HOMOLOGICAL METHODS IN COMMUTATIVE ALGEBRA

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PREFACE

This pamphlet contains a revised version of the lectures given at a Summer School on Homological Methods in Commutative Algebra organised by the Tata Institute of Fundamental Research in 1971. The audience consisted of teachers and research students from Indian universities who desired to have a general introduction to the subject. The lectures were given by S.Raghavan, Balwant Singh and R.Sridharan.

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1. RIEMANN SURFACES.
2. ALGEBRAIC TOPOLOGY.
3. GALOIS THEORY.
4. ALGEBRAIC NUMBER THEORY.
5. HOMOLOGICAL METHODS IN COMMUTATIVE ALGEBRA.

HOMOLOGICAL METHODS IN COMMUTATIVE ALGEBRA

Contents

Prerequisites and Notation

We assume that the reader is familiar with elementary algebra, in particular, with the concepts of groups, rings and modules.

The following notation will be used in the sequel. We denote the set of natural numbers (resp. non-negative integers, integers, rational numbers) by $\mathbf{N}$ (resp. $\mathbf{Z}^+, \mathbf{Z}, \mathbf{Q}$). For $n \in \mathbf{N}$, we write "$n \gg 1$" for "all sufficiently large integers n". If X is a set, we denote by 1_X the identity map of X. Let $\mathcal{F}$ be a set of subsets of a set X. By a minimal (resp. maximal) element of $\mathcal{F}$, we mean a minimal (resp. maximal) element under set-theoretic inclusion. By a ring, we shall always mean a commutative ring with identity. All ring homomorphisms considered are supposed to be unitary and, in particular, all modules considered are unitary. If A is a ring, $\operatorname{Spec}(A)$ denotes the set of all prime ideals of A. If $\phi\colon A \to B$ is a ring homomorphism, then we regard any B-module M as an A-module by setting $ax = \phi(a)x$ for $x \in M$ and $a \in A$; in particular, B can be regarded as an A-module and B becomes an A-algebra.

Let A be a ring and let M, N be A-modules. Then $\operatorname{Hom}_A(M, N)$ denotes the A-module of all A-homomorphisms from M to N. If $f \in \operatorname{Hom}_A(M, N)$, we denote by $\ker f$(resp. $\operatorname{Im} f$, $\operatorname{coker} f$) the kernel of f (resp. $f(M)$, $N/f(M)$).

Chapter 0

Preliminaries

In this chapter, we recall some concepts and state (mostly without proofs) some results from algebra which will be used without explicit reference in the subsequent chapters.

In this, as well as in the subsequent chapters, by a ring we always mean a commutative ring with 1, by a homomorphism of rings, a unitary homomorphism and by a module, a unitary module.

0.1 Functors

Let A, B be rings. A *covariant functor F from A-modules to B-modules* is an assignment of

(i) a B-module $F(M)$ to each A-module M, and

(ii) a map $F = F_{M,N}\colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(F(M), F(N))$ to each pair M, N of A-modules.

such that

(a) $F(1_M) = 1_{F(M)}$,

(b) $F(gf) = F(g)F(f)$, for $f \in \operatorname{Hom}_A(M, N)$ and $g \in \operatorname{Hom}_A(N, P)$ where M, N, P are A-modules.

If (ii) and (b) above are replaced respectively by

(ii)′ a map $F = F_{M,N}\colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(F(N), F(M))$ to each pair M, N of A-modules, and (b)′ $F(gf) = F(f)F(g)$, for $f \in \operatorname{Hom}_A(M,N)$ and $g \in \operatorname{Hom}_A(N,P)$, where M, N, P are A-modules,

then we say that F is a *contravariant functor from A-modules to B-modules*.

A (covariant or contravariant) functor F from A-modules to B-modules is said to be *additive* if $F_{M,N}$ is a homomorphism of groups for every pair M, N of A-modules. A functor F from A-modules is said to be *A-linear* if, for every pair M, N of A-modules, $F_{M,N}$ is an A-homomorphism.

Let F and G be covariant functors from A-modules to B-modules. A collection $\{\varphi_M : F(M) \to G(M)\}$ of B-homomorphisms, where M runs over all A-modules is said to be *functorial in M*, if, for every $f \in \operatorname{Hom}_A(M,N)$ the diagram

$$\begin{array}{rcl} F(M) & \xrightarrow{\varphi_M} & G(M) \\ F(f)\downarrow & & \downarrow \mathrm{G(f)} \\ F(N) & \xrightarrow{\varphi_N} & F(N) \end{array}$$

is commutative. A similar definition can be given for contravariant functors.

In the sequel, unless explicitly stated otherwise, by a functor we mean a covariant functor. Also, while describing a functor, we shall sometimes not explicitly mention the assignment (ii) of the definition.

0.2 Exact sequences

Let A be a ring and $M,\ M',\ M''$ be A-modules. A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of A-homomorphisms is said to be *exact* if $\ker g = \operatorname{im} f$. Note that this is equivalent to saying that "$gf = 0$ and $\ker g \subset \operatorname{im} f$". Let

$$M_0 \to M_1 \to M_2 \to \ldots \to M_n$$

be an exact sequence of A-homomorphisms. Let i be an integer with $1 \le i \le n-1$. We say that the sequence is *exact at M_i* if $M_{i-1} \to M_i \to M_{i+1}$ is exact. If the sequence is exact at M_i for every i, with $1 \le i \le n-1$, then we say it is *exact*.

Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules. (This precisely means that f is injective, g is surjective and $\operatorname{im} f = \ker g$). We refer to its as a *short exact sequence*. We say that a short exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *splits* or that it is a *split exact sequence* if there exists an A-homomorphism $t\colon M'' \to M$ such that $gt = 1_{M''}$. Then, clearly, t is injective and $M = f(M') \oplus t(M'')$.

Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a split exact sequence of A-modules and let F be an additive functor from A-modules to B-modules. Then the sequence $0 \to F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \to 0$ is again a split exact sequence.

Let F be an additive functor from A-modules to B-modules. Then F is said to be *right exact*, if for every sequence $0 \to M' \to M \to M'' \to 0$ the sequence $F(M') \to F(M) \to F(M'') \to 0$ is exact. We say F is *exact*, if for every exact sequence $0 \to M' \to M \to M'' \to 0$, the sequence $0 \to F(M') \to F(M) \to F(M'') \to 0$ is exact. If F is exact and if $M_0 \to M_1 \to \cdots \to M_n$ is an exact sequence, then it is easily verified that $F(M_0) \to F(M_1) \to \cdots \to F(M_n)$ is exact.

0.3 Tensor Products

let A be a ring and M, N be A-modules. A *tensor product* of M and N over A, is a pair (T, φ), where T is an A-module and $\varphi\colon M \times N \to T$ is an A-bilinear map such that, for any A-module P and any A-bilinear map $f\colon M \times N \to P$, there exists a *unique* A-homomorphism $\tilde{f}\colon T \to P$ which makes the diagram

$$\begin{tikzcd} M \times N \arrow[d, "\varphi"'] \arrow[dr, "f"] & \\ T \arrow[r, "\tilde{f}"] & P \end{tikzcd}$$

commutative. (Recall that a map $\psi\colon M \times N \to P$ is A-*bilinear* if $\psi(ax + by,\ z) = a\psi(x, z) + b\psi(y, z)$ and $\psi(x, az + bt) = a\psi(x, z) + b\psi(x, t)$ for $x, y \in M,\ z, t \in N$ and $a, b \in A$.)

If $(T, \varphi),\ (T', \varphi')$ are tensor products of M and N over A, there exist unique A-homomorphisms $\tilde{\varphi}\colon T' \to T,\ \tilde{\varphi}\colon T \to T'$ such that the diagram

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\varphi"'] \arrow[d, "\varphi'"] \arrow[dr, "\varphi"] & \\
T \arrow[r, "\tilde{\varphi}"] & T' \arrow[r, "\tilde{\varphi}"] & T
\end{tikzcd}$$

is commutative. Since the diagrams

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
T \arrow[rr, "\tilde{\varphi}\tilde{\varphi}'"] & & T
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M \times N \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
T \arrow[rr, "1_T"] & & T
\end{tikzcd}$$

are both commutative, it follows from the uniqueness condition that $\widetilde{\phi}'\tilde{\varphi} = 1_T$. Similarly, we have $\widetilde{\varphi'}\tilde{\varphi} = 1_{T'}$. This proves that, upto isomorphism, the tensor product of M and N over A is unique, if it exists. We now show that the tensor product of M and N over A exists. Let F be the free A-module with the set $M \times N$ as a basis and let H be the submodule of F generated by elements of the form $(ax + by,\ z) - a(x, z) - b(y, z),\ (x, az + bt) - a(x, z) - b(x, t)$, where $x, y \in M,\ z, t \in N$ and $a, b \in A$. Let $T = F/H$ and $\varphi\colon M \times N \to T$ be the composite $M \times N \hookrightarrow F \to T$, where $F \to T$ is the canonical epimorphism. Then, it is easily seen that (T, φ) is a tensor product of M and N over A.

We denote T by $M \otimes_A N$ and call $M \otimes_A N$ itself *the* tensor product of M and N over A. For $(x, y) \in M \times N$, we denote its image under φ in $M \otimes_A N$ by $x \otimes y$. Note that any element of $M \otimes_A N$ is of the form $\sum_i x_i \otimes y_i$, with $x_i \in M,\ y \in N$.

Let $f\colon M \to M',\ g\colon N \to N'$ be homomorphisms of A-modules. Then the map $M \times N \to M' \otimes_A N'$ given by $(x, y) \mapsto f(x) \otimes g(y)$ is clearly A-bilinear and hence induces an A-homomorphism.

$$f \otimes g\colon M \otimes_A N \to M' \otimes_A N'$$

such that $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$ for $x \in M,\ y \in N$. It is easily verified that for a fixed A-module N, the assignment $M \mapsto M \otimes_A N$ for every A-module M and $f \mapsto f \otimes 1_N$ for every $f \in \mathrm{Hom}_A(M, M')$ is an A-linear functor from A-modules to A-modules. Similarly, by fixing M, we get an A-linear functor $N \mapsto M \otimes_A N$.

0.4 Some properties of tensor products

(i) Let M be an A-module. Then the A-bilinear map $A \times M \to M$ given by $(a, x) \mapsto ax$ induces an A-isomorphism $A \otimes_A M \xrightarrow{\sim} M$, which is functorial in M.

(ii) *Commutativity of tensor products.* For A-modules M, N the A-bilinear map $M \times N \to N \otimes_A M$ given by $(x, y) \mapsto y \otimes x$ induces an A-isomorphism $M \otimes_A N \simeq N \otimes_A M$, which is functorial in both M and N.

(iii) Both the functors $M \mapsto M \otimes_A N$(for fixed N) and $N \mapsto M \otimes_A N$ (for fixed M) are right exact.

(iv) Let $\mathfrak{a}$ be an ideal of A and M be an A-module. Then the map $A/\mathfrak{a} \times M \mapsto M/\mathfrak{a}M$ given by $(\bar{a}, x) \mapsto \overline{ax}$ is a well-defined A-bilinear map and induces an isomorphism $A/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M$, which is functorial in M.

(v) Let $M = \bigoplus_i M_i,\ N = \bigoplus_j N_j$ be direct sums of A-modules. Then the A-bilinear map $M \times N \to \bigoplus_{i,j} M_i \otimes_A N_j$. given by $((x_i),\ (y_j)) \mapsto (x_i \otimes y_j)$ induces an isomorphism $M \otimes_A N \simeq \bigoplus_{ij} M_i \otimes_A N_j$.

(vi) Let F be a free A-module with $\{e_i\}_{i \in I}$ as a basis and let $A \to B$ be a ring homomorphism. Then $B \otimes_A F$ is a free B-module with $\{1 \otimes e_i\}_{i \in I}$ as a basis.

(vii) *Associativity of tensor products.* Let A, B be commutative rings, M an A-module, P a B-module and N, an A–B-bimodule. (Recall that an *A–B-bimodule* N is an A-module which is also a B-module such that $a(by) = b(ay)$, for all $y \in N,\ a \in A$ and $b \in B$.) Then, $M \otimes_A N$ is an A-B-bimodule for the B-module structure given by $b(x \otimes y) = x \otimes by$, for $x \in M,\ y \in N$ and $b \in B$. Similarly, $N \otimes_B P$ is an A–B-bimodule. There exists an isomorphism $(M \otimes_A N) \otimes_B P \simeq M \otimes_A (N \otimes_B P)$ given by $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$ for $x \in M,\ y \in N$ and $z \in P$ and this is functorial in M, N and P.

In view of the associativity of tensor products, we may talk, without ambiguity, of the tensor product $M_1 \otimes_A \cdots \otimes_A M_r$ for A-modules $M_1, M_2, \ldots, M_r$.

0.5 Exterior products

Let A be a ring, M an A-module and $p \geq 1$ an integer. Let $\otimes^p M$ denote the tensor product of M with itself, p times. An A-homomorphism f of $\otimes^p M$ into an A-module N is said to be *alternating*, if for $x_1, \ldots, x_p \in M$, we have $f(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = 0$, whenever $x_i = x_j$ for some $i, j,\ i \neq j,\ 1 \leq i,\ j \leq p$.

A *p-fold exterior product* of M is a pair (E_p, ψ) where E_p is an A-module and $\psi\colon \bigotimes^p M \to E_p$ is an alternating A-homomorphism, such that for any alternating A-homomorphism f of $\otimes^p M$ into an A-module N, there exists a unique A-homomorphism $\tilde{f}\colon E_p \to N$ which makes the diagram

$$\begin{tikzcd} \bigotimes^p M \arrow[d, "\psi"'] \arrow[dr, "f"] & \\ E_p \arrow[r, "\tilde{f}"] & N \end{tikzcd}$$

commutative. As in the case of tensor products, it is easy to see that the p-fold exterior product of M is unique, if it exists. Let P be the submodule of $\otimes^p M$ generated by all elements of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ where $x_1, x_2, \ldots, x_p \in M$ and $x_i = x_j$ for some i, j with $i \neq j$. Let $\bigwedge^p M = \bigotimes^p M/P$ and $\psi\colon \bigotimes^p M \to \bigwedge^p M$ be the canonical homomorphism. Then it is easily seen that the pair $(\bigwedge^p M, \psi)$ (or briefly $\bigwedge^p M$) is the p-fold exterior product of M. For $x_1, \ldots, x_p \in M$, the image of $x_1 \otimes \cdots \otimes x_p$ in $\bigwedge^p M$ under ψ is denoted by $x_1 \wedge \cdots \wedge x_p$.

We set $\bigwedge^0 M = A$ so that $\bigwedge^p M$ is defined for all $p \in \mathbf{Z}^+$. Note that $\bigwedge^1 M = M$.

In the sequel, we need the following properties of exterior products.

(i) Let $A \to B$ be a ring homomorphism and let M be an A-module. Composing the B-isomorphism

$$\bigotimes^p (B \otimes_A M) (= (B \otimes_A M) \otimes_B (B \otimes_A M) \otimes_B \cdots \otimes_B (B \otimes_A M)) \simeq B \otimes_A (\otimes^p M)$$

with the B-homomorphism $1_B \otimes \psi\colon B \otimes_A (\otimes^p M) \to B \otimes_A (\bigwedge^p M)$, we have an alternating B-homomorphism $\bigotimes^p (B \otimes_A M) \to B \otimes_A (\bigwedge^p M)$, which induces a B-isomorphism $\bigwedge^p (B \otimes_A M) \simeq B \otimes_A (\bigwedge^p M)$.

(ii) Let M, N be A-modules. Then for $0 \leq i \leq p$, the map

$$\underbrace{(M \times \cdots \times M)}_{i \text{ times}} \times \underbrace{(N \times \cdots \times N)}_{p-i \text{ times}} \to \bigwedge^p (M \oplus N)$$

given, for $x_1, \ldots, x_i \in M$ and $y_1, \ldots, y_{p-i} \in N$, by

$$((x_1, \ldots, x_i),\ (y_1, \ldots, y_{p-i})) \mapsto (x_1, 0) \wedge \cdots \wedge (x_i, 0) \wedge (0, y_1) \wedge \cdots \wedge (0, y_{p-i})$$

induces an A-homomorphism $(\bigwedge^i M) \otimes (\bigwedge^{p-i} N) \to \bigwedge^p (M \oplus N)$ and we have an isomorphism

$$\bigwedge^p (M \oplus N) \simeq \bigoplus_{0 \leq i \leq p} (\bigwedge^i M \otimes \bigwedge^{p-i} N).$$

(iii) If F is a free A-module with a basis of n elements, then $\bigwedge^n F \simeq A$ and $\bigwedge^i F = 0$ for $i > n$, as may be easily deduced from (ii) above.

Chapter 1

Some Results from Commutative Algebra

In this chapter, by a ring we always mean a commutative ring with unit element (denoted by 1), and by a module, we mean a unitary module.

1.1 Ring of fractions and localization

Let A be a (commutative) ring (with 1). A subset S of A is said to be *multiplicative* if $1 \in S$ and for $s,\ s' \in S$, we have $ss' \in S$. Let M be an A-module and let S be a multiplicative subset of A. On the set $M \times S$ we define a relation as follows: $(m, s) \sim (m', s')$ if there exists s'' in S such that $s''(s'm - sm') = 0$. It is easy to verify that $\sim$ is an equivalence relation. We denote the set of equivalence classes by $S^{-1}M$. If $(m, s) \in M \times S$, the equivalence class containing (m, s) will be denoted by m/s.

In particular, if $M = A$, the above construction leads to the set $S^{-1}A$. We define in $S^{-1}A$, addition and multiplication by

$$a/s + a'/s' = (s'a + sa')/ss'$$

$$a/s \cdot a'/s' = aa'/ss'.$$

It is easily verified that these are well-defined and that $S^{-1}A$ is a (commutative) ring under these operations. The zero element of $S^{-1}A$ is $0/1$ and the unit element is $1/1$. We call the ring $S^{-1}A$ the *ring of fractions* of A with respect to S. Note that $S^{-1}A = 0 \Longleftrightarrow 0 \in S$.

If M is an A-module, the set $S^{-1}M$ is easily verified to be an $S^{-1}A$-module under the (well-defined) operations:

$$m/s + m'/s' = (s'm + sm')/ss'$$

$$a/s \cdot m/s' = am/ss'.$$

The $S^{-1}A$ module $S^{-1}M$ will be called the *module of fractions* of M with respect to S.

Let $\mathfrak{p}$ be a prime ideal of a ring A. Then $S = A - \mathfrak{p}$ is clearly a multiplicative subset of A. In this case, we denote $S^{-1}A$ by $A_{\mathfrak{p}}$, and for an A-module M, the module $S^{-1}M$ is denoted by $M_{\mathfrak{p}}$. (If A is an integral domain, then (0) is a prime ideal and $A_{(0)}$ is the quotient field of A.)

Let M, N be A-modules and $f \in \operatorname{Hom}_A(M, N)$. We define $S^{-1}f$: $S^{-1}M \to S^{-1}N$ by $(S^{-1}f)(m/s) = f(m)/s$. It is easily seen that $S^{-1}f$ is well-defined and that it belongs to $\operatorname{Hom}_{S^{-1}A}(S^{-1}M,\ S^{-1}N)$.

Proposition 1.1 *The assignments $M \mapsto S^{-1}M$, $f \mapsto S^{-1}f$ define an exact functor from A-modules to $S^{-1}A$-modules.*

PROOF: The only non-trivial thing to be verified is the following: given an exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

of A-modules, the sequence

$$0 \to S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \to 0$$

of $S^{-1}A$-modules is exact.

Exactness at $S^{-1}M''$. Any element of $S^{-1}M''$ is of the form m''/s, with $m'' \in M''$, $s \in S$. Let $m \in M$ be such that $g(m) = m''$. Then $m''/s = S^{-1}g(m/s)$.

Exactness at $S^{-1}M$. First $g \circ f = 0$ implies $S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$, so that $\operatorname{Im} S^{-1}f \subset \ker S^{-1}g$. Let now $m/s \in S^{-1}M$ be such that $S^{-1}g(m/s) = g(m)/s = 0$. Then there exists $t \in S$ such that $g(tm) = tg(m) = 0$. Therefore, there exists $m' \in M'$ such that $tm = f(m')$. Now $S^{-1}f(m'/ts) = f(m')/ts = tm/ts = m/s$.

Exactness at $S^{-1}M'$. Let $m'/s \in S^{-1}M'$ be such that $f(m')/s = S^{-1}f(m'/s) = 0$. Then there exists $t \in S$ such that $f(tm') = tf(m') = 0$ which implies that $tm' = 0$. Hence $m'/s = 0$.

This completes the proof of the proposition.

We have a map $i_M\colon M \to S^{-1}M$ given by $m \mapsto m/1$. The map $i_A\colon A \to S^{-1}A$ is easily seen to be a ring homomorphism. Thus any $S^{-1}A$-module can be regarded as an A-module through i_A. In particular, $S^{-1}M$ is an A-module and it is easily checked that i_M is an A-homomorphism, which is functorial in M.

The map $S^{-1}A \times M \to S^{-1}M$ given by $(a/s,\ m) \mapsto am/s$ is well-defined and is A-bilinear. This induces an A-homomorphism

$$\varphi\colon S^{-1}A \otimes_A M \to S^{-1}M \text{ given by } \varphi(a/s \otimes m) = am/s.$$

Proposition 1.2 *The map φ is an $S^{-1}A$-isomorphism and is functorial in M.*

PROOF: Since $\varphi\Big(\dfrac{a'}{s'}\Big(\dfrac{a}{s} \otimes m\Big)\Big) = \Big(\dfrac{a'a}{s's} \otimes m\Big) = \dfrac{a'am}{s's} = \Big(\dfrac{a'}{s'}\Big)\Big(\dfrac{am}{s}\Big) = \Big(\dfrac{a'}{s'}\Big)\varphi\Big(\dfrac{a}{s} \otimes m\Big)$, φ is an $S^{-1}A$-homomorphism. The assignment $m/s \mapsto 1/s \otimes m$ is easily seen to be a well-defined map and is the inverse of φ. The functoriality of φ is easily checked and the proposition is proved.

Let $\mathfrak{a}$ be an ideal of A. Then the ideal of $S^{-1}A$ generated by $i_A(\mathfrak{a})$ is denoted by $\mathfrak{a}S^{-1}A$. Note that $\mathfrak{a}S^{-1}A = S^{-1}\mathfrak{a}$ if we regard $S^{-1}\mathfrak{a}$ as a subset of $S^{-1}A$.

If $\mathfrak{a} \cap S \neq \emptyset$, then $\mathfrak{a}S^{-1}A = S^{-1}A$, for if $s \in \mathfrak{a} \cap S$ then $1 = s \cdot 1/s \in \mathfrak{a}S^{-1}A$.

Proposition 1.3 *The map $\varphi\colon \mathfrak{p} \mapsto S^{-1}\mathfrak{p}(= \mathfrak{p}S^{-1}A)$ is an inclusion-preserving bijection of the set of prime ideals $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \emptyset$ onto the set of all prime ideals of $S^{-1}A$.*

PROOF: For a prime ideal $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \emptyset$, we first assert that if an element a/s of $S^{-1}A$ is in $S^{-1}\mathfrak{p}$, then $a \in \mathfrak{p}$. For $a/s \in S^{-1}\mathfrak{p} \Longrightarrow a/s = p/t$ for $p \in \mathfrak{p},\ t \in S \Longrightarrow$ there exists $u \in S$ such that $uta = usp \in \mathfrak{p} \Longrightarrow a \in \mathfrak{p}$ since $ut \in S \subset A - \mathfrak{p}$. This implies that $S^{-1}\mathfrak{p} \neq S^{-1}A$. Now $(a/s)(a'/s') \in S^{-1}\mathfrak{p} \Longrightarrow aa'/ss' \in S^{-1}\mathfrak{p} \Longrightarrow aa' \in \mathfrak{p} \Longrightarrow$ either a or a' is in $\mathfrak{p} \Longrightarrow$ either a/s or a'/s' is in $S^{-1}\mathfrak{p}$. Further clearly, $\mathfrak{p}_1 \subset \mathfrak{p}_2 \Longleftrightarrow S^{-1}\mathfrak{p}_1 \subset S^{-1}\mathfrak{p}_2$. Let $\wp$ be a prime ideal of $S^{-1}A$ and let $\mathfrak{p} = i_A^{-1}(\wp)$. Obviously, $\mathfrak{p}$ is a prime ideal of A and, further, $\mathfrak{p} \cap S = \emptyset$. We claim that the map $\wp \mapsto i_A^{-1}(\wp)$ is the inverse of φ. First, for a prime ideal $\wp$ of $S^{-1}A$, we evidently have $i_A^{-1}(\wp)S^{-1}A \subset \wp$. On the other hand, $a/s \in \wp \Longrightarrow a/1 \in \wp \Longrightarrow a \in i_A^{-1}(\wp) \Longrightarrow a/s \in i_A^{-1}(\wp)S^{-1}A$. Thus,

$\wp = i_A^{-1}(\wp)S^{-1}A$. Let now $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p} \cap S = \emptyset$. Clearly, $\mathfrak{p} \subset i_A^{-1}(\mathfrak{p}S^{-1}A)$. On the other hand, $a \in i_A^{-1}(\mathfrak{p}S^{-1}A) \Longrightarrow a/1 \in \mathfrak{p}S^{-1}A \Longrightarrow a \in \mathfrak{p}$. This shows that $\mathfrak{p} = i_A^{-1}(S^{-1}\mathfrak{p})$.

A ring A is called a *local ring* if $A \neq 0$ and has a unique maximal ideal.

Corollary 1.4 *Let A be a commutative ring and let $\mathfrak{p}$ be a prime ideal of A. Then $A_\mathfrak{p}$ is a local ring with $\mathfrak{p}A_\mathfrak{p}$ as its unique maximal ideal.*

PROOF: Observe first that $\mathfrak{p}A_\mathfrak{p}$ is a prime ideal. Let now $\wp$ be any prime ideal of $A_\mathfrak{p}$. Then $i_A^{-1}(\wp)$ is a prime ideal of A contained in $\mathfrak{p}$. Hence $\wp = i_A^{-1}(\wp)A_\mathfrak{p} \subset \mathfrak{p}A_\mathfrak{p}$ and the corollary follows.

We call $A_\mathfrak{p}$ the *localization* of A at $\mathfrak{p}$.

1.2 Noetherian modules

Proposition 1.5 *Let A be a ring. For an A-module M, the following conditions are equivalent:*

(i) Every submodule of M is finitely generated:

(ii) M satisfies the ascending chain condition for submodules i.e. every sequence $M_0 \underset{\neq}{\subset} M_1 \underset{\neq}{\subset} M_2 \underset{\neq}{\subset} \cdots$ of submodules of M is finite:

(iii) every nonempty set of submodules of M has a maximal element.

PROOF: $(i) \Longrightarrow (ii)$. Let $N = \bigcup_{i \geq 0} M_i$; it is easy to see that N is a submodule of M. Let $\{n_1, n_2, \ldots, n_r\}$ be a set of generators of N. There exists M_p such that $n_1, n_2, \ldots, n_r$ are all contained in M_p. Thus $N \subset M_p \subset N \Longrightarrow N = M_p \Longrightarrow M_p = M_{p+1} = \cdots$.

(ii) $\Longrightarrow$ (iii). Let $\mathcal{F}$ be a nonempty set of submodules of M. Take $M_0 \in \mathcal{F}$. If M_0 is maximal in $\mathcal{F}$, we are through. Otherwise, choose M_1 in $\mathcal{F}$ such that $M_0 \underset{\neq}{\subset} M_1$. If M_1 is maximal, we are done. Otherwise, there exists M_2 in $\mathcal{F}$ such that $M_1 \underset{\neq}{\subset} M_2$. Proceeding this way, we get a sequence $M_0 \underset{\neq}{\subset} M_1 \underset{\neq}{\subset} M_2 \underset{\neq}{\subset} \cdots$ which by (ii), is necessarily finite. The last element in the sequence is maximal in $\mathcal{F}$.

(iii) $\Longrightarrow$ (i). Since condition (iii) holds also if M is replaced by any submodule, it is enough to show that M is finitely generated. Let $\mathcal{F}$ be the family of all finitely generated submodules of M; clearly, $\mathcal{F}$ is nonempty. By (iii), there exists a maximal element N in $\mathcal{F}$. If $N \neq M$, there exists $m \in M$, $m \notin N$ the submodule generated by N and m belongs to $\mathcal{F}$ and contains N property. This contradiction proves that M is finitely generated.

An A-module M is said to be *noetherian* if it satisfies any one of the three equivalent conditions of Proposition 1.5. A ring A is *noetherian* if it is a noetherian A-module.

Proposition 1.6 *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be an exact sequence of* A*-modules. Then* M *is noetherian if and only if* M', M'' *are noetherian.*

PROOF: Let M be noetherian. Since any submodule N' of M' can be identified with a submodule of M, it follows that N' is finitely generated and hence M' is noetherian. Since any submodule of M'' is the image of a submodule of M, it is finitely generated. Hence M'' is noetherian.

Let M', M'' be noetherian. Let N be any submodule of M. Let $n_1, \ldots, n_s \in N$ be such that $g(n_1), \ldots, g(n_s)$ generate the submodule $g(N)$ of M'' and let $n_{s+1}, \ldots, n_r \in N$ be such that $f^{-1}(n_{s+1}), \ldots, f^{-1}(n_r)$ generate the submodule $f^{-1}(N)$ of M'. It is easy to see that $n_1, \ldots, n_r$ generate N. Hence M is noetherian.

Proposition 1.7 *Let* A *be a noetherian ring and* M *a finitely generated* A*-module. Then* M *is noetherian.*

PROOF: Let M be generated by n elements. The proposition is proved by induction on n. If $n = 1$, then M is isomorphic to a quotient of the noetherian module A and hence is noetherian, by Proposition 1.6. Let $n > 1$, and let M be generated by $x_1, \ldots, x_n$. Let $M' = Ax_1$ and $M'' = M/M'$. By our earlier remark M' is noetherian. Since M'' is generated by the images of $x_2, \ldots, x_n$ in M'', it is noetherian by the induction hypothesis. From Proposition 1.6. it follows that M is noetherian.

