Riemann Surfaces

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Editorial Note

THIS series of mathematical pamphlets is issued in response to a widespread demand from university teachers and research students in India who want to acquire a knowledge of some of those branches of mathematics which are not a part of the curricula for ordinary university degrees. While some of these pamphlets are based on lectures given by members of the Tata Institute of Fundamental Research at summer schools organized by the Institute, in cooperation with University of Bombay and the University Grants Commission, it is not the intention to restrict the series to such lectures. Pamphlets will be issued from time to time which are of interest to students.

K. CHANDRASEKHARAN

Preface

This pamphlet contains the notes of lectures given at a Summer School on Riemann Surfaces at the Tata Institute of Fundamental Research in 1963. The audience consisted of teachers and students from Indian Universities who desired to have a general knowledge of the subject, without necessarily having the intention of specializing in it.

Chapter I, which briefly sets out preliminaries from set topology and algebra which are indispensable, and Chapter II, which deals with the Monodromy Theorem (from a purely topologicl point of view, namely in the context of the lifting of curves on a manifold to one which is spread over it) are the notes of lectures given by M.S.Narasimhan. Chapter III, which gives a survey of function theory in the complex plane, the definition of a Riemann surface and holomorphic and meromorphic functions and differentials, represent the lectures given by R.R.Simha. Chapter IV, which presents analytic continuation and the construction of the Riemann surface of an irreducible algebraic equation $P(z, w) = 0$, represent lectures of Raghavan Narasimhan. Finally, Chapter V, where the Riemann-Roch theorem is stated and various corollaries derived, was presented by C.S. Seshadri.

Contents

Chapter 1

Preliminaries

1.1 Set-theoretic Preliminaries

1.1.1 Sets and Maps

We shall adopt the point of view of naive set theory. A *set* is a collection of objects which are called the *elements* of the set. The set of all rational integers (i.e. integers positive, negative, and zero) is denoted by $\mathbb{Z}$, the set of all non-negative integers by $\mathbb{Z}^+$, the set of all rational numbers by $\mathbb{Q}$, the set of all real numbers by $\mathbb{R}$, and the set of all complex numbers by $\mathbb{C}$.

If x is an element of a set A, we write $x \in A$. If x is not an element of A, we write $x \notin A$. Thus $x \in \mathbb{R}$ will mean that x is a real number. If P is a property, the set of all objects with the property P will be denoted $\{x \mid x \text{ satisfies } P\}$. Thus $\{x \mid x \in \mathbb{Z},\ x < 0\}$ is the set of all integers which are negative. The set which does not contain any element is called the empty set and is denoted by the symbol $\emptyset$.

Let X and Y be two sets. If every element of X is an element of Y, we say X is a *subset* of Y and write $X \subset Y$ or $Y \supset X$. It is clear that if $X \subset Y$ and $Y \subset X$, we must have $X = Y$. If X and Y are two sets, we define

(i) the *union* of X and Y, denoted by $X \cup Y$, as the set $\{z \mid z \in X$ or $z \in Y\}$;

(ii) the *intersection* of X and Y, denoted by $X \cap Y$, as the set $\{z \mid z \in X \text{ and } z \in Y\}$;

(iii) the *cartesian product* $X \times Y$ as $\{(x, y) \mid x \in X \text{ and } y \in Y\}$.

If $X \cap Y = \emptyset$, we say X and Y are *disjoint*.

If $X \subset Y$, we define the complement of X in Y, denoted by $Y - X$, as the set $\{z \mid z \in Y \text{ and } z \notin X\}$.

Suppose that J is a set and, for every $i \in J$, is given a set X_i. We say X_i is a *family of sets indexed by the set J*. Then we define

(i) the union of the family $\{X_i\}$ denoted by $\bigcup_{i\in J} X_i$ as the set $\{x \mid x \in X_i \text{ for at least one } i \in J\}$;

(ii) the intersection of the family $\{X_i\}$ denoted by $\bigcap_{i\in J} X_i$, as the set $\{x \mid x \in X_i \text{ for every } i \in J\}$.

It is easy to verify the following:

(a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$,
or more generally

$$X \cup (\bigcap_{i\in J} Y_i) = \bigcap_{i\in J} (X \cup Y_i).$$

(b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$,
or more generally

$$X \cap (\bigcup_{i\in J} Y_i) = \bigcup_{i\in J} (X \cap Y_i).$$

(c) If $(Y_i)_{i\in J}$ is a family of subsets of a set X, then

$$X - \bigcup_{i\in J} Y_i = \bigcap_{i\in J} (X - Y_i), \text{ and } X - \bigcap_{i\in J} Y_i = \bigcup_{i\in J} (X - Y_i).$$

Let X and Y be two sets. A *map* $f\colon X \to Y$ is an assignment to each $x \in X$ of an element $f(x) \in Y$. If A is a subset of X, the *image* $f(A)$ is the set $\{f(x) \mid x \in A\}$. The *inverse image* of a subset B of Y, denoted $f^{-1}(B)$, is the set $\{x \mid x \in X \text{ and } f(x) \in B\}$. The map f is said to be *onto* if $f(X) = Y$, *one-one* if no two distinct elements of X have the same image by f, i.e. $f(x) = f(y)$ implies $x = y$. If $f\colon X \to Y$, $g : Y \to Z$ are two maps, we define the *composite* $(g \circ f)\colon X \to Z$ as follows: $(g \circ f)(x) = g(f(x))$ for $x \in X$. The map $X \to X$ which associates to each $x \in X$, the element x itself is called the *identity* map of X, denoted by I_X. If $f\colon X \to Y$ is both one-one and onto, there is a map, denoted by $f^{-1}\colon Y \to X$ such that $f \circ f^{-1} = I_Y$, $f^{-1} \circ f = I_X$. This map f^{-1} is called the *inverse* of f. If A is a subset of X, the map $j\colon A \to X$ which associates to each $a \in A$ the same element a in X, is called the *inclusion* map of A in X. if $f\colon X \to Y$ is any map, the map $f \circ j\colon A \to Y$ is called the *restriction* of f to A and is often denoted by $f|A$.

1.1.2 Equivalence Relations

Definition 1.1 Let X be a set, An equivalence relation in X is a subset R of $X \times X$ such that

(i) for every $x \in X$, $(x, x) \in R$;

(ii) if $(x, y) \in R$, then $(y, x) \in R$;

(iii) if $(x, y) \in R$, $(y, z) \in R$, then $(x, z) \in R$.

We say x is *equivalent* to y under R, and write xRy if $(x, y) \in R$. Then the above conditions simply require that (i) every element is equivalent to itself (reflexivity); (ii) if x is equivalent to y, then y is equivalent to x (symmetry) ; (iii) if x is equivalent to y, y is equivalent to z, then x is equivalent to z (transitivity).

Example 1.2 The subset $R \subset X \times X$ consisting of elements (x, x), $x \in X$ is an equivalence relation. This is called the identity relation.

Example 1.3 $R = X \times X$ is also an equivalence relation in which all elements are equivalent.

Example 1.4 Let $q \in \mathbb{Z}$; consider the set in $\mathbb{Z} \times \mathbb{Z}$ consisting of pairs (m, n) of integers such that $m - n$ is divisible by q. This is again an equivalence relation under which two integers are equivalent if and only if they are congruent modulo q.

Example 1.5 In $\mathbb{R}^2$, consider the subset $\{(x, y) \mid x \leq y\}$. This satisfies (i) and (iii) but not (ii) and is therefore not an equivalence relation.

Example 1.6 If $f : X \to Y$ is a map, consider the subset $R_f \subset X \times X$ consisting of (x_1, x_2) such that $f(x_1) = f(x_2)$. It is easy to check that this is an equivalence relation.

Let now $x \in X$, and R be an equivalence relation in X. The set of all elements of X equivalent to x under R is called the *equivalence class* R_x. Consider the family of distinct equivalence classes of X under R. It is easily verified that they are pairwise disjoint and their union is X. We shall define the set of residue classes modulo R, or the quotient of X by R, as the set whose elements are these equivalence classes. This set is denoted by X/R. The natural map $\eta : X \to X/R$ which associates to each $x \in X$, the equivalence class R_x which contains x, is onto. Finally, let $f: X \to Y$ be a map, and R_f the equivalence relation in X defined by f. We shall now define a map $q_f: X/R_f \to Y$ by setting $q_f(R_x) = f(x)$. This is a map by our definition of R_f. Clearly q_f is one-one. Moreover we have $q_f \circ \eta = f$. We have therefore proved the

Theorem 1.7 *Let $f: X \to Y$ be a map. Then there exists an equivalence relation R_f on X, and a one-one onto map $q_f: X/R_f \to f(X)$, such that $f = j \circ q_f \circ \eta$, where j is the inclusion $f(X) \to Y$, and η is the natural map $X \to X/R_f$.*

1.2 Topological Preliminaries

1.2.1 Topological spaces

A topological space is a set X together with a collection T of subsets of X (called *open sets*) with the following properties:

(i) the empty set $\emptyset$ and X are in T;

(ii) any *finite* intersection of sets in T is again in T;

(iii) an *arbitrary* union of sets in T is again in T.

1. Let X be a set : the set T consisting only of X and $\emptyset$ defines a topology on X.

2. Let X be a set and let T consist of *all* subsets of X. This topology is called the *discrete* topology on X.

3. METRIC SPACES. A *metric space* is a set X together with a function d (called the *metric* or the *distance*) from $X \times X$ to the non-negative real numbers, such that

(i) $d(x, y) = 0$ if and only if $x = y$, where $x, y \in X$;

(ii) $d(x, y) = d(y, x)$ for $x, y \in X$; and

(iii) $d(x, z) \leq d(x, y) + d(y, z)$, for every $x, y, z \in X$ (triangle inequality).

Let $x \in X$ and ρ a positive real number. By the open ball around x of radius ρ we mean the set : $\{y \in X \mid d(x, y) < \rho\}$. We define a topology on the metric space by defining the open sets to be all sets which are unions of open balls (together with the empty set).

4. Let $\mathbb{R}$ be the set of real numbers. $\mathbb{R}$ has a natural metric defined by $d(x, y) = |x - y|$. This metric defines a topology on $\mathbb{R}$.

5. Let $\mathbb{C}$ be the set of complex numbers. $\mathbb{C}$ has a natural metric defined by $d(z_1, z_2) = |z_1 - z_2|$ where $|z_1 - z_2|$ denotes the modulus of the complex number $z_1 - z_2$. We shall always consider $\mathbb{C}$ as a topological space with the topology defined by this metric.

6. Let $\mathbb{R}^n$ be the n-dimensional euclidean space consisting of n-tuples $(x_1, \ldots, x_n)$ of real numbers. $\mathbb{R}^n$ is a metric space with the metric defined by $d(x, y) = \|x - y\| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{1/2}$, where $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$. We will suppose always that $\mathbb{R}^n$ is endowed with this topology.

7. INDUCED TOPOLOGY. Let X be a topological space and $A \subset X$. We define a topology on A by taking for the collection of open sets, the sets of the form $U \cap A$, where U is an open set of X. This topology is called the *induced topology.*

Let X be a topological space. The complement of an open set is called a *closed* set.

From the axioms for open sets we obtain at once the following properties of closed sets:

(i) the empty set $\emptyset$ and X are closed sets;
(ii) the union of a finite number of closed sets is closed;
(iii) any intersection of closed sets is closed.

Example 1.8 In $\mathbb{R}$ the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ $a, b \in \mathbb{R},\ a \leq b$, is closed

Definition 1.9 Let X be a topological space and $x \in X$. By a neighbourhood of x we mean an open set containing x.

Example 1.10 Let $X = \mathbb{R}$. For $\epsilon > 0$, the set $\{x \in \mathbb{R} \mid\ |x| < \epsilon\}$ is a neighbourhood of the origin 0.

Definition 1.11 Let A be a subset of a topological space X. A point $x \in X$ is said to be an adherent point of A, if every neighbourhood of x contains a point of A. The set of adherent points of A is called the closure of A, and is denoted by $\bar{A}$.

Remark 1.12 $\bar{A}$ is the intersection of all closed sets containing A, i.e., the smallest closed set containing A. The closure of a closed set is itself.

Example 1.13 Let $A = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$, the open unit ball in $\mathbb{R}^n$. Then $\bar{A}$ is the closed ball $\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Definition 1.14 A subset A of a topological space X is said to be dense in X if $\bar{A} = X$.

Example 1.15 The set of rational numbers is dense in $\mathbb{R}$.

Definition 1.16 The interior of set A in X is the union of all open sets contained in A, so that it is the largest open set contained in A.

Example 1.17 The interior of the closed unit ball in $\mathbb{R}^n$ is the open unit ball. The interior of the set consisting of just one point, say 0, in $\mathbb{R}^n$ is empty.

The interior of an open set U is U itself.

Definition 1.18 We say that a topological space is a Hausdorff space if every pair of distinct points have disjoint neighbourhoods; that is, if $x, y \in X$ with $x \neq y$, then there exist open sets U_1 and U_2 of X with $x \in U_1,\ y \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Example 1.19 A set with the discrete topology is Hausdorff.

Example 1.20 $\mathbb{R}^n$ with its natural topology is Hausdorff. For, let $x, y \in \mathbb{R}^n$, with $x \neq y$. Since $x \neq y$, the distance between them is positive, i.e. $\|x-y\| > 0$. Then the open balls around x and y of radius $\frac{1}{2}\|x-y\|$ do not intersect.

By the same reasoning we see that *any metric space is Hausdorff.*

Example 1.21 Let X be a set consisting of more than one point. The topological space in which the only open sets are the empty set and X, is not Hausdorff.

Example 1.22 Let X be a Hausdorff space. A subset A of X, provided with the induced topology, is Hausdorff.

1.2.2 Compact spaces

Definition 1.23 A family $\{V_\alpha\}_{\alpha \in I}$ of subsets of a set X is said to be a covering of X if $\bigcup_{\alpha \in I} V_\alpha = X$, i.e. if each point of X belongs to at least one V_α. If, further, X is a topological space, and each V_α is an open subset of X, we say that $\{V_\alpha\}$ is an open covering of X.

Definition 1.24 A topological space X is said to be compact, if the following condition is satisfied: if $\{V_\alpha\}_{\alpha \in I}$ is any open covering of X, then some finite sub-collection of $\{V_\alpha\}$ is already a covering; that is if $\{V_\alpha\}_{\alpha \in I}$ is any open covering of X, then there exists a finite number of elements $\alpha_1, \ldots, \alpha_n$ of I such that $\bigcup_{1 \leq j \leq n} V_{\alpha_j} = X$.

Definition 1.25 A subset A of a topological space is said to be compact if it is compact in the induced topology.

Remark 1.26 A closed subset of a compact space is compact.

Remark 1.27 Let $a, b \in \mathbb{R}$ with $a \leq b$. Then the set $A = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is compact. This fact is known as the Heine-Borel Theorem.

The space $\mathbb{R}$ is not compact.

Remark 1.28 We say that a set $A \subset \mathbb{R}^n$ is bounded if there exists a real number a, such that $\|x\| < a$ for each $x \in A$. Then we have the following result: a closed bounded set in $\mathbb{R}^n$ is compact. This result may be deduced from Remarks 1.26 and 1.27 above. This is also sometimes called the Heine-Borel theorem.

Remark 1.29 Let X be a topological space. A subset A of X consisting of a finite number of points is compact.

Proposition 1.30 *A compact subset of a Hausdorff space is closed.*

PROOF: Let X be a Hausdorff space, and A any compact subset of X. We have to show that $X \setminus A$ is open. For this it is sufficient to prove that every point $x \in X \setminus A$ has a neighbourhood which does not intersect A. Let $x \in X \setminus A$. If $y \in A$, we can find, since X is Hausdorff, a neighbourhood V_y of x and a neighbourhood U_y of y in X, such that $V_y \cap U_y = \emptyset$. If $U'_y = U_y \cap A$, then $\{U'_y\}_{y \in A}$ is an open covering of A, and since A is compact, we can find $y_1, \ldots, y_n \in A$, such that $U'_{y_1} \cup \cdots \cup U'_{y_n} = A$. Then $V = V_{y_1} \cap \cdots \cap V_{y_n}$ is an open set containing x, which does not intersect A.

1.2.3 Connected spaces

Definition 1.31 A topological space X is said to be connected if it is not the union of two non-empty disjoint open sets. A subspace of a topological space is said to be connected if it is connected in the induced topology.

Remark 1.32 A topological space X is connected if and only if X is not the union of two non-empty disjoint closed sets. X is connected if and only if there is no subset of X which is both open and closed except the whole set X and the empty set.

Proposition 1.33 *A subset of $\mathbb{R}$ is connected if and only if it is an interval. A subset A of $\mathbb{R}$ will be called an interval if the following condition is satisfied : if $a_1, a_2 \in A$ with $a_1 < a_2$, then any a in $\mathbb{R}$ such that $a_1 < a < a_2$ also belongs to A.*

Proof: Let A be a connected subset of $\mathbb{R}$. If A were not an interval, there would exist $a_1, a_2 \in A,\ a_1 < a_2$, and a real number a not in A with $a_1 < a < a_2$. Then the sets $B = \{b \in A \mid b < a\}$ and $C = \{c \in A \mid c > a\}$ are non-empty disjoint open sets whose union is A. This contradicts our assumption that A is connected.

Let now A be an interval. If possible, let $A = U \cup V, U, V$ open, non-empty and $U \cap V = \emptyset$. Let $a_1 \in U,\ a_2 \in V$. We can assume, without loss of generality, that $a_1 < a_2$. Consider $b = \sup\{r \in U \mid a_1 \leq r < a_2\}$. Then it is easily seen that $b \in \bar{U} = U$, and $b \in \bar{V} = V$, i.e. $b \in U \cap V = \emptyset$ a contradiction.

1.2.4 Continuous mappings

Definition 1.34 Let X and Y be two topological spaces, and $f\colon X \to Y$ be a map. Let $x \in X$. We say that f is continuous at x, if the following condition is satisfied: for every neighbourhood U of $f(x)$, there exists a neighbourhood V of x, such that $f(V) \subset U$. We say that f is continuous (or f is a continuous map), if f is continuous at every point of X.

Remark 1.35 We can easily prove that $f\colon X \to Y$ is continuous if and only if the inverse image by f of every open set in Y is an open set of X.

Remark 1.36 Let $f\colon X \to Y,\ g\colon Y \to Z$ be continuous maps. Then the composite $(g \circ f)\colon X \to Z$ is continuous.

Example 1.37 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map (i.e. f is a "real-valued function of a real variable"). Let $x_0 \in \mathbb{R}$. Then f is continuous at x_0, if and only if for every $\epsilon > 0$, we can find a real number $\delta > 0$ (depending on ϵ) such that for each x with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.

Let X be a discrete space, and Y a topological space. Then any map $f\colon X \to Y$ is continuous.

Proposition 1.38 *Let X be a compact space, Y a topological space, and $f\colon X \to Y$ a continuous map. Then $f(X)$ is compact. (That is, a continuous image of a compact set is compact.)*

PROOF: Let $\{V_\alpha\}_{\alpha \in I}$ be an open covering of $f(X)$ and let U_α be open in Y with $U_\alpha \cap f(X) = V_\alpha$. Since f is continuous, and U_α are open in Y, $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ forms an open covering of X. Since X is compact, there are $\alpha_1, \ldots, \alpha_n \in I$ such that $\bigcup_{1 \le i \le n} f^{-1}(U_{\alpha_i}) = X$. Then $\bigcup_{1 \le i \le n} V_{\alpha_i} = f(X)$.

