - \tau + t\log\frac{\tau}{2\pi mn} - \frac{\pi}{4}\right) + O\left(G^2L^2T^{-1}\right), \\
&(2\theta' - \log mn)^{-1} = \left(\log\frac{\tau}{2\pi mn}\right)^{-1} + O\left(GL\left(\log\frac{T}{2\pi mn}\right)^{-2}\right),
\end{aligned}$$

which follows by Taylor's formula after some simplifying. The total contribution of the error terms will be again $O(GL)$. Thus using (4.78) we obtain, for $T^{1/4} \leq G \leq T^{1/3}$, $\tau = T + GL$, $L = \log T$,

$$\begin{aligned}
&E(T) \leq CGL^2 + 2\sum_{n\leq\sqrt{\frac{T}{2T}}}\ \sum_{m\leq\sqrt{\frac{T}{2T}},m\neq n} \\
&\frac{1}{\sqrt{mn}}\left\{\frac{\sin\left(J\log\frac{n}{m}\right)}{\log\frac{n}{m}}e^{-\frac{1}{4}G^2\log^2\frac{n}{m}} + \frac{\sin\left(J\log\frac{\tau}{2Tmn} - J - \frac{\pi}{4}\right)}{\log\frac{J}{2\pi mn}}e^{-\frac{1}{4}G^2\log^2\frac{J}{2\pi mn}}\right\}.
\end{aligned} \tag{4.79}$$

The bound for $E(T)$ given by (4.79) allows an easy application of exponent pairs (and also of two-dimensional techniques for the estimation of exponential sums). The sum over n can be split into $O(\log T)$

subsums in which $K_1 < n \leq 2K_1 (\ll T^{1/2})$, and then we may suppose $1/2K_1 < m \leq 1/2(5K_1)$, for otherwise the exponential factors are very small. By this and by symmetry we may set $m = n + r$, $1 \leq r \leq 1/2K_1$, the contribution of r for which $r \geq K_1 G^{-1} L$ being negligible by the rapid decay of the exponential function (the contribution of $K_1 \leq 2GL^{-1}$ is also negligible). Thus by using partial summation we get

$$
\begin{aligned}
E(T) &\ll GL^2 + L \max_{K_1 \ll T^{1/2}} \sum_{r \leq K_1 G^{-1} L} r^{-1} \max_{K_1 \leq K_1' \leq K'' \leq 2K_1} \left| \sum_{K_1' < n \leq K_1''} \exp\left(i\tau \log\left(1 + \frac{r}{n}\right)\right) \right| \\
&\ll GL^2 + L \max_{K_1 \ll T^{1/2}} \sum_{r \leq K_1 G^{-1} L} r^{-1} \left(\frac{T r^{\chi}}{K_1^2}\right) K_1^{\lambda},
\end{aligned}
$$

on estimating the first sine terms in (4.79), and the other sine terms will **203**
give at the end the same upper bound. Thus

$$
\begin{aligned}
E(T) &\ll GL^2 + L \max_{K_1 \ll T^{1/2}} T^x K_1^{\lambda - 2x} (K_1 G^{-1} L)^x \\
&\ll L^2 \left(G + T^{1/2(x+\lambda)} G^{-x}\right).
\end{aligned}
$$

Taking $G = T^{(x+\lambda)/(2x+2)}$ we obtain

$$
E(T) \ll T^{\frac{x+\lambda}{2(x+1)}} \log^2 T. \tag{4.80}
$$

This is the same as (4.74), only T^{ϵ} is now replaced by the sharper $\log^2 T$. The reason for this is the use of the specific smoothing factor $\exp(-t^2 G^{-2})$, which decays very fast, and by (4.78) enables one to evaluate explicitly the integrals in question. The had work is, in this case, contained in Balasubramanian's formula (4.75) for $E(T)$. In the previous approach, which was the evaluation of $I_1(T)$ via $I_{1,f}(T)$, the smoothing factor $f(\cdot)$ was built in from the very beginning of proof. This was useful, since it saved some hard work, but as pointed out before, it resulted in the fact that the could not get an explicit formula for $E(T)$ itself, but just an upper bound.

4.6 The Fourth Power Moment

In Chapter 5 we shall study extensively the asymptotic evaluation of

$$I_2(T) = \int\limits_0^T |\zeta(1/2 + it)|^4 dt$$

by the powerful methods developed by Y. Motohashi. In this section we wish to investigate $I_2(T)$ by using Theorem 4.3, and indicate how the bounds $E_2(T) = O(T^{7/8+\epsilon})$ and $E_2(T) = O(T^{2/8+\epsilon})$, due to Hewth-
204 Brown and Zavorotnyi, respectively, may be derived. So far we have pushed Zavorotnyi's method to the extent of establishing Theorem 4.3 for $k \le 4$, and not only for $k \le 2$, as was done by Zavorotnyi. Now we shall use first Heath-Brown's approach in conjunction with Theorem 4.3, and then we shall briefly sketch the salient points of Zavorotnyi's method. In view of (4.61) we need only to evaluate the sum

$$S(T) = S(T; \delta, f) := \sum_{\substack{m,n=1; m \ne n \\ 1-\delta \le m/n \le 1+\delta}}^{\infty} d(m)d(n)(mn)^{-1/2} \operatorname{Re} \left\{ \int\limits_0^T f(t) v\left(\frac{t}{2\pi m}\right)\left(\frac{m}{n}\right)^{it} dt \right\}.$$

This sum is split into three subsums, according to the conditions:

(i) $m > n, m/n \le 1 + \delta$, ii) $m < n, n/m \le 1 + \delta$, iii) $1 + \delta < n/m \le \frac{1}{1-\delta}$, and the subsums are denoted by $S_i(T), S_{ii}(T), S_{iii}(T)$, respectively. Integrating by parts, as in the proof of Theorem 4.3, and using $f'(t) \ll T_{-1}$, we obtain

$$S_{iii}(T) \ll T^{1+\epsilon} T_0^{-1}.$$

In $S_i(T)$ we set $m = n + r$, and in $S_{ii}(T) n = m + r$. Changing then m into n in $S_{ii}(T)$ we obtain, for $0 < \delta < 1/2$,

$$S(T) = O\left(T^{1+\epsilon} T_0^{-1}\right) + \sum_{n=1}^{\infty} \sum_{1 \le r \le \delta n} d(n)d(n+r)(n(n+r))^{-1/2}$$
$$\int\limits_0^{\infty} f(t)\left(v\left(\frac{t}{2\pi(n+r)}\right) + v\left(\frac{t}{2\pi n}\right)\right) \cos\left(t \log\left(1 + \frac{r}{n}\right)\right) dt.$$

Note that $n \leq bT$ in the range of summation of n in view of the properties of $\nu(x)$. By (4.57) we see that the terms for which $n \leq T_0$ contribute

$$\ll \sum_{n \leq T_0} \sum_{r \leq \delta n} \frac{d(n)d(n+r)}{n} \cdot \frac{n}{r} \ll T^{\epsilon} T_0.$$

Now choose $0 < \delta < 1/2$ fixed, $\epsilon > 0$ arbitrarily small but fixed, and $T_1 = T_1(\epsilon, \delta)$ so large that $T^{\epsilon} T_0^{-1} \leq \delta$ for $T \geq T_1$. For $r > T^{\epsilon} T_0^{-1} n$ we have $\log\left(1 + \frac{r}{n}\right) \ll T^{\epsilon} T_0^{-1}$. Thus integrating sufficiently many times by 205
parts it is seen that the contribution of the terms for which $r > T^{\epsilon} T_0^{-1} n$ to $S(T)$ is negligible. We have

$$S(T) = O(T^{\epsilon} T_0) + O(T^{1+\epsilon} T_0^{-1}) + \sum_{n \geq T_0} \sum_{r \leq T^{\epsilon} T_0^{-1} \pi} \tag{4.81}$$

$$\frac{d(n)d(n+r)}{(n(n+r))^{1/2}} \int\limits_0^{\infty} f(t) \left(\nu\left(\frac{t}{2\pi(n+r)}\right) + \nu\left(\frac{t}{2\pi n}\right) \right) \cos\left(t \log\left(1 + \frac{r}{n}\right)\right) dt.$$

Next, for $\nu(\cdot)$ in the above sum we have, by Taylor's formula,

$$\nu\left(\frac{t}{2\pi(n+r)}\right) = \nu\left(\frac{t}{2\pi n}\right) - \frac{tr}{2\pi n^2} \nu'\left(\frac{t}{2\pi n}\right) + O\left(\frac{t^2 r^2}{n^4}\right).$$

The contribution of the O-term to $S(T)$ is trivially

$$\ll \sum_{n \geq T_0} \sum_{r \leq T^{\epsilon} T_0^{-1} n} \frac{d(n)d(n+r)}{(n(n+r))^{1/2}} \cdot \frac{T^3 t^2}{n^4}$$

$$\ll T^{3+\epsilon} \sum_{n \geq T_0} n^{-5} T^{3\epsilon} T_0^{-3} n^3 \ll T^{3+4\epsilon} T_0^{-4}.$$

Therefore (4.81) becomes

$$S(T) = 2 \sum_{n \geq T_0} \sum_{r \leq T^{\epsilon} T_0^{-1} n} \frac{d(n)d(n+r)}{(n(n+r))^{1/2}} \tag{4.82}$$

$$\int\limits_0^{\infty} f(t) \nu\left(\frac{t}{2\pi n}\right) \cos\left(t \log\left(1 + \frac{r}{n}\right)\right) dt - \sum_{n \geq T_0} \frac{1}{2\pi n^2} \sum_{r \leq T^{\epsilon} T_0^{-1} n}$$

$$\frac{rd(n)d(n+r)}{(n(n+r))^{1/2}}\int\limits_0^\infty f(t)tv'\left(\frac{t}{2\pi n}\right)\cos\left(t\log\left(1+\frac{r}{n}\right)\right)dt$$
$$+O\left(T^{1+\epsilon}T_0^{-1}\right)+O(T^\epsilon T_0)+O\left(T^{3+4\epsilon}T_0^{-4}\right).$$

The contribution of the second double sum in (4.82) is, on applying Lemma 2.1,

$$\ll T^\epsilon\sum_{T_0\le n\le bT}n^{-2}\sum_{r\le T^\epsilon T_0^{-1}n}\frac{r}{n}\cdot T\cdot\frac{n}{r}\ll T^{1+3\epsilon}T_0^{-1}.$$

206 Write now

$$(n+r)^{-1/2}=n^{-1/2}-\frac{1}{2}rn^{-3/2}+O\left(r^2n^{-5/2}\right).$$

The contribution of the terms $-r/(2n^{3/2})$ to $S(T)$ will be

$$O\left(T^{1+3\epsilon}T_0^{-1}\right)$$

by Lemma 2.1. The 0-term will contribute

$$\ll T^\epsilon\sum_{T_0\le n\le bT}n^{-1}\sum_{r\le T^\epsilon T_0^{-1}n}Tr^2n^{-5/2}\ll T^{1+\epsilon}\sum_{T_0\le n\le bT}n^{-7/2}T^{3\epsilon}T_0^{-3}n^3$$
$$\ll T^{3/2+4\epsilon}T_0^{-3}.$$

Therefore

$$S(T)=O\left\{T^{4\epsilon}\left(TT_0^{-1}+T_0+T^3T_0^{-4}+T^{3/2}T_0^{-3}\right)\right\}\tag{4.83}$$
$$+2\sum_{T_0\le n\le bT}\sum_{r\le T^\epsilon T_0^{-1}n}d(n)d(n+r)n^{-1}\int\limits_0^\infty f(t)v\left(\frac{t}{2\pi n}\right)\cos\left(t\log\left(1+\frac{r}{n}\right)\right)dt.$$

We may further simplify (4.83) by using

$$\cos\left(t\log\left(1+\frac{r}{n}\right)\right)=\cos\left(\frac{tr}{n}\right)+\frac{tr^2}{2n^2}\sin\left(\frac{tr}{n}\right)+O\left(\frac{t^2r^4}{n^4}\right).$$

The contribution of the error term is

$$\ll \sum_{T_0 \leq n \leq bT} T^{\epsilon} n^{-1} \sum_{r \leq T^{\epsilon} T_0^{-1} n} T^3 r^4 n^{-4} \ll T^{4+6\epsilon} T_0^{-5}.$$

The contribution of the sine terms will be, by Lemma 2.1,

$$O(T^{2+4\epsilon} T_0^{-2}).$$

Hence we have

$$S(T) = \sum(T) + O\left\{T^{6\epsilon}\left(T_0 + TT_0^{-1} + T^2 T_0^{-2} + T^3 T_0^{-4} + T^4 T_0^{-5}\right)\right\}, \quad (4.84)$$

where

$$\sum(T) := 2 \sum_{T_0 \leq n \leq bT} \sum_{r \leq T^{\epsilon} T_0^{-1}} \frac{d(n)d(n+r)}{n} \int\limits_0^{\infty} f(t) v\left(\frac{t}{2\pi n}\right) \cos\left(\frac{tr}{n}\right) dt. \quad (4.85)$$

Writing $A_r = \max(T_0, rT_0 T^{-\epsilon})$, we may change the order of summation in $\sum(T)$ and obtain

$$\sum(T) = 2 \sum_{\gamma \leq bT^{1+\epsilon} T_0^{-1}} \sum_{A_r \leq n \leq bT} \frac{d(n)d(n+r)}{n} \int\limits_0^{\infty} f(t) v\left(\frac{t}{2\pi n}\right) \cos\left(\frac{tr}{n}\right) dt. \quad (4.86)$$

It would be good if the above sum could be evaluated as a double (exponential) sum with divisor coefficients. This, unfortunately, does not seem possible at the moment. One can, however, perform summation over n and then over r. Some loss is bound to occur in this approach, used both by Heath-Brown and Zavorotnyi. The former shows that **207**

$$\sum_{n \leq x} d(n)d(n+r) = m(x, r) + E(x, r),$$

where for some absolute constants c_{ij}

$$m(x, r) = \sum_{i=0}^{2} c_i(r) x (\log x)^i, c_i(r) = \sum_{j=0}^{2} c_{ij} \sum_{j=0}^{2} c_{ij} \sum_{d|r} d^{-1} (\log d)^j,$$

and uniformly for $1 \le r \le x^{5/6}$

$$E(x,r) \ll x^{5/6+\epsilon}, \tag{4.87}$$

while uniformly for $1 \le r \le X^{3/4}$

$$\int\limits_X^{2X} E^2(x,r)dx \ll X^{5/2+\epsilon}. \tag{4.88}$$

The estimates (4.87) and (4.88) are of independent interest. They are obtained by techniques from analytic number theory and T. Estermann's estimate [35]

$$|S(u,v;q)| \le d(q)q^{1/2}(u,v,q)^{1/2} \tag{4.89}$$

for the Kloosterman sum ($e(y) = \exp(2\pi iy)$)

$$S(u,v;q) = \sum_{n\le q,(n,q)=1,nn'\equiv 1 \pmod q} e\left(\frac{un+vn'}{q}\right). \tag{4.90}$$

In (4.86) we write the sum as

$$F(T,r) := \sum_{A_r\le n\le bT} d(n)d(n+r)h(n,r),$$

$$h(x,r) := \frac{1}{x}\int\limits_0^\infty f(t)v\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)dt.$$

Hence

$$\begin{aligned} F(T,r) &= \int\limits_{A_r}^{bT} h(x,r)d\{m(x,r)+E(x,r\} \\ &= \int\limits_{A_r}^{bT} m'(x,r)h(x,r)dx + E(x,r)h(x,r)\Big|_{A_r}^{bT} - \int\limits_{A_r}^{bT} h'(x,r)E(x,r)dx. \end{aligned}$$

208 By Lemma 2.1 we have $h(x,r) = O(1/r)$ uniformly in r. Thus the

total contribution of the integrated terms to $S(T)$ is $\ll T^{5/6+\epsilon}$, the condition $r \leq T^{5/6}$ being trivial if we suppose that $T_0 \geq T^{1/2}$. Next

$$\frac{dh(x,r)}{dx} = -x^{-2}\int_0^\infty f(t)\nu\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)dt$$

$$-x^{-3}\int_0^\infty f(t)\frac{t}{2\pi}\nu'\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)dt + x^{-3}\int_0^\infty f(t)tr\nu\left(\frac{t}{2\pi x}\right)\sin\left(\frac{tr}{x}\right)dt$$

$$\ll (rx)^{-1} + Tr(rx^2) - 1 \ll Tx^2.$$

Using (4.88) (the condition $r \ll T^{3/4}$ holds again for $T_0 \geq T^{1/2}$) we obtain, for $A_r \leq Y \leq bT$,

$$\int_Y^{2Y} h'(x,r)E(x,r)dx \ll T\left(\int_Y^{2Y} x^{-4}dx\right)^{1/2}\left(\int_Y^{2Y} E^2(x,r)dx\right)^{1/2}$$

$$\ll TY^{-3/2}Y^{5/4+\epsilon} \ll TY^{\epsilon-1/4}.$$

Hence

$$\sum_{1\leq r\leq bT^{1+\epsilon}T_0^{-1}}\int_{A_r}^{bT} h'(x,r)E(x,r)dx \ll \sum_{1\leq r\leq bT^{1+\epsilon}T_0^{-1}} T^{1+\epsilon}A_r^{-1/4}$$

$$\ll \sum_{r\leq T^\epsilon} T^{1+\epsilon}T_0^{-1/4} + \sum_{T^\epsilon<r\leq bT^{1+\epsilon}T_0^{-1}} T^{1+\epsilon}T_0^{-1/4}r^{-1/4}$$

$$\ll T^{1+2\epsilon}T_0^{-1/4} + T^{7/4+\epsilon}T_0^{-1}.$$

It is the contribution of the last term above which is large, and sets the limit of Heath-Brown's method to $E_2(T) \ll T^{7/8+\epsilon}$, on choosing $T_0 = T^{7/8}$. With this choice of T_0 we have

$$\sum(T) = 2\sum_{1\leq r\leq bT^{1+\epsilon}T_0^{-1}}\int_{A_r}^{bT} m'(x,r)h(x,r)dx + O\left(T^{7/8+\epsilon}\right),$$

with 209

$$m'(x,r) = (c_0(r) + c_1(r)) + (c_1(r) + 2c_2(r))\log x + c_2(r)\log^2 x$$
$$= d_0(r) + d_1(r)\log x + d_2(r)\log^2 x,$$

say. Thus

$$\sum(T) = 2\sum_{j=0}^{2}\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)\int_{A_r}^{bT} h(x,r)\log^j x\,dx + O\left(T^{7/8+\epsilon}\right).$$

By the second mean value theorem for integrals it is seen that $h(x,r)$ equals at most four expressions of the form

$$\frac{1}{r}\sin(\tau_1 r x^{-1})\nu(\tau_2/(2\pi x)) \qquad \left(\frac{T}{2}\le \tau_1,\tau_2\le \frac{5T}{2}\right),$$

each multiplied by a factor containing the f-function. Hence by Lemma 2.1

$$\int_0^{A_r} h(x,r)\log^j x\,dx \ll T^{\epsilon-1}r^{-2}A_r^2 \ll T^{2\epsilon-1}T_0^2,$$

and so

$$\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)\int_0^{A_r} h(x,r)\log^j x\,dx \ll T^{2\epsilon-1}T_0^2T^{1+\epsilon}T_0^{-1} = T^{3\epsilon}T_0.$$

Thus

$$\sum(T) = 2\sum_{j=0}^{2}\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)\int_0^{bT} h(x,r)\log^j x\,dx + O(T^{7/8+\epsilon}).$$

Now write

$$h(x,r) = \frac{1}{x}\int_T^{2T}\nu\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)dt + \frac{1}{x}\left(\int_{T-T_0}^{T} + \int_{2T}^{2T+T_0}\right)f(t)\nu\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)dt$$

in case $f(t) = \overline{f}(t)$, and the case when $f(t) = \underline{f}(t)$ is analogous. The contribution of the last two integrals is estimated in a similar way. By using Lemma 2.1 we have

$$\sum_{r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)\int\limits_0^{bT}\int\limits_{T-T_0}^{T} f(t)\nu\left(\frac{t}{2\pi x}\right)x^{-1}\cos\left(\frac{tr}{x}\right)dt\,dx$$

$$= \sum_{r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)\int_{T-T_0}^{T} f(t)\left(\int_0^{bT}\nu\left(\frac{t}{2\pi x}\right)x^{-1}\cos\left(\frac{tr}{x}\right)dx\right)dt$$

$$\ll T_0 \sum_{r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)r^{-1}T_0,$$

which is negligible. The same bound holds for $\int\limits_{2T}^{2T+T_0}$. Hence combining 210
this estimate with (4.84)-(4.86) we infer that, for $T_0 = T^{7/8}$,

$$S(T) = 0(T^{\epsilon}T_0) + 2\sum_{j=0}^{2}\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} \tag{4.91}$$
$$d_j(r)\int\limits_0^{bT}\int\limits_T^{2T} x^{-1}\nu\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)\log^j x\,dt\,dx.$$

We have

$$\int\limits_T^{2T}\nu\left(\frac{t}{2\pi x}\right)\cos\left(\frac{tr}{x}\right)dt = \frac{x}{r}\nu\left(\frac{t}{2\pi x}\right)\sin\left(\frac{tr}{x}\right)\Bigg|_T^{2T} - \frac{x}{r}\int\limits_T^{2T}\frac{1}{2\pi x}\nu'\left(\frac{t}{2\pi x}\right)\sin\left(\frac{tr}{x}\right)dt.$$

Hence

$$S(T) = 2\sum_{j=0}^{2}\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)r^{-1}\left\{\int\limits_0^{bT}\sin\left(\frac{tr}{x}\right)\nu\left(\frac{t}{2\pi x}\right)\log^j x\cdot dx\right\}\Bigg|_T^{2T}$$

$$-\frac{1}{\pi}\sum_{j=0}^{2}\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)r^{-1}\int\limits_T^{2T}\int\limits_0^T x^{-1}\sin\left(\frac{tr}{x}\right)\nu'\left(\frac{t}{2\pi x}\right)$$
$$\log^j x\,dx\,dt + O(T^\epsilon T_0).$$

Note that $\nu(y)$, $\nu'(y) = 0$ for $y \le 1/b$, and also $\nu'(y) = 0$ for $y \ge b$. Hence $\int\limits_0^{bT}$ becomes $\int\limits_0^{bt/2\pi}$ in the first sum above and $\int\limits_{t/2\pi b}^{bt/2\pi}$ in the second sum. We make the change of variable $t/(2\pi x) = y$ to obtain

$$S(T) = 2\sum_{j=0}^{2}\sum_{1\le r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)r^{-1}\left\{\int\limits_{1/b}^{\infty}\sin(2\pi yr)\nu(y)\frac{t}{2\pi}\log^j\left(\frac{t}{2\pi y}\right)\frac{dy}{y^2}\right\}\Bigg|_T^{2T}$$
$$-\frac{1}{\pi}\sum_{j=0}^{2}\sum_{r\le bT^{1+\epsilon}T_0^{-1}} d_j(r)r^{-1}\int\limits_T^{2T}\left(\int\limits_{1/b}^{b}\nu'(y)\sin(2\pi yr)\log^j\left(\frac{t}{2\pi y}\right)\frac{dy}{y}\right)dt + O(T^\epsilon T_0).$$

211 It remains to write

$$\log\frac{t}{2\pi y} = \log t - \log(2\pi y),$$
$$log^2\frac{t}{2\pi y} = \log^2 t - 2\log t\log(2\pi y) + \log^2(2\pi y),$$

observe that the integral

$$\int\limits_{1/b}^{\infty}\sin(2\pi yr)\nu(y)\log^j(2\pi y)\frac{dy}{y^2}$$

converges and that, by lemma 2.1, it is uniformly $O(1/r)$. Similar analysis holds for $\int\limits_{1/b}^{b}$. This means that in the expression for $S(T)$ we can extend summation over r to ∞, making an error which is $O(T^{1+\epsilon}T_0^{-1})$. In the second integral we can integrate $\log^j t$ from T to $2T$. This finally gives

$$S(T) = \left(C_0 t\log^2 t + C_1 t\log t + C_2 t\right)\Big|_T^{2T} + O(T^{7/8+\epsilon}), \tag{4.92}$$

where each C_i is a linear combination of integrals of the type

$$\sum_{r=1}^{\infty} e_j(r)r^{-1}\int\limits_{1/b}^{\infty}\sin(2\pi yr)\nu(y)\log^j(2\pi y)\frac{dy}{y^2},$$

$$\sum_{r=1}^{\infty} f_j(r)r^{-1}\int\limits_{1/b}^{b}\sin(2\pi yr)\nu'(2\pi yr)\nu'(y)\log^j(2\pi y)\frac{dy}{y},$$

with suitable $e_j(r)$, $f_j(r)$. When combined with (4.61) ($k = 2$), (4.92) proves Heath-Brown's result

$$E_2(T) \ll T^{7/8+\epsilon}. \tag{4.93}$$

The only problem which remains is the technical one, namely to show that $(tH_{2,\nu}(\log t))\Big|_T^{2T}$ from (4.61) cancels with all terms containing the ν-function in (4.92). This can be achieved in two ways: first by following Zavorotnyi [167], who actually shows that this must be the case, i,e. that the terms with the ν-function actually cancel each other. This fact is a difficult part of Zavorotnyi's proof of (4.6). Equally difficult, but feasible, is to show this fact directly. We can identify the coefficients a_j **212**
in (4.3) in the final formula for $I_2(T)$ by going through Heath-Brown's proof. Essentially these will come from $\int\limits_{1/b}^{\infty}$ if integration is from 1 to ∞, and no ν-function is present. But $\nu(y) = 1$ for $y \geq b$ and in the integral from 1 to b write $\nu(y) = 1+(\nu(y)-1)$ and show that $\nu(y)-1 = -\nu(1/y)$ can be cancelled with the corresponding part from $1/b$ to b. The integrals

$$\int\limits_{1/b}^{b}\sin(2\pi yr)\nu'(y)\log^j(2\pi y)\frac{dy}{y}$$

will be the "true" integrals containing the ν-function. It can be, however, shown that the coefficients of $H_{2,\nu}$ in (4.61) will contain exactly the coefficients representable by these integrals. This requires a lot of additional (technical) work, and will not be carried out explicitly here, since Zavorotnyi's work already established that eventually all the terms containing the ν-function cancel each other. Naturally, Heath-Brown's proof of (4.93) is in several ways technically simpler than the above one, since it does not use the ν-function but smoothing with the exponential factor $\exp(-t^2G^{-2})$, which is easier to handle.

N. Zavorotnyi's improvement of (4.93), namely (4.6), is based on the convolution formula of N.V. Kuznetsov [106]. This gives an explicit representation of

$$W_N(s,\gamma;w_0,w_1) = N^{s-1}\sum_{n=1}^{\infty}\tau_\nu(n) \tag{4.94}$$
$$\left\{\sigma_{1-2s}(n-N)w_0\left(\sqrt{\frac{n}{N}}\right)+\sigma_{1-2s}(n+N)w_1\left(\sqrt{\frac{n}{N}}\right)\right\},$$

where

$$\sigma_a(n) = \sum_{d|n} d^a, \tau_\nu(m) =| m |^{\nu-1/2}\ \sigma_{1-2\nu}(m), \sigma_{1-2s}(0) = \zeta(2s-1),$$

w_0, w_1 are sufficiently smooth functions with rapid decay. Crudely speaking, Heath-Brown used (4.89) as the pointwise estimate of the absolute value of the Kloosterman sum, whereas Kuznetsov's formula for
213 (4.94) involves an average of Kloosterman sums, where a much larger cancellation of terms occurs, and at the end the sharp result (4.6) is obtained. That massive cancellation occurs in sums of Kloosterman sums may be seen from N.V. Kuznetsov's bound

$$\sum_{c\le T} S(m,n;c)c^{-1} \ll T^{1/6}(\log T)^{1/3}. \tag{4.95}$$

Actually in application to the fourth moment Kuznetsov's formula (4.94) is used with $w_0(x) = 0$, $w_1(x) = w_N(x) \in C^\infty(0,\infty)$ and with $s, \nu \to 1/2$. This is written as

$$\begin{aligned} W_N\left(\frac{1}{2},\frac{1}{2};0,w_N\right) &:= N^{-1/2}\sum_{n=1}^{\infty} d(n)d(n+N)w_N\left(\sqrt{\frac{n}{N}}\right)\\ &= Z_N^{(d)}\left(\frac{1}{2},\frac{1}{2};h_0\right) + Z_{-N}^{(d)}\left(\frac{1}{2},\frac{1}{2};h_0-h_1\right)\end{aligned}$$

$$+ Z_N^{(c)}\left(\frac{1}{2}, \frac{1}{2}; h_0 - h_1\right) + Z_N^{(p)}\left(\frac{1}{2}, \frac{1}{2}; h^*\right) + G_N.$$

The Z_N's are explicit, but complicated expressions, involving the Hecke series and spectral values of the corresponding Laplacian. These functions will contribute to the error term (in the application to $I_2(T)$, $N \leq T^{1/3}$ with the choice $T_0 = T^{2/3}$). The main term will arise, after a long and complicated calculation, from

$$\begin{aligned} G_N = \lim_{\substack{s\to 1/2\\ \nu\to 1/2}} \Big\{&\zeta_N(s,\nu)V_N(1/2,\nu) + \zeta_N(s,1-\nu)V_N(1/2,1-\nu)\\ &+ \zeta_N(1-s,\nu)V_N(s,\nu) + \zeta_N(1-s,1-\nu)V_N(s,l-\nu)\Big\} \end{aligned}$$

where

$$\zeta_N(s,\nu) = \frac{\zeta(2s)\zeta(2\nu)}{\zeta(2s+2\nu)}\tau_{s+\nu}(N),$$

$$V_N(s,\nu) = \int\limits_0^\infty (1+x^2)^{1-2s} w_N(x) x^{2\nu} dx.$$

The complete details of the proof are given by N. Zavorotnyi [167], and will not be reproduced here. Another reason for not giving the details of proof is that in Chapter 5 spectral theory will be extensively used in presenting Motohashi's formula for the fourth moment (Theorem 5.1), which is more powerful than Heath-Brown's or Zavorotnyi's.

Also in Section 5.3 more details on spectral theory and hecke series **214**
may be found.

4.7 The Sixth Power Moment

We shall conclude this chapter by giving a discussion of $I_3(T)$, based on Theorem 4.3 and (4.61). This problem is considerably more difficult than the problem of evaluation of $I_1(T)$ or $I_2(T)$, and it does not seen possible at this moment to prove by any existing method even the weak upper bound $I_3(T) \ll T^{1+\epsilon}$, much less an asymptotic formula for $I_3(T)$.

As in the case of $I_2(T)$ we define similarly (see (4.59) with $k = 3$)

$$S(T) = \sum_{\substack{m \neq n; m,n=1 \\ 1-\delta \leq m/n \leq 1+\delta}} d_3(m)d_3(n)(mn)^{-1/2} \operatorname{Re}\left\{\int_0^\infty f(t)\nu\left(\left(\frac{t}{2\pi}\right)^{3/2} m^{-1}\right)\left(\frac{m}{n}\right)^{it} dt\right\}$$

$$= S_i(T) + S_{ii}(T) + S_{iii}(T),$$

say. We have $0 < \delta < 1/2$, in $S_i(T) m > n$, $m/n \leq 1 + \delta$, in $S_{ii}(T)$ we have $m < n$, $n/m \leq 1 + \delta$, and in $S_{iii}(T)$ $1 + \delta < m \leq 1/(1 - \delta)$. The property that $\nu(y) = 0$ for $y \leq 1/b$ gives the condition $m, n \leq bT^{3/2}$ in $S(T)$, since $1 - \delta \leq \frac{m}{n} \leq 1 + \delta$. We integrate $S_{iii}(T)$ by parts and use $f'(t) \ll T_0^{-1}$ to obtain

$$S_{iii}(T) \ll \sum_{m,n \leq bT^{3/2}} d_3(m)d_3(n)(mn)^{-1/2} T_0^{-1} \ll T^{3/2} T_0^{-1} \log^4 T.$$

In $S_i(T)$ we set $m = n + r$, and in $S_{ii}(T) n = m + r$. Changing m into n in $S_{ii}(T)$ we obtain

$$S(T) = O(T^{3/2} T_0^{-1} \log^4 T) + \sum_{n=1}^{\infty} \sum_{r \leq \delta n} d_3(n) d_3(n+r) n^{-1/2} (n+r)^{-\frac{1}{2}}$$

$$\int_0^\infty f(t)\left\{\nu\left(\frac{(t/2\pi)^{3/2}}{n+r}\right) + \nu\left(\frac{(t/2\pi)^{3/2}}{n}\right)\right\} \cos(t \log(1 + \frac{r}{n}))dt.$$

215 As in the case of $I_2(T)$, the terms for which $n \leq T_0$ by Lemma 2.1 make a contribution which is

$$\ll \sum_{n \leq T_0} \sum_{r \leq \delta n} \frac{d_3(n) d_3(n+r)}{n} \frac{n}{r} \ll T^\epsilon T_0.$$

Now choose a fixed δ such that $0 < \delta < \frac{1}{2}, \epsilon > 0$ arbitrarily small but fixed, and $T_1 = T_1(\delta, \epsilon)$ so large that $T^\epsilon T_0^{-1} \leq \delta$ for $T \geq T_1$. For $r > T^\epsilon T_0^{-1} n$ and $n > T_0$ we have $\log(1 + r/n) \gg T^\epsilon T_0^{-1}$. Thus integrating R times by parts we have

$$\sum_{T_0 < n \leq bT^{3/2}} \sum_{T^\epsilon T_0^{-1} n < r \leq \delta n} d_3(n) d_3(n+r) n^{-1/2} (n+r)^{-1/2}$$

$$\begin{aligned}
&\int_0^\infty f(t)\left\{\nu\left(\frac{(d/2\pi)^{3/2}}{n+r}\right)+\nu\left(\frac{(t/2\pi)^{3/2}}{n}\right)\right\}\cos\left(t\log\left(1+\frac{r}{n}\right)\right)dt\\
&\ll T^\epsilon\sum_{T_0\le n\le bT^{3/2}}\frac{1}{n}\sum_{T^\epsilon T_0^{-1}n<r\le\delta n}TT_0^{-R}\left(\log\left(1+\frac{r}{n}\right)\right)^{-R}\\
&\ll T^{1+\epsilon}\sum_{T_0\le n\le bT^{3/2}}\frac{1}{n}T_0^{-R}\sum_{T^\epsilon T_0^{-1}n<r\le\delta n}n^R r^{-R}\\
&\ll T^{1+\epsilon}T_0^{-R}\sum_{n\le bT^{3/2}}n^{R-1}(T^\epsilon T_0^{-1})^{1-R}\ll T^{1+\epsilon+\epsilon(1-R)}T_0^{-1}T^{3/2}\\
&\ll 1
\end{aligned}$$

if $R=[5/(2\epsilon)+1]$, say. We thus have

$$\begin{aligned}
S(T) &= O(T^\epsilon T_0)+O(T^{3/2+\epsilon}T_0^{-1})\\
&\quad+\sum_{T_0\le n\le bT^{3/2}}\sum_{r\le T^\epsilon T_0^{-1}n}d_3(n)d_3(n+r)n^{-1/2}(n+r)^{-1/2}\\
&\quad\int_0^\infty f(t)\left\{\nu\left(\frac{(t/2\pi)^{3/2}}{n+r}\right)+\nu\left(\frac{(t/2\pi)^{3/2}}{n}\right)\right\}\cos\left(t\log\left(1+\frac{r}{n}\right)\right)dt.
\end{aligned}\tag{4.96}$$

Now we use

$$\nu\left(\frac{(t/2\pi)^{3/2}}{n+r}\right)=\nu\left(\frac{(t/2\pi)^{3/2}}{n}\right)-\left(\frac{t}{2\pi}\right)^{3/2}\frac{r}{n^2}\nu'\left(\frac{(t/2\pi)^{3/2}}{n}\right)+O\left(\frac{t^3r^2}{n^4}\right)$$

and Lemma 2.1 to find that the total error terms coming from the O-term and ν' are $\ll T^{4+4\epsilon}T_0^{-4}$ and $T^{3/2+2\epsilon}T_0^{-1}$, respectively. Hence for $T^{1/2}\le T_0\ll T^{1-\epsilon}$ (4.96) becomes

$$\begin{aligned}
S(T) &= O(T^\epsilon T_0)+O\left(T^{3/2+2\epsilon}T_0^{-1}\right)+O\left(T^{4+4\epsilon}T_0^{-4}\right)\\
&\quad+2\sum_{T_0\le n\le bT^{3/2}}\sum_{r\le T^\epsilon T_0^{-1}\pi}d_3(n)d_3(n+r)n^{-1/2}(n+r)^{-1/2}\\
&\quad\int_0^\infty f(t)\nu\left(\frac{(t/2\pi)^{3/2}}{n}\right)\cos\left(t\log\left(1+\frac{r}{n}\right)\right)dt.
\end{aligned}\tag{4.97}$$

The last sum can be further simplified if we use 216

$$(n+r)^{-1/2} = n^{-1/2} - \frac{1}{2} rn^{-3/2} + O\left(r^2 n^{-5/2}\right).$$

The contribution of the terms $-\frac{1}{2} rn^{3/2}$ to $S(T)$ will be, by using Lemma 2.1,

$$\ll T^{\epsilon} \sum_{T_0 \le n \le bT^{3/2}} n^{-2} \sum_{r \le T^{\epsilon} T_0^{-1} n} rnr^{-1} \ll T^{3/2+2\epsilon} T_0^{-1}.$$

The O-term will contribute

$$\ll T^{\epsilon} \sum_{T_0 \le n \le bT^{3/2}} n^{-1} \sum_{r \le T^{\epsilon} T_0^{-1} n} T r^2 n^{-5/2}$$

$$\ll T^{1+\epsilon} \sum_{T_0 \le n \le bT^{3/2}} n^{-1/2} T^{3\epsilon} T_0^{-3} \ll T^{7/4+4\epsilon} T_0^{-3} \ll T^{3/2+4\epsilon} T_0^{-1}$$

if $T_0 \ge T^{1/2}$. We further simplify (4.97) by using

$$\cos\left(t \log\left(1 + \frac{r}{n}\right)\right) = \cos\left(\frac{tr}{n}\right) + \frac{tr^2}{2n^2} \sin\left(\frac{tr}{n}\right) + O\left(\frac{t^2 r^4}{4}\right).$$

The contribution of the last error term is $\ll T^{9/2+6\epsilon} T_0^{-5}$, and the sine term contributes $\ll T^{5/2+3\epsilon} T_0^{-2}$ if we again use Lemma 2.1. For $T^{1/2} \le T_0 \ll T^{1-\epsilon}$ we have

$$T^{9/2} T_0^{-5} \le T^4 T_0^{-4},$$

and we obtain

$$S(T) = \sum(T) + O\left\{T^{6\epsilon}\left(T_0 + T^{3/2} T_0^{-1} + T^{5/2} T_0^{-2} + T^4 T_0^{-4}\right)\right\},$$

where we set

$$\sum(T) = 2 \sum_{T_0 \le n \le bT^{3/2}} \sum_{\gamma \le T^{\epsilon} T_0^{-1} n} d_3(n) d_3(n+r) n^{-1}$$

$$\int_0^{\infty} f(t) v\left(\frac{(t/2\pi)^{3/2}}{n}\right) \cos\left(t \log\left(1 + \frac{r}{n}\right)\right) dt.$$

217 Therefore if we collect all previous estimates we obtain

Theorem 4.5. *For $T^{1/2} \leq T_0 \ll T^{1-\epsilon}$ there exist polynomials $Q_9(y)$ and $H_{3,\nu}(y)$ of degree nine and seven, respectively, such that the coefficients of $H_{3,\nu}$ depend on the smoothing function $\nu(\cdot)$, while those of Q_9 do not, and*

$$\begin{aligned} I_{3,f}(T) &= (tQ_9(\log t))\Big|_T^{2T} + (tH_{3,\nu}(\log t))\Big|_T^{2T} \\ &+ O\left\{T^{6\epsilon}\left(T_0 + T^{3/2}T_0^{-1} + T^{5/2}T_0^{-2} + T^4T_0^{-4}\right)\right\} \\ &+ 4 \sum_{T_0 \leq n \leq bT^{3/2}} \sum_{1 \leq r \leq T^\epsilon T_0^{-1} n} d_3(n)d_3(n+r)n^{-1} \\ &\times \int_0^\infty f(t)\nu\left(\frac{(t/2\pi)^{3/2}}{n}\right)\cos\left(\frac{tr}{n}\right)dt. \end{aligned} \tag{4.98}$$

Obviously, the best error term one can get here is $O(T^{5/6+\epsilon})$ with the choice $T_0 = T^{5/6}$. But the main difficulty lies in the evaluation of the double sum in (4.98). I expect the double sum in question to equal

$$(tR_{3,\nu}(\log t) + tS_3(\log t))\Big|_T^{2T}$$

plus an error term, where $R_{3,\nu}(y)$ is a polynomial of degree seven in y whose coefficients depend on ν, and actually equals $-H_{3,\nu}(y)$. The coefficients of the polynomial $S_3(\log t)$, of degree ≤ 7 in $\log t$, should not depend on ν. It is hard to imagine that one could take advantage of the fact that the sum in (4.98) is a double sum, when the same situation was difficult to exploit in the simpler case of $I_2(T)$. One thing seems clear: there is hope of getting $I_3(T) \leq T^{1+\epsilon}$ (weak form of the sixth moment) by this method only if one can take advantage in (4.98) of special properties of the function $d_3(\cdot)$. Specifically, one should try to establish an **218**
asymptotic formula for the summatory function of $d_3(n)d_3(n+r)$, where $n \ll T^{3/2}$, $r \ll T^{1+\epsilon}$. Trivial estimation of $\int_0^\infty \ldots dt$ in (4.98) by Lemma 2.1 produces only the trivial bound

$$I_3(T) \ll T^{3/2+\epsilon}.$$

It should be of interest even to reprove

$$I_3(T) \ll T^{5/4+\epsilon},$$

which is (up to "ϵ") the best currently known upper bound for $I_3(T)$. Finally, it was pointed out by Y. Motohashi that the key to solving the sixth power moment problem lies probably in the use of spectral theory of $SL(3, \mathbb{Z})$. Motohashi was led to this assertion by analogy with the study of $I_2(T)$. His powerful method will be fully explained in Chapter 5. In any case, the formula (4.98) may serve as the basis for an attack on the sixth moment. The double sum appearing in it has the merit that the sum over r is "short", in the sense that the range is $\ll T^{3/2+\epsilon}T_0^{-1}$, while is the range for n is $\ll T^{3/2}$, and heuristically some saving should result from this situation.

Notes For Chapter 4

A proof of A.E. Ingham's classical result (4.2), given in his work [74], is given in Chapter 5 of Ivić [75]. The latter proof is based on the ideas of K. Ramachandra [136] and the use of the mean value theorem (1.15). 219

The asymptotic formula of H. Kober [101] for $\int\limits_0^\infty e^{-\delta t}\left|\zeta(\frac{1}{2}+it)\right|^2 dt$ is proved in Chapter 7 of Titchmarsh [155].

F.V. Atkinson proved, as $\delta \to 0+$,

$$\int\limits_0^\infty e^{-\delta t}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \frac{1}{\delta}\left(A\log^4\frac{1}{\delta}+B\log^3\frac{1}{\delta}+C\log^2\frac{1}{\delta}+D\log\log\frac{1}{\delta}+E\right) + O\left(\left(\frac{1}{\delta}\right)^{13/14+\epsilon}\right) \tag{4.99}$$

with suitable constants a, B, C, D, E. In particular, he obtained

$$A = \frac{1}{2\pi^2}, \quad B = -\frac{1}{\pi^2}\left(2\log(2\pi) - 6\gamma + \frac{24\zeta'(2)}{\pi^2}\right).$$

His proof uses T. Estermann's formula [34] for the sum

$$S(x, r) := \sum_{n\le x} d(n)d(n+r);$$

see Section 4.6 for Heath-Brown's results (4.87) and (4.88), proved in his paper [61]. Observe that $A = a_4$ in (4.4), but $B \neq a_3$. Atkinson remarked that improved estimates for $S(x, r)$, already available in his time, would improve the exponent in the error term in (4.99) to $8/9+\epsilon$, and further improvements would result from the best known estimates for $S(x, r)$ (see Th. 4.1 of N.V. Kuznetsov [106]). Thus, through the use of Estermann's formula for $S(x, r)$, Kloosterman sums appear in the asymptotic formula for the integral in (4.99). In [34] Estermann acknowledges the influence of E. Hecke in the approach to treat $S(x, r)$ by considering Dirichlet series of $d(n)e\left(\frac{h}{k}n\right)$. Clearly one can see the genesis of "Kloostermania" here.

The work of N.I. Zavorotnyi [167] in which he proves (4.6) exists in the form of a preprint, and to the best of my knowledge it has not been published in a periodical.

220 For N.V. Kuznetsov's convolution formula and related results involving the use of spectral theory of automorphic functions see his papers [103] - [107]. In [107] he claims to have proved

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^8 dt \ll T(\log T)^B, \tag{4.100}$$

but although his paper contains some nice and deep ideas, (4.100) is not proved. Not only is the proof in the text itself not complete, but Y. Motohashi kindly pointed out that e.g. the change of triple summation in (3.6) of [127] needs certain conditions under which $\Phi_0(x)$ does not seem to satisfy the conditions stated in Theorem 4.

The function $E_k(T)$, defined by (4.9), should not be confused with $E_\sigma(T)$ (defined by (2.2) for $\frac{1}{2} < \sigma < 1$), which was discussed in Chapter 2 and Chapter 3.

The definition of $c(k)$ in (4.12) is made by using the gamma - function, so that the expression for $c(k)$ makes sense even when k is not an integer, although in that case I have no conjecture about the correct value of $c(k)$.

For some of the approximate functional equations for $\zeta^k(s)$ see Chapter 4 of Ivić [75]. This contains also an account of the Riemann-Siegel formula, for which one also see C.L. Siegel [152] and W. Gabcke [38]. For Y. Motohashi's analogue of the Riemann-Siegel formula for $\zeta^2(s)$, see his workds [124] and [126]. M. Jutila [93], [94] also obtained interesting results concerning the approximate functional equation for $\zeta^2(s)$.

The smoothing function $\rho(x)$ of the type given by Lemma 4.3 is used also by Zavorotnyi [167], but his work does not give the construction of such a function, whereas Lemma 4.3 does. Other authors, such as A. Good [44], [45] or N.V. Kuznetsov [105], made use of similar smoothing functions.

The classical results of G.H. Hardy and J.E. Littlewood on the approximate functuional equations for $\zeta(s)$ and $\zeta^2(s)$ are to be found in their papers [56] and [57].

The rather awkward form of the approximate functional equation (4.26) is used in the proof of Theorem 4.3. This is the reason why (4.26) is derived. 221

For the Perron inversion formula for Dirichlet series, used in the proof of (4.54), see the Appendix of Ivić [75] or Lemma 3.12 of E.C. Titchmarsh [155].

