Several Complex Variables

Local Theory

M. HERVÉ

SEVERAL COMPLEX VARIABLES

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PREFACE

THIS book is an introduction to recent work in the theory of functions of several complex variables, especially on complex spaces. Many results of a local character, relating to the ring of germs of holomorphic functions at a given point, holomorphic mappings, analytic continuation, analytic sets and so on are usually assumed known, although they are not proved in the well-known books of Behnke-Thullen and Bochner-Martin and are available only in the original papers of H. Cartan, R. Remmert, K. Stein and others, or in seminar notes. (See [3], [5] in the references).

I thought that it might be useful to put all this material together, and that a new treatment might suggest fresh ideas. The treatment given here is as self-contained as was found possible. The reader needs only to be acquainted with the classical theory of holomorphic functions of a complex variable, and with a few results from Algebra, which are summarized in Chapter II, §2.

The text is based on a course of lectures given early in 1962 at the Tata Institute of Fundamental Research, Bombay. I wish to express my gratitude to Professor K. Chandrasekharan who invited me to give these lectures and decided to have them printed.

The major part of the text was actually written by Mr. R. R. Simha. I thank him warmly for his useful remarks and his wholehearted co-operation. I am also indebted to Dr. Raghavan Narasimhan for important improvements.

NANCY MICHEL HERVÉ

CONTENTS

I

BASIC PROPERTIES OF HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

IN this chapter we present some basic properties of holomorphic functions of several complex variables, mostly without detailed proofs.

1. Holomorphic functions. The space we work in is an m-dimensional vector-space C^m over the field C of complex numbers, $m \geqslant 1$. In general we shall suppose a basis fixed for C^m, and identify C^m with the space of ordered m-tuples $x = (x_1, \ldots, x_m)$ of complex numbers.

Given a point $a = (a_1, \ldots, a_m)$ of C^m, and real numbers $r_1, \ldots, r_m > 0$, we call the set $P = \{(x_1, \ldots, x_m) \in C^m \mid |x_j - a_j| < r_j,\ j = 1, \ldots, m\}$ *the open polydisc with centre a and radii* r_j. Similarly, *the closed polydisc with centre a and radii* r_j is the set

$$\bar{P} = \{(x_1, \ldots, x_m) \in C^m \mid |x_j - a_j| \leqslant r_j, j = 1, \ldots, m\}.$$

The set

$$\Gamma = \{(x_1, \ldots, x_m) \in C^m \mid |x_j - a_j| = r_j,\ j = 1, \ldots, m\}$$

is called the *edge*, or *distinguished boundary*, of P (and of $\bar{P}$). The set of all open polydiscs is a basis for the "usual" topology on C^m. (The topology thus defined does not depend on the choice of the basis for C^m.)

NORMAL CONVERGENCE. A series Σf_n of complex-valued functions defined on a set E is said to converge normally on E if $\sum_n \|f_n\| < +\infty$; here $\|f_n\| = \sup_{x \in E} |f_n(x)|$.

ABEL'S LEMMA. Suppose the power series

$$S = \sum_{k_1, \ldots, k_m \geqslant 0} a_{k_1 \ldots k_m} x_1^{k_1} \ldots x_m^{k_m}$$

in m variables $\in C$ converges† at the point $(b_1, \ldots, b_m)$ of C^m, and let $b_j \neq 0$ for $j = 1$ to m. Then S converges at every point of the open polydisc P with centre the origin of C^m, and radii $|b_j|$; and S converges normally on every compact subset of P.

† Only absolute convergence will be considered in this book.

DEFINITION 1. *A complex-valued function f, defined on an open subset U of C^m, is holomorphic in U if for every point $b \in U$, there exist an open polydisc $P \subset U$ with centre b, and a power series*

$$\sum_{j_1,\ldots,j_m \geqslant 0} a_{j_1\ldots j_m} (x_1 - b_1)^{j_1} \ldots (x_m - b_m)^{j_m}$$

converging to $f(x)$ at every point $x \in P$.

REMARK. Suppose f is holomorphic in U. Then by Abel's lemma, the power series S converges normally on compact subsets of P. Hence a holomorphic function is continuous.

PROPOSITION 1. *Let $f_1, \ldots, f_p$ be holomorphic functions on an open subset U of C^m, and suppose that for every x in U, $(f_1(x), \ldots, f_p(x))$ lies in a given open set V in C^p. Then for every holomorphic function g on V, the function $g(f_1(x), \ldots, f_p(x))$ is a holomorphic function of x on U.*

This follows from associative properties of normally convergent power series.

COROLLARY 1. *The holomorphy of a function on an open set in C^m does not depend on the choice of a basis for C^m.*

COROLLARY 2. *If f and g are holomorphic functions on an open set U in C^m, the functions $f + g$, fg are holomorphic in U. If g does not vanish anywhere in U, the function f/g is holomorphic in U.*

COROLLARY 3. *Let f be a holomorphic function on the open set U in C^m. Then for every point $a = (a_1, \ldots, a_m)$ of U, and every j $(1 \leqslant j \leqslant m)$, the function $f(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_m)$ of one complex variable x_j, defined on the open set*

$$\{x_j \in C \mid (a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_m) \in U\}$$

in C, is holomorphic there.

REMARK. Suppose conversely that f is a complex-valued function on U such that each function of one complex variable obtained from f as above is holomorphic in the corresponding open set in C. Then *f is holomorphic in U* (Theorem of Hartogs-Osgood). This is a deep result, which we shall not prove in this book. For a proof, see Bochner and Martin ([1], p. 140).

PROPOSITION 2. *Let f be a holomorphic function on an open set U in C^m. For any integers $k_1, \ldots, k_m \geqslant 0$, the partial derivative $\dfrac{\partial^{k_1+\cdots+k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$ exists and is holomorphic in U. More precisely let $P \subset U$ be an open polydisc with centre b, and let S be a power series in the $x_j - b_j$, $j = 1, \ldots, m$, converging to f in P. Then the power series obtained by termwise differentiation of S, k_1 times with respect to $x_1, \ldots, k_m$ times with respect to x_m, converges to $\dfrac{\partial^{k_1+\cdots+k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$ in P.*

COROLLARY 1. *With the notation of Prop. 2, the coefficient of $(x_1 - b_1)^{k_1} \ldots (x_m - b_m)^{k_m}$ in S is $\dfrac{1}{k_1! \ldots k_m!} \dfrac{\partial^{k_1+\cdots+k_m} f(b)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$. In particular, the power series S is uniquely determined by the values of f in a neighbourhood of b. We call it the Taylor expansion of f at b.*

COROLLARY 2. *Let f and g be holomorphic functions on a connected open set U in C^m. Suppose $f = g$ on a non-empty open subset of U. Then $f = g$ everywhere in U* (Principle of Analytic Continuation).

PROOF. Let $V = \{x \in U \mid f = g \text{ in a neighbourhood of } x\}$. V is non-empty by assumption, and by definition it is open. Plainly a point $x \in U$ belongs to V if and only if f and g have the same Taylor expansion at x. By Corollary 1, this means that

$$V = \bigcap_{k_1, \ldots, k_m \geqslant 0} \left\{ x \in U \;\middle|\; \frac{\partial^{k_1+\cdots+k_m} f(x)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} = \frac{\partial^{k_1+\cdots+k_m} g(x)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} \right\},$$

hence V is closed in U. U being connected, $V = U$, q.e.d.

REMARK. We shall see later (Chapter III, § 1) that if $f = g$ on a subset F of U such that $U - F$ is open and disconnected, or locally disconnected (i.e., there exists an open connected set $W \subset U$ such that $W \cap (U - F)$ is disconnected), then $f = g$ on U.

2. Germs of holomorphic functions. Let X be an arbitrary subset of C^m. We consider the set $E(X) = E$ of pairs (U, f) where U is an open set in C^m containing X and f a function holomorphic on U. We define a relation R on E as follows : $(U, f)\ R\ (V, g)$ if and only if $f = g$ on an open set $(\subset U \cap V)$ containing X. Clearly R is an equivalence relation.

Definition 2. *A germ of a holomorphic function on X is an equivalence class of $E(X)$ with respect to the relation R.*

We denote by $\mathscr{H}(X)$ the set of germs of holomorphic functions on X. With the obvious addition and multiplication, $\mathscr{H}(X)$ is a commutative ring with identity. It is clear that each element of $\mathscr{H}(X)$ has a well-defined value at each point of X ; however, in general, distinct elements of $\mathscr{H}(X)$ may have the same value at all points of X.

Remark 1. Suppose the set X is the closure of its interior ; or suppose each connected component of X has an interior point. Then any element of $\mathscr{H}(X)$ is uniquely determined by its values on X. In fact, suppose (U, f), $(V, g) \in E(X)$, and $f = g$ on X. Then, in both cases, it is clear from Corollary 2 to Prop. 2 that $f = g$ on all connected components of $U \cap V$ meeting X. Hence $(U, f)\ R\ (V, g)$.

Remark 2. Suppose X contains just one point a. In this case we write $\mathscr{H}(X) = \mathscr{H}_a^m$. Two functions holomorphic in an open neighbourhood of a coincide in a neighbourhood of a if and only if they have the same Taylor expansion at a. Thus $\mathscr{H}_a^m$ is isomorphic to the ring of convergent power series in m complex variables. (A power-series $\sum_{k_1, \ldots, k_m \geqslant 0} a_{k_1 \ldots k_m} x_1^{k_1} \ldots x_m^{k_m}$ is said to be convergent if it converges in an open polydisc with centre the origin of C^m.) The value of an element of $\mathscr{H}_a^m$ at a is, of course, the constant term of its Taylor series.

3. Cauchy's integral formula. Let f be (a germ of) a holomorphic function on the closed polydisc $\bar{P}$ in C^m with centre a and radii r_j. Then for every point x of the open polydisc P with the same centre and radii, we have

$$f(x) = \frac{1}{(2\pi i)^m} \int_{|y_m - a_m| = r_m} dy_m \dots \int_{|y_1 - a_1| = r_1} \frac{f(y_1, \dots, y_m)}{(y_1 - x_1) \dots (y_m - x_m)} \, dy_1$$

where, for the integrations, the circles $|y_j - a_j| = r_j$ are assumed positively oriented.

This follows immediately from Cauchy's integral formula for holomorphic functions of one complex variable.

Corollary 1. *With the above notation the Taylor series* $S = \sum_{k_1, \dots, k_m \geqslant 0} a_{k_1 \dots k_m} (x_1 - a_1)^{k_1} \dots (x_m - a_m)^{k_m}$ *of* f *at* a *converges in* P.

In fact, the integrand in the Cauchy integral formula can be expanded in a power series in the $x_j - a_j$, converging in P, and with coefficients which are functions of y on the edge Γ of P. Since f is continuous, hence bounded, on Γ, this series (for each x in P) converges normally on Γ. The series can therefore be integrated termwise, and yields a power series in the $x_j - a_j$ converging to f in P.

Remark. Suppose that f is a holomorphic function on an open set U in C^m. The above result implies that the Taylor series of f at any point $a \in U$ converges in any polydisc with centre a, contained in U.

Corollary 2. *With the notation of Corollary* 1, *we have, for any* $k_1, \dots, k_m \geqslant 0$,

$$|a_{k_1 \dots k_m}| \leqslant \frac{1}{r_1^{k_1} \dots r_m^{k_m}} \sup_{y \in \Gamma} |f(y)|,$$

where Γ *is the edge of* P (Inequalities of Cauchy).

In fact the integral formula yields

$$a_{k_1 \dots k_m} = \frac{1}{(2\pi i)^m} \int_{|y_m - a_m| = r_m} dy_m \dots \int_{|y_1 - a_1| = r_1} \frac{f(y_1, \dots, y_m)}{(y_1 - a_1)^{k_1 + 1} \dots (y_m - a_m)^{k_m + 1}} \, dy_1,$$

leading to the given majorisations.

REMARK. Let $\mathscr{F}$ be a family of holomorphic functions on an open set U of C^m, uniformly bounded on compact subsets of U. Since the Taylor series of the partial derivatives of a holomorphic function may be obtained by termwise differentiation, the inequalities of Cauchy imply that for every $k_1, \ldots, k_m \geqslant 0$ the family $\left\{ \frac{\partial^{k_1+\ldots+k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} \middle| f \in \mathscr{F} \right\}$ is also uniformly bounded on compact subsets of U. In particular, the family of all first partial derivatives of members of $\mathscr{F}$ is uniformly bounded on compact subsets of U, hence the family $\mathscr{F}$ is *equicontinuous*. It follows from Ascoli's Theorem (see [2], p. 43), that one may extract, from any infinite sequence of members of $\mathscr{F}$, a subsequence which converges uniformly on every compact subset of U.

COROLLARY 3. (The Maximum Principle.) *Let f be a holomorphic function on an open set U in C^m. Let ∂U be the boundary of U—if U is not relatively compact in C^m, ∂U is to include the point at infinity of C^m. Suppose that, for every point y of ∂U, $\limsup_{x \to y, x \in U} |f(x)| \leqslant M$. Then* (i) $|f| \leqslant M$ *in* U, (ii) *if* $|f(x_0)| = M$ *for a point* x_0 *in* U, *then* $f(x) \equiv f(x_0)$ *on the connected component of* U *containing* x_0.

This is proved as in the case of one complex variable, using the integral formula.