Corollary 1.39 *Let X be compact, Y Hausdorff and $f\colon X \to Y$ continuous. Then $f(X)$ is closed in Y.*

Proposition 1.40 *Let X be a connected topological space, Y a topological space, and $f\colon X \to Y$ a continuous map. Then $f(X)$ is connected.*

PROOF: If $f(X) = A$ were not connected, there would exist open sets U_1 and U_2 of Y, such that

(i) $U_1 \cap A \neq \emptyset,\ U_2 \cap A \neq \emptyset,\ (U_1 \cap A) \cap (U_2 \cap A) = \emptyset$, and

(ii) $(U_1 \cap A) \cup (U_2 \cap A) = A$.

Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open : further $f^{-1}(U_1) \neq \emptyset,\ f^{-1}(U_2) \neq \emptyset,\ f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$ and $f^{-1}(U_1) \cup f^{-1}(U_2) = X$. Thus X is the union of two non-empty disjoint open sets. This contradicts the hypothesis that X is connected.

Proposition 1.41 *Let X be a topological space, and Y a Hausdorff space. Let f and g be a continuous maps of X into Y. Then the set $E = \{x \mid x \in X,\ f(x) = g(x)\}$ is closed in X.*

PROOF: It is sufficient to prove that the set $F = \{x \mid x \in X,\ f(x) \neq g(x)\}$ is open in X. Let $x_0 \in F$. Since $f(x_0) \neq g(x_0)$ and Y is Hausdorff, there exist neighbourhoods U_1 and U_2 of $f(x_0)$ and $g(x_0)$ respectively with $U_1 \cap U_2 = \emptyset$. By the continuity of f and g, we can find neighbourhoods V_1 and V_2 of x_0 in X, such that $f(V_1) \subset U_1,\ g(V_2) \subset U_2$. If $V = V_1 \cap V_2$, V is a neighbourhood of x_0, and $f(x) \neq g(x)$ for $x \in V$ so that $V \subset F$. Hence F is open in X.

1.2.5 Homeomorphisms

Definition 1.42 Let X and Y be two topological spaces. A map $f\colon X \to Y$ is said to be a homeomorphism, if f is one-one onto, and if f and f^{-1} are both continuous. Two topological spaces X and Y are said to be homeomorphic, if there is a homeomorphism f of X onto Y.

Example 1.43 Let $X = \mathbb{R}$ and $Y = \mathbb{R}$. The map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a homeomorphism. The map $f(x) = -x$ is also a homeomorphism.

Example 1.44 The space $\mathbb{R}$ and the open set $\{x \in \mathbb{R} \mid -1 < x < 1\}$ are homeomorphic. (Consider the map $f(x) = x/(1 + |x|)$). But $\mathbb{R}$ and the closed set $I = \{x \in R \mid -1 \leq x \leq 1\}$ are not homeomorphic, since $\mathbb{R}$ is not compact, and I is compact. $\mathbb{R}^n$ and the open unit ball $B = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ are homeomorphic.

1.2.6 Product spaces

Let X and Y be two topological spaces. Consider the cartesian product $X \times Y$. We define a topology on $X \times Y$, called the *product topology* as follows. An open set in $X \times Y$ will be, by definition, a union of sets of the form $U \times V$, where U is an open set in X, and V an open set in Y.

If $p_1: X \times Y \to X$ and $p_2: X \times Y \to Y$ are the projections, defined by $p_1(x, y) = x$ and $p_2(x, y) = y$ respectively, then p_1 and p_2 are continuous maps.

In a similar way the product of a finite number of topological spaces can be defined.

Example 1.45 The metric topology on $\mathbb{R}^n$ is the product topology of $\mathbb{R}$ (n times).

Problem 1.1 The product of two topological spaces is compact, if each component is compact.

1.2.7 Arc-wise connected spaces. Homotopy.

Definition 1.46 A curve in a topological space X is a continuous map $f: [0, 1] \to X$. $f(0)$ and $f(1)$ are called, respectively, the initial point, and end-point of the curve. [Here $[0, 1]$ denotes the closed interval $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.]

Definition 1.47 A topological space is said to be arc-wise connected if, given $x_0, x_1 \in X$, there exists a curve in X with x_0 and x_1 as its initial and end-points.

Problem 1.2 Every arc-wise connected space is connected. (The converse is not true. Example?)

Definition 1.48 Let X be a topological space; let γ_1 and γ_2 be two curves in X with the same initial point p_1 and end-point p_2. We say that γ_1 and γ_2 are homotopic in X, if there exists a continuous map f from the square $S = \{(t,u) \mid 0 \leq t \leq 1,\ 0 \leq u \leq 1\}$ into X, such that

(i) $f(t,0) = \gamma_1(t),\ f(t,1) = \gamma_2(t)$,

(ii) $f(0,u) = p_1,\ f(1,u) = p_2$, for $0 \leq u \leq 1$.

f is called a homotopy between γ_1 and γ_2.

Definition 1.49 The product $\gamma_1\gamma_2$ of two curves γ_1 and γ_2 such that the initial point of γ_2 is the same as the end point of γ_1 is the curve γ defined by

$$\gamma(t) = \gamma_1(2t) \text{ for } 0 \leq t \leq \frac{1}{2},\ \gamma(t) = \gamma_2(2t-1) \text{ for } \frac{1}{2} \leq t < 1.$$

1.2.8 Connected components

Proposition 1.50 *Let A be a connected subset of a topological space X (i.e. A is connected in the induced topology). Then the closure of A in X is connected.*

Proof: Let $\bar{A}$ be the closure of A. If $\bar{A}$ were not connected, there would exist open sets U_1 and U_2 in X, such that $U_1 \cap \bar{A} \neq \emptyset,\ U_2 \cap \bar{A} \neq \emptyset,\ U_1 \cap U_2 \cap \bar{A} = \emptyset$ and $(U_1 \cup U_2) \cap \bar{A} = \bar{A}$. Then $U_1 \cap A$ and $U_2 \cap A$ would be non-empty disjoint open sets in A, whose union is A, contradicting the hypothesis on A.

Proposition 1.51 *The union of a family of connected sets, whose intersection is not empty, is a connected set.*

Proof: Let $\{A_\alpha\}_{\alpha \in I}$ be a family of connected sets in a topological space X, such that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$. Let $A = \bigcup_{\alpha \in I} A_\alpha$. If A were not connected, there would exist open sets U_1 and U_2 in X, such that $U_1 \cap A \neq \emptyset,\ U_2 \cap A \neq \emptyset,\ A \subset U_1 \cup U_2$ and $A \cap U_1 \cap U_2 = \emptyset$. Let $x \in \bigcap_{\alpha \in I} A_\alpha$. Then x belongs to one of the sets U_1 or U_2 say $x \in U_1$. There exists an index α such that $U_2 \cap A_\alpha \neq \emptyset$. Since $x \in A_\alpha,\ U_1 \cap A_\alpha \neq \emptyset$, one would therefore have $A_\alpha \subset U_1 \cup U_2,\ A_\alpha \cap U_1 \cap U_2 = \emptyset,\ U_1 \cap A_\alpha \neq \emptyset$ and $U_2 \cap A_\alpha \neq \emptyset$, contradicting the hypothesis that the A_α are connected.

Let X be a topological space and $x \in X$. Because of the proposition just proved, the union of all connected subsets of X containing x is also a connected subset. This is therefore the largest connected subset of X containing x.

Definition 1.52 Let X be a topological space. By the connected component of a point of X we mean the largest connected subset of X containing that point. By the connected components of X we mean the connected components of points of X.

Since the closure of a connected set is connected, it follows that the connected component of any point is closed. As the union of connected sets having a point in common is connected, the relation "y belongs to the connected component of x" is an equivalence relation in X.

Problem 1.3 Let X be an open subset of $\mathbb{R}^n$. Prove that the connected components of X are open in X.

1.2.9 Quotient spaces

Let X be a topological space, and R an equivalence relation in the underlying set X. Let $Y = X/R$ be the quotient set, and $\eta: X \to Y$ the natural map. We put on Y the following topology. A set in Y is open if and only if its inverse image by η is open in X. This topology in Y is called the *quotient topology* on X/R. Now X/R, endowed with this topology will be referred to as the *quotient space.* The map $\eta: X \to X/R$ is continuous by definition.

Let X and Y be two topological spaces and $f: X \to Y$ a continuous map. Let R_f denote the equivalence relation defined by f. Then it is easy to check that the map $q_f: X/R \to Y$ is continuous and f admits the decomposition by continuous maps:

$$X \xrightarrow{\eta} X/R_f \xrightarrow{q_f} f(X) \xrightarrow{j} Y.$$

Remark 1.53 q_f is 1–1, onto and continuous, but not necessarily a homeomorphism onto $f(X)$.

Example 1.54 Let $X = \mathbb{R}$ be the group of real numbers, and $\mathbb{Z}$ the subgroup of integers. Let R be the equivalence relation in $\mathbb{R}$ defined by the subgroup $\mathbb{Z}$. The quotient space is homeomorphic to the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$.

1.3 Algebraic Preliminaries

An *operation* $\circ$ on a set X is a mapping $\circ : X \times X \to X$ from the product set $X \times X$ into X. It is usual to denote $\circ(x, y)$ by $x \circ y$ (or simply xy). A *group* is a couple $(G, \circ)$ where G is a set and $\circ$ is an operation, such that

1. for $x, y, z \in G, (x \circ y) \circ z = x \circ (y \circ z)$ (associativity);

2. there exists a unique element 1, called the identity element of G, such that $a \circ 1 = 1 \circ a = a$, for all $a \in G$;

3. given $a \in G$, there exists a unique element a^{-1}, called the inverse of a, such that $a \circ a^{-1} = a^{-1} \circ a = 1$.

If $a \circ b = b \circ a$, for $a, b \in G$, the group is said to be *abelian.* If the group is abelian, the operation $\circ$ is often denoted by $+$; then the identity element is denoted by 0, and the inverse of an element a is denoted by $-a$.

Often the set G is itself refered to as a group, it being understood that the operation is also given.

A subset G' of G is said to be a *subgroup* of G if (1) for $x, y \in G', x \circ y \in G'$ and (2) $(G', \circ)$ is a group. Let G_1, G_2 be two groups, and $f : G_1 \to G_2$ a mapping of G_1 into G_2. Then f is said to be a *homomorphism* if $f(xy) = f(x)f(y), x, y \in G_1$. It follows easily that $f(1) = 1$, (i.e. the image of the identity element in G_1 is the identity element in G_2) and $f(x^{-1}) = (f(x))^{-1}$. The set of elements x of G_1 such that $f(x) = 1$ is easily verified to be a subgroup of G_1, called the *kernel* of f. Similarly the image of G_1 by f in G_2 is verified to be a subgroup of G_2. The homomorphism f is said to be an *isomorphism* if f is onto and the kernel of f is $\{1\}$.

Let G be an abelian group and H a subgroup. Then the relation: $x \sim y$ $(x, y \in G)$ if and only if $x - y \in H$, is an equivalence relation. Let $\bar{x}$ be the equivalence class containing x. We can define the structure of an abelian group on the set of equivalence classes, by setting

$$\bar{x} + \bar{y} = \overline{(x + y)}.$$

This group is denoted by G/H. There is a natural mapping of G onto G/H, namely the one which maps x to $\bar{x}$, and this is a homomorphism whose kernel is H.

We say that we are given the structure of a *field* on a set of K, if there are two operations $+$ and $\circ$ on K satisfying the following conditions:

(1) $(K, +)$ is an abelian group (with identity element 0);

(2) if $K^* = K - \{0\}, x, y \in K^*, x \circ y \in K^*$ and $(K^*, \circ)$ is an abelian group (with identity element 1);

(3) $x \circ y = y \circ x$, for, $x, y \in K$;

(4) $x \circ (y + z) = x \circ y + x \circ z$, for, $x, y, z \in K$.

It follows immediately that $x \circ 0 = 0$ for all $x \in K$.

We say that L is a *subfield* of a field K, if L is a subset of K closed with respect to the operations $+, \circ$ and such that L is itself a field with respect to these operations.

Example 1.55 The set of integers $\mathbb{Z}$ with the usual operation of addition forms an abelian group.

Example 1.56 If m is a given integer, the set of multiples of m forms a subgroup (m) of $\mathbb{Z}$, and the quotient group is the group $\mathbb{Z}/(m)$ of integers modulo m.

Example 1.57 The set of rational numbers (resp. real numbers, complex numbers) with the usual addition and multiplication forms a field.

Example 1.58 Let $\mathbb{Z}/(m)$ be the group of integers modulo m. We denote by $\bar{x}$ the class containing an integer x. We can define an operation $\circ$ on $\mathbb{Z}/(m)$ by $\bar{x} \circ \bar{y} = \overline{(x \circ y)}$, where $x \circ y$ denotes the usual product of the integers x and y. If p is a prime number, $\mathbb{Z}/(p)$ is a field for the operations $+$ and $\circ$.

It is easily proved that either the rational number field or a unique field of the form $\mathbb{Z}/(p)$ introduced above, is a subfield of K. In the former case K is said to be of *characteristic zero*; in the latter case, the field is said to be of *characteristic* p $(p \neq 0)$.

We say that there is the structure of a *vector space over a field* K on a set V, if

1. V is an abelian group (operation $+$);

2. there exists a mapping $\circ : K \times V \to V, (\circ(\lambda, v)$ being denoted by $\lambda v)$ such that (i) $\lambda(u + v) = \lambda u + \lambda v$; (ii) $(\lambda + \mu)v = \lambda v + \mu v$; (iii) $(\lambda\mu)v = \lambda(\mu v)$; (iv) $1 \cdot v = v$ for $v \in V$; here $\lambda, \mu \in K$ and $u, v \in V$.

We say that a subset V_1 of V is *a vector subspace or a linear subspace*, if (1) V_1 is a subgroup; (2) $\lambda v \in V_1$ whenever $\lambda \in K$ and $v \in V_1$. Then V_1 is a vector space over K.

A *linear mapping* $f : V_1 \to V_2$ of two vector spaces V_1 and V_2 over K is a homomorphism f of the underlying groups, having the further property that $f(\lambda v) = \lambda f(v)$ for $\lambda \in K, v \in V_1$. It is then easily verified that the kernel of f is a linear subspace of V_1, and that the image of f is a linear subspace of V_2. The mapping is said to be an *isomorphism*, if f is onto and the kernel of f is 0.

Let $e_1, \ldots, e_n$ be a finite set of elements of a vector space defined over a field K. We say that $\{e_i\}$ *generates* V, if every element $v \in V$ can be expressed in the form $v = \sum_{i=1}^{n} \lambda_i e_i, \lambda_i \in K$. We say that the e_i are *linearly independent*, if the relation $\sum_{i=1}^{n} \lambda_i e_i = 0, \lambda_i \in K$, implies $\lambda_i = 0$ for each i. We say that $\{e_i\}$ is a *basis*, if e_i generate V, and are linearly independent. It can be proved that the number of elements in a basis of V is independent of the basis chosen, and this integer is called the the *dimension* of V; if V is the zero vector space, the dimension of V is defined to be 0. It can be proved that if $\{e_i\}$ generates V, there exists a subset of $\{e_i\}$ which is, in fact, a basis of V (V being assumed to be $\neq 0$.) In particular, if V is generated by a finite number of elements, V is of finite dimension.

Let K be a field and k a subfield of K. K is in a natural way a vector space over k because $xy \in K$ for $x \in k,\ y \in K$. If the dimension of K as a vector space over k is finite, K is said to be a *finite extension of* k, and then the dimension is denoted by $(K : k)$. An element θ of K is said to be algebraic over k, if there exists a polynomial $F(X) = X^n + a_1 X^{n-1} + \cdots + a_n,\ a_i \in k,\ n \geq 0$ such that $F(\theta) = 0$. It can be proved that an element θ is algebraic over k if and only if the field $k(\theta)$ (i.e. the elements of K which can be expressed as rational functions of θ with coefficients in k) is a finite extension of k. If θ is algebraic over k, there exists a unique polynomial $F(X)$ of the form $X^n + a_1 X^{n-1} + \cdots + a_n,\ a_i \in k,\ n \geq 1$ (called the minimum polynomial of θ) such that if $G(X)$ is any other polynomial with co-efficients in k, with $G(\theta) = 0$, then $G(X) \equiv H(X)F(X)$ where $H(X)$ is a polynomial with coefficients in k. The polynomial $F(X)$ is *irreducible*, i.e. if $F(X) \equiv H_1(X) \cdot H_2(X)$, where H_1, H_2 have coefficients in k, H_1 or H_2 reduces to a constant polynomial.

Let K be a field of characteristic zero which is a finite extension of a subfield k. Then it can be proved that there exists an element θ of K, such that $K = k(\theta)$.

Let K be a field, k_2 a subfield of K, and k_1 a subfield of k_2. Then if K is a finite extension over k_1, it can be easily proved that k_2 is a finite extension of k_1, that K is a finite extension over k_2, and that $(K : k_1) = (K : k_2)(k_2 : k_1)$.

For the proofs of these results, see for example B. L. van der Waerden – *Modern Algebra*, Volume I, or N. Bourbaki – *Algèbre.*

Chapter 2

The Monodromy Theorem

2.1 Manifolds

Definition 2.1 A topological space M is said to be an n-dimensional manifold if

(i) M is Hausdorff and arc-wise connected, and

(ii) every point of M has a neighbourhood homeomorphic to an open subset of the Euclidean n-space $\mathbb{R}^n$.

According to a theorem of L.E.J. Brouwer (which is too difficult to be proved here) an open set in $\mathbb{R}^n$ can be homeomorphic to an open in subset in $\mathbb{R}^k$ only if $k = n$, so that the *dimension* n of a manifold is uniquely defined.

Definition 2.2 *A* 2-dimensional manifold will be called a surface.

Example 2.3 Any connected open subset of $\mathbb{R}^n$ is an n-dimensional manifold, in particular, arc-wise connected.

Example 2.4 Let S^n be the sphere $\{x \mid x \in \mathbb{R}^{n+1}, \|x\| = 1\}$, with the topology induced from $\mathbb{R}^{n+1}$. S^n is an n-dimensional manifold for $n \geq 1$.

Example 2.5 If M_1 is an n-dimensional manifold and M_2 an m-dimensional manifold, then $M_1 \times M_2$, with the product topology, is an $(m+n)$-dimensional manifold.

2.2 Simply-connected Manifolds

Definition 2.6 Let M be an n-dimensional manifold and $m \in M$. By a closed curve in M at the point m we mean a curve γ in M such that $\gamma(0) = \gamma(1) = m$.

Remark 2.7 If $m \in M$, the curve γ defined by $\gamma(t) = m$, for all t in the interval $[0, 1]$ is a closed curve at m. We say that γ is the constant curve at m.

Definition 2.8 Let M be an n-dimensional manifold. We say that M is simply-connected if for every point $m \in M$, every closed curve at m is homotopic to the curve which maps the whole closed interval $[0, 1]$ into m.

(i) It is clear from the definition that if M_1 and M_2 are homeomorphic manifolds and if M_1 is simply connected, so is M_2.

(ii) It is sufficient to assume in the definition that every closed curve at some point m is homotopic to the constant curve at m.

Example 2.9 $\mathbb{R}^n$ is simply connected. For let $m \in \mathbb{R}^n$, and γ be a closed curved at m. Then γ and the constant curve at m are homotopic, the homotopy being given by

$$F(t, u) = (1 - u)\gamma(t) + um.$$

[If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $a \in \mathbb{R}$, by ax we mean the point $(ax_1, \dots, ax_n)$ in $\mathbb{R}^n$.]

