ALGEBRAIC TOPOLOGY

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Editorial Note

THIS series of mathematical pamphlets is issued in response to a widespread demand from university teachers and research students in India who want to acquire a knowledge of some of those branches of mathematics which are not a part of the curricula for ordinary university degrees. While some of these pamphlets are based on lectures given by members of the Tata Institute of Fundamental Research at summer schools organized by the Institute, in cooperation with University of Bombay and the University Grants Commission, it is not the intention to restrict the series to such lectures. Pamphlets will be issued from time to time which are of interest to students.

K. CHANDRASEKHARAN

PREFACE

THIS pamphlet contains the notes of lectures given at a Summer School on Algebraic Topology at the Tata Institute of Fundamental Research in 1962. The audience consisted of teachers and students from Indian Universities who desired to have a general knowledge of the subject, without necessarily having the intention of specializing it. The speakers were M.S. Narasimhan, S. Ramanan, R. Sridharan and K. Varadarajan.

Chapter I introduces the elements of Set Theory. Chapter II deals with abelian groups and proves the structure theorem for finitely generated abelian groups. Chapter III covers some topological preliminaries. Singular homology groups are defined and their invariance under homotopy is proved in Chapter IV. The homology of simplical complexes is treated in Chapter V, where proofs of the Euler-Poincaré formula and Brouwer's fixed point thoerem are given (assuming the invariance theorem for simplical homology).

Contents

Chapter 1

Set-theoretic Preliminaries

1.1 Sets and maps

We shall adopt the point of view of naive set theory. A *set* is a collection of objects which are called *elements* or *points* of the set. The set of all rational integers (i.e. integers positive, negative and zero) is denoted by $\mathbf{Z}$, the set of all non-negative integers by $\mathbf{Z}^+$, the set of all rational numbers by $\mathbf{Q}$, the set of all real numbers by $\mathbf{R}$, and the set of all complex numbers by $\mathbf{C}$.

If x is an element of a set A, we write $x \in A$. If x is not an element of A, we write $x \notin A$. (Thus $x \in \mathbf{R}$ will mean that x is a real number). If P is a property, the set of all objects with property P will be denoted by $\{x \mid x \text{ satisfies } P\}$. Thus $\{x \mid x \in \mathbf{Z},\ x < 0\}$ is the set of all negative integers. The set which does not contain any element is called the *empty set* and is denoted by the symbol $\emptyset$.

Let X and Y be two sets. If every element of X is an element of Y, we say X is a subset of Y and write $X \subset Y$ or $Y \supset X$. If, in addition, $X \neq Y$, we say that X is a *proper subset* of Y. If $y \in Y$, $\{y\}$ will denote the subset of Y consisting of the element y. It is clear that if $X \subset Y$ and $Y \subset X$, we must have $X = Y$.

(i) The *union* $X \cup Y$ of two sets X and Y is defined to be the set $\{z \mid z \in X \text{ or } z \in Y\}$.

(ii) The *intersection* $X \cap Y$ of two sets X and Y is defined to be the set $\{z \mid z \in X \text{ and } z \in Y\}$.

If $X \cap Y \neq \emptyset$, we say X and Y are *disjoint*.

(iii) Let $X \subset Y$. The *complement* $Y - X$ of X in Y is defined to be the set $\{z \mid z \in Y \text{ and } z \notin X\}$.

(iv) The *cartesian product* $X \times Y$ of two sets X and Y is defined to be the set $\{(x, y) \mid x \in X \text{ and } y \in Y\}$.

The cartesian product $\prod_{i=1}^{n} X_i$ of sets X_i, $1 \leq i \leq n$, is defined in a similar manner. If $X_i = \mathbf{R}$, we denote $\prod_{i=1}^{n} X_i$ by $\mathbf{R}^n$.

Suppose J is a set and, for each $i \in J$ we are given a set X_i. We say $\{X_i\}$ *is a family of sets indexed by the set* J. We define

(i) the *union* of the family $\{X_i\}$ denoted by $\bigcup_{i \in J} X_i$, as the set $\{x \mid x \in X_i \text{ for at least one } i \in J\}$;

(ii) the *intersection* of the family X_i denoted by $\bigcap_{i \in J} X_i$, as the set $\{x \mid x \in X_i \text{ for every } i \in J\}$.

It is easy to verify the following:

(i) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$,

or more generally

$$X \cup (\bigcap_{i \in J} Y_i) = \bigcap_{i \in J} (X \cup Y_i).$$

(ii) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$,

or more generally

$X \cap (\bigcup_{i \in J} Y_i) = \bigcup_{i \in J} (X \cap Y_i)$.

(iii) If $(Y_i)_{i \in J}$ is a family of subsets of a set X, then

(a) $X - \bigcup_{i \in J} Y_i = \bigcap_{i \in J} (X - Y_i)$

(b) $X - \bigcap_{i \in J} Y_i = \bigcup_{i \in J} (X - Y_i)$.

Let X and Y be two sets. A *map* $f\colon X \to Y$ is an assignment to each $x \in X$ of an element $f(x) \in Y$. If A is a subset of X, the *image* $f(A)$ is the set $\{f(x) \mid x \in A\}$. The *inverse image* of a subset B of Y denoted $f^{-1}(B)$, is the set $\{x \mid x \in X \text{ and } f(x) \in B\}$. The map f is said to be *onto* if $f(X) = Y$, *one-one* if $f(x) = f(y)$ implies $x = y$. If $f\colon X \to Y$, $g\colon Y \to Z$ are two maps, we define the *composite* $(g \circ f)\colon X \to Z$ by setting $(g \circ f)(x) = g(f(x))$ for every $x \in X$. The map

$X \to X$ which associates to each $x \in X$ the element x itself is called the *identity map* of X, denoted by I_X. Let $f: X \to Y$ be a map. It is easy to see that f is one-one and onto if and only if there exists a map $g: Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. If such a map g exists, it is unique and we shall denote it by f^{-1}; f^{-1} is called the *inverse* of f. If A is a subset of X, the map $j: A \to X$, which associates to each $a \in A$ the same element a in X, is called the *inclusion* map of A in X. If $f: X \to Y$ is any map, the map $f \circ j: A \to Y$ is called the *restriction* of f to A and is often denoted $f|A$.

1.1.1 Equivalence relations

Definition 1.1 Any subset $R \subset X \times X$ is said to be a relation in X. We write $x\ R\ y$ if $(x, y) \in R$. An equivalence relation in a set X is a relation R in X such that the following conditions are satisfied.

(i) For every $x \in X$, $(x, x) \in R$.

(ii) If $(x, y) \in R$ then $(y, x) \in R$.

(iii) If $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

We say x is *equivalent to y under R*, if xRy i.e. $(x, y) \in R$. The above conditions simply require that

(i) every element is equivalent to itself (*reflexivity*),

(ii) if x is equivalent to y, then y is equivalent to x (*symmetry*),

(iii) if x is equivalent to y, and y is equivalent to z, then x is equivalent to z (*transitivity*).

Example 1.2 The subset $R \subset X \times X$ consisting of elements (x, x), $x \in X$, is an equivalence relation in X. This is called the *identity relation* in X.

Example 1.3 $R = X \times X$ is an equivalence relation in X, in which all elements are equivalent to one another.

Example 1.4 Let $q \in \mathbf{Z}$. Consider the subset of $\mathbf{Z} \times \mathbf{Z}$ consisting of pairs (m, n) of integers such that $m - n$ is divisible by q. This is an equivalence relation under which two integers are equivalent if and only if they are congruent modulo q.

Example 1.5 Consider the subset $\{(x, y) \mid (x, y) \in \mathbf{R}^2,\ x \leq y\}$ of $\mathbf{R}^2$. This satisfies (i) and (iii) but not (ii) and is therefore *not* an equivalence relation.

Example 1.6 If $f: X \to Y$ is a map, the subset $R_f \subset X \times X$ consisting of (x_1, x_2) such that $f(x_1) = f(x_2)$ is an equivalence relation.

Let now $x \in X$ and R an equivalence relation in X. The set of all elements of X equivalent to x under R is called an *equivalence class* $\bar{x}$. Consider the family of distinct equivalence classes of X under R. It is easily verified that they are pairwise disjoint and that their union is X. We shall define the *quotient* X/R of X by R as the set whose elements are these equivalence classes. The natural map $\eta: X \to X/R$ which associates to each $x \in X$, the equivalence class $\bar{x}$ which contains x, is clearly onto. In example 1.4 above, the residue classes are the usual congruence classes modulo q and we denote the quotient set by $\mathbf{Z}/(q)$. Finally, let $f: X \to Y$ be a map and R_f the equivalence relation defined by f. We define a map $q_f: X/R_f \to Y$ by setting $q_f(\bar{x}) = f(x)$. By definition of R_f, q_f is well defined. Clearly q_f is one-one. Moreover we have $q_f \circ \eta = f$. We have therefore proved the

Theorem 1.7 *Let $f: X \to Y$ be a map. Then there exists an equivalence relation R_f on X and a one-one onto map $q_f: X/R_f \to f(X)$ such that $f = j \circ q_f \circ \eta$ where j is the inclusion $f(X) \to Y$ and η is the natural map $X \to X/R_f$.*

Chapter 2

Abelian Groups

2.1 Groups and homomorphisms

Definition 2.1 An abelian group (G, ψ) is a non-empty set G together with a mapping $\psi: G \times G \to G$, $\psi(x, y)$ being denoted by $x+y$, satisfying the following conditions.

(a) $x + (y + z) = (x + y) + z$, for every x, y, z in G (associativity).

(b) There exists an element 0, called the zero element of G, which satisfies $0 + x = x + 0 = x$ for every $x \in G$.

(c) For every $x \in G$, there exists an element $-x$ called the negative of x such that $x + (-x) = (-x) + x = 0$.

(d) $x + y = y + x$ for every x, y in G (commutativity).

Remark 2.2 (i) We often write G for a group instead of (G, ψ), when it is clear from the context to which map ψ we are referring.

(ii) The map ψ is called the "*addition*" in G.

(iii) The zero element is unique. In fact, if there is an element $0' \in G$ such that (b) is valid for every $x \in G$ with 0 replaced by $0'$ we have, in particular, $0 = 0 + 0' = 0'$.

(iv) The negative of any element is unique (proof is easy).

(v) A group consisting of only a finite number of elements will be called a *finite group*. The number of elements in a finite group is called its order. A group which is not finite will be called an *infinite group*.

(vi) In view of (a), for $x, y, z \in G$, the element $x + y + z \in G$ is defined without parenthesis. More generally, for $x_i \in G$, $i = 1, 2, \ldots, n$, $\sum_{1 \leq i \leq n} x_i$ can be defined without ambiguity.

(vii) In what follows, for convenience, we sometimes drop the word "abelian" and refer to abelian groups simply as groups.

Example 2.3 The set $\mathbf{Z}(\text{ resp.}\,\mathbf{Q},\ \text{ resp.}\,\mathbf{R},\ \text{ resp.}\,\mathbf{C})$ of integers (resp. rationals, resp. reals, resp. complex numbers) with the usual operation of addition is an abelian group.

Example 2.4 Let $G = \mathbf{Z}/(m)$ where m is any integer ≥ 0. Set $\bar{k} + \bar{l} = \overline{k + l}$. It is easy to check that this defines an operation which satisfies our axioms. G becomes thus an abelian group and is finite if $m > 0$.