4. Weierstrass' Theorem. *If a sequence $\{f_n\}$ of functions, holomorphic on an open subset U of C^m, converges uniformly on every compact subset of U, then* (i) *the limit function f is holomorphic in U,* (ii) *for any $k_1, \ldots, k_m \geqslant 0$, the sequence $\left\{ \frac{\partial^{k_1+\ldots+k_m} f_n}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} \right\}$ converges to $\frac{\partial^{k_1+\ldots+k_m} f}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}$ on U, uniformly on every compact subset of U.*

The first statement is proved by using Cauchy's integral formula, the second one then follows from Cauchy's inequalities.

COROLLARY 1. *If a power series in $x_1, \ldots, x_m$ converges in an open polydisc P with the origin of C^m as centre, then the sum is a holomorphic function on P.*

COROLLARY 2. *Let U be an open set in C^m, K a compact space, and μ a Radon measure on K. Suppose $(x, y) \to f(x, y)$ is a continuous function on $U \times K$, and $x \to f(x, y)$ is a holomorphic function on U for each fixed y in K. Then* (i) *the function $F(x) = \int_K f(x, y)d\mu(y)$ is holomorphic in U,* (ii) *for any $k_1, \ldots, k_m \geq 0$,*

$$\frac{\partial^{k_1+\cdots+k_m}F(x)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} = \int_K \frac{\partial^{k_1+\cdots+k_m}f(x,y)}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}} d\mu(y).$$

We thus obtain, in particular, the Cauchy integral formulas for the partial derivatives of a function holomorphic on a closed polydisc.

5. Holomorphic mappings.

DEFINITION 3. *A mapping $f(x) = (f_1(x), \ldots, f_p(x))$ of an open subset U of C^m into C^p is holomorphic if its coordinates $f_1(x), \ldots, f_p(x)$ are holomorphic functions on U. If $p = m$, then the Jacobian $J_f(x)$ of f at $x \in U$ is the determinant of the matrix $\left(\frac{\partial f_i(x)}{\partial x_j}\right)$.*

We shall give a complete proof of the following theorem.

THEOREM 1. *Let f be a holomorphic mapping of an open subset U of C^m into C^m, and suppose $J_f(a) \neq 0$ at a point $a \in U$. Then there exist open neighbourhoods $V(\subset U)$ and V' of a and $f(a)$ respectively such that* (i) *the restriction $f \mid V$ of f to V is one-one in V and maps V onto V',* (ii) *the inverse mapping of $f \mid V$ is holomorphic in V'.*

PROOF. We may assume that $a = f(a) = \mathfrak{o}$, the origin of C^m. Since $J_f(\mathfrak{o}) \neq 0$, we may also assume a basis for C^m so chosen that the matrix of $J_f(\mathfrak{o})$ is the identity matrix. For $x \in U$, let us write $f(x) = x - g(x)$. Then $g(x)$ defines a holomorphic mapping of U into C^m, and all the coordinates $g_j(x)$, with all their first partial derivatives, vanish at $\mathfrak{o}$. Using the mean-value theorem of the differential calculus, we can therefore find an open polydisc $P \subset U$, with centre $\mathfrak{o}$ and radii r, such that for all x, x' in P,

$$\sup_{1 \leq j \leq m} |g_j(x) - g_j(x')| < \tfrac{1}{2} \sup_{1 \leq j \leq m} |x_j - x'_j|. \qquad (*)$$

Let P' be the open polydisc with centre 0 and radii $r/2$; (*) implies in particular that $g(P) \subset P'$. We shall show that the assertions of Theorem 1 are valid with $V = P \cap f^{-1}(P')$, and $V' = P'$.

By definition of V, V', we have $f(V) \subset V'$. We assert first that $f \mid V$ is one-one. Suppose in fact that $x, x' \in V$, $x \neq x'$, and $f(x) = f(x')$. Then $x - x' = g(x) - g(x')$, so that

$$\sup_{1 \leqslant j \leqslant m} |g_j(x) - g_j(x')| = \sup_{1 \leqslant j \leqslant m} |x_j - x_j'| \,(> 0, \text{ since } x \neq x').$$

However, $V \subset P$, so this is a contradiction. Now for any $y \in V$, we set $x^{(0)}(y) = y$, and define $x^{(n)}(y)$ for $n \geqslant 1$ inductively by $x^{(n)}(y) = y + g(x^{(n-1)}(y))$ — since $g(P) \subset P'$, it is easily checked, by induction, that $x^{(n)}(y) \in P$ for all n. We have, for $n \geqslant 2$,

$$x^{(n)}(y) - x^{(n-1)}(y) = g(x^{(n-1)}(y)) - g(x^{(n-2)}(y)),$$

and hence (*) easily leads to the majorisation

$$\sup_{1 \leqslant j \leqslant m} |x_j^{(n)}(y) - x_j^{(n-1)}(y)| \leqslant \frac{r}{2^n}, \; n \geqslant 1.$$

Hence the sequences $\{x_j^{(n)}(y)\}$ converge uniformly on V'. Plainly the $x_j^{(n)}(y)$ are holomorphic functions on V', hence the limit $x_j(y)$ of the $x_j^{(n)}(y)$ is a holomorphic function on V, for $j = 1, \ldots, m$. Set $x(y) = (x_1(y), \ldots, x_m(y))$. Then the mapping $x(y)$ of V into C^m is holomorphic. Since, for every n, $x^{(n)}(y) - y \in P'$, $x(y) - y$ lies in $\overline{P'}$. Since $y \in P'$, this means that $x(y)$ lies in P. Again, for every n, $x^{(n)}(y) - g(x^{(n-1)}(y)) = y$, hence $x(y) - g(x(y)) = y$, i.e., $f(x(y)) = y$. Since $f \mid V$ is one-one, this shows that $f(V) = V'$, and that $x(y)$ is the inverse of $f \mid V$, q.e.d.

Remark. Conversely, suppose f is a one-one holomorphic mapping of an open set U of C^m onto an open subset of C^p. Then $p = m$, and the Jacobian of f never vanishes in U. This will be proved later (Chapter IV, §5).

II

THE RING OF GERMS OF HOLOMORPHIC FUNCTIONS AT A POINT

In this chapter, we shall be mainly concerned with the "Preparation Theorem of Weierstrass" and some of its consequences. This theorem is an important tool in the local study of the zeros of holomorphic functions.

1. Preparation theorems. Let $\mathscr{H}_a^m$ denote, as before, the ring of germs of holomorphic functions at the point $a \in C^m$. If f is a holomorphic function on some open neighbourhood of a in C^m, we denote by $\mathbf{f}$ the element of $\mathscr{H}_a^m$ induced by f. In particular, $\mathbf{0}$ and $\mathbf{1}$ are respectively the zero element and the identity of $\mathscr{H}_a^m$.

Proposition 1. *$\mathscr{H}_a^m$ is an integral domain.*

Proof. As observed before, $\mathscr{H}_a^m$ is isomorphic to the ring of convergent power series in m variables over C, which is a subring of the ring $\mathscr{F}^m$ of formal power series in m variables over C. Since $\mathscr{F}^m$ is isomorphic (for $m \geqslant 2$) to the ring of formal power-series in one variable over $\mathscr{F}^{m-1}$, we can deduce, by induction on m, that $\mathscr{F}^m$ is an integral domain, from the following fact : the ring $A[[X]]$ of formal power series in one variable X over an integral domain A is an integral domain. To prove this last fact, let $f = \sum_{k \geqslant p} a_k X^k$, $g = \sum_{k \geqslant q} b_k X^k$, $p, q \geqslant 0, a_p \neq 0, b_q \neq 0$, be two non-zero elements of $A[[X]]$. Then $fg = \sum_{k \geqslant p+q} c_k X^k$ with $c_{p+q} = a_p b_q \neq 0$, q.e.d.

Remark. Proposition 1 may also be deduced from the principle of analytic continuation : if f is holomorphic on an open connected set U in C^m, and vanishes on a non-empty open subset of U, then f vanishes identically on U.

An element $\mathbf{f} \in \mathscr{H}_a^m$ is invertible if and only if $f(a) \neq 0$. Hence the set of non-invertible elements of $\mathscr{H}_a^m$ is an ideal (which is therefore the unique (proper) maximal ideal of $\mathscr{H}_a^m$). We denote it by $\mathscr{H}'^m_a$.

DEFINITION 1. *Two elements* $\mathbf{f}, \mathbf{g} \in \mathscr{H}_a^m$ *are equivalent*: $\mathbf{f} \sim \mathbf{g}$, *if there exists an invertible element* $\mathbf{h} \in \mathscr{H}_a^m$ *such that* $\mathbf{f} = \mathbf{hg}$.

Clearly $\sim$ is an equivalence relation in $\mathscr{H}_a^m$; and $\mathbf{f} \sim \mathbf{g}$ implies that f and g have the same zeros in a neighbourhood of a. The trivial equivalence-classes in $\mathscr{H}_a^m$ with respect to this relation are those of $\mathbf{0}$ and $\mathbf{1}$, consisting, respectively, of $\mathbf{0}$ alone and of all invertible elements of $\mathscr{H}_a^m$.

If $x = (x_1, \ldots, x_m)$ is any point of C^m $(m \geqslant 2)$, we shall denote by x' the point $(x_1, \ldots, x_{m-1}) \in C^{m-1}$; in particular, $\mathfrak{o}$ and $\mathfrak{o}'$ are respectively the origins of C^m and C^{m-1}. Conversely, if $x' = (x_1, \ldots, x_{m-1}) \in C^{m-1}$, and $x_m \in C$, (x', x_m) denotes the point $(x_1, \ldots, x_{m-1}, x_m) \in C^m$. We shall also write $\mathscr{H}_\mathfrak{o}^m = \mathscr{H}^m$, $\mathscr{H}'^m_\mathfrak{o} = \mathscr{H}'^m$.

DEFINITION 2. *A distinguished pseudo-polynomial in* x_m, *of degree* p, *is an expression of the form* $x_m^p + \sum_{k=1}^{p} c_k(x')\, x_m^{p-k}$, $p \geqslant 1$, *where the* $c_k(x')$ *are holomorphic functions on open neighbourhoods of* $\mathfrak{o}'$ *in* C^{m-1}, *vanishing at* $\mathfrak{o}'$.

A distinguished pseudo-polynomial induces a non-zero and non-invertible element in $\mathscr{H}^m$.

THEOREM 1. (The Weierstrass Preparation Theorem.) *Suppose given an element* $\mathbf{f} \in \mathscr{H}'^m$, $\mathbf{f} \neq \mathbf{0}$. *Then*: (i) *we can choose a basis for* C^m *in such a way that* $f(\mathfrak{o}', x_m)$ *does not vanish identically in any neighbourhood of* $x_m = 0$ *in* C^1; (ii) *if the basis for* C^m *is such that* $f(\mathfrak{o}', x_m)$ *does not vanish identically in any neighbourhood of* $x_m = 0$ *in* C^1, *there exists a distinguished pseudo-polynomial* $\phi = x_m^p + \sum_{k=1}^{p} c_k(x')\, x_m^{p-k}$ *such that* $\mathbf{f} \sim \boldsymbol{\varphi}$; (iii) *if, with respect to the same basis as in* (ii), $\psi = x_m^q + \sum_{k=1}^{q} d_k(x')\, x_m^{q-k}$ *is any distinguished pseudo-polynomial such that* $\boldsymbol{\psi} \sim \mathbf{f}$, *then* $q = p$, *and for every* k, $1 \leqslant k \leqslant p$, c_k *and* d_k *induce the same element of* $\mathscr{H}'^{m-1}_\mathfrak{o} = \mathscr{H}'^{m-1}$.

PROOF. (i) *Choice of the basis for* C^m. Let U be an open convex neighbourhood of $\mathfrak{o}$ on which f is defined and holomorphic. Since $\mathbf{f} \neq \mathbf{0}$, there is a point $a \in U$, $a \neq \mathfrak{o}$, such that $f(a) \neq 0$. If we choose any basis for C^m whose mth element is a, $f(\mathfrak{o}', x_m)$ is defined and holomorphic on the connected (in fact convex) open set $\{x_m \in C^1 \mid (\mathfrak{o}', x_m) \in U\}$ and does not vanish at the point $x_m = 1$ of that set, q.e.d.