Example 2.10 The same proof shows that an open convex set in $\mathbb{R}^n$ is simply connected. In particular the open ball $\|x\| < 1$ is simply connected.

Example 2.11 The sphere S^n is simply connected for $n > 1$. The circle S^1 is *not* simply connected. The complement of the origin in $\mathbb{R}^2$ is not simply connected. We do not prove these results here. (See, however, the example at the end of the chapter.)

Example 2.12 If M_1 and M_2 are simply connected manifolds, so is $M_1 \times M_2$.

2.3 Spreads

Definition 2.13 A spread is a triple $(\tilde{M}, M, f)$ where $\tilde{M}$ and M are n-dimensional manifolds and $f\colon \tilde{M} \to M$ a map with the following property: every point of x of $\tilde{M}$ has a neighbourhood which is mapped homeomorphically by f onto a neighbourhood of $f(x)$ in M (i.e. f is a local homeomorphism).

We say that a point x of $\tilde{M}$ lies over a point m of M if $f(x) = m$.

Remark 2.14 f is a continuous map.

Remark 2.15 f maps open sets of $\tilde{M}$ into open sets of M.

Example 2.16 Take $\tilde{M} = \mathbb{C}$, the complex plane and $M = \mathbb{C}^*$, the space of non-zero complex numbers. Define $f(z) = e^{2\pi i z}$, $z \in \mathbb{C}$. The triple $(\tilde{M},\ M,\ f)$ is a spread.

Example 2.17 The triple $(\tilde{M},\ M,\ f)$, where $\tilde{M} = M = \mathbb{C}^*, f(z) = e^{2\pi i z}, z \in \mathbb{C}^*$, is a spread.

Proposition 2.18 *Let $(\tilde{M},\ M,\ f)$, be a spread and X a connected topological space. Let g_1 and g_2 be two continuous maps from X to $\tilde{M}$, such that (i) $g_1(x_0) = g_2(x_0)$ for some $x_0 \in X$, and (ii) $f \circ g_1 = f \circ g_2$. Then $g_1(x) = g_2(x)$, for all $x \in X$.*

Proof: Let E be the set $E = \{x \mid x \in X,\ g_1(x) = g_2(x)\}$. E is non-empty by hypothesis. Since $\tilde{M}$ is a Hausdorff space, by an earlier proposition, E is closed. If we prove that E is open, it will follow that $E = X$, since X is connected.

To prove that E is open, let $x_1 \in E$. Let $\tilde{V}$ be a neighbourhood of $g_1(x_1) = g_2(x_1)$ which is mapped homeomorphically by f onto a neighbourhood V of $f \circ g_1(x_1)$. By the continuity of g_1 and g_2 we can find a neighbourhood U of x_1 in X such that $g_1(U) \subset \tilde{V}$ and $g_2(U) \subset \tilde{V}$. Since $f \circ g_1 = f \circ g_2$ it follows that $g_1(x) = g_2(x)$ for $x \in U$, i.e. $U \subset E$. This prove that E is open.

Definition 2.19 Let $(\tilde{M}, M, f)$ be a spread and γ a curve in M. Let $\gamma(0) = m$ and $\tilde{m}$ a point in $\tilde{M}$ lying over m. A curve $\tilde{\gamma}$ in $\tilde{M}$ is said to be a lifting of γ with $\tilde{m}$ as the initial point if $\tilde{\gamma}(0) = \tilde{m}$ and $f \circ \tilde{\gamma} = \gamma$.

The existence of liftings cannot always be asserted. But we have uniqueness. Since the interval $[0,1]$ is connected the previous proposition yields the

Proposition 2.20 *Any two liftings of a curve in M with the same initial point in $\tilde{M}$ are identical.*

Example 2.21 Consider the spread $(\tilde{M}, M, f), \tilde{M} = M = \mathbb{C}^*, f(z) = e^{2\pi i z}$. The closed curve at 1 in M defined by $\gamma(t) = e^{2\pi i(1-t)}$ does not have a lift with initial point 1 in $\tilde{M}$.

Definition 2.22 A spread $(\tilde{M}, M, f)$ will be called a spread without relative boundary, if for every curve γ in M and every point p lying over the initial point of γ, there is a lifting of γ with p as the initial point.

2.4 The Monodromy Theorem

Theorem 2.23 *Let $(\tilde{M}, M, f)$ be a spread without relative boundary. Let γ_1 and γ_2 be two homotopic curves in M from m_1 to m_2. Let p be a point in $\tilde{M}$ lying over m_1. If $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the liftings of γ_1 and γ_2 with initial point p, then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same end point in $\tilde{M}$, and are homotopic. More precisely, if F is a homotopy between γ_1 and γ_2, then there is a homotopy $\tilde{F}$ between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, such that $f \circ \tilde{F} = F$.*

Proof: For fixed u in $[0,1]$ define $\tilde{F}(t,u) = \tilde{\gamma}_u(t)$, where $\tilde{\gamma}_u$ is the unique curve obtained by lifting the path $t \to F(t,u)$ with initial point p. Clearly $f \circ \tilde{F} = F$, and $\tilde{F}(0,u) = p$. If we prove that $\tilde{F}$ is a continuous function of the square $0 \leq t \leq 1, 0 \leq u \leq 1$, the theorem will be proved. For then the function $g(u) = \tilde{F}(1,u)$ will be a continuous function and $g(u) \in f^{-1}(m_2)$, for u in $[0,1]$. By the uniqueness of the lift for the constant curve at $m_2, g(u)$ must reduce to a constant. Since $\tilde{F}$ is continuous, all the assertions of the theorem will be proved.

To prove that $\tilde{F}$ is continuous, let (a,b) be a point in the square. Consider the path $\tilde{F}(t,b)$. By the compactness of the closed interval $[0,1]$, we can find points $t_1, \ldots, t_n, 0 = t_1 < t_2 < \cdots < t_n = 1$ such that each of the closed intervals $t_i \leq t \leq t_{i+1}$ is mapped by $\tilde{F}(t,b)$ into an open set $\tilde{U}_i$ in $\tilde{M}$, which is mapped by homeomorphically f onto an open set U_i in M. If $a \neq 1$ and $a \neq 0$, we may suppose that a belongs to the open interval $t_v < t < t_{v+1}$ for some v. Let $f_i^{-1} : U_i \to \tilde{U}_i$ be the inverse of the map $f|\tilde{U}_i : \tilde{U}_i \to U_i$.

Since $F(t,b) = f \circ \tilde{F}(t,b) \in U_i$ for $t_i \le t \le t_{i+1}$, and F is continuous in the square, we can find $\epsilon > 0$, such that for every $i, F(t,u) \in U_i$ for $t_i \le t \le t_{i+1}$ and $|u-b| < \epsilon$ (we consider only points (t,u) with $0 \le t \le 1, 0 \le u \le 1$). We shall now prove that for $t_i \le t \le t_{i+1}$ and $|u-b| < \epsilon, \tilde{F}(t,u) = f_i^{-1} \circ F(t,u)$; since f_i^{-1} and F are continuous, this will prove the continuity of $\tilde{F}$ at the point (a,b). Using, for fixed u with $|u-b| < \epsilon$, the uniqueness of the lift of the map $t \to F(t,u), 0 \le t \le t_2$, with initial point p, we see that $\tilde{F}(t,u) = f_i^{-1} \circ F(t,u)$, for $0 \le t \le t_2$, $|u-b| < \epsilon$. Assuming that $\tilde{F} = f_j^{-1} \circ F^1, t_j \le t \le t_{j+1}, |u-b| < \epsilon$ for $j = 1, \ldots, i-1 (i \ge 2)$, we shall now prove that $\tilde{F} = f_i^{-1} \circ F$ for $t_i \le t \le t_{i+1}, |u-b| < \epsilon$. This will complete the proof of the theorem. By hypothesis, the function $\phi(u) = \tilde{F}(t_i,u) = f_{i-1}^{-1} \circ F(t_i,u)$ is continuous in $N = \{u \mid 0 \le u \le 1, |u-b| < \epsilon\}$. Now $\phi(u)$ and $f_i^{-1} \circ F(t_i,u)$ are two continuous maps from N into $\tilde{M}$, such that $f \circ \phi(u) = f(f_i^{-1} \circ F(t_i,u))$ and $\phi(b) = f_i^{-1} \circ F(t_i,b) (= \tilde{F}(t_i,b))$. Hence $\phi(u) = f_i^{-1} \circ F(t_i,u), u \in N$. So $\tilde{F}(t_i,u) \in \tilde{U}_i$. For fixed $u \in N$, the continuous maps $\tilde{F}(t,u)$ and $f_i^{-1} \circ F(t,u)$ from the interval $t_i \le t \le t_{i+1}$ into $\tilde{M}$ are such that $(f \circ \tilde{F})(t,u) = f \circ f_i^{-1} \circ F(t,u)$ and $\tilde{F}(t_i,u) = f_i^{-1} \circ F(t_i,u)$. Hence $\tilde{F}(t,u) = f_i^{-1} \circ F(t,u)$ for $t_i \le t \le t_{i+1}$ and $u \in N$, *q.e.d.*

Remark 2.24 The above theorem is known as the monodromy theorem. The next theorem also is sometimes refered to as the monodromy theorem.

Theorem 2.25 *Let $(\tilde{M}, M, f)$ be a spread without relative boundary and M be simply-connected. Then f is a homeomorphism of $\tilde{M}$ onto M.*

Proof: Let $x \in \tilde{M}$. Let $m \in M$. Let γ be a curve in M such that $\gamma(0) = f(x)$ and $\gamma(1) = m$ and $\tilde{\gamma}$ be the lift of γ through x. Then $f(\tilde{\gamma}(1)) = m$. Hence f is onto.

We prove that f is one-to-one. Let $x, y \in \tilde{M}$ such that $f(x) = f(y)$. Join x and y by a path $\tilde{\gamma}$ in $\tilde{M}$. Then $\gamma = f \circ \tilde{\gamma}$ is a closed curve at $f(x)$. Since M is simply connected, γ is homotopic to the constant curve at $f(x)$. Hence, by the previous theorem, $\tilde{\gamma}$ is a closed path at x, i.e. $x = y$.

Since f is one-one onto, and is a local homeomorphism, f is a homeomorphism of $\tilde{M}$ onto M.

2.5 Covering Spaces

Definition 2.26 A triple $(\tilde{M}, M, f)$ where $\tilde{M}$ and M are n-dimensional manifolds, and $f : \tilde{M} \to M$ is a continuous mapping, is said to be a covering if every point $m \in M$ has a neighbourhood U such that $f^{-1}(U)$ is a disjoint union of open sets in $\tilde{M}$ each of which is mapped homeomorphically by f onto U. We say that $\tilde{M}$ is a covering manifold of M; U will be called a special neighbourhood of m.

Remark 2.27 A covering is a spread.

Example 2.28 $\tilde{M} = \mathbb{R}, M = S^1, f(x) = e^{2\pi i x}, x \in \mathbb{R}$.

Example 2.29 $\tilde{M} = \{z \mid z \in \mathbb{C}, 0 < |z| < r\}(r > 0),\ M = \{z \in \mathbb{C} \mid 0 < |z| < r^n\},\ f(z) = z^n$, where n is a positive integer.

Example 2.30 $\tilde{M} = \mathbb{C},\ M = \mathbb{C}^*,\ f(z) = e^{2\pi i z},\ z \in \mathbb{C}$.

Example 2.31 $\tilde{M} = \mathbb{R}^2,\ M = S^1 \times S^1$ (Torus), $f(x, y) = (e^{2\pi i x}, e^{2\pi i y})$, $(x, y) \in \mathbb{R}^2$.

Proposition 2.32 *Let $(\tilde{M}, M, f)$ be a covering. Then $(\tilde{M}, M, f)$ is a spread without relative boundary.*

Proof: Let γ be a curve in M, and m a point lying over $\gamma(0)$. We can find points $0 = t_1 < t_2 < \cdots < t_n = 1$ in $[0, 1]$ and special neighbourhoods $U_1, \ldots, U_{n-1}$ (of points $m_1, \ldots, m_{n-1}$) such that the closed interval $t_i \le t \le t_{i+1}$ is mapped by γ into U_i. Now $\tilde{m} \in f^{-1}(U_1)$, and by hypothesis there is an open set $\tilde{U}_1$, such that $\tilde{m} \in \tilde{U}_1$, and $f|\tilde{U}_1 : \tilde{U}_1 \to U_1$ is a homeomorphism Let $f^{-1} : U_1 \to \tilde{U}_1$ be the inverse of $f|\tilde{U}_1$. Then $\tilde{\gamma}(t) = f^{-1} \circ \gamma(t),\ 0 \le t \le t_2$ is continuous in $0 \le t \le t_2$ and $f \circ \tilde{\gamma}(t) = \gamma$ in this interval. Suppose now that we have found a continuous function $\tilde{\gamma}_{i-1}$ from the interval $0 \le t \le t_i$ into $\tilde{M}$ with $\tilde{\gamma}_{i-1}(0) = \tilde{m}$ and $f \circ \tilde{\gamma}_{i-1} = \gamma$ in this interval. Since $\tilde{\gamma}_{i-1}(t_i) \in f^{-1}(U_i)$, there is an open set $\tilde{U}_i$ in $\tilde{M}$ such that $\tilde{\gamma}_{i-1}(t_i) \in \tilde{U}_i$ and $f|\tilde{U}_i : \tilde{U}_i \to U_i$ is a homeomorphism. Let $f_i^{-1} : U_i \to \tilde{U}_i$ be the inverse of $f|\tilde{U}_i$. Then the function

$$\tilde{\gamma}_i(t) = \begin{cases} \tilde{\gamma}_{i-1}(t), & \text{for } 0 \le t \le t_i, \\ f_i^{-1} \circ \gamma(t), & \text{for } t_i \le t \le t_{i+1}, \end{cases}$$

is a continuous function in $0 \leq t \leq t_{i+1}$, such that $\tilde{\gamma}_i(0) = m$, and $f \circ \tilde{\gamma}_i(t) = \gamma(t)$ for $0 \leq t \leq t_{i+1}$. After a finite number of steps we arrive at a curve $\tilde{\gamma}$ with initial point $\tilde{m}$ such that $f \circ \tilde{\gamma} = \gamma$.

This proposition and the monodromy theorem imply the

Theorem 2.33 *If $(\tilde{M}, M, f)$ is a covering, and M is simply connected, then f is a homeomorphism of $\tilde{M}$ onto M.*

Remark 2.34 Using the monodromy theorem we can easily prove the following result: if $(\tilde{M}, M, f)$ is a spread without relative boundary, then $(\tilde{M}, M, f)$ is a covering. (If U is a neighbourhood of $m \in M$ homeomorphic to an open ball in $\mathbb{R}^n$ then the connected components of $f^{-1}(U)$ are open in $\tilde{M}$ and are mapped homeomorphically by f onto U.) Thus the notions of a covering and a spread without relative boundary are identical.

Example 2.35 Let $f\colon \mathbb{C} \to \mathbb{C}^*$ be the map $f(z) = e^{2\pi i z}$. Then this is a covering, and $\mathbb{C}$ is simply connected. The curve $\gamma^h\colon z = ae^{2\pi i h t}, 0 \leq t \leq 1$, in $\mathbb{C}^*$ has a lift $\tilde{\gamma}$ at a point b with $f(b) = a$ given by $\tilde{\gamma}(t) = b + ht, 0 \leq t \leq 1$. Hence γ^h is homotopic to constant in $\mathbb{C}^*$ if and only if $h = 0$. Moreover, for any closed curve σ in $\mathbb{C}^*$, if $\tilde{\sigma}$ is a lift of σ, then $\tilde{\sigma}(1) = \tilde{\sigma}(0) + h$ where h is an integer (since $f(\tilde{\sigma}(1)) = f(\tilde{\sigma}(0))$). Hence $\tilde{\sigma}$ is homotopic to the curve $\tilde{\Gamma}(t) = \tilde{\sigma}(0) + ht$ $(0 \leq t \leq 1)$ so that σ is homotopic in $\mathbb{C}^*$ to the curve $\gamma^h(t) = \sigma(0)e^{2\pi i h t}$. Thus any *closed curve in $\mathbb{C}^*$ is homotopic to a unique curve γ^h.*

Chapter 3

Riemann Surfaces

3.1 Holomorphic Functions in the Complex Plane

We begin by recalling some definitions and results concerning holomorphic functions in the complex plane.

3.1.1 Holomorphic Functions

Definition 3.1 A complex-valued function f, defined on an open set U in $\mathbb{C}$, is holomorphic in U if, for every $z_0 \in U$, the limit $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

Remark 3.2 A holomorphic function is continuous.

Remark 3.3 The sum and product of two holomorphic functions is holomorphic.

Remark 3.4 The composite of two holomorphic functions is holomorphic. More precisely, let U, V be open sets in $\mathbb{C}$, and let $f\colon U \to V$ be holomorphic. Then for any holomorphic function g on V, $g \circ f$ is holomorphic in U.

Remark 3.5 A holomorphic function has first partial derivatives, which satisfy the Cauchy-Riemann equations $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$.

3.1.2 Differentiable Curves

Definition 3.6 A curve $z(t) = x(t) + iy(t),\ a \leq t \leq b$, in the complex plane is piecewise differentiable if the interval $a \leq t \leq b$ can be divided into finitely many closed subintervals, in each of which $x(t)$ and $y(t)$ have continuous derivatives.

Example 3.7 The line segment $\overline{z_0 z_1}$ joining $z_0, z_1 \in \mathbb{C}$ i.e. $z(t) = z_0 + t(z_1 - z_0),\ 0 \leq t \leq 1$.

Example 3.8 The circle $|z - z_0| = r$ i.e. $z(t) = z_0 + re^{2\pi it},\ 0 \leq t \leq 1$.

Example 3.9 If γ_1 and γ_2 are piecewise-differentiable curves with end-point of $\gamma_1 =$ initial point of γ_2 the curve $\gamma_1 \cdot \gamma_2$ is piecewise differentiable. In particular, if R is the rectangle $a \leq x \leq b,\ c \leq y \leq d$, its boundary ∂R will always be regarded as the closed piecewise-differentiable curve $\overline{z_1 z_2} \cdot \overline{z_2 z_3} \cdot \overline{z_3 z_4} \cdot \overline{z_4 z_1}$, where $z_1 = a+ic,\ z_2 = b+ic,\ z_3 = b+id,\ z_4 = a+id$.

If $\gamma = z(t),\ a \leq t \leq b$ is a piecewise-differentiable curve in $\mathbb{C}$ and f is a continuous function on the set $\underline{\gamma} = \{z(t),\ a \leq t \leq b\}$, then $\int_\gamma f(z)\, dz$ is by definition $\int_a^b f(z(t)) \frac{dz(t)}{dt} dt$.

Similarly, if u and v are continuous complex-valued functions on $\underline{\gamma}$, $\int_\gamma (u\, dx + v\, dy)$ is, by definition $\displaystyle\int_a^b \left(u(z(t)) \frac{dx(t)}{dt} + v(z(t)) \frac{dy(t)}{dt} \right) dt.$

Example 3.10

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{dz}{z - \zeta} = \begin{cases} 1, & |\zeta - z_0| < r \\ 0, & |\zeta - z_0| > r \end{cases}.$$

3.1.3 Cauchy's Theorem

Let $f(z)$ be a holomorphic function in a disc $D\colon \{|z - a|, \rho\}$, and γ any closed piecewise differentiable curve in D. Then $\int_\gamma f(z) dz = 0$. This assertion would still be true if $f(z)$ were not defined at finitely many points z_i in D provided $\lim_{z \to z_i} (z - z_i) f(z) = 0$ for all the z_i and γ does not pass through any z_i.