Example 2.5 The non-zero real numbers denoted by $\mathbf{R}^*$ (resp. the non-zero rational numbers denoted by $\mathbf{Q}^*$, resp. the non-zero complex numbers denoted by $\mathbf{C}^*$) become an abelian group under the usual operation of multiplication.

Definition 2.6 Let G, G' be two groups. A homomorphism $f\colon G \to G'$ is a map such that $f(x + y) = f(x) + f(y)$ for every $x, y \in G$.

Let $f\colon G \to G'$ be a homomorphism. Then $f(0) = 0$. In fact, $f(0) = f(0 + 0) = f(0) + f(0)$ Adding $-f(0)$ to both sides we get $0 = f(0)$. If $f\colon G \to G'$ and $g\colon G' \to G''$ are homomorphisms, then $g \circ f\colon G \to G''$ is also a homomorphism. For any group G, the identity map $I_G\colon G \to G$ is a homomorphism.

Definition 2.7 A homomorphism $f\colon G \to G'$ is called an isomorphism if there exists a homomorphism $g\colon G' \to G$ such that $f \circ g = I_{G'}$ (the identity map of G') and $g \circ f = I_G$ (identity map of G).

It is easily seen that a homomorphism $f\colon G \to G'$ is an isomorphism if and only if it is both one-one and onto.

Remark 2.8 If G, G' are groups such that there exists an isomorphism of G onto G' then G and G' are said to be *isomorphic*. If G and G' are isomorphic, we sometimes write $G \approx G'$. In some sense, isomorphic groups have "indistinguishable structures".

Example 2.9 The natural map $\eta: \mathbf{Z} \to \mathbf{Z}/(m)$ is an onto homomorphism. If $m \neq 0$, it is not one-one and hence not an isomorphism.

Example 2.10 The map $\mathbf{Z} \to \mathbf{Z}$ given by $x \mapsto 2x$ is a one-one homomorphism. It is not onto and hence not an isomorphism.

Example 2.11 The map $\mathbf{Q} \to \mathbf{Q}$ given by $x \mapsto 2x$ is an isomorphism.

Let G be a group. A non-empty subset H of G is called a *subgroup* of G if for every $x, y \in H$, we have $x - y \in H$. In particular $0 \in H$ and for any $x \in H$, we have $-x \in H$. It can be easily checked that H, with the 'addition' induced by that of G, is a group with 0 as the zero-element and the negative of $x \in H$ being $-x$.

Let H be a subgroup of G. Then the inclusion map $j: H \to G$ is a one-one homomorphism.

For any group G, G itself and the subset $\{0\}$ are subgroups.

Example 2.12 Let H_1, H_2 be two groups of a group G. Then $H_1 \cap H_2$ is a subgroup of G. Also, the set of elements of the form $h_1 + h_2$, $h_1 \in H_1$, $h_2 \in H_2$, is a subgroup, $H_1 + H_2$, of G. If $H_1 \cap H_2 = (0)$ and $H_1 + H_2 = G$, we say that G is the *direct sum* of H_1 and H_2 and write $G = H_1 \oplus H_2$.

Example 2.13 $\mathbf{Z}$ is a subgroup of $\mathbf{Q}$, $\mathbf{Q}$ a subgroup of $\mathbf{R}$ and $\mathbf{R}$ a subgroup of $\mathbf{C}$.

Example 2.14 For $m \geq 0$, the subset $m\mathbf{Z} = \{m \cdot n \mid n \in \mathbf{Z}\}$ is a subgroup of $\mathbf{Z}$.

Proposition 2.15 *The only subgroups of* $\mathbf{Z}$ *are* $m\mathbf{Z}$, $m \in \mathbf{Z}^+$.

PROOF: Let H be a subgroup of $\mathbf{Z}$. If $H = 0$, m can be taken to be 0. If $H \neq (0)$ let $n \in H$, $n \neq 0$. Since $-n \in H$, we may assume without loss of generality that $n > 0$. Let m be the smallest strictly positive integer contained in H. (This exists since every non-empty subset of the set of natural numbers has a least element.) We now assert $H = m\mathbf{Z}$. In fact, let $h \in H$. Then $h = m \cdot q + r$ with $0 \leq r < m$. Since $m \in H$, we have $m \cdot q \in H$ and hence $r = h - m \cdot q \in H$. Since $r \in H$ and $r < m$ we must have $r = 0$, which proves our assertion.

Let $f: G \to G'$ be a homomorphism of groups. Clearly, the image $f(G)$ of G by f is a subgroup of G'.

Definition 2.16 The subset $\{x \mid x \in G,\ f(x) = 0 \in G'\}$ is called the kernel of f, usually denoted $\ker f$.

It is easily seen that the kernel of any homomorphism $f: G \to G'$ is a subgroup of G.

Let G be an abelian group and H a subgroup. The relation $R_H = R$ defined in G by $x\ R\ y$ if and only if $x - y \in H$ is easily seen to be an equivalence relation. The quotient set G/R is usually denoted by G/H. We now define the structure of a group on G/H as follows. For $\bar{x}, \bar{y} \in G/H$, set $\bar{x} + \bar{y} = \overline{x + y}$. It is easily seen that the element thus defined depends only on the classes $\bar{x}$ and $\bar{y}$ and not on the particular representatives x and y. It can be checked that G/H becomes a group under this operation as addition. It is called the *quotient group* of G by H. The natural map $\eta: G \to G/H$ is clearly a homomorphism which is onto.

Let $f: G \to G'$ be a homomorphism. Then the equivalence relation R_f defined by f (Example 1.6, Chapter 1) is easily seen to be $R_{\ker f}$. The induced map $q_f: G/R_f \to G'$ (defined by $q_f(x) = f(x)$) is trivially checked to be a homomorphism. f then admits the following decomposition into homomorphisms

$$G \xrightarrow{\eta} G/R_f \xrightarrow{q_f} f(G) \xrightarrow{j} G'$$

where η is the natural map, q_f is an isomorphism and j is the inclusion of $f(G)$ in G'.

Remark 2.17 (i) The above result is sometimes called the "Fundamental theorem of homomorphisms".

(ii) Let $f: G \to G'$ be a homomorphism of abelian groups. Let H be a subgroup of G and H' a subgroup of G' such that $f(H) \subset H'$. There exists then a homomorphism $\tilde{f}: G/H \to G'/H'$ such that $\tilde{f} \circ \eta = \eta' \circ f$.

2.1.1 Finitely generated abelian groups

Definition 2.18 A torsion element of an abelian group A is an element $a \in G$ such that $a + \cdots + a$ (n times), denoted by na, is 0 for some integer $n > 0$.

Remark 2.19 The element 0 is always a torsion element of an abelian group. A non-zero torsion element may not exist in an abelian group.

For instance, the group $\mathbf{Q}$ of rational numbers does not possess any non-zero torsion element. On the other hand, every element of $\mathbf{Z}/(n)$ $(n > 0)$ is a torsion element.

If a, b are torsion elements, then $a - b$ is also a torsion element. In fact, if $ma = 0,\ nb = 0$ with $m \neq 0,\ n \neq 0$ then $m \cdot n(a - b) = 0$, and $m \cdot n \neq 0$. The torsion elements of an abelian group A thus form a subgroup, called the *torsion subgroup* of A denoted by $T(A)$.

Definition 2.20 An abelian group A for which $T(A) = (0)$ is called torsion-free. A group A for which $T(A) = A$ is called a torsion group.

Proposition 2.21 *For any abelian group A, $A/T(A)$ is torsion-free.*

In fact, let $\bar{a} \in A/T(A)$ be such that $n\bar{a} = 0$ for some $n \in \mathbf{Z}^+,\ n \neq 0$. Then we have $na \in T(A)$. Therefore there exists an $m \in \mathbf{Z}^+,\ m \neq 0$ such that $m(na) = 0$. Since $m \cdot n \neq 0$, it follows that $a \in T(A)$ i.e. $\bar{a} = 0$.

Definition 2.22 Let A be an abelian group and B a subgroup of A. A subset S of B is said to be a set of generators for B if every element of B can be written as $\sum_{1\leq i\leq n} m_i a_i,\ m_i \in \mathbf{Z},\ a_i \in S$. An abelian group A is said to be finitely generated if there exists a finite set of generators for A.

Remark 2.23 Any quotient group of a finitely generated abelian group A is finitely generated. In fact, if $a_1, \ldots, a_n$ is a set of generators for A, their images under the natural map form a set of generators for the quotient.

Remark 2.24 Every finitely generated torsion group is finite. In fact, let $a_1, \ldots, a_n$ generate a torsion group A. If m_i is the least positive integer such that $m_i a_i = 0,\ 1 \leq i \leq n$ then clearly every element of A can be written as $\sum_{1\leq i\leq n} l_i a_i$, with $0 \leq l_i \leq m_i$. Hence A is finite of order $\leq \prod_{1\leq i\leq n} m_i$.

Example 2.25 $(\mathbf{Z}, +)$ is generated by 1 (or -1).

Example 2.26 $(\mathbf{Z}/(n),\ +)$ is generated by 1 (remark above).

Example 2.27 $(\mathbf{Q},\ +)$ is not finitely generated.

Theorem 2.28 *Every subgroup of a finitely generated abelian group is finitely generated.*

PROOF: The proof is by induction on the number n of generators of the given group. The theorem being trivially true for $n = 0$, we shall assume the theorem for all groups which can be generated by less than n elements. Let now A be a group generated by $e_1, \dots, e_n$ and let B be a subgroup of A. The set $\{k \mid k \in \mathbf{Z}$, such that there exists $b \in B$ of the form $\sum_{1\le i\le n-1} k_i e_i + k e_n\}$ is obviously a subgroup of $\mathbf{Z}$ and is hence of the form $l \cdot \mathbf{Z}$ for some $l \in \mathbf{Z}$ (see Proposition 2.15). Let $b_0 \in B$ such that $b_0 = \sum_{1\le i\le n-1} k_i e_i + l e_n$. The intersection of the subgroup generated by $e_1, \dots, e_{n-1}$ with B, which we denote by B' is finitely generated, by induction hypothesis. Let $f_1, \dots, f_q$ be a set of generators for B'. We assert that $f_1, \dots, f_q,\ b_0$ generate B. In fact, let $b = \sum_{1\le i\le n} m_i e_i \in B$. By our choice of l, $m_n = p \cdot l$ for some $p \in \mathbf{Z}$. Hence $b - p \cdot b_0 = \sum_{1\le i\le n-1}(m_i - p k_i) e_i \in B'$. Thus $b - p b_0 = \sum_{1\le i\le q} t_i f_i$ for $t_i \in \mathbf{Z}$ or, what is the same $b = p b_0 + \sum_{1\le i\le q} t_i f_i$. This proves our assertion.

Remark 2.29 The above proof actually yields the following stronger assertion. If an abelian group is generated by n elements, every subgroup can also be generated by n elements.

Definition 2.30 A set $a_1, \dots, a_n$ of elements of an abelian group is said to be free if $\sum_{1\le i\le n} m_i a_i = 0,\ m_i \in \mathbf{Z}$, implies $m_i = 0$ for every i with $1 \le i \le n$. A subset S of an abelian group is said to be free if every finite subset of S is free.

Definition 2.31 An abelian group A which admits of a free set S of generators is said to be free. S is then called a base for A.

Remark 2.32 By a maximal free set S of A we mean a free set S which is not a proper subset of any other free set. It is clear that a base of a free abelian group is a maximal free set.