Proposition 1.8 *Let* S *be a multiplicative subset of a noetherian ring* A*. Then* $S^{-1}A$ *is noetherian. In particular, the localization of a noetherian ring at a prime ideal is noetherian.*

PROOF: Let $\mathfrak{b}$ be an ideal of $S^{-1}A$ and let $a_1, \ldots, a_r$ in A generate the ideal $i_A^{-1}(\mathfrak{b})$ of A. Clearly, $a_1/1, \ldots, a_r/1$ generate the ideal $\mathfrak{b}$.

Theorem 1.9 *(Hilbert basis theorem) Let* A *be a noetherian ring. Then the polynomial ring* $A[X_1, \ldots, X_n]$ *in* n *variables over* A *is noetherian.*

PROOF: By induction on n, it is sufficient to prove the theorem for $n = 1$, i.e. that the polynomial ring $B = A[X]$ in one variable is noetherian. Let $\mathfrak{b}$ be an ideal of B, we will show that $\mathfrak{b}$ is finitely generated. Let $\mathfrak{a} = \{0\} \bigcup \{$leading coefficients of elements of $\mathfrak{b}\}$. It is clear that $\mathfrak{a}$ is an ideal of A. If $\mathfrak{a} = 0$, then $\mathfrak{b} = 0$ and there is nothing to prove. Let then $\mathfrak{a} \neq 0$. Since $\mathfrak{a}$ is finitely generated, there exist in A non-zero elements $c_1, \ldots, c_r$ such that $\mathfrak{a} = (c_1, \ldots, c_r)$; let $f_i \in \mathfrak{b}$ with leading coefficient c_i for $1 \leq i \leq r$. Let $N = \max_i(\deg f_i)$. We claim that $\mathfrak{b} = (f_1, \ldots, f_r) + \mathfrak{b}'$ where $\mathfrak{b}' = \mathfrak{b} \cap (A + AX + \cdots + AX^{N-1})$. To prove this, it is enough to show that any $f = a_m X^m + \cdots + a_0$ in $\mathfrak{b}$ belongs to $(f_1, \ldots, f_r) + \mathfrak{b}'$. If $m \leq N-1$, this is clear. Let then $m \geq N$ and let $a_m = \sum_{1 \leq i \leq r} d_i c_i,\ d_i \in A$. Then $\deg(f - \sum_{1 \leq i \leq r} d_i X^{m - \deg f_i} f_i) \leq m - 1$ and hence, by induction on m, $f - \sum_{1 \leq i \leq r} d_i X^{m - \deg f_i} f_i$ belongs to $(f_1, \ldots, f_r) + \mathfrak{b}'$ and consequently, so does f. Being an A-submodule of $A + AX + \cdots + AX^{n-1}$, $\mathfrak{b}'$ has, by Proposition 1.7, a finite set of generators $g_1, \ldots, g_s$ over A. It is clear that $f_1, \ldots, f_r,\ g_1, \ldots, g_s$ generate the ideal $\mathfrak{b}$.

Corollary 1.10 *Let A be a noetherian ring and B a finitely generated A-algebra. Then B is noetherian.*

PROOF: We first remark that if A is a noetherian ring and $A \to B$ is a surjective ring homomorphism then B is noetherian. The proof is one the same lines as in Proposition 1.6. Since any finitely generated A-algebra is a quotient of a polynomial ring $A[X_1, \ldots, X_n]$, the corollary follows.

1.3 Some lemmas

Let A be a ring. The intersection of all maximal ideal of A is called the *Jacobson radical* of A and is denoted by $\underline{r}(A)$ or simply, by $\underline{r}$.

Note that if $a \in \underline{r}(A)$, then $1 - a$ is invertible.

Lemma 1.11 *(Nakayama). Let M be a finitely generated A-module. If $\underline{r}M = M$, then $M = 0$.*

PROOF: Let, if possible, $M \neq 0$ and let $x_1, \ldots, x_n$ be a minimal set of generators for M. Since $M = \underline{r}M$, we have $x_1 = \sum_{1 \leq i \leq n} a_i x_i$ with $a_i \in \underline{r}$. This gives $(1 - a_1)x_1 = \sum_{2 \leq i \leq n} a_i x_i$, i.e. $x_1 = \sum_{2 \leq i \leq n} (1 - a_1)^{-1} a_i x_i$ so that $x_2, \ldots, x_n$ generate M, which is a contradiction.

Lemma 1.12 *Let* $\mathfrak{a}, \mathfrak{b}_0, \mathfrak{b}_1, \ldots, \mathfrak{b}_n$ *be ideals of a ring* A *with* $\mathfrak{b}_0$ *prime and* $\mathfrak{a} \subset \bigcup_{0 \leq i \leq n} \mathfrak{b}_i$. *Then there exists a proper subset* J *of* $\{0, 1, 2, \ldots, n\}$ *such that* $\mathfrak{a} \subset \bigcup_{j \in J} \mathfrak{b}_j$.

PROOF: If the result is false, then, for every i with $0 \leq i \leq n$, there exists $a_i \in \mathfrak{a} - \bigcup_{j \neq i} \mathfrak{b}_j$. Clearly $a_i \in \mathfrak{b}_i$. Let $a = a_0 + a_1 a_2 \cdots a_n$. Then $a \in \mathfrak{a}$ and hence $a \in \mathfrak{b}_i$ for some i. If $a \in \mathfrak{b}_0$ then $a_1 a_2 \cdots a_n \in \mathfrak{b}_0$. Since $\mathfrak{b}_0$ is prime, this implies that $a_i \in \mathfrak{b}_0$ for some $i \geq 1$, which is impossible. Let than $a \in \mathfrak{b}_i$ for some $i \geq 1$. Then $a_0 \in \mathfrak{b}_i$ which is again impossible.

Lemma 1.13 *Let* M, N *be non-zero finitely generated modules over a local ring* A. *Then* $M \otimes_A N \neq 0$.

PROOF: Let $\mathfrak{m}$ be the maximal ideal of A. By Nakayama's lemma, $M/\mathfrak{m}M \neq 0$ and $N/\mathfrak{m}N \neq 0$. Since these are vector spaces over the field $A/\mathfrak{m}$, we have $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (N/\mathfrak{m}N) \neq 0$, i.e. $(M \otimes_A N)/\mathfrak{m}(M \otimes_A N) \neq 0$. Hence $M \otimes_A N \neq 0$.

1.4 Primary decomposition

Let A be a (commutative) ring (with 1) and let M be an A-module. For any $a \in A$, the map $a_M : M \to M$, defined by $a_M(x) = ax$ for $x \in M$, is an A-homomorphism and is called the *homothesy by* a. Let N be a submodule of M. We say that N is *primary* in M (or a *primary submodule of* M) if $N \neq M$ and for any $a \in A$, the homothesy $a_{M/N}$ is either injective or nilpotent. By a *primary ideal* of A, we mean a primary submodule of A.

Proposition 1.14 *Let* N *be a primary submodule of an* A*-module* M *and let* $\mathfrak{p} = \{a \in A \mid a_{M/N}$ *is not injective*$\}$. *Then* $\mathfrak{p}$ *is a prime ideal of* A.

PROOF: Since N is primary in M, we have $\mathfrak{p} = \{a \in A \mid a_{M/N}$ is nilpotent $\}$. From this, it trivially follows that $\mathfrak{p}$ is a proper ideal. Let $a, b \in A$ with $a \notin \mathfrak{p}$, $b \notin \mathfrak{p}$. Then $a_{M/N}$ and $b_{M/N}$ are both injective and hence, so is $ab_{M/N} = a_{M/N} \circ b_{M/N}$. Thus $ab \notin \mathfrak{p}$, so that $\mathfrak{p}$ is prime.

For a primary submodule N of M, the prime ideal $\mathfrak{p}$ defined in Proposition 1.14, is called the *prime ideal belonging to* N in M and we say N is $\mathfrak{p}$-*primary* (in M).

Let N be a submodule of M. A decomposition of the form $N = N_1 \cap N_2 \cap \cdots N_r$ where N_i, $1 \leq i \leq r$ are primary submodules of M, is called a *primary decomposition* of N (in M). This decomposition is said to be *reduced* (or *irredundant*) if (i) N cannot be expressed as the intersection of a proper subset of $\{N_1, N_2, \ldots, N_r\}$ and (ii) the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ belonging respectively to $N_1, \ldots, N_r$ in M are distinct.

Proposition 1.15 *Let M be a noetherian A-module. Then any proper submodule of M admits of a reduced primary decomposition.*

To prove this proposition, we need a few lemmas.

Lemma 1.16 *Let $N_1, \ldots, N_r$ be $\mathfrak{p}$-primary submodule of M. Then $N = N_1 \cap \cdots \cap N_r$ is $\mathfrak{p}$-primary.*

Proof: Let $a \in \mathfrak{p}$. Then there exists $n_i \in N$ such that $(a_{M/N_i})^{n_i} = 0$, for $1 \leq i \leq r$. Let $n = \max_{1 \leq i \leq r} n_i$. Clearly, $(a_{M/N})^n = 0$. Let now $a \notin \mathfrak{p}$. If $x \in M$, $x \notin N$, then there exists i such that $x \notin N_i$. Since N_i is $\mathfrak{p}$-primary, we have $ax \notin N_i$ and *a fortiori*, $ax \notin N$. This proves that $a_{M/N}$ is injective. Hence N is $\mathfrak{p}$-primary.

A submodule N of M is said to be *irreducible* if (i) $N \neq M$ and (ii) N cannot be expressed as $N = N_1 \cap N_2$ with submodules N_1, N_2 containing N properly.

Lemma 1.17 *Any irreducible submodule N of noetherian module M is primary.*

Proof: Let, if possible, N be not primary. Then there exists $a \in A$ such that $a_{M/N}$ is neither injective nor nilpotent. The sequence of submodules $\ker a^r_{M/N}$, $r = 1, 2, \ldots$, is clearly increasing. Since M is noetherian, there exists r such that $\ker a^r_{M/N} = \ker a^{r+1}_{M/N} = \cdots$. Let $\varphi = a^r_{M/N}$. Then $\ker \varphi = \ker \varphi^2$. We claim that $\ker \varphi \cap \operatorname{im} \varphi = 0$. In fact, $x \in \ker \varphi \cap \operatorname{Im} \varphi \Longrightarrow \varphi(x) = 0$ and $x = \varphi(y)$ for $y \in M/N \Longrightarrow y \in \ker \varphi^2 = \ker \varphi \Longrightarrow x = \varphi(y) = 0$. Further, since $a_{M/N}$ is neither injective nor nilpotent, we have $\ker \varphi \neq 0$, $\operatorname{Im} \varphi \neq 0$. Let $\eta \colon M \to M/N$ be the canonical homomorphism. Then $N = \eta^{-1}(0) = \eta^{-1}(\ker \varphi) \cap \eta^{-1}(\operatorname{Im} \varphi)$. Since $\eta^{-1}(\ker \varphi)$ and $\eta^{-1}(\operatorname{Im} \varphi)$ both contain N properly, this contradicts irreducibility of N and proves the lemma.

Lemma 1.18 *Let M be a noetherian A-module. Then any proper submodule of M is finite intersection of irreducible submodules.*

Proof: Let $\mathcal{F}$ be the set of proper submodules of M which cannot be expressed as a finite intersection of irreducible submodules. If possible, let $\mathcal{F} \neq \emptyset$. Then, M being noetherian $\mathcal{F}$ contains maximal element N. Clearly N is not irreducible. Let $N = N_1 \cap N_2$ with submodules N_1, N_2 containing N properly. The maximality of N implies that both N_1 and N_2 can be expressed as finite intersection of irreducible submodules. Hence N itself is such a finite intersection. This contradiction shows that $\mathcal{F} = \emptyset$ and proves the lemma.

Proof of Proposition 1.15. From Lemmas 1.18 and 1.17, it follows that any proper submodule N admits of a primary decomposition, $N = N_1 \cap \cdots \cap N_r$. In view of Lemma 1.16, we may, after grouping together all the $\mathfrak{p}$-primary submodules with the same prime ideal $\mathfrak{p}$, assume that the prime ideals belonging to N_i are distinct. Now by deleting some of the N_i'if necessary, we get a reduced primary decomposition for N. This completes the proof of the proposition.

Let M be an A-module. A prime ideal $\mathfrak{p}$ of A is said to be *associated* to M, if there exists $x \neq 0$ in M such that $\mathfrak{p}$ is the annihilator of x, i.e. $\mathfrak{p} = \{a \in A \mid ax = 0\}$. We denote by $\operatorname{Ass}(M)$ the set of all prime ideals of A associated to M. Note that $\mathfrak{p} \in \operatorname{Ass}(M)$ if and only if there exists an A-monomorphism $A/\mathfrak{p} \to M$.

Proposition 1.19 *Let A be a noetherian ring and M a finitely generated A-module. Let $0 = N_1 \cap \cdots \cap N_r$ be a reduced primary decomposition of 0 in M, with N_i being $\mathfrak{p}_i$-primary for $1 \leq i \leq r$. Then* $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$. *In particular,* $\operatorname{Ass}(M)$ *is finite, moreover,* $M = 0$ *if and only if* $\operatorname{Ass}(M) = \emptyset$.

Proof: Let $\mathfrak{p} \in \operatorname{Ass}(M)$. Then there exists $x \neq 0$ in M such that $\mathfrak{p}$ is the annihilator of x. Since $x \neq 0$, we may assume that $x \notin N_1 \cup \cdots \cup N_j$, $x \in N_{j+1} \cap \cdots \cap N_r$ for some j with $1 \leq j \leq r$. For any $a \in \mathfrak{p}_i$, the homothesy a_{M/N_i} is nilpotent. Since $\mathfrak{p}_i$ is finitely generated, there exists $n_i \in \mathbf{N}$ such that $\mathfrak{p}_i^{n_i} M \subset N_i$. Clearly, $\prod_{1 \leq i \leq j} \mathfrak{p}_i^{n_i} x \subset (N_1 \cap \cdots \cap N_j) \bigcap (N_{j+1} \cap \cdots \cap N_r) = 0$. Thus $\prod_{1 \leq i \leq j} \mathfrak{p}_i^{n_i} \subset \mathfrak{p}$ which implies $\mathfrak{p}_k \subset \mathfrak{p}$ for some k with $1 \leq k \leq j$. On the other hand, $\mathfrak{p}x = 0$ implies that the homothesy a_{M/N_k} is not injective, for every $a \in \mathfrak{p}$. Therefore $\mathfrak{p} \subset \mathfrak{p}_k$, i.e. $\mathfrak{p} = \mathfrak{p}_k$. This proves that $\operatorname{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

We now show that $\mathfrak{p}_i \in \operatorname{Ass}(M)$ for $1 \leq i \leq r$. It is enough to prove that $\mathfrak{p}_1 \in \operatorname{Ass}(M)$. Since the given primary decomposition is reduced there exists $x \in N_2 \cap \cdots \cap N_r$, $x \notin N_1$. As N_1 is $\mathfrak{p}_1$-primary, it is easy to see as above that there exists $n \in \mathbf{N}$ such that $\mathfrak{p}_1^n x \subset N_1$ and

$\mathfrak{p}_1^{n-1}x \not\subset N_1$. Let $y \in \mathfrak{p}_1^{n-1}x,\ y \notin N_1$. Then clearly $\mathfrak{p}_1$ is contained in the annihilator of y. On the other hand, if $a \in A$ is such that $ay = 0$, then a_{M/N_1}, is not injective which implies that a is in $\mathfrak{p}_1$. Thus $\mathfrak{p}_1$ is the annihilator of y which shows that $\mathfrak{p}_1 \in \operatorname{Ass}(M)$.

Corollary 1.20 *Let A be a noetherian ring and N be a submodule of a finitely generated A-module M with reduced primary decomposition $N = N_1 \cap \cdots \cap N_r$. Then $\operatorname{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ where $\mathfrak{p}_i$ are the prime ideals belonging to N_i in M, for $1 \leq i \leq r$. In particular, the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of prime ideals corresponding to a reduced primary decomposition of N is independent of the decomposition.*

Proof: This follows from the proposition above, by observing that $0 = (N_1/N) \cap \cdots \cap (N_r/N)$ is a reduced primary decomposition of 0 in M/N and that N_i/N are $\mathfrak{p}_i$-primary, in M/N for $1 \leq i \leq r$.

An element $a \in A$ is said to be a zero divisor of an A-module M, if there exists $x \in M,\ x \neq 0.$ such that $ax = 0$.

Proposition 1.21 *Let A be a noetherian ring and M, a finitely generated A-module. Then the set of zero-divisors of M is $\bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$.*

Proof: Let $a \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(M)$. Let $\mathfrak{p}$ be the annihilator of $x \in M,\ x \neq 0$. Then $ax = 0$. Conversely, let a be a zero-divisor of M and let $x \in M,\ x \neq 0$ be such that $ax = 0$. Let $0 = N_1 \cap \cdots N_r$ be a reduced primary decomposition of 0 in M. Then $x \notin N_i$ for some N_i. Let $\mathfrak{p}_i$ be the prime ideal belonging to N_i in M. By Proposition 1.19, $\mathfrak{p}_i \in \operatorname{Ass}(M)$. Now since $ax = 0$ the homothesy a_{M/N_i} is not injective and $a \in \mathfrak{p}_i$.

Let A be a ring. We recall that Spec (A) denotes the set of all prime ideals of A.

The *nilradical* $\mathfrak{n}(A)$ of a ring A is defined to be the subset $\{a \in A \mid a^n = 0 \text{ for some } n \in \mathbf{N}\}$ of A; clearly $\mathfrak{n}(A)$ is an ideal of A.

Proposition 1.22 *Let A be a ring. Then*

$$\mathfrak{n}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p}$$

Proof: Let $a \in \mathfrak{n}(A)$. Then $a^n = 0$ for some $n \in \mathbf{N} \Rightarrow a^n \in \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec}(A) \Rightarrow a \in \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Conversely, let $a \in \mathfrak{p}$ for every $\mathfrak{p} \in$ Spec (A) and let S be the multiplicative subset $\{1, a, a^2, \cdots\}$ of A. Since $S \cap \mathfrak{p} \neq \emptyset$ for every $\mathfrak{p} \in$ Spec (A) we have by

Proposition 1.3 that Spec $(S^{-1}A) = \emptyset$. Hence $S^{-1}A = 0$ which implies that $0 \in S$ i.e., $a \in \mathfrak{n}(A)$.

Let $\mathfrak{a}$ be an ideal of a ring A. The *radical* $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ is defined by $\sqrt{\mathfrak{a}} = \{a \in A \mid a^n \in \mathfrak{a} \text{ for some } n \in \mathbf{N}\}$. Clearly $\sqrt{\mathfrak{a}}$ is an ideal of A containing $\mathfrak{a}$ and $\sqrt{\mathfrak{a}}/\mathfrak{a} = \mathfrak{n}(A/\mathfrak{a})$.

Corollary 1.23 to Proposition 1.22. *Let $\mathfrak{a}$ be an ideal of A. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \in \text{ Spec } (A) \\ \mathfrak{p} \supset \mathfrak{a}}} \mathfrak{p}$$

PROOF: Immediate.

For an A-module M, we define the *annihilator* ann (M), by ann $(M) = \{a \in A \mid aM = 0\}$. Clearly ann M is an ideal of A.

Proposition 1.24 *Let A be a noetherian ring, M, a finitely generated A-module and let $\mathfrak{a} =$ ann M. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$$

PROOF: If $M = 0$, $\text{Ass}(M) = \emptyset, \mathfrak{a} = A$ and there is nothing to prove. Let $M \neq 0$ and let $0 = N_1 \cap \cdots \cap N_r$ be a reduced primary decomposition of 0 in M, N_i being $\mathfrak{p}_i$-primary for $1 \leq i \leq r$. By Proposition 1.19, $\text{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$.

Let $a \in \sqrt{\mathfrak{a}}$. Then $a^n M = 0$ for some $n \in \mathbf{N} \Rightarrow$ the homothesy a_{M/N_i} is nilpotent for every $i \Rightarrow a \in \bigcap_{\mathfrak{p} \in \text{Ass} M} \mathfrak{p}$. Conversely, $a \in \bigcap_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p} \Rightarrow$ there exists $n_i \in \mathbf{N}$ such that $a_{M/N_i}^{n_i} = 0,\ 1 \leq i \leq r$. Let $n = \max_{1 \leq i \leq r} n_i$. Then $a^n M \subset \bigcap_{1 \leq i \leq r} N_i = 0$, i.e. $a \in \sqrt{\mathfrak{a}}$.

Corollary 1.25 *For a noetherian ring A, we have $\mathfrak{n}(A) = \bigcap_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p}$.*

PROOF: Note that ann $(A) = 0$ and $\mathfrak{n}(A) = \sqrt{0}$.

Let M be an A-module. The set $\{\mathfrak{p} \in \text{Spec}\,(A) \mid M_{\mathfrak{p}} \neq 0\}$ is called the *support* of M and is denoted Supp (M).

Proposition 1.26 *Let M be an A-module. Then the A-homomorphism $M \xrightarrow{\varphi} \prod_{\mathfrak{p} \in \text{ Spec}\,(A)} M_{\mathfrak{p}}$ induced by the canonical homomorphisms $M \to M_{\mathfrak{p}}$ is injective. In particular, $M = 0$ if and only if* Supp $(M) = \emptyset$.

PROOF: Let $x \in M$ be such that $\varphi(x) = 0$. This means that, for every $\mathfrak{p} \in$ Spec (A), there exists $s_{\mathfrak{p}} \in A - \mathfrak{p}$ such that $s_{\mathfrak{p}} x = 0$. Thus ann $(Ax) \not\subset \mathfrak{p}$, for every $\mathfrak{p} \in$ Spec (A), so that ann $(Ax) = A$ and $x = 0$.

Proposition 1.27 *Let A be a noetherian ring and M a finitely generated A-module. For $\mathfrak{p} \in \operatorname{Spec} A$, the following conditions are equivalent:*

(i) $\mathfrak{p} \in \operatorname{Supp}(M)$*;*

(ii) there exists $\mathfrak{p}' \in \operatorname{Ass}(M)$ *such that* $\mathfrak{p} \supset \mathfrak{p}'$*;*

(iii) $\mathfrak{p} \supset \operatorname{ann}(M)$.

PROOF: $(i) \Rightarrow (ii)$. Let $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. If (ii) does not hold, $\mathfrak{p} \not\supset \bigcap_{1 \leq i \leq r} \mathfrak{p}_i$. This implies, by Proposition 1.24, that $\mathfrak{p} \not\supset \operatorname{ann}(M)$ and consequently $M_{\mathfrak{p}} = 0$, contradicting (i).

$(ii) \Rightarrow (iii)$. Clearly $\mathfrak{p} \supset \mathfrak{p}' \longrightarrow \mathfrak{p} \supset \sqrt{\operatorname{ann}(M)}$ by Proposition 1.24,

$(iii) \Rightarrow (i)$. Let $M_{\mathfrak{p}} = 0$. Let $\{x_1, \ldots, x_r\}$ be a set of generators of M. There exists $s_i \in A - \mathfrak{p}$ such that $s_i x_i = 0$. Then $s = s_1 \cdots s_r \in \operatorname{ann}(M)$. This contradicts (iii), since $s \notin \mathfrak{p}$.

Corollary 1.28 *We have* $\operatorname{Ass}(M) \subset \operatorname{Supp}(M)$*. The minimal elements of Supp* (M) *belongs to* $\operatorname{Ass}(M)$ *and they are precisely the minimal elements of Supp* (M)*.*

PROOF: Immediate from (ii) of Proposition 1.27.

Proposition 1.29 *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of A-modules. Then* $\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$*. If* N_1, N_2 *are finitely generated A-modules, then* $\operatorname{Supp}(N_1 \otimes_A N_2) = \operatorname{Supp}(N_1) \cap \operatorname{Supp}(N_2)$.

PROOF: For $\mathfrak{p} \in \operatorname{Spec}(A)$, we have by Proposition 1.1, the exact sequence $0 \to M'\mathfrak{p} \to M\mathfrak{p} \to M''\mathfrak{p} \to 0$. Now $\mathfrak{p} \in \operatorname{Supp}(M) \Leftrightarrow M_{\mathfrak{p}} \neq 0 \Leftrightarrow$ either $M'_{\mathfrak{p}} \neq 0$ or $M''_{\mathfrak{p}} \neq 0 \Leftrightarrow \mathfrak{p} \in \operatorname{Supp}(M') \cup \operatorname{Supp}(M'')$. We now prove the second assertion. For $\mathfrak{p} \in \operatorname{Spec}(A)$, we have

$$\begin{aligned}(N_1)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (N_2)_{\mathfrak{p}} \simeq (A_{\mathfrak{p}} \otimes_A N_1) \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A N_2) &\simeq A_{\mathfrak{p}} \otimes_A (N_1 \otimes_A N_2) \\ &\simeq (N_1 \otimes_A N_2)_{\mathfrak{p}}\end{aligned}$$

Thus $\mathfrak{p} \in \operatorname{Supp}((N_1 \otimes_A N_2) \Leftrightarrow (N_1 \otimes_A N_2)_{\mathfrak{p}} \neq 0 \Leftrightarrow (N_1)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (N_2)_{\mathfrak{p}} \neq 0 \Leftrightarrow (N_1)_{\mathfrak{p}} \neq 0,\ (N_2)_{\mathfrak{p}} \neq 0$, by Lemma 1.13 $\Leftrightarrow \mathfrak{p} \in \operatorname{Supp}(N_1) \cap \operatorname{Supp}(N_2)$.

1.5 Artinian modules and modules of finite length

An A-module M *is artinian* if M satisfies the *descending chain condition* for submodules, i.e. every sequence $M_0 \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} M_2 \underset{\neq}{\supset}$ of submodules of M is finite. A ring A is *artinian*, if it is artinian as an A-module.

An A-module M is of *finite length* if it possesses a Jordan-Hölder series (i.e. there exists a sequence $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ of submodules of M such that M_i/M_{i+1} is a simple A-module for $i = 0, 1, 2, \ldots, n-1$).

It is well-known that if a module M has two Jordan-Hölder series $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ and $M = M'_0 \supset M'_1 \supset \cdots \supset M'_m = 0$ then $m = n$ and there exists a permutation σ of $\{0, 1, 2, \ldots, n-1\}$ such that $M_i/M_{i+1} \simeq M_{\sigma(i)}/M_{\sigma(i)+1}$. The integer n is called the *length* of the module M and denoted $\ell_A(M)$.

Proposition 1.30 *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *be an exact sequence of A-modules. Then (i) M is artinian if and only if M' and M'' are artinian, and (ii) M is of finite length if and only if M' and M'' are of finite length and, in this case,* $\ell_A(M) = \ell_A(M') + \ell_A(M'')$.

PROOF: (i) Let M be artinian. It is clear that M' is artinian. Let $M''_0 \underset{\neq}{\supset} M''_1 \underset{\neq}{\supset} \cdots$ be an infinite sequence of submodules of M'' and let $M_i = g^{-1}(M''_i)$. Then $M_i \underset{\neq}{\supset} M_{i+1}$ and we get an infinite sequence $M_0 \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} \cdots$ of submodules of M, which is a contradiction.

Conversely, let M' and M'' be both artinian. Let $M_0 \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} \cdots$ be a descending sequence of submodules of M. Let $M''_i = g(M_i),\ i = 0, 1, \ldots$. Since M'' is artinian, there exists i_0 such that $M''_i = M''_{i+1}$ for $i \geq i_0$, i.e. $M_i + M' = M_{i+1} + M'$ for $i \geq i_0$. Since M' is artinian, we also have $f^{-1}(M_i) = f^{-1}(M_{i+1})$ for $i \geq i_1$, for some i_1. It is easy to see that $M_i = M_{i+1}$ for $i \geq \max(i_0, i_1)$.

(ii) is well-known.

Proposition 1.31 *Let M be an A-module. The following conditions are equivalent:*

(i) M is artinian;

(ii) every non-empty family of submodules of M contains a minimal element.

PROOF: (i)$\Rightarrow$ (ii) Let $\mathcal{F}$ be a non-empty family of submodules of M. Let $M_0 \in \mathcal{F}$. If $M_0$is not minimal, there exists $M_1 \in \mathcal{F}$ such that $M_0 \underset{\neq}{\supset} M_1$. If M_1 is not minimal in $\mathcal{F}$ there exists $M_2 \in \mathcal{F}$ such that $M_1 \underset{\neq}{\supset} M_2$. Since M is artinian, this process must terminate, i.e. $\mathcal{F}$ contains a minimal element.

(ii)$\Rightarrow$ (i) Let, if possible, $M_0 \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} \cdots$ be an infinite sequence of submodules of M. Let M_n be a minimal element in the family $\{M_i \mid i \in \mathbf{Z}^+\}$. Then $M_n = M_{n+1} = \cdots$, a contradiction. Thus M is artinian.

Proposition 1.32 *Let A be a noetherian ring and M a finitely generated A-module. Then M is of finite length if and only if every $\mathfrak{p} \in$* Supp (M) *is a maximal ideal.*

PROOF: Let M be of finite length. If $M = 0$, then Supp $(M) = \emptyset$ and the assertion is trivial. Let $\ell_A(M) = 1$. Then $M \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ and Supp$(M) = \{\mathfrak{m}\}$. Assume now that $\ell_A(M) > 1$. Let $M' \neq 0$ be a proper submodule of M. Then the exact sequence $0 \to M' \to M \to M/M' \to 0$ gives by Proposition 1.29 that Supp (M) = Supp$(M') \cup$ Supp(M/M'). Since by Proposition 1.30 $\ell_A(M')$ and $\ell_A(M/M')$ are both strictly less than $\ell_A(M)$, it follows by induction on $\ell_A(M)$ that any $\mathfrak{p} \in \operatorname{Supp} M$ is maximal.