Several interesting results on mean values of $|\zeta(\frac{1}{2}+it)|$ are obtained by J.B. Conrey and A. Ghosh [23] - [26] and Conrey et al. [23]. In particular, in [26] Conrey and Ghosh consider $c(k)$, as defined by (4.11), for integral and non-integral values of $k > 0$. Assuming that $c(k)$ exists they show that

$$c(k) \geq F_k \Gamma^{-1}(k^2+1) \prod_p \left\{ \left(1-\frac{1}{p}\right)^{k^2} \left(\sum_{j=0}^{\infty} \left(\frac{\Gamma(k+j)}{j!\Gamma(k)} \right)^2 p^{-j} \right) \right\}$$

with specific values $F_3 = 10.13$, $F_4 = 205$, $F_5 = 3242$, $F_6 = 28130$, and with even sharper conditional bounds. None of these lower bounds contradict my conjectural value (4.12) for $c(k)$.

The discussion on $E(T)$ in Section 4.5 complements Chapter 2. R. Balasubramanian's formula [6] is presented here to show how a different smoothing technique, namely one with the exponential function, can be also effectively used. The bound (4.79) seems to be particularly well-suited for the application of two-dimensional techniques for the estimation of exponential sums, but it seems unlikely that these techniques can improve on the result of Heath-Brown and Huxley (Theorem 2.7) that $E(T) \ll T^{7/22+\epsilon}$. This is in distinction with (4.86), where one does not see how to take advantage of the fact that the exponential sum in question is two dimensional.

If instead of Heath-Brown's results (4.87) and (4.88) one uses Th. 4.1 of N.V. Kuznetsov [106] for $E(x,r)$, one can get $E_2(T) \ll T^{5/7+\epsilon}$ directly by the method used in the text. N.I. Zavorotnyi's approach [167], which we also briefly discussed, is in the same vein, but it is more sophisticated and leads to (4.6).

From the discussion of $E(T)$ made in this chapter and previously in 222

Chapter 2, and from the estimation of $E_2(T)$ in Chapter 5, it transpires that

$$E_k(T) \ll_\epsilon T^{k/3+\epsilon} \tag{4.101}$$

can be proved for $k = 1, 2$ by trivial estimation. In fact, for $k = 2$ no non-trivial estimation of exponential sums with the quantities $\alpha_j H_j^3(\frac{1}{2})$ is known. The same situation is expected when $k \geq 3$, so at most that we can heuristically hope for is (4.101) for $k = 3$, which is of course the sixth moment. But if (4.101) holds for $k = 3$, then for $k > 3$ it trivially holds by using the bound for $k = 3$ and $\zeta(\frac{1}{2} + it) \ll t^{1/6}$. A strong conjecture of A.I. Vinogradov [159] states that

$$\sum_{n \leq x} d_k(n) d_k(n + r) = x Q_{2k-2}(\log x; r) + 0\left(x^{(k-1)/k}\right), \tag{4.102}$$

where $k \geq 2$ is fixed, Q_{2k-2} is a polynomial in $\log x$ of degree $2k - 2$ whose coefficients depend on r, and r lies in a certain range (although this is not discussed by Vinogradov). The asymptotic formula (4.102) suggests that perhaps on could have

$$E_k(T) \ll_\epsilon T^{(k-1)k+\epsilon} \quad (k \geq 2),$$

which is stronger than my conjecture (4.21) for $k \geq 3$. Vinogradov's paper [159] stresses the importance of spectral theory of $SL(k; \mathbb{Z})$ in the problem of the evaluation of $I_k(T)$.

Additive divisor problems, of which (4.102) is an example, are especially difficult when r is not fixed, but may be depending on x. When r is fixed, a large literature exists on the asymptotic formulas for sums of $d_k(n) d_m(n + r)$. For example, Y. Motohashi [123] shows that for any fixed $k \geq 2$

$$\sum_{n \leq x} d_k(n) d(n + 1) = x \sum_{j=0}^{k} \xi_k(j) \log^{k-j} x + O\left(x \frac{(\log\log x)^{c(k)}}{\log x}\right) \tag{4.103}$$

with $c(k) \geq 0$ and constants $\xi_k(j)$ which may be effectively calculated.
223 When $k = 2$ J.M. Deshouillers and H. Iwaniec [30] obtain the asymptotic formula with the error term $0(x^{2/3+\epsilon})$, and show explicitly

$$\xi_2(0) = 6\pi^{-2}, \xi_2(1) = \sum_{n=1}^{\infty} \mu(n) n^{-2} (4\gamma - 4\log n - 2),$$

$$\xi_2(2) = 4\sum_{n=1}^{\infty} \mu(n)n^{-2}\{(\gamma - \log n)(\gamma - \log n - 1) + 2\}.$$

Their technique, based on the use of Kuznetsov's trace formula, was used by N.V. Kuznetsov himself in [106] to yield that, for $k = 2$ the error term in (4.103) is $O((x\log x)^{2/3})$. For $k = 3$ D.R. Heath-Brown [63] obtained (4.103) with the error term $O(x^{1-1/102+\epsilon})$, while for $k \ge 4$ E. Fouvry and G. Tenenbaum [37] have shown that the error term is

$$O\left(x\exp\left(-c_1(k)\sqrt{\log x}\right)\right).$$

The bound (4.95) is proved by N.V. Kuznetsov [104], while the proof of this result (with $(\log T)^{1/3}$ replaced by T^{ϵ}) has been alternatively obtained by D. Goldfeld and P. Sarnak [43], whose method is simpler than Kuznetsov's. In fact, Y.V. Linnik [113] and A. Selberg [150] independently conjectured that, for any $\epsilon > 0$ and $T > (m,n)^{1/2+\epsilon}$, one has

$$\sum_{c\le T} c^{-1}S(m,n;c) \ll_{\epsilon} T^{\epsilon}.$$

If true, this conjecture is close to being best possible, since M. Ram Murthy [146] proved that, for some $C_1 > 0$,

$$\sum_{c\le T} c^{-1}S(m,n;c) = \Omega\left(\exp\left(\frac{C_1\log T}{\log\log T}\right)\right).$$

As shown by Theorem 4.5, sums of $d_3(n)d_3(n+r)$ play a fundamental rôle in the study of $I_3(T)$. They were studied, together with analytic properties of the associated zeta-function

$$Z_3(s) = \sum_{n=1}^{\infty} d_3(n)d_3(n+r)n^{-s} \quad (\mathrm{Re}\, s > 1),$$

by A.I. Vinogradov and L. Tahtadžjan [160]. This topic is also investigated by A.I. Vinogradov [157], [158].

Chapter 5
Motohashi's formula for the fourth moment

5.1 Introduction and Statement fo results

IN THIS CHAPTER we shall study in detail Y. Motohashi's recent important work on the critical line. Instead of working directly with **224**

$$I_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt$$

we shall deal with

$$\int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + iT + it\right)\right|^4 e^{-(t/\Delta)^2} dt \quad \left(0 < \Delta \leq \frac{T}{\log T}\right), \tag{5.1}$$

and then use later averaging techniques to obtain information about $I_2(T)$ itself, or about $E_2(T)$. The last function was defined in Chapter 4 as

$$E_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt - TP_4(\log T), \tag{5.2}$$

where $P_4(y)$ is a suitable polynomial in y of degree four with the leading coefficient equal to $1/(2\pi^2)$. The basic idea is to consider the following function of four complex variables u, v, w, z, namely

$$I(u, v, w, z; \Delta) := \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \zeta(u + it)\zeta(v - it)\zeta(w + it)\zeta(z - it)e^{-(t/\Delta)^2} dt. \quad (5.3)$$

For $\mathrm{Re}\, u$, $\mathrm{Re}\, v$, $\mathrm{Re}\, w$, $\mathrm{Re}\, z > 1$ we have by absolute convergence

$$\begin{aligned} &I(u, v, w, z; \Delta) \\ &= \frac{1}{\Delta\sqrt{\pi}} \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k^{-u}\ell^{-v}m^{-w}n^{-z} \int_{-\infty}^{\infty} \exp\left(it \ \log\frac{\ell n}{km} - t^2\Delta^{-2}\right) dt \\ &= \sum_{k,\ell,m,n=1}^{\infty} k^{-u}\ell^{-v}m^{-w}n^{-z} \exp\left(-\left(\frac{\Delta}{2}\log\frac{\ell n}{km}\right)^2\right), \end{aligned}$$

since

$$\int_{-\infty}^{\infty} e^{Ax-Bx^2} dx = \sqrt{\frac{\pi}{B}} e^{A^2/(4B)} \quad (\mathrm{Re}\, B > 0).$$

225 Following the initial step in Atkinson's treatment of $E(T)$ (see (2.7) and (2.8)) one writes then

$$\begin{aligned} I(u, v, w, z; \Delta) &= \left\{\sum_{km=\ell n} + \sum_{km<\ell n} + \sum_{km>\ell n}\right\} k^{-u}\ell^{-v}m^{-w}n^{-z} \times \exp\left(-\left(\frac{\Delta}{2}\log\frac{\ell n}{km}\right)^2\right) \\ &= I_1(u, v, w, z; \Delta) + I_2(u, v, w, z; \Delta) + I_3(u, v, w, z; \Delta), \end{aligned} \quad (5.4)$$

say. In I_1 we have $n = km/\ell$, hence

$$\begin{aligned} I_1(u, v, w, z; \Delta) &= \sum_{k,m,\ell,n\ge1; km=\ell n} \ell^{z-v} m^{u-w} (km)^{-u-z} \\ &= \sum_{r=1}^{\infty}\left(\sum_{\ell n=r} \ell^{z-v}\right)\left(\sum_{km=r} m^{u-w}\right) r^{-u-z} = \sum_{r=1}^{\infty} \sigma_{z-v}(r)\sigma_{u-w}(r)r^{-u-z}. \end{aligned}$$

To evaluate the last series we recall a classical identity of S. Ramanujan. This is

$$\sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)n^{-s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \quad (5.5)$$

It is valid for $\operatorname{Re} s > \max(1, \operatorname{Re} a+1, \operatorname{Re} b+1, \operatorname{Re} a+\operatorname{Re} b+1)$ and can be proved by expanding both sides into an Euler product, since $\sigma_z(n)$ is a multiplicative function of n for any z. Setting $s = u + z$, $a = z - v$, $b = u - w$ we obtain

$$I_1(u, v, w, z; \Delta) = \frac{\zeta(u + v)\zeta(u + z)\zeta(v + w)\zeta(w + z)}{\zeta(u + v + w + z)}, \tag{5.6}$$

thereby providing analytic continuation of I_1 as a meromorphic function of u, v, w, z over $\mathbb{C}^4$. By symmetry

$$I_3(u, v, w, z; \Delta) = I_2(v, u, z, w; \Delta), \tag{5.7}$$

and the main task is to prove that $I_2(u, v, w, z; \Delta)$ also exists as a meromorphic function of u, v, w, z on $\mathbb{C}^4$. We are interested in (5.3) for $u = \frac{1}{2} + iT$, $v = \frac{1}{2} - iT$, $w = \frac{1}{2} + iT$, $z = \frac{1}{2} - iT$, that is, in the an- **226**
alytic continuation of the function defined initially by (5.3). Henceforth we shall assume that $I(u, v, w, z; \Delta)$ stands for the meromorphic function of u, v, w, z on $\mathbb{C}^4$ defined initially by (5.3).

It is not difficult to show that

$$I\left(\frac{1}{2} + iT, \frac{1}{2} - iT, \frac{1}{2} + iT, \frac{1}{2} - iT; \Delta\right) = \tag{5.8}$$
$$\frac{1}{\Delta\sqrt{x}} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + iT + it\right)\right|^4 e^{-(t/\Delta)^2} dt + \frac{\sqrt{x}}{\Delta} \operatorname{Re}$$
$$\left\{\left(\gamma - \log 2\pi + \Delta^{-2}\left(\frac{1}{2} + iT\right)\right) \exp\left(\left(\frac{\frac{1}{2} + iT}{\Delta}\right)^2\right)\right\},$$

and the last expression is $O\left(\exp\left(-\frac{1}{2}\log^2 T\right)\right)$ for $0 < \Delta \leq \dfrac{T}{\log T}$. The analytic continuation of $I_2(u, v, w, z; \Delta)$ is carried out in several steps, to a region in which it admits an expression in terms of sums of Kloosterman sums

$$S(m, n; c) = \sum_{1 \leq d \leq c, (d, c) = 1, dd' \equiv 1 (\bmod c)} e\left(\frac{md + nd'}{c}\right), \tag{5.9}$$

where $e(x) = e^{2\pi ix}$. This analytic continuation will be given in detail in Section 5.2. It turns out that one can successfully apply the powerful machinery of Kuznetsov's trace formula and the spectral theory of automorphic forms to the resulting expression for analytic continuation. This will be discussed in Section 5.3, while in Section 5.4 we shall obtain an explicit formula for (5.1) which requires a considerable amount of technical work. This is Motohashi's main result, which we state here as

Theorem 5.1. *If* $0 < \Delta \leq I/\log T$, *then*

$$\frac{1}{\Delta\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iT+it\right)\right|^4 e^{-(t/\Delta)^2}dt \tag{5.10}$$

$$= F_0(T,\Delta)+\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{\left|\zeta\left(\frac{1}{2}+i\xi\right)\right|^6}{|\zeta(1+2i\xi)|^2}\theta(\xi;T,\Delta)d\xi$$

$$= \sum_{j=1}^{\infty}\alpha_j H_j^3\left(\frac{1}{2}\right)\theta(x_j;T,\Delta)$$

$$+\sum_{k=6}^{\infty}\sum_{j=1}^{d_{2k}}\alpha_{j,2k}H_{j,2k}^3\left(\frac{1}{2}\right)\Lambda(k;T,\Delta)+O\left(T^{-1}\log^2 T\right),$$

227 *where* $\alpha_j, a_{j,2k}$ *and the functions* H_j, $H_{j,2k}$ *are defined in Section 5.3;* F_0, θ, Λ *by (5.112) and (5.110). The* 0*-constant in (5.10) is absolute.*

One can evaluate explicitly $F_0(T,\Delta)$ as follows. From Stirling's formula one has

$$\Gamma'(s) = \Gamma(s)\left(\log s - \frac{1}{2s} + a_2 s^{-2} + \ldots + a_r s^{-r} + F_r(s)\right)$$

with $F_r(s) \ll_r |s|^{-r-1}$. Hence by successive differentiation

$$\frac{\Gamma^{(k)}(s)}{\Gamma(s)} = \sum_{j=0}^{k} b_{j,k}(s)\log^j s + c_{-1,k}s^{-1} + \ldots + c_{-r,k}s^{-r} + 0\left({}_r|s|^{-r-1}\right)$$

for any fixed integers $k \geq 1$, $r \geq 0$, where each $b_{j,k}(s)$ ($\sim b_{j,k}$ for a suitable constant $b_{j,k}$) has an expansion in nonpositive powers of s. From (5.112) it is seen that in evaluating $F_0(T,\Delta)$ we encounter sums which involve, for $0 \leq r \leq 4$,

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{1}{\Delta\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\log^r\left(\frac{1}{2}+iT+it\right)e^{-(t/\Delta)^2}dt\right\}\\
&= \pi^{-\frac{1}{2}}\operatorname{Re}\left\{\int\limits_{-\infty}^{\infty}\log^r\left(\frac{1}{2}+iT+iu\Delta\right)e^{-u^2}du\right\}\\
&= \pi^{-\frac{1}{2}}\operatorname{Re}\left\{\int\limits_{-\log T}^{\log T}\log^r\left(\frac{1}{2}+iT+iu\Delta\right)e^{-u^2}du\right\}+O_A(T^{-A})
\end{aligned}$$

for any fixed $A > 0$. For $|u| \leq \log T$ one has the power series expansion

$$\begin{aligned}
\log^r\left(\frac{1}{2}+iT+iu\Delta\right) = (\log iT)^r + \sum_{k=1}^{r}\binom{r}{k}(\log iT)^{r-k}\\
\left(\frac{u\Delta}{T}+\frac{1}{2iT}-\frac{1}{2}\left(\frac{u\Delta}{T}+\frac{1}{2iT}\right)^2+\ldots\right)^k.
\end{aligned}$$

Thus in the range $0 < \Delta \leq T\exp(-\sqrt{\log T})$ we have

$$F_0(T,\Delta) = Q_4(\log T) + O\left(\exp\left(-\frac{1}{2}\sqrt{\log T}\right)\right), \tag{5.11}$$

where $Q_4(y)$ is a polynomial of degree four whose coefficients may be explicitly evaluated. **228**

A remarkable feature of Theorem 5.1 is that the error term appearing in it is quite small. This makes it a powerful instrument in the study of $\zeta(s)$ in general, with greater, with greater potential and applicability than the formulas for the fourth moment of Heath-Brown and Zavorotnyi that were discussed in Chapter 4. By more careful analysis, involving the evaluation of certain exponential integrals by the saddle-point method, it is possible to transform the right-hand side of (5.10) into a more explicit form, at the cost of a poorer error term and a shorter range for Δ. This will be effected in Section 5.5, and the result is the following theorem (all the relevant notation is defined in Section 5.3):

Theorem 5.2. *If $T^{\frac{1}{2}}\log^{-A} T \le \Delta \le T\exp(-\sqrt{\log T})$ where $A > 0$ is arbitrary but fixed, then there exists $C = C(A) > 0$ such that uniformly in Δ*

$$\frac{1}{\Delta\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iT+it\right)\right|^4 e^{-(t/\Delta)^2}dt \tag{5.12}$$

$$= \pi 2^{-\frac{1}{2}}T^{-\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_j x_j^{-\frac{1}{2}}H_j^3\left(\frac{1}{2}\right)\sin\left(x_j\log\frac{x_j}{4eT}\right)e^{-(\Delta x_j/2T)^2} + O(\log^C T).$$

By (5.11) we have $F_0(T,\Delta) \gg \log^4 T$, so that $C \ge 4$ in (5.12). It is also useful to have an integrated version of (5.12).

Theorem 5.3. *If $V^{\frac{1}{2}}\log^{-A} V \le \Delta \le V\exp(-\sqrt{\log V})$, where $A > 0$ is arbitrary but fixed, then there exists $C = C(A) > 0$ such that uniformly in Δ*

$$\frac{1}{\Delta\sqrt{\pi}}\int\limits_{0}^{V}\int\limits_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iT+it\right)\right|^4 e^{-(t/\Delta)^2}dt\,dT \tag{5.13}$$

$$= VP_4(\log V) + \pi\left(\frac{V}{2}\right)^{\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_j H_j^3\left(\frac{1}{2}\right)x_j^{-3/2}\cos\left(x_j\log\frac{x_j}{4eV}\right)e^{-(\Delta x_j/2V)^2}$$

$$+ O\left(V^{\frac{1}{2}}\log^C V\right) + O(\Delta\log^5 V),$$

229 *where $P_4(y)$ is a suitable polynomial of degree four in y. Moreover, if $V^{\frac{1}{2}}\log^{-A} V \le \Delta \le V^{\frac{3}{4}}$, then*

$$\frac{1}{\Delta\sqrt{\pi}}\int\limits_{0}^{V}\int\limits_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iT+it\right)\right|^4 e^{-(t/\Delta)^2}dt\,dT = VP_4(\log V)$$

$$+ \pi\left(\frac{V}{2}\right)^{\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_j H_j^3\left(\frac{1}{2}\right)c_j\cos\left(x_j\log\frac{x_j}{4eV}\right)$$

$$e^{-(\Delta x_j/2V)^2} + O(V^{\frac{1}{2}}) + O(\Delta \log^5 V), \tag{5.14}$$

where, as $j \to \infty$,

$$c_j = (1 + o(1))x_j^{-3/2}.$$

In fact, it is not difficult to see that P_4 is the same polynomial which appears in the definition (5.2) of $E_2(T)$ (see also (4.3) and (4.4)), and it is possible to deduce an upper bound for $E_2(T)$ from Theorem 5.3. This is

Theorem 5.4. *There is an absolute constant $C > 0$ such that*

$$E_2(T) \ll T^{2/3} \log^C T. \tag{5.15}$$

This upper bound is a slightly sharper form of the result (4.6) of N. Zavorotnyi. We can also obtain integral averages of $E_2(T)$ which correspond to Theorem 3.1 and Theorem 3.7 for $E_1(T) = E(T)$. This is

Theorem 5.5. *We have*

$$\int_0^T E_2(t)dt \ll T^{3/2} \tag{5.16}$$

and

$$\int_0^T E_2(t)\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll T^{3/2}\log^4 T. \tag{5.17}$$

We also have a mean square result on $E_2(T)$, contained in

Theorem 5.6. *There is a suitable constant $C > 0$ such that uniformly for $T^{2/3} \ll H \leq T$*

$$\int_T^{T+H} E_2^2(t)dt \ll T^{3/2}H^{3/4}\log^C T. \tag{5.18}$$

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The previous theorems dealt with upper bound results. However, we can also obtain an omega-result for $E_2(T)$. The result, which is one of the most important applications of the explicit formula for the fourth moment, is

Theorem 5.7.

$$E_2(T) = \Omega(T^{\frac{1}{2}}). \tag{5.19}$$

In the proof of Theorem 5.7, given in Section 5.7, we shall make use of a non-vanishing type of result for automorphic L-functions, proved recently by Y. Motohashi. However, if one could prove a certain linear independence over the integers of spectral values, then we shall that (5.19) may be replaced by the stronger

$$\limsup_{T\to\infty} |E_2(T)|T^{-\frac{1}{2}} = +\infty.$$

Finally, we shall reprove an important result of H. Iwaniec on sums of integrals of $|\zeta\left(\frac{1}{2}+it\right)|^4$ in short intervals. In fact we improve on Iwaniec's result by obtaining log-powers where we had T^{ϵ}. This is

Theorem 5.8. *Suppose $R < t_1 < t_2 < \ldots < t_R \le 2T$ with $t_{r+1} - t_r > \Delta$ $(r = 1, 2, \ldots, R-1)$ and $T^{\frac{1}{2}} \le \Delta \le T$. Then there exist constants $C_1, C_2 > 0$ such that*

$$\sum_{r\le R} \int\limits_{t_r}^{t_r+\Delta} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll R\Delta \log^{C_1} T + R^{\frac{1}{2}} T \Delta^{-\frac{1}{2}} \log^{C_2} T.$$

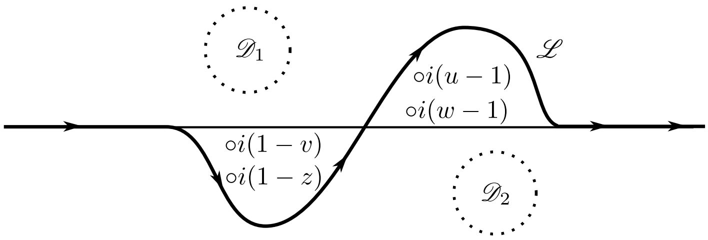
We shall prove Theorems 5.4 - 5.6 and Theorem 5.8 in Section 5.1. We remark that Theorem 5.8 yields

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll T^2 \log^D T$$

231 as a corollary with an effectively computable $D > 0$. The above estimate, with $D = 17$, was proved first by D.R. Heath-Brown.

5.2 Analytic Continuation

As the first step we shall establish (5.8), which is important because the integral (5.1) appears on the right-hand side. In (5.3) we move the line of integration to $\mathscr{L}$, as follows:



Then we have, for $\operatorname{Re} u$, $\operatorname{Re} v$, $\operatorname{Re} w$, $\operatorname{Re} z > 1$, by the residue theorem

$$I(u, v, w, z; \Delta) = \frac{1}{\Delta\sqrt{\pi}} \int_{\mathscr{L}} \zeta(u+it)\zeta(v-it)\zeta(w+it)\zeta(z-it)e^{-(t/\Delta)^2}\,dt$$
$$+ \frac{2}{\Delta}\pi^{\frac{1}{2}} \left\{ \zeta(u+v-1)\zeta(w-u+1)\zeta(u+z-1)\exp\left(\left(\frac{u-1}{\Delta}\right)^2\right) \right.$$
$$+ \zeta(u+v-1)\zeta(v+w-1)\zeta(z-v+1)\exp\left(\left(\frac{v-1}{\Delta}\right)^2\right)$$
$$+ \zeta(u-w+1)\zeta(v+w-1)\zeta(w+z-1)\exp\left(\left(\frac{w-1}{\Delta}\right)^2\right)$$
$$\left. +\zeta(u+z-1)\zeta(v-z+1)\zeta(w+z-1)\exp\left(\left(\frac{z-1}{\Delta}\right)^2\right) \right\}.$$

Namely, there are simple poles of the integrand at the points $t = (1-u)/i$, $(1-w)/i$, $(v-1)/i$, $(z-1)/i$ and e.g.

$$\lim_{t\to(1-u)/i} \zeta(u+it)\left(t - \frac{1-u}{i}\right) = \frac{1}{i},$$

hence the above relation follows. It obviously holds for those u, v, w, z **232** such that $i(1-v)$, $i(1-z) \in \mathscr{D}_1$, and $i(u-1)$, $i(w-1) \in \mathscr{D}_2$. Then $\mathscr{L}$ can be replaced by the real axis, and we obtain for any $T > 0$ and $\operatorname{Re} u$, $\operatorname{Re} v$, $\operatorname{Re} w$, $\operatorname{Re} z < 1$,

$$
\begin{aligned}
&I(u+iT, v-iT, w+iT, z-iT; \Delta) \qquad (5.20)\\
&= \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \zeta(u+iT+it)\zeta(v-iT-it)\zeta(w+iT+it)\\
&\qquad\qquad \zeta(z-iT-it)e^{-(t/\Delta)^2}dt\\
&+ \frac{2\pi^{\frac{1}{2}}}{\Delta}\Bigg\{\zeta(u+v-1)\zeta(w-u+1)\zeta(u+z-1)\exp\left(\left(\frac{u+iT-1}{\Delta}\right)^2\right)\\
&+ \zeta(u+v-1)\zeta(v+w-1)\zeta(z-v+1)\exp\left(\left(\frac{v-iT-1}{\Delta}\right)^2\right)\\
&+ \zeta(u-w+1)\zeta(v+w-1)\zeta(w+z-1)\exp\left(\left(\frac{w+iT-1}{\Delta}\right)^2\right)\\
&+ \zeta(u+z-1)\zeta(v-z+1)\zeta(w+z-1)\exp\left(\left(\frac{z-iT-1}{\Delta}\right)^2\right)\Bigg\}.
\end{aligned}
$$

When (u, v, w, z) is in the neighbourhood of the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, the points $u+v-1$, $w-u+1$ etc. appearing in curly braces in (5.20) are close to 0 and 1. Using the power series expansion of $\zeta(s)$ near these points and simplifying we obtain (5.8) from (5.20).

The next step is to seek analytic continuation of the functions I_2 and I_3, defined by (5.4), in the case $u = \frac{1}{2} + iT$, $v = \frac{1}{2} - iT$, $w = \frac{1}{2} + iT$, $z = \frac{1}{2} - iT$. We can write, for $\operatorname{Re} u$, $\operatorname{Re} v$, $\operatorname{Re} w$, $\operatorname{Re} z > 1$,

$$
\begin{aligned}
&I_2(u+iT, v-iT, w+iT, z-iT; \Delta)\\
&= \sum_{m_1,m_2,n_1,n_2=1; m_1m_2<n_1n_2}^{\infty} m_1^{-u-iT} m_2^{-w-iT} n_1^{-v+iT} n_2^{-z+iT} \exp\left(-\left(\frac{\Delta}{2}\log\frac{n_1n_2}{m_1m_2}\right)^2\right).
\end{aligned}
$$

We group together terms with $m_1m_2 = m$, set $n_1n_2 = m+n$, so that $m, n \geq 1$. Thus we obtain

$$I_2(u+iT, v-iT, w+iT, z-iT; \Delta)$$

$$= \sum_{m,n=1}^{\infty} m^{-u}(m+n)^{-v} \left(\sum_{m_2 | m} m_2^{u-w}\right)\left(\sum_{n_2|(m+n)} n_2^{v-z}\right)$$
$$\left(1+\frac{n}{m}\right)^{iT} \exp\left(-\left(\frac{\Delta}{2}\log\left(1+\frac{n}{m}\right)\right)^2\right)$$
$$= \sum_{m,n=1}^{\infty} \sigma_{u-w}(m)\sigma_{v-z}(m+n)\left(1+\frac{n}{m}\right)^{-v+iT}$$
$$m^{-u-v} \exp\left(-\left(\frac{\Delta}{2}\log\left(1+\frac{n}{m}\right)\right)^2\right).$$

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To transform further the last expression we set $p = s$, $q = w - s$ in the beta-integral formula

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}} dx \quad (\operatorname{Re} p, \operatorname{Re} q > 0)$$

to obtain

$$\frac{\Gamma(s)\Gamma(w-s)}{\Gamma(w)} = \int_0^{\infty} (1+x)^{-w} x^{s-1} dx.$$

This means that $F(s) = \Gamma(s)\Gamma(w-s)/\Gamma(w)$ (for w fixed) is the Mellin transform of $f(x) = (1+x)^{-w}$. Hence by the Mellin inversion formula

$$\Gamma(w)(1+x)^{-w} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s)\Gamma(w-s)x^{-s} ds, \tag{5.21}$$

where $\operatorname{Re} w > a > 0$, $x > 0$. We divide (5.21) by $\Gamma(w)$, replace w by $w + it$, multiply by $\exp(-(t/\Delta)^2)$ and integrate over t. Using the exponential integral as in the transformation of (5.3) we have

$$(1+x)^{-w} \exp\left(-\left(\frac{\Delta}{2}\log(1+x)\right)^2\right) \tag{5.22}$$
$$= \frac{1}{2\pi i} \frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-(t/\Delta)^2}}{\Gamma(w+it)} \int_{a-i\infty}^{a+i\infty} \Gamma(s)\Gamma(w+it-s)x^{-s} ds\, dt$$

$$= \frac{1}{2\pi i} \int\limits_{a-i\infty}^{a+i\infty} M(s, w; \Delta) x^{-s} ds,$$

where the interchange of integration is justified by absolute convergence, and where we set, for $\mathrm{Re}\, w > \mathrm{Re}\, s > 0$,

$$M(s, w; \Delta) := \frac{1}{\Delta\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\Gamma(s)\Gamma(w + it - s)}{\Gamma(w + it)} e^{-(t/\Delta)^2} dt. \tag{5.23}$$

By the Mellin inversion formula one has from (5.22), for $\mathrm{Re}\, s > 0$ and any w,

$$M(s, w; \Delta) = \int\limits_0^{\infty} y^{s-1}(1 + y)^{-w} \exp\left(-\left(\frac{\Delta}{2}\log(1 + y)\right)^2\right) dy. \tag{5.24}$$

234 This implies that $M(s, w; \Delta)$ is an entire function of w and a meromorphic function of s which decays rapidly with respect to s. Namely, we have for all s and w

$$M(s, w; \Delta) = (e^{2\pi is} - 1)^{-1} \int\limits_{\mathscr{L}} z^{s-1}(1 - z)^{-w} \exp\left(-\frac{\Delta^2}{4}\log^2(1 + z)\right) dz,$$

where $\mathscr{L}$ is the loop contour consisting of the real axis from $+\infty$ to ϵ, the circular arc of radius ϵ with center at the origin ($0 < \epsilon < 1$), and again the real axis from ϵ to $+\infty$. By performing $[C] + 1$ integrations by parts in the above integral it is seen that

$$M(s, w; \Delta) = O(1 + |s|)^{-C}$$

uniformly for any fixed $C > 0$, bounded w and $\mathrm{Re}\, s$ bounded, as long as s stays away from nonpositive integers.

Now we recall that Ramanujan's sum $c_r(n)$, defined by

$$c_r(n) = \sum_{h=1,(h,r)=1}^{r} e\left(\frac{h}{r}n\right), \tag{5.25}$$

may be written alternatively as

$$c_r(n) = \sum_{d|r, d|n} \mu\left(\frac{r}{d}\right) d = \sum_{\ell m = r, m|n} \mu(\ell) m.$$

Thus

$$c_r(n) r^{-s} = \sum_{\ell m = r, m|n} \mu(\ell) \ell^{-s} m^{1-s},$$

and consequently for $\operatorname{Re} s > 1$

$$\sum_{r=1}^{\infty} c_r(n) r^{-s} = \sum_{\ell=1}^{\infty} \left(\sum_{m|n} m^{1-s} \right) \mu(\ell) \ell^{-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)}.$$

In this formula set $s = 1 - \alpha$ with $\operatorname{Re} \alpha < 0$. Then we obtain

$$\sigma_\alpha(n) = \zeta(1-\alpha) \sum_{r=1}^{\infty} c_r(n) r^{\alpha - 1} \quad (\operatorname{Re} \alpha < 0), \tag{5.26}$$

which is a variant of a classical formula of S. Ramanujan. With (5.22) and (5.26) we have

$$I_2 = \sum_{m,n=1}^{\infty} \sigma_{u-w}(m) \sigma_{v-z}(m+n) m^{-u-v} \cdot \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} M(s, v-iT; \Delta)$$

$$\left(\frac{n}{m}\right)^{-s} ds = \zeta(1+z-v) \sum_{r=1}^{\infty} \sum_{m,n=1}^{\infty} \sigma_{u-w}(m) c_r(m+n) r^{v-z-1} \cdot \frac{1}{2\pi i}$$

$$\int_{a-i\infty}^{a+i\infty} M(s, v-iT; \Delta) m^s n^{-s} ds$$

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$$= \zeta(1+z-v) \sum_{r=1}^{\infty} r^{v-z-1} \sum_{h=1, (h,r)=1}^{r} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e\left((m+n)\frac{h}{r}\right)$$

$$\sigma_{u-w}(m) m^{-u-v} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} m^s n^{-s} M(s, v-iT; \Delta) ds$$

$$= \zeta(1+z-v)\sum_{r=1}^{\infty} r^{v-z-1} \sum_{h=1,(h,r)=1}^{r} \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} M(s, v-iT; \Delta)$$
$$\left(\sum_{n=1}^{\infty} e(\frac{h}{r}n)n^{-s}\right)\left(\sum_{m=1}^{\infty} e(\frac{h}{r}m)\sigma_{u-w}(m)m^{-u-v+s}\right)ds.$$

The introduction of Ramanujan's sums via the identity (5.26) played a crucial rôle, because it enabled us to separate the variables m and n, since $e\left((m+n)\frac{h}{r}\right) = e\left(m\frac{h}{r}\right)e\left(n\frac{h}{r}\right)$. To write the last two sums above in brackets in closed form we introduce the zeta-functions

$$D\left(s;\alpha, e\left(\frac{h}{r}\right)\right) := \sum_{m=1}^{\infty} \sigma_{\alpha}(m) e\left(\frac{h}{r}m\right) m^{-s} \quad (\sigma > \max(1, \operatorname{Re}\alpha + 1)) \tag{5.27}$$

and

$$\zeta\left(s; e\left(\frac{h}{r}\right)\right) := \sum_{n=1}^{\infty} e\left(\frac{h}{r}n\right) n^{-s} \quad (\operatorname{Re} s > 1), \tag{5.28}$$

the latter being a special case (with $x = h/r$, $a = 0$) of the so-called Lerch zeta-function

$$\varphi(x, a, s) = \sum_{n=0}^{\infty} e(nx)(n+a)^{-s}. \tag{5.29}$$

With this notation we obtain, for $\operatorname{Re} u$, $\operatorname{Re} w > 1$ and $\operatorname{Re} v > \operatorname{Re} z > a > 1$,

$$I_2(u+iT, v-iT, w+iT, z-iT; \Delta) \tag{5.30}$$
$$= \zeta(1+v-z)\sum_{r=1}^{\infty} r^{z-v-1} \sum_{\substack{h=1\\(h,r)=1}}^{r} \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}$$
$$D\left(u+z-s; u-w, e\left(\frac{h}{r}\right)\right)\zeta\left(s; e\left(\frac{h}{r}\right)\right) M(s, z-iT; \Delta) ds.$$

Actually here we changed places of v and z, for technical reasons,
236 which is permissible because from the definition of I_2 it follows that

$$I_2(u+iT, v-iT, w+iT, z-iT; \Delta) = I_2(u+iT, z-iT, w+iT, v-iT; \Delta).$$

A common property of zeta-functions is that they possess in many cases functional equations resembling (and often generalizing) the classical functional equation (1.8) for $\zeta(s)$. This is the case with the zeta-function $D(\cdot)$, defined by (5.27). If $(h,r) = 1$ and $h\bar{h} \equiv 1 (\operatorname{mod} r)$, then we have the functional equation

$$\begin{aligned} D\left(s; \alpha, e\left(\frac{h}{r}\right)\right) &= 2(2\pi)^{2s-2-\alpha} r^{\alpha-2s+1} \Gamma(1-s)\Gamma(1+\alpha-s) \qquad (5.31) \\ &\left\{-\cos\left(\pi s - \frac{\pi\alpha}{2}\right) D\left(1-s; -\alpha, e\left(-\frac{\bar{h}}{r}\right)\right)\right. \\ &\left. + D\left(1-s; -\alpha, e\left(\frac{\bar{h}}{r}\right)\right)\cos\left(\frac{\pi\alpha}{2}\right)\right\}. \end{aligned}$$

The functional equation (5.31) follows similarly as does the functional equation ($h\bar{h} \equiv 1 (\mod k)$ here)

$$\begin{aligned} E\left(s, \frac{h}{k}\right) &= 2(2\pi)^{2s-2}\Gamma^2(1-s)k^{1-2s} \\ &\times \left\{E\left(1-s, \frac{\bar{h}}{k}\right) - \cos(\pi s) E\left(1-s, -\frac{\bar{h}}{k}\right)\right\}, \qquad (5.32) \end{aligned}$$

where

$$E\left(s, \frac{h}{k}\right) := \sum_{m=1}^{\infty} d(m) e\left(m\frac{h}{k}\right) m^{-s} \quad (\operatorname{Re} s > 1),$$

so that

$$E\left(s, \frac{h}{r}\right) = D\left(s; 0, e\left(\frac{h}{r}\right)\right)$$

and (5.32) is a special case of (5.31). To obtain (5.31) one writes

$$\begin{aligned} D\left(s; \alpha, e\left(\frac{h}{r}\right)\right) &= \sum_{m,n=1}^{\infty} e\left(\frac{h}{r} mn\right) m^{\alpha} (mn)^{-s} \qquad (5.33) \\ &= \sum_{a,b=1}^{r} e\left(\frac{h}{r} ab\right) \sum_{m \equiv a (\operatorname{mod} r),\ n \equiv b (\operatorname{mod} r)} m^{\alpha-s} n^{-s} \end{aligned}$$

$$= \sum_{a,b=1}^{r} e\left(\frac{h}{r}ab\right) \sum_{\mu,\nu=0}^{\infty} (a+\mu r)^{\alpha-s}(b+\nu r)^{-s}$$

$$= r^{\alpha-2s} \sum_{a,b=1}^{r} e\left(\frac{h}{r}ab\right) \zeta\left(s-\alpha, \frac{a}{r}\right) \zeta\left(s, \frac{b}{r}\right),$$

237 where

$$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s} \quad (0 \le a \le 1,\ \mathrm{Re}\, s > 1)$$

is the Hurwitz zeta-function (it should not be confused with $\zeta\left(s; e\left(\frac{h}{r}\right)\right)$ of (5.28)). The representation (5.33) provides analytic continuation of $D(\cdot)$, showing that it has simple poles at $s = 1$ and $s = \alpha + 1$ ($\alpha \neq 0$) with residues $r^{\alpha-1}\zeta(1-\alpha)$ and $r^{-\alpha-1}\zeta(1+\alpha)$, respectively.

We can write $\zeta(s,a)$ as

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-(n+a)x} dx = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}} dx \tag{5.34}$$

$$= \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}e^{-az}}{1-e^{-z}} dz,$$

where C is the loop contour as in the proof of Theorem 2.10, and where we have used $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$. The last integral representation provides the analytic continuation of $\zeta(s,a)$ over the whole complex plane. Expanding the loop to infinity it is seen that the residues at $2m\pi i$ and $-2m\pi i$ contribute together

$$-2(2m\pi)^{s-1} e^{+i\pi s} \sin\left(\frac{1}{2}\pi s + 2m\pi a\right),$$

hence it follows that

$$\zeta(s,a) = 2(2\pi)^{s-1}\Gamma(1-s) \sum_{m=1}^{\infty} \sin\left(\frac{1}{2}\pi s + 2m\pi a\right) m^{s-1} \quad (\mathrm{Re}\, s < 0). \tag{5.35}$$

Inserting (5.35) in (5.33) and simplifying one obtains (5.31) for $\operatorname{Re} s < 0$, and by analytic continuation (5.31) holds also for other values of s.

Now we return to (5.30) and shift the line of integration to $\operatorname{Re} s = b$, where $b > a + 1$ and the conditions

$$\operatorname{Re} u > 1, \quad \operatorname{Re} w > 1, \quad \operatorname{Re} v > \operatorname{Re} z > a > 1 \tag{5.36}$$

and

$$\operatorname{Re}(u + z) < b, \quad \operatorname{Re}(z + w) < b, \quad 2(b + 1) < \operatorname{Re}(u + v + w + z) \tag{5.37}$$

are satisfied. There is a domain in $\mathbb{C}^4$ where both (5.36) and (5.37) are **238**
satisfied, and assuming $u \neq w$ there are simple poles at $s = -1 + u + z$ and $s = -1 + w + z$. By using (5.26) and

$$\sum_{n=1}^{\infty} \sigma_a(n) n^{-s} = \zeta(s)\zeta(s - a) \quad (\sigma > \max(1, \operatorname{Re} a + 1)),$$

it is seen that the contribution of the residues from these poles to (5.30) will be

$$\begin{aligned} &\frac{\zeta(u+v)\zeta(z+w-1)\zeta(1+v-z)\zeta(1+u-w)}{\zeta(u+v-z-w+2)} M(z+w-1, z-iT;\Delta) \quad (5.38)\\ &+ \frac{\zeta(w+v)\zeta(u+z-1)\zeta(1+v-z)\zeta(1+w-u)}{\zeta(w+v-u-z+2)} M(u+z-1, z-iT;\Delta). \end{aligned}$$

There remains

$$\begin{aligned} &\zeta(1+v-z) \sum_{r=1}^{\infty} r^{z-v-1} \sum_{\substack{h=1\\(h,r)=1}}^{r} \int\limits_{b-i\infty}^{b+i\infty} D\left(u+z-s; u-w, e\left(\frac{h}{r}\right)\right) \quad (5.39)\\ &\qquad \zeta\left(s; e\left(\frac{h}{r}\right)\right) M(s, z-iT;\Delta)\, ds. \end{aligned}$$

By (5.31) it is seen that the double sum in (5.39) converges absolutely in the region defined by (5.37), hence there it is regular. Therefore in (5.37) I_2 possesses analytic continuation and in that region it may be decomposed into the sum of the expressions (5.38) and (5.39), so we may write

$$\begin{aligned} & I_2(u+iT, v-iT, w+iT, z-iT; \Delta) \\ & = I_2^{(1)}(u,v,w,z;T,\Delta) + I_2^{(2)}(u,v,w,z;T,\Delta), \end{aligned} \tag{5.40}$$

where $I_2^{(1)}$ denotes the first summand in (5.38), and $I_2^{(2)}$ denotes the second. At this point we shall use the functional equation (5.31) and simplify the resulting expression. The exponential factors $e\left(\frac{h}{r}\right)$ and $e\left(-\frac{\bar{h}}{r}\right)$ will lead to the appearance of the Kloosterman sums

$$S(m,n;c) = \sum_{1 \le d \le c, (d,c)=1,\ d\bar{d} \equiv (\mathrm{mod}\ c)} e\left(\frac{md + n\bar{d}}{c}\right) \tag{5.41}$$

and the related Kloosterman-sum zeta-function

$$Z_{m,n}(s) := (2\pi\sqrt{mn})^{2s-1} \sum_{\ell=1}^{\infty} S(m,n;\ell)\ell^{-2s}. \tag{5.42}$$

This is one of the most important features of the whole approach, since sums of Kloosterman sums can be successfully treated by the Kuznetsov trace formula and spectral theory of automorphic forms. In
239 view of the bound $S(m,n;c) \ll_{m,n,\epsilon} c^{\frac{1}{2}+\epsilon}$ (see (4.89)) it is seen that the series representation (5.42) is valid for $\sigma > 3/4$.

So we use the functional equation (5.31) in (5.39), writing

$$I_2^{(2)}(u,v,w,z;T,\Delta) = I_{2,+}^{(2)}(u,v,w,z;T,\Delta) + I_{2,-}^{(2)}(u,v,w,z;T,\Delta), \tag{5.43}$$

where $I_{2,+}^{(2)}$ refers to $D\left(1-s; -\alpha, e\left(\frac{\bar{h}}{s}\right)\right)\cos\left(\frac{\pi\alpha}{2}\right)$ in (5.31), and $I_{2,-}^{(2)}$ to $-\cos\left(\pi s - \frac{\pi\alpha}{2}\right) D\left(1-s; -\alpha, e\left(-\frac{\bar{h}}{r}\right)\right)$. Then we obtain

$$I_{2,-}^{(2)} = \zeta(1+v-z)\sum_{r=1}^{\infty} r^{z-v-1} \sum_{\substack{h=1 \\ (h,r)=1}}^{r} \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} -2(2\pi)^{2u+2z-2s-2-u+w_r u-w-2u-2z+2s+1}$$

$$\times\Gamma(1-u-z-s)\Gamma(1-w-z+s)\cos\left(\pi(u+z-s)-\frac{\pi}{2}(u-w)\right)$$
$$D\left(1-u-z+s;w-u,e\left(-\frac{\bar{h}}{r}\right)\right)\zeta\left(s;e\left(\frac{h}{r}\right)\right)M(s,z-iT;\Delta)ds$$
$$=\zeta(1+v-z)\sum_{r=1}^{\infty}r^{-u-v-w-z}\frac{-1}{2\pi i}\sum_{h=1,(h,r)=1}^{r}\int_{b-i\infty}^{b+i\infty}2(2\pi)^{u+w+2z-2s-2}r^{2s}$$
$$\Gamma(1-u-z+s)\Gamma(1-w-z+s)\times\cos\left(\pi\left(z+\frac{u+w}{2}-s\right)\right)$$
$$M(s,z-iT;\Delta)\sum_{m=1}^{\infty}e\left(\frac{h}{r}m\right)m^{-s}\sum_{n=1}^{\infty}\sigma_{w-u}(n)e\left(-\frac{\bar{h}}{r}n\right)n^{-1+u+z-s}ds$$
$$=\zeta(1+v-z)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}2i\sigma_{w-u}(n)n^{-1+u+z}\sum_{r=1}^{\infty}r^{-u-v-w-z}\left(\sum_{h=1,(h,r)=1}^{r}e\left(\frac{hm-\bar{h}n}{r}\right)\right)$$
$$\times\int_{b-i\infty}^{b+i\infty}\cos\left(\pi\left(z+\frac{u+w}{2}-s\right)\right)\Gamma(1-u-z+s)\Gamma(1-w-z+s)$$
$$M(s,z-iT;\Delta)(2\pi)^{u+w+2z-2s-3}r^{2s}(mn)^{-s}ds,$$

where the interchange of summation and integration is justified by absolute convergence because of the choice of b. If h runs over a reduced system of residues mod r, so does $-\bar{h}$, so that the sum over h is $S(-m,n;r)$. Hence after some rearrangement, which will be useful later for technical reasons (i.e. application of (5.64)), we obtain

$$I_{2,-}^{(2)}(u,v,w,z;T,\Delta)=2i(2\pi)^{2-v-2}\zeta(1+v-2)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tag{5.44}$$
$$\delta_{w-u}(n)n^{-\frac{1}{2}(w+v-u-z+l)}m^{-\frac{1}{2}(u+v+w+z-1)}$$
$$\times\sum_{r=1}^{\infty}\frac{S(-m,n;r)}{r}\left(\frac{2\pi\sqrt{mn}}{r}\right)^{u+v+w+z-1}$$
$$\times\int_{b-i\infty}^{b+i\infty}\cos\left(\pi\left(z+\frac{u+w}{2}-s\right)\right)\Gamma(1-u-z+s)\Gamma(1-w-z+s)$$
$$M(s,z-iT;\Delta)\left(\frac{2\pi\sqrt{mn}}{r}\right)^{-2s}ds.$$

In a similar way it follows that

$$
\begin{aligned}
I^{(2)}_{2,+}(u,v,w,z;T,\Delta) = -2i(2\pi)^{z-v-z}\cos\left(\frac{\pi}{2}(u-w)\right)\zeta(1+v-z) \\
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-\frac{1}{2}(u+v+w+z-1)}\sigma_{w-u}(n)n^{-\frac{1}{2}(w+v-u-z+1)} \\
\times \int\limits_{b-i\infty}^{b+i\infty} Z_{m,n}\left(\frac{u+v+w+z}{2}-s\right)\Gamma(1-u-z+s) \\
\times\,\Gamma(1-w-z+s)M(s,z-iT;\Delta)ds, \qquad (5.45)
\end{aligned}
$$

where the Kloosterman-sum zeta-function $Z_{m,n}$ is defined by (5.42). This ends our present transformation of I_2. Further analysis will be carried out by the means of spectral theory of automorphic functions and the Kuznetsov trace formula.