Remark 3.11 This theorem follows easily from the corresponding theorem for rectangles, and involves no topological difficulties.

3.1.4 Cauchy's Integral Formula

Let $f(z)$ be holomorphic in a disc $\{|z-z_0|<R_0\}$. Then, for $|z-z_0| < R < R_0$,

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Again, $f(z)$ may be undefined at finitely many points z_i (not lying on $|z-z_0| = R$), provided $\lim_{z\to z_i}(z-z_i)f(z) = 0$ for each i.

Consequences

1. $f(z)$ can be made holomorphic in the disc $|z-z_0| < R_0$ by defining it at the exceptional points if any, by means of the integral formula. This is Riemann's Theorem on removable singularities.

2. $f(z)$ has derivatives of all orders (given by the integral formulae), and is represented by its Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{in } |z-z_0| < R_0.$$

As an immediate consequences, we have the principle of analytic continuation: if a holomorphic function on a *connected* open set U in $\mathbb{C}$ vanishes on a non-empty open subset of U, it vanishes identically in U.

3. If a sequence $\{f_n\}$ of holomorphic functions on an open set U converges uniformly in U, then the limit function f is holomorphic in U and $\{df_n/dz\}$ converges to df/dz (uniformly on compact subsets of U.)

3.1.5 Order of zeroes

Let $f(z)$ be holomorphic (and non-constant) in the connected open set U in $\mathbb{C}$. Then for any $z_0 \in U$, there exists an integer $k \geq 0$ such that $f(z) = (z-z_0)^k g(z)$, with $g(z)$ holomorphic in U and $g(z_0) \neq 0$; in fact k is the least integer n such that $d^n f(z_0)/dz^n \neq 0$. This k is called the *order of* f at z_0. By (3.1.4, k is finite.) From the representation $f(z) = (z-z_0)^k g(z)$, it follows that the *zeros* of f in U are *isolated.*

Now let the disc $D = \{|z-z_0| \leq R\} \subset U$, and suppose that $f(z)$ has no zeros on $|z-z_0| = R$. Then $f(z)$ has finitely many zeros in $D = \{|z-z_0| < R\}$, say $\zeta_1, \ldots, \zeta_n$, with orders $k_1, \ldots, k_n$ respectively. We now have

The Argument Principle. For any holomorphic function $\phi(z)$ in U, we have

$$\frac{1}{2\pi i}\int_{|z-z_0|=R} \phi(z)\frac{f'(z)}{f(z)}\, dz = \sum_{i=1}^{n} k_i \phi(\zeta_i).$$

In particular, if $\phi \equiv 1$, we have

$$\frac{1}{2\pi i}\int_{|z-z_0|=R}\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} k_i.$$

Now let $f(z_0) = 0$, and let R above be such that $f(z) \neq 0$, $f'(z) \neq 0$ in $0 < |z - z_0)| \leq R$. Then for any complex number w, $|w| < \inf_{|z-z_0|=R}|f(z)|$ we have $\frac{1}{2\pi i}\int_{|z-z_0|=R}\frac{f'(z)}{f(z-w)}dz = k =$ order of f at z_0.

Consequences

1. $f\colon U \to \mathbb{C}$ is an open mapping. The *maximum modulus principle,* namely that for $z \in U$, we have

$$|f(z)| \leq \sup_{\eta}\{\limsup_{\zeta\to\eta\in\partial U}|f(\zeta)|\}$$

follows easily from this.

2. For any $z_0 \in U$, f is one-one in some neighbourhood of z_0 if and only if $f'(z_0) \neq 0$. Thus, using 1, we see that if f is one-one on $U, f(U) = V$ is an open set in $\mathbb{C}$ and $f^{-1}\colon V \to U$ is holomorphic.

3.1.6 n-th roots of holomorphic functions

Let the notation be as in 3.1.5, and suppose, for $z_0 \in U$, that $k =$ order of f at $z_0 > 1$. Then, in a neighbourhood of z_0 we can write $f(z) = [h(z)]^k$, with $h(z)$ holomorphic, and having a zero of order 1 at z_0. This follows from the

Theorem 3.12 *If $f(z)$ is holomorphic and never zero in a disc D, there exists a holomorphic function $g(z)$ in D such that $e^g \equiv f$. Hence, for each integer $n > 0$, there exists a holomorphic function $h_n(z)(= e^{g(z)/n})$ such that $(h_n)^n \equiv f$.*

In fact, $z_0 \in D$ being fixed, $\displaystyle\int_{z_0}^{z}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta$ is well-defined by Cauchy's Theorem and, but for an additive constant, is the $g(z)$ we want.

3.1.7 Isolated singularities

Let $f(z)$ be holomorphic in $0 < |z - z_0| < R$.

(i) If $\lim_{z\to z_0}\{(z - z_0)f(z)\} = 0$ (in particular if $f(z)$ is bounded in $(0 < |z - z_0| < \rho$ for some $\rho > 0)$, we know that f can be defined at z_0 so that it is holomorphic in $|z - z_0| < R$.

(ii) If $|f(z)| \to \infty$ as $z \to z_0$, f is said to have a *pole* at z_0. In this case, for a sufficiently small $\rho > 0$, $g(z) = \frac{1}{f(z)}$ for $0 < |z - z_0| < \rho$, $g(z_0) = 0$ defines holomorphic function g in $|z - z_0| < \rho$. The order of g at z_0 is called *the order of the pole* of f at z_0. If $f(z)$ has a pole of order k at z_0 we can write $f(z) = (z - z_0)^{-k} h(z)$ in $0 < |z - z_0| < R, h(z)$ holomorphic in $|z - z_0| < R$, $h(z_0) \neq 0$. Thus $f(z)$ can be expanded in a Laurent series $f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$, $a_{-k} \neq 0$. *The residue* of f at z_0 is a_{-1}; clearly $a_{-1} = \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z)\, dz$ for any r, $0 < r < R$.

A *meromorphic function* f in an open set U in $\mathbb{C}$ is a holomorphic function in the complement of a discrete (possible empty) set E in U, such that $|f(z)| \to \infty$ as z $(\in U - E)$ tends to any point of E. If U is connected, the meromorphic functions in U form a field. If $f(\not\equiv 0)$ is a meromorphic function on the connected open set U in $\mathbb{C}$, f' is also meromorphic in U, and f'/f is a meromorphic function in U whose poles are precisely the poles and zeros of f, with a residue which equals order f at a zero of f, and equals negative of the order of the pole at a pole of f.

3.1.8 Green's Theorem

Let u, v be complex-valued functions, defined and having continuous first partial derivatives in an open set containing the rectangle $R = \{a \leq x \leq b,\ c \leq y \leq d\}$ in $\mathbb{C}$. Then

$$\int_{\partial R} (u\, dx + v\, dy) = \int_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\ dy.$$

If $R' = \{a' \leq x \leq b',\ c' \leq y \leq d'\} \subset R$, and u, v are defined and have continuous first partial derivatives in an open set containing $\overline{R - R'}$, we deduce that

$$\int_{\partial R} u\, dx + v\, dy - \int_{\partial R'} u\, dx + v\, dy = \int_{\overline{R-R'}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\ dy.$$

3.2 Riemann Surfaces

3.2.1

Let M be a Hausdorff space.

Definition 3.13 (i) A coordinate system (U, z) on M consists of a non-empty open subset U of M and a homeomorphism z of U onto an open set in the complex plane.

(ii) Given two coordinate systems (U_1, z_1), (U_2, z_2) on M, (U_1, z_1) is said to be compatible with $(U_2, z_2))$ if $U_1 \cap U_2 = \emptyset$ or if $z_2 \circ z_1^{-1} \colon z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2)$ is a holomorphic mapping. (Compatibility is a symmetric relation.)

(iii) A complex structure (or holomorphic structure) on M is a set $\Phi = \{(U_i, z_i)\}_{i \in I}$ of coordinate systems on M, such that (a) $\bigcup_{i \in I} U_i = M$, (b) any two members of Φ are compatible, (c) if a coordinate system (U, z) on M is compatible with every member of Φ, then $(U, z) \in \Phi$.

(iv) A Riemann surface $(M, \Phi) = M$ is a Hausdorff space M with a complex structure Φ. A coordinate system on a Riemann surface always means one belonging to the complex structure.

Remark 3.14 Condition (iii)(a) implies that a connected Riemann surface is a two-dimensional manifold.

Remark 3.15 Let Φ' be a set of coordinate systems on a Hausdorff space M satisfying (a) and (b) of (iii) above. Then there is a unique complex structure $\Phi \supset \Phi'$ on M, namely the one consisting of all coordinate systems on M compatible with all members of Φ'.

Remark 3.16 According to a theorem of T. Rado, any connected Riemann surface is a union of countably many compact subsets.

Definition 3.17 Let M, N be Riemann surfaces. A holomorphic mapping $f \colon M \to N$ is a continuous mapping such that, for any coordinate systems $(U, z), (V, w)$ on M, N respectively with $f(U) \subset V$, the mapping $w \circ f \circ z^{-1} \colon z(U) \to w(V)$ is holomorphic.

It is easy to verify that a mapping $f \colon M \to N$ of Riemann surfaces is holomorphic if and only if, for any $p \in M$, there exist coordinate systems (U, z), (V, w) on M, N respectively, such that $p \in U$ and $f(U) \subset V$, while $w \circ f \circ z^{-1} \colon z(U) \to w(V)$ is holomorphic.

3.2.2 Examples of Riemann Surfaces

Example 3.18 Non-empty open sets in the complex plane are Riemann surfaces in a natural way. A holomorphic function on a Riemann surface is by definition a holomorphic mapping of it into the complex plane.

Example 3.19 Any non-empty open subset U of a Riemann surface M has a natural complex structure, such that the inclusion of U in M is holomorphic. If a mapping $z: U \to \mathbb{C}$ is one-one and holomorphic on $U, (U, z)$ is a coordinate system on M, and conversely. Also a mapping $f: M \to N$ of Riemann surfaces is holomorphic if and only if every $p \in M$ has a neighbourhood U such that $f|U$ is holomorphic.

Example 3.20 $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can be made a Riemann surface as follows. The topology on $\bar{\mathbb{C}}$, i.e. its set of open sets on $\mathbb{C}$, together with sets of the form $U \cup \{\infty\}$, where U is an open set in $\mathbb{C}$, which contains $\{|z| > R\}$ for some R. (This topology is compact Hausdorff.) The mapping $w: \bar{\mathbb{C}} - \{0\} \to \mathbb{C}$ defined by $w(z) = 1/z$ if $z \neq \infty$, $w(\infty) = 0$, is a homeomorphism onto $\mathbb{C}$, and the coordinate systems $(\mathbb{C},\ I_{\mathbb{C}}(=$ identitymap$))$, $(\bar{\mathbb{C}} - \{0\},\ w)$ define a complex structure on $\bar{\mathbb{C}}$. The verifications are easy.

Example 3.21 Let (N, M, f) be a spread (§2.3), where M is a surface. Then any complex structure on M defines one on N in a natural way, such that f becomes holomorphic. This structure is defined by coordinate systems $(V, z \circ f)$, where $f|V$ is one-one, and $(f(V), z)$ is a coordinate system with respect to the complex structure of M.

Example 3.22 Let ω_1, ω_2 be non-zero complex numbers with non-real ratio, and let R be the relation on $\mathbb{C}$ defined by "$z_1\ R\ z_2$ if and only if there exist integers m and n such that $z_1 - z_2 = m\omega_1 + n\omega_2$." Let T be the quotient space $\mathbb{C}/R$ and $\pi: \mathbb{C} \to T$ the natural map. T is compact. It is easy to see that π is an open mapping. Let $\delta = \inf |m\omega_1 + nw_2|,\ m,\ n$ integral and not both zero. It is easily verified that $\delta > 0$. It follows that T is a Hausdorff space, and that π is one-one on any open disc of diameter δ. The pairs $(\pi(U),\ (\pi|U)^{-1})$, with U any disc of diameter δ, are coordinate systems on T and make it a Riemann surface such that π is holomorphic. T is called a complex torus.

We could also consider the equivalence relation R on $\mathbb{C}$ defined by a single complex number $\omega \neq 0$: $z_1\ R\ z_2$ if and only if $z_1 - z_2 = n\omega$ for some integer n. The function $e^{2\pi i z/\omega}$ on $\mathbb{C}$ "passes down" to one on $\mathbb{C}/R$ and maps it one-one holomorphically onto $\mathbb{C} - \{0\}$.

3.2.3

The following properties of holomorphic mappings of Riemann surfaces are immediate consequences of the analogous statements for holomorphic functions in the plane.

1. The composite of holomorphic mappings is holomorphic.

2. If f, g are two holomorphic mappings of a connected Riemann surface M into a Riemann surface N, and if $f = g$ on a non-empty open set in M, then $f \equiv g$. This is called the Principle of Analytic Continuation. Similarly, if f is not constant, then for every $q \in N$, $f^{-1}(q)$ is a discrete set in M.

3. Let M be a connected Riemann surface. Then any non-constant holomorphic mapping of M (into any Riemann surface) is an open mapping. The maximum principle for holomorphic functions on M follows as before. We also deduce that, on a *compact connected* Riemann surface, the only holomorphic functions are the constants.

A *meromorphic function* on a Riemann surface M is a holomorphic function on the complement of a discrete set E in M (E may be empty), such that for every q in E, $|f(p)| \to \infty$ as $p \to q$. The mapping $f\colon M \to \bar{\mathbb{C}}$ defined by putting $f(q) = \infty$ is then holomorphic, for it follows easily from Riemann's theorem on removable singularities that a continuous mapping $h\colon M \to N$ of Riemann surfaces is holomorphic if it is so in the complement of a discrete set in M.

Conversely, if M is connected, every holomorphic mapping $f\colon M \to \bar{\mathbb{C}}$ with $f(M) \neq \{\infty\}$ defines a meromorphic function on M. Again, the meromorphic functions on a connected Riemann surface constitute a field.

Example 3.23 Meromorphic functions on $\bar{\mathbb{C}}$ are just the rational functions (i.e. quotients of polynomials).

Example 3.24 With the notation of Example 3.22 every meromorphic function f on $\mathbb{C}$ with periods ω_1 and ω_2 defines a meromorphic function g on T such that $f = g \circ \pi$; and conversely.

Let f be a non-constant holomorphic function on the connected Riemann surface M. Then the order of f at $p \in M$ is well-defined. Namely, if (U, z) is any coordinate system at p, the order of $f \circ z^{-1}$ at $z(p) \in z(U)$ is determined by f (and does not depend on the coordinate system at p chosen), and we define $\mathrm{ord}_p f = \mathrm{ord}_{z(p)} f \circ z^{-1}$. Similarly, for a meromorphic function f on M, the order k of any pole p is well-defined, and we sometimes write $\mathrm{ord}_p f = -k$.

Let $f\colon M \to N$ be a non-constant holomorphic mapping of the connected Riemann surface M into N. Then for any $p \in M$, we can choose coordinate systems at p and $q = f(p)$, say (U, z) and (V, w) respectively, such that (i) $z(p) = w(q) = 0$, (ii) $f(U) \subset V$, (iii) $w \circ f \circ z^{-1} \equiv z^k$ on $z(U)$, for an integer $k \geq 1$ (in fact $k = \mathrm{ord}_p(w \circ f - w(f(p)))$.

3.2.4

Let U, V be open sets in $\mathbb{C}$, $F\colon U \to V$ a one-one holomorphic mapping of U onto V (or more generally let F be a one-one mapping of U onto V, and suppose that F and F^{-1} are differentiable). For any piecewise differentiable curve γ in U, $F(\gamma) = F \circ \gamma$ is a piecewise differentiable curve in V. Also, for any continuous complex-valued functions f, g on U, $f^* = f \circ F^{-1}$, $g^* = g \circ F^{-1}$ are continuous on V, and $\int_{F(\gamma)} f^* dx + g^* dy$ is defined. If $F = u + iv$, it is easy to see that

$$\int_{F(\gamma)} (f^* dx + g^* dy) = \int_\gamma \left(\left(f \frac{\partial u}{\partial x} + g \frac{\partial v}{\partial x} \right) dx + \left(f \frac{\partial u}{\partial y} + g \frac{\partial v}{\partial y} \right) dy \right).$$

Similarly, if f is a continuous function on V vanishing outside a compact set in V, we have

$$\int_V f \, dx \, dy = \int_U (f \circ F) |J| dx \, dy,$$

where J is the Jacobian of F (if F is holomorphic, its Jacobian is $|F'|^2$).

These transformation formulae suggest that we should consider the so-called differential forms, or forms, which are defined below.

Definition 3.25 A differential 1-form (resp. 2-form) ω on a Riemann surface M is the assignment to each coordinate system (U, z) on M of an ordered pair (ω_x, ω_y) of complex-valued functions on $z(U)$ (resp. a complex-valued function ω on $z(U)$), such that if (U_1, z_1), $(U_2, z_2))$ are any coordinate systems on M with $U_1 \cap U_2 \neq \emptyset$, and $\phi_{12} = z_2 \circ z_1^{-1} = u + iv$ on $z_1(U_1 \cap U_2)$, then.

$$\begin{aligned} \omega_{x_1} &= (\omega_{x_2} \circ \phi_{12}) \frac{\partial u}{\partial_{x_1}} + (\omega_{y_2} \circ \phi_{12}) \frac{\partial v}{\partial_{x_1}} \\ \omega_{y_1} &= (\omega_{x_2} \circ \phi_{12}) \frac{\partial u}{\partial_{y_1}} + (\omega_{y_2} \circ \phi_{12}) \frac{\partial v}{\partial_{y_1}} \end{aligned}$$

(resp. $\omega_1 = (\omega_2 \circ \phi_{12}) |\phi'_{12}|^2$ on $z_1(U_1 \cap U_2)$).

Remark 3.26 To define a 1-form ω on M, it is sufficient to assign the (ω_x, ω_y) for each of a set Φ of coordinate systems on M such that the U, $(U, z) \in \Phi$, cover M, if the transformation laws above are satisfied for members of Φ. The same fact is true in the case of 2-forms.

Remark 3.27 A form (i.e. 1-form or 2-form) on M induces one on any open subset of M in an obvious way.

Remark 3.28 A 1-form ω on M is of class C^k, if for every coordinate system (U, z) on M, "components" $\omega_x,\ \omega_y$ of ω are of differentiability class C^k on $z(U)(k = 0, 1, 2, \ldots)$. Again it is sufficient to verify this on a set of coordinate neighbourhoods covering M. A similar definition can be given for 2-forms.

Remark 3.29 For any 1-form ω and any $p \in M$, $\omega_x(z(p)) = 0 = \omega_y(z(p))$ for every coordinate system at p, if it is so for one such system. We say in this case that $\omega(p) = 0$. The *support* of ω is the closure of the set $\{p \in M \mid \omega(p) \neq 0\}$. Similar considerations are valid for a 2-form ω; in this case the statement $\omega(p) > 0$ also has an intrinsic meaning.

3.2.5 Examples of 1- and 2-forms

Example 3.30 Let f be a complex-valued function of class C^k on M, $k \geq 1$ (i.e. for every coordinate system (U, z), $f \circ z^{-1}$ is of class C^k on $z(U)$). Then the assignment to (U, z) of $\left(\frac{\partial(f\circ z^{-1})}{\partial x}, \frac{\partial(f\circ z^{-1})}{\partial y}\right)$ can be verified to be a 1-form of class C^{k-1} on M. This 1-form is denoted by df. If M is connected $df \equiv 0$ implies that f is a constant.