Conversely, suppose that every $\mathfrak{p} \in \operatorname{Supp}(M)$ is maximal. If $M = 0$, there is nothing to prove. Let $M \neq 0$ be generated by $x_1, \ldots, x_n$. We use induction on n. If $n = 1$, then $M \simeq A/\mathfrak{a}$ for an ideal $\mathfrak{a}$ of A. Let $\mathfrak{a} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r$ be a reduced primary decomposition of $\mathfrak{a}$ with $\mathfrak{a}_i$ being $\mathfrak{p}_i$-primary. By hypothesis, all the $\mathfrak{p}_i$ are maximal. We have a monomorphism $A/\mathfrak{a} \to \bigoplus_i A/\mathfrak{a}_i$. It is therefore enough to show that each $A/\mathfrak{a}_i$ is of finite length. Since A is noetherian and since for any $x \in \mathfrak{p}_i$ some power of x is in $\mathfrak{a}_i$, we have $\mathfrak{p}_i^m \subset \mathfrak{a}_i$ for some $m \in \mathbf{N}$. We have an epimorphism $A/\mathfrak{p}_i^m \to A/\mathfrak{a}_i$. By Proposition 1.30, it is enough to prove that $\ell_A(A/\mathfrak{p}_i^m)$ is finite. We do this by induction on m. Since $\mathfrak{p}_i$ is maximal, $A/\mathfrak{p}_i$ is of finite length. In view of the exact sequence $0 \to \mathfrak{p}_i^{m-1}/\mathfrak{p}_i^m \to A/\mathfrak{p}_i^m \to A/\mathfrak{p}_i^{m-1} \to 0$ it is enough to show that $\mathfrak{p}_i^{m-1}/\mathfrak{p}_i^m$ is of finite length. Since $\mathfrak{p}_i$ is finitely generated, $\mathfrak{p}_i^{m-1}/\mathfrak{p}_i^m$ is finitely generated over $A/\mathfrak{p}_i$ and hence is of finite length as an $A/\mathfrak{p}_i$-module and therefore as an A-module.

Let $n > 1$. Let $M' = Ax_1$. Then we have the exact sequence $0 \to M' \to M \to M/M' \to 0$ and by Proposition 1.29, $\mathrm{Supp}\,(M) = \mathrm{Supp}\,(M') \cup \mathrm{Supp}(M/M')$. Since M' and M/M' are both generated by less than n elements, it follows by induction that both M' and M/M' are of finite length and therefore by Proposition 1.30 so is M.

Proposition 1.33 *Every artinian ring is of finite length.*

PROOF: Let A be an artinian ring, and let $\underline{r} = \underline{r}(A)$ be its Jacobson radical. We claim that $\underline{r}^n = 0$ for some $n \in \mathbf{N}$. Consider the descending sequence $\underline{r} \supset \underline{r}^2 \supset \underline{r}^3 \supset \cdots$. Since A is artinian, there exists $n \in \mathbf{N}$ such that $\mathfrak{a} = \underline{r}^n = \underline{r}^{n+1} = \cdots$. If $\mathfrak{a} \neq 0$, then the set of ideals $\mathfrak{b}$ such that $\mathfrak{ab} \neq 0$ is nonempty and has, by Proposition 1.31, a minimal element $\mathfrak{c}$. We claim that $\mathfrak{c}$ is principal. Let $x \in \mathfrak{c}$ with $\mathfrak{a}x \neq 0$. Then by the minimality of $\mathfrak{c}$ we have $\mathfrak{c} = Ax$. Also, we have $\mathfrak{a}(\mathfrak{ac}) = \mathfrak{a}^2\mathfrak{c} = \mathfrak{ac} \neq 0$. Again by the minimality of $\mathfrak{c}$, we have $\mathfrak{ac} = \mathfrak{c}$. By Nakayama's lemma we have $\mathfrak{c} = 0$ which is a contradiction. Thus $\underline{r}^n = 0$. Let $\mathcal{F}$ be the family of finite intersections of maximal ideals. By Proposition 1.31, $\mathcal{F}$ has a minimal element which is clearly $\underline{r}$. Let $\underline{r} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s$ with $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ maximal. We then have an A-monomorphism $A/\underline{r} \to \bigoplus_i A/\mathfrak{m}_i$. Since $A/\mathfrak{m}_i$ is simple for every i, it follows by Proposition 1.30, that $A/\underline{r}$ is of finite length. Since A is artinian, we have $\underline{r}^j$ is artinian for every integer $j \geq 0$ and hence $\underline{r}^j/\underline{r}^{j+1}$ is artinian as an $A/\underline{r}$-module. Since $A/\underline{r}$ is semi-simple, $\underline{r}^j/\underline{r}^{j+1}$ is a semi-simple artinian module and is hence of finite length. Considering the sequence $A \supset \underline{r} \supset \underline{r}^2 \supset \cdots \supset \underline{r}^n = 0$, we conclude that A is of finite length.

Corollary 1.34 *Every artinian ring is noetherian.*

PROOF: In fact, any strictly ascending sequence of ideals can be refined to a Jordan-Hölder series for the ring (as a module over itself).

Corollary 1.35 *Any finitely generated module over an artinian ring is of finite length.*

PROOF: Easy induction on the number of generators for the module.

1.6 Graded and filtered modules, Artin-Rees Theorem

Let A be a ring. A *gradation* on A is a decomposition $A = \bigoplus_{n\geq 0} A_n$ of A as a direct sum of subgroups A_n of A, where n runs over the set $\mathbf{Z}^+$ of all non-negative integers, such that $A_m A_n \subset A_{m+n}$ for all $m, n \in \mathbf{Z}^+$. A ring with a gradation is called a *graded ring.* Let $A = \bigoplus_n A_n$ be a graded ring. The non-zero elements of A_n are called *homogeneous* elements of A of *degree* n. We call A_n the n^{th} *homogeneous component of* A.

Proposition 1.36 *A_0 is a subring and $1 \in A_0$. Moreover, each A_n is an A_0-module and A is an A_0-algebra.*

Proof: Let $1 = e_0 + e_1 + \cdots + e_n$ where $e_i \in A_i$. For any $a \in A_j$, we have $a = a \cdot 1 = ae_0 + ae_1 + \cdots + ae_n$ with $ae_i \in A_{j+i}$. It follows that $a = ae_0$ and consequently $b = be_0$, for every $b \in A$. This proves that $1 = e_0 \in A_0$. The rest of the proposition is clear since $A_0 A_n \subset A_n$ for every $n \in \mathbb{Z}$.

Let A be as above and let $k \to A_0$ be a ring homomorphism. Then it is clear from the above proposition that A is a k-algebra and we refer to A as a *graded k-algebra.*

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded ring and let M be an A-module. An *A-gradation on M* is a decomposition $M = \bigoplus_{n\geq 0} M_n$ of M as a direct sum of subgroups M_n of M such that $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbf{Z}^+$. Such a module is called a *graded A-module.* We note that each M_n is an A_0-module.

Let $M = \bigoplus_{n\geq 0} M_n$ and $N = \bigoplus_{n\geq 0} N_n$ be two graded A-modules. A *homomorphism $f\colon M \to N$ of graded A-modules of degree r* is an A-homomorphism such that $f(M_n) \subset M_{n+r}$, for every $n \in \mathbf{Z}^+$. If $r = 0$, f is simply called a *homomorphism of graded modules.*

Let $A = \bigoplus_n A_n$ and $B = \bigoplus_n B_n$ be two graded rings. A ring homomorphism $f\colon A \to B$ is called a *homomorphism of graded rings* if $f(A_n) \subset B_n$, for every $n \in \mathbf{Z}^+$.

Let $A = \bigoplus_n A_n$ be a graded ring and $M = \bigoplus_n M_n$, a graded A-module. A submodule N of M is called a *graded submodule* if $N = \bigoplus_n (N \cap M_n)$. An ideal which is a graded submodule of A is called a homogeneous ideal of A. If N is a graded submodule of M, then M/N has an A-gradation induced from that of M, namely $M/N = \bigoplus_n (M_n + N)/N$. Let $f\colon M \to M'$ be a homomorphism of graded A-modules M, M'. Then ker f and im f are clearly graded submodules of

M and M' respectively.

Proposition 1.37 *(i) Let $A = \bigoplus_n A_n$ be a graded ring and $M = \bigoplus_n M_n$ a graded A-module. If M is noetherian, then each M_n is a finitely generated A_0-module.*

(ii) Assume that A is generated by A_1 as an A_0-algebra. Then A is noetherian if and only if A_0 is noetherian and $A - i$ is a finitely generated A_0-module.

Proof:

(i) Let $n \in \mathbf{Z}^+$ and let $N = \bigoplus_{m \geq n} M_m$. Then N is clearly a submodule of M. Since M is noetherian, N is finitely generated. Let $x_1, \ldots, x_r$ generate N. Write $x_i = y_i + z_i$ with $y_i \in M_n$ and $z_i \in \bigoplus_{m \geq n+1} M_m$, $1 \leq i \leq r$. We claim that M_n is generated over A_0 by $y_1, \ldots, y_r$. For, let $t \in M_n$. Then $t = \sum_{1 \leq i \leq r} a_i x_i$ with $a_i \in A$. Let us write $a_i = b_i + c_i$ with $b_i \in A_0$ and $c_i \in \bigoplus_{m \geq 1} A_m$, $1 \leq i \leq r$. Clearly $t = \sum_{1 \leq i \leq r} b_i y_i$.

(ii) Let $A_+ = \bigoplus_{m \geq 1} A_m$. Then A_+ is an ideal of A and $A_0 \simeq A/A_+$. Therefore if A is noetherian, A_0 is noetherian. Also, by taking $M = A$, it follows from (i) that each A_n (in particular, A_1) is finitely generated over A_0.

Conversely, suppose A_0 is noetherian and A_1 is a finitely generated A_0-module. Let $x_1 \ldots x_r$ generate A_1 over A_0. Since A_1 generates A as an A_0-algebra, we have $A = A_0[x_1, \ldots, x_r]$ and it follows, by Corollary to Theorem 1.10, that A is noetherian.

This completes the proof of the proposition.

Let A be a ring. A *filtration* on A is sequence $A = A_0 \supset A_1 \supset A_1 \supset \cdots \supset A_n \supset \cdots$ of ideals of A such that $A_m A_n \subset A_{m+n}$ for all $m, n \in \mathbf{Z}^+$. A ring with a filtration is called a *filtered ring.* Let A be a filtered ring. A *filtration* on an A-module M is a sequence $M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$ of submodules M_n such that $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbf{Z}^+$. An A-module with a filtration is called a *filtered module.*

Let A be a filtered ring and $\mathfrak{a}$ an ideal of A. Let M be a filtered A-module with filtration $M = M_0 \supset M_1 \supset \cdots$. We say that this filtration if *compatible with* $\mathfrak{a}$ if $\mathfrak{a}M_n \subset M_{n+1}$ for every $n \geq 0$. We say that the filtration is $\mathfrak{a}$-good if it is compatible with $\mathfrak{a}$ and for $n \gg 1$ (i.e. for n sufficiently large), $\mathfrak{a}M_n = M_{n+1}$.

Let A be a ring and M an A-module. Let $\mathfrak{a}$ be an ideal of A. Then $\mathfrak{a}$ defines a filtration $A = \mathfrak{a}^0 \supset \mathfrak{a} \supset \mathfrak{a}^2 \supset \cdots$ on A and a filtration $M = \mathfrak{a}^0 M \supset \mathfrak{a}M \supset \mathfrak{a}^2 M \supset \cdots$ on M, called the $\mathfrak{a}$-adic *filtrations*. Clearly an $\mathfrak{a}$-adic filtration is $\mathfrak{a}$-good.

Let A be a ring and $\mathfrak{a}$ an ideal of A. Let M be an A-module with a filtration $M = M_0 \supset M_1 \supset \cdots$ compatible with $\mathfrak{a}$. Consider the direct sum $\bar{A} = A \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots$. We make $\bar{A}$ into a graded ring under the multiplication induced by that in A. Let $\bar{M} = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$. The A-module structure on M induces the structure of a graded $\bar{A}$-module on $\bar{M}$.

Lemma 1.38 *Let $\mathfrak{a}$ be an ideal of A and let M be a noetherian A-module with a filtration $M = M_0 \supset M_1 \supset \cdots$ compatible with $\mathfrak{a}$. Then $\bar{M}$ is finitely generated as an $\bar{A}$-module if and only if the filtration is $\mathfrak{a}$-good.*

PROOF: Let $\bar{M}$ be finitely generated over $\bar{A}$. Then there exists $n \in \mathbf{Z}^+$ such that $M_0 \oplus \cdots \oplus M_n$ generates $\bar{M}$ over $\bar{A}$. We claim that $\mathfrak{a}M_m = M_{m+1}$, for $m \geq n$. Let $x \in M_{m+1}$. Then we can write $x = \sum_i a_i x_i$ where $x_i \in M$ are homogeneous of degree $d_i \leq n$ and $a_i \in \mathfrak{a}^{m+1-d_i}$ for every i. We can write $a_i = \sum_j b_{i,j} c_{i,j}$ with $b_{i,j} \in \mathfrak{a}$ and $c_{i,j} \in \mathfrak{a}^{m-d_i}$. Thus $x = \sum_{i,j} b_{i,j}(c_{i,j} x_i) \in \mathfrak{a}M_m$. It follows that the filtration is $\mathfrak{a}$-good. Conversely, let the filtration be $\mathfrak{a}$-good. Then there exists $n \in \mathbf{Z}^+$ such that for $m \geq n$, we have $\mathfrak{a}M_m = M_{m+1}$. Since M is noetherian, the A-module $M_0 \oplus M_1 \oplus + \cdots \oplus M_n$ is finitely generated. Since $\mathfrak{a}M_m = M_{m+1}$ for $m \geq n$, $M_0 \oplus M_1 \oplus \cdots M_n$ generates $\bar{M}$ as an $\bar{A}$-module. It follows that $\bar{M}$ is finitely generated over $\bar{A}$ and the lemma is proved.

Let M be an A-module with a filtration $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ and let N be a submodule of M. Then $N_n = N \cap M_n (n \geq 0)$ defines a filtration on N, called the *induced filtration* on N.

Theorem 1.39 *(Artin-Rees). Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module and N submodule of M. Then, for any $\mathfrak{a}$-good filtration on M, the induced filtration on N is $\mathfrak{a}$-good.*

PROOF: Let $M = M_0 \supset M_1 \supset \cdots$ be an $\mathfrak{a}$-good filtration on M. Let $\bar{A} = A \oplus \mathfrak{a} \oplus \mathfrak{a}^2 \oplus \cdots$, $\bar{M} = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$ and $\bar{N} = N \oplus (M_1 \cap N) \oplus (M_2 \cap N) \oplus \cdots$. Since the filtration on M is $\mathfrak{a}$-good, it follows from Lemma 1.38, that $\bar{M}$ is a finitely generated $\bar{A}$-module. Since A is noetherian, $\mathfrak{a}$ is finitely generated and hence, by Proposition 1.37,

$\bar{A}$ is noetherian. Therefore, by Proposition 1.7, $\bar{M}$ is a noetherian $\bar{A}$-module so that $\bar{N}$ is finitely generated over $\bar{A}$. By Lemma 1.38 again, the induced filtration on N is $\mathfrak{a}$-good.

Corollary 1.40 *Let A be a noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module and N, a submodule of M. Then there exists $n_0 \in \mathbf{Z}^+$ such that for every $n \geq n_0$,we have $\mathfrak{a}(\mathfrak{a}^n M \cap N) = \mathfrak{a}^{n+1} M \cap N$.*

Proof: Apply the theorem to the $\mathfrak{a}$-adic filtration on M.

Corollary 1.41 *Let A be a noetherian ring and $\underline{r}$ its Jacobson radical. Then $\bigcap_{n\geq 0} \underline{r}^n = 0$.*

Proof: Let $N = \bigcap_{n\geq 0} \underline{r}^n = 0$. Applying Corollary 1.41 to $M = A$ and $\mathfrak{a} = \underline{r}$, we get $\underline{r}N = N$. Hence, by Nakayama's lemma, we get $N = 0$ and the corollary is proved.

Let A be a filtered ring with a filtration $A = A_0 \supset A_1 \supset A_2 \supset \cdots$ and let M be a filtered A-module with a filtration $M = M_0 \supset M_1 \supset M_2 \supset \cdots$. Consider the direct sum $G(A) = \bigoplus_{n\geq 0} A_n/A_{n+1}$ of abelian groups A_n/A_{n+1}. We make $G(A)$ a graded ring by defining a multiplication as follows. Let $\bar{a} \in A_n/A_{n+1},\ \bar{b} \in A_m/A_{m+1}$ be homogeneous elements of degrees n and m respectively and let $a \in A_n,\ b \in A_m$ be such that $\bar{a}, \bar{b}$ are the images of a, b under the natural maps $A_n \to A_n/A_{n+1},\ A_m \to A_m/A_{m+1}$ respectively. then $ab \in A_{n+m}$ and we define $\bar{a}\bar{b}$ to be the image of ab under the natural map $A_{n+m} \to A_{n+m}/A_{n+m+1}$. This is clearly well-defined and extends to a multiplication in $G(A)$ making it a graded ring. In a similar manner, we make $G(M) = \bigoplus_{n\geq 0} M_n/M_{n+1}$ a graded $G(A)$-module. We say that $G(A)$ is the *graded ring associated to the filtration* $A = A_0 \supset A_1 \supset \cdots$ and $G(M)$ is *the graded module associated the filtration* $M = M_0 \supset M_1 \supset \cdots$.

Let A be a ring, $\mathfrak{a}$ an ideal of A and M an A-module. The graded ring and the graded module associated to the $\mathfrak{a}$-adic filtrations on A and M are denoted by $G_{\mathfrak{a}}(A)$ and $G_{\mathfrak{a}}(M)$ respectively.

Lemma 1.42 *Let A be a noetherian ring and $\mathfrak{a}$ an ideal of A contained in $\underline{r}(A)$. If $G_{\mathfrak{a}}(A)$ is an integral domain, then so is A.*

Proof: Let $a, b \in A,\ a \neq 0,\ b \neq 0$. Since, by Corollary 1.41, we have $\bigcap_{n\geq 0} \mathfrak{a}^n = 0$, there exists $m, n \in \mathbf{Z}^+$ such that $a \in \mathfrak{a}^m,\ a \notin \mathfrak{a}^{m+1}$ and $b \in \mathfrak{a}^n,\ b \notin \mathfrak{a}^{n+1}$. Let $\bar{a}, \bar{b}$ be the images of a, b in $\mathfrak{a}^m/\mathfrak{a}^{m+1},\ \mathfrak{a}^n/\mathfrak{a}^{n+1}$ respectively. Then $\bar{a} \neq 0,\ \bar{b} \neq 0$. Since $G_{\mathfrak{a}}(A)$ is an integral domain, we have $\bar{a}\bar{b} \neq 0$ and *a fortiori*, $ab \neq 0$.

Chapter 2

Some Results from Homological Algebra

In this chapter, A denotes a commutative ring with 1 and by a module we mean a unitary module.

2.1 Complexes and homology

By a *complex* $\underline{X}$ of A-modules, we mean a sequence

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

of A-modules X_n and A-homomorphisms d_n such that $d_n \circ d_{n+1} = 0$ for every $n \in \mathbf{Z}$. We say that $\underline{X}$ is a left (resp. *right*) *complex if* $X_n = 0$ for $n < 0$ (resp.$n > 0$). The condition $d_n \circ d_{n+1} = 0$ implies that $\operatorname{Im} d_{n+1} \subset \ker d_n$. We define the n^{th} *homology module* $\underline{X}$ to be $\ker d_n / \operatorname{im} d_{n+1}$ and denote it by $H_n(\underline{X})$. We sometimes write X^n for X_{-n} and H^n for H_{-n}. If $\underline{X}$ is a right complex, we usually denote it by $0 \to X^0 \to X^1 \to \cdots$

Let $\underline{X}, \underline{Y}$ be complexes of A-modules. A *morphism* $f \colon \underline{X} \to \underline{Y}$ *of complexes* is a family $\{f_n \colon X_n \to Y_n\}_{n \in \mathbf{Z}}$ of A-homomorphisms such that, for every $n \in \mathbf{Z}$, the diagram

$$\begin{tikzcd}
X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d_{n+1}"] & Y_{n+1} \arrow[d, "d'_{n+1}"] \\
X_n \arrow[r, "f_n"] & Y_n
\end{tikzcd}$$

is commutative.

Let $f\colon \underline{X} \to \underline{Y}$ be a morphism of complexes. Since the diagram

$$\begin{tikzcd}
X_n \arrow[r, "f_n"'] \arrow[d, "d_n"] & Y_n \arrow[d, "d'_n"] \\
X_{n-1} \arrow[r, "f_{n-1}"] & Y_{n-1}
\end{tikzcd}$$

is commutative, we have $f_n(\ker d_n) \subset \ker d'_n$. Similarly, we have $f_n(\operatorname{Im} d_{n+1}) \subset \operatorname{Im} d'_{n+1}$. Thus, f_n induces an A-homomorphism $H_n(f)\colon H_n(\underline{X}) \to H_n(\underline{Y})$.

If $g\colon \underline{Y} \to \underline{Z}$ is another morphism of complexes then the morphism $gf\colon \underline{X} \to \underline{Z}$ of complexes is defined in an obvious way, and we have $H_n(gf) = H_n(g)H_n(f)$, for every $n \in \mathbf{Z}$. It is also clear that $H_n(1_{\underline{X}}) = 1_{H_n(\underline{X})}$. We will denote by 0 the complex X with $X_n = 0$ for every $n \in \mathbf{Z}$. A sequence

$$0 \to \underline{X} \overset{f}{\to} \underline{Y} \overset{g}{\to} \underline{Z} \to 0$$

of complexes is said to be *exact* if, for every $n \in \mathbf{Z}$ the sequence

$$0 \to X_n \overset{f_n}{\to} Y_n \overset{g_n}{\to} Z \to 0$$

is exact.

Let $0 \to \underline{X} \overset{f}{\to} \underline{Y} \overset{g}{\to} \underline{Z} \to 0$ be an exact sequence of complexes of A-modules. For $n \in \mathbf{Z}$ we have the commutative diagram

$$\begin{tikzcd}
 & & Y_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d'_{n+1}"] & Z_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "d_n"] & Y_n \arrow[r, "g_n"] \arrow[d, "d'_n"] & Z_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d_{n-1}"] & Y_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d'_{n-1}"] & Z_{n-1} \arrow[r] & 0 \\
0 \arrow[r] & X_{n-2} \arrow[r, "f_{n-2}"] & Y_{n-2} & &
\end{tikzcd}$$

Let $\bar{z} \in H_n(\underline{Z})$, and let $z \in \ker d''_n$ represent $\bar{z}$. Choose $y \in Y_n$ such that $g_n(y) = z$. Let $y' = d'_n(y)$. Then $g_{n-1}(y') = 0$. Therefore, there exists $x \in X_{n-1}$ such that $y' = f_{n-1}(x)$. Since $(d'_{n-1}f_{n-1})(x) = (d'_{n-1}d'_n)(y) = 0$, it follows that $(f_{n-2}d_{n-1})(x) = 0$ i.e. $d_{n-1}(x) \in \ker f_{n-2} = 0$ i.e. $x \in \ker d_{n-1}$. Let $\bar{x}$ be the canonical image of x in $H_{n-1}(\underline{X})$. It is easily seen that $\bar{x}$ does not depend on the choice of z and y. We define $\partial_n\colon H_n(\underline{Z}) \to H_{n-1}(\underline{X})$ by $\partial_n(\bar{z}) = \bar{x}$. Clearly, ∂_n is an A-homomorphism.

Proposition 2.1 *Let $0 \to \underline{X} \overset{f}{\to} \underline{Y} \overset{g}{\to} \underline{Z} \to 0$ be an exact sequence of complexes of A-modules. Then the sequence*

$$\cdots \to H_n\underline{X} \overset{H_n(f)}{\longrightarrow} H_n\underline{Y} \overset{H_n(g)}{\longrightarrow} H_n\underline{Z} \overset{\partial_n}{\longrightarrow} H_{n-1}\underline{X} \overset{H_{n-1}(f)}{\longrightarrow} H_{n-1}(\underline{Y})$$

is exact.

Proof: If $\bar{z} \in \operatorname{Im} H_n(g)$, then, with the notation immediately preceding the proposition, we can choose $y \in \ker d'_n$ so that $y' = 0$ and hence $\partial_n(\bar{z}) = \bar{x} = 0$. Conversely, suppose $\bar{x} = \partial_n(\bar{z}) = 0$. Then there exists $x' \in X_n$ such that $x = d_n(x')$. Let $y'' = y - f_n(x')$. Then $y'' \in \ker d'_n$ and $g_n(y'') = z$. It follows that $\bar{z} \in \operatorname{Im} H_n(g)$. This proves exactness of the sequence at $H_n(\underline{Z})$.

With the same notation, we have $H_{n-1}(f)\partial_n(\bar{z}) = H_{n-1}(f)(\bar{x})$ is the canonical image of $y' = f(x)$ in $H_{n-1}(Y)$, which is zero, since $y' \in \operatorname{Im} d'_n$.

Conversely, let $x_1 \in H_{n-1}(\underline{X})$ be such that $H_{n-1}(f)(x_1) = 0$. This means that $f_{n-1}(x_1) \in \operatorname{Im} d'_n$. Let $y_1 \in Y_n$ be such that $f_{n-1}(x_1) = d'_n(y_1)$. Let $\bar{z}_1$ be the class of $g_n(y_1)$ in $H_n(\underline{Z})$. Then $\partial_n(\bar{z}_1) = \bar{x}_1$. This proves the exactness of the sequence at $H_{n-1}(\underline{X})$.

Finally, $H_n(g)H_n(f) = H_n(gf) = H_n(0) = 0$, clearly. Let $\bar{y} \in H_n(\underline{Y})$ be such that $H_n(g)(\bar{y}) = 0$. Let $y \in \ker d'_n$ be a representative of $\bar{y}$. Then there exists $z \in Z_{n+1}$ such that $g_n(y) = d''_{n+1}(z)$. Choose $y' \in Y_{n+1}$ such that $z = g_{n+1}(y')$. Then $g_n(y - d'_{n+1}(y')) = 0$. Therefore, there exists $x \in X_n$ such that $y - d'_{n+1}(y') = f_n(x)$. Clearly, the class of $f_n(x)$ in $H_n(\underline{Y})$ is the same as that of y. To complete the proof, it is enough to prove that $x \in \ker d_n$. But $d'_n(f_n(x)) = d'_n(y - d'_{n+1}(y)) = 0 \Rightarrow f_{n-1}(d_n(x)) = 0 \Rightarrow d_n(x) = 0$. This proves the exactness at $H_n(\underline{Y})$ and the proposition is proved.

The homomorphisms $\{\partial_n\}_{n\in\mathbf{Z}}$ defined above are called *connecting homomorphisms.*

Proposition 2.2 *Let*

$$\begin{tikzcd}
0 \arrow[r] & \underline{X} \arrow[r, "f"] \arrow[d, "\alpha"] & \underline{Y} \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \underline{X'} \arrow[r, "f'"] & \underline{Y'} \arrow[r, "g'"] & \underline{Z'} \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of complexes with rows exact. Then the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(\underline{X}) \arrow[r] \arrow[d, "H_n(\alpha)"] & H_n(\underline{Y}) \arrow[r] \arrow[d, "H_n(\beta)"] & H_n(\underline{Z}) \arrow[r, "\partial_n"] \arrow[d, "H_n(\gamma)"] & H_{n-1}(\underline{X}) \arrow[r] \arrow[d, "H_{n-1}(\alpha)"] & H_{n-1}(\underline{Y}) \arrow[r] \arrow[d, "H_{n-1}(\beta)"] & \cdots \\
\cdots \arrow[r] & H_n(\underline{X'}) \arrow[r] & H_n(\underline{Y'}) \arrow[r] & H_n(\underline{Z'}) \arrow[r, "\partial'_n"] & H_{n-1}(\underline{X'}) \arrow[r] & H_{n-1}(\underline{Y'}) \arrow[r] & \cdots
\end{tikzcd}$$

of homology modules is commutative.

PROOF: We need only prove that $H_{n-1}(\alpha)\partial_n = \partial'_n H_n(\gamma)$ for every $n \in \mathbf{Z}$. Let $\bar{z} \in H_n(\underline{Z})$ with z as a representative. Let $y \in Y_n$ be such that $g_n(y) = z$. Then $\gamma_n(z)$ is a representative of $H_n(\gamma)(\bar{z})$ and $\beta_n(y)$ is a lift of $\gamma_n(z)$ in Y'_n. Using the lift y to compute $\partial_n(\bar{z})$ and the lift $\beta_n(y)$ to compute $\partial'_n(H_n(\gamma)(\bar{z}))$, it is clear that $(H_{n-1}(\alpha)\partial_n)(\bar{z}) = (\partial'_n H_n(\gamma))(\bar{z})$. Since $\bar{z} \in H_n(\underline{z})$ is arbitrary, the proposition follows.

Let $f, g: \underline{X} \to \underline{Y}$ be two morphisms of complexes X, Y of A-modules. A *homotopy h between f and g* is a family $h = \{h_n\}_{n\in\mathbf{Z}}$ of A-homomorphisms $h_n: X_n \to Y_{n+1}$ such that $h_{n-1}d_n + d'_{n+1}h_n = f_n - g_n$ for every $n \in \mathbf{Z}$. We then say that f and g are homotopic. Clearly homotopy is an equivalence relation.

Proposition 2.3 *Let $f, g: X \to Y$ be homotopic morphisms of complexes of A-modules. Then $H_n(f) = H_n(g)$, for every $n \in \mathbf{Z}$.*

PROOF: For $\bar{x} \in H_n(X)$, let $x \in \ker d_n$ be a representative. We then have $f_n(x) = g_n(x) = h_{n-1}d_n(x) + d'_{n+1}h_n(x) = d'_{n+1}h_n(x) \in \operatorname{Im} d'_{n+1}$. Thus $H_n(f)(\bar{x}) = H_n(g)(\bar{x})$.

2.2 Projective modules

Proposition 2.4 *For an A-module P, the following conditions are equivalent:*

(i) P is a direct summand of a free A-module:

(ii) given any diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "\varphi"] & M'' \arrow[r] & 0 \end{tikzcd}$$

of A-homomorphisms with exact row, there exists an A-homomorphism $\bar{f}: P \to M$ such that $\varphi\bar{f} = f$;

(iii) every exact sequence $0 \to M' \to M \overset{\varphi}{\to} P \to 0$ of A-modules splits.