(ii) We now suppose that $f(\mathfrak{o}', x_m)$ does not vanish identically in any neighbourhood of $x_m = 0$. Let P be an open polydisc, with centre $\mathfrak{o}$ and radii r, on which f is defined and holomorphic. Then there exists a number ρ, $0 < \rho < r$, such that $f(\mathfrak{o}', x_m) \neq 0$ if $0 < |x_m| \leqslant \rho$. By the continuity of f in P, we may find an open polydisc P' in C^{m-1}, with centre $\mathfrak{o}'$ and radii $< r$, such that $f(x', x_m) \neq 0$ if $x' \in P'$ and $|x_m| = \rho$—for, f does not vanish on the compact set $\{(\mathfrak{o}', x_m) \mid |x_m| = \rho\}$. It is useful to notice that ρ and the radii of P can be chosen, in that order, arbitrarily small. Let γ be the positively oriented circle $|t| = \rho$ in C^1; define, for $x' \in P'$,

$$\sigma_0(x') = \frac{1}{2\pi i}\int_\gamma \frac{\partial f(x', t)}{\partial t}\,\frac{dt}{f(x', t)}\,.$$

Since, for each fixed x' in P', $f(x', t)$ is a holomorphic function of t in $|t| \leqslant \rho$ and does not vanish on γ, $\sigma_0(x')$ is precisely the number of zeros (counted with multiplicity) of $f(x', t)$ in $|t| < \rho$. Plainly σ_0 is continuous on P'. Since it is integer-valued, and P' is connected, we must have $\sigma_0 \equiv \sigma_0(\mathfrak{o}') = p$ say. Since $f(\mathfrak{o}) = 0$, while $f(\mathfrak{o}', x_m) \neq 0$ in $0 < |x_m| \leqslant \rho$, p is precisely the multiplicity with which $f(\mathfrak{o}', x_m)$ vanishes at $x_m = 0$, and $p \geqslant 1$. Now $f(x', t)$ has, for every x' in P', precisely p zeros in $|t| < \rho$, say $t_1(x'), \ldots, t_p(x')$. For $k \geqslant 1$, and x' in P', let $\sigma_k(x') = \{t_1(x')\}^k + \ldots + \{t_p(x')\}^k$. The formulas

$$\sigma_k(x') = \frac{1}{2\pi i}\int_\gamma t^k\,\frac{\partial f(x', t)}{\partial t}\,\frac{dt}{f(x, t)}, \quad k \geqslant 1,$$

show that all the σ_k are holomorphic functions on P'. We now consider the elementary symmetric functions of the $t_k(x')$: for $1 \leqslant k \leqslant p$, let

$$c_k(x') = (-1)^k \sum_{1 \leqslant j_1 < \ldots < j_k \leqslant p} t_{j_1}(x') \ldots t_{j_k}(x').$$

Each c_k can be expressed as a polynomial in the σ_j (with rational coefficients) and hence is holomorphic on P'. It is evident that all the c_k vanish at $0'$. Hence $\phi(x', x_m) = x_m^p + \sum_{k=1}^{p} c_k(x')\, x_m^{p-k}$ is a distinguished pseudo-polynomial. We shall show that f and ϕ induce equivalent germs at every point of $P_1 = \{(x', x_m) \mid x' \in P', |x_m| < \rho\}$.

For each x' in P', $f(x', t)$ and $\phi(x', t)$ are holomorphic in $|t| \leqslant \rho$, and do not vanish on $|t| = \rho$, and $\phi(x', t)$ has, by definition, the same zeros, with the same multiplicities, as $f(x', t)$ in $|t| < \rho$. Hence $f(x', t)/\phi(x', t)$ and $\phi(x', t)/f(x', t)$ are, for fixed x' in P', holomorphic on $|t| < \rho$. Hence we have, by Cauchy's integral formula,

$$\frac{f(x', x_m)}{\phi(x', x_m)} = \frac{1}{2\pi i}\int_\gamma \frac{f(x', t)}{\phi(x', t)}\frac{dt}{t - x_m}$$

$$\frac{\phi(x', x_m)}{f(x', x_m)} = \frac{1}{2\pi i}\int_\gamma \frac{\phi(x', t)}{f(x', t)}\frac{dt}{t - x_m}$$

for all points (x', x_m) in P_1. However the integrands in these formulas are continuous functions of (x, t) on $P_1 \times \gamma$, holomorphic on P_1 for each $t \in \gamma$. Hence $\dfrac{f(x)}{\phi(x)}$ and $\dfrac{\phi(x)}{f(x)}$ are holomorphic in P_1 and this implies that f and ϕ induce equivalent germs at every point of P_1, proving (ii).

(iii) Let $\psi(x', x_m) = x_m^q + \sum_{k=1}^{q} d_k(x')x_m^{q-k}$ be any distinguished pseudo-polynomial such that $\boldsymbol{\psi} \sim \mathbf{f}$. Then $\boldsymbol{\psi} \sim \boldsymbol{\varphi}$, where ϕ is as in (ii). Since the P_1 of (ii) can be chosen to have as small radii as desired, we may assume that : (a) the d_k, $k = 1, \dots, q$, are all defined and holomorphic on P'; (b) there exists a holomorphic function h on P_1, vanishing nowhere in P_1, such that $\psi(x', x_m) = h(x', x_m)\,\phi(x', x_m)$. We then have $\psi(0', x_m) = h(0', x_m)\phi(0', x_m)$ in $|x_m| < \rho$, i.e., $x_m^q = h(0', x_m)\, x_m^p$ in $|x_m| < \rho$. Since $h(0', x_m)$ vanishes nowhere in $|x_m| < \rho$, we must have $q = p$. The equation $\psi = h\phi$ shows that for each x' in P', $\psi(x', t)$ has p roots in $|t| < \rho$, viz. those of $\phi(x', t)$. Since $\psi(x', t)$ and $\phi(x', t)$ are monic polynomials of degree p, their corresponding coefficients, which are elementary symmetric functions of these p roots, are the same, i.e., $d_k(x') = c_k(x')$ on P', $k = 1, \dots, p$, q.e.d.

REMARK 1. Given a finite family of functions f_j, each of which satisfies the assumptions of Theorem 1 (i.e., $\mathbf{f}_j \in \mathscr{H}'^m$, $\mathbf{f}_j \neq \mathbf{0}$), we can choose a basis for C^m in such a way that, for each j, $f_j(\mathfrak{o}', x_m)$ does not vanish identically in any neighbourhood of $x_m = 0$ in C^1, and therefore there exist distinguished pseudo-polynomials ϕ_j in x_m such that $\mathbf{f}_j \sim \boldsymbol{\varphi}_j$.

REMARK 2. Given a function f holomorphic in some neighbourhood of $\mathfrak{o}$ such that $f(\mathfrak{o}) = 0$ and $f(\mathfrak{o}', x_m) \not\equiv 0$ in any neighbourhood of $x_m = 0$ in C^1, we can find, as in the proof of (ii), arbitrarily small polydiscs P with centre $\mathfrak{o}$ (on which f is holomorphic), with the property : for every $x' \in P'$, there exists an $x = (x', x_m) \in P$ such that $f(x) = 0$.

THEOREM 2. (The Späth-Cartan Preparation Theorem.) *Suppose given an element* $\mathbf{f} \in \mathscr{H}'^m$, $\mathbf{f} \neq \mathbf{0}$, *and the basis for* C^m *so chosen that* $f(\mathfrak{o}', x_m)$ *does not vanish identically in any neighbourhood of* $x_m = 0$ *in* C^1. *Then there exist a sequence* $\{P_n\}$ *of open polydiscs with centre* $\mathfrak{o}$ *and radii decreasing to zero, and real numbers* $\alpha_n > 0$, *with the following properties : if* g *is any holomorphic function on* P_n, *there exists a holomorphic function* h *on* P_n *such that* : (i) $g - fh$ *is a pseudo-polynomial in* x_m (*i.e. a polynomial in* x_m *with holomorphic functions of* x' *as its coefficients*), *of degree* $< p$, *—here* p *is the degree of the unique distinguished polynomial* ϕ *such that* $\mathbf{f} \sim \boldsymbol{\varphi}$; (ii) $\sup_{x \in P_n} |h(x)| \leqslant \alpha_n \sup_{x \in P_n} |g(x)|$. *Moreover,* h *is uniquely determined by* g *and the condition* (i).

REMARK. The existence, given a g holomorphic in a neighbourhood of $\mathfrak{o}$, of a holomorphic function h on a neighbourhood of $\mathfrak{o}$ satisfying (i), and the fact that g and (i) uniquely determine h, were first proved by Späth (*Journal für die reine und angewandte Mathematik*, volume 161, (1929)). In the form stated here the theorem (in particular the majorisation (ii) which we shall need) is due to H. Cartan (*Annales de l'Ecole Normale Supérieure*, t. 61, (1944)).

PROOF. Obviously, we may suppose that $\mathbf{f} = \boldsymbol{\varphi}$ where $\phi = x_m^p + \sum_{k=1}^{p} \phi_k(x') x_m^{p-k}$ is a distinguished pseudo-polynomial. From the proof of Theorem 1, we see that we can find real numbers $\rho_n > 0$, decreasing to zero, and polydiscs P'_n in C^{m-1} with centre $\mathfrak{o}'$ and radii decreasing to zero, such that : (a) all the c_k are defined and holomorphic in $\bar{P}'_1$; (b) $\phi(x', x_m) \neq 0$ if $x' \in \bar{P}'_n$ and $\rho_n/2 \leqslant |x_m| \leqslant \rho_n$. Let $\delta_n = \inf |\phi(x', x_m)|$ for $x' \in \bar{P}'_n$ and $\rho_n/2 \leqslant |x_m| \leqslant \rho_n$, and let P_n be the polydisc $\{(x', x_m) \mid x' \in P'_n, |x_m| < \rho_n\}$ $(n = 1, 2, \ldots)$.

Suppose now that g is a holomorphic function on P_n. For any r with $\rho_n/2 < r < \rho_n$, consider the function $h(x', x_m)$ defined, for $x' \in P'_n$ and $|x_m| < r$, by $h(x', x_m) = \dfrac{1}{2\pi i} \displaystyle\int_{|t|=r} \frac{g(x', t)}{\phi(x', t)} \frac{dt}{t - x_m}$, where the circle $|t| = r$ is assumed positively oriented. Since $\phi(x', t)$ vanishes nowhere in $\rho_n/2 < |t| < \rho_n$ for any x' in P'_n, it is clear that $h(x', x_m)$ is a holomorphic function on $\{(x', x_m) \mid x' \in P'_n, |x_m| < r\}$, and that its value at any $(x', x_m) \in P_n$ is independent of the $r > |x_m|$ used in its definition. Thus there is a well-defined holomorphic function h on P_n, given at any (x', x_m) by the above integral formula, with any $r > |x_m|$ in $\rho_n/2 < r < \rho_n$. We shall prove that h has the desired properties (i) and (ii).

Consider, for $(x', x_m) \in P_n$, $g(x', x_m) - h(x', x_m)\, \phi(x', x_m)$. With an $r > |x_m|$, $\rho_n/2 < r < \rho_n$, we have

$$g(x', x_m) - h(x', x_m)\, \phi(x', x_m)$$

$$= \frac{1}{2\pi i} \int_{|t|=r} \frac{g(x', t)\, dt}{t - x_m} - \frac{1}{2\pi i} \int_{|t|=r} \frac{\phi(x', x_m)\, g(x', t)}{\phi(x', t)} \frac{dt}{t - x_m}$$

$$= \frac{1}{2\pi i} \int_{|t|=r} \frac{g(x', t)}{\phi(x', t)} \frac{\phi(x', t) - \phi(x', x_m)}{t - x_m}\, dt.$$

Now

$$\frac{\phi(x', t) - \phi(x', x_m)}{t - x_m} = \frac{t^p - x_m^p}{t - x_m} + \sum_{k=1}^{p} c_k(x') \frac{t^{p-k} - x_m^{p-k}}{t - x_m}$$

$$= \sum_{k=0}^{p-1} d_k(x', t)\, x_m^k$$

say, where the $d_k(x', t)$ are polynomials of degree $< p$ in t, the coefficients of which are integral linear combinations of the $c_k(x')$. Let β_n be a real number such that for all $x' \in P'_n$ and $|t| < \rho_n$, and all $k = 0, \ldots, p-1$, $|d_k(x', t)| \leqslant \beta_n$.

We have

$$g(x', x_m) - h(x', x_m)\,\phi(x', x_m)$$
$$= \sum_{k=0}^{p-1} \left(\frac{1}{2\pi i} \int_{|t|=r} \frac{g(x', t)}{\phi(x', t)}\, d_k(x', t)\, dt \right) x_m^k,$$

proving (i). Further, if we assume $\rho_1 < 1$, we obtain from the above:

$$|\, g(x', x_m) - h(x', x_m)\,\phi(x', x_m)\,|$$
$$\leqslant \frac{p}{\delta_n}\, \beta_n \sup_{x \in P_n} |g(x)| = \gamma_n \sup_{x \in P_n} |g(x)|$$

say, and this is valid for every $(x', x_m) \in P_n$. Hence, for every $x \in P_n$,

$$|h(x)\ \phi(x)| \leqslant (1 + \gamma_n) \sup_{x \in P_n} |g(x)|.$$

For $x = (x', x_m) \in P_n$ with $|x_m| \geqslant \rho_n/2$, this yields

$$|h(x)| \leqslant \frac{1 + \gamma_n}{\delta_n} \sup_{x \in P_n} |g(x)|.$$

The maximum principle now shows that $\sup_{x \in P_n} |h(x)| \leqslant \frac{1+\gamma_n}{\delta_n} \sup_{x \in P_n} |g(x)|$.

Plainly $\alpha_n = \dfrac{1 + \gamma_n}{\delta_n}$ is independent of g, hence (ii) is proved.

It only remains to prove that, given g, condition (i) determines h uniquely. For this purpose it is sufficient to show that the Taylor expansion at $\mathfrak{o}$ of h is uniquely determined.

