CONTRIBUTIONS TO FUNCTION THEORY

Papers communicated to the
International Colloquium on Function Theory
held at the
Tata Institute of Fundamental Research, Bombay
on 12-19 January 1960

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BOMBAY

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INTERNATIONAL COLLOQUIUM ON FUNCTION THEORY

BOMBAY, 12-19 JANUARY 1960

REPORT

An International Colloquium on Function Theory was held at the Tata Institute of Fundamental Research, Bombay, on 12-19 January, 1960. The Colloquium was a closed meeting of experts and serious-minded workers in function theory. It was attended by twentyone members, and twenty other participants, from seven countries : France, West Germany, India, Japan, Switzerland, the United Kingdom, and the United States.

The Colloquium was jointly sponsored and financially supported by the Tata Institute of Fundamental Research and the Sir Dorabji Tata Trust.

An Organizing Committee consisting of Professor K. Chandrasekharan (Chairman), Professor K. G. Ramanathan, and Professor Carl L. Siegel was in charge of the scientific programme. The object of the Colloquium was to lay stress on the geometrical, topological, and number-theoretic aspects of function theory. Attention was focussed on problems and trends in (i) the theory of functions of several complex variables, (ii) automorphic functions, and (iii) geometrical methods in the theory of functions of one complex variable. It was the policy of the Organizing Committee to invite not only acknowledged experts in these subjects, but also wherever possible, a few other distinguished mathematicians interested in related subjects. The Committee was specially keen on securing the participation of younger mathematicians from inside India as well as from abroad. It was decided that English should be the language of the Colloquium.

The following seventeen mathematicians accepted invitations to speak in the Colloquium: Professor Walter L. Baily, Jr. (Chicago), Professor Lipman Bers (New York), Professor Arne Beurling

(Princeton), Professor S. Bochner (Princeton), Professor H. Cartan (Paris), Professor Hans Grauert (Göttingen), Dr. Karl B. Gundlach (Göttingen), Professor Heinz Huber (Basel), Professor James A. Jenkins (St. Louis), Dr. R. Kawai (Kyoto), Mr. Raghavan Narasimhan (Bombay), Professor Louis Nirenberg (New York), Professor Hans Petersson (Münster), Dr. Chr. Pommerenke (Göttingen), Mr. S. Raghavan (Bombay), Professor H. E. Rauch (New York), Professor A. Selberg (Princeton).

Professors Bochner and Selberg could not attend the Colloquium. Professor Selberg's paper was, however, read by Mr. Raghavan Narasimhan. Professor Maurice Heins, who was unable to accept the invitation to attend the colloquium, sent in a paper. Professor Beurling gave two lectures in the Colloquium, but his papers have not been received for publication.

The following mathematicians attended the Colloquium by invitation : Professor Y. Akizuki (Kyoto), Professor M. Kuranishi (Nagoya), and Professor M. H. A. Newman (Manchester). With the approval of the Organizing Committee, several research workers from the Institute participated in the Colloquium.

The Colloquium met in closed sessions. Professor Chandrasekharan and Professor Siegel served as Chairmen. Nineteen lectures were given. Each lecture lasted fifty minutes, and was followed by a discussion. Informal lectures and discussions continued for a period of two weeks, since members of the Colloquium were invited to attend the Second South Asian Conference on Mathematical Education held at the Institute during the following week.

The Council of the Tata Institute of Fundamental Research gave a Tea Party in honour of the members of the Colloquium on Monday, 11 January 1960, which was attended by about 200 invitees. Mr. J. R. D. Tata, Chairman of the Council, extended a cordial welcome to the guests, and wished the Colloquium success. Dr. H. J. Bhabha, Director of the Institute, then addressed the gathering. He described the development of the Institute since its foundation

in 1945, and referred to "the strong, unstinted, and continuously increasing support of the Government of India". He said: "The purpose of the Institute is to enable Indian scientists to play their part in pushing forward the frontiers of human knowledge in the field of nuclear science—especially physics—and mathematics. In this respect this National Centre for Advanced Study and Fundamental Research in Nuclear Science and Mathematics differs from the National Laboratories, which even though they may incidentally do some fundamental research, must finally be judged by the contribution they make to the application of modern science and technology towards solving practical problems. In technology it may be useful and significant to establish and perfect processes in one's own country, which have already been developed elsewhere. In fundamental research it is all or nothing. The research has to make a contribution, however small, to an increase in the sum total of human knowledge or it is no research at all. The Institute's work must therefore be judged by international standards.... This judgement can only be made by our colleagues and leaders in the field all over the world, for in the words of an old Indian saying, 'only the learned can recognise the learned'. We shall be more than content if our distinguished colleagues feel that we have made some little contribution to the advancement of knowledge". He said that this was the third international colloquium organized by the Institute. The first was in 1950 on elementary particles, and the second in 1956 on zeta-functions. The purpose of this Colloquium was to create an international forum for the survey of some of the latest developments in function theory, and to bring research workers in the Institute into contact with some of the foremost mathematicians in that field. Speaking about the usefulness of international contacts, he said: "Our School of Mathematics has enjoyed and benefited from collaboration with many foreign mathematicians...The Institute hopes that these contacts will grow in the future and bring forth good results not only for us, but for mathematics as a whole." He then read out a special message received from the Prime Minister Jawaharlal Nehru. Professor Lipman Bers acknowledged the

welcome on behalf of the members of the Colloquium, and expressed his thanks to the hosts.

The social programme during the Colloquium week included a special performance of classical Indian dances, Bharata Natyam and Kuchipudi, on 12 January; a dinner by Dr. H. J. Bhabha on 12 January ; a Kathak recital on 13 January ; a special show of cultural films on 14 January; an excursion to Elephanta on 16 January ; and a dinner at Juhu on 19 January.

MEMBERS

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24. Professor C. L. Siegel
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West Germany

* Did not attend the Colloquium

OTHER PARTICIPANTS

1. Miss V. N. Bhat
2. Dr. J. R. Choksi
3. Mr. N. S. Gopalakrishnan
4. Mr. K. N. Gowrisankaran
5. Mrs. P. Mohandas
6. Mr. M. Pavman Murthy
7. Mr. M. K. Venkatesha Murthy
8. Mr. V. C. Nanda
9. Mr. Ramabhadran Narasimhan
10. Mr. N. Ramabhadran
11. Mr. S. Ramanan
12. Mr. K. Ramachandra
13. Mr. C. P. Ramanujam
14. Dr. S. S. Rangachari
15. Mr. R. R. Simha
16. Mr. A. Srinivasa Rao
17. Mr. R. C. Shukla
18. Mr. Sunder Lal
19. Mrs. V. Venkatesh Rau
20. Miss K. B. Vedak

PROGRAMME

Tuesday, January 12, 1960

10.00 A.M. — 10.50 A.M. H. Cartan : Quotients of complex analytic spaces

4.15 P.M. — 5.05 P.M. J. A. Jenkins : The general coefficient theorem and its applications

Wednesday, January 13, 1960

10.00 A.M. — 10.50 A.M. L. Bers : Uniformization and moduli

11.30 A.M. — 12.20 P.M. A. Selberg : On discontinuous groups in higher dimensional symmetric spaces

4.15 P.M. — 5.05 P.M. W. L. Baily Jr. : On the moduli of Jacobian varieties and curves

Thursday, January 14, 1960

10.00 A.M. — 10.50 A.M. A. Beurling : On analytic continuation in several variables

11.30 A.M. — 12.20 P.M. L. Nirenberg : On rigidity of holomorphic imbeddings

4.15 P.M. — 5.05 P.M. R. Kawai : On the construction of a holomorphic function in the neighbourhood of a critical point of a ramified domain

Friday, January 15, 1960

10.00 A.M. — 10.50 A.M. H. Huber : Riemann surfaces of hyperbolic type in the Euclidean 3-space

Friday, January 15, 1960

11.30 A.M. — 12.20 P.M. K.-B. Gundlach : Some new results in the theory of Hilbert's modular group

4.15 P.M. — 5.05 P.M. Raghavan Narasimhan : Imbedding of open Riemann surfaces

Saturday, January 16, 1960

10.00 A.M. — 10.50 A.M. A Beurling : A property of quasi-conformal mapping

11.30 A.M. — 12.20 P.M. H. Grauert : On the number of moduli of complex structures

Sunday, January 17, 1960

10.00 A.M. — 10.50 A.M. L. Bers : Uniformization and Moduli II

Monday, January 18, 1960

10.00 A.M. — 10.50 A.M. H. Grauert : On point modifications

11.30 A.M. — 12.20 P.M. S. Raghavan : Modular forms of degree n and representation by quadratic forms

4.15 P.M. — 5.05 P.M. Chr. Pommerenke : On the capacity of plane sets

Tuesday, January 19, 1960

10.00 A.M. — 10.50 A.M. H. Petersson : The properties of the representation of the abelian differentials by Poincaré's series

11.30 A.M. — 12.20 P.M. H. E. Rauch : Variational methods in the problem of the moduli of Riemann surfaces

MESSAGE

I am glad to learn that the Tata Institute of Fundamental Research has organised an International Colloquium on Function Theory as well as a Conference on Mathematical Education in South Asia. This Colloquium is of high importance as the most eminent mathematicians from other countries are attending it. I should like to extend a warm welcome to them not only on my own behalf, but also on behalf of the Government of India.

The study of mathematics has always been of great importance. Today its importance has increased even more and it has become the basis of the great advances that science has made in recent years. I am sure that the International Colloquium that is going to be held in Bombay in the presence of so many eminent mathematicians will advance the cause of both mathematical research and science. I wish it success.

Jawaharlal Nehru

Anand Bhawan,
Allahabad,
December 24, 1959

QUOTIENTS OF COMPLEX ANALYTIC SPACES

By HENRI CARTAN

1. Complex analytic spaces, ringed spaces. First we have to recall the notion of a complex analytic space. It is a special case of a *ringed space*, a notion of basic importance in modern Algebraic Geometry after the work of J. P. Serre.

A ringed space is defined by giving a topological space X and a *sheaf of rings* $\mathcal{O}$ on this space X. We recall that a "sheaf"† means the following: we have for each point $x \in X$ a commutative ring $\mathcal{O}_x$ with a unit, and a topology is given on the union $\mathcal{O}$ of all rings $\mathcal{O}_x$, in such a way that: (1) the map which assigns to each $a \in \mathcal{O}$ the unique $x \in X$ such that $a \in \mathcal{O}_x$, is locally a homeomorphism; (2) if we denote by $\Gamma(U, \mathcal{O})$ the set of continuous "sections" of $\mathcal{O}$ over any open set $U \subset X$, then $\Gamma(U, \mathcal{O})$ is a ring with unit (this means that the unit section is continuous, and the addition and the multiplication are continuous operations). Then, for each $x \in U$, we have a natural homomorphism $\Gamma(U, \mathcal{O}) \to \mathcal{O}_x$, which is a unitary homomorphism of rings with unit element. The knowledge of the rings $\Gamma(U, \mathcal{O})$ for all open sets U and of the homomorphisms $\Gamma(U, \mathcal{O}) \to \Gamma(V, \mathcal{O})$ (for $V \subset U$) determines completely the sheaf $\mathcal{O}$.

We will restrict ourselves to the case where $\mathcal{O}$ is a subsheaf of the sheaf of all germs of continuous complex-valued functions. Hence $\mathcal{O}_x$ is a subring of the ring of all germs of continuous functions on X at the point x, and $\Gamma(U, \mathcal{O})$ is a subring of the ring of all continuous functions on U. In such a case, the ringed spaces are the objects of a category $\mathcal{R}$, the "morphisms" of $\mathcal{R}$ being defined in the following way: given two ringed spaces $(X, \mathcal{O})$ and $(X', \mathcal{O}')$, a *morphism* is a continuous map $f: X \to X'$ having the following property: for each $x \in X$, and each $\phi \in \mathcal{O}'_{f(x)}$, the composed germ of function $\phi \circ f$ belongs to $\mathcal{O}_x$. Thus a morphism $f: X \to X'$ induces a ring homomorphism $\mathcal{O}'_{f(x)} \to \mathcal{O}_x$ for each point $x \in X$. Clearly the composition of two morphisms $f: X \to X'$ and $f': X' \to X''$ is a

†Concerning the theory of sheaves, see [4].

morphism $X \to X''$; and the identity map $X \to X$ is a morphism. We see that $\mathscr{R}$ is a category, as announced; in this category, an *isomorphism* $(X, \mathcal{O}) \to (X', \mathcal{O}')$ is a homeomorphism $f : X \to X'$ which carries the ring $\mathcal{O}'_{f(x)}$ isomorphically onto the ring $\mathcal{O}_x$, for each $x \in X$.

The most classical examples of ringed spaces are : (1) differentiable manifolds (X being then a topological manifold satisfying the axiom of Hausdorff, and $\mathcal{O}_x$ being the ring of germs of differentiable functions at x ; then $\Gamma(U, \mathcal{O})$ is nothing but the ring of differentiable functions in the open set U) ; (2) complex manifolds (X being then a topological even-dimensional manifold, and $\mathcal{O}_x$ being the ring of germs of holomorphic functions at x ; $\Gamma(U, \mathcal{O})$ is the ring of holomorphic functions in U). In the first case, the morphisms are the differentiable mappings ; in the second case, they are the holomorphic mappings.

Given a ringed space $(X, \mathcal{O})$, a subset M of X is *distinguished* if each point $x \in M$ has an open neighbourhood U such that $M \cap U$ is exactly the set of solutions (in U) of a finite number of equations $f_i = 0$, with $f_i \in \Gamma(U, \mathcal{O})$. For instance, if $(X, \mathcal{O})$ is a complex manifold the distinguished subsets are nothing but the "analytic subsets" of X.

In a ringed space $(X, \mathcal{O})$, any distinguished subset M carries naturally a structure of a ringed space $(M, \mathscr{A})$, namely : for $x \in M$ the ring $\mathscr{A}_x$ will consist of those germs of functions on M which are induced by germs belonging to $\mathcal{O}_x$. With this definition, the injection $M \to X$ is a morphism of ringed spaces.

We are now in a position to define the category of complex analytic spaces. For each positive integer n, consider the space $\mathbf{C}^n$ ($\mathbf{C}$ being the field of complex numbers), with its standard ringed structure : for each $x \in \mathbf{C}^n$, take for $\mathcal{O}_x$ the ring of germs of functions holomorphic at x. Now the analytic subsets of $\mathbf{C}^n$ (for all n) build a category $\mathscr{A}'$ of ringed spaces.

DEFINITION†. *A complex analytic space* is a ringed space $(X, \mathscr{H})$, such that : (i) X satisfies the axiom of Hausdorff; (ii) X may be

† Many definitions have been proposed for analytic spaces. We adopt here the definition of Serre [12].

covered by open subsets, each of which is isomorphic (as a ringed space) to some object of the category $\mathscr{A}'$. In other words, an analytic space is a Hausdorff ringed space which can be *locally* realized (as a ringed space) by some analytic subset of some space $\mathbf{C}^n$. In the category $\mathscr{A}$ of analytic spaces, the morphisms are simply the morphisms of ringed spaces; they are called *analytic* mappings, or *holomorphic* mappings. The holomorphic (scalar) functions on X are simply the sections of the sheaf $\mathscr{H}$.

Any distinguished subset of an analytic space carries a structure of ringed space, and is actually an analytic space.

2. Equivalence relations in ringed spaces. Let X be a topological space, and R an equivalence relation on X. As is well known, we have a topology on the set X/R of equivalence classes, having the following universal property: a mapping $f: X/R \to X'$ (X' being any topological space) is continuous if and only if the composed map $X \xrightarrow{p} X/R \xrightarrow{f} X'$ is continuous. With this topology, X/R is called the quotient space of X by R. Now, assume that we have a ringed space $(X, \mathcal{O})$; on the quotient space X/R we define a sheaf of rings $\mathcal{O}_1$ in the following way: for any open subset U of X/R, $\Gamma(U, \mathcal{O}_1)$ is the ring of (continuous) functions $\phi: U \to \mathbf{C}$ such that $\phi \circ p$ belongs to $\Gamma(p^{-1}(U), \mathcal{O})$. We shall denote this sheaf $\mathcal{O}_1$ by $\mathcal{O}/R$. It is easily seen that the ringed space $(X/R, \mathcal{O}/R)$ has the following universal property: the mapping $p: X \to X/R$ is a morphism of ringed spaces, and for any ringed space $(X', \mathcal{O}')$ a map $X/R \to X'$ is a morphism if and only if the composition $X \xrightarrow{p} X/R \to X'$ is a morphism.

Thus any quotient of a ringed space carries naturally a structure of ringed space. And the question arises: assuming that we have an *analytic* space $(X, \mathscr{H})$, what conditions must be imposed on an equivalence relation R on X, in order that the space $(X/R, \mathscr{H}/R)$ be an *analytic* space?

The first known example of a quotient of an analytic space is the following one: consider a group G of analytic automorphisms of

an analytic space X, and assume that G is *properly discontinuous*; this means that for any compact subset $K \subset X$ there are only a finite number of $s \in G$ such that $(sK) \cap K \neq \emptyset$. Now let X/G be the quotient of X by the equivalence relation defined by G; X/G is Hausdorff, and it is known [1, 3] that X/G, with its ringed structure, is actually an analytic space. In fact, the problem of proving this has a local character: let x be a point of X; the isotropy subgroup $G(x)$ (consisting of those $s \in G$ satisfying $sx = x$) is finite, and x has a neighbourhood U, stable under $G(x)$, such that the equivalence relation induced on U is just defined by the operations of the finite subgroup $G(x)$; thus we are led to study the quotient of an analytic space U by a finite group of analytic automorphisms. We shall not recall here the proof of the fact that X/G is an analytic space; we just want to observe that, for $x \in X$, the ring attached to the point $p(x) \in X/G$ may be identified with the subring of $\mathscr{O}_x$ consisting of those elements which are invariant under $G(x)$. To give a very simple example, consider $X = \mathbf{C}^2$, G consisting of the identity and the symmetry $(u, v) \to (-u, -v)$; then the ringed space $\mathbf{C}^2/G$ may be realized (in the large) as the analytic subspace $Y \subset \mathbf{C}^3$ consisting of points (x, y, z) which satisfy the relation $x^2 + y^2 = z^2$; the correspondence is defined by

$$x = u^2 - v^2,\ y = 2uv,\ z = u^2 + v^2.$$

Observe that any function holomorphic in u and v at the origin, and invariant under G, can be expressed as a holomorphic function of x, y, z at the origin.

3. Proper equivalence relations. The equivalence relation defined on a locally compact space X by a finite group of automorphisms is a particular case of a *proper* equivalence relation. Let R be an equivalence relation on a *locally compact* space X; let us call *fibres* the equivalence classes of R. Consider the following properties:

(i) for any compact $K \subset X$, the R-saturated subset of K (i.e. the union of all fibres meeting K) is also compact;

(ii) every fibre is compact and has a fundamental system of open neighbourhoods which are saturated;

(iii) the quotient space X/R is locally compact (in particular, is Hausdorff), and the natural projection $p: X \to X/R$ is a *proper* map (i.e. the inverse image of any compact of X/R is compact).

It is easy to see that the three preceding properties are equivalent. If they are satisfied, we shall say that R is a *proper* equivalence relation.

Let X and Y be locally compact spaces, and let f be a continuous proper map $X \to Y$; let R denote the equivalence relation defined by f (namely: $x \sim x'$ if and only if $f(x) = f(x')$); clearly R is proper. It is a classical and easy result, that the image $f(X)$ is closed in Y, and the map $X/R \to f(X)$ induced by f is a homeomorphism. Now consider two complex analytic spaces X and Y, and a proper holomorphic mapping $f: X \to Y$; in this situation we have two important results:

THEOREM 1. *If f is a proper, holomorphic mapping of an analytic space X into an analytic space Y, then the image $f(X)$ is a (closed) analytic subspace of Y.* (This theorem is due to Remmert [11]; its proof requires a careful and difficult analysis.)

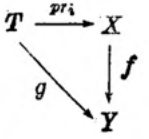
THEOREM 2. *If f is a proper, holomorphic mapping of an analytic space X into an analytic space Y, and if R denotes the equivalence relation defined by f, then the quotient space X/R is an analytic space.*

REMARK. The map $X/R \to f(X)$ induced by f is a holomorphic homeomorphism, but is not, in general, an isomorphism of analytic spaces. We have a counter-example by taking $X = Y$ and f the identity map, the structural sheaf of X being richer than the structural sheaf of Y. Nevertheless, if the analytic space $f(X)$ is *normal* (i.e. the ring attached to each point is an integrally closed domain of integrity), then $X/R \to f(X)$ is an isomorphism of analytic spaces.

We shall now sketch the proof of Theorem 2. By Theorem 1, it suffices to consider the case where f is a proper, holomorphic and *surjective* map $X \to Y$; thus f induces a homeomorphism $X/R \to Y$, and we may identify X/R with Y as topological spaces. Then the

sheaf $\mathscr{H}(X)/R$ may be considered as a sheaf on Y: a section of $\mathscr{H}(X)/R$ on an open set $U \subset Y$ is a function ϕ such that $\phi \circ f$ be holomorphic on $f^{-1}(Y)$. Clearly $\mathscr{H}(Y)$ is a subsheaf of $\mathscr{H}(X)/R$. We shall prove that $\mathscr{H}(X)/R$ is a *coherent*† $\mathscr{H}(Y)$-sheaf. Assuming this for a moment, take a point $y_0 \in Y$; there is a finite number of sections ϕ_i of $\mathscr{H}(X)/R$, in an open neighbourhood U of y_0, which generate the $\mathscr{H}(Y)_y$-module $(\mathscr{H}(X)/R)_y$ in each point y sufficiently near to y_0 (this is true because $\mathscr{H}(X)/R$ is coherent). Consider the functions $f_i = \phi_i \circ f$ holomorphic on $f^{-1}(U) = V$; together with f they define a holomorphic mapping g of V into a product $\mathbf{C}^n \times Y$, and g is proper; by Theorem 1, the image $g(V)$ is an analytic subspace Z of $\mathbf{C}^n \times Y$, and we have a factorization $V \xrightarrow{g} Z \xrightarrow{h} U$ of f, where h is a holomorphic homeomorphism. It is easily seen that the sheaf of germs of holomorphic functions on Z, considered as a sheaf on U by means of h, is nothing but the restriction to U of $\mathscr{H}(X)/R$. Thus the open set U of the ringed space X/R has been realized by an analytic space Z, which proves Theorem 2.

It remains to prove that $\mathscr{H}(X)/R$ is a coherent $\mathscr{H}(Y)$-sheaf. The following proof is due to Grothendieck (unpublished): let T be the subspace of $X \times X$, inverse image of the diagonal of $Y \times Y$ by the map $f \times f : X \times X \to Y \times Y$. Consider the holomorphic map $pr_i : T \to X$ induced by the projection $X \times X \to X$ on the ith factor $(i = 1, 2)$. Consider both commutative diagrams



where g is induced by $f \times f$, using the diagonal map of Y. Let $\mathscr{A}$ be the sheaf on Y, direct image‡ of $\mathscr{H}(X)$ by f; and let $\mathscr{B}$ the

†About coherent sheaves, see [13].

‡Given a continuous map $f : X \to Y$ of topological spaces, and a sheaf $\mathscr{A}$ on X, the direct image $f(\mathscr{A})$ is the sheaf $\mathscr{C}$ on Y, such that for any open set $U \subset Y$ the set of sections $\Gamma(U, \mathscr{C})$ be simply $\Gamma(f^{-1}(U), \mathscr{A})$.

sheaf on Y, direct image of $\mathscr{H}(T)$ by g. Then the preceding diagrams define two homomorphisms $u_i : \mathscr{A} \to \mathscr{B}$ of $\mathscr{H}(Y)$-sheaves. It is easy to check that $\mathscr{H}(X)/R$ is the subsheaf of $\mathscr{A}$, kernel of the homomorphism $u_1 - u_2$. Now $\mathscr{A}$ and $\mathscr{B}$ are coherent, by a recent theorem of Grauert [6] (asserting that the direct image of a coherent sheaf by a *proper* holomorphic map is a coherent sheaf). It follows that the kernel of the $\mathscr{H}(Y)$-homomorphism $u_1 - u_2$ is coherent. Q.E.D.

We are now in a position to state and prove our main theorem concerning proper equivalence relations in analytic spaces :

MAIN THEOREM. *Consider a proper equivalence relation R on an analytic space X. In order that the ringed space X/R be an analytic space, each of the following conditions is necessary and sufficient :*

(a) *each point of X/R has an open neighbourhood V such that the sections of $\mathscr{H}(X)/R$ over V separate the points of V* (i.e. given any two distinct points y_1, y_2 in V, there is a section ϕ of $\mathscr{H}(X)/R$ over V, such that $\phi(y_1) \neq \phi(y_2)$);

(b) *each point of X/R has an open neighbourhood V such that the ringed maps $V \to Z$ (Z* being any analytic space) *separate the points of V.*

PROOF. Clearly (a) and (b) are necessary. Conversely, assume that (a) *or* (b) is satisfied; call U the open set $p^{-1}(V)$ of X; the assumption (a) (resp. (b)) states that those holomorphic functions on U (resp. those holomorphic mappings of U into the complex analytic spaces) which are constant on the classes of R, actually separate the classes of R. But we have the following useful

LEMMA. *Let X be an analytic space; consider any family of analytic maps $f_i : X \to Y_i$ ($i \in I$, the Y_i's being analytic spaces), and denote by R_I the equivalence relation defined on X by these maps. Then for any compact subset K of X there is a finite subset J of I such that R_J and R_I induce the same relation on K.*

The proof of this lemma follows a well-known argument: for any finite subset J of I, let Δ_J be the subset of $X \times X$, inverse

image of the diagonal of $Y_J \times Y_J$ by $f_J \times f_J : X \times X \to Y_J \times Y_J$ (Y_J denotes the product of the spaces Y_i for $i \in J$, and f_J the map $X \to Y_J$ defined by the f_i's for $i \in J$). Each Δ_J is an analytic subset of the space $X \times X$; if $J \subset J'$, then $\Delta_{J'} \subset \Delta_J$. Hence the Δ_J's build a filtered decreasing family of analytic subsets, and it is well known that such a family becomes constant on a given compact subset of $X \times X$. The lemma is proved.

We shall now achieve the proof of the main theorem. Returning to U, take in U a saturated open neighbourhood U' of the class of a point $y \in U$, such that the closure $\overline{U}'$ be compact and contained in U (cf. property (iii) of proper mappings). By the lemma, we have a *finite* set of holomorphic mappings $f_i : U \to Z_i$ such that the equivalence relation defined by the f_i's coincides with R on $\overline{U}'$. Consider the product map of the f_i's into the product space Z of the Z_i's. The image $f(\dot{U}')$ of the boundary $\dot{U}'$ of U' is closed in Z and does not meet the image $f(U')$. Thus $Y = Z - f(\dot{U}')$ is an analytic space, and f induces a holomorphic map g of U' into Y; it is obvious that g is proper. On U' the equivalence relation R is defined by the map g; we know from Theorem 2 that U'/R is an analytic space, and the main theorem is proved.

REMARK. Let X be an analytic space, and consider the equivalence relation defined on X by a family of analytic mappings $X \to Y_i$. If R is *proper*, the main theorem may be applied: *the space X/R is an analytic space.*

4. **Examples.** We recall the classical notion of holomorphic convexity: an analytic space X is *holomorphically convex* if, for any compact subset K of X, the set of all $x \in X$ such that

$$|f(x)| \leqslant \sup_K |f| \text{ for every } f \text{ holomorphic on } X,$$

is compact. Consider on X the equivalence relation R defined by all holomorphic functions; clearly if X is holomorphically convex, then R is proper (the converse is not true: take for X an open subset of $\mathbf{C}^n$ which is not a domain of holomorphy; then R is the

identity relation, but X is not holomorphically convex). Obviously any compact analytic space is holomorphically convex.

EXAMPLE 1. Let X be a holomorphically convex analytic space. Take for R the relation defined by all holomorphic functions on X. By the main theorem, X/R is an analytic space. Obviously the natural projection $X \to X/R$ induces an isomorphism of the ring of all holomorphic functions on X and on X/R. Moreover X/R is holomorphically convex; since the functions holomorphic on X/R separate the points of the space, it follows that X/R is "holomorphically complete" in the sense of Grauert. This space X/R is naturally attached to X and was discovered by Remmert [10]. We recall that a holomorphically complete complex manifold is nothing but a "Stein manifold"; and the class of all holomorphically complete spaces is exactly the class for which the fundamental theorems concerning coherent sheaves are valid [2].

EXAMPLE 2. The preceding procedure is not interesting when the analytic space X is compact. In that case, consider the equivalence relation R defined by all holomorphic mappings of X into complex projective spaces. By the main theorem, the quotient space X/R is an analytic space; but by the lemma, there is a holomorphic injective mapping f of X/R into a finite product of projective spaces; hence we have finally a holomorphic injection $f: X/R \to P_n$ into some projective space P_n. By Theorem 1 the image of f is a closed analytic subspace of P_n, hence is an algebraic subspace of P_n by the classical theorem of Chow. Thus we have a holomorphic homeomorphism of X/R onto a projective algebraic variety; a known result [8] says that X/R *is itself a projective algebraic variety.* Thus we have a projective algebraic variety X' and a holomorphic surjection $p: X \to X'$ enjoying the following universal property: given any holomorphic map $g: X \to Y$, where Y is a projective algebraic variety, there is a unique factorization $X \xrightarrow{p} X' \xrightarrow{g'} Y$ of g, with g' holomorphic.

EXAMPLE 3. Let X be again a compact analytic space. Consider the equivalence relation R defined by all holomorphic mappings

of X into compact algebraic (not necessarily projective) varieties. The same argument as before applies : X/R is a compact algebraic variety and possesses a universal property with respect to all holomorphic mappings of X into compact algebraic varieties.

5. A result of Stein. The following result was proved by K. Stein [15] for complex manifolds, but is true in the general case of analytic spaces.

THEOREM 3. *Let $f : X \to Y$ be a holomorphic mapping. Assume that the connected components of the fibres of f are compact. Then the equivalence relation R' defined by these connected components is proper, and the quotient ringed space X/R' is an analytic space.*

PROOF. The fact that R' is proper is of a purely topological nature, and may be proved for a continuous map of locally compact spaces when the connected components of the fibres are compact. In view of condition (ii) of §3, we have to prove that any (compact) fibre N of R' has a fundamental system of neighbourhoods which are open and saturated. Now let V be any open neighbourhood of N; if V is small enough, then $f(N) \notin f(\dot{V})$, $\dot{V}$ denoting the boundary of V. Let U be the open subset $Y - f(\dot{V})$ of Y; then $V' = V \cap f^{-1}(U)$ is open and contains N ; any fibre F' of R' meeting V' does not meet $\dot{V}$, hence is contained in V (since F' is connected) and thus is contained in V'. In other words, V' is R'-saturated. Q.E.D.

Now we know that X/R' is Hausdorff (since R' is proper), and the question of proving that X/R' is an analytic space has a local character. Since f induces a holomorphic map $V' \to U$, which is proper, we are reduced to the case where the map f is proper. Now assume that $f : X \to Y$ is proper ; replacing Y by $f(X)$ we may assume that f is also surjective. The direct image of the sheaf $\mathscr{H}(X)$ by f is a sheaf $\mathscr{A}$ on Y ; it is a coherent $\mathscr{H}(Y)$-sheaf, by Grauert's theorem. But $\mathscr{A}$ may be identified with $\mathscr{H}(X)/R'$.

Now, in order to prove that X/R' is an analytic space, we shall show that $(X/R', \mathscr{H}(X)/R')$ satisfies condition (a) of the main theorem. Let us denote by p' the projection $X/R' \to Y$; let us take

$x'_0 \in X/R'$, and $y_0 = p'(x_0)$; let U be an open set of Y containing y_0, and let $\phi_i \in \Gamma(U, \mathscr{A})$ be sections in finite number, which generate the $\mathscr{H}(Y)_y$-module $\mathscr{A}_y$ for each point $y \in U$ (such an U exists because $\mathscr{A}$ is coherent). Moreover, let us choose a finite set of functions ψ_j holomorphic on U, which separate the points of U (such a set exists if U is small enough). Then the set $\{\phi_i, \psi_j\}$ separate the points of $p'^{-1}(U) \subset X'/R$, and condition (a) of the main theorem is satisfied. Hence Theorem 3 is proved.

DEFINITION. Let X and Y be two analytic spaces ; a map $f : X \to Y$ is called a *ramified covering* if f is a proper surjective holomorphic mapping, and if the inverse image of each point of Y is finite. One also says that X is a ramified covering of Y by means of f. It is known† that a ramified covering of a holomorphically complete space (resp. a projective algebraic variety, resp. a compact algebraic variety) is of the same type.

Now, returning to the three examples treated in §4, Theorem 3 implies the following results :

(1) Let X be a holomorphically convex analytic space ; take on X the equivalence relation R defined by all functions holomorphic on X ; then the fibres of R are *connected*.

(2) Let X be a compact analytic space ; take on X the equivalence relation R defined by all holomorphic maps of X into projective spaces. Then the fibres of R are *connected*. There is a corresponding result for the relation defined by all holomorphic mappings of X into compact algebraic varieties.

6. A generalization. Consider a class $\mathscr{C}$ of analytic spaces, satisfying the following conditions :

(P_1) *the product of two spaces of $\mathscr{C}$ belongs to $\mathscr{C}$* ;

†In the case of holomorphically complete spaces, this follows from [5]. In the case of projective algebraic varieties, it follows from ([8], Théorème 1). The case of compact (not necessarily projective) algebraic varieties cannot be found in the literature, but the result may be deduced from a theorem of Grothendieck, asserting that any analytic coherent sheaf on a compact algebraic variety comes from an algebraic coherent sheaf (see Séminaire H. Cartan, 1956-57, Exposé 2, of Grothendieck).

(P_2) *if X is an analytic space and if, for each point $x \in X$, there exists a proper holomorphic mapping $f : X \to Y$ into a space Y of the class $\mathscr{C}$, such that the fibre $f^{-1}(f(x))$ contains x as an isolated point, then X belongs to $\mathscr{C}$.*

The condition (P_2) obviously implies :

(P'_2) *any analytic subspace of a space belonging to $\mathscr{C}$ belongs to $\mathscr{C}$.*

Moreover, if the spaces of $\mathscr{C}$ are compact, (P_2) is equivalent (modulo (P_1)) to the conjunction of (P'_2) and

(P''_2) *if $f : X \to Y$ is a ramified covering and if Y belongs to $\mathscr{C}$, then X belongs to $\mathscr{C}$.*

Now the reasoning of §§3 and 4 leads to the following :

THEOREM 4. *Let $\mathscr{C}$ be a class satisfying (P_1) and (P_2). Let X be an analytic space such that there exists a holomorphic and proper map $X \to Y$, Y belonging to $\mathscr{C}$. Then, if R denotes the equivalence relation defined on X by all holomorphic mappings of X into the spaces of the class $\mathscr{C}$, the space X/R is an analytic space belonging to $\mathscr{C}$, and the fibres of R are connected.*

We shall only sketch the proof. First, the relation R is proper because of the existence of a proper map $f : X \to Y$, where $Y \in \mathscr{C}$. Thus X/R is an analytic space (main theorem and remark). Applying the lemma and the property (P_1), we see that each point $x \in X$ possesses an open saturated neighbourhood U on which R may be defined by a holomorphic map $g : X \to Z$, where Z belongs to $\mathscr{C}$; R may also be defined on U by the map $f \times g : X \to Y \times Z$, which is proper because f is proper. Then, applying (P_2) to the space X/R, we see that X/R belongs to $\mathscr{C}$. Finally, the fibres of R are connected, because if we denote by R' the equivalence relation defined by the connected components of the fibres of R, X/R' is an analytic space (Theorem 3) and the natural map $X/R' \to X/R$ is proper (because $X \to X/R$ is proper). By (P_2) it follows that X/R' belongs to $\mathscr{C}$, which implies $R' = R$.

The examples treated in § 4 correspond to the following classes $\mathscr{C}$:

— the class $\mathscr{C}_1$ of all holomorphically complete spaces;

— the class $\mathscr{C}_2$ of all projective algebraic varieties;

— the class $\mathscr{C}_3$ of all compact algebraic varieties.

We have a further example: the class $\mathscr{C}_4$ of all "analytically complete" spaces in the sense of Grauert-Remmert [8], namely: X is analytically complete if X is holomorphically convex and if, for each $x \in X$, there is a projective space P_n and a holomorphic map $f: X \to P_n$, such that x be isolated in the fibre $f^{-1}(f(x))$. Thus Theorem 4 may be applied to the class $\mathscr{C}_4$: for each analytic space X which is holomorphically convex, there exists a quotient space X/R in the class $\mathscr{C}_4$, such that any holomorphic map of X into a space of $\mathscr{C}_4$ may be factored through X/R; moreover the fibres of R are connected.

7. Conclusion. Not very much is known about equivalence relations which are *not* proper. Consider for instance the equivalence R defined on an analytic space X by a holomorphic map $f: X \to Y$; it is *not* true, in general, that the ringed space X/R is an analytic space. If R' denotes the relation defined by the connected components of the fibres of f, then X/R' is an analytic space in some cases (cf. [16]), even if the fibres of R' are not compact. It would be worthy of interest to make further investigations in this direction.

There is another situation in which an equivalence relation arises. Consider a connected, complex Lie group G, and let R be the relation defined on G by all holomorphic functions. It is almost obvious that the fibres of R are cosets of a closed, invariant, complex Lie subgroup H of G, and that the holomorphic functions separate the points of G/H. Actually, it follows from recent work of Matsushima and Morimoto [9] that G/H is a Stein variety, and H is a connected subgroup of the centre of G. But it may happen that H be not compact, and hence the relation R be not proper; for instance, there is an abelian connected complex Lie group G

having a closed complex subgroup isomorphic to $\mathbf{C}^*$ (multiplicative group of complex numbers $\neq 0$), with $G/\mathbf{C}^*$ compact of complex dimension 1, such that any function holomorphic on G is a constant.

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VARIATIONAL METHODS IN THE PROBLEM OF THE MODULI OF RIEMANN SURFACES†

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1. Introduction. The problem of the moduli of Riemann surfaces is not a well-defined, precisely formulated problem but rather a loose complex of questions, some rather vaguely put, arising from some observations in Riemann's memoir, "Theorie der Abelschen Funktionen", and subsequent critical examination.

Riemann considered all conformally equivalent Riemann surfaces of fixed genus p as constituting one object, a conformal equivalence class, and indicated by several methods that the totality of these classes could be considered in some sense or senses as a continuum described by $3p-3$ (when $p>1$) complex parameters. Later attempts to assign precise meaning to this statement have brought, among others, four questions to the fore: (i) describe an intrinsic topology (preferably that of a manifold) for the conformal classes with real dimension $6p-6$; (ii) introduce complex functions, preferably $3p-3$ in number, on the classes, functions which are continuous or differentiable in the topology of (i) and describe an equivalent topology; and which, further, either globally or locally are in 1-1 correspondence with the conformal classes and form a complex analytic structure (iii) describe the totality of classes as an analytic, or part of an algebraic, locus in an affine or projective space; and (iv) justify the constant counting in the algebraic geometry of curves.

The solution to (i) is due to Teichmueller (Teichmueller [14]) and forms the foundation of the other investigations described herein. A short description thereof is in section 2.

The quantities desired in (ii) are the "moduli", and in section 3 it is shown how the periods of the abelian integrals of first kind can

† Some of the work herein contained was done while the author held a National Science Foundation Senior Post-doctoral Fellowship.

serve the purpose in at least a limited sense. Section 3 also indicates how the local moduli shed some light on (iii).

Finally, the first tentative steps into (iv) are taken in section 4.

2. The Teichmueller and Torelli spaces. Instead of considering individual closed Riemann surfaces of genus $p > 1$ and their conformal equivalence classes, one considers new objects, surfaces with a supplementary topological structure and a relation of equivalence among them.

Let S_0 be a particular fixed closed Riemann surface of genus p. A *topologically determined surface* is a pair (S, α) consisting of a closed Riemann surface S of genus p and an orientation preserving homeomorphism α of S_0 on S. Two topologically determined surfaces, (S, α) and (S', α') are *equivalent* if there is a conformal homeomorphism f of S on S' lying in the homotopy class of $\alpha'\alpha^{-1}$. A class of equivalent topologically determined surfaces forms a *Teichmueller surface*, denoted by $S(T)$, where the S indicates that there is a representative in the class of the form (S, α'). The totality of Teichmueller surfaces (of genus p) will be denoted by T_p and forms a pre-space, which will be topologized in a manner indicated in the following.

Before continuing, the reader who is not acquainted with the material is advised to consult a straightforward exposition of Teichmueller's original method of topologizing T_p as an aid to intuitive comprehension of what follows. The introductory sections of Ahlfors [1] contain such an exposition. The method to be followed, due to Bers (Bers [4]), is a modification of Teichmueller's original method and is based on ideas implicit and explicit in his work. It, of course, generates the same topology, but it does so without making use of Teichmueller's deep results on quasi-conformal mapping (but also without yielding as much information).

Let $S_0(T)$ be a fixed Teichmueller surface and assume that $(S_0, 1)$, 1 being the identity map of S_0 on itself, is a representative so that S_0 is the fixed reference surface used to define T_p. Let $S(T)$ be any other element of T_p with (S, α) as representative. One can show that in the homotopy class of $\alpha\ 1^{-1} = \alpha\colon\ S_0 \to S$ there is a C_1

homeomorphism g(i.e. because of compactness, a quasi-conformal homeomorphism). Assuming, as one may, that the conformal structure of S is defined by a conformal metric, $\Lambda \mid dw \mid^2$, w being a local parameter on S, and writing g locally as $w = g(z, \bar{z})$, z a local parameter on S_0, one can by means of undetermined coefficients explicitly calculate that

$$\Lambda \mid dw \mid^2 = \lambda \mid dz + \mu d\bar{z} \mid^2, \tag{2.1}$$

where μ is a tensor satisfying

$$\mid \mu \mid \leqslant k < 1$$

and

$$\mu(z) \frac{d\bar{z}}{dz} = \mu'(z') \frac{d\bar{z}'}{dz'} \tag{2.2}$$

under change of local parameter on S_0.

(2.1) and its implied context will be called *the change of structure due to* μ or *the deformation due to* μ, henceforth. Thus, given $S(T)$ there exists a μ or *Beltrami differential* for which $S(T)$ is derived from $S_0(T)$ by the change of structure due to μ.

Conversely given a Beltrami differential μ on S_0 and a conformal metric $\lambda \mid dz \mid^2$ on S_0, (2.1) defines the conformal structure of a new Riemann surface S with the same base topological space as S_0. The Teichmueller surface $S(T)$ of which $(S, 1)$ is a representative is thus defined by the deformation due to μ. The new surface $S(T)$ is realized concretely as follows. Choose a distinguished covering of S_0 by parameter disks, and in each disk with variable z apply the solution theory of the Beltrami equation

$$w_{\bar{z}} = \mu w_z \tag{2.3}$$

for μ with suitable regularity (see Bers [4] for details and references) to obtain a solution $w(z, \bar{z})$ with $w(0, 0) = 0$, $w_z(0, 0) = 1$, which is homeomorphic for $\mid z \mid$ small. The theory of the Beltrami equation shows that where two z-disks overlap the corresponding w solutions are analytic functions of one another. Choosing the w's as a distinguished covering of the base space of S_0 one has defined $S(T)$.

A set of Beltrami differentials $\mu_1, \ldots, \mu_k$ on S_0 will be termed linearly independent if no k-vector $(a_1, \ldots, a_k)$ with complex

components can be found for which

$$a_1\mu_1 + \cdots + a_k\mu_k = h_z^- \tag{2.4}$$

for some contravariant (tangent) vector h on S_0, i.e. for h satisfying

$$\frac{h(z)}{dz} = \frac{h'(z')}{dz'}. \tag{2.5}$$

A μ satisfying (2.4) is called *trivial* (Bers: locally trivial). The intuitive meaning of this concept will be found in Ahlfors [1].

Now, a maximal set of linearly independent $\mu_1, \cdots, \mu_k$ has $k = 3p - 3$. This follows immediately from the duality between the μ and the *quadratic differentials* (analytic and finite) ϕ satisfying

$$\phi\, dz^2 = \phi' dz'^2 \tag{2.6}$$

on S_0. The duality is established by the existence of the bilinear inner product

$$(\phi, \mu) \equiv i \iint_{S_0} \phi\, \mu\, dz \wedge dz$$

which is assured by (2.2) and (2.6), and by the lemma (see Ahlfors [1] for example) that a necessary and sufficient condition for μ to be trivial is that $(\phi, \mu) = 0$ for all ϕ. Finally, there are $3p - 3$ linearly independent ϕ in the conventional sense over the complex field, according to the Riemann-Roch theorem.

To topologize T_p one now takes $3p - 3$ linearly independent $\mu_1, \cdots, \mu_{3p-3}$ and forms

$$\mu = a_1\mu_1 + \cdots + a_{3p-3}\,\mu_{3p-3}, \tag{2.8}$$

which for sufficiently small $\Sigma |a_i|^2$ defines a Beltrami differential. Inserting μ in (2.1) defines then a $6p - 6$ real dimensional neighbourhood of $S_0(T)$. However, T_p does not depend, as one sees, on the particular reference surface, and so one can introduce neighbourhoods about every point and assume them homeomorphic with euclidean solid spheres. T_p with this topology is a $6p - 6$ real dimensional manifold once connectedness is established; but the argument leading to (2.1) establishes connectedness when one observes that the insertion of the factor ϵ before μ in (2.1) leads to the one-parameter family of surfaces due to deformation by $\epsilon\mu$ as

ϵ runs from 0 to 1. In the next section it will be shown that T_p is in fact a complex-analytic manifold.

It will be necessary to consider the set U_p of hyperelliptic Teichmueller surfaces *per se*.

Let $S(T) \in U_p$ and let z be a function on S with one double pole and no other, whose presence characterizes the hyperelliptic surfaces. S is mapped by z on a two-sheeted surface over the z plane. Given $P \in S$ let $E(P)$ be the point in the other sheet lying over P. E is clearly an involution and it is obviously conformal because the same local parameter, by projection, serves both P and $E(P)$. The reasoning in the middle of p. 394 of Hurwitz [8] shows that any other function with one double pole and no others on S takes equal values at P and $E(P)$. Thus E is a uniquely defined operation on S.

Consider a fixed $S_0(T) \in U_p$ with involution E_0 and another $S(T) \in U_p$ with involution E. If (S_0, α_0) and (S, α) are representatives of $S_0(T)$ and $S(T)$ respectively, then the homotopy class of $\alpha\alpha_0^{-1}: S_0 \to S$ may or may not contain a *symmetric* homeomorphism g, i.e. one for which $g(E_0(P)) = Eg(P)$. If the affirmative holds then $S_0(T)$ and $S(T)$ are called *symmetrically equivalent*. Since there is no guarantee that the affirmative holds, U_p is spilt up into disjoint equivalence classes.

LEMMA 1. *Each symmetric equivalence class of U_p is a $4p - 2$ real dimensional manifold with the Teichmueller topology generated by the deformations due to $\mu = a_1\mu_1 + \ldots + a_{2p-1}\mu_{2p-1}$, where the a's are complex constants and $\mu_1, \ldots, \mu_{2p-1}$ are a maximal set of linearly independent even Beltrami differentials. U_p is then a union of such manifolds.*

Before proving this statement one must define *even* Beltrami differential: μ is even if $\mu(P) = \mu(E(P))$ on S, where $S(T)$ is the point of U_p whose neighbourhood is to be constructed.

To prove Lemma 1 simply retrace the steps at the beginning of this section, being careful to respect symmetry. Thus, referring to the paragraph before Lemma 1, if $S_0(T)$ is symmetrically equivalent to $S(T)$, then the same argument that proves that every homotopy

class contains a quasi-conformal homeomorphism, namely, triangulation, simplicial approximation, and smoothing at the edges and vertices, shows that the symmetric g is homotopic to a *symmetric* quasi-conformal map. One has only to use triangulations obtained by " pressing through " triangulations of the sphere with vertices at the branch points. Returning to (2.1) derived now from a symmetric g one sees that the equation is exactly the same at P as at $E_0(P)$ since $g(E_0(P)) = Eg(P)$ and the same z and w parameters are used at P and $E_0(P)$ and $g(P)$ and $Eg(P)$ respectively. *Hence* μ *is even.* Conversely, one sees that a deformation due to an even μ on S leads to a symmetric map on the new S, so that $S(T)$ is symmetrically equivalent to $S_0(T)$. One simply re-examines Beltrami's equation (2.3) and the associated theory and sees that the evenness of μ implies that the normalized solutions u are identical at P and $E_0(P)$. The connectedness follows as before.

The Riemann surface over the z-plane on which S is mapped by z is the Riemann surface of

$$w^2 = \psi_{2p+1}(z)$$

where infinity and the roots of the polynomial ψ_{2p+1} of degree $2p+1$ are the branch points. In this representtion the quadratic differentials on S split into two sets :

$$\frac{dz^2}{w^2}, \frac{z\,dz^2}{w^2}, \cdots, \frac{z^{2p-2}\,dz^2}{w^2}, \tag{2.9}$$

which are $2p-1$ in number and *even*; and

$$\frac{dz^2}{w}, \cdots, \frac{z^{p-3}\,dz^2}{w}, \tag{2.10}$$

which are $p-2$ in number and *odd*. The *even* μ are manifestly dual under the inner product to the *even* ϕ in (2.9) since the inner product of an even μ and an odd ϕ must equal its negative, and, hence, vanish. A basis for the even μ thus has $2p-1$ members. Lemma 1 is proved.

For the purposes of section 3, a discussion of the *Torelli* space is in order. A Torelli surface is an equivalence class of pairs, $(S, \{\Gamma, \Delta\})$,

each of a Riemann surface S of genus $p > 1$ and an ordered canonical set of homology *classes* generating the homology group of dimension one, $H^1(S)$. Two pairs $(S_0, \{\Gamma_0, \Delta_0\})$ and $(S, \{\Gamma, \Delta\})$ are equivalent if there exists a conformal homeomorphism of S_0 on S carrying $\{\Gamma_0, \Delta_0\}$ onto $\{\Gamma, \Delta\}$ with order preserved. An equivalent definition is to operate as in the case of Teichmueller surfaces with topologically determined surfaces, (S, α), where α is an orientation preserving homeomorphism of a fixed surface S_0 on S, where two such pairs (S, α), and (S', α') are considered equivalent if there exists a conformal homeomorphism $f: S \to S'$ such that $f^{-1}\alpha'\alpha^{-1}$ acts as the identity on $H^1(S)$. One sees the equivalence by simply choosing fixed canonical basis of $H^1(S_0)$ and assigning the image thereof by α in $H^1(S)$ to S.

Let $\mathfrak{T}_p$ denote the set (as yet untopologized) of Torelli surfaces. Recalling the definition of a Teichmueller surface in the equivalent form that the condition for equivalence of (S, α), (S', α') is that $f^{-1}\alpha'\alpha^{-1}$ be homotopic to the identity and recalling, too, that "homotopic implies homologous", one sees that every Teichmueller surface $S(T)$ determines a unique Torelli surface $S(\mathfrak{T})$, and one can state: T_p *covers* $\mathfrak{T}_p$, where cover is meant here in the purely point-set sense of the existence of a projection.

To exhibit the group of the covering one first defines the *mapping class group* G on T_p. Given a homotopy class g of homeomorphisms of S_0 on itself and an element $\beta \in g$, it is clear that the mapping $(S, \alpha) \to (S, \alpha\beta)$ of the set of topologically determined surfaces on itself uniquely determines a map $g: T_p \to T_p$ and that the totality of such g forms a group G in a natural manner.

It is now obvious that the group of T_p over $\mathfrak{T}_p$ is the sub-group $H \subset G$ associated with the homotopy classes of self-homeomorphisms of S_0 that act as the identity on $H^1(S_0)$.

LEMMA 2. *T_p is an unramified covering of $\mathfrak{T}_p$.*

PROOF. It is sufficient, as is well known, to show that H has no fixed points on T_p. But if $h\,S(T) = S(T)$, where $h \in H$, then, (S, α) representing $S(T)$, $(S, \alpha\beta)$, $\beta \in h$, is equivalent to (S, α), which

means there exists $f : S \to S$, conformal and homotopic to $\alpha\beta\alpha^{-1} : S \to S$, which, because $\beta \in h$, acts as the identity on $H^1(S)$. But a theorem of Hurwitz (Hurwitz [8], p. 416) asserts that f is then necessarily the identity. Lemma 2 is established.

COROLLARY : *The Torelli space T_p has the same local manifold structure as T_p the Teichmueller space.*

If one now calls the totality of hyperelliptic Torelli surfaces $\mathfrak{U}_p$, then the concept of symmetric equivalence can be introduced by analogy and clearly one has

LEMMA 3. *Appropriately analogous statements to those in Lemma* 1 *about U_p apply to $\mathfrak{U}_p$.*

The concept of symmetric equivalence and the essence of Lemma 1 and its proof are contained in Rauch [11].

Additional considerations on the topology of U_p relative to T_p (see Ahlfors [2]) show that one can state

LEMMA 1 (*bis*). *U_p is a disconnected real $4p-2$ dimensional sub-manifold of T_p, being the union of the isolated sub-manifolds which are the symmetric equivalence classes of U_p.*

Clearly one has

LEMMA 3 (*bis*). *A statement analogous to Lemma* 1 (*bis*) *applies to $\mathfrak{U}_p$.*

3. The abstract variational formulae for the periods of the abelian integrals of first kind. The following material, unless otherwise explicitly noted, is based on Rauch [10] and [11].

Consider a fixed Riemann surface S of genus $p > 1$ and on it a canonical homology basis consisting of two conjugate sets of cycles γ_i, δ_i; $K(\gamma_i, \delta_j) = \delta_{ij}$, $i, j = 1, \ldots, p$. $K(\gamma, \delta)$ is the Kronecker index.

The basis for the abelian differentials of first kind $d\zeta_i$, $i = 1, \ldots, p$, for which

$$\int_{\gamma_j} d\zeta_i = \delta_{ij},\ i, j = 1, \ldots, p,$$

consists of the *normal* differentials of first kind. The remaining periods

$$\pi_{ij} = \int_{\delta_j} d\zeta_i, \quad i, j = 1, \ldots, p, \tag{3.1}$$

are fixed by the normalization and are the elements of the $p \times p$ *Riemann matrix* attached to S and the canonical homology basis γ_i, δ_i, $i = 1, \ldots, p$. A Riemann matrix is symmetric and has positive definite imaginary part (the Riemann matrix as defined here is the right half of the traditional Riemann matrix, which is the $p \times 2p$ matrix of all periods). If one topologizes the Riemann matrices with the usual euclidean topology of $p(p+1)$ real dimensional euclidean space then one may speak of the *space of Riemann matrices.*

The first observation is that the π_{ij} are single valued functions on the Torelli and Teichmueller spaces. In fact let $S(\mathfrak{T})$ be the Torelli surface of which $(S, \{\Gamma, \Delta\})$ is a representative, where, $\{\Gamma, \Delta\}$ is the ordered basis of $H^1(S)$ determined by (γ_i, δ_i), $i = 1, \ldots, p$. If $(S', \{\Gamma', \Delta'\})$ is another representative then there exists $f : S \to S'$, conformal and homeomorphic for which $\gamma_i' = f(\gamma_i)$, $\delta'_i = f(\delta_i)$, $i = 1, \ldots, p$, lie respectively in the ordered basis $\{\Gamma', \Delta'\}$ for the classes of $H^1(S')$. If one writes $z' = f(z)$ then one has

$$\delta_{ij} = \int_{\gamma_j'} d\zeta_i(f^{-1}(z')).$$

But this normalization determines the set of normal differentials of first kind $d\zeta_i'$ of S' with respect to the basis γ_i', δ_i', $i = 1, \ldots, p$, completely and one has

$$\pi_{ij} = \int_{\delta_j} d\zeta_i = \int_{\delta_j'} d\zeta_i(f^{-1}(z')) = \int_{\delta_j'} d\zeta'_i = \pi'_{ij}.$$

Finally, however, π_{ij} and π'_{ij} depend respectively, only on the classes $\{\Gamma, \Delta\}$ and $\{\Gamma', \Delta'\}$, which together with the preceding establishes the single valued nature of π_{ij} on $\mathfrak{T}_p$ or equivalently that (3.1) establishes a single valued map $\pi : \mathfrak{T}_p \to$ Riemann matrices.

The definition of π may be extended to T_p by setting $\pi(S(T)) = \pi(S(\mathfrak{T}))$ where $S(\mathfrak{T})$ is the image of $S(T)$ under the covering map of T_p over $\mathfrak{T}_p$.

Now the central fact of the lecture is that the map π established by (3.1) is differentiable with respect to any particular set of local coordinates $(a_1, \ldots, a_{3p-3})$ at any one $S(\mathfrak{T}) \in \mathfrak{T}_p$ or $S(T) \in T_p$ (one must remember that only continuous structures, not differentiable structures, have been imposed on $S(T)$ and $S(T)$ in section 2). This is established quite simply by exhibiting explicit formulae for the differential of π. In fact, given $S(T) \in T_p$ and its projection $S(\mathfrak{T}) \in \mathfrak{T}_p$ and given coordinates $(a_1, \ldots, a_{3p-3})$ which describe neighbourhoods of both $S(T)$ and $S(\mathfrak{T})$ as described in section 2 then one has, both at $S(T)$ and $S(\mathfrak{T})$

$$\delta\pi_{ij} = \frac{1}{i}\left(\frac{d\zeta_i}{dz}\frac{d\zeta_j}{dz}, \mu\right) = \frac{1}{i}\sum_{\alpha=1}^{3p-3} a_\alpha\left(\frac{d\zeta_i}{dz}\frac{d\zeta_j}{dz}, \mu_\alpha\right), \qquad (3.2)$$

where one observes immediately that the quantities in the left hand entry of each inner product are quadratic differentials. A proof of (3.2) will be found in the appendix.

It is convenient to call (3.2) the *abstract variational formulae* because they describe the result of an abstract variation of an abstract surface as contrasted with the concrete variations of concrete models to be found in the next section.

Now a key theorem, due to Noether (Hensel-Landsberg [7], p. 502), states that the quadratic differentials on a *non-hyperelliptic* surface S are generated by products of the elements of any basis for the abelian differentials of the first kind on S. More precisely given $S(\mathfrak{T})$, there exists at least one set I of $3p-3$ index pairs (i, j) for which $d\zeta_i\, d\zeta_j/dz^2$, $(i,j) \in I$, are a basis for the ϕ on the base surface S, provided S is not hyperelliptic.