5.3 Application of Spectral theory and Kuznetsov's trace formula

We begin first by introducing briefly some notions and results which will be used later. A detailed study would lead us too much astray.

On the upper complex half-plane $\mathbb{H}$ the modular group

$$
\Gamma = SL(2,\mathbb{Z}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : (a,b,c,d\in\mathbb{Z}) \wedge (ad-bc=1)\right\}
$$

acts in the obvious way. Namely, if $\gamma \in \Gamma$, then $\gamma z = \dfrac{az+b}{cz+d}$. The non-Euclidean Laplace operator

$$
L = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)
$$

is invariant under Γ, as is the measure $d\mu(z) = y^{-2}dxdy$, $z = x+iy$. The terminology non-Euclidean comes from the fact that $\mathbb{H}$ is the Poincaré model of Lobačevskyi's hyperbolic geometry, where straight lines are

241 either semicircles with centers on the real axis or lines perpendicular to it. The classical cusp forms of even integral weight are regular functions $f(z)$ on $\mathbb{H}$ such that

$$f(\gamma z) = (cz+d)^k f(z)$$

for any $\gamma \in \Gamma$ and $y^{\frac{1}{2}k}|f(z)|$ is bounded as $y \to \infty$. As a generalization of these, H. Maass introduced non-holomorphic cusp forms (cusp forms of weight zero), the so-called Maass wave forms. They are the eigenfunctions of the discrete spectrum of L (which has the form $\{\lambda_j\}_{j=1}^{\infty}$ with $\lambda_j = x_j^2 + \dfrac{1}{4}$ and $x_j > 0$), and they satisfy the partial differential equation $L\Psi(z) = \lambda\Psi(z)$. Every solution $\Psi(z)$ satisfies also $\Psi(\gamma z) = \Psi(z)$ for $\gamma \in \Gamma$, and the finiteness condition

$$\int\limits_{\mathscr{D}} |\Psi(z)|^2 d\mu(z) = \int\limits_{\mathscr{D}} |\Psi(x+iy)|^2 y^{-2} dx\, dy < +\infty,$$

where $\mathscr{D}$ is the fundamental domain of the modular group Γ. In standard form one usually takes ($z = x + iy$)

$$\mathscr{D} = \left\{z : y > 0, |z| > 1, -\frac{1}{2} \leq x < \frac{1}{2}\right\} \cap \left\{z : |z| = 1, -\frac{1}{2} \leq x < 0\right\}.$$

Since $\Psi(z)$ is periodic in $x = \operatorname{Re} z$ with period 1, it has a Fourier series of the form

$$\Psi(z) = \sum_{m=-\infty}^{\infty} c_m(y) e^{2\pi imx}.$$

If we insert this relation in the equation $L\Psi(z) = \lambda\Psi(z)$, multiply by $e(-nx)$ and integrate over x from 0 to 1, we obtain the differential equation

$$-y^2 c_n''(y) + 4\pi^2 n^2 y^2 c_n(y) = \lambda c_n(y),$$

which determines $c_n(y)$ up to a constant factor. For $n \neq 0$ and $y > 0$ the solution of this equation is

$$c_n(y) = \rho(n) y^{\frac{1}{2}} K_{ix}(2\pi|n|y) + \bar{\rho}(n) y^{\frac{1}{2}} I_{ix}(2\pi|n|y),$$

where $x = \left(\lambda - \frac{1}{4}\right)^{\frac{1}{2}}$, $\rho(n)$ and $\tilde{\rho}(n)$ are constants, I and K are the Bessel functions. The above finiteness condition forces $\tilde{\rho}(n) = 0$ and yields also $c_0(y) = 0$. Thus if φ_j is the eigenfunction attached to x_j, then we **242**
have the Fourier expansion

$$\varphi_j(z) = \sum_{n \neq 0} \rho_j(n) e(nx) y^{\frac{1}{2}} K_{ix_j}(2\pi|n|y) \quad (z = x + iy),$$

and moreover for real-valued φ_j we have $\rho_j(n) = \overline{\rho_j(-n)}$, in which case φ_j is either an even or an odd function of x.

The function $K_s(y)$ is an even function of s and is real for s purely imaginary and $y > 0$. For $\operatorname{Re} z > 0$ one defines

$$K_s(z) = \frac{1}{2} \int_0^\infty t^{s-1} \exp\left(-\frac{z}{2}\left(t + \frac{1}{2}\right)\right) dt,$$

which gives $K_s(z) = K_{-s}(z)$ by changing t to $1/t$ in the above integral. We have the Mellin transform

$$\int_0^\infty K_r(x) x^{s-1} dx = 2^{s-2} \Gamma\left(\frac{s+r}{2}\right) \Gamma\left(\frac{s-r}{2}\right) \ (\operatorname{Re} s > |\operatorname{Re} r|), \tag{5.46}$$

and if $\operatorname{Re} s, \operatorname{Re} z > 0$, then

$$K_s(z) \sim 2^{s-1} \Gamma(s) z^{-s} \ (z \to 0), K_2(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \ (z \to \infty).$$

Setting $s = ir$, $z = y$ in the definition of $K_2(z)$ we obtain

$$K_{ir}(y) = \int_0^\infty e^{-ycht} \cos(rt) dt \sim \left(\frac{\pi}{2y}\right)^{\frac{1}{2}} e^{-y} \quad (y \to \infty)$$

for r fixed, while for y fixed

$$K_{ir}(y) \sim \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\pi r} \sin\left(\frac{\pi}{4} + r \log r - r - r \log \frac{y}{2}\right) \qquad (r \to \infty).$$

From the Mellin transform formula (5.46) one obtains by the inversion formula for Mellin transforms

$$K_{2ir}(x) = \frac{1}{4\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} 4^s\Gamma(s+ir)\Gamma(s-ir)x^{-2s}\,ds (x>0,\ \sigma>0).$$

Henceforth let φ_j denote the Maass wave form attached to x_j, so that $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal basis (with respect to the Petersson inner product $(f_1, f_2) = \int\limits_{\mathscr{D}} f_1\bar{f_2}d\mu(z))$ of all cusp forms of $\mathbb{H}$, and φ_j is an eigenfunction of every Hecke operator. The Hecke operator T_n acts on $\mathbb{H}$, for a given $n \in \mathbb{N}$, by the relation

$$(T_nf)(z) = n^{-\frac{1}{2}} \sum_{ad=n,\ d>0}\ \sum_{b(\operatorname{mod} d)} f\left(\frac{az+b}{d}\right),$$

and we also have $(T_{-1}f)(z) = f(-\bar{z})$. The Hecke operators are commutative, which follows from the relation (omitting f and z) 243

$$T_nT_m = \sum_{d|(n,m)} T_{nm/d^2}, \tag{5.47}$$

so that $T_{nm} = T_nT_m$ if $(n,m) = 1$, that is, T_n is a multiplicative function of n. Thus T_n is determined by its values at prime powers $n = p^\alpha$, and (5.47) yields for $\alpha \geq 2$

$$T_{p^\alpha} = T_pT_{p^{\alpha-1}} - T_{p^{\alpha-2}}.$$

This recurrence relation shows that if we define the Čebyshev polynomial $U_n(x) (n = 0, 1, \ldots)$ by the identity

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

then for any integer $r \geq 1$

$$T_{p^r} = U_r\left(\frac{1}{2}T_p\right) = \sum_{0\leq k\leq\frac{1}{2}r} \frac{(-1)^k(r-k)!}{k!(r-2k)!}T_p^{r-2k}.$$

Let, for each $n \geq 1$, $t_j(n)$ denote the eigenvalue corresponding to φ_j with respect to T_n, i.e. $T_n\varphi_j(z) = t_j(n)\varphi_j(z)$, and assume that $\varphi_j(z)$ is an eigenfunction of the reflection operator. This means that $\varphi_j(-\bar{z}) = \epsilon_j\varphi_j(z)$, where $\epsilon_j = \pm 1$ is the parity sign of the form corresponding to x_j. Thus $\epsilon_j = +1$ if φ_j is an even function of x, and $\epsilon_j = -1$ if φ_j is an odd function of x. Calculating $T_n\varphi_j(z)$ by using the Fourier expansion of $\varphi_j(z)$, it follows that

$$T_n\varphi_j(z) = \sum_{m\neq 0}\left(\sum_{d|(n,m),d>0}\rho_j\left(\frac{nm}{d^2}\right)\right)e(mx)y^{\frac{1}{2}}K_{ix_j}(2\pi|m|y),$$

while by comparing this result with the expansion of $t_j(n)\varphi_j(z)$ we obtain

$$t_j(n)\rho_j(m) = \sum_{d|(n,m),d>0}\rho_j\left(\frac{nm}{d^2}\right), \quad (m \neq 0)$$

244 hence in particular

$$\rho_j(1)t_j(n) = \rho_j(n) \quad (n \geq 1,\ j \geq 1).$$

The Fourier coefficient $\rho_j(1)$ is of special importance in the sequel. Following standard usage we set

$$\alpha_j := |\rho_j(1)|^2(ch(\pi x_j))^{-1}.$$

One always has $\alpha_j > 0$, for otherwise the corresponding Maass wave form would be identically zero (since $\rho_j(n) = \rho_j(1)t_j(n)$), which is impossible.

N.V. Kuznetsov proved the asymptotic formula

$$\sum_{x_j\leq T}|\rho_j(n)|^2(ch\pi x_j)^{-1} = \left(\frac{T}{\pi}\right)^2 + O\left(T\log T + Tn^{\epsilon} + n^{\frac{1}{2}+\epsilon}\right), \quad (5.48)$$

so that (5.48) for $n = 1$ provides an asymptotic formula for $\sum\limits_{x_j\leq T}\alpha_j$.

From the theory of the Selberg zeta-function it follows that

$$\{x_j : x_j \leq T\} = \frac{T^2}{12} + O(T), \quad \left\{x_j : |x_j - T| \leq 1\right\} \ll T, \quad (5.49)$$

whence

$$x_j \sim \sqrt{12j} \quad (j \to \infty).$$

By the Rankin-Selberg convolution method one can deduce a functional equation and obtain analytic continuation of the function

$$R_j(s) = \sum_{n=1}^{\infty} |\rho_j(n)|^2 n^{-s},$$

and furthermore deduce

$$\sum_{n \le N} |\rho_j(n)|^2 = 6\pi^{-2} ch(\pi x_j)N + O_{j,\epsilon}(N^{3/5+\epsilon}). \tag{5.50}$$

This immediately implies $\rho_j(n) \ll_{\epsilon,j} n^{3/10+\epsilon}$, but even $\rho_j(n) \ll_{\epsilon,j} n^{1/5+\epsilon}$ is known.

At this point we define the Hecke series **245**

$$H_j(s) := \sum_{n=1}^{\infty} t_j(n)n^{-s} = \prod_p \left(1 - t_j(p)p^{-s} + p^{-2s}\right)^{-1} \tag{5.51}$$

in the region of absolute convergence of the above series and product (which includes the region $\sigma > 2$, since $t_j(n) \ll \sigma_1(n)$), and otherwise by analytic continuation. From the properties of Hecke operators T_n it follows that $t_j(n)$ is a multiplicative function of n, and in fact

$$t_j(m)t_j(n) = \sum_{d|(m,n)} t_j\left(\frac{mn}{d^2}\right) \quad (m, n \ge 1). \tag{5.52}$$

Writing out the series representation for $H_j(s)H_j(s-a)$ and using (5.52) it is seen that, in the region of absolute convergence, we have

$$\sum_{n=1}^{\infty} \sigma_a(n)t_j(n)n^{-s} = (H_j(s)H_j(s-a))/\zeta(2s-a). \tag{5.53}$$

This identity is the counterpart of the classical Ramanujan identity (5.5).

The Hecke series satisfy the functional equation

$$\begin{aligned} H_j(s) = \pi^{-1}(2\pi)^{2s-1}\Gamma(1-s+ix_j)\Gamma(1-s-ix_j) \\ \{-\cos(\pi s) + \epsilon_j ch(\pi x_j)\}H_j(1-s), \end{aligned} \tag{5.54}$$

where ϵ_j is the parity sign corresponding to φ_j. This functional equation is analogous to the functional equation $\zeta(s) = x(s)\zeta(1-s)$ for the Riemann zeta-function. The function $H_j(s)$ is an entire function of s and satisfies $H_j(s) \ll x_j^c$ for some $c > 0$ and bounded s.

One has the following formula, due to Y. Motohashi:

$$\sum_{x_j \le T} \alpha_j H_j^2\left(\frac{1}{2}\right) = \frac{2T^2}{\pi^2}\left(\log T + \gamma - \frac{1}{2} - \log(2\pi)\right) + O(T\log^6 T). \tag{5.55}$$

We shall now briefly discuss holomorphic cusp forms and introduce analogous notions and notation. Let $\{\varphi_{j,2k}\}$, $1 \le j \le d_{2k}$, $k \ge 6$ be the orthonormal basis, which consists of eigenfunctions of Hecke operators
246 $T_{2k}(n)$, of the Petersson unitary space of holomorphic cusp forms of weight $2k$ for the full modular group. Thus for every $n \ge 1$ there is a $t_{j,2k}(n)$ such that

$$\begin{aligned} &T_{2k}(n)(\varphi_{j,2k}(z)) \\ &= n^{-\frac{1}{2}} \sum_{ad=n,\ d>0} \left(\frac{a}{d}\right)^k \sum_{b(\mod d)} \varphi_{j,2k}\left(\frac{az+b}{d}\right) = t_{j,2k}(n)\varphi_{j,2k}(z). \end{aligned} \tag{5.56}$$

The corresponding Hecke series

$$H_{j,2k}(s) := \sum_{n=1}^{\infty} t_{j,2k}(n)n^{-s} \tag{5.57}$$

converges absolutely, as in the case of $H_j(s)$, at least for $\sigma > 2$. The function $H_{j,2k}(s)$ is entire. It satisfies a functional equation which implies that, uniformly for bounded s,

$$H_{j,2k}(s) \ll_j k^c \tag{5.58}$$

nor some $c > 0$. The analogue of (5.52) is

$$\sum_{n=1}^{\infty} \sigma_a(n) t_{j,2k}(n) n^{-s} = (H_{j,2k}(s) H_{j,2k}(s-a))/\zeta(2s-a). \tag{5.59}$$

If $\rho_{j,2k}(1)$ is the first Fourier coefficient of $\varphi_{j,2k}$, then let

$$\alpha_{j,2k} := (2k-1)! 2^{-4k+2} \pi^{-2k-1} |\rho_{j,2k}(1)|^2.$$

Setting

$$p_{m,n}(k) := (2k-1) \sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,n;\ell) J_{2k-1}\left(4\pi \sqrt{mn}\ell^{-1}\right), \tag{5.60}$$

where J is the Bessel function of the first kind, we have for every $m, n \geq 1$ Petersson's formula

$$p_{m,n}(k) = \frac{\pi}{2}(-1)^k \sum_{j \leq d_{2k}} \alpha_{j,2k} t_{j,2k}(m) t_{j,2k}(n) + (-1)^{k-1} \delta_{m,n} \frac{(2k-1)}{\pi}, \tag{5.61}$$

where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ if $m \neq n$. From the standard representation

$$J_\nu(x) = \frac{2\pi^{-\frac{1}{2}}(x/2)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_0^1 \cos(xt)(1-t^2)^{\nu - \frac{1}{2}} dt \quad \left(\operatorname{Re} \nu > -\frac{1}{2}\right)$$

one obtains, for $x > 0$,

$$J_{2k-1}(x) \ll \frac{1}{\Gamma(2k - \frac{1}{2})} \left(\frac{x}{2}\right)^{2k-1}.$$

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Hence using the trivial bound $|S(m,n;\ell)| \leq \ell$ it follows that uniformly

$$p_{m,n}(k) \ll \frac{1}{\Gamma(2k - 3/2)} (2\pi \sqrt{mn})^{2k-1}, \tag{5.62}$$

and setting $m = n = 1$ in (5.61) and (5.62) we obtain

$$\sum_{j \leq d_{2k}} \alpha_{j,2k} \ll k \quad (k \geq 6).$$

After these preparations we are going to state a variant of the Kuznetsov trace formula, which connects a sum of Kloosterman sums with a sum over the eigenvalues t_j. Let

$$\check{\mathrm{h}}\,(w) := 2ch(\pi w)\int\limits_0^\infty h(y)K_{2iw}(y)\frac{dy}{y} \tag{5.63}$$

be the Bessel transform of $h(y)$, where $h(y) \in C^3(0,\infty)$, $h(0) = h'(0) = 0$, $h^{(j)}(y) \ll y^{-2-\epsilon}$ as $y \to \infty$ for some $\epsilon > 0$ and $0 \le j \le 3$. Then for every $m, n \ge 1$

$$\begin{aligned} &\sum_{\ell=1}^\infty \frac{1}{\ell} S(-m,n;\ell)h\left(4\pi\sqrt{mn}\ell^{-1}\right) \qquad (5.64)\\ &= \sum_{j=1}^\infty \alpha_j\epsilon_j t_j(m)t_j(n)\,\check{\mathrm{h}}\,(x_j) + \frac{1}{\pi}\int\limits_{-\infty}^\infty \frac{\sigma_{2iw}(m)\sigma_{2iw}(n)\,\check{\mathrm{h}}\,(w)}{(mn)^{iw}|\zeta(1+2iw)|^2}dw. \end{aligned}$$

Another form of the trace formula deals with sums of $S(m,n;\ell)$ when $mn > 0$. We shall need a consequence of it, which is the following decomposition of the zeta-function $Z_{m,n}(x)$, defined by (5.42). Namely for any s and $m, n \ge 1$ we have

$$Z_{m,n}(s) = Z^{(1)}_{m,n}(s) + Z^{(2)}_{m,n}(s) + Z^{(3)}_{m,n}(s) - \delta_{m,n}\cdot\frac{\Gamma(s)}{2\pi\Gamma(1-s)}, \tag{5.65}$$

where

$$Z^{(1)}_{m,n}(s) = \frac{1}{2}\sin(\pi s)\sum_{j=1}^\infty \alpha_j t_j(m)t_j(n)\Gamma\left(s-\frac{1}{2}+ix_j\right)\Gamma\left(s-\frac{1}{2}-ix_j\right), \tag{5.66}$$

$$\begin{aligned} Z^{(2)}_{m,n}(s) = \frac{1}{\pi}\sin(\pi s)\sum_{k=1}^\infty (-1)^k p_{m,n}(k)\Gamma\left(s-\frac{1}{2}+\left(k-\frac{1}{2}\right)\right)&\\ \Gamma\left(s-\frac{1}{2}-\left(k-\frac{1}{2}\right)\right),& \qquad (5.67) \end{aligned}$$

248 and $Z^{(3)}_{m,n}(s)$ is the analytic continuation of the function defined, for Re

$s > 1/2$, by the formula

$$Z^{(3)}_{m,n}(s) = \frac{\sin(\pi s)}{2\pi} \int_{-\infty}^{\infty} \left(\frac{n}{m}\right)^{iw} \sigma_{2iw}(m)\sigma_{-2iw}(n) \tag{5.68}$$
$$|\zeta(1+2iw)|^{-2}\Gamma\left(s-\frac{1}{2}+iw\right)\Gamma\left(s-\frac{1}{2}-iw\right)dw.$$

The decomposition formula (5.65) will be applied to (5.45). Note that (5.37) holds, hence $\mathrm{Re}\left(\frac{u+v+w+z}{2}-r\right) > 1$ for $\mathrm{Re}\, r = b$. The integral in (5.45) becomes then

$$\int_{b-i\infty}^{b+i\infty} \ldots M(s, z-iT; \Delta)ds = \int_{b-i\infty}^{b+i\infty} \left\{ \left(Z^{(1)}_{m,n} + Z^{(2)}_{m,n} + Z^{(3)}_{m,n}\right) \right.$$
$$\left. \left(\frac{u+v+w+2}{2} - s\right) - \frac{\delta_{m,n}}{2\pi} \cdot \frac{\Gamma\left(\frac{u+v+w+2}{2}-s\right)}{\Gamma\left(1+s-\frac{u+v+w+z}{2}\right)} \right\}$$
$$\Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s, z-iT; \Delta)ds$$
$$= (C^{(1)}_{m,n} + C^{(2)}_{m,n} + C^{(3)}_{m,n} + C^{(4)}_{m,n})\,(u, v, w, z), \tag{5.69}$$

say. Here obviously $C^{(j)}_{m,n}$ corresponds to $Z^{(j)}_{m,n}$ for $1 \le j \le 3$, and $C^{(4)}_{m,n}$ to the portion with $\delta_{m,n}$.

From (5.66), $t_j(n) \ll \sigma_1(n)$, $\sum_{x_j \le x} \alpha_j \ll x^2$ and Stirling's formula we find that

$$Z^{(1)}_{m,n}(s) \ll \sigma_1(m)\sigma_1(n)|s|^{2\,\mathrm{Re}\, s}.$$

This implies that we may change the order of summation and integration in $C^{(1)}_{m,n}(u, v, w, z)$ to obtain

$$C^{(1)}_{m,n}(u, v, w, z) = \frac{1}{2}\sum_{j=1}^{\infty} \alpha_j t_j(m) t_j(n) U(u, v, w, z; ix_j), \tag{5.70}$$

where

$$U(u, v, w, z; \xi) := \int_{b-i\infty}^{b+i\infty} \sin\left(\frac{\pi}{2}(u+v+w+z-2s)\right) \tag{5.71}$$

$$\Gamma\left(\frac{u+v+w+z-1}{2}-s+\xi\right)\Gamma\left(\frac{u+v+w+z-1}{2}-s-\xi\right)$$
$$\Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s,z-iT;\Delta)ds.$$

Note that U is well defined if (5.37) holds and $|\operatorname{Re}\xi|<1$. Moreover
249 for compacta of (5.37) and $\xi\in\mathbb{R}$

$$U(u,v,w,z;\xi)\ll e^{-\pi|\xi|}. \tag{5.72}$$

Next consider $C^{(3)}_{m,n}$ and observe that we may change the order of integration by appealing to Stirling's formula. It follows that

$$C^{(3)}_{m,n}(u,v,w,z)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left(\frac{n}{m}\right)^{i\xi}\sigma_{2i\xi}(m)\sigma_{-2i\xi}(n)\frac{U(u,v,w,z;i\xi)}{|\zeta(1+2i\xi)|^2}d\xi. \tag{5.73}$$

Similarly we may change the order of summation and integration in $C^{(2)}_{m,n}$ to obtain first

$$C^{(2)}_{m,n}(u,v,w,z)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k p_{m,n}(k)U\left(u,v,w,z;k-\frac{1}{2}\right). \tag{5.74}$$

Then we shift the line of integration in (5.71) to $\operatorname{Re}s=b_1>0$, where b_1 may be arbitrarily large, but is fixed. In doing this no poles of the integrand will be encountered, and by Stirling's formula we infer that, if (5.37) holds and $C>0$ is arbitrary but fixed, then uniformly

$$U\left(u,v,w,z;k-\frac{1}{2}\right)\ll k^{-C}. \tag{5.75}$$

With this estimate it is seen that we may write (5.74) as

$$C^{(2)}_{m,n}(u,v,w,z)=\frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\left(p_{m,n}(k)-(-1)^{k-1}\delta_{m,n}\frac{(2k-1)}{2\pi}\right)$$
$$U\left(u,v,w,z;k-\frac{1}{2}\right)-\frac{1}{2\pi}\delta_{m,n}$$
$$\sum_{k=1}^{\infty}(2k-1)U\left(u,v,w,z;k-\frac{1}{2}\right). \tag{5.76}$$

To transform the second sum in (5.76), note that using $z\Gamma(z) = \Gamma(z+1)$ we have

$$(2k-1)\frac{\Gamma(k-1+s)}{\Gamma(1+k-s)} = \frac{\Gamma(k+s)}{\Gamma(1+k-s)} + \frac{\Gamma(k-1+s)}{\Gamma(k-s)}, \tag{5.77}$$

and write

$$\sum_{k=1}^{\infty} = \sum_{k\le k_0} + \sum_{k>k_0} = \sum_1 + \sum_2,$$

say, where k_0 is a large integer. Using (5.77) we find that

$$\sum_1 = -\pi \int\limits_{b-i\infty}^{b+i\infty} \Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s, z-iT; \Delta) \tag{5.78}$$
$$\times \left\{ \frac{\Gamma(\frac{u+v+w+z}{2} - s)}{\Gamma(1 - \frac{u+v+w+z}{2} + s)} + (-1)^{k_o+i} \frac{\Gamma(k_0 + \frac{u+v+w+z}{2} - s)}{\Gamma(k_0 + 1 - \frac{u+v+w+z}{2} + s)} \right\} ds.$$

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Now by using again (5.77) and inverting the order of summation and integration by appealing to (5.75), we have with b_1 arbitrarily large

$$\sum_2 = \pi \sum_{k>k_0} (-1)^k (2k-1) \int\limits_{b_1-i\infty}^{b_1+i\infty} \Gamma(1-u-z+s)\Gamma(1-w-z+s)$$
$$M(s, z-iT; \Delta) \frac{\Gamma(k-1-s+\frac{1}{2}(u+v+w+z))}{\Gamma(k+1+s-\frac{1}{2}(u+v+w+z))} ds$$
$$= \pi(-1)^{k_0+1} \int\limits_{b_1-i\infty}^{b_1+i\infty} \Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s, z-iT; \Delta)$$
$$\frac{\Gamma(k_0+\frac{1}{2}(u+v+w+z)-s)}{\Gamma(k_0+1+s-\frac{1}{2}(u+v+w+z))} ds,$$

since all the terms in the sum will cancel, except the one corresponding to $k_0 + 1$, because of the presence of the factor $(-1)^k$. In the last integral we shift the line of integration back to $\operatorname{Re} s = b$, and add the results to

(5.76). We find then that the second term on the right-hand side of (5.76) is equal to $-C^{(4)}_{m,n}(u, v, w, z)$. Thus using Petersson's formula (5.61) we have

$$\begin{aligned} & C^{(2)}_{m,n}(u, v, w, z) + C^{(4)}_{m,n}(u, v, w, z) \qquad (5.79)\\ &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k\left(p_{m,n}(k) - (-1)^{k-1}\delta_{m,n}\cdot\frac{2k-1}{2\pi}\right)U\left(u, v, w, z; k-\frac{1}{2}\right)\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j\le d_{2k}}\alpha_{j,2k}t_{j,2k}(m)t_{j,2k}(n)U\left(u, v, w, z; k-\frac{1}{2}\right). \end{aligned}$$

This ends our transformation of the integral in (5.69). We are going to insert the transformed formula into the right-hand side of (5.45) and transform it further. To achieve this note that from the above expressions we have

$$C^{(1)}_{m,n}(u, v, w, z) \ll \sigma_1(m)\sigma_1(n)\sum_{j=1}^{\infty}\alpha_j e^{-\pi x_j} \ll \sigma_1(m)\sigma_1(n), \tag{5.80}$$

$$C^{(3)}_{m,n}(u, v, w, z) \ll d(m)d(n)\int_{-\infty}^{\infty}|\zeta(1+2i\xi)|^{-2}e^{-\pi|\xi|}d\xi, \tag{5.81}$$

$$C^{(2)}_{m,n}(u, v, w, z) + C^{(4)}_{m,n}(u, v, w, z) \ll \sigma_1(m)\sigma_1(n)\sum_k k^{-C} \ll \sigma_1(m)\sigma_1(n). \tag{5.82}$$

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These estimates are all uniform for bounded u, v, w, z satisfying (5.37).

Now temporarily assume that (u, v, w, z) satisfies (5.37) and that $\mathrm{Re}\, v$ is large. Then (5.80)-(5.82) ensure that all multiple sums that arise, after inserting the transformed formula for (5.69) in (5.45), are absolutely convergent. Performing the summation and taking into account (5.52) and (5.59) we obtain the following spectral decomposition formula:

$$\begin{aligned} I^{(2)}_{2,+}(u, v, w, z; T, \Delta) = \frac{1}{\pi}\int_{-\infty}^{\infty}\zeta\left(\frac{u+v+w+z-1}{2}+i\xi\right) \qquad (5.83)\\ \zeta\left(\frac{u+v+w+z-1}{2}-i\xi\right)\times\zeta\left(\frac{u+v-w-z+1}{2}+i\xi\right) \end{aligned}$$

$$
\zeta\left(\frac{u+v-w-z+1}{2}-i\xi\right)\zeta\left(\frac{v+w-u-z+i}{2}-i\xi\right)
$$
$$
\times\frac{\Psi(u,v,w,z;i\xi)}{|\zeta(1+2i\xi)|^{w}}d\xi
$$
$$
+\sum_{j=1}^{\infty}\alpha_j H_j\left(\frac{u+v+w+z-1}{2}\right)H_j\left(\frac{u+v-w-z+1}{2}\right)
$$
$$
H_j\left(\frac{v+w-u-z+1}{2}\right)\Psi(u,v,w,z;ix_j)+\sum_{k=6}^{\infty}\sum_{j\le d_{2k}}\alpha_{j,2k}
$$
$$
H_{j,2k}\left(\frac{u+v+w+z-1}{2}\right)H_{j,2k}\left(\frac{u+v-w-z+1}{2}\right)
$$
$$
H_{j,2k}\left(\frac{v+w-u-z+1}{2}\right)\Psi\left(u,v,w,z;k-\frac{1}{2}\right),
$$

where we have set

$$
\Psi(u,v,w,z;\xi)=\Psi(u,v,w,z;\xi,T,\Delta):=-i(2\pi)^{z-v-2} \tag{5.84}
$$
$$
\cos\left(\frac{\pi u-\pi w}{2}\right)\int_{-i\infty}^{i\infty}\sin\left(\pi\left(\frac{u+v+w+z}{2}-s\right)\right)
$$
$$
\Gamma\left(\frac{u+v+w+z-1}{2}-s+\xi\right)\Gamma\left(\frac{u+v+w+z-1}{2}-s-\xi\right)
$$
$$
\Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s,z-iT;\Delta)ds.
$$

In the last integral the path of integration is curved (indented) to ensure that the poles of the first two gamma-factors of the integrand lie to the right of the path and the poles of the other two gamma-factors are on the left of the path, with the condition that u, v, w, z, are such that the contour may be drawn. Obviously when (5.37) is satisfied and ξ is
purely imaginary, then the path may be taken as $\operatorname{Re} s = b$. Since for **252**
bounded s we have uniformly

$$
H_j(s)\ll x_j^c,\ H_{j,2k}(s)\ll k^c \qquad (c>0), \tag{5.85}
$$

which follows from the functional equations for H_j and $H_{j,2k}$, we infer that the condition that $\operatorname{Re} v$ be large may be dropped. Hence (5.83) holds for u, v, w, z satisfying (5.37).

The integral in (5.84) bears resemblance to the so-called Mellin-Barnes type of integral. A classical example is the Barnes lemma, which states that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)ds \tag{5.86}$$
$$=\frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)},$$

and the line of integration is to be taken to the right of the poles of $\Gamma(\alpha+s)\Gamma(\beta+s)$, and to the left of the poles of $\Gamma(\gamma-s)\Gamma(\delta-s)$.

Now we shall transform $I_{2,-}^{(s)}$ in (5.44). The presence of the Kloosterman sums $S(-m,n;r)$ makes the application of the Kuznetsov trace formula (5.64) a natural tool. We are going to transform the sum

$$D_{m,n}(u,v,w,z) := \sum_{\ell=1}^{\infty}\frac{1}{\ell}S(-m,n;\ell)\varphi\left(4\pi\sqrt{mn}\ell^{-1}\right),$$

where $m,n\ge 1$ and

$$\varphi(x) := \left(\frac{x}{2}\right)^{u+v+w+z-1}\int_{b-i\infty}^{b+i\infty}\cos\left(\pi\left(z+\frac{u+w}{2}-s\right)\right) \tag{5.87}$$
$$\Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s,z-iT;\Delta)\left(\frac{x}{2}\right)^{-2s}ds$$

with (u,v,w,z) satisfying (5.37). The function $\varphi(x)$ defined by (5.87) satisfies the regularity and decay condition needed in the application of (5.64), which may be seen by suitably shifting the line of integration. Then applying (5.64) we have

$$D_{m,n}(u,v,w,z) = \sum_{j=1}^{\infty}\alpha_j\epsilon_j t_j(m)t_j(n)\check{\varphi}(x_j)$$
$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\sigma_{2i\xi}(m)\sigma_{2i\xi}(n)(mn)^{-i\xi}|\zeta(1+2i\xi)|^{-2}\check{\varphi}(\xi)d\xi.$$

253 Similarly as in the case of $I^{(x)}_{2,+}$, it follows that for $I^{(2)}_{2,-}$ we have the following spectral decomposition formula

$$
\begin{aligned}
I^{(2)}_{2,-}(u,v,w,z;T,\Delta) = &-\frac{1}{\pi}\int_{-\infty}^{\infty}\zeta\left(\frac{u+v+w+z-1}{2}-\xi\right) \qquad (5.88)\\
&\times\zeta\left(\frac{u+v-w-z+1}{2}+i\xi\right)\zeta\left(\frac{u+v-w-z+1}{2}-i\xi\right)\\
&\zeta\left(\frac{v+w-u-z+1}{2}+i\xi\right)d\zeta\left(\frac{v+w-u-z+1}{2}-i\xi\right)\\
&\frac{\Phi(u,v,w,z;i\xi)}{|\zeta(1+2i\xi)|^2}d\xi-\sum_{j=1}^{\infty}\epsilon_j\alpha_j H_j\left(\frac{u+v+w+z-1}{2}\right)\\
&H_j\left(\frac{u+v-w-z+1}{2}\right)H_j\left(\frac{v+w-u-z+1}{2}\right)\Phi(u,v,w,z;ix_j),
\end{aligned}
$$

where we have set

$$
\begin{aligned}
\Phi(u,v,w,z;\xi) = \Phi(u,v,w,z;\xi,T,\Delta) := &-i(2\pi)^{z-v-2}\cos(\pi\xi) \qquad (5.89)\\
&\int_{-i\infty}^{i\infty}\cos\left(\pi\left(z+\frac{u+w}{2}-s\right)\right)\Gamma\left(\frac{u+v+w+z-1}{2}-s+\xi\right)\\
&\Gamma\left(\frac{u+v+w+z-1}{2}-s-\xi\right)\Gamma(1-u-z+s)\\
&\Gamma(1-w-z+s)M(s,z-iT;\Delta)ds.
\end{aligned}
$$

The same remark about the path of integration should be made here as for (5.84), and (5.89) also holds for u,v,w,z satisfying (5.37).

Thus finally we may collect the preceding formulas and obtain an expression for $I_2(u+iT,v-iT,w+iT,z-iT;\Delta)$ (the omission of iT, that is, the consideration of $I_2(u,v,w,z;\Delta)$ results only in the omission of iT in the M-factor; thus it is no loss of generality if one worked with $I_2(u,v,w,z;\Delta)$), which is valid for u,v,w,z satisfying (5.37). This is

$$I_2(u+iT,v-iT,w+iT,z-iT;\Delta) \qquad (5.90)$$

$$
\begin{aligned}
&= \zeta(u+v)\zeta(w+z-1)\zeta(u-w+1)\zeta(v-z+1)\\
&\frac{M(w+z-1,z-iT;\Delta)}{\zeta(u+v-w-z+2)} + \zeta(v+w)\zeta(u+z-1)\zeta(w-u+1)\\
&\zeta(v-z+1)\frac{M(u+z-1,z-iT;\Delta)}{\zeta(w+v-u-z+2)}\frac{1}{i\pi}\int\limits_{-i\infty}^{i\infty}\zeta\left(\frac{u+v+w+z-1}{2}+\xi\right)\\
&\zeta\left(\frac{u+v+w+z-1}{2}-\xi\right)\zeta\left(\frac{u+v-w-z+1}{2}+\xi\right)\\
&\times\zeta\left(\frac{u+v-w-z+1}{2}-\xi\right)\zeta\left(\frac{v+w-u-z+1}{2}+\xi\right)\\
&\zeta\left(\frac{v+w-u-z+1}{2}-\xi\right)\frac{(\Psi-\Phi)(u,v,w,z;\xi)}{\zeta(1+2\xi)\zeta(1-2\xi)}d\xi\\
&+\sum_{j=1}^{\infty}\alpha_j H_j\left(\frac{u+v+w+z-1}{2}\right)H_j\left(\frac{u+v-w-z+1}{2}\right)\\
&H_j\left(\frac{v+w-u-z+1}{2}\right)(\Psi-\epsilon_j\Phi)(u,v,w,z;ix_j)\\
&+\sum_{k=6}^{\infty}\sum_{j\le d_{2k}}\alpha_{j,2k}H_{j,2k}\left(\frac{u+v+w+z-1}{2}\right)H_{j,2k}\left(\frac{u+v-w-z+1}{2}\right)\\
&H_{j,2k}\left(\frac{v+w-u-z+1}{2}\right)\Psi\left(u,v,w,z;k-\frac{1}{2}\right),
\end{aligned}
$$

254 where we changed the variable so that the ξ-integral is teken along the imaginary axis. This ends our transformation of I_2. In the next section it will be shown that (5.90) actually provides the analytic continuation of $I_2(u+iT, v-iT, w+iT, z-iT;\Delta)$ to the entire four-dimensional space $\mathbb{C}^4$.

5.4 Further Analytic Continuation and the explicit formula

In this section it will be shown that (5.90) provides analytic continuation of $I_2(u+iT, v-iT, w+iT, z-iT;\Delta)$ to $\mathbb{C}^4$. Hence specializing $(u,v,w,z) = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ one will obtain eventually (5.10) (Theorem 5.1) from (5.4) and (5.6) - (5.8). Theorem 5.1 is the fundamental result on which all subsequent results of this chapter (Theorems 5.2 - 5.8) are founded. However, before we make the passage from (5.90) to (5.10), some difficult technical work has to be done. We have to show that $\Psi(u,v,w,z;\xi)$ and $\Phi(u,v,w,z;\xi)$ possess meromorphic continuation to the entire space $\mathbb{C}^5$. This will be achieved by showing that Ψ and Φ can be expressed in terms of the function

$$\Xi(u,v,w,z;\xi) := \int\limits_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2}(u+v+w+z-1)-s+\xi)}{\Gamma(\frac{1}{2}(3-u-v-w-z)+s+\xi)} \tag{5.91}$$
$$\Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s,z-iT;\Delta)ds,$$

and then proving the analytic continuation of Ξ. In (5.91) the path of integration, as before in analogous situations, has to be suitably curved. **255**

Using $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$ one obtains the identities

$$\sin(\pi s)\Gamma\left(s-\frac{1}{2}+\eta\right)\Gamma\left(s-\frac{1}{2}-\eta\right) = \frac{\pi}{2\sin(\pi\eta)}$$
$$\left\{\frac{\Gamma(s-\frac{1}{2}-\eta)}{\Gamma(-s+\frac{3}{2}-\eta)} - \frac{\Gamma(s-\frac{1}{2}+\eta)}{\Gamma(-s+\frac{3}{2}+\eta)}\right\} \tag{5.92}$$

and

$$\cos(\pi s)\Gamma\left(s-\frac{1}{2}+\eta\right)\Gamma\left(s-\frac{1}{2}-\eta\right) = \frac{-\pi}{2\cos(\pi\eta)}$$
$$\left\{\frac{\Gamma(s-\frac{1}{2}+\eta)}{\Gamma(-s+\frac{3}{2}+\eta)} - \frac{\Gamma(s-\frac{1}{2}-\eta)}{\Gamma(-s+\frac{3}{2}-\eta)}\right\} \tag{5.93}$$

for any s and η. From (5.92) and (5.93) it follows that

$$\Psi(u,v,w,z;\xi) = i(2\pi)^{z-v-1}\frac{\cos(\frac{1}{2}(\pi u-\pi w))}{4\sin(\pi\xi)}$$
$$\{\Xi(u,v,w,z;\xi)-\Xi(u,v,w,z,-\xi)\} \tag{5.94}$$

and

$$\Phi(u,v,w,z;\xi) = \frac{i(2\pi)^{z-v-1}}{4\sin(\pi\xi)}\left\{\sin\left(\pi\left(\frac{v-z}{2}+\xi\right)\right)\right. \tag{5.95}$$

$$\xi(u,v,w,z;\xi) - \sin\left(\pi\left(\frac{v-z}{2}-\xi\right)\right)\Xi(u,v,w,z;-\xi)\Big\}.$$

To investigate the analytic continuation of Ξ, consider the region $D(P)$, which consists of (u,v,w,z) satisfying (5.37), ξ satisfying $|\operatorname{Re}\xi| < \frac{1}{2}$ and $|u|, |v|, |w|, |z|, |\xi| < P$, where $P > 0$ is a (large) parameter. Assuming this we may take $\operatorname{Re} s = b$ (see (5.37)) as the line of integration in (5.91). We split Ξ into two parts by noting that, using (5.24) and integrating by parts ν times, we have

$$\begin{aligned} M(s,z;\Delta) &= \int_0^\infty x^{s-1}(1+x)^{-z}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx \qquad (5.96)\\ &= \frac{\Gamma(s)}{\Gamma(s+\nu)}\left(\int_0^1+\int_1^\infty\right)x^{s+\nu-1}f^{(\nu)}(x)dx\\ &= \frac{\Gamma(s)}{\Gamma(s+\nu)}(M_1^{(\nu)}(s,z;\Delta)+M_2^{(\nu)}(s,z;\Delta)), \end{aligned}$$

say, where

$$f^{(\nu)}(x) = \left(\frac{d}{dx}\right)^\nu\left\{(1+x)^{-z}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)\right\}.$$

Accordingly we may write

$$\Xi(u,v,w,z;\xi) = \Xi_1^{(\nu)}(u,v,w,z;\xi) + \Xi_2^{(\nu)}(u,v,w,z;\xi), \qquad (5.97)$$

where $\sum_1$ corresponds to $\int_0^1$, and $\sum_2$ to $\int_1^\infty$. Note that, uniformly for
256 $\operatorname{Re} s > 0$,

$$M_1^{(\nu)}(s,z;\Delta) \ll 1,$$

with the $\ll$-constant depending on ν, z, Δ. Thus by Stirling's formula the integrand of $\Xi_1^{(\nu)}$ is, for $\operatorname{Re} s \geq b$,

$$\ll |s|^{-\nu-1+\operatorname{Re}(\nu-z)},$$

as long as $\frac{1}{2}(u+v+w+z-1)+\xi-s$ stays away from nonpositive integers. Thus we may shift the line of integration from $\operatorname{Re} s = b$ to $+\infty$ if $\nu > 3P$, say. If this is assumed, then by the residue theorem we obtain

$$\Xi_1^{(\nu)}(u,v,w,z;\xi) = -2\pi i \sum_{q=0}^{\infty} (-1)^q M_1^{(\nu)}\left(\frac{u+v+w+z-1}{2}+\xi+q, z; \Delta\right)$$
$$\frac{\Gamma(\frac{1}{2}(v+w-u-z+1)+\xi+q)\Gamma(\frac{1}{2}(u+v-w-z+1)+\xi+q)\Gamma(\frac{1}{2}(u+v+w+z-1)+\xi+q)}{\Gamma(q+1)\Gamma(q+1+2\xi)\Gamma(\frac{1}{2}(u+v+w+z-1)+q+\xi+\nu)}. \tag{5.98}$$

This series provides meromorphic continuation of $\Xi_1^{(\nu)}$ to compacta of $\mathbb{C}^5$, since the summands are by Stirling's formula

$$\ll q^{-\nu-1+\operatorname{Re}(v-z)}$$

uniformly for all bounded (u,v,w,z,ξ) regardless of whether they belong to the region $D(P)$ or not, and ν may be taken sufficiently large. By analogous considerations one may also obtain a series representation for $\Xi_2^{(\nu)}(u,v,w,z;\xi)$ which is of a similar type as the one we have in (5.98). We conclude that $\Xi(u,v,w,z;\xi)$, and then by (5.94) and (5.95), the functions $\Psi(u,v,w,z;\xi)$ and $\Phi(u,v,w,z;\xi)$ are meromorphic over $\mathbb{C}^5$. To show that the double sum over k and j in (5.83) is a meromorphic function over $\mathbb{C}^4$, it is enough to show that

$$\Psi\left(u,v,w,z;k-\frac{1}{2}\right) \ll k^{-C} \tag{5.99}$$

uniformly for $|u|,|v|,|w|,|z| < P$ and $C > 0$ an arbitrary, but fixed constant. By (5.94) we see that (5.99) follows from

$$\Xi\left(u,v,w,z;k-\frac{1}{2}\right) \ll k^{-C}, \tag{5.100}$$

since it gives, for k an integer,

$$\Psi\left(u,v,w,z;k-\frac{1}{2}\right) = \frac{i}{2}(-1)^{k+1}(2\pi)^{z-v-1}$$
$$\cos\left(\frac{\pi}{2}(z-w)\right)\Xi\left(u,v,w,z;k-\frac{1}{2}\right), \tag{5.101}$$

since by using $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ and (5.91) we find that 257

$$\Xi\left(u, v, w, z; -\left(k - \frac{1}{2}\right)\right) = -\Xi\left(u, v, w, z; k - \frac{1}{2}\right) \qquad (k \in \mathbb{Z}).$$

To obtain (5.100), shift the line of integration in (5.91) to $\operatorname{Re} s = Q$, where $2P - 1 < Q < k - 1 + 2P$. Using Stirling's formula and choosing Q appropriately it is seen that (5.100) holds. Also if $|u| < P$, $|v| < P$, $|w| < P$, $|z| < P$ and $|\xi| < P$, then

$$\Xi(u, v, w, z; \xi) \ll |\xi|^{-C} \tag{5.102}$$

for any fixed $C > 0$, and consequently also

$$\Psi(u, v, w, z; \xi) \ll |\xi|^{-C} e^{-\pi|\xi|} \tag{5.103}$$

and

$$\Phi(u, v, w, z; \xi) \ll |\xi|^{-C}. \tag{5.104}$$

Therefore it is seen that the sums over the discrete spectrum in (5.90) admit meromorphic continuation over $\mathbb{C}^4$, and it remains to deal with the continuous spectrum in (5.90) and (5.88). Using (5.94) and (5.95) we have that the term pertaining to the continuous spectrum in (5.90) is equal to

$$\begin{aligned}
I_c(u, v, w, z) := 2(2\pi)^{z-v-3} \int\limits_{-i\infty}^{i\infty} \zeta\left(\frac{u+v+w+z-1}{2} + \xi\right) \\
\zeta\left(\frac{u+v+w+z-1}{2} - \xi\right) \times \zeta\left(\frac{u+v-w-z+1}{2} + \xi\right) \\
\zeta\left(\frac{u+v-w-z+1}{2} - \xi\right) \zeta\left(\frac{v+w-u-z+1}{2} + \xi\right) \\
\zeta\left(\frac{v+w-u-z+1}{2} - \xi\right) \times (2\pi)^{2\xi} \Gamma(1-2\xi) \left\{\cos\left(\frac{\pi}{2}(u-w)\right)\right. \\
\left. - \sin\left(\pi\left(\frac{v-z}{2} + \xi\right)\right)\right\} \frac{\Xi(u, v, w, z; \xi)}{\zeta(1+2\xi)\zeta(2\xi)} d\xi,
\end{aligned}$$

where we assume that u, v, w, z satisfy (5.37), and the path of integration is the imaginary axis. The singularities of the integrand are the poles coming from a) poles of the function Ξ, b) poles of the product of the six zeta-factors, c) poles of $(\zeta(1+2\xi)\zeta(2\xi))^{-1}$, d) poles of $\Gamma(1-2\xi)$. If we assume again that P is large, $|u|, |v|, |w|, |z| < P$ and $\zeta(s) \neq 0$ on the **258** line $|\operatorname{Im} s| = 3P$, then we can replace the line of integration $\operatorname{Re}\xi = 0$ for I_c by the contour which is piecewise linear and is formed by the points $-i\infty$, $-3Pi$, $3P - 3Pi$, $3P + 3Pi$, $3Pi$, $i\infty$. Denote this contour by $\mathscr{L}$. By the residue theorem we obtain a contribution of $0(P \log P)$ residues, all of which are meromorphic functions on $\mathbb{C}^4$. Recalling (5.102), we have that $\Xi(u, v, w, z; \xi)$ decays rapidly if $|u|, |v|, |w|, |z| < P$, which is our case. Hence the integral over $\mathscr{L}$ is a regular function, and since P may be arbitrary this shows that I_c admits meromorphic continuation to $\mathbb{C}^4$. Hence the decomposition formula for $I(u, v, w, z; \Delta)$, given by (5.4) in terms of I_1, I_2, I_3, holds for $(u, v, w, z) \in \mathbb{C}^4$.