PROOF: *(i)* $\Rightarrow$ *(ii)*. Let Q be an A-module such that $P \oplus Q = F$ is free with basis $(e_i)_{i\in I}$. Define $g: F \to M''$ by $g|P = f$ and $g \mid Q = 0$. Let $x_i \in M$ be such that $\varphi(x_i) = g(e_i)$. Define $\bar{g}: F \to M$ by $\bar{g}(e_i) = x_i$. Then, it is clear that $\bar{f} = \bar{g}|P$ satisfies $\varphi\bar{f} = f$.

$(ii) \Rightarrow (iii)$. In view of the diagram

$$\begin{tikzcd} & P \arrow[d, "1_P"] & \\ M \arrow[r, "\varphi"] & P \arrow[r] & 0 \end{tikzcd}$$

we have an A-homomorphism $f\colon P \to M$ such that $\varphi f = 1_P$,i.e. the sequence $0 \to M' \to M \to P \to 0$ splits.

$(iii) \Rightarrow (i)$. Let $F \to P \to 0$ be an exact sequence with F an A-free module. (For instance, take F to be the free A-module on a set of generators of P.) If $K = \ker(F \to P)$ then the exact sequence $0 \to K \to F \to P \to 0$ splits by (iii) and P is a direct summand of F.

An A-module P which satisfies any of the equivalent conditions of Proposition 2.4 is called a *projective* A-module.

Corollary 2.5 *A free A-module is projective. Direct sums and direct summands of projective modules are projective.*

Corollary 2.6 *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of A-modules. If P is projective, then the sequence*

$$0 \to P \otimes_A N' \to P \otimes_A N \to P \otimes_A N'' \to 0$$

is exact.

PROOF: Let Q be an A-modules such that $F = P \oplus Q$ is free. Since direct sum commutes with tensor products, it is clear that

$$0 \to F \otimes_A N' \to F \otimes_A N \to F \otimes_A N'' \to 0$$

is exact i.e.

$$0 \to (P\otimes_A N')\oplus(Q\otimes_A N') \to (P\otimes_A N)\oplus(Q\otimes_A N) \to (P\otimes_A N'')\oplus(Q\otimes_A N'') \to 0$$

is exact, and the result follows.

Corollary 2.7 *If P is a projective A-module and if $A \to B$ is a ring homomorphism, then $B \otimes_A P$ is a projective B-module.*

PROOF: Let P be a direct summand of a free A-module F. Then, $B \otimes_A P$, being a direct summand of the free B-module $B \otimes_A F$ is a projective B-module.

2.3 Projective resolutions

Let M be an A-module. A *projective resolution* of M is a pair $(\underline{P}, \varepsilon)$, where P is a left complex with each P_i a projective A-module and where $\varepsilon\colon P_0 \to M$ is an A-homomorphism such that the sequence

$$\cdots \to P_n \overset{d_n}{\to} P_{n-1} \to \cdots \to P_0 \overset{\varepsilon}{\to} M \to 0$$

is exact.

Proposition 2.8 *Every A-module M admits of a projective resolution.*

Proof: Let $P_0 \to M \to 0$ be an exact sequence of A-modules with P_0 a free A-module. We define P_i and d_i inductively as follows. Suppose we already have an exact sequence

$$P_n \overset{d_n}{\to} P_{n-1} \to \cdots \to P_0 \to M \to 0$$

with each P_i being A-free. Let $K_n = \ker d_n$. There exists an exact sequence of A-modules $P_{n+1} \overset{\varphi_{n+1}}{\longrightarrow} K_n \to 0$ with P_{n+1} being A-free. Define $d_{n+1} = j\varphi_{n+1}$, where $j\colon K_n \to P_n$ is the canonical inclusion. It is trivially seen that

$$P_{n+1} \overset{d_{n+1}}{\longrightarrow} P_n \to \cdots \to P_0 \to M \to 0$$

is exact, and this proves the proposition.

Corollary 2.9 *Every A-module admits of a projective resolution $(\underline{P}, \varepsilon)$ with P_i being A-free.*

Corollary 2.10 *Let A be an noetherian ring and M a finitely generated A-module. Then M admits of a projective resolution $(\underline{P}, \epsilon)$ with each P_i a finitely generated free A-module.*

Proof: Let us use the notation of Proposition 2.8. Since M is finitely generated, we can choose P_0 to be a finitely generated free A-module. Then $K_0 = \ker \varepsilon$ is finitely generated by Proposition 1.7. We therefore choose inductively each P_n to be finitely generated, and the corollary follows.

Let $f\colon M \to M'$ be a homomorphism of A-modules and let $(\underline{P}, \varepsilon)$, $(\underline{P}', \varepsilon')$ be projective resolutions of M, M' respectively. A morphism $F\colon \underline{P} \to \underline{P}'$ of complexes is said to be *over* f if $f\varepsilon = \varepsilon' F_0$.

Proposition 2.11 *Let $f: M \to M'$ be a homomorphism of A-modules and let $(\underline{P}, \varepsilon), (\underline{P}', \varepsilon')$ be projective resolutions of M, M' respectively. Then there exists a morphism $F: \underline{P} \to \underline{P}'$ over f. Moreover, if $F, G: \underline{P} \to \underline{P}'$ are morphisms over f then F and G are homotopic.*

PROOF: *Existence of F.* Consider the diagram

$$\begin{tikzcd} & P_0 \arrow[d, "f\varepsilon"] & \\ P_0' \arrow[r, "\varepsilon'"] & M' \arrow[r] & 0 \end{tikzcd}$$

where the row is exact. Since P_0 is projective, there exists an A-homomorphism $F_0: P_0 \to P_0'$ such that $f\varepsilon = \varepsilon' F_0$. We now define F_n by induction on n, assuming F_m to be defined for $m < n$. We have $d_{n-1}' \circ F_{n-1} \circ d_n = F_{n-2} \circ d_{n-1} \circ d_n = 0$, where we set $F_{-1} = f$, $d_0 = \varepsilon$ and $d_0' = \varepsilon'$. Therefore $\operatorname{Im}(F_{n-1}d_n) \subset \ker d_{n-1}' = \operatorname{Im} d_n'$. Thus we have the diagram

$$\begin{tikzcd} & P_n \arrow[d, "F_{n-1}d_n"] & \\ P_n' \arrow[r, "d_n'"] & \operatorname{Im} d_n' \arrow[r] & 0 \end{tikzcd}$$

where the row is exact. Since P_n is projective, there exists $F_n: P_n \to P_n'$ such that $d_n' F_n = F_{n-1} d_n$. This proves the existence of F.

Homotopy between F and G. Since $\varepsilon' F_0 = f\varepsilon = \varepsilon' G_0$, we have $\varepsilon'(F_0 - G_0) = 0$, i.e. $\operatorname{Im}(F_0 - G_0) \subset \ker \varepsilon' = \operatorname{Im} d_1'$. Thus we get a diagram

$$\begin{tikzcd} & P_0 \arrow[d, "F_0 - G_0"] & \\ P_1' \arrow[r, "d_1'"] & \operatorname{Im} d_1' \arrow[r] & 0 \end{tikzcd}$$

Since P_0 is projective, there exists an A-homomorphism $h_0: P_0 \to P_1'$ such that $d_1' h_0 = F_0 - G_0$.

Assuming inductively that, for $m < n$, $h_m\colon P_m \to P'_{m+1}$ has been defined such that $d'_{m+1}h_m + h_{m-1}d_m = F_m - G_m$, we define $h_n\colon P_n \to P'_{n+1}$ as follows: since $d'_n(F_n - G_n) = (F_{n-1} - G_{n-1})d_n = (d'_n h_{n-1} + h_{n-2}d_{n-1})d_n = d'_n h_{n-1}d_n$, we have $d'_n(F_n - G_n - h_{n-1}d_n) = 0$, i.e. $\operatorname{Im}(F_n - G_n - h_{n-1}d_n) \subset \ker d'_n = \operatorname{Im} d'_{n+1}$. Thus we get a diagram

$$\begin{tikzcd} & P_n \arrow[d, "F_n - G_n - h_{n-1}d_n"] & \\ P'_{n+1} \arrow[r, "d'_{n+1}"] & \operatorname{Im} d'_{n+1} \arrow[r] & 0 \end{tikzcd}$$

with exact row. Since P_n is projective, there exists a homomorphism $h_n\colon P_n \to P'_{n+1}$ such that $d'_{n+1}h_n = F_n - G_n - h_{n-1}d_n$. Thus $h = \{h_n\}$ is a homotopy between F and G.

Proposition 2.12 *Let* $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ *be an exact sequence of A-modules. Let* $(\underline{P}', \varepsilon'), (\underline{P}'', \varepsilon'')$ *be projective resolutions of* M', M'' *respectively. Then there exists a projective resolution* $(\underline{P}, \varepsilon)$ *of M such that we have an exact sequence*

$$0 \to \underline{P}' \xrightarrow{f} \underline{P} \xrightarrow{g} \underline{P}'' \to 0$$

of complexes and such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & P'_0 \arrow[r, "f_0"] \arrow[d, "\varepsilon'"] & P_0 \arrow[r, "g_0"] \arrow[d, "\varepsilon"] & P''_0 \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "i"] & M \arrow[r, "j"] & M'' \arrow[r] & 0 \end{tikzcd} \qquad (*)$$

is commutative.

Proof: For $n \in \mathbf{Z}^+$, we define $P_n = P'_n \oplus P''_n$ and $f_n : P'_n \to P_n$, $g_n : P_n \to P''_n$ by $f_n(x') = (x', 0), g_n(x', x'') = x''$, respectively. Assume, for the moment, that there exist A-homomorphisms $1\colon P''_0 \to M$ and for every $n > 0$, $k_n\colon P''_n \to P'_{n-1}$ satisfying the following conditions:

$$\varepsilon'' = j1,$$

$$i\varepsilon' k_1 + l d''_1 = 0 \qquad (**)$$

$$d'_{n-1}k_n + k_{n-1}d''_n = 0, \quad \text{for } n > 1.$$

Now define, $d_n\colon P_n \to P_{n-1}$, for $n > 0$ and $\varepsilon\colon P_0 \to M$ by

$$d_n(x', x'') = (d'_n x' + k_n x'',\ d''_n x'') \tag{$***$}$$

$$\varepsilon(x', x'') = i\varepsilon' x' + l x''.$$

It is then easily verified that $(\underline{P}, \varepsilon)$ is a projective resolution of M and that the diagram $(*)$ is commutative.

We now prove the existence of l and k_n. The existence of l is trivial, since P''_0 is projective and j is an epimorphism. We construct k_n by induction on n. Consider the diagram

$$\begin{tikzcd}
 & P''_1 \arrow[d, "-ld''_1"] & & \\
P'_0 \arrow[r, "i\varepsilon'"] & M \arrow[r, "j"] & M'' \arrow[r] & 0
\end{tikzcd}$$

Since $j(-ld''_1) = -\varepsilon'' d''_1 = 0$, we have $\mathrm{Im}\,(-1d''_1) \subset \ker j = \mathrm{Im}\,(i\varepsilon')$. Since P''_1 is projective, the existence of $k_1\colon P''_1 \to P'_0$ is proved. Assume, by induction, that $k_{n-1}\colon P''_{n-1} \to P'_{n-2}$ has been constructed. Consider the diagram

$$\begin{tikzcd}
 & P''_n \arrow[d, "-k_{n-1}d''_n"] & \\
P'_{n-1} \arrow[r, "d'_{n-1}"] & P'_{n-2} \arrow[r, "d'_{n-2}"] & P'_{n-3}
\end{tikzcd}$$

where we set $d'_0 = i\varepsilon'$ and $P'_{-1} = M$. Now $d'_{n-2}(-k_{n-1}d''_n) = k_{n-2}d''_{n-1}d''_n = 0$, where $k_0 = 1$. Therefore, $\mathrm{Im}\,(-k_{n-1}d''_n) \subset \ker d'_{n-2} = \mathrm{Im}\, d'_{n-1}$. Since P''_n is projective, the existence of $k_n\colon P''_n \to P'_{n-1}$ is proved. This completes the proof of the proposition.

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. A *projective resolution* of this sequence is an exact sequence $0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$ where $(\underline{P}', \varepsilon')$, $(\underline{P}, \varepsilon)$, $(\underline{P}'', \varepsilon'')$ are projective resolutions of M', M, M'' respectively, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & P_0' \arrow[r] \arrow[d, "\varepsilon'"] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & P_0'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

is commutative.

The proposition above shows that every exact sequence admits of a projective resolution.

Proposition 2.13 *Let*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i_1"] \arrow[d, "f'"] & M \arrow[r, "p_1"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "i_2"] & N \arrow[r, "p_2"] & N'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of A-modules with exact rows. Let $0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$, $0 \to \underline{Q}' \to \underline{Q} \to \underline{Q}'' \to 0$ be projective resolutions of $0 \to M' \to M \to M'' \to 0$, $0 \to N' \to N \to N'' \to 0$. respectively and let F': $\underline{P}' \to \underline{Q}'$, F'': $\underline{P}'' \to \underline{Q}''$ be morphisms over f', f'',respectively. Then there exists a morphism F: $\underline{P} \to Q$ over f such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}$$

is commutative.

PROOF: Since P_n'' is projective, we can assume that, for $n \in \mathbf{Z}$, $P_n = P_n' \oplus P_n''$ and that the maps $P_n' \to P_n$, $P_n \to P_n''$ are respectively the natural inclusion and the natural epimorphism. We can make similar assumptions for the Q's. Then it is easy to see that d_n: $P_n \to P_{n-1}$ and

$\varepsilon\colon P_0 \to M$ are given by formulæ $(***)$ of the proof of Proposition 2.12 with conditions $(**)$ fulfilled; similarly for the Q's.

It may be checked inductively that there exist A-homomorphisms $c_n\colon P''_n \to Q'_n$ satisfying, for $n > 0$ the conditions

$$i_2\varepsilon' c_0 + l^{\underline{Q}} F''_0 = f l^{\underline{P}}$$

$$d'_n c_n - c_{n-1} d''_n = F'_{n-1} k_n^{\underline{P}} - k_n^{\underline{Q}} F''_n.$$

Now define $F\colon \underline{P} \to \underline{Q}$ by

$$F_n(x', x'') = (F'_n(x') + c_n(x''),\ F''_n(x''))$$

It is easy to verify that this is the required F.

2.4 The functors Tor

Let M be an A-module and let $\underline{P} = (\underline{P}, \varepsilon)$ be a projective resolution of M. Then, for any A-module N, we denote by $\underline{P} \otimes_A N$ the left complex

$$\cdots \to P_N \otimes_A N \xrightarrow{d_n \otimes 1_N} P_{n-1} \otimes_A N \to \cdots \to P_0 \otimes_A N \to 0.$$

We denote the homology modules $H_n(\underline{P} \otimes_A N)$ of this complex by $H_n(M, N; P)$.

Let $f\colon M \to M'$ be a homomorphism of A-modules and let $\underline{P}$, $\underline{P}'$ be projective resolutions of M, M' respectively. Let $F\colon \underline{P} \to \underline{P}'$ be a morphism over f. Note that such an F exists, by Proposition 2.11. This morphism F defines, for every $n \in \mathbf{Z}^+$, an A-homomorphism

$$H_n(f, N; \underline{P},\ \underline{P}')\colon H_n(M, N; \underline{P}) \to H_n(M', N; \underline{P}').$$

In view of Proposition 2.3, the homomorphism $H_n(f, N; \underline{P}, \underline{P}')$ does not depend on the choice of the morphism over f. For, by Proposition 2.11, by two morphisms $\underline{P} \to \underline{P}'$ over f are homotopic.

Let now $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ be an exact sequence of A-modules and let

$$0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0 \qquad (*)$$

be a projective resolution of this exact sequence. Let, for $n \geq 1$,

$$\partial_n(N, (*))\colon H_n(M'', N; \underline{P}'') \to H_{n-1}(M', N; \underline{P}')$$

be the connecting homomorphisms defined by $(*)$. We then have the following

Lemma 2.14 *(i) Let $f\colon M \to M'$, $g\colon M' \to M''$ be homomorphisms of A-modules and let $\underline{P}, \underline{P}', \underline{P}''$ be projective resolutions of M, M', M'' respectively. Then, for every $n \in \mathbf{Z}^+$ we have*

$$H_n(gf, N; \underline{P}, \underline{P}'') = H_n(g, N; \underline{P}', \underline{P}'')H_n(f, N; \underline{P}, \underline{P}')$$

Moreover,

$$H_n(1_M, N; \underline{P}, \underline{P}) = 1_{H_n}(M, N; \underline{P}).$$

(ii) If

$$0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0 \tag{$*$}$$

is a projective resolution of an exact sequence $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ of A-modules, then the sequence

$$\cdots \to H_n(M'', N; \underline{P}'') \xrightarrow{\partial_n} H_{n-1}(M', N; \underline{P}') \xrightarrow{H_{n-1}(i)} H_{n-1}(M, N; \underline{P}) \to$$

$$\xrightarrow{H_{n-1}(j)} H_{n-1}(M'', N; \underline{P}'') \to \cdots \to H_0(M'', N; \underline{P}'') \to 0$$

is exact (where we have written ∂_n for $\partial_n(N, (*)), H_{n-1}(i)$ for $H_{n-1}(i, N; \underline{P}', \underline{P})$, etc.).

(iii) if

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & L' \arrow[r] & L \arrow[r] & L'' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of A-modules with exact rows and if

$$\begin{tikzcd}
0 \arrow[r] & \underline{P}' \arrow[r] \arrow[d, "F'"] & \underline{P} \arrow[r] \arrow[d, "F"] & \underline{P}'' \arrow[r] \arrow[d, "F''"] & 0 & (*) \\
0 \arrow[r] & \underline{Q}' \arrow[r] & \underline{Q} \arrow[r] & \underline{Q}'' \arrow[r] & 0 & (**)
\end{tikzcd}$$

is a commutative diagram of complexes where $0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$, $0 \to \underline{Q}' \to \underline{Q} \to \underline{Q}'' \to 0$ are projective resolutions of $0 \to M' \to M \to M'' \to 0$, $0 \to L' \to L \to L'' \to 0$, respectively and where $\underline{P}' \to$

$\underline{Q}'$ *(resp.*$\underline{P} \to \underline{Q}$, $\underline{P}'' \to \underline{Q}''$*) is a morphism over* $M' \to L'$*(resp.*$M \to L$, $M'' \to L''$*) then the diagram*

$$\begin{tikzcd} H_n(M'', N; \underline{P}'') \arrow[r, "{H_n(f'', N; \underline{P}'', \underline{Q}'')}"] \arrow[d, "{\partial_n(N, (*))}"'] & H_n(L'', N; \underline{Q}'') \arrow[d, "{\partial_n(N, (**))}"] \\ H_{n-1}(M', N; \underline{P}') \arrow[r, "{H_{n-1}(f', N; \underline{P}', \underline{Q}')}"] & H_{n-1}(L', N; \underline{Q}') \end{tikzcd}$$

is commutative, for every $n \geq 1$.

Proof: (i) If $F: \underline{P} \to \underline{P}'$, $G: \underline{P}' \to \underline{P}''$ are morphisms over f, g respectively, then GF is clearly over gf. Further, $1_{\underline{P}}$ is over 1_M, this proves (i). The assertions (ii) and (iii) follow from Propositions 2.1 and 2.2 respectively.

Proposition 2.15 *Let* M *be an* A*-module and let* $\underline{P}$, $\underline{Q}$ *be two projective resolutions of* M*. Then*

$$H_n(1_M, N; \underline{P}, \underline{Q}): H_n(M, N; \underline{P}) \to H_n(M, N; \underline{Q})$$

is an isomorphism, for every $n \in \mathbf{Z}^+$*. If* $f: M \to M'$ *is a homomorphism of* A*-modules and if* $\underline{P}'$, $\underline{Q}'$ *are projective resolutions of* M'*, then the diagram*

$$\begin{tikzcd} H_n(M'', N; \underline{P}) \arrow[r, "{H_n(1_M, N; \underline{P}, \underline{Q})}"] \arrow[d, "{H_n(f, N, \underline{P}, \underline{P}')}"'] & H_n(M, N; \underline{Q}) \arrow[d, "{H_n(f, N; \underline{Q}, \underline{Q}')}"] \\ H_n(M', N; \underline{P}') \arrow[r, "{H_n(1_{M'}, N; \underline{P}', \underline{Q}')}"] & H_n(M', N; \underline{Q}') \end{tikzcd}$$

is commutative. Further, if $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of* A*-modules and if*

$$0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0 \tag{$*$}$$

$$0 \to \underline{Q}' \to \underline{Q} \to \underline{Q}' \to 0 \tag{$**$}$$

are two projective resolutions of this sequence, then the diagram

$$\begin{tikzcd}
H_n(M'', N; \underline{P}'') \arrow[r, "{H_n(1_{M''}, N; \underline{P}'', \underline{Q}'')}"] \arrow[d, "{\partial_n(N, (*))}"'] & H_n(M'', N; \underline{Q}'') \arrow[d, "{\partial_n(N, (**))}"] \\
H_{n-1}(M', N; \underline{P}') \arrow[r, "{H_{n-1}(1_{M'}, N; \underline{P}', \underline{Q}')}"] & H_{n-1}(M', N; \underline{Q}')
\end{tikzcd}$$

is commutative for $n \geq 1$.

PROOF: We have

$$H_n(1_M, N; \underline{Q}, \underline{P})H_n(1_M, N; \underline{P}, \underline{Q}) = H_n(1_M, N; \underline{P}, P) = 1_{H_n(M,N;\underline{P})}$$

by the lemma above. Similarly

$$H_n(1_M, N; \underline{P}, Q)H_n(1_M, N; \underline{Q}, \underline{P}) = 1_{H_nM,N;\underline{Q})}.$$

Hence $H_n(1_M, N; \underline{P}, \underline{Q})$ is an isomorphism. Now

$$\begin{aligned} H_n(f, N; \underline{Q}, \underline{Q}')H_n(1_M, N; \underline{P}, \underline{Q}) &= H_n(f,\ N; \underline{P}, \underline{Q}') \\ &= H_n(1_{M'}, N; \underline{P}', \underline{Q}')H_n(f, N; \underline{P}, \underline{P}'), \end{aligned}$$

by the lemma above. This proves the commutativity of the first diagram of the proposition.

In order to prove the commutativity of the second diagram, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "1_{M'}"'] & M \arrow[r] \arrow[d, "1_M"] & M'' \arrow[r] \arrow[d, "1_{M''}"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

By Proposition 2.13, there exist morphisms F', F, F'' over $1_{M'}, 1_M, 1_{M''}$ respectively such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"'] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"'] & 0 \\
0 \arrow[r] & \underline{Q}' \arrow[r] & \underline{Q} \arrow[r] & \underline{Q}'' \arrow[r] & 0
\end{tikzcd}$$

is commutative. Now the assertion follows from Lemma 2.14(iii). This completes the proof of the proposition.

Let, for $n \in \mathbb{Z}, \mathrm{Tor}_n^A(M,N) = H_n(M,N;\underline{P})$ where P is a projective resolution of M. In view of Proposition 2.15, we have, for a fixed A-module N, a sequence $\{\mathrm{Tor}_n^A(M,N)\}_{n\in\mathbf{Z}^+}$ of functors from A-modules to A-modules, defined independently of the projective resolutions $\underline{P}$ of M. Moreover, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, we have A-homomorphisms $\{\partial_n\colon \mathrm{Tor}_n^A(M'',N) \to \mathrm{Tor}_{n-1}^A(M',N)\}_{n\geq 1}$ called the *connecting homomorphisms.*

Let now M be an A-module and $\underline{P}$ a projective resolution of M. If $f\colon N \to N'$ is a homomorphism of A-modules, then the morphism $1_{\underline{P}} \otimes f\colon \underline{P} \otimes N \to \underline{P} \otimes N'$ of complexes induces, for every $n \in \mathbb{Z}$ an A-homomorphism.

$$\mathrm{Tor}_n^A(M,f)\colon \mathrm{Tor}_n^A(M,N) \to \mathrm{Tor}_n^A(M,N').$$

If $0 \to N' \to N \to N'' \to 0$ is an exact sequence A-modules, then by Corollary 2.6 to Proposition 2.4, the sequence

$$0 \to \underline{P} \otimes_A N' \to \underline{P} \otimes_A N \to \underline{P} \otimes_A N'' \to 0$$

of complexes is exact. Therefore, this defines *connecting homomorphisms*

$$\partial_n\colon \mathrm{Tor}_n^A(M,N'') \to \mathrm{Tor}_{n-1}^A(M,N')$$

We then have

Theorem 2.16 *(i) For a fixed A-module N, the assignments $\{M \mapsto \mathrm{Tor}_n^A(M,N)\}_{n\in\mathbf{Z}^+}$ and $\{M \mapsto \mathrm{Tor}_n^A(N,M)\}_{n\in\mathbf{Z}^+}$ are sequences of functors from A-modules to A-modules.*

(ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then the sequences

$$\cdots \to \mathrm{Tor}_n^A(M'',N) \xrightarrow{\partial_n} \mathrm{Tor}_{n-1}^A(M',N) \to \mathrm{Tor}_{n-1}^A(M,N) \to$$
$$\to \mathrm{Tor}_{n-1}^A(M'',N) \to \cdots \to \mathrm{Tor}_0^A(M'',N) \to 0$$

and

$$\cdots \to \mathrm{Tor}_n^A(N,M'') \xrightarrow{\partial_n} \mathrm{Tor}_{n-1}^A(N,M') \to \mathrm{Tor}_{n-1}^A(N,M) \to$$
$$\mathrm{Tor}_{n-1}^A(N,M'') \to \cdots \to \mathrm{Tor}_0^A(N,M'') \to 0$$

are exact (iii) If

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{K'} \arrow[r] & \underline{K} \arrow[r] & \underline{K''} \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of A-modules with exact rows, then for every $n \geq 1$, the induced diagrams

$$\begin{tikzcd} \operatorname{Tor}_n^A(M'', N) \arrow[r, "\partial_n"] \arrow[d] & \operatorname{Tor}_{n-1}^A(M', N) \arrow[d] \\ \operatorname{Tor}_n^A(K'', N) \arrow[r, "\partial_n"] & \operatorname{Tor}_{n-1}^A(K', N) \end{tikzcd}$$

and

$$\begin{tikzcd} \operatorname{Tor}_n^A(N, M'') \arrow[r, "\partial_n"] \arrow[d] & \operatorname{Tor}_{n-1}^A(N, M') \arrow[d] \\ \operatorname{Tor}_n^A(N, K'') \arrow[r, "\partial_n"] & \operatorname{Tor}_{n-1}^A(N, K') \end{tikzcd}$$

are commutative.

(iv) For every $n \in \mathbf{Z}^+$ the functor $\operatorname{Tor}_n^A(M, N)$ is A-linear in both M and N.

(v) There exists an A-isomorphism $\operatorname{Tor}_0^A(M, N) \simeq M \otimes_A N$ which is functorial in M and N.

Proof: In view of Proposition 2.15, (i) and (ii) are clear.

The assertion (iii) for the functors $\{M \mapsto \operatorname{Tor}_n^A(M, N)\}$ follows from Propositions 2.15 and 2.2. To prove (iii) for the functors $\{M \mapsto \operatorname{Tor}_n^A(N, M)\}$ note that if $\underline{P}$ is any projective resolution of M, then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{P} \otimes_A M' \arrow[r] \arrow[d] & \underline{P} \otimes_A M \arrow[r] \arrow[d] & \underline{P} \otimes_A M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{P} \otimes_A K' \arrow[r] & \underline{P} \otimes_A K \arrow[r] & \underline{P} \otimes_A K'' \arrow[r] & 0
\end{tikzcd}$$

induced by $(*)$ is commutative, so that Proposition 2.2 gives the commutativity of $(**)$.

We now prove (iv). Let $f, g\colon N \to N'$ be homomorphisms of A-modules and let $\underline{P}$ be a projective resolution of M. Since, clearly, for $a, b \in A$,we have $a(1_{\underline{P}} \otimes f) + b(1_{\underline{P}} \otimes g) = 1_{\underline{P}} \otimes (af + bg)$, it follows that $a\mathrm{Tor}_n^A(M, f) + b\mathrm{Tor}_n^A(M, g) = \mathrm{Tor}_n^A(M, af + bg)$. This proves that $\mathrm{Tor}_n^A(M, N)$ is A-linear in N. Let now $\underline{Q}, \underline{Q}'$ be projective resolutions of N, N' respectively. Then, clearly, for $a, b \in A$, the morphism $aF + bG$ is over $af + bg$ and the A-linearity of $\mathrm{Tor}_n^A(M, N)$ in M follows.

Finally, to prove (v), let $\underline{P}$ be a projective resolution of M. Since $\mathrm{Tor}_0^A(M, N) = (P_0 \otimes_A N)/\mathrm{Im}\,(d_1 \otimes 1_N)$, the A-homomorphism $\varepsilon \otimes 1_N\colon P_0 \otimes_A N \to M \otimes_A N$ induces an A-homomorphism $\varepsilon(M, N)\colon \mathrm{Tor}_0^A(M, N) \to M \otimes_A N$. We shall prove that $\varepsilon(M, N)$ is an isomorphism functorial in M and N. Since $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$ is exact, the sequence $P_1 \otimes_A N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_A N \xrightarrow{\varepsilon \otimes 1_N} M \otimes_A N \to 0$ is exact, so that $\varepsilon(M, N)$ is an isomorphism. If now $f\colon M \to M'$ is a homomorphism of A-modules and $F\colon \underline{P} \to \underline{P}'$ is a morphism over f, where $\underline{P}, \underline{P}'$ are projective resolutions of M, M' respectively, then the diagram

$$\begin{tikzcd}
P_1 \otimes_A N \arrow[r, "d_1 \otimes 1_N"] \arrow[d, "F_1 \otimes 1"] & P_0 \otimes_A N \arrow[r, "\varepsilon \otimes 1_N"] \arrow[d, "F_0 \otimes 1"] & M \otimes_A N \arrow[r] \arrow[d, "f \otimes 1"] & 0 \\
P_1 \otimes_A N' \arrow[r, "d_1 \otimes 1_{N'}"] & P_0 \otimes_A N' \arrow[r, "\varepsilon \otimes 1_{N'}"] & M \otimes_A N' \arrow[r] & 0
\end{tikzcd}$$

is commutative and therefore so is the diagram

$$\begin{tikzcd}
\mathrm{Tor}_0^A(M, N) \arrow[r, "\varepsilon(M, N)"] \arrow[d, "\mathrm{Tor}_0^A(f, N)"'] & M \otimes_A N \arrow[d, "f \otimes 1_N"] \\
\mathrm{Tor}_0^A(M', N) \arrow[r, "\varepsilon(M', N)"] & M' \otimes_A N
\end{tikzcd}$$

This proves that $\varepsilon(M,N)$ is functorial in M. Next, let $g\colon N \to N'$ be a homomorphism of A-modules and let $\underline{P}$ be a projective resolution of M. Then the commutativity of the diagram

$$\begin{tikzcd}
P_1 \otimes_A N \arrow[r, "d_1 \otimes 1_N"] \arrow[d, "1 \otimes g"] & P_0 \otimes_A N \arrow[r, "\varepsilon \otimes 1_N"] \arrow[d, "1 \otimes g"] & M \otimes_A N \arrow[r] \arrow[d, "1 \otimes g"] & 0 \\
P_1 \otimes_A N' \arrow[r, "d_1 \otimes 1_{N'}"] & P_0 \otimes_A N' \arrow[r, "\varepsilon \otimes 1_{N'}"] & M \otimes_A N' \arrow[r] & 0
\end{tikzcd}$$

implies that $\varepsilon(M,N)$ is functorial in N. This completes the proof of Theorem 2.16.