Let us write the Taylor expansion $\mathbf{g}$ of g at $\mathfrak{o}$ in the form $\mathbf{g} = \sum_{k=0}^{\infty} \mathbf{g}_k\, x_m^k$, where the $\mathbf{g}_k$ are elements of $\mathscr{H}^{m-1}$. Similarly let $\mathbf{c}_k$ be the germ of c_k at $\mathfrak{o}'$, $k = 1, \ldots, p$. We shall show that a (formal) power series h in $x_1, \ldots, x_m$ is uniquely determined by the condition : $\mathbf{g} - h\,\boldsymbol{\varphi}$ contains no terms of degree $> p - 1$ in x_m. In fact let h satisfy this condition, and let us write $h = \sum_{k=0}^{\infty} h_k\, x_m^k$ where the h_k are formal power series in $x_1, \ldots, x_{m-1}$ over C. The coefficient of x_m^{p+k} (for any $k \geqslant 0$) in $\mathbf{g} - h\boldsymbol{\varphi}$ is

$$\mathbf{g}_{p+k} - h_k - \sum_{j=1}^{p} \mathbf{c}_j h_{k+j}$$

Hence, for all $k \geqslant 0$,

$$h_k = \mathbf{g}_{p+k} - \sum_{j=1}^{p} \mathbf{c}_j h_{k+j} \tag{*}$$

Now, if $m = 1$, the $\mathbf{c}_j$ are all equal to zero (in C) and the h_k and $\mathbf{g}_k = g_k$ are complex numbers ; (*) determines the h_k uniquely : $h_k = g_{p+k}$ for $k \geqslant 0$. Suppose therefore that $m \geqslant 2$. The equations (*) hold in the ring of formal power series in $x_1, \ldots, x_{m-1}$ over C. Let us write $h_k = \sum_{s=0}^{\infty} h_k^{(s)}$, where $h_k^{(s)}$ is a homogeneous polynomial of degree s in $x_1, \ldots, x_{m-1}$; similarly $\mathbf{g}_k = \sum_{s=0}^{\infty} g_k^{(s)}$, $\mathbf{c}_k = \sum_{s=1}^{\infty} c_k^{(s)}$ — note that since $c_k(\mathfrak{o}') = 0$, c_k has no constant term, $k = 1, \ldots, p$.

The equations (*) are equivalent to the equations

$$h_k^{(0)} = g_{p+k}^{(0)}, \; k = 0, 1, 2, \ldots$$

and

$$h_k^{(s)} = g_{p+k}^{(s)} - \sum_{j=1}^{p} \sum_{i=0}^{s-1} h_{k+j}^{(i)} c_j^{(s-i)}, \; s = 1, 2, \ldots ; \; k = 0, 1, 2, \ldots$$

The first set of equations determine $h_k^{(0)}$ for all $k \geqslant 0$, and the second set of equations show that the $h_k^{(i)}$, $0 \leqslant i \leqslant s-1$, $k \geqslant 0$ determine all the $h_k^{(s)}$, $k \geqslant 0$. This concludes the proof of Theorem 2.

REMARKS. (A) One can show that the formal power series $h = \sum_{k=0}^{\infty} h_k x_m^k$ defined above actually converges in a neighbourhood of $\mathfrak{o}$; this would prove the result of Späth. For the details, see Bochner-Martin ([1], pp. 183-187).

(B) Theorem 2 is true with *any* sequence of open polydiscs P_n having $\mathfrak{o}$ as their centre, radii decreasing to zero, and satisfying conditions (a) and (b) in the first paragraph of the proof. Therefore, given a finite family of functions f_j each of which satisfies the assumptions of Theorem 2 (i.e., $\mathbf{f}_j \in \mathscr{H}'^m$, $f_j(\mathfrak{o}', x_m)$ does not vanish identically in any neighbourhood of $x_m = 0$ in C^1), a sequence of open polydiscs P_n with centre $\mathfrak{o}$ and radii $r_1^{(n)}, \ldots, r_m^{(n)}$ decreasing to zero can be found, with the following property: not only does it meet the requirements of Theorem 2 for *each* function f_j in the given family, but so does any sequence of polydiscs P^*_n with centre $\mathfrak{o}$ and radii $r_1^{*(n)} \leqslant r_1^{(n)}, \ldots, r_{m-1}^{*(n)} \leqslant r_{m-1}^{(n)}, r_m^{*(n)} = r_m^{(n)}, r_1^{*(n)}, \ldots, r_{m-1}^{*(n)}$ decreasing.

2. Algebraic preliminaries. Before using Theorems 1 and 2 to derive some algebraic properties of the ring $\mathscr{H}^m$, we state some definitions and theorems from algebra.

In this article A always stands for a commutative ring, with identity $1\ (\neq 0)$, and $A[x]$ for the polynomial ring in one variable x over A. If A is an integral domain, we denote its field of quotients by K.

(a) Resultant and Discriminant. Let $P = \sum_{k=0}^{p} a_k\, x^{p-k}$ and $Q = \sum_{k=0} b_k\, x^{q-k}$ be elements of $A\,[x]$, $p, q \geqslant 1$. The determinant

$$\rho(P,Q) = \begin{vmatrix} \overbrace{\begin{matrix} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & \cdot & \\ \cdot & \cdot & \cdot & a_0 \\ \cdot & \cdot & \cdot & a_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_p & \cdot & \cdot & \cdot \\ & a_p & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & a_p \end{matrix}}^{q \text{ columns}} & \overbrace{\begin{matrix} b_0 & & & & \\ b_1 & b_0 & & & \\ b_2 & b_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & b_0 \\ b_q & \cdot & \cdot & \cdot & b_1 \\ & b_q & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & b_q \end{matrix}}^{p \text{ columns}} \end{vmatrix}$$

of order $p + q$ is called *Sylvester's Resultant* of P and Q. Let $u_0, \ldots, u_{q-1}, v_0, \ldots, v_{p-1}$ be the co-factors of the elements of the last row of $\rho(P, Q)$, and let

$$U = u_0\, x^{q-1} + \ldots + u_{q-1},\ V = v_0\, x^{p-1} + \ldots + v_{p-1}.$$

Then we have the identity

$$PU + QV = \rho(P, Q)$$

with the consequence : $\rho(P, Q)$ belongs to the ideal generated in $A[x]$ by P and Q.

Suppose now that A is a field.

(1) $\rho(P, Q) = 0$ if and only if there exist U, V (not both 0) $\in A[x]$, of degrees $< q, p$ respectively, such that $PU + QV = 0$.

(2) If $a_0 \neq 0$, then $\rho(P, Q) = 0$ if and only if $Q = 0$, or $Q \neq 0$ and P and Q have a common divisor of degree > 0 in $A[x]$. In particular, if a_0 or $b_0 \neq 0$ and if A is an algebraically closed field, $\rho(P, Q) = 0$ if and only if P and Q have a common root in A.

Remark. If A is only an integral domain (and $a_0 \neq 0$), then $\rho(P, Q) = 0$ if and only if $Q = 0$, or $Q \neq 0$ and P and Q have a common divisor of degree > 0 in $K[x]$.

(For proofs of (1) and (2) see van der Waerden [6], pp. 83-85.)

Let $p \geqslant 2$ and $P' = \sum_{k=0}^{p-1} (p - k)\, a_k\, x^{p-k-1}$. We denote by $\delta(P)$ the determinant obtained when, in the first row of $\rho(P, P')$, a_0 is replaced by 1. $\delta(P)$ is called the *discriminant* of P.

(b) Ideals in $A[x]$. If A is a field, then every ideal in $A[x]$ is a principal ideal. If A is only an integral domain, then, for any ideal I in $A[x]$, there exists an element P_0 of I with the property : for every $P \in I$, there is an $\alpha \in A$, $\alpha \neq 0$, such that P_0 divides αP.

(c) Prime and Primary Ideals.

Definition. *An ideal I in A is prime if, for any α, $\beta \in A$, $\alpha\beta \in I$ and $\alpha \notin I$ imply $\beta \in I$.*

Proposition. *An ideal I ($\neq A$) in A is prime if and only if the ring A/I of residue-classes of A modulo I is an integral domain.*

Definition. *If I is any ideal in A, then the radical of I is the set* rad $I = \{\alpha \in A \mid \alpha^n \in I$ *for some integer* $n \geqslant 0\}$.

The radical of any ideal I is an ideal which contains I.

PROPOSITION. *If I is prime, then $I = \text{rad } I$.*

DEFINITION. *An ideal I in A is primary if, for any $\alpha, \beta \in A$, $\alpha\beta \in I$ and $\alpha \notin I$ imply $\beta \in \text{rad } I$.*

PROPOSITION. *If I is primary,* rad *I is prime.*

DEFINITION. *A finite family $\mathcal{F}$ of primary ideals in A is canonical if*: (i) *no member of $\mathcal{F}$ contains the intersection of the remaining*; (ii) *distinct members of the family have distinct radicals.*

PROPOSITION. *The intersection of a finite family of primary ideals with the same radical is primary and has the same radical.* (For a proof see van der Waerden [6 a], pp. 32-33.)

COROLLARY. *Every finite family of primary ideals has the same intersection as a certain canonical family.*

THEOREM. *If two canonical families of primary ideals have the same intersection, then there is a one-one correspondence between the two families such that corresponding elements have the same radical.* (For a proof see van der Waerden [6 a], pp. 35-36.)

(d) NOETHERIAN RINGS.

THEOREM. *The following statements are equivalent.*

(i) *Every ideal I in A is generated over A by finitely many elements of I.*

(ii) *Every strictly increasing sequence of ideals in A is finite.* (For the proof see van der Waerden [6 a], pp. 20-21.)

DEFINITION. *A is Noetherian if* (i) *or* (ii) *of the above theorem is valid.*

THEOREM. (Hilbert's Basis Theorem.) *If A is Noetherian, $A[x]$ is Noetherian.*

PROPOSITION. *If A is Noetherian, and I ($\neq A$) is any ideal in A, then A/I is Noetherian.*

THEOREM. (Primary Decomposition Theorem.) *Every ideal in a Noetherian ring is the intersection of a canonical family of primary ideals.*

(For the proof, see van der Waerden [6 a], pp. 31-34.)

REMARK (cf. (b)). If A is a Noetherian integral domain, then, for every ideal I in $A[x]$, there exist $P_0 \in I$ and $\alpha_0 \in A$, $\alpha_0 \neq 0$, such that if $P \in I$, P_0 divides $\alpha_0 P$.

(e) UNIQUE FACTORIZATION. A is now supposed to be an integral domain.

DEFINITION. *An element of A is reducible if it is a product of two non-invertible elements of A. An element of A is irreducible if it is not reducible.*

DEFINITION. *A is a unique factorization domain (or a factorial ring) if every non-invertible element $\alpha \in A$, $\alpha \neq 0$, is the product of finitely many non-invertible irreducible elements of A, determined uniquely by α except for invertible factors.*

If A is factorial, the greatest common divisor of any finitely many elements of A is well defined upto an invertible factor.

DEFINITION. *Suppose A is factorial. An element of $A[x]$ is primitive if the greatest common divisor of its coefficients is* 1.

GAUSS' LEMMA. *If A is factorial, then the product of primitive elements of $A[x]$ is primitive.*

This lemma has the following consequences.

(i) *If A is factorial, and $P \in A[x]$ is primitive, then P divides an element $Q \in A[x]$ in $A[x]$ if (and only if) it does so in $K[x]$.*

In particular, with the notation of (a), let $a_0 \neq 0$. Then $\rho(P, Q) = 0$ if and only if $Q = 0$, or $Q \neq 0$ and P and Q have a common divisor of degree > 0 in $A[x]$. If, further, A has characteristic 0, then $\delta(P) = 0$ if and only if P is divisible by the square of an element of degree > 0 of $A[x]$.

(ii) GAUSS' THEOREM. *If A is factorial, then $A[x]$ is factorial.*

For proofs, see van der Waerden [6], pp. 70-72.

3. $\mathscr{H}^m$ is a factorial ring.

LEMMA 1. *$\mathscr{H}^1$ is a factorial ring.*

PROOF. If $\mathbf{f} \neq \mathbf{0}$ is an element of $\mathscr{H}^1$ whose Taylor expansion at $\mathfrak{o}$ is $\sum_{k \geqslant p} a_k x^k$ with $a_p \neq 0$, then $\mathbf{f}$ is invertible if and only if $p = 0$, and reducible if and only if $p \geqslant 2$. The lemma follows easily from this.

Let $m \geqslant 2$. The polynomial ring $\mathscr{H}^{m-1}[x_m]$ is a subring of $\mathscr{H}^m$. We shall say that an element $P = \sum_{k=0}^{p} \alpha_k x_m^{p-k} \in \mathscr{H}^{m-1}[x_m]$ is *distinguished* if (i) $p \geqslant 1$, (ii) $\alpha_0 = 1$, and (iii) the $\alpha_k (k > 0)$ are non-invertible. According to the Weierstrass Preparation Theorem, every element $\mathbf{f} \in \mathscr{H}'^m$ such that $f(\mathfrak{o}', x_m)$ does not vanish identically in any neighbourhood of $x_m = 0$ is equivalent to precisely one distinguished element of $\mathscr{H}^{m-1}[x_m]$. A finite product of distinguished elements of $\mathscr{H}^{m-1}[x_m]$ is again distinguished.

LEMMA 2. *Let $P_1, \ldots, P_n$ be elements of $\mathscr{H}^{m-1}[x_m]$ and suppose that $P = P_1 P_2 \ldots P_n$ is distinguished. Then, for each j, the leading coefficient $\alpha^{(j)}$ of P_j is invertible, and $Q_j = (\alpha^{(j)})^{-1} P_j$ is distinguished or $\mathbf{1}$. Also $P = Q_1 Q_2 \ldots Q_n$.*

PROOF. The $\alpha^{(j)}$ are invertible since the leading coefficient $\alpha^{(1)} \alpha^{(2)} \ldots \alpha^{(n)}$ of P is $\mathbf{1}$; hence, if the Q_j are as defined, we also have $Q_1 Q_2 \ldots Q_n = P$. Now let p_j be the smallest power of x_m which appears in Q_j with an invertible coefficient. Then the coefficient of $x_m^{\Sigma p_j}$ in P is invertible. Since P is distinguished, this means that $\Sigma p_j =$ degree of P. Hence, for each j, $p_j =$ degree of Q_j. Since each Q_j is monic, this means that $Q_j = \mathbf{1}$ or Q_j is distinguished, q.e.d.