Remark 2.33 It is easily seen that a finitely generated free abelian group admits of a finite base.

Remark 2.34 Let S be any set. Consider the set $F(S)$ of all formal linear combinations $\sum_{s\in S} n_s s,\ n_s \in \mathbf{Z}$, where $n_s = 0$ for all but a finite number of $s \in S$. We define addition in this set by

$$\Sigma n_s s + \Sigma m_s s = \Sigma(n_s + m_s)s.$$

Under this operation $F(S)$ becomes an abelian group. Every $s \in S$ will be identified with the element $\Sigma n_t t$ where $n_t = 0$ for $t \neq s$ and $n_s = 1$. With this identification, S clearly is a base for $F(S)$. $F(S)$ will be called the *free abelian group generated* by S. Let S, S' be two sets and $f\colon S \to S'$ any map. It is easily seen that there exists a unique homomorphism $F(f)$: $F(S) \to F(S')$ such that $F(f)/S = f$.

Remark 2.35 There exist abelian groups which are not free. For instance, the group $\mathbf{Z}/(n)$ is not free if $n \neq 0$.

Remark 2.36 The group $\mathbf{Z}$ is free with 1 as base.

Remark 2.37 Every free abelian group is torsion-free.

Theorem 2.38 *If an abelian group has a base consisting of n elements, then every other base also consists of n elements.*

Proof: This theorem is a consequence of the following lemmas and Remark 2.32 above.

Lemma 2.39 *Let A be a finitely generated abelian group and S a maximal free set in A. If B is the subgroup generated by S, then A/B is finite group.*

Proof: Since A is finitely generated, so is A/B. We shall now prove that A/B is a torsion group. Let $\bar{a} \in A/B$ with $a \in A$ as its representative. Since S is a maximal free set, $\{a\} \bigcup S$ is not free and there is a nontrivial relation $na + \Sigma m_i s_i = 0,\ n,\ m_i \in \mathbf{Z},\ s_i \in S$. Since S is free, $n \neq 0$ and we have $na \in B$. This proves that $n\bar{a} = 0$. By Remark 2.24 following Proposition 2, A/B is finite.

Lemma 2.40 *Let S be a maximal free set of a finitely generated abelian group A and B the subgroup generated by S. For every prime p which does not divide the order of A/B, the group A/pA is finite if and only if S is finite. Moreover, if S consists of n elements, the order of A/pA is p^n.*

Proof: Let $\xi\colon A \to A/B, \eta\colon A \to A/pA$ be the natural maps and let q denote the order of A/B. Then, for any $a \in A$, $qa \in B$. Let $l, m \in \mathbf{Z}$ such that $lp + mq = 1$. Then $mqa = a - pla$. That is $\eta(a) = \eta(mqa) \in \eta(B)$ and hence $\eta(S)$ generates A/pA. Moreover, $\sum m_i\eta(s_i) = \sum n_i\eta(s_i), s_i \in S$ implies that $m_i \equiv n_i(p)$. For $\sum(m_i - n_i)\eta(s_i) = 0$ implies $\sum(m_i - n_i)s_i = pa$ for some $a \in A$. As $\sum(m_i - n_i)s_i \in B, \xi(pa) = p\xi(a) = 0$ and since p does not divide $q, \xi(a) = 0$, or what is the same, $a = \sum l_j s_j$. Since S is a free set, this proves that $m_i - n_i = pl_i$. In other words, every element in A/pA can be written uniquely in the form $\sum n_i\eta(s_i)$, with $0 \le n_i < p, s_i \in S$. This proves that A/pA is infinite if S is infinite and if S is finite, consisting of n elements, then A/pA consists of p^n elements.

Lemma 2.41 *Let S be a maximal free set with n elements in a finitely generated abelian group. Then any other maximal free set S' also has n elements.*

Proof: We denote by B, B' the subgroups of A generated by S, S' respectively. Choose a prime p not dividing the orders of A/B and A/B'. By Lemma 2.40, A/pA has p^n elements which implies that S' is finite and that it has n elements.

Definition 2.42 The number of n of the above theorem will be called the rank of free abelian group.

Theorem 2.43 *A finitely generated torsion-free abelian group is free (i.e. it has a base consisting of a finite number of elements).*

Proof: We proceed by induction on the number n of generators. If $n = 0$ there is nothing to prove. Let us assume the theorem to be true for all groups which can be generated by $n-1$ elements. Let now A be a torsion-free group with generators $a_1, \dots a_n$. Let B denote the subgroup generated by $a_1, \dots a_{n-1}$.
Case i. Assume that there exists an integer $m \neq 0$ such that $ma_n \in B$. The map $\varphi\colon A \to A$ defined by $\varphi(a) = ma$ is a homomorphism. It is one-one since A is torsion-free. Moreover, $\varphi(A) \subset B$ by our assumption. Thus A is isomorphic to $\varphi(A)$ and hence by the remark following Theorem 2.28, can be generated by $n-1$ elements. The induction assumption now yields that A is free.
Case ii. Assume that $ma_n \in B$ implies $m = 0$. B is torsion-free and is generated by $n-1$ elements. By induction assumption, B is free. Let

$b_1, \ldots, b_k$ be a base for B. We assert that $b_1, \ldots, b_k,\ a_n$ is a base for A. In fact, evidently they form a set of generators. If $\Sigma l_i b_i + l a_n = 0$ then $l a_n \in B$ and this by our assumption implies that $l = 0$. Hence $\Sigma l_i b_i = 0$ which yields $l_i = 0$.

Remark 2.44 The above theorem is false if the group is not finitely generated. For instance, $(\mathbf{Q},\ +)$ is torsion-free, but not free.

Definition 2.45 Let A be a finitely generated abelian group. The rank of A is by definition the rank of the (torsion-free and hence) free group $A/T(A)$.

If $\bar{a}_1, \ldots, \bar{a}_n$ is a base of $A/T(A)$ with $a_1, \ldots, a_n$ as representatives in A, then $a_1, \ldots, a_n$ clearly form a free set. Moreover, if $a \in A$ then $\bar{a} = \Sigma m_i \bar{a}_i,\ m_i \in \mathbf{Z}$. This means that $ka - \Sigma k m_i a_i = 0$ for some $k \in \mathbf{Z},\ k \neq 0$. Thus $(a_1, \ldots, a_n)$ is a maximal free set. In view of the above lemmas, this means that the rank of a finitely generated abelian group is the same as the number of elements in any maximal free set.

Remark 2.46 The above considerations show that the subgroup B generated by $a_1, \ldots, a_n$ is a free subgroup of A such that every element of A can be uniquely written as $b + x,\ b \in B,\ x \in T(A)$. In other words, A is the direct sum of a free group and a finite group.

Proposition 2.47 *Let A be a finitely generated abelian group and B be a subgroup. Then*

$$\text{rank } A = \text{rank } B + \text{rank } A/B.$$

Proof: Since B is a subgroup of a finitely generated group and A/B a quotient , it is clear, in view of our earlier results, that B and A/B have finite ranks. Let rank $B = l'$ and rank $A/B = l''$. Then there exist maximal free sets $b_1, \ldots, b_{l'}$ in B and $\bar{a}_1, \ldots, \bar{a}_{l''}$ in A/B. Let $a_i \in A$ represent $\bar{a}_i \in A/B,\ 1 \leq i \leq l''$. We claim that the set $S = \{a_1, \ldots, a_{l''},\ b_1, \ldots, b_{l'}\}$ is a maximal free set. Suppose that $\sum_{1 \leq i \leq l''} n_i a_i + \sum_{1 \leq j \leq l'} m_j b_j = 0$. We then have $\sum_{1 \leq j \leq l''} n_i \bar{a}_i = 0$ and this means that $n_i = 0$. This means that $\sum_{1 \leq j \leq l'} m_j b_j = 0$, which implies $m_j = 0,\ 1 \leq j \leq l'$. This shows that S is a free set. On the other hand, if $a \in A$, there exists an integer $n \neq 0$ such that $n\bar{a} = \sum_{1 \leq j \leq l''} n_i \bar{a}_i$. This means that $na - \sum_{1 \leq i \leq l''} n_i a_i \in B$. Hence there exists an integer $m \neq 0$ such that $m(na - \sum_{1 \leq i \leq l''} n_i a_i) = \sum_{1 \leq j \leq l'} m_j b_j$, which proves that S is maximal. By what we have said above, it now follows that rank $A = l' + l''$.

Chapter 3

Set Topology

3.1 Topological spaces

A *topological space* is a set X together with a collection T of subsets of X (called *open sets*) with the following properties.

(i) The empty set $\emptyset$ and X are in T.

(ii) Any *finite* intersection of sets in T is again in T.

(iii) *Any* union of sets in T is again in T.

Example 3.1 Let X be a set. Let T consist only of X and $\emptyset$. This is called the trivial topology on X.

Example 3.2 Let X be a set. Let T consist of all subsets of X. This topology is called the discrete topology on X.

Example 3.3 *Metric space.* A metric space is a set X together with a function d (called the metric or the distance) from $X \times X$ to the non-negative real numbers such that the following conditions are satisfied:
(a) $d(x, y) = 0$ if and only if $x = y$, where $x, y \in X$;
(b) $d(x, y) = d(y, x)$ for $x, y \in X$ (symmetry)
(c) $d(x, z) \leq d(x, y) + d(y, z)$, for every $x, y, z \in X$ (triangle inequality).

Let $x \in X$ and r a positive real number. By the open ball around x of radius r we mean the set $\{y \mid y \in X,\ d(x, y) < r\}$. We define a topology on the metric space by defining the open sets to be all sets which are unions of open balls (together with the empty set).

Example 3.4 Let $\mathbf{R}$ be the set of real numbers. $\mathbf{R}$ has a natural metric defined by $d(x, y) = |x - y|$. This metric defines a topology on $\mathbf{R}$.

Example 3.5 Let $\mathbf{R}^n$ be the n-fold cartesian product of $\mathbf{R}$, that is, the set of n-tuples of real numbers $(x_1, \ldots, x_n)$. $\mathbf{R}^n$ is a metric space with the metric defined by, $d(x, y) = \|x - y\| = \{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\}^{\frac{1}{2}}$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Thus $\mathbf{R}^n$ has a natural topology. Whenever we refer to $\mathbf{R}^n$ as a topological space, we assume that it is provided with this topology.

Example 3.6 *Induced topology.* Let X be a topological space and $A \subset X$. We define a topology on A by defining the collection of open sets to be sets of the form $U \cap A$ where U runs through all open sets of X. This topology is called the induced topology on A.

Let X be a topological space. A subset of X is said to be *closed* if its complement is open.

The axioms for open sets are equivalent to the following.

(i) The empty set $\emptyset$ and X are closed sets.

(ii) The union of a finite number of closed sets is closed.

(iii) Any intersection of closed sets is closed.

Example 3.7 In $\mathbf{R}$, the set $\{x \mid x \in \mathbf{R},\ a \leq x \leq b\}$ $a, b \in \mathbf{R},\ a \leq b$, is closed.

Definition 3.8 Let X be a topological space and $x \in X$. By a neighbourhood of x we mean an open set containing x.

Example 3.9 Let $X = \mathbf{R}$. The set $\{x \mid x \in \mathbf{R},\ |x| < \varepsilon\}$ is a neighbourhood of the origin 0, if $\varepsilon > 0$.