Lemma 2.17 *Let P be a projective A-module. Then* $\mathrm{Tor}_n^A(P,M) = 0$ *and $Tor_n^A(M,P) = 0$, for every A-module M and every $n \geq 1$.*

PROOF: Since P is projective, we have a projective resolution $0 \to P \stackrel{1_P}{\to} P \to 0$ of P. Using this resolution to compute Tor we have $\mathrm{Tor}_n^A(P,M) = 0$ for every $n \geq 1$. On the other hand, if $(\underline{Q}, \varepsilon)$ is any projective resolution of M, then the sequence

$$\ldots \to Q_n \otimes_A P \to Q_{n-1} \otimes_A P \to \ldots \to Q_0 \otimes_A P \stackrel{\varepsilon \otimes 1}{\to} M \otimes_A P \to 0$$

is exact by Corollary 2.6 to Proposition 2.4. It now follows that $\mathrm{Tor}_n^A(M,P) = 0$ for $n \geq 1$.

Proposition 2.18 *Let M, N be A-modules. Then, for every $n \in \mathbf{Z}^+$ there exists an isomorphism* $\mathrm{Tor}_n^A(M,N) \simeq \mathrm{Tor}_n^A(N,M)$*, which is functorial in M and N.*

PROOF: We prove the result by induction on n. For $n = 0$, we have by Theorem 2.16(v), functorial isomorphisms $\mathrm{Tor}_0^A(M,N) \simeq M \otimes_A N$ and $\mathrm{Tor}_0^A(N,M) \simeq N \otimes_A M$. Since $M \otimes_A N \simeq N \otimes_A M$ functorially, the result is proved for $n = 0$. Let $n > 0$. Let then $0 \to K \to F \to M \to 0$ be an exact sequence of A-modules where F is A-free. This induces the exact sequences

$$\mathrm{Tor}_n^A(F,N) \to \mathrm{Tor}_n^A(M,N) \stackrel{\partial_n}{\to} \mathrm{Tor}_{n-1}^A(K,N) \to \mathrm{Tor}_{n-1}^A(F,N)$$

$$\mathrm{Tor}_n^A(N,F) \to \mathrm{Tor}_n^A(N,M) \stackrel{\partial_n}{\to} \mathrm{Tor}_{n-1}^A(N,K) \to \mathrm{Tor}_{n-1}^A(N,F).$$

Since $n > 0$, we have $\mathrm{Tor}_n^A(F, N) = 0 = \mathrm{Tor}_n^A(N, F)$ by Lemma 2.17. Therefore, by induction hypothesis, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_n^A(M,N) \arrow[r, "\partial_n"] & \mathrm{Tor}_{n-1}^A(K,N) \arrow[r] \arrow[d, "\varphi"] & \mathrm{Tor}_{n-1}^A(F,N) \arrow[d, "\psi"] \\
0 \arrow[r] & \mathrm{Tor}_n^A(N,M) \arrow[r, "\partial_n"] & \mathrm{Tor}_{n-1}^A(N,K) \arrow[r] & \mathrm{Tor}_{n-1}^A(N,F)
\end{tikzcd}$$

where φ and ψ are isomorphisms. Therefore, φ induces an isomorphism $\varphi' \colon \mathrm{Tor}_n^A(M, N) \simeq \mathrm{Tor}_n^A(N, M)$. Since φ is functorial by induction hypothesis and since ∂_n is functorial, by Theorem 2.16, the isomorphism φ' is also functorial.

2.5 The functors Ext

Let M, N be A-modules and let $(\underline{P}, \varepsilon)$ be a projective resolution of M. We denote by $\mathrm{Hom}_A(\underline{P}, N)$ the complex

$$0 \to \mathrm{Hom}_A(P_0, N) \to \mathrm{Hom}_A(P_1, N) \to \ldots \to \mathrm{Hom}_A(P_n, N) \to \ldots$$

and by $\mathrm{Ext}_A^n(M, N)$, the homology module $H^n(\mathrm{Hom}_A(\underline{P}, N))$. If $f \colon M \to M'$ and $g \colon N \to N'$ are homomorphisms of A-modules, then we define

$$\mathrm{Ext}_A^n(f, N) \colon \mathrm{Ext}_A^n(M', N) \to \mathrm{Ext}_A^n(M, N)$$

and

$$\mathrm{Ext}_A^n(M, g) \colon \mathrm{Ext}_A^n(M, N) \to \mathrm{Ext}_A^n(M, N')$$

in the same manner as in the case of Tor. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules and N is any A-module, we define, as in the previous section, connecting homomorphisms

$$\partial^{n-1} \colon \mathrm{Ext}_A^{n-1}(M', N) \to \mathrm{Ext}_A^n(M'', N)$$

and similarly for exact sequences in the second variable. We then have

Theorem 2.19 *(i) For a fixed A-module N, the assignment $\{M \mapsto \mathrm{Ext}_A^n(M, N)\}_{n \in \mathbf{Z}^+}$ is a sequence of contravariant functors from A modules to A-modules and the assignment $\{M \mapsto \mathrm{Ext}_A^n(N, M)\}_{n \in \mathbf{Z}^+}$ is a sequence of functors from A-modules to A-modules.*

(ii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then the sequences

$$0 \to \mathrm{Ext}^0_A(M'', N) \to \ldots \to \mathrm{Ext}^{n-1}_A(M'', N) \to \mathrm{Ext}^{n-1}_A(M, N) \to$$

$$\to \mathrm{Ext}^{n-1}_A(M', N) \xrightarrow{\partial^{n-1}} \mathrm{Ext}^n_A(M'', N) \to \ldots$$

and

$$0 \to \mathrm{Ext}^0_A(N, M') \to \ldots \to \mathrm{Ext}^{n-1}_A(N, M') \to \mathrm{Ext}^{n-1}_A(N, M) \to$$

$$\mathrm{Ext}^{n-1}_A(N, M'') \stackrel{\partial^{n-1}}{\to} \mathrm{Ext}^n_A(N, M') \to \ldots$$

are exact.

(iii) If

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K' \arrow[r] & K \arrow[r] & K'' \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of A-modules with exact rows, then, for every $n \geq 1$ the induced diagrams

$$\begin{tikzcd} \mathrm{Ext}^{n-1}_A(M', N) \arrow[r, "\partial^{n-1}"] & \mathrm{Ext}^n_A(M'', N) \\ \mathrm{Ext}^{n-1}_A(K', N) \arrow[r, "\partial^{n-1}"] \arrow[u] & \mathrm{Ext}^n_A(K'', N) \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Ext}^{n-1}_A(N, M'') \arrow[r, "\partial^{n-1}"] \arrow[d] & \mathrm{Ext}^n_A(N, M') \arrow[d] \\ \mathrm{Ext}^{n-1}_A(N, K'') \arrow[r, "\partial^{n-1}"] & \mathrm{Ext}^n_A(N, K') \end{tikzcd}$$

are commutative.

(iv) For every $n \in \mathbf{Z}^+$ *the functor* $\operatorname{Ext}^n_A(M, N)$ *is A-linear in both* M *and* N*.*

(v) There exists an A-isomorphism

$$\operatorname{Ext}^0_A(M, N) \simeq \operatorname{Hom}_A(M, N)$$

which is functorial in M *and* N*.*

(vi) Let P *be a projective A-module. Then* $\operatorname{Ext}^n_A(P, N) = 0$*, for every A-module* N *and for every* $n \geq 1$*.*

PROOF: On the same lines as for the functors Tor in the previous section.

Chapter 3

Dimension Theory

3.1 The Hilbert-Samuel polynomial

Let $f\colon \mathbb{Z} \to \mathbf{Q}$ be a map. We define $\Delta f\colon \mathbb{Z} \to \mathbf{Q}$ by

$$(\Delta f)(n) = f(n+1) - f(n); \quad n \in \mathbb{Z}.$$

By induction on r, we define $\Delta^r f$ for every $r \in \mathbb{Z}$ by

$$\Delta^0 f = f;$$

$$\Delta^r f = \Delta(\Delta^{r-1} f), \quad r \geq 1.$$

Let $Q[X]$ denote the ring of polynomials in one variable X over $\mathbf{Q}$. A map $f\colon \mathbf{Z}^+ \to \mathbf{Q}$ is called a *polynomial function* if there exists $g \in \mathbf{Q}[X]$ such that $f(n) = g(n)$ for $n \gg 1$. Note that if $g_1, g_2 \in \mathbf{Q}[X]$ are such that $g_1(n) = g_2(n)$ for $n \gg 1$, then $g_1 = g_2$. Therefore for a polynomial function f, the corresponding polynomial g is uniquely determined. The degree of g is called the *degree of* f and the leading coefficient of g is called the *leading coefficient* of f. If $f \neq 0$ and if $f(n) \geq 0$ for $n \gg 1$, then the leading coefficient of f is clearly positive. Let f_1, f_2 be two polynomial functions. We say $f_1 \leq f_2$ if $f_1(n) \leq f_2(n)$ for $n \gg 1$.

Lemma 3.1 *Let $r \in \mathbf{N}$. Then a map $f\colon \mathbf{Z}^+ \longrightarrow \mathbf{Q}$ is a polynomial function of degree r if and only if $\Delta f\colon \mathbb{Z} \to \mathbf{Q}$ is a polynomial function of degree $r-1$.*

Proof: If f is a polynomial function of degree r, then clearly Δf is a polynomial function of degree $r-1$. We prove the converse by induction on r. If $r = 1$, there exists $p \in \mathbf{Q}, p \neq 0$ such that $(\Delta f)(n) = p$ for $n \gg 1$. Thus

$$f(n+1) - f(n) = p = p(n+1) - p(n),$$

that is

$$f(n+1) - p(n+1) = f(n) - pn = q$$

for $n \gg 1$ and for some $q \in \mathbf{Q}$. this implies that $f(n) = pn + q$ for $n \gg 1$ and the assertion is proved in this case.

Now, let $r > 1$. Let $g(X) = a_0 X^{r-1} + \ldots + \ldots + a_{r-1} \in \mathbf{Q}[X], a_0 \neq 0$ be such that $\Delta f(n) = g(n)$, for $\gg 1$. Then

$$f(n+1) - f(n) = \Delta f(n) = \frac{a_0}{r}\{(n+1)^r - n^r + h(n)\}$$

where $h \in \mathbf{Q}[X]$ and $\deg h \leq r - 2$. Setting $f^*(n) = f(n) - \frac{a_0}{r} n^r$, we have for $n \gg 1$

$$\Delta f^*(n) = f^*(n+1) - f^*(n) = h(n)$$

and by induction hypothesis, $f^*(n)$ is a polynomial function of degree less than or equal to $r - 1$. Since $f(n) = \frac{a_0}{r} n^r + f^*(n)$ for $n \gg 1$ and $a_0 \neq 0$, the result follows.

Remark 3.2 If we assign the degree -1 to the zero polynomial, it is clear that Δf is a polynomial function of degree $-1 \Leftrightarrow f$ is a polynomial function of degree ≤ 0.

Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring such that R_0 is artinian and R is generated as an R_0- algebra by r elements $x_1, \ldots, x_r$ of R_1. Then, R being a finitely generated R_0-algebra, is noetherian (by Corollary to Theorem 1.10). Let $N = \bigoplus_{n \geq 0} N_n$ be a finitely generated graded R-module. Then, as an R_0-module, each N_n is finitely generated (by Proposition 1.37) and hence of finite length (since R_0 is artinian) in view of Corollary 1.35 to Proposition 1.33. Define $\chi(N, .) \colon \mathbb{Z} \to \mathbf{Z}$ by setting $\chi(N, \cdot)(n),\ \chi(N, n) = \ell_{R_0}(N - n)$.

Proposition 3.3 *(Hilbert). The map $\chi(N, .)$ is a polynomial function of degree $\leq r - 1$ where r is as above.*

Proof: We prove the proposition by induction on r. If $r = 0$, then $R = R_0$. Let S be a finite set of homogeneous generators of N over

$R = R_0$, and let $m = \sup_{s \in S}(\deg s)$. Then $N_n = 0$ for $n > m$. Hence $\chi(N, n) = 0$ for $n \gg 1$, i.e. $\chi(N, n)$ is a polynomial function of degree -1. Assume now that $r > 0$ and that the result is true for all finitely generated graded modules over graded rings R which are generated as R_0-algebras by less than r elements of R_1.

Let K and C be respectively the kernel and cokernel of the graded endomorphism $\varphi: N \to N$ of degree 1, given by $\varphi = (x_r)_N$, the homothesy by x_r. We then have, for every n, an exact sequence

$$0 \to K_n \to N_n \xrightarrow{\varphi} N_{n+1} \to C_{n+1} \to 0.$$

Since N is noetherian, the R-modules K and C are noetherian so that $\chi(K, .)$ and $\chi(C, .)$ are defined. It follows from Proposition 1.30 that $\chi(K, n) - \chi(N, n) + \chi(N, n+1) - \chi(C, n+1) = 0$, i.e.

$$\Delta\chi(N, n) = \chi(C, n+1) - \chi(K, n).$$

Since x_r annihilates both K and C, these are finitely generated (graded) modules over the graded subring $R' = R_0[x_1, \ldots, x_{r-1}]$ of R. By the induction hypothesis, $\chi(C, n)$ and $\chi(K, n)$ are both polynomial functions of degree less than $r - 1$ and so is $\Delta\chi(N, n)$. Now the proposition is a consequence of Lemma 3.1 and the remark following it.

The polynomial associated to the polynomial function $\chi(N, n)$ is called the *Hilbert polynomial* of N and is also denoted by $\chi(N, n)$.

Till the end of this chapter, we assume that A is a noetherian local ring. We denote by $\mathfrak{m}$ the maximal ideal of A. By an A-module, we mean a finitely generated A-module.

An ideal $\mathfrak{a}$ of A is said to be an *ideal of definition* of A if $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$ for some integer $n \geq 1$.

Let $\mathfrak{a}$ be an ideal of definition of A. As in Chapter 1, we denote by $G_{\mathfrak{a}}(A)$ the graded ring $\oplus_{n \geq 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$ associated to the $\mathfrak{a}$-adic filtration on A. Similarly, for an A-module M, we denote by $G_{\mathfrak{a}}(M)$ the graded $G_{\mathfrak{a}}(A)$-module $\bigoplus_{n \geq 0} \mathfrak{a}^n M / \mathfrak{a}^{n+1} M$ corresponding to the $\mathfrak{a}$-adic filtration on M. By our hypothesis on $\mathfrak{a}, \operatorname{Supp}(A/\mathfrak{a}) = \{\mathfrak{m}\}$. Thus by Proposition 1.32, $A/\mathfrak{a}$ is of finite length and hence artinian. Since A is noetherian, the ideal $\mathfrak{a}$ is finitely generated, say, by r elements and the conditions of Proposition 3.3 are satisfied for $R = G_{\mathfrak{a}}(A), N = G_{\mathfrak{a}}(M)$. Therefore, $\chi(G_{\mathfrak{a}}(M), n) = \ell_{A/\mathfrak{a}}(\mathfrak{a}^n M / \mathfrak{a}^{n+1} M)$ is a polynomial of degree less than or equal to $r - 1$.

Since $\operatorname{Supp}(M/\mathfrak{a}^n M) = \{\mathfrak{m}\}$, the A-module $M/\mathfrak{a}^n M$ is of finite length by Proposition 1.32. We set $P_{\mathfrak{a}}(M, n) = \ell_A(M/\mathfrak{a}^n M)$. Since

$\ell_A(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M) = \ell_{A/\mathfrak{a}}(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M)$, the exact sequence

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^n M \to 0$$

gives $\chi(G_\mathfrak{a}(M), n) = P_\mathfrak{a}(M, n+1) - P_\mathfrak{a}(M, n) = \Delta P_\mathfrak{a}(M, n)$. From Lemma 3.1 (and the remark following it), we get

Theorem 3.4 ***(Samuel)*** *Let A be a local ring, M a finitely generated A-module and $\mathfrak{a}$ an ideal of definition of A generated by r elements. Then $P_\mathfrak{a}(M, n)$ is a polynomial function of degree less than or equal to r.*

Lemma 3.5 *Let M be an A-module and $\mathfrak{a}, \mathfrak{a}'$ ideals of definition of A. Then $P_\mathfrak{a}(M, n)$ and $P_{\mathfrak{a}'}(M, n)$ have the same degree.*

PROOF: It is sufficient to prove that $P_\mathfrak{a}(M, n)$ and $P_\mathfrak{m}(M, n)$ have the same degree. Since $\mathfrak{a}$ is an ideal of definition of A, there exists $m \in \mathbf{N}$ such that $\mathfrak{m}^m \subset \mathfrak{a} \subset \mathfrak{m}$. Hence, for every $n \in \mathbf{N}$,we have $\mathfrak{m}^{mn} \subset \mathfrak{a}^n \subset \mathfrak{m}^n$ so that $P_\mathfrak{m}(M, nm) \geq P_\mathfrak{a}(M, n) \geq P_\mathfrak{m}(M, n)$ and the lemma follows.

Proposition 3.6 *Let $\mathfrak{a}$ be an ideal of definition of A and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then we have*

$$P_\mathfrak{a}(M', n) + P_\mathfrak{a}(M'', n) = P_\mathfrak{a}(M, n) + R(n),$$

where $R(n)$ is a polynomial function of degree less than $\deg P_\mathfrak{a}(M, n)$ and the leading coefficient of $R(n)$ is non-negative.

PROOF: For every $n \in \mathbf{N}$, we have an exact sequence

$$0 \to M'/M' \cap \mathfrak{a}^n M \to M/\mathfrak{a}^n M \to M''/\mathfrak{a}^n M'' \to 0$$

induced by the given exact sequence. (In writing $M' \cap \mathfrak{a}^n M$, we have tacitly identified M' with a submodule of M). This gives

$$\ell_A(M'/M' \cap \mathfrak{a}^n M) + \ell_A(M''/\mathfrak{a}^n M'') = \ell_A(M/\mathfrak{a}^n M).$$

Setting $M'_n = M' \cap \mathfrak{a}^n M$, we have

$$\ell_A(M'/M'_n) = P_\mathfrak{a}(M, n) - P_\mathfrak{a}(M, n) \tag{3.1}$$

which shows that $\ell_A(M'/M'_n)$ is a polynomial function. By the theorem of Artin-Rees (Theorem 1.39), there exists $m \in \mathbf{N}$ such that

$\mathfrak{a}M'_n = M'_{n+1}$ for $n \geq m$. It follows that for any $n \in \mathbf{N}, \mathfrak{a}^{m+n}M' \subset M'_{n+m} = \mathfrak{a}^n M'_m \subset \mathfrak{a}^n M'$, so that $\ell_A(M'/\mathfrak{a}^{m+n}M') \geq \ell_A(M'/M'_{n+m}) \geq \ell_A(M'/\mathfrak{a}^n M')$, i.e.

$$P_{\mathfrak{a}}(M', n+m) \geq \ell_A(M'/M'_{n+m}) \geq P_{\mathfrak{a}}(M', n). \tag{3.2}$$

These inequalities show that $P_{\mathfrak{a}}(M', n)$ and $\ell_A(M'/M'_n)$ have the same degree and the same leading coefficient. Thus $R(n) = P_{\mathfrak{a}}(M', n) - \ell_A(M'/M'_n)$ is a polynomial function of degree less than the degree of $\ell_A(M'/M'_n)$ which, by (3.1), is less than or equal to the degree of $P_{\mathfrak{a}}(M, n)$ since $\deg P_{\mathfrak{a}}(M', n) \leq \deg P_{\mathfrak{a}}(M, n)$. Since, by (3.2), $R(n) \geq 0$ for $n \gg 1$, its leading coefficient is non-negative. This completes the proof of the proposition.

Corollary 3.7 *Let M' be a submodule of M. Then* $\deg P_{\mathfrak{a}}(M', n) \leq \deg P_{\mathfrak{a}}(M, n)$.

PROOF: Let $M'' = M/M'$. Since $\deg P_{\mathfrak{a}}(M'', n) \leq \deg aP_{\mathfrak{a}}(M, n)$, the corollary follows from Proposition 3.6.

The above results apply in particular to the case $M = A$. Let $G(A) = G_{\mathfrak{m}}(A)$ and let $k = A/\mathfrak{m}$. If $\mathfrak{m}$ is generated by r elements $x_1, \ldots, x_r$, we have $\deg \chi(G(A), n) \leq r - 1$. We have a graded k-algebra epimorphism $\varphi: k[X_1, \ldots, X_r] \to G(A)$ defined by $\varphi(X_i) = \bar{x}_i$, where $\bar{x}_i$ denotes the class of $x_i \bmod \mathfrak{m}^2$.

Proposition 3.8 *With the above notation, we have* $\deg \chi(G(A), n) = r - 1$ *if and only if* $\varphi: k[X_1, \ldots, X_r] \to G(A)$ *is an isomorphism.*

PROOF: Let $B = k[X_1, \ldots, X_r]$. Let φ be an isomorphism. Then we have an isomorphism $B_n \simeq \mathfrak{m}^n/\mathfrak{m}^{n+1}$ of k-vector spaces, where B_n denotes the n^{th} homogeneous component of B. Therefore $\chi(G(A), n) = \ell_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \ell_k(B_n) = \binom{n+r-1}{r-1}$, since B_n is a k-vector space of rank $\binom{n+r-1}{r-1}$. The map $n \mapsto \binom{n+r-1}{r-1}$ is clearly a polynomial function of degree $r-1$ and hence $\deg \chi(G(A), n) = r - 1$.

Conversely, let φ be not an isomorphism and let $N = \ker \varphi \neq 0$. Then, for every $n \in \mathbb{Z}$, we have an exact sequence

$$0 \to N_n \to B_n \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to 0$$

of k-vector spaces, which gives

$$\chi(G(A), n) = \binom{n+r-1}{r-1} - \ell_k(N_n). \tag{$*$}$$

Choose a non-zero homogeneous element $f \in N$ and let $\deg f = d$. Then, for every $n \in \mathbb{Z}$, we have $fB_n \subset N_{n+d}$, which implies that $\ell_k(N_{n+d}) \geq \ell_k(fB_n) = \ell_k(B_n) \geq \ell_k(N_n)$. Thus $\ell_k(N_n)$ and $\ell_k(B_n) = \binom{n+r-1}{r-1}$ have he same degree $r-1$ and the same leading coefficient. Now $(*)$ implies that $\deg \chi(G(A), n) < r-1$ and the proposition is proved.

Corollary 3.9 *We have* $\deg P_{\mathfrak{m}}(A, n) = r$ *if and only if*

$$\varphi: k[X_1, \ldots, X_r] \to G(A)$$

is an isomorphism.

Proof: This is immediate from the above Proposition, since $\Delta P_{\mathfrak{m}}(A, n) = \chi(G(A), n)$.

3.2 Dimension theorem

By a *chain* in A, we mean sequence $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals $\mathfrak{p}_i$ of A such that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $0 \leq i \leq r-1$ and we say that this chain is of *length* r. The *height* of a prime ideal $\mathfrak{p}$, denoted $\operatorname{ht} \mathfrak{p}$, is defined by

$$\operatorname{ht} \mathfrak{p} = \sup\{r \mid \text{ there exists in } A \text{ a chain } \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}\}.$$

If S is a multiplicative subset of A with $\mathfrak{p} \cap S = \phi$, then it follows from Proposition 1.3 that $\operatorname{ht} \mathfrak{p} = \operatorname{ht} S^{-1}\mathfrak{p}$. The *coheight* of a prime ideal $\mathfrak{p}$, denoted $\operatorname{coht} \mathfrak{p}$, is defined by

$$\operatorname{coht} \mathfrak{p} = \sup\{r \mid \text{there exists in } A \text{ a chain } \mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r\}.$$

Let M be a nonzero A-module. The *Krull dimension of* M denoted $\dim_A M$ (or, simply $\dim M$, if no confusion is likely) is defined by

$$\dim M = \sup_{\mathfrak{p} \in \operatorname{Supp}(M)} \operatorname{coht} \mathfrak{p}.$$

Thus, $\dim M$ is the supremum of lengths of chains of prime ideals belonging to $\operatorname{Supp}(M)$. Since the minimal elements of $\operatorname{Supp}(M)$ belong to $\operatorname{Ass}(M)$ (by Corollary 1.28 to Proposition 1.27), we also have $\dim M = \sup_{\mathfrak{p} \in \operatorname{Ass}(M)} \operatorname{coht} \mathfrak{p}$. If $M = 0, \operatorname{Supp}(M) = \theta$ and we define $\dim M = -1$. *The Krull dimension of a ring* A is defined to be $\dim_A A$ and denoted $\dim A$. Thus, $\dim A$ is the supremum of lengths of all chains of prime ideals in A. It is clear that if $\mathfrak{p}$ is a prime ideal of $A, \dim A/\mathfrak{p} = $ coht $\mathfrak{p}$ and $\dim A\mathfrak{p} = $ ht $\mathfrak{p}$.

The *Chevalley dimension* $s(M)$ of an A-module $M \neq 0$ is defined to be the least integer r for which there exist r elements $a_1, \ldots, a_r$ in $\mathfrak{m}$ such that $M/(a_1, \ldots, a_r)M$ is of finite length as an A-module. Note that since $M/\mathfrak{m}M$ is of finite length, such an integer r exists. If $M = 0$, we define $s(M) = -1$. We define the *Chevalley dimension of a ring* A to be its Chevalley dimensions as an A-module.

Let $\mathfrak{a}$ be an ideal of definition of A, i.e. $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$ for some $m \in \mathbf{N}$. For an A.module M, we denote as before, the length of the A-module $M/\mathfrak{a}^n M$ by $P_{\mathfrak{a}}(M, n)$. By Theorem 3.4, we know that if $\mathfrak{a}$ is generated by r elements, then $P_{\mathfrak{a}}(M, n)$ is a polynomial function of degree less than or equal to r. By Lemma 3.5, $\deg P_{\mathfrak{a}}(M, n)$ is independent of the choice of the ideal $\mathfrak{a}$ of definition and is equal to $P_{\mathfrak{m}}(M, n)$. We define

$$d(M) = \deg P_{\mathfrak{m}}(M, n).$$

Theorem 3.10 *(Dimension theorem). Let M be a finitely generated module over a noetherian local ring A. Then* $\dim M = d(M) = s(M)$.

Proof: We prove the theorem by showing that $\dim M \leq d(M) \leq s(M) \leq \dim M$. First, we prove the inequality $\dim M \leq d(M)$. If $d(M) = -1$, then $P_{\mathfrak{m}}(M, n) = 0$ for $n \gg 1$ which implies that $M = \mathfrak{m}^n M$ for $n \gg 1$. Since M is finitely generated, it follows from Nakayama's lemma that $M = 0$ and $\dim M = -1$. Assume now that $d(M) \geq 0$. Since, by Proposition 1.19, Ass(M) is finite, there exists $\mathfrak{p} in \operatorname{Ass}(M)$ such that $\dim M = $ coht $\mathfrak{p} = \dim A/\mathfrak{p}$. Since $\mathfrak{p} \in $ Ass(M), we have a monomorphism $A/\mathfrak{p} \hookrightarrow M$ and by Corollary 3.7, $d(A/\mathfrak{p}) \leq d(M)$. It is therefore sufficient to prove that $\dim A/\mathfrak{p} \leq d(A/\mathfrak{p})$. In order to prove this, we have only to show that if $\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ is a chain in A, then $r \leq d(A/\mathfrak{p})$. Since $A/\mathfrak{p} \neq 0$, we have $d(A/\mathfrak{p}) \neq -1$ and there is nothing to prove if $r = 0$. We assume therefore that $r \geq 1$. Let us make the following induction by hypothesis : if $\mathfrak{p}' = \mathfrak{p}'_0 \subset \ldots \subset \mathfrak{p}'_{r-1}$ is a chain of length $r - 1$ in A, then $r - 1 \leq d(A/\mathfrak{p}')$. Choose $a \in \mathfrak{p}_1, a \notin \mathfrak{p}$ and let $\mathfrak{p}'$ be a minimal prime ideal containing $Aa + \mathfrak{p}$ and contained in $\mathfrak{p}_1$. We then have a chain $\mathfrak{p}' \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_r$ of length $r - 1$ which gives $r - 1 \leq d(A/\mathfrak{p}')$. Further, since $\mathfrak{p}' \in $ Ass$(A/Aa + \mathfrak{p})$, we have monomorphism $A/\mathfrak{p}' \hookrightarrow A/Aa + \mathfrak{p}$ which implies, by Corollary 3.7 that $d(A/\mathfrak{p}') \leq d(A/Aa + \mathfrak{p})$. Consider now the exact sequence

$$0 \to A/\mathfrak{p} \xrightarrow{\varphi} A/\mathfrak{p} \to A/Aa + \mathfrak{p} \to 0$$

where the map $\varphi\colon A/\mathfrak{p} \to A/\mathfrak{p}$ is the homothesy by a. By Proposition 3.6, we have

$$P_{\mathfrak{m}}(A/\mathfrak{p}, n) + P_{\mathfrak{m}}(A/Aa + \mathfrak{p}, n) = P_{\mathfrak{m}}(A/\mathfrak{p}, n) + R(n),$$

where $R(n)$ is a polynomial function of degree less than $d(A/\mathfrak{p})$. Therefore $d(A/Aa + \mathfrak{p}) < d(A/\mathfrak{p})$ and it follows that $r \leq d(A/\mathfrak{p})$.