If S is hyperelliptic then Noether's theorem fails. Indeed, returning to the representation $w^2 = \psi_{2p+1}(z)$ as the equation of the curve of which S is the two sheeted surface, one can write the differentials of first kind explicitly as

$$\frac{dz}{w}, \ldots, \frac{z^{p-1}\,dz}{w}, \tag{3.3}$$

and all possible products of these are precisely the *even* quadratic differentials (2.9), which are only $2p-1$ in number. However, as pointed out in section 2, these are the interesting ones if one is content to work within the set of hyperelliptic S only. Thus one can assert, given $S(\mathfrak{T})$, where S is hyperelliptic, there exists at least one set I_E of $2p-1$ index pairs (i,j) for which $d\zeta_i d\zeta_j/dz^2$, $(i,j) \in I_E$, are a basis for the even quadratic differentials on S.

On the basis of the two preceding prescriptions and (3.2) one can describe an important property of the map π and consequently introduce distinguished complex coordinates on $\mathfrak{T}_p - \mathfrak{U}_p$, $T_p - U_p$, $\mathfrak{U}_p$ and U_p, so that each of them becomes a complex analytic manifold. Observe that because of Lemmas 1(bis) and 3(bis) that in case $p > 2$, $T_p - U_p$ and $\mathfrak{T}_p - \mathfrak{U}_p$ are connected and open and hence, $3p-3$ dimensional submanifolds of T_p and $\mathfrak{T}_p$ respectively. It is known that all S of genus $p = 2$ are hyperelliptic and so $\mathfrak{T}_2 = \mathfrak{U}_2$, $T_2 = U_2$.

One more preliminary to enunciation of the first theorem : the image of one manifold in another under a *local* homeomorphism will be called a *weak submanifold*. If the mapping is also $1-1$, i.e. a homeomorphism, then the image is a *submanifold*. If the manifolds and the mapping are differentiable and the rank is the same number m at every point then the mapping will be said to have rank m. In particular if the rank is maximal the submanifold, be it weak or not, will be called *regular*.

THEOREM 1. *The map π defined by* (3.2) *maps the $6p-6$ real dimensional manifold $\mathfrak{T}_p - \mathfrak{U}_p$ ($T_p - U_p$) of non-hyperelliptic Torelli (Teichmueller) surfaces on a complex analytic regular weak submanifold of the space of Riemann matrices, and this map induces on $\mathfrak{T}_p - \mathfrak{U}_p$ ($T_p - U_p$) the structure of a complex analytic manifold of $3p-3$ complex dimensions. Given $S(\mathfrak{T}) \in \mathfrak{T}_p - \mathfrak{U}_p$ ($S(T) \in T_p - U_p$), the distinguished co-ordinates of this structure in the neighbhourhood of $S(\mathfrak{T})$ ($S(T)$) are π_{ij}, $(i,j) \in I$, where $d\zeta_i\, d\zeta_j/dz^2$, $(i,j) \in I$, are a basis for the quadratic differentials on S.*

π maps the union of $4p-2$ real dimensional manifolds $\mathfrak{U}_p(U_p)$, in particular $\mathfrak{T}_2(T_2)$, onto the union of complex analytic regular weak submanifolds of $2p-1$ complex dimensions and thus induces the structure of disconnected complex analytic manifold on $\mathfrak{U}_p(U_p)$. Given $S(\mathfrak{T})$ $(S(T))$ in $\mathfrak{U}_p(U_p)$, the distinguished coordinates in the neighbourhood of $S(\mathfrak{T})$ $(S(T))$ are π_{ij}, $(i,j)\in I_E$, where $d\zeta_i\, d\zeta_j/dz^2$, $(i,j)\in I_E$, generate the even quadratic differentials on S.

For the proof of the first part one must first show that the π_{ij} chosen by the prescription of the theorem are local coordinates at $S(\mathfrak{T})$. It will be sufficient by the implicit function theorem to show the non-vanishing of the jacobian of the particular π_{ij} with respect to the a_α at $S(\mathfrak{T})$. But the real jacobian is the absolute value squared of

$$\frac{\partial(\pi_{ij})}{\partial(a_\alpha)} = \left|\frac{1}{i}\left(\frac{d\zeta_i}{dz}\frac{d\zeta_j}{dz}, \mu_\alpha\right)\right|, (i,j)\in I, \alpha = 1, \ldots, 3p-3, \quad (3.4)$$

a determinant of Gram type. (3.4) follows from (3.2). If it vanished at $S(\mathfrak{T})$ there would exist a linear combination of the columns which vanishes. The linear combination of the μ_α with the same coefficients would then be orthogonal to all $d\zeta_i\, d\zeta_j/dz^2$, $(i,j)\in I$ and, therefore, trivial, contrary to the assumption of linear independence of the μ_α.

The next step in the proof is to show that when two coordinate systems overlap at $S(\mathfrak{T})$ then one is analytic in the other. In the present case then it is necessary to show that π_{kl} for arbitrary indices k, l is an analytic function of the π_{ij}, $(i,j)\in I$ at $S(\mathfrak{T})$. But that follows trivially from (3.2) since

$$\delta\pi_{kl} = \frac{1}{i}\left(\frac{d\zeta_k}{dz}\frac{d\zeta_l}{dz}, \mu\right)$$

and

$$\frac{d\zeta_k}{dz}\frac{d\zeta_l}{dz} = \sum_{(i,j)\in I} a_{ij}\frac{d\zeta_i}{dz}\frac{d\zeta_j}{dz},$$

whence

$$\delta\pi_{kl} = \frac{1}{i}\sum_{(i,j)\in I} a_{ij}\left(\frac{d\zeta_i}{dz}\frac{d\zeta_j}{dz}, \mu\right) = \sum_{(i,j)\in I} a_{ij}\,\delta\pi_{ij},$$

and, therefore,

$$\frac{\partial \pi_{kl}}{\partial \pi_{ij}} = a_{ij}, \ (i, j) \in I.$$

Being differentiable in the complex sense with respect to π_{ij}, $(i, j) \in I$, π_{kl} is analytic in these variables.

Recalling the definition of regular, weak submanifold the first part is proved.

The reader will have no difficulty in amending the argument to account for the second part about hyperelliptic surfaces.

Finally, the alternative statements about T_p and U_p follow in the usual way for unramified coverings.

Now if one invokes Torelli's theorem (cf. Andreotti [3]), which in the language of the present paper takes the form:

TORELLI'S THEOREM : *π maps $\mathfrak{T}_p$ 1-1 into the Riemann matrices*; then one can strengthen Theorem 1 to

THEOREM 2. *If in Theorem* 1 *one considers only $\mathfrak{T}_p$ and $\mathfrak{U}_p$ then the word weak can be dropped, i.e. π is a homeomorphism into a regular submanifold on both $\mathfrak{T}_p - \mathfrak{U}_p$ and $\mathfrak{U}_p$.*

There still remains the problem of making a complex analytic manifold out of *all* of T_p and $\mathfrak{T}_p$ and not just $T_p - U_p$, U_p, $\mathfrak{T}_p - \mathfrak{U}_p$, and $\mathfrak{U}_p$. For that purpose it is necessary to introduce $3p - 3$ new independent, intrinisically defined, function theoretic parameters in the neighbourhood of the points of U_p and $\mathfrak{U}_p$ such that they remain independent even on U_p and $\mathfrak{U}_p$. One set of possible local moduli, suggested before such a concept was made rigorous by present developments (for example, Schiffer & Spencer [13]), is deduced from an Abelian differential of first kind $d\zeta$ with simple zeros and fixed periods over the γ_i on $S(\mathfrak{T})$ by using the δ-periods π_i, $i = 1, \ldots, p$, and the integrals α_i, $i = 1, \ldots, 2p - 3$ from one fixed zero z_1, to the remaining zeros $z_2, \ldots, z_{2p-2}$.

In Ahlfors [2], which is otherwise based on the ideas in Rauch [10] and [11], it is shown that the suggested quantities are indeed local moduli by computing abstract variational formulae for them.

From (3.2) one obtains immediately

$$\delta\pi_i = \frac{1}{i}\left(\frac{d\zeta}{dz}\frac{d\zeta_i}{dz}, \mu\right);$$

and the computation in Ahlfors [2] gives

$$\delta\alpha_j = \frac{1}{i}\left(\frac{d\zeta}{dz}\frac{d\eta_j}{dz}, \mu\right),$$

where $d\eta_j$ is the normal differential of third kind with poles of residue 1 at z_{1+j} and -1 at z_1.

The $3p-3$ quadratic differentials (finite because the poles of $d\eta_j$ are cancelled by the zeros of $d\zeta$) $d\zeta\, d\zeta_1/dz^2, \ldots, d\zeta\, d\eta_1/dz^2, \ldots, d\zeta\, d\eta_{2p-3}/dz^2$, are clearly linearly independent and the arguments used to establish Theorem 1 show immediately that π and α are admissible co-ordinates in the complex analytic structures of $T_p - U_p$ and $\mathfrak{T}_p - \mathfrak{U}_p$ and that they are analytic functions of the co-ordinates on U_p and $\mathfrak{U}_p$ since the new variational formulae are valid at any point of T_p or $\mathfrak{T}_p$ hyperelliptic or not. Introducing the new co-ordinates at the points of U_p and $\mathfrak{U}_p$ and invoking Theorem 1 one obtains

THEOREM 3. *The Teichmueller space T_p and the Torelli space $\mathfrak{T}_p$ are complex analytic manifolds of $3p-3$ dimensions with the structures described in Theorem 1 and the preceding paragraph.*

The hyperelliptic sub-spaces U_p and $\mathfrak{U}_p$ are regular complex analytic submanifolds (disconnected) of T_p and $\mathfrak{U}_p$, respectively, of $2p-1$ dimensions.

The last observation of the section is that if one identifies all points of T_p equivalent under any one element of G and does so for all elements of G one obtains, by Theorem 2, the set of all equivalence classes of Riemann surfaces of fixed genus $p > 1$ as an analytic subvariety of the Riemann matrices identified under the higher modular group, the so-called Siegel space. This variety is the image under π of the *complex space* M_p of all equivalence classes obtained as indicated from T_p by identification. It is M_p that was conceived of by Riemann and the algebraic geometers. It has singularities caused by the presence of curves with self transformations

among which are the fixed points of G. $\pi(M_p)$ is a faithful realization except at the hyperelliptic locus on M_p, which locus is mapped on a singular sub-locus of $2p-1$ complex dimensions.

An examination of the special considerations on the hyperelliptic surfaces should disclose how similar, if not as complete information, can be obtained about the sets of surfaces admitting conformal self-transformations.

Alternative proofs of Theorem 3 without the information about the hyperelliptic sub-loci have been obtained in Bers [4] and, inferentially, in Kodaira and Spencer [9]. Neither method is applicable to the problems of the next section. However, work, as yet unpublished, by Röhrl on M_p seems to have intimate connection with the next section.

4. Concrete variational formulae and the principle of counting constants. The abstract variational formulae of the last section gave some insight into the general abstract global structure of the set of Riemann surfaces. In particular they made possible the introduction of a distinguished set of local moduli compatible with the Teichmueller topology.

The question then arises as to how to relate the classical concrete algebraic parameters, such as the coordinates of the branch points of a concrete model or the coefficients of the algebraic equation defining the model, to the Teichmueller topology. In this section this will be done for a particular problem by exhibiting variational formulae, appropriately called *concrete variational formulae*, for the π_{ij} and related quantities in terms of the branch points. The abstract variational formulae then relate the π_{ij} and, hence, the branch points to the Teichmueller topology.

Unlike the abstract formulae some related concrete formulae have been in the literature for some time, first in Douglas's papers on the Plateau problem and then, independently, in Schiffer's work with subsequent developments by Schiffer, Spencer, and Garabedian. The derivation of the formulae used here is patterned after some calculations in Garabedian [6].

Consider a fixed abstract Riemann surface S of genus $p > 1$ and an analytic function w on S. The function w maps S onto a conformally equivalent, many sheeted concrete Riemann surface S_w over the w-sphere. Let the total number of branch points, multiplicity counted be N. Then the following process describes how to vary S_w into a neighbouring concrete surface, call it $S_{w+\delta w}$, with the same number N of branch points and the same number of sheets. Given any one branch point u of multiplicity μ, take a small circle in the w-plane about the projection of u and let γ be the closed curve about u on S_w that projects on the circle. Excise the interior of γ from S_w and replace the former parametrization there, $w = t^{\mu+1}$ by $w = t^{*\mu+1} + c_\mu t^{*\mu} + \ldots + c_0$ defined in an appropriate cell of the t^* plane.

Now let $S(\mathfrak{T})$ be a Torelli surface and let S be a representative of its associated conformal structure, and let representatives of the distinguished homology classes be chosen such that none of them has an image under w in S_w passing through a branch-point. Since nothing outside sufficiently small neighbourhoods of the branch points is varied in passing to $S_{w+\delta w}$ the latter is also a representative of a Torelli surface $S_{w+\delta w}(\mathfrak{T})$. If now $S_{w+\delta w}$ is the surface obtained by making variations of all branch points in the indicated manner, if proper symbolic conventions are made about *treating a branch point of order μ as μ simple branch points in the formulae without actually assuming all branch points to be simple* and if a_α, $\alpha = 1, \ldots, N$ are constants differing from the c at the various branch points by fixed numerical factors (at each branch point c_μ is irrelevant to first order) then one has

$$\delta \pi_{ij} = \sum_{\alpha=1}^{N} a_\alpha \zeta'_i(u_\alpha)\, \zeta'_j(u_\alpha), \tag{4.1}$$

where the prime denotes differentiation with respect to the local variable t_α at the branch point u_α and the difference between $\delta\pi_{ij}$ and the actual change of π_{ij} is $o(\epsilon)$ where $\epsilon \equiv \max_\alpha |a_\alpha|$.

For an explanation of the conventions used in and a proof of (4.1) see Rauch [12].

It is important to note again explicitly that in this method of concrete variation, a vector $(a_1, \ldots, a_N)$ completely specifies the neighbouring surface—the a_α are the *local coordinates* in the branched representation.

Now, again referring to Rauch [12] for details, it is known that at any point $P \in S$, S any surface of genus p, the *Weierstrass sequence*, the sequence of those positive integers which are the orders of poles at P of analytic functions on S having no other poles, consists of all the positive integers except for p missing ones, the *gaps*. Let the p abelian integrals of first kind be normalized so that in a local parameter z at P they appear as

$$\left.\begin{aligned} Z_1 &= 1 + \ldots \\ Z_2 &= z^{r_2} + \ldots \\ &\cdot \\ &\cdot \\ &\cdot \\ Z_p &= z^{r_p} + \ldots \end{aligned}\right\} \tag{4.2}$$

in the neighbourhood of P. Then the gaps are precisely $1, r_2, \ldots, r_p$. If P is such that the Weierstrass sequence is $p+1, p+2, \ldots$, then the point is *ordinary*. Otherwise there is a first number $n \leqslant p$, and P is a *Weierstrass* point. The Weierstrass points are finite in number on S.

Given an $S(\mathfrak{T})$ let $P \in S$ be a Weierstrass point and let $n \leqslant p$ be the first member of the Weierstrass sequence of P. A function w *belonging to* n, i.e. having nth order pole at P and no other, maps S on S_w having $(n-1)$th order branch point at infinity and $(2n + 2p - 2) - (n - 1) = n + 2p - 1$ *finite* branch points. One can normalize two of these by a linear transformation of w leaving infinity fixed, leaving $n + 2p - 3$ finite branch points at one's disposal. Since a variation of these produces another n-sheeted surface whose Weierstrass sequence at infinity, then, contains n and by the *principle of non-degeneracy* (or better the principle of lower semicontinuity), Rauch [12], begins with n. One is then

tempted to say, by invoking the time-honoured *principle of counting constants* used by high school algebra students and algebraic geometers alike, that the set of surfaces neighbouring a given one, all having Weierstrass points (corresponding in the indicated sense to one another) whose sequences begin with n, depends on $n+2p-3$ constants. This is not always true. The correct statement is given by Theorem 3 of Rauch [12] which states that the map $\pi(S_w(\mathfrak{T}))$ into the Riemann matrices is of rank $n+2p-3$ as a function of the a_α at S_w if $n+1$ is a gap but of rank $n+2p-4$ if $n+1$ is not a gap.

Extrapolating one can state a rule of thumb :

Principle of correct counting of constants. To determine the dimension, at a point, of a locus of algebraic curves of fixed genus depending on some parameters, relate the parameters to the π_{ij} by means of concrete variational formulae and compute the rank at the point in question.

As a further, more cogent, illustration of this principle some more results stemming from the same problem that motivated Rauch [12], namely, some indications on normal forms of equations of curves in Hensel-Landsberg ([7], pp. 540-550), will be given. This is research still in a very early stage, and no final form is claimed here for it.

Accordingly, having found a rigorous definition of the number of conditions imposed on a set of concrete surfaces in order that at one point (say at infinity) on all there should be a Weierstrass point whose sequence begins with a fixed n, one may ask, with Hensel and Landsberg, what is the number of conditions imposed on those same surfaces if one demands, in addition, that they possess functions with fixed principal part belonging to a higher member of the Weierstrass sequence?

To attempt to answer this question consider the *normal* differential of second kind $d\tau_w^\nu$ for which $d\tau_w^\nu \sim -(\nu+1)\,dz_w/z_w^{\nu+2}$ in a local variable z_w at $w \in S$, and for which

$$\int_{\gamma_i} d\tau_w^\nu = 0,\ i = 1, \ldots, p;$$

$$\pi_i^{\nu}(w) = \int_{\delta_i} d\tau_w^{\nu} = \frac{2\pi i}{\nu!}\,\zeta_i^{(\nu+1)}(w), \quad i = 1, \ldots, p, \tag{4.3}$$

where the derivative is with respect to z_w.

In Rauch [3] one will find, with the same conventions as before, the concrete variational formulae

$$\delta\pi_i^{\nu}(w) = \sum_{\alpha=1}^{N} a_\alpha\, \tau_w^{\nu\prime}(u_\alpha)\, \zeta_i'(u_\alpha), \quad i = 1, \ldots, p. \tag{4.4}$$

From (4.4) one deduces immediately for the periods $\Pi_i(w)$ of $dT_w \equiv A_0\, d\tau_w^0 + \ldots + A_\nu\, d\tau_w^{\nu}$ the formulae

$$\delta\Pi_i(w) = \sum_{\alpha=1}^{N} a_\alpha\, T_w'(u_\alpha)\, \zeta_i'(u_\alpha), \quad i = 1, \ldots, p. \tag{4.5}$$

Now let S_w be the n-sheeted surface with $n + 2p - 3$ free finite branch points and $n - 1$ branch points at infinity used above. Let $N = n + 2p - 3$ and $w = \infty$, and fix ν and $A_0, \ldots, A_\nu$ — all in (4.5). Suppose then that the equations

$$\Pi_i(\infty) = 0, \quad i = 1, \ldots, p, \tag{4.6}$$

admit a single valued solution,

$$\left.\begin{aligned} a_1 &= a_1(a_l, \ldots, a_{n+2p-3}) \\ &\cdot \\ &\cdot \\ &\cdot \\ a_{l-1} &= a_{l-1}(a_l, \ldots, a_{n+2p-3}), \end{aligned}\right\} \tag{4.7}$$

$1 \leqslant l \leqslant n + 2p - 3$, analytic in the neighbourhood of $(0, \ldots, 0)$ with $a_1(0) = \ldots = a_{l-1}(0) = 0$.

Formulae (4.5) imply the analyticity of $\Pi_i(w)$ in the a_α and also that

$$\sum_{\alpha=1}^{n+2p-} a_\alpha\, T'_\infty(u_\alpha)\, \zeta'_i(u_\alpha) = 0, \quad i = 1, \ldots, p, \tag{4.8}$$

where the substitutions (4.7) have been made.

Examine now the significance of equations (4.6), (4.7) and (4.8). The first and second simply express, according to (4.3), that there

is an analytic family of the n-sheeted surfaces having functions with the prescribed principal part $A_0/z_\infty + \ldots + A_\nu/z_\infty{}^{\nu+1}$ at ∞ in the local parameter z_∞, whereas (4.8) is the differential equation of (4.6), if one views the a_α in (4.8) as differentials. On the other hand, adopting the differential point of view, (4.8) is a set of linear equations whose coefficients are the values of a *set of p not necessarily finite quadratic differentials* at the branch points. These quadratic differentials, $T'_\infty(u)\,\zeta'_i(u)$, $i = 1, \ldots, p$, have poles at most at ∞. Since $Z'_1, \ldots, Z'_p$, where the Z are defined in (4.2), form a basis equivalent to $\zeta'_1, \ldots, \zeta'_p$ one may, for a study of linear dependence, replace the $T'_\infty(u)\,\zeta'_i(u)$ by the $T'_\infty(u)\,Z'_i(u)$ where $z = z_\infty$ in (4.2).

The polar order of $T'_\infty(u)$ at ∞ is $\nu + 2$. Therefore, when the zero of Z_i at ∞ has order at least $\nu + 2$, then $T'_\infty(u)Z'_i(u)$ is a *finite* quadratic differential. Thus there are, in the bracket notation customary in the Riemann-Roch theorem,

$$\left\{\frac{d\zeta}{P_\infty{}^{\nu+2}}\right\}$$

independent finite quadratic differentials among $T'_\infty(u)\,\zeta'_i(u)$, where $P_\infty \equiv \infty$, and they are precisely

$$Q_i(u) = T'_\infty(u)\,Z_i(u), \quad i = i_0, \ldots, p, \tag{4.9}$$

where $r_{i_0} \geqslant \nu + 2$.

If S_w is not hyperelliptic (the modifications necessary when S_w is hyperelliptic are obvious) one can write

$$Q_i(u) = \sum_{(j,k)\in I} b_{ijk}\,\zeta'_j(u)\,\zeta'_k(u), \quad i = 1, \ldots, p. \tag{4.10}$$

One can also write

$$Z'_i(u) = \sum_{j=1}^{p} d_{ij}\,\zeta'_j(u), \quad i = 1, \ldots, p.$$

Multiplying the jth equation of (4.8) by d_{ij} and summing, then substituting (4.10) in the new equations one obtains

$$0 = \sum_{(j,k)\in I} b_{ijk} \sum_{\alpha=1}^{n+2p-3} a_\alpha \zeta'_j(u_\alpha)\,\zeta'_k(u_\alpha)$$

$$= \sum_{(j,k)\in I} b_{ijk}\delta\pi_{jk}, \quad i = i_0, \ldots, p, \tag{4.11}$$

where the last equation results from (4.1).

Now the $p - i_0 + 1 = \{d\zeta/P_\infty^{\nu+2}\}$ Q_i defined by (4.9) are manifestly linearly independent. Let m be the number of them not vanishing at all branch points; then (4.11) shows that the dimension of the set of surfaces for which (4.6) and (4.7) are satisfied is not $n + 2p - 3 - (l - 1)$ in the significant case when $l \geqslant m$ as one would believe but $n + 2p - 3 - m$ if $n + 1$ is a gap and $n + 2p - 4 - m$ if $n + 1$ is not a gap (see Rauch [12], section 5), and

$$m \leqslant \{d\zeta/P_\infty^{\nu+2}\}$$

by definition.

Summarizing, one has (see Theorem 3 of Rauch [12])

THEOREM 4. *If S_w is part of an analytic family* (4.7) *satisfying* (4.6) *with $l \geqslant m$, i.e. having functions belonging to $\nu + 1$ with assigned principal part then the dimension of the family, assuming S_w is as described earlier, is $n + 2p - 3 - m$ or $n + 2p - 4 - m$ at S_w according as $n + 1$ is or is not a gap. Here, m is the number of linearly independent quadratic differentials of the form* (4.9) *not vanishing at all branch points. Finally, $m \leqslant \{d\zeta/P_\infty^{\nu+2}\}$.*

One sees how (4.8) and (4.11) represent the principle of correct counting of constants.

In particular, in comparing with the quoted reference to Hensel-Landsberg, one sees that when $\nu + 1 = n + r$ where $n + r$ is the smallest member of the Weierstrass sequence relatively prime to n one obtains their count only if $m = \{d\zeta/P_\infty^{n+r+1}\}$ and $n + r + 1$ is not a gap (the usual situation). It would be interesting to know if in this case m must necessarily equal $\{d\zeta/P_\infty^{n+r+1}\}$.

If $\nu + 1$ is not prime to n then the functions whose existence is supposed are really defined on a surface whose number of sheets is a divisor of n, and new phenomena occur.

5. Appendix: proof of abstract variational formulae. To prove (3.2) it will be convenient to refer to Rauch [12] for notation and some formulae.

Let S of $S(T)$ be uniformized by a Schottky uniformization (cf. Fatou [5]) in the z-plane. Under the uniformization μ on S is defined in the entire plane minus the limit points of the Schottky group G and transforms properly under G. Bers (oral communication based on forthcoming paper) has shown that the Beltrami equation

$$w_{\bar{z}} = \epsilon \mu w_z$$

under reasonable regularity assumptions admits a solution which is a homeomorphism on the plane minus the limit points; admits G, i.e. $w(g(z)) = g'w(z)$ where $g \in G$ and $g' \in G'$, the Schottky group of the new Riemann surface generated by μ; is properly normalized and reduces to the identity for $\epsilon = 0$; and is C^∞ in ϵ.

Now pick a fixed fundamental domain Δ and perform a Riemann dissection (not a canonical dissection) using arcs in Δ and its boundary curves. Let Γ be the dissection run through in the usual order. The interior of Γ is then defined and simply connected.

In Δ consider $\zeta^*_i(z) - \zeta_i(z)$ where ζ^*_i and in general all starred quantities refer to the surface S^* generated by μ.

By the bilinear identity and Cauchy's theorem one has

$$\begin{aligned}&(\zeta^*_i(z) - \zeta_i(z)) - (\zeta^*_i(w) - \zeta_i(w)) \\ &= -\frac{1}{2\pi i}\int_\Gamma (\zeta^*_i(t) - \zeta_i(t))\,\frac{\partial \eta_{zw}}{\partial t}\,(t)\,dt. \end{aligned} \tag{5.1}$$

Now

$$\int_\Gamma \zeta_i(t)\,\frac{\partial \eta_{zw}}{\partial t}\,(t)\,dt = \int_{\delta_i} \frac{\partial \eta_{zw}}{\partial t}\,(t)\,dt. \tag{5.2}$$

Now let $t^*(t)$ be the solution of the Beltrami equation described above (with t^* and t replacing w and z). One has

$$\zeta^*_i(t^*) = \zeta^*_i(t) + \epsilon\,\omega\zeta^{*\prime}_i(t) + O(\epsilon^2), \tag{5.3}$$

where $t^*(t) = t + \epsilon\,\omega(t) + O(\epsilon^2)$.

Now the bilinear identity also proves that

$$\int_\Gamma \zeta^*(t^*(t))\,\frac{\partial \eta_{zw}}{\partial t}\,(t)\,dt = \int_{\delta_i} \frac{\partial \eta_{zw}}{\partial t}\,(t)\,dt, \tag{5.4}$$

because t^* lies on Δ^*, the fundamental domain of S^*. Hence, substituting (5.4), (5.2) and (5.3) in (5.1), one obtains

$$\begin{aligned}&(\zeta^*{}_i(z) - \zeta_i(z)) - (\zeta^*{}_i(w) - \zeta_i(w))\\ &= \frac{\epsilon}{2\pi i}\int_\Gamma \zeta^{*\prime}{}_i(t)\,\omega(t)\,\frac{\partial \eta_{zw}}{\partial t}\,(t)\,dt + O(\epsilon^2).\end{aligned} \tag{5.5}$$

Differentiating (5.5) with respect to z, one finds that $\zeta^{*\prime}{}_i(z) - \zeta'_i(z)$ is $O(\epsilon)$ so that one can substitute back in (5.5) and replace $\zeta^{*\prime}{}_i$ there by ζ'_i.

Differentiating the revised (5.5) with respect to z and integrating over δ_j one obtains

$$\delta\pi_{ij} = \frac{\epsilon}{2\pi i}\int_\Gamma \zeta'_i(t)\,\omega(t)\left[\int_{\delta_j} \frac{\partial^2 \eta_{zw}}{\partial z\,\partial t}\,(t)dz\right] dt, \tag{5.6}$$

which, by the exchange of argument and parameter,

$$\eta_{zw}(t) - \eta_{zw}(u) = \eta_{ut}(w) - \eta_{ut}(z),$$

and the resulting

$$\frac{\partial^2 \eta_{zw}(t)}{\partial z\,\partial t} = -\frac{\partial^2 \eta_{ut}(z)}{\partial t\,\partial z}$$

and formula (14) of Rauch [12] becomes

$$\delta\pi_{ij} = -\epsilon\int_\Gamma \zeta'_i(t)\,\zeta'_j(t)\,\omega(t)\,dt.$$

Applying Stokes's theorem, one obtains

$$\delta\pi_{ij} = \epsilon\iint_S \zeta'_i(t)\,\zeta'_j(t)\;\omega_{\bar{t}}(t)\,dt\wedge d\bar{t}. \tag{5.7}$$

Now substituting $t^*(t) = t + \epsilon\,\omega(t) + O(\epsilon^2)$ in $t^*_{\bar{t}} = \epsilon\,\mu\,t^*_t$ one obtains

$$\epsilon\,\omega_{\bar{t}} + O(\epsilon^2) = \epsilon\,\mu(1 + \epsilon\,\omega_t + O(\epsilon^2)).$$

Comparing, one has $\omega_{\bar{t}} = \mu$, and substituting in (5.7) one obtains (3.2).

This is a revision of the proof in Rauch [10].

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UNIFORMIZATION AND MODULI*

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BELTRAMI equations with discontinuous coefficients, first considered by Morrey (cf. [3] and the references given there), are a useful tool in studying uniformization and moduli of Riemann surfaces. The resulting theory of moduli is, of course, restricted to the classical case; on the other hand, some theorems can be proved also for open Riemann surfaces.

A complete presentation is impossible in the time allotted. We shall, therefore, only state a central result pertaining to closed surfaces, (§1), make some comments on it (§2), and derive a new uniformization theorem which is crucial for the proof and is of independent interest (§3). Proofs are outlined in [6, 7, 8]; a detailed presentation is in preparation.

1. Statement of the main theorem. The theorem stated below asserts, roughly speaking, the possibility of simultaneous uniformization of all algebraic curves of a given genus $g > 1$. It should be compared with the corresponding statements on the Weierstrass functions $\wp(z, 1, \tau)$ and $\wp'(z, 1, \tau)$, $|z| < \infty$, $\operatorname{Im} \tau > 0$, which uniformize the curves of genus 1.

THEOREM. *Let $g > 1$ be a given integer. There exist*

(1) *a bounded domain T in the complex number space C^{3g-3} (with coordinates $\tau_1, \ldots, \tau_{3g-3}$) which is homeomorphic to a cell,*

(2) *a domain $M \subset C^{3g-2}$ (with coordinates $z, \tau_1, \ldots, \tau_{3g-3}$) which is homeomorphic to a cell and holomorphically equivalent to a bounded domain,*

(3) *a continuous complex-valued function $\sigma(t, \tau)$, $-\infty < t + \infty$, $\tau \in T$, such that $\sigma(t, \tau)$ is holomorphic in τ for every fixed t and $\sigma(t, \tau) \to \infty$ for τ fixed and $|t| \to \infty$,*

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(4) *a properly discontinuous fixed-point-free group G of holomorphic automorphisms of M,*

(5) *a properly discontinuous (but not fixed-point-free) group Γ of holomorphic automorphisms of T,*

(6) *a holomorphic mapping $\tau \to Z(\tau)$ of T into the Siegel space of complex symmetric $g \times g$ matrices $Z = X + iY$, $Y > 0$, and*

(7) *a finite number of meromorphic functions $F_j(z, \tau)$ defined in M and automorphic under G,*

such that the following statements are valid.

(8) *For every $\tau \in T$, the curve $\gamma(\tau): z = \sigma(t, \tau)$, $-\infty < t < +\infty$ is the boundary curve of a simply connected domain $\mathcal{D}(\tau)$ in the z-plane.*

(9) *A point $(z, \tau_1, \dots, \tau_{3g-3}) = (z, \tau)$ belongs to M if and only if $\tau \in T$ and $z \in \mathcal{D}(\tau)$.*

(10) *Every element of G is of the form*

$$\tau \to \tau,$$

$$z \to \frac{a(\tau)\, z + b(\tau)}{c(\tau)\, z + d(\tau)}$$

where a, b, c, d, are holomorphic functions on T and $ad - bc = 1$. G is generated by $2g$ elements $A_1, \dots, A_g, B_1, \dots, B_g$ with $\Pi[A_j, B_j] = 1$, where $[A, B] = ABA^{-1}B^{-1}$.

(11) *Denote by $G(\tau)$ the restriction of G to a fixed τ, (so that $G(\tau)$ is a group of Möbius transformation of $\mathcal{D}(\tau)$ onto itself). The Riemann surface $S(\tau) = \mathcal{D}(\tau)/G(\tau)$ is a closed surface of genus g.*

(12) *Every closed surface of genus g is conformally equivalent to an $S(\tau)$, and $S(\tau')$ and $S(\tau'')$ are conformally equivalent if and only if τ' and τ'' are equivalent under Γ.*

(13) *The matrix $Z(\tau)$ is a 'Riemann period matrix' of $S(\tau)$.*

Finally,

(14) *the restrictions of the functions F_j to a fixed τ generate the field of automorphic functions in $\mathcal{D}(\tau)$, under $G(\tau)$, i.e. the field of meromorphic functions on $S(\tau)$.*

We note the

COROLLARY I. *T/Γ is a normal complex space.*

This follows from (5) and a theorem of H. Cartan [9]. Quite different proofs are due to Röhrl [12] and to Baily [4]. Baily proved also that T/Γ is a Zariski open set in an algebraic variety. T/Γ is, of course, the space of conformal equivalence classes of Riemann surfaces of genus g (cf. 12).

COROLLARY II. *M/G is a complex manifold, the natural mapping $M/G \to T$ is holomorphic, and the inverse image of $\tau \in T$ is a regularly embedded subvariety of M/G conformally equivalent to $S(\tau)$.*

In the same way one can construct complex fiber spaces over T for which the fiber over τ is $S(\tau) \times \ldots \times S(\tau)$ or the Jacobian variety of $S(\tau)$.

2. Comments. We shall describe the elements entering in the proof.

(a) NOTATIONS. The letter S will always denote an abstract Riemann surface. S is called exceptional if it admits conformal self-mappings homotopic to but distinct from the identity. A *non-exceptional* S can be represented as U/G where U is the upper half plane and G a Fuchsian group. S is said to be of the *first kind* if the fixed points of G are dense on the real axis.

In particular, let S be *of type* (g, n), i.e. obtained from a closed surface of genus $g \geqslant 0$ by removing $n \geqslant 0$ distinct points. Then S is non-exceptional (and of the first kind) if $3g - 3 + n > 0$.

Let m be a differential of type $(-1, 1)$ on S_0; locally, $m = \mu(\zeta)d\bar{\zeta}/d\zeta$ where μ is measurable and ζ a local uniformizer. Since $|\mu|$ is a scalar we may define: $\|m\|$ = essential supremum of $|\mu|$. If $\|m\| < 1$ we write $m \in B(S_0)$ and call m a *Beltrami differential.* In this case S_0^m denotes the surface S_0 with the conformal structure redefined by the conditon: every solution of the *Beltrami equation* $\omega_{\bar{\zeta}} = \mu\omega_\zeta$ is holomorphic on S_0^m. (A solution is required to be continuous and to have locally square-integrable generalized derivatives.) The natural mapping of S_0^m onto S_0 will be denoted by 1.

A homeomorphism onto, $S \xrightarrow{f} S_0$, is called *quasiconformal* if it can be factored as follows: $S \xrightarrow{h} S_0^m \xrightarrow{1} S_0$, h being conformal. Assuming that this is so, let $[f]$ denote the homotopy class of f. We call $(S, [f], S_0)$ an *even coupled pair*. Two such pairs, $(S, [f], S_0)$ and $(S', [f'], S'_0)$ are called *equivalent* if there exist conformal mappings $S' \xrightarrow{h} S$, $S'_0 \xrightarrow{h_0} S_0$ with $[h_0^{-1} f h] = [f']$, *strongly equivalent* if $S'_0 = S_0$ and h_0 may be chosen as the identity. Every even pair is equivalent to one of the form $(S_0^m, [1], S_0)$.

(b) Let S_0 be non-exceptional. The set of strong equivalence classes of even pairs $(S, [f], S_0)$ is the *Teichmüller space* $T(S_0)$. The *Teichmüller distance* between $(S, [f], S_0)$ and $(S', [g], S_0)$ is $\log [(1+k)/(1-k)]$ where $k = \inf \| m \|$ for $m \in B(S)$, $(S^m, [1], S)$ strongly equivalent to $(S', [f^{-1}g], S)$.

A continuous complex valued function Φ on $T(S_0)$ will be called *complex-analytic* or holomorphic if for every set $(m_1, \ldots, m_r) \subset B(S)$, where $(S, [f], S_0) \in T(S_0)$, the mapping of a neighborhood of $0 \in C^r$ into C given by

$$(\zeta_1, \ldots, \zeta_r) \to p = (S^{\zeta_1 m_1 + \ldots + \zeta_r m_r}, [f], S_0) \to \Phi(p)$$

is holomorphic. *Real analyticity* is defined similarly.

A quasiconformal mapping $S_1 \xrightarrow{g} S_0$ induces an 'allowable mapping' g_* of $T(S_1)$ onto $T(S_0)$:

$$g_*((S, [f], S_1)) = (S, [gf], S_0);$$

g_* depends only on $[g]$ and preserves the Teichmüller distance and real and complex analyticity. The group of allowable *self-mappings* of $T(S_0)$ will be denoted by $\Gamma(S_0)$.

Using uniformization by Fuchsian groups one proves: $T(S_0)$ is a *complete metric space*; if the fundamental group of S_0 is finitely generated $\Gamma(S_0)$ is a *properly discontinuous* group. Real analytic functions on $T(S_0)$ *separate points*. Using the new uniformization theorem stated in §3 one proves: if S_0 is of the *first kind*, then holomorphic functions on $T(S_0)$ *separate points*.

(c) If S_0 is of type (g, n) we write $T(S_0) = T_{g,n}$, $\Gamma(S_0) = \Gamma_{g,n}$. This notation is justified by the observation that any two surfaces of type (g, n) are *quasiconformally equivalent*. We set $\rho = 3g - 3 + n$, and assume that $\rho > 0$.

The Teichmüller theory [1, 5, 13, 14] of extremal quasiconformal mappings implies that $T_{g,n}$ *is a* 2ρ*-cell*. Also, it is known that $T_{g,n}$ *is a complex analytic-manifold* of dimension ρ (proved first by Ahlfors [2], cf. also [6, 10, 11, 15]).

In our theorem T stands for $T_{g,0}$, Γ for $\Gamma_{g,0}$, and M for $T_{g,1}$. The preceding remarks contain the justification of some of our statements. The existence of the mapping described in (6), (13) follows from Rauch's variational formula [11].

(d) In the statement of our theorem $T = T_g$ appears not as an abstract complex-analytic manifold but as a bounded domain. This is a special case of a more general result: $T_{g,n}$ *is* (holormophically equivalent to) *a bounded domain* in C^ρ. The proof (sketched in [7]) is rather complicated. It is based on the possibility of uniformizing every closed Riemann surface by *Schottky groups*, and involves a detailed geometrical analysis of the ' Schottky space ' of which T_g is the universal covering space. The proof proceeds by induction on g and in the induction argument a special role is played by hyperelliptic surfaces.

(e) The representation of $T_{g,1}$ in the form M, i.e. in the form described by statements (8) and (9), and the construction of the group G having the properties (4), (10), (11), are based on the uniformization theorem of § 3.

(f) Assuming the preceding steps to be accomplished, the completion of the proof, i.e. the construction of the functions F_j having properties (7), (14) is not difficult. We choose a fixed set of generators A_j, B_j, of G (cf. 10)). These define on each $S(\tau)$ a homology basis, which we denote by the same letters. Let ω_j be the Abelian differential of the first kind which has A_k-periods δ_{jk} (so that, by the way, the B_k-period of ω_j is the element Z_{jk} of $Z(\tau)$). Let Ω_{jk} be the Abelian differential of the third kind on

$S(\tau)$ which has all A_j-periods 0 and such that at any point of $S(\tau)$ the residue of Ω_{jk} equals the order of ω_j/ω_k. The set of functions ω_j/ω_k, Ω_{jk}/ω_l, $j, k, l, = 1, 2, \ldots, g$, considered as functions on $\mathscr{D}(\tau)$, has the required properties.

REMARK. The theorem of § 1 is, unfortunately, rather "existential". One would like to have explicit expressions for the domains and functions described. I hesitate to state whether there is much hope for obtaining such formulas.

3. A new uniformization theorem. A group G of Möbius transformations will be called *quasi-Fuchsian* if there exists on the Riemann sphere a directed Jordan curve γ_G such that γ_G is fixed under G, and G is fixed-point-free and properly discontinuous in the domains $I(\gamma_G)$ and $E(\gamma_G)$ interior and exterior to γ_G, respectively.

THEOREM I. *Let S_1 and S_2 be two Riemann surfaces. Assume that S_1 and S_2 have hyperbolic universal covering surfaces, and that S_1 is quasiconformally equivalent to the mirror image $\overline{S}_2$ of S_2. Then there exists a quasi-Fuchsian group G such that $I(\gamma_G)/G$ is conformally equivalent to S_1 and $E(\gamma_G)/G$ to S_2.*

REMARK. $\overline{S}$ is defined by replacing each local uniformizer ζ on S by its complex conjugate $\bar{\zeta}$. The hypothesis for I are satisfied if S_1 and S_2 are closed and of the same genus > 1.

PROOF. Set $S_0 = \overline{S}_2$. Then $S_1 = S_0^m$ for some $m \in B(S_0)$. By hypothesis $S_0 = U/G_0$ where U is the upper half-plane and G_0 a fixed-point-free Fuchsian group. Then $L/G_0 = \overline{S}_0$, L being the lower half-plane. Set $\mu(z) \equiv 0$ for $\operatorname{Im} z \leqslant 0$ and define $\mu(z)$ for $\operatorname{Im} z > 0$ by the condition: $\mu(z)\, d\bar{z}/dz = m$. Then $|\mu(z)| < k < 1$ and

$$\mu(A(z))\ \overline{A'(z)} = \mu(z)\ A'(z) \text{ for } A \in G_0.$$

There exists one and only one homeomorphism w^μ of the plane onto itself which leaves 0 and 1 fixed and is μ-conformal, i.e. a solution of the Beltrami equation:

$$\partial w/\partial \bar{z} = \mu(z)\, \partial w/\partial z.$$

If $A \in G_0$, then the functional equation for μ implies that $w^\mu(A(z))$ is also a μ-conformal automorphism of the Riemann sphere so that

$$A^\mu = w^\mu A (w^\mu)^{-1}$$

is a Möbius transformation. One verifies that $G = w^\mu G_0 (w^\mu)^{-1}$ is the desired quasi-Fuchsian group.

We denote the natural mapping of $\bar{S}_0$ onto S_0 by ι. A homeomorphism $S \xrightarrow{h} \bar{S}_0$ is called *anti-quasiconformal* if it can be factored as follows: $S \xrightarrow{h} S_0^m \xrightarrow{\iota} \bar{S}_0$, h being conformal. Such an f defines an *odd coupled pair* $(S, [f], \bar{S}_0)$. Every odd coupled pair is equivalent, in the sense of § 2 (a), to one of the form $(S_0^m\ [\iota], \bar{S}_0)$.

Let G be a quasi-Fuchsian group and set $\Lambda = \Lambda_1/\Lambda_0$ where Λ_1 is the group of automorphisms of G and Λ_0 the subgroup of inner automorphisms. Since G can be identified with the fundamental groups of $I(\gamma_G)/G$ and of $E(\gamma_G)/G$ every homeomorphism

$$I(\gamma_G)/G \xrightarrow{\phi} E(\gamma_G)/G$$

induces an element λ_ϕ of Λ, and $\lambda_\phi = \lambda_\psi$ if and only if $[\phi] = [\psi]$. If $\lambda_\phi = 1$ and ϕ is anti-quasiconformal we say that G *represents* the pair $(I(\gamma_G)/G, [\phi], E(\gamma_G)/G)$, and any pair equivalent to it.

Re-examining the proof of I, we see that we actually proved the first part of the

THEOREM II. *Any odd coupled pair* $(S, [f], \bar{S}_0)$ *can be represented by a quasi-Fuchsian group* G, *provided that* S_0 *has a hyperbolic universal covering surface. If* S_0 *is of the first kind, then every quasi-Fuchsian group* G_1 *representing this pair is of the form* $G_1 = QGQ^{-1}$, Q *a Möbius transformation.*

In proving the second statement we may assume that $S = S_0^m$ $f = \iota$, and G is the group constructed above. The assumption on G_1 implies the existence of conformal homeomorphisms:

$$I(\gamma_G) \cup \gamma_G \xrightarrow{\phi} I(\gamma_{G_1}) \cup \gamma_{G_1},\ E(\gamma_G) \cup \gamma_G \xrightarrow{\psi} E(\gamma_{G_1}) \cup \gamma_{G_1}$$

with

$$\phi\, A\, \phi^{-1} = \psi\, A\, \psi^{-1} \text{ on } \gamma_G \text{ for } A \in G.$$

The last relation implies that $\phi = \psi$ at all fixed points of G and thus, by continuity everywhere on γ_G. Set $Q(z) = \phi(z)$ for $z \in I(\gamma_G) \cup \gamma_G$, $Q(z) = \psi(z)$ for $z \in E(\gamma_G)$. Then Q is an automorphism of the plane and $G_1 = QGQ^{-1}$.

It remains to show that $Q(z)$ is holomorphic. This would be immediate if γ_G were rectifiable. Under the present circumstances, however, we must consider $Q(z)$ as a function of $\zeta = w^\mu(z)$ and use the properties of w^μ given, for instance, in [3]. These properties imply that mes $\gamma_G = 0$ and that $Q(z)$ has generalized L_2 derivatives in a neighborhood of γ_G. Since $\partial Q/\partial \bar{z} = 0$ off γ_G, the analyticity of Q everywhere follows by well-known arguments.

We are now in a position to construct the domain M described in §1. In the proof of I let S_0 be a fixed closed surface of genus $g > 1$. For every

$$(S_0^m, [f], S_0) = \tau \in T_g = T(S_0)$$

let w^μ, A^μ and G have the same meaning as before. If we choose G_0 so that 0, 1, ∞ are fixed points, the uniqueness part of II shows that A^μ and G and depend only on τ and not on the particular choice of m. We may set $G = G(\tau)$, $\gamma_G = \gamma(\tau)$, $I(\gamma_G) = \mathcal{D}(\tau)$. Then (11) holds and $\sigma(t, \tau) = w^\mu(t)$.

The fact that A^μ and σ depend holomorphically on τ follows from a result proved in [3] : if μ depends holomorphically on *complex parameters* so does $w^\mu(z)$.

We conclude by stating two problems. Does every quasi-Fuchsian group represent a coupled pair ? Can one prove Theorem I "classically", i.e. by using only conformal transformations ?

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ON THE MODULI OF JACOBIAN VARIETIES AND CURVES

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INTRODUCTION. It is our purpose here to realize the family of normally polarized Abelian varieties of a given dimension n as being a Zariski open subset of a certain irreducible, projective, algebraic family, and then to show that the subfamily consisting of those Abelian varieties of dimension n which are the canonically polarized Jacobian varieties of Riemann surfaces of genus n is an irreducible algebraic subfamily of the family of Abelian varieties (or, more properly speaking, a Zariski open subset of such a sub-family). In view of Torelli's theorem, which says that two curves of genus n are conformally equivalent if and only if their canonically polarized Jacobians are isomorphic, we have as a corollary the fact that the space of moduli of Riemann surfaces of genus n is (naturally identifiable with) a Zariski open subset of a projective algebraic variety.

Our principal tools in this proof, aside from the theory of θ-functions and the Chow point, are a criterion of Matsusaka [16] for an Abelian variety to be a Jacobian variety and some useful generalizations of Matsusaka's criterion due to Hoyt [9]; we shall also have occasion to use results of Ahlfors, Bers, Teichmüller, and Weil [1, 5, 22, 23] in showing that the algebraic family of Jacobian varieties is irreducible.

1. The algebraic family of Abelian varieties. By a polarized Abelian variety we mean an Abelian variety A together with a distinguished family of positive divisors on A which are numerically equivalent to the rational scalar multiples of a fixed non-degenerate positive divisor [24]. The choice of such a distinguished family of divisors is equivalent to choosing a family of principal matrices for A which are all positive rational scalar multiples of a fixed principal matrix. If the elementary divisors of the latter are all equal, A is said to be normally polarized. It is well known that every canonically polarized Jacobian variety is normally polarized [24].

We first construct an irreducible algebraic family of projective varieties, whose general member is an Abelian variety, such that every normally polarized Abelian variety of dimension n occurs at least once in the family, and such that there is a finite group G of automorphisms of the parameter variety having the property that two points of the parameter variety correspond to isomorphic polarized Abelian varieties if and only if they belong to the same orbit of G.

Let H_n be the space of symmetric, complex $n \times n$ matrices $Z = X + iY$ such that Y is positive definite. Let M_n denote the generalized modular group of degree n consisting of the real, integral, unimodular $2n \times 2n$ symplectic matrices acting on H_n as follows: If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$ (A, B, C, D being $n \times n$), $Z \in H_n$, then $gZ = (AZ + B)(CZ + D)^{-1}$. If m is a positive integer, we let $M_n(m) = \{g \in M_n \mid g \equiv E \bmod m\}$, E being the $2n \times 2n$ identity matrix. We denote H_n/M_n by V_n and $H_n/M_n(m)$ by $V_n(m)$.

It has been shown in [2, 21, 6] that $V_n(m)$ has a natural compactification $V_n(m)^*$ which is a normal general analytic space and which may be isomorphically imbedded as such in a complex projective space of sufficiently high dimension. Indeed, if $\Phi_0, \ldots, \Phi_M$ is a basis of modular forms of a suitable high weight with respect to $M_n(m)$, then such an imbedding

$$\Phi : V_n(m)^* \to CP^M$$

may be defined by $\Phi(x) = [\Phi_0(x) : \ldots : \Phi_m(x)]$. Since $M_n(m)$ is trivially a normal subgroup of $M_n = M_n(1)$, it is not hard to see that the quotient group $G = M_n/M_n(m)$ is a finite group of automorphisms of $V_n(m)^*$ and that $V_n(m)^*/G = V_n^*$ (we let $V_n(1)^* = V_n^*$).

For $Z \in H_n$ we denote by L_Z the lattice in C^n generated by the columns of the $n \times 2n$ matrix (E, Z), E being the $n \times n$ identity matrix. We consider the product space $H_n \times C^n$ and form the quotient $(H_n \times C^n)/L$, where we identify (Z, ζ) and (Z', ζ') if and only if $Z = Z'$ and $\zeta - \zeta' \in L_Z$. Then it may be seen that

the group M_n acts naturally on $H_n \times C^n$ as follows: If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\in M_n$, we define $g(Z, \zeta) = (gZ, {}^t(CZ + D)^{-1}\zeta)$, ^{t} denoting the transpose of a matrix. Since g is unimodular and symplectic, it follows easily that $\zeta \in L_Z$ if and only if ${}^t(CZ + D)^{-1}\zeta \in L_{gZ}$. Therefore g acts naturally on $(H_n \times C^n)/L$.

Now let m be any positive integer and let $\Theta(\zeta; Z)$ be any mth order θ-function with respect to the period lattice generated by the columns of (E, Z). Then $\Theta = \Theta(\zeta; Z)$ is an analytic function in the product space $H_n \times C^n$ and satisfies the quasiperiodicity conditions

$$\Theta(\zeta + a; Z) = \Theta(\zeta; Z)$$

$$\Theta(\zeta + Za; Z) = \epsilon(-m\,{}^t a\,\zeta - \tfrac{1}{2}mZ[a])\,\Theta(\zeta; Z),$$

where ζ is the vector with components $(\zeta_1, \ldots, \zeta_n)$, a is any integral n-vector, $Z[a] = {}^t aZa$, and for any matrix X, $\epsilon(X) = e^{2\pi i tr(X)}$. It is well known [15] that for $m \geqslant 3$ a basis $\Theta_0, \ldots, \Theta_n$ of the θ-functions of the mth order can be used to define an imbedding T of the Abelian variety C^n/L_Z into CP^N by

$$\zeta \to [\Theta_0(\zeta; Z) : \ldots : \Theta_N(\zeta; Z)].$$

Now it follows from the classical theory of θ-functions [13, 14] that if m' is a suitable multiple of m (e.g. $m' = 2(m!)^4$), if $g \in M_n(m')$, and if Θ_1, Θ_2 are any θ-functions of the mth order $(\Theta_2 \not\equiv 0)$ then we have $\Theta_1(g(\zeta; Z)) / \Theta_2(g(\zeta; Z)) = \Theta_1(\zeta; Z) / \Theta_2(\zeta; Z)$; it follows easily from this that if $\Theta_1, \ldots, \Theta_{n+1}$ are a maximal set of algebraically independent mth order θ-functions and if Θ is any other mth order θ-function, then $\Theta, \Theta_1, \ldots, \Theta_{n+1}$ satisfy a non-trivial homogeneous algebraic equation with coefficients which are homogeneous polynomials in modular forms of a suitable weight with respect to $M_n(m')$.

It is a simple consequence of these and the preceding facts that we may define a mapping $\Phi \times T$ of $H_n \times C^n$ into $CP^M \times CP^N$ such that if U denotes the image of $H_n \times C^n$ under this mapping and if ρ denotes the projection of $CP^M \times CP^N$ onto its first factor, then

(1) U is a Zariski-open subset of an algebraic variety V of dimension $n + \frac{1}{2}n(n+1)$.

(2) The set $\rho(V)$ is naturally identifiable with $V_n(m')^*$.

(3) If $c \in V_n(m') \subset V_n(m')^*$, then $\mathfrak{A}(c) = \rho^{-1}(c) \cap V$ is the Abelian variety C^n/L_Z for a representative Z of c in H_n.

2. The main result. The following theorems are due respectively to Matsusaka [16] and Hoyt [9]:

(1) *Let A be an n-dimensional Abelian variety. Suppose X is an irreducible divisor and C a positive (complex) 1-cycle on A such that $X^{(n-1)} \equiv (n-1)!\ C$ and $\deg(X^{(n)}) = n!$. Then C is irreducible and non-singular, A is the Jacobian variety of C, C is canonically imbedded in A, and X is a canonical divisor on A.*

(2) *If (A, C, X) are as above, and if the triple (B, D, Y) is a specialization of (A, C, X) such that B is an Abelian variety of dimension n, then the above numerical equivalences are preserved, D is a union of irreducible non-singular curves $D_1, \ldots, D_t$, and B is the product of the Jacobian varieties of $D_1, \ldots, D_t$.*

Here $\equiv$ denotes numerical equivalence and $X^{(\nu)}$ is the ν-fold self-intersection of X. The converse of (1) is also true [16].

Going back to the notation of the preceding section, we first of all imbed $CP^M \times CP^N$ in CP^L, $L = MN+M+N$, in the usual way and then call all objects in the image of $CP^M \times CP^N$ by the same names as they had before in $CP^M \times CP^N$. Let $E(m')$ denote the subset of $V_n(m')$ consisting of images of those $Z \in H_n$ such that C^n/L_Z is a Jacobian variety, and denote by $J(m')$ the smallest subvariety of $V_n(m')^*$ containing $E(m')$. Let $c \in E(m')$ and let C be a canonically imbedded curve on its Jacobian variety $\mathfrak{A}(c)$. Let $\Theta(\zeta; Z)$ be the standard Riemann θ-function of 1st order. It may be easily seen that the divisor of Θ induces a divisor D on $\rho^{-1}(V_n(m')) \cap V$, that the closure of D is an analytic variety (Θ) on all of V, and that $(\Theta) \cap \mathfrak{A}(c)$ is a canonical divisor $X = X(c)$ on $\mathfrak{A}(c)$. We have $\dim C = 1$, $\dim \mathfrak{A}(c) = n$, $\dim X(c) = n - 1$, and it may be easily verified that as

varieties in the projective space CP^L we have $\deg C = mn$, $\deg X(c) = m^{n-1}n!$, and $\deg \mathfrak{A}(c) = m^n n!$. For the sake of convenience we call the sets $\rho^{-1}(c) \cap V$ for $c \in V_n(m')^*$ the fibres of the mapping ρ. Then it may be verified by known properties of the Chow point [7] that the totality of Chow points of varieties of dimension μ and of degree ν in CP^L which lie on a fibre of ρ for all fibres of ρ forms an algebraic variety which we denote by $\Gamma_{\mu\nu}$. We let $\Gamma_{1,mn} = \mathscr{C}$, $\Gamma_{n,m^n n!} = \mathscr{A}$, and denote by $\mathscr{X}$ the subvariety of $\Gamma_{n-1,m^{n-1}n!}$ consisting of just those Chow points of varieties which lie wholly on (Θ). For each μ, ν we have a natural regular mapping $\tau_{\mu\nu}$: $\Gamma_{\mu,\nu} \to V_n(m')^*$ which assigns to each Chow point p the base point of the fibre in which the variety lies of which p is the Chow point. We denote the corresponding maps for $\mathscr{A}, \mathscr{C}, \mathscr{X}$ by τ_1, τ_2, and τ_3 respectively. We let R^* be the analytic subvariety of $\mathscr{A} \times \mathscr{C} \times \mathscr{X}$ defined by

$$R^* = \{ (\alpha, \gamma, \xi) \mid \tau_1(\alpha) = \tau_2(\gamma) = \tau_3(\xi) \}.$$

There is a natural mapping $\tau : R^* \to V_n(m')^*$, and if $(\alpha, \gamma, \xi) \in R^*$ is such that $\tau(\alpha, \gamma, \xi) = c \in V_n(m')$, then (α, γ, ξ) is the Chow point of a triple (A, C, X), where $A = \mathfrak{A}(c)$ is an Abelian variety, C is a curve on A, and X is the divisor of the "standard" Riemann θ-function on A. Let R^{**} be the smallest subvariety of R^* containing the set S of all triples (α, γ, ξ) such that A is the Jacobian variety of C, C is canonically imbedded in A, and X is a canonical divisor on A.