LEMMA 3. *Let P be a distinguished element of $\mathscr{H}^{m-1}[x_m]$. Then P is reducible in $\mathscr{H}^m$ if and only if it is so in $\mathscr{H}^{m-1}[x_m]$.*

PROOF. Suppose first that $P = P_1 P_2$, where P_1, P_2, are non-invertible elements of $\mathscr{H}^{m-1}[x_m]$. By Lemma 2, we also have $P = Q_1 Q_2$, where Q_1, Q_2 are distinguished polynomials—since the P_i are non-invertible, neither of the Q_i formed from the P_i as in Lemma 2 can be **1**. Since distinguished elements of $\mathscr{H}^{m-1}[x_m]$ are not invertible in $\mathscr{H}^m$, this means that P is reducible in $\mathscr{H}^m$.

Conversely suppose P is reducible in $\mathscr{H}^m$: $P = \mathbf{f}_1 \mathbf{f}_2$ where $\mathbf{f}_1$ and $\mathbf{f}_2$ are non-invertible elements of $\mathscr{H}^m$. We must have then $x_m^p = f_1(0', x_m) f_2(0', x_m)$ with $p =$ degree of $P \geqslant 1$. Hence neither of the $f_i(0', x_m)$ vanishes identically in any neighbourhood of $x_m = 0$. By the preparation theorem of Weierstrass, we have $\mathbf{f}_1 \sim P_1$, $\mathbf{f}_2 \sim P_2$ where P_1, P_2 are distinguished elements of $\mathscr{H}^{m-1}[x_m]$. Hence $P \sim P_1 P_2$. The uniqueness assertion of the preparation theorem implies $P = P_1 P_2$, i.e., P is a reducible element of $\mathscr{H}^{m-1}[x_m]$, q.e.d.

We now prove the

THEOREM 3. *$\mathscr{H}^m$ is a factorial ring.*

PROOF. By Lemma 1, $\mathscr{H}^1$ is a factorial ring. We proceed by induction: for $m \geqslant 2$, we assume that $\mathscr{H}^{m-1}$ is factorial, and prove that $\mathscr{H}^m$ is factorial.

Suppose given an element $\mathbf{f} \in \mathscr{H}'^m$, $\mathbf{f} \neq \mathbf{0}$. We can choose a basis for C^m in such a way that $\mathbf{f} \sim P$, where P is a distinguished element of $\mathscr{H}^{m-1}[x_m]$. By the induction hypothesis and by Gauss' Theorem (§2, (e)), $\mathscr{H}^{m-1}[x_m]$ is factorial. Let $P = P_1 \dots P_n$ where the P_j are non-invertible irreducible elements of $\mathscr{H}^{m-1}[x_m]$. By Lemma 2 (and since the P_j are non-invertible), the P_j may be assumed distinguished (hence non-invertible in $\mathscr{H}^m$); and by Lemma 3, they are irreducible in $\mathscr{H}^m$. Thus $\mathbf{f}$ is equivalent to a product of non-invertible irreducible elements in $\mathscr{H}^m$.

Now suppose $\mathbf{f} \sim \mathbf{f}_1 \dots \mathbf{f}_q$, where the $\mathbf{f}_k$ are non-invertible irreducible elements of $\mathscr{H}^m$. Since $\mathbf{f} \sim P$, we may suppose that $P = \mathbf{f}_1 \dots \mathbf{f}_k$ (after multiplying say $\mathbf{f}_1$ by an invertible element of $\mathscr{H}^m$). Then $f_1(0, x_m) \dots f_q(0, x_m) = x_m^p$, where p (the degree of P) $\geqslant 1$. Hence none of the $f_k(0, x_m)$ can vanish identically in any neighbourhood of $x_m = 0$. By the preparation theorem of Weierstrass, each $\mathbf{f}_k$ is equivalent in $\mathscr{H}^m$ to a distinguished element $P'_k \in \mathscr{H}^{m-1}[x_m]$, and P'_k is irreducible in $\mathscr{H}^m$ (hence by Lemma 3, in $\mathscr{H}^{m-1}[x_m]$) since $\mathbf{f}_k$ is. Now $P \sim P'_1 \dots P'_q$; actually, equality holds, by the uniqueness assertion of the Weierstrass preparation theorem. Since $\mathscr{H}^{m-1}[x_m]$ is factorial, we must have $q = n$, and, for a suitable ordering, $P'_j \sim P_j (j = 1, \ldots, n)$, i.e. $\mathbf{f}_j \sim P_j$. Thus the factorization $\mathbf{f} \sim P_1 \ldots P_n$ is, except for invertible factors, the unique factorization of $\mathbf{f}$ into irreducible non-invertible elements of $\mathscr{H}^m$, q.e.d.

4. Relatively prime germs of holomorphic functions. To state the next result, due again to Weierstrass, we introduce the following notation. Suppose f is a function holomorphic on an open set U of C^m. If a is any point of U, we shall denote by $\mathbf{f}_a$ the element of $\mathscr{H}^m_a$ induced by f.

THEOREM 4. *Let f and g be holomorphic functions on an open set U in C^m, and suppose, for a point $a \in U$, $\mathbf{f}_a$ and $\mathbf{g}_a$ are relatively prime in $\mathscr{H}^m_a$. Then there exists a neighbourhood $V \subset U$ of a such that, for every $x \in V$, $\mathbf{f}_x$ and $\mathbf{g}_x$ are relatively prime in $\mathscr{H}^m_x$.*

PROOF. The cases when either $\mathbf{f}_a$ or $\mathbf{g}_a$ is $\mathbf{0}_a$, or is invertible in $\mathscr{H}^m_a$, are easily disposed of (and the case $m = 1$ is included in them). Suppose then that both $\mathbf{f}_a$ and $\mathbf{g}_a$ are non-invertible and different from $\mathbf{0}_a$, and $m \geqslant 2$. By Remark 1 on the preparation theorem we may assume that $\mathbf{f}_a$ and $\mathbf{g}_a$ are then respectively equivalent to distinguished polynomials P and Q in $y_m = x_m - a_m$ over $\mathscr{H}^{m-1}_{a'}$; and it follows from Lemma 2 of § 3 that P and Q have no common divisor of degree > 0 in $\mathscr{H}^{m-1}_{a'}[y_m]$. Since $\mathscr{H}^{m-1}_{a'}$ is factorial, and P, Q are monic, it follows (§ 2,(e)) that $\rho(P, Q) \neq \mathbf{0}_{a'}$.

We can find an open connected set $V' \ni a'$, and monic polynomials in y_m, say $\phi(x', y_m)$ and $\psi(x', y_m)$, whose coefficients are holomorphic functions for $x' \in V'$, such that $\boldsymbol{\varphi}_a = P$, $\boldsymbol{\psi}_a = Q$. Then $\rho(\phi, \psi) = R$ is a polynomial in the coefficients of ϕ and ψ, hence is a holomorphic function on V'. And since $\mathbf{R}_{a'} = \rho(P, Q) \neq \mathbf{0}_{a'}$, and V' is connected, $\mathbf{R}_{x'} \neq \mathbf{0}_{x'}$ for every $x' \in V'$.

We now assert that, for every point b of C^m such that $b' \in V'$, $\boldsymbol{\varphi}_b$ and $\boldsymbol{\psi}_b$ are relatively prime. Suppose, on the contrary, that $\mathbf{h}_b (\neq \mathbf{0}_b)$ is a non-invertible common factor of $\boldsymbol{\varphi}_b$ and $\boldsymbol{\psi}_b$ in $\mathscr{H}^m_b$. Then $h(b', b_m + z)$ does not vanish identically in any neighbourhood of

$z = 0$ (since $\phi(b', b_m - a_m - z)$ is monic in z). Since $h(b) = 0$, Remark 2 on Weierstrass' preparation theorem implies the existence of an open polydisc W with centre b and the following properties: (i) on W, ϕ or ψ is the product of h by a holomorphic function; (ii) for every $x' \in W'$ there exists at least one $x_m \in C$ such that $(x', x_m) \in W$ and $h(x', x_m) = 0$. Since the zeros of h in W are common zeros of ϕ and ψ, this means that $\rho(\phi, \psi) = R$ vanishes identically in a neighbourhood of b'. This is a contradiction and proves our assertion.

Finally $\mathbf{f}_a \sim P = \boldsymbol{\varphi}_a$ and $\mathbf{g}_a \sim Q = \boldsymbol{\psi}_a$, hence $\mathbf{f}_x \sim \boldsymbol{\varphi}_x$ and $\mathbf{g}_x \sim \boldsymbol{\psi}_x$ for all x in a sufficiently small neighbourhood V of a. If V is so chosen that $x \in V$ implies $x' \in V'$, then for all $x \in V$, $\mathbf{f}_x$ and $\mathbf{g}_x$ are relatively prime, q.e.d.

Remark. Theorem 4 says that if f and g are two holomorphic functions in an open set U in C^m, then the set $\{x \in U \mid \mathbf{f}_x$ and $\mathbf{g}_x$ are relatively prime in $\mathscr{H}_x^m\}$ is open. A stronger result will be proved in Chapter IV, § 3.

5. Meromorphic functions. Let $U(\neq \varnothing)$ be an open set in C^m, $m \geqslant 2$. We consider sets $\mathscr{T}$ such that

(i) each element of $\mathscr{T}$ is a triple (V, f, g), where V is a connected open subset of U, and f, g are holomorphic functions on V with $g \not\equiv 0$;

(ii) for any (V_1, f_1, g_1) and $(V_2, f_2, g_2) \in \mathscr{T}$ with $V_1 \cap V_2 \neq \varnothing$; $f_1 g_2 = f_2 g_1$ on $V_1 \cap V_2$;

(iii) $\bigcup_{(V,f,g)\in T} V = U.$

Let $\mathbf{T}(U)$ be the set of all such $\mathscr{T}$. For $\mathscr{T}, \mathscr{T}' \in \mathbf{T}(U)$, we shall write $\mathscr{T} \sim \mathscr{T}'$ if $\mathscr{T} \cup \mathscr{T}' \in \mathbf{T}(U)$, i.e. if (ii) holds for $\mathscr{T} \cup \mathscr{T}'$ ((i) and (iii) hold trivially). In particular $\mathscr{T} \subset \mathscr{T}'$ implies $\mathscr{T} \sim \mathscr{T}'$. $\sim$ is an equivalence relation on $\mathbf{T}(U)$. In fact the symmetry and reflexivity of $\sim$ are evident. To prove the transitivity, let $\mathscr{T}, \mathscr{T}', \mathscr{T}'' \in \mathbf{T}(U)$ and $\mathscr{T} \sim \mathscr{T}'$, $\mathscr{T} \sim \mathscr{T}''$. For any $(V', f', g') \in \mathscr{T}'$ and $(V'', f'', g'') \in \mathscr{T}''$ with $V' \cap V'' \neq \varnothing$, we must show that $f'g'' = f''g'$ on $V' \cap V''$. But on account of (iii) it is sufficient to check this on all the non-empty $V' \cap V'' \cap V$, $(V, f, g) \in \mathscr{T}$. Since $\mathscr{T} \sim \mathscr{T}'$, and $\mathscr{T} \sim \mathscr{T}''$, we have on such $V' \cap V'' \cap V$,

$fg' = f'g$ and $fg'' = f''g$. Hence $f'g\,g'' = fg'\,g'' = f''\,g\,g'$. Since g cannot vanish identically on any connected component of $V' \cap V'' \cap V$, we have $f'\,g'' = f''\,g'$ on $V' \cap V'' \cap V$; this proves the transitivity of $\sim$. Now, a *meromorphic function on* U is, by definition, an equivalence class in the set $\mathbf{T}(U)$ with respect to this equivalence relation.

DEFINITION 3. *A meromorphic function F on U is determinate at $x \in U$ if there exist $\mathcal{T} \in \mathbf{T}(U)$ defining F, and $(V, f, g) \in \mathcal{T}$, such that $x \in V$, and f, g do not both vanish at x. In this case, the value $F(x)$ of F at x is ∞ if $g(x) = 0$, and $f(x)/g(x)$ if $g(x) \neq 0$ (it is easily verified that $F(x)$ then depends only on F and x). F is indeterminate at $x \in U$ if it is not determinate at x.*

The set of points in U at which F is determinate is an open subset of U, and $x \to F(x)$ a continuous mapping from that subset into the compactified complex plane.

REMARKS. (1) Any holomorphic function f on U defines, in a natural (and one-one) way, a meromorphic function on U (also denoted by f) whose value at any $x \in U$ is $f(x)$ — for instance by means of the element $\{(U_\alpha, f_\alpha, 1)_{\alpha \in I}\} \in \mathbf{T}(U)$, where the U_α, $\alpha \in I$ are the connected components of U, and f_α is the restriction of f to U_α.