We next prove that $d(M) \leq s(M)$. If $s(M) = -1$, then $M = 0$ and $d(M) = -1$. So, let $s = s(M) \geq 0$ and let $a_1, \ldots, a_s \in \mathfrak{m}$ be such that $M/(a_1, \ldots, a_s)M$ is of finite length. Let $\mathfrak{a} = \operatorname{ann} M$ and let $\mathfrak{b} = (a_1, \ldots, a_s) + \mathfrak{a}$. We claim that $\operatorname{Supp}(A/\mathfrak{b}) = \{\mathfrak{m}\}$. For since $M/(a_1, \ldots, a_s)M = M \otimes_A A/(a_1, \ldots, a_s)$, we have by Proposition 1.29, $\operatorname{Supp}(M/(a_1, \ldots, a_s)M) = \operatorname{Supp}(M) \cap \operatorname{Supp}(A/(a_1, \ldots, a_s))$ and $\operatorname{Supp}(M/(a_1, \ldots, a_s)M) = \{\mathfrak{m}\}$, by Proposition 1.32. It follows from Corollary 1.28 to Proposition 1.27 that $\operatorname{Ass}(A/\mathfrak{b}) = \{\mathfrak{m}\}$. Thus $\mathfrak{b}$ is $\mathfrak{m}$-primary and hence $\mathfrak{m}^n \subset \mathfrak{b}$ for some $n \in \mathbf{N}$. Thus $\mathfrak{b}$ is an ideal of definition of A. Let $\bar{A} = A/\mathfrak{a}$ and let $\bar{\mathfrak{b}} = \mathfrak{b}/\mathfrak{a}$. Then $\bar{A}$ is a local ring and $\bar{\mathfrak{b}}$ is an ideal definition of $\bar{A}$ and generated by the elements $\bar{a}_1, \ldots, \bar{a}_s$ where $\bar{a}_i$ denotes the image of a_i in $\bar{A}$. Considering M as an $\bar{A}$-module, it follows from Theorem 3.4 that $P_{\bar{\mathfrak{b}}}(m, n)$ is of degree less than or equal to s. Since $\ell_{\bar{A}}(M/\bar{\mathfrak{b}}^n M) = \ell_A(M/\bar{\mathfrak{b}}^n M)$, we have $P_{\bar{\mathfrak{b}}}(M, n) = P_{\mathfrak{b}}(M, n)$ and $d(M) \leq s$.

Finally, we prove that $s(M) \leq \dim M$, by induction on $\dim M$ which is finite, since $\dim M \leq d(M)$. If $\dim M = -1$, then $M = 0$ and $s(M) = -1$. If $\dim M = 0$, then $\operatorname{Supp}(M) = \{\mathfrak{m}\}$, so that M is of finite length. It follows that $s(M) = 0$. Let then $\dim M > 0$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$ be those elements of $\operatorname{Ass}(M)$ for which $\dim M = \operatorname{coht} \mathfrak{p}_i, 1 \leq i \leq g$. Since $\dim M > 0$, we have $\mathfrak{p}_i \neq \mathfrak{m}$, for every i and hence $\mathfrak{m} \not\subset \bigcup_{1 \leq i \leq g} \mathfrak{p}_i$. Choose $a \in \mathfrak{m}$, $a \notin \bigcup_{1 \leq i \leq g}^{i} \mathfrak{p}_i$ and let $M' = M/aM$. Then $\operatorname{Supp}(M') \subset \operatorname{Supp}(M) - \{\mathfrak{p}_1, \ldots, \mathfrak{p}_g\}$ and it follows that $\dim M' < \dim M$. Let $t = s(M')$ and let $a_1, \ldots, a_t \in \mathfrak{m}$ be such that $M'/(a_1, \ldots, a_t)M'$ is of finite length. Then $M/(a, a_1, \ldots, a_t)M$ is of finite length so that $s(M) \leq t+1$. By induction hypothesis, $t \leq \dim M'$. Hence $s(M) \leq \dim M$ and the theorem is proved.

Corollary 3.11 *Let M be a finitely generated module over a noetherian local ring A. Then $\dim_A M < \infty$.*

We call the common value $\dim M = d(M) = s(M)$ the *dimension* of M and denote it by $\dim_A M$ or $\dim M$.

Corollary 3.12 *Let B be any noetherian ring and $\mathfrak{a}$ an ideal of B generated by r elements. Then, for any minimal prime ideal $\mathfrak{p}$ of B containing $\mathfrak{a}$ we have* $\operatorname{ht}\mathfrak{p} \leq r$.

PROOF: Consider the local ring $B_{\mathfrak{p}}$. Since $\mathfrak{p}$ is a minimal prime ideal containing $\mathfrak{a}$ we have $\operatorname{Supp}(B_{\mathfrak{p}}/\mathfrak{a}B_{\mathfrak{p}}) = \{\mathfrak{p}B_{\mathfrak{p}}\}$. Therefore $\ell_{B_{\mathfrak{p}}}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) < \infty$ and it follows that $s(B_{\mathfrak{p}}) \leq r$. Hence

$$\operatorname{ht}\mathfrak{p} = \dim B_{\mathfrak{q}} = s(B_{\mathfrak{p}}) \leq r.$$

Corollary 3.13 *(Principal ideal Theorem). Let B be a noetherian ring and Ba a principal ideal of B. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$ be the minimal prime ideals of B containing Ba. Then* $\operatorname{ht}\mathfrak{p}_i \leq 1$ *for* $1 \leq i \leq g$. *Moreover, if a is not a zero divisor of B, then* $\operatorname{ht}\mathfrak{p}_i = 1$ *for* $1 \leq i \leq g$.

PROOF: The first assertion is a particular case of the corollary above. Suppose a is not a zero divisor of B. Then, by Proposition 1.21, a cannot belong to any minimal prime ideal of B. Hence, for every $\mathfrak{p} \in \operatorname{Ass}(B/Ba)$, we have $\operatorname{ht}\mathfrak{p} \geq 1$. Thus $\operatorname{ht}\mathfrak{p}_i = 1$, for $1 \leq i \leq g$.

Chapter 4

Homological Characterisation of Regular Local Rings

4.1 Homological dimension

In this section, as before, A denotes a commutative ring with 1 and all modules are assumed to be unitary.

Let M be a non-zero A-module and let

$$\ldots \to P_n \xrightarrow{d} P_{n-1} \to \ldots \to P_0 \xrightarrow{\varepsilon} M \to 0$$

be a projective resolution of M. We say that this resolution is of *length* n, if $P_n \neq 0$ and $P_i = 0$, for $i > n$. The *homological dimension* of a non-zero A-module M, denoted $\operatorname{hd}_A M$, is the least integer n, if it exists, such that there exists a projective resolution of M of length n. If no such integer exists, we set $\operatorname{hd}_A M = \infty$. If $M = 0$, we set $\operatorname{hd}_A M = -1$. It is clear that an A-module M is projective if and only if $\operatorname{hd}_A M \leq 0$.

The *global dimension* of a ring A, denoted $\operatorname{gl.dim} A$, is defined by

$$\operatorname{gl.dim} A = \sup_M \operatorname{hd}_A M,$$

where the supremum is taken over all A-modules M.

Proposition 4.1 *For an A-module M, the following conditions are equivalent*:

(i) M is projective;

(ii) $\operatorname{Ext}_A^j(M,N) = 0$ *for all A-modules N and all $j \geq 1$;*
(iii) $\operatorname{Ext}_A^1(M,N) = 0$ *for all A-modules N.*

Proof: (i) $\Rightarrow$ (ii). Since M is projective, M has a projective resolution

$$0 \to M \xrightarrow{1_M} M \to 0.$$

Using this resolution to compute Ext, we see that $\operatorname{Ext}_A^j(M,N) = 0$ for all N and $j \geq 1$.

$(ii) \Rightarrow (iii)$. Trivial.

$(iii) \Rightarrow (i)$. Let

$$\begin{tikzcd} & M \arrow[d, "f"] & \\ N \arrow[r, "\varphi"] & N'' \arrow[r] & 0 \end{tikzcd}$$

be a given diagram of A-homomorphisms, where the row is exact. If $N' = \ker \varphi$, we have an exact sequence $0 \to N' \to N \xrightarrow{\varphi} N'' \to 0$ which, by Theorem 2.19, induces an exact sequence

$$\operatorname{Hom}_A(M,N) \xrightarrow{\varphi^*} \operatorname{Hom}_A(M,N'') \to \operatorname{Ext}_A^1(M,N').$$

Since $\operatorname{Ext}_A^1(M,N') = 0$ by hypothesis, it follows that φ^* is an epimorphism. Hence there exists an A-homomorphism $g: M \to N$ such that $\varphi \circ g = f$. This proves that M is projective.

Proposition 4.2 *For an A-module M and $n \in \mathbf{Z}^+$, the following conditions are equivalent.*

(i) $\operatorname{hd}_A M \leq n$;
(ii) $\operatorname{Ext}_A^j(M,N) = 0$ *for all A-modules N and all $j \geq n+1$;*
(iii) $\operatorname{Ext}_A^{n+1}(M,N) = 0$ *for all A-modules N;*
(iv) if $0 \to K_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ is exact with P_j being A-projective for $0 \leq j \leq n-1$, then K_n is A-projective.

Proof: $(i) \Rightarrow (ii)$. By hypothesis, there exists a projective resolution of M of length less than or equal to n. Using this resolution to compute Ext, (ii) trivially follows.

$(ii) \Rightarrow (iii)$. Trivial.

(iii) $\Rightarrow$ *(iv)*. If $n = 0$, it follows from Proposition 4.1 that M is projective and the assertion is clear in this case. Let then $n \geq 1$. The given exact sequence induces short exact sequences

$$0 \to K_{j+1} \to P_j \to K_j \to 0, \qquad 0 \leq j \leq n-1,$$

where $K_{j+1} = \operatorname{Im}(P_{j+1} \to P_j), 0 \leq j \leq n-2$ and $K_0 = M$. For any A-module N, these sequences yield the exact sequences $\operatorname{Ext}^{n-j}(P_j, N) \to \operatorname{Ext}^{n-j})(K_{j+1}, N) \to \operatorname{Ext}^{n-j+1}(K_j, N) \to \operatorname{Ext}^{n-j+1}(P_j, N), 0 \leq j \leq n-1$. Since P_j is projective, we have $\operatorname{Ext}_A^{n-j}(P_j, N) = 0$, and $\operatorname{Ext}^{n-j+1}(P_j, N) = 0$, for $0 \leq j \leq n-1$, by Proposition 4.1. Thus $\operatorname{Ext}^1(K_n, N) \simeq \operatorname{Ext}^2(K_{n-1}, N) \simeq \ldots \simeq \operatorname{Ext}_A^{n+1}(K_0, N)$. Since $K_0 = M$, we have $\operatorname{Ext}_A^{n+1}(K_0, N) = 0$, and hence $\operatorname{Ext}_A^1(K_n, N) = 0$. Since N is arbitrary, it follows from Proposition 4.1 that K_n is projective.

(iv) $\Rightarrow$ *(i)*. This is clear from the proof of the Proposition 2.8.

Corollary 4.3 *For a non-zero A-module M, we have*

$$\mathrm{hd}_A M = \sup_n \{n \mid \exists \text{ an } A\text{-module } N \text{ with } \operatorname{Ext}_A^n(M, N) \neq 0\}.$$

PROOF: Immediate.

Corollary 4.4 *If M' is a direct summand of M, then $\mathrm{hd}_A M' \leq \mathrm{hd}_A M$.*

PROOF: This follows from the corollary above, since, for $n \in \mathbf{Z}^+$ and an A-module N, $\operatorname{Ext}_A^n(M', N)$ is a direct summand of $\operatorname{Ext}_A^n(M, N)$ by Theorem 2.19(iv).

Proposition 4.5 *Let $0 \to M' \to P \to M'' \to 0$ be an exact sequence of A-module with P projective. Then*

(i) if M'' is projective, so is M'.

(ii) if $\mathrm{hd}_A M'' \geq 1$, then $\mathrm{hd}_A M'' = \mathrm{hd}_A M' + 1$ where both sides may be infinite.

PROOF: If M'' is projective, the sequence $0 \to M' \to P \to M'' \to 0$ splits so that M' is a direct summand of P and hence projective. This proves (i). We now prove (ii). For an A-module N, and $n \in \mathbf{N}$, we have an exact sequence

$$0 \to \operatorname{Ext}_A^n(M', N) \to \operatorname{Ext}_A^{n+1}(M'', N) \to 0,$$

since, P being projective, we have $\operatorname{Ext}_A^n(P, N) = 0 = \operatorname{Ext}_A^{n+1}(P, N)$. The assertion (ii) now follows from Corollary 4.3 to Proposition 4.2.

Lemma 4.6 *Let M be an A-module with $\mathrm{hd}_A M < \infty$. If $a \in A$ is a non-zero divisor of both A and M, then $\mathrm{hd}_{A/Aa} M/aM < \infty$.*

PROOF: We prove the lemma by induction on $\mathrm{hd}_A M$. We may clearly assume $M \neq 0$. If $\mathrm{hd}_A M = 0$, then M is A-projective and hence, by Corollary 2.7, $M/aM = M \otimes A/Aa$ is A/Aa-projective.

Let $\mathrm{hd}_A M > 0$ and let

$$0 \to N \to P \to M \to 0 \qquad (*)$$

be an exact sequence where P is A-projective. By Proposition 4.5, we have $\mathrm{hd}_A N = \mathrm{hd}_A M - 1$. The exact sequence $(*)$ induces the exact sequence

$$\mathrm{Tor}_1^A(M, A/Aa) \to N/aN \to P/aP \to M/aM \to 0$$

of A/Aa-modules. Since a is not a zero divisor of A, we have an exact sequence $0 \to A \xrightarrow{\varphi} A \to A \to A/Aa \to 0$ where φ is the homothesy a_A. This induces the exact sequence $0 \to \mathrm{Tor}_1^A(M, A/Aa) \to M \xrightarrow{a} M$. Since a is not a zero divisor of M, it follows that $\mathrm{Tor}_1^A(M, A/Aa) = 0$. Therefore the sequence,

$$0 \to N/aN \to P/ap \to M/aM \to 0$$

is exact. By induction hypothesis, $\mathrm{hd}_{A/Aa} N/aN < \infty$ and since P/aP is A/Aa-projective, we have $\mathrm{hd}_{A/Aa} M/aM < \infty$, by Proposition 4.5.

4.2 Injective dimension and global dimension

An A-module Q is said to be injective, if, given any diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[dr, "f"'] & M \arrow[d] \\
 & & Q
\end{tikzcd}$$

of A-homomorphisms with exact row, there exists an A-homomorphism $\bar{f}\colon M \to Q$ such that $\bar{f} \circ i = f$.

Proposition 4.7 *An A-module N is injective if and only if any A-homomorphism from any ideal of A into N can be extended to an A-homomorphism of A into N.*

PROOF: Clearly, any injective module has the property stated in the proposition. Suppose now that N is an A-module which has the above property. Let M be any A-module, M' an A-submodule and $f: M' \to N$ an A-homomorphism. We shall prove that f can be extended to an A-homomorphism of M into N.

Let $\mathcal{F}$ be the family of all pairs (P, g), where P is a submodule of M containing M' and $g: P \to N$ an A-homomorphism extending f. This family is non-empty since $(M', f) \in \mathcal{F}$. We introduce a partial order in $\mathcal{F}$ by setting $(P_1, g_1) \leq (P_2, g_2)$ if $P_1 \subset P_2$ and $g_2|P_1 = g_1$. If $(P_\alpha, g_\alpha)_{\alpha \in I}$ is a totally ordered subfamily, let $P = \bigcup_{\alpha \in I} P_\alpha$. Define $g: P \to N$ by setting for $x \in P_\alpha \subset P$, $g(x) = g_\alpha(x)$. It is easily verified that (P, g) belongs to $\mathcal{F}$ and that it is an upper bound of this totally ordered subfamily. By Zorn's lemma, $\mathcal{F}$, has a maximal element (M_1, f_1). We claim that $M_1 = M$. Suppose $M_1 \neq M$ and let $x \in M$, $x \notin M_1$. The map $a \mapsto ax$ of A into M induces an A-isomorphism, $A/\mathfrak{b} \xrightarrow{\sim} Ax$, where $\mathfrak{b}$ is an ideal of A. Under this isomorphism, $M_1 \cap Ax$ corresponds to an ideal $\mathfrak{a}/\mathfrak{b}$ of $A/\mathfrak{b}$. The restriction of f_1 to $M_1 \cap Ax$ induces an A-homomorphism $\mathfrak{a}/\mathfrak{b} \to N$. Composing this with the canonical map $\mathfrak{a} \to \mathfrak{a}/\mathfrak{b}$, we get a homomorphism $\mathfrak{a} \to N$ which vanishes on $\mathfrak{b}$. By our assumption on N, this homomorphism can be extended to a homomorphism $A \to N$ which vanishes on $\mathfrak{b}$ so that we have a homomorphism $f_2: Ax \xrightarrow{\sim} A/\mathfrak{b} \to N$. Define a map $g: M_1 + Ax \to N$ by setting $g|M_1 = f_1$ and $g|Ax = f_2$ so that $(M_1 + Ax, g)$ is in $\mathcal{F}$ contradicting the maximality of (M_1, f_1). The proposition is proved.

We recall that a module M over an integral domain A is said to be divisible if for any $m \in M$ and $0 \neq a \in A$, there exists $n \in M$ such that $m = an$.

Proposition 4.8 *If A is an integral domain, any injective A-module is divisible. If A is a principal ideal domain, any divisible A-module is injective.*

PROOF: Let M be an injective A-module and let $m \in M$, $a \in A$, $a \neq 0$. Define an A-homomorphism $f: Aa \to M$ by setting $f(ba) = bm$. Since, M is injective, f can be extended to a homomorphism $\bar{f}: A \to M$. If $n = \bar{f}(1)$, we clearly have $m = an$.

Suppose next that A is a principal ideal domain and that M is a divisible A-module. Let $f\colon \mathfrak{a} \to M$ be an A-homomorphism, where $\mathfrak{a}$ is any ideal of A. If $\mathfrak{a} = 0$, $f = 0$ and can be trivially extended to a homomorphism $A \to M$. Suppose $\mathfrak{a} = Aa \neq 0$. Let $f(a) = m$. Since M is divisible, there exists $n \in M$ such that $m = an$. Define $\bar{f}\colon A \to M$ by $\bar{f}(b) = bn$. Clearly $\bar{f}$ extends f. Proposition 4.7 now shows that M is injective.

Corollary 4.9 *The $\mathbf{Z}$-modules $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are $\mathbf{Z}$ injective.*

PROOF: Note that $\mathbf{Z}$ is a principal ideal domain and that both $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are divisible.

Proposition 4.10 *Any module is isomorphic to a submodule of an injective module.*

PROOF: Let M be any A-module. We define

$$M^* = \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z}).$$

We make M^* into an A-module by defining for $a \in A, f \in M^*, af \in \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ by $(af)(m) = f(am)$. We have an A-homomorphism $i_M\colon M \to (M^*)^*$ defined by $i_M(m)(f) = f(m), m \in M, f \in M^*$, which is functorial in M. We claim that i_M is injective. In fact, if $x \in M, x \neq 0$, we have a $\mathbf{Z}$-homomorphism $h\colon \mathbf{Z}x \to \mathbf{Q}/\mathbf{Z}$ such that $h(x) \neq 0$; if x is of infinite order (M treated as an abelian group), then choose $h(x)$ to be any non-zero element of $\mathbf{Q}/\mathbf{Z}$ and if x is of finite order n, choose $h(x)$ to be the class $1/n$ in $\mathbf{Q}/\mathbf{Z}$. Since $\mathbf{Q}/\mathbf{Z}$ is $\mathbf{Z}$-injective (Corollary to Proposition 4.8), this extends to a homomorphism $\bar{h}\colon M \to \mathbf{Q}/\mathbf{Z}$ and $\bar{h}(x) = h(x) \neq 0$ i.e. $i_M(x) \neq 0$. This proves that i_M is injective. Let $F \xrightarrow{j} M^* \to 0$ be exact with F a free A-module. We then have an exact sequence $0 \to (M^*)^* \xrightarrow{j^*} F^*$ of A-modules, so that M is isomorphic to a submodule of F^*. The proposition is proved if we show that for any projective A-module P, the module P^* is injective.

Let P be a projective A-module and suppose we are given a diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[dr, "g"'] & M \\
& & P^*
\end{tikzcd}$$

of A-homomorphisms. Since $\mathbf{Q}/\mathbf{Z}$ is $\mathbf{Z}$-injective, $i^*\colon M^* \to M'^*$ is surjective and we have the diagram

$$\begin{tikzcd}
 & P \arrow[r, "i_P"] \arrow[dr] & (P^*)^* \arrow[d, "g^*"] & \\
 & M^* \arrow[r, "i^*"'] & M'^* \arrow[r] & 0
\end{tikzcd}$$

Since P is A-projective, there exists an A-homomorphism $h\colon P \to M^*$ such that $g^* \circ i_P = i^* \circ h$. We then have an A-homomorphism $h^*\colon (M^*)^* \to P^*$. It is easily seen that $h^* \circ i_M \circ i = g$ (where $i_M\colon M \to M^{**}$ is the obvious A-homomorphism defined earlier). This proves that P^* is injective.

Proposition 4.11 *For an A-module N, the following conditions are equivalent : (i) N is injective*

(ii) $\mathrm{Ext}_A^1(M, N) = 0$ *for all A-modules M;*

(iii) $\mathrm{Ext}_A^i(M, N) = 0$ *for all integers $i \geq 1$ and for all A-modules M;*

(iv) $\mathrm{Ext}_A^1(M, N) = 0$ *for all finitely generated A-modules M;*

(v) $\mathrm{Ext}_A^1(A\mathfrak{a}, N) = 0$ *for all ideals $\mathfrak{a}$ of A.*

Proof: (i) $\Rightarrow$ (ii). Let M be any A-module and let $0 \to R \xrightarrow{i} P \xrightarrow{j} M \to 0$ be an exact sequence with P, A-projective. This gives rise to an exact sequence

$$\mathrm{Hom}_A(P, N) \xrightarrow{i^*} \mathrm{Hom}_A(R, N) \to \mathrm{Ext}_A^1(M, N) \to \mathrm{Ext}_A^1(P, N).$$

Since P is projective, $\mathrm{Ext}_A^1(P, N) = 0$. The map i^* is surjective, since N is injective. Hence $\mathrm{Ext}_A^1(M, N) = 0$.

(ii) $\Rightarrow$ (iii). Assume by induction that $\mathrm{Ext}_A^i(M, N) = 0$ for all $i, 1 \leq i \leq n-1, n \geq 2$ and for all A-modules M. We prove $\mathrm{Ext}_A^n(M, N) = 0$ for any A-module M. If $0 \to R \xrightarrow{i} P \xrightarrow{j} M \to 0$ is exact with P projective, we have the exact sequence

$$\mathrm{Ext}_A^{n-1}(R, N) \to \mathrm{Ext}_A^n(M, N) \to \mathrm{Ext}_A^n(P, N).$$

By induction assumption, $\mathrm{Ext}_A^{n-1}(R, N) = 0$. Since P is projective, $\mathrm{Ext}_A^n(P, N) = 0$. Hence $\mathrm{Ext}_A^n(M, N) = 0$. (iii) $\Rightarrow$ (iv). Trivial. (iv)

$\Rightarrow$ (v). Trivial. (v) $\Rightarrow$ (i). For any ideal $\mathfrak{a}$ of A, the exact sequence $o \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ gives rise to an exact sequence

$$\mathrm{Hom}_A(A, N) \to \mathrm{Hom}_A(\mathfrak{a}, N) \to \mathrm{Ext}^1_A(A/\mathfrak{a}, N).$$

By assumption $\mathrm{Ext}^1_A(A/\mathfrak{a}, N) = 0$, so that any A-homomorphism $\mathfrak{a} \to N$ can be extended to an A-homomorphism $A \to N$. By Proposition 4.7, it follows that N is injective.

Let N be a non-zero A-module. The *injective dimension* of N, denoted $\mathrm{inj\,dim}_A N$, is defined by

$$\mathrm{inj\,dim}_A N = \sup\{n \mid \exists \text{ an } A\text{−module } M \text{ with } \mathrm{Ext}^n_A(M, N) \neq 0\},$$

if it exists and otherwise ∞. If $N = 0$, we set $\mathrm{inj\,dim}_A N = -1$. It follows from Proposition 4.11 that an A-module N is injective if and only if $\mathrm{inj\ dim}_A N \leq 0$.

Proposition 4.12 *For any A-module N,*

$$\begin{aligned}\mathrm{inj\ dim}_A N \quad = \quad & \sup\{n \mid \exists \text{a finitely generated } A\text{-module} \\ & \qquad M \text{such that } \mathrm{Ext}^n_A(M, N) \neq 0\}.\end{aligned}$$

PROOF: It suffices to prove that for any integer $i \geq 0$, if $\mathrm{Ext}^i_A(M, N) = 0$ for all finitely generated A-modules M, then $\mathrm{Ext}^i_A(M, N) = 0$ for all A-modules M. We prove this statement by induction on i. If $i = 0$, $N \stackrel{\sim}{\to} \mathrm{Hom}_A(A, N) = \mathrm{Ext}^0_A(A, N)$ so that $\mathrm{Ext}^0_A(A, N) = 0$ implies $N = 0$. For $i = 1$, the assertion follows from Proposition 4.11. Suppose then $i \geq 2$. By Proposition 4.10, there exists an exact sequence $0 \to N \to Q \to Q/N \to 0$, with Q injective. For any A-module M there is an induced exact sequence

$$\mathrm{Ext}^{i-1}_A(M, Q) \to \mathrm{Ext}^{i-1}_A(M, Q/N) \to \mathrm{Ext}^i_A(M, N) \to \mathrm{Ext}^i_A(M, Q).$$

Since Q is injective, it follows, by Proposition 4.11, that $\mathrm{Ext}^{i-1}_A(M, Q) = \mathrm{Ext}^i_A(M, Q) = 0$, so that we have an isomorphism $\mathrm{Ext}^{i-1}_A(M, Q/N) \stackrel{\sim}{\to} \mathrm{Ext}^i_A(M, N)$. Since $\mathrm{Ext}^i_A(M, N) = 0$ for all finitely generated A-modules M. Hence by induction $\mathrm{Ext}^{i-1}_A(M, Q/N) = 0$ for *all* A-modules M which by the above isomorphism again implies that $\mathrm{Ext}^i_A(M, N) = 0$ for all A-modules M. This proves the proposition.

Proposition 4.13 *For any ring A,*

$$\mathrm{gl\ dim}_A = \sup_N \mathrm{inj\ dim}_A N.$$

PROOF:

$$\begin{aligned}
\mathrm{gl}\,.\dim_A &= \sup_M \mathrm{hd}_A M \\
&= \sup_M \sup\{n \mid \exists \text{an } A\text{-module } N \text{ such that } \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup\{n \mid \exists A\text{-modules } M \text{ and } N \text{ such that } \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup_N \sup\{n \mid \exists \text{an } A\text{-module } M \text{ such that } \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup_N \text{inj } \dim_A N.
\end{aligned}$$

Theorem 4.14 *For any ring A*

$$\mathrm{gl}\,.\dim A = \sup_M \{\,\mathrm{hd}_A M \mid M \text{ finitely generated}\}.$$

PROOF:

$$\begin{aligned}
\mathrm{gl}\,.\dim_A &= \sup_N \text{inj } \dim_A N \text{ by Proposition 4.13} \\
&= \sup_N \sup\{n \mid \exists \text{a finitely generated } A\text{-module } M \\
&\qquad\qquad \text{with} \operatorname{Ext}^n_A(M,N) \neq 0\}, \text{ by Proposition 4.12} \\
&= \sup_{\substack{M \text{ finitely} \\ \text{generated}}} \sup\{n \mid \exists \text{an} A\text{-module } N \text{with} \operatorname{Ext}^n_A(M,N) \neq 0\} \\
&= \sup_{\substack{M \text{ finitely} \\ \text{generated}}} \mathrm{hd}_A M \text{ by Corollary4.3Proposition4.2}
\end{aligned}$$

4.3 Global dimension of noetherian local rings

In this section, A denotes a local ring $\mathfrak{m}$ its maximal ideal and $k = A/\mathfrak{m}$ its residue field. All A-modules that we consider are assumed to be finitely generated.

Lemma 4.15 *Let M be an A-module. A set of elements $x_1, \ldots, x_n$ of M is a minimal set of generators of M if and only if their canonical images $\bar{x}_1, \ldots, \bar{x}_n$ in $M/\mathfrak{m}M$ form a basis of the k-vector space $M/\mathfrak{m}M$. In particular, the cardinality of any minimal set of generators of M is equal to the rank of the k-vector space $M/\mathfrak{m}M$.*

PROOF: Clearly, it is enough to prove that $x_1, \ldots, x_n \in M$ generate M over A if and only if $\bar{x}_1, \ldots, \bar{x}_n$ generate $M/\mathfrak{m}M$ over k. If

$x_1, \ldots, x_n \in M$ generate M, then obviously $\bar{x}_1, \ldots, \bar{x}_n$ generate $M/\mathfrak{m}M$ over k. Conversely, suppose $x_1, x_2, \ldots, x_n$ are such that $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ generate $M/\mathfrak{m}M$ over k. Let M' be the submodule of M generated by $x_1, \ldots, x_n$. If $M'' = M/M'$, we have an exact sequence $0 \to M' \stackrel{i}{\to} M \to M'' \to 0$, which induces an exact sequence

$$M'/\mathfrak{m}M' \stackrel{\bar{i}}{\to} M/\mathfrak{m}M \to M''/\mathfrak{m}M'' \to 0.$$

Since $x_1, \ldots, x_n$ are in M' and since $\bar{x}_1, \ldots, \bar{x}_n$ generate $M/\mathfrak{m}M$, $\bar{i}$ is an epimorphism; it follows that $M''/\mathfrak{m}M'' = 0$. Since M'' is finitely generated, we have $M'' = 0$, by Nakayama's lemma and $M' = M$.