Definition 3.10 Let A be a subset of a topological space X. A point $x \in X$ is said to be a closure point of A if every neighbourhood of x contains a point of A. The set of closure points of A is called the closure of A, and is denoted by $\bar{A}$.

Remark 3.11 $\bar{A}$ is the intersection of all closed sets containing A. In particular, the closure of a closed set is itself.

Example 3.12 Let $A = \{x \mid x \in \mathbf{R}^n,\ \|x\| < 1\}$, the open unit ball in $\mathbf{R}^n$. Then $\bar{A}$ is the closed ball $\{x \mid x \in \mathbf{R}^n,\ \|x\| \leq 1\}$.

Definition 3.13 A subset A of a topological space X is said to be dense in X if $\bar{A} = X$.

Example 3.14 The set of rational numbers is dense in $\mathbf{R}$.

Definition 3.15 Let A be a subset of a topological space X. A point $x \in X$ is said to be an interior point of A if there exists a neighbourhood of x in X contained in A. The set of all interior points of A is called the interior of A and is denoted by A°.

Remark 3.16 A° is the union of all open sets contained in A. In particular, the interior of an open set is itself.

Example 3.17 The interior of the closed unit ball in $\mathbf{R}^n$ is the open unit ball. The interior of the set consisting of only one point, say 0 in $\mathbf{R}^n$ is empty.

Definition 3.18 A topological space is said to be a Hausdorff space if every pair of distinct points have disjoint neighbourhoods; that is if $x, y \in X$ with $x \neq y$ then there exist open sets U_1 and U_2 of X with $x \in U_1,\ y \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Example 3.19 A set with the discrete topology is Hausdorff.

Example 3.20 $\mathbf{R}^n$ with its natural topology is Hausdorff. For, let $x, y \in \mathbf{R}^n$ with $x \neq y$. Since $x \neq y$, the distance between them is positive, i.e. $\|x - y\| > 0$. Then the open balls around x and y of radius $\frac{1}{2}\|x - y\|$ do not intersect. By a similar reasoning, we see that any metric space is Hausdorff.

Example 3.21 Let X be a set consisting of more than one point. The trivial topology on X is not Hausdorff.

Example 3.22 Let X be a Hausdorff space. A subset A of X, provided with the induced topology, is Hausdorff.

3.1.1 Compact spaces

Definition 3.23 A family $\{U_\alpha\}_{\alpha\in I}$ of subsets of a set X is said to be a covering of X if $\bigcup_{\alpha\in I} U_\alpha = X$, i.e. if each point of X belongs to at least one U_α. If, further, X is a topological space and each U_α is an open set of X, we say that $\{U_\alpha\}_{\alpha\in I}$ is an open covering of X.

Definition 3.24 A topological space X is said to be compact if the following condition is satisfied: if $\{U_\alpha\}_{\alpha\in I}$ is any open covering of X, then some finite sub-collection of $\{U_\alpha\}_{\alpha\in I}$ is already a covering; that is, if $\{U_\alpha\}_{\alpha\in I}$ is any open covering of X, then there exist a finite number of elements $\alpha_1, \ldots, \alpha_n \in I$ such that $\bigcup_{1\leq i\leq n} U_{\alpha_i} = X$.

Definition 3.25 A subset A of a topological space is said to be compact if it is compact in the induced topology.

Remark 3.26 (i) The space $\mathbf{R}$ is not compact.

(ii) Let X be a topological space. A subset A of X consisting of a finite number of points is compact.

(iii) A closed subset of a compact space is compact.

Proposition 3.27 *A compact subset of a Hausdorff space is closed.*

Proof: Let X be a Hausdorff space and A a compact subset of X. We have to show that $X - A$ is open. For this, it is sufficient to prove that every point $x \in X - A$ has a neighbourhood which does not intersect A. Let $x \in X - A$. If $y \in A$, since X is Hausdorff, we can find a neighbourhood V_y of x and a neighbourhood U_y of y in X such that $V_y \cap U_y = \emptyset$. If, $U'_y = U_y \cap A$,then $\{U'_y\}_{y\in A}$ is an open covering of A and since A is compact, we can find $y_1, \ldots, y_n \in A$ such that $\bigcup_{1\leq i\leq n} U'_{y_i} = A$. Then $V = \bigcap_{1\leq i\leq n} V_{y_i}$ is a neighbourhood of x which does not intersect A.

Remark 3.28 Since the space $\mathbf{R}$ is Hausdorff, it follows that every compact subset of $\mathbf{R}$ is closed. On the other hand, it is easy to see that every compact subset of $\mathbf{R}$ is bounded. Conversely, every closed bounded set of $\mathbf{R}$ can be proved to be compact. This result is usually known as the Heine-Borel Theorem.

3.1.2 Connected spaces

Definition 3.29 A topological space X is said to be connected if it is not the union of two non-empty disjoint open sets. A subset of a topological space is said to be connected if it is connected in the induced topology.

Clearly a topological space X is connected if and only if either of the following conditions is satisfied.

(i) X is not the union of two non-empty disjoint closed sets.

(ii) There is no subset of X which is both open and closed except the whole set and the empty set.

Proposition 3.30 *A subset of* $\mathbf{R}$ *is connected if and only if it is an interval. (A subset A of* $\mathbf{R}$ *will be called an interval if the following condition is satisfied: if $a_1, a_2 \in A$ with $a_1 < a_2$, then any $a \in \mathbf{R}$ such that $a_1 < a < a_2$ also belongs to A).*

Proof: Let A be a connected subset of $\mathbf{R}$. If A were not an interval, there would exist $a_1, a_2 \in A,\ a_1 < a_2$, and a real number a not in A with $a_1 < a < a_2$. Then the sets $B = \{b \mid b \in A,\ b < a\}$ and $C = \{c \mid c \in A,\ c > a\}$ are non-empty disjoint open sets of A whose union is A. This contradicts our assumption that A is connected.

Let now A be an interval. If possible, let $A = U \cup V$ with U, V open, non-empty with $U \cap V = \emptyset$. Let $a_1 \in U,\ a_2 \in V$. We may assume without loss of generality that $a_1 < a_2$. Consider $b = \sup\{r \mid r \in U,\ a_1 \le r \le a_2\}$. Since $a_1 \le b \le a_2$, it follows that $b \in \bar{U} = U$. On the other hand, since U is open, we can choose an $\varepsilon > 0$ sufficiently small so that the closed interval $[b - \varepsilon,\ b + \varepsilon]$ is contained in U. This contradicts the choice of b.

3.1.3 Continuous maps

Definition 3.31 Let X and Y be two topological spaces and $f\colon X \to Y$ be a map. Let $x \in X$. We say that f is continuous at x if the following condition is satisfied: for every neighbourhood U of $f(x)$ there exists a neighbourhood V of x such that $f(V) \subset U$. We say that f is continuous (or f is a continuous map) if f is continuous at every point of X.

Remark 3.32 We can easily prove that $f: X \to Y$ is continuous if and only if the inverse image by f of every open (resp. closed) set in Y is an open (resp. closed) set of X.

Remark 3.33 Let $f: X \to Y$, $g: Y \to Z$ be continuous maps. Then the composite $(g \circ f): X \to Z$ is continuous.

Remark 3.34 The identity map $I_X: X \to X$ is continuous.

Example 3.35 Let $f: \mathbf{R} \to \mathbf{R}$ be a map (i.e., f is a "real-valued function of a real variable"). Let $x_0 \in \mathbf{R}$. Then f is continuous at x_0 if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

Example 3.36 Let X be a discrete space and Y a topological space. Then any map $f: X \to Y$ is continuous.

Example 3.37 Let X be a topological space and Y a set with trivial topology. Then any map $f: X \to Y$ is continuous.

Definition 3.38 Let X and Y be two topological spaces. A continuous map $f: X \to Y$ is said to be a homeomorphism if there exists a continuous map $g: Y \to X$ such that $g \circ f = I_X$, $f \circ g = I_y$. Two topological spaces are said to be homeomorphic if there exists a homeomorphism f of X onto Y.

In a sense, homeomorphic spaces have "indistinguishable structures".

Remark 3.39 A one-one continuous map f of a topological space onto another is not necessarily a homeomorphism. This is because the set theoretic inverse of f is not in general continuous. For instance, let X be any set with more than one element. Let T be the discrete topology on X and T' the trivial topology on X. Then the identity map $I_X: (X, T) \to (X, T')$ is continuous, but is not a homeomorphism.

Remark 3.40 Properties of topological spaces like compactness, connectedness etc. are "preserved" under homeomorphisms. Such properties are said to be "topologically invariant".

Example 3.41 Let $X = Y = \mathbf{R}$. The map $f: X \to Y$ defined by $f(x) = x^3$ is a homeomorphism. The map $f(x) = -x$ is also a homeomorphism.

Example 3.42 The space $\mathbf{R}$ and the open subset $\{x \mid -1 < x < 1\}$ are homeomorphic $\left(\text{consider the map } f(x) = \frac{x}{1+|x|}\right)$. But $\mathbf{R}$ and the closed interval $I = \{x \mid -1 \leq x \leq 1\}$ are not homeomorphic since I is compact and $\mathbf{R}$ is not. $\mathbf{R}^n$ and the open unit ball $B = \{x \mid x \in \mathbf{R}^n,\ \|x\| < 1\}$ are homeomorphic.

Example 3.43 Let X be the unit circle $\{(x,y) \mid (x,y) \in \mathbf{R}^2,\ x^2+y^2 = 1\}$ and Y the ellipse $\{(x,y) \mid (x,y) \in \mathbf{R}^2,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ ($a \neq 0,\ b \neq 0$ being real numbers). The map $f\colon X \to Y$ defined by $f(x,y) = (ax, by)$ is a homeomorphism. (We put on X and Y the topology induced from $\mathbf{R}^2$.) But the unit circle and the parabola $\{(x,y) \mid (x,y) \in \mathbf{R}^2,\ y^2 = 4ax\}$ $a > 0$, are not homeomorphic. The parabola and the hyperbola $\{(x,y) \mid (x,y) \in \mathbf{R}^2,\ x^2 - y^2 = 1\}$ are not homeomorphic. (The parabola is connected whereas the hyperbola is not.)

Example 3.44 Let X be the complement of the origin in $\mathbf{R}^2$ (with the induced topology). Let Y be the cylinder in $\mathbf{R}^3$ defined by $Y = \{(x,y,z) \mid (x,y,z) \in \mathbf{R}^3,\ x^2+y^2 = 1,\ z > 0\}$ with the induced topology. X and Y are homeomorphic (Use polar coordinates in $\mathbf{R}^2 - (0)$).

Proposition 3.45 *Let X be a compact space, Y a topological space and $f\colon X \to Y$ a continuous map. Then $f(X)$ is compact. (That is, a continuous image of a compact space is compact.)*

PROOF: Let $\{U_\alpha\}_{\alpha \in I}$ be a family of open sets of $f(X)$ such that $\bigcup_{\alpha \in I} U_\alpha = f(X)$. Since clearly $f\colon X \to f(X)$ is continuous, $\{f^{-1}(U_\alpha)\}$ forms an open covering of X. Since X is compact, there exist $\alpha_1, \ldots, \alpha_n \in I$ such that $\bigcup_{1 \leq i \leq n} f^{-1}(U_{\alpha_i}) = X$. Then it is clear that $\bigcup_{1 \leq i \leq n} U_{\alpha_i} = f(X)$.