(2) Let U_0 be an open subset of U. Then if F is any meromorphic function on U, the restriction $F_0 = F \mid U_0$ of F to U_0 is the meromorphic function on U_0 defined as follows. Let $\mathcal{T} \in \mathbf{T}(U)$ define F on U. Then an element $\mathcal{T}_0 \in \mathbf{T}(U_0)$ is defined by requiring that $(V_0, f_0, g_0) \in \mathcal{T}_0$ if and only if there exists $(V, f, g) \in \mathcal{T}$ such that $V \supset V_0$, and f_0, g_0 are the restrictions of f, g to V_0. F_0 is the meromorphic function defined by $\mathcal{T}_0$ on U_0. F_0 depends only on F. The verifications are easy. F_0 is determinate at $x \in U_0$ if and only if F is, and then $F_0(x) = F(x)$: in fact, if another element $\mathcal{T}'_0 \in \mathbf{T}(U_0)$ defines F_0, then $\mathcal{T} \cup \mathcal{T}'_0 \in \mathbf{T}(U)$, and defines F. For example, if U_0 is the set of points of U at which F is determinate and finite, $F \mid U_0$ is holomorphic in U_0.

(3) Suppose U is connected, and let $\mathcal{F} \in \mathbf{T}(U)$ contain a triple (V_0, f_0, g_0) with $f_0 \equiv 0$. Then for every $(V, f, g) \in \mathcal{T}$, $f \equiv 0$. In fact

suppose first that $V \cap V_0 \neq \varnothing$. Then on $V \cap V_0$, $fg_0 = f_0 g \equiv 0$. Since $g_0 \not\equiv 0$, g_0 cannot vanish identically on any connected component of $V \cap V_0$. Hence it follows that $f \equiv 0$. Now, since U is connected, we can find, for any $(V, f, g) \in \mathcal{T}$, finitely many $(V_1, f_1, g_1), \dots, (V_n, f_n, g_n) \in \mathcal{T}$ such that $V_0 \cap V_1, \dots, V_{n-1} \cap V_n$, $V_n \cap V$ are all non-empty. Applying the above argument successively we conclude that $f \equiv 0$.

Thus, if a meromorphic function F on a connected open set U in C^m has the value 0 at every point of a non-empty open subset of U, then $F \equiv 0$ on U.

(4) The sum and product of two meromorphic functions on U are easily obtained as follows: if $\mathcal{T}_1 = \{(V_1, f_1, g_1)\}$ and $\mathcal{T}_2 = \{(V_2, f_2, g_2)\}$ define F_1 and F_2 respectively, define $\mathcal{T} \in \mathbf{T}(U)$ by requiring that $(V, f, g) \in \mathcal{T}$ if and only if there exist $(V_1, f_1, g_1) \in \mathcal{T}_1$ and $(V_2, f_2, g_2) \in \mathcal{T}_2$ such that $V \subset V_1 \cap V_2$ and f, g are the restrictions of $f_1 g_2 + f_2 g_1$ (resp. $f_1 f_2$), $g_1 g_2$ to V; then $F_1 + F_2$ (resp. $F_1 F_2$) is the meromorphic function on U defined by $\mathcal{T}$, a function which depends only on F_1 and F_2. If all functions F_1, F_2, $F_1 + F_2$ (resp. $F_1 F_2$) are determinate at a point $x \in U$, then $F_1(x) + F_2(x) = (F_1 + F_2)(x)$ (resp. $F_1(x) F_2(x) = (F_1 F_2)(x)$).

(5) Let F be a meromorphic function on U, not vanishing identically on any connected component of U, and let $\mathcal{T}$ be any element of $\mathbf{T}(U)$ defining F. By Remark (3), for every $(V, f, g) \in \mathcal{T}$, $f \not\equiv 0$ on V. Hence $\mathcal{T}' = \{(V, g, f) \mid (V, f, g) \in \mathcal{T}\} \in \mathbf{T}(U)$, and it is easily verified that the meromorphic function F' which $\mathcal{T}'$ defines on U, depends only on F. Clearly $FF' \equiv 1$; we write $F' = 1/F$. F and F' are determinate at the same points of U.

Theorem 5. *Any meromorphic function F on U can be defined by a $\mathcal{T}_0 \in \mathbf{T}(U)$ with the property: for any $(V_0, \phi, \psi) \in \mathcal{T}_0$, ϕ and ψ have relatively prime germs at every point of V_0. Further, if a $\mathcal{T}_0$ defining F has this property, then, for any $(V_0, \phi, \psi) \in \mathcal{T}_0$ and any $x \in V_0$, $\phi(x) = \psi(x) = 0$ implies that F is indeterminate at x.*

Proof. We may assume that U is connected. Also if $F \equiv 0$, we may take for $\mathcal{T}_0$ the single triple $(U, 0, 1)$. Hence let $F \not\equiv 0$. Let

$\mathscr{T}$ be any element of $\mathbf{T}(U)$ defining F. Given any $x \in U$, there is a $(V, f, g) \in \mathscr{T}$ such that $x \in V$. Since $F \not\equiv 0$, we have $f \not\equiv 0$ on V (Remark (3) above), and we know that $g \not\equiv 0$ on V. Hence $\mathbf{f}_x, \mathbf{g}_x \neq \mathbf{0}_x$ in $\mathscr{H}_x^m$. Let $\mathbf{d}_x$ be the greatest common divisor of $\mathbf{f}_x$ and $\mathbf{g}_x$ in $\mathscr{H}_x^m$, and let $\boldsymbol{\varphi}_x = \mathbf{f}_x/\mathbf{d}_x$, $\boldsymbol{\psi}_x = \mathbf{g}_x/\mathbf{d}_x$. Then $\boldsymbol{\varphi}_x$ and $\boldsymbol{\psi}_x$ are relatively prime elements of $\mathscr{H}_x^m$. By Theorem 4, there exists a connected open neighbourhood $V_x \subset V$ of x and holomorphic functions $\phi_x, \psi_x \not\equiv 0$ on V_x which induce the germs $\boldsymbol{\varphi}_x$, $\boldsymbol{\psi}_x$ at x and have relatively prime germs at every point of V_x. We then have also $f\psi_x = g\phi_x$: V_x being connected this is implied by $\mathbf{f}_x\boldsymbol{\psi}_x = \mathbf{g}_x\boldsymbol{\varphi}_x$. This means that the single triple (V_x, ϕ_x, ψ_x) defines on V_x the meromorphic function $F \mid V_x =$ restriction of F to V_x. It follows that the set $\mathscr{T}_0 = \{(V_x, \phi_x, \psi_x) \mid x \in U\} \in \mathbf{T}(U)$, and defines F. $\mathscr{T}_0$ has the desired property by construction.

Now let $\mathscr{T}_0 \in \mathbf{T}(U)$, define F, and have the property mentioned in the theorem. For a $(V_0, \phi, \psi) \in \mathscr{T}_0$ and an $x \in V_0$, suppose $\phi(x) = \psi(x) = 0$. We must show that F is indeterminate at x, i.e., for any $\mathscr{T}$ defining F and any $(V, f, g) \in \mathscr{T}$ such that $x \in V$, $f(x) = g(x) = 0$. Now, we have $f\psi = \phi g$ on $V \cap V_0$; in particular, $\mathbf{f}_x\boldsymbol{\psi}_x = \boldsymbol{\varphi}_x\mathbf{g}_x$. Since $\boldsymbol{\varphi}_x$ and $\boldsymbol{\psi}_x$ are relatively prime in $\mathscr{H}_x^m$, this means that $\boldsymbol{\varphi}_x$ divides $\mathbf{f}_x$ and $\boldsymbol{\psi}_x$ divides $\mathbf{g}_x$. But $\phi(x) = \psi(x) = 0$, hence $f(x) = g(x) = 0$, q.e.d.

6. Analytic sets and germs of analytic sets.

DEFINITION 4. *An analytic set (resp. a principal analytic set) in an open set U in C^m is a subset S of U with the following property : for every $a \in U$, there exists an open (resp. open connected) neighbourhood $V \subset U$ of a and finitely many holomorphic functions $f_1, \dots, f_r$ on V (resp. a holomorphic function $f \not\equiv 0$ on V) such that $S \cap V = \{x \in V \mid f_1(x) = \dots = f_r(x) = 0\}$ (resp. $S \cap V = \{x \in V \mid f(x) = 0\}$).*

Obviously, an analytic set in U is closed in U.

Let S and T be analytic sets in U. Let V, W $(\subset U)$ be open neighbourhoods of $a \in U$ and $f_1, \dots, f_r$ (resp. $g_1, \dots, g_s$) holomorphic functions on V(resp. W) such that $S \cap V = \{x \in V \mid f_1(x) = \dots = f_r(x) = 0\}$ and $T \subset W = \{x \in W \mid g_1(x) = \dots = g_s(x) = 0\}$. Then

$S \cap T \cap (V \cap W) = \{x \in V \cap W \mid f_1(x) = \ldots = f_r(x) = g_1(x) = \ldots = g_s(x) = 0\}$ and $(S \cup T) \cap (V \cap W) = \{x \in V \cap W \mid f_i(x) g_j(x) = 0, 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s\}$. Hence a finite intersection of analytic sets in U is an analytic set in U and a finite union of (principal) analytic sets in U is a (principal) analytic set in U. (An arbitrary intersection of analytic sets in U is also an analytic set in U : this will be proved in Chapter IV, §1.)

DEFINITION 5. *An analytic set S in U is reducible in U if it is the union of two analytic sets in U, both distinct from S. It is irreducible if it is not reducible.*

In particular, an irreducible analytic set in U is contained in a single connected component of U.

By Theorem 5, the set S_0 of points of U at which a meromorphic function on U is indeterminate is an analytic set in U ; so is the union of S_0 with the set of points of U where F is determinate and assumes a given value.

Let a be a given point of C^m; let S, S' be analytic sets in open neighbourhoods V, V' of a respectively. We shall write $(V, S) \sim (V', S')$ if there exists an open neighbourhood $W \subset V \cap V'$ such that $S \cap W = S' \cap W$. Clearly $\sim$ is an equivalence relation in the set of all (V, S) where V is an open neighbourhood of a and S an analytic set in V. A *germ of an analytic set at a* is, by definition an equivalance class of this relation.

If S is an analytic set in an open neighbourhood V of $a \in C^m$ we shall denote the germ of analytic set which (V, S) defines at a by $\mathbf{S}_a$(or $\mathbf{S}$ if no confusion is possible). If T is another analytic set in an open neighbourhood W of a, then the germ of analytic set at a defined by the analytic set $(S \cup T) \cap (V \cap W)$ (resp. $(S \cap T) \cap (V \cap W)$) in $V \cap W$ depends only on $\mathbf{S}_a$ and $\mathbf{T}_a$; it is called the *union* (resp. *intersection*) of $\mathbf{S}_a$ and $\mathbf{T}_a$, and is denoted by $\mathbf{S}_a \cup \mathbf{T}_a$ (resp. $\mathbf{S}_a \cap \mathbf{T}_a$). Similarly if $\mathbf{S}_a$ and $\mathbf{T}_a$ are germs of analytic sets at $a \in C^m$, we shall write $\mathbf{S}_a \subset \mathbf{T}_a$, if there exist analytic sets S and T in some open neighbourhood V of a, which induce the germs $\mathbf{S}_a$ and $\mathbf{T}_a$ at a, such that $S \subset T$. Evidently, $\mathbf{S}_a \subset \mathbf{T}_a$ and $\mathbf{T}_a \supset \mathbf{S}_a$ imply $\mathbf{S}_a = \mathbf{T}_a$.

EXAMPLES. $\mathbf{C}^m$ and $\mathbf{a}$ will denote the germs at a induced by the analytic sets C^m and $\{a\}$; $\boldsymbol{\varnothing}$ will denote the empty germ, induced by the empty analytic set.

DEFINITION 6. *A germ of analytic set* $\mathbf{S}$ *at* $a \in C^m$ *is reducible if it is the union of two germs of analytic sets at a both distinct from* $\mathbf{S}$; *it is irreducible if it is not reducible.*

EXAMPLE. The germ induced by C^m itself at any point of it is irreducible. From this it follows that every affine sub-space L of C^m (L is obviously an analytic set in C^m) induces an irreducible germ at every point of C^m.

7. Germs of principal analytic sets. Every element of $\mathscr{H}_a^m - \{\mathbf{0}\}$ defines a germ of principal analytic set at a in the following way. Let f be a holomorphic function on an open neighbourhood V of a, $\mathbf{f}_a \neq \mathbf{0}$, and S the set of zeros of f in V. Obviously, the germ of analytic set $\mathbf{S}_a$ which S induces at a depends only on $\mathbf{f}_a$. We call $\mathbf{S}_a$ the *germ of principal analytic set* at a defined by $\mathbf{f}_a \in \mathscr{H}_a^m$ (or by the principal ideal generated by $\mathbf{f}_a$ in $\mathscr{H}_a^m$).

THEOREM 6. *Let* $\mathbf{S}$ *be the germ of principal analytic set at* $a \in C^m$ *defined by* $\mathbf{f} \in \mathscr{H}_a^m$, $\mathbf{f} \neq \mathbf{0}$. *Let* $\mathbf{T}$ *be the germ of principal analytic set at a defined by* $\mathbf{g} \in \mathscr{H}_a^m$, $\mathbf{g} \neq \mathbf{0}$, *and suppose* $\mathbf{S} \subset \mathbf{T}$. *Then every irreducible factor of* $\mathbf{f}$ *divides* $\mathbf{g}$.