Now by the results of [5, 17, 22, 23] $E(m')$ is the holomorphic image of an irreducible complex analytic space. Therefore, as follows easily from its definition, $J(m')$ is irreducible. Clearly $J(m') \subset \tau(R^{**})$. Let R denote an irreducible component of R^{**} such that $J(m') \subset \tau(R)$. It is trivial from our definitions of R^{**} and $E(m')$ that $J(m') = \tau(R)$ and $E(m') \subset \tau(S)$. Our desired result will follow easily from the criteria (1) and (2) (above) if we show that for any countable field of definition k of R there exists a generic point $x = (\alpha, \gamma, \xi)$ of R with respect to k such that $x \in S$. In fact, any other point (β, δ, η) is then a specialization of (α, γ, ξ) over k, so that the triple (B, D, Y) of which (β, δ, η) is the Chow point is a specialization

of the triple (A, C, X) for which (α, γ, ξ) is the Chow point. Hence if B is an Abelian variety of dimension n, then D is a sum of non-singular curves and B is the product of their Jacobians; and D will be irreducible and B its Jacobian precisely when Y is irreducible. Therefore it is not hard to see that $E(m')$ is a Zariski-open subset of $\tau(R)$.

In order to find a generic point x of R relative to a countable field of definition, one may proceed as follows [4]. For any $(\alpha, \gamma, \xi) \in R$ denote by $\Delta(\alpha)$ the collection of 2-dimensional cohomology classes D on $A = A(\alpha)$ representing divisors on A; then an algebraic 1-cycle on A is numerically equivalent to zero if and only if $D(C)=0$ for all $D \in \Delta(\alpha)$. One may show the existence of a subset T' of $J(m')$ consisting locally of a countable collection of proper analytic subvarieties of $J(m')$ such that $\Delta(\alpha)$ is constant $= \Delta$ for $\tau_1(\alpha) = \tau(\alpha, \gamma, \xi) \notin T'$, and such that if $\tau_1(\alpha) \in T' \cap V_n(m')$, then $\Delta \subsetneq \Delta(\alpha)$. We let Σ denote the union of the set of singular points of R with the subvariety of R consisting of those (α, γ, ξ) such that the curve of which γ is the Chow point has singularities; Σ is clearly a proper subvariety of R. Then one may show by an argument involving continuous deformation of cycles together with Matsusaka's criterion (1) that if $(\alpha, \gamma, \xi) \in R - \Sigma - \tau^{-1}(T')$, then $(\alpha, \gamma, \xi) \in S$. Obtaining the desired generic point of R relative to any countable field of definition is then quite easy.

However, it now appears that another approach to this part of the problem may be employed so as to eliminate certain rather undesirable features of the above proof. Namely, Matsusaka has proved [16] that if C is a (algebraic) 1-cycle on the Abelian variety A, if X is a positive non-degenerate divisor on A, and if $\alpha(X, C)$ is the endomorphism of A canonically associated to C and X (defined by $u \to S(X.(C_u - C))$, then for C to be numerically equivalent to zero it is necessary and sufficient that $\alpha(X, C) = 0$. On the other hand, we let

$$D = \{(v_1, v_2, v_3) \in V \times V \times V \mid \rho(v_1) = \rho(v_2) = \rho(v_3)\}$$

and define G to be the closure in $V \times V \times V$ of the set

$$\{(v_1, v_2, v_3) \mid (v_1, v_2, v_3) \in D,\ \rho(v_1) \in V_n,\ v_1 + v_2 = v_3\},$$

$+$ denoting the group law in $\mathfrak{A}(\rho(v_1))$; then by using the addition theorem for θ-functions, it may be seen that G is an algebraic subvariety of $V \times V \times V$. Using this in connection with Matsusaka's criteria, it now appears nearly certain that the whole proof that $\overline{\mathscr{E}}$ is an algebraic variety can be reduced to more or less purely algebraic considerations. We hope to publish a proof along these lines in the near future. Still missing, however, is a proof, not using transcendental tools, of the irreducibility of $\overline{\mathscr{E}}$.

As remarked previously, the group $G = M_n/M_n(m')$ is a group of automorphisms of $V_n(m')^*$, and $V_n(m')^*/G \simeq V_n^*$. It is not hard to see that $E(m')$ is stable under G, and therefore the same is true of $J(m')$. Therefore, $J(m')/G = J$ is a subvariety of V_n^* and the points of a certain Zariski open subset $\mathscr{E}$ of J are in a natural one-to-one correspondence with the canonically polarized Jacobian varieties of dimension n. In view of Torelli's theorem [24] and the results of [1, 5, 22, 23] we may, with considerable justification, refer to $\mathscr{E}$ as the variety or moduli of Riemann surfaces of genus n. We shall come back to the problem of fully justifying this designation in the next section.

We now come to the problem of determining the smallest field of definition for $\overline{\mathscr{E}}$. We shall first attack the corresponding problem for $\overline{\mathscr{E}}(m')$. We have the mapping $\rho : V \to V_n(m')^*$. Suppose ρ, V, $V_n(m')^*$ are all defined over the countable field k. $\overline{\mathscr{E}}(m') \subset V_n(m')^*$. Let $K \supset k$ be a countable, finitely generated field of definition for $\overline{\mathscr{E}}(m')$. Let k_0 be the smallest field of definition for $\overline{\mathscr{E}}(m')$ which contains k (so k_0 is the composition of k and the smallest field of definition for $\overline{\mathscr{E}}(m')$). We shall prove $k_0 = k$. We may take K to be a finite algebraic extension of a purely transcendental extension K_0 of k. Let $\bar{K}$ be the Galois closure of K over K_0 and let $\mathfrak{G}$ be the finite Galois group of K over K_0. It is sufficient to show that $k_0 \subset K_0$. For let $x \in \bar{K} - k$. If $x \in \bar{K} - K_0$, then $x \notin k_0$; on the other hand, if $x \in K_0 - k$, we may extend x to a transcendence

basis $x, x_1, \ldots, x_{n-1}$ of K_0 over k, let K'_0 be the fixed field of the automorphism $(x, x_1, \ldots, x_{n-1}) \to (-x, x_1, \ldots, x_{n-1})$, replace K_0 by K'_0, K by K_0, and begin again, thus concluding that $x \notin k_0$. To show $k_0 \subset K_0$, it is sufficient to show that given any $\sigma \in \mathfrak{G}$, we have $\overline{\mathscr{E}}(m')^\sigma = \overline{\mathscr{E}}(m')$, because K_0 is of characteristic zero. So let $\sigma \in \mathfrak{G}$ let $K^* \supset \overline{K}$ be a countable field of definition for R and τ, and let (α, γ, ξ) be a generic point of R relative to K^* such that $\tau(\alpha, \gamma\, \xi)$ is a generic point of $\overline{\mathscr{E}}(m')$ relative to K^*. Extend σ to an isomorphism of K^* onto $K^{*\sigma} \subset \mathbf{C}$ (= field of complex numbers) and then extend σ further to an isomorphism of $K^*(\alpha, \gamma, \xi)$ onto $K^{*\sigma}(\alpha^\sigma, \gamma^\sigma, \xi^\sigma)$ Since $\tau(\alpha, \gamma, \xi)$ is a generic point of $\overline{\mathscr{E}}(m')$ over K^* and therefore over $\overline{K}$, it follows that $\tau^\sigma(\alpha^\sigma, \gamma^\sigma, \xi^\sigma)$ is a generic point of $\overline{\mathscr{E}}(m')^\sigma$ over $\overline{K}^\sigma = \overline{K}$. Moreover, $A^\sigma = (\rho^\sigma)^{-1}\ (\tau^\sigma(\alpha^\sigma, \gamma^\sigma, \xi^\sigma)) \cap V^\sigma$. Since ρ and V are defined over $k \subset K_0$, it follows that $A^\sigma = \rho^{-1}(\tau^\sigma(\alpha^\sigma, \gamma^\sigma, \xi^\sigma)) \cap V$. By the main result of Hoyt's thesis [9], A^σ is the Jacobian variety of C^σ which is canonically imbedded in A^σ and X^σ is a canonical divisor on A^σ. Therefore, $\tau^\sigma(\alpha^\sigma, \gamma^\sigma, \xi^\sigma) \in \overline{\mathscr{E}}(m')$. Therefore $\tau^\sigma(\alpha^\sigma, \gamma^\sigma, \xi^\sigma)$ is a specialization of $\tau(\alpha, \gamma, \xi)$ over $\overline{K}$, so $\overline{\mathscr{E}}(m')^\sigma \subset \overline{\mathscr{E}}(m')$. Since dim $\overline{\mathscr{E}}(m')^\sigma = \dim \overline{\mathscr{E}}(m')$ and $\overline{\mathscr{E}}(m')$ is irreducible, we have $\overline{\mathscr{E}}(m')^\sigma = \overline{\mathscr{E}}(m')$, as desired. To determine the smallest field of definition for $V_n(m')^*$, V, and ρ, one might first of all determine the smallest field containing the Fourier coefficients of the Eisenstien series of "Stufe" m' and then examine the transformation theory of the θ-functions. On the other hand, we may proceed as follows to determine the smallest field of definition for $\overline{\mathscr{E}}$. Namely, the group G may be viewed as a group of automorphisms of V as well as of $V_n(m')^*$, in such a manner that the elements of G commute with ρ. Dividing out by the group G, we have a natural mapping $\bar{\rho}: V/G \to V_n{}^*$. Applying considerations similar to those above, we see that $\overline{\mathscr{E}}$ is defined over any common field of definition for $\bar{\rho}$, V/G, and $V_n{}^*$. $V_n{}^*$ is known to be defined over the rational numbers, and it is easily seen from the transformation theory of θ-functions that $\bar{\rho}$ and V/G may be defined over the field of m'th roots of unity. In

any case, it can be shown that we may choose $m' = 72$. However, it may be hoped that with a little more effort it can be proved that $\bar{\rho}$, V/G, V_n^*, and therefore $\overline{\mathscr{E}}$ can all be defined over the field of rational numbers.

3. Some unsolved problems. At this point it may be appropriate to mention certain problems which naturally arise in connection with the preceding developments and which, to the best of the author's knowledge, remain as yet unsolved, or at best only partly solved.

First of all, let us consider the variety $\mathscr{A}$ defined in the preceding section. There is a natural map $\tau_1 : \mathscr{A} \to V_n(m')^*$, and since τ_1 is " in general " one-to-one, it is clear that $\mathscr{A}$ is birationally equivalent to $V_n(m')^*$. We define W to be the graph of the correspondence between $\mathscr{A}$ and V in which to each $\alpha \in \mathscr{A}$ correspond those points of V lying on the variety of which α is the Chow point. Then there is a natural mapping $\psi : W \to \mathscr{A}$. If $\alpha \in \mathscr{A}$ is such that $\tau_1(\alpha) \in V_n(m')$, then $\psi^{-1}(\alpha)$ is a certain Abelian variety. Suppose, however, $\tau_1(\alpha) \in V_n(m')^* - V_n(m')$. $V_n(m')^* - V_n(m')$ consists of the union of a finite number of replicas of each of the spaces V_r, $0 \leqslant r \leqslant n-1$. If $\tau_1(\alpha)$ belongs to one of the replicas of V_r, it may be conjectured that $\psi^{-1}(\alpha)$ is the Zariski closure of a certain algebraic complex Lie group, that Lie group being a group extension of a product of $n - r$ replicas of C^* (the multiplicative group of non-zero complex numbers) by the r-dimensional Abelian variety corresponding to $\tau_1(\alpha)$. Certain preliminary investigations of the author based on the theory of θ-functions tend to indicate that this is the case and that perhaps $\mathscr{A}$ may in fact be the variety obtained from $V_n(m')^*$ by a certain succession of monoidal transformations in the sense of [8] and normalizations. In particular, one may conjecture that if $(\alpha, \gamma, \xi) \in R$ and if $\tau_1(\alpha) \in V_n(m')^* - V_n(m')$, then $A(\alpha)$ is the generalized, Jacobian variety of C in the sense of Rosenlicht [20]. Certain work of Igusa along these lines tends to give support to this idea [10]. If all of these conjectures should prove to be correct, then it would follow, for example, that the closure of $\mathscr{E} = \mathscr{E}(n)$ in V_n^* would be contained in $\bigcup_{r=0}^{n} \mathscr{E}(r) \subset V_n^* = \bigcup_{r=0}^{n} V_r$. Along these same lines and in

connection with Hoyt's criterion (2), it would also be of interest to know which products (of dimension n) of Jacobian varieties of lower dimension are actually specializations of Jacobian varieties of dimension n.

A second type of problem has to do with the analytic subvariety J of V_n^*. Is it a normally imbedded subvariety of V_n^*? What, if any, are its singularities?

Thirdly, in order to call $\mathfrak{E}$ the variety of moduli of Riemann surfaces of genus n, one should be able to state that it is unique and in some sense universal among normal parameter varieties of algebraic systems of curves of genus n (if $\mathfrak{E}$ is not a normal analytic space, one should perhaps consider its normal model). Namely, given any normal algebraic system of curves of genus n (by which we mean that the parameter variety is a normal variety) there should exist a natural map of the parameter variety of the non-singular members of this system into $\mathfrak{E}$. For work in this direction, see Röhrl [19].

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ON THE NUMBER OF MODULI OF COMPLEX STRUCTURES

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INTRODUCTION

The one dimensional case

IN 1857 B. Riemann found in his famous work on abelian functions that the number $m(p)$ of independent parameters on which the Riemann surfaces R_p of genus p depend† is $3p-3$, 1, or 0 if $p>1$, $p=1$ or $p=0$, respectively. Riemann called $m(p)$ the number of moduli.

In the time of Riemann there was no clear idea what these parameters really are. So it remained for nearly 90 years until in 1942 Teichmüller [16] succeeded in making precise the concept. Teichmüller considered the set $M_p=\{R_p\}$ and proved that it carries a natural topology, a natural metric and a natural complex structure. It is a complex manifold with singular points, expressed in modern terminology : a complex space. The number of moduli $m(p)$ is nothing but the complex dimension of M_p.

Teichmüller also considered the set $X_p=\bigcup_{R_p\in M_p} R_p$ and tried to introduce in X_p a natural topology and complex structure such that X_p becomes a complex space and the map $\pi_p: X_p\to M_p$ which attaches to each point $x\in R_p\subset X_p$ the point $R_p\in M_p$ is a proper holomorphic map and some further conditions are satisfied. Each Riemann surface of genus p then appears exactly once as the inverse image $R(y)=\pi_p^{-1}(y)$ of a point $y\in M_p$ and so as a subspace of X_p : X_p is fibred by the surfaces $R(y)$, X_p can be considered as a family of Riemann surfaces with M_p for parameter space.‡

† R_p really is a class of analytically equivalent Riemann surfaces. In what follows we do not distinguish between analytically equivalent Riemann surfaces.

‡ More precisely, the family X_p exists only if certain Riemann surfaces are replaced by certain 'degenerate' ones.

Teichmüller's paper contains many gaps. Most of the proofs are outlined only. The merit of completion is due to Ahlfors [1] and Bers, so that today at least partial results have been proved. Ahlfors covered this subject in his talk given at the Conference on Analytic Functions in Princeton, 1957.

In the papers of Teichmüller and Ahlfors the concept of topologically fixed Riemann surface appears. This is a class of triples (R_0, ϕ, R) of an oriented compact 2-dimensional manifold R_0 of genus p, a compact Riemann surface R of genus p and a topological mapping $\phi\colon R_0 \to R$ which preserves the orientation.† Two triples (R_0, ϕ_1, R_1), (R_0, ϕ_2, R_2) are said to belong to the same class or in other words to define the same topologically fixed Riemann surface if R_1, R_2 are analytically equivalent and if simultaneously the map $\phi_1^{-1} \circ \phi_2\colon R_0 \to R_0$ is homotopic to the identity map.

The space $\hat{M}_p$ of all topologically fixed Riemann surfaces of genus p carries a natural topology and complex structure. The map τ which is defined by the correspondence $(R_0, \phi, R) \to R$ is a holomorphic mapping of $\hat{M}_p$ onto M_p which is nowhere degenerate: $\hat{M}_p$ is a branched covering over M_p. As Ahlfors proved, $\hat{M}_p$ is a complex manifold without singularities, which is topologically equivalent to the hyperball of the same dimension as M_p. In the case $n = 1$ the relation:

$$M_1 \approx \text{upper half plane} \approx \text{unit circle}$$

has been known since many years.

A question which remained open was: Is it possible to add a lower dimensional set A_p of points to M_p such that M_p becomes a compact projective algebraic space $\overline{M}_p$? An old conjecture in this direction was, that this is possible and that the points of $A_p \approx \overline{M}_{p-1}$ correspond to the Riemann surfaces of lower genus. Last summer the problem was partially solved by W. Baily. He uses a method which was inspired by A. Weil. It uses the results of

† In some papers it is required that R_0 is a Riemann surface and that ϕ is a quasiconformal mapping.

Teichmüller and Ahlfors, but apart from this is totally different. Essential is the theorem of Torelli which enables us to attach to each Riemann surface R_p of genus p a point $z \in F_p = H_p/\Gamma_p$, in which H_p denotes the Siegel upper half plane of degree p and Γ_p the Siegel modular group. The map $\psi: M_p \to F_p$ obtained by this is one-to-one (injective) and holomorphic. Thus $\psi(M_p) \approx M_p$ is an analytic surface in F_p. Two years ago Satake, Baily and Cartan proved that F_p can be compactified to a normal projective space $\bar{F}_p$. In this the set of infinite points $\bar{F}_p - F_p$ is analytically equivalent to $\bar{F}_{p-1}$. Baily proved that the closure $\overline{\psi(M_p)}$ of $\psi(M_p)$ is a compact analytic subset of F_p and that $\overline{\psi(M_p)} - \psi(M_p)$ is a lower dimensional analytic subset of $\overline{\psi(M_p)}$. By a theorem of Chow, it follows that $\overline{\psi(M_p)}$ is a projective algebraic space. $\psi(M_p) \approx M_p$ is a Zariski open subset of $\overline{\psi(M_p)}$. Thus $\overline{\psi(M_p)}$ can be considered as the desired compactification of M_p.

The question of what dimension the set of the added points has and if each added point corresponds to a Riemann surface of lower dimension remains still open.

Regular families

In higher dimensions the problem was first considered by M. Noether [14] in 1888. M. Noether looked at compact algebraic surfaces V with vanishing irregularity† q. He denoted the number of linearly independent holomorphic differentials of dimension 2 by p_2 and called $p_a = p_2 - q$ the arithemetic genus of V. The linear genus of V was denoted by $p^{(1)}$. This is roughly speaking nothing but the genus of each curve which can be obtained as the set of zeros of a 2 dimensional holomorphic differential in V. M. Noether required that $p_a > 3$, $p^{(1)} > 5$, made some further assumptions and found that the number of moduli for V is $m(V) = 10\,(p_a + 1) - 2(p^{(1)} - 1)$. Later on this result was generalized by Enriques. Beyond that no classical papers on this subject are known.

† In modern terminology the irregularity is nothing but $b_1/2$, where b_1 is the first Betti number of V.

A precise definition of the number of moduli was neither known to Noether nor to Enriques. In Noether's formula one observes that p_a, $p^{(1)}$ are birational invariants. Therefore $m(V)$ has to be considered as the number of independent parameters of the function field on V. An intimation of a definition is given in the Ergebnisbericht [17] of Zariski on algebraic surfaces.

Last year Kodaira and Spencer have applied totally new methods [11]. They define the number of moduli as a biregular and not a a birational invariant. The definition is based on the concept of regular family of complex structures.

DEFINITION. *Let X and M be complex manifolds, M be connected, $\pi: X \to M$ a proper, regular, holomorphic map of X onto M. Then (X, π, M) is called a regular† family of complex structures‡.*

Because π is a regular holomorphic map, each fibre $X_y = \pi^{-1}(y)$, $y \in M$ is a non-singularly embedded analytic submanifold of X. The dimension of X_y does not depend on y. Because π is proper it follows that each X_y is compact. Introducing a C^∞-Riemann metric using a partition of unity one can prove that all X_y have the same C^∞-differentiable structure: by means of surfaces which are of complementary dimension and orthogonal with respect to X_y, invertible C^∞-differentiable maps of X_y onto fibres $X_{y'}$ which are near to X_y can be constructed. From a result of [4] one sees that there exists a real-analytic metric on X. Repeating the same argument it can be proved therefore, that all X_y possess the same real-analytic structure. Locally one can look upon (X, π, M) as a deformation of the complex structure on a fixed topological (or real-analytic) manifold.

Some examples may show in what way the complex structure of the fibres X_y can depend on $y \in M$.

† "Regular" means that the Jacobian of π has the rank $rk(\pi) = \dim_C M$ everywhere. A map is proper if the inverse image of every compact subset of M is compact.

‡ Kodaira and Spencer consider also families of complex structures which depend on a differentiable parameter. However, the complex analytic case seems to be the more important one.

(a) Let $Z = X + iY$ denote the square matrices of n rows and n columns with $|Y| \neq 0$, E be the unit matrix and $Q = (E, Z)$. The set M of all Q can be considered as an open subset of C^{n^2}. Each $Q \in M$ defines a period torus T_Q of dimension n. The set $X = \{T_Q\}$ can be topologized in a canonical way and has a natural complex structure. The mapping which attaches to each point $P \in T_Q$ the point $Q \in M$ is a proper, regular, holomorphic map of X onto M. Therefore if M' denotes any connected component of M, $X' = \pi^{-1}(M')$, $\pi' = \pi \mid M'$ then (X', π', M') is a complex analytic family of complex tori.

Let us call two points Q_1, $Q_2 \in M$ equivalent ($Q_1 \sim Q_2$) if the torus T_{Q_1} is analytically equivalent with T_{Q_2}. The relation $\sim$ defines a decomposition Γ of M into equivalence classes. The space M/Γ can be considered as the natural modular space of complex tori : there is a one-to-one correspondence between the analytic equivalence classes of complex tori and the points of M/Γ.

What about M/Γ ? In the case $n = 1$, M/Γ is nothing but the fundamental domain of the modular group which is analytically equivalent to the open z-plane and is a nice complex manifold. In the higher dimensional case $n > 1$ by a very general theorem of C. L. Siegel [15] one obtains (see [11]) that the following proposition is true :

Let $U \subset M$ be an arbitrary small open subset of M. Then there is always a class $\{Q\}$ of equivalent points of M which has at least one accumulation point in U.

On the other hand, each equivalence class is countable. This means that certainly M/Γ is not a Hausdorff space and does not possess a natural complex structure which makes M/Γ a complex space. The general construction of modular spaces therefore will not be possible.

Looking at the example (a) a result recently proved by Kodaira, Spencer and Andreotti may be of interest :

Let (X, π, M) be a regular family of complex manifolds, let X_{y_0}, $y_0 \in M$ a point, be a torus of dimension n. Then each fibre X_y, $y \in M$ is a torus.

This means that any deformation of a torus structure leads to a torus structure.

In order to consider another phenomenon which does not appear in the deformation of one dimensional complex manifolds a second example is given.

(b) Let us denote by $\mathcal{O}$ the cartesian product $P^1 \times P^1$ of the Riemann sphere with itself. $\mathcal{O}$ often is called the Osgood closure of the complex number space of two dimensions, C^2. If (X, π, M) is a regular family, and X_{y_0}, $y_0 \in M$, is the space $\mathcal{O}$, then it can be proved that there is a neighborhood $U = U(y_0) \subset M$ such that for each point $y \in U$ the fibre $X_y = \mathcal{O}$. This means a rigidity of the complex structure of $\mathcal{O}$.

The topological space $|\mathcal{O}|$, which underlies $\mathcal{O}$, has still further complex structures. It is not known how many. However there is a conjecture that the Hirzebruch structures [9] are the only complex structures on $\mathcal{O}$. These Hirzebruch structures form an infinite sequence S_ν, $\nu = 0, 1, 2, \ldots$. $|\mathcal{O}|$ equipped with S_ν is a complex manifold $\mathcal{O}_\nu$. One has $\mathcal{O}_0 = \mathcal{O}$. It can be proved, that for a properly chosen family (X, π, M) the following phenomenon can happen (if $\dim_C M = 1$):

Let $y_0 \in M$ be a point, $U = U(y_0)$ a neighbourhood, ν_0, ν_1 integers with $0 \leqslant \nu_1 < \nu_0$. Then $X_{y_0} = \mathcal{O}_{\nu_0}$ but $X_y = \mathcal{O}_{\nu_1}$ for each point $y \in U - y_0$.

This means a jumping of complex structure under deformation. The complex manifolds $\mathcal{O}_{\nu_0}$, $\mathcal{O}_{\nu_1}$ considered as points of a modular space Q that might exist, have the "distance" zero, e.g. there does not exist a metric in Q as in the case of the Teichmüller space. It is possible to prove that the jumping of complex structure has a direction: the complex manifold $\mathcal{O}_{\nu_0}$ can jump to the complex manifold $\mathcal{O}_{\nu_1}$, if $\nu_1 < \nu_0$, but $\mathcal{O}_{\nu_1}$ may never jump to $\mathcal{O}_{\nu_0}$.

The following theorem analogous to the proposition in the case of the example (a) is due to Andreotti:

Let (X, π, M) be a regular family, X_{y_0}, $y_0 \in M$, a Hirzebruch manifold. Then each fibre X_y, $y \in M$, is a Hirzebruch manifold†.

Sheaf theory

For further investigations on regular families the theory of coherent analytic sheaves is a necessary tool. The most important concepts of this theory have to be remembered in this section.

Let X be a complex space‡. Then there is the notion of analytic and coherent analytic sheaf over X (see [2], where an introduction to sheaf theory is given). An example of a coherent analytic sheaf over X is the space of all germs of local holomorphic functions on X. This sheaf generally is denoted by $\mathcal{O}$. Each set $\mathcal{O}_x$ of elements of $\mathcal{O}$, which are germs of functions at x, forms a ring. A general analytic sheaf is a specially topologized space S, which is mapped by a continuous mapping ϕ onto X. The stalks $S_x = \phi^{-1}(x)$ are modules over $\mathcal{O}_x$. The module operations are continuous with respect to the topology of $\mathcal{O}$ and S. The coherence is a local property on S, it implies that the stalks S_x of S are finitely generated over $\mathcal{O}_x$. A coherent analytic sheaf S is called free¶ if all the stalks S_x are free modules over $\mathcal{O}_x$.

For the theory of complex functions only the coherent analytic sheaves are important. However, if an analytic sheaf S is given in general it will not be self-evident that S is coherent or moreover free. The proofs of coherence of analytic sheaves belong to the most difficult tasks in modern complex analysis!

If S is a given analytic sheaf over X, the concept of cross-section in S and of cohomology group with coefficients in S is well known. Let B be an open subset of X. Then a cross-section out of $\Gamma(B, S)$

† Under some additional assumptions.

‡ An introduction to the theory of complex spaces is given in [6].

¶ This is equivalent with the proposition that S is isomorphic to the sheaf of germs of local holomorphic cross-sections in a vectorbundle.

is a continuous mapping s of B into S such that $\phi \circ s$ is the identity map $B \to B$. The cohomology groups $H^\nu(B, S)$, $\nu = 0, 1, 2, \ldots$ are defined by a procedure due to Chech. Both, $\Gamma(B, S)$ and $H^\nu(B, S)$ are complex vector spaces, there is a natural isomorphism $\Gamma(B, S) \approx H^0(B, S)$.

Cartan and Serre have proved† :

Let X be a compact complex space, S a coherent analytic sheaf on X. Then the vector spaces $H^\nu(X, S)$ are of finite dimension and $H^\nu(X, S) = 0$ for $\nu > \nu_0$.

Now, let π be a holomorphic mapping of a complex space X into a complex space Y. If S is an analytic sheaf over X then over Y there is defined a sequence of analytic image sheaves $\pi_\nu(S)$, $\nu = 0, 1, 2, \ldots$. The definition‡ is such that each element $(\pi_\nu(S))_y$ can be considered as a germ of a ν-dimensional cohomology class with coefficients in S, which is formed along the fibre $\pi^{-1}(y)$. Therefore to each class $\xi \in H^\nu(X, S)$ belongs a cross-section $\pi_\nu{}^*(\xi) \in \Gamma(Y, \pi_\nu(S))$. $\pi_\nu{}^*$ is a homomorphism of $H^\nu(X, S)$ into $\Gamma(Y, \pi_\nu(S))$. As can be proved easily $\pi_0{}^*$ is an isomorphism : $\Gamma(X, S) \approx \Gamma(Y, \pi_0(S))$.

Let (X, π, M) be a regular family. Kodaira and Spencer considered the sheaf Θ of germs of local holomorphic fields of such contravariant vectors in X, which have the direction of the fibres. Θ is a free analytic sheaf on X. A proposition of Kodaira and Spencer is :

Let $\widetilde{U} \subset M$ be an open subset, $V = \pi^{-1}(\widetilde{U}) \subset X$, v a holomorphic field of contravariant vectors in $\widetilde{U}$. Then there belongs a cohomology class $\xi(v) \in H^1(V, \Theta)$ to v.

The construction of $\xi(v)$ runs as follows. First let $\mathfrak{U} = \{U_\iota\}$ be an open covering of V, which is so fine that each U_ι is contained in a patch of (special) local coordinates of X. Then in each U_ι a holomorphic vector field ξ_ι can be defined such that each vector $\xi_\iota(x)$, $x \in U_\iota$ is projected by π onto the vector $v(y)$, $y = \pi(x)$. $\xi_{\iota_1\iota_2} = \xi_{\iota_2} - \xi_{\iota_1}$ are holomorphic vectorfields in the intersections $U_{\iota_1} \cap U_{\iota_2}$ which

† See the Cartan Séminaire E.N.S. notes of 1953/54.

‡ For definition see [5].

have the direction of the fibres. By the methods of sheaf theory it is clear that the collection $\{\xi_{\iota_1\iota_2}\}$ is a cocycle out of $Z^1(\mathfrak{U}, \Theta)$ which defines a cohomology class $\xi(v) \in H^1(V, \Theta)$. As can be seen very easily the definition of $\xi(v)$ is independent of the choice of the covering U and the vectorfields ξ_ι.

An important result of Kodaira and Spencer is :

A regular family (X, π, M) is a fibre bundle if and only if for each open subset $B \subset M$ and each holomorphic vectorfield v over B the class $\xi(v) = 0 \in H^1(\pi^{-1}(B), \Theta)$.

If (X, π, M) is a fibre bundle then, in particular (X, π, M) is locally trivial. This means that for each point $y_0 \in M$ there exists a neighbourhood $U(y_0)$ and a biholomorphic map α of $V = \pi^{-1}(U)$ onto $U \times F$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & U & \end{array}$$

is commutative. Here F denotes a compact complex manifold and τ the projection $U \times F \to U$. Because $\tau \circ \alpha = \pi$ each fibre X_y, $y \in U$, is mapped biholomorphically into the complex manifold $y \times F \approx F$. All X_y, $y \in U$, are analytically equivalent to F, we can say, equal to F. The complex structure does not depend on $y \in M$.†

Let Θ_y denote the sheaf of germs of local holomorphic fields of contravariant vectors on X_y, $U = U(y)$ be a neighborhood of a point $y \in M$, $V = \pi^{-1}(U)$. There exists a restriction map r^*: $H^1(V, \Theta) \to H^1(X_y, \Theta_y)$. If v is a holomorphic vectorfield in U, the restriction $r^*(\xi(v))$ is a cohomology class of $H^1(X_y, \Theta_y)$. As can easily be proved $r^*(\xi(v))$ depends only on the vector $v(y)$ but not on the whole vectorfield v. Thus there is a mapping ρ_y of the vector space T_y of tangent vectors in y into the vector space $H^1(X_y, \Theta_y)$. ρ_y is a linear mapping.

† It is possible to prove the following : If the complex structure of X_y is independent of $y \in M$ then (X, π, M) is a fibre bundle.

If M' is a complex manifold, ψ a holomorphic mapping of M' into M, it is possible to construct a complex manifold X', which is mapped by a holomorphic mapping π' onto M' and by a holomorphic mapping $\hat{\psi}$ onto X, such that the diagram

$$\begin{array}{ccc} X & \xleftarrow{\hat{\psi}} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xleftarrow[\psi]{} & M' \end{array}$$

is commutative. $\mathfrak{X}' = (X', \pi', M')$ is a regular family, each fibre X'_y, is analytically equivalent with the fibre X_y, $y = \psi(y')$ of $\mathfrak{X}$. $\mathfrak{X}'$ therefore is called the lifted family. It is now possible to give the following definition:

DEFINITION. *A regular family $\mathfrak{X} = (X, \pi, M)$ is called complete at a point $y \in M$ if for any regular family $\mathfrak{X}' = (X', \pi', M')$ with $X'_y = X_y$ there exists a neighbourhood $U = U(y')$ and a holomorphic map $\psi : U \to M$ with $\psi(y') = y$ such that $\mathfrak{X}' \mid U = (\pi'^{-1}(U), \pi', U)$ is obtained by lifting with respect to ψ. A regular family (X, π, M) will be called complete if it is complete at each point $y \in M$.*

Now the definition of the number of moduli can be given:

DEFINITION. *Let X_0 be a compact complex manifold. Then the number $m(X_0)$ of moduli of X_0 exists, if there is a regular family $\mathfrak{X} = (X, \pi, M)$ with the following properties:*

(1) *$X_0 = X_{y_0}$ for at least one point $y_0 \in M$,*

(2) *$\mathfrak{X}$ is complete,*

(3) *the mapping ρ_y is injective for each $y \in M$.* (We say: X is effectively parametrized.)

Under the assumption of existence $m(X_0)$ is equal to the complex dimension of M (i.e. $m(X_0) = \dim_C M$).

One sees immediately that $m(X_0)$ is defined independently of the choice of (X, π, M) and the point $y_0 \in M$, if it exists. The main conjecture of Kodaira and Spencer states :

If $m(X_0)$ exists, then $m(X_0) = \dim_C H^1(X_0, \Theta_0)$, where Θ_0 denotes the sheaf of local holomorphic vector fields (= tangent sheaf) of X_0.

Like $H^1(X_0, \Theta_0)$ the number of moduli $m(X_0)$ is a biregular invariant. In Noether's case of certain regular algebraic surfaces V one obtains—under the additional assumption, that V is a minimal model in the sense of Zariski—that $p^{(1)} = c_1^2[V] + 1$, where c_1 denotes the first Chern class of V, and therefore

$$m(V) = 10(p_a + 1) - 2(p^{(1)} - 1) = 10(p_a + 1) - 2c_1^2[V] = -T(V, \Theta_0),$$

$T(V, \Theta_0)$ being the Todd characteristic of V with coefficients in the tangent sheaf Θ_0. By Hirzebruch's formula

$$T(V, \Theta_0) = \chi(V, \Theta_0)$$
$$= {}_{def.} \dim_C H^0(V, \Theta_0) - \dim_C H^1(V, \Theta_0) + \dim_C H^2(V, \Theta_0)$$

(see Riemann-Roch [10]) we get: $m(V) = \dim_C H^1(V, \Theta_0) - (\dim_C H^0 + \dim_C H^2)$. H^0 is the vector space of infinitesimal holomorphic transformations of V, which is 0 in the case considered by Noether. Because under Noether's assumption also $\dim_C H^2$ should be 0, Noether's formula coincides with the conjecture of Kodaira and Spencer.

Meanwhile the main problem was solved in a special case:

THEOREM. (Kodaira, Nirenberg, Spencer [12]). *Let X_0 be a compact complex manifold, Θ_0 its tangent sheaf, $H^2(X_0, \Theta_0) = 0$. Then there is a regular family $\mathfrak{X} = (X, \pi, M)$ such that $X_0 = X_y$, $y \in M$ and the mapping $\rho_y: T_y \to H^1(X_y, \Theta_y)$ is bijective. If moreover $\dim_C H^0(X_0, \Theta_0) = 0$ then X is effectively parametrized in a neighborhood of y and the number of moduli exists. One has $m(V) = \dim_C M = \dim_C H^1(X_0, \Theta_0)$.*

The proof of this theorem is very difficult. It uses the solution of a complex Frobenius theorem which is due to Newlander and Nirenberg [13]. If $H^2(X_0, \Theta_0) \neq 0$, in general a regular family $\mathfrak{X}$ does not exist such that $X_0 = X_y$ and ρ_y is surjective. However, in many interesting cases of algebraic surfaces and manifolds the condition is satisfied.

Further results

Most of Kodaira's and Spencer's results on the local theory of deformations of complex structures are based on the following two theorems ((X, π, M) = always a regular family, S any free sheaf over X).

(I) *The functions* $d_\nu(y) = \dim_C H^\nu(X_y, S_y)$, $\nu = 0, 1, 2, \ldots$ *are upper semi-continuous.*†

(II) *If* $d_\mu(y)$ *is a constant function for* $\mu \leqslant \nu$ *then the image sheaf* $\pi_\nu(S)$ *is free at each point* $y \in M$, *i.e. is isomorphic to the sheaf* Λ *of germs of local holomorphic cross-sections in a complex analytic vector bundle* V *over* M. *There exists a canonical bilinear mapping of the vector space* V_y *over each point* y *into* $H^\nu(X_y, S_y)$ *such that the points of* V_y *can be considered as cohomology classes of* $H^\nu(X_y, S_y)$.

An immediate consequence of I and II is the rigidity theorem of Fröhlicher-Nijenhuis, for instance [4]:

Let $H^1(X_0, \Theta_0)$, $0 \in M$, *be* 0. *Then there is a neighbourhood* $U(0) \subset M$ *such that* $(\pi^{-1}(U), \pi, U)$ *is a fibre bundle.*

This generalizes the result on the Osgood space $P^1 \times P^1$.

In the work of Kodaira and Spencer the proof of I and II is established using the theory of harmonic differentials (potential theory) which is based on the theory of elliptic equations. The fact that real analysis is used ensures that proofs do not get too difficult. However, there are some disadvantages too.

(1) Till now generalized potential theory is only available in manifolds and not in complex spaces with singular points. But every expert in algebraic geometry knows that it is not enough to consider only manifolds. In general modular spaces and especially compactified modular spaces will have non-uniformizable points!

† S_y is the analytic restriction of S to X_y. S_y can be constructed in the same way as Θ_y.

(2) The results derived by potential theory are too weak. In the case of Theorem I for instance one would like to have a stronger proposition of the following kind:

(I*) *The function $d_\nu(y)$ is constant outside a lower dimensional analytic subset $A \subset M$. On A it may have a bigger value, it can jump up.*

In order to carry over the Theorems I and II to complex spaces one has to develop a method which does not use differentiation. Such a method can be built up from the Cartan theory of Stein spaces (= holomorphically complete spaces) and a generalization of the theory of formal power series. Doing this the following result [8] was obtained:

MAIN THEOREM. *Let X, Y be complex spaces, $\pi : X \to Y$ a proper holomorphic mapping, S a coherent analytic sheaf over X. Then all image sheaves $\pi_\nu(S)$, $\nu = 0, 1, 2, \ldots$, are coherent analytic sheaves over Y.*†

From the main theorem the Kodaira-Spencer results I and II can easily be derived. I is obtained with the desired improvement. This implies for instance the proposition that the complex structure of the fibres of a regular family can only jump on an analytic subset $A \subset M$. Also II is obtained in an improved form: it is no longer necessary to require that $d_\mu(y) =$ constant for $\mu \leqslant \nu$, the assumption $d_\nu(y) =$ constant is enough.

Beyond that it is now allowed that X, M are complex spaces with singular points. The map π has to be proper and holomorphic. The assumption that S is a free sheaf and that π is a regular map can be replaced by a weaker condition: The sheaf S has to be flat over M. This means that in each point $x \in X$ the module $\mathrm{Tor}_1^{\mathcal{O}_y(M)}(C, S_x)$ is zero,‡ where $y = \pi(x)$ and in a natural way the complex field C and the stalk S_x are considered as modules over the ring $\mathcal{O}_y(M)$ of germs of holomorphic functions at $y \in M$.

† Under a very restrictive assumption, which implies that all fibres X_y are projective algebraic spaces, the theorem was already proved in [5].

‡ For definition see [3].

The global theory

The results which were discussed here in the theory of the parameters of complex and algebraic manifolds of higher dimension are of local nature. Therefore the question arises: what can be done in the global theory, for instance in constructing the modular spaces. Because of the jumping of complex structure, in general a metric space or a Hausdorff space will not be obtained. The phenomenon considered at the torus deformation is still worse. But in many special cases the existence of the modular space will not give any difficulty. However, the construction has been done only in nearly trivial cases till now.

The first non-trivial deformation, which was considered (by Andreotti et al) is given by the (topologically fixed) Kummer manifolds. Let Γ be a lattice in C^n and Γ_1 be the group which is generated by Γ and the reflection of C^n at the zero point. Then C^n/Γ is a torus T and C^n/Γ_1 a Kummer manifold $K = K(T)$. We say: K is induced by the torus T.

In the case $n \neq 2$ the (local) deformation of Kummer manifolds $K(T)$ leads always to a Kummer manifold which is induced by a torus. This is not further interesting. But if $n=2$ the following propositions are obtained :†

(1) $H^0(K(T), \Theta) = H^2(K(T), \Theta) = 0$, *where* Θ *denotes the tangent sheaf of* $K(T)$. *Therefore the number of moduli exists.*

(2) $m(K(T)) = 20$. *Because* $m(T) = 4$ *the local deformation leads to complex manifolds, which are not induced by a torus.*

(3) *If* $\mathfrak{X} = (X, \pi, M)$ *is the deformation of Kummer surfaces* $K(T)$ *given by the deformation of 2-dimensional complex tori considered in section 2, there is an extension of* X *to a regular complete, effectively parametrized family* $\hat{\mathfrak{X}} = (\hat{X}, \hat{\pi}, \hat{M},)$ *such that* $\dim_C \hat{M} = 20$.

(4) $\hat{M}$ *is an open (canonically embedded) subset of the Siegel manifold* G :

† These results of Andreotti are not yet published.

$$z \cdot H \cdot z = 0,\ z \cdot H \cdot \bar{z} > 0,$$

where $z = (z_1, \ldots, z_{22})$ *denotes the homogeneous coordinates of the complex projective space* P^{21}, *and* H *is a real symmetric non-singular* 22*-square matrix with exactly* 3 *negative eigenvalues.*

It is not known if $\hat{\mathfrak{X}}$ can be analytically extended to G such that G can be considered as the modular space of Kummer surfaces (for each Kähler manifold obtained by a deformation of a $K(T)$ corresponds a point of G in a canonical way). Also the question: "What points $z \in G$ corrrespond to algebraic manifolds ? ", will be of interest. A conjecture is that these points form analytic surfaces $A \subset G$ of dimension 19.

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HARTOGS' THEOREM IN EUCLIDEAN SPACE AND A RELATED COUSIN THEOREM ON THE TORUS†

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1. Introduction. The title of this paper refers to the following theorem of Hartogs (which Hartogs himself proved but very faultily). In the space of 2 or more complex variables if a bounded domain has a connected exterior, then any function which is defined and holomorphic in a neighborhood of the boundary of the domain can be continued analytically into the domain itself.

Our own proof of the theorem, as given in [1], applied immediately to a more comprehensive proposition set in real Euclidean E_n, and we generalized it further in [2] and [3], and then also in [4].

A version in [2], with which we would like to resume the topic reads as follows. Take a suitable elliptic operator with constant coefficients and of arbitrary order :

$$\Delta f = \sum_{p_1+\ldots+p_n \leqslant 2h} a_{p_1 \ldots p_n} \frac{\partial^{p_1+\ldots+p_n} f}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}}, \tag{1.1}$$

and some second (non-elliptic) operator with constant coefficients

$$\Lambda f = \sum_{q_1+\ldots+q_m \leqslant m} b_{q_1 \ldots q_m} \frac{\partial^{q_1+\ldots+q_m} f}{\partial x_1^{q_1} \ldots \partial x_m^{q_m}}, \tag{1.2}$$

which acts only on a proper part of the variables, that is

$$m < n. \tag{1.3}$$

Now, if $f(x)$ is defined real-analytic in a neighborhood of the boundary, and if $\Delta f = 0$, $\Lambda f = 0$ then f can be continued analytically into the domain.

At present we generalize this as follows. We form the functions

$$\rho(x) = \Delta f, \quad \sigma(x) = \Lambda f \tag{1.4}$$

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‡ Did not attend the Colloquium.

and we do not at all demand that $\rho(x)$ and $\sigma(x)$ shall be zero. All we demand is that the functions $\rho(x)$ and $\sigma(x)$ shall be continuable into the domain. This is enough to conclude that $f(x)$ itself is also so continuable.

We also have theorems along these lines if our domain is bounded by an outer hypersurface and several interior ones.

If one goes over from Euclidean space onto torus then there are no " outer " and " inner " domains, and Hartogs' theorem goes over into a Riemann-Cousin theorem. For instance, we may decompose the torus into two domains D_1, D_2 which are separated by a hypersurface B. Let $f(x)$ be an analytic function with certain properties which is given in a neighborhood of the boundary. The problem becomes one of putting $f(x)$ in the form $f(x) = F_1(x) - F_2(x)$ where $F_1(x)$ is continuable into D_1 and $F_2(x)$ into D_2, and such that F_1, F_2 retain certain desirable properties of $f(x)$ itself. Now, for an elliptic operator on the torus, the Green's function no longer satisfies $\Delta G(\xi) = 0$ but only $\Delta G(\xi) = \text{constant}$ and this introduces certain " residues " whose vanishing is sufficient, and frequently also necessary for an affirmative answer to the existence question connected with the problem. We previously introduced in [1] a residue for the case corresponding to $\rho(x) = 0$, $\sigma(x) = 0$, of the present setting. This residue in the identical form will reappear for non-vanishing $\rho(x)$, $\sigma(x)$, and it will be a primary one, but a further residue, of more secondary importance, will be present as well.

" Inhomogeneous " systems of equations of the type (1.4) with non-vanishing given functions $\rho(x)$, $\sigma(x)$ were introduced in and formed the subject matter of our last paper [4], but the results and outlooks of the present paper are in a different direction. Also, we expect to extend some of the statements of the present paper from the torus to some other homogeneous compact manifolds, but this is being deferred to another occasion.

2. One closed hypersurface in Euclidean space. In real Euclidean E_n, any $n \geqslant 2$, we take a bounded domain D whose exterior is

connected. In our leading statements the domain can be deformed a little, and therefore we assume immediately that its boundary B is a combinatorially closed $(n-1)$-dimensional hypersurface in simplicial form, and that each simplex together with its pointset closure is part of a larger "open" $(n-1)$-dimensional simplex, of differentiability class $C^{(1)}$ say. We take some n-dimensional neighborhood of B and denote it always by $\tilde{B}$. In any one case the neighborhood can be replaced by a subneighborhood. Therefore we may think of $\tilde{B}$ as being an ϵ-neighborhood of B for some $\epsilon > 0$, say.

If $n = 2k$, and E_n is viewed as the space of complex variables $z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1,\ldots, k$, and if $k \geqslant 2$, then Hartogs' own theorem states as follows. If $f(z)$ is given holomorphic in $\tilde{B}$ then it can be continued into D. That is, in the domain

$$\tilde{D} \equiv D \cup \tilde{B} \tag{2.1}$$

there exists a holomorphic function whose restrictions to $\tilde{B}$ is the given function $f(z)$.

A holomorphic function is a simultaneous solution of the system of equations

$$\frac{\partial f}{\partial \bar{z}_1} = 0,\ldots, \frac{\partial f}{\partial \bar{z}_k} = 0. \tag{2.2}$$

Each of these k equations is of the form $\Lambda f = 0$, where in the real space coordinates (x_α, y_α), Λf is a linear partial differential operator with complex constant coefficients, and for $k \geqslant 2$ it is *defective* in the sense that it "effectively" operates on fewer than all coordinates. Our generalization of it is as follows.

I. Usually, the symbol Λf will denote, for $n \geqslant 2$, any fixed operator of the form (1.2) subject to the condition (1.3). The coefficients $b_{q_1 \ldots q_m}$ are complex constants and at least one coefficient is $\neq 0$.

Returning to the holomorphic case for guidance we note that the system (2.2) also implies

$$\frac{1}{4}\left(\frac{\partial^2 f}{\partial \bar{z}_1 \partial \bar{z}_1} + \ldots + \frac{\partial^2 f}{\partial z_k \partial \bar{z}_k}\right) = 0, \tag{2.3}$$

that is $\Delta f = 0$, where Δf is the classical Laplacean. Our generalization of this is any elliptic operator Δf with constant coefficients and " customary " properties. Rather than derive and analyze these properties we prefer to re-state them " axiomatically " from our previous papers [2], [3], [4], in the manner needed.

II. The symbol Δf will always denote an operator of the form (1.1) with constant coefficients having the following properties among others.

III. It gives rise to a *Green's formula*

$$\left.\begin{matrix} f(x) \\ 0 \end{matrix}\right\} = \int_B \Omega(f(\xi), \xi - x, d\xi) - \int_D \Delta f(\xi).\ G(\xi - x)\, dv_\xi \qquad (2.4)$$

of the following description.

IV. The function $G(\xi)$ is defined and (real-) analytic in E_n after deletion of the origin $\xi = 0$, and it is Lebesgue integrable in the neighborhood of the origin,

$$\int_{0<|\xi|<1} |\, G(\xi)\,|\, dv_\xi < \infty. \qquad (2.5)$$

V. As regards its behavior at infinity, there is an integer $r_0 \geqslant 0$ such that for every system of integers

$$r_1 \geqslant 0, \ldots, r_n \geqslant 0, \quad r_1 + \ldots + r_n \geqslant r_0, \qquad (2.6)$$

we have

$$\frac{\partial^{r_1+\ldots+r_n} G}{\partial \xi_1^{r_1} \ldots \partial \xi_n^{r_n}} = O(|\, \xi\,|^{-n-1}), \quad |\, \xi\,| \to \infty. \qquad (2.7)$$

VI. The expression $\Omega(f(\xi),\ \xi - x, d\xi)$ which is the integrand under the surface integral in (2.4) is a certain external differential form

$$\Sigma_{(\alpha)}\, \Omega_{\alpha_1 \ldots \alpha_{n-1}}\, d\xi_{\alpha_1} \wedge \ldots \wedge d\xi_{\alpha_{n-1}}, \qquad (2.8)$$

in which each component $\Omega_{\alpha_1 \ldots \alpha_{n-1}}$ is a certain finite expression

$$\Sigma_{(p)}\, G_{p_1 \ldots p_n}(\xi - x)\, \frac{\partial^{p_1+\ldots+p_n} f(\xi)}{\partial \xi_1^{p_1} \ldots \partial \xi_n^{p_n}} \qquad (2.9)$$

with a generic function $f(\xi)$. The coefficients $G_{p_1 \cdots p_n}(\xi)$ have "axiomatically" the same properties as $G(\xi)$ itself. They are defined and analytic in $E_n - (0)$ and there is an $r_0 \geqslant 0$, which we may assume to be common to all, so that for (2.6) we have

$$\frac{\partial^{r_1+\cdots+r_n} G_{p_1 \cdots p_n}}{\partial \xi_1^{p_1} \cdots \partial \xi_n^{p_n}} = O(|\xi|^{-n-1}), \quad |\xi| \to \infty. \tag{2.10}$$

In actual fact these coefficients are each a finite combination with constant coefficients of partial derivatives of the Green's function $G(\xi)$, but we will have no opportunity to use this explicitly.

VII. Identically in the symbol $f(\xi)$ and the "parameter" x we have

$$d_\xi \Omega(f(\xi),\ \xi - x,\ d\xi) = \Delta f(\xi).\ G(\xi - x) dv_\xi, \tag{2.11}$$

where $d_\xi \Omega$ is the external derivative relative to the space coordinates (ξ), and dv_ξ is the external form

$$d\xi_1 \wedge \cdots \wedge d\xi_n. \tag{2.12}$$

Also, $G(\xi)$ is the Green's function of the operator Δ so that in particular

$$\Delta G(\xi) = 0 \tag{2.13}$$

but this again will not be used expressly.

VIII. Relation (2.11) automatically implies the *lower* half of (2.4) in the following sense. If D is a bounded domain whose boundary B consists of one or several disjoint combinatorially closed hypersurfaces, and if x is a point of D_∞ which is the complement of $D \cup B$ in E_n, then the right side of (2.4) has the value 0. Now, the *upper* half of (2.4) refers to points x in D, and in this case the fact is only this, that the right side is a point functional of $f(\xi)$ which is commutative with translations, see [3] and [4]. We could, as we did in [4], operate to an extent with this functional unrestricted, but we prefer, in order to avoid complications, to restrict ourselves to the case in which this functional is the identity. Thus we have as follows. If D is a bounded domain whose boundary B consists of one or several closed hypersurfaces, all suitably oriented, and if $f(\xi)$ is defined in a neighborhood of $D \cup B$ and belongs there to differentiability

class $C^{(\infty)}$, (or only to differentiability class $C^{(N)}$, for N sufficiently large); then for x in D the right side in (2.4) has value $f(x)$, and for x in D_∞ it has value 0.

Our last "axiomatic" property is the following known basic result.

IX. If $\rho(x)$ is (real-) analytic in a neighborhood and $f(x)$ is a $C^{(\infty)}$ solution of

$$\rho(x) = \Delta f(x), \tag{2.14}$$

then $f(x)$ itself is analytic.

We will now have our first theorem which is nearest to Hartogs' classical version (and subsumes it), but in actual trend is already rather different from it.

THEOREM 1. *If D is a bounded domain whose boundary is one closed hypersurface B, if $\widetilde{B}$ is a neighborhood of B, and Δf, Λf are operators as described, then*

(i) *If $f(x)$ is given (real-) analytic in $\widetilde{B}$, and if the functions $\rho(x)$, $\sigma(x)$ as defined by*

$$\rho(x) = \Delta f(x), \quad \sigma(x) = \Lambda f(x) \tag{2.15}$$

in $\widetilde{B}$ have analytic continuations into $\widetilde{D}$, (see (2.1)), then $f(x)$ itself has an analytic continuation into $\widetilde{D}$.

(ii) *Especially if $f(x)$ in $\widetilde{B}$ is an (analytic) solution of*

$$\Delta f = 0, \quad \Lambda f = 0, \tag{2.16}$$

then it has an analytic continuation into $\widetilde{D}$.

(iii) *If $f(x)$ in $\widetilde{B}$ belongs only to class $C^{(\infty)}$, and if the $C^{(\infty)}$ functions $\rho(x)$, $\sigma(x)$ as defined by (2.15) have $C^{(\infty)}$ continuations into $\widetilde{D}$ and if these satisfy in $\widetilde{D}$ the integrability condition*

$$\Lambda\rho(x) = \Delta\,\sigma(x), \tag{2.17}$$

then $f(x)$ as a solution of (2.15) has a $C^{(\infty)}$ continuation into $\widetilde{D}$.

REMARK 1. In $\widetilde{B}$ we have

$$\Lambda\rho - \Delta\sigma = (\Lambda\Delta - \Delta\Lambda)f.$$

However operators with constant coefficients are commutative and thus this is 0. Now in the analytic case the relation (2.17) which was automatic in $\widetilde{B}$ continues by itself into $\widetilde{D}$, but in the $C^{(\infty)}$ case it is necessary to stipulate the latter expressly. This necessity is emphasized by the fact that $C^{(\infty)}$ continuations of $\rho(x)$, $\sigma(x)$, even if subject to (2.17), if at all possible are *never* unique.

PROOF. It will suffice to deal with part (iii). With the given functions $f(\xi)$ in $\widetilde{B}$ and $\rho(\xi)$ in $\widetilde{D}$ we set up the expression,

$$\int_{\widetilde{B}} \Omega(f(\xi), \xi - x, d\xi) - \int_{\widetilde{D}} \rho(\xi).\ G(\xi - x)\, dv_\xi, \tag{2.19}$$

in which Ω and G are the data of formula (2.4) and we denote it now and later more briefly by

$$[B, D; f(\xi), \rho(\xi)]\,(x). \tag{2.20}$$

It defines two separate functions,

$$\left.\begin{matrix} F(x) \\ F_\infty(x) \end{matrix}\right\} = [B, D; f(\xi), \rho(\xi)]\,(x), \tag{2.21}$$

where the upper function $F(x)$ is intended for (and restricted to) x in D and the lower function $F_\infty(x)$ for x in D_∞. The previous axioms IV, V, VI imply immediately as follows.

LEMMA 1. *The function $F_\infty(x)$ is analytic and for* (2.6) *we have*

$$\frac{\partial^{r_1+\cdots+r_n} F_\infty}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = O(|x|^{-n-1}),\ |x| \to \infty. \tag{2.22}$$

The next step is less obvious.

LEMMA 2. *The function $F(x)$ is $C^{(\infty)}$ and for any linear operator with constant coefficients*

$$Lf = \sum_{(s)} l_{s_1 \ldots s_n} \frac{\partial^{s_1 + \cdots + s_n} f}{\partial x_1^{s_1} \ldots \partial x_n^{s_n}}, \tag{2.23}$$

we have

$$\left.\begin{matrix} L_x\, F(x) \\ L_x\, F_\infty(x) \end{matrix}\right\} = [B, D;\ L_\xi f(\xi),\ L_\xi \rho(\xi)]\,(x). \tag{2.24}$$

REMARK 2. In the paper [4] the symbol Lf had a specialized meaning which it does not have in the present paper.

PROOF. Firstly, due to (2.5), $F(x)$ is certainly continuous. Next, if in

$$F(x) = [B,\ D;\ f(\xi),\ \rho(\xi)]\ (x) \tag{2.25}$$

we hold x fixed in D, then for a sufficiently small constant vector

$$h = (h_1, \ldots, h_n), \tag{2.26}$$

we can envisage the translated function

$$F(x+h) = [B,\ D;\ f(\xi),\ \rho(\xi)\ ]\ (x+h), \tag{2.27}$$

and since the right side of (2.25) is the expression (2.19) we have

$$F(x+h) = [\ B(h),\ D(h);\ f(\xi+h),\ \rho(\xi+h)\ ](x), \tag{2.28}$$

where $[\ B(h),\ D(h)\ ]$ arise from $[\ B,\ D]$ if we translate the latter by $-h$, that is,

$$B(h) = E\{\xi \mid \xi + h \in B\},\ D(h) = E\{\xi \mid \xi + h \in D\}. \tag{2.29}$$

Now $B - B(h)$ is the combinatorial boundary of the combinatorial complex $D - D(h)$, and for x fixed in D it follows from the lower half of (2.4) that for h sufficiently small we have

$$[B - B(h),\ D - D(h);\ f(\xi+h),\ \rho(\xi+h)]\ (x) = 0. \tag{2.30}$$

Thus we can replace (2.28) by

$$F(x+h) = [B,\ D;\ f(\xi+h),\ \rho(\xi+h)]\ (x). \tag{2.31}$$

We now introduce the special vector

$$h = (h,\ 0,\ 0,\ \ldots,\ 0), \tag{2.32}$$

and the difference quotients

$$F_h(x) = \frac{1}{h}\,(F(x+h) - F(x)), \tag{2.33}$$

$$f_h(x) = \frac{1}{h}\,(f(x+h) - f(x)),\ \rho_h(x) = \frac{1}{h}\,(\rho(x+h) - \rho(x)). \tag{2.34}$$

From (2.25) and (2.31) we then obtain

$$F_h(x) = [B,\ D;\ f_h(\xi),\ \rho_h(\xi)](x). \tag{2.35}$$

Due to (2.5) we can let $h \to 0$ on the right side. Therefore we can also let $h \to 0$ in (2.33), and we obtain

$$\frac{\partial F}{\partial x_1} = \left[B, D; \frac{\partial f}{\partial \xi_1}, \frac{\partial \rho}{\partial \xi_1} \right](x), \tag{2.36}$$

and we also find that $\dfrac{\partial F}{\partial x_1}$ is even continuous. The reasoning also applies to other coordinates, and the outcome is that $F(x)$ belongs at any rate to $C^{(1)}$ and that (2.24) holds for the first order operators

$$Lf \equiv \frac{\partial f}{\partial x_m}, \quad m = 1,\ldots,n. \tag{2.37}$$

But, we can iterate the procedure arbitrarily often and the lemma follows for $F(x)$. For $F_\infty(x)$ the proof is quite similar and in fact much easier.