Corollary 3.46 *Let X be compact, Y Hausdorff and $f\colon X \to Y$ continuous. Then $f(X)$ is closed in Y.*

Corollary 3.47 *Let X be a compact space and Y a Hausdorff space. Let $f\colon X \to Y$ be a one-one continuous onto map. Then f is a homeomorphism.*

Contrast the above with the Remark 3.39 following the definition of homeomorphism.

Proposition 3.48 *Let X be a connected topological space, Y a topological space and $f\colon X \to Y$ a continuous map. Then $f(X)$ is connected.*

Proof: If $f(X) = A$ were not connected, there would exist non-empty open sets U_1, U_2 of $f(X)$ such that $f(X) = U_1 \cup U_2,\ U_1 \cap U_2 = \emptyset$. Since clearly $f: X \to f(X)$ is continuous, $f^{-1}(U_i),\ i = 1, 2$ are open (non-empty) sets of X with $X = f^{-1}(U_1) \cup f^{-1}(U_2)$ and $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$. This contradicts the connectedness of X.

3.1.4 Some applications to real-valued functions

Let X be a set. A map from X to the real numbers is called a real-valued function on X.

Proposition 3.49 *Let X be a compact topological space and f a continuous real-valued function on X. Then we have the following two properties.*

1. *The function f is bounded on X i.e. there exists a positive real number a such that $|f(x)| \leq a$ for every $x \in X$.*
2. *The function f attains its least upper bound (denoted by sup) and its greatest lower bound (denoted by inf) on X, i.e. if $M = \sup_{x \in X} f(x)$ and $m = \inf_{x \in X} f(x)$, then there exist elements x_1, x_2 in X such that $f(x_1) = M,\ f(x_2) = m$.*

Proof: The result follows from the fact that $f(X)$ is compact and that a compact subset of $\mathbf{R}$ is bounded.

Since $f(X)$ is compact it is a closed and bounded set. Hence the greatest lower bound and the least upper bound of $f(X)$ belong to $f(X)$.

Proposition 3.50 *Let X be a connected topological space and f a continuous real-valued function on X. Then f attains every value between any two of its values, i.e. if $f(x_1) = a_1$ and $f(x_2) = a_2,\ a_2 \leq a_1$ and a any real number such that $a_2 \leq a \leq a_1$, then there exists an element $y \in X$ such that $f(y) = a$.*

Proof: Since f is continuous, $f(X)$ is connected and hence is an interval. Since a_1 and a_2 belong to $f(X),\ a \in f(X)$.

Remark 3.51 If X is an interval in $\mathbf{R}$, the above property reduces to a well-known property of continuous real-valued functions of a real variable.

3.1.5 Product spaces

Let X and Y be two topological spaces. Consider the cartesian product $X \times Y$. We define a topology on $X \times Y$, called the product topology, as follows: an open set in $X \times Y$ will be, by definition, a union of sets of the form $U \times V$ where U is an open set in X and V an open set in Y. (Verify that these sets satisfy the axioms for open sets).

If $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the *projections*, defined by $p_1(x, y) = x$ and $p_2(x, y) = y$ respectively, then p_1 and p_2 are continuous maps, by the definition of topology on $X \times Y$.

In a similar way, the product of a finite number of topological spaces can be defined.

Remark 3.52 The metric topology on $\mathbf{R}^n$ is the product topology on the n-fold product of the topological space $\mathbf{R}$.

Problem 3.1 The product of two topological spaces is compact if each component is compact.

3.1.6 Quotient spaces

Let X be a topological space and R an equivalence relation in X. Let $Y = X/R$ be the quotient set and $\eta : X \to Y$ the natural map. We put on Y the following topology: a set in Y is open if and only if its inverse image by η is open in X. This topology on Y is called the *quotient topology* on X/R. X/R endowed with this topology will be referred to as the *quotient space*. The map $\eta : X \to X/R$ is continuous, be definition.

Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. Let R_f denote the equivalence relation defined by F. Then it is easy to check that the map $q_f : X/R \to Y$ is continuous and f admits the following decomposition into continuous maps:

$$X \xrightarrow{\eta} X/R_f \xrightarrow{q_f} f(X) \xrightarrow{j} Y.$$

Remark 3.53 q_f is one-one, onto and continuous, but not necessarily a homeomorphism onto $f(X)$.

Example 3.54 Let $X = \mathbf{R}$ be the group of real numbers and $\mathbf{Z}$ the subgroup of integers. Let R be the equivalence relation in $\mathbf{R}$ defined by the subgroup $\mathbf{Z}$. The quotient space is homeomorphic to the circle $\{(x, y) \mid (x, y) \in \mathbf{R}^2,\ x^2 + y^2 = 1\}$, with the topology induced from that of $\mathbf{R}^2$.

3.1.7 Homotopy of maps

Definition 3.55 A path in a topological space X is a continuous map $f: [0,1] \to X$. $f(0)$ and $f(1)$ are called, respectively, the initial and end points of the path. We say that "f joins $f(0)$ and $f(1)$."

Definition 3.56 A topological space is said to be arcwise connected if given $x_0, x_1 \in X$, there exists a path in X with x_0 and x_1 as its initial and end points respectively.

Problem 3.2 Every arcwise connected space is connected. (The converse is not true. Example ?).

Problem 3.3 If every point of a topological sauce can be joined by a path to a fixed point $x_0 \in X$, then X is arcwise connected.

Definition 3.57 Let X, Y be topological spaces and θ, $\theta': X \to Y$ two continuous maps. We say that θ is homotopic to θ' if there exists a continuous map $F: X \times I \to Y$ with $F(x,0) = \theta(x)$ and $F(x,1) = \theta'(x)$ for every $x \in X (I$ denotes the closed interval $[0,1]$ and $X \times I$ is provided with the product topology). We say F is a homotopy of θ on θ'.

Geometrically, this means that there exists a continuous map F from the "cylinder" on X of "length 1" into Y which coincides with θ on the "bottom edge" and with θ' on the "top edge".

Remark 3.58 A space X is arcwise connected if and only if any two maps from a space consisting of a single point into X are homotopic.

Remark 3.59 The relation "θ is homotopic to θ'" is an equivalence relation.

Definition 3.60 Let X and Y be two topological spaces. A continuous map $f: X \to Y$ is said to be a homotopy equivalence if there exists a continuous map $g: Y \to X$ such that $g \circ f$ is homotopic to I_X and $f \circ g$ is homotopic to I_Y. Two topological spaces X and Y are said to be of the same homotopy type if there exists a continuous map $f: X \to Y$ which is a homotopy equivalence.

Definition 3.61 A topological space X is said to be contractible if the identity map I_X is homotopic to a constant map of X into X.

Remark 3.62 A contractible space is arcwise connected. In fact, let $F: X \times I \to X$ be a homotopy of I_X into a constant map, i.e. $F(x, 0) = x$ and $F(x, 1) = x_0$ for every $x \in X$. Clearly, for every $x \in X$, the path $\gamma: I \to X$ defined by $\gamma(t) = F(x, t)$ joins x and x_0. Hence X is arcwise connected. Problem 3.3

Remark 3.63 It is clear that a space X is contractible if and only if it is of the same homotopy type as a space consisting of a single point.

Example 3.64 $\mathbf{R}^n$ is contractible. In fact the map $F: \mathbf{R}^n \times I \to \mathbf{R}^n$ given by $F((x_1, x_2, \ldots, x_n), t) = ((1-t)x_1, (1-t)x_2, \ldots, (1-t)x_n)$, $x_i \in \mathbf{R}$, $t \in I$ is a homotopy between the identify map and the constant map 0. Similarly it can be proved that the closed (resp. open) unit ball in $\mathbf{R}^n$ is contractible.

Chapter 4

Singular Homology

4.1 Notation

Let $\mathbf{R}^\infty$ denote the set whose elements are sequences $x = (x_0, x_1, x_2, \ldots, x_j, \ldots)$ of real numbers with $x_j = 0$ except for a finite number of j. For $\lambda \in \mathbf{R}$, $x \in \mathbf{R}^\infty$, we write λx for the element $(\lambda x_0, \lambda x_1, \ldots) \in \mathbf{R}^\infty$. For $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \mathbf{R}^\infty$, we define $x + y = (x_1 + y_1,\ x_2 + y_2, \ldots) \in \mathbf{R}^\infty$. $\mathbf{R}^\infty$ is provided with the following metric: if $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$ are any two elements of $\mathbf{R}^\infty$, $d(x, y) = \{\sum_{j=0}^{\infty}(x_j - y_j)^2\}^{\frac{1}{2}}$, where by square root we mean the positive square root. $\sum_{j=o}^{\infty}(x_j - y_j)^2$, is a finite sum and the verification that $d(x, y)$ is a metric on $\mathbf{R}^\infty$ is left to the reader. Let n be any integer ≥ 0. Consider the map $\mathbf{R}^{n+1} \to \mathbf{R}^\infty$ given by $(x_0, \ldots, x_n) \to (x_0, \ldots, x_n,\ 0, 0, \ldots)$. This is one-one and we shall from now on identify $\mathbf{R}^{n+1}$ with its image in $\mathbf{R}^\infty$. For each $j \in \mathbf{Z}^+$, let e_j be the element $(x_0, x_1, \ldots)$ given by $x_j = 1$ and $x_i = 0$ for $i \neq j$.

Definition 4.1 By the standard Euclidean n-simplex we mean the subset of $\mathbf{R}^{n+1}$ consisting of points $(x_0, x_1, \ldots, x_n)$ with $x_j \geq 0$ and $\sum_{0 \leq jleqn} x_j = 1$ with the induced topology.

The standard Euclidean n-simplex is denoted by Δ_n.

Trivially, the elements e_j, $j = 0, 1, \ldots, n$ lie in Δ_n and any element x of Δ_n can uniquely written as $x = \sum_{0 \leq j \leq n} x_j e_j$, $x_j \geq 0$ and $\sum_{0 \leq j \leq n} x_j = 1$. The elements $\{e_j\}_{j=0,1,\ldots,n}$ are known as the vertices of Δ_n.

For instance, Δ_0 is a point, Δ_1 is a segment, Δ_2 is a triangle and Δ_3 a tetrahedron.

4.1.1 Singular homology groups

Let q be any integer ≥ 1. Consider the maps $\epsilon_q^i\colon \Delta_{q-1} \to \Delta_q,\ i = 0, 1, \dots, q$, defined as follows:

$$\epsilon_q^i(x_0, \dots, x_{q-1}) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{q-1})$$

for every $(x_0, \dots, x_{q-1}) \in \Delta_{q-1}$. Actually the maps ϵ_q^i can be described as follows. The vertices $e_j,\ j = 0, 1, 2, \dots, q-1$, of Δ_{q-1} are mapped into vertices of Δ_q by the following assignment:

$$\epsilon_q^i(e_j) = e_j \text{ for } j < i$$

$$\epsilon_q^i(e_j) = e_{j+1} \text{ for } j \geq i.$$

With this vertex assignment the maps are extended linearly, that is to say the point $x = \sum_{0 \leq j \leq q-1} x_j e_j$ of Δ_{q-1} is mapped into the point $\sum_{0 \leq j \leq i} x_j e_j + \sum_{i \leq j \leq q-1} x_j e_{j+1}$ of Δ_q by ϵ_q^i. These maps ϵ_q^i are trivially continuous.