Proposition 4.16 *Let A be a local ring and M a finitely generated A-module. Then the following conditions are equivalent:*

(i) M *is free;*

(ii) M *is projective.*

Moreover, if A is noetherian, then (i) and (ii) are also equivalent to

(iii) $\mathrm{Tor}_j^A(M, N) = 0$ *for all A-modules N and all $j \geq 1$;*

(iv) $\mathrm{Tor}_1^A(M, k) = 0$.

Proof: $(i) \Rightarrow (ii)$. Trivial. $(ii) \Rightarrow (i)$. Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of M and let $\varphi\colon F \to M$ be an A-epimorphism, where F is a free A-module with a basis of n elements. If $K = \ker \varphi$, we have the exact sequence

$$0 \to K \to F \stackrel{\varphi}{\to} M \to 0. \qquad (*)$$

Since M is projective, this sequence splits, so that

$$0 \to K/\mathfrak{m}K \to F/\mathfrak{m}F \stackrel{\bar{\varphi}}{\to} M/\mathfrak{m}M \to 0. \qquad (**)$$

is exact. By Lemma 4.15, $\bar{\varphi}$ is an isomorphism, and $K/\mathfrak{m}K = 0$. Since the sequence $(*)$ splits, we have that K is finitely generated. Hence, by Nakayama's lemma, $K = 0$, φ is an isomorphism and M is free.

$(ii) \Rightarrow (iii)$. Since M is projective, M has a projective resolution $0 \to M \stackrel{1_M}{\to} M \to 0$. Using this resolution to compute Tor, (iii) follows.

$(iii) \Rightarrow (iv)$. Trivial.

(iv) $\Rightarrow$ *(i)*. The proof is on the same lines as that of '*(ii)* $\Rightarrow$ *(i)*'. We have only to note that the exactness of $(**)$ is, in this case, a consequence of the hypothesis $\mathrm{Tor}_1^A(M,k)=0$, and K is finitely generated because A is noetherian.

Proposition 4.17 *Let A be a noetherian local ring, M a finitely generated A-module and $n \in \mathbf{Z}^+$. Then the following conditions are equivalent:*

(i) $\mathrm{hd}_A M \leq n$:

(ii) $\mathrm{Tor}_j^A(M,N)=0$ *for all A-modules N and all $j \geq n+1$;*

(iii) $\mathrm{Tor}_1^A(M,k)=0$.

Proof: *(i)* $\Rightarrow$ *(ii)*. Using a projective resolution of length less than or equal to n to compute Tor, we find that $\mathrm{Tor}_j^A(M,N)=0$ for all A-modules N and all $j \geq n+1$.

(ii) $\Rightarrow$ *(iii)*. Trivial. *(iii)* $\Rightarrow$ *(i)*. We prove this by induction on n. If $n=0$, then $\mathrm{Tor}_1^A(M,k)=0$ and by Proposition 4.16, M is free. Hence $\mathrm{hd}_A M \leq 0$. We may therefore assume $n \geq 1$. There exists an exact sequence

$$0 \to M' \to P \to M \to 0,$$

where P is A-projective. This induces the exact sequence

$$\mathrm{Tor}_{n+1}^A(M,k) \to \mathrm{Tor}_n^A(M',k) \to \mathrm{Tor}_n^A(P,k).$$

By assumption, $\mathrm{Tor}_{n+1}^A(M,k)=0$, and by Proposition 4.16, $\mathrm{Tor}_n^A(P,k) = 0$ so that we have $\mathrm{Tor}_n^A(M',k)=0$. Hence by induction hypothesis, we have $\mathrm{hd}_A M' \leq n-1$ and, by Proposition 4.5, we get $\mathrm{hd}_A M \leq n$.

Proposition 4.18 *Let A be a noetherian local ring. For any $n \in \mathbf{Z}^+$, the following statements are equivalent:*

(i) $\mathrm{gl}\,.\dim A \leq n$;

(ii) $\mathrm{Tor}_j^A(M,N)=0$ *for all A-modules M,N and $j \geq n+1$;*

(iii) $\mathrm{Tor}_{n+1}^A(k,k)=0$.

Proof: (i) $\Rightarrow$ (ii). This follows from the above proposition. (ii) $\Rightarrow$ (iii). Trivial. (iii) $\Rightarrow$ (i). Suppose (iii) holds. Then, by the above proposition, $\mathrm{Tor}_{n+1}^A(k, M) = 0$ for all A-modules M. Since by Proposition 2.18, we have $\mathrm{Tor}_{n+1}^A(M, k) \simeq \mathrm{Tor}_{n+1}^A(k, M)$, it follows from the above proposition that if M is finitely generated, then $\mathrm{hd}_A M \leq n$. Since this holds for all finitely generated A-modules M, (i) follows from Theorem 4.14.

Corollary 4.19 *For a noetherian local ring A, we have* $\mathrm{gl}\,.\dim_A = \mathrm{hd}_A k$.

4.4 Regular local rings

In this section, A denotes a noetherian local ring and $\mathfrak{m}$ its maximal ideal. Let $k = A/\mathfrak{m}$ denote the residue field.

Let $\dim_A = r$. By Theorem 3.4, we know that $\mathfrak{m}$ cannot be generated by less than r elements.

A noetherian local ring A of dimension r is said to be *regular* if its maximal ideal can be generated by r elements.

Theorem 4.20 *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $k = A/\mathfrak{m}$. Then the following conditions are equivalent :*

(i) A is regular.

(ii) the rank of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ is equal to $\dim A$;

(iii) the k-algebra $G(A) = \bigoplus_{j \geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$ is isomorphic as a graded k-algebra to a polynomial algebra $k[X_1, \ldots, X_s]$;

(iv) $G(A)$ is isomorphic as a graded k-algebra to the polynomial algebra $k[X_1, \ldots, X_r]$ with $r = \dim A$.

Proof: *(ii)* $\Rightarrow$ *(i)*. Immediate from Lemma 4.15.

(iii) $\Rightarrow$ *(ii)*. Let $\varphi\colon k[X_1, \ldots, X_s] \to G(A)$ be an isomorphism of graded k-algebras and let $x_1, \ldots, x_s \in \mathfrak{m}$ be such that $\varphi(X_j) = x_j$ modulo $\mathfrak{m}^2$, for $1 \leq j \leq s$. By Lemma 4.15, the elements $x_1, \ldots, x_s$ generate $\mathfrak{m}$ and we have, by Corollary to Proposition 3.8, $\deg P_{\mathfrak{m}}(A, n) = s$. On the other hand, since $\mathfrak{m}/\mathfrak{m}^2 = G(A)_1 \simeq k[X_1, \ldots, X_s]_1$ we have $\mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2 = s$. This proves (ii).

(iv) $\Rightarrow$ (iii). Trivial.

(i) $\Rightarrow$ (iv). Let $\{x_1, \dots, x_r\}$ be a set of generators of $\mathfrak{m}$, with $r = \dim_A$. Let $\varphi: k[X_1, \dots, X_r] \to G(A)$ be the graded k-algebra homomorphism defined by $\varphi(X_j) = x_j$ modulo $\mathfrak{m}^2$, $1 \leq j \leq r$. Since $\deg P_{\mathfrak{m}}(A, n) = r$, we have, by Corollary to Proposition 3.8, that φ is an isomorphism.

Corollary 4.21 *A regular local ring is an integral domain.*

Proof: Since $G(A) \simeq k[X_1, \dots, X_r]$ is an integral domain, the Corollary follows from Lemma 1.42.

Let A be a regular local ring of dimension r. Any set of generators for $\mathfrak{m}$ consisting of r elements is called a *regular system of parameters of* A.

Proposition 4.22 *Let A be a regular local ring of dimension r and let $a_1, \dots, a_j$ be any j elements of $\mathfrak{m}$, $0 \leq j \leq r$. Then the following statements are equivalent:*

(i) $\{a_1, \dots, a_j\}$ is a part of a regular system of parameters of A;

(ii) the images $\bar{a}_1, \dots, \bar{a}_j$ of $a_1, \dots, a_j$ under the canonical map $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k;

(iii) $A/(a_1, \dots, a_j)$ is a part of a regular local ring of dimension $r - j$.

Proof: (i) $\Leftrightarrow$ (ii). Trivial consequences of Lemma 4.15.

(i) $\Rightarrow$ (iii). Let $\bar{A} = A/(a_1, \dots, a_j)$ and let $\bar{\mathfrak{m}} = \mathfrak{m}/(a_1, \dots, a_j)$. Let $a_1, \dots, a_j, a_{j+1}, \dots, a_r$ be a regular system of parameters of A. Then the canonical images of $a_{j+1}, \dots, a_r$ in $\bar{\mathfrak{m}}$ obviously generate $\bar{\mathfrak{m}}$ and hence, by Theorem 3.4, $\dim \bar{A} \leq r - j$. Let $s = \dim \bar{A}$ and let $b_1, \dots, b_s \in \mathfrak{m}$ be such that if $\bar{b}_1, \dots, \bar{b}_s$ are their canonical images in $\bar{\mathfrak{m}}$, then $\bar{A}/(\bar{b}_1, \dots, \bar{b}_s)$ is of finite length. Since $A/(a_1, \dots, a_j,\ b_1, \dots, b_s) \simeq \bar{A}/(\bar{b}_1, \dots, \bar{b}_s)$, we have $s + j \geq \dim A = r$. Thus $\dim \bar{A} = r - j$. Since $\bar{\mathfrak{m}}$ is generated by $r - j$ elements, $\bar{A}$ is regular and (iii) is proved.

(iii) $\Rightarrow$ (i). Let $a_{j+1}, \dots, a_r \in \mathfrak{m}$ be such that their canonical images in $\bar{\mathfrak{m}} = \mathfrak{m}/(a_1, \dots, a_j)$ generate $\bar{\mathfrak{m}}$. Then $a_1, \dots, a_j,\ a_{j+1}, \dots, a_r$ generate $\mathfrak{m}$ and (i) is proved.

Corollary 4.23 *Let $\{a_1, \dots, a_j\}$ be a part of a regular system of parameters of a regular local ring A. Then $\mathfrak{p} = (a_1, \dots, a_j)$ is a prime ideal of A of height j.*

PROOF: Since, by the above Proposition, $A/\mathfrak{p}$ is a regular local ring, it is an integral domain by Corollary to Theorem 4.20, and hence $\mathfrak{p}$ is a prime ideal. We show by induction on j that $\operatorname{ht}\mathfrak{p} = j$. If $j = 0$, then $\mathfrak{p} = 0$ and $\operatorname{ht}\mathfrak{p} = 0$. Let $j > 0$. By induction hypothesis, the ideal $(a_1, \ldots, a_{j-1})$ is a prime ideal of height $j-1$. Moreover, it is properly contained in $\mathfrak{p}$, since $\{a_1, \ldots, a_j\}$ is a minimal system of generators of $\mathfrak{p}$. Thus $\operatorname{ht}\mathfrak{p} \geq j$. On the other hand, by Corollary 4.4 to Theorem 3.10, $\operatorname{ht}\mathfrak{p} \leq j$. Thus $\operatorname{ht}\mathfrak{p} = j$ and this completes the proof.

Let M be a non-zero A-module. A sequence $a_1, \ldots, a_r$ of elements of $\mathfrak{m}$ is called an *M-sequence* if a_i is not a zero-divisor of $M/(a_1, \ldots, a_{i-1})M$ for $1 \leq i \leq r$. (For $i = 1$, the condition means that a_1 is not a zero-divisor of M.)

Proposition 4.24 *Let M be a non-zero A-module and $a_1, \ldots, a_r$ an M-sequence. Then $r \leq \dim M$.*

PROOF: We use induction on r. For $r = 0$, there is nothing to prove. Assume $r > 0$ and let $M'' = M/a_1M$. Since a_1 is not a zero-divisor of M, we have an exact sequence $0 \to M \xrightarrow{\varphi} M \to M'' \to 0$, where φ is the homothesy by a_1. Applying Proposition 3.6 to this exact sequence, we get $P_{\mathfrak{m}}(M'', n) = R(n)$, where $R(n)$ is a polynomial function of degree less than $\deg P_{\mathfrak{m}}(M, n)$, i.e. $\dim M'' < \dim M$. Since clearly $a_2, \ldots, a_r$ is an M''-sequence, we have by induction hypothesis, $r - 1 \leq \dim M'' \leq \dim M - 1$. This proves the proposition.

Corollary 4.25 *A noetherian local ring A is regular if and only if its maximal ideal is generated by an A-sequence.*

PROOF: Let A be regular and let $\{a_1, \ldots, a_r\}$ be a regular system of parameters of A. Then from Proposition 4.22, it follows that $a_1, \ldots, a_r$ is an A-sequence. Conversely, suppose $a_1, \ldots, a_r$ is an A-sequence generating $\mathfrak{m}$. Then, by Proposition 4.24, we have $r \leq \dim A$. Also, by Theorem 3.4, $\dim A \leq r$. Thus $r = \dim A$ and the corollary is proved.

4.5 A homological characterisation of regular local rings

The aim of this section is to prove that a local ring A is regular if and only if $\operatorname{gl.dim} A < \infty$.

In this section, A denotes, as before a noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field $A/\mathfrak{m}$.

Lemma 4.26 *If $a \in \mathfrak{m} - \mathfrak{m}^2$, then the exact sequence*

$$0 \to Aa/\mathfrak{m}a \to \mathfrak{m}/\mathfrak{m}a \to \mathfrak{m}/Aa \to 0$$

of A/Aa-modules splits.

PROOF: Let $d = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2$. Since $a \notin \mathfrak{m}^2$ there exist by Lemma 4.15, $a_1, \ldots, a_{d-1} \in \mathfrak{m}$ such that $\{a, a_1, \ldots, a_{d-1}\}$ is a minimal set of generators of $\mathfrak{m}$. Let $\mathfrak{a} = (a_1, \ldots, a_{d-1})$. Let $b \in A$ be such that $ba \in \mathfrak{a}$. Then by the minimality of $\{a, a_1, \ldots, a_{d-1}\}$ as a set of generators for $\mathfrak{m}$, b cannot be a unit, so that $b \in \mathfrak{m}$. Thus $\mathfrak{a} \cap Aa \subset \mathfrak{a} \cap \mathfrak{m}a$. Clearly $\mathfrak{a} \cap Aa \supset \mathfrak{a} \cap \mathfrak{m}a$, so that $\mathfrak{a} \cap Aa = \mathfrak{a} \cap \mathfrak{m}a$. We now have $\mathfrak{a} + \mathfrak{m}a/\mathfrak{m}a \simeq \mathfrak{a}/\mathfrak{a} \cap \mathfrak{m}a = \mathfrak{a}/\mathfrak{a} \cap Aa \simeq \mathfrak{a} + Aa/Aa = \mathfrak{m}/Aa$, which shows that the canonical homomorphism $\mathfrak{m}/\mathfrak{m}a \to \mathfrak{m}/Aa$ maps $\mathfrak{a} + \mathfrak{m}a/\mathfrak{m}a$ is isomorphically onto $\mathfrak{m}/Aa$. Hence the exact sequence splits.

Corollary 4.27 *Let A be a noetherian local ring with $\operatorname{gl.dim} A < \infty$. If $a \in \mathfrak{m} - \mathfrak{m}^2$ is not a zero divisor of A, then $\operatorname{gl.dim} A/Aa < \infty$.*

PROOF: We have an A/Aa-isomorphism $(A/Aa)/(\mathfrak{m}/Aa) \simeq A/\mathfrak{m} = k$, and hence an exact sequence

$$0 \to \mathfrak{m}/Aa \to A/Aa \to k \to 0$$

of A/Aa-modules. By Corollary to Proposition 4.18, we now have $\operatorname{gl.dim} A/Aa = \operatorname{hd}_{A/Aa} k$. To prove the corollary, it suffices, in view of Proposition 4.5, to show that $\operatorname{hd}_{A/Aa} \mathfrak{m}/Aa < \infty$. By hypothesis, $\operatorname{gl.dim} A < \infty$ and hence $\operatorname{hd}_A \mathfrak{m} < \infty$. By Lemma 4.6, we have $\operatorname{hd}_{A/Aa} \mathfrak{m}/\mathfrak{m}a < \infty$. By the above lemma, $\mathfrak{m}/Aa$ is a direct summand of $\mathfrak{m}/\mathfrak{m}a$. Now Corollary 4.4 to Proposition 4.2 shows that $\operatorname{hd}_{A/Aa} \mathfrak{m}/Aa < \infty$ and this completes the proof.

Lemma 4.28 *Let M be a non-zero A-module and let $a \in \mathfrak{m}$ be not a zero divisor of M. Then $\operatorname{hd}_A M/aM = \operatorname{hd}_A M + 1$, where both sides may be infinite.*

PROOF: The exact sequence

$$0 \to M \xrightarrow{a_M} M \longrightarrow M/aM \to 0$$

induces an exact sequence

$$\operatorname{Tor}_{n+1}^A(M,k) \longrightarrow \operatorname{Tor}_{n+1}^A(M/aM,k) \longrightarrow \operatorname{Tor}_n^A(M,k) \xrightarrow{\operatorname{Tor}_n^A(a_M,k)} \operatorname{Tor}_n^A(M,k)$$

for every $n \in \mathbf{Z}^+$. Now $\operatorname{Tor}_n^A(a_M,k) = a\operatorname{Tor}_n^A(1_M,k) = \operatorname{Tor}_n^A(1_M,a_k) = 0$, since a being in $\mathfrak{m}$, a_k is zero. Thus the sequence

$$\operatorname{Tor}_{n+1}^A(M,k) \to \operatorname{Tor}_{n+1}^A(M/aM,k) \to \operatorname{Tor}_n^A(M,k) \to 0$$

is exact. The lemma now follows from Proposition 4.17.

Lemma 4.29 *Let A be a noetherian local ring such that $\mathfrak{m} \neq \mathfrak{m}^2$ and such that every element of $\mathfrak{m} - \mathfrak{m}^2$ is a zero-divisor. Then any A-module of finite homological dimension is free.*

PROOF: By Proposition 1.21, we have

$$\mathfrak{m} - \mathfrak{m}^2 \subset \bigcup_{\mathfrak{p} \in \operatorname{Ass}(A)} \mathfrak{p}$$

This means that $\mathfrak{m} \subset \bigcup_{\mathfrak{p}\in \operatorname{Ass}(A)} \mathfrak{p} \cup \mathfrak{m}^2$ and since $\mathfrak{m} \neq \mathfrak{m}^2$, we have by Lemma 1.12 that $\mathfrak{m} \in \operatorname{Ass}(A)$ and we have an A-monomorphism $k = A/\mathfrak{m} \hookrightarrow A$. Let M be any A-module with $\operatorname{hd}_A M = n < \infty$. If $n = -1$, then $M = 0$ and there is nothing to prove. Let $n \geq 0$. The exact sequence $0 \to k \to A \to A/k \to 0$ induces the exact sequence

$$\operatorname{Tor}_{n+1}^A(M, A/k) \to \operatorname{Tor}_n^A(M,k) \to \operatorname{Tor}_n^A(M,A).$$

By Proposition 4.17, we have $\operatorname{Tor}_{n+1}^A(M,A/k) = 0$ and $\operatorname{Tor}_n^A(M,k) \neq 0$. This implies that $\operatorname{Tor}_n^A(M,A) \neq 0$ which implies $n = 0$. Hence M is projective. Proposition 4.16 now proves the lemma.

Theorem 4.30 *Let A be a noetherian local ring. Then A is regular if and only if $\operatorname{gl.dim} A < \infty$ and moreover, if $\operatorname{gl.dim} A < \infty$, then $\operatorname{gl.dim} A = \dim A$.*

PROOF: By Corollary to Proposition 4.24, it is enough to show that the maximal ideal $\mathfrak{m}$ of A is generated by an A-sequence if and only if $\operatorname{gl.dim} A < \infty$ and that, in this case, $\operatorname{gl.dim} A = \dim A$.

Let $\mathfrak{m}$ be generated by an A-sequence $a_1, \ldots, a_r$. Then by repeated applications of Lemma 4.28, it follows that $\operatorname{hd}_A A/\mathfrak{m} = r$. Therefore by Corollary to Proposition 4.18, we have $\operatorname{gl.dim} A = r < \infty$. Moreover by Proposition 4.24, we have $r \leq \dim A$. Now Theorem 3.4 implies that $\dim A \leq r$, so that $\mathrm{gl}\,.\dim A = r = \dim A$.

To complete the proof of the theorem, it is now enough to show that if $\mathrm{gl}\,.\dim A < \infty$, then $\mathfrak{m}$ is generated by an A-sequence. Let then $\mathrm{gl}\,.\dim A < \infty$. We use induction on $r = \mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2$ to prove that $\mathfrak{m}$ is generated by an A-sequence. If $r = 0$, then $\mathfrak{m} = \mathfrak{m}^2$ and, by Nakayama's lemma, $\mathfrak{m} = 0$ showing that $\mathfrak{m}$ is generated by the empty sequence. Let now $r > 0$. If every element of $\mathfrak{m} - \mathfrak{m}^2$ is a zero-divisor then, since $\mathrm{hd}_A A/\mathfrak{m} < \infty$, we have, by Lemma 4.29, that $A/\mathfrak{m}$ is free; therefore $\mathfrak{m} = 0$, contradicting the assumption that $r > 0$. Thus there exists $a \in \mathfrak{m} - \mathfrak{m}^2$ which is not a zero divisor. Then, by Corollary to Lemma 4.26, we have $\mathrm{gl}\,.\dim A/Aa < \infty$. Let $\bar{\mathfrak{m}} = \mathfrak{m}/Aa$; since $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$ is a k-vector space of rank$r-1$, we see by induction that $\mathfrak{m}/Aa$ is generated by an A/Aa-sequence $\bar{a}_1, \ldots, \bar{a}_{r-1}$ where $a_i \in \mathfrak{m}$ and $\bar{a}_i$ is the class of a_i modulo Aa, for $1 \leq i \leq r-1$. Then clearly, $a, a_1, \ldots, a_{r-1}$ is an A-sequence which generates $\mathfrak{m}$, and the theorem is proved.

Corollary 4.31 *Let A be a regular local ring and let $\mathfrak{p}$ be a prime ideal of A. Then $A_\mathfrak{p}$ is a regular local ring.*

PROOF: In view of the above theorem, it is enough to show that $\mathrm{gl}\,.\dim A_\mathfrak{p} \leq \mathrm{gl}\,.\dim A$. Let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to A/\mathfrak{p} \to 0 \qquad (*)$$

be an A-free resolution of the A-module $A/\mathfrak{p}$ with $n \leq \mathrm{gl}\,.\dim A$. In view of Proposition 1.1 and 1.2, we obtain by tensoring $(*)$ with $A_\mathfrak{p}$, an $A_\mathfrak{p}$-free resolution

$$0 \to F_n \otimes_A A_\mathfrak{p} \to F_{n-1} \otimes_A A_\mathfrak{p} \to \cdots \to F_0 \otimes_A A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to 0$$

of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ which shows that $\mathrm{hd}_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \leq n$. Now Corollary 4.19 shows that $\mathrm{gl}\,.\dim A_\mathfrak{p} = \mathrm{hd}_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \leq n \leq \mathrm{gl}\,.\dim A$.

Chapter 5

Unique Factorisation in Regular Local Rings

5.1 Locally free modules and a "cancellation lemma"

Lemma 5.1 *Let A be a noetherian ring and let S be a multiplicative subset of A. Let M be a finitely generated A-module. Then for any A-module N, the canonical map*

$$\varphi_M \colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \text{ given by } f \mapsto S^{-1}f$$

induces an $S^{-1}A$-isomorphism

$$\bar{\varphi}_M \colon S^{-1}\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N),$$

which is functorial both in M and N.

Proof: If $M = A$, and if $\operatorname{Hom}_A(A, N)$ and $\operatorname{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N)$ are identified respectively with N and $S^{-1}N$, φ_A is simply the canonical map i_N and hence $\bar{\varphi}_A = 1_{S^{-1}N}$. Since both Hom and S^{-1} are additive functors, it follows that $\bar{\varphi}_M$ is an isomorphism for any finitely generated free A-module M. Let now M be any finitely generated A-module. Since A is noetherian, we have an exact sequence

$$F_1 \to F_0 \to M \to 0$$

where F_0 and F_1 are finitely generated free A-modules. We then have the exact sequences

$$0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(F_0, N) \to \operatorname{Hom}_A(F_1, N)$$

and

$$S^{-1}F_1 \to S^{-1}F_0 \to S^{-1}M \to 0$$

and hence we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 \to S^{-1}\operatorname{Hom}_A(M,N) & \longrightarrow & S^{-1}\operatorname{Hom}_A(F_0,N) & \longrightarrow & S^{-1}\operatorname{Hom}_A(F_1,N) \\
\Big\downarrow \bar{\varphi}_M & & \Big\downarrow \bar{\varphi}_{F_0} & & \Big\downarrow \bar{\varphi}_{F_1} \\
0 \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}F_0, S^{-1}N) & \longrightarrow & \operatorname{Hom}_{S^{-1}A}(S^{-1}F_1, S^{-1}N)
\end{array}$$

Since $\bar{\varphi}_{F_0}$ and $\bar{\varphi}_{F_1}$ are isomorphisms, it follows easily that $\bar{\varphi}_M$ is an isomorphism.

Lemma 5.2 *Let A be a noetherian ring and P a finitely generated A-module. Then P is projective if and only if, $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free for every $\mathfrak{p} \in \operatorname{Spec}(A)$.*

Proof: Let P be projective. It follows immediately from Corollary 2.7 to Proposition 2.4 and Proposition 4.16 that $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Conversely, let $P_{\mathfrak{p}}$ be $A_{\mathfrak{p}}$-free for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $F \to P \to 0$ be any exact sequence, where F is a finitely generated free A-module. To prove that P is projective, we need to show that

$$\operatorname{Hom}_A(P,F) \xrightarrow{\psi} \operatorname{Hom}_A(P,P) \to 0$$

is exact. Let $C = \operatorname{coker}\psi$. Since P is finitely generated, $\operatorname{Hom}_A(P,P)$ and hence C is finitely generated. To prove the exactness of the above sequence, we need, in view of Proposition 1.26, only show that $C_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Let then $\mathfrak{p}$ be in $\operatorname{Spec}(A)$. By Proposition 1.1, we have $C_{\mathfrak{p}} = \operatorname{coker}\psi_{\mathfrak{p}}$. By Lemma 5.1, we have the commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}_A(P,F)_{\mathfrak{p}} & \xrightarrow{\psi_{\mathfrak{p}}} & \operatorname{Hom}_A(P,P)_{\mathfrak{p}} \\
\bar{\varphi}_P \Big\downarrow \simeq & & \simeq \Big\downarrow \bar{\varphi}_P \\
\operatorname{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, F_{\mathfrak{p}}) & \xrightarrow{\theta} & \operatorname{Hom}_{A_{\mathfrak{p}}}(P_{\mathfrak{p}}, P_{\mathfrak{p}})
\end{array}$$

where the vertical maps are isomorphisms. Since $P_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-free by hypothesis, it follows that the sequence $F_{\mathfrak{p}} \to P_{\mathfrak{p}} \to 0$ splits and hence θ is surjective. Therefore $\psi_{\mathfrak{p}}$ is surjective, $C_{\mathfrak{p}} = 0$ and the lemma follows.

Lemma 5.3 *Let F be a finitely generated free module over a non-zero ring A. Then, any two bases of F have the same cardinality.*

PROOF: Let $\mathfrak{m}$ be a maximal ideal of A. If $\{e_1, \ldots, e_n\}$ is an A-basis of F, then $\{1 \otimes e_1, \ldots, 1 \otimes e_n\} \subset A/\mathfrak{m} \otimes_A F = \bar{F}$ is clearly an $A/\mathfrak{m}$-basis of $\bar{F}$. Since $A/\mathfrak{m}$ is a field, the lemma follows from the corresponding result for vector spaces.

The number of elements in any basis of a finitely generated free A-module F is called the rank of F (over A) and denoted $\operatorname{rank}_A F$.

Let P be a finitely generated projective A-module. Then, if $\mathfrak{p}$ is any prime ideal of A, we know by Proposition 4.16 that $P_{\mathfrak{p}}$ is a (finitely generated) free A-module. We define the *rank of* P at $\mathfrak{p}$ to be the rank of the free module $P_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$. We thus have a map $\operatorname{rank}_A P$: $\operatorname{Spec}(A) \to \mathbf{Z}^+$ defined by $\mathfrak{p} \mapsto \operatorname{rank}_{A_{\mathfrak{p}}} P_{\mathfrak{p}}$. We say that P has *constant rank* n if $\operatorname{rank}_A P$ is the constant map n. Note that any free module has constant rank.

Proposition 5.4 *If $0 \to P' \to P \to P'' \to 0$ is an exact sequence of finitely generated projective A-modules, then*

$$\operatorname{rank}_A P = \operatorname{rank}_A P' + \operatorname{rank}_A P''.$$

PROOF: For any $\mathfrak{p} \in \operatorname{Spec}(A)$, we have, by Proposition 1.1, the exact sequence $0 \to P'_{\mathfrak{p}} \to P_{\mathfrak{p}} \to P''_{\mathfrak{p}} \to 0$ of finitely generated free $A_{\mathfrak{p}}$-modules. Since $P_{\mathfrak{p}} \simeq P'_{\mathfrak{p}} \oplus P''_{\mathfrak{p}}$, it follows that $\operatorname{rank}_{A_{\mathfrak{p}}} P_{\mathfrak{p}} = \operatorname{rank}_{A_{\mathfrak{p}}} P'_{\mathfrak{p}} + \operatorname{rank}_{A_{\mathfrak{p}}} P''_{\mathfrak{p}}$. This proves the proposition.