LEMMA 3. *We have*

$$\Delta F(x) = \rho(x), \ \Lambda F(x) = \sigma(x), \tag{2.38}$$

$$\Delta F_\infty(x) = 0, \ \Lambda F_\infty(x) = 0. \tag{2.39}$$

PROOF. For $L = \Delta$ in (2.24) we obtain, because of (2.15),

$$\left.\begin{matrix} \Delta F \\ \Delta F_\infty \end{matrix}\right\} = [B, D; \rho(\xi), \Delta\rho(\xi)](x), \tag{2.40}$$

and for $L = \Lambda$ we obtain, because of (2.15) and (2.17),

$$\left.\begin{matrix} \Lambda F \\ \Lambda F_\infty \end{matrix}\right\} = [B, D; \sigma(\xi), \Delta\sigma(\xi)](x). \tag{2.41}$$

Now, $\rho(\xi)$ and $\sigma(\xi)$ are defined in $\tilde{D}$, and the conclusions of the lemma follow from (2.40) and (2.41) if we apply both parts of (2.4) for $f(x) = \rho(x)$ and $f(x) = \sigma(x)$.

LEMMA 4. *The function $F_\infty(x)$ has an analytic continuation into all of E_n; in fact it is a polynomial in $(x_1, \ldots, x_n)$ as is every function $\phi(x_1, \ldots, x_n)$ of the following kind. It is defined and analytic in a spherical exterior $|x| > R$; there is an $r_0 \geqslant 0$ such that for* (2.6)

$$\frac{\partial^{r_1+\ldots+r_n}\phi}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} = O(|x|^{-n-1}), \quad |x| \to \infty,$$

and there is an operator Λf satisfying our axiom I *for which*

$$\Lambda\phi = 0.$$

PROOF. See [3], pages 8-9.

REMARK 3. Any function of which we prove that it can be continued beyond its original domain of definition will after continuation be denoted by the same symbol as before.

LEMMA 5. *In the intersection*

$$D \cap \widetilde{B}, \tag{2.42}$$

which is an "interior boundary layer" of D we have

$$F(x) - F_\infty(x) = f(x). \tag{2.43}$$

PROOF. We deform B into a closed hypersurface B' within (2.42), so that B' decomposes E_n into domains D', D'_∞ with $D' \subset D$, $D'_\infty \supset D_\infty$. The pointset difference

$$D'_\infty - D_\infty \tag{2.44}$$

is then contained in $\widetilde{B}$. Now in D'_∞ we can introduce the function

$$F'_\infty(x) = [B', D'; f(\xi), \rho(\xi)]\,(x). \tag{2.45}$$

At first it only "corresponds" to $F_\infty(x)$, but in D_∞, which is a subdomain of D'_∞, the two are equal. In fact, for x in D_∞,

$$F_\infty(x) - F'_\infty(x) = [B - B', D - D'; f(\xi), \rho(\xi)]\,(x), \tag{2.46}$$

and by lower (2.4) this is 0. Thus (2.45) may be viewed as a continuation of F_∞ from D_∞ into D'_∞. Hence, for x in the domain (2.44), we have for $F(x) - F_\infty(x)$ the same expression

$$[B - B', D - D'; f(\xi), \rho(\xi)]\,(x), \tag{2.47}$$

but now x being a point of $D - D'$. By the upper (2.4) this is indeed $f(x)$, as claimed. Finally we note that any point of (2.42) can be made to lie in a domain (2.44) by proper choice of B', which completes the proof of the lemma.

We can now finally deal with Theorem 1 itself by proving its part (iii) as announced. Since $F(x)$ and $F_\infty(x)$ both exist in D, formula (2.43) gives a $C^{(\infty)}$ continuation of $f(x)$ from (2.42) into D and hence

also from $\widetilde{B}$ into $\widetilde{D}$. Also relations (2.38) and (2.39) imply for the difference (2.43) in D the relations $\Delta f = \rho$, $\Lambda f = \sigma$, which completes the proof of our assertions.

REMARK 4. Relation (2.43) implies that if $f(x)$ and $\rho(x)$ are analytic in $\widetilde{B}$, $F(x)$ can be continued "outwardly" from D into $\widetilde{D}$. This by itself can also be obtained "directly", that is without first proving the continuability of F_∞, as follows. We deform B into a hypersurface B'' which is located in the "exterior" boundary layer $D \cap \widetilde{D}_\infty$, and we set up the function

$$F''(x) = [B'', D''; f(\xi), \rho(\xi)]\,(x) \tag{2.48}$$

in the bounded domain $D'' \supset D$ bounded by B''. It follows by the lower (2.4) that, for x in D, the expression (2.48) is equal to (2.25) itself, so that in D, $F''(x) = F(x)$. Next, by Lemma 3, $\Delta F'' = \rho(x)$ in D'', so that $F''(x)$ is analytic in D'', just as $F(x)$ is analytic in D. Hence $F''(x)$ is an analytic continuation of $F(x)$ outwardly.

3. Several closed hypersurfaces in Euclidean space. In the case of Theorem 1 we have introduced and even emphasized its part (iii) which refers to non-analytic data, mainly in order to delineate the "true" range and scope of the entire theory. From now on however, in order to avoid certain inconveniences of formulation and reasoning we will be dealing with the analytic case exclusively.

We are hereby postulating that henceforth all *given* functions are to be analytic (in the real components of the underlying space). For *constructed* functions it will not be immediately obvious whether they are analytic and it will be expressly stated if they are. Also the word "continuation" by itself will mean analytic continuation, both in assumptions and conclusions.

We take in E_n a finite number of strictly disjoint combinatorially closed simplicial hypersurfaces, say in their "natural" orientation and denote them by

$$B_1, B_2, \ldots, B_s. \tag{3.1}$$

They decompose E_n into a finite number of domains which in their natural orientation we denote by

$$D_1, \ldots, D_{t-1}, D_t. \tag{3.2}$$

The last domain, that is D_t, shall be the one which is unbounded (and which corresponds to D_∞ of the previous section). We introduce the incidence matrix

$$\eta_{q\tau} = 0, +1, -1; \; 1 \leqslant q \leqslant s, \; 1 \leqslant \tau \leqslant t, \tag{3.3}$$

where $\eta_{q\tau} = 0$ if B_q is not on the boundary of D_τ; $= 1$ if B_q is on the boundary of D_τ and is co-oriented with it; and $= -1$ if it is counter-oriented with it. Each B_q " separates " two of our domains (3.2), and we can denote these by D_u, D_v in such a way that

$$\eta_{qu} = +1; \; \eta_{qv} = -1; \; \eta_{q\tau} = 0 \text{ for } \tau \neq u, v. \tag{3.4}$$

With each B_q we associate a pointset neighbhorhood $\widetilde{B}_q$, and these neighborhoods shall be so small that any two are disjoint. Also, we denote by $\widetilde{D}_\tau$ a neighbhorhood of the closure of D_τ, and it is convenient to take

$$\widetilde{D}_\tau = D_\tau \cup \sum_{\eta_{q\tau} \neq 0} \widetilde{B}_q. \tag{3.5}$$

We now assume that there are given (analytic) functions

$$\rho_1(x), \; \rho_2(x), \; \ldots, \; \rho_{t-1}(x), \; \rho_t(x) \tag{3.6}$$

and

$$f_1(x), f_2(x), \ldots, f_s(x) \tag{3.7}$$

of the following interlocking descriptions. Each ρ_τ is given in $\widetilde{D}_\tau$ and f_q in $\widetilde{B}_q$, and for each q we have

$$\Delta f_q(x) = \sum_{\tau=1}^{t} \eta_{q\tau} \, \rho_\tau(x) = \rho_u(x) - \rho_v(x), \tag{3.8}$$

in $\widetilde{B}_q$. Finally for the present we are imposing the rigorous restriction, which in the next section will be relaxed, that the $\rho(x)$ in the infinite domain D_t is strictly 0, that is,

$$\rho_t(x) = 0, \; x \in \widetilde{D}_t. \tag{3.9}$$

In the set-up of Theorem 1 this corresponds to the fact that no $\rho_\infty(x)$ was assigned to $\widetilde{D}_\infty$.

We are introducing again the expression (2.20) and now we are defining it as follows :

$$[B,\ D;\ f(\xi),\ \rho(\xi)]\,(x) = \sum_{q=1}^{s} \int_{B_q} \Omega(f_q(\xi),\ \xi - x,\ d\xi) - \\ - \sum_{\tau=1}^{t} \int_{D_\tau} \rho_\tau(\xi).\, G(\xi - x).\, dv_\xi. \qquad (3.10)$$

The function

$$F(x) = [\,B,\ D\,;\, f(\xi),\ \rho(\xi)\,]\,(x) \qquad (3.11)$$

is now defined in the union $\bigcup_\tau D_\tau$, and separately we have a function

$$F_\tau(x) \text{ in } D_\tau,\ 1 \leqslant \tau \leqslant t. \qquad (3.12)$$

LEMMA 6. *For given functions ρ_τ, f_q with* (3.8) *if there is also given a function $\phi_\tau(x)$ in $\widetilde{D}_\tau$, $1 \leqslant \tau \leqslant t$, such that*

$$\rho_\tau(x) = \Delta\,\phi_\tau(x), \quad \phi_t(x) = 0, \qquad (3.13)$$

$$f_q(x) = \sum_{\tau=1}^{t} \eta_{q\tau}\,\phi_\tau(x) = \phi_u(x) - \phi_v(x), \qquad (3.14)$$

then the "operator" (3.11) *reproduces $\{\phi(x)\}$, that is*

$$F_\tau(x) = \phi_\tau(x), \quad \tau = 1, \ldots, t. \qquad (3.15)$$

PROOF. Follows by a judicious application of (2.4).

LEMMA 7. *Let x be given fixed in one of the D_{τ_0}. If B'_q is a sufficiently small deformation of B_q, $1 \leqslant q \leqslant s$, and $\{D'_\tau\}$ are the resulting deformations of $\{D_\tau\}$, then*

$$[B,\ D\,;\, f(\xi),\ \rho(\xi)\,]\,(x) = [B',\ D'\,;\, f(\xi),\ \rho(\xi)]\,(x). \qquad (3.16)$$

PROOF. Any small deformation can be built up "additively" from deformations in neighborhoods of single points. For instance, let B_q separate D_u, D_v as in (3.4). Let $\bar{B}_q$ be a small piece of B_q and let the deformation consist of replacing it by a small piece of $\bar{B}'_q$ which lies in D_u, and let R denote the piece of D_u bounded by $\bar{B}_q - \bar{B}'_q$. In this case we have on the one hand, by (2.4) and (3.8),

$$\sum_q \Big[\int_{B_q} \Omega(f_q(\xi), \xi - x, d\xi) - \int_{B'_q} \Omega(f_q(\xi), \xi - x, d\xi) \Big]$$
$$= \int_R (\rho_u - \rho_v) \cdot G(\xi - x)\, dv_\xi.$$

But, on the other hand, also

$$\sum_\tau \Big[\int_{D_\tau} \rho_\tau \cdot G \cdot dv_\xi - \int_{D'_\tau} \rho_\tau \cdot G\, dv_\xi \Big]$$
$$= \int_{D_u} \rho_u(\xi) \cdot G(\xi - x)\, dv_\xi - \int_{D_u - R} \rho_u(\xi) \cdot G(\xi - x)\, dv_\xi$$
$$+ \int_{D_v} \rho_v(\xi) \cdot G(\xi - x)\, dv_\xi - \int_{D_v + R} \rho_v(\xi) \cdot G(\xi - x)\, dv$$
$$= \int_R (\rho_u - \rho_v) \cdot G(\xi - x) \cdot dv_\xi,$$

and this verifies (3.16) for our deformation.

Lemmas 6 and 7 lead by the same steps as in Section 2 to the following proposition.

THEOREM 2. (i) *The functions* $F_\tau(x)$, (see (3.12)), *are immediately seen to be continuous, and the function* $F_t(x)$ *to be analytic.*

(ii) *But in fact each* $F_\tau(x)$ *is* $C^{(\infty)}$ *and for any linear operator* (2.23) *we have*

$$L_x F(x) = [B, D; L_\xi f(\xi), L_\xi \rho(\xi)]\,(x). \tag{3.17}$$

(iii) *Especially for* $L = \Delta$ *we have*

$$\Delta F_\tau(x) = \rho_\tau(x), \tag{3.18}$$

so that every $F_\tau(x)$ *is analytic.*

(iv) *Furthermore, each* $F_\tau(x)$ *has an analytic continuation into* $\widetilde{D}_\tau$, *and also*

(v) *For each* q, *we have in* $\widetilde{B}_q$

$$f_q(x) = \sum_{\tau=1} \eta_{q\tau} F_\tau(x) = F_u(x) - F_v(x), \tag{3.19}$$

(see (3.4)).

(vi) *Finally, if there is given some operator* Λf, *at first not necessarily defective as in axiom* I, *and if there are given functions*

$$\sigma_1(x), \ldots, \sigma_t(x), \tag{3.20}$$

σ_τ *in* $\widetilde{D}_\tau$, *such that*

$$\Lambda \rho_\tau = \Delta \sigma_\tau, \quad \tau = 1, \ldots, t,$$

and

$$\Lambda f_q = \sum_{\tau=1}^{t} \eta_{q\tau} \sigma_\tau, \quad q = 1, \ldots, s,$$

then

$$\Lambda F_\tau = \sigma_\tau, \quad \tau = 1, \ldots, t.$$

The defectiveness of Λf comes in only at the very end to give the following conclusion à la Hartogs.

THEOREM 3. *If* Λf *is indeed as prescribed in axiom* I, *and if* $\sigma_t = 0$ (*only for* $\tau = t$); *then* $F_t = 0$ (*only for* $\tau = t$), *and for any* q *for which* $\eta_{qt} \neq 0$, *the function* $f_q(x)$ *can be continued from* $\widetilde{B}_q$ *into the domain* $\widetilde{D}_\tau$ *for which* $\eta_{q\tau} = -\eta_{qt}$.

REMARK 5. We have imposed the rigorous restriction (3.9) only in order to avoid discussing the integrals

$$\int_{D_t} \rho_t(\xi - x)\, dv_\xi, \quad \int_{D_t} L_\xi\, \rho_t(\xi) \cdot G(\xi - x) dv_\xi$$

over the unbounded domain D_t, and such integrals for $\phi_t(x)$ in the place of $\rho_t(x)$ in Lemma 6. However, due to (2.5) and (2.10), all such integrals as are actually required in Theorem 2 would become manageable and admissible provided the function $\rho_t(x)$ and certain of its partial derivatives are sufficiently small at infinity.

4. A Cousin type theorem in Euclidean space. The present section consists of two distinct parts. In the first part we will state for Theorem 2, without proof, a generalization of the topological set-up which we have already introduced and featured in our first paper [1]. In the second part we will give an interpretation of Theorem 2 which is fairly meaningful for this theorem itself but becomes appreciably

more so after the topological generalizations of the first part has been superimposed.

We are retaining the previous decomposition of E_n into finitely many domains with simplicial boundaries,

$$D_1, \dots, D_{t-1}, D_t, \tag{4.1}$$

with D_t being unbounded and the other domains bounded. However, the bounding $(n-1)$-dimensional simplices need no longer combine into finitely many distinct hypersurfaces each with combinatorial boundary 0. We only assume at present, for better visualization, that each such simplex separates two different domains, although in [1] even this was not required. These $(n-1)$-dimensional *simplices* themselves, *individually*, and each occurring only once, shall now be denoted by

$$B_1, \dots, B_s. \tag{4.2}$$

The pointset closures of any two B_q shall be either disjoint or their intersection shall be the pointset closure of an entire $(n-2)$ dimensional face common to the two. The set of all these faces, each counted only once, shall be denoted by

$$C_1, \dots, C_r. \tag{4.3}$$

The domains (4.1) have "natural" orientations and we adopt those, so that the "external" monomial (2.12) represents the positive volume element dv_ξ. However, for the simplices (4.2) and (4.3) no "natural" orientations are available now. We impart some fixed orientations to them and we introduce two incidence matrices: firstly the matrix

$$\eta_{q\tau} = 0, +1, -1, \quad 1 \leqslant q \leqslant s, \tag{4.4}$$

between B_q, D_τ as before, and secondly the corresponding matrix

$$\epsilon_{pq} = 0, +1, -1, \quad 1 \leqslant p \leqslant r, \quad 1 \leqslant q \leqslant s \tag{4.5}$$

between C_p, B_q. We recall the basic elementary relation

$$\sum_{q=1}^{s} \epsilon_{pq}\, \eta_{q\tau} = 0. \tag{4.6}$$

We again introduce ϵ-neighborhoods $\widetilde{B}_q$ of B_q, then neighborhoods $\widetilde{D}_\tau$ of D_τ, say as defined by (3.5), and also the neighborhoods

$$\widetilde{C}_p \equiv \bigcap_{|\epsilon_{pq}|\neq 0} \widetilde{B}_q \tag{4.7}$$

of C_p.

The incidence matrix (4.5), which we did not need in Section 3, will now enter by the following opening. If in each $\widetilde{D}_\tau$ there is defined a function $\phi_\tau(x)$, $1 \leqslant \tau \leqslant t$, and if we introduce the "saltuses"

$$f_q(x) = \sum_{\tau=1}^{t} \eta_{q\tau}\, \phi_\tau(x) = \phi_u(x) - \phi_v(x) \tag{4.8}$$

in the $\widetilde{B}_q$, then (4.6) implies that the latter satisfy the relations

$$\sum_{q=1}^{s} \epsilon_{pq} f_q(x) = 0 \tag{4.9}$$

in the $\widetilde{C}_p$. These relations, which were introduced and featured for the first time (apparently) in our paper [1], were not stated in Section 3 because they were there "vacuous", but now they will have to be stipulated expressly.

DEFINITION 1. *We again introduce and define the functions*

$$F(x) = [B, D; f(\xi), \rho(\xi)]\,(x) \tag{4.10}$$

by the same expressions (3.10) *as before, for any data of the following description. Each function $f_q(x)$ is defined in its $\widetilde{B}_q$, and, what is important, the relations* (4.9) *holds. Each function $\rho_\tau(x)$ is defined in its $\widetilde{D}_\tau$, and again we impose the rigorous restriction $\rho_t = 0$. And the functions $\{f_q\}$ and $\{\rho_\tau\}$ are again connected by the relations* (3.8).

This leads to the following generalizations.

LEMMA 6′. *Lemma 6 holds literally as before.*

The proof is the same.

LEMMA 7′. *Lemma 7 holds literally as before.*

This lemma is a serious claim, but we will not go into details. We are referring to pages 654-5 of [1] for a basic lemma needed for the generalization.

THEOREM 2′. *Theorem 2 holds literally as before.*

THEOREM 3′. *Theorem 3 holds literally as before.*

REMARK 5′. Remark 5 applies again.

These things stated, we are now turning to an interpretation of Theorem 2 (and Theorem 2′) which will be the second half of this section. We take in each $\widetilde{D}_\tau$ a function $\phi_\tau(x)$ which, as always, is real-analytic, however we are permitting, as an exception, that each $\phi_\tau(x)$ have "singularities." We are not concerned with the actual nature of the singularities, so that singularities are for us points at which the functions are not defined. Specifically we take a closed set E in E_n, and we assume that each ϕ_τ is defined and analytic in the pointset difference

$$\widetilde{D}_\tau - E = \widetilde{D}_\tau - E_\tau, \tag{4.11}$$

where

$$E_\tau = \widetilde{D}_\tau \cap E. \tag{4.12}$$

It is an open set, and we do not have to assume that it is connected, that is, a subdomain of $\widetilde{D}_\tau$. With such functions we form the "saltuses"

$$f_q = \sum_{\tau=1}^{\ell} \eta_{q\tau}\, \phi_\tau, \tag{4.13}$$

each of which is defined at first not in $\widetilde{B}_q$ but in

$$\widetilde{B}_q - E = \widetilde{B}_q - \widetilde{B}_q \cap E. \tag{4.14}$$

Now, we expressly assume that the singularities of f_q are "removable", meaning that there exists an (analytic) function which we also denote by f_q, in all of $\widetilde{B}_q$ which in (4.14) coincides with (4.13). If (4.14) is connected, this "larger" $f_q(x)$ is unique, otherwise perhaps not. We need not analyze the possibilities, but for the

general topological set-up we do have to make expressly the indispensable assumptions

$$\Sigma \epsilon_{pq} f_q = 0, \quad 1 \leqslant p \leqslant r, \tag{4.15}$$

for the enlarged functions $\{f_q\}$. If the functions ϕ_τ have no singularities, (4.15) is a formal consequence of (4.14), as previously noted, but with our very general notion of singularities it is safest to stipulate (4.15) expressly.

Next we form the functions

$$\rho_\tau(x) = \Delta \phi_\tau(x). \tag{4.16}$$

To begin with, they are defined only in (4.11), however we assume that each $\rho_\tau(x)$ shall have some analytic continuation into all of $\widetilde{D}_\tau$, and that (in all of $\widetilde{B}_q$) the relation

$$\Delta f_q = \sum_{\tau=1}^{t} \eta_{q\tau} \rho_\tau \tag{4.17}$$

holds. Finally we are also imposing the rigorous restriction $\phi_t = 0$, and hence also $\rho_t = 0$.

The conclusion from Theorem 2 (2′) is now as follows.

INTERPRETATION. *There exist fixed functions $F_\tau(x)$, $1 \leqslant \tau \leqslant t$, each analytic in all of $\widetilde{D}_\tau$, such that*

$$\Delta F_\tau(x) = \rho_\tau(x) \equiv \Delta \phi_\tau(x), \tag{4.18}$$

and also

$$\Sigma \eta_{q\tau} F_\tau(x) = f_q(x) \equiv \Sigma \eta_{q\tau} \phi_\tau(x). \tag{4.19}$$

If for some further operator Λf the functions

$$\sigma_\tau = \Lambda \phi_\tau \tag{4.20}$$

are also continuable into all of $\widetilde{D}_\tau$ and

$$\Delta \sigma_\tau = \Lambda \rho_\tau, \quad 1 \leqslant \tau \leqslant t, \tag{4.21}$$

then also

$$\Lambda F_\tau(x) = \sigma_\tau(x) \equiv \Lambda \phi_\tau(x). \tag{4.22}$$

5. A lemma from the theory of integral invariants. We now take an n-dimensional oriented real torus of total Euclidean volume 1,

$$\det |g_{ij}| = 1, \tag{5.1}$$

and we represent it in E_n by the fundamental domain

$$-\tfrac{1}{2} < x_m \leqslant \tfrac{1}{2}, \quad m = 1, 2, \ldots, n, \tag{5.2}$$

or any other pointset which arises from this by a homogeneous affine transformation with determinant $+1$. We will denote by S_n either the torus as a compact space, or one of these fundamental domains, depending on the context.

Given any constants

$$a^1, a^2, \ldots, a^n, \tag{5.3}$$

we can interpret them on S_n as a contravariant vector. With each such set of constants we will associate on S_n two objects which will be denoted by the use of the letter A; firstly the linear partial differential operator of first order

$$AF(x) \equiv a^1 \frac{\partial F}{\partial x_1} + \ldots + a^n \frac{\partial F}{\partial x_n}, \tag{5.4}$$

and secondly the $(n-1)$-dimensional external form

$$A(d\xi) = (\det |g_{ij}|)^{1/2} \sum_{\alpha=1}^{n} a^\alpha \, d\xi_1 \wedge \ldots \wedge d\xi_{\alpha-1} \wedge d\xi_{\alpha+1} \wedge \ldots \wedge d\xi_n. \tag{5.5}$$

The factor in front, $\det |g_{ij}|$, has numerical value $+1$ by (5.1), but it is technically needed in order to "transform" the vector component a^α into a tensor component $b_{1\ldots\alpha-1,\,\alpha+1\ldots n}$ of a covariant skewtensor of dimension $n-1$.

Finally we associate with our constants (5.3)—if they are real valued—the one parametric group of translations

$$x'_\alpha = x_\alpha + a^\alpha h, \; \alpha = 1, \ldots, n; \; -\infty < h < \infty \tag{5.6}$$

on S_n. If B is a closed oriented hypersurface on S_n, then (5.6) "deforms" it into another such hypersurface, which we denote by B^h, and we introduce the n-dimensional combinatorial complex whose boundary is $B^h - B$. The complex will be denoted by R^h.

If $\widetilde{B}$ is a neighborhood of B then for sufficiently small h the pointset underlying R^h is contained in a compact subset of $\widetilde{B}$.

LEMMA 8. *If $f(\xi)$ is defined and continuous in $\widetilde{B}$, then*

$$\lim_{h\to 0}\frac{1}{h}\int_{R^h} f(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n = \int_B f(\xi)\, A(d\xi), \tag{5.7}$$

the limit existing.

PROOF. For the purpose of proof one can replace the neighborhood $\widetilde{B}$ by a smaller neighborhood and denote the smaller by the same symbol. We take some oriented domain D whose oriented boundary consists of B and some other hypersurface B_0 which is disjoint from the closure of $\widetilde{B}$. We can find in all of S_n a continuous function $g(x)$ which in $\widetilde{B}$ equals $f(x)$ and in a neighborhood of B_0 has value 0. Now, in (5.7) itself, we may replace $f(x)$ by $g(x)$ and for the proof of (5.7) utilize properties of the latter. Now for $g(x)$, if D^h is the image of D under (5.6) we have identically

$$\frac{1}{h}\int_{R^h} g(\xi) d\xi_1 \wedge \cdots \wedge d\xi_n = \frac{1}{h}\left(\int_{D^h} - \int_D\right) g(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n. \tag{5.8}$$

However, by a theorem from the theory of invariants, see [5], the right side in (5.8) has indeed a limit, as $h \to 0$, and its value is

$$\int_{B^h} g(\xi).\, A(d\xi) - \int_{B_0^h} g(\xi).\, A(d\xi). \tag{5.9}$$

But the second integral is 0, and in the first we may replace the symbol $g(\xi)$ by $f(\xi)$, which proves the lemma.

6. Residues on toruses. Any differential operator with constant coefficients can be interpreted as acting on S_n for any fundamental domain. With the operator (1.1) we associate for a brief moment the ordinary polynominal in the real variables $t_1, \ldots, t_n$,

$$T(t_1, \ldots, t_n) = \sum_{(p)} a_{p_1 \ldots p_n} (2\pi i t_1)^{p_1} \ldots (2\pi i t_n)^{p_n}, \tag{6.1}$$

and we decompose it in the form

$$T(t) = T_{2h}(t) + T^0(t), \tag{6.2}$$

where $T_{2h}(t)$ consists of the highest terms with

$$p_1 + \ldots + p_n = 2h,$$

and $T^0(t)$ of the additional term with

$$p_1 + \ldots + p_n < 2h.$$

For our purposes the operator Δf is elliptic on every S_n, if firstly the principal part is positive definite, that is

$$T_{2h}(t_1, \ldots, t_n) > 0 \text{ for } t_1^2 + \ldots + t_n^2 > 0, \tag{6.3}$$

and also, (if T^0 is not $\equiv 0$,)

$$(T(t) \equiv)\ T_{2h}(t) + T^0(t) \neq 0 \text{ for } t_1^2 + \ldots + t_n^2 > 0. \tag{6.4}$$

We have shown ourselves in [6], that for instance for the fundamental domain (5.2) the limit

$$G(\xi) = \lim_{\epsilon \downarrow 0} \sum_{(r)}{}' \frac{e^{-\epsilon(r_1^2 + \ldots + r_n^2)}\, e^{2\pi i(r_1 x_1 + \ldots + r_n x_n)}}{T(r_1, \ldots, r_n)} \tag{6.5}$$

exists for $(\xi_1 \ldots, \xi_n) \neq (0, \ldots, 0)$ and constitutes a Green's function with which a Green's formula can be erected. However, due to the compactness of S_n the Green's function does not satisfy relation (2.13) but instead the relation

$$\Delta G(\xi) = -1, \tag{6.6}$$

and this introduces a certain " complication " into the set-up. The Green's formula becomes

$$\left.\begin{matrix} f(x) \\ 0 \end{matrix}\right\} = [B,\ D;\ f(\xi),\ \Delta f(\xi)]\,(x) + \int_D f(\xi)\, dv_\xi, \tag{6.7}$$

where, as before, the first term on the right is again the difference of integrals (2.19). However there is the additional term

$$\int_D f(\xi)\, dv_\xi \tag{6.8}$$

which is the same for the upper and lower half and depends on the configuration, that is on D, and on the given function $f(\xi)$, and linearly on the latter. However it does not depend on the variable

point x, and in this sense it is a constant. This kind of " additional " quantity we term a *residue*. In the Green's formula, the residue is a volume integral. In the subsequent theorems it will engender other residues, and these will be integrals over $(n-1)$-dimensional hypersurfaces (or simplices). We will not be led to introducing residues which are integrals over lower dimensional simplices but it would be possible to undertake elaboration of the theory that would lead to such.

Proceeding as in Section 3 we are introducing on S_n a decomposition into a finite number of domains

$$D_1, \ldots, D_{t-1}, D \tag{6.9}$$

with separating closed surfaces

$$B_1, \ldots, B_s, \tag{6.10}$$

and with the incidence matrix

$$\eta_{q\tau}, \ 1 \leqslant q \leqslant s, 1 \leqslant \tau \leqslant t. \tag{6.11}$$

The domains (6.9) have natural orientations and we endow them with those, for simplicity's sake. The hypersurfaces, if in general positions on the torus, do no longer have " natural " orientations, but this causes no syllogistic inconvenience whatsoever. We impart to each B_q some fixed orientation, and we introduce the incidence matrix (6.11) accordingly.

REMARK 6. On the torus we do *not* single out a "last" domain D_t in any manner whatsoever, and there will be no analog whatsoever to the " rigorous restriction " (3.9).

We again introduce neighborhoods $\widetilde{B}_q$, $\widetilde{D}_\tau$ and, with function f_q in $\widetilde{B}_q$, ρ_τ in $\widetilde{D}_\tau$, which are linked by

$$\Delta f_q(\xi) = \sum_\tau \eta_{q\tau} \rho_\tau(\xi), \tag{6.12}$$

we again set up the functions

$$F_\tau(x) \text{ in } D_\tau, \tag{6.13}$$

defined collectively by

$$F(x) = [B,\ D;\ f(\xi),\ \rho(\xi)\,](x). \tag{6.14}$$

The right side is identically the same formula as in Section 3, namely

$$\sum_{q=1}^{s} \int_{B_q} \Omega(f_q(\xi),\ \xi - x,\ d\xi) \ - \sum_{\tau=1}^{t} \int_{D_\tau} \rho_\tau(\xi)\ G(\xi - x)\ dv_\xi. \tag{6.15}$$

The functions (6.13) are at first sight certainly continuous. Our first formal statement is an analog to Lemma 6. It will differ from Lemma 6 by the occurrence of a residue, and because of this novelty we will state it as a full theorem. But the proof is quite simple and will be left to the reader.

THEOREM 4. *If there are given functions $\phi_\tau(x)$ in the $\widetilde{D}_\tau$ (without any singularities), and we again introduce the functions*

$$\rho_\tau(\xi) = \Delta\,\phi_\tau(\xi), \tag{6.16}$$

$$f_q(\xi) = \sum_\tau \eta_{q\tau}\ \phi_\tau(\xi), \tag{6.17}$$

which satisfy (6.12), *then*

$$F_\tau(x) = \phi_\tau(x) - \text{residue}, \tag{6.18}$$

where, at present,

$$\text{residue} = \sum_{\tau=1}^{t} \int_{D_\tau} \phi_\tau(\xi)\ dv_\xi. \tag{6.19}$$

The next theorem is the analog to Lemma 7. The proof will also be left to the reader although it requires a little more attention.

THEOREM 5. *If x is held fixed in one of the domains, if B'_q is a sufficiently small deformation of B_q, $1 \leqslant q \leqslant s$, and D'_τ is the resulting deformation of D_τ, $1 \leqslant \tau \leqslant t$, and if we denote by R_q the n-dimensional chain whose boundary is $B'_q - B_q$, that is,*

$$\text{boundary of } R_q = B'_q - B_q, \tag{6.20}$$

then

$$[B',\ D';\ f(\xi),\ \rho(\xi)\,](x) - [B,\ D;\ f(\xi),\ \rho(\xi)\,](x) = -\,\text{residue}, \tag{9.21}$$

where at present,

$$\text{residue} = \sum_{q} \int_{R_q} f_q(\xi)\, d\xi_1 \wedge \ldots \wedge d\xi_n. \tag{6.22}$$

Herein $\{f_q\}$, $\{\rho_\tau\}$ *are general functions satisfying* (6.12).

The next two theorems taken together are the analog to part (ii) of Theorem 3.

THEOREM 6. *Each function* $F_\tau(x)$ *has continuous first derivatives, and if for any constants* (5.3) *we introduce the differential operator*

$$A_x F(x) = a^1 \frac{\partial F}{\partial x_1} + \ldots + a^n \frac{\partial F}{\partial x_n}, \tag{5.4}$$

then

$$A_x F(x) = [\, B,\ D;\ A_\xi f(\xi),\ A_\xi \rho(\xi)\,]\,(x) - \text{residue}, \tag{6.23}$$

where at present

$$\text{residue} = \sum_{q=1}^{s} \int_{B_q} f(\xi).A(d\,\xi), \tag{6.24}$$

and $A(d\,\xi)$ *is defined by* (5.5).

PROOF. We first take the particular constants

$$a^1 = 1, a^2 = 0, \ldots, a^n = 0, \tag{6.25}$$

so that (5.4) is

$$A_x F = \frac{\partial F}{\partial x}. \tag{6.26}$$

As in Section 2 we introduce the vector $h = (h, 0, \ldots, 0)$, the shifted set $B_q(h)$, $D_\tau(h)$, and the difference quotients $F_{\tau h}(x)$, $f_{qh}(x)$, $\rho_{\tau h}(x)$ as defined by (2.33) and (2.34). We then obtain

$$F_h(x) = \frac{1}{h}\{[\, B(h),\ D(h);\ f(\xi + h),\ \rho(\xi + h)\,]\,(x) - [B,\ D;\, f(\xi),\ \rho(\xi)]\,(x)\}. \tag{6.27}$$

If now $R_q(h)$ is the chain bounded by $B_q - B_q(h)$, then by Theorem 5 we obtain

$$F_h(x) = [\, B,\ D;\ f_h(\xi),\ \rho_h(\xi)]\,(x) - \text{residue}, \tag{2.28}$$

where

$$\text{residue} = \frac{1}{h}\sum_{q}\int_{R_q^{(h)}} f_q(\xi)\, d\xi_1 \wedge \ldots \wedge d\xi_n. \tag{6.29}$$

We now wish to let $h \to 0$ in (6.28). Obviously

$$\lim_{h\to 0}\,[B, D; f_h(\xi), \rho_h(\xi)]\,(x) = \left[B, D; \frac{\partial f}{\partial \xi_1}, \frac{\partial \rho}{\partial \xi_1}\right](x),$$

the limit existing. Now, in consequence of Lemma 8, we have

$$\lim_{h\to 0}\,(\text{residue}) = \sum_{q}\int_{B_q} f_q(\xi).\, d\xi_2 \wedge \ldots \wedge d\xi_n, \tag{6.30}$$

this limit also existing. Therefore the limit

$$\lim_{h\to 0} F_h(x) = \frac{\partial F}{\partial x_1} \tag{6.31}$$

exists too, and the limit of (6.28) is the relation (6.23) corresponding to the particular constants (6.25). The same reasoning gives formula (6.23) for the constants $a^m = 1$, $a^\nu = 0$, $\nu \neq m$, any m, and since (6.23) is additive in the vector $(a^1, \ldots, a^2)$, the theorem follows.

THEOREM 7. *Each function $F_\tau(x)$ is $C^{(\infty)}$ and for any linear operator Lf, see* (2.23), *we have*

$$L_x F(x) = [B, D; L_\xi f(\xi), L_\xi \rho(\xi)]\,(x) - \text{residue}. \tag{6.32}$$

REMARK 7. This "residue" is a quantity which is independent of x and τ. A formula for it will be given in the next section.

PROOF. We apply induction on the order of Lf. If the order is 1, Theorem 7 is a consequence of Theorem 6. Now an operator of an order $s+1$ can be represented as a sum

$$L = \frac{\partial}{\partial x_1} L^1 + \ldots + \frac{\partial}{\partial x_n} L^n + L^0 \tag{6.33}$$

in which $L^0, L^1, \ldots, L^n$ is each of order $\leqslant s$. If we assume that the theorem holds for order $\leqslant s$, then

$$L^m F(x) = [B, D; L_\xi^m f(\xi), L_\xi^m \rho(\xi)]\,(x) - \text{constant},$$
$$m = 0, 1, \ldots n, \tag{6.34}$$

and by differentiation we obtain

$$\left(\frac{\partial}{\partial x_m} L^m\right) F = \frac{\partial}{\partial x_m} \{[B, D; L^m f(\xi), L^m \rho(\xi)](x)\}, \quad 1 \leqslant m \leqslant n. \tag{6.35}$$

But if we apply Theorem 6 to the functions $\{L^m f(\xi), L^m \rho(\xi)\}$ in the place of the functions $\{f(\xi), \rho(\xi)\}$ with $A = \dfrac{\partial}{\partial x_m}$, then we finally obtain

$$\left(\frac{\partial}{\partial x_m} L^m\right) F = \left[B, D; \left(\frac{\partial}{\partial \xi_m} L^m\right) f(\xi), \left(\frac{\partial}{\partial \xi_m} L^m\right) \rho(\xi)\right](x) - \\ - \text{residue}. \tag{6.36}$$

Relation (6.32) itself follows now if we add up relations (6.36) for $m = 1, \ldots, n$ and relation (6.34) for $m = 0$.

THEOREM 8. *In particular*

$$\Delta F_\tau(x) = \rho_\tau(x) - \text{residue}, \quad 1 \leqslant \tau \leqslant t \tag{6.37}$$

so that each $F_\tau(x)$ is even analytic.

REMARK 8. The residue in (6.37) is a constant in x which does not even depend on τ.

PROOF. If in Theorem 7 we put $L = \Delta$ we obtain

$$\Delta F = [B, D; \Delta f(\xi), \Delta \rho(\xi)](x) - (\text{res})_1, \tag{6.38}$$

and, by Theorem 4 in connection with (6.12),

$$[B, D; \Delta f(\xi), \Delta \rho(\xi)] = \rho(\xi) - (\text{res})_2, \tag{6.39}$$

where, as it happens,

$$(\text{res})_2 = \sum_\tau \int_{D_\tau} \rho_\tau(\xi)\, dv_\xi. \tag{6.40}$$

This proves (6.37) with

$$\text{residue} = (\text{res})_1 + (\text{res})_2. \tag{6.41}$$

THEOREM 9. (i) *Each $F_\tau(x)$ has a continuation from D_τ into $\tilde{D}_\tau$ and*

(ii) *we have*

$$\sum_{\tau=1}^{t} \eta_{q\tau} F_\tau(x) = f_q(x), \tag{6.42}$$

in $\widetilde{B}_q$.

PROOF. We single out one of the hypersurfaces. We assume that it is B_1 and that it separates the domain D_u with which it is co-oriented, from the domain D_v with which it is counteroriented. We replace a small piece of B_1 by a small piece of hypersurface located in D_v and let this be the total deformation of our system of hypersurfaces. Let R denote the small piece of D_v bounded by $B'_1 - B_1$. For x in D_u we introduce our original function

$$F_u(x) = [B, D; f(\xi), \rho(\xi)](x), \tag{6.43}$$

and the second function

$$F'_u(x) = [B', D'; f(\xi), \rho(\xi)](x). \tag{6.44}$$

By Theorem 5 we have

$$F'_u(x) = F_u(x) - c, \tag{6.45}$$

where, as it happens, the constant c has the value

$$c = \int_R f_1(\xi)\, dv_\xi. \tag{6.46}$$

Now, by Theorem 8, F'_u is defined and analytic in D'_u which is D_u plus R, and thus, by (6.45), $F_u(x)$ itself has a continuation into D'_u. This proves part (i).

Let the symbol $F_u(x)$ denote the continued function in R, and for x in R we wish to determine the value of the difference

$$F_u(x) - F_v(x). \tag{6.47}$$

It is

$$F'_u(x) - F_v(x) + c = c + [B', D'; f, \rho](x) - [B, D; f, \rho](x)$$

$$= c + \int_{B'_1 - B_1} \Omega(f_1, \xi - x, d\xi) - \int_{D_u + R} \rho_u \, dv -$$

$$- \int_{D_v - R} \rho_v \, dv + \int_{D_u} \rho_u \, dv - \int_{D_v} \rho_u \, dv$$

$$= c + \int_{B'_1 - B_1} \Omega(f_1, \xi - x, d\xi) - \int_R (\rho_u(\xi) - \rho_v(\xi)) \, dv_\xi.$$

Now, by the upper part of (6.7) and (6.46) this is

$$c + f_1(x) - c = f_1(x),$$

which proves part (ii) of the theorem.

THEOREM 9. *If furthermore for some operator* Λf *and functions* $\sigma_\tau(x)$ *in the* $\widetilde{D}_\tau(x)$ *we have*

$$\Lambda \rho_\tau(x) = \Delta \sigma_\tau(x), \quad 1 \leqslant \tau \leqslant t, \tag{6.48}$$

and

$$\Lambda f_q(x) = \sum_{\tau=1}^{t} \eta_{q\tau} \sigma_\tau(x), \quad 1 \leqslant q \leqslant s, \tag{6.49}$$

then

$$\Lambda F_\tau(x) = \sigma_\tau(x) - \text{residue} \tag{6.50}$$

for $1 \leqslant \tau \leqslant t$.

PROOF. By Theorem 7 we have

$$\Lambda F_\tau = [B, D; \Lambda f, \Lambda \rho] - (\text{res})_1$$

$$= [B, D; \Lambda f, \Delta \sigma] - (\text{res})_1. \tag{6.51}$$

However, by Theorem 4, in connection with (6.49),

$$[B, D; \Lambda f, \Delta \sigma] = \sigma_\tau(x) - (\text{res})_2, \tag{6.52}$$

where

$$(\text{res})_2 = \sum_{\tau=1}^{t} \int_{D_\tau} \sigma_\tau(\xi) \, dv_\xi. \tag{6.53}$$

This proves (6.50) with

$$\text{residue} = (\text{res})_1 + (\text{res})_2. \tag{6.54}$$

7. A Cousin type theorem on the torus. In the expression (6.54) and also in the earlier expression (6.41) which can be subsumed under it, the total residue appears as a sum of two addends. The second addend, which we denoted by $(\text{res})_2$, is a volume integral. It was easy to give immediately a formula for it, as we did, and it is of secondary importance in as much as it is easy to state when its value is zero.

THEOREM 10. *We have, corresponding to the operator* Λf,

$$(\text{res})_2 = \sum_{\tau} \int_{D_\tau} \sigma_\tau \, dv_\xi \tag{7.1}$$

so that

$$(\text{res})_2 = 0 \tag{7.2}$$

if all $\sigma_\tau = 0$. *In particular,* (7.2) *holds for the operator* Δf *itself, if all* $\rho_\tau = 0$.

Thus, this part of the residue does not occur effectively whenever the functions $f_q(x)$ are solutions of the " homogeneous " equations

$$\Lambda f = 0 \tag{7.3}$$

and

$$\Delta f = 0 \tag{7.4}$$

respectively. And these " homogeneous " forms of the equations are the ones which are basic to holomorphy and to such generalizations of holomorphy as our approach suggests and implies.

But the first addend in the sum (6.54), which we have denoted by $(\text{res})_1$, is the really *primary* part of it, and it is an evolution of Cauchy's own concept of a residue.

If $\Lambda_x F$ is an isobaric operator of the first order, see (5.4), then $(\text{res})_1$ is given again "uniquely " by a simple expression, namely by (6.24). Building on this, one can give an expression for the residue in the general case, but it is no longer unique in its form or in its implications.

THEOREM 11. *If one takes any representation*

$$\Lambda f = \sum_{\mu=1}^{\nu} A_\mu(L^\mu f) + b_0 f \tag{7.5}$$

in which $A_1,\ldots,A_\nu$ are a finite number of isobaric operators of first order, if their factors L^μ are any other linear operators, and if b_0 is a constant, then this implies the formula

$$(\mathrm{res})_1 = \sum_{\mu=1}^{\nu} \sum_{q=1}^{s} \int_{B_q} L^\mu f(\xi).A_\mu(d\xi). \tag{7.6}$$

PROOF. "Usually" the constant b_0 in (7.5) will be 0, but even if it is $\neq 0$ it obviously does not contribute anything to the value of $(\mathrm{res})_1$ in (6.51), therefore it does not enter (7.6).

Formula (6.35) implies

$$\Lambda_x F = \sum_{\mu=1}^{\nu} A_\mu\{[B, D; L^\mu f, L^\mu \rho](x)\}, \tag{7.7}$$

but by Theorem 6 we have for each μ

$$A_\mu\{[B, D; L^\mu f, L^\mu \rho](x)\} = [B, D; A_\mu L^\mu f, A_\mu L^\mu \rho](x) - \\ - \sum_{q=1}^{s} \int_{B_q} L^\mu f_q(\xi)\cdot A_\mu(d\xi). \tag{7.8}$$

If we insert this in (7.7) it follows that (7.6) is the correct value with which to validate (6.51).

DEFINITION 2. *For given functions $\{f_q\}$ if Λf (or Δf) is some operator, and if for this operator we choose some representation of the form* (7.5), *then the corresponding value* (7.6) *shall be called a principal residue of the operator.*

We now take a set-up as in the second half of Section 4. We take a closed set E in S_n and we assume that there is given a function $\phi_\tau(x)$ in each $\widetilde{D} - E$, such that the functions

$$\rho_\tau = \Delta\phi_\tau \tag{7.9}$$

are continuable into all of $\widetilde{D}_\tau$ and the saltuses

$$f_q = \sum_{\tau=1}^{t} \eta_{q\tau} \phi_\tau \tag{7.10}$$

into all of $\widetilde{B}_q$. If we introduce the functions $F_\tau(x)$ as defined by (6.15) then they have the following properties.

THEOREM 12. *Each $F_\tau(x)$ is defined and analytic in all of $\widetilde{D}_\tau$ and the functions $\{F_\tau\}$ have the same saltuses as the functions with singularities $\{\phi_\tau\}$,*

$$\sum_\tau \eta_{q\tau} F_\tau = f_q = \Sigma \eta_{q\tau} \phi_\tau. \tag{7.11}$$

However,

$$\Delta F_\tau = \rho = \Delta \phi_\tau, \tag{7.12}$$

whenever (i) *a principal residue of the operator (relative to the given functions* (7.10)) *is zero and* (ii) *also*

$$\sum_{\tau=1}^{t} \int_{D_\tau} \rho_\tau(\xi)\, dv_\xi = 0. \tag{7.13}$$

If for some other operator Λf the functions

$$\sigma_\tau = \Lambda \phi_\tau \tag{7.14}$$

are also continuable into all of D_τ so that also

$$\Delta \sigma_\tau = \Lambda \rho_\tau, \tag{7.15}$$

then the same functions $\{F_\tau\}$ also satisfy

$$\Lambda F_\tau = \sigma_\tau = \Lambda \phi_\tau, \tag{7.16}$$

provided a principal residue of this operator Λ relative to the functions f_q is zero and also

$$\sum_{\tau=1}^{t} \int_{D_\tau} \sigma_\tau(\xi)\, dv_\xi = 0. \tag{7.17}$$

A situation of considerable interest arises if in the second half of this theorem there is given not just one operator Λf but several operators simultaneously, which we distinguish by an upper index, thus,

$$\Lambda^1, \ldots, \Lambda^k, \tag{7.18}$$

and if furthermore this set of operators "generate" the elliptic operator Δ in the sense that for some linear operators $M_1, \ldots, M_k$ we have

$$\Delta = M_1 \Lambda^1 + \ldots + M_k \Lambda^k. \tag{7.19}$$

Now, instead of starting out with an elliptic operator Δ, and then adding operators (7.18) by which Δ is presentable in a form (7.19), we may also adopt the approach that we start out initially with a system of operators (7.18) which are the only ones given. But the *system* shall be *elliptic* in the sense that among all linear combinations (7.19) for " variable " $M_1,\ldots,M_k$ there is at least one for which Δ is elliptic in our previous sense. We can now state as a corollary to Theorem 12 an "existence" theorem in which the operator Δ no longer enters *explicitly* at all.

THEOREM 13. *If there is given a system of functions with singularities* $\{\phi_\tau\}$, *whose saltuses* $\{f_q\}$ *are without singularities, and an elliptic system of operators* (7.18) *such that the functions*

$$\sigma_\tau^j = \Lambda^j \phi_\tau, \quad 1 \leqslant j \leqslant k, \quad 1 \leqslant \tau \leqslant t,$$

are also without singularities; and if the principal and secondary residues for these operators are zero; then there exist functions F_τ *without singularities which have the same saltuses* f_q *as the* ϕ_τ *and which satisfy the same equations*

$$\sigma_\tau^j = \Lambda^j F_\tau$$

for the given operators (7.18).

8. Meromorphic functions on the torus. We will now apply Theorem 13 to obtain a statement which, except for an accidental mistake in a formula, was already made in [1].

We consider a complex torus. Thus the space dimension is an even number $n = 2k$, and there are complex coordinates $z_j = x_j + \sqrt{(-1)}\, y$; $j = 1,\ldots,k$. Our operators (7.18) shall be

$$\Lambda^j f \equiv \frac{\partial f}{\partial \bar{z}_j}, \; j = 1, \ldots, k, \tag{8.1}$$

and they constitute an elliptic system since

$$\tfrac{1}{4}\sum_{j=1}^{k}\frac{\partial}{\partial z_j}\cdot\frac{\partial}{\partial \bar{z}_j} \tag{8.2}$$

is the classical Laplacean.

We are only interested in holomorphic functions with singularities and thus we consider a system of functions $\{\phi_\tau(z)\}$ with singularities whose saltuses $f_q(z)$ have holomorphic continuations into the entire sets $\tilde{B}_q$.

Now, it can be computed that the equations for the vanishing of the principal residues are

$$\sum_{q=1}^{s}\int_{B_q} f_q(\xi).\, d\zeta_j \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{k-1} = 0, \quad j = 1, \ldots, k, \tag{8.3}$$

where $\zeta_j = \xi_j + \sqrt{(-1)}\,\eta_j$ and

$$\omega = d\zeta_1 \wedge d\zeta_1 + \ldots + d\zeta_k \wedge d\zeta_k. \tag{8.4}$$

Thus, if the residues (8.3) are all zero, there are holomorphic functions without singularities having the same saltuses $\{f_q\}$. That is the differences $\phi_\tau - F_\tau$ have saltuses 0, and thus constitute a function on $S_{2k} - E$, and this function has the same singularities as were given by the functional elements $\phi_\tau(z)$.

In the case of one complex variable the result is as follows. If on a complex torus in one variable one wishes to have a function, which is holomorphic except for a finite number of isolated singularities say (which may be essential singularities), and if the principal parts of the singularities are prescribed, then such a singular function exists provided only that the sum of the residues of the principal parts is 0. This theorem in one variable is easily proven, although rarely stated in books on elliptic functions. Our work above for the torus may be therefore viewed as an elaboration of this simple theorem, as it were.

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ON THE CONSTRUCTION OF A HOLOMORPHIC FUNCTION IN THE NEIGHBOURHOOD OF A CRITICAL POINT OF A RAMIFIED DOMAIN

By RYOICHIRO KAWAI

INTRODUCTION. The problem of the existence of a holomorphic function in the neighbourhood of a critical point has a very important position in the theory of analytic functions of several complex variables. Indeed it is indispensable to the further investigation of analytic functions of several complex variables. It was conjectured by Oka, 15 years ago, that this problem could be treated as a problem of the integral equation. I took up the integral equation of Oka in the form of his Mem. IX† and found that it is possible to estimate the solution of this integral equation in the same way as the estimation of the sum of Cousin integrals which appears in the construction of theta functions.* Therefore all the estimations of the solution are supposed to be carried out in this way although it is not written here explicitly. We shall treat the problem for domains over the complex number space in two variables. The general case of n variables has already been solved by Grauert and Remmert§ by entirely different methods.

1. The main problem. Let $\mathcal{D}$ be a ramified covering of m-sheets of the polycylinder $\mathcal{Q}: |x_1| < 1 + d_1,\ |x_2| < 1 + d_2$ in the space of 2 variables (x_1, x_2) and let S be the set of its critical points. To prove the existence of a holomorphic function in the neighbourhood of a critical point of a domain which contains critical points as inner

†K. Oka : Sur les fonctions analytiques de plusieurs variables. IX Domaines finis sans point critique intérieur. *Japanese J. Math.* 23 (1953), 97-155.

*C. L. Siegel : Analytic Functions of Several Complex Variables, Princeton Lectures, 1948-1949 (mimeographed notes).

§H. Grauert u. R. Remmert : Sur les revêtements analytiques des variétés analytiques *Comptes Rendus Acad Sci.* (Paris) 245, (1957) 918-921; Komplexe Räume, *Math. Annalen* 136, (1958) 245-318.

points, we may assume that the following condition is satisfied for the base points (Grundpunkt) $\underline{S}$ of S.

The only singular point of $\underline{S}$ in $\underline{\mathscr{D}}$ is the origin, because this assumption will be satisfied, if the neighbourhood is chosen sufficiently small.

DEFINITION OF THE DOMAIN $\mathscr{D}_n$. Let E_0 be the set of points of $\mathscr{D}$ which lie over $x_1 = 0$, and let $\mathscr{D}_n$ be the part of $\mathscr{D}$ lying over

$$\underline{\mathscr{D}_n}: \frac{1-\delta_n}{2^n} < |x_1| < \frac{1-\delta_{n-1}}{2^{n-1}}, \quad |x_2| < 1 + d_2 \, (n = 1, 2, \dots),$$

where these positive numbers δ_n satisfy the condition $\delta_1 \geqslant \delta_2 \geqslant \dots \geqslant \delta_n \geqslant \dots$. Then the domain $\mathscr{D}$ is considered to be composed of the following two domains $\mathscr{D}'$, $\mathscr{D}''$

$$\mathscr{D}' = \mathscr{D}_1 \cup \mathscr{D}_3 \cup \mathscr{D}_5 \cup \dots, \quad \mathscr{D}'' = \mathscr{D}_2 \cup \mathscr{D}_4 \cup \mathscr{D}_6 \dots.$$

"PROBLÈME AUXILIAIRE" D'OKA QUANTITATIVE. Suppose that E be the part of $\mathscr{D}$ which covers the closed polycylinder $|x_1| \leqslant 1$, $|x_2| \leqslant 1$ and that E_n be the part of $\mathscr{D}$ which covers the following closed cylindrical domain

$$\underline{E_n}: \frac{1}{2^n} \leqslant |x_1| \leqslant \frac{1}{2^{n-1}}, \quad |x_2| \leqslant 1 \ (n = 1, 2, \dots).$$

Then we have $E - E_0 = \overset{\infty}{\underset{n=1}{\cup}} E_n$.

Now we shall set our main problem.

Assume that a holomorphic function $\phi_n(P)$ be given on each $\mathscr{D}_n \cap \mathscr{D}_{n+1}$ such that

$$|\phi_n(P)| < M \ \textit{(uniformly for all } n).$$

To find a holomorphic function $\psi_n(P)$ on each E_n which remains holomorphic on both boundaries $|x_1| = 1/2^n$, $|x_1| = 1/2^{n-1}$ and satisfies

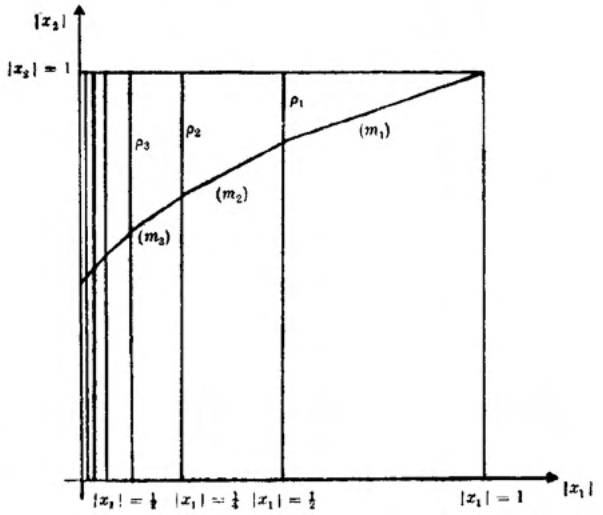
$$\psi^n(P) - \psi_{n+1}(P) = \phi_n(P) \text{ on } |x_1| = 1/2^n$$

with an additional quantitative condition

$$|\psi_n(P)| < \frac{C\,M}{|x_1|^\alpha}, \quad \alpha : \textit{a positive integer},$$

where C is a constant independent of n.

This is " problème auxiliaire " of Oka, (with an additional quantitative condition) between $\mathscr{D}'$, $\mathscr{D}''$. It is easy to prove the existence of a holomorphic function, if we can find a solution of this problem. We shall show that we can find a solution $\psi_n(P)$, for which $\alpha = 1$, in the domain $|x_1| < 1$, $|x_2| < 1 - r_0$, $0 < r_0 < 1$. The process of the construction of such a solution will be shown in the following.