Lemma 4.2 *For $q \geq 2, \epsilon_q^i \circ \epsilon_{q-1}^j = \epsilon_q^j \circ \epsilon_{q-1}^{i-1},\ 0 \leq j < i \leq q$. In fact, for $0 \leq j < i \leq q$, we have*

$$\begin{aligned} \epsilon_q^i \circ \epsilon_{q-1}^j(x_0, \dots, x_{q-2}) &= (x_0, \dots, x_{j-1},\ 0, \dots, x_{i-2},\ 0, \dots, x_{q-2}) \\ &= \epsilon_q^i \circ \epsilon_{q-1}^{i-1}(x_0, \dots, x_{q-2}). \end{aligned}$$

Definition 4.3 For any integer $q \geq 0$, a singular q-simplex in a topological space X is a continuous map $f\colon \Delta_q \to X$.

For any $q \geq 1$ and for any singular q-simplex $f\colon \Delta_q \to X$ the singular $(q-1)$-simplex $f \circ \epsilon_q^i\colon \Delta_{q-1} \to X$ is called the i-th face of f and is denoted by $f^{(i)}$, for $i = 0, 1, \dots, q$.

Lemma 4.4 $\{f^{(i)}\}^{(j)} = \{f^{(j)}\}^{(i-1)}$ *for any singular q-simplex f of X and $0 \leq j < i \leq q$, with $q \geq 2$.*

This is an immediate consequence of the above lemma.

Example 4.5 For any $a \in X$, the constant map $\tilde{a}_q\colon \Delta_q \to X$ given by $\tilde{a}_q(y) = a$ for every $y \in \Delta_q$ is a singular q-simplex. Clearly, $\tilde{a}_q^{(i)} = \tilde{a}_{q-1}$ for $0 \leq i \leq q$, if $q \geq 1$.

Let $f\colon \Delta_q \to X$ be a singular q-simplex. $f(\Delta_q)$ is called the *support* of f.

Definition 4.6 The free abelian group generated by the set of singular q-simplices of X is called the group of singular q-chains of X and is denoted by $C_q(X)$. Any element of this group is called a singular q-chain of X.

The singular q-simplices in X are defined only for $q \in \mathbf{Z}^+$. For any negative integer q, we set $C_q(X) = (0)$ so that $C_q(X)$ is defined for $q \in \mathbf{Z}$.

Let $q \geq 1$. For any singular q-simplex $f\colon \Delta_q \to X$,let

$$\partial_q f = \sum_{0 \leq i \leq q} (-1)^i f^{(i)}.$$

Since the singular q-simplices form a base for $C_q(X)$, we get a unique extension of ∂_q into a homomorphism $\partial_q\colon C_q(X) \to C_{q-1}(X)$. If $q \leq 0$ we set $\partial_q = 0$. ∂_q is called the *boundary homomorphism.*

Proposition 4.7 *For every $q \in \mathbf{Z}$,we have $\partial_{q-1} \circ \partial_q = 0$.*

PROOF: If $q \leq 1$, by definition $\partial_{q-1} = 0$ and hence $\partial_{q-1} \circ \partial_q = 0$. Let then $q \geq 2$. It is sufficient to verify that $\partial_{q-1} \circ \partial_q(f) = 0$ for every singular q-simplex, since the singular q-simplices generate $C_q(X)$.

$$\begin{aligned}
\partial_{q-1} \circ \partial_q(f) &= \partial_{q-1}\left(\sum_{0 \leq i \leq q} (-1)^i f^{(i)}\right) \\
&= \sum_{0 \leq i \leq q} (-1)^i \partial_{q-1}(f^{(i)}) \\
&= \sum_{0 \leq i \leq q} (-1)^i \sum_{0 \leq j \leq q-1} (-1)^j (f^{(i)})^{(j)} \\
&= \sum_{0 \leq j < i \leq q} (-1)^{i+j} \{f^{(i)}\}^{(j)} + \sum_{0 \leq i \leq j < q} (-1)^{i+j} \{f^{(i)}\}^{(j)}.
\end{aligned}$$

By the preceding lemma, the first expression on the right is

$$\sum_{0 \leq j < i \leq q} (-1)^{i+j} \{f^{(j)}\}^{(i-1)}$$

If we write j' for $(i-1)$ and i' for j we see that

$$\sum_{0\le j<i\le q}(-1)^{i+j}\{f^{(j)}\}^{(i-1)} = \sum_{0\le i'\le j'<q}(-1)^{i'+j'}\{f^{(i')}\}^{(j')}$$

Thus

$$\partial_{q-1}\circ\partial_q(f) = \sum_{0\le i'\le j'<q}(-1)^{i'+j'+1}\{f^{(i')}\}^{(j')} + \sum_{0\le i\le j<q}(-1)^{i+j}\{f^{(i)}\}^{(j)} = 0$$

Definition 4.8 The kernel of the homomorphism $\partial_q\colon C_q(X)\to C_{q-1}(X)$ is called the group of singular q-cycles and is denoted by $Z_q(X)$. Any element of $Z_q(X)$ is called a singular q-cycle of X. The image of the homomorphism $\partial_{q+1}\colon C_{q+1}(X)\to C_q(X)$ is denoted by $B_q(X)$ and is called the group of singular q-boundaries of X. Any element of $B_q(X)$ is known as a singular q-boundary.

Since $\partial_q\circ\partial_{q+1}=0$, it is clear that $B_q(X)$ is a subgroup of $Z_q(X)$.

Remark 4.9 For any topological space, since $\partial_0 = 0$, we have $C_0(X) = Z_0(X)$.

Definition 4.10 The quotient group $Z_q(X)/B_q(X)$ is called the singular q-th homology group of the topological space X and is denoted by $H_q(X)$.

Clearly, for any topological space X, we have $H_q(X) = 0$ for $q < 0$.

Example 4.11 *The singular homology groups of a space X consisting of a single point a.* It is obvious that for any integer $q\ge 0$, $\tilde{a}_q$ defined earlier is the only singular simplex of X. Hence for $q \ge 0, C_q(X)$ is the free abelian group with $\tilde{a}_q$ as its basis.

$$\text{For}\quad q\ge 1,\ \partial_q(\tilde{a}_q) = \sum_{0\le i\le q}(-1))^i\tilde{a}_{q-1} = \begin{cases} 0 \text{ if } q \text{ is odd} \\ a_{q-1} \text{ if } q \text{ is even.}\end{cases}$$

$$\text{If}\quad q\ge 1,\ \text{we have } Z_q(X) = \begin{cases} C_q(X) \text{ if } q \text{ is odd} \\ 0 \text{ if } q \text{ is even.}\end{cases}$$

Also since $\partial_1 = 0$, it follows that $B_0(X) = (0)$.

$$\text{For}\quad q\ge 1,\ B_q(X) = \begin{cases} C_q(X) \text{ if } q \text{ is odd} \\ (0) \text{ if } q \text{ is even.}\end{cases}$$

It now trivially follows that $H_q(X) = 0$ for $q \ne 0$. On the other hand, we have $Z_0(X) = C_0(X) \approx \mathbf{Z}$ and $B_0(X) = 0$. Thus $H_0(X) \approx \mathbf{Z}$.

Example 4.12 The 0-th homology group $H_0(X)$ of an arcwise connected space X. We first remark that if $I = \{t \mid t \in \mathbf{R},\ 0 \leq t \leq 1\}$ with the induced topology, we can define a homeomorphism $\phi: \Delta_1 \to I$ as follows:

$$\phi(x_0, x_1) = 1 - x_0 = x_1.$$

Any zero simplex $f: \Delta_0 \to X$ is uniquely determined by its support (which has to be a point, since Δ_0 consists of one point). Hence $C_0(X)$ is the free abelian group generated by the set X. Given any two points $a, b \in X$ since X is arcwise connected, there exists a continuous map $\theta: I \to X$ with $\theta(0) = a,\ \theta(1) = b$. Then $f = \theta \circ \phi: \Delta_1 \to X$ is a singular 1-simplex with $\partial f = f^{(0)} - f^{(1)}$. But $f^{(0)}(1) = f(0,1) = \theta(1) = b$, and $f^{(1)}(1) = f(1,0) = \theta(0) = a$. Thus when we consider the group $C_0(X)$ as the free abelian group generated by $X, \partial f$ is the element $b - a$. This shows that for any $a, b \in X$, the element $b - a$ is in $B_0(X)$. Let a_0 be a fixed element of X. Consider the mapping $\alpha: C_0(X) \to \mathbf{Z}$ given by $\sum_i m_i b_i \to (\sum_i m_i)$. Clearly α is onto and we claim that the kernel of this homomorphism is $B_0(X)$. For, if $\sum m_i b_i$ is in the kernel, we have $\sum_i m_i = 0$ and hence

$$\sum m_i b_i = \sum m_i b_i - \left(\sum m_i\right) a_0 = \sum_i m_i(b_i - a_0) \in B_0(X),$$

since $b_i - a_0 \in B_0(X)$. Conversely, since $B_0(X)$ is generated by elements of the form $b - a$ and since $\alpha(b - a) = 0$, we have $\alpha(B_0(X)) = (0)$. Hence $H_0(X) = Z_0(X)/B_0(X) = C_0(X)/B_0(X) \approx \mathbf{Z}$, using the Fundamental theorem of homomorphisms.

4.1.2 Effect of a continuous map on the homology groups

We will show that any continuous map $\theta: X \to Y$ induces natural homomorphisms $H_n(\theta): H_n(X) \to H_n(Y)$ for every integer n.

Let $f: \Delta_n \to X$ be any singular n-simplex in X. Then $\theta \circ f: \Delta_n \to Y$ is a singular n-simplex in Y, which we denote by $C_n(\theta)(f)$. The association $f \to C_n(\theta)(f)$ can be extended to a unique homomorphism $C_n(\theta): C_n(X) \to C_n(Y)$. Trivially, for any $f: \Delta_n \to X$ and $n \geq 1$ we have $\{C_n(\theta)(f)\}^{(i)} = C_{n-1}(\theta)(f^{(i)})$ for $0 \leq i \leq n$. From this it follows that $\partial_n \circ C_n(\theta) = C_{n-1}(\theta) \circ \partial_n$. If now $\mu \in Z_n(X)$, we have

$$\partial_n \circ C_n(\theta)(\mu) = C_{n-1}(\theta)(\partial_n(\mu)) = 0.$$

Hence $C_n(\theta)(\mu) \in Z_n(Y)$. Similarly we can show that if $\nu \in B_n(X)$, we have $C_n(\theta)(\nu) \in B_n(Y)$. Hence $C_n(\theta)$ induces a homomorphism of $Z_n(X)/B_n(X) = H_n(X)$ into $Z_n(Y)/B_n(Y) = H_n(Y)$, which we denote by, $H_n(\theta)$.

Remark 4.13 If $I: X \to X$ is the identity map, $H_n(I): H_n(X) \to H_n(X)$ is the identity homomorphism.

Remark 4.14 If $\theta: X \to X'$ and $\theta': X' \to X''$ are continuous maps, the homomorphism $H_n(\theta' \circ \theta): H_n(X) \to H_n(X'')$ induced by $\theta' \circ \theta: X \to X''$ is the same as $H_n(\theta') \circ H_n(\theta)$.

Remark 4.15 Let $\theta: X \to X'$ be a homeomorphism From (i) and (ii) it can be easily seen that $H_n(\theta): H_n(X) \to H_n(X')$ is an isomorphism of groups.