PROOF. We may assume that $a = \mathbf{0}$ and $m \geqslant 2$. Again if $\mathbf{f}$ is invertible, there is nothing to prove. Hence let $\mathbf{f}$ be non-invertible. Since $\mathbf{f} \neq \mathbf{0}$, we may after a suitable choice of a basis for C^m, assume that $\mathbf{f} \sim \boldsymbol{\varphi}$ where $\boldsymbol{\varphi} \in \mathscr{H}^{m-1}[x_m]$ is distinguished. The irreducible factors of $\mathbf{f}$ in $\mathscr{H}^m$ are (upto invertible factors in $\mathscr{H}^m$) precisely the irreducible factors of $\boldsymbol{\varphi}$ in $\mathscr{H}^{m-1}[x_m]$. Hence if $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$ are all the distinct irreducible distinguished polynomials dividing $\boldsymbol{\varphi}$ in $\mathscr{H}^{m-1}[x_m]$, it is sufficient to show that $\boldsymbol{\psi} = \boldsymbol{\varphi}_1 \ldots \boldsymbol{\varphi}_n$ divides $\mathbf{g}$. Now $\boldsymbol{\psi}$, as the product of distinguished polynomials, is itself distinguished, and being the product of inequivalent irreducible polynomials, it is not divisible by the square of any non-invertible element of

$\mathscr{H}^{m-1}[x_m]$. Since $\mathscr{H}^{m-1}$ is factorial, we must have $\delta(\boldsymbol{\psi}) \neq \mathbf{0}_{0'}$ where $\delta(\boldsymbol{\psi})$ is the discriminant of $\boldsymbol{\psi}$(§ 3, (e)).

Now, there exists an open polydisc $P = \{(x', x_m) \in C^m \mid x' \in P', |x_m| < \rho\}$ in C^m with centre 0, and a distinguished pseudo-polynomial $\psi(x', x_m) = x_m^p + \sum_{k=1}^{p} a_k(x')\, x_m^{p-k}$, such that: (i) the a_k are all holomorphic functions on P', and $\boldsymbol{\psi}_0 = \boldsymbol{\psi}$, and (ii) for every $x' \in P'$, $\psi(x', t) = 0$ implies $|t| < \rho$, i.e., $(x', t) \in P$. Further ρ, and the radii of P', may be chosen as small as we please. Hence we may assume that (iii) $\mathbf{f}, \mathbf{g}$ are defined by holomorphic functions f, g on P, (iv) $x \in P$, $f(x) = 0$ imply $g(x) = 0$, (v) $\boldsymbol{\psi}_x$ divides $\mathbf{f}_x$ for every $x \in P$, (in particular $\psi(x) = 0$ implies $f(x) = 0$).

If P is chosen sufficiently small, after Theorem 2, there exists a holomorphic function h on P such that, on P, $r = g - h\psi$ can be written in the form $\sum_{k=0}^{p-1} r_k(x')\, x_m^{p-k}$, where the r_k are holomorphic functions on P'. We shall prove that $r \equiv 0$ on P', and this will prove the theorem.

$\psi(x', t)$ is a monic polynomial in t over the ring $\mathscr{H}(P')$ of holomorphic functions on P'. Consider the discriminant $D = \delta(\psi)$ of ψ. D does not vanish identically on P', since $\mathbf{D}_{0'} = \delta(\boldsymbol{\psi}) \neq \mathbf{0}$; hence $W' = \{x' \in P' \mid D(x') \neq 0\}$ is non-empty. We claim that, for every $x' \in W'$, all the coefficients $r_k(x')$ of the polynomial $r(x', t)$ in t vanish. In fact, for any given $x' \in W'$, $\psi(x', t)$, as a polynomial in t, has p distinct roots, and by assumption all these roots lie in $|t| < \rho$. All these roots of $\psi(x', t)$ are also zeros of $g(x', t)$, since they are zeros of $f(x', t)$. Thus for each $x' \in W'$, the polynomial $r(x', t)$ in t, of degree $p-1$, has p distinct roots ; hence all its coefficients must vanish. This proves our assertion. Finally, the r_k are holomorphic functions on the connected set P', and vanish identically on the non-empty open subset W' of P', hence they must vanish identically on P', q.e.d.

Corollary 1. *Let the meromorphic function F on the open set U in $C^m (m \geqslant 2)$ be indeterminate at $a \in U$. Then for every $\lambda \in C$ (or for $\lambda = \infty$), there exist points of U, arbitrarily near a, at which F is determinate and has the value λ.*

PROOF. We may assume that U is connected. Since F is indeterminate at $a \in U$, $F \not\equiv 0$ on U, and $F' = 1/F$ is well-defined. F' is also indeterminate at a, and at a point $x \in U$, $F(x)$ is defined and equals ∞ if and only if $F'(x)$ is defined and equals 0. Hence it is sufficient to consider $\lambda \in C$.

Let $\mathscr{T}$ be an element of $\mathbf{T}(U)$, defining F, which has the property mentioned in Theorem 5. Let (V, f, g) be any element of $\mathscr{T}$ such that $a \in V$. Consider the holomorphic function $h = f - \lambda g$ on V. h does not vanish identically on $V : h \equiv 0$ implies $F \equiv \lambda$ on V, while F is indeterminate at $a \in V$. Also, $h(a) = 0$, since $g(a) = f(a) = 0$. Hence, by the Weierstrass preparation theorem, h has zeros arbitrarily close to a. Now for any $x \in V$, $h(x) = 0$, $g(x) \neq 0$ imply $F(x)$ (is defined and) $= \lambda$. Hence to prove the theorem, it is sufficient to show that the germ of analytic set at a defined by $\mathbf{h}_a$ is not contained in that defined by $\mathbf{g}_a$. If this were not so, then, by Theorem 6, every irreducible factor of $\mathbf{h}_a$ divides $\mathbf{g}_a$ in $\mathscr{H}_a^m$, hence it also divides $\mathbf{h}_a + \lambda\,\mathbf{g}_a = \mathbf{f}_a$. But $\mathbf{h}_a$ is non-invertible in $\mathscr{H}_a^m$ since $h(a) = 0$, and $\mathbf{f}_a$ and $\mathbf{g}_a$ are relatively prime in $\mathscr{H}_a^m$ by assumption. This contradiction proves the corollary.

COROLLARY 2. *The germ of principal analytic set* $\mathbf{S}$ *at* a *defined by* $\mathbf{f} \in \mathscr{H}_a^m$, $\mathbf{f} \neq \mathbf{0}$ *is reducible if and only if* $\mathbf{f}$ *has at least two inequivalent non-invertible irreducible factors in* $\mathscr{H}_a^m$.

PROOF. Let $\mathbf{f} \sim \mathbf{f}_1^{k_1} \dots \mathbf{f}_n^{k_n}$, where the $\mathbf{f}_j$ are mutually inequivalent, non-invertible and irreducible, and the k_j are integers $\geqslant 1$. We shall first suppose $n \geqslant 2$, and show that $\mathbf{S}$ is reducible. Let $\mathbf{S}'$ and $\mathbf{S}''$ be respectively the germs of analytic sets at a defined by $\mathbf{f}_1$ and $\mathbf{f}_2 \dots \mathbf{f}_n$. Clearly $\mathbf{S} = \mathbf{S}' \cup \mathbf{S}''$. Also, $\mathbf{S}', \mathbf{S}'' \neq \mathbf{S}$: $\mathbf{S} = \mathbf{S}'$ for instance implies, by Theorem 6, that $\mathbf{f}_1 \dots \mathbf{f}_n$ divides $\mathbf{f}_1$, which is impossible.

Suppose now that $\mathbf{f} \sim \mathbf{f}_1^{k_1}$, i.e., $n = 1$. We must show that $\mathbf{S}$ is irreducible. Suppose on the contrary that $\mathbf{S} = \mathbf{S}' \cup \mathbf{S}''$, $\mathbf{S}', \mathbf{S}'' \neq \mathbf{S}$. Let S', S'' be analytic sets in an open neighbourhood V of a such that $\mathbf{S}'_a = \mathbf{S}'$, $\mathbf{S}''_a = \mathbf{S}''$. We may assume that S' (resp. S'') is the set of common zeros of finitely many holomorphic functions $g_1, \dots, g_r$ (resp. $h_1, \dots, h_s$) on V. Since $\mathbf{S}', \mathbf{S}'' \neq \mathbf{S}$, there exist g_{i_0}, h_{j_0} such that

$\mathbf{f}_1$ does not divide the germs of either g_{i_0} or h_{j_0} at a ($1 \leqslant i_0 \leqslant r$, $i < j_0 \leqslant s$). Now $\mathbf{S}' \cup \mathbf{S}''$ is the germ at a induced by the set of common zeros of the $g_i h_j$ in V, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Hence $\mathbf{S}' \cup \mathbf{S}'' = \mathbf{S}$ implies, by Theorem 6, that $\mathbf{f}_1$ divides the germs of all the $g_i h_j$ at a, in particular of $g_{i_0} h_{j_0}$. But $\mathbf{f}_1$ does not divide the germs of either g_{i_0} or h_{j_0} at a. Since $\mathbf{f}_1$ is irreducible, this is a contradiction and the corollary is proved.

Corollary 3. *Any germ of a principal analytic set is a finite union of irreducible germs of principal analytic sets, none of which is contained in the union of the remaining.*

Proof. Let the germ of principal analytic set $\mathbf{S}$ at $a \in C^m$ be defined by $\mathbf{f} \in \mathscr{H}_a^m$, $\mathbf{f} \neq \mathbf{0}$. If $\mathbf{f}$ is invertible, then $\mathbf{S}$ is already irreducible. Otherwise let $\mathbf{f} \sim \mathbf{f}_1^{k_1} \dots \mathbf{f}_n^{k_n}$, where the $\mathbf{f}_j$ are non-invertible, irreducible, and mutually inequivalent and the k_j are integers $\geqslant 1$. Let $\mathbf{S}_j$ be the germ of analytic set at a defined by $\mathbf{f}_j$, $j = 1, \dots, n$. By Corollary 2, each $\mathbf{S}_j$ is irreducible, and clearly $\bigcup_j \mathbf{S}_j = \mathbf{S}$. Finally if $n \geqslant 1$, no $\mathbf{S}_j$ is contained in the union of the remaining. For otherwise, by Theorem 6, $\mathbf{f}_j$ must divide $\mathbf{f}_1 \dots \mathbf{f}_{j-1} \mathbf{f}_{j+1} \dots \mathbf{f}_n$, and this is impossible since the $\mathbf{f}_j$ are irreducible and mutually inequivalent.

8. $\mathscr{H}_a^m$ is a Noetherian ring. Some consequences. We shall now prove that $\mathscr{H}^m (= \mathscr{H}_0^m)$ is Noetherian. We shall in fact prove the following stronger result.

Theorem 7. *Given an ideal I in $\mathscr{H}^m$, we can find :* (i) *finitely many holomorphic functions $f_0, f_1, \dots, f_q$ on an open neighbourhood U of $\mathfrak{o}$, whose germs at $\mathfrak{o}$ belong to I;* (ii) *(a basis for C^m and) a sequence $\{P_n, n = 1, 2, \dots\}$ of open polydiscs with centre $\mathfrak{o}$ and radii decreasing to zero ($\bar{P}_1 \subset U$); and* (iii) *real numbers $\delta_n > 0$, $n = 1, 2, \dots$, with the following property : for any function g holomorphic on a P_n whose germ at $\mathfrak{o}$ lies in I, there exist holomorphic functions $h_0, \dots, h_q$ on P_n such that $g = \sum_{j=0}^{q} h_j f_j$ on P_n and*

$$\|h_j\|_{P_n} (= \sup_{x \in P_n} |h_j(x)|) \leqslant \delta_n \|g\|_{P_n}, \; j = 0, \dots, q.$$

In order to carry out the proof by induction on m, we need the

SUPPLEMENT TO THEOREM 7. *Given finitely many ideals in $\mathscr{H}^m$, we may, in Theorem 7, choose the same sequence $\{P_n\}$ of polydiscs to serve for all the given ideals.*

PROOF. We first consider the case $m = 1$. For an $\mathbf{f} \in \mathscr{H}^1$, $\mathbf{f} \neq \mathbf{0}$, we shall say that $\mathbf{f}$ has order p if its Taylor expansion at $\mathfrak{o}$ is of the form $\sum_{k \geqslant p} a_k x^k$, $a_p \neq 0$. Now suppose given finitely many ideals $I^{(1)}, \ldots, I^{(s)}$ in $\mathscr{H}^1$. Obviously, we may suppose that none of the $I^{(\sigma)}$ is $\{\mathbf{0}\}$ or $\mathscr{H}^1$. Let $\mathbf{f}_\sigma$ be a non-zero element of minimal order in $I^{(\sigma)}$, $\sigma = 1, \ldots, s$. For a sufficiently small $\rho > 0$, there exist holomorphic functions $f_1, \ldots, f_s$ on $U = \{|x| < \rho\}$, inducing respectively the germs $\mathbf{f}_1, \ldots, \mathbf{f}_s$ at the origin, and vanishing nowhere in $0 < |x| < \rho$. Let $\{\rho_n\}$ be any sequence of real numbers $< \rho$, decreasing to zero. Let P_n be the open disc $\{|x| < \rho_n\}$ and $\delta_n = \sup_{1 \leqslant \sigma \leqslant s} \{1/ \inf_{|x| = \rho_n} |f_\sigma(x)|\}$. Suppose now that g is a function holomorphic on a P_n, with $\mathbf{g}$(= germ induced by g at the origin) $\in I^{(\sigma)}$, $\mathbf{g} \neq \mathbf{0}$. Then the order of $\mathbf{g}$ is at least that of $\mathbf{f}_\sigma$. Since further $f_\sigma(x) \neq 0$ if $0 < |x| < \rho$, it follows that $h = g/f_\sigma$ is holomorphic on P_n. Further by the maximum principle,

$$\|h\|_{P_n} = \limsup_{x \in P_n, |x| \to \rho_n} |h(x)| \leqslant \delta_n \|g\|_{P_n}.$$

Hence Theorem 7, with the supplementary condition, is proved in the case $m = 1$.