Example 3.31 Let ω be a 1-form of class C^k on M, $k \geq 1$. Then the assignment to (U, z) of the function $\left(\frac{\partial\omega_y}{\partial x} - \frac{\partial\omega_x}{\partial y}\right)$ is a 2-form of class C^{k-1} on M, denoted by $d\omega$. Clearly the support of $d\omega \subset$ the support of ω. Now ω is said to be closed if $d\omega = 0$, exact if $\omega = df$ for some function f of class C^1. An exact 1-form (of class C^1) is always closed. Conversely, it follows easily from Green's theorem that any closed 1-form on a disc in $\mathbb{C}$ is exact (cf.[1, p.85]).

Example 3.32 The sum $\omega_1 + \omega_2$ of a pair of forms can be defined in an obvious way, since the transformation laws affect the components linearly. If the ω_i are of class C^k, so is their sum. If they are 1-forms of class C^1, $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$. Similarly, if $f_1,\ f_2$ are functions of class C^1, $d(f_1 + f_2) = df_1 + df_2$.

Example 3.33 Let ω be a 1-form on M, f a function on M. The 1-form $f\omega$ on M is defined as follows : if ω has components (ω_x, ω_y) in the coordinate system (U, z), those of $f\omega$ are $((f \circ z^{-1}))\omega_x,\ (f \circ z^{-1})\omega_y)$. If

f and ω are of class C^k, so is $f\omega$; it is sufficient that f be defined in a neighbourhood of the support of ω. Obviously, $\omega|U = (\omega_x \circ z)dx + (\omega_y \circ z)dy$.

Example 3.34 The real and imaginary parts of a form ω are defined in the obvious way, and we have $\omega = \text{Re } \omega + i \text{ Im } \omega$.

Example 3.35 If ω_1 and ω_2 are two 1-forms, a 2-form $\omega_1 \wedge \omega_2$ is defined by assigning to any coordinate system (U, z) the function $\omega_{1x}\omega_{2y} - \omega_{2x}\omega_{1y}$, ω_{ix}, ω_{iy} being the components of ω_i on (U, z).

3.2.6 Integration of 1-forms

Definition 3.36 Let γ be a curve on a Riemann surface M and suppose that there exists a coordinate system (U, z) on M such that $\gamma \subset U\cdot$ γ is said to be the piecewise differentiable if the curve $z \circ \gamma = z(\gamma)$ in $z(U)$ is. In this case, if ω is a continuous 1-form on M,

$$\int_\gamma \omega = \int_{z(\gamma)} (\omega_x \, dx + \omega_y \, dy).$$

We note that in the above definition, if (U_1, z_1) is another coordinate system on M such that $\gamma \subset U_1$, $z_1(\gamma)$ is piecewise differentiable in $z_1(U_1)$ and by the transformation formulae for the components of ω, $\int_{z_1(\gamma)}(\omega_{x_1} dx_1 + \omega_{y_1} dy_1) = \int_{z(\gamma)}(\omega_x dx + \omega_y dy)$. Thus $\int_\gamma \omega$ is well-defined.

Any curve γ on M can be written as $\gamma_1, \ldots, \gamma_n$ where each γ_i is contained in a coordinate neighbourhood of M. We say that γ is piecewise differentiable if each γ_i is (this is a property of γ which does not depend on a choice of the decomposition $\gamma = \gamma_1 \ldots \gamma_n$). If ω is a continuous 1-form on M, we define $\int_\gamma \omega = \Sigma \int_{\gamma_i} \omega$; again the value of $\int_\gamma \omega$ is independent of the decomposition of γ.

3.2.7 Integration of 2-forms

An analogous theory of integration of (continuous) 2-forms involves the use of "differentiable singular 2-simplexes" (see [2]). Here, we shall only indicate how the integral of a 2-form with compact support (over the whole Riemann surface) is defined.

Definition 3.37 If ω is a continuous 2-form with compact support on a Riemann surface M, and (U, z) a coordinate system on M such that U contains the support of ω then $\int_M \omega = \int_{z(U)} \omega \; dx \; dy$.

As in the case of the integral of 1-forms, $\int_M \omega$ is then well-defined. To define $\int_M \omega$ for a continuous 2-form with arbitrary compact support, we express ω as a finite sum of continuous 2-forms ω_i each of which has compact support contained in a coordinate neighbourhood of M, and define $\int_M \omega = \Sigma \int_M \omega_i$. That ω can be written as such a sum is proved below, and the fact that the value of $\int_M \omega$ is independent of the choice of such a decomposition then follows easily.

Lemma 3.38 *Given a compact set K on the Riemann surface M, there exist finitely many continuous functions $f_i \geq 0$ on M, such that each f_i has compact support contained in a coordinate neighbourhood of M, while $\Sigma f_i > 0$ on K.*

PROOF: Plainly, it is sufficient, given $p \in M$ and a coordinate system (U, z) at p, to construct a continuous function $f \geq 0$ with compact support contained in U, with $f(p) > 0$. This is trivial, since U may be identified with $z(U)$ by means of z.

[For the same reason, the above lemma is still valid if we require the f_i to be of class C^k for any $k \geq 0$. We need this with $k = 1$. In this case we may use the following statement, which will be required later. If a_1, a_2 are real numbers, $0 < a_1 < a_2$ and S_i is the square $\{|x|,\ |y| \leq a_i\}$ in $\mathbb{C}$, there exists a function f of class C^1 on $\mathbb{C}$, such that $f \equiv 1$ on S_1 and $f \equiv 0$ outside S_2. Such an f is, for instance, given by $f(x, y) = g(x)g(y)$, where $g(t) = 1$ for $|t| \leq a_1$, 0 for $|t| \geq a_2$ and $\cos^2\left(\frac{\pi}{2}\frac{t^2 - a_1^2}{a_2^2 - a_1^2}\right)$ otherwise.]

To get the desired decomposition of a continuous form ω with compact support into continuous forms with "small" supports, let K, in the lemma above, be the support of ω and let $\omega_i = (\dfrac{f_i}{\Sigma f_i})\omega$ if $\Sigma f_i > 0$ and 0 otherwise. Then clearly ω_i is a continuous form on M with support contained in (support f_i) $\cap$ (support ω), and $\omega = \Sigma\omega_i$. Note that if $\omega = \Sigma g_i \omega = \Sigma h_j \omega$, where the g_i, h_j are functions defined in a neighbourhood of the support of ω we also have $\omega = \sum_{i,j}(g_i h_j)\omega$. This remark is used in verifying that the integral of a 2-form as we have defined it does not depend on the decomposition chosen.

Finally, we note that the integrations we have defined have the natural properties $\int(\omega_1 + \omega_2) = \int \omega_1 + \int \omega_2$, $\int c\omega = c\int \omega$ for any $c \in \mathbb{C}$ and (in the case of 1-forms), $\int_{\gamma_1\gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$.

3.2.8 Holomorphic forms

Definition 3.39 A 1-form ω on a Riemann surface M is holomorphic, if for any coordinate system (U, z) on M, $\omega|U = \omega_z dz$ with ω_z holomorphic in U (i.e. the components ω_x, ω_y of ω are holomorphic, and $\omega_y = i\omega_x$; we shall identify ω_x with $\omega_z \circ z^{-1}$ and write $\omega_x = i\omega_y = \omega_z$).

Remark 3.40 If f is a holomorphic function on M, df is holomorphic: in any coordinate system (U, z), $df = \left(\dfrac{d(f \circ z^{-1})}{dz} \circ z\right) dz$. It is usual to write $df = \frac{df}{dz}\, dz$.

Remark 3.41 If the holomorphic 1-form ω on M equals $\omega_{z_1} dz_1$ on (U_1, z_1), and $\omega_{z_2} dz_2$ on (U_2, z_2), we have $\omega_{z_1} = \omega_{z_2} \dfrac{dz_2}{dz_1}$ on $U_1 \cap U_2$, where $\dfrac{dz_2}{dz_1}$ means, as before $\dfrac{d(z_2 \circ z_1^{-1})}{dz_1} \circ z_1$. Conversely, if we have holomorphic functions ω_{zi} on (U_i, z_i) such that $\omega_{z_j} \frac{dz_j}{dz_i}$ on $U_i \cap U_j$ a unique holomorphic 1-form ω is defined on $\bigcup U_i$, whose restriction to U_i is $\omega_{z_i}\, dz_i$, for each i.

Remark 3.42 Let M be connected, and let ω be a holomorphic 1-form $\not\equiv 0$. Then the set of zeros of ω in M is a discrete set. Further, the order of ω at any point $p \in M$ is well-defined; if (U, z) is any coordinate system at p, and $\omega|U = \omega_z dz$, the order of ω_z at p does not depend on the coordinate system. For instance, if f is a holomorphic function in a neighbourhood of $p \in M$, (V, f) is a coordinate system for some neighbourhood V of p if and only if $\mathrm{ord}_p df = 0$.

Remark 3.43 On account of the Cauchy-Riemann equations, a holomorphic 1-form is always closed.

3.2.9

Definition 3.44 A meromorphic 1-form ω on a Riemann surface M is a holomorphic 1-form ω on the complement of a discrete set E in M with the following property : if (U, z) is any coordinate system at any $p \in E$ with $U \cap E = \{p\}$, and if $\omega = \omega_z dz$ in $U - \{p\}$, then ω_z has a pole at p.

Remark 3.45 It is sufficient to assume that for one coordinate system at p as above, the stated condition is satisfied.

Remark 3.46 With the notation as above, the order of ω_z at p is independent of the coordinate system at p chosen. Moreover, the same is true of the residue of $\omega_z \circ z^{-1}$, at $z(p)$; for, this is equal to $\frac{1}{2\pi i}\int_\gamma \omega$ where γ is any closed curve in U such that $z \circ \gamma$ is for instance a sufficiently small circle $|z - z(p)| = r$. Of course this fact can also be checked by computation.

Remark 3.47 If f is a meromorphic function on M, the meromorphic 1-form df, and the meromorphic 1-form $f\omega$ (ω a meromorphic 1-form on M) are defined in the obvious manner. Thus, if ω is a meromorphic 1-form, and (U, z) a coordinate system on M, then $\omega|U = \omega_z dz$, where ω_z is a meromorphic function on U. Similarly the sum of two meromorphic 1-forms is meromorphic.

Remark 3.48 If f is a non-constant meromorphic function on a connected Riemann surface M, $\frac{df}{f}$ is a meromorphic 1-form on M, whose residue at any $p \in M = \mathrm{ord}_p f$.

Remark 3.49 Let $\omega_0 \not\equiv 0$ be a meromorphic 1-form on the connected Riemann surface M. Then any meromorphic 1-form ω on M is of the form $f\omega_0$ with f a meromorphic function on M. If, in a coordinate system $(U, z), \omega = \omega_z dz$ and $\omega_0 = \omega_{0z} dz$, then f is defined by $f|U = \omega_z/\omega_{0z}$.

Problem 3.1 If M is a compact, connected Riemann surface, and ω is a nowhere vanishing holomorphic 1-form on M, then any holomorphic 1-form on M is of the form $c\omega,\ c \in \mathbb{C}$.

3.2.10 Green's Theorem

The general form of Cauchy's theorem and Green's theorem is outside the scope of this seminar. We only mention a version which can be proved fairly simply.

Theorem 3.50 (Green's Theorem) *Let M be a simply connected Riemann surface (chapter 2, section 2), and ω a closed 1-form on M. Then for any (piecewise differentiable) closed curve γ in M, $\int_\gamma \omega = 0$.*

In particular, if ω above is holomorphic, this can be proved using the Monodromy Theorem of Chapter II (see [11], p.55)

3.2.11 Residue Theorem

Theorem 3.51 *Let M be a compact Riemann surface, and ω a meromorphic 1-form on M. Then the sum of the residues of ω is zero.*

Remark 3.52 The sum in question is finite; the set of poles of ω is discrete, hence finite, by the compactness of M.

The proof uses the following:

Lemma 3.53 *Let ω be a 1-form of class C^1 with compact support, on a Riemann surface M. Then $\int_M d\omega = 0$.*

PROOF: If we write ω as a finite sum of C^1 1-forms ω^i with compact supports contained in coordinate neighbourhoods of M, we have $\int_M d\omega_i = 0$ by Green's Theorem in the plane; hence $\int_M d\omega = \Sigma \int_M d\omega_i = 0$. *q.e.d.*

PROOF OF THE THEOREM : Let $E = \{p_1, \ldots, p_n\}$ be the set of poles of ω if $E = \emptyset$ there is nothing to prove. Let (U_i, z_i) be a coordinate system at p_i, $i = 1, \ldots, n$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$; we may suppose that $z_i(p_i) = 0$, $i = 1, \ldots, n$. Let $0 < a_i < b_i$ be real numbers such that $R_i^* = \{|x|, |y| \leq b_i\} \subset z_i(U_i)$, and let $S_i^* = \{|x|, |y| < a_i\}$. Let f_i^* be a function of class C^1 on $\mathbb{C}$, such that $f_i^* \equiv 1$ on S_i^*, $\equiv 0$ outside R_i^*. Finally let $f_i = f_i^* \circ z_i$, $f = \Sigma f_i$, $\gamma_i = z_i^{-1}(\partial S_i^*)$, $S_i = z_i^{-1}(S_i^*)$, $S = \bigcup S_i$.

We now note that for any continuous 2-form ϕ on $M \backslash E$, we can define $\int_{M\backslash S} \phi$ as $\int_M (1-f)\phi + \int_{M\backslash S} f\phi$ (note that $f\phi$ has support contained in $\bigcup U_i$). If ϕ has support $\subset M \setminus S$, we then have $\int_{M\backslash S} \phi = \int_M \phi$. We also have $\int_{M\backslash S}(\phi_1 + \phi_2) = \int_{M\backslash S} \phi_1 + \int_{M\backslash S} \phi_2$.

In particular we have, since $d\omega \equiv 0$ in $M \setminus S$, $\int_{M\backslash S} d\omega = 0 = \int_{M\backslash S} d(f\omega) + \int_{M\backslash S} d((1-f)\omega)$. Now, $(1-f)\omega$ is a C^1 1-form on M, with support contained in $M - S$, hence by the lemma proved above, $\int_M d((1-f)\omega) = 0 = \int_{M\backslash S} d((1-f)\omega)$. On the other hand, using Green's Theorem as stated in §8 of part A, and the fact that $f_i^* = 1$ on ∂S_i^* and 0 on ∂R_i^*, we see that $\int_{M\backslash S} d(f\omega) = -\Sigma \int_{\gamma_i} \omega = -2\pi\, i\, \Sigma\ \mathrm{Res}_{p_i}\, \omega$, q.e.d.

3.2.12

An immediate consequence of the theorem of 3.51 is the

Theorem 3.54 *A non-constant meromorphic function f on a compact connected Riemann surface M assumes every value (in $\bar{\mathbb{C}}$) equally often, multiplicity being taken into account.*

[The multiplicity with which f takes the value a at a point p is $\mathrm{ord}_p(f-a)$ if a is finite, $\mathrm{ord}_p(1/f)$ if a is infinite.]
PROOF: For any complex number a, $\frac{df}{f-a}$ is a meromorphic 1-form on M, whose residue at any $p \in M$ is $\mathrm{ord}_p(f-a)$. Hence by the theorem of 3.2.11 assumes the value a often as it assumes ∞, *q.e.d.*

Chapter 4

Analytic Continuation and Algebraic Functions

4.1 Analytic Continuation

The results of this section consist merely in an application of the general results on the lifting of curves proved for one surface spread out over another to a special case. We begin with some definitions.

Let U be a connected open set in the complex plane and f a function holomorphic in U. The pair (f, U) is called (after Weierstrass) a *function element*. Let $a \in \mathbb{C}$. We introduce an equivalence relation $\sim$ in the set of all function elements (f, U) with $a \in U$ as follows. $(f, U) \sim (f', U')$ if there is an open set $V \subset U \cap U'$, $a \in V$, such that $f(z) = f'(z)$ for all $z \in V$. (Note that we do not require that $f(z) = f'(z)$ for all $z \in U \cap U'$.) The equivalence class with respect to $\sim$ containing the element (f, U) is called the *germ of* f at a and is denoted by (a, f_a). Note that the *value of this germ* at a, $f_a(a) = f'(a)$ for any $(f', U') \in (a, f_a)$, is defined independently of (f', U').

Analytic Continuation along an Arc. Let (a, f_a) be a germ of holomorphic function, and γ a curve starting at $a, \gamma(0) = a$. We say that (a, f_a) *can be analytically continued along* γ if the following holds.

To every point t is associated a germ $(\gamma(t),\ f_{\gamma(t)})$ (which is equal to (a, f_a) for $t = 0$) and such that for every t_0, there is a neighbourhood $U = \{z \mid |z - \gamma(t_0)| < \rho\}$ and a function F holomorphic in U with $(F, U) \in (\gamma(t_0),\ f_{\gamma(t_o)})$, such that for all t near enough to t_0, the germ induced at $\gamma(t)$ by (F, U) is precisely the given germ $(\gamma(t), f_{\gamma(t)})$.

We have the following proposition, whose proof is not very easy, and will be omitted.

Proposition 4.1 *A germ (a, f_a) can be continued along a curve γ if and only if there exists a finite family of discs $D_0, \ldots, D_{q-1}, a \in D_1, \gamma(1) \in D_{q-1}$ and points $z_0 = a, z_1, \ldots, z_q = \gamma(1)$, on $\gamma, z_i = \gamma(t_i)$, such that*

(i) $t_i < t_{i+1}$ *and for* $t_i \leq t \leq t_{i+1}$, $\gamma(t) \in D_i$, $0 \leq i \leq q-1$;

(ii) *there is a holomorphic function f_i in D_i with $f_i(z) = f_{i+1}(z)$ in $D_i \cap D_{i+1}$;*

(iii) $(f_0, D_0) \in (a, f_a)$.

We now introduce a topological space $\mathcal{O}$ which is spread over the complex plane $\mathbb{C}$.

Let $\mathcal{O}$ denote the set of all germs (a, f_a) where $a \in \mathbb{C}$. We define a topology on $\mathcal{O}$ as follows. A fundamental system of neighbourhoods of (a, f_a) is formed by the sets $\mathcal{U} = \{(b, f_b) \mid b \in U\}$ where $(f, U) \in (a, f_a)$ and, for $b \in U$, (b, f_b) is the germ induced by f at b.

Lemma 4.2 *With this topology, the space $\mathcal{O}$ is Hausdorff.*

PROOF: Let $(a, f_a),\ (b, g_b) \in \mathcal{O},\ (a, f_a) \neq (b, g_b)$.
Case (i). $a \neq b$. Let $(f, U) \in (a, f_a),\ (g, V) \in (b, g_b),\ U \cap V = \emptyset$. Let $\mathcal{U} = \{(c, g_c) \mid c \in U\}$, the set of germs induced by f at points $c \in U$, and similarly we define $\mathcal{V} = \{(d, g_d) \mid d \in V\}$. Clearly $\mathcal{U} \cap \mathcal{V} = \emptyset$.
Case (ii). $a = b$. In this case let U be a *connected* open set, $a \in U,\ (f, U) \in (a, f_a),\ (g, U) \in (b, g_b)$, and let $\mathcal{U} = \{(c, f_c) \mid c \in U\},\ \mathcal{V} = \{(c, g_c) \mid c \in U\}$ consist of the germs induced by f, g respectively at points of U. Then we assert that $\mathcal{U} \cap \mathcal{V} = \emptyset$. In fact, if $(c, f_c) = (c, g_c)$, then $f(z) = g(z)$ for all z in a neighbourhood of c, hence, by the principle of analytic continuation, $f = g$ on U, hence $(a, f_a) = (a, g_a) = (b, g_b)$, which is not the case.