Remark 4.16 Let $\theta: X \to Y$ be a constant map. Since $H_n(\theta(X)) = (0)$ for $n \neq 0$, $H_n(\theta) = 0$ for $n \neq 0$.

4.1.3 Homomorphisms induced by homotopic maps

Our main aim in this section is to show that if θ and θ' are homotopic, the induced homomorphisms $H_n(\theta)$ and $H_n(\theta')$ are the same for every integer n. To this end, we prove first the following

Lemma 4.17 *If θ, θ' are two continuous maps of X into Y, which are homotopic, then there exists a sequence $\phi_q: C_q(X) \to C_{q+1}(Y)$ of homomorphisms such that*

$$\partial_{q+1} \circ \phi_q + \phi_{q-1} \circ \partial_q = C_q(\theta') - C_q(\theta).$$

PROOF: Let $\Delta_q \times I$ denote the cylinder over Δ_q. As usual, the vertices of Δ_q will be denoted by $e_0, \ldots, e_q$. Let the vertices of $\Delta_q \times (0)$ be denoted by $le_0, \ldots, le_q$ and the vertices of $\Delta_q \times (1)$ by $ue_0, \ldots, ue_q$. For each $i = 0, 1, 2, \ldots, q$, we denote the linear map of Δ_{q+1} in $\Delta_q \times I$ which takes e_j into le_j for $j \leq i$ and e_j into ue_{j-1} for $j > i$ by $(le_0, le_1, \ldots, le_i, ue_i, ue_{i+1}, \ldots, ue_q)$. Clearly it is a singular $(q+1)$-simplex of $\Delta_q \times I$.

Consider the singular $(q+1)$-chain γ_q of $\Delta_q \times I$ given by

$$\gamma_q = \sum_{0 \leq i \leq q} (-1)^i (le_0, le_1, \ldots, le_i, ue_i, ue_{i+1}, \ldots, ue_q).$$

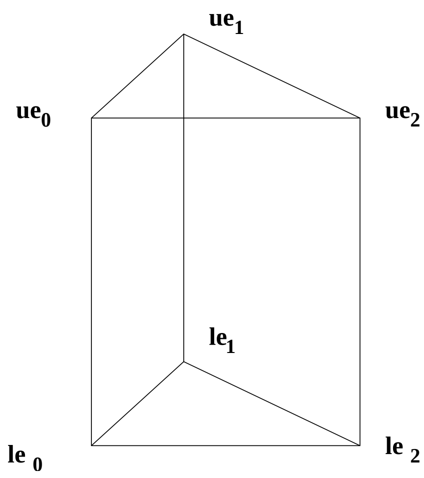

Figure 4.1:

The map $\epsilon_q^i : \Delta_{q-1} \to \Delta_q$(defined in §2) gives a map $\bar{\epsilon_q^i} : \Delta_{q-1} \times I \to \Delta_q \times I$ defined by $\bar{\epsilon_q^i}(\lambda, t) = (\epsilon_q^i \lambda, t), \ \lambda \in \Delta_{q-1}, \ t \in I$. This in turn induces a homomorphism $C(\bar{\epsilon_q^i}) : C(\Delta_{q-1} \times I) \to C(\Delta_q \times I)$. We denote by $\gamma_q^{(i)}$ the image of the q-chain γ_{q-1} by $C_q(\bar{\epsilon_q^i})$. We can verify that

$$(*) \qquad \partial_{q+1}\gamma_q = (ue_0, \ldots, ue_q) - (le_0, \ldots, le_q) - \sum_{0 \le i \le q} (-1)^i \gamma_q^{(i)},$$

where $(le_0, \ldots, le_q)$[resp.$(ue_0, \ldots, ue_q)$] denotes the singular q-simplex of $\Delta_q \times I$ which sends e_i to le_i(resp. e_i to ue_i) for $0 \le i \le q$ and which is linear. {The above formula makes precise the following geometric fact which is intuitively clear : the boundary of the cylinder over the simplex Δ_q consists of the cylinder over the boundary of Δ_q and the upper and lower faces of the cylinder, with proper signs.}

Let now $f : \Delta_q \to X$ be any singular q-simplex. We have a map $\Phi(f) : \Delta_q \times I \to Y$ defined by $\Phi(f) \ (\lambda, t) = F(f(\lambda), \ t)), \ \lambda \in \Delta_q, \ t \in I$, where F is a homotopy between θ and θ'. This induces a homomorphism $C(\Phi(f)) : C(\Delta_q \times I) \to C(Y)$. We set $\phi_q(f) = C_{q+1}(\Phi(f)) \ (\gamma_q)$. (Here $\phi_q(f)$ is the 'cylinder' constructed over the singular simplex f by means of the homotopy F.) Using (*) one easily verifies that

$$\partial_{q+1} \circ \phi_q(f) + \phi_{q+1} \circ \partial_q(f) = (C_q(\theta') - C_q(\theta)) \ (f);$$

If ϕ_q is extended as a homomorphism $C_q(X) \to C_{q+1}(Y)$, it is clear that (ϕ_q) has the required property.

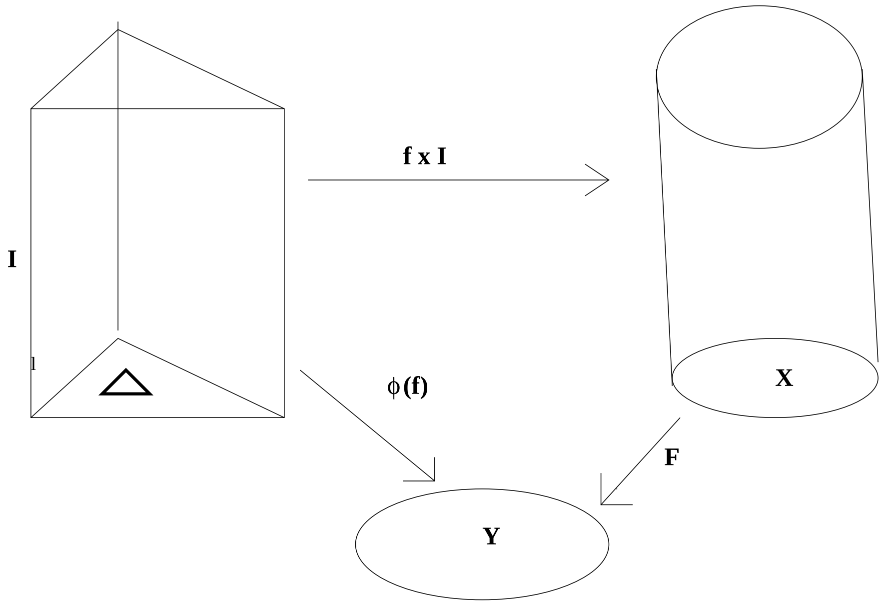

Figure 4.2:

Theorem 4.18 *Let X and Y be two topological spaces. If $\theta: X \to Y$ and $\theta': X \to Y$ are homotopic maps, then the induced homomorphisms $H_n(\theta): H_n(X) \to H_n(Y)$ and $H_n(\theta'): H_n(X) \to H_n(Y)$ are the same.*

PROOF: Let $\phi_q: C_q(X) \to C_{q+1}(Y)$ be a sequence of homomorphisms such that $\partial_{q+1} \circ \phi_q + \phi_{q-1} \circ \partial_q = C_q(\theta') - C_q(\theta)$. Let now $\mu \in Z_q(X)$ be an element representing $\bar{\mu} \in H_q(X)$. Then $\partial_q \mu = 0$. Thus $\partial_{q+1} \circ \phi_q \mu = C_q(\theta')\mu - C_q(\theta)\mu$ or, what is the same, $H_q(\theta')(\bar{\mu}) - H_q(\theta)(\bar{\mu}) = 0$. This proves the theorem.

Theorem 4.19 *If X and Y are topological spaces of the same homotopy type, they have isomorphic singular homology groups.*

PROOF: Let $f: X \to Y$, $g: Y \to X$ be two continuous maps such that $g \circ f$ is homotopic to I_X and $f \circ g$ is homotopic to I_Y. Then we have $H_n(g)) \circ H_n(f) = H_n(g \circ f) = H_n(I_X) = I_{H_n(X)}$ and similarly $H_n(f) \circ H_n(g) = I_{H_n(Y)}$.

Corollary 4.20 *If X is a contractible topological space, then $H_n(X) = 0$ for $n \neq 0$.*

PROOF: In fact, I_X and any constant map of X into itself are homotopic, and the constant map induces 0 on the homology groups $H_n(X)$ for $n \neq 0$ (Remark 4.16).

In particular, the corollary applies to closed and open balls of $\mathbf{R}^n$.

Chapter 5

Simplicial Complexes

5.1 Simplicial decomposition

Definition 5.1 Let $v_0, \ldots, v_q \in \mathbf{R}^n$. The set Δ of all points $\sum_{0\le i\le q} \lambda_i v_i$, $\lambda_i \in \mathbf{R}^+$, $\sum_{0\le i\le q} \lambda_i = 1$ is called the convex hull of the set $\{v_0, \ldots, v_q\}$ If, moreover, every point of Δ can be uniquely written in the form $\sum_{0\le i\le q} \lambda_i v_i$, $\lambda_i \ge 0$, $\sum_{0\le i\le q} \lambda_i = 1$ then we call Δ a simplex and denote it by $[v_0, \ldots, v_q]$.

Proposition 5.2 *If $\Delta = [v_0, \ldots, v_q] = [w_0, \ldots, w_p]$ is a simplex, then we have $p = q$ and $\{v_0, \ldots, v_q\} = \{w_0, \ldots, w_p\}$.*

Proof: By assumption, for $i = 0, \ldots, p$, $w_i = \sum_{0\le j\le q} \lambda_{ij} v_j$ with $\sum_{0\le j\le q} \lambda_{ij} = 1, \lambda_{ij} \ge 0$ and similarly for $j = 0, \ldots, q$, $v_j = \sum_{0\le k\le p} \mu_{jk} w_k$ with $\sum_{0\le k\le p} \mu_{jk} = 1$, $\mu_{jk} \ge 0$. We then have on substitution, $v_i = \sum_{k,j} \mu_{ik} \lambda_{kj} v_j$. By uniqueness, $\sum_{0\le k\le p} \mu_{ik} \lambda_{kj} = \delta_{i,j}$ where $\delta_{i,j} = 0$ if $i \ne j$ and $\delta_{i,j} = 1$ if $i = j$. Let $\lambda_j = \max(\lambda_{ij})$. Then $\sum_{0\le k\le p} \mu_{ik} \lambda_{kj} \le \lambda_j \sum_{0\le k\le p} \mu_{ik} = \lambda_j$. Hence $\lambda_j \ge \delta_{i,j}$ for every i and in particular, $\lambda_j \ge 1$. On the other hand, since each $\lambda_{ij} \le 1$, λ_j is also ≤ 1. Hence $\lambda_j = 1$. i.e. for some $i = i_0$, $\lambda_{i_0 j} = 1$. Then for every $k \ne j$, $\lambda_{i_0 k} = 0$ since $\sum_{0\le k\le q} \lambda_{i_0 k} = 1$. i.e. $w_{i_0} = v_j$. In other words, for every j, there exists some i such that $w_i = v_j$. i.e. $\{w_0, \ldots, w_p\} \supset \{v_0, \ldots, v_q\}$. Similarly $\{v_0, \ldots, v_q\} \supset \{w_0, \ldots, w_p\}$ which proves our proposition.