We are going now to take the special value

$$(u, v, w, z) = \left(\frac{1}{2} + iT, \frac{1}{2} - iT, \frac{1}{2} + iT, \frac{1}{2} - iT\right) \tag{5.105}$$

in (5.4), or equivalently in the expression (5.90) for I_2 we take $u = v = w = z = \frac{1}{2}$. More precisely, if P_T is the point given by (5.105), then we have to study $I_2(u, v, w, z; \Delta)$ in the vicinity of P_T.

Consider first the contribution of the discrete spectrum in (5.90). The functions H_j are entire, and for $\xi = ix_j$ the functions Ψ and Φ are regular near P_T, since

$$\Xi(u, v, w, z; ix_j) = \int\limits_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\Gamma(\frac{1}{2}(u+v+w+z-1) - s + ix_j)}{\Gamma(\frac{1}{2}(3-u-v-w-z) + s + ix_j)}$$
$$\Gamma(1-u-z+s)\Gamma(1-w-z+s)M(s, z; \Delta)ds,$$

and the right-hand side is regular near P_T. Hence the first term in (5.90) that belongs to the discrete spectrum equals

$$\sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right)(\Psi - \epsilon_j\Phi)(P_T; ix_j),$$

with the obvious abuse of notation. Similarly it follows that the other sum in (5.90) involving the discrete spectrum is equal to

$$\sum_{k=6}^{\infty} \sum_{j \le d_{2k}} \alpha_{j,2k} H_{j,2k}^3 \left(\frac{1}{2}\right) \psi\left(P_T; k - \frac{1}{2}\right).$$

259 To deal with the continuous-spectrum term I_c, suppose first that (u, v, w, s) satisfies (5.37). Then we replace the line of integration $\operatorname{Re} \xi = 0$ in the integral for I_c by the contour $\mathscr{L}$, as defined above. By the residue theorem it follows that

$$I_c(u, v, w, z) = I_c^{(1)}(u, v, w, z) + I_c^{(2)}(u, v, w, z) + \int_{\mathscr{L}}, \tag{5.106}$$

where (ρ denotes complex zeros of $\zeta(x)$)

$$I_c^{(1)}(u, v, w, z) = -2\pi i \sum_{|\operatorname{Im} \rho| < 3P} \operatorname*{Res}_{\xi = \frac{1}{2}\rho}$$

and

$$I_c^{(2)}(u, v, w, z) = -2\pi i \sum \operatorname{Res},$$

the residues being taken here at the points $\xi = \frac{1}{2}(u + v + w + z - 3)$, $\frac{1}{2}(u + v - w - z - 1)$, $\frac{1}{2}(v + w - u - z - 1)$, all of which have positive real parts for suitable u, v, w, z satisfying (5.37). But, as stated before, $\int_{\mathscr{L}}$ provides analytic continuation of I_c, so we may consider (u, v, w, z) to belong to a neighbourhood of P_T. Then we obtain

$$\int_{\mathscr{L}} = I_c^{(3)}(u, v, w, z) - I_c^{(1)}(u, v, w, z) + \int_{-i\infty}^{i\infty}, \tag{5.107}$$

where

$$I_c^{(3)}(u, v, w, z) = 2\pi i \sum \operatorname{Re} s,$$

the residues being now taken at the points $\xi = \frac{1}{2}(3 - u - v - w - z)$, $\frac{1}{2}(w + z - u - v + 1)$, $\frac{1}{2}(u + z - v - w + 1)$, which have positive real parts in a neighbourhood of P_T. Combining (5.106) and (5.107) we obtain

$$I_c(u,v,w,z) = I_c^{(2)}(u,v,w,z) + I_c^{(3)}(u,v,w,z) + \int\limits_{-i\infty}^{i\infty}.$$

The last integral is over the imaginary axis. It is regular at P_T and equals there

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+i\xi)|^2}{|\zeta(1+2i\xi)|^2}(\Psi-\Phi)(P_T;i\xi)d\xi.$$

Now we compute $I_c^{(2)}$ and $I_c^{(3)}$, noting that

$$\Psi(u,v,w,z;\xi) = \Psi(u,v,w,z;-\xi), \Phi(u,v,w,z;\xi) = \Phi(u,v,w,z;-\xi),$$

so that $I_c^{(2)}$ and $I_c^{(3)}$ are of equal absolute value, but of opposite sign. We **260**
insert the preceding estimates in the expression (5.90) for I_2, and recall that by (5.4), (5.6) and (5.7) we have

$$\begin{aligned} I(u,v,w,z;\Delta) = {} & \frac{\zeta(u+v)\zeta(u+z)\zeta(v+w)\zeta(w+z)}{\zeta(u+v+w+z)} \\ & + I_2(u,v,w,z;\Delta) + I_2(v,u,z,w;\Delta) \end{aligned} \tag{5.108}$$

Hence

$$\begin{aligned} I(P_T;\Delta) = {} & F(\dot{T},\Delta) + \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+i\xi)|^2}{|\zeta(1+2i\xi)|^2}\theta(\xi;T,\Delta)d\xi \\ & + \sum_{j=1}^{\infty}\alpha_j H_j^3\left(\frac{1}{2}\right)\theta(x_j;T,\Delta) + \sum_{k=6}^{\infty}\sum_{j\le d_{2k}}\alpha_{j,2k}H_{j,2k}^3\left(\frac{1}{2}\right)\Lambda(k,T;\Delta), \end{aligned} \tag{5.109}$$

where we have (note that we may assume $\epsilon_j = +1$, for if $\epsilon_j = -1$, then $H_j(\frac{1}{2}) = 0$ by (5.54))

$$\theta(\xi;T,\Delta) := 2\,\mathrm{Re}\{(\Psi-\Phi)(P_T;i\xi)\}, \Lambda(k;T,\Delta) := 2\,\mathrm{Re}\left\{\Psi\left(P_T;k-\frac{1}{2}\right)\right\}, \tag{5.110}$$

and $F(T, \Delta)$ is the value at P_T of the function

$$f(u,v,w,z) := \frac{\zeta(u+v)\zeta(u+z)\zeta(v+w)\zeta(w+z)}{\zeta(u+v+w+z)} \tag{5.111}$$

$$+ \frac{\zeta(u+v)\zeta(w+z-1)\zeta(u-w+1)\zeta(v-z+1)}{\zeta(u+v-w-z+2)} M(w+z-1, z; \Delta)$$

$$+ \frac{\zeta(v+w)\zeta(u+z-1)\zeta(w-u+1)\zeta(v-z+1)}{\zeta(v+w-u-z+2)} M(u+z-1, z; \Delta)$$

$$+ \frac{\zeta(u+v)\zeta(w+z-1)\zeta(u-w+1)\zeta(v-z+1)}{\zeta(u+v-w-z+2)} M(w+z-1; w, \Delta)$$

$$+ \frac{\zeta(u+z)\zeta(v+w-1)\zeta(z-v+1)\zeta(u-w+1)}{\zeta(u+z-v-w+2)} M(v+w-1; w, \Delta)$$

$$+ 4\frac{\zeta(u+z)\zeta(v+w-1)\zeta(u-w+1)\zeta(v-z)}{\zeta(2+u+z-v-w)} (\Psi - \Phi)$$

$$\left(u, v, w, z; \frac{1}{2}(v+w-u-z-1)\right)$$

$$+ 4\frac{\zeta(w+z)\zeta(u+v-1)\zeta(w-u+1)\zeta(v-z)}{\zeta(2+wz-u-v)} (\Psi - \Phi)$$

$$\left(u, v, w, z; \frac{1}{2}(u+v-w-z-1)\right)$$

$$+ 4\frac{\zeta(2-u-z)\zeta(2-w-z)\zeta(u+v-1)\zeta(v+w-1)}{\zeta(4-u-v-w-z)} (\Psi - \Phi)$$

$$\left(u, v, w, z; \frac{1}{2}(u+v+w+z-3)\right)$$

$$+ 4\frac{\zeta(v+w)\zeta(u+z-1)\zeta(v-z+1)\zeta(u-w)}{\zeta(2+v+w-u-z)} (\Psi - \Phi)$$

$$\left(v, u, z, w; \frac{1}{2}(u+z-v-w-1)\right)$$

$$+ 4\frac{\zeta(w+z)\zeta(u+v-1)\zeta(z-v+1)\zeta(u-w)}{\zeta(2+w+z-u-v)} (\Psi - \Phi)$$

$$\left(v, u, z, w; \frac{1}{2}(u+v-w-z-1)\right)$$

$$+ 4\frac{\zeta(2-v-w)\zeta(2-w-z)\zeta(u+v-1)\zeta(u+z-1)}{\zeta(4-u-v-w-z)} (\Psi - \Phi)$$

$$\left(v, u, z, w; \frac{1}{2}(u+v+w+z-3)\right).$$

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This expression contains 11 terms, of which the first is the zeta-function term in (5.108), and the others (denoted by I to X) are formed as follows. I and II are the first two terms (without $-iT$ in $M(\cdot)$) on the right-hand side of (5.90), and III, IV come form I, II by changing u to v and w to z, as is clear from (5.108). The terms V, VI, VII come from the residues of $I_c^{(2)}$ at the points $\frac{1}{2}(v+w-u-z-1)$, $\frac{1}{2}(u+v-w-z-1)$, $\frac{1}{2}(u+v+w+z-3)$, respectively. Finally VIII, IX, X come from V, VI, VII by changing again u to v and w to z. What remains is the technical task to evaluate $f(u, v, w, z)$ at the point P_T. It will be shown that, with the obvious abuse of notation, there are absolute constants $c_0, \ldots, c_{11}$ such that

$$\begin{aligned} F(T, \Delta) = H_0(T, \Delta) + c_0 + \operatorname{Re}\Bigg((\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} \{c_1 \frac{\Gamma'}{\Gamma} + \qquad\qquad (5.112) \\ + c_2 \frac{\Gamma''}{\Gamma} + c_3 \left(\frac{\Gamma'}{\Gamma}\right)^2 + c_4 \frac{\Gamma'''}{\Gamma} + c_5 \frac{\Gamma'\Gamma''}{\Gamma^2} + c_6 \left(\frac{\Gamma'}{\Gamma}\right)^3 + c_7 \frac{\Gamma^{(4)}}{\Gamma} + c_8 \left(\frac{\Gamma''}{\Gamma}\right)^2 \\ c_9 \frac{\Gamma'\Gamma'''}{\Gamma^2} + c_{10} \left(\frac{\Gamma'}{\Gamma}\right)^4 + c_{11} \frac{(\Gamma')^2\Gamma''}{\Gamma^2} \} \left(\frac{1}{2} + iT + it\right) e^{-(t/\Delta)^2} dt\Bigg) \\ = H_0(T, \Delta) + F_0(T, \Delta), \end{aligned}$$

say. We already simplified $F_0(T, \Delta)$ in (5.11), and it will turn out that

$$H_0(T, \Delta) \ll T^{-1} \log^2 T + e^{-(T/\Delta)^2} \qquad (5.113)$$

for any $T, \Delta > 0$, so that the contribution of $H_0(T, \Delta)$ is negligible. 262

In the neighbourhood of P_T the points $\xi = \frac{1}{2}(v+w-u-z-1)$ etc. in $V - X$ in (5.11) are close to $-\frac{1}{2}$. Thus the integrals (5.84) and (5.89), which define Ψ and ϕ, cannot be used, since the path of integration cannot be drawn. For Ψ the singularity comes from the factor $\Gamma(\frac{1}{2}(u+v+w+z) - s + \xi)$, which has a simple pole at $s = \frac{1}{2}(u+v+w+z) + \xi$. We make a small indentation around the path in (5.84), so that the last point lies to the left of the indented contour, and then use the theorem of residues, moving eventually the contour to $\operatorname{Re} s = \frac{1}{4}$. This can be done for Φ also, and therefore when (u, v, w, z) is in the neighbourhood of P_T and ξ is close to $-\frac{1}{2}$ we have

$$
\begin{aligned}
(\Psi - \Phi)(u, v, w, z; \xi) &= H(u, v, w, z; \xi) + (2\pi)^{z-v-1}\left\{\cos\left(\frac{\pi u - \pi w}{2}\right)\right. \\
&\left. - \cos\left(\frac{\pi}{2}(z - v + 1) - \pi\xi\right)\right\}\cos(\pi\xi)\Gamma(-2\xi)\Gamma\left(\frac{1 + v + w - u - z}{2} + \xi\right) \\
&\Gamma\left(\frac{1 + u + v - w - z}{2} + \xi\right) M\left(\frac{u + v + w + z - 1}{2} + \xi, z; \Delta\right) \qquad (5.114)
\end{aligned}
$$

where

$$
\begin{aligned}
H(u, v, w, z; \xi) := -i(2\pi)^{z-v-2}\cos\left(\frac{\pi u - \pi w}{2}\right) \\
\times \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \sin\left(\frac{\pi}{2}(u + v + w + z) - \pi s\right) \\
\times \Gamma\left(\frac{u + v + w + z - 1}{2} - s + \xi\right) \\
\times \Gamma\left(\frac{u + v + w + z - 1}{2} s - \xi\right)\Gamma(1 - u - z + s) \\
\times \Gamma(1 - w - z + s) M(s, z; \Delta)\, ds \\
+ i(2\pi)^{z-v-2}\cos(\pi\xi) \\
\times \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \cos\left(\pi z + \frac{\pi}{2}(u + w) - \pi s\right) \\
\times \Gamma\left(\frac{u + v + w + z - 1}{2} s + \xi\right) \\
\times \Gamma\left(\frac{u + v + w + z - 1}{2} s - \xi\right)\Gamma(1 - u - z + s) \\
\times \Gamma(1 - w - z + s) < (s, z; \Delta)\, ds. \qquad (5.115)
\end{aligned}
$$

In (5.114) we may set $\xi = \frac{1}{2}(v + w - u - z - 1)$ etc., and then insert the resulting expression in $V - X$ of (5.111). Then we set

$$(u, v, w, z) = P_T + (\delta_1, \delta_2, \delta_3, \delta_4),$$

or

$$u = \frac{1}{2} + iT + \delta_1, v = \frac{1}{2} - iT + \delta_2, w = \frac{1}{2} + iT + \delta_3, z = \frac{1}{2} - iT + \delta_4,$$

where the δ's are small in absolute value, and will eventually tend to **263**
zero. In the process we take into account that

$$M(r, s; \Delta) = \Gamma(r)(\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} \frac{\Gamma(s - r + it)}{\Gamma(s + it)} e^{-(t/\Delta)^2} dt \quad (\operatorname{Re} s > \operatorname{Re} r), \qquad (5.116)$$

so that the gamma-factor terms are introduced. The resulting expression for

$$f\left(\frac{1}{2} + iT + \delta_1, \frac{1}{2} - iT + \delta_2, \frac{1}{2} + iT + \delta_3, \frac{1}{2} - iT + \delta_4\right)$$

is cumbersome. It contains 11 terms without H (of which all but the first contain integrals with the gamma-function), plus six terms with H. The simplification of the expression in question is tedious, albeit straightforward. Eventually one obtains (5.112), with $H_0(T, \Delta)$ coming from all the terms containing the H-functions. Putting (5.116) in (5.115) it is seen that at the points $P_T + (\delta_1, \delta_2, \delta_3, \delta_4)$ and $\xi = \frac{1}{2}(\delta_2 + \delta_3 - \delta_1 - \delta_4 - 1)$ the function H may be expressed as a double integral, whose estimation leads to (5.113). By restricting finally Δ to the range $0 < \Delta \leq T/\log T$ we obtain (5.10), so that Theorem 5.1 is proved.

5.5 Deductions from the explicit formula

Theorem 5.1 provides a precise expression for the integral in (5.10), but the right-hand side contains the functions Θ and Λ whose asymptotic behaviour is not simple. Thus in this section we shall give a detailed deduction of Theorem 5.2 from Theorem 5.1. The result is the asymptotic formula (5.12), which is quite explicit, although the error term is not sharp. The same type of analysis with appropriate modifications, which will be indicated, permits also the deduction of Theorem 5.3, which contains the asymptotic formulas (5.13) and (5.14), valid for different ranges of V.

To begin with, note that by (5.94) and (5.95), for real ξ we have

$$\psi(P_T; i\xi) = \frac{1}{8\pi sh(\pi\xi)}(\Xi(P_T; i\xi) - \Xi(P_T; -i\xi)) \tag{5.117}$$

and

$$\Phi(P_T; i\xi) = \frac{i}{8\pi}(\Xi(P_T; i\xi) + \Xi(P_T; -i\Xi)), \tag{5.118}$$

264 where by (5.91)

$$\Xi(P_T; i\xi) = \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\Gamma(\frac{1}{2} - s + i\xi)}{\Gamma(\frac{1}{2} + s + i\xi)}\Gamma^2(s)M\left(s, \frac{1}{2} - iT; \Delta\right)ds. \tag{5.119}$$

Moreover by (5.101) we have, for $k = 6, 7, \ldots$

$$\Psi\left(P_T; k - \frac{1}{2}\right) = \frac{(-1)^{k+1}}{4\pi} \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\Gamma(k-s)}{\Gamma(k+s)}\Gamma^2(s)M\left(s, \frac{1}{2} - iT; \Delta\right)ds. \tag{5.120}$$

To estimate $\Xi(P_T; i\xi)$ in (5.119) and $\Psi(P_T; k - \frac{1}{2})$ in (5.120) we need a bound for $M(s, \frac{1}{2} - iT; \Delta)$. In (5.24) we replace the ray of integration and write

$$M\left(s, \frac{1}{2} - iT; \Delta\right) = \int_0^{\infty \exp(iT - 1\cdot gn(Ims))} x^{s-1}(1+x)^{-\frac{1}{2}+iT} \tag{5.121}$$

$$\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx \ll e^{-|Ims|/TM(\mathrm{Re}\, s, \frac{1}{2}; \Delta)} \ll e^{-|Ims|/T_\Delta - \mathrm{Re}\, s},$$

where the $\ll$-constant depends only on $\mathrm{Re}\, s$, since by a change of variable we obtain, for $\mathrm{Re}\, s > 0$,

$$M\left(\mathrm{Re}\, s, \frac{1}{2}; \Delta\right) = \Delta^{-\mathrm{Re}\, s} \int_0^\infty u^{\mathrm{Re}\, s - 1}\left(1 + \frac{u}{\Delta}\right)^{-\frac{1}{2}}$$

$$\exp\left(-\frac{\Delta^2}{4}\log^2\left(1+\frac{u}{\Delta}\right)\right) \ll \Delta^{-\operatorname{Re} s},$$

the last integral above being bounded.

By shifting the line of integration integration in (5.120) sufficiently to the right and using (5.121) it is seen that for any fixed $C_1 > 0$

$$\Psi\left(P_T; k-\frac{1}{2}\right) \ll \begin{cases} \Delta^{-k} & \text{if } k \le C, \\ (k\Delta)^{-C_1} & \text{if } k > C, \end{cases} \tag{5.122}$$

where C depends on C_1. Therefore we have, uniformly for $0 < \Delta < \dfrac{T}{\log T}$,

$$\sum_{k=6}^{\infty}\sum_{j\le d_{2k}} \alpha_{j,2k} H^3_{j,2k}\left(\frac{1}{2}\right)\Lambda(k; T, \Delta) \ll \Delta^{-6}. \tag{5.123}$$

Next we indicate how one obtains

$$\Xi(P_T; i\xi) \ll |\xi|^{-C} \tag{5.124}$$

for any fixed $C > 0$ if $T^\delta \le \Delta \le T$ for any fixed $0 < \delta < 1$, $|\xi| \ge$ 265
$C_1 T\log T$ with $C_1 > 0$ a sufficiently large constant. To see this we move the line of integration in (5.119) to $\operatorname{Re} s = m + \dfrac{1}{4}$ for a large integer m. Thus by the residue theorem

$$\begin{aligned}\Xi(P_T; i\xi) &= 2\pi i \sum_{k=0}^{m-1} \frac{(-1)^k\Gamma^2(k+\frac{1}{2}+i\xi)}{\Gamma(k+1)\Gamma(k+1+2i\xi)} M\left(k+\frac{1}{2}+i\xi, \frac{1}{2}-iT; \Delta\right) \\ &+ \int\limits_{m+\frac{1}{4}-i\infty}^{m+\frac{1}{4}+i\infty} \frac{\Gamma(\frac{1}{2}-s+i\xi)}{\Gamma(\frac{1}{2}+s+i\xi)}\Gamma^2(s) M\left(s, \frac{1}{2}-iT; \Delta\right) ds.\end{aligned}$$

On using (5.121) and Stirling's formula (5.124) then follows. Therefore we have shown that, for $T^\delta \le \Delta \le T/\log T$,

$$(\Delta\sqrt{\pi})^{-1}\int\limits_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iT+it\right)\right|^4 e^{-(t/\Delta)^2}\,dt \tag{5.125}$$

$$= F_0(T,\Delta) + \frac{1}{\pi} \int\limits_{-CT\log T}^{CT\log T} \frac{|\zeta(\frac{1}{2}+i\xi)|^2}{|\zeta(1+2i\xi)|^2} \theta(\xi;T,\Delta)d\xi$$

$$+ \sum_{x_j \le CT\log T} \alpha_j H_j^3\left(\frac{1}{2}\right)\theta(x_j;T,\Delta) + o(\Delta^{-6}) + o(T^{-1}\log^2 T).$$

We want to show that in (5.125) both the sum and the integral can be further truncated with a small error. To this end we transform $\Xi(P_T;i\xi)$ by noting that, for $|\xi| \ll T\log T$ and $T^\delta \le \Delta \le T/\log T$ ($\delta > 0$ arbitrarily small, but fixed) we have

$$\Xi(P_T;i\xi) = \int\limits_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\Gamma(\frac{1}{2}-\Gamma+i\xi)}{\Gamma(\frac{1}{2}+\Gamma+i\xi)}\Gamma^2(r)$$

$$\int\limits_0^\infty x^{r-1}(1+x)^{-\frac{1}{2}+iT} e^{-\frac{1}{4}\Delta^2\log^2(1+x)}dx\, dr \tag{5.126}$$

$$= \int\limits_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \ldots \int\limits_0^{\frac{1}{2}} \ldots dx\, dr + o(e^{-\Delta^2/100})$$

$$= \Xi^o(P_T;i\xi) + o(e^{-\Delta^2/100}),$$

say. We have

$$\Xi^o(P_T;i\xi) := \int\limits_0^{\frac{1}{2}} x^{-1}(1+x)^{-\frac{1}{2}+iT} \tag{5.127}$$

$$\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)\left\{\int\limits_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\Gamma(\frac{1}{2}-r+i\xi)}{\Gamma(\frac{1}{2}+r+i\xi)}\Gamma^2(r)x^r dr\right\}dx$$

$$= 2\pi i \int\limits_0^{\frac{1}{2}} R(x,\xi)x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx,$$

where **266**

$$R(x,\xi) := \int_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi}(1+xy)^{-\frac{1}{2}-i\xi}dy. \tag{5.128}$$

The proof of the identity (5.127), with $R(x,\xi)$ given by (5.128), is equivalent to showing that

$$\begin{aligned}(2\pi i)^{-1} &\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \frac{\Gamma(\frac{1}{2}-r+i\xi)}{\Gamma(\frac{1}{2}+r+i\xi)}\Gamma^2(r)x^r dr \\ &= x^{\frac{1}{2}+i\xi}\int_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi}(1+xy)^{-\frac{1}{2}-i\xi}dy.\end{aligned} \tag{5.129}$$

The right-hand side of (5.129) may be written as $x^{\frac{1}{2}+i\xi}I(-x)$, where

$$I(x) := \int_0^1 z^{\alpha-1}(1-z)^{\gamma-\alpha-1}(1-zx)^{-\beta}dz$$

with $\alpha = \beta = \frac{1}{2} + i\xi$, $\gamma = 1 + 2i\xi$. Setting

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1),\ (a)_0 = 1,$$

we have, for $|x| < 1$,

$$\begin{aligned} I(x) &= \int_0^1 z^{\alpha-1}(1-z)^{\gamma-\alpha-1}\sum_{k=0}^{\infty}\binom{-\beta}{k}(-zx)^k dz \\ &= \sum_{k=0}^{\infty}\left\{\frac{(\beta)_k}{k!}\int_0^1 z^{\alpha+k-1}(1-z)^{\gamma-\alpha-1}dz\right\}x^k \\ &= \sum_{k=0}^{\infty}\left\{\frac{(\beta)_k}{k!}\cdot\frac{\Gamma(\alpha+k)\Gamma(\gamma-\alpha)}{\Gamma(\gamma+k)}\right\}x^k = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F(\alpha,\beta;\gamma;x),\end{aligned} \tag{5.130}$$

where

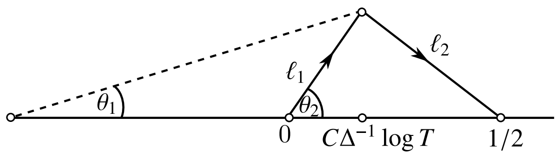
$$F(\alpha, \beta, \gamma; x)\cdot = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} x^k$$

267 is the hypergeometric function. Now on the left-hand side of (5.129) we shift the line of integration to $\operatorname{Re} r = N + \dfrac{1}{4}$, where N is a large integer. There are simple poles of the integrand at $r = \frac{1}{2} + i\xi + n$, $n = 0, 1, \ldots, N$. By using the identity

$$\Gamma(s + n) = s(s + 1)\ldots(s + n - 1)\Gamma(s)$$

and the fact that $(-1)^n/n!$ is the residue of $\Gamma(s)$ at $s = -n$ we see that, on letting $N \to \infty$, the left-hand side of (5.129) equals the right-hand side of (5.130).

In (5.127) we replace the segment of integration $[0, \frac{1}{2}]$ by $\ell_1 \cup \ell_2$, as follows:



$H \in e$ $\theta_1 = (T \log T)^{-1}$, $\theta_2 \asymp \Delta(T \log^2 T)^{-1}$, $C > 0$ is a large constant and $\xi > 0$ (the case $\xi < 0$ is similar). On ℓ_1 we have $x^{i\xi} \ll \exp(-C\xi\Delta/(T \log^2 T))$, and on ℓ_2 we have $|x| \geq C\Delta^{-1} \log T$. Hence we obtain, for some $C' > 0$,

$$\Xi^{\circ}(P_T; i\xi) \ll \exp\left(-C' \frac{|\xi|\Delta}{T \log^2 T}\right) + \exp(-C' \log^2 T), \tag{5.131}$$

providing $|\xi| \leq C_1 T \log T$. Therefore

$$(\Delta \sqrt{\pi})^{-1} \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + iT + it\right)\right|^4 e^{-(t/\Delta)^2} dt \tag{5.132}$$

$$= F_0(T, \Delta) + \frac{1}{\pi} \int_{-CT\Delta^{-1} \log^3 T}^{CT\Delta^{-i} \log^3 T} \frac{|\zeta(\frac{1}{2} + i\xi)|^6}{|\zeta(1 + 2i\xi)|^2} \theta(\xi; T, \Delta) d\xi$$

$$+ \sum_{x_j \le CT\Delta^{-1}\log^3 T} \alpha_j H_j^3\left(\frac{1}{2}\right)\theta(x_j; T, \Delta) + o(T^{-1}\log^2 T),$$

and henceforth we restrict Δ to the range

$$T^{\frac{1}{2}}\log^{-A} T \le \Delta \le T\exp(-\sqrt{\log T}), \tag{5.133}$$

with $A > 0$ an arbitrary, but fixed constant. One can consider also the 268
values of Δ outside this range, but for all practical purposes the range (5.133) appears to be sufficient. We proceed now to evaluate

$$\Xi^\circ(P_T; i\xi) = 2\pi i \int_0^1 \int_0^{\frac{1}{2}} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}(1+xy)^{-\frac{1}{2}-i\xi} y^{-\frac{1}{2}+i\xi}$$
$$(1-y)^{-\frac{1}{2}+i\xi} e^{-\frac{1}{4}\Delta^2\log^2(1+x)} dx\, dy \tag{5.134}$$

by using the following procedure. For $|\xi| \le \log^{3A} T$ we integrate first over y (simplifying $(1+xy)^{-\frac{1}{2}-i\xi}$ by Taylor's formula), getting essentially a beta-integral, that is,

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (\operatorname{Re} p > 0,\ \operatorname{Re} q > 0). \tag{5.135}$$

For the remaining range $\log^{3A} T \le |\xi| \le CT\Delta^{-1}\log^3 T$ we use the saddle-point method, evaluating first the integral over x, and then integrating over y.

Thus we begin the evaluation of $\Xi^\circ(P_T; i\xi)$, supposing $|\xi| \le \log^{3A} T$. By (5.134) we have, with some $c > 0$,

$$\Xi^\circ(P_T; i\xi) = O(e^{-clog^2 T}) + 2\pi i \int_0^{\Delta^{-1}\log T} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}$$
$$\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right) dx \int_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi}(1+xy)^{-\frac{1}{2}-i\xi} dy$$

$$= 2\pi i \int\limits_0^{\Delta^{-1}\log T} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx$$

$$\int\limits_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi}dy + 2\pi i \int\limits_0^{\Delta^{-1}\log T} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}$$

$$x\left(-\frac{1}{x}-i\xi\right)\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx \int\limits_0^1 y^{\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi}dy$$

$$+ O\left(\int\limits_0^{\Delta^{-1}\log T} x^{3/2}\log^{6A} T \cdot dx\right) + O(e^{-c\log^2 T}) = I_1 + I_2 + O(T^{-1}),$$

say. We have

$$I_1 = 2\pi i \int\limits_0^{\Delta^{-1}\log T} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}$$

$$\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx \int\limits_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi}dy$$

$$= 2\pi i \frac{\Gamma^2(\frac{1}{2}+i\xi)}{\Gamma(1+2i\xi)} \int\limits_0^\infty x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}$$

$$\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx + O(e^{-c\log^2 T}).$$

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Next we use

$$\int\limits_{-\infty}^{\infty} \exp(\alpha x - \beta x^2)dx = \sqrt{\frac{\pi}{\beta}}\exp\left(\frac{\alpha^2}{4\beta}\right) \quad (\mathrm{Re}\,\beta > 0)$$

to write the last integral above as

$$(\Delta\sqrt{\pi})^{-1}\int_0^\infty x^{-\frac12+i\xi}(1+x)^{-\frac12+iT}\left(\int_{-\infty}^{\infty}(1+x)^{iu}e^{-(u/\Delta)^2}du\right)dx$$

$$=(\Delta\sqrt{\pi})^{-1}\int_0^1 v^{-\frac12+i\xi}(1-v)^{-1-i\xi-iT}\int_{-\infty}^{\infty}(1-v)^{-iu}e^{-(u/\Delta)^2}du\,dv$$

after change of variable $x/(1+x)=v$. This may be further written as

$$(\Delta/\sqrt{\pi})^{-1}\lim_{\alpha\to-1+0}\int_0^1 v^{-\frac12+i\xi}(1-v)^{\alpha-i\xi-iT}\int_{-\infty}^{\infty}(1-v)^{-iu}e^{-(u/\Delta)^2}du\,dv$$

$$=(\Delta\sqrt{\pi})^{-1}\lim_{\alpha\to-1+0}\int_{-\Delta\log T}^{\Delta\log T}e^{-(u/\Delta)^2}du$$

$$\int_0^1 v^{-\frac12+i\xi}(1-v)^{\alpha-i\xi-iT-iu}dv+O(e^{-c\log^2 T})$$

$$=(\Delta\sqrt{\pi})^{-1}\lim_{\alpha\to-1+0}\int_{-\Delta\log T}^{\Delta\log T}e^{-(u/\Delta)^2}\cdot$$

$$\frac{\Gamma(\frac12+i\xi)\Gamma(\alpha+1-i\xi-iT-iu)}{\Gamma(\alpha+\frac32-iT-iu)}du+O(e^{-c\log^2 T})$$

$$=(\Delta\sqrt{\pi})^{-1}\int_{-\Delta\log T}^{\Delta\log T}e^{-(u/\Delta)^2}\cdot\frac{\Gamma(\frac12+i\xi)\Gamma(-\xi-iT-iu)}{\Gamma(\frac12-iT-iu)}du+O(e^{-c\log^2 T}),$$

where uniform convergence was used and the fact that $\xi+T+u\neq 0$ for $|u|\le\Delta\log T$, $|\xi|\le\log^{3A}T$. By Stirling's formula, in the form stated after (3.18), we find that

$$\frac{\Gamma(\frac12+i\xi)\Gamma(-i\xi-iT-iu)}{\Gamma(\frac12-iT-iu)}=\Gamma\left(\frac12+i\xi\right)(T+u+\xi)^{-\frac12}$$

$$\exp\left\{-\frac{\pi}{2}\xi + \frac{\pi i}{4} - i\xi\log(T+u) + O\left(\frac{|\xi|}{T}\right)\right\}\cdot\left(1+O\left(\frac{1}{T}\right)\right).$$

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Since

$$\log(T+u) = \log T + \frac{u}{T} + O\left(\frac{\Delta^2\log^2 T}{T^2}\right),$$
$$(T+u+\xi)^{-\frac{1}{2}} = T^{-\frac{1}{2}} + O(T^{-3/2}\Delta\log T)$$

and $\Delta \le T\exp(\sigma\sqrt{\log T})$, it follows that

$$\frac{\Gamma(\frac{1}{2}+i\xi)\Gamma(-i\xi-iT-iu)}{\Gamma(\frac{1}{2}-iu-iT)} = \Gamma\left(\frac{1}{2}+i\xi\right)$$
$$e^{\frac{1}{4}\pi i-\frac{1}{2}\pi\xi}T^{-\frac{1}{2}-i\xi}\cdot\left\{1+0\left(\exp\left(-\frac{1}{2}\sqrt{\log T}\right)\right)\right\}.$$

This means that the last expression is negligible if $\xi > 0$ and ξ is large, in which case $\Gamma(\frac{1}{2}+i\xi) \asymp \exp(-\frac{1}{2}\pi\xi)$. Therefore

$$I_1 = 2\pi i\frac{\Gamma^3(\frac{1}{2}+i\xi)}{\Gamma(1+2i\xi)}e^{\frac{1}{4}\pi i-\frac{1}{2}\pi\xi}T^{-\frac{1}{2}-i\xi}\left\{1+0\left(\exp\left(-\frac{1}{2}\sqrt{\log T}\right)\right)\right\},$$

and by the same method it is established that

$$I_2 \ll T^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\sqrt{\log T}\right).$$

Hence uniformly for $|\xi| \le \log^{3A} T$ we have

$$\Xi^\circ(P_T; i\xi) = 2\pi i e^{\frac{1}{4}\pi i-\frac{1}{2}\pi\xi}\frac{\Gamma^3(\frac{1}{2}+i\xi)}{\Gamma(1+2i\xi)}T^{-\frac{1}{2}-i\xi}\left\{1+O\left(\exp\left(-\frac{1}{2}\sqrt{\log T}\right)\right)\right\}. \tag{5.136}$$

It may be remarked that the expression in (5.136) can be integrated in applications whenever this is needed. Now we suppose additionally

$$D < |\xi| \le \log^{3A} T, \tag{5.137}$$

where $D > 0$ is a large constant. Then we use again Stirling's formula to obtain

$$\Xi^\circ(P_T; i\xi) \ll T^{-\frac{1}{2}} e^{-\pi\xi} \quad (\xi > 0), \tag{5.138}$$

and

$$\Xi^\circ(P_T; i\xi) = -4\pi^2 2^{-\frac{1}{2}} (T|\xi|)^{-\frac{1}{2}} \exp(i\xi \log|\xi| - i\xi \log(4eT)) \cdot \left(1 + O\left(\frac{1}{|\xi|}\right)\right) \tag{5.139}$$

for $\xi < 0$. Finally from (5.138) and (5.139) we can obtain an asymptotic formula for $\theta(\xi; T, \Delta)$ when (5.137) holds. On using (5.117) and (5.118) it is seen that

$$\begin{aligned}
\theta(\xi; T, \Delta) &= 2\,\mathrm{Re}\{\Psi(P_T; i\xi) - \Phi(P_T; i\xi)\} \qquad (5.140)\\
&= \frac{1}{4\pi} \mathrm{Re}\left\{\left(\frac{1}{sh(\pi\xi)} - i\right)\Xi(P_T; i\xi) - \left(\frac{1}{sh(\pi\xi)} + i\right)\Xi(P_T; -i\xi)\right\}\\
&= \frac{1}{4\pi} \mathrm{Im}\left\{\Xi^\circ(P_T; i\xi) + \Xi^\circ(P_T; -i\xi)\right\} + O(T^{-\frac{1}{2}}|\xi|^{-1}).
\end{aligned}$$

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Thus $\theta(\xi; T, \Delta) \ll |\xi|^{-\frac{1}{2}} T^{-\frac{1}{2}}$ in the continuous spectrum, making the total contribution of the continuous spectrum $\ll T^{-\frac{1}{2}} \log^C T$ for $|\xi| \le \log^{3A} T$. On the other hand, in the discrete spectrum one has to take $\xi > 0$, so that the main contribution to $\theta(x_j; T, \Delta)$ (with error which is $0(T^{-\frac{1}{2}} x_j^{-1})$) will be

$$\begin{aligned}
&\frac{1}{4\pi} \mathrm{Im}\{\Xi^\circ(P_T; -ix_j)\}\\
&= -\pi(2Tx_j)^{-\frac{1}{2}} \sin\left(x_j \log \frac{4eT}{x_j}\right) = \pi(2Tx_j)^{-\frac{1}{2}} \sin\left(x_j \log \frac{x_j}{4eT}\right).
\end{aligned}$$

Then we can, in view of $\Delta \le T \exp(-\sqrt{\log T})$, $x_j \le \log^{3A} T$, insert the exponential factor $\exp(-(\Delta x_j/2T)^2)$ in the relevant term in (5.132), making a negligible error. This is done, because these (harmless) factors are present in the series appearing in Theorem 5.2 and Theorem 5.3. Hence it remains to consider the ranges $|\xi| \le D$ and $\log^{3A} T \le |\xi| \le CT\Delta^{-1} \log^3 T$. In the range $|\xi| \le D$ we trivially have $\Xi^\circ(P_T; i\xi) \ll T^{-\frac{1}{2}}$, so that the contribution to both the continuous and discrete spectrum is $\ll T^{-\frac{1}{2}}$, more than required by Theorem 5.2.

We turn now to the range

$$\log^{3A} T \leq |\xi| \leq CT\Delta^{-1} \log^3 T, \tag{5.141}$$

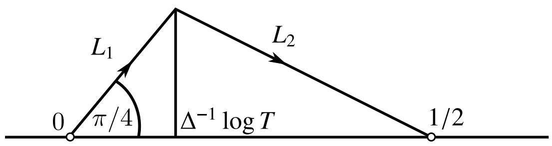
provided that (5.133) holds. We rewrite (5.134) as

$$\Xi^\circ(P_T; i\xi) = 2\pi i \int\limits_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi} S(y;\xi)dy, \tag{5.142}$$

where

$$S(y;\xi) := \int\limits_0^{\frac{1}{2}} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}(1+xy)^{-\frac{1}{2}-i\xi} e^{-\frac{1}{4}\Delta^2 \log^2(1+x)} dx, \tag{5.143}$$

and proceed to evaluate by the saddle-point method first $S(y;\xi)$, and
272 then Ξ° itself. First we show that, in case $\xi > 0$ and (5.141) holds, the contribution of $S(y;\xi)$ is negligible uniformly in y. Namely we replace the segment of integration $[0, \frac{1}{2}]$ by $L_1 \cup L_2$, as follows:



On L_1 we have $x^{i\xi} \ll \exp\left(-\frac{1}{4}\pi\xi\right)$, hence the integral over L_1 is $\ll \exp\left(-\dfrac{\pi}{8}\xi\right)$. On L_2 we have $|x| \gg \Delta^{-1} \log T$, consequently we obtain, for $\xi > 0$,

$$\Xi^\circ(P_T; i\xi) \ll \exp\left(-\frac{\pi}{8}\xi\right) + \exp(-C_1 \log^2 T). \tag{5.144}$$

In case $\xi < 0$ write

$$S(y;\xi) = \int\limits_0^{\frac{1}{2}} x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \exp\left(-\frac{\Delta^2}{4} \log^2(1+x)\right) \exp(if(x))dx,$$

where

$$f(x) = f(x;\xi,T,y) := \xi\log x + T\log(1+x) - \xi\log(1+xy),$$

so that

$$f'(x) = \frac{\xi}{x} + \frac{T}{1+x} - \frac{\xi y}{1+xy}, f''(x) = -\frac{\xi}{x^2} - \frac{T}{(1+x)^2} + \frac{\xi y^2}{(1+xy)^2}.$$

The saddle point is the solution of the equation $f'(x_\circ) = 0$, giving

$$\begin{aligned} x_\circ &= \frac{2|\xi|}{T - |\xi| + ((T-|\xi|)^2 + 4Ty\xi)^{\frac{1}{2}}} \\ &= \frac{|\xi|}{T-|\xi|}\left(1 - \frac{y|\xi|}{T} + O\left(\frac{\xi^2}{T^2}\right)\right) \sim \frac{|\xi|}{T}. \end{aligned} \tag{5.145}$$

We can now evaluate $S(y;\xi)$ by a general result on exponential integrals, like Theorem 2.3. This will be sufficient for the proof of Theorem 5.2 and the first part of Theorem 5.3. On the other hand, we may use a special contour, suited to the particular structure of the integral $S(y;\xi)$, and evaluate $S(y;\xi)$ from first principles. The latter approach, which is naturally more delicate, leads to slightly better error terms and enables one to deduce the second part of Theorem 5.3.