We have now a mapping $\pi\colon \mathcal{O} \to \mathbb{C}$ defined by $\pi((a, f_a)) = a$.

Lemma 4.3 *π is a local homeomorphism.*

PROOF: Let $(f, U) \in (a, f_a)$, and $\mathcal{U} = \{(c, f_c) \mid c \in U\}$ the open set of germs induced by f at points of U. Then $\pi(\mathcal{U}) = U$ is a neighbourhood of a. Hence π is open. If V is a given neighbourhood of a and $U \subset V$, $(f, U) \in (a, f_a)$ and $\mathcal{U}$ is as above, then $\pi(\mathcal{U}) \subset V$. Hence π is continuous. Also $\pi|\mathcal{U}$ ($\mathcal{U}$ as above) is one-one onto U. This proves Lemma 4.3.

Lemma 4.4 *Let $a \in \mathbb{C}$, γ a curve starting at a, and (a, f_a) a germ at a. Then (a, f_a) can be continued along γ if and only if there is a curve $\tilde{\gamma}$ in $\mathcal{O}$ with $\tilde{\gamma}(0) = (a, f_a), \pi \circ \tilde{\gamma}(t) = \gamma(t)$ for each t, $0 \leq t \leq 1$. Moreover, the continuation $(\gamma(t),\ f_{\gamma(t)})$ along γ is determined by $\tilde{\gamma}$ by the rule $(\gamma(t),\ f_{\gamma(t)}) = \tilde{\gamma}(t)$.*

This is a trivial reformulation of the definition of analytic continuation along an arc.

Corollary 4.5 *Analytic continuation along an arc is uniquely determined by the initial germ and the arc.*

Corollary 4.6 *If continuation of (a, f_a) along γ leads to (b, g_b),then continuation of (b, g_b) along γ^{-1} leads to (a, f_a).*

We mention the following consequence of Corollary 4.6. Let D be any domain and (a, f_a) a germ, $a \in D$. Suppose that continuation of (a, f_a) along any curve γ in D is possible and that continuation along any curve γ leads back to (a, f_a), i.e. $(\gamma(1),\ f_{\gamma(1)}) = (a, f_a)$. Then there is a holomorphic function F in D which induces (a, f_a). In fact, for any $z \in D$, let γ be a curve joining a to z and let $(\gamma(t),\ f_{\gamma(t)})$ be the continuation of (a, f_a) along γ. Set

$$F(z) = f_{\gamma(1)}(\gamma(1)) \quad (\gamma(1) = z).$$

Then, if γ' is any other curve joining a to z and continuation along $\gamma'\gamma^{-1}$ leads back to (a, f_a), we have

$$(\gamma(1),\ f_{\gamma(1)}) = (\gamma'(1),\ f_{\gamma'(1)}).$$

Hence $F(z)$ is independent of the curve γ used, and our result follows at once.

Finally, we mention the

Theorem 4.7 *MONODROMY THEOREM. Let D be a domain, and (a, f_a) a germ at $a \in D$ which can be continued along any curve in D. Then continuation along any closed curve γ starting at a which is homotopic to the constant curve at a leads back to (a, f_a).*

PROOF: Let X be the connected component of $\pi^{-1}(D)(\pi\colon \mathcal{O} \to \mathbb{C})$ which contains (a, f_a). We assert that $\pi\colon X \to D$ is a covering. To prove this, it is enough to prove that any curve can be lifted to X with a given initial point. Let $p \in X$ be given $z = \pi(p)$ and Γ a curve in D starting at z. Let $\tilde{\gamma}'$ be a curve in X joining (a, f_a) to p and $\gamma' = \pi(\tilde{\gamma}')$. By 4.4, continuation of (a, f_a) along γ' leads to the germ p at z. Since continuation of (a, f_a) along $\gamma'\ \Gamma$ is possible, p can be continued along Γ. Hence (Lemma 4.4 again), Γ can be lifted to a curve $\tilde{\Gamma}$ in X starting at p.

Hence, if γ is homotopic to the constant curve at a, then the monodromy theorem of §4, Chapter II shows that the lift $\tilde{\gamma}$ of γ starting at (a, f_a) is a closed curve in X. This means precisely that continuation of (a, f_a) along γ leads back to (a, f_a).

4.2 Algebraic Functions

4.2.1 Regular points

An algebraic function w is "defined" by an equation

$$P(z, w) = 0, \tag{4.1}$$

where P is an irreducible polynomial in the two (complex) variables z, w. To make precise the meaning of this statement and to study the behaviour of these algebraic functions, we begin with functions defined implicitly by an equation of the type (4.1).

Theorem 4.8 *Let $P(z, w)$ be a polynomial in the two variables z, w and let a, b be complex numbers such that*

$$P(a, b) = 0,\ \frac{\partial P}{\partial w}(a, b) \neq 0.$$

Then there is a disc $D\colon |z - a| < \rho$ $(\rho > 0)$, and in this disc a unique holomorphic function $w = w(z)$ such that

$$w(a) = b,\ P(z, w(z)) \equiv 0 \text{ in } D.$$

PROOF: Let $\delta > 0$ be so chosen that for $0 < |w - b| \leq \delta$, we have $P(a, w) \neq 0$. We can then find $\rho > 0$ such that for $|z - a| < \rho$ and $|w - b| = \delta$, we have $P(z, w) \neq 0$. Now, b is a simple root of the polynomial $P(a, w)$. Hence, we have

$$\frac{1}{2\pi i} \int_{|w-b|=\delta} \frac{\partial P}{\partial w}(a, w) \frac{dw}{P(a, w)} = 1.$$

Consider, for $|z-a|<\rho$, the integral

$$n(z) = \frac{1}{2\pi i}\int_{|w-b|=\delta} \frac{\partial P}{\partial w}(z,w)\frac{dw}{P(z,w)}.$$

Since $P(z,w)\neq 0$ for $|z-a|<\rho$, $|w-b|=\delta$, it follows that $n(z)$ is a continuous function of z. However, since it is the number of zeros of $P(z,w)$ which satisfy $|w-b|<\delta$, $n(z)$ is an integer for each z. Hence, it is a constant. Thus, *for any* z, $|z-a|<\rho$ *there is a unique* $w=w(z)$, $|w-b|<\delta$, *such that* $P(z,\ w(z))=0$. We assert that $w(z)$ is a holomorphic function of z. This follows at once from the formula

$$w(z) = \frac{1}{2\pi i}\int_{|w-b|=\delta} w\cdot\frac{\partial P}{\partial w}(z,w)\cdot\frac{dw}{P(z,w)}.$$

The uniqueness is easy. Suppose $w_1(z)$ is holomorphic in $|z-a|<\rho$, $w_1(a)=b$, $P(z,\ w_1(z))=0$. Let $0<\rho_1<\rho$ be such that for $|z-a|<\rho_1$, we have $|w_1(z)-b|<\delta$ (δ being as above.) We have shown that for $|z-a|<\rho$, the polynomial $P(z,w)$ in w has only one root in the disc $|w-b|<\delta$. Hence, for $|z-a|<\rho_1$, we have $w(z)=w_1(z)$. By the principle of analytic continuation, $w(z)=w_1(z)$ for all z with $|z-a|<\rho$.

In what follows, we write

$$P(z,w)\equiv p_0(z)w^k+p_1(z)w^{k-1}+\cdots+p_k(z), \tag{4.2}$$

where $p_0,\ldots,p_k$ are polynomials in z, $p_0(z)\not\equiv 0$, which have no common factor (since P is irreducible).

Corollary 4.9 *Suppose that a is such that $\frac{\partial P}{\partial w}(a,w)\neq 0$ for any w for which $P(a,w)=0$, and moreover that $p_0(a)\neq 0$. Then, if ρ is sufficiently small, there exist k holomorphic functions $w_i(z)$, $1\le i\le k$, in $|z-a|<\rho$ with the following properties:*

(i) $w_i(z)\neq w_j(z')$ *for* $i\neq j$, $|z-a|<\rho$, $|z'-a|<\rho$;

(ii) $P(z,\ w_i(z))=0$;

(iii) If $P(z,w_0)=0$, $|z-a|<\rho$, *then* $w_0=w_i(z)$ *for some* i, $1\le i\le k$.

Proof: Let $b_1,\ldots,b_k$ be the roots of the polynomial $P(a,w)$. Because of our assumptions, the b_i are all distinct. Let $w_i(z)$ be a holomorphic function in $|z-a|<\rho$ with $w_i(a)=b_i$, $P(z,\ w_i(z))=0$. Further, since the b_i are distinct, we may choose ρ so small that condition (i) above is fulfilled. Condition (iii) then follows from (i) and (ii) since the polynomial $P(z,w)$ in w of degree $\le k$ can have at most k roots.

Corollary 4.10 *Let a be a complex number, and $R > 0$ be such that for $|z-a| < R$ we have $p_0(z) \ne 0$, the equations $P(z,w) = \frac{\partial P}{\partial w}(z,w) = 0$ have no complex solution w. Let b be such that $P(a,b) = 0$.*

Then there is a unique holomorphic function $w(z)$ in $|z-a| < R$, such that

$$w(a) = b,\ P(z,\ w(z)) \equiv 0.$$

Proof: Let r be the upper bound of the values $\rho \le R$, such that there is a holomorphic function $w_\rho(z)$ in

$$D_\rho = \{z \mid |z-a| < \rho\},$$

with $w_\rho(a) = b$, $P(z,\ w_\rho(z)) = 0$. By the uniqueness assertion of Theorem of 4.8, and the principle of analytic continuation, we see that if $\rho_1 \le \rho_2$, we have $w_{\rho_1}(z) = w_{\rho_2}(z)$ for $z \in D_{\rho_1}$. Hence, there is $w(z)$, holomorphic in D_r with $w(a) = b$, $P(z,\ w(z)) = 0$. We assert that $r = R$. Suppose that $r < R$. For every ξ with $|\xi - a| = r$, there are, by Cor.4.9, k functions $w_1(\xi,z), \dots, w_k(\xi,z)$ in a disc $K_\xi\colon |z-\xi| < \delta_\xi$ with the properties stated in 4.9. Let $z_0 \in D_r \cap K_\xi$. 4.9 , we have $w(z_0) = w_i(\xi, z_0)$ for some i, say $i = 1$. Then, again by the uniqueness assertion of Theorem 4.8 and analytic continuation, we conclude that

$$w(z) = w_1(\xi,z) \ \text{ for } z \in D_r \cap K_\xi. \qquad (*)$$

Let now ξ_1, ξ_2 be two points such that $K_{\xi_1} \cap K_{\xi_2} \ne \emptyset$. This intersection is connected, so that by (*) and analytic continuation, we have

$$w_1(\xi_1, z) = w_1(\xi_2, z) \ \text{ for } z \in K_{\xi_1} \cap K_{\xi_2}.$$

Hence, $w(z)$ has an analytic continuation $W(z)$ to the domain $D_r \cup \bigcup_\xi K_\xi = D$ say. Let r', $r < r' \le R$ be such that $D_{r'} \subset D$. Clearly $W(a) = b$, $P(z,\ W(z)) = 0$ in $D_{r'}$ which contradicts the definition of r. Hence $r = R$ and Cor.4.10 is proved.

Our next theorem is as follows.

Theorem 4.11 *If $P(z,w)$ is an irreducible polynomial in z, w, then there are only finitely many values z for which the equations*

$$P(z,w) = \frac{\partial P}{\partial w}(z,w) = 0,$$

have a solution w.

PROOF: Let $Q(z,w) = \frac{\partial P}{\partial w}(z,w)$. Since P is irreducible, P and Q have no common factors. By the division algorithm, if

$$Q(z,w) = q_0(z)w^{k-1} + \cdots + q_{k-1}(z),$$

we can find polynomials $Q = Q_0, Q_1, \ldots, Q_r$, $r \leq k$, and integers $\nu_0, \ldots, \nu_r$ such that the degree of Q_{i+1} in w is $<$ the degree of Q_i in w and

$$\begin{aligned}
(e_0)\ q_0^{\nu_0}P \quad &= A_0Q + Q_1,\\
(e_1)\ q_1^{\nu_1}Q \quad &= A_1Q_1 + Q_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
(e_{r-1})\ q_{r-1}^{\nu_{r-1}}Q_{r-2} \quad &= A_{r-1}Q_{r-1} + Q_r,
\end{aligned}$$

where $q_s(z)$ is the coefficient of the power of w of highest degree in $Q_s(z,w)$. We may suppose that Q_r is a polynomial in z alone. We assert that $Q_r \not\equiv 0$; in fact if it were, any irreducible factor $B(z,w)$, of degree > 0 in w of Q_{r-1} must, because of (e_{r-1}), divide Q_{r-2}, hence, by (e_{r-2}), Q_{r-3} and so on, so that $B(z,w)$ is a common factor of P and Q, which is not possible. Hence $Q_r \not\equiv 0$.

Suppose a is such that $P(a,w) = Q(a,w) = 0$ for some w. By (e_0), we deduce that $Q_1(a,w) = 0$. This, and (e_1) give $Q_2(a,w) = 0$. We continue this process and conclude that $Q_r(a,w) = 0$. But Q_r is independent of w. Hence $Q_r(a) = 0$. Since $Q_r \not\equiv 0$, there are only finitely many zeros of the polynomial $Q_r(z)$. This proves Theorem 4.11.

We shall call the finite number of zeros of $p_0(z)$, the points a for which the equations $P(a,w) = \frac{\partial P}{\partial w}(a,w) = 0$ have a solution, and the point at infinity *critical points* of the equation $P(z,w) = 0$. All other points are called *regular points.*

Let $a_1, \ldots, a_p$ be the critical points of the equation $P(z,w) = 0$, and let G be the complement of these points on the Riemann sphere.

We can now easily prove the following

Theorem 4.12 *Let $D = \{z \mid |z-a| < \rho\}$ be a disc contained in G and let $w(z)$ be a holomorphic function in D with $P(z,\ w(z)) = 0$. Then $w(z)$ can be continued analytically along any curve starting at a and contained in G.*

Proof: Let γ be a curve starting at a and lying in G. Let $R > 0$ be the minimum distance of a critical point a_i from γ. Let $0 = t_0 < t_1 < \ldots < t_q = 1$ be such that $|\gamma(t) - \gamma(t')| < R$ whenever $t_i \leq t, t' \leq t_{i+1}$ and let $z_i = \gamma(t_i)$. Let D_i be the disc $|z - z_i| < R$. Let $w_0(z)$ be the holomorphic function in D_0 with $w_0(a) = w(a), P(z, w_0(z)) = 0$. This exists, by Corollary 4.10 to Theorem 4.8, and $w_0(z) = w(z)$ in $D \cap D_0$. Let $w_1(z)$ be the holomorphic function in D_i with $w_1(z_1) = w_0(z_1)$ and $P(z, w_1(z)) = 0$, and in general, $w_i(z)$ the holomorphic function in D_i with $w_i(z_i) = w_{i-1}(z_i), P(z, w_i(z)) = 0$. By the uniqueness assertion of Theorem 4.8 and analytic continuation, we see that $w_i(z) = w_{i-1}(z)$ in $D_i \cap D_{i-1}$. This proves Theorem 4.12.

4.2.2 Critical points

We shall now study the behaviour of the function-elements defined by Theorem 4.12 in the neighbourhood of the critical points. We begin with the points a for which

$$p_o(a) \neq 0,\ P(a, w)\frac{\partial P}{\partial w}(a, w) = 0 \text{ for some } w.$$

Let $\rho > 0$ be such that, for $0 < |z - a| < \rho$, we have $p_0(z) \neq 0,\ z$ is not a critical point of the equation (4.1).

Let $D = \{z \mid |z - z_0| < \delta\}$ be a disc which is contained in $K_\rho = \{0 < |z - a| < \rho\}$. Let γ be the positively oriented circle $|z - a| = |z_0 - a|$ (i.e. the circle $z = a + (z_0 - a)e^{2\pi i t},\ 0 \leq t \leq 1$) and γ^h the circle γ described h times (i.e. γ^h is given by the equation $z = a + (z_0 - a)e^{2\pi i h t},\ 0 \leq t \leq 1$). Suppose δ so chosen that there are k functions $w_1(z), \ldots, w_k(z)$ in D having the properties stated in Corollary 4.9 to Theorem 4.8. By Theorem 4.12, $w_1(z)$ can be continued along γ^h for each h. Because of Corollary 4.9 to Theorem 4.8, continuation of $w_1(z)$ along γ^h leads to one of the functions $w_1(z), \ldots, w_k(z)$ say $w_{i_h}(z)$. Let m be the smallest integer > 0 such that for some $h \geq 0$, we have $i_h = i_{h+m}$. We then assert that continuation of w_1 along γ^m leads back to w_1. This is simple : w_1 is obtained by continuation of w_{i_h} along γ^{-h}; hence continuation of w_1 along γ^m is the continuation of w_{i_h} along γ^{m-h}, hence it is also the continuation of $w_{i_{h+m}}$ along γ^{-h}, which is again w_1. Moreover, m is the smallest integer > 0, such that $i_m = 1$. We suppose the indices i so chosen that $i_l = l + 1$ for $0 \leq l \leq m - 1$. Now we assert that $i_h = 1$ if and only if h is divisible by m. In fact let $h = m \cdot q + r,\ 0 \leq r < m$; if $i_h = i_m = 1$, then $i_r = 1$, which , since $0 \leq r < m$, is not possible unless $r = 0$. This integer m is independent of the point z_0 in the following sense. Let $z_1 \in K_\rho$ and σ be any curve joining z_0 to z_1 in K. Let (z_1, w_{z_1}) be the germ at z_1 obtained by continuation of w_1 along σ. Then continuation of w_{z_1} along Γ^h (Γ^h being the (h-tuply described) circle $z_1 + (z - z_1)e^{2\pi iht}$) leads back to w_{z_1} if and only if m divides h. This follows from what we have shown above and the monodromy theorem since γ is homotopic, in K_ρ, to the curve $\sigma^{-1}\Gamma\sigma$.