Thus q depends only on the simplex Δ and is called the dimension of Δ. Any simplex $[w_0, \ldots, w_p]$ with $\{w_0, \ldots, w_p\} \subset \{v_0, \ldots, v_q\}$ is called a face of Δ. In particular the points $v_0, \ldots, v_q$ are faces of Δ and Δ is a face of itself. The 0-dimensional faces $v_0, \ldots, v_q$ of Δ are called vertices of Δ.

Definition 5.3 A simplicial decomposition of a set $X \subset \mathbf{R}^n$ is a finite collection T of subsets $\{s_\alpha\}$ of X such that

(i) $\bigcup_\alpha s_\alpha = X$;

(ii) each s_α is a simplex;

(iii) if s_α is in the collection, so is every face of s_α;

(iv) if s_1 and s_2 in the collection have a non-empty intersection, then $s_1 \cap s_2$ is a common face of both s_1 and s_2.

The largest integer n such that there exists an n-dimensional simplex occurring in T is called the *dimension* of the decomposition. Note that there can be more than one decomposition for the same set.

Remark 5.4 The space X (with the topology induced from $\mathbf{R}^n$) is compact, since it is a finite union of compact sets.

Remark 5.5 Let $v_0, \ldots, v_N$ be the 0-simplices of T. For every simplex $s_\alpha = [v_{i_0}, \ldots, v_{i_n}]$ in T, consider the face $\sigma_\alpha = [e_{i_0}, \ldots, e_{i_r}]$ of the standard Euclidean simplex Δ_N. Let $|T|$ be the union of all faces σ_α, for $s_\alpha \in T$, provided with the topology induced from that of Δ_N. We define a continuous map $(x_0, \ldots, x_N) \to \sum_{0 \le i \le N} x_i v_i$ of Δ_N into $\mathbf{R}^n$. It is easily seen that the restriction φ of this map to $|T|$ is a continuous map of $|T|$ onto X. In view of condition (iv) in the definition of a simplicial decomposition, φ is one-one and since $|T|$ is compact, φ is a homeomorphism.

5.1.1 Homology of a simplicial decomposition

Let $X \subset \mathbf{R}^n$ and T a simplicial decomposition of X. Let r be the dimension of T. We consider the set of all vertices of the simplices of the decomposition. Order the set of all vertices in some way. If $\{v_i\}_{i=1,\ldots,N}$ is the set of all vertices, every q-simplex in the decomposition is simply $[v_{i_0}, \ldots, v_{i_q}.]$ for some $i_0 < \cdots <, i_q$. We shall call $[v_{i_0}, \ldots, \hat{v}_{i_r}, \ldots, v_{i_q}]$, where $\hat{}$ implies that the corresponding vertex is omitted, the rth face of $[v_{i_0}, \ldots, v_{i_q}]$.. For $q \ge 0$, let $C_q(T)$ be the free abelian group generated by set of all q-simplices. For every q-simplex $\sigma (q > 0)$ we define $\partial_q \sigma = \sum_{0 \le i \le q} (-1)^i F_i \sigma$, where $F_i \sigma$ is the ith face of σ. We extend ∂_q as a homomorphism $\partial_q \colon C_q(T) \to C_{q-1}(T)$. We set $\partial_q = 0$ for $q \le 0$. It is easy to verify that $\partial_{q-1} \circ \partial_q = 0$.

We thus have a sequence of abelian groups $C_q(T)$ and homomorphisms $\partial_q: C_q(T) \to C_{q-1}(T)$ such that $\partial_{q-1} \circ \partial_q = 0$. Let $Z_q(T)$ be the kernel of $\partial_q: C_q(T) \to C_{q-1}(T)$. Let $B_q(T)$ be the image $\partial_{q+1}: C_{q+1}(T) \to C_q(T)$. Then since $\partial_q \circ \partial_{q+1} = 0$, we have $B_q(T) \subset Z_q(T)$ and we define $H_q(T) = Z_q(T)/B_q(T)$. $H_q(T)$ is called the qth *simplicial homology group* of (X, T).

Remark 5.6 It is not obvious that $H_q(T)$ is a topological invariant of X(i.e. invariant under homeomorphisms). The definition depends on the particular simplicial decomposition we take, as also on the order in which the vertices are written. However, it can be proved that $H_q(T)$ is isomorphic in a natural way to the qth singular homology group of X. (See [6], Ch. VII, §10). Since singular homology groups are topological invariants, so are the simplicial homology groups. But while the singular homology groups are defined for any topological space, simplicial homology groups are defined only for spaces which admit of a simplicial decomposition. This enables one to compute the homology groups of spaces which are homeomorphic to those admitting of a simplicial decomposition. Such spaces are said to be triangulable.

Remark 5.7 It is clear that $H_q(T) = 0$ for $q > \dim T$.

Since $C_q(T)$ is finitely generated for every q, so is $Z_q(T)$ and hence $H_q(T)$.

Definition 5.8 The rank of $H_q(T)$ is called the q-th Betti number of the simplicial decomposition.

In view of the topological invariance of the homology groups, it follows that the Betti numbers are topological invariants.

5.1.2 The Euler-Poincaré Characteristic

Let X be a triangulable space. The Euler-Poincaré Characteristic of X is, by definition, the integer

$$\chi(X) = \Sigma(-1)^i \text{ rank } H_i(X).$$

Proposition 5.9 *Let T be a simplicial decomposition of X and let n_i be the number of i-simplices in the decomposition, then $\chi(T) = \sum(-1)^i n_i$.*

PROOF: $\partial\colon C_q(T) \to C_{q-1}(T)$ is a homomorphism. The kernel is Z_q and the image is B_{q-1}. By the Fundamental theorem of homomorphisms, we have $C_q/Z_q \approx B_{q-1}$. By the proposition on ranks of abelian groups, we have

$$n_q = (\text{rank } Z_q) + (\text{rank } B_{q-1}).$$

Moreover, since $H_q(X) \approx Z_q/B_q$, we have rank $H_q(X) =$ rank $Z_q -$ rank B_q. Thus $n_q -$ rank $H_q(X) =$ rank $B_q +$ rank B_{q-1}. Hence

$$\sum(-1)^q(n_q - \text{ rank } H_q(X)) = 0 \text{ or } \chi(X) = \sum(-1)^q n_q.$$

Remark 5.10 The above formula is a generalization of the well-known Euler formula connecting the number of vertices, edges and faces of a convex polyhedron ($V + F = E + 2$).

5.1.3 Homology of the sphere

Let us compute the homology of the sphere $S^2 = \{x \mid x \in \mathbf{R}^3,\ \|x\| = 1\}$. The sphere is homeomorphic to the surface of a tetrahedron. The surface of the tetrahedron Δ_3 consists of four vertices e_0, e_1, e_2, e_3. The 1-simplices are

$$\begin{aligned}
\sigma_1 &= (e_0, e_1),\ \sigma_2 = (e_0, e_2),\ \sigma_3 = (e_0, e_3),\ \sigma_4 = (e_1, e_2),\ \sigma_5 = (e_1, e_3),\\
\sigma_6 &= (e_2, e_3),\ \text{The 2-simplices are } \tau_1 = (e_1, e_2, e_3),\ \tau_2 = (e_0, e_2, e_3),\\
\tau_3 &= (e_0, e_1, e_3),\ \tau_4 = (e_0, e_1, e_2).
\end{aligned}$$

We have

$$\begin{aligned}
\partial_2\tau_1 &= \sigma_6 - \sigma_5 + \sigma_4\\
\partial_2\tau_2 &= \sigma_6 - \sigma_3 + \sigma_2\\
\partial_2\tau_3 &= \sigma_5 - \sigma_3 + \sigma_1\\
\partial_2\tau_4 &= \sigma_4 - \sigma_2 + \sigma_1
\end{aligned}$$

It is easy to see that $\partial_2\left(\sum_{i=1}^4 n_i\tau_i\right) = 0$ if and only if $n_1 = -n_2 = n_3 = -n_4$; i.e. Z_2 consists of element of the form $m(\tau_1 - \tau_2 + \tau_3 - \tau_4)$. Thus Z_2 is isomorphic to $\mathbf{Z}$. Since there are no 3-simplices, $B_2 = (0)$. Hence $H_2 \approx \mathbf{Z}$. It is also easy to see that $H_1 = (0)$ and $H_0 \approx \mathbf{Z}$. Similarly for the n-sphere S^n, we have $H_i(S^n) = (0)$ if $i \neq 0,\ n;\ H_0(S^n) \approx H_n(S^n) \approx \mathbf{Z}$. ($S^n = \{x \mid x \in \mathbf{R}^{n+1},\ \|x\| = 1\},\ n \geq 1$)

5.1.4 Brouwer's fixed point theorem

Theorem 5.11 *Let f be any continuous map of a simplex into itself. Then there exists a point x in the simplex such that $f(x) = x$.*

PROOF: Since any n-simplex is homeomorphic to the closed unit $(n+1)$-ball E^{n+1}, it is enough to prove the theorem for the ball. If possible, let $f\colon E^{n+1} \to E^{n+1}$ be a continuous map without any fixed point. Consider the map $g\colon E^{n+1} \to S^n$ defined by sending each $x \in E^{n+1}$ to the intersection of S^n and the ray $(f(x),\ x)$. It is easily checked that g is continuous and that $g(x) = x$ for every $x \in S^n$. We then have $g \circ j = I_{S^n}$ where $j\colon S^n \to E^{n+1}$ is the inclusion map. If $H_n(j),\ H_n(g)$ are the induced mappings $H_n(S^n) \to H_n(E^{n+1}),\ H_n(E^{n+1}) \to H_n(S^n)$ we have $H_n(g) \circ H_n(j) = H_n(g \circ j) =$Identity. This is impossible, since

$$H_n(E^{n+1}) = (0) \text{ and } H_n(S^n) \approx \mathbf{Z}.$$

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Bibliography

[1] Alexandroff, P. and Hopf, H., *Topologie*, Springer, Berlin (1935).

[2] Bourbaki, N., *Théorie des Ensembles,* Hermann, Paris, (1960).

[3] Bourbaki, N., *Toplogie Générale* Chap.I, Hermann, Paris, (1960).

[4] Cartan, H., *Séminaire, Topologie Algébrique*, Paris (1948-49).

[5] Cartan, H. and others, *Structures Algébriques et Structures Topologiques*, Paris(1958).

[6] Eilenberg, S and Steenrod, N., *Foundations of Algebraic Topology*, Chap. VII, Princeton, (1952).

[7] Eilenberg, S., Singular Homology Theory, *Annals of Mathematics*, Vol. 45 (1944) 407–447.

[8] Halmos, P.R., *Naive Set Thoery,* Van Nostrand Co., Princeton, (1960).

[9] Kelley, J.L., *General Topology*, Van Nostrand Co., Princeton, (1955).

[10] Lefschetz, S., *Algebraic Topology*, Chap.I, Amer. Math. Soc. Colloq. Publ. (1942).

[11] Pontrjagin, L.S., *Combinatorial Topology*, Graylock (1952).

[12] van der Waerden, B.L., *Modern Algebra*, Ungar (1949).

[13] Eilenberg, S. and Steenrod, N., *Foundations of Algebraic Topology*, Chap. II, VII and XI, Princeton (1952).

[14] LEFSCHETZ, S., *Introduction to Topology*, Chap. III and IV, Princeton (1949).