Thus we shall first use Theorem 2.3, but before it is applied we shall truncate $S(y;\xi)$. So consider **273**

$$\begin{aligned} &\int\limits_0^{T^{-1}} x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)\exp(if(x))dx \\ &= \sum_{k=0}^{\infty}\int\limits_{T^{-1}2^{-k-1}}^{T^{-1}2^{-k}}\ldots = \sum_{k=0}^{\infty} I(k), \end{aligned}$$

say. In each $I(k)$ we have $f'(x) \gg |\xi|/x$, hence by Lemma 2.1

$$I(k) \ll (T^{-1}2^{-k})^{-\frac{1}{2}}(|\xi|T2^k)^{-1} \ll |\xi|^{-1}T^{-\frac{1}{2}}2^{-\frac{1}{2}}k,$$

and consequently

$$\int\limits_0^{T^{-1}} x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)\exp(if(x))dx \ll \quad |\xi|^{-1}T^{-\frac{1}{2}}$$

Also we have trivially

$$\int\limits_{|\xi|^{1/2}(10T)^{-1/2}}^{\frac{1}{2}} x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)\exp(if(x))dx$$
$$\ll \exp\left(-\frac{\Delta^2|\xi|}{100T}\right) \ll \exp\left(-\frac{|\xi|\log^{-2A} T}{100}\right) \ll \exp\left(-\frac{\log^A T}{100}\right)$$

in view of (5.141), and thus the integral is negligible if $A > 1$, which may be assumed. In the remaining integral

$$I := \int\limits_{T^{-1}}^{|\xi|^{1/2}(10T)^{-1/2}} x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)\exp(if(x))dx$$

we have $f'(x) \ll \min(|\xi|x^{-1}, T)$, $f''(x) \gg |\xi|x^{-2}$. To evaluate I we apply Theorem 2.3 with $a = T^{-1}$, $b = |\xi|^{\frac{1}{2}}(10T)^{-\frac{1}{2}}$, $\Phi(x) = x^{-\frac{1}{2}}\exp(-C\Delta^2x^2)$, $\mu(x) = x/10$, $\varphi(x) = x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)$ $F(x) = \min(|\xi|, Tx)$. We have $|f''(z)|^{-1} \ll x^2|\xi|^{-1} \ll \mu^2(x)F^{-1}(x)$ trivially if $F(x) = |\xi|$, and $x^2|\xi|^{-1} \ll x^2T^{-1}x^{-1} \ll \mu^2(x)F^{-1}(x)$ if $x \le |\xi|T^{-1}$. For the error terms we obtain

$$\Phi_a\left(|f_a'| + f''^{\frac{1}{2}}_a\right)^{-1} \ll T^{\frac{1}{2}}(|\xi|T)^{-1} = |\xi|^{-1}T^{-\frac{1}{2}},$$
$$\Phi_b\left(|f_b'| + f''^{\frac{1}{2}}_b\right)^{-1} \ll e^{-c\log^2 T}, \quad \Phi_0\mu_0F_0^{-3/2} \ll |\xi|^{-1}\cdot T^{-\frac{1}{2}},$$

274 and the contribution of the exponential error term in Theorem 2.3 is clearly negligible. Therefore Theorem 2.3 gives

$$I = \sqrt{2\pi}\varphi_0(f_0'')^{-\frac{1}{2}}\exp\left(iT\log(1+x_0) + i\xi\log\frac{x_0}{1+x_0y} + \frac{i\pi}{4}\right)$$

$$+ O\left(|\xi|^{-1}T^{-\frac{1}{2}}\right) = \sqrt{2\pi}T^{-\frac{1}{2}} \exp\left(-\left(\frac{\Delta\xi}{2T}\right)^2\right)$$
$$\exp\left(iT\log(1+x_0) + i\xi\log\frac{x_0}{1+x_0y} + \frac{i\pi}{4}\right) + O\left(|\xi|^{-1}T^{-\frac{1}{2}}\right).$$

Since $\xi < 0$ we also obtain, by using (5.145),

$$T\log(1+x_0) + \xi\log\frac{x_0}{1+x_0y} = \xi\log\frac{|\xi|}{eT} + \left(y - \frac{1}{2}\right)\frac{\xi^2}{T} + O\left(\frac{|\xi|^2}{T^2}\right),$$

which gives

$$I = \sqrt{2\pi}T^{-\frac{1}{2}}\exp\left(-\left(\frac{\Delta\xi}{2T}\right)^2\right)\exp\left(i\xi\log\frac{|\xi|}{eT} + i\left(y-\frac{1}{2}\right)\frac{\xi^2}{T} + \frac{i\pi}{4}\right)$$
$$+ O\left(|\xi|^3T^{-5/2}\right) + O\left(|\xi|^{-1}T^{-\frac{1}{2}}\right).$$

Then

$$\Xi^\circ(P_T; i\xi) = O\left(|\xi|^3T^{-5/2}\right) + O\left(|\xi|^{-1}T^{-\frac{1}{2}}\right) + (2\pi)^{3/2}e^{\frac{3}{4}\pi i}T^{-\frac{1}{2}} \tag{5.146}$$
$$\exp\left(i\xi\log\frac{|\xi|}{eT} - \frac{i\xi^2}{2T}\right)e^{-(\frac{\Delta\xi}{2T})^2} \cdot \int_0^1 (y(1-y))^{-\frac{1}{2}}$$
$$\exp\left\{i\xi\log(y(1-y)) + \frac{i\xi^2}{T}y\right\}dy.$$

The last integral may be truncated at $|\xi|^{-2}$ and $1 - |\xi|^2$ with an error of $O(|\xi|^{-1})$. The remaining integral

$$J := \int_{|\xi|^{-2}}^{1-|\xi|^{-2}} (y(1-y))^{-\frac{1}{2}}\exp(i\bar{f}(y))dy,$$

$$\bar{f}(y) = \bar{f}(y;\xi,T) := \xi\log(y(1-y)) + \frac{\xi^2}{T}y$$

may be evaluated again by Theorem 2.3. We have

$$\bar{f}'(x) = \frac{\xi}{x} - \frac{\xi}{1-x} + \frac{\xi^2}{T},\ \bar{f}''(x) = -\frac{\xi}{x^2} - \frac{\xi}{(1-x)^2},$$

and the saddle point $x_\circ$ satisfies $\bar{f}'(x_\circ) = 0$, whence

$$x_\circ = \left(1 - \frac{\xi}{2T} + \left(1 + \frac{\xi^2}{4T^2}\right)^{\frac{1}{2}}\right)^{-1} = \frac{1}{2} + \frac{\xi}{8T} - \frac{\xi^3}{128T^3} + O\left(\frac{|\xi|^5}{T^5}\right).$$

275 The error terms from Theorem 2.3 will be all $\ll \xi^{-2}$. Thus with $\varphi(x) = (x(1-x))^{-\frac{1}{2}}$ we obtain

$$\begin{aligned} J &= \sqrt{2\pi}\varphi_\circ(\bar{f}_\circ'')\exp\left(if_\circ + \frac{i\pi}{4}\right) + O(\xi^{-2}) \\ &= \pi^{\frac{1}{2}} e^{\frac{1}{4}\pi i} |\xi|^{-\frac{1}{2}} \left\{1 + O\left(|\xi|T^{-1}\right) + O\left(|\xi|^3 T^{-2}\right)\right\} \\ &\qquad \exp\left(-i\xi\log 4 + \frac{i\xi^2}{T}\right) + O(\xi^{-2}). \end{aligned}$$

Inserting the expression for J in (5.146) we obtain

$$\begin{aligned} \Xi^\circ(P_T; i\xi) = -2^{3/2}\pi^2(|\xi|T)^{-\frac{1}{2}} &\exp\left(i\xi\log\frac{|\xi|}{4eT}\right) \\ &\exp\left(\left(-\frac{\Delta\xi}{2T}\right)^2\right)\left(1 + O\left(|\xi|T^{-1}\right) + O\left(|\xi|^3T^{-2}\right)\right) \\ &+ O\left(|\xi|^3 T^{-5/2}\right) + O\left(|\xi|^{-1}T^{-\frac{1}{2}}\right) \quad (\xi < 0), \end{aligned} \tag{5.147}$$

provided that (5.141) holds. The error terms $O(|\xi|T^{-1})$ and $O(|\xi|^3T^{-2})$ in (5.147) will make a negligible contribution. The main term in (5.147) is the same main term as in (5.139), only with the additional factor $\exp(-(\Delta\xi/2T)^2)$. Hence the analysis concerning the main term in (5.147) will be as in the previous case. To deal with the remaining two error terms in (5.147) we shall use the bounds

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^6 dt \ll T^{5/4}\log^{29/4} T \tag{5.148}$$

and

$$\sum_{x_j \le x} \alpha_j \left|H_j^3\left(\frac{1}{2}\right)\right| \ll x^2 \log^C x. \tag{5.149}$$

The last bound (with $C = \frac{1}{2}(B+1)$) follows on using the Cauchy-Schwarz inequality and the bounds

$$\sum_{x_j \le x} \alpha_j H_j^2\left(\frac{1}{2}\right) \ll x^2 \log x, \tag{5.150}$$

$$\sum_{x_j \le x} \alpha_j H_j^4(\frac{1}{2}) \ll x^2 \log^B x. \tag{5.151}$$

Note that (5.150) is a weak consequence of (5.55), while (5.151) has been claimed by N. Kuzmetsov. A proof was found recently by Y. Motohashi, which shows that actually (5.151) holds with $B = 20$. Therefore on using (5.148) it follows that the contribution of the error **276**
terms $O(|\xi|^3 T^{-5/2})$ and $O(|\xi|^{-1} T^{-\frac{1}{2}})$ to the continuous spectrum (i.e. integral) in (5.132) is negligible. The contribution of these error terms to the discrete spectrum (i.e. sum) in (5.132) is, by (5.149) and partial summation,

$$\begin{aligned}
&\ll \sum_{x_j \le CT\Delta^{-1}\log^3 T} \alpha_j \left|H_j^3\left(\frac{1}{2}\right)\right| \left(x_j^3 T^{-5/2} + x_j^{-1} T^{-\frac{1}{2}}\right)\\
&\ll \left(T\Delta^{-1}\log^3 T\right)^5 T^{-5/2}\log^C T + \left(T\Delta^{-1}\log^3 T\right) T^{-\frac{1}{2}}\log^C T\\
&\ll (\log T)^{D(A)}
\end{aligned}$$

with $D(A) = 15 + 5A + C$, providing that (5.137) holds. This established then Theorem 5.2 in view of (5.11).

We pass now to the discussion of the proof of the first part of Theorem 5.3. We integrate Theorem 5.1 over T for $V \le T \le 2V$. First we note that from (5.24) we have

$$\begin{aligned}
&\int_V^{2V} M\left(s, \frac{1}{2} - iT; \Delta\right) dT\\
&= -i \int_0^\infty x^{s-1}(1+x)^{-\frac{1}{2}+iV} \frac{(1+x)^{iV} - 1}{\log(1+x)} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right) dx.
\end{aligned}$$

If $\mathrm{Re}\, s > 1$ and $T^\epsilon \le \Delta \le T\exp(-\sqrt{\log T})$ the right-hand side is

$$\ll \Delta^{1-\mathrm{Re}\, s} e^{-|\,\mathrm{Im}\, s|/V}$$

similarly as in the proof of (5.121). Hence we obtain

$$\int\limits_V^{2V} \Lambda(k; T, \Delta) dT \ll \begin{cases} \Delta^{1-k} & k \le C, \\ (k\Delta)^{-C_1} & k > C, \end{cases}$$

for any fixed $C_1 > 0$. This means that the contribution of the holomorphic part in (5.10) to the integral in (5.13) is negligible. From the formula (see Section 5.1)

$$\frac{\Gamma^{(k)}(s)}{\Gamma(s)} = \sum_{j=0}^{k} b_{j,k}(s) \log^j s + c_{-1,k} s^{-1} + \ldots + c_{-r,k} s^{-r} + O_r\left(|s|^{-r-1}\right)$$

277 it follows that, for a suitable polynomial $R_4(y)$ of degree four in y,

$$\int\limits_V^{2V} F_0(T, \Delta) dT = V R_4(\log V) + O(\Delta \log^5 V),$$

hence both (5.13) and (5.14) will contain the error term $O(\Delta \log^5 V)$.

The proof of (5.13) resembles the foregoing proof, only in (5.134) we integrate for $V \le T \le 2V$, and eventually we shall replace V by $V2^{-j}$ and sum over $j = 1, 2, \ldots$. We have now also to consider the additional factor $i^{-1} \log^{-1}(1+x) \sim i^{-1} x^{-1}$ at $x = x_0$, which appears in the integrals in (5.134) after integration over T. Since $x_0 \sim |\xi| T^{-1}$, the factor $i^{-1} T |\xi|^{-1}$ will be essentially reproduced in the whole analysis, both in the main term and error terms. Thus we shall have, as the analogue of (5.147),

$$\int\limits_V^{2V} \Xi^\circ(P_T; i\xi) dT = O\left(|\xi|^2 V^{-3/2}\right) + O\left(|\xi|^{-2} V^{1/2}\right)$$
$$+ \left| i 2^{3/2} \pi^2 |\xi|^{-3/2} T^{\frac{1}{2}} \exp\left(i\xi \log \frac{|\xi|}{4eT}\right) e^{-(\Delta\xi/2T)^2} \right|_V^{2V}$$

$$\left(1 + O\left(|\xi|V^{-1}\right) + O\left(|\xi|^3 V^{-2}\right)\right),$$

and the factor i accounts for the change of sine into cosine in Theorem 5.3. The error terms above will all contribute a total of $O(V^{\frac{1}{2}} \log^{D(A)} V)$, and the first part of Theorem 5.3 follows, since similar arguments apply to (5.136).

As mentioned earlier, there is a possibility to evaluate $S(y;\xi)$ in (5.143) (for $\xi < 0$ and (5.141) being true) directly by treating it as a complex integral, choosing a suitable contour in the x-plane to replace the segment of integration $[0, \frac{1}{2}]$. Such an evaluation is necessary for the proof of the second part of Theorem 5.3, namely (5.14). This was the approach used originally by Motohashi, who replaced the segment $[0, \frac{1}{2}]$ by $\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$. Here $\ell_1, \ell_2, \ell_3, \ell_4$ denote segments formed by the points $0, \frac{1}{2}x_0(1-i), x_0(1+\epsilon+\epsilon i), \Delta^{-1}\log^2 T + x_0\epsilon i, \frac{1}{2}$ respectively, and $\epsilon > 0$ is a small constant. On ℓ_1 the integrand of $S(y;\xi)$ is, uniformly in y,

$$\begin{aligned}
&\ll |x|^{-\frac{1}{2}} \exp\left(-\frac{\pi}{4}|\xi| + T \arctan \frac{x_0}{2+x_0} + |\xi| \arctan \frac{x_0 y}{2 + x_0 y}\right)\\
&\ll |x|^{-\frac{1}{2}} \exp\left(-\frac{1}{5}|\xi|\right)
\end{aligned}$$

since $x_0 \sim |\xi| T^{-1}$. Hence the contribution of ℓ_1 is negligible if, say, **278**
$A > 1/3$ in (5.141). Clearly the contribution of ℓ_4 can also be neglected. As for ℓ_3, let $Q = x_0(1 + u + \epsilon + \epsilon i)$ with $u \geq 0$. The for $0 \leq y \leq 1$, $u \geq 0$ the integrand of $S(y;\xi)$ at Q is uniformly

$$\begin{aligned}
\ll |x|^{-\frac{1}{2}} \exp\Bigg(&|\xi| \arctan \frac{\epsilon}{1+u+\epsilon} - T \arctan \frac{x_0\epsilon}{1 + x_0(1+u+\epsilon)}\\
&-|\xi| \arctan \frac{y x_0 \epsilon}{1 + y x_0(1+u+\epsilon)}\Bigg) \ll |x|^{-\frac{1}{2}} \exp\left(-\frac{\epsilon^2}{2}|\xi|\right) \qquad (5.152)
\end{aligned}$$

if ϵ is sufficiently small. Thus the contribution on ℓ_3 will be also negligible if A in (5.141) is large enough.

On ℓ_2 we may use (5.152) with $u = 0$ and ϵ a small, positive constant. It follows that we have $x = x_0 + r x_0$, $dx = x_0 dr$, where r is the variable that runs over the segment $\left[-\epsilon e^{\frac{1}{4}\pi i}, \epsilon e^{\frac{1}{4}\pi i}\right]$. However, if we restrict r to the segment $[-|\xi|^{-2/5}e^{\frac{1}{4}\pi i}, |\xi|^{-2/5}e^{\frac{1}{4}\pi i}]$ then the integrand on the remaining two segments of ℓ_2 is bounded by the last bound in (5.152) with $\epsilon = |\xi|^{-2/5}$. Hence we have a total error which is

$$\ll |x|^{-\frac{1}{2}} \exp\left(-C'|\xi|^{1/5}\right) \quad (C' > 0), \tag{5.153}$$

and this is certainly negligible if $A \geq 2$ in (5.141). Therefore it follows that, for $\xi < 0$ and an absolute 0-constant,

$$\begin{aligned} S(y;\xi) = {} & O\left(\exp\left(-C'|\xi|^{1/5}\right)\right) + \left(\frac{x_0}{(1+x_0)(1+x_0y)}\right)^{\frac{1}{2}} \\ & \times \exp\left(-\frac{\Delta^2}{4}\log^2(1+x_0)\right)\exp\left(iT\log(1+x_0) + i\xi\log\frac{x_0}{1+x_0y}\right) \\ & \times \int\limits_{-|\xi|^{-2/5}e^{\pi i/\mu}}^{|\xi|^{-2|5}e^{\pi i/\mu}} (1+r)^{-\frac{1}{2}}\left(1+\frac{x_0yr}{1+x_0y}\right)^{-\frac{1}{2}}\left(1+\frac{x_0r}{1+x_0}\right)^{-\frac{1}{2}} \\ & \times \exp\left\{-\frac{\Delta^2}{4}\left(2\log(1+x_0)\log\left(1+\frac{x_0r}{1+x_0}\right)\right.\right. \\ & \left.\left. + \log^2\left(1+\frac{x_0r}{1+x_0}\right)\right) + i\sum_{j=2}^{\infty} a_j r^j\right\} dr. \end{aligned} \tag{5.154}$$

Here we have, by Taylor's formula,

$$a_j = \frac{x_0^j f^{(j)}(x_0)}{j!} = O_j(|\xi|),$$

279 and the term with $j = 1$ vanishes because x_0 is the saddle point of $f(x)$, that is, $f'(x_0) = 0$. Since

$$f''(x) = -\frac{\xi}{x^2} - \frac{T}{(1+x)^2} + \frac{\xi y^2}{(1+xy)^2},$$

it follows that

$$a_2 = \frac{|\xi|}{2}\left(1 + \frac{Tx_0^2}{\xi(1+x_0)^2} - \left(\frac{x_0y}{1+x_0y}\right)^2\right) = \frac{|\xi|}{2}\left(1 + \frac{\xi}{T} + O\left(\frac{\xi^2}{T^2}\right)\right).$$

With the change of variable $r = e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u$ the last integral above is equal to

$$e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \int_{-A_\xi}^{A_\xi} \left(1 + e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u\right)^{-\frac{1}{2}} (1 + \ldots)^{-\frac{1}{2}} (1 + \ldots)^{-1/2}$$
$$\exp\left(-\frac{\Delta^2}{4}(2 \log \ldots)\right) \exp\left(-u^2 + ia_3 e^{\frac{3}{4}\pi i} a_2^{-3/2} u^3 - ia_4 a_2^{-2} u^4 + \ldots\right) du,$$

where $A_\xi \sim \frac{1}{2}|\xi|^{1/10}$ as $T \to \infty$. Expanding the last exponential into its Taylor series it is seen that this becomes

$$I(0) + \sum_{j=3}^{\infty} b_j I(j), \quad b_j \ll |\xi|^{1-\frac{1}{2}j},$$
$$I(j) = e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \int_{-A_\xi}^{A_\xi} \left(1 + e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u\right)^{-\frac{1}{2}} (1 + \ldots)^{-\frac{1}{2}} (1 + \ldots)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{\Delta^2}{4}\left(2 \log \ldots + \log^2\left(1 + \frac{x_0 r}{1 + x_0}\right)\right)\right) \cdot u^j e^{-u^2} du.$$

We truncate the series $\sum\limits_{j=3}^{\infty}$ at $j = J$ so large that the trivial estimation of the terms with $j > J$ makes a total contribution of order $O(T^{-\frac{1}{2}})$. The integrals $I(j)$ for $j \geq 3$ will make the same type of contribution as $I(0)$, only each will be by a factor of $|\xi|^{1-\frac{1}{2}j}$ smaller than $I(0)$. Thus we may consider only $I(0)$ in detail. We have

$$I(0) = e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \int_{-A_\xi}^{A_\xi} \left(1 + e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u\right)^{-\frac{1}{2}} \left(1 + \frac{x_0 y e^{\frac{1}{4}\pi i} a^{-\frac{1}{2}2} u}{1 + x_0 y}\right)^{-\frac{1}{2}}$$
$$\left(1 + \frac{x_0 e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u}{1 + x_0}\right)^{-\frac{1}{2}} \times \exp\left\{-\left(\frac{\Delta}{2}\right)^2 (2 \log(1 + x_0) \log\right.$$

$$\left(1 + \frac{e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u x_0}{1 + x_0} + \left(\log\left(1 + \frac{x_0 e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}u}}{1 + x_0}\right)\right)^2\right)\Bigg\} e^{-u^2} du.$$

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In $I(0)$ we have $0 \leq y \leq 1$, $a_2 \asymp |\xi|$, $x_0 \sim |\xi| T^{-1}$. Thus we repeat what we already did: we expand all three square roots above into power series, keeping as many terms as is necessary so that the trivial estimation of the remaining terms makes a total contribution which is $O(T^{-\frac{1}{2}})$. Thus we are left with the integral

$$J = e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \int\limits_{-A_\xi}^{A_\xi} \exp\Bigg\{-u^2 - \frac{\Delta^2}{4}\Bigg(2\log(1 + x_0)\log \tag{5.155}$$

$$\left(1 + \frac{e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u x_0}{1 + x_0}\right) + \log^2\left(1 + \frac{e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u x_0}{1 + x_0}\right)\Bigg)\Bigg\} du$$

plus integrals of similar type, each of which is by an order of magnitude of $|\xi|^{\frac{1}{2}}$ smaller than the previous one. When we expand the integrand in (5.155) into power series we obtain the integral

$$J_1 = e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \int\limits_{-A_\xi}^{A_\xi} \exp\left\{-\left(1 + \frac{i}{4}\Delta^2 x_0^2 a_2^{-1}\right)u^2 - \frac{\Delta^2}{2} x_0^2 e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u\right\} du \tag{5.156}$$

plus again a number of integrals of similar type, which are of a lower order of magnitude than J_1. Since

$$\Delta^2 x_0^2 a_2^{-1} \ll \Delta^2 \xi^2 T^{-2} |\xi|^{-1} \ll T^{-1} \Delta \log^3 T \ll \exp\left(-\frac{1}{2}\sqrt{\log T}\right),$$

we have

$$J_1 = e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \int\limits_{-\infty}^{\infty} \exp\left\{-\left(1 + \frac{i}{4}\Delta^2 x_0^2 a_2^{-1}\right)u^2 - \frac{\Delta^2}{2} x_0^2 e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} u\right\} du$$
$$+ O\left(e^{-c_o|\xi|^{1/5}}\right).$$

By using

$$\int_{-\infty}^{\infty} \exp\left(\alpha x - \beta x^2\right) dx = \sqrt{\frac{\pi}{\beta}} \exp\left(\frac{\alpha^2}{4\beta}\right) \quad (\mathrm{Re}\,\beta > 0)$$

we find that

$$\begin{aligned} J_1 &= e^{\frac{1}{4}\pi i} a_2^{-\frac{1}{2}} \pi^{\frac{1}{2}} \left(1 + \frac{i}{4}\Delta^2 x_0^2 a_2^{-1}\right)^{-\frac{1}{2}} \exp\left(\frac{i\Delta^4 x_0^4 a_2^{-1}}{8\left(1 + \frac{i}{4}\Delta^2 x_0^2 a_2^{-1}\right)}\right) \\ &\quad + O\left(\exp\left(-c_0|\xi|^{1/5}\right)\right) \\ &= e^{\frac{1}{4}\pi i} \pi^{\frac{1}{2}} a_2^{-\frac{1}{2}} \left(1 - \frac{i}{8}\Delta^2 x_0^2 a_2^{-1} + \ldots\right)\left(1 + \frac{i\Delta^4 x_0^4 a_2^{-1}}{8\left(1 + \frac{i}{4}\Delta^2 x_0^2 a_2^{-1}\right)} + \ldots\right) \\ &\qquad + O\left(\exp\left(-c_0|\xi|^{1/5}\right)\right). \end{aligned}$$

If we further make the restriction $\Delta \le T^{\frac{3}{4}}$, then all the terms involv- **281**
ing powers of $\Delta^2 x_0^2 a_2^{-1}$ will eventually make a negligible contribution. On noting that with suitable constants $e_j = e_j(y)$ we have

$$a_2^{-\frac{1}{2}} = 2^{\frac{1}{2}} |\xi|^{-\frac{1}{2}} \left(1 + \sum_{j=1}^{\infty} e_j \xi^j T^{-j}\right),$$

we may collect all the expressions for J_1 in (5.156) and insert them in (5.154). Expanding the quantity

$$\left(\frac{x_0}{(1+x_0)(1+x_0 y)}\right)^{\frac{1}{2}} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x_0)\right)$$

into Taylor series and using

$$T\log(1+x_0) + \xi\log\frac{x_0}{1+x_0 y} = \xi\log\frac{|\xi|}{eT} + \left(y - \frac{1}{2}\right)\frac{\xi^2}{T} + O\left(\frac{|\xi|^3}{T^2}\right)$$

we arrive at the expression for $\Xi^\circ(P_T; i\xi)$, where the main term is like in (5.147), only it is multiplied by a finite sum of the form $1 + \sum_{a,b,c} |\xi|^a T^b \Delta^c$ with each term satisfying

$$|\xi|^a T^b \Delta^c \ll \exp\left(-\frac{1}{2}\sqrt{\log T}\right),$$

and the contribution of the error terms is negligible (i.e. it is $O(T^{-\frac{1}{2}})$). When we integrate then $\Xi^\circ(P_T; i\xi)$ over T, $V \le T \le 2V$, replace V by $V2^{-j}$, sum over j, then we obtain the second assertion of Theorem 5.3. From the above discussion it follows that the total error term for the range $|\xi| \ge \log^{3A} T$ will be $O(V^{\frac{1}{2}})$, while for the range $D < |\xi| \le \log^{3A} T$ it will be sufficient to use

$$\int\limits_V^{2V} \Xi^\circ(P_T; i\xi) dT = i2^{3/2}\pi^2 |\xi|^{-3/2} T^{\frac{1}{2}} \exp\left(i\xi \log \frac{|\xi|}{4eT}\right)\Bigg|_V^{2V} \cdot \left(1 + O(|\xi|^{-1})\right)$$

for $\xi < 0$, which is the analogue of (5.139). For $|\xi| \le D$ the trivial bound

$$\int\limits_V^{2V} \Xi^\circ(P_T; i\xi) dT \ll V^{\frac{1}{2}}$$

suffices.

In concluding, it may be remarked that the quantities c_j which ap-
282 pear in (5.14), satisfy $c_j = (1 + o(1)) x_j^{-3/2} (j \to \infty)$ as asserted. However, in the relevant range

$$V^{\frac{1}{2}} \log^{-A} V \le \Delta \le V^{\frac{3}{4}}, \quad x_j \le V\Delta^{-1} \log V,$$

one can replace the term $\circ(1)$ in the above expression for c_j by an explicit 0-term and obtain

$$c_j = \left\{1 + O\left(V^{-\frac{1}{4}} \log^3 V\right) + O\left(x_j^{-\frac{1}{2}}\right)\right\} x_j^{-3/2}.$$

5.6 Upper Bounds

In this section we shall deduce the upper bound results for $E_2(T)$ contained in Theorems 5.4 - 5.6, and we shall also prove Theorem 5.8. To begin with, we shall apply an averaging technique to the asymptotic formulas of Theorem 5.3. Rewrite Theorem 5.3 as

$$\int\limits_0^T I_4(t, \Delta) dt = TP_4(\log T) + S(T, \Delta) + R(T, \Delta), \tag{5.157}$$

where

$$S(T,\Delta) := \pi\left(\frac{T}{2}\right)^{\frac{1}{2}} \sum_{j=1}^{\infty} \alpha_j x_j^{-3/2} H_j^3\left(\frac{1}{2}\right)\cos\left(x_j \log\frac{4eT}{x_j}\right) e^{-(\Delta x_j/2T)^2} \tag{5.158}$$

and

$$R(T,\Delta) \ll T^{\frac{1}{2}} \log^{C(A)} T \tag{5.159}$$

for $T^{\frac{1}{2}} \log^{-A} T \le \Delta \le T\exp(-\sqrt{\log T})$ and any fixed $A > 0$. Suppose henceforth that $T^{\epsilon} \le \Delta \le T\exp(-\sqrt{\log T})$ and put first $T_1 = T - \Delta\log T$, $T_2 = 2T + \Delta\log T$. Then

$$\int_{T_1}^{T_2} I_4(t,\Delta)dt = \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 \left((\Delta\sqrt{\pi})^{-1}\int_{T_1}^{T_2} e^{-(t-u)^2/\Delta^2}dt\right)du \tag{5.160}$$

$$\ge \int_{T}^{2T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 \left((\Delta\sqrt{\pi})^{-1}\int_{T-\Delta\log T}^{2T+\Delta\log T} e^{-(t-u)^2}/\Delta^2 dt\right)du.$$

But for $T \le u \le 2T$ we have, by the change of variable $t-u = \Delta v$,

$$(\Delta\sqrt{\pi})^{-1}\int_{T-\Delta\log T}^{2T+\Delta\log T} e^{-(t-u)^2/\Delta^2}dt = \pi^{-\frac{1}{2}}\int_{(T-u)\Delta^{-1}-\log T}^{(2T-u)\Delta^{-1}+\log T} e^{-v^2}dv$$

$$= \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-v^2}dv + O\left(\int_{\log T}^{\infty} e^{-v^2}dv + \int_{-\infty}^{-\log T} e^{-v^2}dv\right) = 1 + O\left(e^{-\log^2 T}\right),$$

since $T-u \le 0$, $2T-u \ge 0$. Therefore **283**

$$\int_{T}^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \le \int_{T_1}^{T_2} I_4(t,\Delta)dt + O(1) = 2TP_4(\log 2T)$$
$$- TP_4(\log T) + O(\Delta\log^5 T) + S(2T+\Delta\log T,\Delta)$$
$$- S(T-\Delta\log T,\Delta) + R(2T+\Delta\log T,\Delta) - R(T-\Delta\log T,\Delta)$$

by using the mean value theorem.

Consider now the equality in (5.160) with $T_1 = T + \Delta \log T$, $T_2 = 2T - \Delta \log T$. We obtain

$$\int_{T_1}^{T_2} I_4(t,\Delta)dt = \int_T^{2T} \left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 \left((\Delta\sqrt{\pi})^{-1}\int_{T_1}^{T_2} e^{-(t-u)^2/\Delta^2}dt\right)du + O(1),$$

since with $u - t = \Delta v$ we have

$$\int_{-\infty}^{T} \left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 \left((\Delta\sqrt{\pi})^{-1}\int_{T+\Delta\log T}^{2T-\Delta\log T} e^{-(t-u)^2/\Delta^2}dt\right)du$$

$$= (\Delta\sqrt{\pi})^{-1}\int_{T+\Delta\log T}^{2T-\Delta\log T}\left(\int_{-\infty}^{T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 e^{-(t-u)^2/\Delta^2}du\right)dt$$

$$= \pi^{-\frac{1}{2}}\int_{T+\Delta\log T}^{2T-\Delta\log T}\left(\int_{(t-T)/\Delta}^{\infty}\left|\zeta\left(\frac{1}{2}+it-i\Delta v\right)\right|^4 e^{-v^2}dv\right)dt$$

$$\ll T^2\int_{\log T}^{\infty}(1+v)e^{-v^2}dv \ll e^{-\frac{1}{2}\log^2 T}.$$

The same upper bound holds for the integral from $2T$ to ∞. But

$$\int_T^{2T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4\left((\Delta\sqrt{\pi})^{-1}\int_{T_1}^{T_2} e^{-(t-u)^2/\Delta^2}dt\right)du$$

$$\leq \int_T^{2T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4\left((\Delta\sqrt{\pi})^{-1}\int_{T-\Delta\log T}^{2T+\Delta\log T} e^{-(t-u)^2/\Delta^2}dt\right)du$$

$$= \int_T^{2T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 du + 0(1)$$

284 as in the previous case. Thus we have proved

Lemma 5.1. *For $0 < \epsilon < 1$ fixed and $T^{\epsilon} \le \Delta \le Te^{-\sqrt{\log T}}$ we have uniformly*

$$\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \le 2TP_4(\log 2T) - TP_4(\log T) + O(\Delta \log^5 T)$$
$$+ S(2T+\Delta\log T, \Delta) - S(T-\Delta\log T, \Delta)$$
$$+ R(2T+\Delta\log T, \Delta) - R(T-\Delta\log T, \Delta)$$

and

$$\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ge 2TP_4(\log 2T) - TP_4(\log T) + O(\Delta \log^5 T)$$
$$+ S(2T-\Delta\log T, \Delta) - S(T+\Delta\log T, \Delta)$$
$$+ R(2T-\Delta\log T, \Delta) - R(T+\Delta\log T, \Delta).$$

To obtain an upper bound for $E_2(T)$ from Lemma 5.1 note that, for $\tau \asymp T$, we have uniformly for $A > 0$ sufficiently large

$$S(\tau,\Delta) = \pi\left(\frac{\tau}{2}\right)^{\frac{1}{2}} \sum_{x_j \le T\Delta^{-1}\log^{\frac{1}{2}} T} \alpha_j x_j^{-3/2} H_j^3\left(\frac{1}{2}\right)\cos\left(x_j \log\frac{4e\tau}{x_j}\right) e^{-(\Delta x_j/2\tau)^2} \tag{5.161}$$
$$+ O(1) = O\left(T^{\frac{1}{2}} \sum_{x_j \le AT\Delta^{-1}\log^{\frac{1}{2}} T} \alpha_j x_j^{-3/2}\left|H_j^3\left(\frac{1}{2}\right)\right|\right) + O(1)$$

by trivial estimation. To estimate the last sum we use (5.149) with $C = \frac{1}{2}(B+1)$ to obtain by partial summation

$$S(\tau,\Delta) \ll T\Delta^{-\frac{1}{2}}(\log T)^{\frac{1}{2}B+\frac{3}{4}} \qquad (\tau \asymp T).$$

Therefore from Lemma 5.1 and Theorem 5.3 we infer that, for $T^{\frac{1}{2}} \le$ **285**
$\Delta \le T\exp(-\sqrt{\log T})$, uniformly in Δ

$$\int_T^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = 2TP_4(\log 2T) - TP_4(\log T)$$

$$+ O(\Delta \log^5 T) + O\left(T^{\frac{1}{2}} \log^{C(A)} T\right) + O\left(T\Delta^{-\frac{1}{2}} (\log T)^{\frac{1}{2}B+\frac{3}{4}}\right).$$

We equalize the first and the third O-term by choosing

$$\Delta = T^{2/3} (\log T)^{B/3-17/6}.$$

Then we obtain

$$E_2(2T) - E_2(T) \ll T^{2/3} (\log T)^{B/3+13/6}.$$

Hence replacing T by $T2^{-j}$ and summing over $j = 1, 2, \ldots$ we have

$$E_2(T) \ll T^{2/3} (\log T)^{B/3+13/6}, \tag{5.162}$$

which proves Theorem 5.4 with $C = \dfrac{B}{3} + \dfrac{13}{6}$, where B is the constant appearing in (5.151). Note that any non-trivial estimation of the sum in (5.161) would lead to improvements of (5.162). Such a non trivial estimation would presumably lead to a bound of the form $E_2(T) \ll T^{n+\epsilon}$ with some $\eta < 2/3$. By Theorem 1.1 this would imply $\zeta(\frac{1}{2} + iT) \ll T^{\theta+\epsilon}$ with $\theta = \dfrac{\eta}{4} < \dfrac{1}{6}$, thus providing a nontrivial estimate for $\zeta(\frac{1}{2} + iT)$.

We pass now to the proof of Theorem 5.5. From Lemma 5.1 and (5.10) we have, with a slight abuse of notation,

$$\begin{aligned} E_2(2T) - E_2(T) &\le S(2T + \Delta \log T, \Delta) - S(T - \Delta \log T, \Delta) \\ &\quad + O(T^{\frac{1}{2}}) + O(\Delta \log^5 T), \end{aligned} \tag{5.163}$$

where this time

$$S(T, \Delta) = \pi \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{j=1}^{\infty} \alpha_j c_j H_j^3 \left(\frac{1}{2}\right) \cos\left(x_j \log \frac{4eT}{x_j}\right) e^{-(\Delta x_j/2T)^2}$$

with $c_j \sim x_j^{-3/2}$, $T^{\frac{1}{2}} \log^{-A} T \le \Delta \le T^{\frac{3}{4}}$. An expression of similar type holds also for the lower bound. We replace in (5.163) T by $t2^{-\ell}$ and sum for $\ell = 1, 2, \ldots, L$, where L is chosen in such a way that $t2^{-L} <$
286 $t^{5/7} \le t2^{1-L}$. Then we take $\Delta = T^{\frac{1}{2}} \log^{-5} T$ and $T \le t \le 2T$, so that the

condition $T^{\frac{1}{2}} \log^{-5} T \leq \Delta \leq T^{\frac{3}{4}}$ is satisfied if T is replaced by $t2^{-\ell}$ for each $\ell = 1, 2, \ldots, L$. By Theorem 5.4 we have

$$E_2(t^{5/7}) \ll (T^{5/7})^{2/3} \log^C T \ll T^{1/2},$$

hence we obtain

$$E_2(t) \leq \sum_{\ell=1}^{L} \left\{ S\left(2^{1-\ell} t + \Delta \log T, \Delta\right) - S\left(2^{-\ell} t - \Delta \log T, \Delta\right)\right\} + O(T^{\frac{1}{2}}), \quad (5.164)$$

since the analysis of the proof of Lemma 5.1 shows that $\log(t2^{-j})$ may be replaced by $\log T$ with the same effect. Therefore integration of (5.164) gives

$$\int_T^{2T} E_2(t)dt \leq O(T^{3/2}) \quad (5.165)$$
$$+ \sum_{\ell=1}^{L} \left\{ \int_T^{2T} S\left(2^{1-\ell} t + \Delta \log T, \Delta\right) dt - \int_T^{2T} S\left(2^{-\ell} t - \Delta \log T, \Delta\right) dt \right\},$$

and a lower bound of similar type will hold for the left-hand side of (5.165). We break the series for $S(T, \Delta)$ at $T\Delta^{-1} \log^T$ with a negligible error to obtain

$$\int_T^{2T} S\left(2^{-\ell} t - \Delta \log T, \Delta\right) dt = \pi 2^{-\frac{1}{2}} \int_T^{2T} \left(2^{-\ell} t - \Delta \log T\right)^{\frac{1}{2}}$$
$$\times \sum_{x_j \leq T\Delta^{-t} \log T} \alpha_j c_j H_j^3 \left(\frac{1}{2}\right) \cos\left(x_j \log \frac{4e(2^{-\ell} t - \Delta \log T)}{x_j}\right)$$
$$\times \exp\left(-\left(\frac{\Delta x_j}{2^{-\ell} t}\right)\right) dt \ll (T2^{-\ell})^{\frac{1}{2}} T$$
$$\times \sum_{x_j \leq T\Delta^{-1} \log T} \alpha_j c_j x_j^{-1} \left| H_j^3 \left(\frac{1}{2}\right)\right| \ll T^{3/2} 2^{-\frac{1}{2}\ell}.$$

Here we used Lemma 2.1 and the fact that by (5.162) and partial summation

$$\sum_{x_j \le T\Delta^{-1}\log T} \alpha_j c_j x_j^{-1} \left|H_j^3\left(\frac{1}{2}\right)\right| \ll \sum_{j=1}^{\infty} \alpha_j x_j^{-5/2} \left|H_j^3\left(\frac{1}{2}\right)\right| < +\infty.$$

Summing over ℓ we have

$$\int_T^{2T} E_2(t)dt \le CT^{3/2} \qquad (C > 0),$$

287 and the analogue of (5.165) for the lower bound will show that the above integral is also $-CT^{3/2}$. This proves (5.16). In view of Lemma 5.4 it may well be true that

$$\int_0^T E_2(t)dt \sim C(T)T^{3/2} \qquad (T \to \infty), \tag{5.166}$$

where $C(T)$ is a function represented by infinite series of the type appearing in (5.206), and where

$$C(T) = O(1), \quad C(T) = \Omega_{\pm}(1)$$

holds. It does not appear easy, however, to prove (5.166).

To prove (5.17) write

$$I_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 (dt) = TP_4(\log T) + E_2(T).$$

Then we have

$$\int_T^{2T} E_2(t) \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt = \int_T^{2T} E_2(t) I_2'(t)dt$$

$$= \int_T^{2T} E_2(t)\Big(P_4(\log t) + P_4'(\log t) + E_2'(t)\Big)dt$$

$$= \int_{T}^{2T} E_2(t)\left(P_4(\log t) + P_4'(\log t)\right) dt + \frac{1}{2}E_2^2(2T) - \frac{1}{2}E_2^2(T).$$

For the last integral we use (5.16) and integration by parts. Since $E_2(T) \ll T^{2/3}\log^C T$, (5.17) follows. Recall that $E(T)(= E_1(T))$ had mean value π (Theorem 3.1). On the other hand, if the conjecture (5.166) is true, then $E_2(T)$ certainly can have no mean value.

We proceed now to prove the mean square estimate given by Theorem 5.6. Before giving the proof we remark that (5.18) gives

$$\int_0^T E_2^2(t)dt \ll T^{9/4}\log^C T.$$

This bound shows that, in mean square, $E_2(t)$ is $\ll t^{5/8}$, and this is better than what one gets from the pointwise estimate (5.15). I expect that

$$\int_0^T E_2^2(t)dt = CT^2 + H(T) \tag{5.167}$$

holds for some $C > 0$ and $H(T) = \circ(T^2)$ as $T \to \infty$. This conjecture 288
seems to be fairly deep, since proving even

$$\int_T^{2T} E_2^2(t)dt \gg T^2$$

seems difficult (this lower bound trivially implies Theorem 5.7). In fact it is even plausible to conjecture that

$$H(T) = O\left(T^{3/2+\epsilon}\right), \quad H(T) = \Omega\left(T^{3/2-\delta}\right) \tag{5.168}$$

for any $\delta, \epsilon > 0$, and the above omega-result can be proved if the strong omega-result (5.212) is true. Incidentally, (5.18) implies $E_2(T) \ll T^{2/3}\log^C T$, which is Theorem 5.4. To see this recall that by Lemma 4.2 with

$k = 2$ we have

$$E_2(T) \le H^{-1} \int\limits_T^{T+H} E_2(t)dt + C_1 H \log^4 T \quad (0 < H \le T) \tag{5.169}$$

and

$$E_2(T) \ge H^{-1} \int\limits_{T-H}^{T} E_2(t)dt - C_1 H \log^4 T \quad (0 < H \le T). \tag{5.170}$$

From (5.18), (5.169), (5.170) and the Cauchy-Schwarz inequality it follows that

$$\begin{aligned}
|E_2(T)| &\le H^{-1} \int\limits_{T-H}^{T+H} |E_2(t)|dt + C_1 H \log^4 T \\
&\ll H^{-\frac{1}{2}} \left(\int\limits_{T-H}^{T+H} E_2^2(t)dt \right)^{\frac{1}{2}} + H \log^4 T \\
&\ll T^{\frac{3}{4}} H^{-1/8} \log^{\frac{1}{2}C} T + H \log^4 T \ll T^{2/3} (\log T)^{(4C+4)/9}
\end{aligned}$$

with $H = T^{2/3}(\log T)^{(4C-32)/9}$. By the same argument $E_2(T) \ll T^{\frac{1}{2}+\epsilon}$ is true if (5.167) and the 0-result in (5.168) are true.

To prove (5.18) we use (5.163) provided that $E(t) > 0$, and an analogous lower bound when $E(t) < 0$. Thus squaring and integrating we
289 obtain

$$\begin{aligned}
\int\limits_T^{T+H} E_2^2(t)dt &\ll TH + H\Delta^2 \log^{10} T + \log T \sum_{\ell=1}^{L} \\
&\times \left\{ \int\limits_T^{T+H} S^2\left(2^{1-\ell} t + \Delta \log T, \Delta\right) dt + \int\limits_T^{T+H} S^2\left(2^{-\ell} t - \Delta \log T, \Delta\right) dt \right. \\
&\left. + \int\limits_T^{T+H} S^2\left(2^{1-\ell} t - \Delta \log T, \Delta\right) dt + \int\limits_T^{T+H} S^2\left(2^{-\ell} t + \Delta \log T, \Delta\right) dt \right\}.
\end{aligned} \tag{5.171}$$

All four integrals in (5.171) are estimated similarly, the first being majorized by 0(log T) subintegrals of the type

$$J_N = J_N(T,H,\ell)$$
$$:= \int\limits_T^{T+H} \tau \left| \sum_{N<x_j\le 2N} \alpha_j c_j H_j^3\left(\frac{1}{2}\right) \cos\left(x_j \log\frac{4e\tau}{x_j}\right) e^{-(\Delta x_j/2\tau)^2}\right|^2 dt,$$

where $\tau = 2^{1-\ell}t + \Delta\log T \asymp 2^{-\ell}T$, $N \ll T\Delta^{-1}\log T$. Squaring out the integrand one obtains

$$J_N \ll T2^{-\ell} \sum_{N<x_j,x_m\le 2N} \alpha_j\alpha_m(x_jx_m)^{-3/2}\left|H_j^3\left(\frac{1}{2}\right)H_m^3\left(\frac{1}{2}\right)\right|$$
$$\left|\int\limits_T^{T+H}\cos\left(x_j\log\frac{4e(2^{1-\ell}t+\Delta\log T)}{x_j}\right)\cos\left(x_m\log\frac{2^{1-\ell}t+\Delta\log T}{x_m}\right)dt\right|$$
$$= T2^{-\ell}\left(\sum_1 + \sum_2\right),$$

say, where in $\sum_1$ we fix x_j and sum over x_m such that $|x_j - x_m| \le V$, and in $\sum_2$ we sum over $|x_j - x_m| > V$. Here V ($\gg 1$) is a parameter satisfying $V \ll N$, and which will be suitably chosen a little later. To estimate $\sum_1$ we use the bound

$$\sum_{x-V\le x_j\le x+V} \alpha_j\left|H_j^3\left(\frac{1}{2}\right)\right| \ll x^{3/2}V^{1/2}\log^C x \tag{5.172}$$

for $\log^{C_1} x \ll V \ll x$, which follows by the Cauchy-Schwarz inequality from (5.55) and (5.151) with $C = \frac{1}{2}(B+1)$, where B is the constant in (5.151). Alternatively, one can use the bound

$$\sum_{x-V\le x_j\le x+V} \alpha_j H_j^2\left(\frac{1}{2}\right) \ll xV\log x \ \left(B\log^{\frac{1}{2}} x \le V \le x, B > 0\right),$$

proved very recently by Y. Motohashi [129]. This gives, again by (5.151), the same bound as in (5.172).