Consider now a new complex variable s and consider the punctured disc $0 < |s| < \rho^{1/m}$. The function $w_1(a + s^m) = \eta_1(s)$ is defined in a neighbourhood of the point $s_0 = (z_0 - a)^{1/m}$. Moreover $\eta_1(s)$ can be continued analytically along any curve starting at s_0 and lying in $0 < |s| < \rho^{1/m}$, and continuation along the curve $\gamma' \colon s = s_0 e^{2\pi it},\ 0 \leq t \leq 1$ of $\eta_1(s)$ is equivalent to continuation of $w_1(z)$ along γ^m, so that continuation of $\eta_1(s)$ leads to a (single valued) holomorphic function $\eta_1(s)$ defined in the entire punctured disc $0 < |s| < \rho^{1/m}$ (by the monodromy theorem and the example at the end of Chapter II). We note that if a choice of s_0 is made, continuation of $\eta_1(s)$ along the curve $s = s_0 e^{2\pi it},\ 0 \leq t \leq 1/m$ corresponds to continuation of $w_1(z)$ along γ. Hence, we deduce that, *in a neighbourhood of* s_0 *we have*

$$w_{\nu+1}(z) = \eta_1(e^{2\pi i\nu/m} \cdot\ s) \quad \text{for } 0 \leq \nu \leq m-1.$$

Consequently, the values taken by η_1 at the points $s,\ se^{2\pi i/m}, \ldots,$ $se^{2\pi i(m-1)/m)}$ are all distinct for any s with $0 < |s| < \rho^{1/m}$. Hence, the functions $w_1, \ldots, w_m$ are "described" by the equations

$$\begin{cases} z = a + s^m, \\ w = \eta_1(s). \end{cases}$$

We now assert that $\eta_1(s)$ has a continuation which is *holomorphic in the disc* $|s| < \rho^{1/m}$. To prove this, it suffices to prove that $\eta_1(s)$ remains bounded in a disc $0 < |s| < \epsilon$, where ϵ is sufficiently small. It is clearly enough to show that if ϵ is sufficiently small, there is $M > 0$ such that for any root w of the equation $P(z, w) = 0,\ |z - a| < \epsilon$, we have $|w| < M$. Since $p_0(z) \neq 0$ for $|z - a| < \rho$, this is an immediate consequence of the following

Lemma 4.13 *If* $a_1, \ldots, a_k, \zeta$ *are complex numbers such that*

$$\zeta^k + a_1\zeta^{k-1} + \cdots + a_k = 0,$$

then we have

$$|\zeta| \leq 2 \max_i |a_i|^{1/i}.$$

Proof: Let $c = \max |a_i|^{1/i}$; then $|a_i| \le c^i$, and, assuming that $c \neq 0$,

$$\eta^k + \frac{a_1}{c}\eta^{k-1} + \cdots + \frac{a_k}{c_k} = 0$$

(where $\eta = \zeta/c$). Then

$$|\eta|^k \le 1 + |\eta| + \cdots + |\eta|^{k-1}.$$

If $|\eta| \ge 2$, $|\eta|^k \le \frac{|\eta|^k - 1}{|\eta| - 1} \le |\eta|^k - 1$, a contradiction.

Hence $|\eta| \le 2$, i.e. $|\zeta| \le 2c$.

We now consider the points a with $p_0(a) = 0$. Write $y = \frac{1}{w}$. We obtain

$$P(z, w) \equiv \frac{1}{y^k}\{p_0(z) + \cdots + p_k(z) y^k\} = y^{-k}\ A(z, y).$$

Then $A(z, y)$ is also irreducible, so that if $p_k(a) \neq 0$, we see that in a disc $|z - a| < \rho$, the function elements y given by the equation $A(z, y) = 0$, satisfy the relations

$$\begin{cases} z = a + s^m, \\ y = \phi_1(s) = \alpha_0 + \alpha_p s^p + \cdots, \quad \alpha_p \neq 0. \end{cases}$$

Now, the α_0 occurring in these relations satisfy the equation $A(a, y) = 0$. Since $p_0(a) = 0$, at least one of these numbers α_0 is zero. Thus the function elements $w = 1/y$ which satisfy the equation $P(z, w) = 0$ are of two kinds, viz. those which, when continued indefinitely in a neighbourhood $0 < |z - a| < \rho$ lead to representations

$$\begin{cases} z = a + s^m, \\ w = \eta(s), \text{ where } \eta(s) \text{ is meromorphic in } |s| < \delta \text{ with a pole at } s = 0. \end{cases}$$

and those which lead to representations

$$\begin{cases} z = a + s^m, \\ w = \eta(s), \text{ where } \eta(s) \text{ is meromorphic in } |s| < \delta. \end{cases}$$

We remark that by a simple formal change, we may always ensure that the hypothesis $p_k(a) \neq 0$ is fulfilled. In fact if $u = w + \beta$, we have

$$P(z, w) = P_1(z, u) = p_{0,1}(z) u^k + \cdots + p_{k,1}(z),$$

where

$$p_{0,1}(z) \equiv p_0(z),\ p_{k,1}(z) = p_0(z)\beta^k + \cdots + p_k(z).$$

Since P is irreducible, the $p_0, \dots, p_k$ cannot have a common factor, so that not all the $p_i(a)$ are zero. Hence we may choose β such that $p_{k,1}(a) \neq 0$. The above method then leads to similar representation for the function elements defined by $P(z, w) = 0$.

The discussion for $z = \infty$ is similar : we merely set $x = 1/z$ and consider the point $x = 0$. We then obtain the representation

$$\begin{cases} z = s^{-m}, \\ w = \eta(s), \ \text{where } \eta(s) \text{ is meremorphic in } |s| < \delta. \end{cases}$$

4.2.3 The Riemann surface of an algebraic function

We will now construct a compact Riemann surface X associated with the equation

$$P(z, w) = 0, \tag{4.3}$$

such that both z and w may be interpreted as meromorphic functions on X.

Let G be the set of regular points of the equation (4.3). We define a topological space Y in the following way.

Y is the (open) subset of $\mathcal{O}$ consisting of all germs (a, w_a), $a \in G$, defined by a function element (w, U) with $a \in U$, $P(z,\ w(z)) \equiv 0$ in U.

We have a continuous map $\pi\colon Y \to G$ defined by $\pi(a, w_a) = a$. Then, π is a *local homeomorphism,* and so defines on Y the structure of a Riemann surface (which we do not know to be connected; we shall later show that Y is connected). Our aim is to complete Y by adding to it certain points which represent the critical points of (4.3).

Let, then, a be a critical point. [If $a = \infty$, such inequalities as $|z - a| < \delta$ are to be interpreted as $|z| > \frac{1}{\delta}$.] Let $K = \{0 < |z - a| < \delta\}$ be a punctured disc which contains no critical points, and $z_0 \in K$. Let $w_1(z), \dots, w_k(z)$ be holomorphic functions in a neighbourhood of z_0 having the properties stated in Corollary 4.9 to Theorem 4.8. We suppose the indices so chosen that continuation of w_1 along γ^h for some h (γ being the curve $z = a + (z_0 - a)e^{2\pi i t}$) leads to $w_1, \dots, w_{m_1}$ and no others (m_1 is the m of our previous notation), continuation of w_{m_1+1} along γ^h to w_i, $m_1 < i \leq m_2$ and no others and so on. Then, as we have seen, we have representations

$$\begin{cases} z = a + s^{m_i - m_{i-1}}, \ (m_0 = 0), \\ w = \eta_i(s), \ 1 \leq i \leq q, \ m_q = k. \end{cases}$$

(The equation $z = a + s^{m_i - m_{i-1}}$ is to be interpreted as $z = s^{-(m_i - m_{i-1})}$ if $a = \infty$.) Note that these representations correspond one-one with the connected components U_i of $\pi^{-1}(K)$. Further, from our remark that the m_i are independent of the point $z_0 \in K$, it follows that if $K' = \{0 < |z - a| < \epsilon\} \subset K$ and U_i is a connected component of $\pi^{-1}(K)$, then the inverse image $(\pi|U_i)^{-1}(K') \cap U_i$ is also connected.

To each set of indices $i, m_{j-1} < i \leq m_j, 1 \leq j \leq q$, we make correspond a "point" $(a, \mathfrak{p}_j)$. Let C be the set of all these points $(a, \mathfrak{p}_j)$, when a runs over the critical points.

Let X denote the set $Y \cup C$. We define a topology on X by defining a fundamental system of neighbourhoods of each point $(a, \mathfrak{p}_j)$ as follows.

Let $K_\epsilon = \{0 < |z - a| < \epsilon\}$, $\epsilon > 0$, be any punctured disc with $K_\epsilon \subset G$. Let the point $(a, \mathfrak{p}_j)$ correspond to the indices $m_{j-1} < i \leq m_j$, i.e. to the connected component U_j of $\pi^{-1}(K)$ containing the germs of the functions w_i with $m_{j-1} < i \leq m_j$. Let $E_{\epsilon,j} = \pi^{-1}(K_\epsilon) \cap U_j$. The sets $U_j(\epsilon) = E_{\epsilon,j} \cup \{(a, \mathfrak{p}_j)\}$ form, by definition, a fundamental system of neighbourhoods of $(a, \mathfrak{p}_j)$.

We now assert that there is a homeomorphism F of the disc $D_\epsilon = \{|s| < \varepsilon^{1/(m_j - m_{j-1})}\}$ onto $U_j(\epsilon)$ which is an analytic isomorphism of $D_\epsilon - \{0\}$ onto $E_{\varepsilon,j}$ with the complex structure induced from Y. To prove this, we suppose that we have, for $|s| < \varepsilon^{1/(m_i - m_{j-1})}$ representation

$$\begin{cases} z = a + s^{m_j - m_{j-1}}, \\ w = \eta(s), \end{cases}$$

of the functions w_i, $m_{j-1} < i \leq m_j$ defined in a neighbourhood of the point z_0, $0 < |z_0 - a| < \epsilon$ and that w_i goes into w_{i+1} by continuation along the curve $\gamma: z = a + (z_0 - a)e^{2\pi it}$. We also suppose a choice of $s_0 = (z_0 - a)^{1/m}$ made. We then define the mapping F by $F(0) = (a, \mathfrak{p}_j)$ and, for $s \neq 0$, $F(s) = (z, w_z)$ where $z = a + s^{m_j - m_{j-1}}$ and w_z is the germ at z induced by the function obtained by continuation of $w_{m_{j-1}+1}$ along a curve Γ which is the image under the mapping $s \to z = a + s^{m_j - m_{j-1}}$ of a curve Γ' in the s-plane joining s_0 to s.

We remark that $F(s) = (z, w_z)$, where the germ w_z is the one satisfying $P(z, w_z) = 0$ for which $w_z(z) = \eta_j(s)$. Since the values

$$\eta_j(se^{2\pi iv/(m_j - m_{j-1})}), \quad 0 \leq \nu \leq m_j - m_{j-1} - 1$$

are all distinct, we deduce that F is a one-one continuous map of D_ϵ onto $U_j(\epsilon)$. It is easy to show that F has the other required properties.

Thus every connected component of X is endowed with the structure of a Riemann surface.

We have defined the mapping $\pi: Y \to G$ which is a holomorphic map which is a local homeomorphism. We extend π to a mapping

$$\pi: X \to \bar{\mathbb{C}} \text{ (the Riemann sphere)},$$

by putting

$$\pi((a, \mathfrak{p}_j)) = a \text{ for } (a, \mathfrak{p}_j) \in C.$$

Then, this defines a holomorphic map $X \to \bar{\mathbb{C}}$. (This is the function z considered as a meromorphic function on X). w gives rise to a meromorphic function Φ on X. In fact, for $(z, w_z) \in Y$, we set $\Phi((z, w_z)) = w_z(z)$ (which has a unique meaning). For a point $(a, \mathfrak{p}_j) \in C$, let F be an analytic isomorphism of the disc D_ϵ onto $U_j(\epsilon)$ as above. We set $\Phi = \eta_j \circ F^{-1}$ on $U_j(\epsilon)$, $z = a + s^{m_j - m_{j-1}}$, $w = \eta_j(s)$ being the representation of the functions w_i as above. Thus on X, we have two meromorphic functions π, Φ which satisfy the relation

$$P(\pi(p),\ \Phi(p)) = 0 \text{ for } p \in X.$$

Theorem 4.14 *X is a compact, connected topological space.*

Proof: (i) X is compact. Let $U(a, \mathfrak{p}_j)$ be a neighbourhood of $(a, \mathfrak{p}_j)$ as constructed above. It is sufficient to prove that $X - \bigcup U(a, \mathfrak{p}_j)$ is compact. Hence, it is enough to show that if $\{|z-a| < \rho\}$ are sufficiently small neighbourhood of critical point, and K is the complement of these sets on $\bar{\mathbb{C}}$ then $\pi^{-1}(K)$ is compact. For this, let $D_\nu,\ \nu = 1, \ldots, r$ be discs covering K such that in each D_ν, there are k functions $w_{1,\nu}, \ldots, w_{k,\nu}$, satisfying the conditions of Corollary 4.9 to Theorem 4.8. Let $U_{j,\nu}$ be the open set $\{(z, w_z) \mid z \in D_\nu,\ w_z$, being the germ at z induced by $W_{j,\nu}\}$. Clearly, if D_ν is sufficiently small, $U_{j,\nu}$ has compact closure in X. Also $\pi^{-1}(K) \subset \bigcup_{j,\nu} U_{j,\nu}$. Hence $\pi^{-1}(K)$ is compact.

(ii) X is *connected.* We shall prove that Y is connected. It follows then that X is also connected. To prove this, let (ξ, w_ξ), $(\xi',\ w'_{\xi'})$ be two points of Y. We suppose $w_\xi,\ w'_{\xi'}$, are represented by the functions $w(\xi, z),\ w(\xi', z)$. We shall show that *continuation* of $w(\xi, z)$ *along some curve* Γ *starting at* ξ *and lying in* G *leads to* $w(\xi', z)$.

Let Γ_1 be any curve joining $\xi,\ \xi'$. By Theorem 4.12, continuation of $w(\xi, z)$ along Γ_1 is possible, and, by Corollary 4.9 to Theorem 4.8, leads to one of the k functions $w_1(\xi', z), \ldots, w_k(\xi', z)$ at ξ' having the properties stated in Corollary 4.9. Also $w(\xi', z)$ is one of these k functions.

Thus it is enough to prove that *any one of the functions* $w_j(\xi', z)$ *can be obtained from* $w_1(\xi', z)$ *by analytic continuation along some closed curve* Γ_2 *in* G *starting at* ξ'.

Suppose that this false. Let

$$w_1(\xi', z), \ldots, w_l(\xi', z),\ l < k,$$

be the functions which are obtained by continuation of $w_1(\xi', z)$ along a closed curve. Set

$$\begin{aligned}
a_1(z) &= -(w_1(\xi', z) + \cdots + w_l(\xi', z)) \\
a_2(z) &= +(w_1(\xi', z) \cdot w_2(\xi', z) + \cdots + w_{l-1}(\xi', z) \cdot w_l(\xi', z)) \\
&\ldots\ldots \\
a_l(z) &= (-1)^l w_1(\xi', z) \cdots w_l(\xi', z).
\end{aligned}$$

Since, by assumption, continuation along any closed curve starting at ξ' permutes the $w_j(\xi', z)$ the $a_j(z)$ are unaltered by this process ($j \leq l$). Hence continuation of the $a_j(z)$ to any point of G is uniquely determined, and the $a_j(z)$ *are therefore holomorphic in* G. For any critical point $a \neq \infty$ there is a neighbourhood $|z - a| < \rho$ and an integer $N_1 > 0$, and a constant $M_1 > 0$ so that, for any root w of the equation $P(z, w) = 0,\ 0 < |z - a| < \rho$, we have

$$|(z - a)^{N_1} w| \leq M_1.$$

This follows at once from Lemma 4.13. Hence there is an integer $N > 0$ and a constant $M > 0$, such that

$$|(z - a)^N\ a_j(z)| \leq M \qquad \text{for} \qquad 0 < |z - a| < \epsilon,$$

and

$$\left|\frac{a_j(z)}{z^N}\right| \leq M \qquad \text{for} \qquad |z| > \frac{1}{\epsilon}.$$

Hence the $a_j(z)$ have meromorphic continuations to the whole Riemann sphere, and so are rational functions.

Let

$$a_j(z) = \frac{s_j(z)}{s_0(z)}$$

where the $s_j(z),\ s_0(z)$ are polynomials without common factors.

Thus $w_1(\xi', z), \ldots, w_l(\xi', z)$ satisfy the equation

$$H(z, w_j(\xi', z)) = 0,$$

where $H(z,w) = s_0(z)w^l + \cdots + s_l(z)$.

Now $P(z,w)$ and $H(z,w)$ have no common factor since P is irreducible and $1 \le l \le k-1$. Hence, as in Theorem 4.11, there are only finitely many z for which the equations

$$P(z,w) = H(z,w) = 0$$

have a solution w. This however contradicts the fact that $w_1(\xi', z)$ is a root of both $P(z,w)$ and $H(z,w)$ for *all* z *near* ξ'. Hence we must have $l = k$ and this proves our theorem.

This compact, connected Riemann surface X is the surface of the "algebraic function Φ defined by $P(z,w) = 0$". Φ is a meromorphic function on X, there is a holomorphic map $\pi\colon X \to \bar{\mathbb{C}}$, such that for all $p \in X$, we have

$$P(\pi(p),\ \Phi(p)) = 0.$$

It can be proved that *every meromorphic function on* X *is a rational function of* π *and* Φ and that X has the following universal property. If M *is a Riemann surface and* f, g *two meromorphic functions on* M *with* $P(f,g) = 0$, *then there is a holomorphic map* $F\colon M \to X$ *such that* $f = \pi \circ F,\ g = \Phi \circ F$.

Chapter 5

The Riemann-Roch Theorem and the Field of Meromorphic Functions on a Compact Riemann Surface

5.1 Divisors on a Riemann Surface

Let X be a compact connected Riemann surface.

A divisor D on X is a mapping $D: X \to \mathbb{Z}$ from X into the set of integers $\mathbb{Z}$ such that $D(P) = 0$ for all but a finite number of points of X. The divisor is usually represented by the formal sum $D = \sum_P n_P \cdot P$, where $n_P = D(P)$.

If D_1 and D_2 are any two divisors on X, we define the sum of D_1 and D_2 denoted by $D_1 + D_2$, as $(D_1 + D_2)(P) = D_1(P) + D_2(P)$. We see easily that the set of divisors under this operation is an abelian group. The zero element of this group, called the zero divisor (denoted by 0), is the divisor which associates to every P of X, the integer 0. The inverse of the divisor D is denoted by $-D$ and we have $(-D)(P) = -D(P)$.

There is a natural relation of order in the set of divisors on X; we define $D_1 \geq D_2$ if $D_1(P) \geq D_2(P)$ for every P of X. We say that a divisor D is *positive* (resp. negative) if $D \geq 0$ (resp. $-D \geq 0$).

The *degree* of a divisor D on X is defined to be the integer $\deg D =$

$\sum_{P \in X} D(P)$. This is well defined because $D(P)$ is zero for all but a finite number of points of X.

Let f be a meromorphic function (or a meromorphic differential) on X, which is not identically zero. Then we define the divisor $D(f)$ (called the *divisor of* f) as follows:

(i) if f is regular and non-vanishing at P, the value of $D(f)$ at P is zero;

(ii) if f is regular and vanishes at P, the value of $D(f)$ at P is the order of the zero of f at P;

(iii) if f is not regular at P, the value of $D(f)$ at P is minus the order of the pole of f at P.

Since the set of zeros and poles of f is a finite subset of $X, D(f)$ takes the value zero for all but a finite subset of X and therefore defines a divisor on X. The degree of $D(f)$ is zero if f is a meromorphic function $\not\equiv 0$ (by the theorem of 3.2.12). We deduce immediately the following properties.

(i) $D(\lambda) = 0$, where λ is a non-zero, constant function.

(ii) $D(fg) = D(f) + D(g)$, where f is a meromorphic function, and g is either a meromorphic function or differential, it being assumed that neither f nor g is $\equiv 0$.