(m_n) are direction coefficients

Now we shall make $\mathscr{D}$ smaller.

Consider the part of $\mathscr{D}$, denoted by E'_n, which covers the following closed domain

$$E'_n : \frac{1}{2^n} \leqslant |x_1| \leqslant \frac{1}{2^{n-1}},$$

$$|x_2| \leqslant m_n\left(|x_1| - \frac{1}{2^n}\right) + 1 - \rho_n$$

and $E' = \bigcup_{n=1}^{\infty} E_n'$ and denote $E' \cap \mathscr{D}_n$ by D_n.

Assume that $m_n = \lambda^{n-1} m_1$, then ρ_n in the above figure is

$$\rho_n = \tfrac{1}{2} m_1 \left(1 + \frac{\lambda}{2} + \left(\frac{\lambda}{2}\right)^2 + \ldots + \left(\frac{\lambda}{2}\right)^{n-1}\right);$$

therefore ρ_n converges, for $n \to \infty$, if $1 < \lambda < 2$, and we have

$$\rho_\infty = \lim_{n\to\infty} \rho_n = \frac{m_1}{2-\lambda}.$$

Hence, if m_1 be sufficiently small, ρ_∞ will be smaller than a given positive number. The domain $D = \bigcup_{n=1}^{\infty} D_n$, which is obtained from $\mathscr{D} - E_0$ by the above process of *edge-making*, is a pseudoconvex domain.

In this procedure m_n are so chosen that we can obtain a meromorphic function which has a pole of the lowest order at $x_1 = 0$ directly from a solution of this "problème auxiliaire d'Oka".

2. A fundamental system of holomorphic functions in a domain of holomorphy (*general concept*). Let $\mathscr{D}$ be a domain of holomorphy over the space of n-complex variables $x_1, \ldots, x_n$. We shall introduce the following concept of the *fundamental system*, suggested by the "Lemme fondamental" of Oka, Mem VIII.†

Let Δ be an analytic polyhedron which is contained completely in $\mathscr{D}$, defined by

$$|x_i| \leqslant r_i \quad i = 1, 2, \ldots, n,$$

$$|f_j(P)| \leqslant 1, \quad j = 1, 2, \ldots, \nu, \quad P \in \mathscr{D}.$$

Let the system of holomorphic functions $x_1, x_2, \ldots, x_n, f_1(P), f_2(P), \ldots f_\nu(P)$ satisfy the following two conditions

(1) Consider the analytic variety $\Sigma: y_j = f_j(P)$, $j = 1, 2, \ldots, \nu$ in the complex space $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_\nu)$ and consider the part Σ_γ of Σ which lies in the polycylinder

$$\gamma: |x_i| \leqslant r_i, i = 1, 2, \ldots, n, \quad |y_j| \leqslant 1, j = 1, 2, \ldots, \nu.$$

Then the correspondence

$$P \longleftrightarrow (x_1, x_2, \ldots, x_n, f_1(P), \ldots, f_\nu(P))$$

gives a one-to-one correspondence between Δ and Σ_γ.

†K. Oka: Sur les fonctions analytiques de plusieurs Variables, VIII Lemme Fondamental, *J. of the Math. Soc. Japan* 3 No. 2 (1951), 204-214, 259-278.

(2) For any holomorphic function $\phi(P)$ given in the neighbourhood of Δ, we can find a holomorphic function $\Phi(x, y)$ in the neighbourhood of γ such that

$$\Phi(x, f_1(P), \ldots, f_\nu(P)) = \phi(P) \text{ on } \Sigma_\gamma.$$

If these two conditions are satisfied, we shall say that this system of holomorphic functions $x_1, x_2, \ldots, x_n, f_1(P), f_2(P), \ldots, f_\nu(P)$ constitutes a *fundamental system* of Δ (or of a closed domain contained in Δ).

For this fundametal system $x_1, \ldots, x_n, f_1, \ldots, f_\nu$ and sufficiently small $\epsilon > 0$, any system of holomorphic functions $x_1, \ldots, x_n, f'_1, \ldots, f'_\nu$ such that

$$|f'_j(P) - f_j(P)| < \epsilon, \quad j = 1, 2, \ldots, \nu$$

constitutes a fundamental system of $\Delta \cap \Delta'$, where Δ' denotes the analytic polyhedron

$$|f'_j(P)| \leqslant 1, \quad j = 1, 2, \ldots, \nu.$$

Using this concept of fundamental system, the method of the integral equation of Oka Mem. IX., pp. 138-139, § 27-29, which is formulated for domains without critical points, can be applied to domains which contain critical points as inner points, if we replace the condition 3 on $f_1, f_2, \ldots, f_\nu$ in Oka Mem. IX, p. 138 by the condition 3* that these $f_1, f_2, \ldots, f_\nu$ constitute a fundamental system of Δ.

3. Fundamental system of $\mathscr{D}_n$. Assume that the base-points $\underline{S}$ of the critical points S is composed of irreducible components $\underline{S}_k (k = 1, \ldots, \lambda)$. which have a point in common only at the origin in the polycylinder $|x_1| < 1 + d_1$, $|x_2| < 1 + d_2$, because of the initial assumption. These $\underline{S}_k$ can be considered to be expressed as the zeros of holomorphic functions $g_k(x)$ in the polycylinder $\underline{\mathscr{D}}$, where

$$g_k(x) = x_2^{\alpha_k} + a_1^{(k)}(x_1)\ x_2^{\alpha_k - 1} + \ldots + a^{(k)}(x_1), \ k = 1, \ldots, \lambda.$$

Consider one of $g_k(x)$ $(k = 1, \ldots, \lambda)$ and denote it simply by $g(x)$. The equation $g(x) = 0$, determines a finite number of branches

$$x_2 = \eta(x_1)$$

of an algebroid function. Denoting by e the order of this branch, $\eta(x_1)$ has a Puiseux expansion

$$\eta(x_1) = ax_1 + a^{(1)} x_1^{1+1/e} + a^{(2)} x_1^{1+2/e} + \cdots .$$

It is possible to assume, without loss of generality, that

$$|\eta(x_1)| < N_1 < 1, \text{ for } |x_1| < 1 + \delta_0 \text{ for all } \eta(x_1)$$

and that

$$|a| < N_1 < 1 \text{ for the leading coefficient } a \text{ of all } \eta(x_1).$$

There are finitely many complex straight lines $x_2 = a_l x_1 (l = 1, 2, \ldots, \lambda')$ which meet only at the origin, such that each of these straight lines $x_2 = ax_1$ is in contact with several branch curves $x_2 = \eta(x_1)$ at the origin.

If we restrict the domain of definition of $\eta(x_1)$ to $(1-\delta_n)/2^n < |x_1| < (1+\delta_{n-1})/2^{n-1}$, it is easy to see that each $\mathfrak{D}_n$ has the same number of critical surfaces which do not meet anywhere in $\mathfrak{D}_n$. Let $S_1^{(n)}, S_2^{(n)}, \ldots, S_\lambda^{(n)} (n = 1, 2, \ldots, \lambda)$ be these critical surfaces of $\mathfrak{D}_n$ and let $\underline{S}_1^{(n)}, \underline{S}_2^{(n)}, \ldots, \underline{S}_\lambda^{(n)} (n = 1, 2, \ldots, \lambda)$ be their base-points, then each $\underline{S}_k^{(n)} (k = 1, 2, \ldots, \lambda;\ n = 1, 2, \ldots)$ is a regular analytic variety. Take a sequence of points $x_1^0/2^n$, $\frac{1}{2} \leqslant |x_1^0| \leqslant 1$ $(n = 1, 2, \ldots)$ one in each x_1-component of the domain $\underline{\mathfrak{D}}_n$ such that $(1-\delta_n)/2^n < |x_1| < (1+\delta_{n-1})/2^{n-1}$, and consider small polycylinders

$$\gamma_n : |x_1/2^n - x_1^0/2^n| < \epsilon/2^n,\ (1-\delta_n)/2 < |x_1^0| < 1 + \delta_{n-1} \qquad n = 1, 2, \ldots,$$

in $\underline{\mathfrak{D}}_n$. If the points $x_1/2^n$ vary in $|x_1/2^n - x_1^0/2^n| < \epsilon/2^n$, the corresponding parts of $S_k^{(n)}$ are considered to be included in a small polycylinder

$$|x_1/2^n - x_1^0/2^n| < \epsilon/2^n,\ |x_2 - a_l x_1^0/2| < \epsilon/2^n,\ n = 1, 2, \ldots .$$

Therefore if we divide the x_2-component $|x_2| < 1 + d_2$ into λ distinct parts $\gamma_n', \gamma_n'', \ldots, \gamma_n^{(\lambda)}$, then each cylindrical domain $(\gamma_n, \gamma_n^{(k)})$, includes only one critical surface $\underline{S}_k^{(n)}$. Let $\mathfrak{D}_n(\gamma_n, \gamma_n^{(k)})$ be the part of $\mathfrak{D}_n$ lying over $(\gamma_n, \gamma_n^{(k)})$. These $\mathfrak{D}_n(\gamma_n, \gamma_n^{(k)})$ are cyclic ramified domains, which are topologically equivalent to each other for $n = 1, 2, \ldots$. Therefore $\mathfrak{D}_n(\gamma_n, \gamma_n^{(k)})$ have a fundamental

system composed of the same number of holomorphic functions for all n.

Considering that $\mathscr{D}_n$ is a domain of holomorphy, and that $\mathscr{D}_n(\gamma_n, \gamma_n^{(k)})$ is holomorph convex with respect to $\mathscr{D}_n$, each holomorphic function of the fundamental system of $\mathscr{D}_n(\gamma_n, \gamma_n^{(k)})$ can be approximated by a holomorphic function on $\mathscr{D}_n$. The system of holomorphic functions thus obtained in $\mathscr{D}_n$ constitutes a fundamental system of $\mathscr{D}_n(\gamma_n, \gamma_n^{(k)})$. Consequently, putting all the holomorphic functions of the fundamental systems of $\mathscr{D}_n(\gamma_n, \gamma_n^{(k)})$, we get a fundamental system of $\mathscr{D}_n$. Thus we have obtained

Each $\mathscr{D}_n$ has a fundamental system composed of the same number of holomorphic functions.

Denote this number by μ.

4. Evaluation of the constant "K" of Oka Mem IX, Lemme I. Since E_0 can be regarded as a covering of the polycylinder $|x_2| < 1 + d_2$ of one variable x_2, there exists on E_0 an m-valued holomorphic function $f_0(P)$, $P \in E_0$. Consider the function $x_1^\beta f_0(P)$ with a large positive integer β and denote it again by $f_0(P)$.

The possibility of the estimation of K is based upon the fact that these domains $\mathscr{D}_n$ are similar to each other with respect to the state of the existence of holomorphic functions. For the precise observation of the domain $\mathscr{D}_n$, we take the following two kinds of enlargement at the aim of getting suitable fundamental systems of $\mathscr{D}_n$.

1°. domain $\mathscr{D}'_n$: $\mathscr{D}'_n$ arises from $\mathscr{D}_n$ by putting $x'_1 = 2^{n-1} x_1$. It is considered to be a covering of the cylindrical domain

$$\frac{1-\delta_n}{2} < |x'_1| < 1 + \delta_{n-1},\ |x_2| < 1 + \delta_{n-1}$$

in the complex space (x'_1, x_2).

2°. domain $\mathscr{D}_n''$: $\mathscr{D}_n''$ arises from $\mathscr{D}_n$ by putting $x_1' = 2^{n-1} x_1$, $x_2' = 2^{n-1} x_2$.

It is considered to be a covering of the cylindrical domain

$$\frac{1-\delta_n}{2} < |x_1'| < 1 + \delta_{n-1},\ |x_2'| < 2^{n-1}(1 + \delta_{n-1}).$$

Now consider a point $(x_1'^0, x_2'^0)$ on the tangent $x_2 = a x_1$, and take a sufficiently small polycylinder $\gamma : |x_1' - x_1'^0| < r_1, |x'_2 - x_2'^0| < r_2$ in the above cylindrical domain $\underline{\mathscr{D}''}_n$, and let $\mathscr{D}''_n(\gamma)$ be the part of $\mathscr{D}''_n$ which covers the polycylinder γ, then $\mathscr{D}''_n(\gamma)$ includes the same number of critical surfaces for each n, which tend uniformly to $x'_2 = a\, x'_1$ for $n \to \infty$. Using the integral equation of Oka finitely many times in $\mathscr{D}''_n(\gamma)$, we shall obtain a holomorphic function $f_n(P'')$, $P'' \in \mathscr{D}''_n(\gamma)$ such that

$$\lim_{n\to\infty} f_n(P'') = f_0(P'') \text{ uniformly.}$$

The same construction is possible for the other tangents. For λ' polycylinders γ_l: $|x'_1 - x'^0_1| < r_1$, $|x'_2 - a_l x'^0_1| < r_2$ $(l = 1, 2, \ldots, \lambda')$ and corresponding $\mathscr{D}''_n(\gamma_l)$ $(l = 1, 2, \ldots, \lambda')$, we have holomorphic functions $f_{n,l}(P'')$, $P'' \in \mathscr{D}''_n(\gamma_l)$ such that

$$\lim_{n\to\infty} f_{n,l}(P'') = f_0(P'').$$

Further between these $\mathscr{D}''_n(\gamma_l)$ and the part of $\mathscr{D}''_n$ which is given by

$$|x'_1 - x'^0_1| < r_1,\ |x'_2 - a_l\, x'^0_1| > r^*_2,\ |x'_2| < 1 + \delta_{n-1};$$

$$r_2 > r_2^* > r_2 - |a_l| r_1 > 0$$

using the integral equation once again, we have a holomorphic function $f_n(P'')$, $P'' \in \mathscr{D}''_n$ such that $\lim_{n\to\infty} f_n(P'') = f_0(P'')$ in the polycylinder $|x'_1 - x'^0_1| < r_1$, $|x'_2| < 1 + \delta_{n-1}$, and finally a holomorphic function with this property in $\mathscr{D}''_n$.

The above process of constructing a holomorphic function $f_n(P)$ such that $\lim_{n\to\infty} f'_n(P') = f'_0(P')$ is based upon the fundamental property of the solution of the "problème auxiliaire" of Oka Mem IX. This property mentioned may be expressed as follows.

If $\phi(P)$ *satisfies* $|\phi(P)| < M$ *in* A, *then the solution* $\Psi_1(P)$, $\Psi_2(P)$ *satisfies*

$$|\Psi_1(P)| < C_1 M,\ |\Psi_2(P)| < C_2 M,$$

where C_1, C_2 *are constants.*

In the case we are considering, these constants C_1, C_2 are independent of n.

By means of the repeated use of this "problème auxiliaire", and if we construct $f'_n(P')$ simultaneously in each $\mathscr{D}'_n$, we can obtain finally a holomorphic function $f'_n(P')$ in $\mathscr{D}'_n$ such that for given $\epsilon_n > 0$,

$$|f'_n(P') - f'_0(P')| < \epsilon_n \text{ uniformly.} \tag{C}$$

Each holomorphic function $f_n(P)$ can be considered as a holomorphic function $f'_n(P')$ of P'. We denote this holomorphic function by $f'_n(P')$, $P' \in \mathscr{D}'_n$, and similarly for $f''_n(P'')$. From among all the holomorphic functions $f_n(P)$ which satisfy the above conditions (C), we can choose a fundamental system of $\mathscr{D}_n$. It is easy to see that the fundamental system $f_1^{(n)}(P), \ldots, f_\mu^{(n)}(P)$ thus chosen has the following property.

(1) Let $\epsilon_1 \geqslant \epsilon_2 \geqslant \ldots \geqslant \epsilon_n \ldots$ be an arbitrarily given sequence of positive numbers, then we have

$$|f_j^{(n)}(P) - f_j^{(n+1)}(P)| < \epsilon_n \text{ uniformly in } \mathscr{D}_n \cap \mathscr{D}_{n+1} \ (j = 1, \ldots, \mu).$$

Applying the estimation of the constant K of Oka Mem IX to this fundamental system $f_1^{(n)}(P), \ldots, f_\mu^{(n)}(P)$, it is proved that K is independent of n, since the number of equivalent relations in the *problem* (C_2) of Oka are equal for all n. We have therefore

(2) Consider in the complex space $(x_1, x_2, y_1, \ldots y_\mu)$ an analytic variety Σ_n

$$\Sigma_n : y_j = f_j^{(n)}(P), \ j = 1, 2, \ldots, \mu,$$

then given a holomorphic function $\phi_n(P)$ on $\mathscr{D}_n \cap \mathscr{D}_{n+1}$, we can construct a holomorphic function $\Phi_n(x, y)$ in the cylindrical domain

$$\frac{1-\delta_n}{2^n} < |x_1| < \frac{1+\delta_{n-1}}{2^{n-1}}, \quad |x_2| < 1 + \delta'_0$$

$$|y_j| < 1, \ j = 1, 2, \ldots, \mu,$$

with the following properties : if $\phi_n(P)$ are holomorphic functions such that $|\phi_n(P)| < M$, then $\Phi_n(x, y) = \phi_n(P)$ on Σ_n and $|\Phi_n(x, y)| < KM$ uniformly for all n in this cylindrical domain, K being independent of ϕ_n.

5. The system of holomorphic functions for edge-cutting off. Since the estimation of δ_n is important to the convergence of the solution, a precise study of this quantity will be made in this paragraph. By means of the general theory, for each point $(e^{-\alpha_1}/2^n,\ (1-\rho_n)\ e^{\alpha_2})$ of the boundary of $\underline{E'}_n \cap \underline{E'}_{n+1}$ there exists a characteristic variety of the second order which has no point in common with $\underline{E'}_n$, $\underline{E'}_{n+1}$ except this boundary point. Because the precise knowledge of this characteristic variety is necessary for the estimation of δ_n, we shall adopt here the following process of inversion.

Transform the two domains E'_n, E'_{n+1} which neighbour each other on the boundary surface $|x_1| = 1/2^n$ by the change of variable $x_1 = 1/(2^n x'_1)$. Then E'_n is transformed into a domain E''_n which is a covering of

$$\tfrac{1}{2} \leqslant |x'_1| \leqslant 1,\ |x_2| \leqslant \frac{m_n}{2^n}\left(\frac{1}{|x'_1|} - 1\right) + 1 - \rho_n$$

and E'_{n+1} into E''_{n+1} which is a covering of

$$1 \leqslant |x'_1| \leqslant 2,\ |x_2| \leqslant \frac{m_{n+1}}{2^n}\left(\frac{1}{|x'_1|} - 1\right) + 1 - \rho_n.$$

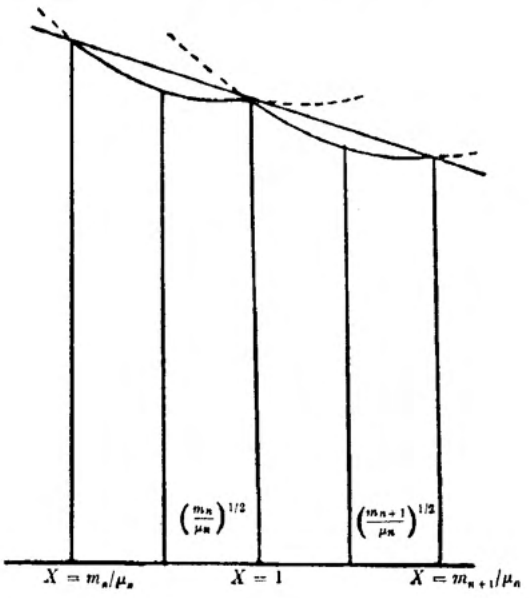
Through a point $(e^{\alpha_1},\ (1-\rho_n)\ e^{\alpha_2})$ on the boundary of both $\underline{E''}_n$, $\underline{E''}_{n+1}$ consider a complex straight line

$$l(x) = (1-\rho_n)\ e^{\alpha_2} - x_2 - \frac{\mu_n}{2^n}(x'_1 - e^{\alpha_1}) = 0$$

which has no point in common, except this point, with these domains $\underline{E''}_n$, $\underline{E''}_{n+1}$ in the neighbourhood of $|x'_1| = 1$. There are infinitely many such straight lines. But we shall choose among them, for the simplicity of computation, a special one whose direction coefficent μ_n is

$$\mu_n = \frac{m_n + m_{n+1}}{2}.$$

For the simplicity of notations, we write X, Y instead of $|x'_1|$, $|x_2|$. The straight line $Y = -\frac{\mu_n}{2^n}(X-1) + 1 - \rho_n$ meets the lines



$$Y = \frac{m_n}{2^n}\left(\frac{1}{X} - 1\right) + 1 - \rho_n,\ Y = \frac{m_{n+1}}{2^n}\left(\frac{1}{X} - 1\right) + 1 - \rho_n$$

respectively at $X = \dfrac{m_n}{\mu_n}$, 1 and $X = 1, \dfrac{m_{n+1}}{\mu_n}$ and we have

$$1 - \frac{m_n}{\mu_n} = \frac{m_{n+1}}{\mu_n} - 1 = \frac{\lambda - 1}{\lambda + 1} \text{ (constant).}$$

And further

$$Y = \frac{m_n}{2^n}\left(\frac{1}{X} - 1\right) + 1 - \rho_n \text{ and } Y = \frac{m_{n+1}}{2^n}\left(\frac{1}{X} - 1\right) + 1 - \rho_n$$

have respectively at $X = \sqrt{(m_n/\mu_n)}$, $\sqrt{(m_{n+1}/\mu_n)}$ the tangents whose direction coefficients are $-\mu_n/2^n$. Therefore the complex straight line

$$l(x) = (1 - \rho_n)\, e^{\alpha_2} - x_2 - \frac{\mu_n}{2^n}(x'_1 - e^{\alpha_1}) = 0$$

goes through the boundary point $(e^{\alpha_1}, (1 - \rho_n)\, e^{\alpha_2})$ and has no points in common with either of E''_n, E''_{n+1} except this point, if we restrict the domain of x'_1 to

$$1-\delta_n<|x'_1|<1+\delta_n,\ \delta_n\leqslant 1-\sqrt{\left(\frac{1+\lambda}{2\lambda}\right)}.$$

And after simple calculation, we have the following result.

Take a function

$$g(x)=\frac{\sigma_n}{(1-\rho_n)e^{\alpha_1}-x_2-(\mu_n/2^n)\,(1/(2^n x_1)-e^{\alpha_1})}$$

with

$$\sigma_n<\frac{(\sqrt{m_n}-\sqrt{m_{n+1}})^2}{2^n}\quad(n=1,2,\ldots)$$

and

$$\delta_n=c_0,\ \delta''_n=c_2\ (0<c_0<1-\sqrt{\left(\frac{1+\lambda}{2\lambda}\right)};\ n=1,2,\ldots),$$

where c_2 is a sufficiently small positive constant compared with c_0. Then for a sufficiently small constant $\epsilon_0>0$, the region $|g(x)|\geqslant 1-\epsilon_0$ lies completely exterior to the part of $\underline{D}$ which satisfies the condition

$$\frac{1-\delta_n-\delta''_n}{2^n}<|x_1|<\frac{1-\delta_n+\delta''_n}{2^n},$$

$$\frac{1+\delta_n-\delta''_n}{2^n}<|x_1|<\frac{1+\delta_n+\delta''_n}{2^n},$$

if σ_n is taken suitably under the condition $\sigma_n<\frac{(\sqrt{m_n}-\sqrt{m_{n+1}})^2}{2^n}$.

Therefore by choosing the points $(e^{-\alpha_1}/2^n,\ (1-\rho^n)\,e^{\alpha_2})$ sufficiently densely on the boundary of $E'_n\cap E'_{n+1}$, $|x'_1|=1/2^n$ and choosing σ_n sufficiently large, but such that the corresponding $g(x)$ satisfies the condition stated above on the region $|g(x)|\geqslant 1-\epsilon_0$, in each of these boundary points $(e^{-\alpha_1}/2^n,\ (1-\rho_n)e^{\alpha_2})$, we can find a finite number of such functions $g_1(x),\ \ldots\ g_{\nu_n}(x)$, (by means of the lemma of Borel-Lebesgue), for which the following domain A_n defined by

$$\frac{1-\delta'_n}{2^n}<|x_1|<\frac{1+\delta'_n}{2^n},\quad |g_j(x)|<1\ (j=1,2,\ldots,\nu_n)$$

is completely in $D_n\cap D_{n+1}$, where δ'_n can be chosen as a constant c_1 such that $0<c_1<c_0-c_2$. This A_n corresponds to the domain A in Oka Mem IX. It is important here that these constants δ_n, δ'_n, δ''_n depend neither on n nor on m_1.

Corresponding to the Oka Mem IX, introduce the following:

Domain Δ : Δ includes all points of $D - E_0$ which do not belong to $D_n \cap D_{n+1}$ for some n, and a point of $D_n \cap D_{n+1}$ belongs to Δ if and only if

$$|g_j(x)| < 1 \ (j = 1, 2, \ldots, \nu_n).$$

Domain Δ^0: this domain is the part of Δ which satisfies the condition

$$|g_j(x)| < \rho_0 \ (j = 1, 2, \ldots, \nu_n), \text{ where } 1 - \epsilon_0 < \rho_0 < 1.$$

Domain Δ_n^0 : $\Delta_n^0 = \Delta^0 \cap \{1/2^n < |x_1| < 1/2^{n-1}\}$, $n = 1, 2, \ldots$. Restricting the function $\phi_n(P)$ in $D_n \cap D_{n+1}$ to A_n, the initial problem is transformed into the following.

Assume that a holomorphic function $\phi_n(P)$ is given on A_n. To find holomorphic functions $\psi_n(P)$ in $\Delta_n{}^0$ which remain holomorphic on $|x_1| = 1/2^n$, $|x_1| = 1/2^{n-1}$ and satisfy

$$\psi_n(P) - \psi_{n+1}(P) = \phi_n(P) \text{ on } |x_1| = \frac{1}{2^n} \quad (n = 1, 2, \ldots)$$

with an additional quantitative condition $|\psi_n(P)| < \dfrac{CM}{|x_1|}$, where C is a constant. The possibility of finding a solution of the problem under the condition $|x_2| < r_0 < 1 - \rho_\infty$ will be shown in the next paragraph. If we can find a solution $\psi_n(P)$ of this problem the construction of the holomorphic function can be made as follows.

Give a holomorphic function $f_n(P)$, such that $|f_n(P)| < M_1$ uniformly, on each $\mathscr{D}_n$. And let $x_1{}^0$ be a point of a certain $\mathscr{D}_{n_0}$ which belongs to neither

$$\frac{1 - \delta_{n_0}}{2^{n_0}} < |x_1| < \frac{1 + \delta_{n_0}}{2^{n_0}} \text{ nor } \frac{1 - \delta_{n_0+1}}{2^{n_0+1}} < |x_1| < \frac{1 + \delta_{n_0+1}}{2^{n_0}},$$

and give on $\mathscr{D}_n \cap \mathscr{D}_{n+1}$ the holomorphic function

$$\phi_n(P) = \frac{f_n(P) - f_{n+1}(P)}{x_1 - x_1{}^0},$$

then we have $|\phi_n(P)| < M$. Then the function

$$\psi(P) = \psi_n(P) - \frac{f_n(P)}{x_1 - x_1^0} = \psi_{n+1}(P) - \frac{f_{n+1}(P)}{x_1 - x_1^0} \quad (n = 1, 2, \ldots)$$

is a meromorphic function which has a pole of at most first order on $x_1 = x_1^0$ and $x_1 = 0$ from the additional quantitative condition of the problem. Therefore the existence of a holomorphic function is proved.

6. Solution of the 'problème auxiliaire'. Now suppose that a holomorphic function $\phi_n(P)$ is given on each A_n such that $|\phi_n(P)| < M$ for all n. And let $f_1^{(n)}(P)$, $f_2^{(n)}(P)$, ..., $f_\mu^{(n)}(P)$ be a fundamental system of holomorphic functions in $\mathscr{D}_n \cap \mathscr{D}_{n+1}$, which is considered to constitute a fundamental system of the domain A_n. Assume that

$$|f_j^{(n)}(P)| \leqslant N < 1, \quad j = 1, 2, \ldots, \mu, \quad n = 1, 2, \ldots.$$

and

$$|f_j^{(n)}(P) - \mathfrak{f}_j^{(n)}(P)| < \delta'_n, \quad j = 1, 2, \ldots, \mu, \quad n = 1, 2, \ldots,$$

where $\mathfrak{f}_1^{(n)}(P)$, $\mathfrak{f}_2^{(n)}(P)$, ..., $\mathfrak{f}_\mu^{(n)}(P)$ form the system of holomorphic functions, constructed in paragraph 4 as a fundamental system of $\mathscr{D}_n$. Consider a domain A'_n, which is contained completely in A_n, defined by

$$\frac{1 - \delta'''_n}{2^n} < |x_1| < \frac{1 + \delta'''_n}{2^n}, \; |g_k^{(n)}(x)| < \rho, \quad k = 1, 2, \ldots, \nu_n,$$

where $\rho_0 < \rho < 1$, $\delta'''_n < \delta'_n$. Since $\phi_n(P)$ is holomorphic in A_n, and therefore in A'_n, $\phi_n(P)$ is expressed as $\phi_n(P) = c_1^{(n)}(x)\, f_1^{(n)}(P) + c_2^{(n)}(x)\, f_2^{(n)}(P) + \ldots + c_\mu^{(n)}(x)\, f_\mu^{(n)}(P)$ using *configuration of Oka*, where $c_j^{(n)}(x)$ is a holomorphic function in $\underline{A}_n$. Denote the function $\sum_{j=1}^{\mu} c_j^{(n)}(x)\, y_j$ by $\Phi_n(x, y)$. Because the fundamental system $f_1^{(n)}(P)$, ..., $f_\mu^{(n)}(P)$ of $\mathscr{D}_n \cap \mathscr{D}_{n+1}$ is taken as in paragraph 4, we have $|\Phi_n(x, y)| < KM$, $x \in \underline{\mathscr{D}}_n \cap \underline{\mathscr{D}}_{n+1}$, $|y_j| < 1$ $(j = 1, 2, \ldots, \mu)$ for all n $(n = 1, 2, \ldots)$.

Now consider the sum of Cousin integrals

$$\Psi(x, y) = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{C_n} \frac{\Phi_n(x, y)}{x'_1 - x_1} dx'_1,$$

where the path of integration C_n is taken to be $|x_1| = 1/2^n$ in the positive sense. Then we have

$$|\Psi(x, y)| < KM \left(\frac{\delta_{n_0}}{2^{n_0}}\right)^{-1},$$

if $x \in \underline{E'}_{n_0}$, $|x_2| < 1 - \rho_\infty$, $|y_j| < 1$ $(j = 1, 2, \ldots, \mu)$. Therefore $\Psi(x, y)$ expresses a holomorphic function $\Psi_{n_0}(x, y)$ which is also holomorphic on both boundaries $|x_1| = 1/2^{n_0}$, $|x_1| = 1/2^{n_0-1}$, if $x \in \underline{E'}_{n_0}$, $|x_2| < 1 - \rho_\infty$, $|y_j| < 1$, $j = 1, 2, \ldots, \mu$, and we have

$\Psi_{n_0}(x, y) - \Psi_{n_0+1}(x, y) = \Phi_{n_0}(x, y)$ on $|x_1| = 1/2^{n_0}$, $n_0 = 1, 2, \ldots$. Since δ_n can be considered to be a constant for sufficiently large n, we have for $x_1 \to 0$

$$|\Psi_{n_0}(x, y)| < \frac{CM}{|x_1|}, \quad C: \text{constant independent of } n_0.$$

Now, under the assumption $P \in \mathscr{D}_{n_0} \cap \mathscr{D}_{n_0+1}$, consider the holomorphic function

$$\Psi_{n_0}(P) = \Psi(x, f_1^{(n_0)}(P), f_2^{(n_0)}(P), \ldots, f_\mu^{(n_0)}(P)).$$

This function has two branches $\Psi'_{n_0}(P)$, $\Psi''_{n_0}(P)$ in the neighbourhood of $|x_1| = 1/2^{n_0}$, holomorphic in $|x_1| > 1/2^{n_0}$, $|x_1| < 1/2^{n_0}$ respectively and

$$\Psi'_{n_0}(P) - \Psi''_{n_0}(P) = \phi_{n_0}(P).$$

Denote by Γ_j $(j = 1, 2, \ldots,)$ a circle of radius ρ_0 in the plane of y_j, then by means of the integral formula of Cauchy

$$\Phi_n(x, y) = \frac{1}{(2\pi i)^\mu} \int_{\Gamma_1} \frac{dy_1'}{y_1' - y_1} \int_{\Gamma_2} \frac{dy_2'}{y_2' - y_2} \cdots \int_{\Gamma_\mu} \frac{\Phi_n(x, y')}{y_\mu' - y_\mu} dy'_\mu,$$

where the integrations are in the positive sense on Γ_j. As in Oka Mem. IX, $\Psi_{n_0}(P)$ is transformed into

$$\Psi_{n_0}(P) = \sum_{n=1}^{\infty} \frac{1}{(2\pi i)^{\mu+1}} \int_{(C_n, \Gamma)} \chi_{n_0}(x'_1, y', P)\, \Phi_n(x'_1, x_2, y')\, dx'_1\, dy',$$

where

$$\chi_{n_0}(x'_1, y', P)$$

$$= \frac{1}{(x_1'-x_1)\,(y_1'-f_1^{(n_0)}(P))\,(y_2'-f_2^{(n_0)}(P))\,\dots\,(y_\mu'-f_\mu^{(n_0)}(P))} \times$$

$$\times \frac{1}{(2\pi i)^{\nu_{n_0}}} \int_{\Gamma'_1}\int_{\Gamma_2'}\cdots\int_{\Gamma'_{\nu_{n_0}}} \frac{dz_1'\dots dz'_{\nu_{n_0}}}{(z_1'-g^{(n_0)}(x))\dots(z'_{\nu_{n_0}}-g^{(n_0)}_{\nu_{n_0}}(x))}.$$

Here Γ'_k is a circle of radius ρ_0 in the complex number plane z'_k whose centre is the origin of this plane; the g's are the functions introduced in the definition of the domain Δ in §5.

In this expression of $\chi_{n_0}(x'_1,\ y',\ P)$ replace $f_1^{(n_0)}(P)$, $f_2^{(n_0)}(P)$, ..., $f_\mu^{(n_0)}(P)$ by the fundamental system $\mathfrak{f}_1^{(n_0)}(P)$, $\mathfrak{f}_2^{(n_0)}(P)$, ..., $\mathfrak{f}_\mu^{(n_0)}(P)$ of $\mathscr{D}_{n_0}$, then we get

$$X_{n_0}(x'_1, y', P)$$

$$= \frac{1}{(x'_1-x_1)(y'_1-\mathfrak{f}_1^{(n_0)}(P))(y'_2-\mathfrak{f}_2^{(n_0)}(P))\dots(y'_\mu-\mathfrak{f}_\mu^{(n_0)}(P))} \times \dots .$$

We may assume here that

$$|f_j^{(n_0)}(P) - \mathfrak{f}_j^{(n_0)}(P)| < \epsilon_{n_0}$$

$$|f_j^{(n_0-1)}(P) - \mathfrak{f}_j^{(n_0)}(P)| < \epsilon_{n_0} \quad j = 1, 2, \dots, \mu$$

by the results of paragraph 4, where $\epsilon_1 > \dots > \epsilon_{n_0} > \dots$. Consider the sum of integrals

$$I_{n_0}(P) = \sum_{n=1}^{\infty} \frac{1}{(2\pi i)^{\mu+1}} \int_{(C_n,\Gamma)} X_{n_0}(x'_1, y', P)\,\Phi_n(x'_1, x_2, y')\,dx_1\,dy'.$$

Then $I_{n_0}(P)$ is a holomorphic function for $P \in \mathscr{D}_{n_0}$, $|x_2| < 1-\rho_\infty$ and we have the following estimate :

$$|\Psi'_{n_0}(P) - I_{n_0}(P)|$$

$$= \left| \sum_{n=1}^{\infty} \frac{1}{(2\pi i)^{\mu+1}} \int_{(C_n,\Gamma)} \{\chi_{n_0}(x'_1, y', P) - X_{n_0}(x'_1, y', P)\}\Phi_n(x'_1, x_2, y')\,dx'_1\,dy' \right|$$

$$\leqslant KM \left(\frac{\delta_{n_0}}{2^{n_0}}\right)^{-1} \rho_0^{\mu}\,\epsilon_{n_0}\,C.$$

Similarly

$$|\Psi''_{n_0-1}(P) - I_{n_0}(P)| < KM(\delta_{n_0-1}/2^{n_0-1})^{-1}\,\rho_0^{\mu}\,\epsilon_{n_0-1}\,C.$$

Therefore we have, if $P \in D_{n_0} \cap D_{n_0+1}$, $|x_2| < 1-\rho_\infty$,

$$I_{n_0}(P) - I_{n_0+1}(P) = \phi_{n_0}(P) + I_{n_0}(P) - \Psi'_{n_0}(P) + \Psi''_{n_0}(P) - I_{n_0+1}(P)$$

and

$$|\Psi'_{n_0}(P) - I_{n_0}(P)| < KM\left(\frac{\delta_{n_0}}{2^{n_0}}\right)^{-1}\rho_0^{\mu}\,\epsilon_{n_0}\,C,$$

$$|\Psi''_{n_0}(P) - I_{n_0+1}(P)| < KM\left(\frac{\delta_{n_0+1}}{2^{n_0+1}}\right)^{-1}\rho_0^{\mu}\,\epsilon_{n_0+1}\,C.$$

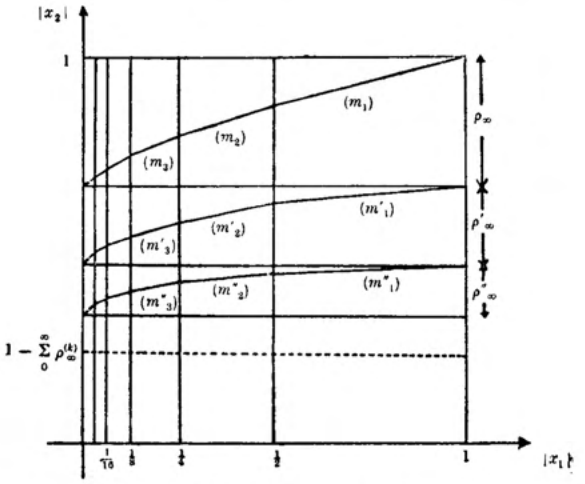
Hence the function

$$\phi'_{n_0}(P) = I_{n_0}(P) - \Psi'_{n_0}(P) + \Psi''_{n_0}(P) - I_{n_0+1}(P)$$

satisfies

$$|\phi'_{n_0}(P)| < 2\max\left(\frac{2^{n_0}}{\delta_{n_0}}\rho_0^{\mu}\,\epsilon_{n_0}, \frac{2^{n_0+1}}{\delta_{n_0+1}}\rho_0^{\mu}\epsilon_{n_0+1}\right)KCM$$

$$< 2\max(2^{n_0}\,\epsilon_{n_0}, 2^{n_0+1}\,\epsilon_{n_0+1})\,\frac{KC}{\delta_0}\,M,\ \delta_{n_0} \geqslant \delta_0 > 0.$$



The (m_n), (m'_n),... are direction coefficients.

Since the fundamental system of $\mathscr{D}_{n_0}(n_0 = 1, 2, \ldots)$ is so chosen that $2^{n_0}\epsilon_{n_0}$ is sufficiently small, we have $2\max(2^{n_0}\epsilon_{n_0}, 2^{n_0+1}\epsilon_{n_0+1})KC/\delta_0 < \lambda < 1$, therefore $|\phi'_{n_0}(P)| < \lambda M (0 < \lambda < 1)$. The holomorphic function $I_{n_0}(P) - I_{n_0+1}(P) = \phi_{n_0,0}(P)$ is the first approximation of $\phi_{n_0}(P)$ in the domain $A_{n_0,1} = A_{n_0} \cap \{|x_2| < 1 - \rho_\infty\}$.

In the next step, taking up the part $\mathscr{D}'$ of $\mathscr{D}$ which satisfies $|x_1| < 1 + d_1$, $|x_2| < 1 - \rho_\infty$, instead of the initial $\mathscr{D}$, and repeating the same process for this $\mathscr{D}'$ and $\phi'_{n_0}(P)$, we get the sum of integrals $I'_{n_0}(P)$ such that

$$I'_{n_0}(P) - I'_{n_0+1}(P) = \phi_{n_0,1}(P),$$

and $\phi_{n_0,0} + \phi_{n_0,1}$ is a second approximation of $\phi_{n_0}(P)$ whose error is $\phi''_{n_0}(P)$, for which we have $|\phi''_{n_0}(P)| < \lambda^2 M$. We can continue this procedure infinitely many times, because the quantities δ_n, δ'_n, δ''_n depend neither on n nor on m_1. On the other hand the initial value m_1, m_1', m_1'',...which corresponds respectively to the domains $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{D}''$, ... can be chosen so small that $\sum_{k=0}^{\infty} m_1^{(k)} < 2 - \lambda$. Then we have $\sum_{k=0}^{\infty} \rho_\infty^{(k)} < 1$, and $\sum_{k=0}^{\infty} I_{n_0}^{(k)}(P)$ gives a solution of the "problème auxiliaire" in the polycylinder $|x_1| \leqslant 1$, $|x_2| \leqslant 1 - \sum_{k=0}^{\infty} \rho_\infty^{(k)}$.

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ON RIGIDITY OF HOLOMORPHIC IMBEDDINGS†

By L. NIRENBERG‡ and D. C. SPENCER

LET V be a compact complex analytic manifold of (complex) dimension n and denote by (W, ϕ) a holomorphic imbedding ϕ of V into a complex manifold W of dimension $n+q$, $q \geqslant 1$, as a nonsingular complex submanifold. Two holomorphic imbeddings of V, (W, ϕ), (W', ϕ') with dim $W' =$ dim W are called equivalent if there are open neighborhoods $\widetilde{W}$, $\widetilde{W}'$ of $\phi(V)$, $\phi'(V)$ and a biholomorphic map h of $\widetilde{W}$ onto $\widetilde{W}'$ such that $h \circ \phi = \phi'$. For convenience we shall sometimes denote $\phi(V)$ and $\phi'(V)$ simply by V, so that, loosely speaking, h keeps V pointwise fixed. Furthermore we shall restrict ourselves here to the case that $q = 1$, though the results hold for arbitrary q.

We may introduce local coordinates in a neighborhood of a point on $\phi(V)$ so that some represent "tangential" coordinates and others "normal" coordinates: let $\phi(V)$ be covered by a finite number of neighborhoods U_i in each of which we have complex coordinates $z_i = (z_i^1, \ldots, z_i^n)$, w_i with $w_i = 0$ on $\phi(V)$ and such that the closure $\overline{U}_i$ has the form $|z_i^\alpha| \leqslant 1$, $\alpha = 1, 2, \ldots, n$, $|w_i| \leqslant 1$. In overlapping patches U_i, U_j the coordinates are related by the transition formulas:

$$(z_i, w_i) = e_{ij}(z_j, w_j) = (a_{ij}(z_j, w_j), b_{ij}(z_j, w_j)).$$

The imbedding (W, ϕ) is said to be *transversely foliated* if a neighborhood U of $\phi(V)$ is covered by a family of analytic varieties of dimension q (one, in our case) which are pairwise disjoint and each of which intersects $\phi(V)$ at only one point. These varieties are called the transverse sheets to V. Transversely foliated imbeddings

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‡ L. Nirenberg is a Sloan Fellow.

are of special interest because the local coordinates (z_i, w_i) above may then be chosen in such a way that the functions a_{ij} are independent of w_j, $z_i = a_{ij}(z_j)$.

A second imbedding $\phi'(V)$ in W' may also be covered by an equal number of neighborhoods U'_i with coordinates z'_i, w'_i in such a way that the points $(z_i, 0)$ of $\phi(V)$ and $(z'_i = z_i, 0)$ of $\phi'(V)$ correspond to the same point of V. By choosing the U_i suitably we see easily that (W', ϕ') is equivalent to (W, ϕ) if and only if there exist biholomorphic maps g_i of U_i into U_i which keep V pointwise fixed and satisfy

$$e'_{ij} \circ g_j = g_i \circ e_{ij} \text{ in } U_i \cap U_j.$$

Here e'_{ij} are the transition mappings for the second imbedding.

We now introduce the notion of *μ-equivalent imbeddings*. (W, ϕ), (W', ϕ') are called μ-equivalent if there exist biholomorphic maps g_i^μ of U_i into U_i, keeping V pointwise fixed, such that

$$e'_{ij} \circ g_j^\mu - g_i^\mu \circ e_{ij}$$

vanishes of order $(\mu + 1)$ in w_j. We may describe this in another way. Denote by G the sheaf of germs of biholomorphic mappings of neighborhoods of points of $\phi(V)$ into W which are equal to the identity on $\phi(V)$. Then the transition maps e'_{ij} may be thought of as an element of the first cohomology set $H^1(V, G)$, the distinguished element of $H^1(V, G)$ corresponding to the germ e_{ij} represented by the given imbedding. Denote by G_μ those maps of G which differ from the identity map by terms vanishing to order $\mu + 1$ at least on $\phi(V)$; G_μ is a sheaf of normal subgroups of G. If $G^\mu = G/G_\mu$ is the quotient sheaf, then

$$1 \to G_\mu \to G \to G^\mu \to 1$$

is an exact sequence of sheaves of groups, and we can define the cohomology sets $H^0(V, G_\mu)$, $H^1(V, G_\mu)$, $H^0(V, G^\mu)$, $H^1(V, G^\mu)$. Among the various natural mappings between these we mention only the mappings

$$H^1(V, G_\mu) \to H^1(V, G) \to H^1(V, G^\mu);$$

this sequence is exact at the middle term. We note that the kernel of the map $H^1(V, G) \to H^1(V, G^\mu)$ is precisely equivalent to the subset of germs of imbeddings which are μ equivalent to the initial imbedding (W, ϕ).

Our aim is to give conditions which ensure that there is a μ_0 such that if (W', ϕ') is μ_0-equivalent to (W, ϕ) then it is equivalent to it, i.e. $H^1(V, G_{\mu_0})$ is trivial.

In order to formulate our conditions we let L denote the restriction of the holomorphic tangent bundle of W on V, and T denote the holomorphic tangent bundle of V. The quotient (vector) bundle $N = L/T$ is the *normal bundle* of the imbedding. Let N^* be the dual vector bundle of N and denote by $N^{*\mu}$ the μ-tuple symmetric product of N^* : $N^{*\mu} = \bigodot^\mu N^*$.

CONJECTURE. *Suppose that the dimension n of V exceeds one and that (W, ϕ) is an imbedding which induces over V the bundles L, N with N a positive bundle (in the sense of K. Kodaira* [1]*). Then any other imbedding (W', ϕ') which is μ_0-equivalent to (W, ϕ) is equivalent to it; here μ_0 is the smallest positive integer such that the bundle $L \otimes N^{*\mu+1}$ is negative for $\mu \geqslant \mu_0$.* (See [1]).

At this Colloquium H. Grauert announced the proof of a corresponding conjecture for the case that N is a negative bundle.

We have proved

THEOREM A. *The conjecture holds in case (W, ϕ) is a transversely foliated imbedding.*

The proof of the theorem involves writing down a series expansion (in w_i) for the mappings g_i — that the terms of the series expansion are uniquely determined is proved with the aid of the " vanishing theorem " of Kodaira [1]. The principal difficulty is the proof of convergence of the series. In estimating its terms we make use of the maximum principle for second order linear elliptic equations, as used by Bochner (see [2]), as well as other potential theoretic inequalities. We mention one of these in a simple form.

LEMMA. *Let $w(z)$ be a complex analytic function defined in the unit circle $|z| < 1$. Let $h(z)$ be a positive function of class $C^{1+\alpha}$ in $|z| < 1$; to be precise, we assume that there is a positive number $\alpha < 1$ and a constant k such that h satisfies*

$$|h^{-1}(z)\, Dh(z)| \leqslant k, \quad \frac{|h^{-1}(z)\, Dh(z) - h^{-1}(\zeta)\, Dh(\zeta)|}{|z-\zeta|^{\alpha}} \leqslant k^{1+\alpha} \qquad (1)$$

for all z, ζ in the unit circle. Here D represents either partial derivative $\frac{\partial}{\partial z}$ or $\frac{\partial}{\partial \bar{z}}$ and both of these are assumed to satisfy the hypotheses. Then there is a constant C depending only on α such that for $r < 1$,

$$\text{l.u.b.}_{|z| \leqslant r} \left| h(z) \frac{dw}{dz}(z) \right| \leqslant C \left(k + \frac{1}{1-r} \right) \text{l.u.b.}_{|z|<1} |hw|.$$

We remark that a function h satisfying $|h^{-1}\, Dh| \leqslant k$, and $|h^{-1}\, D^2 h| \leqslant k^2$ for every second order partial derivative $D^2 h$ also satisfies (1).

We conclude with some results that are analogous to Theorem A but simpler. The first was suggested by a question of M. F. Atiyah.

THEOREM. *Let (W, ϕ) be transversely foliated and let another holomorphic imbedding (W', ϕ') be topologically equivalent to (W, ϕ). Then for any integer $k > 0$ there are neighborhoods U, U' of $\phi(V)$, $\phi'(V)$ in W, W' and a bidifferentiable (of class C^k) map h of U onto U' which is holomorphic along the transverse sheets to V, and such that $h \circ \phi = \phi'$.*

We note that if W is an arbitrary holomorphic vector bundle over V then the natural imbedding of V in W as 0-section is transversely foliated by the fibres of W.

Let (W, ϕ) be a given imbedding and let B be a fixed holomorphic vector bundle defined over a neighborhood of $\phi(V)$ in W. Another such vector bundle B' is called equivalent to B if there is a neighborhood V of $\phi(V)$ in W and a holomorphic bundle map h of the restriction $B|U$ onto the restriction $B'|U$ which leaves U pointwise fixed and such that the restriction of h to $B|V$ is the identity map. As above there is also a notion of μ-equivalence of B' to B.

THEOREM. *Given (W, ϕ) satisfying the conditions of the Conjecture, and given a bundle B, there is a μ_0 such that any bundle B' which is μ_0-equivalent to B is equivalent to it.*

As an illustration let V be a hypersurface of dimension n with positive normal bundle imbedded in an abelian variety T, and take for B the tangent bundle of W. This is holomorphically trivial and we can assert that : *any holomorphic vector bundle on a neighborhood of V in W which coincides over V with the tangent bundle of W is necessarily equivalent to it*, and therefore trivial.

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ON POINT MODIFICATIONS†

By HANS GRAUERT

1. Let $G \subset \mathbf{C}^n$ be a domain, $p = p(z)$ a twice continuously differentiable real-valued function in G. Then p is called *strongly plurisubharmonic* if the Levi form $\sum_{\nu,\mu=1}^{n} \frac{\partial^2 p}{\partial z_\nu \partial \bar{z}_\mu} dz_\nu d\bar{z}_\mu$ is positive definite, everywhere in G. The concept of these functions can be carried over to complex spaces X.

DEFINITION 1. *A real-valued function $p = p(x)$ on X is called strongly plurisubharmonic if for each point $x_0 \in X$ there is a neighbourhood $U = U(x_0)$, a domain $G \subset \mathbf{C}^n$, a biholomorphic imbedding $\phi : U \to G$, and a strongly plurisubharmonic function $\hat{p}$ in G such that $p|U = \hat{p} \circ \phi$.*

As in the analysis of complex manifolds we introduce the following definition.

DEFINITION 2. *A subdomain $G \subset\subset X$ is called strongly pseudoconvex if for each point $x_0 \in \partial G$ there is a neighbourhood $U = U(x_0)$ and a strongly plurisubharmonic function $p = p(x)$ in U such that $G \cap U = \{x \in U : p(x) < 0\}$.*

By a theorem of Narasimhan, any strongly pseudoconvex subdomain G is holomorphically convex. This makes it possible to apply the reduction theory of Remmert to G (see the paper of H. Cartan in this volume). The results of section 2 are obtained by use of this theory.

2. Let $A \subset X$ be a compact analytic subset. Then, as is well known in topology, we can find a topological space X^* and a continuous mapping $\phi : X \to X^*$ which has the following properties :

†Abstract of a paper presented to the International Colloquium on Function Theory held at the Tata Institute of Fundamental Research, Bombay, in January 1960.

(1) $\phi(A) = x^*_0 \in X^*$ is a single point,

(2) ϕ maps $X - A$ topologically onto $X^* - x^*_0$.

Because ϕ is topological in $X-A$, the complex structure can be carried over to $X^* - x^*_0$. The question which arises is: can this complex structure be extended to the whole space X^* ? If this is the case, X^* is a complex space, ϕ becomes holomorphic and (X, ϕ, X^*) is a modification in the sense of Behnke and Stein which replaces the set A by a point. We say that A can be *shrunk* to a point.

THEOREM 1. *A can be shrunk to a point if and only if there exists a domain $G \subset\subset X$ with $A \subset G$ such that G is strongly pseudoconvex.*

It may be mentioned that this theorem is of transcendental nature. If X is algebraic, X^* will not in general be algebraic. One thus constructs a 2-dimensional compact normal complex space Y which has the following properties.

(1) Y has only one singular point y_0.

(2) On Y exist 2 analytically independent meromorphic functions.

This result means that the theorem of Chow-Kodaira for compact complex surfaces S is no longer true, if S contains even one singular point.

3. It may be assumed now, that X is a complex manifold and $A \subset X$ is imbedded without singularities. Because we can blow up A by a monoidal transformation, we also may assume without loss of generality that A has codimension one. Thus the normal bundle N of A is a complex-analytic line bundle over A.

THEOREM 2. *A strongly pseudoconvex neighbourhood $G = G(A) \subset\subset X$ exists if and only if N is negative (in the sense of Kodaira).*

Thus, by Theorem 1, A can be shrunk to a point if and only if N is negative.

The point $x^*_0 \in X^*$ which is obtained by shrinking A will, in general, be a singular point. Of course, the structure of the singularity

depends only on the structure of X in the neighbourhood of A. In order to determine this, we proceed as follows.

Let $m^k = m^k(A)$ denote the sheaf of germs of local holomorphic functions which vanish at least of order k on A. If G is the sheaf of germs of local holomorphic functions on X, then $H_k = (G/m^k) \mid A$ is a sheaf of local rings on A. The pair (A, H_k) is a generalized complex space in the sense of Grothendieck (see the paper of the author in *Publications mathématiques, Institute des Hautes Études Scientifique*, Paris, (1960)).

THEOREM 3. *Let X, X_1 be complex manifolds, $A \subset X$, $A_1 \subset X_1$ compact analytic subsets of codimension* 1 *without singularities such that A can be shrunk to a point. Then if all the complex spaces (A, H_k) and $(A_1, H_k^{(1)})$ are isomorphic, there are neighbourhoods $U(A)$, $U(A_1)$ which are analytically equivalent.*

A sufficient condition for the isomorphism of (A, H_k), $(A_1, H_k^{(1)})$, $k = 0, 1, 2, \ldots$, is that A and A_1 are analytically equivalent and that all cohomology groups

$$H^1(A, {}^*\underline{N}^l), \qquad l = 1, 2, 3, \ldots$$

$$H^1(A, \Theta \otimes {}^*\underline{N}^l), \quad l = 2, 3, \ldots$$

vanish. Here Θ denotes the tangent sheaf of A, ${}^*\underline{N}^l$ is the sheaf of germs of local holomorphic cross-sections in the line bundle ${}^*N^l$, ${}^*N^l$ itself is the dual of the lth tensor power of N.

The structure of many singularities which are obtained by shrinking a set A can be computed by means of this result. A problem of Hirzebruch (see F. Hirzebruch, *Annals of Mathematics*, 60 (1954)) is solved by this method.

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IMBEDDING OF OPEN RIEMANN SURFACES*

By RAGHAVAN NARASIMHAN

THE main theorem is the following :

THEOREM. *Any open Riemann surface has a one-one, regular proper, holomorphic imbedding in* C^3.

As usual regular means that the Jacobian has maximal rank and proper means that the inverse image of any compact set is also compact ; the theorem states simply that any open Riemann surface may be realized as a closed, nonsingular analytic locus in complex 3-space.

We begin by sketching a proof of the existence of a one-one holomorphic mapping.

Let g_1 be a nonconstant holomorphic function on the open Riemann surface X. Let $Y = X \times X$. Then the set of $(x, y) \in Y$ where $g_1(x) = g_1(y)$ forms an analytic set containing the diagonal Δ of Y. It has dimension 1. Let A_p, $p = 1, 2, \ldots$, be the irreducible components of this set different from Δ and let $(x_p, y_p) \in A_p - \Delta$. Then it can be shown that there is a holomorphic function g_2 on X such that $g_2(x_p) \neq g_2(y_p)$, $p = 1, 2, \ldots$. The set of $(x, y) \in Y$ where $g_i(x) = g_i(y)$, $i = 1, 2$ has the property that any irreducible component $B_k \neq \Delta$ is a point (in fact, if $B_k \subset A_p$, $(x_p, y_p) \in A_p$, $\notin B_k$ so that dim $B_k <$ dim $A_p = 1$). If (x'_q, y'_q) are these points $(q = 1, 2, \ldots)$ and if g_3 is holomorphic on X and satisfies $g_3(x'_q) \neq g_3(y'_q)$, then (g_1, g_2, g_3) is one-one. A very simple modification arranges also that this mapping be regular.

The existence of a proper mapping is based on two results, which are as follows.

I. X can be written as the union of three open sets V_λ, $\lambda = 1, 2, 3$, such that V_λ is the union of a sequence of relatively compact,

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mutually disjoint simply connected domains D_n^λ such that any compact set of X meets only finitely many D_n^λ.

(The proof of this statement uses a triangulation of the surface X.)

II. Let $D_1, D_2, \ldots$ be a sequence of mutually disjoint, Runge domains of X such that any compact set of X meets only finitely many D_n. Suppose that $D = \bigcup_{n=1}^{\infty} D_n$ is holomorphically convex in X (on an open Riemann surface X, this simply means that $X - D$ has no compact connected component). Let K_n be a compact set in D_n, $\epsilon_n > 0$, and f_n be a holomorphic function on D_n. Then there is a holomorphic function f on X satisfying

$$|f(x) - f_n(x)| < \epsilon_n, \quad x \in K_n, \quad n = 1, 2, \ldots$$

The proof of II is, roughly speaking, as follows. We may suppose that each D_n is relatively compact. Next one constructs a sequence of Runge domains $\{H_k\}$, $H_k \subset\subset H_{k+1}$ which exhaust X, whose boundaries do not meet D, such that $H_k \cup D$ is holomorphically convex in X. If $D_1, \ldots, D_{n_1}$, lie in H_1, D_n in H_k for $n_k < n \leqslant n_{k+1}$ then we approximate to f_n on K_n for $n \leqslant n_1$ by F_1 holomorphic on X, then to F_1 on H_1 and f_n on K_n for $n_1 < n \leqslant n_2$ by F_2 holomorphic on X and so on. f is then defined to be $\lim F_r$.