Hence by trivial estimation and (5.172) we have

$$\sum_1 \ll H \sum_{N<x_j\le 2N} \alpha_j x_j^{-3/2} \left| H_j^3\left(\frac{1}{2}\right)\right| V^{\frac{1}{2}} \log^C T \tag{5.173}$$
$$\ll HV^{\frac{1}{2}} \log^C T \sum_{x_j\le 2N} \alpha_j \left| H_j^3\left(\frac{1}{2}\right)\right| N^{-3/2} \ll HV^{\frac{1}{2}}N^{\frac{1}{2}} \log^{2C} T.$$

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To estimate $\sum_2$ we use Lemma 2.1 and

$$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}.$$

It follows that

$$\sum_2 \ll T \sum_{N<x_j,x_m\le 2N;|x_j-x_m|>V} \alpha_j\alpha_m(x_jx_m)^{-3/2} \frac{H_j^3\left(\frac{1}{2}\right)H_m^3\left(\frac{1}{2}\right)}{|x_j-x_m|} \tag{5.174}$$
$$\ll TV^{-1}\left(\sum_{N<x_j\le 2N} \alpha_j x_j^{-3/2}\left|H_j^3\left(\frac{1}{2}\right)\right|\right)^2 \ll TV^{-1}N\log^{B+1} T,$$

with B as in (5.151). From (5.173) and (5.174) we have

$$J_N \ll T2^{-\ell}\log^{B+1} T\left(HV^{\frac{1}{2}}N^{\frac{1}{2}} + TV^{-1}N\right)$$
$$\ll 2^{-\ell}T^{\frac{4}{3}}N^{\frac{2}{3}}H^{\frac{2}{3}}\log^{B+1} T \tag{5.175}$$

for

$$V = T^{2/3}N^{1/3}H^{-2/3} \ll N.$$

The condition $V \ll N$ is satisfied for $N \ge TH^{-1}$. If $N < TH^{-1}$, then it suffices to use (5.149) and obtain, by trivial estimation,

$$J_N \ll 2^{-\ell}THN\log^{2C} T \ll 2^{-\ell}T^2\log^{2C} T.$$

If $N \ge TH^{-1}$, then in view of $N \ll T\Delta^{-1}\log T$ we must have $\Delta \ll H\log T$. Therefore combining the preceding estimates we have

$$\int_T^{T+H} E_2^2(t)dt \ll \left(T^2 + \Delta^2 H + T^{4/3}H^{2/3} \max_{N\le T\Delta^{-1}\log TN^{2/3}}\right)\log^D T$$

$$\ll \left(T^2 + \Delta^2 H + T^2 H^{2/3}\Delta^{-2/3}\right)\log^{D+1} T \ll T^{3/2}H^{3/4}\log^{D+1} T$$

with a suitable $D > 0$ and

$$\Delta = T^{3/4}H^{-1/8}.$$

With the above Δ the condition $\Delta \ll H\log T$ holds for $H \ll T^{\frac{2}{3}}$ $\log^{-\frac{8}{9}} T$. This is actually the relevant range for (5.18), since for $H \ll$ 291
$T^{2/3}$
$\log^{-8/9} T$ the trivial estimate

$$\int\limits_T^{T+H} E_2^2(t)dt \leq H\left(\mathrm{Max}_{T\leq t\leq 2T}|E_2(t)|\right)^2 \ll HT^{4/3}\log^{2C} T,$$

which comes from Theorem 5.4, is better than (5.18) if one disregards the log-factors. The condition $(2^{-\ell}t)^{1/2} \ll \Delta \ll (2^{-\ell})^{3/4}$ will be satisfied if L in (5.171) is chosen to satisfy $2^{-L}t \leq T^{\frac{3}{4}-\epsilon} \leq 2^{1-L_t}$. This completes the proof of Theorem 5.6. We remark that, if instead of (5.151), one also had

$$\sum_{x-V\leq x_j\leq x+V} \alpha_j H_j^4\left(\frac{1}{2}\right) \ll xV\log^C x \quad \left(x^\epsilon \ll V \leq x^{\frac{1}{2}}\right),$$

then instead of (5.172) the Cauchy-Schwarz inequality would give

$$\sum_{x-V\leq x_j\leq x+V} \alpha_j \left|H_j^3\left(\frac{1}{2}\right)\right| \ll xV\log^C x \quad \left(x^\epsilon \ll V \leq x^{\frac{1}{2}}\right).$$

Proceeding as above we would obtain

$$\int\limits_T^{T+H} E_2^2(t)dt \ll \left(T^2 + \Delta^2 H + T^2 H^{\frac{1}{2}}\Delta^{-\frac{1}{2}}\right)\log^C T$$

$$\ll \left(T^2 + T^{8/5}H^{3/5}\right)\log^C T$$

for $\Delta = T^{4/5}H^{-1/5}$, and again $\Delta \ll H$ for $H \gg T^{2/3}$. Hence we would get

$$\int\limits_T^{T+H} E_2^2(t)dt \ll T^{8/5}H^{3/5}\log^C T \quad \left(T^{2/3} \ll H \leq T\right),$$

which improves Theorem 5.6. In particular, we would obtain

$$\int_0^T E^2(t)dt \ll T^{11/5}\log^C T.$$

It remains yet in this section to prove Theorem 5.8, namely

$$\sum_{r\le R}\int_r^{t_r+\Delta}|\zeta\left(\frac{1}{2}+it\right)|^4 dt \ll R\Delta\log^{C_1} T + R^{\frac{1}{2}}T\Delta^{-\frac{1}{2}}\log^{C_2} T, \qquad (5.176)$$

292 provided that

$$T < t_1 < \ldots < t_R \le 2T,\ t_{r+1}-t_r > \Delta (r=1,\ldots,R-1),$$
$$T^{\frac{1}{2}} \le \Delta \le T^{2/3},$$

since for $T^{2/3}\le\Delta\le T$ the bound (5.176) is trivial in virtue of Theorem 5.4. We have

$$\sum_{r\le R}\int_{t_r}^{t_r+\Delta}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \le \sum_{r\le R}\int_{-\Delta}^{\Delta}\left|\zeta\left(\frac{1}{2}+it_r+it\right)\right|^4 dt$$
$$\le \sqrt{\pi}e\Delta\sum_{r\le R}\left((\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+it_r+it\right)\right|^4 e^{-(t/\Delta)^2}dt\right)$$
$$\le e\pi\sqrt{\frac{\pi}{2}}\Delta\sum_{r\le R}t_r^{-\frac{1}{2}}\sum_{j=1}^{\infty}\alpha_j x_j^{-\frac{1}{2}}H_j^3\left(\frac{1}{2}\right)\sin\left(x_j\log\frac{x_j}{4et_r}\right)$$
$$e^{-(\Delta x_j/2t_r)^2} + O(R\Delta\log^C T)$$
$$\le e\pi\sqrt{\frac{\pi}{2}}\Delta\sum_{x_j\le 20T\Delta^{-1}\log^{\frac{1}{2}}T}\alpha_j x_j^{-\frac{1}{2}}H_j^3\left(\frac{1}{2}\right)\sum_{r\le R}t_r^{-\frac{1}{2}}\sin\left(x_j\log\frac{x_j}{4et_r}\right)$$
$$e^{-(\Delta x_j/2t_r)^2} + O(R\Delta\log^C T),$$

where Theorem 5.2 was used. For technical reasons it is convenient to remove $e^{-(\ldots)}$ by partial summation from the last sum with a total error which is certainly $\ll R\Delta \log^C T$. Then we have to majorize

$$\sum := \sum_{K=20T\Delta^{-1}\log^{\frac{1}{2}} T 2^{-m}; m=1,2,\ldots} S_K,$$

$$S_K := \sum_{K<x_j\le 2K} \alpha_j x_j^{-\frac{1}{2}} \left| H_j^3\left(\frac{1}{2}\right)\right| \left|\sum_{r\le R} t_r^{-\frac{1}{2}-ix_j}\right|.$$

By using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} S_K &\le \left(\sum_{K<x_j\le 2K} \alpha_j x_j^{-1} H_j^4\left(\frac{1}{2}\right)\right)^{\frac{1}{2}} \left(\sum_{K<x_j\le 2K} \alpha_j H_j^2\left(\frac{1}{2}\right)\left|\sum_{r\le R} t_r^{-\frac{1}{2}-ix_j}\right|^2\right)^{\frac{1}{2}} \\ &= (S'_K S''_K)^{\frac{1}{2}}, \end{aligned}$$

say. By using (5.151) we immediately obtain **293**

$$S'_K \ll K \log^B K.$$

To bound S''_K one needs a "large sieve" type of inequality for sums with $\alpha_j H^2(\frac{1}{2})$. Such a result has been recently established by Y. Motohashi (see (3.9) of [129]), and we state it here without proof as

Lemma 5.2. *Let $T < t_1 < \ldots < t_R \le 2T$, $t_{r+1} - t_r > AT\Delta^{-1}\log^{\frac{1}{2}} T$ for $r = 1, \ldots, R-1$ and suitable $A > 0$, $\log T \asymp \log K$, $\log^{\frac{1}{2}} K \ll \Delta \le K/\log K$. Then for arbitrary complex numbers $c_1, \ldots, c_R$ we have*

$$\sum_{K<x_j\le K+\Delta} \alpha_j H_j^2\left(\frac{1}{2}\right)\left|\sum_{r=1}^{R} c_r t_r^{ix_j}\right|^2 \ll \sum_{r=1}^{R} |c_r|^2 K\Delta \log K.$$

In this result, which is of considerable interest in itself, we replace K by $K+\Delta$, $K+2\Delta, \ldots$ etc. and take $\Delta = K/\log K$. Then we obtain

$$\sum_{K<x_j\le 2K} \alpha_j H_j^2\left(\frac{1}{2}\right)\left|\sum_{r=1}^{R} c_r t_r^{ix_j}\right|^2 \ll \sum_{r=1}^{R} |c_r|^2 K^2 \log K \tag{5.177}$$

provided that

$$t_{r+1} - t_r \gg TK^{-1}\log^{3/2} T, \tag{5.178}$$

in which case we obtain from (5.177) (with $c_r = t_r^{-\frac{1}{2}}$)

$$S_K'' \ll RT^{-1}K^2\log K.$$

However, our assumption is that $t_{r+1} - t_r > \Delta$ and K satisfies $K \ll T\Delta^{-1}\log^{\frac{1}{2}} T$, so that (5.178) does not have to hold. In that case we may choose from our system of points $\{t_r\}_{r=1}^{R}$ subsystems of points for which (5.178) is satisfied for consecutive points of each subsystem. There are then $\ll TK^{-1}\Delta^{-1}\log^{3/2} T$ subsystems in question, and we obtain, on using the above bound for S_K'' for each of these subsystems,

$$S_K'' \ll RT^{-1}K^2\log K \cdot TK^{-1}\Delta^{-1}\log^{3/2} T \ll RK\Delta^{-1}\log^{5/2} T.$$

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Hence we obtain in any case

$$S_K \ll R^{\frac{1}{2}}K\Delta^{-\frac{1}{2}}(\log T)^{\frac{1}{4}(2B+5)},$$

and consequently

$$\sum \ll R^{\frac{1}{2}}T\Delta^{-3/2}(\log T)^{\frac{1}{4}(2B+7)}.$$

This finally gives

$$\sum_{r\le R}\int\limits_{t_r}^{t_r+\Delta}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll R\Delta\log^C T + R^{\frac{1}{2}}T\Delta^{-\frac{1}{2}}(\log T)^{\frac{1}{4}(2B+7)},$$

which is (5.176) with $C_1 = C$ (≥ 4, the constant in Theorem 5.2), $C_2 = \frac{1}{4}(2B+7)$.

We recall that the result

$$\int\limits_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll T^2\log^D T \tag{5.179}$$

was proved by D.R. Heath-Brown with $D = 17$. We shall show now that (5.179) follows also from Theorem 5.8. Write

$$\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt = \int_{|\zeta|\le T^{1/8}\log^{(C_4+4)/4}T} + \int_{|\zeta|>T^{1/8}\log^{(C_4+1)/4}T} = I_1 + I_2,$$

say, where C_1 is the constant appearing in Theorem 5.8. Trivially we have

$$I_1 \le T\log^{2C_1+2} T\cdot\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll T^2\log^{2C_1+6} T.$$

We put

$$I_2 = \sum_{V=T^{1/6}2^{-j},j\ge1} I_2V, I_2(V) := \int_{\substack{V<|\zeta|\le 2V\\ T\le t\le 2T}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt.$$

Then

$$I_2(V) \ll \sum_{r\le R_V}\left|\zeta\left(\frac{1}{2}+it_r\right)\right|^{12} \ll R_V V^{12},$$

where $\{t\}_{r=1}^{R_V}$ is a system of points such that $T\le t_r\le 2T$ and $t_{r+1}-t_r\ge 1$. **295**
By Theorem 1.2 we have

$$R_V V^4 \ll \log T\sum_r \int_{t_r-1/3}^{t_r+1/3}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll \log T\sum_{r=1}^{R_1}\int_{\tau_r}^{\tau_r+\Delta}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt$$

for a system of points $\{\tau_r\}_{r=1}^{R_1}$ such that $T\le\tau_r\le 2T$ and $\tau_{r+1}-\tau_r\ge 1$, $R_1\le R_V$. An application of Theorem 5.8 gives, with $T^{\frac{1}{2}}\le\Delta\le T^{2/3}$,

$$R_V V^4 \ll \log T\left(R_V\Delta\log^{C_1} T + R_V^{\frac{1}{2}}T\Delta^{-\frac{1}{2}}\log^{C_2} T\right) \ll R_V^{\frac{1}{2}}T\Delta^{-\frac{1}{2}}\log^{C_2+1} T$$

if with a suitable $c>0$

$$\Delta = cV^4(\log T)^{-C_1-1} > T^{\frac{1}{2}}.$$

This happens for

$$V \gg T^{1/8}(\log T)^{\frac{1}{4}(C_1+1)},$$

which is the range of V in I_2. It follows that

$$R_V \ll \left(TV^{-6}(\log T)^{\frac{1}{2}(1+C_1)+C_2+12}\right) = T^2V^{-12}(\log T)^{3+C_1+2C_2},$$
$$I_2(V) \ll T^2(\log T)^{3+C_1+2C_1}, I_2 \ll T^2(\log T)^{4+C_1+2C_2}.$$

From the estimates for I_1 and I_2 (5.179) follows with

$$D = \operatorname{Max}(2C_1 + 6,\ 4 + C_1 + 2C_2).$$

5.7 The Omega-Result

One of the most striking features of the explicit formula (5.10) is the fact that is can be used to prove the Ω-result of Theorem 5.7, namely

$$E_2(T) = \Omega(T^{\frac{1}{2}}).$$

The proof of this result requires first some technical preparation. We begin with a lemma, which is the counterpart of Lemma 5.1. From (5.160) of Section 5.6 it follows, with $T_1 = T$, $T_2 = 2T$ and $T^\epsilon \leq \Delta \leq T/\log T$

$$\int\limits_T^{2T} I_4(t,\Delta)dt \geq \int\limits_{T+\Delta\log T}^{2T-\Delta\log T} \left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 \left((\Delta\sqrt{\pi})^{-1}\int\limits_T^{2T} e^{-(t-u)^2/\Delta^2}dt\right)du$$
$$\int\limits_{T+\Delta\log T}^{2T-\Delta\log T} \left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 du + O(1)$$
$$= tP_4(\log t)\Big|_T^{2T} + O(\Delta\log^5 T) + E_2(2T-\Delta\log T) - E_2(T+\Delta\log T),$$

296 similarly as in the proof of the first inequality in Lemma 5.1. On the

other hand, we have

$$
\begin{aligned}
\int_T^{2T} I_4(t,\Delta)dt &= \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4\left((\Delta\sqrt{\pi})^{-1}\int_T^{2T} e^{-(t-u)^2/\Delta^2}dt\right)dy\\
&= \int_{T-\Delta\log T}^{2T+\Delta\log T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4\left((\Delta\sqrt{\pi})^{-1}\int_T^{2T} e^{-(t-u)^2/\Delta^2}dt\right)du + O(1)\\
&\le \int_{T-\Delta\log T}^{2T+\Delta\log T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4\left((\Delta\sqrt{\pi})^{-1}\int_{T-2\Delta\log T}^{2T+2\Delta\log T} e^{-(t-u)^2/\Delta^2}dt\right)du + O(1)\\
&= \int_{T-\Delta\log T}^{2T+\Delta\log T}\left|\zeta\left(\frac{1}{2}+iu\right)\right|^4 du + O(1)\\
&= tP_4(\log t)\Big|_T^{2T} + O\left(\Delta\log^5 T\right) + E_2(2T+\Delta\log T) - E_2(T-\Delta\log T).
\end{aligned}
$$

Therefore we have proved

Lemma 5.3. *For $0<\epsilon<1$ fixed and $T^\epsilon \le \Delta \le T/\log T$, we have uniformly*

$$
\begin{aligned}
\int_T^{2T} I_4(t,\Delta)dt \ge{}& 2TP_4(\log 2T) - TP_4(\log T)\\
&+ O\left(\Delta\log^5 T\right) + E_2(2T-\Delta\log T) - E_2(T+\Delta\log T)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_T^{2T} I_4(t,\Delta)dt \le{}& 2TP_4(\log 2T) - TP_4(\log T) + O\left(\Delta\log^5 T\right)\\
&+ E_2(2T+\Delta\log T) - E_2(T-\Delta\log T).
\end{aligned}
$$

The idea of the proof of Theorem 5.7 is as follows. With 297

$$I_4(T,\Delta) = (\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+iT+it\right)\right|^4 e^{-(t/\Delta)^2}dt$$

rewrite (5.157) as

$$\int_T^{2T} I_4(t,\Delta)dt = TQ_4(\log T) + U(T,\Delta), \tag{5.180}$$

where $Q_4(y)$ is a suitable polynomial of degree four in y. Consider then

$$H(V,\Delta) := \int_V^{2V} U(T,\Delta)dT. \tag{5.181}$$

If it can be shown that

$$H(V,\Delta) = \Omega(V^{3/2}) \tag{5.182}$$

with a suitable $\Delta = \Delta(V)$, then this means that $H(V,\Delta) = \circ(V^{3/2})$ cannot hold as $V \to \infty$, and so by (5.181) we see that, as $T \to \infty$, $U(T,\Delta) = \circ(T^{\frac{1}{2}})$ cannot hold. But if Theorem 5.7 were false, namely if $E_2(T) = \circ(T^{\frac{1}{2}})$ were true, then by Lemma 5.3 and (5.180) it would follows that

$$U(T,\Delta) = \circ(T^{\frac{1}{2}}),$$

which we know is impossible. We shall actually prove that, for suitable Δ, we have $H(V,\Delta) = \Omega_\pm(V^{3/2})$. This does not seem to imply that

$$E_2(T) = \Omega_\pm(T^{\frac{1}{2}}), \tag{5.183}$$

although I am convinced that (5.183) must be true. The reason is that in Lemma 5.3 we have differences with the function E_2, which hinders the efforts to obtain the above result from $H(V,\Delta) = \Omega_\pm(V^{3/2})$.

Therefore by (5.180), (5.181) and Theorem 5.1 our problem is first reduced to the study of the integrals

$$\int_T^{2T}\theta(\xi;t,\Delta)dt,\quad \int_T^{2T}\Lambda(k;t,\Delta)dt.$$

But in Section 5.5 we have shown that

$$\int_T^{2T} \Lambda(k;t,\Delta)dt \ll \begin{cases} \Delta^{1-k} & k \le C, \\ (k\Delta)^{-C_1} & k \ge C, \end{cases} \tag{5.184}$$

for any $C_1 > 0$. Thus by (5.184) the contribution of the holomorphic cusp-forms is negligible. Hence it remains to study the integral **298**

$$\int_T^{2T} \theta(\xi;t,\Delta)dt. \tag{5.185}$$

This is carried out by the analysis developed in Section 5.5. If $U(T,\Delta)$ is defined by (5.180) and (5.181), then uniformly for $T^\epsilon \le \Delta \le T^{1-\epsilon}$ we have

$$U(T,\Delta) = U_1(T,\Delta) + O\left(\Delta \log^5 T\right) \tag{5.186}$$

with

$$U_1(T,\Delta) := \frac{1}{\pi} \int\limits_{|\xi| \le T\Delta^{-1}\log^3 T} \frac{|\zeta(\frac{1}{2}+i\xi)^6|}{|\zeta(1+2i\xi)|^2} \int_T^{2T} \sum(\xi;t,\Delta)dt\, d\xi \tag{5.187}$$

$$+ \sum_{x_j \le T\Delta^{-1}\log^3 T} \alpha_j H_j^3\left(\frac{1}{2}\right) \int_T^{2T} \sum(x_j;t,\Delta)dt = U_{11}(T,\Delta) + U_{12}(T,\Delta),$$

say, where

$$\sum(\xi;T,\Delta) := \frac{-1}{4\pi sh(\pi\xi)} \operatorname{Re}\{\Xi^\circ(P_T;-i\xi) - \Xi^\circ(P_T;i\xi)\} \tag{5.188}$$

$$+ \frac{1}{4\pi} \operatorname{Im}\{\Xi^\circ(P_T;i\xi) + \Xi^\circ(P_T;-i\xi)\}$$

$$= \frac{1}{4\pi} \operatorname{Re}\left\{\left(\frac{1}{sh(\pi\xi)} - i\right)\Xi^\circ(P_T;i\xi) - \left(\frac{1}{sh(\pi\xi)} + i\right)\Xi^\circ(P_T;-i\xi)\right\},$$

and as in Section 5.5

$$\Xi^{\circ}(P_T; i\xi) = 2\pi i \int\limits_0^1 y^{-\frac{1}{2}+i\xi}(1-y)^{-\frac{1}{2}+i\xi} S(y;\xi)dy \tag{5.189}$$

with

$$S(y;\xi) = \int\limits_0^{\frac{1}{2}} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iT}(1+xy)^{-\frac{1}{2}-i\xi} \exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx. \tag{5.190}$$

Integrating $U(T,\Delta)$ in (5.181) we obtain

$$H(V,\Delta) = H_{11}(V,\Delta) + H_{12}(V,\Delta) + O\left(V\Delta\log^5 V\right), \tag{5.191}$$

where H_{11} and H_{12} come from the integration of U_{11} and U_{12}, respectively, and $V^{\epsilon} \le \Delta \le V^{1-\epsilon}$. Thus in view of (5.188) - (5.190) the problem is reduced to the study of

$$\int\limits_V^{2V}\int\limits_T^{2T} \Xi^{\circ}(P_t; i\xi)dt\, dT = 2\pi i \int\limits_0^1 y^{-\frac{1}{2}+i\xi}(1+y)^{-\frac{1}{2}+i\xi} S^*(y;V,\xi)dy + O(e^{-c\Delta^2}) \tag{5.192}$$

299 with suitable $c > 0$, where

$$S^*(y;V,\xi) := -\int\limits_0^{\infty} x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iV} \frac{\left((1+x)^{3iV} - 3(1+x)^{iV} + 2\right)}{2\log^2(1+x)}(1+xy)^{-\frac{1}{2}-i\xi} e^{-\frac{1}{4}\Delta^2\log^2(1+x)}dx. \tag{5.193}$$

Here we used the representation (5.189)–(5.190) and the fact that in (5.192) one can replace S by S^* with the total error which is $\ll \exp(-c\Delta^2)$. Henceforth we assume that

$$V^{\epsilon} \le \Delta \le V^{\frac{1}{2}}\log^2 V.$$

In the range

$$\Delta/\log V \le |\xi| \le V\Delta^{-1}\log^3 V \tag{5.194}$$

we have, uniformly for $V^{\epsilon} \le \Delta \le V^{\frac{1}{2}}\log^2 V$,

$$\int_V^{2V}\int_V^{2T} \Xi^{\circ}(P_t; i\xi)dt\, dT \ll V^{3/2}|\xi|^{-5/2}. \tag{5.195}$$

This is obtained similarly as the evaluation of $S(y;\xi)$ and $\Xi^{\circ}(P_T; i\xi)$ in Section 5.5. The bound (5.195) comes essentially from integration of the main terms in the expression for $\Xi^{\circ}(P_t; i\xi)$, namely

$$-2^{3/2}\pi^2 \exp\left(i\xi\log\frac{\xi}{4et}\right)(t\xi)^{-\frac{1}{2}}e^{-(\Delta\xi/2t)^2} \qquad (\xi > 0)$$

over $T \le t \le 2T$, $V \le T \le 2V$. The integrals of the error terms will be $\ll V^{3/2}|\xi|^{-5/2}$. Another possibility to obtain (5.195) is to note that $S^*(y; V, \xi)$ differs from $S(y; V, \xi)$ by a factor $\log^{-2}(1 + x)$, which at the saddle point $x = x_0$ changes the final expression by a factor of order $x_0^{-2} \asymp \xi^{-2}V^2$. Thus when we multiply by $V^{-\frac{1}{2}}$ (the order of the integral for S) and integrate over y (which makes a contribution of order $|\xi|^{-\frac{1}{2}}$ in the expression for ξ°), we obtain again (5.195).

It will turn out that for the range (5.194) the estimate (5.195) is sufficient. For the remaining range 300

$$|\xi| \le \Delta/\log V \tag{5.196}$$

we shall establish an asymptotic formula for the integral in (5.195). First note that the integral in (5.193) may be truncated at $x = \Delta^{-1}\log^{\frac{3}{4}} V$ with a negligible error because of the presence of the presence of the exponential factor. Then (5.196) implies that in the remaining integral

$$(1 + xy)^{-\frac{1}{2}-i\xi} = 1 + O(x(1 + |\xi|)).$$

Consequently

$$S^*(y; V, \xi) = S_1^*(y; v, \xi) + O\left((1 + |\xi|)V\Delta^{-\frac{1}{2}}\log^{3/8} V\right), \tag{5.197}$$

where

$$S_1^*(y;V,\xi) := -\int\limits_0^\infty x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iV} \tag{5.198}$$

$$\frac{\left((1+x)^{3iV}-3(1+x)^{iV}+2\right)}{2\log^2(1+x)}\exp\left(-\frac{\Delta^2}{4}\log^2(1+x)\right)dx,$$

since the integral from 0 to $\Delta^{-1}\log^{\frac{3}{4}}V$ containing the error term $O(x(1+|\xi|))$ may be estimated trivially as

$$\ll (1+|\xi|)\int\limits_0^{\Delta^{-1}\log^{\frac{3}{4}}V} x^{-\frac{1}{2}}x\left|\int\limits_V^{2V}\int\limits_T^{2T}(1+x)^{it}dt\,dT\right|dx$$

$$\ll (1+|\xi|)\int\limits_0^{\Delta^{-1}\log^{\frac{3}{4}}V}\frac{x^{+\frac{1}{2}}}{\log(1+x)}\left|\int\limits_V^{2V}\left((1+x)^{2iT}-(1+x)^{iT}\right)dT\right|dx$$

$$\ll V(1+|\xi|)_0^{\Delta^{-1}\log^{\frac{3}{4}}V}x^{-\frac{1}{2}}dx \ll (1+|\xi|)V\Delta^{-\frac{1}{2}}\log^{3/8}V,$$

and so (5.197) follows. In the range (5.196) the total contribution of the error term in (5.197) will be

$$\ll V\Delta^{-\frac{1}{2}}\log^{3/8}V\left\{\int\limits_{|\xi|\le\Delta/\log V}\frac{|\zeta(\frac{1}{2}+i\xi)^6|}{|\zeta(1+2i\xi)|^2}(1+|\xi|)d\xi+\sum_{x_j\le\frac{\Delta}{\log V}}\alpha_j x_j\left|H_j^3\left(\frac{1}{2}\right)\right|\right\}$$

$$\ll V\Delta^{5/2}\log^C V$$

301 on using (5.148) and (5.149). The integral in (5.198) is evaluated similarly as we deduced (5.136) from (5.134). Thus we have

$$S_1^*(y;V,\xi) = -(\Delta\sqrt{\pi})^{-1}\int\limits_{-\infty}^{\infty}e^{-(u/\Delta)^2}\int\limits_0^\infty x^{-\frac{1}{2}+i\xi}(1+x)^{-\frac{1}{2}+iV+iu}$$

$$\frac{\left((1+x)^{3iV}-3(1+x)^{iV}+2\right)}{2\log^2(1+x)}dx\,du$$

$$= -(\Delta\sqrt{\pi})^{-1}\int\limits_{-\Delta\log V}^{\Delta\log V} I(u)e^{-(u/\Delta)^2}du + O\left(e^{-\frac{1}{2}\log^2 V}\right),$$

say. Because of uniform convergence we have

$$\begin{aligned}
I(u) &= \lim_{\alpha\to\frac{1}{2}+0}\int\limits_0^\infty x^{-\frac{1}{2}+i\xi}(1+x)^{-\alpha+iV+iu}\frac{\left((1+x)^{3iV}-3(1+x)^{iV}+2\right)}{2\log^2(1+x)}dx\\
&= -\lim_{\alpha\to\frac{1}{2}+0}\int\limits_0^\infty x^{-\frac{1}{2}+i\xi}(1+x)^{-\alpha+iu}\int\limits_V^{2V}\int\limits_T^{2T}(1+x)^{it}dt\,dT\,dx\\
&= -\lim_{\alpha\to\frac{1}{2}+0}\int\limits_0^{2V}\int\limits_T^{2T}\int\limits_0^\infty x^{-\frac{1}{2}+i\xi}(1+x)^{-\alpha+iu+it}dx\,dt\,dT\\
&= -\lim_{\alpha\to\frac{1}{2}+0}\int\limits_V^{2V}\int\limits_T^{2T}\frac{\Gamma\left(\frac{1}{2}+i\xi\right)\Gamma\left(\alpha-\frac{1}{2}-iu-it-i\xi\right)}{\Gamma(\alpha-iu-it)}dt\,dT\\
&= -\int\limits_V^{2V}\int\limits_T^{2T}\frac{\Gamma\left(\frac{1}{2}+i\xi\right)\Gamma(-iu-it-i\xi)}{\Gamma\left(\frac{1}{2}-iu-it\right)}dt\,dT.
\end{aligned}$$

We use Stirling's formula, as in the derivation of (5.136), to find that uniformly for $|u| \le \Delta\log V$, $\xi \le \Delta/\log V$, $V \asymp T$, $\Delta \le V^{\frac{1}{2}}\log^2 V$

$$\begin{aligned}
I(u) = {}&-e^{\frac{1}{4}\pi i}\Gamma\left(\frac{1}{2}+i\xi\right)e^{-\frac{1}{2}\pi\xi}\int\limits_V^{2V}\int\limits_T^{2T}t^{-\frac{1}{2}-i\xi}dt\,dT\\
&+O\left(e^{-\frac{1}{2}\pi(\xi+|\xi|)}V^{\frac{1}{2}}\Delta^2\log^2 V\right).
\end{aligned}\tag{5.199}$$

Therefore (5.199) gives

$$S_1^*(y;V,\Delta) = e^{\frac{1}{4}\pi i-\frac{1}{2}\pi\xi}\Gamma\left(\frac{1}{2}+i\xi\right)\int\limits_V^{2V}\int\limits_T^{2T}t^{-\frac{1}{2}-i\xi}dt\,dT$$

$$+ O\left(e^{-\frac{1}{2}\pi(\xi+|\xi|)} V^{\frac{1}{2}} \Delta^2 \log^2 V\right). \tag{5.200}$$

302 Hence from (5.192), (5.197) and (5.200) we have

$$\int_V^{2V}\int_T^{2T} \Xi^\circ(P_t; i\xi) dt\, dT = 2\pi i e^{\frac{1}{4}\pi i - \frac{1}{2}\pi\xi} \frac{\Gamma^3(\frac{1}{2}+i\xi)}{\Gamma(1+2i\xi)} \int_V^{2V}\int_T^{2T} t^{-\frac{1}{2}-i\xi} dt\, dT$$
$$+ O\left(e^{-\frac{1}{2}\pi(\xi+|\xi|)} V^{\frac{1}{2}} \Delta^2 \log^2 V\right) \tag{5.201}$$

uniformly for $V^\epsilon \le \Delta \le V^{\frac{1}{2}} \log^2 V$, $|\xi| \le \Delta/\log V$, and we note that the main term on the right-hand side of (5.201) is $\ll V^{3/2}(|\xi|+1)^{-5/2}$, similarly as in (5.195). The total contribution of the integrals in (5.195), in the range (5.194), is

$$\ll V^{3/2} \int\limits_{\frac{\Delta}{\log V} \le |\xi| \le \frac{V}{\Delta}\log^3 V} \frac{|\zeta(\frac{1}{2}+i\xi)|^6}{|\zeta(1+2i\xi)|^2} |\xi|^{-5/2} d\xi + V^{3/2} \sum_{x_j \ge \frac{\Delta}{\log V}} \alpha_j |H_j^3\left(\frac{1}{2}\right)| x_j^{-5/2}$$
$$\ll V^{3/2}\Delta^{-\frac{1}{2}} \log^C V$$

on using (5.148) and (5.149). Hence we obtain from (5.191)

$$H(V,\Delta) = H_{11}^*(V,\Delta) + H_{12}^*(V,\Delta) + O\left(V\Delta^{5/2}\log^C V\right) \tag{5.202}$$
$$+ O\left(V^{\frac{1}{2}}\Delta^2 \log V\right) + O\left(V^{3/2}\Delta^{-\frac{1}{2}} \log^C V\right),$$

where

$$H_{11}^*(V,\Delta) := \frac{1}{\pi} \int\limits_{|\xi| \le \Delta/\log V} \frac{|\zeta(\frac{1}{2}+i\xi)|^6}{|\zeta(1+2i\xi)|^2} \int_V^{2V}\int_T^{2T} \overset{\circ}{\sum}(\xi; t,\Delta) dt\, dT\, d\xi, \tag{5.203}$$

$$H_{12}^*(V,\Delta) := \sum_{x_j \le \Delta/\log V} \alpha_j H_j^3\left(\frac{1}{2}\right) \int_V^{2V}\int_T^{2T} \overset{\circ}{\sum}(x_j; t,\Delta) dt\, dT, \tag{5.204}$$

and $\overset{\circ}{\sum}(\xi; t,\Delta)$ is obtained if in the definition (5.188) of $\sum(\xi; t,\Delta)$ one replaces ξ° by

$$2\pi i e^{\frac{1}{4}\pi i - \frac{1}{2}\pi\xi} \frac{\Gamma^3\left(\frac{1}{2}+i\xi\right)}{\Gamma(1+2i\xi)} t^{-\frac{1}{2}-i\xi}$$

303 in view of (5.201). Since $t \asymp V$ in (5.203) and $t^{-i\xi} = e^{-i\xi \log t}$, it is seen that, performing an integration by parts over ξ in $H^*_{11}(V, \Delta)$, we obtain

$$H^*_{11}(V, \Delta) \ll V^{3/2}/\log V \tag{5.205}$$

Note that all the error terms in (5.202) are $\ll V^{3/2}/\log V$ for $V^\delta \leq \Delta \leq V^{1/5-\delta}$, when $0 < \delta < \dfrac{1}{10}$ is any fixed constant. Also we may replace the sum in (5.204) by the infinite series, producing na error term $\ll V^{3/2}/\log V$. It follows from (5.202) – (5.205) that we have proved

Lemma 5.4. *Let $0 < \delta < \dfrac{1}{10}$ be arbitrary, but fixed. Then for $V^\delta \leq \Delta \leq V^{1/5-\delta}$ we have uniformly*

$$H(V, \Delta) = 2V^{\frac{3}{2}} \operatorname{Im}\left\{\sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right)(F(x_j)V^{ix_j} + F(-x_j)V^{-ix_j})\right\} + O\left(\frac{V^{\frac{3}{2}}}{\log V}\right), \tag{5.206}$$

where $H(V, \Delta)$ is defined by (5.181) and

$$F(z) := \left(\frac{1}{sh(\pi z)} + i\right) e^{\frac{1}{2}\pi(z+i)} \frac{\Gamma^3\left(\frac{1}{2} - iz\right)\left(2^{\frac{1}{2}+iz} - 1\right)\left(2^{\frac{3}{2}+iz-1}\right)}{\Gamma(1 - 2iz)(1 + 2iz)(3 + iz)}.$$

Having at our disposal Lemma 5.4, we may proceed to prove $E_2(T) = \Omega(T^{\frac{1}{2}})$, or equivalently by (5.182) and (5.206) that

$$\operatorname{Im}\left\{\sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right)(F(x_j)V^{ix_j} + F(-x_j)V^{-x_j})\right\} = \Omega(1) \tag{5.207}$$

by fixing $\Delta = V^{1/6}$, say. This will follow (even with $\Omega_\pm(1)$ in (5.207)) from

Lemma 5.5. *Let $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ be two complex sequences such that $\sum\limits_{j=1}^{\infty}(|a_j| + |b_j|) < \infty$ and $|a_1| > |b_1|$, and let $\{\omega_j\}_{j=1}^\infty$ be a strictly increasing sequence of positive numbers. Then, as $x \to \infty$*

$$\operatorname{Im}\left\{\sum_{j=1}^{\infty}\left(a_j x^{i\omega_j} + b_j x^{-i\omega_j}\right)\right\} = \Omega_\pm(1).$$

Before giving the proof of Lemma 5.5, which is not difficult, let us show how Lemma 5.5 implies (5.207), and consequently Theorem 5.7. Denote by $\{\mu_h\}_{h=1}^{\infty}$, arranged in increasing order, the set of all distinct $x_j's$ and put, for a given μ_h, 304

$$G_h := \sum_{x_j=\mu_h} \alpha_j H_j^3\left(\frac{1}{2}\right).$$

With this notation (5.207) becomes

$$\operatorname{Im}\left\{\sum_{h=1}^{\infty} G_h\left(F(\mu_h)V^{i\mu_h} + F(-\mu_h)V^{-i\mu_h}\right)\right\} = \Omega(1).$$

Set in Lemma 5.5 $a_j = G_jF(\mu_j)$, $b_j = G_jF(-\mu_j)$, $\omega_j = \mu_j$, $x = V$. Since the series in (5.206) is majorized by a multiple of

$$\sum_{j=1}^{\infty} \alpha_j \left|H_j^3\left(\frac{1}{2}\right)\right| x_j^{-5/2} = O(1),$$

we obviously have

$$\sum_{j=1}^{\infty}\left(|a_j| + |b_j|\right) < +\infty.$$

We have yet to show that

$$|G_1|\,|F(\mu_1)| > |G_1|\,|F(-\mu_1)|,$$

which is true if $G_1 \neq 0$ and

$$|F(\mu_1)| > |F(-\mu_1)|. \tag{5.208}$$

From the definition of $F(z)$ it follows that, for $x > 0$,

$$|F(x)| = e^{-x}|F(-x)|,$$

so that (5.208) is true. Now to have $G_1 \neq 0$, we can relabel the $\mu_h's$, if necessary, and by actual numerical calculation find a h such that $G_h \neq 0$. As this is tedious, we appeal to the asymptotic formula

$$\sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right) h(x_j) = (1 + O(1))\frac{8}{3}\pi^{-3/2}K^3\Delta\log^3 K \quad (K \to \infty), \tag{5.209}$$

305 which is uniform for $K^{\frac{1}{2}+\epsilon} \le \Delta \le K^{1-\epsilon}$, where $0 < \epsilon < \frac{1}{4}$ is fixed and

$$h(x) = \left(x^2 + \frac{1}{4}\right)\left\{\exp\left(-\left(\frac{x-K}{\Delta}\right)\right)^2 + \exp\left(-\left(\frac{x+K}{\Delta}\right)^2\right)\right\}.$$

This has been proved by Y. Motohashi [129], and it clearly implies that there are actually infinitely many h such that $G_h \neq 0$. So all there remains is the

PROOF OF LEMMA 5.5. Let

$$f_0(x) := \operatorname{Im}\left\{\sum_{j=1}^{\infty}\left(a_j x^{i\omega_j} + b_j x^{-i\omega_j}\right)\right\},$$

and for $n \ge 1$

$$f_n(x) := \int\limits_x^{\tau x} f_{n-1}(u)\frac{du}{u},$$

where $\tau = \exp(\pi/\omega_1)$ (> 1). Then by induction it follows that

$$f_n(x) = \operatorname{Im}\left\{\sum_{j=1}^{\infty}\left(a_j\left(\frac{\tau^{i\omega_j}-1}{i\omega_j}\right)^n x^{i\omega_j} + b_j\left(\frac{\tau^{-i\omega_j-1}}{-i\omega_j}\right)x^{-i\omega_j}\right)\right\}.$$

Since $\tau^{i\omega_1} = -1$ we obtain

$$f_n(x) = 2^n \operatorname{Im}\left\{a_1\left(\frac{i}{\omega_1}\right)^n x^{i\omega_1} + b_1\left(\frac{-i}{\omega_1}\right)^n x^{-i\omega}\right\} + R_n(x), \tag{5.210}$$

where uniformly

$$R_n(x) = O(2^n\omega_2^{-n}),$$

because $\sum\limits_{j=1}^{\infty}(|a_j| + |b_j|) < \infty$. If $f_n(x) = \Omega_+(1)$ is not true, then for any $\epsilon > 0$ and $x \ge x_0(\epsilon)$ we have $f_n(x) < \epsilon$. Since $\omega_2 > \omega_1$, we have for any $\delta > 0$

$$|R_n(x)| < \delta|a_1|\omega_1^{-n}2^n$$

for $n \geq n_0(\epsilon)$, uniformly in x. Now for $n = 4N + 1 (\geq n_0)$ we have $i^{4N+1} = i^n = -1$, so if we take

$$x = \exp\left(\frac{2M\pi - \arg a_i}{\omega_1}\right)$$

306 and M is any large positive integer, we have

$$\operatorname{Im}\left\{a_1\left(\frac{i}{\omega_1}\right)^n x^{i\omega_1}\right\} = \operatorname{Im}\left\{i|a_1|\omega_1^{-n}\right\} = |a_1|\omega_1^{-n}. \tag{5.211}$$

Therefore (5.210) yields, if $f_n(x) < \epsilon$,

$$\epsilon > 2^n|a_1|\omega_1^{-n} - 2^n|b_1|\omega_1^{-n} - \delta|a_1|\omega_1^{-n2n} = 2^n\omega_1^{-n}((1-\delta)|a_1| - |b_1|).$$

But for sufficiently small δ we have $(1-\delta)|a_1| > |b_1|$, so fixing $n(\geq n_0(\delta))$ the last inequality produces a contradiction, since ϵ may be arbitrarily small. Analogously we obtain a contradiction if $f_n(x) = \Omega_-(1)$ does not hold by noting that $i^{4N+2} = -i$, so we can obtain that the left-hand side of (5.211) equals $-|a_1|\omega_1^{-n}$. Thus, by the defining property of $f_n(x)$, we have also that

$$f_{n-1}(x) = \Omega_\pm(1), \quad f_{n-2}(x) = \Omega_\pm(1), \ldots,$$

until finally we conclude that $f_0(x) = \Omega_\pm(1)$, and the lemma is proved. We remark that the proof of Lemma 5.5 actually yields the following: There exist constants $A > 0$ and $B > 1$ such that every interval $[V, BV]$ for $V \geq V_0$ contains two points V_1 and V_2 such that for $V^\delta \leq \Delta \leq V^{1/5-\delta}$

$$H(V_1, \Delta) > AV_1^{3/2}, \quad H(V_2, \Delta) < -AV_2^{3/2}.$$

The foregoing proof was self-contained, but there is another, quicker way to deduce (5.207). Let

$$L_1(V) := \sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right) F(x_j) V^{ix_j}, \quad L_2(V) := \sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right) \overline{F(-x_j)} V^{ix_j},$$

and set

$$\ell_1(x) = L_1(e^x), \quad \ell_2(x) = L_2(e^x).$$

Then $\ell_1(x)$ and $\ell_2(x)$ are almost periodic functions in the sense of H. Bohr [15] by the Fundamental Theorem for almost periodic functions, because e.g.

$$\ell_1(x) = \sum_{h=1}^{\infty} G_h F(\mu_h) e^{i\mu_h x} = \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j H_j^3\left(\frac{1}{2}\right) F(x_j) e^{ix_j x}$$

uniformly for all x, since **307**

$$\sum_{j=1}^{\infty} \left|\alpha_j H_j^3\left(\frac{1}{2}\right) F(x_j)\right| \ll \sum_{j=1}^{\infty} \alpha_j \left|H_j^3\left(\frac{1}{2}\right)\right| x_j^{-5/2} \ll 1.$$

Then the function

$$\begin{aligned}\ell(x) &:= \operatorname{Im}\left\{\ell_1(x) + \overline{\ell_2(x)}\right\}\\ &= \frac{1}{2i}\left\{(\ell_1(x) - \ell_2(x)) - \left(\overline{\ell_1(x)} - \overline{\ell_2(x)}\right)\right\} = \sum_{h=-\infty}^{\infty} G_h a_h e^{i\mu_h x},\end{aligned}$$

where we have set

$$\mu_{-h} = -\mu_h, \quad G_{-h} = G_h, a_h = \frac{1}{2i}\left(F(\mu_h) - F(-\mu_h)\right),$$

is also almost periodic, since conjugation, multiplication by constants and addition preserve almost-periodicity. By Parseval's identity for almost periodic functions

$$\sum_{h=-\infty}^{\infty} G_h^2 |a_h|^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\ell(x)|^2 dx.$$

Since $|F(x)| = e^{-x}|F(-x)|$ for $x > 0$ and $G_h \neq 0$ for at least one h, it follows that the series on the left-hand side above is positive. Thus by continuity we cannot have $\ell(x) \equiv 0$. Let x_0 be a point such that $\ell(x_0) \neq 0$. By the definition of an almost periodic function, to every $\epsilon > 0$ there corresponds a length $L = L(\epsilon)$ such that each interval of length $L(\epsilon)$ contains a translation number τ for which

$$|\ell(x) - \ell(x+\tau)| < \epsilon$$

for all real x, and so in particular this holds for $x = x_0$. If $\epsilon = \frac{1}{2}|\ell(x_0)|(> 0)$, then either $\ell(x_0+\tau) > \frac{1}{2}|\ell(x_0)|$ or $\ell(x_0+\tau) < -\frac{1}{2}|\ell(x_0)|$ holds for some arbitrarily large values of τ. Hence (5.207) follows, because $\ell(\log V)$ is the left-hand side of (5.207).

Note that this argument gives $\ell(x) = \Omega_{\pm}(1)$ if we can find (say by
308 numerical computation) two values x_1, x_2 such that $\ell(x_1) > 0$, $\ell(x_2) < 0$, and this is regardless of any non-vanishing hypothesis for G_h.

We shall now explore another possibility of obtaining omega-results for $E_2(T)$, by a method which could possible give

$$\limsup_{T\to\infty} |E_2(T)|T^{-\frac{1}{2}} = +\infty, \tag{5.212}$$

which is stronger than $E_2(T) = \Omega(T^{\frac{1}{2}})$ of Theorem 5.7.

Suppose that

$$F(y) = \sum_{j=1} f(j)\cos\left(a_j y + b_j\right), \quad \sum_{j=1}^{\infty} |f(j)|_{a_j}^{-1} < +\infty$$

is a trigonometric series with $a_j > 0$, and a_j nondecreasing. Let

$$K_J(y) = K_{a_{\frac{1}{2}J}}(y) = \frac{a_J}{2\pi}\left(\frac{\sin(\frac{1}{2}a_J y)}{\frac{1}{2}a_J y}\right)^2$$

be the Fejér kernel of index $\frac{1}{2}a_J$. Then

$$\int_{-1}^{1} e^{ia_j y} k_J(y)dy = \begin{cases} 1 - a_j/a_J + O(1/a_j) & \text{if } j \le J, \\ O(1/a_j) & \text{if } j > J. \end{cases}$$

Hence

$$\begin{aligned} \tilde{F}_J(y) &:= \int_{-1}^{1} F(y+u)k_J(u)du \\ &= \sum_{j\le J}\left(1 - \frac{a_j}{a_J}\right) f(j)\cos(a_j y + b_j) + O(1), \end{aligned} \tag{5.213}$$

which is essentially the transformation used in the proof of the first part of Theorem 3.4. However, one has

$$k_J(y) \geq 0,\ 0 < \int_{-1}^{1} k_J(u)du < 1$$

for all $J \geq 1$, so that if $\tilde{F}_J(y)$ is large, then $F(y_1)$ is large for some y_1 such that $|y - y_1| \leq 1$.