We say that a divisor D_1 is *linearly equivalent* to D_2 (denoted by $D_1 \sim D_2$) if $D_1 - D_2$ is the divisor of a meromorphic function $\not\equiv 0$. This defines an equivalence relation in the set of divisors on X and an equivalence class is called a *divisor class.* It is to be remarked that the divisors of meromorphic functions $\not\equiv 0$ form a subgroup of the group of divisors on X, and the quotient group can be identified with the set of divisor classes on X, and therefore there is a natural structure of an abelian group on the set of divisor classes on X. If $D_1 \sim D_2$ we have $\deg D_1 = \deg D_2$, and therefore we can speak of the *degree of a divisor class.*

If ω_1 and ω_2 are two meromorphic differentials $\not\equiv 0$ we have $D(\omega_1) \sim D(\omega_2)$, because ω_1/ω_2 is a meromorphic function and $D(\omega_1) = D(\omega_2) + D(\omega_1/\omega_2)$. Therefore the divisors of all meromorphic differentials $\not\equiv 0$ lie in the same divisor class, called the canonical divisor class of X. we shall see a little later that the *canonical divisor class* always exists, or equivalently that there exists a meromorphic differential on X which does not vanish identically.

Let D be a divisor on X. Then we define the set $M(D)$ of meromorphic functions (resp. the set $N(D)$ of meromorphic differentials) as follows:

(i) the function (resp. differential) which is identically zero belongs to $M(D)$ (resp, to $N(D)$);

(ii) if $f \not\equiv 0$, $f \in M(D)$ (resp.$N(D)$), if and only if $D(f) \geq D$.

Now if $f \in M(D)$ (resp.$N(D)$), $\lambda f \in M(D)$ (resp.$N(D)$), λ being a complex number and if f_1, $f_2 \in M(D)$(resp.$N(D)$), $f_1 + f_2 \in M(D)$ (resp.$N(D)$). From this it follows that there is a natural structure of vector space over the complex numbers on $M(D)$(resp.$N(D)$).

Theorem 5.1 *If D_1 and D_2 are two divisors on X such that $D_1 \sim D_2$, there is a linear isomorphism of $M(D_1)$ (resp. $N(D_1)$) onto $M(D_2)$ (resp. $N(D_2)$).*

Let $D_1 - D_2 = D(f)$, the divisor of a meromorphic function $f \not\equiv 0$ on X. Then we see that $\theta \in M(D_2)$ (resp.$N(D_2)$) if and only if $f \cdot \theta \in M(D_1)$ (resp.$N(D_1)$). To every element θ of $M(D_2)$ (resp.$N(D_2)$), we associate the element $f \cdot \theta$ of $M(D_1)$ (resp.$N(D_1)$) and we check easily that this is a linear isomorphism of $M(D_1)$ onto $M(D_2)$ (resp. $N(D_1)$, onto $N(D_2)$), q.e.d.

Theorem 5.2 *Let K be the divisor of a meromorphic differential $\not\equiv 0$ on X. Then if D is a divisor on X, there is a linear isomorphism of $M(D)$ onto $N(K + D)$.*

Let K be the divisor of a meromorphic differential $\omega \not\equiv 0$. Then we see that $f \in M(D)$ if and only if $f.\ \omega \in N(K + D)$. To every element $f \in M(D)$, we associate the element $f\omega$ of $N(K + D)$ and we check easily that this is a linear isomorphism of $M(D)$ onto $N(K + D)$, q.e.d.

Theorem 5.3 *For every divisor D, the dimension of the vector space $M(D)$ (resp.$N(D)$) is finite.*

If the only meromorphic differential on X is the zero differential, the dimension of $N(D) = 0$, and is therefore finite. Otherwise, we can apply Theorem 5.2, and it suffices to prove that $M(D)$ is of finite dimension for every divisor D on X.

If E is a divisor such that $D \geq E$, $M(D)$ is a linear subspace of $M(E)$, and if $M(E)$ is of finite dimension, a *fortiori* $M(D)$ is. Now there exists a negative divisor E such that $D \geq E$. Therefore to prove the theorem, we may assume without loss of generality that D is negative.

We prove the theorem by induction on $\deg(-D)$. If $\deg D = 0$, then $D = 0$ and the dimension of the vector space $M(0)$ is 1 since it reduces to the space of constants. Assume now that the theorem is true for all $D \leq 0$ for which $-\deg D \leq n$ and let us prove it for those D for which $-\deg D = (n+1)$.

Let P_0 be a point such that $D(P_0) \neq 0$. Let D' be the divisor defined by : $D(P) = D'(P)$ if $P \neq P_0$ and $D'(P_0) = D(P_0)+1$. Now D' is again negative, $-\deg D' = -\deg D - 1$ and $D' \geq D$. The vector space $M(D')$ is a subspace of $M(D)$ and by the induction hypothesis, $M(D')$ is of finite dimension. If $M(D) = M(D')$ there is nothing to prove, otherwise there exists a function $g \in M(D)$ such that the order of the pole of g at P_0 is precisely $k = -D(P_0)$. Now if g_1 is any element of $M(D)$, we can find a complex constant λ such that $g_1 - \lambda g \in M(D')$. This is easily proved as follows.

Let (U, z) be a co-ordinate system at P_0; then we can find constants $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$, such that the functions

$$g - \left\{ \frac{\alpha_1}{z - z(P_0)} + \frac{\alpha_2}{(z - z(P_0)))^2} + \cdots + \frac{\alpha_k}{(z - z(P_0))^k} \right\}$$

and

$$g_1 - \left\{ \frac{\beta_1}{z - z(P_0)} + \frac{\beta_2}{(z - z(P_0))^2} + \cdots + \frac{\beta_k}{(z - z(P_0))^k} \right\}$$

are holomorphic in a sufficiently small neighbourhood of P_0. Now the order of the pole of $g_1 - (\beta_k/\alpha_k)g$ at P_0 is $\leq (k-1)$(we note that $\alpha_k \neq 0$). This proves the assertion.

Thus we see that $M(D)$ is the sum of the spaces $M(D')$ and the subspace of dimension 1 generated by g. This proves that the dimension of $M(D)$ is finite.

The set of holomorphic differentials on X forms a vector space over the complex number, and this space is precisely $N(0)$, and by the above theorem, $N(0)$ is of finite dimension.

Definition 5.4 The genus of X is defined to be the non-negative integer equal to the dimension of the vector space (over the complex numbers) of holomorphic differentials on X.

Let X and Y be two isomorphic Riemann surfaces, i.e. there exists a 1-1 mapping f of X onto Y, such that f and f^{-1} are holomorphic; then genus of X = genus of Y, for it is immediately seen that there exists a linear isomorphism of the space of holomorphic differentials of X onto that of Y.

Theorem 5.5 *The genus of the Riemann sphere (i.e. the Example 3.20) is zero.*

If $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere, it can be covered by two coordinate systems (U, z) and (V, ζ) such that $U = \mathbb{C}$, $V = \bar{\mathbb{C}} - \{0\}$, z is the identity mapping of $\mathbb{C}$ onto $\mathbb{C}$ and $\zeta(\infty) = 0$ and $\zeta = 1/z$ in $\mathbb{C} - \{0\}$. Therefore a holomorphic differential on $\bar{\mathbb{C}}$ is defined by a pair of entire functions $f(z)$, $g(\zeta)$ in the z and ζ planes respectively, such that

$$f(z) = g(1/z) \cdot (-1/z^2) \text{ (cf.2, §9, Chap.III)}.$$

If $f(z) = \sum_{n \geq 0} a_n z^n$, and $g(\zeta) = \sum_{n \geq 0} b_n \zeta^n$, we have

$$-\frac{1}{z^2} \left(\sum_{n \geq 0} b_n \frac{1}{z^n} \right) = \left(\sum_{n \geq 0} a_n z^n \right),$$

which shows that $f \equiv 0$.

Therefore every holomorphic differential on X is identically zero which proves the theorem.

Remark 5.6 It can be proved that for every integer $g \geq 0$, there exists a compact Riemann surface with genus g. We have seen that two Riemann surfaces which are isomorphic (in the holomorphic sense) have the same genus. It can be shown that the converse is not true for $g \geq 1$. One can show, however, that any two Riemann surfaces of the same genus g are homeomorphic. In fact each of them is homeomorphic to the Riemann sphere to which g handles are attached.

Problem 5.1 Show that on the Riemann sphere, every divisor of degree 0 is the divisor of a meromorphic function.

2. Show that if D is a divisor on the Riemann sphere, dimension of $M(D) = -\deg D + 1$, if $\deg D \leq 0$.

3. Show that the genus of a complex torus T (3.2.4 for the definition of a complex torus) is 1. (Hint: Let $\pi\colon \mathbb{C} \to T$ be the mapping of the complex plane onto T as in 3.2.4. Show that the non-vanishing differential dz "goes down" to a differential on T so that there exists a non-vanishing holomorphic differential on T; or use the fact that if $\theta(z)$ is an elliptic function. $\theta'(z)$ is again an elliptic function so that the divisor of the differential $d\theta$ on T is of degree 0.)

5.1.1 The Riemann-Roch theorem and applications

Let X be a compact, connected, Riemann surface. If D is a divisor on X, we denote by $m(D)$ (resp. $n(D)$) the dimension of the vector space $M(D)$ (over the complex numbers) of meromorphic functions which are multiples of D (resp. the vector space $N(D)$, of meromorphic differentials which are multiples of D).

We now state the important theorem of Riemann-Roch, whose proofs are, however, too complicated to be given here.

Theorem 5.7 *(Riemann-Roch)*[1] *Let D be any divisor on X. Then the integer*

$$k = m(-D) - n(D) - \deg D,$$

is independent of the divisor D.

Let D be the zero divisor. Then $m(-D) = 1,\ n(D) = g$ (the genus of X) and $\deg D$ (i.e. the degree of D) $= 0$. Therefore $k = 1 - g$. Thus the Riemann-Roch theorem can be stated as follows:

$$m(-D) - n(D) = \deg D - g + 1.$$

Theorem 5.8 *On X there exist non-constant meromorphic functions and meromorphic differentials not identically zero.*

If D is a negative divisor, $m(-D) = 0$, and if further D is chosen so that $-\deg D \geq 2,\ -\deg D + g - 1 \geq 1$. Therefore by the Riemann-Roch theorem, $n(D) > 0$ which proves that there exist meromorphic differentials not identically zero. If $\deg D \geq g + 1,\ m(-D) \geq n(D) + 2$, which shows that there exist non-constant meromorphic functions.

Theorem 5.9 *The degree of the canonical divisor class of X is $2g-2$, g being the genus of X.*

Let ω be a meromorphic differential on X which is $\not\equiv 0$, and K the divisor of ω. Then by the Riemann-Roch theorem, we have

$$m(-K) - n(K) = \deg K - g + 1.$$

By Theorem 5.2, $m(-K) = n(0) = g$ and $m(0) = n(K) = 1$. Therefore, $\deg K = 2g - 2$, *q.e.d.*

[1] The usual proofs of the Riemann-Roch theorem (cf. for example [11]) use the existence of meromorphic functions and differentials with certain properties. For a direct proof see [9].

Theorem 5.10 *Every Riemann surface of genus* 0 *is isomorphic to the Riemann-sphere.*

Let X be a Riemann surface of genus 0. Let P and Q be two distinct points of X, and D the divisor $1 \cdot P - 1 \cdot Q$. Then by the Riemann-Roch theorem $m(-D) \geq \deg D - g + 1 = 1$. Therefore, there exists a meromorphic function f which is a multiple of $-D$. Now f cannot be constant (non-zero) for it has a zero at Q. Therefore, f is a non-constant meromorphic function whose only zero is a simple zero at Q (and only pole is a simple pole at P). Hence f assumes every value on the Riemann sphere only once (cf. Theorem 3.53,§13, Chap.III); therefore f defines a holomorphic 1-1 mapping $f: X \to \bar{\mathbb{C}}$ of X onto the Riemann sphere $\bar{\mathbb{C}}$. Now for every $p \in X$, $\text{ord}_p(f - f(p)) = 1$. Therefore there exist coordinate systems (U, z) at p and (V, w) at $q = f(p)$, such that

(i) $z(p) = w(q) = 0$,

(ii) $f(U) = V$, and

(iii) $w \circ f \circ z^{-1} = z$ on $z(U)$ (cf.[6], §4,Chap. III).

Therefore $w \circ f \circ z^{-1}: z(U) \to w(V)$ is the identity mapping; in particular it is an isomorphism. It follows that f^{-1} maps V onto U, and in fact that it is an analytic mapping of V onto U. This shows that $f^{-1}: \bar{\mathbb{C}} \to X$ is holomorphic. Therefore $f: X \to \bar{\mathbb{C}}$ is an isomorphism, *q.e.d.*

Remark 5.11 It can be proved that every Riemann surface of genus 1 is isomorphic to a complex torus.

Problem 5.2 Show that given two distinct points P and Q on X (resp. one point P), there exists a meromorphic differential whose only poles are simple poles at P and Q (resp. whose only pole is a double pole at P).

5.1.2 Field of algebraic functions on a compact Riemann surface

It is well-known result in the theory of elliptic functions (i.e. meromorphic functions on a complex torus) that every elliptic function can be expressed as a rational function of $\wp$ and $\wp'$ with complex coefficients, $\wp$ being the Weierstrassian elliptic function. One has further $\wp'^2 = 4\wp^3 - g_2\wp - g_3$. We shall now generalize this result to the case of an arbitrary compact Riemann surface.

Let k be a subfield of a field K. Then the field generated by k and elements $z_1, \ldots, z_n$ of K is denoted by $k(z_1, \ldots, z_n)$ and it is precisely the set of elements of K which can be expressed as rational functions of $z_1, \ldots, z_n$ with coefficients in k.

Definition 5.12 Let K be a field and let $\mathbb{C}$, the field of complex numbers, be a subfield of k. Then K is said to be an algebraic function field of one variable over $\mathbb{C}$ if (1) there exists an element z of K not belonging to $\mathbb{C}$ and (2) there exists an element w of k which is algebraic over $\mathbb{C}(z)$(i.e. there exist $\lambda_1, \ldots, \lambda_p \in \mathbb{C}(z)$, $\lambda_1 \neq 0$ such that $\sum_{i=1}^{p} \lambda_i w^i = 0$) such that every element of the field K is a rational function of z and w with coefficients in $\mathbb{C}$.

Remark 5.13 The condition (2) is equivalent to stating that K is a finite extension of $\mathbb{C}(z)$.

The meromorphic functions on a (connected) Riemann surface form a field. This field contains $\mathbb{C}$. We have now the following theorem.

Theorem 5.14 *Let K be the field of meromorphic functions on a compact (connected) Riemann surface X. Then K is an algebraic function field of one variable over $\mathbb{C}$.*

Let z be a non-constant meromorphic function on X and n the number of zeros (or poles) of z. Now z defines a holomorphic mapping $z\colon X \to \bar{\mathbb{C}}$ of X onto the Riemann sphere, such that every value of $\bar{\mathbb{C}}$ is assumed n times by z (a point $p \in X$ is counted k times, k being $\mathrm{ord}_p(z - z(p))$). Given $P \in \bar{\mathbb{C}}$, let $P_1, \ldots, P_n$ be the points of X over P(i.e. $z^{-1}(P)$). Let S_1 be the set of poles of f and $S = z(S_1)$. Now consider the elementary symmetric functions

$$r_1(P) = f(P_1) + \cdots + f(P_n),$$

$$\cdots\cdots\cdots\cdots$$

$$r_n(P) = f(P_1)f(P_2)\cdots f(P_n),$$

for $P \in \bar{\mathbb{C}} - S$. We shall now prove that the r_i are meromorphic on $\bar{\mathbb{C}}$.

Let E be the (finite) set of points $Q \in \bar{\mathbb{C}}$ such that there is a point $P \in z^{-1}(Q)$ for which $\mathrm{ord}_P(z - z(P)) > 1$ if P is not a pole of z and $\mathrm{ord}_P(1/z) > 1$ if P is a pole of z, and let $A = z^{-1}(E)$.

Let $Q \in \bar{\mathbb{C}}$ and $Q_1, \dots, Q_l$ be the distinct points of X in the set $z^{-1}(Q)$. Then, given any neighbourhood V of $z^{-1}(Q)$, there is a neighbourhood U of Q so that $z^{-1}(U) \subset V,\ z^{-1}(U) = \bigcup_{i=1}^{l} U_i$ where U_i is a neighbourhood of Q_i.

PROOF: We have only to find U with $z^{-1}(U) \subset V$. To do this, we remark that $X - V$ is a compact set so that $K = z(X - V)$ is a compact (hence closed) set of $\bar{\mathbb{C}}$ not containing Q and we may take $U = \bar{\mathbb{C}} - K$.

Since z is a local homeomorphism on $X - A$, if $Q \in E$, we may in addition suppose that $z_i = z|U_i$ is an isomorphism onto U. If $Q \notin E$, we have, on U, $r_1(Q) = \sum_{i=1}^{n} f(z_i^{-1}(Q))$ so that r_1 (and similarly $r_2, \dots, r_n$) are meromorphic in U, hence in $\bar{\mathbb{C}} - E$. Let now $Q \in E$. To prove that r_p is meromorphic at Q, let ϕ be any holomorphic function in U which vanishes at Q. It suffices to show that there is an integer N so that $\phi^N r_p$ is bounded in a neighbourhood of Q. Let $\theta = \phi \circ z$. Then θ vanishes on $z^{-1}(Q)$ and we may therefore choose V and an integer M so that $\theta^M f$ is bounded in V (f being meromorphic on X). The existence of the integer N clearly follows from this.

Now every meromorphic function h on $\bar{\mathbb{C}}$ can be naturally identified with a meromorphic function on X; in fact we identify h with $(h \circ z)$. We therefore identify the meromorphic function $I : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ which maps every point of $\bar{\mathbb{C}}$ onto itself, with z. Now the r_i are rational functions in I and can therefore be naturally identified with rational functions in z (with complex coefficients).

Now consider the meromorphic function

$$A(f) = f^n - r_1 f^{n-1} + \cdots + (-1)^n r_n,$$

on X. Then there exists a non-empty open set U of $\bar{\mathbb{C}}$ (in fact $\bar{\mathbb{C}}$ minus a finite number of points) such that f and $A(f)$ are holomorphic in $V = z^{-1}(U)$. Now if $Q \in V,\ P = z(Q)$ and $P_1, \dots, P_n$ are the points of V over P, we have $Q = P_i$ for some i and

$$A(f)(Q) = \prod_{i=1}^{n} (f(Q) - f(P_i)) = 0.$$

Therefore $A(f)$ is the zero function in V, therefore in X. Thus f is algebraic over $\mathbb{C}(z)$, and is the zero of a polynomial of degree n with coefficients in $\mathbb{C}(z)$.

Now every element $f \in K$ is the zero of an irreducible polynomial over $\mathbb{C}(z)$ and its degree $\leq n$, for this polynomial divides $A(f)$. Choose now an element f_0 of K such that the degree of the irreducible polynomial over $\mathbb{C}(z)$, satisfied by f_0, is of maximum degree, say equal to m. Let f be any element of K. Then if $k = \mathbb{C}(z)$, we have

$$(k(f, f_0)\colon k) = (k(f, f_0)\colon k(f_0)) \cdot (k(f_0)\colon k).$$

(Here $(k(f_0)\colon k)$, for example, denotes the degree of $k(f_0)$ over k.) Since $k(f, f_0)$ is a finite extension over k (which has characteristic zero), $k(f, f_0) = k(g)$ for some $g \in K$. Therefore $(k(f, f_0)\colon k) \leq m$. But $(k(f_0)\colon k) = m$. This implies that $k(f, f_0) = k(f_0)$, i.e. that $f \in k(f_0)$. This proves the theorem.

Remark 5.15 As a consequence of 5.14, it can be proved that any compact Riemann surface X is isomorphic to the Riemann surface of an algebraic function as defined in Chap.IV.

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