The hypothesis that D is holomorphically convex in X, is satisfied if all but one D_n are simply connected.

The existence of a proper map follows from I and II. In fact, since D_n^λ are relatively compact and $\bigcup_{n,\lambda} D_n^\lambda = X$, we may find compact sets $K_n^\lambda \subset D_n^\lambda$ with $\bigcup_{n,\lambda} K_n^\lambda = X$. By II, there is a holomorphic function ϕ_λ on X such that $|\phi_\lambda - (n+1)| < 1$ on K_n^λ (λ fixed, $n = 1, 2, \ldots$). Then (ϕ_1, ϕ_2, ϕ_3) is proper since the set where $|\phi_\lambda| \leqslant n$ ($\lambda = 1, 2, 3$) is clearly contained in $\bigcup_{m=1}^{n} \bigcup_{\lambda=1}^{3} K_m^\lambda$.

From the proofs given above, we see that the main theorem is an immediate consequence of the following lemma.

LEMMA. *Let* $\{(x_p, y_p)\}$ *be a discrete set of* $Y = X \times X, x_p \neq y_p$. *Let* $\{D_n\}$ *be a locally finite, mutually disjoint sequence of simply connected domains of* X, K_n *a compact set of* D_n. *Then, there is a holomorphic function* ϕ *on* X *with*

(i) $\phi(x_p) \neq \phi(y_p)$, ϕ *is univalent in a neighbourhood of* K_n;

(ii) $|\phi(x)| \geqslant n$ *for* $x \in K_n$.

The proof will not be given here.

The following result is a consequence of the imbedding theorem.

On any open Riemann surface X, *there are three holomorphic functions* ϕ_1, ϕ_2, ϕ_3 *such that if* D *is any Runge domain of* X *and* ϕ *is holomorphic in* D, *then* ϕ *can be approximated, uniformly on any compact set of* D, *by polynomials in* ϕ_1, ϕ_2, ϕ_3.

POSTSCRIPT. Analogues of the imbedding theorem above have since been proved for higher dimensional complex spaces. Thus, any holomorphically complete complex space of dimension n admits a one-one, proper, holomorphic map in C^{2n+1} which is regular at every uniformizable point.

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ON DISCONTINUOUS GROUPS IN HIGHER-DIMENSIONAL SYMMETRIC SPACES

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1. It is well known that the theory of uniformization makes it possible to construct families of discontinuous groups of motions in the hyperbolic plane, which in a non-trivial way depend continuously and even analytically on as many parameters as we might prescribe, and at the same time have a compact fundamental domain in the hyperbolic plane.

For the higher-dimensional symmetric spaces, no uniformization theory is available, and if we exclude certain product spaces involving compact spaces, euclidean spaces, or hyperbolic planes as factors, the only discontinuous groups known, that have compact fundamental domain, or at least finite volume of the fundamental domain are the groups that have an arithmetical definition, or that are equivalent to these under an inner automorphism of the continuous group of isometries.

One is therefore led to suspect, that if one excludes certain product spaces, it might be true that there are no families of discontinuous groups with fundamental domain which is compact or of finite volume, and which depend non-trivially on a continuous parameter. And if we restrict ourselves to the symmetric spaces where the isometries are represented by matrices, it might be true that a discontinuous group with fundamental domain which is compact or of finite volume, by an inner automorphism of the continuous group of isometries can be brought to such a form that all the elements of the discontinuous group are represented by algebraic matrices. By an *algebraic matrix* we mean here and later a matrix formed by algebraic numbers only.

A few years back I became interested in these questions and obtained first some results affirming the first conjecture in certain cases, later I was able to extend these results and also answer

the second question in the affirmative, for some larger classes of spaces. Later this was supplemented by certain results of E. Calabi and E. Vesentini, a more complete account of these results will be given in section 5.

I shall in this lecture give a fairly complete account of the proof of the above-mentioned results for a particularly simple case, and later in section 5 list the further results obtainable by these methods, as well as some still outstanding problems.

2. We shall in this section prove some lemmas, since the two first of these are as simple to prove and state in their general form as for the special case we shall later study, we state them here in their general form.

LEMMA 1. *Let G be a locally compact group and Γ a discrete subgroup such that the quotient $\Gamma \backslash G$ has finite Haar-measure (this implies that the measure is bi-invariant). Then, if g is an element of G and Ω an open neighborhood of the identity element in G, there exists always a positive integer n and elements ω_1 and ω_2 in Ω such that $\omega_1 g^n \omega_2^{-1}$ is an element of Γ.*

We prove this lemma quite simply by considering the sets Ωg^m; they all have the same positive measure as Ω, and if g does not lie in a compact subgroup of G (in which case nothing needs to be proved), these sets are non-overlapping if Ω is chosen small enough. Because the quotient $\Gamma \backslash G$ has finite measure, it is easily seen that we must have that there are two sets $\gamma_1 \Omega g^{m_1}$ and $\gamma_2 \Omega g^{m_2}$ with $m_1 \neq m_2$ and γ_1 and γ_2 in Γ, that overlap. This gives the lemma.

LEMMA 2. *We make in addition to the assumptions about G and Γ in Lemma 1, the assumption that the quotient $\Gamma \backslash G$ is compact. If γ is an element of Γ we denote by Γ_γ the subgroup of Γ formed by the elements that commute with γ, similarly we denote by G_γ the subgroup of G formed by the elements that commute with γ. Then the quotient $\Gamma_\gamma \backslash G_\gamma$ is compact.*

It should be remarked that this holds if we do not assume that γ is an element of Γ, but just assume that it is some element of G whose conjugates with respect to Γ form a discrete set.

It is obvious that G_γ is a locally compact subgroup of G. Also since $\Gamma \backslash G$ is compact we can give a compact region R in G such that for any element ρ in G_γ there exists an element γ_ρ in Γ such that $\gamma_\rho \rho$ lies in R. Therefore also $\gamma_\rho \gamma \gamma_\rho^{-1}$ lies in a certain compact region of G, and so there are only a finite number of these conjugates of γ which are distinct as ρ ranges through G_γ. This means that the γ_ρ's are all obtained by multiplying a finite number of them on the right by elements from Γ_γ. From this we easily see that there exists a compact region in G_γ such that any element ρ of G_γ can be brought into this region by multiplication on the left with some element from Γ_γ, which proves the lemma.

We now turn to the special symmetric space and the group G we are to consider in the following, namely the symmetric space S formed by the positive definite n by n symmetric matrices Y, with the normalization $|Y| = 1$, where $|Y|$ is the determinant. The invariant riemannian metric is given by $ds^2 = \sigma(Y^{-1}\,dY\,Y^{-1}\,dY)$ where σ denotes the trace.

$G = G_n$ shall from now on mean the group usually denoted by $Sl(n, R)$, that is the group of all n by n real matrices of determinant 1, its action on S being defined by $Y \to TYT'$, where T is an element of G and T' its transposed.

If Γ is a discrete subgroup of G with a quotient $\Gamma \backslash G$ of finite measure, then Γ acts discontinuously on S and has a fundamental domain of finite volume. The converse also holds. These statements remain true if we replace "of finite measure" and "of finite volume" by "compact".

LEMMA 3. *If the quotient $\Gamma \backslash G$ has finite measure, then there exist n^2 elements of Γ, say A_i, $i = 1, \ldots, n^2$, such that the linear combinations $\sum_{i=1}^{n^2} \lambda_i A_i$ with λ_i real numbers, span the full space of real n by n matrices.*

The proof of this depends on Lemma 1. It is clearly enough to prove that the closure of all expressions of the form $\Sigma\mu_j B_j$, where the B_j are elements of Γ, with μ_j real numbers, contains all real n by n matrices. From Lemma 1, we easily obtain that the closure of the expressions μB contains all diagonalizable matrices with one eigenvalue equal to 1 and all the others equal to zero. Since these span the full set of real n by n matrices, the lemma is proved.

We shall in the following, when Γ is a discrete subgroup of G, with quotient $\Gamma\backslash G$ of finite measure, say that Γ, whose elements we denote by A, is *deformable* if it can be imbedded in a family of groups $\Gamma^{(t)}$ with elements $A^{(t)}$, depending continuously on a real parameter t in an interval $0 \leqslant t \leqslant \vartheta$, in such a way that $\Gamma^{(0)} = \Gamma$, and that the structure of the group, the discreteness, and the finiteness of the measure of $\Gamma^{(t)}\backslash G$ is preserved as t varies in this interval. If the quotient $\Gamma\backslash G$ is compact the last assumption is redundant, since the quotient $\Gamma^{(t)}\backslash G$ would then obviously be compact for t sufficiently small.

If $\Gamma^{(t)} = T\Gamma T^{-1}$, where T is an element of G that depends continuously on t, we say that the deformation is *trivial*, otherwise that it is *non-trivial*.

LEMMA 4. *A deformation of* Γ *which preserves the traces of all elements of* Γ *is a trivial deformation.*

We first show that if we have a linear combination of elements of Γ which represents the zero matrix, $\Sigma\lambda A = 0$, then we have also $\Sigma\lambda A^{(t)} = 0$.

Writing $W^{(t)} = \Sigma\lambda A^{(t)}$, we have that $\sigma(W^{(0)}\Sigma\mu B) = 0$, for any linear combination of elements B of Γ. Therefore we have also $\sigma(W^{(t)}\Sigma\mu B^{(t)}) = 0$, or because of Lemma 3, we get $\sigma(W^{(t)} M) = 0$, for an arbitrary n by n matrix M, which shows that $W^{(t)} = 0$. Conversely if $W^{(t)} = 0$ for some t in our interval $0 \leqslant t \leqslant \vartheta$, we have also $W^{(0)} = 0$.

Because of this we can extend the deformation of Γ to a deformation of G, by expressing an element of G, say R in the form $R = \Sigma\,\mu A$ and writing $R^{(t)} = \Sigma\,\mu A^{(t)}$.

This deformation preserves the group structure of G and its topology and must therefore be produced by an inner automorphism of G, which proves our lemma.

LEMMA 5. *A deformation which preserves the traces of those elements of* Γ *which have real eigenvalues with distinct absolute values, is a trivial deformation.*

We first observe that Γ contains elements whose eigenvalues are real and have distinct absolute values. This follows from Lemma 1, by taking for g an element P of G which has this property. Then if Ω is sufficiently small and m sufficiently large $\omega_1 P^m \omega_2^{-1}$ has also only real eigenvalues with distinct absolute values.

Secondly if A is an arbitrary element of Γ and B another element of Γ which has only real eigenvalues with distinct absolute values, one can easily prove that there are infinitely many positive integers m_i such that $B^l A^{m_i}$, for l positive and large enough, has only real eigenvalues with distinct absolute values.

If we now, by a similarity transformation $\Gamma^{(t)} \to T\Gamma^{(t)}T^{-1}$ where T is an element of G depending on t, arrange that $B^{(t)}$ is in diagonal form, so that $B = B^{(t)}$ (because its eigenvalues are preserved), we observe that the diagonal-elements of $(A^{(t)})^{m_i}$ must remain fixed during the deformation, thus the traces of the $(A^{(t)})^{m_i}$ remain fixed, and therefore also the eigenvalues of $A^{(t)}$ and thus its trace. The conclusion now follows from the preceding lemma.

LEMMA 6. *If the quotient* $\Gamma \backslash G$ *is compact, and* A *is an element of* Γ *with real and distinct eigenvalues,* Γ_A *(in the notation of Lemma 2) is abelian; if we denote by* $\overset{+}{\Gamma}_A$ *the subgroup of* Γ_A *which is formed by the elements of* Γ_A *which have only positive eigenvalues, then* $\overset{+}{\Gamma}_A$ *is generated by* $(n-1)$ *independent elements of infinite order. By a continuous deformation of* Γ *the ratios between the logarithms of the eigenvalues of elements in* $\overset{+}{\Gamma}_A$ *are preserved, as are the ratios between the logarithms of the absolute values of eigenvalues of elements in* Γ_A *and in particular of* A.

That Γ_A is abelian is obvious, also that $\overset{+}{\Gamma}_A$ has finite index in Γ_A. Since $\overset{+}{\Gamma}_A$ is discrete it possesses at most $(n-1)$ independent generators. That the number is not less, follows from Lemma 2, since, if we (as we may without loss of generality) assume A to be in diagonal form, G_A consists of all n by n diagonal real matrices with determinant 1.

To prove the latter part of the lemma, we observe that if B is an element of Γ with real distinct eigenvalues, then during a deformation of Γ, these eigenvalues have to remain distinct (and therefore also real). Namely if $B^{(t)}$ had two or more eigenvalues equal, $\Gamma^{(t)}{}_{B(t)}$ would still have to be abelian because of the preservation of the structure, but $G_{B(t)}$ is a non-abelian group, such that the quotient with its maximal abelian subgroup is non-compact, therefore also $\Gamma^{(t)}{}_{B(t)} \backslash G_{B(t)}$ would be non-compact, contradicting Lemma 2. More generally we can see in a similar way, that if B is an element of Γ with real eigenvalues, eigenvalues that are distinct must remain distinct, and eigenvalues that are equal must remain equal during the deformation of Γ.

Consider now $\overset{+}{\Gamma}_A$ and let its generators be A_i, $i = 1, \ldots, n-1$; where A_i is represented by the diagonal matrix with elements $\rho_{i,j}$ $j = 1, 2, \ldots, n$, with $\prod_{j=1}^{n} \rho_{i,j} = 1$.

We then have that if, the m_i being integers, $j_1 \neq j_2$,

$$\prod_{i=1}^{n-1} \rho_{i,j_1}^{m_i} \gtreqless \prod_{i=1}^{n-1} \rho_{i,j_2}^{m_i},$$

then this inequality or equality holds with the same sign throughout the deformation, so that if we denote the corresponding eigenvalues of the $A_i^{(t)}$ by $\rho_{i,j}^{(t)}$, we must have, as one easily concludes,

$$\rho_{i,j}^{(t)} = (\rho_{i,j})^{\tau},$$

where $\tau = \tau_A(t)$ is independent of i and j, and depends on A and the parameter t only. Its dependence on t is of course continuous. Since the square of any element of Γ_A lies in $\overset{+}{\Gamma}_A$, and A is an element

of Γ_A, this proves our lemma. The eigenvalues of A of course keep their sign under the deformation.

LEMMA 7. *Under the assumptions of the preceding lemma, and that* $n \geqslant 3$, *if* $\overset{+}{\Gamma}_A$ *contains an element* B *such that two of its eigenvalues are equal, whereas the others are distinct from these two and from each other, then the eigenvalues of* A *are preserved under the deformation.*

If the diagonal elements of B (we may again assume A to be in diagonal form, and to remain so during the deformation), are $\sigma_i, i = 1, 2, \ldots, n$, with $\sigma_1 = \sigma_2 = \sigma$ and $\sigma_i \neq \sigma_j$ for $i < j$ and $j > 2$, G_B consists of all real matrices of the form

$$\begin{bmatrix} \alpha & \beta & \\ \gamma & \delta & 0 \\ & & \\ 0 & & D^{(n-2)} \end{bmatrix}$$

with determinant 1, where $D^{(n-2)}$ denotes an $(n-2)$ by $(n-2)$ diagonal matrix. The fact that $\Gamma_B \backslash G_B$ is compact, can be seen to imply that if we form the set of 2 by 2 matrices obtained by forming, for those elements of Γ_B for which $\alpha\delta - \beta\gamma > 0$, the matrices

$$\frac{1}{\sqrt{(\alpha\delta - \beta\gamma)}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix};$$

then these form a group which we shall denote by $\Gamma_B(2)$, and the quotient $\Gamma_B(2) \backslash G_2$ is compact. As Γ is deformed, $\Gamma_B(2)$ is deformed, preserving the compactness of this quotient.

Let us consider a hyperbolic element of $\Gamma_B(2)$ (that is, an element with real eigenvalues $\neq \pm 1$); this element can always arise by the above construction from an element C of Γ_B which has all its eigenvalues distinct and real, namely if the eigenvalues of C were not all distinct, we could form CB^l, with some integral exponent l which would make them all distinct, and C and CB^l give rise to the same element in $\Gamma_B(2)$.

Since B is not the identity element, and lies in both $\overset{+}{\Gamma}_A$ and $\overset{+}{\Gamma}_C$, we see that the expression $\tau_A(t)$ defined in the proof of the preceding lemma, must be identical with $\tau_C(t)$. This implies that during the deformation of $\Gamma_B(2)$, the logarithms of the absolute values of the

eigenvalues of the hyperbolic elements are multiplied by the factor $\tau = \tau_A(t)$.

From this we easily get a contradiction unless $\tau_A(t) = 1$ throughout the deformation. This proves our lemma.

It is possible to prove directly from the preceding lemmas (in fact using only Lemma 6) that the deformation of Γ for $n > 2$ is necessarily a trivial one, and in fact this was how the proof was first given. However, this is rather involved and cannot be generalized as widely as the method we shall follow here, where we postpone the proof of this fact until after the next section, where we shall prove some lemmas of a rather different nature.

3. We shall in this section show that a discrete group Γ with compact quotient $\Gamma \backslash G$, if it is not already a group for which all the elements A of Γ are algebraic matrices, can be deformed into a group where all matrices are algebraic.

Because it is somewhat useful, and serves to introduce some of the ideas used in the more complicated proof of Lemma 9, we first prove a lemma that we actually could do without in this connection.

LEMMA 8. *Let H be a finitely generated group of n by n matrices (they need not be real, nor does H need to be discrete), then H possesses a normal subgroup of finite index, which contains no element of finite order other than the identity element.*

Let H be generated by matrices L_i, $i = 1, 2, \ldots, l$, with $L_i = (a^i_{j,k})$, $1 \leqslant j, k \leqslant n$.

Let us form a field by adjoining to the rational field all the numbers $a^i_{j,k}$. This gives us a field which is either an algebraic field $K(\alpha)$ or a field arising by adjoining to its maximal algebraic subfield certain transcendental elements say $\theta^{(0)}_i$, $1 \leqslant i \leqslant r$, between which there may be algebraic relations, say $P_i(\theta^{(0)}) = 0$ with $1 \leqslant i \leqslant s$ where $s < r$, form a set of such relations from which all others may be derived. The P_i here denote polynomials with coefficients from the maximal algebraic subfield. Leaving for the moment the first case, that all the matrices L_i were algebraic,

we consider the algebraic variety defined by the equations $P_i(\theta) = 0$, where $\theta = \{\theta_1, \ldots, \theta_r\}$ of which $\theta^{(0)}$ is a generic point. We choose a point on this variety with algebraic coordinates, which is always possible. We now form a new group H_1 by replacing in the matrices $L = (a_{jk})$ denoting the elements of H, and where the a_{jk} are expressible as rational functions of the $\theta_i^{(0)}$ with algebraic coefficients, the numbers $\theta_i^{(0)}$ by the algebraic numbers θ_i, and obtain in this way the finitely generated group H_1, which is a representation of H; H_1 has only algebraic matrices as elements, and the orders of all elements of finite order are preserved, since if L is an element of finite order, all its eigenvalues are roots of unity and all of these would be preserved by the representation because they are algebraic numbers. H_1 is generated by the elements that represent the L_i. If we were in the first case, left aside above, we would simply put $H_1 = H$. There is an upper bound on the orders of the elements of finite order that occur in H_1. Namely let $K(\beta)$ be the algebraic field that contains all numbers that enter into the matrices representing the generators of H_1. If this field is of degree m, no root of unity that is an algebraic number of degree greater than nm can occur as an eigenvalue in an element of H_1. From this we see for instance that there is no element of finite order greater than $(2nm)^{2n} = N$. Further there exists a positive integer q such that the generators of H_1 and their inverses become matrices with integral algebraic elements, when multiplied with q. Now let p be a rational prime number greater than N and than q. The subgroup of H_1 formed by the elements of H_1 that are congruent to the identity matrix modulo p, is a normal subgroup of finite index, which is easily seen to contain no element of finite order other than the identity. Since no element of H of finite order greater than one is mapped into the identity in H_1, the subgroup of H that is represented by the constructed subgroup of H_1 fulfils the requirements of our lemma.

Before we prove the main lemma of this section, we shall first for convenience introduce a different model of our space S, formed by the positive definite symmetric n by n matrices $Y > 0$ with $|Y| = 1$; namely we map this by the transformation

$$X = \frac{1}{\sigma(Y)}\, Y,$$

into the convex bounded region S^* of the $\frac{n(n+1)}{2} - 1$ dimensional hyperplane $\sigma(X) = 1$, defined by $X > 0$.

If X_1 and X_2 are two points in the space the expression

$$D(X_1, X_2) = \sigma(X_1 X_2^{-1}) \, |X_2 X_1^{-1}|^{1/n}, \tag{3.1}$$

is an invariant of the two points when the group G acts on S^*. We shall use this as a "distance function" in our space, when constructing fundamental domains for discrete subgroups of G, in the same way one usually uses the geodesic distance (which cannot be used because it is a transcendental function in the coordinates). That $D(X_1, X_2)$ is asymmetric in the two points does no harm.

Let now Γ be a discrete subgroup of G with compact quotient $\Gamma \backslash G$, and we may without loss of generality assume that the point

$$X_0 = (1/n)E, \ (E = \text{identity matrix}),$$

is not a fixpoint of any element of Γ. We now construct a fundamental domain D of Γ in S^* which has as inner points all points X for which

$$D(X, X_0) < D(X, X_0^A),$$

for all elements A different from E in Γ, where X^A is short for the transformation of the point X under the element A of G. The above inequality is seen to be equivalent to

$$\sigma(X X_0^{-1}) < \sigma(X(A X_0 A')^{-1}). \tag{3.2}$$

The fundamental domain $\mathfrak{D}$ obtained in this way, will be a polyhedron with a finite number of faces lying completely inside S^*, whose faces are formed by the $(\frac{1}{2}n(n+1) - 2)$ dimensional hyperplanes given by the intersection $\sigma(X) = 1$ and

$$\sigma(X X_0^{-1}) = \sigma(X(A_i X_0 A_i')^{-1}),$$

where the A_i, $i = 1, \ldots, l$, are a certain set of elements of Γ, that generate the full group. Actually the set will also contain the inverse of each A_i.

All images of the fundamental domain under transformations from Γ are similarly polyhedra and together they cover S^* completely and without overlapping. If we write $A_i = (a_{jk}^i)$, in the linear equations that determine the boundary elements of various dimensions of these polyhedra, all coefficients are expressed as polynomials in the a_{jk}^i with integral rational coefficients.

We now form a field by adjoining all the numbers a_{jk}^i to the field of rational numbers. If we assume that not all a_{jk}^i are algebraic numbers, this gives us a field of transcendentality degree at least 1, which can be produced by adjoining to its maximal algebraic subfield $K(\alpha)$, (which in this case is real) a certain number of transcendental real numbers $\theta_i^{(0)}$, $i = 1, \ldots, r$, which we shall denote in short by $\theta^{(0)}$. Between these numbers there may be algebraic relations, and let $P_i(\theta^{(0)}) = 0$, $i = 1, \ldots, s$, with $s < r$ be a set of such relations from which all others can be derived. The P_i are here polynomials in the $\theta_i^{(0)}$ with coefficients from $K(\alpha)$. These equations define an algebraic variety of complex dimension $r - s > 0$. Since $\theta^{(0)}$ is a generic point on this variety, it cannot be an isolated real point, actually one easily sees that $\theta^{(0)}$ is an inner point on the real variety which we shall denote by V and that V has real dimension $r - s$. For all elements A of Γ, the elements of the matrix $A = (a_{jk})$ are expressible as rational functions of the $\theta_i^{(0)}$ with coefficients from $K(\alpha)$, which we may express by writing $A = A(\theta^{(0)})$.

Now let Ω denote an open neighborhood of $\theta^{(0)}$ on V, and let θ denote a point in this neighborhood, where θ stands for $\theta_i, i = 1, \ldots, r$. If we now replace in each element $A = A(\theta^{(0)})$ of Γ, the point $\theta^{(0)}$ by θ, we get the matrices $A(\theta)$ which we shall denote by A^θ. These form a group of real matrices with determinant 1, which we shall denote by Γ^θ. Γ^θ is generated by the elements A_i^θ, $i = 1, \ldots, l$. We now want to prove the following

LEMMA 9. *If the neighborhood Ω of $\theta^{(0)}$ on V is chosen sufficiently small, Γ^θ is again a discrete subgroup of G, with compact quotient $\Gamma^\theta \backslash G$, that is isomorphic to Γ, and therefore represents a continuous deformation of Γ.*

This is somewhat simpler to prove if we deal with a group without elements with fixpoints. From Lemma 8 we know that we can find a normal subgroup of Γ of finite index, which has no element of finite order, and therefore no element with fixpoints. Denoting such a subgroup by $\widetilde{\Gamma}$, we shall now consider $\widetilde{\Gamma}$, and the corresponding subgroup of Γ^θ which we call $\widetilde{\Gamma}^\theta$.

We construct in the way indicated above a fundamental polyhedron $\widetilde{\mathfrak{D}}$ of $\widetilde{\Gamma}$. We now look at the set of all the images of $\widetilde{\mathfrak{D}}$ under transformation from $\widetilde{\Gamma}$ other than the identity. The subset of these formed by those which have at least one boundary point in common with $\widetilde{\mathfrak{D}}$, we shall refer to as the "first shell" of $\widetilde{\mathfrak{D}}$. The polyhedra in the first shell separate $\widetilde{\mathfrak{D}}$ completely from all the other images. The configuration of the polyhedron $\widetilde{\mathfrak{D}}$ and the polyhedra in the first shell of $\widetilde{\mathfrak{D}}$ is described or determined by hyperplanes, whose linear equations have as coefficients rational functions of the $\theta_i^{(0)}$ with coefficients from the field $K(\alpha)$.

Now let Ω be a neighborhood of $\theta^{(0)}$ on V which is so small that this configuration is "stable" if we replace the point $\theta^{(0)}$ by θ in Ω, in the equations that define the configuration. By the configuration being stable we mean that no two boundary elements will be brought into coincidence, that were not originally so (it should be noted that no coincidence can be destroyed because all algebraic relations are being preserved), and that the configuration should remain completely inside S^*. This will be true if every vertex in the configuration stays on the same side of each of the hyperplanes that form the faces in the configuration, which did not originally pass through it, and furthermore that each vertex lies in S^*. Since the coordinates of the vertices are expressible as rational functions of the θ_i, these conditions give a finite set of inequalities that θ has to satisfy, since these inequalities hold strictly for the point $\theta^{(0)}$, they will also hold strictly (that is, with equality excluded) in some sufficiently small open neighborhood of $\theta^{(0)}$ on V.

Let us denote the deformed fundamental domain $\widetilde{\mathfrak{D}}$ by $\widetilde{\mathfrak{D}}^\theta$. If the transformation A from $\widetilde{\Gamma}$ carries the polyhedron $\mathfrak{D}$ into one of the

polyhedra in the first shell of $\widetilde{\mathfrak{D}}$, the transformation A^θ from $\widetilde{\Gamma}^\theta$ carries the polyhedron $\widetilde{\mathfrak{D}}$ into the corresponding polyhedron in the deformed configuration. These images of $\widetilde{\mathfrak{D}}^\theta$ will form a " first shell " of $\widetilde{\mathfrak{D}}^\theta$, with all coincidences of boundary elements of all dimensions preserved from the original configuration.

We now try to make a covering of S^*, by joining to this configuration images of $\widetilde{\mathfrak{D}}^\theta$ produced by the transformations of $\widetilde{\Gamma}^\theta$, joining them across the faces and boundary elements of lower dimension, that correspond to those where the corresponding images of $\widetilde{\mathfrak{D}}$ under transformations of $\widetilde{\Gamma}$ are joined, since these still remain in coincidence, the covering space S^{xx} which we obtain in this way is locally everywhere a simple covering of S^*, we see also easily that it is a complete covering, and S^{xx} is a complete riemannian space if we take over the metric from S^*. But since S^* is simply connected such a covering space must be identical to S^*.†

Therefore $\widetilde{\Gamma}^\theta$ is discrete and $\widetilde{\mathfrak{D}}^\theta$ is a fundamental domain of this group, and $\widetilde{\Gamma}^\theta$ and $\widetilde{\Gamma}$ are isomorphic. Hence $\widetilde{\Gamma}^\theta$ is also discrete, isomorphic to Γ and since its fundamental domain in S^* is compact, the quotient $\Gamma^\theta \backslash G$ is compact, which proves our lemma.

We should remark here that this lemma can be proved to remain true, if we replace the words " compact " in the assumption and the conclusion by " of finite measure ". The proof of this is rather more complicated, and since we anyway need the compactness assumption for later application of the Lemmas 6 and 7, we cannot make much use of this stronger form of Lemma 9 at present.

4. We can now prove the main result of this paper.

THEOREM. *If $n \geqslant 3$ and Γ is a discrete subgroup of G with compact quotient $\Gamma \backslash G$, then we can by an inner automorphism of G bring Γ to such a form $T\Gamma T^{-1}$, that all the elements of $T\Gamma T^{-1}$ are algebraic matrices.*

† This can also be seen from the fact that, in the configuration of $\widetilde{\mathfrak{D}}^\theta$ and its first shell, we have all that is needed to show that if we identify corresponding faces and corresponding boundary elements of lower dimensions in $\widetilde{\mathfrak{D}}^\theta$, it becomes a compact locally symmetric riemannian manifold.

It is clear from Lemma 9 that if Γ is not already of this form, it can be continuously deformed into a group with only algebraic matrices as elements. Namely if we in the deformation Γ^θ defined above, chose for θ a point in Ω which has algebraic coordinates, Γ^θ will be a group with this property. It remains to show that for $n \geqslant 3$, the deformation represented by Γ^θ is a trivial one.

Let A be an element of Γ all of whose eigenvalues are real and distinct, and A^θ the corresponding element in Γ^θ, where θ is a point in Ω. The absolute values of the eigenvalues of A^θ are algebraic functions of the coordinates of θ on V, and so are the eigenvalues of the A_j^θ which generate $\overset{+}{\Gamma}{}_A^\theta$ in the notation of Lemma 6. From Lemma 6 it follows that the ratios between the logarithms of these algebraic functions are constant as θ varies. This leaves two possibilities, either these algebraic functions are all constants, in which case the eigenvalues of A^θ remain constant during the deformation; or all the ratios referred to above are rational numbers so that these algebraic functions are all integral powers of a single algebraic function say $\rho(\theta)$. If $n \geqslant 3$, we can deduce from this and from the fact that there are $(n-1)$ independent generators A_j^θ in $\overset{+}{\Gamma}{}_A^\theta$, that $\overset{+}{\Gamma}{}_A^\theta$ and in particular $\overset{+}{\Gamma}{}_A$ contains an element two of whose eigenvalues are equal, and the others distinct from each other and from these two, denoting such an element of $\overset{+}{\Gamma}{}_A$ by B the application of Lemma 7 shows that again the eigenvalues of A^θ remain constant during the deformation. Thus our deformation satisfies the assumptions of Lemma 5, and is therefore a trivial one.

5. We shall here give some indications of to what extent these methods carry over to groups operating on other symmetric spaces. We consider here only matrix groups† G and leave aside the exceptional semi-simple lie groups and the symmetric spaces arising from these.

† Where the constants involved in the eventual algebraic conditions that the matrices in G have to satisfy are algebraic numbers, this can always be arranged.

The analogue of Lemma 9 holds in general, with the modification that if the group G is not real, we have to consider if $A_1, \ldots, A_l$ are the generators of Γ, the field formed by adjoining to the rational field, the numbers occurring in these matrices and their complex conjugates as well (and algebraic numbers occurring in the definition of G).

Our space S will in this case be given by the quotient G/K where K is a maximal compact subgroup of G. Coordinates in S can be chosen such that the transformations of G are algebraic in the coordinates and the elements of the matrix representing the transformation. It is further essential that the distance function, used to define the fundamental domains, be such that they are bounded by hypersurfaces, defined by equations that are algebraic in the coordinates and with coefficients lying in the field defined above. Such a distance function can always be constructed. The stability requirements referred to in the proof of Lemma 9 above, which were used to define the neighborhood Ω, have to be formulated in a more complicated way, since it is not enough to look at the relationships of the vertices and the hypersurfaces defining the faces only.

The replacement of the words " compact " by " of finite measure " in the formulation of Lemma 9, is however more difficult and has only been proved to hold for special classes of groups.

One may conjecture that much more is true, namely it is reasonable to believe that if in general $\Gamma \backslash G$ is compact, and one replaces the generators of Γ by another set of elements of G which lie sufficiently close to the original generators and which satisfy all the group relations in Γ, then the new set of elements generate a discrete group Γ^* which is isomorphic to Γ and where the quotient $\Gamma^* \backslash G$ is again compact.

The analogue of Lemma 3 holds in general if G is not a product group with a compact factor; then the linear combinations of elements of Γ span the same set of matrices that the linear combinations of elements of G do. Lemma 4 therefore holds unchanged.

The Lemmas 5, 6 and 7, can of course not be carried over in general in essentially the same form, since first of all for many of the

groups there are no elements with all eigenvalues real and distinct (although for the symplectic group we could use essentially this procedure). However, we worked with these elements only, mostly for reasons of convenience.

Our Lemma 2 gives us essentially two pieces of information, namely first that during a continuous deformation of Γ, the quotient of G_A with its maximal compact subgroup cannot change dimension. This prevents an element A from being deformed in certain ways. Secondly it gives us some information about the structure of the group Γ_A, in order that this information should be non-trivial it is necessary that Γ should contain sufficiently many elements A for which the dimension referred to above is >1. This can be established by means of Lemma 1, only if the elements for which this is the case do not have measure zero in G.

The latter shows that our methods *cannot* be applied to the isotropic spaces of negative curvature.

There are other cases where only partial results can be obtained at present ; if G is the group $Sl(n, C)$, that is the group of complex n by n matrices with determinant one, all that we can prove is that a discrete subgroup Γ with compact quotient is equivalent by an inner automorphism of G to a group, which is such that if we adjoin to the rational field all the real and imaginary parts of the numbers occurring in the generators, the real field that we get has transcendentality degree at most equal to 1. If the stronger version of Lemma 9 conjectured earlier in this section (or even a considerably weakened form of it, which I shall not state here) could be proved, we would be able to prove the analogue of the theorem of the preceding section fully in this case.

We shall below list the essential cases for which our methods work satisfactorily so that we obtain the complete analogue of our theorem. We first consider the case that the symmetric space is irreducible, that is that the connected component of G is not a direct product of other groups.

For the groups operating on the irreducible bounded symmetric complex domains of E. Cartan, if we exclude the exceptional ones, and those which are isotropic and of negative curvature, our methods work for the domains of type I, III and IV, and presumably for type II but I have never checked this last case in detail.† The proof is in the case of type I much more complicated than in the two other cases.

If G is the orthogonal group of an indefinite quadratic form of signature (n, m), and we require $n > 1$, $m > 1$, and exclude the case (2, 2) (which is reducible and corresponds to a product of two hyperbolic planes), we can again prove the analogue of the theorem of the preceding section.

For the reducible spaces, that can then be expressed as product spaces of lower dimensional symmetric spaces, we exclude those that have euclidean or compact components or any of the spaces arising from the exceptional semisimple lie groups as components.

It is sufficient here to investigate the groups Γ that are not commensurate with a direct product of two discrete groups acting independently on two components of the space. Such a group Γ we say is irreducible. It is possible to show that if the connected component of G is a direct product of G' and G'' and $\Gamma\backslash G$ has finite measure, then the G'-components of the elements of Γ are dense in G'. Also the G'-component of an element in Γ is not the identity element in G' unless the element of Γ is the identity element of G.

For the reducible spaces, and with irreducible Γ with compact quotient $\Gamma\backslash G$, we can obtain the analogue of the theorem of the preceding section, if all irreducible factor spaces are of the types mentioned above for which the method works, and in addition we may allow as factor spaces isotropic spaces of negative curvature and even real dimension.

For the product of two or more hyperbolic planes, we can actually prove the analogue of our theorem, assuming only that Γ is irreducible and $\Gamma \backslash G$ of finite measure.

† For this classification, see C. L. Siegel, *Lectures on functions of several complex variables*, Princeton, (1948) (mimeographed notes).

It should finally be mentioned that after most of these results were already obtained, E. Calabi proved by methods from differential geometry that the groups with compact fundamental domain in the n-dimensional hyperbolic space cannot be deformed non-trivially for $n \geqslant 3$. Further joint work by E. Calabi and E. Vesentini provided the same conclusion for groups with compact fundamental domain acting on the bounded symmetric complex domains, including the exceptional ones, but excluding of course those that contain the unit disc (hyperbolic plane) as factor [cf. *Annals of Math.* 71 (1960), 472-507].

Combining their results with the appropriate case of Lemma 9 (excluding however the cases where the exceptional spaces occur), one gets the analogue of the theorem of section 4, in particular for the isotropic spaces of negative curvature which are completely inaccessible to my methods.

It seems probable that the differential geometric methods should generalize to give the rigidity of groups with compact fundamental domain (that is, that no non-trivial deformation is possible), also for the symmetric spaces that remain.

The case of non-compact fundamental domain with finite volume is doubtless much more difficult, but similar results are probably true.

A final conjecture is that the eigenvalues of the matrices that occur as elements in these groups that cannot be deformed non-trivially, always are units in some algebraic number field. This problem seems also very difficult to attack.

ADDED IN PROOF. According to communications from A. Weil, he has proved the conjecture stated in lines 13-7 from bottom of p. 161. Also he has given a general proof by differential geometric methods (such as conjectured in lines 16-19 on this page) of rigidity of groups with compact fundamental domain.

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SOME NEW RESULTS IN THE THEORY OF HILBERT'S MODULAR GROUP

By KARL-BERNHARD GUNDLACH

One of the most annoying shortcomings in the theory of Hilbert's modular group was the fact that only little was known about the structure of the quotient space in the neighbourhood of the critical points, the parabolic cusps. While the quotient space $\mathfrak{R}$ of the common modular group or of a subgroup of finite index of the modular group is not compact there exists a natural compactification of $\mathfrak{R}$; completion of $\mathfrak{R}$ by addition of the classes of parabolic fixed points of the group yields a compact Riemann surface. The behaviour of the generalizations to several variables of the modular group is more complicated. At the beginning of the theory of Hilbert's modular group half a century ago Blumenthal tried to overcome these difficulties by a complicated investigation of the invariant functions in a neighbourhood of the cusps but some of his arguments do not seem very convincing nowadays, and even his proposed results would not give sufficient insight into the structure of the cusps. Later on the existence could be shown of a natural compactification of the quotient space of Hilbert's modular group Γ the result of which is a compact Hausdorff space $\overline{\mathfrak{R}}$ which is a complex manifold save for a finite number of points, namely the classes of the elliptic and parabolic fixed points. By the general theory of discontinuous groups the neighbourhood of an elliptic fixed point mod Γ is a normal complex space in the sense of Serre ; but to the neighbourhood of the cusps the general theory was not directly applicable. The corresponding problem for Siegel's modular group has recently been solved by Baily [1]. Some of his results have been generalized by Cartan who proved a general theorem on continuation of complex spaces [2]. This theorem and Baily's method are easily applicable to $\overline{\mathfrak{R}}$ showing that $\overline{\mathfrak{R}}$ is a normal complex space, analytically equivalent to a projective algebraic variety [3]. These general results are, however, not satisfactory. Any hope, e.g. of applying

a generalization of the Riemann-Roch theorem to the Hilbert modular group up to now depends on the possibility of finding groups for which $\overline{\mathfrak{R}}$ not only is a normal complex space but a complex manifold. A detailed investigation therefore was necessary the first results of which will be outlined here.

For the sake of simplicity some restrictions will be made, and most of the proofs will only be sketched. The following has to be regarded as a preliminary survey of methods and of some results selected from those already obtained.

1. Definitions and general remarks. Let

P be the rational number-field,

K a totally-real algebraic number-field,

$[\mathsf{K} : \mathsf{P}] = n > 1$, n a natural number.

The group

$$\Gamma = \{A \mid A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a, b, c, d \text{ algebraic integers from } \mathsf{K},\ |A| = 1\}$$

is called the Hilbert modular group of K. For each of the n conjugate fields K_j, $j = 1, \ldots, n$, of K over P choose a complex variable $\tau^{(j)}$ and define τ, $A(\tau)$ $\{A \in \Gamma\}$, as follows:

$$\tau = (\tau^{(1)}, \ldots, \tau^{(n)})$$

$$A(\tau) = (A^{(1)}(\tau^{(1)}), \ldots, A^{(n)}(\tau^{(n)}),$$

$$A^{(j)} = \begin{pmatrix} a^{(j)} & b^{(j)} \\ c^{(j)} & d^{(j)} \end{pmatrix},\ A^{(j)}(\tau^{(j)}) = \frac{a^{(j)}\tau^{(j)} + b^{(j)}}{c^{(j)}\tau^{(j)} + d^{(j)}}.$$

$a^{(j)}$ is the conjugate of number j of a, i.e. the image in K_j of a under the canonical isomorphism $\mathsf{K} \longleftrightarrow \mathsf{K}_j$. Denote by $\mathfrak{H}^{(j)}$ the half-plane $y^{(j)} > 0$ ($\tau^{(j)} = x^{(j)} + iy^{(j)}$). Γ operates on the product space

$$\mathfrak{H} = \mathfrak{H}^{(1)} \times \ldots \times \mathfrak{H}^{(n)}$$

as a properly discontinuous group of analytic automorphisms (I shall use the symbol Γ for the group of matrices as well as for the corresponding group of mappings of $\mathfrak{H}$ onto itself, $A_1(\tau) \equiv A_2(\tau)$ is equivalent to $A_1 = \pm A_2$). Remove from $\mathfrak{H}$ all fixed points of Γ, i.e. all those points τ_0 for which in Γ a transformation other than the identity exists that leaves τ_0 fixed, and call the rest $\mathfrak{H}^*$. The quotient space $\mathfrak{H}^*/\Gamma$ is a complex manifold. A point $\tau = \alpha =$

$(\alpha^{(1)}, \ldots, \alpha^{(n)})$, $\alpha \in \mathsf{K}$, on the boundary of $\mathfrak{H}$ can be carried to $\infty = (\infty, \ldots, \infty)$ by a transformation $B = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, a_1, b_1, c_1, d_1 from K, $|B| = 1$, and the subgroup $(B\Gamma B^{-1})_\infty$ of those elements of $B\Gamma B^{-1}$ that leave ∞ fixed is generated by

n independent translations of the form $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, $\omega \in \mathsf{K}$

and

$n-1$ multiplicatively independent multipliers of the form $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ where ϵ is a unit of K.

Such a point α is called a parabolic cusp of Γ. $(B\Gamma B^{-1})_\infty$ operates on the regions

$$N(y) = y^{(1)} \ldots y^{(n)} > q \geqslant 0, \ \tau \in \mathfrak{H}.$$

If q is greater than some constant q_0, a transformation of $B\Gamma B^{-1}$ not contained in $(B\Gamma B^{-1})_\infty$ always transforms the region $N(y) > q$ into a region having no point in common with $N(y) > q$. Addition to $\mathfrak{H}$ of these points with the obvious topology gives a space $\overline{\mathfrak{H}}$ for which

$\overline{\mathfrak{H}}/\Gamma$ is a compact Hausdorff space,

$\overline{\mathfrak{H}}/\Gamma - \mathfrak{H}^*/\Gamma$ consists of a finite number of points.

Nothing essential is changed if, instead of Γ, some other group Γ^* is taken, Γ^*, in the same way as Γ, arising from a group of matrices with coefficients in K, such that $\Gamma \cap \Gamma^*$ has a finite index in Γ as well as in Γ^*, and if furthermore some of the $\mathfrak{H}^{(j)}$ do not denote the upper half-plane of the variable $\tau^{(j)}$ but the lower half-plane.

2. The elliptic fixed points. The fixed points of Γ in $\mathfrak{H}$ are isolated points and are called elliptic fixed points. If $\tau_0 = (\tau_0^{(1)}, \ldots, \tau_0^{(n)})$ is such a point, the subgroup Γ_{τ_0} of Γ leaving τ_0 fixed is a finite group as Γ is properly discontinuous in $\mathfrak{H}$. On the other hand, all the finite subgroups of Γ are of this type. Here use can be made of the fact that there is an obvious canonical isomorphism

$$\Gamma \longleftrightarrow \Gamma^{(1)} \longleftrightarrow \Gamma^{(j)}, \ j = 1, \ldots, n$$

$\Gamma^{(j)}$ denoting the component of number j :

$$\Gamma^{(j)} = \{ A^{(j)} \mid A \in \Gamma \}.$$

For a finite subgroup H of Γ the group $\mathsf{H}^{(j)}$ is a finite group of linear transformations of $\mathfrak{H}^{(j)}$ onto itself. It is well known that the only finite groups of linear transformations of a half-plane onto itself are cyclic groups of elliptic transformations. Therefore H is cyclic with a fixed point $\tau_0 = (\tau_0^{(1)}, \ldots, \tau_0^{(n)})$, $\tau_0^{(j)}$ being the fixed point of $\mathsf{H}^{(j)}$ in $\mathfrak{H}^{(j)}$. The order l of H is called the order of the fixed point τ_0. This is true for all hyperabelian groups constructed in the way Γ is, but not necessarily for general hyperabelian groups of the type of Γ, i.e. with compact $\bar{\mathfrak{H}}/\Gamma$ and $\bar{\mathfrak{H}}/\Gamma - \mathfrak{H}^*/\Gamma$ consisting of a finite number of points. But no non-trivial example is known for such a general group.

A suitable linear transformation $z = B(\tau)$ $(z_j = B^{(j)}(\tau^{(j)}), j = 1, \ldots, n)$ yields

$$B(\tau_0) = 0,\ B(\bar{\tau}_0) = \infty,$$

and $B.\,\Gamma_{\tau_0}.\,B^{-1}$ is a group of rotations generated by a rotation

$$R(z),\ R^{(j)}(z_j) = \zeta_j z_j,\quad j = 1, \ldots, n.$$

Each ζ_j is a primitive lth root of unity, l is the order of Γ_{τ_0} (the order of the fixed point τ_0). The ring of germs of functions holomorphic at τ_0/Γ on $\bar{\mathfrak{H}}/\Gamma$ is canonically isomorphic to the ring of convergent power series in $z_1, \ldots, z_n$ which satisfy $f(z_1, \ldots, z_n) = f(\zeta_1 z_1, \ldots, \zeta_n z_n)$. The existence of uniformizing parameters for a neighbourhood of τ_0/Γ on $\bar{\mathfrak{H}}/\Gamma$ implies the existence in this ring of n functions $f_1, \ldots, f_n$ such that any other function from this ring can be expressed as a power series in $f_1, \ldots, f_n$, these n functions vanishing at the origin.

Now

$$f_k(z) = \sum_{i=1}^{n} a_{ki}\, z_i^l + \text{mixed and higher terms.}$$

The functions z_m^l must be power series in $f_1, \ldots, f_n$,

$$z_m^l = \sum_{k=1}^{n} c_{mk}\, f_k + \text{mixed and higher terms}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} c_{mk}\, a_{ki}\, z_i^l + \text{mixed and higher terms.}$$

Hence $(c_{mk})\,(a_{ki}) = (\delta_{mi})$ and the Jacobian

$$\frac{\partial(z_1^l, \ldots, z_n^l)}{\partial(f_1 \ldots, f_n)} \neq 0$$

in a neighbourhood of the origin. Consequently $z_1^l, \ldots, z_n^l$ also is a system of uniformizing parameters. But $\zeta_2 = \zeta_1^r$, r satisfying $1 \leqslant r \leqslant l-1$, and $z_1^{l-r} z_2$ is an invariant function certainly not expressible as a power series in $z_1^l, \ldots, z_n^l$. The result is

THEOREM 1. *The classes of elliptic fixed points for Hilbert's modular group (for $n > 1$) are non-uniformizable points of the quotient space $\bar{\mathfrak{H}}/\Gamma$.*

This theorem remains true for those more general groups mentioned at the end of part 1.

In order to find groups whose quotient spaces are complex manifolds it is therefore necessary to select subgroups, preferably principal congruence subgroups, without elliptic fixed points. For a suitable choice of such subgroups a knowledge of the possible orders of elliptic fixed points turns out to be indispensable.

If

$$A(\tau),\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a transformation of order l, the fixed point τ_0 of A and the conjugate fixed point $\bar{\tau}_0$ are solutions of $A(\tau_0) = \tau_0$ which is a quadratic equation. $[\mathsf{K}(\tau_0) : \mathsf{K}] = 2$, and

$$\frac{1}{\tau_0 - \bar{\tau}_0} \begin{pmatrix} 1 & -\tau_0 \\ 1 & -\bar{\tau}_0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\bar{\tau}_0 & \tau_0 \\ -1 & 1 \end{pmatrix} = BAB^{-1}$$

where $B(\tau_0) = 0$, $B(\bar{\tau}_0) = \infty$. A can be assumed to have

$$A^l = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

otherwise l is odd and $-A$ has the required property. This and $|BAB^{-1}| = 1$ implies

$$BAB^{-1} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

ζ being a primitive $2l$th root of unity. Consequently ζ lies in $\mathsf{K}(\tau_0)$ and $\zeta + \zeta^{-1}$ lies in K as the transformation by B does not alter the trace of the matrix A. If, on the other hand, K contains $\zeta + \zeta^{-1}$,

$$\begin{pmatrix} \zeta + \zeta^{-1} & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$$

is the matrix of a transformation of order l. Put

$$\sigma_{2l} = \zeta + \zeta^{-1}.$$

The field $\mathsf{P}(\sigma_{2l})$ is the maximal real subfield of the field of the $2l$th roots of unity. The degree of $\mathsf{P}(\sigma_{2l})$ is

$$[\mathsf{P}(\sigma_{2l}) : \mathsf{P}] = \tfrac{1}{2}\phi(2l).$$

Thus one obtains

THEOREM 2. *For the Hilbert modular group of a field* K *there exists a fixed point of order* l *if and only if* $\mathsf{P}(\sigma_{2l})$ *is a subfield of* K.

Now

$$\mathsf{P}(\sigma_4) = \mathsf{P}(\sigma_6) = \mathsf{P}, \quad \mathsf{P}(\sigma_{2l}) \neq \mathsf{P} \text{ if } l > 3,$$

and $\mathsf{P}(\sigma_{2l}) \subset \mathsf{K}$ implies $\frac{1}{2}\phi(2l) \mid n$, which gives

THEOREM 2a. *A Hilbert modular group of a field* K *of order* n *always has fixed points of order* 2 *and* 3. *Fixed points of orders* l *other than* 2 *or* 3 *can occur only for those* l *for which* $\phi(2l) \mid 2n$, *and do occur only in the exceptional cases of those fields* K *that are extensions of* $\mathsf{P}(\sigma_{2l})$.

In the special case $n = 2$ this works out as follows. The solutions of $\phi(2l) \mid 2n$ $(2n = 4)$ are

$$l = 2, 3 \text{ and } l = 4, 5, 6.$$

The fields $\mathsf{P}(\sigma_{2l})$ are

$$\mathsf{P}(\sigma_{2\cdot 4}) = \mathsf{P}(\sqrt{2}), \quad \mathsf{P}(\sigma_{2\cdot 5}) = \mathsf{P}(\sqrt{5}), \quad \mathsf{P}(\sigma_{2\cdot 6}) = \mathsf{P}(\sqrt{3}).$$

Therefore

A fixed point of order 4 only occurs for $\mathsf{K} = \mathsf{P}(\sqrt{2})$.

A fixed point of order 5 only occurs for $\mathsf{K} = \mathsf{P}(\sqrt{5})$.

A fixed point of order 6 only occurs for $\mathsf{K} = \mathsf{P}(\sqrt{3})$.

A Hilbert modular group of a real-quadratic number field different from these three fields has fixed points of order 2 and 3 only.

Similar results can be obtained for the other groups mentioned at the end of part 1.

3. The parabolic cusps. Here I shall restrict myself to the case $n = 2$ as for $n > 2$ certain additional complications arise, especially for the construction of uniformizing variables, and not all the final results could yet be established.

The parabolic cusp α is supposed to be transformed to ∞. The subgroup $(B\Gamma B^{-1})_\infty$ leaving ∞ fixed operates on the regions

$$N(y) = y'y'' > \delta \geqslant 0, \quad \tau = (x' + iy', x'' + iy'') \in \mathfrak{H};$$

(For a general hyperabelian group) the group of translations is

$$T = \{L \mid L(\tau) = \tau + \omega = (\tau' + \omega', \ \tau'' + \omega''), \ \omega \in \Omega\},$$

the $\omega \in \Omega$ defining a lattice in the euclidean plane of the real parts of τ', τ'', the lattice being generated by two independent translations ω_1, ω_2; for the modular group, Ω is the lattice of all numbers of a certain ideal of K. T is a normal subgroup of $(B\Gamma B^{-1})_\infty$, $(B\Gamma B^{-1})_\infty/T$ is isomorphic to M,

$$M = \{L \mid L(\tau) = \epsilon^{2m}\tau = (\epsilon'^{2m}\tau', \ \epsilon''^{2m}\tau''), \ m \text{ a rational integer}\},$$

where ϵ is a certain unit of K (for Γ, ϵ is the fundamental unit).

The ring of germs of functions holomorphic at α/Γ on $\overline{\mathfrak{H}}/\Gamma$ is canonically isomorphic to the ring F of Fourier series

$$f(\tau) = \sum_{\nu \in \mathfrak{m}} a^*(\nu) \ e^{2\pi i S\nu\tau}, \ (S\nu\tau = \nu'\tau' + \nu''\tau'')$$

convergent for $N(y) > \delta$ for some constant $\delta \geqslant 0$.

$\mathfrak{m}$ is the part of the polar lattice to Ω containing the origin and those $\nu = (\nu', \nu'')$ for which both $\nu' > 0$ and $\nu'' > 0$. $a^*(\nu)$ satisfies $a^*(\epsilon^{2m}\nu) = a^*(\nu)$ for all rational integers m. By the multiplier ϵ^{2m} a lattice point ν is translated along the hyperbola $N(\mu) = \mu'\mu'' = N(\nu)$ $(\mu = \epsilon^{2m}\nu)$. There are exactly one or two values of m such that $S(\epsilon^{2m}\nu) = \epsilon'^{2m}\nu' + \epsilon''^{2m}\nu''$ is minimal. If there are two values $\nu_1 = \epsilon^{m_1}\nu$ and $\nu_2 = \epsilon^{m_2}\nu$, choose $\nu'_1 > \nu''_1$, $\nu'_2 = \epsilon''^2\nu'_1 < \nu''_2$. If there is only one minimum, take $\nu_1 = \nu_2$ (here $\nu_1' < \nu''_1$ is possible!). In any case

$$\nu_1 \in \mathfrak{F}\,, \quad \mathfrak{F} : |\epsilon''/\epsilon'| < \nu'_1/\nu''_1 \leqslant |\epsilon'/\epsilon''|\,,$$

$\nu_1 \neq \nu_2$ if and only if $\nu'_1/\nu_1'' = |\epsilon'/\epsilon''|$, and ν given, ν_1 is uniquely determined by $\nu_1 \in \mathfrak{F}$ and $\nu_1 = \epsilon^{2m}\nu$.

Put

$$f(\nu_1, \nu_2; \tau) = \sum_{m=-\infty}^{\infty} \exp(2\pi i S \nu_1 \epsilon^{2m} \tau).$$

Then $f(\nu_1, \nu_2; \tau)$ is a function from F and any function from F is given by an expansion

$$f(\tau) = \sum_{\nu_1 \in \overline{\mathfrak{F}}} a(\nu_1, \nu_2) f(\nu_1, \nu_2; \tau)$$

the $a(\nu_1, \nu_2)$ being uniquely determined by $f(\tau)$.

The existence of uniformizing parameters for a neighbhourhood of α/Γ on $\overline{\mathfrak{F}}/\Gamma$ implies the existence in F of two functions $g(\tau)$, $h(\tau)$, without constant terms in their Fourier expansions, such that any other function from F can be expressed as a convergent power series in $g(\tau)$, $h(\tau)$. Let $\beta_1 \neq 0$ be from $\mathfrak{F}$, $S(\beta_1)$ minimal, and $\gamma_1 \neq 0$ from $\mathfrak{F}$, $\gamma_1 \neq k\,\beta_1$ for natural numbers k, $S(\gamma_1)$ minimal under that condition. Then

$$g(\tau) = b(\beta_1) f(\beta_1, \beta_2; \tau) + b(\gamma_1) f(\gamma_1, \gamma_2; \tau) + \\ + \text{terms with } \nu_1 = k\beta_1 \text{ or } S(\nu_1) \geqslant S(\gamma_1);\ \nu_1 \neq \gamma_1,$$

$$h(\tau) = c(\beta_1) f(\beta_1, \beta_2; \tau) + c(\gamma_1) f(\gamma_1, \gamma_2; \tau) + \\ + \text{terms with } \nu_1 = k\beta_1 \text{ or } S(\nu_1) \geqslant S(\gamma_1);\ \nu_1 \neq \gamma_1.$$

Now $S(\mu_1 + \kappa_1) = S\mu_1 + S\kappa_1$, and, if $\mu_1 \neq \mu_2$, $\kappa_1 \neq \kappa_2$,

$$f(\mu_1, \mu_2; \tau) f(\kappa_1, \kappa_2; \tau) = f(\mu_1 + \kappa_1, \mu_2 + \kappa_2; \tau) + \\ + f(\mu_1 + \kappa_2, \mu_1 + \kappa_2; \tau) + f(\mu_2 + \kappa_1, \mu_2 + \kappa_1; \tau) + \\ + \text{terms with } S\nu_1 > S(\mu_1 + \kappa_1).$$

(If $\mu_1 = \mu_2$ or $\kappa_1 = \kappa_2$ only one or two of the first three terms are present.) As $f(\beta_1, \beta_2; \tau)$ must be expressible by $g(\tau)$ and $h(\tau)$, $b(\beta_1)$ or $c(\beta_1)$ must be different from zero. Interchanging $g(\tau)$ and $h(\tau)$, if necessary, and taking $g(\tau)$, $h(\tau) - c(\beta_1)\, g(\tau)/b(\beta_1)$ as a pair of uniformizing parameters one may assume

$$b(\beta_1) \neq 0,\ c(\beta_1) = 0$$

and

$$f(\beta_1, \beta_2; \tau) = b^{-1}(\beta_1)\, g(\tau) + \text{mixed and higher terms in } g(\tau),\ h(\tau).$$

The multiplication formula for $f(\mu_1, \mu_2; \tau)$, $f(\kappa_1, \kappa_2; \tau)$ readily gives

$$f(\gamma_1, \gamma_2; \tau) = c^{-1}(\gamma_1)\, h(\tau) + \text{mixed and higher terms in } g(\tau),\ h(\tau)$$

again after possibly changing $h(\tau)$ into $h(\tau)$ minus a polynomial in $g(\tau)$. Therefore the Jacobian

$$\frac{\partial(f(\beta_1, \beta_2; \tau), f(\gamma_1, \gamma_2; \tau))}{\partial(g(\tau), h(\tau))} \neq 0$$

in a neighbourhood of $g = h = 0$.

$$f(\beta_1, \beta_2; \tau),\ f(\gamma_1, \gamma_2; \tau)$$

is a new pair of uniformizing parameters. If $f(\nu_1, \nu_2; \tau)$ is a power series in $f(\beta_1, \beta_2; \tau)$, $f(\gamma_1, \gamma_2; \tau)$, one can easily show that this implies the existence of two non-negative rational integers m_1 and m_2 satisfying $\nu_1 = m_1\beta_1 + m_2\gamma_1$. This must be true for all lattice vectors ν_1 within $\mathfrak{F}$ which is impossible by elementary analytic geometry as β_1 and γ_1 are vectors from $\mathfrak{F}$, and $\mathfrak{F}$ is not closed to the left. From this follows

THEOREM 3. *The classes of parabolic cusps of a Hilbert modular group (and, in fact, of any hyperabelian group) for $n = 2$ are non-uniformizable points of the quotient space $\bar{\mathfrak{H}}/\Gamma$.*

In order to find a group supplying a quotient space $\bar{\mathfrak{H}}/\Gamma$ that is a manifold one must therefore look out for some non-hyperabelian extension of Γ.