Now we use (5.14) of Theorem 5.3, namely

$$\int_0^T I_4(t, \Delta)dt = O(T^{\frac{1}{2}}) + O\left(\Delta \log^5 T\right) \tag{5.214}$$

$$= TP_4(\log T) + \pi\left(\frac{T}{2}\right)^{\frac{1}{2}} \sum_{j=1}^{\infty} \alpha_j c_j H_j^3\left(\frac{1}{2}\right)\cos\left(x_j \log \frac{4eT}{x_j}\right) e^{-(\Delta x_j/2T)^2}$$

uniformly for $T^{\frac{1}{2}} \log^{-A} T \leq \Delta \leq T^{\frac{3}{4}}$ and any fixed $A > 0$. In (5.214) **309**
take $\Delta = T^{\frac{1}{2}} \log^{-6} T$. To show that (5.212) holds it suffices to show that

$$\limsup_{T\to\infty} \left|\sum_{j=1}^{\infty} \alpha_j c_j H_j^3\left(\frac{1}{2}\right)\cos\left(x_j \log \frac{4eT}{x_j}\right) e^{-(\Delta x_j/2T)^2}\right| = +\infty. \tag{5.215}$$

Namely, if the lim sup in (5.212) were finite, then so would be the limsup in (5.215) by Lemma 5.3. To prove (5.215) one can prove

$$\limsup_{T\to\infty} \left|\sum_{j=1}^{\infty} \alpha_j c_j H_j^3\left(\frac{1}{2}\right)\cos\left(x_j \log T + x_j \log \frac{4e}{x_j}\right)\right| = +\infty. \tag{5.216}$$

since if the limsup in (5.215) is finite, then the partial sum of any length of the series in (5.215) is bounded. But by using

$$e^{-(\Delta x_j/2T)^2} = 1 + O\left(\Delta^2 x_j^2 T^{-2}\right)$$

it is seen that any partial sum of the series in (5.216) is bounded, which is impossible. We rewrite (5.216) as

$$\limsup_{T\to\infty} \left|\sum_{h=1}^{\infty} g_h \cos\left(\mu_h \log T + \mu_h \log \frac{4e}{\mu_h}\right)\right| = +\infty, \tag{5.217}$$

where as before $\{\mu_h\}_{h=1}^{\infty}$ is the set of distinct x_j's in increasing order, and for a given μ_h

$$g_h := \sum_{x_j=\mu_h} \alpha_j c_j H_j^3\left(\frac{1}{2}\right)$$

with $c_j \sim x_j^{-3/2}$ as $j \to \infty$. We shall consider $\tilde{F}_J(y)$ in (5.213) with $\log T = y$, $a_j = \mu_j$, $b_j = \mu_j \log(\frac{4e}{\mu_j})$, $f(j) = g_j$. It will be assumed that $g_j > 0$, for it there are negative values the proof can be modified by making the relevant cosines near to 1 for such j (there are infinitely many values of j such that $g_j \neq 0$ by (5.209)). At this point the following
310 effective version of L. Kronecker's theorem on Diophantine approximation is used: Let $b_1, b_2, \ldots, b_J$ be given real numbers, $a_1, a_2, \ldots, a_J$ given real numbers linearly independent over the integers, and q a given positive number. Then there is an arbitrarily large positive number t and integers $x_1, x_2, \ldots, x_J$ such that

$$|ta_j - b_j - x_j| \leq 1/q \quad (j = 1, 2, \ldots, J)$$

In this result take (with a slight abuse of notation for a_j and b_j) $a_j = \mu_j/(2\pi)$, $b_j = \mu_j(2\pi)^{-1}\log(4e/\mu_j)$, $q = \mu_j$. Then we have with $y = t$

$$\cos\left(\mu_j y + \mu_j \log\frac{4e}{\mu_j}\right) = 1 + O\left(\frac{1}{\mu_J}\right) \quad (j \leq V).$$

Therefore

$$\begin{aligned}\tilde{F}_J(y) &= \sum_{j\leq J}\left(1 - \frac{\mu_j}{\mu_J}\right) g_j \cos\left(\mu_j y + \mu_j \log\frac{4e}{\mu_j}\right) + O(1)\\ &= \sum_{j\leq J'}\left(1 - \frac{x_j}{x_{J'}}\right)\alpha_j c_j H_j^3\left(\frac{1}{2}\right) + O(1)\end{aligned}$$

where $J' \geq J$ and the O-constant is absolute. Since $x_j \sim \sqrt{12j}$ we have $1 - x_j/x_{J'} \geq 1/2$ for $j \leq J^{\frac{1}{2}}$ and J large. Therefore

$$\tilde{F}_J(y) \gg \sum_{j\leq J^{\frac{1}{2}}} \alpha_j x_j^{-3/2}\left|H_j^3\left(\frac{1}{2}\right)\right| + O(1),$$

and the last sum over j is unbounded as $J \to \infty$. Thus (5.212) follows if we can prove that the μ_j*'s are linearly independent over the integers*, which does not seem obvious. Very likely even better Ω-results than (5.212) are true, but their proofs will depend on arithmetic properties of the x_j's, which so far appear to be rather obscure. The problem of linear independence is not trivial even when one tries to get Ω_--results for the (relatively simple) function

$$\Delta(x) = \pi^{-1} 2^{-\frac{1}{2}} x^{\frac{1}{4}} \sum_{n=1}^{\infty} d(n) n^{-\frac{3}{4}} \cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(1)$$

by the classical method of A.E. Ingham, which rests on certain integral (Laplace) transforms. The method requires the arithmetic fact that (roughly) the square roots of squarefree numbers are linearly independent over the rationals. The last assertion, although not very difficult, certainly requires proof. Perhaps Ingham's method could be also used in connection with Ω-results for $E_2(T)$, but very likely a "linear inde- **311**
pendence condition" would be required again.

NOTES FOR CHAPTER 5

The fundamental result of this chapter is Theorem 5.1. The groundwork for it laid in Y. Motohashi [127], the result (together with variants of Theorems 5.2 and 5.3) was announced in [3, Part VI] and proved in detail in [128]. The proof given in the text is based on a draft version of Motohashi [128], and for this reason some of the notation differs from the one used in Motohashi [128]. A conditional proof of Theorem 5.7 is sketched in Ivić-Motohashi [79], namely if not all G_h vanish, where for a given μ_h

$$G_h := \sum_{x_j=\mu_h} \alpha_j H_j^3\left(\frac{1}{2}\right),$$

and $\{\mu_h\}$ denotes the set of distinct x_j's. Detailed proofs of Theorems 5.4 - 5.8 are to be found in Ivić-Motohashi [80], while the results on spectral mean values needed in the proofs of these theorems are furnished by Motohashi [129], and are quoted at the appropriate places in the text.

In Motohashi [128] the series appearing in the formulation of Theorem 5.1 are written as

$$\sum_{j=1}^{\infty} \alpha_j H_j^3\left(\frac{1}{2}\right)\theta(x_j; T, \Delta) + \sum_{k=6}^{\infty}\sum_{j\le d_{2k}} \alpha_{j,2k} H_{j,2k}^3\left(\frac{1}{2}\right)\theta\left(i\left(\frac{1}{2}-k\right); T, \Delta\right),$$

where the function θ is explicitly written down as

$$\begin{aligned}\theta(r; T, \Delta) = \int_0^\infty & x^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} \cos\left(T\log\left(1+\frac{1}{x}\right)\right)\\ & \Lambda(x,r)\exp\left(-\frac{\Delta^2}{4}\log^2\left(1+\frac{1}{x}\right)\right)dx,\\ \Lambda(x,r) = \operatorname{Re}\Bigg\{ & x^{-\frac{1}{2}-ir}\left(1+\frac{i}{\sinh(\pi r)}\right)\frac{\Gamma^2\left(\frac{1}{2}+ir\right)}{\Gamma(1+2ir)}\\ & F\left(\frac{1}{2}+ir, \frac{1}{2}+ir; 1+2ir; -\frac{1}{x}\right)\Bigg\}\end{aligned}$$

with the hypergeometric function (see the definition after (5.130)). This
312 representation is possible because $\Phi(P_T; k - \frac{1}{2}) = 0$, and it stresses the analogy between the holomorphic and nonholomorphic cusp forms. However, the contribution of the holomorphic part, as shown by (5.125), is small and it is the nonholomorphic part that is difficult to deal with.

The estimate

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{12} dt \ll T^2 \log^{17} T \tag{5.218}$$

was first proved by D.R. Heath-Brown [59] (see also Chapter 7 of Ivić [75]). It can be also deduced from H. Iwaniec's result [82] on the fourth moment of $\zeta(\frac{1}{2} + it)$ in short intervals, i.e. Theorem 5.8 with T^ϵ instead of log-powers. Then one gets $T^{2+\epsilon}$ in (5.218) instead of $T^2 \log^{17} T$. Iwaniec's proof was based on the intricate use of Kloosterman sums (see also his papers [82], [84], [85] and the joint works with J.-M. Deshouillers [29], [31], [32]). Deshouillers-Iwaniec [31], [32] continue the work of Iwaniec [83]. They obtain several interesting results on upper bounds for integrals of the form

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 \left|\sum_{m \le M} a_m m^{-\frac{1}{2} - it}\right|^2 dt, \quad \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left|\sum_{m \le M} a_m m^{-\frac{1}{2} - it}\right|^4 dt,$$

where the a_m's are arbitrary complex numbers satisfying $|a_m| \le 1$. The proofs use deep bounds for various sums of Kloosterman sums, but although the results are strong, they cannot yield the sixth moment that $I_3(T) \ll_\epsilon T^{1+\epsilon}$. The results of H. Iwaniec [82] are reproved by M. Jutila [96], [97], [98] by another method, which is essentially elementary in nature and does not involve the use of Kloosterman sums. It is based on transformations of exponential sums with the divisor function. For this see his paper [89], and the monograph [95] which provides an extensive account of this topic. The method is suitable for generalizations to exponential sums involving Fourier coefficients of modular forms and Maass wave forms. For applications of Jutila's method to the latter, see the papers of T. Meurman [119], [121].

For S. Ramanujan's work on $c_r(n)$ see G.H. Hardy [55] and S. Ramanujan [145]. The identity (5.26) can be extended to $SL(3,\mathbb{Z})$. For
313 this see D. Bump [18], and for generalizations to higher dimensions see the papers of D. Goldfeld [41], [42].

The functional equation (5.32) was proved by T. Estermann [33]. A proof may be also found in M. Jutila [95].

The definition of $Z_{m,n}(s)$ in (5.42) is from N.V. Kuznetsov [104]. A. Selberg, whose paper [150] incited much subsequent research, does not have the factor $(2\pi\sqrt{mn})^{2s-1}$ in the definition of $Z_{m,n}(s)$. Selberg has shown that it is possible to continue the function $Z_{m,n}(s)$ meromorphically onto the entire s-plane, and gave the location of its poles in the half-plane $\operatorname{Re} s \geq \frac{1}{2}$.

The term "non-Euclidean" regarding the upper complex half-plane $\mathbb{H}$ comes from the fact that $\mathbb{H}$ represents the Poincaré model of Lobačevskyi's hyperbolic non-Euclidean geometry. In this model the straight lines are either semicircles with centers on the real axis or semilines perpendicular to it. Detailed accounts on the spectral theory of the non-Euclidean Laplace operator may be found in N.V. Kuznetsov [104] or A. Terras [154].

To prove (5.46) make the change of variables $u = xt$, $v = \frac{x}{t}$. Then

$$\begin{aligned}
\int_0^\infty K_r(x)x^{s-1}dx &= \frac{1}{2}\int_0^\infty\int_0^\infty x^{s-1}t^{r-1}e^{-\frac{1}{2}x(t+t^{-1})}dt\,dx \\
&= \frac{1}{4}\int_0^\infty\int_0^\infty (uv)^{\frac{1}{2}(s-1)}u^{\frac{1}{2}(r-1)}v^{\frac{1}{2}(1-r)}e^{-\frac{1}{2}(u+v)}v^{-1}du\,dv \\
&= 2^{s-2}\Gamma\left(\frac{s+r}{2}\right)\Gamma\left(\frac{s-r}{2}\right).
\end{aligned}$$

The asymptotic formula (5.48) is proved by N.V. Kuznetsov in [104]. It improves a formula obtained by R.W. Bruggeman [17].

The Rankin-Selberg convolution method was introduced independently in the fundamental papers of R.A. Rankin [147] and A. Selberg [149]. See also the papers of P. Ogg [131] and D. Bump [19] for the Rankin-Selberg method in a general setting.

In view of (5.50) and $\rho_j(1)t_j(n) = \rho_j(n)$ it follows, by an application of the Cauchy-Schwarz inequality, that the series in (5.51) is actually **314** absolutely convergent for $\operatorname{Re} s > 1$. Analogously the series in (5.57) converges absolutely for $\operatorname{Re} s > 1$.

The functional equation (5.53) for Hecke series is a special case of Theorem 2.2 of N.V. Kuznetsov [106].

The asymptotic formula (5.55), proved by Y. Motohashi [129], is a corrected version of a result of N.V. Kuznetsov. Namely, in [106] Kuznetsov states without proof that, for fixed real t and $\frac{1}{2} < \sigma < 1$, one has

$$\sum_{x_j \le T} \alpha_j |H_j(\sigma + it)|^2 = \left(\frac{T}{\pi}\right)^2 \left(\zeta(2\sigma) + \frac{\zeta(2-2\sigma)}{2-2\sigma}\left(\frac{T^2}{2\pi}\right)^{1-2\sigma}\right) + O(T\log T),$$

while for $\sigma = \dfrac{1}{2}$ the right-hand side in the above formula has to be replaced by

$$\frac{2T^2}{\pi^2}\big(\log T + 2\gamma - 1 + 2\log(2\pi)\big) + O(T\log T).$$

This is incorrect, as the main term differs from the one given by Motohashi in (5.55). It incidentally also differs from what one gets if in the above formula for $|H_j(\sigma + it)|^2$ one sets $t = 0$ and lets $\sigma \to \dfrac{1}{2} + 0$. Both of the above formulas of Kuznetsov are not correct.

The trace formulas of N.V. Kuznetsov [104], [105] for transforming series of the form

$$\sum_{\ell=1}^{\infty} \ell^{-1} S(-m, n; \ell) h\left(4\pi \frac{\sqrt{mn}}{\ell}\right), \quad \sum_{\ell=1}^{\infty} \ell^{-1} S(m, n; \ell) h\left(4\pi \frac{\sqrt{mn}}{\ell}\right)$$

are fundamental results with far-reaching applications. Their proofs are not given in the text, since that would lead us too much astray. The reader is referred to the papers of Deshouillers-Iwanice [29], M.N. Huxley [66] and Y. Motohashi [128]. The last paper contains a sketch of the proof of (5.63), with a somewhat more stringent condition on the decrease of $h^{(j)}(y)$.

The bound in (5.148) is the strongest known bound, stemming from

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^6 dt = \int\limits_0^T |\zeta|^3|\zeta|^3 dt \le \left(\int\limits_0^T |\zeta|^4 dt\right)^{\frac{3}{4}} \left(\int\limits_0^T |\zeta|^{12} dt\right)^{\frac{1}{4}}$$

315 and the bounds for the fourth and twelfth moment. For the proof in the text the trivial bound

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^6 dt \le \mathrm{Max}_{0\le t\le T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll_\epsilon T^{\frac{4}{3}+\epsilon}$$

is in fact sufficient.

The bound (5.51), with an unspecified B, is indicated in N.V. Kuznetsov [107], but not proof is given. A rigorous proof is due to Y. Motohashi [129], while a proof of the slightly weaker result

$$\sum_{x_j\le x} \alpha_j H_j^4\left(\frac{1}{2}\right) \ll_\epsilon x^{2+\epsilon}$$

is briefly sketched by Vinogradov-Tahtadžjan [161], where upper and lower bounds for α_j are also given.

It should be remarked that Theorems 5.4 and 5.7, namely $E_2(T) \ll T^{2/3}\log^C T$ and $E_2(T) = \Omega(T^{\frac{1}{2}})$, have their counterparts for zeta - functions of cusp forms. Namely, if $a(n)$ is the n-th Fourier coefficient of a cusp form of weight $x = 2m(\ge 12)$ (see e.g. T.M. Apostol [2]), and let

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \quad \left(\mathrm{Re}\, s > \frac{1}{2}(x+1)\right)$$

be the associated zeta-function. E. Hecke [58] proved the functional equation

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^{\frac{1}{2}x}(2\pi)^{-(x-s)}\Gamma(x-s)\varphi(x-s),$$

so that the rôle of the critical line $\sigma = \dfrac{1}{2}$ for $\zeta(s)$ is played in this case by the line $\sigma = \dfrac{x}{2}$. The functional equation for $\varphi(s)$ provides a certain analogy between $\varphi(\frac{1}{2}(x-1)+s)$ and $\zeta^2(s)$, which was systematically exploited by M. Jutila [95]. The function $\varphi(s)$ was studied by A. Good [44], [47], [48], [49]. Let

$$\int_0^T \left|\varphi\left(\frac{x}{2}+it\right)\right|^2 dt = 2c_{-1}\left(\log\frac{T}{2\pi e}+c_0\right)T + E(\varphi;T), \tag{5.219}$$

where c_{-1} denotes the residue, and $c_{-1}c_0$ the constant term in the Laurent expansion of $\sum\limits_{n=1}^{\infty}|a(n)|^2 n^{-s}$ at $s=x$. Then Good [47] proved that $E(\varphi;T)=\Omega(T^{\frac{1}{2}})$ if the non-vanishing of a certain series, too complicated to be stated here in detail, is assumed. In [48] he proved that $E(\varphi;T)\ll(T\log T)^{2/3}$, which corresponds to our result that $E_2(T)\ll T^{2/3}\log^C T$. It should be remarked that $\zeta^2(s)$ has a double pole at $s=1$, while $\varphi(s)$ is an entire function of s. This fact makes the study of $E(\varphi;T)$ in some aspects less difficult than the study of $E_2(T)$. For example, this is reflected in the simpler form of the main term for the integral in (5.219). 316

It was already stated that it seems unlikely that an analogue of Atkinson's formula exists for $E_2(T)$. The Ω-result $E_2(T)=\Omega(T^{\frac{1}{2}})$ supports this viewpoint, in case should such an analogue contain the divisor function $d_4(n)$ (generated by $\zeta^4(s)$). Namely for $\Delta_4(x)$, the error term in the asymptotic formula for $\sum\limits_{n\le x} d_4(n)$, it is known (see Chapter 13 of Ivić [75]) that

$$\Delta_4(x)=\Omega(x^{3/8}), \qquad \int_1^x \Delta_4^2(y)dy \ll_\epsilon x^{7/4+\epsilon},$$

and one conjectures that $\Delta_4(x)=O(x^{3/8+\epsilon})$. Thus the functions $\Delta_4(T)$ and $E_2(T)$ are of a different order of magnitude.

For A.E. Ingham's method for omega-results see his paper [74] and K. Gangadharan [40]. This method is essentially used in the proof of (3.44) of Theorem 3.4.

Chapter 6
Fractional Mean Values

6.1 Upper and Lower Bounds for Fractional mean Values

IN THIS CHAPTER we are going to study the asymptotic behaviour of the integral **317**

$$I_k(T,\sigma) := \int_0^T |\zeta(\sigma + it)|^{2k} dt \tag{6.1}$$

when $\frac{1}{2} \leq \sigma < 1$ and $k \geq 0$ is not necessarily an integer. If k is not an integer, then it is natural to expect that the problem becomes more difficult. In the most important case when $\sigma = \frac{1}{2}$ we shall set $I_k(T, \frac{1}{2}) = I_k(T)$, in accordance with the notation used in Chapter 4. When k is an integer, we can even obtain good lower bounds for $I_k(T + H) - I_k(T)$ for a wide range of H, by using a method developed by R. Balasubramanian and K. Ramachandra. This will be done in Section 6.4. In this section we shall use the method of D.R. Heath-Brown, which is based on a convexity technique. Actually this type of approach works indeed for any real $k \geq 0$, but when k is not rational it requires the Riemann hypothesis to yield sharp results. Since *I* prefer not to work with unproved hypotheses such as the Riemann hypothesis is, henceforth it will be assumed that $k \geq 0$ is rational. Now we shall formulate the main result of this section, which will be used in Section 6.2 to yield an asymptotic formula for certain fractional mean values, and in Section 6.3 for the study of the distribution of values of $|\zeta(\frac{1}{2} + it)|$. This is

Theorem 6.1. *If $k \geq 0$ is a fixed rational number, then*

$$I_k(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \gg T(\log T)^{k_2}. \tag{6.2}$$

318 *If $m \geq 1$ is an integer, then uniformly in m*

$$T(\log T)^{1/m^2} \ll I_{1/m}(T) \ll T(\log T)^{1/m^2}. \tag{6.3}$$

Before we proceed to the proof of Theorem 6.1 we shall make some preliminary remarks and prove several lemmas.

Already in Chapter 1 we encountered the divisor function $d_k(n)$, which represents the number of ways n may be written as a product of k factors, where $k(\geq 2)$ is a fixed integer. In case $k \geq 0$ is not an integer, we define $d_k(n)$ by

$$\zeta^k(s) = \prod_p (1 - p^{-s})^{-k} = \sum_{n=1}^{\infty} d_k(n) n^{-s} (\operatorname{Re} s > 1). \tag{6.4}$$

Here a branch of $\zeta^k(s)$ is defined by the relation

$$\zeta^k(s) = \exp(k \log \zeta(s)) = \exp\left(-k \sum_p \sum_{j=1}^{\infty} j^{-1} p^{-js}\right) (\operatorname{Re} s > 1). \tag{6.5}$$

Note that the above definition makes sense even if k as an arbitrary complex number. It shows that $d_k(n)$ is a multiplicative function of n for a given k. If p^α is an arbitrary prime power, then from (6.4) we have

$$d_k(p^\alpha) = (-1)^\alpha \binom{-k}{\alpha} = \frac{k(k+1)\ldots(k+\alpha-1)}{\alpha!} = \frac{\Gamma(k+\alpha)}{\Gamma(k)\alpha!}, \tag{6.6}$$

so that by multiplicativity

$$d_k(n) = \prod_{p^\alpha \| n} \frac{\Gamma(k+\alpha)}{\Gamma(k)\alpha!}. \tag{6.7}$$

From (6.6) it follows then that always $d_k(n) \geq 0$ for $k \geq 0$, the function $d_k(n)$ is an increasing function of k for fixed n, and moreover for fixed $k \geq 0$ and any $\epsilon > 0$ we have $d_k(n) \ll_\epsilon n^\epsilon$. Now we state

Lemma 6.1. *For fixed $k \geq 0$ there exists $c_k > 0$ such that*

$$\left(\sigma - \frac{1}{2}\right)^{-k^2} \ll \sum_{n=1}^{N} d_k^2(n) n^{-2\sigma} \ll \left(\sigma - \frac{1}{2}\right)^{-k^2} \tag{6.8}$$

uniformly for $1/2 + c_k/\log N \leq \sigma \leq 1$ and **319**

$$\log^{k^2} N \ll \sum_{n=1}^{N} d_k^2(n) n^{-1} \ll \log^{k^2} N. \tag{6.9}$$

Moreover, if $k = 1/m$ and $m \geq 1$ is an integer, then (6.9) holds uniformly in m, and also (6.8) if $1/2 + c/\log N \leq \sigma \leq 1$ for a suitable constant $c > 0$.

PROOF OF LEMMA 6.1. By taking $\sigma = 1/2 + c_k/\log N$ and noting that we then have $n^{-1} \ll n^{-2\sigma} \ll n^{-1}$ for $1 \leq n \leq N$, we see that (6.9) follows from (6.8). By induction on α it follows that

$$d_k^2(p^\alpha) = \left(\frac{k(k+1)\ldots(k+\alpha-1)}{\alpha!}\right)^2 \leq \frac{k^2(k^2+1)\ldots(k^2+\alpha-1)}{\alpha!} = d_{k^2}(p^\alpha),$$

whence by multiplicativity $d_k^2(n) \leq d_{k^2}(n)$ for $n \geq 1$. This gives

$$\begin{aligned}\sum_{n=1}^{N} d_k^2(n) n^{-2\sigma} &\leq \sum_{n=1}^{N} d_{k^2}(n)(n)^{-2\sigma} \leq \sum_{n=1}^{\infty} d_{k^2}(n) n^{-2\sigma} \\ &= \zeta^{k^2}(2\sigma) \ll \left(\sigma - \frac{1}{2}\right)^{-k^2},\end{aligned}$$

and this bound is uniform for $k \leq k_0$. To prove the lower bound in (6.8) write $f(n) := d_k^2(n)\mu^2(n)$, $\sigma = 1/2 + \delta$. Then

$$\begin{aligned}\sum_{n=1}^{N} d_k^2(n) n^{-2\sigma} &\geq \sum_{n=1}^{N} f(n) n^{-1-2\delta} \geq \sum_{n=1}^{\infty} f(n) n^{-1-2\delta} \left(1 - \left(\frac{n}{N}\right)^\delta\right) \\ &= S(1+2\delta) - N^{-\delta} S(1+\delta),\end{aligned} \tag{6.10}$$

where for real $s > 1$

$$S(s) := \sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p \left(1 + k^2 p^{-s}\right) = \zeta^{k^2}(s) g_k(s)$$

with

$$g_k(s) := \exp\left(-\sum_p \left\{\log\left(1 + k^2 p^{-s}\right) + k^2 \log\left(1 - p^2\right)\right\}\right).$$

Thus $g_k(s)$ is continuous, and so it is bounded for $1 \le s \le 2$. If
320 $0 < A_k \le g_k(s) \le B_k$ and $C_k \le (s-1)\zeta(s) \le D_k$ for $1 \le s \le 2$, then we have

$$S(1+2\delta) - N^{-\delta} S(1+\delta) \ge A_k C_k^{k^2} (2\delta)^{-k^2} - N^{-\delta} B_k D_k^{k^2} \delta^{-k^2}.$$

Since $\sigma \ge 1/2 + c_k/\log N$ we have $N^{\delta} \ge \exp(c_k)$, and if c_k is so large that

$$\exp(c_k) \ge 2 \frac{B_k}{A_k} \left(\frac{D_k}{C_k}\right)^{k^2},$$

then we shall have

$$S(1+2\delta) - N^{-\delta} S(1+\delta) \gg \delta^{-k^2},$$

and consequently the lower bound in (6.8) will follow from (6.10). For $k \le k_0$ the argument shows that the lower bound is also uniform, to taking $k = 1/m$, $m \ge 1$ an integer, the last part of Lemma 6.1 easily follows.

The next two lemmas are classical convexity results from complex function theory, in the from given by R.M. Gabriel [39], and their proof will not be given here.

Lemma 6.2. *Let $f(z)$ be regular for $\alpha < \operatorname{Re} z < \beta$ and continuous for $\alpha \le \operatorname{Re} z \le \beta$. suppose $f(z) \to 0$ as $|\operatorname{Im} z| \to \infty$ uniformly for $\alpha \le \operatorname{Re} z \le \beta$. Then for $\alpha \le \gamma \le \beta$ and any $q > 0$ we have*

$$\int_{-\infty}^{\infty} |f(\gamma + it)|^q dt \le \left(\int_{-\infty}^{\infty} |f(\alpha + it)|^q dt\right)^{\frac{\beta-\gamma}{\beta-\alpha}} \left(\int_{-\infty}^{\infty} |f(\beta + it)|^q dt\right)^{\frac{\gamma-\alpha}{\beta-\alpha}}. \quad (6.11)$$

Lemma 6.3. *Let R be the closed rectangle with vertices z_0, $\bar{z}_0$, $-z_0$, $-\bar{z}_0$. Let $F(z)$ be continuous on R and regular on the interior of R. Then*

$$\int_L |F(z)|^q |dz| \leq \left(\int_{P_1} |F(z)|^q |dz| \right)^{\frac{1}{2}} \left(\int_{P_2} |F(z)|^q |dz| \right)^{\frac{1}{2}} \tag{6.12}$$

for any $q \geq 0$, where L is the line segment from $\frac{1}{2}(\bar{z}_0 - z_0)$ to $\frac{1}{2}(z_0 - \bar{z}_0)$, P_1 consists of the three line segments connecting $\frac{1}{2}(\bar{z}_0 - z_0)$, $\bar{z}_0$, z_0 and $\frac{1}{2}(z_0 - \bar{z}_0)$, and P_2 is the mirror image of P_1 in L.

We shall first apply Lemma 6.2 to $f(z) = \zeta(z)\exp((z - i\tau)^2)$, $\alpha =$ **321**
$1 - \sigma, \beta = \sigma, \gamma = \frac{1}{2}$, where $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $q = 2k > 0$ and $\tau \geq 2$. By the functional equation $\zeta(s) = x(s)\zeta(1 - s)$, $x(s) \asymp |t|^{\frac{1}{2} - \sigma}$ we have

$$\zeta(\alpha + it) \ll |\zeta(\beta + it)|(1 + |t|)^{\sigma - \frac{1}{2}},$$

whence

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(\alpha + it)|^{2k} dt &\ll \int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k} (1 + |t|)^{k(2\sigma - 1)} e^{-2k(t-\tau)^2} dt \\
&\ll \left(\int_{-\infty}^{\frac{1}{2}\tau} + \int_{3\tau/2}^{\infty} \right) (1 + |t|)^{2k} e^{-2k(t-\tau)^2} dt + \tau^{k(2\sigma - 1)} \int_{\tau/2}^{3\tau/2} |\zeta(\sigma + it)|^{2k} e^{-2k(t-\tau)^2} dt \\
&\ll e^{-2k\tau^2/5} + \tau^{k(2\sigma - 1)} \int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k} e^{-2k(t-\tau)^2} dt.
\end{aligned}$$

The above bound is also seen to hold uniformly for $k = 1/m$, $m \leq (\log\log T)^{\frac{1}{2}}$, which is a condition that we henceforth assume. Since $(\beta - \gamma)/(\beta - \alpha) = (\gamma - \alpha)/(\beta - \alpha) = 1/2$, (6.11) gives then

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} e^{-2k(t-\tau)^2} dt \\
&\ll e^{-2k\tau^2/5} + \tau^{k(\sigma - \frac{1}{2})} \int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k} e^{-2k(t-\tau)^2} dt.
\end{aligned}$$

If we define

$$w(t) := \int_T^{2T} e^{-2k(t-\tau)^2} d\tau$$

and integrate the last bound for $T \le \tau \le 2T$, then we obtain

Lemma 6.4. *Let* $\frac{1}{2} \le \sigma \le \frac{3}{4}$, $k > 0$ *and* $T \ge 2$. *Then*

$$J\left(\frac{1}{2}\right) \ll T^{k(\sigma-\frac{1}{2})} J(\sigma) + e^{-kT^2/3}, \tag{6.13}$$

where

$$J(\sigma) := \int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k} w(t) dt.$$

Also (6.13) holds uniformly for $k = 1/m$, *where* $m \ge 1$ *is an integer such that* $m \le (\log\log T)^{\frac{1}{2}}$.

322 Next we take $f(z) = (z-1)\zeta(z)\exp((z-i\tau)^2)$ (so that $f(z)$ is entire, because $z-1$ cancels the pole of $\zeta(z)$ at $z=1$), $\gamma = \sigma$, $\alpha = 1/2, \beta = 3/2$, $q = 2k > 0$, $1/2 \le \sigma \le 3/4$, and $\tau \ge 2$. Then

$$\int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right)\right|^{2k} dt \ll \int_{\tau/2}^{3\tau/2} \left| f\left(\frac{1}{2} + it\right)\right|^{2k} dt + e^{-2k\tau^2/5}$$

$$\ll \tau^{2k} \int_{\tau/2}^{3\tau/2} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} e^{-2k(t-\tau)^2} dt + e^{-2k\tau^2/5},$$

and similarly

$$\int_{-\infty}^{\infty} \left| f\left(\frac{3}{2} + it\right)\right|^{2k} dt \ll \tau^{2k} \int_{\tau/2}^{3\tau/2} \left|\zeta\left(\frac{3}{2} + it\right)\right|^{2k} e^{-2k(t-\tau)^2} dt + e^{-2k\tau^2/5}$$

$$\ll \tau^{2k} \int_{\tau/2}^{3\tau/2} e^{-2k(t-\tau)^2} dt + e^{-2k\tau^2/5} \ll \tau^{2k}.$$

From (6.11) we conclude that

$$\int_{-\infty}^{\infty} |f(\sigma+it)|^{2k} dt \ll \tau^{2k} \left(\int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2}+it\right) \right|^{2k} e^{-2k(t-\tau)^2} dt \right)^{\frac{3}{2}-\sigma} + e^{-k\tau^2/3}. \quad (6.14)$$

But we have

$$\begin{aligned} \int_{-\infty}^{\infty} |\zeta(\sigma+it)|^{2k} e^{-2k(t-\tau)^2} dt &\ll \int_{\tau/2}^{3\tau/2} |\zeta(\sigma+it)|^{2k} e^{-2k(t-\tau)^2} dt + e^{-2k\tau^2/5} \\ &\ll \tau^{-2k} \int_{\tau/2}^{3\tau/2} |f(\sigma+it)|^{2k} dt + e^{-2k\tau^2/5} \quad (6.15) \\ &\leq \tau^{-2k} \int_{-\infty}^{\infty} |f(\sigma+it)|^{2k} dt + e^{-2k\tau^2/5}. \end{aligned}$$

In case $k = 1/m$ we have to observe that, for $T \leq \tau \leq 2T$,

$$\int_{\tau/2}^{3\tau/2} e^{-2(t-\tau)^2/m} dt \ll m^{\frac{1}{2}}$$

instead of $\ll 1$. We combine (6.15) with (6.14) and integrate for $T \leq \tau \leq 2T$. By using Hölder's inequality we obtain then

Lemma 6.5. *Let* $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $k > 0$ *and* $T \geq 2$. *Then* 323

$$J(\sigma) \ll T^{\sigma-1/2} J\left(\frac{1}{2}\right)^{3/2-\sigma} + e^{-kT^2/4},$$

and if $k = 1/m$, $m \geq 1$ *is an integer, then uniformly for* $m \leq (\log\log T)^{\frac{1}{2}}$ *we have*

$$J(\sigma) \ll \left(m^{\frac{1}{2}} T\right)^{\sigma-\frac{1}{2}} J\left(\frac{1}{2}\right)^{\frac{1}{2}(3-2\sigma)} + e^{-T^2/(4m)}.$$

Since (6.2) is trivial for $k = 0$, we may henceforth assume that $k = u/v$, where u and v are positive coprime integers, so that for (6.3) we have $k = u/v$ with $u = 1$, $v = m$. In the former case let $N = T^{\frac{1}{2}}$ and in the latter $N = T^{\frac{3}{4}}$. We also write

$$S(s) := \sum_{n \le N} d_k(n) n^{-s}, \quad g(s) := \zeta^u(s) - S^v(s).$$

We apply now Lemma 6.2 to the function $f(z) = g(z)\exp(u(z - i\tau)^2)$ with $\gamma = \sigma$, $\alpha = 1/2$, $\beta = 7/8$ and $q = 2/v$. Hence

$$\int_{-\infty}^{\infty} |f(\sigma + it)|^{\frac{2}{v}} dt \le \left(\int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right)\right|^{\frac{2}{v}} dt \right)^{\frac{7-8\sigma}{3}} \left(\int_{-\infty}^{\infty} \left| f\left(\frac{7}{8} + it\right)\right|^{\frac{2}{v}} dt \right)^{\frac{8\sigma-4}{3}}. \quad (6.16)$$

Note that, for $\frac{1}{2} \le \operatorname{Re} s \le 2$, trivial estimation gives

$$S(s) \ll N^{\epsilon + 1 - \operatorname{Re} s} + 1 \ll T,$$

thus

$$g(s) \ll (T + |t|)^{u+v} \quad \left(\frac{1}{2} \le \operatorname{Re} s \le 2,\ |s - 1| \le \frac{1}{10} \right). \quad (6.17)$$

It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} \left| f\left(\frac{7}{8} + it\right)\right|^{\frac{2}{v}} dt &= \int_{\tau/2}^{3\tau/2} \left| f\left(\frac{7}{8} + it\right)\right|^{\frac{2}{v}} dt \\ &\quad + O\left(\left(\int_{-\infty}^{\frac{1}{2}\tau} + \int_{\frac{3\tau}{2}}^{\infty} \right) (T + |t|)^{2+2k} e^{-2k(t-\tau)^2} dt \right) \\ &= \int_{\tau/2}^{3\tau/2} \left| f\left(\frac{7}{8} + it\right)\right|^{\frac{2}{v}} dt + O\left(T^{2+2k} e^{-k\tau^2/3}\right). \quad (6.18)\end{aligned}$$

At this point we shall use Lemma 6.3, which allows us to avoid the
324 singularity of $\zeta(s)$ at $s = 1$. We take $z_0 = \dfrac{3}{8} + \dfrac{1}{2} i\tau$, $F(z) = f\left(z - \frac{7}{8} - i\tau\right)$

and $q = \frac{2}{v}$. For the integrals in (6.12) we then have

$$\int_{L} |F(z)|^q |dz| = \int_{\tau/2}^{3\tau/2} \left| f\left(\frac{7}{8} + it\right)\right|^{\frac{2}{v}} dt$$

and

$$\int_{P_1} |F(z)|^q |dz| \tag{6.19}$$
$$= \int_{\tau/2}^{3\tau/2} \left| f\left(\frac{5}{4} + it\right)\right|^{\frac{2}{v}} dt + \int_{7/8}^{5/4} \left\{ \left| f\left(\eta + \frac{1}{2} i\tau\right)\right|^{\frac{2}{v}} + \left| f\left(\eta + \frac{3i\tau}{2}\right)\right|^{\frac{2}{v}} \right\} d\eta.$$

By (6.17) we have

$$f\left(\eta + \frac{1}{2} i\tau\right) \ll (T+\tau)^{u+v} e^{-\frac{1}{4} u\tau^2}$$

and similarly for $f\left(\eta + \frac{3i\tau}{2}\right)$. Thus the second integral on the right of (6.19) is $\ll T^{2+2k} \exp(-k\tau^2/3)$ and similarly

$$\int_{P_2} |F(z)|^q |dz| = \int_{\tau/2}^{3\tau/2} \left| f\left(\frac{1}{2} + it\right)\right|^{\frac{2}{v}} dt + O\left(T^{2+2k} e^{-k\tau^2/3}\right).$$

Lemma 6.3 therefore yields

$$\int_{\tau/2}^{3\tau/2} \left| f\left(\frac{7}{8} + it\right)\right|^{\frac{2}{v}} dt$$
$$\ll \left(\int_{-\infty}^{\infty} \left| f\left(\frac{1}{2} + it\right)\right|^{\frac{2}{v}} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \left| f\left(\frac{5}{4} + it\right)\right|^{\frac{2}{v}} dt \right)^{\frac{1}{2}} + T^{2+2k} e^{-k\tau^2/7},$$

since by (6.17)

$$\int\limits_{\tau/2}^{3\tau/2} \left\{ \left| f\left(\frac{1}{2}+it\right) \right|^{\frac{2}{\nu}} dt + \left| f\left(\frac{5}{4}+it\right) \right|^{\frac{2}{\nu}} \right\} dt \ll (T+\tau)^{2+2k}.$$

From (6.16) and (6.18) we now deduce

$$\int\limits_{-\infty}^{\infty} |f(\sigma+it)|^{\frac{2}{\nu}} dt \ll \left(\int\limits_{-\infty}^{\infty} \left| f\left(\frac{1}{2}+it\right) \right|^{\frac{2}{\nu}} dt \right)^{\frac{5-4\sigma}{3}} \left(\int\limits_{\frac{1}{2}\tau}^{\frac{3\tau}{2}} \left| f\left(\frac{5}{4}+it\right) \right|^{\frac{2}{\nu}} dt \right)^{\frac{4\sigma-2}{3}}$$
$$+ \left(\int\limits_{-\infty}^{\infty} \left| f\left(\frac{1}{2}+it\right) \right|^{\frac{2}{\nu}} dt \right)^{\frac{7-8\sigma}{3}} \left(T^{2+2k} e^{-k\tau^2/7} \right)^{\frac{8\sigma-4}{3}}.$$

325 We integrate this for $T \le \tau \le 2T$ and write

$$K(\sigma) := \int\limits_{-\infty}^{\infty} |g(\sigma+it)|^{\frac{2}{\nu}} w(t) dt,$$

whence

$$K(\sigma) \ll K\left(\frac{1}{2}\right)^{\frac{5-4\sigma}{3}} \left\{ \int\limits_{T}^{2T} \int\limits_{\tau/2}^{3\tau/2} \left| g\left(\frac{5}{4}+it\right) \right|^{\frac{2}{\nu}} e^{-2k(t-\tau)^2} d\tau dt \right\}^{\frac{4\sigma-2}{3}} \tag{6.20}$$
$$+ K\left(\frac{1}{2}\right)^{\frac{7-8\sigma}{3}} \left(e^{-kT^2/8} \right)^{\frac{8\sigma-4}{3}}$$
$$\ll K\left(\frac{1}{2}\right)^{\frac{5-4\sigma}{3}} \left\{ \int\limits_{\frac{1}{2}T}^{3T} \left| g\left(\frac{5}{4}+it\right) \right|^{\frac{2}{\nu}} dt \right\}^{\frac{4\sigma-2}{3}} + K\left(\frac{1}{2}\right)^{\frac{7-8\sigma}{3}} e^{-\frac{kT^2(2\sigma-1)}{6}}.$$

From the definition of $g(s)$ we have

$$g(s) = \sum_{n>N} a_n n^{-s} \quad (\operatorname{Re} s > 1)$$

with $0 \leq a_n \leq d_u(n)$, so that $a_n \leq 1$ if $u = 1$, in particular for $k = 1/m$. By the mean value theorem for Dirichlet polynomials (see (1.15)) we find that

$$\int\limits_{\frac{1}{2}T}^{3T} \left| g\left(\frac{5}{4} + it\right)\right|^2 dt \ll T \sum_{n>N} a_n^2 n^{-5/2} + \sum_{n>N} a_n^2 n^{-3/2}$$
$$\ll TN^{-3/2}(\log N)^{u^2-1} + N^{-\frac{1}{2}}(\log N)^{u^2-1} \ll TN^{-3/2}(\log N)^{u^2-1}$$

since $N \leq T$ and
$$\sum_{n \leq x} d_u^2(n) \ll x(\log x)^{u^2-1}.$$

Therefore by Hölder's inequality for integrals we obtain

$$\int\limits_{\frac{1}{2}T}^{3T} \left| g\left(\frac{5}{4} + it\right)\right|^{\frac{2}{v}} dt \ll T^{1-\frac{1}{v}} \left(TN^{-\frac{3}{2}} \log^{u^2-1} T\right)^{\frac{1}{v}} = TN^{-\frac{3}{2v}} (\log T)^{\frac{u^2-1}{v}}.$$

From (6.20) we then obtain

Lemma 6.6. *Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $T \geq 2$. If $k = u/v$, where u and v are positive coprime integers, then* **326**

$$K(\sigma) \ll K\left(\frac{1}{2}\right)^{\frac{5-4\sigma}{3}} \left(TN^{-\frac{3}{2v}} (\log T)^{k-\frac{1}{v}}\right)^{\frac{4\sigma-2}{3}} + K\left(\frac{1}{3}\right)^{\frac{7-8\sigma}{3}} e^{-\frac{kT^2(2\sigma-1)}{6}},$$

where
$$K(\sigma) = \int\limits_{-\infty}^{\infty} |g(\sigma + it)|^{\frac{2}{v}} w(t) dt.$$

If $k = 1/m$, $m \geq 1$ is an integer, then uniformly for $m \leq (\log\log T)^{\frac{1}{2}}$ we have

$$K(\sigma) \ll K\left(\frac{1}{2}\right)^{\frac{5-4\sigma}{3}} \left(m^{\frac{1}{2}} TN^{-\frac{3}{2m}}\right)^{\frac{4\sigma-2}{3}} + K\left(\frac{1}{2}\right)^{\frac{7-8\sigma}{3}} e^{-\frac{T^2(2\sigma-1)}{6m}}.$$

PROOF OF 6.1. As a companion to the integrals $J(\sigma)$ and $K(\sigma)$ we define

$$L(\sigma) := \int_{-\infty}^{\infty} |S(\sigma + it)|^2 w(t) dt \quad \left(\frac{1}{2} \le \sigma \le \frac{3}{4}\right).$$

To estimate $L(\sigma)$ note that

$$w(t) = \int_{T}^{2T} e^{-2k(t-\tau)^2} d\tau \ll \exp\left(-(T^2 + t^2)k/18\right)$$

for $t \le 0$ or $t \ge 3T$. Since $S(\sigma + it) \ll T$ we have

$$L(\sigma) = \int_{0}^{3T} |S(\sigma + it)|^2 w(t) dt + O(1).$$

Moreover $w(t) \ll 1$ for all t, and $w(t) \gg 1$ for $4T/3 \le t \le 5T/3$. Thus the mean value theorem for Dirichlet polynomials yields

$$\int_{0}^{3T} |S(\sigma + it)|^2 dt = \sum_{n \le N} d_k^2(n) n^{-2\sigma} (3T + O(n)) \ll T \sum_{n \le N} d_k^2(n) n^{-2\sigma}$$

and

$$\int_{4T/3}^{5T/3} |S(\sigma + it)|^2 dt = \sum_{n \le N} d_k^2(n) n^{-2\sigma} \left(\frac{T}{3} + O(n)\right) \gg T \sum_{n \le N} d_k^2(n) n^{-2\sigma}.$$

327 Hence we may deduce from (6.8) of Lemma 6.1 that

$$T\left(\sigma - \frac{1}{2}\right)^{-k^2} \ll L(\sigma) \ll T\left(\sigma - \frac{1}{2}\right)^{-k^2} \tag{6.21}$$

for $1/2 + c_k/\log T \le \sigma \le 3/4$, and from (6.9) that

$$T(\log T)^{k^2} \ll L\left(\frac{1}{2}\right) \ll T(\log T)^{k^2}. \tag{6.22}$$

In case $k = 1/m$ we obtain, as in the previous discussion, an extra factor $m^{\frac{1}{2}}$, and (6.21) and (6.22) become, uniformly for $m \leq (\log\log T)^{\frac{1}{2}}$,

$$m^{\frac{1}{2}} T \left(\sigma - \frac{1}{2}\right)^{-1/m^2} \ll L(\sigma) \ll m^{\frac{1}{2}} T \left(\sigma - \frac{1}{2}\right)^{-1/m^2} \tag{6.23}$$

for $1/2 + c/\log N \leq \sigma \leq 3/4$, and

$$m^{\frac{1}{2}} T (\log T)^{1/m^2} \ll L\left(\frac{1}{2}\right) \ll m^{\frac{1}{2}} T (\log T)^{1/m^2}. \tag{6.24}$$

We trivially have

$$\begin{aligned} &|S^v(s)|^{2/v} = |\zeta^u(s) - g(s)|^{2/v} \\ &\leq (2\max(|\zeta^\mu(s)|, |g(s)|))^{2/v} \ll |\zeta(s)|^{2k} + |g(s)|^{2/v}, \end{aligned}$$

whence

$$L(\sigma) \ll J(\sigma) + K(\sigma). \tag{6.25}$$

Similarly

$$J(\sigma) \ll L(\sigma) + K(\sigma) \tag{6.26}$$

and

$$K\left(\frac{1}{2}\right) \ll L\left(\frac{1}{2}\right) + J\left(\frac{1}{2}\right). \tag{6.27}$$

In case $K(\frac{1}{2}) \leq T$ we have from (6.25) (with $\sigma = \frac{1}{2}$) and (6.22)

$$T(\log T)^{k^2} \ll J\left(\frac{1}{2}\right) \tag{6.28}$$

Similarly (6.26) and (6.22) show that $K\left(\frac{1}{2}\right) \leq T$ implies

$$J\left(\frac{1}{2}\right) \ll T(\log T)^{k^2}. \tag{6.29}$$

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Thus we shall assume $K(\frac{1}{2}) \geq T$ and show that (6.28) holds, and for $k = 1/m$ (6.29) also holds (with the additional factor $m^{\frac{1}{2}}$). From these estimates Theorem 6.1 will be deduced. From Lemma 6.6 we have, assuming $K(\frac{1}{2}) \geq T$,

$$\begin{aligned} K(\delta) &\ll K\left(\frac{1}{2}\right)\left\{K\left(\frac{1}{2}\right)^{\frac{2-4\sigma}{3}}\left(TN^{-\frac{3}{2v}}(\log T)^{k-\frac{1}{v}}\right)^{\frac{4\sigma-2}{3}} + K\left(\frac{1}{2}\right)^{\frac{4-8\sigma}{3}} e^{-\frac{kT^2(4\sigma-3)}{12}}\right\} \\ &\leq K\left(\frac{1}{2}\right)\left\{\left(N^{-\frac{3}{2v}}(\log T)^{k-\frac{1}{v}}\right)^{\frac{4\sigma-2}{3}} + \left(T^{-2}e^{-\frac{1}{4}kT^2}\right)^{\frac{4\sigma-2}{3}}\right\} \qquad (6.30) \\ &\ll K\left(\frac{1}{2}\right)N^{\frac{1-2\sigma}{v}}(\log T)^{(k-\frac{1}{v})(\frac{4\sigma-2}{3})}. \end{aligned}$$

Now we turn to the proof of (6.2), noting that the proof of (6.3) is based on very similar ideas. From (6.25), (6.27) and (6.30) we have, since now $N = T^{\frac{1}{2}}$ and $k = u/v$,

$$L(\sigma) \ll J(\sigma) + \left(L\left(\frac{1}{2}\right) + J\left(\frac{1}{2}\right)\right)\left(T\log^B T\right)^{\frac{1-2\sigma}{4v}},$$

where $B = (8 - 8u)/3$. Thus either

$$L(\sigma) \ll L\left(\frac{1}{2}\right)\left(T\log^B T\right)^{\frac{1-2\sigma}{4v}} \qquad (6.31)$$

or

$$L(\sigma) \ll J\left(\frac{1}{2}\right)\left(T\log^B T\right)^{\frac{1-2\sigma}{4v}} + J(\sigma), \qquad (6.32)$$

and by using Lemma 6.5 the latter bound gives

$$L(\sigma) \ll T^{\sigma-\frac{1}{2}}J\left(\frac{1}{2}\right)^{\frac{3}{2}-\sigma} + (T\log^B T)^{\frac{1-2\sigma}{4v}}J\left(\frac{1}{2}\right) + e^{-\frac{1}{4}kT^2}. \qquad (6.32)$$

Write $\sigma = \dfrac{1}{2} + \dfrac{\eta}{\log T}$, where $\eta > 0$ is a parameter. Then (6.21), (6.31) and (6.22) yield, for some constant $c(k) > 0$,

$$T\left(\sigma - \frac{1}{2}\right)^{-k^2} \leq c(k)T(\log T)^{k^2}e^{-\eta/(2v)}.$$

329 Hence
$$e^{-\frac{\eta}{2v}} \leq \eta^{k^2} c(k),$$
which is false if $\eta = \eta(k)$ is sufficiently large. Thus for this value of η (6.32) holds, and using (6.21) it gives
$$T \log^{k^2} T \ll L(\sigma) \ll J\left(\frac{1}{2}\right)^{\frac{1}{2}(3-2\sigma)} + J\left(\frac{1}{2}\right) + 1,$$
and (6.28) follows. But recall that $w(t) \ll 1$ for all t and
$$w(t) \ll \exp(-k(t^2 + T^2)/18)$$
for $t \leq 0$ and $t \geq 3T$. Thus
$$J\left(\frac{1}{2}\right) \ll I_k(3T) + \left(\int\limits_{-\infty}^{0} + \int\limits_{3T}^{\infty}\right) e^{-k(t^2+t^2)/20} dt \ll I_k(3T) + e^{-kT^2/20},$$
so that (6.2) follows from (6.28). Similarly for the case $k = 1/m$, $m \geq 1$ an integer in (6.3) we obtain the result from
$$m^{\frac{1}{2}} T(\log T)^{1/m^2} \ll J\left(\frac{1}{2}\right) \ll m^{\frac{1}{2}} T(\log T)^{1/m^2} \tag{6.33}$$
because in this case $w(t) \ll m^{\frac{1}{2}}$ always, and $w(t) \gg m^{\frac{1}{2}}$ for $4T/3 \leq t \leq 5T/3$. Thus (6.33) gives in the case of the upper bound
$$m^{\frac{1}{2}}\left(I_{1/m}\left(\frac{5T}{3}\right) - I_{1/m}\left(\frac{4T}{3}\right)\right) \ll J\left(\frac{1}{2}\right) \ll m^{\frac{1}{2}} T(\log T)^{1/m^2}.$$

Replacing T by $(4/5)^n T$ and summing over n we obtain the upper bound in (6.3), and the lower bound is proved as in the previous case. One obtains (6.33) from the corresponding estimates of the previous case and chooses $\sigma = \dfrac{1}{2} + \dfrac{Dm}{\log T}$, where $D > 0$ is a suitable large constant.