Proposition 5.5 *Let $\mathfrak{a}$ be an ideal of A which is a projective A-module. Then $\operatorname{rank}_A \mathfrak{a} \leq 1$, i.e. $\operatorname{rank}_{A_{\mathfrak{p}}} \mathfrak{a}_{\mathfrak{p}} \leq 1$ for every $\mathfrak{p}$ in $\operatorname{Spec}(A)$.*

PROOF: Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $\mathfrak{a}_{\mathfrak{p}}$, being a free $A_{\mathfrak{p}}$-module, is a principal ideal of $A_{\mathfrak{p}}$, since any two distinct elements of a commutative ring R are R-dependent. Hence $\operatorname{rank}_{A_{\mathfrak{p}}} \mathfrak{a}_{\mathfrak{p}} \leq 1$.

Let M be an A-module. A *finite free resolution* of M, is an exact sequence

$$0 \to F_n \to \cdots \to F_0 \to M \to 0,$$

where $n \in \mathbf{Z}^+$ and $F_0, \ldots, F_n$ are finitely generated free A-modules.

Lemma 5.6 *Let P be a projective A-module which has a finite free resolution. Then there exists a finitely generated free A-module F such that $P \oplus F$ is a finitely generated free A-module.*

Proof: Let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to P \to 0$$

be finite free resolution of P. We prove the lemma by induction on n. If $n = 0$, $P \simeq F_0$ and the lemma is trivially proved. Let then $n > 0$ and let $K = \ker(F_0 \to P)$. Then $F_0 \simeq P \oplus K$. Thus K is projective and has a finite free resolution

$$0 \to F_n \to \cdots \to F_1 \to K \to 0,$$

so that, by induction hypothesis, there exists a finitely generated free A-module G such that $K \oplus G = F$ is finitely generated and free. Now $P \oplus F \simeq P \oplus (K \oplus G) \simeq (P \oplus K) \oplus G \simeq F_0 \oplus G$ is finitely generated and free.

Lemma 5.7 *("**Cancellation lemma**") Let P be an A-module such that $P \oplus A^n \simeq A^{n+1}$. Then $P \simeq A$.*

Proof: Clearly P is projective and, in view of Proposition 5.4, is of constant rank 1, i.e. $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank 1 for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Therefore for every $\mathfrak{p} \in \operatorname{Spec}(A)$, we have $(\wedge^i P)_{\mathfrak{p}} \simeq A_{\mathfrak{p}} \otimes_A \wedge^i P \simeq \wedge^i (A_{\mathfrak{p}} \otimes_A P) \simeq \wedge^i P_{\mathfrak{p}} = 0$ for $i > 1$. Hence, by Proposition 1.26, we have $\wedge^i P = 0$ for $i > 1$. Now, we have $A \simeq \wedge^{n+1} A^{n+1} \simeq \wedge^{n+1}(P \oplus A^n) \simeq \bigoplus_{0 \leq i \leq n+1} \wedge^i P \otimes A \wedge^{n+1-i} A^n = (\wedge^0 P \otimes_A \wedge^{n+1} A^n) \oplus (\wedge^1 P \otimes_A \wedge^n A^n) \simeq P \otimes_A A \simeq P$, since $\wedge^{n+1} A^n = 0$.

Corollary 5.8 *Let $\mathfrak{a}$ be a non-zero projective ideal of a ring A such that $\mathfrak{a}$ has a finite free resolution. Then $\mathfrak{a} \simeq A$.*

Proof: By Lemma 5.6, there exist finitely generated free A-modules F and F_1 such that $\mathfrak{a} \oplus F \simeq F_1$. It follows from Proposition 5.4 that $\mathfrak{a}$ has constant rank. Further, since $\mathfrak{a}$ is a non-zero ideal, it follows from Proposition 5.5 that rank $\mathfrak{a} = 1$. Hence, if $F \simeq A^n$, then $F_1 \simeq A^{n+1}$, and the corollary now follows from Lemma 5.7.

5.2 Unique factorization in regular local rings

Let A be an integral domain. An element $p \in A$ is said to be a *prime* if Ap is a prime ideal. An integral domain A is called a *unique factorization domain* if every element can be written in the form $u \prod_{1 \leq i \leq n} p_i$ where p_i are primes, $u \in A$ is a unit and $n \in \mathbf{Z}^+$.

Lemma 5.9 *Let A be a noetherian domain. Then A is a unique factorization domain if and only if every prime ideal of height 1 of A is principal.*

PROOF: Let A be a unique factorization domain and let $\mathfrak{p}$ be a prime ideal of height 1. Let $a \in \mathfrak{p}$, $a \neq 0$. Let $p \in A$ be a prime dividing a. Then $Ap \subset \mathfrak{p}$ is a non-zero prime ideal. Since $\operatorname{ht}\mathfrak{p} = 1$, it follows that $\mathfrak{p} = Ap$.

Conversely, suppose that every prime ideal of height 1 is principal. Since A is noetherian, every element can be written as $u\prod_{i\leq i\leq n} p_i$, p_i being irreducible. (Recall that an element $a \in A$ is *irreducible* if it is not a unit and its only divisors are units of A and those of the form ua, where u is a unit of A.) We need therefore only to show that any irreducible element of A is prime. Let $a \in A$ be irreducible and let $\mathfrak{p}$ be a minimal prime ideal containing Aa. By Corollary 3.13 to Theorem 3.10, we have $\operatorname{ht}\mathfrak{p} = 1$. Therefore $\mathfrak{p} = Ap$, for some $p \in A$. Clearly, p is a prime dividing a and hence $p = ua$ for some unit of u of A. Therefore $Aa = Ap$ and a is a prime.

Theorem 5.10 *Any regular local ring is unique factorization domain.*

PROOF: Let A be regular local ring of dimension r. We prove the theorem by induction on r. If $r = 0$, then A is a field, and there is nothing to prove. Let then $r \geq 1$. In view of Lemma 5.9, we need only to show that any prime ideal of height 1 is principal. Let then $\mathfrak{p}$ be a prime ideal of height 1. Since $r \geq 1$, we have $\mathfrak{m} \neq \mathfrak{m}^2$, by Nakayama's Lemma. Let $a \in \mathfrak{m} - \mathfrak{m}^2$. Then, by Lemma 4.15 and Corollary to Proposition 4.22, a is a prime element. Let $S = \{1, a, a^2, \ldots\}$ and $B = S^{-1}A$. If $a \in \mathfrak{p}$, then $Aa = \mathfrak{p}$, since $\operatorname{ht}\mathfrak{p} = 1$. We may therefore assume $a \notin \mathfrak{p}$. Then $\mathfrak{p}B$ is a prime ideal of B of height 1. Let $\mathfrak{q}B$ be a prime ideal of B, where $\mathfrak{q}$ is a prime ideal of A (hence $a \notin \mathfrak{q}$ so that $\mathfrak{q} \neq \mathfrak{m}$). Then, clearly, $B_{\mathfrak{q}B} = A_{\mathfrak{q}}$. Since $\mathfrak{q} \neq \mathfrak{m}$, $B_{\mathfrak{q}B}$ is a local ring of dimension less than r, and by Corollary to Theorem 4.30, $B_{\mathfrak{q}B}$ is regular. By induction hypothesis, $B_{\mathfrak{q}B}$ is a unique factorization domain. Now, if $\mathfrak{p}B_{\mathfrak{q}B} \neq B_{\mathfrak{q}B}$, then $\mathfrak{p}B_{\mathfrak{q}B}$, being a prime ideal of $B_{\mathfrak{q}B}$ of height 1, is principal, by Lemma 5.9. Therefore, by Lemma 5.2, $\mathfrak{p}B$ is B-projective. Since A is regular, its global dimension is finite by Theorem 4.30. Thus by Corollary 2.10 to Proposition 2.8, Proposition 4.2 and Proposition 4.16, $\mathfrak{p}$ admits of a finite free resolution as an A-module. Proposition 1.1 now implies that $\mathfrak{p}B$ has a finite free resolution as a B-module. Therefore, by Corollary 5.8, $\mathfrak{p}B$ is principal. Let $p \in \mathfrak{p}$ be such that $\mathfrak{p}B = Bp$. Let $n \in \mathbf{Z}^+$ be such that $a^n | p$, $a^{n+1} \not| p$, and let $p = a^n q$. Since $a^n \notin \mathfrak{p}$, we have $q \in \mathfrak{p}$. Also $\mathfrak{p}B = Bq$. By replacing p by q, we may therefore assume that $a \not| p$. We claim that $\mathfrak{p} = Ap$. In fact, by the proof of Proposition 1.3, we have $\mathfrak{p}B \cap A = \mathfrak{p}$, so that we need only to show that $Bp \cap A = Ap$. Clearly, $Ap \subset Bp \cap A$. Let $cp/a^m \in A$, with $m \in \mathbf{Z}^+$ and $c \in A$. Since $a \not| p$, and a is a prime, it follows that a^m / c, i.e. $(c/a^m)p \in Ap$, and the theorem is proved.

EXERCISES

(In what follows, A, B denote commutative rings with 1.)

CHAPTER 0

(1) Show that for a fixed A-module N, the assignment $M \mapsto \operatorname{Hom}_A(M, N)$ is a contravariant A-linear functor and the assignment $M \mapsto \operatorname{Hom}_A(N, M)$ is a covariant A-linear functor. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, show that the sequences

$$0 \to \operatorname{Hom}_A(M'', N) \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M', N)$$

and

$$0 \to \operatorname{Hom}_A(N, M') \to \operatorname{Hom}_A(N, M) \to \operatorname{Hom}_A(N, M'')$$

are exact. (We say that $\operatorname{Hom}_A(M, N)$ is *left-exact* in both M and N.) Give examples to show that $\operatorname{Hom}_A(M, N)$ is *not* exact in either variable.

(2) Let M be an A-module. Show that the map $\operatorname{Hom}_A(A, M) \to M$ given by $f \mapsto f(1)$ is an isomorphism of A-modules which is functorial in M.

(3) Let N be an A-module. Show that the functor $M \mapsto M \oplus N$ is additive if and only if $N = 0$.

(4) Let $M_0 \to M_1 \to M_2 \to \cdots \to M_n$ be an exact sequence of A-modules. If T is an exact functor from A-modules to B-modules, show that

$$T(M_0) \to T(M_1) \to T(M_2) \to \cdots \to T(M_n)$$

is exact.

(5) Let $\{ M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i'' \}_{i \in I}$ be a family of sequence of A homomorphisms and let $M' = \bigoplus_i M_i'$, $M = \bigoplus_i M_i$, $M'' = \bigoplus_i M_i''$, $f = \oplus_i f_i$

and $g = \oplus_i g_i$. Show that $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact if and only if $0 \to M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i'' \to 0$ is exact for every $i \in I$.

(6) Let T be an exact functor from A-modules to B-modules. Let M be an A-module. For a submodule N of M, identify $T(N)$ with a submodule of $T(M)$ in a natural way and show that if N_1, N_2 are submodules of M, then $T(N_1 \cap N_2) = T(N_1) \cap T(N_2)$ and $T(N_1 + N_2) = T(N_1) + T(N_2)$.

(7) Let $M' \to M \to M''$ be homomorphisms of A-modules. If for every A-module N, the sequence $0 \to \operatorname{Hom}_A(N, M') \to \operatorname{Hom}_A(N, M) \to \operatorname{Hom}(N, M'') \to 0$ is exact, show that $0 \to M' \to M \to M'' \to 0$ is a split exact sequence. Similarly if, for every A-module N, the sequence $0 \to \operatorname{Hom}_A(M'', N) \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}(M', N) \to 0$ is exact, show that $0 \to M' \to M \to M'' \to 0$ is split exact. What can you say if, in the above $\operatorname{Hom}_A(.,.)$ is replaced by $\otimes_A$?

(8) Show that if M, N are A-modules, then for any ideal $\mathfrak{a}$ of A, we have

$$(M/\mathfrak{a}M) \otimes_{A/\mathfrak{a}} (N/\mathfrak{a}N) = (M \otimes_A N)/\mathfrak{a}(M \otimes_A N).$$

(9) Show that $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) \approx \mathbf{Z}/d\mathbf{Z}$, where $m, n \in \mathbf{Z}$ and d is the greatest common divisor of m and n.

(10) If M is finitely generated A-module and N is a noetherian A-module, show that $M \otimes_A N$ is noetherian.

CHAPTER 1

(11) Show that the set S of all non-zerodivisors of A is multiplicative and that the natural homomorphism $A \to S^{-1}A$ is injective. Give an example of a multiplicative subset T of a ring A such that the natural homomorphism $A \to T^{-1}A$ is not injective.

(12) (i) Let $f: A \to B$ be a homomorphism of rings and S, T be multiplicative subsets of A, B respectively, such that $f(S) \subset T$. Show that there exists a unique ring homomorphism $f': S^{-1}A \to T^{-1}B$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"'] & B \arrow[d, "i_B"] \\ S^{-1}A \arrow[r, "f'"] & T^{-1}B \end{tikzcd}$$

is commutative.

(ii) A multiplicative subset S of A is said to be *saturated* if for $a, b \in A$ with $ab \in S$ we have $a, b \in S$. For a multiplicative subset S of A let $\bar{S} = \{a \in A \mid \text{a divides } s \text{ for some } s \in S\}$. Prove that $\bar{S}$ is the smallest saturated multiplicative subset of A containing S. Prove that the map $1'_A : S^{-1}A \to \bar{S}^{-1}A$ induced by $1_A : A \to A$ as in (i) above, is an isomorphism.

(13) (cf. Exercise (6)). Let S be multiplicative subset of A. Let M be an A-module and let N_1, N_2 be submodules of M. Show that

$$S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2 \text{ and } S^{-1}(N_1 + N_2) + S^{-1}N_1 + S^{-1}N_2.$$

(14) Show that A is a local ring if and only if the non-units of A form an ideal.

(15) Let M be a noetherian A-module and S a multiplicative subset of A. Show that $S^{-1}M$ is a noetherian $S^{-1}A$-module.

(16) An A-module M is said to be *faithful* if $\operatorname{ann} M = 0$. Show that if there exists a faithful noetherian A-module then A is noetherian.

(17) Let A be a noetherian ring and S a multiplicative subset of A.

(i) If $\mathfrak{a}$ is a $\mathfrak{p}$-primary ideal of A, then $\mathfrak{p}^n \subset \mathfrak{a}$ for some $n \in \mathbb{N}$.

(ii) If $\mathfrak{m}$ is a maximal ideal of A and $\mathfrak{a}$ is an ideal of A such that $\mathfrak{m}^n \subset \mathfrak{a} \subset \mathfrak{m}$ for some $n \in N$, then $\mathfrak{a}$ is $\mathfrak{m}$-primary.

(iii) Let $\mathfrak{a}$ be a $\mathfrak{p}$-primary ideal of A. Then $\mathfrak{a} \cap S = \emptyset$ if and only if $\mathfrak{p} \cap S = \emptyset$. If $\mathfrak{a} \cap S = \emptyset$ the $S^{-1}\mathfrak{a}$ is $S^{-1}\mathfrak{p}$-primary. Moreover, if $\mathfrak{p} \cap S = \emptyset$, then $\mathfrak{a} \mapsto S^{-1}\mathfrak{a}$ is a bijective correspondence between $\mathfrak{p}$-primary ideals of A and $S^{-1}\mathfrak{p}$-primary ideals of $S^{-1}A$.

(iv) Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be an irredundant primary decomposition of an ideal $\mathfrak{a}$ of A. Then $S^{-1}\mathfrak{a} = \bigcap_{j \in J} S^{-1}\mathfrak{q}_j$ is an irredundant primary decomposition in $S^{-1}A$, where $J = \{i \mid 1 \le i \le r,\ \mathfrak{q}_i \cap S = \emptyset\}$.

(18) Let A be a noetherian ring and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then $\operatorname{Ass}(M') \subset \operatorname{Ass}(M) \subset \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$. If the sequence splits, then $\operatorname{Ass}(M) = \operatorname{Ass}(M') \cup \operatorname{Ass}(M'')$.

(19) Let $\mathfrak{a}_1 \mathfrak{a}_2 \ldots, \mathfrak{a}_n$ be ideals of A such that $\mathfrak{a}_i + \mathfrak{a}_j = A$ for every i, j with $i \neq j$, $1 \leq i, j \leq n$. Show that $\prod_{1 \leq i \leq n} \mathfrak{a}_i = \bigcap_{1 \leq i \leq n} \mathfrak{a}_i$. If A is noetherian and $\mathfrak{a}$ is an ideal of A such that $\operatorname{Supp}(A/\mathfrak{a})$ consists only of maximal ideals, then $\mathfrak{a}$ is a unique product of primary ideals.

(20) Let $B = A[x_1, \ldots, x_n]$ be a finitely generated A-algebra and let M be a finitely generated B-module. If $x_1, \ldots x_n \in \sqrt{\operatorname{ann}_B M}$ then show that M is a finitely generated A-module.

(21) For an ideal $\mathfrak{a}$ of A, define $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \supset \mathfrak{a}\}$, and for a subset X of $\operatorname{Spec}(A)$, define $I(X) = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. Show that

(i) there is a topology on $\operatorname{Spec}(A)$ for which the closed sets are $V(\mathfrak{a})$, $\mathfrak{a}$ running over all the ideals of A.

(ii) for $X \subset \operatorname{Spec}(A)$, $V(I(X)) =$ closure of X in $\operatorname{Spec}(A)$;

(iii) for any ideal $\mathfrak{a}$ of A, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$;

(iv) the map $X \mapsto I(X)$ is an inclusion-reversing bijection of the set of closed subsets of $\operatorname{Spec}(A)$ onto the set of ideals $\mathfrak{a}$ of A with $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

(22) Show that $\operatorname{Spec}(A)$ is connected if and only if A has no idempotents other than 0 and 1. Deduce that for a local ring A, $\operatorname{Spec}(A)$ is connected.

(23) A subset F of a topological space is said to be *irreducible* if $F \neq \emptyset$, F is closed and cannot be written as $F_1 \cup F_2$ with closed subsets F_1, F_2 properly contained in F. Show that the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec}(A)$ where $\mathfrak{a}$ is an ideal of A, is irreducible if and only if $\sqrt{\mathfrak{a}}$ is a prime ideal.

(24) A topological space is said be *noetherian* if every sequence $U_1 \subsetneq U_2 \subsetneq \cdots$ of open subsets of X is necessarily finite. Show that if A is noetherian ring, then $\operatorname{Spec}(A)$ is noetherian. Is the converse true?

(25) A maximal irreducible subset of a topological space X is called an *irreducible component* of X. Show that a noetherian topological space has only finitely many irreducible components. Let X be a noetherian topological space and let $\{X_i\}_{1 \leq i \leq r}$ be its irreducible components. Show

that $X = \bigcup_i X_i$ and for every j, $1 \leq j \leq r$, $X \neq \bigcup_{i \neq j} X_i$. If A is a noetherian ring, show that the irreducible components of $\operatorname{Spec}(A)$ are precisely $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_r)$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of A.

(26) Let $\phi\colon A \to B$ be a ring homomorphism. Show that the map $\operatorname{Spec}(\phi)\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ defined by $(\operatorname{Spec}(\phi))(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, for $\mathfrak{p} \in \operatorname{Spec}(B)$, is continuous.

(27) Let $\mathfrak{a}$ be an ideal of A and M an A-module. If $\mathfrak{a} \subset \operatorname{ann} M$, then show that $\ell_A(M) = \ell_{A/\mathfrak{a}}(M)$.

(28) (i) Let $0 \to M_1 \to \cdots \to M_n \to 0$ be an exact sequence of A-modules of finite length. Then show that $\sum_{1 \leq i \leq n} (-1)^i \ell_A(M_i) = 0$.

(ii) Let $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ be a sequence of submodules of an A-module of finite length. Then show that $\ell_A(M) = \sum_{0 \leq i \leq n-1} \ell_A(M_i/M_{i+1})$.

(29) Every artinian integral domain is a field. Every prime ideal of an artinian ring is maximal. If A is artinian, then $\mathfrak{n}(A) = \underline{r}(A)$.

(30) Show that there exists a ring A and a non-zero A-module M (not necessarily finitely generated) such that $\underline{r}(A)M = M$.

(31) Let $\mathfrak{a}$ be an ideal of A such that for all finitely generated A-modules M, $\mathfrak{a}M = M$ implies $M = 0$. Show that $\mathfrak{a} \subset \underline{r}(A)$.

(32) Let M be a finitely generated A-module such that $\mathfrak{m}M = M$ for every maximal ideal $\mathfrak{m}$ of A. Show that $M = 0$. Deduce Nakayama's lemma.

(33) Let M be a noetherian A-module. Show that any surjective A-endomorphism of M is an isomorphism.

(34) Let $A = \bigoplus_{i \geq 0} A_i$ be graded ring and let $A_+ = \bigoplus_{i \geq 1} A_i$. Show that if M is a graded A-module such that $A_+ M = M$ then $M = 0$.

(35) Let A be graded ring , M a graded A-module and let N a submodule of M. Then prove that the following conditions are equivalent:

(i) N is a graded submodule;

(ii) $x \in N \Rightarrow$ all the homogeneous components of x are in N;

(iii) N is generated by homogeneous elements.

(36) Let $A = \bigoplus_{i \geq 0} A_i$ be graded ring and $N = \bigoplus_{i \geq 0} N_i$ be a graded A-module . If N is finitely generated and $A = A_0$ then show that $N_i = 0$ for $i \gg 1$.

CHAPTER 2

(37) Let $0 \to \underline{X} \to \underline{Y} \to \underline{Z} \to 0$ be an exact sequence of complexes. Show that if any two of the $\underline{X}, \underline{Y}, \underline{Z}$ are exact, then so is the third.

(38) An A-module M is said to be *flat* if for any exact sequence $0 \to N' \to N \to N'' \to 0$ of A-modules the sequence $0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$ is exact. Show that, for an A-module M, M is free $\Rightarrow$ M is projective $\Rightarrow$ M is flat. Give examples to show that the implications cannot be reversed.

(39) Let $\underline{X}$ be a complex of A-modules. If M is an A-module, let $X \otimes_A M$ be the complex $\cdots \to X_n \otimes_A M \to X_{n-1} \otimes_A M \to \cdots$. Show that if M is flat, then $H_n(\underline{X} \otimes_A M) \approx H_n(\underline{X}) \otimes_A M$, for every $n \in \mathbf{Z}$.

(40) Let M be an A-module. Show that M is flat if and only if $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

(41) For an A-module M, show that the following conditions are equivalent:

(i) a sequence $0 \to N' \to N \to N'' \to 0$ of A-modules is exact if and only if $0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$ is exact;

(ii) M is flat and for an A-module $N, N \otimes_A M = 0$ implies that $N = 0$.

An A-module M which satisfies either of the above conditions is called *faithfully flat*.

(42) Show that a faithfully flat A-module is flat and faithful (i.e. ann $M = 0$). Give an example of a flat faithful module which is not faithfully flat.

(43) Let $\phi\colon A \to B$ be a ring homomorphism. Show that B is faithfully flat over A if and only if ϕ is injective and $B/\phi(A)$ is A-flat.

(44) Let $\phi\colon A \to B$ be a ring homomorphism and let B be faithfully flat over A. Show that if $\mathfrak{a}$ is an ideal of A, then $\phi^{-1}(\phi(\mathfrak{a})B) = \mathfrak{a}$.

(45) Let P be a projective A-module. Show that there exists a free A-module F such that $P \oplus F$ is free. Give an example of a ring A and a

finitely generated projective A-module P such that there does *not* exist a finitely generated free A-module F with $P \oplus F$ free.

(46) Let P, P' be finitely generated projective A-modules such that $P/(\underline{r}(A)P) \approx P'/(\underline{r}(A)P')$. Show that $P \approx P'$.

(47) Let A be a noetherian ring and M be a finitely generated A-module such that $\operatorname{Tor}_1^A(M, M/\underline{r}(A)) = 0$ and $M/\underline{r}(A)M$ is a projective $A/\underline{r}(A)$-module. Show that M is projective.

(48) Every ideal of a ring A is generated by an idempotent if and only if A is a finite direct of fields.

(49) Every A-module is projective if and only if A is a finite direct product of fields.

(50) Let $0 \to N \to P \to M \to 0$ and $0 \to N' \to P' \to M' \to 0$ be exact sequences of A-modules with P, P' being projective. Define A-homomorphisms $f\colon P \oplus N' \to P'$, $g\colon N \to P \oplus N'$ such that $0 \to N \xrightarrow{g} P \oplus N' \xrightarrow{f} P' \to 0$ is exact. Hence deduce that $P \oplus N' \approx P' \oplus N$.

(51) An A-module M is said to be *finitely presented* if there exists an exact sequence $F_1 \to F \to M \to 0$ with F_1 and F finitely generated free A-modules. Let M be a finitely presented A-module and let $f\colon F \to M$ be any epimorphism, where F is a finitely generated free A-module. Show that $\ker f$ is finitely generated.

(52) Let $A \to B$ be a ring homomorphism such that B is flat. Show that for A-modules M and N, we have

$$\operatorname{Tor}_n^A(M, N) \otimes_A B \approx \operatorname{Tor}_n^B(M \otimes_A B, N \otimes_A B).$$

(53) Let M, N be finitely generated modules over a noetherian ring A. Show that, for every $n \in \mathbf{Z}^+$, $\operatorname{Tor}_n^A(M, N)$ and $\operatorname{Ext}_A^n(M, N)$ are both noetherian.

(54) For simple A-modules M and N which are *not* isomorphic show that $\operatorname{Ext}_A^n(M, N) = 0 = \operatorname{Tor}_n^A(M, N)$ for every $n \in \mathbf{Z}^+$.

CHAPTERS 3, 4 & 5

(55) Let $B = A[X_1, \ldots, X_r]$ be the polynomial ring in r variables over A. Show that the subset B_n of B consisting of homogeneous polynomials of degree n is a free A-module with the set of monomials of degree n

as basis. Show that the number of monomials of degree n is $\binom{n+r-1}{r-1}$. Prove also that $B = \bigoplus_{n\geq 0} B_n$ is a graded ring .

(56) Show that for $r \in \mathbf{Z}^+$, the map $n \mapsto \binom{n}{r}$ is a polynomial function of degree r. Deduce that if in Exercise 55, A is artinian, then $\ell_A(B_n)$ is a polynomial of degree $r-1$.

(57) Check that the relation $\sim$ defined in the set of polynomial functions by $f \sim g$ if and only if $f(n) = g(n)$ for $n \gg 1$ is an equivalence relation. Show that for any polynomial function f of degree r, there exist $a_0, a_1, \ldots, a_r \in \mathbf{Q}$ such that $f \sim a_0 + a_1\binom{n}{1} + \cdots a_r\binom{n}{r}$. Show further that $a_0, a_1, \ldots, a_r$ are uniquely determined by f.

(58) (Cf. Exercises 21 and 23). Let X be a topological space and let $F_0 \underset{\neq}{\subset} F_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} F_N$ be a sequence of irreducible subsets of X. Then the integer n is called the *length* of this sequence. We define $\dim X$ to be the supremum of the lengths of all such sequences. Show that if A is a local ring, then $\dim A = \dim \operatorname{Spec}(A)$.

(59) Let A be a noetherian ring and P a finitely generated A-module. Show that P is projective if and only if $\operatorname{Ext}^1_A(P, N) = 0$ for every finitely generated A-module N.

(60) Let S be multiplicative subset of a noetherian ring A and let M, N be A-modules with M finitely generated . Show that, for every $n \in \mathbf{Z}^+$, there exists an $S^{-1}A$-isomorphism $S^{-1}\operatorname{Ext}^n_A(M, N) \approx \operatorname{Ext}^n_{S^{-1}A}(S^{-1}M, S^{-1}N)$, which is functorial both in M and N.

(61) Let A be a noetherian ring and M a finitely generated A-module. Show that

$$\operatorname{hd}_A M = \sup_{\mathfrak{p}\in \operatorname{Spec}(A)} \operatorname{hd}_{A_\mathfrak{p}} M_\mathfrak{p} = \sup_{\substack{\mathfrak{m}\in \operatorname{Spec}(A)\\ \mathfrak{m}\ \text{maximal}}} \operatorname{hd}_{A_\mathfrak{m}} M_\mathfrak{m}.$$

(62) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules such that $\operatorname{hd}_A M'' > \operatorname{hd}_A M$. Show that $\operatorname{hd}_A M'' = 1 + \operatorname{hd}_A M'$.

(63) Let M be an A-module with $\operatorname{hd}_A M = n < \infty$. Then there exists a free A-module F such that $\operatorname{Ext}^n_A(M, F) \neq 0$. If A is noetherian, show further that $\operatorname{Ext}^m_A(M, A) \neq 0$.

(64) Let M be a finitely generated A-module. A set $\{x_1, \ldots, x_n\}$ of generators of M is said to be *minimal* of no set of $n-1$ elements generate M. It is said to be *irredundant* if no proper subset of $\{x_1, \ldots, x_n\}$

generates M. Show that if A is local if and only if, for any A-module M, any irredundant set of generators is also minimal. (Note that a minimal set of generators is always irredundant).

(65) For a local ring A with maximal ideal $\mathfrak{m}$, show that $\mathrm{Tor}_1^A(k,k) = \mathfrak{m}/\mathfrak{m}^2$.

(66) Show that if P is finitely generated projective A-module, then the map $\mathrm{rank}_A P$: $\mathrm{Spec}\,(A) \to \mathbf{Z}$ is continuous for the discrete topology on $\mathbf{Z}$.

(67) Let A be a local ring. Show that $A[X]/(X^2)$ is a local ring and $\mathrm{gl.\,dim}\,A[X]/(X^2) = \infty$.

(68) Let A be an integral domain, K its quotient field and $\mathfrak{a}$ an ideal of A. We define $\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subset A\}$. Show that $\mathfrak{a}^{-1}$ is an A-submodule of K and, if $\mathfrak{a} \neq 0$, the following conditions are equivalent:

(i) $\mathfrak{a}\mathfrak{a}^{-1} = A$;

(ii) there exists $a_1 \ldots, a_r \in \mathfrak{a}$ and $x_1, \ldots, x_r \in \mathfrak{a}^{-1}$ such that

$$\sum_{1 \le i \le r} a_i x_i = 1;$$

(iii) $\mathfrak{a}$ is a finitely generated projective A-module.

An ideal satisfying any of the equivalent conditions above is called an *invertible* ideal.

(69) Show that in a unique factorization domain, every invertible ideal is principal.

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