4. The symmetric modular group. Let $\hat{\Gamma}$ be the extension of Γ (considered as a group of mappings of $\mathfrak{H}$) by the transformation

$$\psi((\tau', \tau'')) = (\tau'', \tau').$$

ψ just interchanges the variables τ', τ''. Γ is a normal subgroup of $\hat{\Gamma}$, and $[\hat{\Gamma} : \Gamma] = 2$. Any finite subgroup $\hat{H}$ of $\hat{\Gamma}$ either already is a finite subgroup of Γ or is an extension of order 2 of such a subgroup.

There are therefore three classes of fixed points of $\widehat{\Gamma}$ in $\mathfrak{H}$, namely :

(i) Fixed points τ_0 that are fixed points of ψ only or of one of its transforms. These fixed points are of the type of

$$\tau_0 = (\tau_0', \tau_0''), \tau_0' = \tau_0''.$$

Carrying over τ_0 to the origin gives the generating transformation

$$\psi^*((z_1, z_2)) = (z_2, z_1)$$

in a neighbourhood of the origin. Uniformizing parameters are

$$z_1 + z_2, \quad z_1 z_2.$$

(ii) Fixed points τ_0 such that $\widehat{\Gamma}_{\tau_0} = \Gamma_{\tau_0} \subset \Gamma$. Such fixed points are non-uniformizable points of the quotient space according to Theorem 1.

(iii) Fixed points τ_0 such that $\widehat{\Gamma}_{\tau_0}$ is an extension of order 2 of Γ_{τ_0} (Γ_{τ_0} not reducing to the identity). Here $\tau_0/\widehat{\Gamma}$ is a regular point of $\mathfrak{H}/\widehat{\Gamma}$ or a singular point of $\mathfrak{H}/\widehat{\Gamma}$ depending on some special properties of Γ_{τ_0}. As $\widehat{\Gamma}$ usually has fixed points τ_0 of both types I shall not give any further details in this connection.

To remove the fixed points of class (ii) and (iii) take a self-conjugate ideal $\mathfrak{a}$ of K. The principal congruence subgroup $\Gamma(\mathfrak{a})$ of Γ defined by

$$\Gamma(\mathfrak{a}) = \{A \mid A \in \Gamma, \ A \equiv \pm E \bmod \mathfrak{a}\}$$

is a normal subgroup of Γ and also a normal subgroup of $\widehat{\Gamma}$. If $\mathfrak{a}$ is suitably chosen $\Gamma(\mathfrak{a})$ has no elliptic fixed points (choose an $\mathfrak{a}$ such that $\sigma_{2l} \not\equiv \pm 2 \bmod \mathfrak{a}$ for the orders l of elliptic fixed points of Γ). Then $\widehat{\Gamma}(\mathfrak{a})$, the extension of $\Gamma(\mathfrak{a})$ by ψ, has fixed points in $\mathfrak{H}$ of type (i) only.

Now take the parabolic cusp ∞. The group $\Gamma(\mathfrak{a})_\infty$ of the type considered in part 3 has to be enlarged by ψ to reach $\widehat{\Gamma}(\mathfrak{a})_\infty$. Here $\widehat{F}$ is the ring of Fourier series consisting of those elements of F the Fourier coefficients $a^*(\nu)$ of which satisfy the additional condition

$$a^*((\nu', \nu'')) = a^*((\nu'', \nu'))$$

giving the invariance under ψ. Among the vectors

$$\epsilon^{2m}\nu = (\epsilon'^{2m}\nu',\ \epsilon''^{2m}\nu''),\quad \epsilon^{2m}\nu^*,\quad \nu^* = (\nu'', \nu')$$

there is one and only one within $\mathfrak{F}$ having $\nu' \geqslant \nu''$ (Note : for $\Gamma(\mathfrak{a})$ instead of Γ, ϵ no longer is the fundamental unit of K but the least power of the fundamental unit of K congruent to ± 1 mod $\mathfrak{a}$).

Put

$$f(\nu_1;\ \tau) = \sum_{m=-\infty}^{+\infty} \exp\,(2\pi i\, S\,\nu_1\,\epsilon^{2m}\,\tau) + \sum_{m=-\infty}^{+\infty} \exp\,(2\pi i\, S\,\nu_1^*\,\epsilon^{2m}\tau).$$

Then $f(\nu_1;\ \tau)$ is from $\widehat{F}$, and any function $f(\tau)$ from $\widehat{F}$ is given by an expansion

$$f(\tau) = \sum_{\nu_1 \in \widehat{\mathfrak{F}}} a(\nu_1)\ f(\nu_1,\ \tau).$$

$a(\nu_1)$ is uniquely determined by $f(\tau)$, and

$$\widehat{\mathfrak{F}}:\ 1 \leqslant \nu'_1/\nu''_1 \leqslant |\,\epsilon'/\epsilon''\,|.$$

The argument of the preceeding section now shows that the existence of uniformizing parameters implies the existence within $\widehat{\mathfrak{F}}$ of two lattice vectors β_1 and γ_1 such that any lattice vector ν_1 from $\widehat{\mathfrak{F}}$ can be expressed in the form $\nu_1 = m_1\beta_1 + m_2\gamma_1$, m_1, m_2 being non-negative rational integers. This is equivalent to the existence of a basis β_1, γ_1 of the polar lattice to Ω, having

$$\beta'_1/\beta''_1 = 1\,,\ \gamma'_1/\gamma''_1 = |\,\epsilon'/\epsilon''\,| = \epsilon'^2.$$

I now proceed to show that this condition also is sufficient for the existence of uniformizing parameters.

THEOREM 4. *Let $\widehat{\Gamma}$ be the symmetric extension of a hyperabelian group Γ, for $n = 2$, having ∞ as a parabolic cusp, Ω being the lattice defined by the subgroup of translations, $\epsilon^2 = (\epsilon'^2, \epsilon''^2)$ being the generating element of the subgroup of multipliers. Then a necessary and sufficient condition for the existence of uniformizing parameters for the cusp ∞ is the following :*

The polar lattice to Ω has a basis β_1, γ_1 such that

$$\beta'_1/\beta''_1 = 1,\ \gamma'_1/\gamma''_1 = |\,\epsilon'/\epsilon''\,| = \epsilon'^2.$$

Using the fact that any lattice vector ν_1 from $\widehat{\mathfrak{F}}$ has a unique representation $\nu_1 = m_1\beta_1 + m_2\gamma_1$, m_1, m_2 being non-negative rational integers, and ordering the terms in

$$f(\tau) = \sum_{\nu_1 \in \widehat{\mathfrak{F}}} a(\nu_1) f(\nu_1; \tau)$$

according to increasing values of $S\nu_1$, and if $S\nu_1 = S\mu_1$ the term with ν_1 coming first if $\nu'_1/\nu''_1 > \mu'_1/\mu''_1$, it is not very hard to see that a reduction of the series for $f(\tau)$ is possible by $g(\tau) = f(\beta_1; \tau)$, $h(\tau) = f(\gamma_1; \tau)$ giving

$$f(\tau) = P_k(g(\tau), h(\tau)) + \sum_{\nu_1 \in \widehat{\mathfrak{F}}(k)} a_k(\nu_1) f(\nu_1; \tau),$$

where $P_k(z_1, z_2)$ is a polynomial in z_1, z_2, and $\widehat{\mathfrak{F}}(k)$ is that part of $\widehat{\mathfrak{F}}$ all γ_1 of which have

$$\gamma'_1 \gamma''_1 > k^2(\beta'_1 \beta''_1 + \gamma'_1 \gamma''_1).$$

Proceeding in this manner a formal power series $P_\infty(z_1, z_2)$ would be reached having the property that insertion into P_∞ for z_1, z_2 of the Fourier series for $g(\tau)$ and $h(\tau)$ would give the Fourier series of $f(\tau)$. If one could prove that $P_\infty(z_1, z_2)$ always converges in some neighbourhood of the origin, $g(\tau)$, $h(\tau)$ would be established as uniformizing parameters taking into account that it is well known that the functions $f(\tau)$ from $\widehat{F}$ separate the points of a neighbourhood of the cusp at ∞, implying that $g(\tau)$, $h(\tau)$ already must separate these points. Unfortunately a direct proof for the convergence of $P_\infty(z_1, z_2)$ seems to be very difficult. I had to ascertain the convergence using a roundabout way.

Now

$$\begin{aligned} f(\beta_1; \tau) &= 2 \sum_{m=-\infty}^{+\infty} \exp(2\pi i S\beta_1 \epsilon^{2m} \tau) \\ &= 2e^{2\pi i S\beta_1 \tau} + \text{terms with } S\beta_1 \epsilon^{2m} > S\beta_1. \end{aligned}$$

It is not hard to see that if

$$\tau \in \widehat{\mathfrak{F}}_1, \widehat{\mathfrak{F}}_1 : |\epsilon''/\epsilon'|^{3/4} \leqslant y'/y'' \leqslant |\epsilon'/\epsilon''|^{3/4}$$

the term $2e^{2\pi i S\beta_1 \tau}$ is the dominant term, giving

$$\left.\begin{aligned} f(\beta_1;\tau) &= O(e^{2\pi i S\beta_1 \tau}) \\ e^{2\pi i S\beta_1\tau} &= O(f(\beta_1;\tau)) \end{aligned}\right\} \quad \tau \in \widehat{\mathfrak{F}}_1,\ y'y'' \to \infty,$$

implying

$$g(\tau) = f(\beta_1;\tau) \neq 0 \text{ for } \tau \in \widehat{\mathfrak{F}}_1, \quad y'y'' > \rho_1,$$

for some constant ρ_1. The same is true for $h(\tau) = f(\gamma_1;\tau)$ with

$$|\epsilon''/\epsilon'| \leqslant y'/y'' \leqslant |\epsilon''/\epsilon'|^{1/4}, \text{ dominant term } e^{2\pi i S\gamma_1\tau},$$

$$|\epsilon'/\epsilon''|^{1/4} \leqslant y'/y'' \leqslant |\epsilon'/\epsilon''|, \text{ dominant term } e^{2\pi i S\gamma_1\epsilon^2\tau}.$$

If therefore $y'y'' > \rho_1$ is assumed, $g(\tau)$, $h(\tau)$ have no common zero in this region (with the exception of $\infty/\widehat{\Gamma}$ itself where both functions vanish). Moreover

$$f(\tau) - P_k(g,h) = O(|g|^k + |h|^k).$$

Now $f(\tau)$ is shown to be an algebraic integer over the ring of convergent power series in $g(\tau)$, $h(\tau)$, and it can be proved that under these circumstances the estimate for $f - P_k$ implies that f is expressible as a power series in g, h.

For the proof put $V_0 = \{\tau \mid y'y'' > \rho_1\}/\widehat{\Gamma}$, $\alpha = \infty/\widehat{\Gamma}$, $V = V_0 + \{\alpha\}$, and observe that V has the following properties.

V is a locally compact Hausdorff space, V_0 differs from V by only one point, α, and V_0 is a complex manifold of dimension 2. A function f, continuous on an open set U of V, is called holomorphic on U if it is holomorphic on $U \cap V_0$. The functions holomorphic on V separate the points of V, i.e. given two points $\tau_1 \neq \tau_2$ from V there exists a function f holomorphic on V such that $f(\tau_1) \neq f(\tau_2)$. Furthermore there are two functions g, h holomorphic on V having α as the only common zero on V.

This is sufficient to show that there exists a neighbourhood U of α such that (g,h) maps U on $\mathfrak{R} : |z_1| < \sigma, |z_2| < \sigma$ in C_2, the mapping

$$(g,h) : U \to \mathfrak{R}$$

defining $U - \{\alpha\}$ as an analytic covering of $\mathfrak{R} - \{0\}$ of some finite degree d. This is in fact a special case of the method developed by Cartan for the continuation of complex spaces. But in my opinion

it is not useless to find out if a special case arising in some theory like this one can be handled by fairly elementary means thus offering the opportunity of avoiding general theory. I therefore remark that the above mentioned properties of the mapping (g, h) can be proved using only some of the most elementary theorems to be found in the book by Osgood. I shall not, however, give any further details here.

As $U - \{\alpha\}$ is an analytic covering of $\mathfrak{K} - \{0\}$ of degree d any function f holomorphic on U satisfies an equation (of degree m at most equal to d)

$$H(f; z_1, z_2) = f^m + H_1(z_1, z_2)\, f^{m-1} + \ldots + H_m(z_1, z_2) = 0,$$

$H_1, \ldots, H_m$ being power series in z_1, z_2 convergent for $|z_1| < \sigma$, $|z_2| < \sigma$. $H(f)$ can be assumed irreducible. Hence the discriminant of $H(f)$ is not identically zero. Let (z^*_1, z^*_2) be a point where the discriminant of $H(f)$ does not vanish. Then

$$H(f; z^*_1 t, z^*_2 t) = f^m + H_1(z^*_1 t, z^*_2 t)\, f^{m-1} + \ldots + H(z^*_1 t, z^*_2 t)$$

is a polynominal in f whose coefficients are functions of t holomorphic for $|t| < 1$; the discriminant of $H(f)$ does not vanish for $0 < |t| < \rho$ for some constant ρ. Put $t = u^r$ with a suitably chosen rational integer r. Then the m different branches of f are holomorphic functions of u for $|u| < \rho^{1/r}$. But from the behaviour of these functions on U it is known that, given any k, there exists a polynomial $P_k(z_1, z_2)$ such that

$$\begin{aligned} f - P_k(z^*_1 u^r, z^*_2 u^r) &= O(|z^*_1 u^r|^k + |z^*_2 u^r|^k) \\ &= O(|u|^{rk}), \end{aligned}$$

$P_k(z_1, z_2)$ being independent of the branch of f selected here. This, however, implies that there cannot be two different branches, and therefore $m = 1$, the desired result.

5. Some examples. Groups the quotient spaces of which are complex manifolds can for instance be found as follows.

For simplicity take a field K having class number one, and consider the group

$\Gamma^* = \{A \mid A$ a 2×2 square matrix whose coefficients are integers from K, $|A| =$ a positive unit of $\mathsf{K}\}$

instead of Γ. Γ^* is an extension of Γ, $[\Gamma^* : \Gamma] = 1$ or 2. Then the multipliers $\epsilon^2 = (\epsilon'^2\ \epsilon''^2)$ are given by $\epsilon^2 =$ a positive unit of K (i.e. ϵ itself is a square root of a positive unit of K). Look out only for normal subgroups of Γ^*, for here any cusp can be carried to ∞ by a mapping $B \in \Gamma^*$, and transforming the normal subgroup by B only gives an automorphism of the group. Hence one need only investigate if uniformizing parameters exist at the cusp ∞. It is convenient to select a congruence subgroup of Γ^*. Then the lattice Ω of translations at ∞ will be the lattice of the numbers of some ideal of K and so will be the polar lattice to Ω. If the ideal defining the polar lattice is generated by a rational number this lattice can be assumed to be the lattice of the integers of K. This lattice has to comply with the condition of Theorem 4. Thus ask for a field K having an integral basis $1, \beta$ satisfying

$$\beta'/\beta'' = \text{some positive unit } \epsilon_0 \text{ of } \mathsf{K}.$$

Supposing for instance the discriminant of K is equal to $4p$, p a prime, such a basis always exists if

$$p + 2 = \text{the square of a rational number.}$$

If, moreover, ϵ_0 is the fundamental unit of K, and

$$\epsilon_0 \equiv +1 \bmod \sqrt{p},$$

the group

$$\Gamma^*(\sqrt{p}) = \{A \mid A \in \Gamma^*,\ A \equiv E \bmod \sqrt{p}\}$$

has no elliptic fixed points (the small discriminants 8, 12 have to be excluded here), and ϵ_0 is the generating multiplier. Thus

$$\overline{\mathfrak{H}}/\widehat{\Gamma^*(\sqrt{p})} \text{ is a complex manifold.}$$

Examples are $p = 7,\ 23,\ 47$.

By a similar method fields can be found having the following property.

No congruence subgroup of Γ defined in the usual way (here I shall not give the precise meaning of the term 'in the usual way')

has a compact quotient space that is a complex manifold. An example is $K = P(\sqrt{11})$.

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MODULAR FORMS OF DEGREE n AND REPRESENTATION BY QUADRATIC FORMS*

By S. RAGHAVAN

LET $\mathfrak{H}_n$ denote the space of n-rowed complex symmetric matrices $Z = X + iY$ where Y is positive-definite. Let $\mathscr{M}_n(s)$ denote the group of modular matrices M of degree n, satisfying $M \equiv E^{(2n)} \pmod{s}$, ($E^{(r)}$ being the r-rowed identity matrix); for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathscr{M}_n(1)$, the mapping $Z \to M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$ is an analytic homeomorphism of $\mathfrak{H}_n$ onto itself. Further, let $\{n, s, -k, v\}$ denote the space of *modular forms of degree* n, *stufe* s, *dimension* $-k$ (k integral) and *belonging to a multiplier-system* $\{v(M),\ M \in \mathscr{M}_n(s)\}$. Let, for $f(Z) \in \{n, s, -k, v\}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathscr{M}_n(1)$, $f(Z)/M = f(M\langle Z\rangle)\,|\,CZ + D\,|^{-k}$.

Following Hecke and Petersson, we define the generalized Eisenstein series in $\{n, s, -k, v\}$, for $k > n + 1$; these have simple Fourier expansions and their Fourier coefficients $b(T)$ satisfy the estimate $b(T) = O(\delta(T)^{k-(h+1)/2})$, h and $\delta(T)$ being respectively the rank and discriminant of T. We then associate with every $f(z) \in \{n, s, -k, v\}$ for $k > n + 1$, a certain linear combination $\phi(Z)$ of these Eisenstein series; explicitly, $\phi(Z) = \sum_N a(O, N^{-1})\,|\,CZ + D\,|^{-k}$, where $(C\ D)$ is the 'second matrix row' of N, $a(O, N^{-1})$ is the constant term in the Fourier expansion of $f(Z)/N^{-1}$ and N runs over a complete set of representatives of the right cosets of $\mathscr{M}_n(1)$ modulo the subgroup consisting of all modular matrices with second matrix row of the form $(O\ U)$, U being a n-rowed unimodular matrix. With some applications to number theory in mind, we are interested in obtaining an estimate for the Fourier coefficients of $f(Z) - \phi(Z)$.

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For the Fourier coefficients $c(T)$ of a cusp form in $\{n, s, -k, v\}$ we can obtain the estimate $c(T) = O(|T|^{k/2})$ similar to Hecke's for $n = 1$. If $f(Z) - \phi(Z)$ were a cusp form as for $n = 1$, this estimate would give us what we wanted. But since for $n > 1$, $f(Z) - \phi(Z)$ is not necessarily a cusp form, we estimate the coefficients of $f(Z) - \phi(Z)$, by using Siegel's generalized Farey dissection and obtain the following

THEOREM. *If* $f(Z) = \sum_{T \geqslant 0} a(T) \exp((2\pi i/s)\, \sigma(TZ)) \in \{n, s, -k, v\}$ (k *integral and* $k > n + 1$) *and* $\phi(Z) = \sum_{T \geqslant 0} b(T) \exp((2\pi i/s)\, \sigma(TZ))$ *is the associated linear combination of Eisenstein series, then for positive definite* T *with* $|T| \geqslant \nu$ (*depending only on* $f(Z)$),

$$a(T) = b(T) + O((\min T^{-1})^{n(n+1-2k)/2} (\min T)^{(n+1-k)/2}). \quad (1)$$

(For a positive definite R, $\min R$ denotes its ' minimum '.)

In order to make the estimate for $a(T) - b(T)$ in (1) comparable with that for $b(T)$ and worthwhile, we impose on T, the condition '$\min T \geqslant c|T|^{1/n}$ for a fixed $c > 0$'. With this condition, (1) becomes $a(T) = b(T) + O(|T|^{(k(2n-1)/2n - (n^2-1)/2n)})$. The constants in the O-estimate will depend on c and $f(Z)$ in general.

Let $A(S,T)$ denote the number of integral representations of an n-rowed positive definite integral matrix T by an m-rowed ($m \geqslant n$) positive definite integral matrix S. A generating function of $A(S,T)$ is the generalized theta-series $f(S, Z) = \sum_{T \geqslant 0} A(S, T) \exp(2\pi i\, \sigma(TZ))$ with $Z \in \mathfrak{H}_n$. Let m be even and greater than $2n + 2$; then we specialize $f(Z)$ to be $f(S, Z)$ which can be shown to be in $\{n, 4|S|, -m/2, v\}$ for a certain multiplier-system. The associated $\phi(Z)$ is Siegel's 'analytic genus-invariant' associated with S. Appealing to the theorem above, we get the following result, viz.

If S *is an* m*-rowed positive definite integral matrix and* T *an* n*-rowed positive definite integral matrix satisfying the conditions* "$\min T \geqslant c|T|^{1/n}$ *for a fixed* $c > 0$" *and* "$|T| \geqslant \nu$ (*a constant*)", *then for even* $m > 2n + 2$.

$$A(S, T) = \mu \mid T \mid^{(m-n-1)/2} \prod_p \alpha_p(S, T) + O(\mid T \mid^{\{m(2n-1)-2(n^2-1)\}/4n}) \quad (2)$$

where μ is a constant independent of T and $\prod_p \alpha_p(S, T)$ is the infinite product of the p-adic densities of representation of T by S, in the sense of Siegel. Formula (2) can be upheld even for *odd* $m > 2n + 2$.

We may investigate the conditions under which formula (2) is ' truly asymptotic ', in the sense that the principal term is of a strictly higher order in $\mid T \mid$ than the O-term. We can give sufficient conditions on S and T, under which the product $\prod_p \alpha_p(S, T)$ is bounded away from zero uniformly in T and under these conditions, (2) is truly asymptotic. We can also obtain an analogue for $n = 2$, of the Hardy-Ramanujan asymptotic formula for $A(E^{(m)}, t)$ (for $m > 4$) and an analogue for $n = 2$ of a well known theorem of Tartakowsky.

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THE PROPERTIES OF THE REPRESENTATION OF THE ABELIAN DIFFERENTIALS BY POINCARE'S SERIES

By H. PETERSSON

1. We consider a uniformization of a given compact Riemann surface R leading to a horocyclic group of the first kind Γ in the upper half plane H of the uniformizing variable τ. The group Γ may always be chosen such that it contains parabolic substitutions; we use this proposition essentially. The mapping effected by τ transforms the abelian differentials of R into the automorphic forms of the "differential class" $\{\Gamma, -2\}$, i.e. the forms belonging to Γ and dimension -2. Therefore the problem named above is identical with that of representing the forms $\{\Gamma, -2\}$ and here, the only analytic procedure sufficiently general to yield representations of all those automorphic forms consists in using Poincaré's series.

In the present case of dimension -2 and horocyclic groups, these series do not converge absolutely. Therefore, first of all, a summation method has to be developed which correlates to these series values representing automorphic forms of the differential class with characteristic properties corresponding to the situation in the case of dimension < -2. There are several possible approaches to such a method ; one of them has been carried out by the author. The result is, roughly speaking, that the analogy named above is realised with a certain modification which prevents a contradiction to the theorem of residues. The characteristic properties concern the meromorphic behaviour and the metrical composition of automorphic forms.

We now formulate some general conditions and notations.

The proposition concerning Γ implies that there are only a finite number of classes of fixed points of Γ congruent to each other mod Γ. We denote : by p the genus of Γ (i.e. of R); by σ the number of the

congruence classes of parabolic fixed points (mostly called " cusps "); by e_0 the number of the congruence classes of elliptic fixed points of Γ; by H (as introduced above) the upper half-plane of the complex variable $\tau = x + iy$ ($\tau \in H$ meaning $y = \operatorname{Im} \tau > 0$); by H' the sum of H and the set of the cusps of Γ (which form a dense set on the boundary of H); by W^1 the multiplicative group of matrices $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real coefficients and determinant 1. Each $S \in W^1$ is correlated with the transformation

$$\tau \to S\tau = S(\tau) = \frac{a\tau + b}{c\tau + d}. \tag{1.1}$$

Γ is regarded as a discrete subgroup of W^1 containing the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The general matrix of Γ is mostly denoted by L (or $L_1, L_2, \ldots$); where $L \in \Gamma$, $M \in W$ appear, we write or assume that $L = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}, M = \begin{pmatrix} * & * \\ m_1 & m_2 \end{pmatrix}$. A matrix $S \in W^1$ is called conjointly with the transformation (1.1) parabolic, elliptic, hyperbolic and the fixed points of the matrix are those of (1.1). The parabolic matrices of W^1 are of the form $S = A^{-1} U^r A$ where $A \in W^1$, $U^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ for a real number r (symbolic power of $U = U^1$); if $r \neq 0$ the only fixed point of S is $A^{-1}\infty$.

2. The differential class $\{\Gamma, -2\}$ is defined as follows: $f = f(\tau) \in \{\Gamma, -2\}$ means firstly that f is a meromorphic function of the local uniformizing parameter of each point of H'; secondly that

$$f(L\tau) = (\gamma\tau + \delta)^2 f(\tau) \text{ for all } L \in \Gamma. \tag{2.1}$$

Let $\tau = z$ be a point in H. We write $l = l(z) = 1$ if z is not a fixed point of Γ; otherwise we take for $l = l(z)$ the order of z as a fixed point of Γ. Now, in a neighbourhood of $\tau = z$, f admits an expansion

$$f(\tau) = (\tau - \bar{z})^{-2} \sum_{m=-m_0}^{\infty} b_m(z, f) w^m, \tag{2.2}$$

where $w = w(\tau, z) = \dfrac{\tau - z}{\tau - \bar{z}}$ and the b_m's are independent of τ. Here, $b_m(z, f) = 0$ if $m \not\equiv -1 \bmod l$.

$$\frac{1}{2\pi i}\rho(z, f) = \frac{1}{l(z - \bar{z})} b_{-1}(z, f) \text{ (if } m_0 \geqslant 1) \tag{2.3}$$

represents the residue of the differential corresponding to f on R at the point corresponding to $\tau = z$ on R. It is easily proved that

$$\rho(Lz, f) = \rho(z, f) \ (L \in \Gamma).$$

We write a cusp of Γ in the form

$$\tau = \zeta = A^{-1}\infty \left(A = \begin{pmatrix} * & * \\ a_1 & a_2 \end{pmatrix} \in W^1 \right).$$

Among the matrices of Γ with the fixed point ζ there is a generating matrix of the form $A^{-1} U^N A$ where $N > 0$. Each $f \in \{\Gamma, -2\}$ admits, in a neighbourhood of $\tau = \zeta$, an expansion

$$f(\tau) = (a_1\tau + a_2)^{-2} \sum_{m=-m_0}^{\infty} b_m(A, f)\, t_A^m, \tag{2.4}$$

where $t_A = t_A(\tau) = \exp \dfrac{2\pi i}{N} A\tau$ and the b_m's are constant. The residue analogous to (2.3) is given by

$$\frac{1}{2\pi i}\rho(\zeta, f) = \frac{N}{2\pi i} b_0(A, f); \tag{2.5}$$

it does not depend on the choice of A for a preassigned ζ and is invariant under Γ : $\rho(L\zeta, f) = \rho(\zeta, f) \ (L \in \Gamma)$.

If $f \not\equiv 0$ and the least value of m_0 is chosen in (2.2, 4) then $-l^{-1} m_0$ or $-m_0$ is the order of f in $\tau = z$ or $\tau = \zeta$ respectively. This order is reproduced by $z \to Lz$ or $\zeta \to L\zeta$ $(L \in \Gamma)$. On a fundamental set F of Γ, there are atmost a finite number of points in which f has a negative order. The sum of the orders of $f \not\equiv 0$ in all points of F is a certain positive number which depends only on Γ. The theorem of residues consists in the equation

$$\sum_{\tau_0 \in F} \rho(\tau_0, f) = 0;$$

the sum on the left hand side contains atmost a finite number of summands different from zero.

We call f a differential of the first kind if f is holomorphic in H and has a positive order in all cusps of Γ; a differential of the second kind, if $\rho(\tau_0, f) = 0$ for each $\tau_0 \in H'$; a differential of the third kind. if the order of f is everywhere $\geqslant -1$, in the cusps even $\geqslant 0$. This coincides with the usual notions. The differentials of each kind form a linear manifold. We denote it according to the sequence given above by C_1, C_2, C_3; clearly $C_1 = C_2 \cap C_3$.

3. The scalar product of two functions belonging to $\{\Gamma, -2\}$ is defined by an integral over a fundamental domain of Γ. We consider special classes of such domains. By a regular fundamental domain F of Γ we understand a connected domain which forms a fundamental set of Γ; which is bounded by a finite number of hyperbolic segments or straight lines; and whose closure does not contain any two fixed points of Γ congruent to each other mod Γ. F may always be locally deformed into another regular fundamental domain of Γ such that a finite number of given congruence classes of points in H have representatives in the interior of F, provided that no fixed point occurs among these points.

We assume that F has this property with respect to the poles of two given forms f, $g \in \{\Gamma, -2\}$ which do not coincide with fixed points of F. For each such pole, we construct a sufficiently small hyperbolic circular disk with the pole as hyperbolic center and cut it out from F. In the case of an elliptic fixed point we cut out the intersection of such a circle with F. In the case of a parabolic fixed point we cut out the intersection of a hyperbolic paracycle with F. Let F^* be the remaining region. We write the scalar product of f and g as

$$(f, g) = (f(*), g(*)) = \lim \iint_{F^*} f(\tau)\, \overline{g(\tau)}\, dx\, dy \tag{3.1}$$

if the limit exists as the radii of the hyperbolic circles and corresponding parameters of the figures cut out from F tend independently to zero. The necessary and sufficient condition for the

existence of the limit prescribes that for each pole $\tau = z$ of f or g in $F \cap H$ and for each cusp $\tau = \zeta$ in F (cf. (2.2), (2.4))

$$b_m(z, f)\, b_m(z, g) = 0 \ (m < 0); \ b_m(A, f)\, b_m(A, g) = 0 \ (m \leqslant 0). \quad (3.2)$$

Therefore the forms of C_1 have scalar products with all forms of $\{\Gamma, -2\}$ and we have

$$C_2 = C_1 \dot{+} \mathcal{O}_2, \ C_3 = C_1 \dot{+} \mathcal{O}_3, \ \{\Gamma, -2\} = C_1 \dot{+} \mathcal{O}_2 \dot{+} \mathcal{O}_3, \quad (3.3)$$

where $\mathcal{O}_i (i = 2, 3)$ denotes the orthogonal manifold in C_i with respect to C_1. C_1 is of algebraic dimension p, $\mathcal{O}_2$ and $\mathcal{O}_3$ are of algebraic dimension ∞. C_2 contains the manifold E_2 of exact differentials; $f \in E_2$ means that a completely invariant meromorphic automorphic function $\Phi = \Phi(\tau)$ such that $\Phi'(\tau) = f(\tau)$ exists.

If the residues of $f \in \{\Gamma, -2\}$ vanish in the points of H, there exists a corresponding unambiguous meromorphic abelian integral

$$u(\tau_0, \tau; f) = \int_{\tau_0}^{\tau} f(z)\, dz \quad (\tau, \tau_0 \in H). \quad (3.4)$$

The periods of u, defined by

$$\eta(L, f) = u(\tau_0, L\,\tau; f) - u(\tau_0, \tau; f) \quad (L \in \Gamma) \quad (3.5)$$

do not depend on τ_0 and are therefore regarded as the periods of f. If f is fixed, $\eta(L, f)$, *qua* function of L, represents an additive abelian character on Γ. If L is fixed, $\eta(L, f)$, *qua* function of f, represents a linear functional on the manifold of those f. The formation of abelian integrals and periods are subject to certain restrictions in case that residues of f in H exist which are different from zero.

4. The results of the summation method are now summarized in the following way. On the one side of the sign $\sim$ we write the (in general not convergent) Poincaré series, on the other the function $\in \{\Gamma, -2\}$ arising from the series by that method. The law of transitivity and homomorphisms with respect to linear combination may be applied to the sign $\sim$. The functions mentioned above are uniquely determined by their properties stated subsequently.

In the notation of (2.4) we write

$$G(\tau, A, \nu) \sim \sum_{M} (m_1\tau + m_2)^{-2} v_M^{\nu}(\tau) \ (\nu \text{ an integer}, \ \nu \neq 0). \quad (4.1)$$

M runs over a complete system of matrices of $A\Gamma$ with different second rows $\{m_1, m_2\}$. If $\nu > 0$ we have $G(\tau, A, \nu) \in C_1$ and

$$(\phi(*), G(*, A, \nu)) = \frac{N^2}{2\pi\nu} b_\nu(A, \phi) \ \ (\phi \in C_1, \nu > 0). \quad (4.2)$$

If $\nu < 0$ we have $G(\tau, A, \nu) \in \mathcal{O}_2$. In all points of H' incongruent to ζ mod Γ, $G(\tau, A, \nu)$ behaves like a form of C_1. On the contrary,

$$(a_1\tau + a_2)^2 \, G(\tau, A, \nu) = 2v_A^{\nu} + \sum_{n=1}^{\infty} \beta_n v_A^n \quad (\nu < 0) \quad (4.3)$$

where the β_n's are constant and the power series converges in H.

In the notation of (2.2) we write

$$\Psi(\tau, z, m) \sim \sum_{M \in \Gamma} (m_1\tau + m_2)^{-2} (M\tau - \bar{z})^{-2} w^m (M\tau, z)$$

$$(m \text{ an integer}, \ m \neq -1). \quad (4.4)$$

If $m \geqslant 0$, we have $\Psi(\tau, z, m) \in C_1$ and

$$(\phi(*), \Psi(*, z, m)) = \frac{2\pi}{(z - \bar{z})^2 (m + 1)} b_m(z, \phi) \ (\phi \in C_1, \ m \geqslant 0). \quad (4.5)$$

If $m < -1$ we have $\Psi(\tau, z, m) \in \mathcal{O}_2$. In all points of H' incongruent to z mod Γ, $\Psi(\tau, z, m)$ behaves like a form of C_1. In a certain neighbourhood of $\tau = z$, an expansion

$$(\tau - \bar{z})^2 \Psi(\tau, z, m) = 2lw^m(\tau, z) + \sum_{n=0}^{\infty} \beta_n w^n(\tau, z) \ (l = l(z), \ m < -1) \quad (4.6)$$

is valid if $m \equiv -1 \bmod l$. Otherwise, $\Psi(\tau, z, m)$ vanishes identically in τ.

The functions (4.1) with $\nu = 0$ and (4.4) with $m = -1$ exist only in a restricted sense, not bearing an invariant meaning. For certain linear combinations of two of such functions, however,

there is such a meaning. Let ζ and $\zeta' = A'^{-1}\infty$ $(A' \in W^1)$ be two cusps of Γ incongruent to each other mod Γ and z, z' two points in H incongruent to each other mod Γ. Then, if N' has the meaning with respect to ζ' analogous to that of N with respect to ζ, we write

$$\left.\begin{aligned} \Phi(\tau\,;\,\zeta,\zeta') &\sim N'^{-1}\,\pi i G(\tau, A', 0) - N^{-1}\pi i G(\tau, A, 0) \\ \Phi(\tau\,;\,\zeta, z) &\sim \tfrac{1}{2}(z-\bar{z})\,\Psi(\tau, z, -1) - N^{-1}\pi i G(\tau, A, 0) \\ \Phi(\tau\,;\,z, z') &\sim -\tfrac{1}{2}(z-\bar{z})\,\Psi(\tau, z, -1) + \tfrac{1}{2}(z'-\bar{z}')\,\Psi(\tau, z', -1) \end{aligned}\right\} \quad (4.7)$$

and use $\Phi(\tau\,;\,\tau_1, \tau_2)$ $(\tau_1, \tau_2 \in H')$ as a common notation for the three cases. Now $\Phi(\tau\,;\,\tau_1, \tau_2) \in \mathfrak{C}_3$ and

$$\rho(\tau_1, \Phi(*\,;\,\tau_1, \tau_2)) = -2\pi i, \quad \rho(\tau_2\,;\,\Phi(*\,;\,\tau_1, \tau_2)) = +2\pi i. \quad (4.8)$$

Moreover,

$$\Phi(\tau\,;\,\tau_1, \tau_2) + \Phi(\tau\,;\,\tau_2, \tau_3) = \Phi(\tau\,;\,\tau_1, \tau_3) \quad (4.9)$$

for arbitrary $\tau_1, \tau_2, \tau_3 \in H'$, where $\Phi(\tau\,;\,\tau_1, \tau_2)$ is defined to be identically zero if $\tau_2 \equiv \tau_1 \bmod \Gamma$.

The fourth type of differentials arise as follows. Let $\Theta = \begin{pmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta'_1 & \vartheta'_2 \end{pmatrix} \in W^1$ be such that $\Theta\,\vartheta = 0$, $\Theta\vartheta' = \infty$ where ϑ, ϑ' denote the pair of fixed points of a hyperbolic matrix of Γ. Then, we write

$$\Xi(\tau, \Theta) \sim \sum_{\mathsf{B}} (\beta_1\tau + \beta_2)^{-1}\ (\beta'_1\tau + \beta'_2)^{-1} \quad (4.10)$$

where $\mathsf{B} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta'_1 & \beta'_2 \end{pmatrix}$ runs over a complete system of matrices of $\Theta\Gamma$ such that no two such matrices differ from each other by a left-hand-side factor $D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ $(\lambda > 0)$. Now let K be the matrix in Γ of the form $K = \Theta^{-1}\,D_\lambda\,\Theta$ with the least $\lambda > 1$. Then $\Xi(\tau, \Theta) \in C_1$ and

$$(\phi(*), \Xi(*, \Theta)) = 2\pi\eta(K, \phi)\ (\phi \in C_1). \quad (4.11)$$

The functions G, Ψ, Φ, Ξ have certain properties of invariance under the substitutions of Γ, applied to parameters different from τ. For L (or L_1, L_2) $\in \Gamma$ we see that

$$G(\tau, AL, \nu) = G(\tau, A, \nu) \ (\nu \neq 0)$$

$$\Psi(\tau, Lz, m) = (\gamma z + \delta)^{-m} \ (\gamma \bar{z} + \delta)^{m+2} \Psi(\tau, z, m) \ (m \neq -1) \quad (4.12)$$

$$\Phi(\tau\,; L_1 \tau_1, L_2 \tau_2) = \Phi \ (\tau\,; \tau_1, \tau_2)$$

$$\Xi(\tau, \Theta L) = \Xi(\tau, \Theta).$$

It is therefore sufficient to consider the following functions of $\{\Gamma, -2\}$: $G(\tau, A_h, \nu)$ $(1 \leqslant h \leqslant \sigma, \ \nu \neq 0)$ where $A_h \in W^1$ and $\zeta_h = A_h^{-1} \infty$ runs over the complete set of the cusps of a fundamental set F of Γ; $\Psi(\tau, z, m)$ $(z \in H \cap F, m \neq -1)$; $\Phi(\tau\,; \tau_1, \tau_2)$ $(\tau_1, \tau_2 \in F, \tau_1 \neq \tau_2)$. The matrices Θ in the functions $\Xi(\tau, \Theta)$ are determined subsequently.

We mention two theorems on bases:

The functions $G(\tau, A_h, \nu)$ $(1 \leqslant h \leqslant \sigma, \ \nu < 0)$ and $\Psi(\tau, z, m)$ $(z \in H \cap F, \ m \leqslant -2, m \equiv -1 \bmod l(z))$ form a basis of $\mathcal{O}_2$. This basis is — apart from constant factors for the individual functions — the only basis of $\mathcal{O}_2$ such that any two different functions of the basis possess a scalar product with each other.

For each preassigned $z_0 \in F$, the functions $\Phi(\tau\,; z_0, z)$ $(z \in F, \ z \neq z_0)$ form a basis of $\mathcal{O}_3$.

We use special notations for two cases of (4.4), viz.

$$\Psi(\tau, z, 0) = \Omega(\tau, z), \ \Psi(\tau, z, -2) = \Psi(\tau, z) \ (z \in H). \quad (4.13)$$

For these, we obtain according to (4.12)

$$\Omega(\tau, Lz) = (\gamma \bar{z} + \delta)^2 \, \Omega(\tau, z), \ \Psi(\tau, Lz) = (\gamma z + \delta)^2 \, \Psi(\tau, z) \quad (4.14)$$

and according to (4.5)

$$(\phi(*), \Omega(*, z)) = -2\pi\phi(z) \quad (\phi \in C_1, z \in H). \quad (4.15)$$

Moreover,

$$\Omega(z, \tau) = \overline{\Omega}(\tau, z), \quad (4.16)$$

the bar denoting, as usual, the conjugate complex value.

5. We need subsequently a canonical fundamental domain of Γ. We draw a system of p pairs of retrosections on R such that any two sections meet one another only in the cross point P_0 which is situated on all sections and is not a branch point of the covering

surface correlated with the uniformization. From P_0, we draw sections to the different branch points, again meeting each other and the retrosections only in P_0. Let R^* be the interior of the surface R, the system of sections being regarded as the boundary. The transformation effected by τ maps R^* into the interior of a canonical fundamental domain K of Γ. It may be supposed that the boundary of K consists of a finite number of hyperbolic segments and straight lines.

We call boundary matrices of K those $L \in \Gamma$ for which K and LK have a path of edges in common. These paths of edges correspond to the sides of the sections on R and therefore the transformations belonging to the boundary matrices correspond to the transgressions of the sections. L is parabolic, elliptic, hyperbolic according to whether the section leads from P_0 to a logarithmic or an algebraic branch point or back to P_0. We denote the boundary matrices by

$P_h = A_h^{-1} U^{N_h} A_h \quad (1 \leqslant h \leqslant \sigma,\ A_h \in W^1,\ N_h > 0)$;

$\zeta_h = A_h^{-1} \infty$ runs over the system of cusps of K;

$E_k (1 \leqslant k \leqslant e_0)$ where $E_k\, \omega_k = \omega_k$ runs over the system of elliptic fixed points of K;

$G_\nu,\ H_\nu = G_{p+\nu} (1 \leqslant \nu \leqslant p)$ are hyperbolic and have positive traces.

It is not difficult to prove that $G_\nu^{\pm 1} (1 \leqslant \nu \leqslant 2p)$ can be represented in the form $\Theta_\nu^{-1} D_{\lambda_\nu} \Theta_\nu$, $\lambda_\nu > 1$, and that every such $G_\nu^{\pm 1}$ is the matrix of that form in Γ with the least value of $\lambda_\nu > 1$. We then consider the $G(\tau, A_h, \nu)$, $\Psi(\tau, z, m)$ named in § 4 for $F = K$ and

$$\Xi_\nu(\tau) = \Xi(\tau, \Theta_\nu) \ (1 \leqslant \nu \leqslant 2p). \tag{5.1}$$

Γ may be regarded as an abstract group generated by the matrices $-I$, P_h, E_k, $G_\nu (1 \leqslant \nu \leqslant 2p)$, the defining relations between them being

$$\left.\begin{aligned} &(-I)^2 = I,\ E_k^{l_k} = -I \ (1 \leqslant k \leqslant e_0), \\ &\prod_{h=1}^{\sigma} P_h \prod_{k=1}^{e_0} E_k \prod_{\nu=1}^{p} K_\nu = (-I)^{\sigma + e_0}, \end{aligned}\right\} \tag{5.2}$$

where $K_\nu = G_\nu H_\nu G_\nu^{-1} H_\nu^{-1} (1 \leqslant \nu \leqslant p)$, $l_k = l(\omega_k)$ $(1 \leqslant k \leqslant e_0)$.

In the following procedures we may assume, that—besides the fixed points—no pole of a preassigned finite system of forms in $\{\Gamma, -2\}$ lies on the boundary of K. Poles in the interior of K are joined to certain of the points $\tau(P_0)$ in K by paths of hyperbolic segments not intersecting each other or the boundary of K except for P_0 itself.

Now let f_1, f_2 be two forms of $\{\Gamma, -2\}$ and $u_i(\tau) = u(\tau_0, \tau; f_i)$ $(i = 1, 2)$ corresponding abelian integrals. If the scalar product (f_1, f_2) exists, we obtain by integration of $u_1 du_2$ and $\overline{u}_1 du_2$ along the boundary of K two formulae which we subsequently write down. We denote by $\omega_k (e_0 + 1 \leqslant k \leqslant d)$ the poles of f_1 and f_2 in the interior of $K (l_k = l(\omega_k) = 1)$; by $s_\nu (1 \leqslant \nu \leqslant \sigma + d)$ the points $\zeta_h (1 \leqslant h \leqslant \sigma)$, $\omega_k (1 \leqslant k \leqslant d)$; by $c_0(z, f)$ the constant term of the expansion of $u(\tau_0, \tau; f)$ in powers of the local uniformizing parameter of $\tau = z \in K$. The symbol $\hat{c}$ has the following meaning in the following formulae: c is a complex number and $\hat{c}$ is either always $= c$ or always $= \bar{c}$, the conjugate complex number of c. In the first case we obtain W_1, W_2, W_3, W_4 in the second W^*_1, W^*_2, W^*_4. Here

$$W_1 \text{ or } W^*_1 = \sum_{\nu=1}^{\sigma+d} \{\hat{c}_0(s_\nu, f_1)\, \rho(s_\nu, f_2) - \hat{\rho}(s_\nu, f_1)\, c_0(s_\nu, f_2)\},$$

$$W_2 \text{ or } W^*_2 = \tfrac{1}{2} \sum_{1 \leqslant \mu \leqslant \nu \leqslant \sigma+d} \{\hat{\rho}(s_\mu, f_1)\, \rho(s_\nu, f_2) - \hat{\rho}(s_\nu, f_1)\, \rho(s_\mu, f_2)\},$$

$$W_3 = \sum_{h=1}^{\sigma} \frac{N_h^2}{2\pi i} \sum_{0<|m| \leqslant m_{0h}} m^{-1}\, b_m(A_h, f_1)\, b_{-m}(A_h, f_2) +$$

$$+ \sum_{k=1}^{d} \frac{2\pi i}{l_k^2 (\omega_k - \overline{\omega}_k)^2} \sum_{0<|m| \leqslant n_{0k}} m^{-1} b_{ml_k - 1}(\omega_k, f_1)\, b_{-ml_k - 1}(\omega_k, f_2),$$

$$W_4 \text{ or } W^*_4 = \sum_{\nu=1}^{p} \{\hat{\eta}(G_\nu, f_1)\, \eta(H_\nu, f_2) - \hat{\eta}(H_\nu, f_1)\, \eta(G_\nu, f_2)\}.$$

Then we have

$$0 = W_1 + W_2 + W_3 + W_4, \quad -2i(f_2, f_1) = W^*_1 + W^*_2 + W^*_4. \quad (5.4)$$

6. The first applications arise from a combination of (4.2, 5) with (5.4). Let

$$m_1 < m_2 < \ldots < m_p \,(m_j = m_j(z),\ 1 \leqslant j \leqslant p;\ m_1 \geqslant 1)$$

be the system of Weierstrass' gap numbers of the point of R corresponding to the point $\tau = z \in H'$. Then, the functions (cf. (4.1))

$$G(\tau,\ A_h,\ m_j(\zeta_h))\ (1 \leqslant h \leqslant \sigma,\ h \text{ being fixed},\ j = 1,\ 2,\ \ldots,\ p) \quad (6.1)$$

as well as the functions (cf. (4.4))

$$\Psi(\tau,\ z,\ l(z)\ m_j(z) - 1)\ (z \in H \cap K,\ z \text{ being fixed},\ j = 1,\ 2,\ \ldots,\ p) \quad (6.2)$$

form a basis of C_1. On the other hand, we consider vectors $\mathbf{z} = (z_k) = (z_1, z_2, \ldots, z_p)$ where $z_k \in H$ $(1 \leqslant k \leqslant p)$ such that $\det \phi_j(z_k) = 0$ $(j, k = 1, 2, \ldots, p)$ for any basis $\phi_j (1 \leqslant j \leqslant p)$ of C_1. The manifold D of these vectors $\mathbf{z}$ is uniquely determined by Γ and is transformed into itself by $z_k \to L_k z_k (L_k \in \Gamma,\ 1 \leqslant k \leqslant p)$ and by the permutations of the z_k. Now, the functions (cf. (4.13))

$$\Omega(\tau,\ z_j)\ (\mathbf{z} = (z_k),\ z_j \in H \cap K \text{ being fixed},\ 1 \leqslant j \leqslant p) \quad (6.3)$$

form a basis of C_1, too, whenever $\mathbf{z} \notin D$.

For any $f \in C_2$ we denote by $\mathbf{p}(f)$ the vector with the components $\eta(G_\nu, f)$ $(\nu = 1, 2, \ldots, 2p)$. We find that the $2p$ vectors $\mathbf{p}(f_j)$ are linearly independent if the f_j's are chosen as follows: For $1 \leqslant j \leqslant p$ we take the system given by (6.1) or (6.2) or (6.3); we take further

$f_{p+j}(\tau) = G(\tau,\ A_h,\ -m_j(\zeta_h))$ in the first case,

$f_{p+j}(\tau) = \Psi(\tau,\ z,\ -l(z)\ m_j(z) - 1)$ in the second case,

$f_{p+j}(\tau) = \Psi(\tau,\ z_j)$ in the third case,

whenever $1 \leqslant j \leqslant p$.

If S denotes a linear manifold in C_2 we denote by $\mathbf{P}(S)$ the set of the vectors $\mathbf{p}(f)$ $(f \in S)$. $\mathbf{P}(S)$ is a vector space over the field of complex numbers. Moreover, $\mathbf{P}(\mathcal{O}_2)$ is the conjugate complex vector space of $\mathbf{P}(C_1)$ and $\mathbf{P}(C_2)$ is the direct sum of $\mathbf{P}(C_1)$ and $\mathbf{P}(\mathcal{O}_2)$, $\mathbf{p}(f) = \bar{\mathbf{p}}(\phi)$ $(f \in \mathcal{O}_2,\ \phi \in C_1)$ defines an anti-homomorphism of $\mathcal{O}_2$ onto C_1. Its kernel is the manifold E_2, because, as is easily seen from (5.4), E_2 is contained in $\mathcal{O}_2$.

Two other bases of C_1 are represented by the functions $\Xi_j(\tau)$ where $1 \leqslant j \leqslant p$ for the first, $p+1 \leqslant j \leqslant 2p$ for the second basis. Apart from a common constant factor, both systems are complementary orthogonal to the two transcendentally normed bases of the canonical dissection. Moreover it turns out that these functions $\Xi_j(\tau)$ represent the differentials of the first kind belonging to the $2p$ retrosections of the canonical system on R in the sense of H. Weyl. We shall return to this fact later on.

7. We now consider the forms in $\{\Gamma, -2\}$ such that all their periods are purely imaginary numbers. A function of C_3 with this property turns out to be contained in $\mathcal{O}_3$ and to be expressible as a linear combination of a finite number of the functions $\Phi(\tau;\ \tau_1,\ \tau_2)$ $(\tau_1,\ \tau_2 \in K$, cf. (4.7)) with real coefficients. Conversely, such a linear combination represents the most general function in C_3 with that property. Therefore, the logarithmic derivative of a multiplicative function of Γ (belonging as is usually supposed to a character of modulus 1) admits a representation as a linear combination of the Φ's. Also this statement is invertible if the coefficients satisfy a certain condition which prescribes that some of them are integers or fractions with a denominator $l_k (1 \leqslant k \leqslant e_0)$.

A function $F \in \mathcal{O}_2$ is uniquely determined by its principal parts at its poles in K and may be explicitly and uniquely (cf. (4.3, 6)) expressed as the corresponding linear combination of the basis functions $G(\tau, A_h, -\nu)$ $(1 \leqslant h \leqslant \sigma, \nu > 0)$, $\Psi(\tau, z, -lm-1)$ $(z \in H \cap K, l = l(z), m > 0)$. There is one and only one $\psi(\tau) \in C_1$ such that all periods of $F - \psi$ are purely imaginary numbers and this ψ is represented as the linear combination of the functions $G(\tau, A_h, \nu)$, $\Psi(\tau, z, +lm-1)$ $(h,\ \nu,\ z, m$ as above) exactly analogous to the one named above with the conjugate-complex coefficients. From this follow the period relations

$$\left.\begin{aligned} &\eta(L,\ G(*,\ A,\ -\nu)) = \eta\,(L,\ G(*,\ A,\ +\nu))\ (L \in \Gamma) \\ &\eta(L,\ \Psi(*, z, -lm-1)) = \bar{\eta}\,(L,\ \Psi(*, z, +lm-1)). \end{aligned}\right\} \tag{7.1}$$

Now, by (4.2, 5, 11) we obtain the following representations of the periods of the functions G, Ψ (cf. (2.2, 4)):

$$\eta(G_j,\ G(*,\ A,\ -\nu)\ ) = \frac{N^2}{4\pi^2\nu}\, b_\nu(A,\ \Xi_j)\ (\nu > 0,\ 1 \leqslant j \leqslant 2p), \quad (7.2)$$

$$\eta(G_j, \Psi(*, z, -lm-1)\ ) = \frac{-1}{l(z-\bar{z})^2}\ b_{ml-1}(z, \Xi_j)\,(m>0, 1 \leqslant j \leqslant 2p).$$

Especially we have, using (7.1, 2) :

$$\begin{aligned} \eta(G_j, \Psi(*, z)\ ) &= -\Xi_j(z), \\ \eta(G_j, \Omega(*, z)\ ) &= -\overline{\Xi}_j(z)\ (1 \leqslant j \leqslant 2p). \end{aligned} \quad (7.3)$$

A further application of (5.4) yields

$$\eta(G_j, \Phi(*; z_0, z)) = -i\ \mathrm{Im} \int_{z_0}^{z} \Xi_j(t)\, dt (z_0, z \in K,\ 1 \leqslant j \leqslant 2p). \quad (7.4)$$

8. Because $E_2 \in \mathfrak{C}_2$, each exact differential admits a unique expression as a linear combination of the basic differentials $G(\tau, A_h, -\nu), \Psi(\tau, z, -lm-1)$, named in §7. Therefore if the principal parts of a completely invariant automorphic function are known the coefficients of its expansions corresponding to (2. 2, 4) can be expressed explicitly by infinite series. In the case of the Fourier coefficients, these series are closely analogous to Rademacher's partition series. The applications always deal with automorphic functions whose principal parts can be determined explicitly.

Considering the general theory, a necessary and sufficient condition is known for the coefficients of an indeterminate system of principal parts to coincide with the system of the principal parts of an automorphic function $E(\tau)$. This condition is a linear equation whose coefficients are of the form $b_\nu(A_h, \psi)$, $b_{ml-1}(z, \psi)$ (h, ν, z, m as in §7) for a general $\psi \in C_1$. Using (5.4), it is very easy to prove the theorem just stated; for the linear equation means exactly that the expression W_3 formed with $f_1 = \phi$, $f_2 = E'$ vanishes. Moreover, the fundamental theorem of Riemann-Roch may be proved by combining (5.4) with the information which is at H. Weyl's disposal in the corresponding passage of his book (Die Idee der Riemannschen Fläche), and this proof is very much simpler than Weyl's.

There is a metrical characterization of the exact differentials: each exact differential is orthogonal to all differentials of C_2 with which it possesses a scalar product. Conversely, a differential of $\{\Gamma, -2\}$ is exact if it is orthogonal to all differentials of C_2 with which it possesses a scalar product. Here the orthogonality is sufficient with respect to the differentials of C_2 which have single poles of the second order, if the order is measured on R.

The systematic application of (5.4) to the functions of C_2 and C_3 yields a set of formulae which, on the one hand, express the scalar products of two functions of $\mathcal{O}_2 \cup \mathcal{O}_3$ by certain coefficients in the expansions (2. 2, 4) of the corresponding functions of C_1; on the other hand, relations of symmetry between these coefficients result: For two functions $f_1, f_2 \in \mathcal{O}_2$, the expression W_4 vanishes, as may be proved from (5.4) by combining f_1 and f_2 with a transcendentally normed basis of C_1. Now, again by (5.4), if f_1 and f_2 possess a scalar product, also W_3 vanishes and this gives the relations of symmetry. In the terminology of (4.7), these formulae are

$$\frac{N'^2}{\nu'} b_{\nu'}(A', G(*, A, -\nu)) = \frac{N^2}{\nu} b_{\nu}(A, G(*, A', -\nu')),$$

$$\frac{N^2}{\nu} b_{\nu}(A, \Psi(*, z, -ml-1)) = \frac{-4\pi^2}{(z-\bar{z})^2 lm} b_{lm-1}(z, G(*, A, -\nu)),$$

$$\frac{1}{(z'-\bar{z}')^2 l'm'} b_{l'm'-1}(z', \Psi(*, z, -ml-1))$$

$$= \frac{1}{(z-\bar{z})^2 lm} b_{lm-1}(z, \Psi(*, z', -m'l'-1)),$$

where $\nu, \nu', m, m' > 0$, $l = l(z)$, $l' = l'(z')$ and $\nu' \neq \nu$ if $\zeta' \equiv \zeta \bmod \Gamma$ $m' \neq m$ if $z' \equiv z \bmod \Gamma$.