In the analysis concerning the case $k = 1/m$ we have made the assumption that $m \leq (\log\log T)^{\frac{1}{2}}$. Therefore it remains to prove (6.3) when $m \geq m_0 = [(\log\log T)^{\frac{1}{2}}]$. In that case

$$I_{1/m}(T) = \int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \geq T + \circ(T) \gg T(\log T)^{1/m^2}$$

330 for $m \geq m_0$ and $T \to \infty$. This follows from Lemma 6.7 of the next section, whose proof is independent of Theorem 6.1. To deal with the upper bound, denote by A_1 the set of numbers $t \in [0, T]$ such that $|\zeta(\frac{1}{2} + it)| \leq 1$, and let A_2 be the complement of A_1 in the interval $[0, T]$. Then

$$I_{1/m}(T) = \left(\int\limits_{A_1} + \int\limits_{A_2}\right)\left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \leq T + \int\limits_{A_2}\left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m_\circ} dt$$

$$\leq T + \int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m_\circ} dt \ll T + T(\log T)^{m_0^{-2}} \ll T(\log T)^{m^{-2}},$$

since the upper bound in (6.3) holds for $m = m_0$.

6.2 An Asymptotic formula for mean values

In this section we shall consider the limit

$$\lim_{T\to\infty} \frac{1}{T}\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{\lambda} dt = 1, \tag{6.34}$$

or equivalently

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{\lambda} dt = T + \circ(T) \quad (T \to \infty) \tag{6.35}$$

when $\lambda \geq 0$. If λ is fixed, then (6.34) (or (6.35)) cannot hold in view of the lower bound in Theorem 6.1. On the other hand, (6.34) is trivial if

$\lambda = 0$. Therefore one feels that for certain $\lambda = \lambda(T)$, tending sufficiently quickly to zero as $T \to \infty$, (6.34) will hold. It seems interesting to determine precisely the range of $\lambda = \lambda(T)$ for which (6.35) holds. This is achieved in

Theorem 6.2. *If $\psi(T)$ denotes an arbitrary positive function such that $\Psi(T) \to \infty$ as $T \to \infty$, then for*

$$0 \leq \lambda \leq (\Psi(T) \log\log T)^{-\frac{1}{2}}$$

we have

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{\lambda} dt = T + \circ(T) \quad (T \to \infty). \tag{6.36}$$

Moreover (6.36) cannot hold for $\lambda \geq C(\log\log T)^{-\frac{1}{2}}$ and a suitable 331
$C > 0$.

To prove Theorem 6.2 we shall prove the corresponding upper and lower bound estimates. The latter is contained in the following lemma, which was already used in the discussion of the lower bound in Theorem 6.1 in Section 6.1.

Lemma 6.7. *For any $\lambda \geq 0$*

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{\lambda} dt \geq T + \circ(T) \quad (T \to \infty). \tag{6.37}$$

PROOF OF LEMMA 6.7. Let, as usual, $N(\sigma, T)$ denote the number of complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ such that $\beta \geq \sigma$, $0 < \gamma \leq T$. From J.E. Littlewood's classical lemma (see Section 9.1 of E.C. Titchmarsh [155]) on the zeros of an analytic function in a rectangle, it follows that

$$2\pi \int\limits_1 \sigma_0 N(\sigma, T) d\sigma = \int\limits_0^T \log|\zeta(\sigma_0 + it)| dt - \int\limits_0^T \log|\zeta(2 + it)| dt$$

$$+\int_{\sigma_0}^{2} \arg \zeta(\sigma + iT) d\sigma + K(\sigma_0),$$

where $K(\sigma_0)$ is independent of T, and $\frac{1}{2} \leq \sigma_0 \leq 1$. However, uniformly for $\sigma \geq \frac{1}{2}$ we have $\arg \zeta(\sigma + it) \ll \log t$ if t is not the ordinate of a zero of $\zeta(s)$. For $\sigma > 1$ we have

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-ms},$$

hence with $|\Lambda_1(n)| \leq 1$ we have

$$\int_0^T \log |\zeta(2 + it)| dt = \operatorname{Re} \left\{ \sum_{n=2}^{\infty} \Lambda_1(n) n^{-2} \frac{n^{-iT} - 1}{\log n} \right\} = O(1).$$

Therefore we obtain

$$2\pi \int_{\sigma_0}^{1} N(\sigma, T) d\sigma = \int_0^T \log |\zeta(\sigma_0 + it)| dt + O(\log T). \tag{6.38}$$

332 Since $N(\sigma, T) \geq 0$ for all σ and T, (6.38) gives with $\sigma_0 = \frac{1}{2}$

$$0 \leq 2\pi \int_{\frac{1}{2}}^{1} N(\sigma, T) d\delta = \int_0^T \log \left| \zeta\left(\frac{1}{2} + it\right) \right| dt + O(\log T),$$

hence for a suitable $C > 0$

$$\int_0^T \log \left| \zeta\left(\frac{1}{2} + it\right) \right| dt \geq -C \log T. \tag{6.39}$$

Now recall that, if $a < b$, $f(t) \geq 0$ for $a \leq t \leq b$, $f \in C[a, b]$, then

$$\frac{1}{b-a} \int_a^b \log f(t) dt \leq \log \left(\frac{1}{b-a} \int_a^b f(t) dt \right), \tag{6.40}$$

which is an easy consequence of the inequality between the arithmetic and geometric means of nonnegative numbers. Hence, for $\lambda > 0$ and $f(t) = |\zeta(\frac{1}{2} + it)|$, (6.40) gives

$$\int\limits_0^T \log\left|\zeta\left(\frac{1}{2} + it\right)\right| dt = \frac{1}{\lambda}\int\limits_0^T \log\left|\zeta\left(\frac{1}{2} + it\right)\right|^\lambda dt$$
$$\leq \frac{T}{\lambda}\log\left(\frac{1}{T}\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^\lambda dt\right),$$

and so using (6.39) we obtain

$$\frac{1}{T}\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^\lambda dt \geq e^{-\frac{\lambda C\log T}{T}} \geq e^{-\frac{2C\log T}{T}} = 1 + O\left(\frac{\log T}{T}\right)$$

for $0 < \lambda \leq 2$. For $\lambda = 0$ (6.37) is trivial, for $\lambda \geq 2$ it follows by Hölder's inequality for integrals and the weak bound

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt \gg T\log T.$$

PROOF OF THEOREM 6.2. We shall show that for

$$0 \leq \lambda \leq (\psi(T)\log\log T)^{-\frac{1}{2}}$$

we have

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^\lambda dt \leq T + \circ(T) \quad (T \to \infty). \tag{6.41}$$

Take $m = [(\log\log T)^{\frac{1}{2}}]$. Then by Hölder's inequality and the upper bound in (6.3) we have, for some $C_1 > 0$, 333

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^\lambda dt \leq \left(\int\limits_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt\right)^{\frac{1}{2}m\lambda} T^{1-\frac{1}{2}m\lambda}$$

$$\leq \left(C_1 T (\log T)^{1/m^2}\right)^{\frac{1}{2}m\lambda} T^{1-\frac{1}{2}m\lambda} = T C_1^{\frac{1}{2}m\lambda} (\log T)^{\lambda/(2m)} = T + \circ(T).$$

Namely

$$0 \leq \lambda/(2m) \leq \left(\frac{1}{2} + \circ(1)\right) \psi^{-\frac{1}{2}}(T)(\log\log T)^{-1},$$

hence, as $T \to \infty$,

$$(\log T)^{\lambda/(2m)} = 1 + \circ(1), \quad C_1^{\frac{1}{2}m\lambda} = 1 + \circ(1).$$

This establishes (6.41) and in view of (6.37) proves the first part of Theorem 6.2.

To obtain that (6.36) cannot hold for $\lambda > C(\log\log T)^{-\frac{1}{2}}$ and a suitably chosen C, we use the lower bound in (6.3) with

$$m = \left[C_3^{-1}(\log\log T)^{-\frac{1}{2}}\right] \quad (C_3 > 0).$$

This gives with some $C_2 > 0$

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \geq C_2 T (\log T)^{1/m^2} = C_2 T \exp\left(C_3^2 + \circ(1)\right) \tag{6.42}$$
$$\geq C_2 T \exp\left(\frac{1}{2} C_3^2\right) \geq 2T$$

for $C_2 \geq (2\log(2/C_2))^{\frac{1}{2}}$ if $C_2 < 2$, and the bound given by (6.42) is trivial otherwise. Hence as $T \to \infty$

$$\lambda_0 = \frac{2}{m} = 2/\left[C_3^{-1}(\log\log T)^{\frac{1}{2}}\right] = (2C_3 + \circ(1))(\log\log T)^{-\frac{1}{2}}$$
$$3C_3(\log\log T)^{-\frac{1}{2}} = \lambda_1,$$

say. But if for some $\lambda_0 > 0$

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{\lambda_0} dt \geq wT,$$

then for $\lambda > \lambda_0$ Hölder's inequality gives

$$2T \leq \int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{\lambda_0} dt \leq \left(\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{\lambda} dt\right)^{\lambda_0/\lambda} T^{1-\lambda_0/\lambda}.$$

and so 334

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{\lambda} dt \geq 2^{\lambda/\lambda_0} T \geq 2T.$$

Therefore (6.36) cannot hold for $\lambda > C(\log\log T)^{-\frac{1}{2}}$, $C = 3C_3$, where C_3 is as above. This completes the proof of Theorem 6.2.

6.3 The values Distribution on the Critical line

From (6.3) we may deduce some results on the distribution of values of $|\zeta(\frac{1}{2}+it)|$. The order of $|\zeta(\frac{1}{2}+it)|$ remains, of course, one of the most important unsolved problems of analytic number theory. But the following two theorems, due to M. Jutila, show that the corresponding "statistical" problem may be essentially solved.

Theorem 6.3. *Let $T \geq 2$, $1 \leq V \leq \log T$, and denote by $M_T(V)$ the set of numbers $t \in [0, T]$ such that $|\zeta(\frac{1}{2}+it)| \geq V$. Then the measure of the set $M_T(V)$ satisfies*

$$\mu(M_T(V)) \ll T\exp\left\{-\frac{\log^2 V}{\log\log T}\left(1+O\left(\frac{\log V}{\log\log T}\right)\right)\right\} \tag{6.43}$$

and also

$$\mu(M_T(V)) \ll T\exp\left(-c\frac{\log^2 V}{\log\log T}\right) \tag{6.44}$$

for some constant $c > 0$.

Corollary. Let $\Psi(T)$ be any positive function such that $\Psi(T) \to \infty$ as $T \to \infty$. Then

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \leq \exp\left(\Psi(T)(\log\log T)^{\frac{1}{2}}\right)$$

for the numbers $t \in [0, T]$ which do not belong to an exceptional set of measure $\circ(T)$.

335 **Theorem 6.4.** *There exist positive constants a_1, a_2 and a_3 such that for $T \geq 10$ we have*

$$\exp\left(a_1(\log\log T)^{\frac{1}{2}}\right) \leq \left|\zeta\left(\frac{1}{2} + it\right)\right| \leq \exp\left(a_2(\log\log T)^{\frac{1}{2}}\right)$$

in a subset of measure at least $a_3 T$ of the interval $[0, T]$.

PROOF OF THEOREM 6.3. It is enough to restrict V to the range

$$(\log\log T)^{\frac{1}{2}} \leq \log V \leq \log\log T.$$

Namely for $\log V < (\log\log T)^{\frac{1}{2}}$ the $\exp(\ldots)$ terms in (6.43) and (6.44) are bounded, so the bounds in question both reduce to $\mu(M_T(V)) \ll T$, which is trivial, and $\log V \leq \log\log T$ is equivalent to our assumption that $V \leq \log T$. Now by (6.3), for any integer $m \geq 1$,

$$\mu(M_T(V))V^{2/m} \leq \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \ll T(\log T)^{1/m^2},$$

whence

$$\mu(M_T(V)) \ll T(\log T)^{1/m^2} V^{-2/m}. \tag{6.45}$$

As a function of m, the right-hand side of (4.45) is minimized for $m = \log\log T/\log V$. Since m must be an integer, we take

$$m = [\log\log T/\log V],$$

so that (6.45) yields then (6.43) and (6.44).

PROOF OF THEOREM 6.4. Let $0 < A < 1$ be a constant to be chosen sufficiently small later, and let $m = [(A\log\log T)^{\frac{1}{2}}] - a$, where a is 0 or 1 so that m is an even integer. We suppose $T \geq \exp\left(\exp\frac{4}{A}\right)$, so that m is positive, and denote by $b_1, b_2, \ldots$ positive, absolute constants.

Denote by E the set of points $t \in [0, T]$ such that

$$\exp\left((A \log\log T)^{\frac{1}{2}}\right) \leq \left|\zeta\left(\frac{1}{2} + it\right)\right| \leq \exp\left(A^{-1}(\log\log T)^{\frac{1}{2}}\right), \quad (6.46)$$

and let $F = [0, T]\backslash E$. If we can show that

$$\int_F \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \leq b_1 T, \quad (6.47)$$

then Theorem 6.4 follows from (6.47) and the lower bound in (6.3). **336**
Indeed the latter gives, by our choice of m,

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \geq b_2 e^{1/A} T,$$

whence by (6.47) with sufficiently small A we have

$$\begin{aligned}\int_E \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt = &\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \\ &- \int_F \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/m} dt \geq \frac{1}{2} b_2 e^{\frac{1}{A}} T.\end{aligned}$$

Consequently, by the defining property (6.46) of the set E,

$$\mu(E)\left\{\exp\left(A^{-1}(\log\log T)^{\frac{1}{2}}\right)\right\}^{2/m} \geq \frac{1}{2} b_2 e^{1/A} T,$$

which gives

$$\mu(E) \geq \frac{1}{2} b_2 \left\{\exp\left(A^{-1} - b_3 A^{-3/2}\right)\right\} T.$$

Thus Theorem 6.4 holds with $a_1 = A^{\frac{1}{2}}$, $a_2 = A^{-1}$, $a_3 = \frac{1}{2} b_2 \exp$ $(A^{-1} - b_3 A^{-3/2})$.

It remains to prove (6.47). Let F' denote the subset of F in which

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| < \exp\left((A\log\log T)^{\frac{1}{2}}\right),$$

and let $F'' = F\backslash F'$. Then trivially

$$\int_{F'} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2/m} dt \leq b_4 T. \tag{6.48}$$

Further, using (6.3) and noting that m is even, we have

$$\begin{aligned}\int_{F''} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2/m} dt &\leq \left\{\exp\left(A^{-1}(\log\log T)^{\frac{1}{2}}\right)\right\}^{-2/m} \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{4/m} dt\\ &\leq b_5 \exp\left(-b_6 A^{-3/2} + b_7 A^{-1}\right) T \leq b_8 T.\end{aligned} \tag{6.49}$$

Here b_8 may be taken to be independent of A, for example $b_8 = b_5$ if $A \leq (b_6/b_7)^2$. Combining (6.48) and (6.49) we obtain (6.47), completing the proof of Theorem 6.4.

One can also ask the following more general question, related to the distribution of $\zeta(\frac{1}{2}+it)$, or equivalently $\log\zeta(\frac{1}{2}+it)$ (if $\zeta(\frac{1}{2}+it) \neq 0$).
337 Namely, if we are given a measurable set $E(\subseteq \mathbb{C})$ with a positive Jordan content, hos often does $\log\zeta(\frac{1}{2}+it)$ belong to E? This problem was essentially solved in an unpublished work of A. Selberg, who showed that

$$\lim_{T\to\infty} \frac{1}{T}\mu\left(0 \leq t \leq T; \frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{\log\log t}} \in E\right) = \frac{1}{\pi}\iint_E e^{-x^2-y^2} dx\, dy, \tag{6.50}$$

where as before $\mu(\cdot)$ is the Lebesgue measure. Roughly speaking, this result says that $\log\zeta(\frac{1}{2}+it)/\sqrt{\log\log t}$ is approximately normally distributed. From (6.50) one can deduce the asymptotic formula

$$\int_0^T \log\left|\zeta\left(\frac{1}{2}+it\right) - a\right| dt = (2\pi)^{-\frac{1}{2}} T(\log\log T)^{\frac{1}{2}} + O_a(T), \tag{6.51}$$

where $a \neq 0$ is fixed.

Independently of Selberg's unpublished work, A. Laurinčikas obtained results on the value distribution of $|\zeta(\frac{1}{2}+it)|$ and $|L(\frac{1}{2}+it,x)|$ in a series of papers. As a special case of (6.50) he proved

$$\lim_{T\to\infty}\frac{1}{T}\mu\left(0\le t\le T:\frac{\log|\zeta\left(\frac{1}{2}+it\right)|}{\sqrt{\frac{1}{2}\log\log T}}\le y\right)=(2\pi)^{-\frac{1}{2}}\int\limits_{-\infty}^{y}e^{-\frac{1}{2}u^2}du, \tag{6.52}$$

which is of course stronger and more precise than Theorem 6.4. It is interesting that Laurinčikas obtains (6.52) by applying techniques from probabilistic number theory to the asymptotic formula

$$I_x(T)=\int\limits_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2x}dt=T(\log T)^{x^2}\left(1+O\left((\log\log T)^{-\frac{1}{4}}\right)\right), \tag{6.53}$$

which holds uniformly for $k_T\le k\le k_0$, where $k_T=\exp(-(\log\log T)^{\frac{1}{2}})$, k_0 is an arbitrary positive number and

$$x=\left(\left[k^{-1}\sqrt{2\log\log T}\right]\pm 5\right)^{-1}. \tag{6.54}$$

The crucial result, which is (6.53), is proved by using the method employed by D.R. Heath-Brown in proving Theorem 6.1. This involves the use of convexity techniques, but since the argument requires asymptotic formulas instead of upper and lower bounds, it is technically much **338**
more involved than the proof of Theorem 6.1, although the underlying ideas are essentially the same. For example, instead of the bounds (6.8) and (6.9) furnished by Lemma 6.1, for the proof of (6.53) one requires the following asymptotic formulas:

$$\begin{aligned}\sum_{n\le N}d_x^2(n)n^{-2\sigma}&=H(x)(2\sigma-1)^{-x^2}+O\left(N^{1-2\sigma}(\log N)^{x^2}\right)\\&+O\left((2\sigma-1)^{-x^2}\Gamma\left(x^2+1,(2\sigma-1)\log N\right)\right)\\&+O\left((2\sigma-1)\log^{\frac{1}{2}}N\right)+O\left((2\sigma-1)^{-x^2}(\log\log T)^{-\frac{1}{2}}\right)\end{aligned} \tag{6.55}$$

and

$$\sum_{n\le N} d_x^2(n)n^{-1} = \frac{H(x)}{\Gamma(x^2+1)}(\log T)^{x^2}\left(1+O\left((\log\log T)^{-1}\right)\right). \tag{6.56}$$

Both (6.55) and (6.56) are uniform in $k \in [k_T, k_0]$, in (6.55) one has $\sigma > 1/2$, and moreover

$$H(x) := \prod_p \left\{(1-p^{-1})^{x^2}\sum_{r=0}^{\infty} d_x^2(p^r)p^{-r}\right\},$$

$$\Gamma(s,y) := \int_y^{\infty} x^{s-1}e^{-x}dx.$$

We shall not reproduce here the details of the arguments leading to (6.55) - (6.56) and eventually to (6.53) and (6.52). We only remark here that, in order to get the asymptotic formulas that are needed, the weighted analogue of $J(\sigma)$ (and correspondingly other integrals) in Theorem 6.1 is defined somewhat differently now. Namely, for $\sigma \ge 1/2$ one sets

$$J_x(\sigma) = \int_{-\infty}^{\infty} |\zeta(\sigma+it)|^{2x}w(t)dt, \quad w(t) = \int_{\log^2 T}^{T} \exp\left(-2\eta(t-2\tau)^2\right)d\tau,$$

where $\eta = xk^{-1}$. The modified form of the weight function $w(t)$ is found to be more expedient in the proof of (6.53) than the form of $w(t)$ used in the proof of Theorem 6.1.

6.4 Mean Values Over Short Intervals

339 In this section we shall investigate lower bounds for

$$I_k(T+H) - I_k(T) = \int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt \tag{6.57}$$

when $T \geq T_0$, $k \geq 1$ is a fixed integer, and this interval $[T, T+H]$ is "short" in the sense that $H = \circ(T)$ as $T \to \infty$. The method of proof of Theorem 6.1, which works for all rational $k > 0$, will not produce good results when H is relatively small in comparison with T. Methods for dealing successfully with the integral in (6.57) have been developed by K. Ramachandra, who alone or jointly with R. Balasubramanian, obtained a number of significant results valid in a wide range for H. Their latest and sharpest result on (6.57), which follows from a general theorem, is the following

Theorem 6.5. *Let $k \geq 1$ be a fixed integer, $T \geq T_0$ and $(\log\log T)^2 \leq H \leq T$. Then uniformly in H*

$$\int\limits_T^{T+H} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \geq \left(c'_k + O(H^{-1/8}) + O\left(\frac{1}{\log H}\right)\right) H \log^{k^2} H, \quad (6.58)$$

where

$$c'_k = \frac{1}{\Gamma(k^2+1)} \prod_p \left\{\left(1-p^{-1}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)^2}{\Gamma(k)m!}\right) p^{-m}\right\}. \quad (6.59)$$

Proof. Note that (6.58) is a remarkable sharpening of Theorem 1.5, and that it is also true unconditionally. Since

$$\sum_{n \leq x} d_k^2(n) \sim b'_k x \log^{k^2-1} x, \quad \sum_{n \leq x} d_k^2(n) n^{-1} \sim c'_k \log^{k^2} x, \quad (6.60)$$

it is seen that instead of (6.58) it is sufficient to prove

$$\int\limits_T^{T+H} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \geq \sum_{n \leq H} \left(H + O(H^{7/8}) + O(n)\right) d_k^2(n) n^{-1}. \quad (6.61)$$

□

To this end let **340**

$$F(t) := \zeta^k\left(\frac{1}{2}+it\right) - \sum_{n \leq H} d_k(n) n^{-\frac{1}{2}-it} - \sum_{H < n \leq H + H^{\frac{1}{4}}} d_k(n) n^{-\frac{1}{2}-it} \quad (6.62)$$

$$= \zeta^k\left(\frac{1}{2} + it\right) - F_1(t) - F_2(t),$$

say, and assume that (6.61) is false. Then it follows that

$$\int_T^{T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \le 2H \sum_{n\le H} d_k^2(n)n^{-1}. \tag{6.63}$$

By using the mean value theorem for Dirichlet polynomials (see (1.15)) we obtain

$$\int_T^{T+H} |F_1(t)|^2 dt \le 2H \sum_{n\le H} d_k^2(n)n^{-1}, \tag{6.64}$$

and also

$$\begin{aligned}\int_T^{T+H} |F_2(t)|^2 dt &\ll H \sum_{H<n\le H+H^{\frac{1}{4}}} d_k^2(n)n^{-1} \\ &\ll \sum_{H<n\le H+H^{\frac{1}{4}}} d_k^2(n) \ll H^{\frac{1}{2}} \ll H\sum_{n\le H} d_k^2(n)n^{-1}\end{aligned} \tag{6.65}$$

since trivially $d_k(n) \le n^{1/8}$ for $n \ge n_0$. Thus by (6.62)

$$\begin{aligned}\int_T^{T+H} |F(t)|^2 dt &\ll \int_T^{T+H} \left(\left|\zeta\left(\frac{1}{2} + it\right)\right|^2 + |F_1(t)|^2 + |F_2(t)|^2\right) dt \\ &\ll H \sum_{n\le H} d_k^2(n)n^{-1},\end{aligned} \tag{6.66}$$

and consequently from (6.63) - (6.65) we obtain by the Cauchy-Schwarz inequality

$$\int_T^{T+H} |F(t)F_2(t)| dt \ll H^{\frac{7}{8}} \sum_{n\le H} d_k^2(n)n^{-1}, \tag{6.67}$$

$$\int_T^{T+H} |F_1(t)F_2(t)|dt \ll H^{\frac{7}{8}} \sum_{n\le H} d_k^2(n)n^{-1}.$$

We now introduce the multiple averaging, which will have the effect of making certain relevant integrals in the sequel small. Let $U = H^{7/8}$ and $r \geq 1$ a large, but fixed integer. Set 341

$$\int_{(r)} \Psi(t)dt := U^{-r} \int_0^U du_r \int_0^U du_{r-1} \ldots \int_0^U du_1 \int_{T+U+u_1+u_2+\ldots+u_r}^{T+H-u_1-u_2-\ldots-u_r} \Psi(t)dt.$$

Thus if $\Psi(t) \geq 0$ for $T \leq t \leq T+H$ and $\Psi(t)$ is integrable, we have

$$\int_T^{T+H} \Psi(t)dt \geq \int_{(r)} \Psi(t)dt \geq \int_{T+(r+1)U}^{T+H-(r+1)U} \Psi(t)du. \tag{6.68}$$

From (6.62) we have

$$\zeta^k\left(\frac{1}{2}+it\right) = F_1(t) + F_2(t) + F(t),$$

hence

$$\begin{aligned}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} &= |F_1(t)|^2 + |F_2(t)|^2 + |F(t)|^2 \\ &\quad + 2\operatorname{Re}\left\{\overline{F_1(t)}F_2(t) + \overline{F_1(t)}F(t) + \overline{F_2(t)}F(t)\right\}.\end{aligned}$$

This gives

$$\begin{aligned}\int_T^{T+H} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt &\geq \int_{(r)} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \int_{(r)} |F_1(t)|^2 dt \geq \qquad (6.69)\\ &+ 2\operatorname{Re}\left\{\int_{(r)} \overline{F_1(t)}F(t)dt\right\} + O\left(\int_{(r)} \{|F_1(t)F_2(t)| + |F_2(t)F(t)|\}\, dt\right).\end{aligned}$$

The integrals in the error terms are majorized by the integrals in (6.67). Thus using once again (1.15) we obtain

$$\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt \geq \int_{T+(r+1)U}^{T+H-(r+1)U}|F_1(t)|^2dt \tag{6.70}$$

$$+2\operatorname{Re}\left\{\int_{(r)}\overline{F_1(t)}F(t)dt\right\}+O\left(H^{7/8}\sum_{n\leq H}d_k^2(n)n^{-1}\right)$$

$$=\sum_{n\leq H}\left(H+O(H^{7/8})+O(n)\right)d_k^2(n)n^{-1}+2\operatorname{Re}\left\{\int_{(r)}\overline{F_1(t)}F(t)dt\right\}.$$

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We assumed that (6.61) is false. Therefore if we can prove that

$$\int_{(r)}\overline{F_1(t)}F(t)dt \ll H^{7/8}\sum_{n\leq H}d_k^2(n)n^{-1}, \tag{6.71}$$

we shall obtain a contradiction, which proves (6.61) and consequently (6.58). Let

$$A(s) := \sum_{n\leq H}d_k(n)n^{-s}, \quad B(s) := \zeta^k(s)-\sum_{n\leq H+H^{\frac{1}{4}}}d_k(n)n^{-s}.$$

Then by Cauchy's theorem

$$\int_D A(1-s)B(s)ds = 0,$$

where D is the rectangle with vertices $\frac{1}{2}+i(T+U+u_1+\ldots+u_r)$, $2+i(T+U+u_1+\ldots+u_r)$, $2+i(T+H-u_1-\ldots-u_r)$, $\frac{1}{2}+i(T+H-u_1-\ldots-u_r)$. Hence

$$U^{-r}\int_0^U du_r\int_0^U du_{r-1}\ldots\int_0^U du_1\int_D A(1-s)B(s)ds$$

$$= U^{-r}\int_0^U du_r \int_0^U du_{r-1}\ldots\int_0^U du_1\left(\int_{I_1}+\int_{I_2}+\int_{I_3}+\int_{I_4}\right)A(1-s)B(s)ds = 0,$$

where I_1 and I_3 are the left and right vertical sides of D, and I_2 and I_4 are the lower and upper horizontal sides, respectively. If we denote, for $1 \leq n \leq 4$,

$$J_4 = U^{-r}\int_0^U du_r \int_0^U du_{r-1}\ldots\int_0^U du_1 \int_{I_n} A(1-s)B(s)ds$$

and observe that

$$J_1 = i\int_{(r)} \overline{F_1(t)}F(t)dt,$$

then (6.71) will follow if we can show that

$$J_2 \ll H^{7/8}\sum_{n\leq H} d_k^2(n)n^{-1}, \tag{6.72}$$

$$J_3 \ll H^{7/8}\sum_{n\leq H} d_k^2(n)n^{-1}, \tag{6.73}$$

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$$J_4 \ll H^{7/8}\sum_{n\leq H} d_k^2(n)n^{-1}. \tag{6.74}$$

To prove (6.73) note that, for Re $s > 1$,

$$B(s) = \zeta^k(s) - \sum_{m\leq H+H^{\frac{1}{4}}} d_k(m)m^{-s} = \sum_{m>H+H^{\frac{1}{4}}} d_k(m)m^{-s}.$$

Therefore

$$J_3 = U^{-r}\int_0^U du_r\ldots\int_0^U du_1 \int_{2+i(T+U+u_1+\ldots+u_r)}^{2+i(T+H-u_1-\ldots-u_r)} A(1-s)B(s)ds$$

$$= iU^{-r}\sum_{n\leq H}\sum_{m>H+H^{\frac{1}{4}}} d_k(m)m^{-2}nd_k(n)\int_0^U du_r\ldots\int_0^U du_1 \int_{T+U+u_1+\ldots+u_r}^{T+H-u_1-\ldots-u_r}\left(\frac{n}{m}\right)^{it}dt$$

$$\ll_r U^{-r} \sum_{n\le H} \sum_{m>H+H^{\frac{1}{4}}} d_k(m)m^{-2}nd_k(n)\left(\log \frac{m}{n}\right)^{-r-1}.$$

But for m, n in the last sum we have

$$\log \frac{m}{n} \ge \log \frac{H+H^{\frac{1}{4}}}{H} \ge \frac{1}{2}H^{-\frac{3}{4}},$$

and since $U = H^{7/8}$ we obtain

$$J_3 \ll_r H^{7/8}H^{-(r+1)/8} \sum_{n\le H} nd_k(n) \sum_{m>H} d_k(m)m^{-2}$$
$$\ll_r H^{7/8}H^{-(r+1)/8}H \log^{2k-2} H \ll_r H^{7/8} \log^{k^2-1} H$$

for any $r \ge 7$.

It remains to prove (6.72) and (6.74). Since both proofs are similar,
344 only the details for the latter will be given. Set for brevity

$$G(s) := A(1-s)B(s),\ T_r := T + U + u_1 + \ldots + u_r.$$

Then we have

$$J_4 = U^{-r} \int_0^U du_r \ldots \int_0^U du_1 \int_{\frac{1}{2}}^2 G(\sigma + iT_r)d\sigma, \tag{6.75}$$

and by the theorem of residues we may write

$$G(\sigma + iT_r) = \frac{1}{2\pi ie} \int_E G(w + \sigma + iT_r) \exp\left(-\cos \frac{w}{A}\right) \frac{dw}{w}, \tag{6.76}$$

where $A > 0$ is a constant, and E is the rectangle with vertices $\frac{1}{2} - \sigma + iT - iT_r$, $2 + iT - iT_r$, $2 + i(T + H - T_r)$, $\frac{1}{2} - \sigma + i(T + H - T_r)$, whose vertical sides are I_5, I_7 and horizontal sides are I_6, I_8, respectively. The kernel function $\exp\left(-\cos \frac{w}{A}\right)$, which is essentially the one used in the proof of Theorem 1.3, is of very rapid decay, since

$$\left|\exp\left(-\cos \frac{w}{A}\right)\right| = \exp\left(-\cos \frac{u}{A} \cdot ch\frac{v}{A}\right) \quad (w = u + iv,\ A > 0).$$

We insert (6.76) in (6.75), obtaining

$$J_4 = J_5 + J_6 + J_7 + J_8,$$

say, where $J_n(n = 5, 6, 7, 8)$ comes from the integral over I_n. For $w = u + iv$ on I_6 and I_8 we have $|v| \geq U = H^{7/8}$, consequently for $A = 10$

$$\begin{aligned} \left|\exp\left(-\cos\frac{w}{A}\right)\right| &= \exp\left(-\cos\frac{u}{10}\cdot ch\frac{v}{10}\right) \leq \exp\left(-\frac{1}{2}\cos\left(\frac{1}{5}\right)e^{|v|/10}\right) \\ &\leq \exp\left(-\frac{1}{2}\cos\left(\frac{1}{5}\right)eH^{7/8}/10\right) \ll_C T^{-C} \end{aligned} \tag{6.77}$$

for any fixed $C > 0$, since $H \geq (\log\log T)^2$. In fact, one could increase the range for H in Theorem 6.5 to $H \geq (\log\log T)^{1+\delta}$ for any fixed $\delta > 0$ at the cost of replacing the error term $O(H^{-1/8})$ by a weaker one. From (6.77) it easily follows that the contribution of $J_6 + J_8$ is negligible.

Consider now 345

$$\begin{aligned} J_5 = \frac{U^{-r}}{2\pi ie}\int\limits_0^U du_r \ldots \int\limits_0^U du_2 \int\limits_{\frac{1}{2}}^{2} \\ \left\{\int\limits_0^U du_1 \int\limits_{I_5} G(w + \sigma + iT_r)\exp\left(-\cos\frac{w}{A}\right)\frac{dw}{w}\right\}. \end{aligned} \tag{6.78}$$

On I_5 we have $w = \frac{1}{2} - \sigma + i(v - T_r)$, $T \leq v \leq T + H$. Thus the integral in curly brackets in (6.78) becomes (with $A = 10$)

$$\begin{aligned} & i\int\limits_T^{T+H} G\left(\frac{1}{2} + iv\right)dv \int\limits_0^U \exp\left\{-\cos\left(\frac{1}{2} - \sigma + i\,(v - T - u_1 - \ldots - u_r)\,/10\right)\right\} \\ & \qquad \frac{du_1}{\frac{1}{2} - \sigma + i(v - T - u_1 - \ldots - u_r)} \\ & \ll \int\limits_T^{T+H}\left|G\left(\frac{1}{2} + iv\right)\right|dv \int\limits_0^U \exp\left\{-\frac{1}{2}\cos\left(\frac{\frac{1}{2} - \sigma}{10}\right)\cdot e^{-|v - T - u_1 - \ldots - u_r|}\right\} \end{aligned}$$

$$\frac{du_i}{((\delta - 1/2)^2 + (u_1 + T + \ldots + u_r - v)^2)^{1/2}}.$$

The presence of the exponential factor makes the portion of the integral over u_1, for which $|u_1 + T + \ldots + u_r - v| \geq 1$, to be $O(1)$. The remaining portion for which $|u_1 + T + \ldots + u_r - v| < 1$ is majorized by

$$\int\limits_{v-T-U-\ldots-u_r-1}^{v-T-U-\ldots-u_r+1} \frac{du_1}{\left((\sigma - \frac{1}{2})^2 + (u_1 + T + \ldots + u_r - v)^2\right)^{\frac{1}{2}}} = \int\limits_{-1}^{1} \frac{dx}{\left((\sigma - \frac{1}{2})^2 + x^2\right)^{\frac{1}{2}}}$$

$$= \log \frac{1 + \left((\sigma - \frac{1}{2})^2 + 1\right)^{\frac{1}{2}}}{-1 + \left((\sigma - \frac{1}{2})^2 + 1\right)^{\frac{1}{2}}} \ll \log \frac{1}{\sigma - \frac{1}{2}}.$$

Hence we obtain

$$J_5 \ll U^{-1} \int\limits_T^{T+H} \left|G\left(\frac{1}{2} + iv\right)\right| dv \int\limits_{\frac{1}{2}}^{2} \left(\log \frac{1}{\sigma - \frac{1}{2}}\right)^{-1} d\sigma$$

$$\ll U^{-1} \int\limits_T^{T+H} \left|G\left(\frac{1}{2} + iv\right)\right| dv = U^{-1} \int\limits_T^{T+H} |F_1(t)F(t)| dt \ll H^{\frac{1}{8}} \sum_{n \leq H} d_k^2(n) n^{-1},$$

on using (6.64) and (6.66) coupled with the Cauchy-Schwarz inequality.

Finally we consider

$$J_7 = \frac{U^{-r}}{2\pi ie} \int\limits_0^U du_r \ldots \int\limits_0^U du_1 \int\limits_{\frac{1}{2}}^{2} d\sigma \int\limits_{I_7} G(w + \sigma + iT_r) \exp\left(-\cos \frac{w}{10}\right) \frac{dw}{w}.$$

346 On I_7 we have $w = 2 + iv$, $T - T_r \leq v \leq T + H - T_r$. Since $|T - T_r| = |U + u_1 + \ldots + u_r| \geq H^{7/8}$, $|T + H - T_r| \geq H^{7/8}$, the presence of $\exp\left(-\cos \frac{w}{10}\right)$ makes it possible to replace, in the integral over w, the range for v by $(-\infty, \infty)$ with a negligible error. In the remaining integral we may interchange, in view of the absolute convergence, the order of integration. We are left with

$$\frac{1}{2\pi e} \int\limits_{-\infty}^{\infty} \exp\left(-\cos\left(\frac{2 + iv}{10}\right)\right) \frac{dv}{2 + iv}$$

$$\left\{U^{-r} \int\limits_{\frac{1}{2}}^{2} d\sigma \int\limits_{0}^{U} du_r \ldots \int\limits_{0}^{U} du_1 G\left(2+\sigma+i(v+T_r)\right)\right\}.$$

But we have

$$G(2+\sigma+i(v+T_r)) = A(-1-\sigma-iv-iT_r)B(2+\sigma+iv+iT_r)$$
$$= \sum_{m>H+H^{\frac{1}{4}}} \sum_{n\le H} d_k(m)m^{-2-\sigma} d_k(n) n^{\sigma+1} \left(\frac{n}{m}\right)^{i(v+T+U+u_1+\ldots+u_r)}.$$

If we proceed as in the case of the estimation of J_3 and carry out the integrations over $u_1, u_2, \ldots u_r$, we readily seen that the contribution of J_7 is $O(1)$ if r is sufficiently large. This completes the proof of the theorem. It may be noted that essentially the multiple averaging accounts for the shape of the lower bound in (6.58), while the kernel function introduced in (6.76) regulates the permissible range for H.

NOTES FOR CHAPTER 6

347 The method of D.R. Heath-Brown [62], where (6.2) was proved, is based on a convexity technique. This is embodied in Lemma 6.2 and Lemma 6.3, both of which are due to R.M. Gabriel [39]. The bounds given by (6.3), which have the merit of being uniform in m, were proved by M. Jutila [88], who modified the method of Heath-Brown [62]. Lemmas 6.1, 6.4, 6.5 and 6.6 are from Heath-Brown [62].

E.C. Titchmarsh was the first to prove that if k is a fixed natural number, then

$$\int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} e^{-\delta t} dt \gg_k \frac{1}{\delta}\left(\log\frac{1}{\delta}\right)^{k^2}$$

for $0 < \delta \leq \frac{1}{2}$ (see Theorem 7.19 of his book [155]). From this it is easy to deduce that

$$\limsup_{T\to\infty}\left\{I_k(T)T^{-1}(\log T)^{k^2}\right\} > 0.$$

Improving upon this K. Ramachandra (see his series of papers [138]) was the first to obtain lower bounds of the type (6.2). His method uses averaging techniques which do not depend on the aforementioned results of Gabriel, and besides it is capable of obtaining results valid over short intervals. In [138, Part II], he shows that

$$\int_T^{T+H} \left|\frac{d^\ell}{ds^\ell}(\zeta k(s))\Big|_{s=\frac{1}{2}+it}\right| dt > C_{k,\ell} H(\log H)^\lambda, \tag{6.79}$$

where $100 \leq (\log T)^{1/m} \leq H \leq T$. Here k, m, ℓ are any fixed natural numbers, $T \geq T_\circ(k,\ell,m)$, $\lambda = \ell + \frac{1}{4}k^2$, $C_{k,\ell} > 0$ is a constant which depends on k, ℓ. If the kernel $\exp(z^{4a+2})$ that Ramachandra used is replaced by his kernel $\exp(\sin^2 z)$, then the range for H can be taken as $C_k \log\log T \leq H \leq T$ (as mentioned in Notes for Chapter 1, the latter type of kernel is essentially best possible). Upper bounds for the integral in (6.79) are obtained by Ramachandra in [138, Part III]; one of the
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results is that

$$\int\limits_T^{T+H} \left|\zeta^{(\ell)}\left(\frac{1}{2}+it\right)\right| dt \ll H(\log T)^{\frac{1}{4}+\ell} \tag{6.80}$$

for $\ell \geq 1$ a fixed integer, $H = T^{\lambda}$, $\frac{1}{2} < \lambda \leq 1$. From (6.79) and (6.80) we infer that

$$T(\log T)^{\frac{1}{4}} \ll \int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right| dt \ll T(\log T)^{\frac{1}{4}},$$

but it does not seem possible yet to prove (or disprove) that, for some $C > 0$,

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right| dt \sim CT(\log T)^{\frac{1}{4}} \quad (T \to \infty).$$

The lower bound results are quite general (for example, they are applicable to ordinary Dirichlet L-series, L-series of number fields and so on), as was established by Ramachandra in [138, Part III]. The climax of this approach is in Balasubramanian-Ramachandra [12] (see also Ramachandra's papers [140], [142], [143]). Their method is very general, and gives results like

$$I_k(T+H) - I_k(T) \geq C_k' H(\log H)^{k^2}\left(1 + O\left(\frac{\log\log T}{H}\right) + O\left(\frac{1}{\log G}\right)\right)$$

uniformly for $c_k \log\log T \leq H \leq T$, if $k \geq 1$ is a fixed integer (and for all fixed $k > 0$, if the Riemann hypothesis is assumed). Bounds for $I_k(T + H) - I_k(T)$ for irrational k were also considered by K. Ramachandra (see [141, Part II]), and his latest result is published in [141, Part III].

Theorem 6.2 is from A. Ivić - A. Perelli [81], which contains a discussion of the analogous problem for some other zeta-functions.

E.J. Littlewood's lemma, mentioned in the proof of Lemma 6.7, is proved in Section 9.9 of E.C. Titchmarsh [155]. Suppose that $\Phi(s)$ is meromorphic in and upon the boundary of the rectangle $\mathscr{D}$ bounded by

the lines $t = 0$, $t = T(> 0)$, $\sigma = \alpha$, $\sigma = \beta\ B > \alpha)$, and regular and not **349**
zero on $\sigma = \beta$. Let $F(s) = \log \Phi(s)$ in the neighborhood of $\sigma = \beta$, and in general let

$$F(s) = \lim_{\epsilon \to 0+} F(\sigma + it + i\epsilon).$$

If $\nu(\sigma', T)$ denotes the excess of the number of zeros over the number of poles in the part of the rectangle for which $\sigma > \sigma'$, including zeros or poles on $t = T$, but not these on $t = 0$, then Littlewood's lemma states that

$$\int_{\mathbb{D}} F(s)ds = -2\pi i \int_{\alpha}^{\beta} \nu(\sigma, T)d\sigma.$$

Theorem 6.3 and Theorem 6.4 were proved by M. Jutila [88].

A comprehensive account on the distribution of values of $\zeta(s)$, including proofs of Selberg's results (6.50) and (6.51), is to be found in the monograph of D. Joyner [86].

A. Laurinčikas' results on the distribution of $|\zeta(\frac{1}{2} + it)|$, some of which are mentioned at the end of Section 6.3, are to be found in his papers [109], [110], [111], [112]. For example, the fundamental formula (6.52) is proved in [111, Part I] under the Riemann hypothesis, while [111, Part II] contains an unconditional proof of this result.

Theorem 6.5 is a special case of a general theorem due to R. Balasubramanian and K. Ramachandra [12]. I am grateful for their permission to include this result in my text.

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