The third of these formulae, when applied to the case $m = m' = 1$, yields $\Psi(z, \tau) = \Psi(\tau, z)$. As $\Psi(\tau, z)$ has been proved by an extension of the summation method to be a continuous function of $(\tau, z) \in H \times H$ for $\tau \not\equiv z \bmod \Gamma$, the symmetry formula establishes the fact that $\Psi(\tau, z)$ represents an analytic function in the two variables τ and z when $(\tau, z) \in H \times H$ and $\tau \not\equiv z \bmod \Gamma$. Near $\tau = z$, $\Psi(\tau, z)$ behaves like

$$\Psi(\tau, z) = \frac{2}{(\tau - z)^2} + f(\tau, z)$$

where $f(\tau, z)$ is holomorphic in the neighbourhood of $\tau = z = z_0$. In the case $p = 0$, $\Psi(\tau, z)$ may easily be expressed by a principal function of Γ. In the case $p = 1$, a representation by the $\wp$-function of Weierstrass is obtained. As the functions $\Psi(\tau, z, -m)$ $(l = 1, m > 1)$ are generated by the application of certain differential operators on $\Psi(\tau, z)$, this latter function may be, to a certain extent, regarded as an analogue of Weierstrass' function $\wp$.

9. The functions $\Phi(\tau; \tau_1, \tau_2)$ correspond exactly to H. Weyl's differentials of the third kind originating in the local potential functions

$$\log|z - z_1| - \log|z - z_2| + \log|z - z'_1| - \log|z - z'_2|.$$

The conjugate potential functions

$$\arg(z - z_1) - \arg(z - z_2) - \arg(z - z'_1) + \arg(z - z'_2) \quad (9.1)$$

yield the more interesting differentials of the third kind on R which lead to integrals of the first kind by Weyl's method of arranging such differentials in a row along a closed curve on R. In order to construct functions $\Phi^*(\tau; z_0, z) \in C_3$ $(z_0, z \in K)$ corresponding to the differentials originating in (9.1) we discuss the following relations :

$$\rho(z_0, \Phi^*(*; z_0, z)) = +2\pi, \ \rho(z, \Phi^*(*; z_0, z)) = -2\pi;$$

and the periods $\eta(G_j, \Phi^*(*; z_0, z))$ $(1 \leqslant j \leqslant 2p)$ are purely imaginary numbers. From these, it is seen that

$$\phi^*(\tau) = \Phi^*(\tau; z_0, z) - i\,\Phi(\tau; z_0, z) \in C_1$$

and ϕ^* may be constructed explicitly by using the period relations (7.1) and applying (5.4) again. The result is expressed in the two formulae

$$\begin{aligned} -2i\,\Phi(\tau; z_0, z) &= \int_{z_0}^{z} \Psi(\tau, t)\,dt - \int_{z_0}^{z} \Omega(\tau, t)\,d\bar{t} \\ 2\Phi^*(\tau; z_0, z) &= \int_{z_0}^{z} \Psi(\tau, t)\,dt + \int_{z_0}^{z} \Omega(\tau, t)\,d\bar{t}. \end{aligned} \quad (9.2)$$

Now, regarding Φ, firstly, the composition formula (4.9) and secondly, the transformation formulae (4.12) are elucidated by (7.1) and (9.2) in a satisfactory manner. Moreover, $\Phi(\tau; z_0, z)$ and $\Phi^*(\tau; z_0, z)$ turn out to be complex potential functions of z which are regular whenever $z \not\equiv \tau \bmod \Gamma$.

H. Weyl's construction of differentials of the first kind dw_β belonging to a closed curve β on R may now be explicitly carried over to the class $\{\Gamma, -2\}$. If β_j corresponds to the transformation effected by the matrix $G_j \in \Gamma$, then

$$dw_{\beta_j} = \frac{1}{2\pi i}\, \Xi_j(\tau) \quad (1 \leqslant j \leqslant 2p).$$

More generally, if β corresponds to the transformation effected by the matrix L of Γ and R_L denotes the uniquely determined function of G_1 satisfying

$$(\phi, R_L) = 2\pi\, \eta(L, \phi) \quad (\phi \in C_1)$$

then we have

$$dw_\beta = \frac{1}{2\pi i}\, R_L(\tau).$$

Finally we consider the relations of symmetry in $\{\Gamma, -2\}$ which correspond to the relations of symmetry developed by H. Weyl in his book. Most of them deal with differentials of the second kind with single poles of higher order on R such that all their periods are purely imaginary numbers. The corresponding functions of $\{\Gamma, -2\}$ which appear are (cf. (2.2, 4), (4.1, 4), (7.1))

$$\mathsf{P}(\tau, z, m) = (2l)^{-1} \{\Psi(\tau, z, -ml-1) - \Psi(\tau, z, ml-1)\}$$

$$\mathsf{P}^*(\tau, z, m) = i\,(2l)^{-1} \{\Psi(\tau, z, -ml-1) + \Psi(\tau, z, ml-1)\}$$

$$\mathsf{P}(\tau, A, m) = \tfrac{1}{2} \{G(\tau, A, -m) - G(\tau, A, m)\}$$

$$\mathsf{P}^*(\tau, A, m) = \tfrac{1}{2}\, i \{G(\tau, A, -m) + G(\tau, A, m)\}$$

($m \geqslant 1$) and besides them, only the functions Φ and Φ^* appear in the relations which can be derived directly from (5.4). They consist in 24 equations of a rather simple form.

10. Regarding number-theoretical applications we have to mention first of all the partition problems ; here a complex system of theorems exists all concerning the Fourier coefficients of modular functions. Secondly, as stated in § 7, the logarithmic derivative of any multiplicative function belonging to a congruence subgroup of the modular group may be represented as a linear combination of the corresponding functions Φ. In some cases, Hecke's method of determining the periods of Eisenstein's series is applicable to those logarithmic derivaties and so, new class-number formulae can be derived which are connected with Dedekind's sums and other periods.

Applications of a different type arise from the fact that there exist infinitely many subgroups of the modular group of a preassigned genus $p \geqslant 0$ possessing fundamental regions of an extremely simple structure. In particular, one obtains for each value of p belonging to an infinite set of integers infinitely many such groups of genus p which are not congruence subgroups of the modular group and whose index in the modular group is a prime number. For all these groups, the relations between the generating matrices can be determined explicitly and here, Poincaré's series offer new possibilities, for example in the calculation of Fourier coefficients and periods.

A detailed presentation of the investigations outlined above has appeared in the *Annals of the Finnish Academy of Sciences*, series *A*, I, Mathematics, No. 276 (1960).

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ON FUCHSOID GROUPS THAT CONTAIN PARABOLIC TRANSFORMATIONS

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1. THE basic facts concerning parabolic transformations belonging to a Fuchsian group are well known. We refer to the behavior, in the neighborhood of a fixed point of a parabolic transformation belonging to a given Fuchsian group acting on the open unit disk, of the conformal mapping of the open unit disk onto the quotient space of the open unit disk modulo the given group that carries each point of the open unit disk into the orbit that contains it. However, to my knowledge, the corresponding facts concerning parabolic transformations belonging to an arbitrary properly discontinuous group of conformal automorphisms of the open unit disk‡ have not been explicitly stated and established. The object of the present paper is to treat the Fuchsoid case. Of course, the work of Ohtsuka [2] concerning the boundary behavior of a conformal universal covering whose domain is the open unit disk covers the case of a Fuchsoid group that contains *no* elliptic transformations.

Actually it turns out that the brunt of the work in treating the unrestricted Fuchsoid case can be put upon the generalization of the big Picard theorem given by Ohtsuka [3]. (A somewhat less refined version is due to H. Huber [1].) Thus the present paper's interest centres about the statement of the pertinent facts and the observation that the theorem of Ohtsuka and Huber is the key to the situation. We have found it possible to give a treatment of the theorem of Ohtsuka and Huber which is based upon the Lindelöf theorem concerning asymptotic values of bounded analytic functions and the Riemann-Hurwitz relation. An account of this treatment is included in the present paper.

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‡We shall call such groups Fuchsoid and shall not insist upon the distinction usually made between "Fuchsian" and "Fuchsoid". It is to be emphasized that elliptic transformations are not excluded.

2. Let us now turn to an explicit statement of the problem with which we are concerned. We let Γ denote a Fuchsoid group and let ϕ denote the map which takes each point z of $\{|z|<1\}$ into the orbit $\{\tau(z) \mid \tau \in \Gamma\}$ containing z. Let F_Γ denote the image of $\{|z|<1\}$ with respect to ϕ. As is well known, the quotient set F_Γ may be so endowed with a Riemann surface structure that ϕ is rendered a conformal map of $\{|z|<1\}$ onto F_Γ. Such a Riemann surface structure on F_Γ is essentially unique in the sense that two Riemann surface structures on F_Γ which render ϕ conformal are equivalent. We suppose that F_Γ has been so endowed with a Riemann surface structure.

Our basic problem is the following : *How does ϕ behave in a neighborhood of the fixed point of a parabolic transformation $\tau \in \Gamma$?*

We recall that a *boundary element* in the sense of Kérékjartó-Stoilow of a non-compact topological surface S may be defined as a function β whose domain consists of the set of compact subsets of S and which satisfies the following two conditions: (1) for each K in the domain of β, $\beta(K)$ is a component of $S - K$, (2) β is monotone non-increasing, i.e. if $K_1 \subset K_2$, then $\beta(K_1) \supset \beta(K_2)$, K_1 and K_2 being compact subsets of S. We say that β is an *isolated planar* boundary element of S provided that for some compact K, $\beta(K)$ is homeomorphic to an annulus. In the case of a Riemann surface, we shall say that an isolated planar boundary element β is *pointlike* provided that for some K containing more than one point, $\beta(K)$ is conformally equivalent to a punctured disk.

The description of the behavior of ϕ in a neighborhood of the fixed point of a parabolic transformation $\tau \in \Gamma$ is rendered somewhat simpler if we consider the map $\psi = \phi \circ \sigma$, where σ is a linear fractional transformation mapping $\{\mathrm{Re}\, z \geqslant 0\}$ onto $\{|z| \leqslant 1\}$ which carries ∞ into the fixed point of τ and is such that, if Γ_0 consists of the subgroup of Γ whose members are the transformations of Γ having as a fixed point the fixed point of τ, then $\sigma^{-1}\Gamma_0\sigma$ is the group of translations : $z \to z + 2\pi i m$, m an integer. We have

THEOREM 1. *Given that the Fuchsoid group* Γ *contains a parabolic transformation* τ. *Then the Riemann surface* F_Γ *is not compact,* $\psi(z)$ *tends to a pointlike isolated planar boundary element of* F_Γ *as* $\operatorname{Re} z \to +\infty$, *and* ψ *is univalent in the half-strip* $\{\operatorname{Re} z > a, 0 \leqslant \operatorname{Im} z < 2\pi\}$ *for a sufficiently large.*

Here the following gloss should be made : $\phi \circ \sigma(z)$ is said to *tend to the boundary element* β of F_Γ as $\operatorname{Re} z \to +\infty$ provided that for each compact $K \subset F_\Gamma$, $\phi \circ \sigma(z) \in \beta(K)$ for $\operatorname{Re} z$ sufficiently large. Clearly at most one boundary element β has the stated property.

3. We first show that F_Γ *is not compact* and simultaneously that $\psi(z)$ *tends to a boundary element of* F_Γ *as* $\operatorname{Re} z \to +\infty$. If either assertion were not true, there would exist a sequence (a_n) of points of the right half-plane satisfying $\operatorname{Re} a_n \to +\infty$ such that $(\psi(a_n))$ tends to a point of F_Γ, say $\phi(b)$. Hence there would exist a sequence (b_n), $|b_n| < 1$, such that $b = \lim b_n$ and b_n is in the same orbit as $\sigma(a_n)$. Since the hyperbolic distance between a_n and $a_n + 2\pi i$ tends to zero as $n \to \infty$, we conclude that there would exist a sequence of *parabolic* members of Γ, say (τ_n) such that $\lim \tau_n(b) = b$ so that $\tau_n(b) = b$ for n sufficiently large. This is impossible. The stated assertions follow.

4. It is to be observed that the domain of a conformal universal covering of F_Γ is conformally equivalent to either the finite plane or the open unit disk. Let us now consider the first possibility. Here F_Γ is conformally equivalent to either the finite plane or the finite plane punctured at the origin. We may therefore reduce our study to that of an analytic function ψ_1 with domain $\{\operatorname{Re} z > 0\}$ which has the following properties : (1) $\psi_1(z + 2\pi i) = \psi_1(z)$, (2) $\psi_1(z) \to \infty$ as $\operatorname{Re} z \to +\infty$. The facts are now classical. We have $\psi_1(z) = \Psi(e^z)$ where Ψ is analytic in $\{1 < |z| < \infty\}$ and Ψ has a pole at infinity. Thus we see in the present case that the boundary element β of F_Γ to which $\psi(z)$ tends as $\operatorname{Re} z \to +\infty$ is a pointlike isolated planar boundary element and that there is a compact K such that ψ is ramified over no point of $\beta(K)$. In particular, *if* F_Γ *is conformally*

equivalent to the finite plane, then ψ is ramified over only a finite number of points of F_Γ.

The statement concerning the univalence of ψ in a suitable half-strip of width 2π will be treated in a uniform manner for all cases (§ 7).

5. We now turn to the case where F_Γ is such that the domain of a conformal universal covering of F_Γ is conformally equivalent to the open unit disk. Here an essential role will be played by the generalization of the big Picard theorem given by Ohtsuka [3] and Huber [1].

Let f be a conformal map of the punctured disk $\{0 < |z| < 1\}$ into a Riemann surface F having conformal universal covering with hyperbolic domain. Then the following statements hold: (1) *Either $f(z)$ tends to the ideal boundary of F as z tends to 0 or else $f(z)$ tends to a point of F as z tends to 0 and in this latter case f is the restriction of a conformal map of $\{|z| < 1|\}$ into F.* (2) *If the first alternative occurs, then $f(z)$ tends to a pointlike isolated planar boundary element of F.*

A proof of the theorem of Ohtsuka and Huber may be given as follows. Let θ denote a conformal universal covering of F with domain $\{|z| < 1\}$ and let g denote a lifting of $f(e^{-z})$, $\mathrm{Re}\, z > 0$, with respect to θ, i.e. g is an analytic function of modulus less than one with domain $\{\mathrm{Re}\, z > 0\}$ satisfying

$$\theta[g(z)] = f(e^{-z}). \tag{5.1}$$

It is immediate that

$$g(z + 2\pi i) = \tau\,[g(z)] \tag{5.2}$$

where τ is a conformal automorphism of $\{|z| < 1\}$ that leaves θ invariant. Now τ cannot be hyperbolic by virtue of the cited theorem of Lindelöf, for if τ were hyperbolic, the two fixed points of τ would be asymptotic values of g associated with paths tending to ∞. Hence τ is either the identity or a parabolic transformation. In the first case, $g(z)$ is of the form $G(e^{-z})$ where G is analytic and of

modulus less than one in the open unit disk. Here $\lim_{z \to 0} f(z) = \theta(G(0))$. In the second case, we let ζ denote the fixed point of τ and note that there is a constant c such that

$$h = c \frac{\zeta + g}{\zeta - g}$$

satisfies

$$h(z + 2\pi i) = h(z) + 2\pi i. \tag{5.3}$$

Let H denote the analytic function with domain $\{1 < |z| < +\infty\}$ satisfying $H(e^z) = e^{h(z)}$. It follows from an examination of the behavior of H at ∞ that $c > 0$, H has a simple pole at ∞ and that $\operatorname{Re} h(z) \to +\infty$ as $\operatorname{Re} z \to +\infty$. This property of h together with the results of §3 applied to θ implies that $f(z)$ tends to an ideal boundary element of F as $z \to 0$. Thus (1) is established.

To prove (2) we proceed in the following manner. Let Ω_0 be a relatively compact region of F whose frontier consists of a finite number of mutually disjoint closed Jordan curves and which satisfies : $fr\,\Omega_0 = fr\,\overline{\Omega}_0$, $f\{|z| = 1/2\} \subset \Omega_0$, and f is not ramified over any point of $fr\,\Omega_0$. Let R denote the component of $F - \overline{\Omega}_0$ that contains $f(z)$ for z sufficiently small. We may assume that there exists a monotone increasing sequence of regions R_n of F having the following properties : (1) for each n, R_n is relatively compact and $fr\,R_n = fr\,\overline{R}_n$ and consists of a finite number of mutually disjoint closed Jordan curves ; (2) for each n, the component curves of $fr\,R$ are component curves of $fr\,R_n$ and $fr\,R_n - fr\,R \subset R_{n+1}$; (3) $R = \lim_{n \to \infty} R_n$; (4) f is not ramified over any point of $\bigcup fr\,R_n$; (5) of the components of $F - \overline{R}_n$, at most the one containing Ω_0 is relatively compact.

We now observe that the valence of g, the restriction of f to $\{0 < |z| < 1/2\}$ is constant, finite and positive on R. Let the constant value in question be m. We consider $\Delta_n = g^{-1}(R_n)$ introducing Δ_n. Since as it is easily seen, $g^{-1}(R)$ is connected, Δ_n is connected for n sufficiently large. We consider only such n and note that Δ_n is a region whose frontier consists of a finite number (independent of n)

of mutually disjoint closed Jordan curves and whose closure lies in $\{0 < |z| < 1/2\}$. Further, precisely one of the components of $fr\Delta_n$ is carried by f onto $frR_n - frR$. Hence $frR_n - frR$ *is connected.* From the Riemann-Hurwitz relation we have

$$\chi(\overline{\Delta}_n) \geqslant m\, \chi(\overline{R}_n),$$

where χ denotes the Euler characteristic. We see that $(\chi(\overline{R}_n))$ is bounded. Since $(\chi(\overline{R}_n))$ is non-decreasing, $\chi(\overline{R}_n)$ is constant for $n \geqslant n_0$. For such n, $R_{n+1} - \overline{R}_n$ is homeomorphic to an annulus. We conclude that $R - \overline{R}_n$ is also homeomorphic to an annulus for such n. It is now readily verified that there exists a unique boundary element β satisfying $\beta(\overline{R}_n) = R - \overline{R}_n$ for such n, and that not only does $f(z)$ tend to β as z tends to zero, but also that β is an isolated boundary element of planar type. If β were not pointlike, on introducing the harmonic measure ω of the ideal boundary of R (relative to R), we should be led to conclude that $\omega[f(z)]$ would be harmonic and less than one for $|z|$ small and positive and, in addition, would satisfy $\lim_{z \to 0} \omega[f(z)] = 1$. This is not possible. Hence β is pointlike.

6. The proof of Theorem 1 for the case where the domain of a conformal universal covering is hyperbolic may now be rapidly established. We have $\psi(z) = \phi(e^{-z})$, where ϕ is a conformal map of $\{0 < |w| < 1\}$ into F_Γ. Now $\phi(w)$ does not tend to a point of F_Γ as $w \to 0$; otherwise some point of F_Γ would be an asymptotic point of ψ. Hence by the Ohtsuka-Huber theorem $\psi(z)$ tends to a pointlike isolated planar boundary element of F_Γ as $\operatorname{Re} z \to +\infty$.

7. There remains to be verified that ψ is univalent in some half-strip $\{\operatorname{Re} z > a,\ 0 \leqslant \operatorname{Im} z < 2\pi\}$. We note that there exists a simply-connected Jordan region Δ satisfying $0 \in \Delta \subset \overline{\Delta} \subset \{|z| < 1\}$ such that $\phi(\Delta - \{0\})$ is conformally equivalent to a punctured disk and ϕ restricted to $\Delta - \{0\}$ has constant valence, say m, on its image. It suffices to show that $m = 1$. We note in passing that ϕ is unramified in $\Delta - \{0\}$. There exists a conformal automophism α of Δ, $\alpha(0) = 0$, having period m such that $\phi \circ \alpha(z) = \phi(z), z \in \Delta - \{0\}$.

There exists a conformal automorphism α^* of the antecedent Ω of $\Delta - \{0\}$ with respect to $z \to e^{-z}$ such that $\alpha(e^{-z}) = e^{-\alpha^*(z)}$, $z \in \Omega$. From

$$\psi(z) = \phi(e^{-z}) = \phi \circ \alpha(e^{-z}) = \phi(e^{-\alpha^*(z)}) = \psi(\alpha^*(z)), \, z \in \Omega,$$

we conclude that α^* is the restriction to Ω of a member of $\sigma^{-1}\Gamma\sigma$. From this and the fact that $\alpha(0)=0$, we conclude that $\alpha^*(z)=z+2\pi il$, where l is an integer. It follows that α is the identity map of Δ onto itself and hence $m=1$. The univalence of ψ in some half-strip $\{\operatorname{Re} z > a, \; 0 \leqslant \operatorname{Im} z < 2\pi\}$ now follows.

8. We are now in a position to characterize Fuchsoid groups Γ that possess parabolic transformations in terms of F_Γ and ϕ. In fact, the following theorem holds.

THEOREM 2 : *A necessary and sufficient condition that a Fuchsoid group Γ possess a parabolic transformation is that F_Γ possess a pointlike isolated planar boundary element β with the property that for some compact $K \subset F_\Gamma$, ϕ is ramified over no point of $\beta(K)$. Further the set of limit boundary elements associated with the fixed points of the parabolic members of Γ is precisely the set of such β. The limits associated with the fixed points of two parabolic members of Γ are equal if and only if they are the fixed points of two conjugate members of Γ distinct from the identity.*

The necessity part of the first assertion is contained in the developments that precede. Suppose that β is a boundary element with the stated property: Let R belong to the image of β, be conformally equivalent to a punctured disk and not contain any points over which ϕ is ramified. It follows from the proof given above for the first statement of the Ohtsuka-Huber theorem with ϕ replacing θ that Γ contains a parabolic member that β is associated with in the sense of Theorem 1. The second assertion and the sufficiency part of the first assertion follow.

If π_1 and π_2 are conjugate parabolic members of Γ, it is immediate that the limit boundary elements associated with the fixed points of π_1 and π_2 are equal. Suppose then that π_1 and π_2 are parabolic

members of Γ which are such that the limit boundary elements associated with their fixed points are equal. Then there exists $\sigma \in \Gamma$ carrying the fixed point of π_1 into the fixed point of π_2. In fact, suppose that ζ_k is the fixed point of π_k, $k = 1, 2$. Let $\Delta_k(c)$, $c > 0$, denote the disk

$$\left\{ \mathrm{Re}\left(\frac{\zeta_k + z}{\zeta_k - z}\right) > c \right\}.$$

Then it follows from Theorem 1 that given c_2 there exists c_1 with the property that there exists a univalent conformal map σ of $\Delta_1(c_1)$ into $\Delta_2(c_2)$ satisfying $\phi(\sigma(z)), = \phi(z), z \in \Delta_1(c_1)$. It is immediate that σ is the restriction of a member of Γ to $\Delta_1(c_1)$, so that if we denote this member of Γ also by σ, we see that $\sigma(\zeta_1) = \zeta_2$ and the proof is complete.

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THE GENERAL COEFFICIENT THEOREM AND ITS APPLICATIONS

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1. We begin with a statement of the General Coefficient Theorem as formulated at present.

GENERAL COEFFICIENT THEOREM. *Let $\mathfrak{R}$ be a finite oriented Riemann surface, $Q(z)dz^2$ a positive quadratic differential on $\mathfrak{R}$, $\{\Delta\}$ an admissible family of domains Δ_j, $j = 1, \ldots, K$, on $\mathfrak{R}$ relative to $Q(z)dz^2$ and $\{f\}$ an admissible family of functions f_j, $j = 1, \ldots, K$, associated with $\{\Delta\}$. Let $Q(z)dz^2$ have double poles $P_1, \ldots, P_r$ and poles $P_{r+1}, \ldots, P_n$ of order greater than two. We allow either of these sets to be void but not both. Let P_j, $j \leqslant r$, lie in the domain Δ_l and in terms of a local parameter z representing P_j as the point at infinity let f_l have the expansion*

$$f_l(z) = a^{(j)}z + a_0^{(j)} + \text{negative powers of } z \tag{1}$$

and Q the expansion

$$Q(z) = \alpha^{(j)} z^{-2} + \text{higher powers of } z^{-1}. \tag{2}$$

Let P_j, $j < r$, a pole of order m_j greater than two lie in the domain Δ_l and in terms of a local parameter z representing P_j as the point at infinity let f_l have the expansion

$$f_l(z) = z + \sum_{i=k_j}^{\infty} \frac{a_i^{(j)}}{z^i}, \tag{3}$$

where k_j is the smallest integer greater than or equal to $\frac{1}{2}m_j - 2$ and Q the expansion

$$Q(z) = \alpha^{(j)}\left[z^{m_j-4} + \sum_{i=k_j+1}^{\infty} \beta_i^{(j)} z^{m_j-i-4}\right]. \tag{4}$$

Then

$$\Re\left\{\sum_{j=1}^{r} \alpha^{(j)} \log\, a^{(j)} + \sum_{j=r+1}^{N} \alpha^{(j)}\left[a_{m_j-3}^{(j)} + \tfrac{1}{2}\left(\tfrac{1}{2}m_j - 2\right)\epsilon_j (a_{k_j}^{(j)})^2 + \right.\right.$$
$$\left.\left. + \epsilon_j\, \beta_{k_j+1}^{(j)}\, a_{k_j}^{(j)}\right]\right\} \leqslant 0, \tag{5}$$

where $\log a^{(j)} = \log |a^{(j)}| - id(F, P_j)$, $j \leqslant r$, *and* $\epsilon_j = 1$ *if* m_j *is even,* $\epsilon_j = 0$ *if* m_j *is odd,* $j > r$.

If equality occurs in (5) *each* f_j, $j = 1, \ldots, K$, *must be an isometry in the Q-metric*

$$|d\zeta| = |Q(z)|^{\frac{1}{2}} |dz|,$$

each trajectory of $Q(z)dz^2$ *in* $\bigcup_{j=1}^{K} \Delta_j$ *must go into another such and the set* $\bigcup_{j=1}^{K} f_j(\Delta_j)$ *must be dense in* $\mathfrak{R}$. *If equality occurs in* (5) f_l *reduces to the identity in a domain* Δ_l *for which any of the following conditions holds.*

(i) *There is in* Δ_l *a pole* P_j, $j > r$, *of order* m_j *such that* $a_i^{(j)} = 0$ *for* $i < \min(k_j + 1,\ m_j - 3)$.

(ii) *There is in* Δ_l *a pole* P_j, $j \leqslant r$, *with the corresponding coefficient* $a^{(j)}$ *equal to one.*

(iii) *There is in* Δ_l *a simple pole of* $Q(z)\,dz^2$ *or a point on a trajectory ending in a simple pole.*

Equality can occur in (5) *when there exists a double pole* P_j, $j \leqslant r$, *such that for the corresponding coefficient* $|a^{(j)}| \neq 1$ *only when* $\mathfrak{R}$ *is conformally equivalent to the sphere and* $Q(z)\,dz^2$ *is a quadratic differential whose only critical points are two poles each of order two. If further* $\{\Delta\}$ *consists of a single domain the corresponding function is conformally equivalent to a linear transformation with the points corresponding to these poles as fixed points.*

Here the terms finite oriented Riemann surface and positive quadratic differential are used in the sense familiar to all. As usual we will denote by C the set of zeros and simple poles of $Q(z)dz^2$, by H the set of poles of $Q(z)dz^2$ of order at least two. An admissible family of domains relative to $Q(z)dz^2$ is obtained by slitting $\mathfrak{R}$ along a finite number of trajectories of $Q(z)dz^2$ (i.e. curves on which $Q(z)\,dz^2 > 0$) which either are closed or join two points of C and along a finite number of arcs on closures of trajectories in $\mathfrak{R} - H$. An admissible family of functions then consists of a family of conformal

mappings of the domains Δ_j onto non-overlapping domains in $\mathfrak{R}$, leaving fixed all poles of $Q(z)\,dz^2$ interior to these domains, normalized by the expansions (3), admitting an admissible homotopy F into the identity such that for a pole of order greater than two on the boundary of a strip domain the deformation degree of F at that pole is zero. As usual $d(F, P_j)$ denotes the deformation degree of F at the pole P_j.

The General Coefficient Theorem was initially stated [1] for a single admissible domain. In [2] the statement was enlarged to allow a family of admissible domains. Both statements required a normalization of the functions of $\{f\}$ at the poles of $Q(z)dz^2$ of order greater than three more stringent than that of the present form. Indeed they required that $a_i^{(j)} = 0$ for $i < m_j - 3$. In treating the problem of finding bounds for the initial coefficients of normalized univalent functions in the unit circle I found that in these special cases the normalization could be relaxed [3]. The present formulation of the General Coefficient Theorem represents the extension of these considerations to the most general situation[4]. It might appear that the corresponding normalization (4) for the quadratic differential at these points represents a stronger restriction than that imposed in the earlier statements. However it is imposed only to achieve formal simplicity and can always be attained by a suitable change of the local parameters used at the various poles in question.

In proving the General Coefficient Theorem the first step, a very essential one, is to analyze the global structure of the trajectories of a positive quadratic differential on a finite oriented Riemann surface. This consists in showing that this structure is made up of end and strip domains (mapped respectively by $\int (Q(z))^{1/2} dz$ on halfplanes and horizontal strips), circle and ring domains (mapped respectively by $\exp (k \int (Q(z))^{1/2} dz)$ for suitable constants k on circles and circular rings) and a finite number of domains in which the trajectories having limiting end points at points of C are collectively everywhere dense. An important fact is that each point of H has a

neighborhood disjoint from these latter domains. Recently I have proved [5] that this portion of the structure can be further decomposed into a finite number of domains such that every trajectory in such a domain is everywhere dense in that domain.

The next step in the proof is to show that it is sufficient to proceed on the assumption that $Q(z)\,dz^2$ has no simple poles on $\mathfrak{R}$. This is done by replacing $\mathfrak{R}$ by a suitable covering surface. Then the double of $\mathfrak{R}-H$ becomes a complete differential geometric surface with the metric obtained by the natural extension of the Q-metric.

The main part of the proof now consists of drawing suitable curves around each point of H and regarding the set obtained by deleting these curves and their interiors from $\bigcup_{j=1}^{K} \Delta_j$. Then the areas of this set and the corresponding portion of $\bigcup_{j=1}^{K} f_j(\Delta_j)$ in the Q-metric are compared in two ways. On the one hand we obtain one inequality by studying the behaviour of the functions f_j on the above curves. A second inequality is obtained by applying the method of the extremal metric in each of the various domains associated with the trajectory structure of $Q(z)\,dz^2$. Combining these inequalities we are led, after formal reduction, to the inequality (5).

The equality statements given here are almost unchanged from those of the statement in [2]. They are obtained in each case by applying the usual equality treatment in the application of the method of the extremal metric.

2. As we have shown earlier [2] the General Coefficient Theorem even in its more restricted form contains as special cases almost all known results in the theory of univalent functions as well as interesting new results. We have already applied the new form in special cases to the problem of finding bounds for the initial coefficients of normalized univalent functions, as we mentioned earlier. We will now give some examples to show how in the general form it leads to consequences extending well beyond those obtained by other methods. The complete proofs of these results are found in [6].

We confine our attention to three families of functions Σ, S and M. These consist respectively of functions meromorphic and univalent for $|z| > 1$ with expansion about the point at infinity

$$z + \sum_{n=0}^{\infty} \frac{c_n}{z^n} \tag{6}$$

functions regular and univalent for $|z| < 1$ with expansion about the origin

$$z + \sum_{n=2}^{\infty} A_n z^n ; \tag{7}$$

and functions meromorphic and univalent for $|z| < 1$ with expansion (7) about the origin.

We take first the quadratic differentials

$$Q(w ; n, \chi)\ dw^2 = e^{-in\chi}\, w^{n-2}\, dw^2$$

(n integral $\geqslant 1$, χ real) and the functions in Σ

$$g(z;\ n,\ \chi) = z\,(1 + e^{n\chi i}\, z^{-n})^{2/n}$$

mapping $|z| > 1$ onto an admissible domain with respect to $Q(w; n, \chi)\ dw^2$. Applying the General Coefficient Theorem for the index $n + 1$ we obtain the following result.

Let $f(z) \in \Sigma$ and have the expansion (6) *about the point at infinity where $c_j = 0$ for $j \leqslant \frac{1}{2}(n-1)$, n an integer $\geqslant 0$. Further for $n = 0$ let $f(z)$ not assume the value zero for $|z| > 1$. Then*

$$|c_n| \leqslant \frac{2}{n+1}. \tag{8}$$

Equality can occur in (8) *only for the functions $g(z ; n + 1, \chi)$, χ real.*

Next we take the quadratic differentials

$$Q(w ; n, \delta, \psi)\ dw^2 = e^{-2n\psi i}\, w^{n-2}\, (\delta\, e^{n\psi i} - w^n)\ dw^2$$

(n integral > 0, ψ real, $0 \leqslant \delta \leqslant 4$) and functions $f(z ; n, \delta, \psi)$ in Σ which map $|z| > 1$ onto admissible domains with respect to $Q(w; n, \delta, \psi)\ dw^2$ which have n-fold symmetry about the origin and expansion about the point at infinity beginning

$$z + \frac{1}{2n}\delta[1 - \log(\delta/4)]\, e^{n\psi i} z^{-n+1} +$$

$$+ \left[\frac{1}{8n^2}\delta^2 - \frac{1}{4n^2}\delta^2 \log(\delta/4) - \frac{(n-1)^2}{8n^2}\delta^2[\log(\delta/4)]^2 - \frac{1}{n}\right] \times$$

$$\times e^{2n\psi i} z^{-2n+1} + \ldots$$

for $0 < \delta \leqslant 4$ and taking the limiting form for $\delta = 0$. Then applying the General Coefficient Theorem for the index $n+1$ and performing various reductions we have the following result.

Let $f(z) \in \Sigma$ and have the expansion (6) *about the point at infinity where $c_j = 0$ for $j < n$, n a non-negative integer. Further for $n = 0$ let $f(z)$ not assume the value zero for $|z| > 1$. Then for $\alpha + \frac{1}{2}n < 0$*

$$|c_{2n+1} - \alpha c_n^2| \leqslant \frac{1 - n - 4\alpha}{(n+1)^2}.$$

For $\alpha + \frac{1}{2}n = 0$

$$|c_{2n+1} + \tfrac{1}{2} n c_n^2| \leqslant \frac{1}{n+1}.$$

For $-\frac{1}{2}n < \alpha < \frac{1}{2}n + 1$

$$|c_{2n+1} - \alpha c_n^2| \leqslant (n+1)^{-1}\left[1 + 2\exp\left(-2\frac{n - 2\alpha + 2}{n + 2\alpha}\right)\right].$$

For $\alpha \geqslant \frac{1}{2}n + 1$

$$|c_{2n+1} - \alpha c_n^2| \leqslant \frac{4\alpha + n - 1}{(n+1)^2}.$$

In each case it is possible to give a complete description of the possibilities of equality. The preceding result has the following corollary.

Let $f(z) \in \Sigma$ and have the expansion (6) *about the point at infinity where $c_j = 0$ for $j < n$, n a positive integer. Then*

$$|c_{2n+1}| \leqslant (n+1)^{-1}\left[1 + 2\exp\left(-2\frac{n+2}{n}\right)\right].$$

Once again the exact possiblities for equality can be given.

The preceding results can be carried over to the families S and M as follows.

Let n be an integer $\geqslant 2$, let $f(z) \in S$ $(n = 2)$, $f(z) \in M (n > 2)$ and have the expansion (7) *about the origin where $A_j = 0$ for $j \leqslant \frac{1}{2}(n+1)$. Then*

$$|A_n| \leqslant \frac{2}{n-1}.$$

Equality can occur here only for the functions $1/g\,(1/z; n-1, \chi)$, χ real.

This result was obtained by Prawitz assuming that $f(z) \in S$ without regard to the value of n. Had he proved it for $f(z) \in M$, $n > 2$, he would in particular have anticipated Schiffer and Golusin in obtaining the bound $|c_2| \leqslant 2/3$ by about ten years.

Let n be an integer $\geqslant 2$, let $f(z) \in S$ $(n = 2)$, $f(z) \in M (n > 2)$ and have the expansion (7) *about the origin where $A_j = 0$ for $j < n$. Then for $\beta > \frac{1}{2}n$*

$$|A_{2n-1} - \beta A_n^2| \leqslant \frac{4\beta - n - 1}{(n-1)^2}.$$

For $\beta = \frac{1}{2}n$

$$|A_{2n-1} - \tfrac{1}{2}n A_n^2| \leqslant \frac{1}{n-1}.$$

For $\frac{1}{2}n > \beta > 1 - \frac{1}{2}n$

$$|A_{2n-1} - \beta A_n^2| \leqslant (n-1)^{-1}\left[1 + 2\exp\left(-2\frac{n + 2\beta - 2}{n - 2\beta}\right)\right].$$

For $\beta \leqslant 1 - \frac{1}{2}n$

$$|A_{2n-1} - \beta A_n^2| \leqslant \frac{1 + n - 4\beta}{(n-1)^2}.$$

In each case the complete possibilities for equality can be given. We have also the corollary.

Let n be an integer $\geqslant 2$, let $f(z) \in S$ $(n = 2)$, $f(z) \in M$ $(n > 2)$ and have the expansion (7) *about the origin where $A_j = 0$ for $j < n$. Then*

$$|A_{2n-1}| \leqslant (n-1)^{-1}\left[1 + 2\exp\left(-2\frac{n-2}{n}\right)\right].$$

Once again the exact possiblities for equality can be given.

The preceding results contain as special cases all known precise bounds for coefficients of functions in S and Σ apart from the bound $|A_4| \leqslant 4$. Further if we take $n = 3$ in the final corollary we obtain the result proved by Fekete and Szegö under the assumptions that f belonged to S and was odd. Their proof made strong use of these assumptions but it is seen that the validity of the result does not depend on them but only on the conditons $f(z) \in M$, $A_2 = 0$. A similar remark applies to Golusin's generalization of the Fekete-Szegö result to functions with higher degrees of symmetry.

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RIEMANN SURFACES OF HYPERBOLIC TYPE IN THE EUCLIDEAN 3-SPACE†

By H. HUBER

1. Let $f(x, y)$ be a real function defined for *all* real values of x and y. Then

$$\mathfrak{F}: z = f(x, y) \tag{1}$$

is the equation of a surface $\mathfrak{F}$ in the euclidean (x, y, z) space which has a simply covered projection onto the (x, y) plane. In what follows such a surface is called "schlicht". A schlicht surface $\mathfrak{F}$ is said to be of class C^n, $1 \leqslant n \leqslant \infty$, if $f(x, y)$ is n times continuously differentiable in the whole (x, y) plane.

If $\mathfrak{F} \in C^n$, $n \geqslant 2$, then the euclidean metric of the (x, y, z) space provides $\mathfrak{F}$ with a differential geometry $ds^2 = dx^2 + dy^2 + (df)^2$ and thus induces on $\mathfrak{F}$, in a natural way, a conformal structure. The open, simply connected Riemann surface $\mathfrak{F}$ is either of parabolic or of hyperbolic type, i.e. $\mathfrak{F}$ can be mapped conformally onto the complex z plane or onto the unit disk in this plane. In the few known cases where this mapping can be given explicitly (e.g. surfaces of revolution), schlicht surfaces are always parabolic. Ch. Loewner raised the question whether there exist schlicht surfaces of hyperbolic type. At the Princeton Conference on Riemann surfaces in 1954, L. Bers stated the conjecture that every schlicht surface is parabolic. In 1956 R. Osserman (*Trans. American Math. Soc.* 82, 217-245) disproved this conjecture by a rather intricate construction of hyperbolic schlicht surfaces. In the present paper we shall give, by a very simple construction, a new class of schlicht surfaces of hyperbolic type belonging to C^∞.

2. If we introduce, in the usual way, polar coordinates ρ, ϕ in the (x, y) plane, then the surface (1) may be represented by the radius vector $\mathfrak{x}(\rho, \phi)$ as follows:

$$\mathfrak{F}: \mathfrak{x}(\rho, \phi) = (\rho \cos \phi, \rho \sin \phi, g(\rho, \phi)), \rho \geqslant 0, \tag{2}$$

†Abstract of a paper presented to the International Colloquium on Function Theory held at the Tata Institute of Fundamental Research, Bombay, in January 1960. The paper has appeared in *Math. Annalen*, 139 (1959), 140-146.

where $g(\rho, \phi) = f(\rho \cos \phi, \rho \sin \phi)$ is periodic in ϕ with period 2π. We are now going to construct functions $g(\rho, \phi)$ for which the corresponding surfaces (2) are schlicht surfaces of class C^∞.

We consider the class Ω of functions $\omega(t)$ having the following properties :

$$\omega(t) \text{ has derivatives of all orders in } (-\infty, +\infty), \tag{3}$$

$$\omega(t) \begin{cases} = 0 \text{ for } t \notin (0, 1) \\ > 0 \text{ for } t \in (0, 1). \end{cases} \tag{4}$$

The simplest function of this kind is

$$\omega_0(t) = \kappa(t).\ \kappa(1 - t)$$

where

$$\kappa(t) = \begin{cases} 0 \text{ for } t \leqslant 0, \\ e^{-1/t} \text{ for } t > 0. \end{cases}$$

Now let be $\omega \in \Omega$, $\{n_k\}_1^\infty$ an arbitrary sequence of positive integers, and

$$g(\rho, \phi) = \sum_{k=1}^{\infty} \omega(\rho - k).\ \sin n_k \phi, \tag{5}$$

then

$$g(\rho, \phi) = 0 \text{ for } 0 \leqslant \rho \leqslant 1. \tag{6}$$

$g(\rho, \phi)$ is periodic in ϕ with period 2π. It is clear from (3) — (5) that g has everywhere derivatives of all orders with regard to the variables ρ, ϕ. Thus it follows from (6) that g has derivatives of all orders also with regard to the variables $x = \rho \cos \phi$, $y = \rho \sin \phi$ in the whole (x, y) plane. Therefore the corresponding surface (2) is a schlicht surface of class C^∞; we denote this surface by $\mathfrak{F}[\omega, \{n_k\}]$.

Now we define

$$I(t, \omega) = \int_0^1 \frac{\log(1 + t\,\omega(\tau))}{t\,\omega(\tau)}\, d\tau, \quad t > 0, \quad \omega \in \Omega.$$

Obviously

$$I(t, \omega) \to 0 \text{ for } t \to +\infty. \tag{7}$$

We prove

THEOREM A. *If* $\sum_{k=1}^{\infty} k^{-1} I(k^{-1} n_k, \omega)$ *converges, then* $\mathfrak{F}[\omega, \{n_k\}]$ *is of hyperbolic type.*

It follows immediately from (7) and Theorem A that to every $\omega \in \Omega$ there exist sequences $\{n_k\}$ so that $\mathfrak{F}[\omega, \{n_k\}]$ is hyperbolic. It is easy to see that $I(t, \omega_0) = O(1/\log t)$ for $t \to +\infty$. Therefore, from Theorem A we obtain

THEOREM B. *If* $\lim_{k\to\infty} k^{-1} n_k = \infty$ and $\sum_{k=1}^{\infty} 1/k \log(k^{-1} n_k)$ *converges, then* $\mathfrak{F}[\omega_0, \{n_k\}]$ *is hyperbolic.*

Thus $\mathfrak{F}[\omega_0, \{2^k\}]$ is a schlicht surface of hyperbolic type.

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ON THE CAPACITY OF PLANE SETS†

By CHRISTIAN POMMERENKE

1. Let E be a plane set that is bounded and closed, and let

$$\kappa_n = \max_{z_1,\dots,z_n \in E} \left(\prod_{\mu \neq \nu}^{1\dots n} |z_\mu - z_\nu| \right)^{1/n(n-1)}$$

Then $\kappa_n \geqslant \kappa_{n+1}$, and κ_n converges. This limit is called the (logarithmic) capacity—or also the transfinite diameter — of E:

$$\operatorname{cap} E = \lim_{n\to\infty} \kappa_n.$$

This concept is due to Fekete [1] who also showed that

$$\operatorname{cap} E = \lim_{n\to\infty} [\min_{g(z) in P_n} (\max_{z\in E} |g(z)|)^{1/n}],$$

where P_n is the class of all polynomials of the form

$$g(z) = z^n + b_{n-1} z^{n-1} + \dots + b_0. \tag{1}$$

The capacity of a set is translation-invariant and homogeneous of the first degree. The capacity of a closed bounded set is equal to the capacity of its outer boundary‡. It is not changed by omitting isolated points and is thus equal to the capacity of its perfect kernel. For $E \subset F$ we have $\operatorname{cap} E \leqslant \operatorname{cap} F$. To give some examples: The capacity of a circle is equal to its radius, the capacity of a segment is one fourth of its length.

There is a close connection between capacity and polynomials. Let $g(z)$ be a polynomial of the form (1). We call the set of all z with $|g(z)| = 1$ (or more generally $|g(z)| = \text{const}$) the lemniscate of $g(z)$. Then this lemniscate $|g(z)| = 1$ has capacity one§. On the other hand, the outer boundary of any closed bounded set with $\operatorname{cap} E = 1$

† The contents of this paper, presented at the International Colloquium on Function Theory held at the Tata Institute of Fundamental Research, Bombay, are largely contained in [8] and [9].

‡ By outer boundary I mean that part of the frontier of E which also bounds the complementary region of E containing ∞.

§ For an application of this fact to the study of lemniscates see for instance [10].

that consists of a finite number of connected components can be approximated arbitrarily closely by lemniscates $|g(z)| = 1$.

2. There is an extensive theory connected with sets of capacity zero†. We will be concerned with sets E of positive capacity, and especially with the problem of finding estimates for cap E if some geometrical quantities of E are given.

Let me first sketch a proof of the following

THEOREM 1. *If C is a simple curve of length Λ, we have $\Lambda \geqslant 4$ cap C.* [8]

We choose n points $z_1, \ldots, z_n$ on C such that

$$\prod_{\mu \neq \nu} |z_\mu - z_\nu| \tag{2}$$

is maximal. Let s_ν be the corresponding values of the arc length. Then $|s_\mu - s_\nu|$ is the length of the arc of C between z_μ and z_ν and therefore

$$|s_\mu - s_\nu| \geqslant |z_\mu - z_\nu|.$$

Since the product (2) was chosen maximal we get

$$\max_{s_1, \ldots, s_n \in S} \prod_{\mu \neq \nu} |s_\mu - s_\nu| \geqslant \max_{z_1, \ldots, z_n \in C} \prod_{\mu \neq \nu} |z_\mu - z_\nu|,$$

where S is the segment $[0, \Lambda]$. Therefore we obtain from the definition of the capacity that

$$\text{cap } S \geqslant \text{cap } C.$$

Because the capacity of a segment is the fourth part of its length, we get

$$\Lambda = 4 \text{ cap } S \geqslant 4 \text{ cap } C.$$

By a similar method we can prove the next theorem. Let E be any closed bounded set, and let K be a closed convex set. We define the projection P of E on K as the set of all points w that satisfy either

(a) $w \in K \cap E$, or

† See for instance [4, Ch. V].

(b) w belongs to the boundary of K and there exists a point $z \in E$ that lies on an outer normal in w on K.

By an outer normal we mean a ray that starts in w, goes into the exterior of K and is normal to a line of support of K in w. Then we have [8]

THEOREM 2. $\operatorname{cap} P \leqslant \operatorname{cap} E$.

As an example, let c_1 and c_2 be two points and let T be the connecting segment. Let C and C^* be two curves joining c_1 and c_2, so that $C \cup T$ is a convex curve and is contained in the closed interior of $T \cup C^*$. We can apply Theorem 2. If P is the projection of C^* on $C \cup T$ we have $\operatorname{cap} P \leqslant \operatorname{cap} C^*$. Because $C \subset P$ we have $\operatorname{cap} C \leqslant \operatorname{cap} P$ and therefore

$$\operatorname{cap} C \leqslant \operatorname{cap} C^*.$$

This result can be expressed in the following manner: The capacity of the convex curves joining two fixed points varies monotonically.

As another example take as K the straight line L. Then the projection P of E onto L as defined above is the usual orthogonal projection, and $\operatorname{cap} P \leqslant \operatorname{cap} E$. If d^* is the linear measure of P, it is easy to see that $d^{*-1} \operatorname{cap} P$ is minimal in the case that P is a segment. Therefore $d^{*-1} \operatorname{cap} P \geqslant 4^{-1}$ and $d^* \leqslant 4 \operatorname{cap} E$. This is a classical result of Pólya [5].

3. Now we come to the connection between the concept of capacity and Green's function. Let again E be bounded and closed. Exactly one of the complementary regions of E has ∞ as an interior point. We call this region the exterior region of E and denote it by G. Let $U(z)$ be the Green's function of G with respect to infinity, that is, $U(z)$ is the smallest positive function harmonic in G except in ∞ where

$$U(z) = \log |z| + \gamma + V(z), \tag{3}$$

with $V(\infty) = 0$. If the boundary of G consists of a finite number of simple curves, we have $U(z) = 0$ on it. Szegö [11] proved that

$$\operatorname{cap} E = e^{-\gamma},$$

where γ is the constant in (3).

This relation is especially interesting if E is a continuum, that is, if E is a bounded, closed and connected set. Then its exterior region G is simply connected. Let

$$z = f(s) = \alpha s + a_0 + a_1 s^{-1} + \ldots \quad (\alpha > 0) \tag{4}$$

be the univalent function that maps $|s| > 1$ conformally onto G. From the well-known connection between $f(s)$ and the Green's function $U(z)$, we obtain that

$$\operatorname{cap} E = \alpha. \tag{5}$$

This equation will be of fundamental importance in what follows. We can of course assume that E is the complement of its exterior region. For if there are "holes" in E we can fill them in, and the capacity is not altered by this.

Now we will drop the assumption that $f(s)$ is univalent. Let us consider any function $f(s)$ that is regular in $1 < |s| < \infty$ and has the form (4). We denote by Q the set of all values that $z = f(s)$ does not assume in $|s| > 1$. Then Hayman [2] proved that

$$\operatorname{cap} Q \leqslant \alpha,$$

with equality only if $f(s)$ is univalent in $|s| > 1$. This is a generalization of (5).

We can generalize (5) also in another direction. Let again $f(s)$ be a function of the form (4), not necessarily univalent. We consider the set R of the limit points of all sequences $f(s_n)$ with $|s_n| \to 1$. If $f(s)$ is continuous on $|s| = 1$, this set R is the image of $|s| = 1$. We will call R the image of $|s| = 1$ also if $f(s)$ is not continuous on it. We can prove [9]

THEOREM 3. *Let $f(s)$ be regular in $1 < |s| < \infty$, and have the form (4). If the image R of $|s| = 1$ is bounded, then*

$$\operatorname{cap} R \geqslant \alpha,$$

with equality only if $f(s)$ is univalent.

To prove this theorem let us assume for simplicity that $f(s)$ is continuous on $|s| = 1$. Let $g_n(z)$ be the polynomial of the form (1) with least maximum value on R. The function

$$h_n(s) = s^{-n} g_n(f(s)) = \alpha^n + \dots$$

is regular in $|s| > 1$ and continuous in $|s| \geqslant 1$. Since $z = f(s)$ maps $|s| = 1$ onto R, and because $h_n(\infty) = \alpha^n$, we have

$$\max_{z \in R} |g_n(z)| = \max_{|s|=1} |g_n(f(s))| = \max_{|s|=1} |h_n(s)| \geqslant \alpha^n.$$

Since $g_n(z)$ was chosen to be the polynomial with minimal maximum on R, we obtain

$$\operatorname{cap} R = \lim_{n \to \infty} \max_{z \in R} |g_n(z)|^{1/n} \geqslant \alpha.$$

To prove the statement about equality is a little more difficult.

4. We can use Theorem 3 to derive an inequality for the capacity of the Minkowski sum of continua. For two sets E and F we denote by $E + F$ the set of all points $u + v$ with $u \in E$ and $v \in F$. The set $E + F$ is called the Minkowski sum of E and F. Let $f(s) = \alpha s + \dots$ $(\alpha > 0)$ and $g(s) = \beta s + \dots$ $(\beta > 0)$ be the univalent functions that map $|s| > 1$ conformally onto the exterior regions of E and F. Then we have $\alpha = \operatorname{cap} E$, $\beta = \operatorname{cap} F$. The function

$$z = f(z) + g(z) = (\alpha + \beta) s + \dots$$

is regular in $1 < |s| < \infty$ but need not be univalent there. Let R be the image of $|s| = 1$ under this mapping. From Theorem 3 we obtain $\operatorname{cap} R \geqslant \alpha + \beta$. It is easy to see that $R \subset E + F$. Therefore

$$\operatorname{cap} (E + F) \geqslant \operatorname{cap} R = \alpha + \beta,$$

and we have [9]

THEOREM 4. *If E and F are continua, then*

$$\operatorname{cap} (E + F) \geqslant \operatorname{cap} E + \operatorname{cap} F.$$

For $0 < \lambda < 1$ this implies

$$\operatorname{cap} (\lambda E + (1 - \lambda) F) \geqslant \lambda \operatorname{cap} E + (1 - \lambda) \operatorname{cap} F.$$

We can thus say that cap E is a concave functional for continua.

Next we will apply this theorem to central symmetrisation. Let E be a continuum. We form the set E^* of all points $\frac{1}{2}(u_1 - u_2)$ with $u_1 \in E$, $u_2 \in E$. We say that E^* has been obtained from E by central symmetrisation. This process of symmetrisation leaves invariant the perimeter of the convex hull, the diameter and the width and does not decrease the area. Because $E^* = \frac{1}{2}(E + (-1)E)$, Theorem 4 gives

$$\operatorname{cap} E^* \geqslant \operatorname{cap} E. \tag{6}$$

There are other sorts of symmetrisation. For instance Steiner's symmetrisation with regard to the real axis is defined as follows: Let $l(x)$ be the measure of the intersection of E with the straight line $z = x$. Then the symmetrised set E' is the set $-\frac{1}{2}l(x) \leqslant y \leqslant \frac{1}{2}l(x)$ for all x for which this intersection is not empty. Steiner's symmetrisation leaves the area invariant. As Pólya and Szegö [7] have shown, $\operatorname{cap} E' \leqslant \operatorname{cap} E$ (compare inequality (6)!).

Using Steiner's symmetrisation it is not difficult to prove[8]

THEOREM 5. *Let K be a convex continuum of diameter d and width b. Then*

$$(b^2 + d^2)^{1/2} \leqslant 4 \operatorname{cap} K, \tag{7}$$

with equality only if K is a segment.

More generally (7) is true if K contains a segment of length d. Probably (7) is valid for any continuum.

5. Now we will bring further applications of Theorem 4. Let the continuum E have the perimeter L. It is known that

$$L \geqslant 2\pi \operatorname{cap} E,$$

with equality only for the circle. Pólya and Schiffer [6] have conjectured that among all polygons with n sides and given capacity the regular polygon has minimal perimeter. Using Theorem 4 it is easy to prove the special case of this conjecture, that all angles of the polygon are equal [9].

In the same paper [6] Pólya and Schiffer proved the following conjecture of Pólya.

If K is convex and has the perimeter L, then

$$L \leqslant 8 \text{ cap } K. \tag{8}$$

I will give a new proof of this inequality. It is enough to prove it for convex polygons. Now every polygon K is the sum of a finite number of triangles and segments [3, p.145] :

$$K = K_1 + \ldots + K_m. \tag{9}$$

The inequalities $L_\mu \leqslant 8 \text{ cap } K_\mu$ for the perimeters of the triangles and segments K_μ can be verified by direct computation. Since the perimeter of the sum of convex sets is equal to the sum of the perimeters,

$$L = \sum L_\mu \leqslant \sum 8 \text{ cap } K_\mu.$$

From Theorem 4 and equation (9) we obtain

$$\sum \text{cap } K_\mu \leqslant \text{cap } K$$

so that $L \leqslant 8 \text{ cap } K$.

Now we shall consider a continuum E that need not be convex. Then inequality (8) does not hold in general because the perimeter of E can become infinite. But there is a finite upper bound for the perimeter L_0 of its convex hull. We can define L_0 as the minimum of the lengths of all curves that enclose E. By using this fact we can prove [8]

THEOREM 6. *For any continuum E we have*

$$2\pi \text{ cap } E \leqslant L_0 < 9.173 \text{ cap } E.$$

This inequality can be written in the form

$$0.684 < \frac{2\pi \text{ cap } E}{L_0} \leqslant 1. \tag{10}$$

Since it is often easy to compute L_0 (for instance, if E is given graphically), inequality (10) furnishes an estimate for cap E. In the special case of the configuration that consists of three equal segments meeting at 0 under the angles $2\pi/3$ we find $L_0 = 27^{1/2}\, 4^{1/3} \text{ cap } E$. If λ_0 is the maximum of $L_0/\text{cap } E$ for all continua E we have therefore

$$8.248 < \lambda_0 < 9.173.$$

It seems probable that λ_0 is assumed for the special configuration described above, that is $\lambda_0 = 27^{1/2} 4^{1/3}$.

It is also possible to obtain bounds for $b/\mathrm{cap}\ E$, where b is the width of the continuum [8]. For the diameter d of E it is known that

$$2 \leqslant d/\mathrm{cap}\ E \leqslant 4, \tag{11}$$

with equality for the circle respectively the segment.

We can finally ask how far these results remain true for closed bounded sets E with more than one connected component. Let us take for instance inequality (11), and let d again be the diameter of E. Then we have, instead of (11),

$$2 \leqslant d/\mathrm{cap}\ E < \infty. \tag{12}$$

The right inequality cannot be sharpened, that is $d/\mathrm{cap}\ E$ can become arbitraily large, as an example is going to show. The lemniscate $|z^2 - a^2| = 1$ has capacity 1. For large positive a it consists of two components which are approximately the circles $|z \pm a| = (2a)^{-1}$. Therefore we have $d \to \infty$ for $a \to \infty$.

In the case that E is connected we could define d also as the maximum of the projections of E onto straight lines. In the general case we will denote this quantity by d^*. Then Pólya has shown [5] that $d^* \leqslant 4\ \mathrm{cap}\ E$, as I have mentioned before, and we have

$$0 < d/\mathrm{cap}\ E \leqslant 4. \tag{13}$$

In the example just considered we have $d^* \to 0$ for $a \to \infty$. Therefore (13) is a best possible inequality. To sum up : If E is not connected, inequality (11) (with $d = d^*$) splits up into the two sharp inequalities (12) and (13).

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