Now let $m \geqslant 2$, and suppose Theorem 7, with the supplementary condition, has been proved for $\mathscr{H}^{m-1}$. Let finitely many ideals $I^{(1)}, \ldots, I^{(s)}$ in $\mathscr{H}^m$ be given. We may again suppose that none of the $I^{(\sigma)}$ is $\{\mathbf{0}\}$ or $\mathscr{H}^m$. Let $\mathbf{f}_0^{(\sigma)}$ be any non-zero element of $I^{(\sigma)}$, $\sigma = 1, \ldots, s$. By Remark (B) at the end of Theorem 2, we may choose a basis for C^m, open polydiscs P_n and real numbers $\alpha_n > 0$, $n = 1, 2, \ldots$, which satisfy the requirements of Theorem 2 for all the $\mathbf{f}_0^{(\sigma)}$. In the rest of the proof, all our considerations apply to any of the $I^{(\sigma)}$, hence we omit the superscripts σ.

Let p be the degree of the unique distinguished element of $\mathscr{H}^{m-1}[x_m]$ to which $\mathbf{f}_0$ is equivalent. By Theorem 2, we have, for any $\mathbf{g} \in \mathscr{H}^m$, a unique $\mathbf{h} \in \mathscr{H}^m$ such that $\mathbf{g} - \mathbf{h}\mathbf{f}_0$ is an element of $\mathscr{H}^{m-1}[x_m]$ of degree $< p$. For $1 \leqslant k \leqslant p$, let I'_k be the set of the coefficients of x_m^{p-k} in all the $\mathbf{g} - \mathbf{h}\mathbf{f}_0 \in \mathscr{H}^{m-1}[x_m]$ of degree $\leqslant p - k$, as $\mathbf{g}$ runs through I. Using the uniqueness assertion of Theorem 2 (and the fact that I is an ideal in $\mathscr{H}^m$), it is easily verified that the I'_k are ideals in $\mathscr{H}^{m-1}$.

By the induction hypothesis, we can find (a basis for C^{m-1} and) a sequence of open polydiscs P'_n in C^{m-1} with centre $\mathfrak{o}'$ and radii decreasing to zero, and a sequence of real numbers $\delta'_n > 0$, which satisfy the requirements of Theorem 7 for all the ideals I'_k, $1 \leqslant k \leqslant p$ (indeed, with the obvious notation, for all the ideals $I_k^{(\sigma)\prime}$, $1 \leqslant k \leqslant p_\sigma$, $1 \leqslant \sigma \leqslant s$). Let $\{f'_{q_{k-1}+1}, \ldots, f'_{q_k}\}$ be a finite set of holomorphic functions on a neighbourhood of $\overline{P}'_1$ (each f'_j inducing at $\mathfrak{o}$ a germ $\mathbf{f}'_j \neq \mathbf{0}$) which has the property of Theorem 7 with respect to the ideal I'_k, $1 \leqslant k \leqslant p$, $q_0 = 0$. By definition of the I'_k, each $\mathbf{f}'_j$ is, for $q_{k-1} < j \leqslant q_k$, the leading coefficient of an element $\mathbf{f}_j = \mathbf{g}^{(j)} - \mathbf{h}^{(j)}\mathbf{f}_0 \in \mathscr{H}^{m-1}[x_m]$ of degree $p - k$, where $\mathbf{g}^{(j)}$ (and $\mathbf{f}_0$) $\in I$, so we have $\mathbf{f}_j \in I$. By discarding finitely many P'_n if necessary, we may suppose that all the $\mathbf{f}_j$, $j = 1, \ldots, q (= q_p)$ are the germs induced at $\mathfrak{o}$ by pseudo-polynomials f_j in x_m whose coefficients are holomorphic functions of x' on $\overline{P}'_1$. Referring again to the remark (B) at the end of Theorem 2, we may also suppose, by passing to a subsequence of $\{P'_n\}$ if necessary, that for every n, the projection of P_n on C^{m-1} is precisely P'_n. We shall show now that, for a suitable choice of the δ_n, the f_j, $j = 0, 1, \ldots, q$, and the P_n satisfy Theorem 7 for I.

Suppose g is a holomorphic function on P_n with $\mathbf{g} \in I$. By Theorem 2, there exists a holomorphic function h_0 on P_n such that $\|h_0\|_{P_n} \leqslant \alpha_n \|g\|_{P_n}$ and $g_1 = g - h_0 f_0$ has the form $g_1(x', x_m) = \sum_{k=1}^{p} g'_{1k}(x')\, x_m^{p-k}$, where the g'_{1k} are holomorphic functions on P'_n. By the definition of I'_1, $\mathbf{g}'_{11}$ (= the germ induced by g'_{11} at $\mathfrak{o}'$) $\in I'_1$. Hence, by the induction hypothesis, there exist holomorphic functions $h_1, \ldots, h_{q_1}$ on P'_n such that $g'_{11} = \sum_{j=1}^{q_1} h_j f'_j$ on P'_n and $\|h_j\|_{P'_n} \leqslant \delta'_n \|g_{11}\|_{P'_n}$, $j = 1, \ldots, q_1$. Now $g_2 = g_1 - \sum_{j=1}^{q_1} h_j f_j$ has the form $g_2(x', x_m) = \sum_{k=2}^{p} g'_{2k}(x') x_m^{p-k}$ where the g'_{2k} are holomorphic on P'_n; and $\mathbf{g}_2 \in I$ since $\mathbf{g}_1 \in I$ and $\mathbf{f}_j \in I, j = 1, \ldots, q_1$. Hence $\mathbf{g}_{22} \in I'_2$. Repeating the argument used above, we obtain at the end of p steps the relation $g_1 = \sum_{j=1}^{q} h_j f_j$, where the h_j are holomorphic functions on P'_n and $\|h_j\|_{P'_n} \leqslant \delta'_n \|g'_{kk}\|_{P'_n}$ for $q_{k-1} < j \leqslant q_k$; here, for $k > 1$,

$$g_k(x, x_m) = \sum_{l=k}^{p} g'_{kl}(x') x_m^{p-l},$$

and

$$g_{k+1} = g_k - \sum_{j=q_{k-1}+1}^{q_k} h_j f_j$$

Using the Cauchy integral formula, we obtain, from the first of these relations,

$$\| g'_{kk} \|_{P'_n} \leqslant \frac{1}{\rho_n^{p-k}} \| g_k \|_{P_n} \leqslant \max \left\{ 1, \frac{1}{\rho_n^{p-1}} \right\} \| g_k \|_{P_n} = \beta_n \| g_k \|_{P_n}$$

say, where ρ_n is the mth radius of P_n. Hence it remains only to obtain majorisations of the type $\|g_k\|_{P_n} \leqslant \gamma_n \|g\|_{P_n}$, $k > 1$. Suppose $\|f_j\|_{P_1} \leqslant M$, $j = 0, \ldots, q$. We have then

$$\|g_1\|_{P_n} = \|g - h_0 f_0\|_{P_n} \leqslant (1 + \alpha_n M) \|g\|_{P_n}$$

and for $k \geqslant 1$,

$$\|g_{k+1}\|_{P_n} = \left\| g_k - \sum_{=q_{k-1}+1}^{q_k} h_j f_j \right\|_{P_n}$$

$$\leqslant (1 + \beta_n q \delta'_n M) \|g_k\|_{P_n}$$

Hence we are inductively led to the majorisations of the desired kind : $\|g_k\|_{P_n} \leqslant \gamma_n \|g\|_{P_n}$, the γ_n depending only on n (and not on $\mathbf{g} \in I$). Since $g = \sum_{j=0}^{q} h_j f_j$, we are through if we choose $\delta_n = \max \{\delta'_n \beta_n \gamma_n, \alpha_n\}$.

Corollary 1. *Any ideal* $\dot{I}$ *in* $\mathscr{H}^m$ *is closed, in the following sense: if* $\{g_n\}$ *is a sequence of holomorphic functions on an open neighbourhood* U *of* $\mathfrak{o}$, *converging uniformly on* U *to the limit* g, *and if* $\mathbf{g}_n \in I$ *for every* n, *then* $\mathbf{g} \in I$.

Proof. Let the notation be that of Theorem 7. Let $P = P_{n_0}$ be any polydisc of the sequence obtained in Theorem 7 such that $P \subset U$. Since $\{g_n\}$ converges uniformly on U, we may assume, by passing to a subsequence of $\{g_n\}$, that $\| g_{n+1} - g_n \|_P \leqslant 1/2^n$, $n = 1, 2, \ldots$. Then, for each n, $\mathbf{g}_{n+1} - \mathbf{g}_n \in I$, hence we have, by Theorem 7, holomorphic functions $h_n^{(j)} (j = 0, \ldots, q)$ on P, such that $g_{n+1} - g_n = \sum_{j=0}^{q} h_n^{(j)} f_j$, and $\| h_n^{(j)} \|_P \leqslant \delta_{n_0} \| g_{n+1} - g_n \|_P \leqslant \delta_{n_0}/2^n$. Hence each of the series $\sum_{n=1}^{\infty} h_n^{(j)}$ converges normally in P, therefore its sum $h^{(j)}$ is a holomorphic function on P. Since $g - g_1 = \sum_{n=1}^{\infty} (g_{n+1} - g_n) = \sum_{j=0}^{q} h^{(j)} f_j$, we have $\mathbf{g} - \mathbf{g}_1 \in I$, hence $\mathbf{g} \in I$, q.e.d.

Corollary 2. *Any ideal in* $\mathscr{H}^m$ *defines a germ of analytic set at* $\mathfrak{o}$.

Proof. Let I be any ideal in $\mathscr{H}^m$. Since Theorem 7 implies in particular that $\mathscr{H}^m$ is Noetherian, I is generated by finitely many of its non-zero elements, say $\mathbf{f}_1, \ldots, \mathbf{f}_r$. Let $\mathbf{S}_f$ be the germ of principal analytic set at $\mathfrak{o}$ defined by $\mathbf{f} \in \mathscr{H}^m$. Then we claim that the germ $\mathbf{S}_I = \mathbf{S}_{f_1} \cap \ldots \cap \mathbf{S}_{f_r}$ at $\mathfrak{o}$ does not depend on the set of generators chosen for I. In fact, if $\mathbf{g}_1, \ldots, \mathbf{g}_s \in \mathscr{H}^m$ generate the ideal $J \subset I$, we have relations of the type $\mathbf{g}_j = \sum_{i=1}^{r} \mathbf{h}_{j_i} \mathbf{f}_i (j = 1, \ldots, s)$, showing that $\mathbf{S}_I \subset \mathbf{S}_J = \mathbf{S}_{g_1} \cap \ldots \cap \mathbf{S}_{g_s}$. We have thus proved that the germ of analytic set $\mathbf{S}_I$ at $\mathfrak{o}$, defined as above by the ideal I, has the property:

Proposition 1(a). *If* I, J *are ideals in* $\mathscr{H}^m$ *and* $J \subset I$, *then* $\mathbf{S}_I \subset \mathbf{S}_J$.

In particular, I determines $\mathbf{S}_I$ uniquely.

Proposition 1(b). *If* I, J *are ideals in* $\mathscr{H}^m$, *then* $\mathbf{S}_{I \cap J} = \mathbf{S}_I \cup \mathbf{S}_J$.

Proof. By Proposition 1(a), $\mathbf{S}_{I \cap J} \supset \mathbf{S}_I \cup \mathbf{S}_J$. Now let the $\mathbf{f}_i$ generate I and the $\mathbf{g}_j$ generate J. Then $\mathbf{S}_I = \bigcap_i \mathbf{S}_{f_i}$ and $\mathbf{S}_J = \bigcap \mathbf{S}_{g_j}$.

Hence $\mathbf{S}_I \cup \mathbf{S}_J = \bigcap_{i,j} \mathbf{S}_{f_i g_j}$. But each $\mathbf{f}_i\, \mathbf{g}_j \in I \cap J$, hence $\mathbf{S}_{I \cap J} \subset \mathbf{S}_{f_i g_j}$. The proposition follows.

REMARK. The converse of Proposition 1 is false. For instance, suppose $\mathbf{f}$ is a non-invertible element of $\mathscr{H}^m$ different from $\mathbf{0}$. Then all the powers of $\mathbf{f}$ generate distinct principal ideals in $\mathscr{H}^m$, but all these ideals define the same germ of analytic set at $\mathfrak{o}$. As will be proved in Chapter IV (§ 1), $\mathbf{S}_I \subset \mathbf{S}_J$ if and only if rad $J \subset$ rad I.

COROLLARY 3. *Given any family $\mathscr{G}$ of holomorphic functions on an open neighbourhood U of $\mathfrak{o}$ in C^m, we can find an open neighbourhood $V \subset U$ of $\mathfrak{o}$, and finitely many members $g_1, \ldots, g_r$ of $\mathscr{G}$, with the property: for every $g \in \mathscr{G}$, there exist holomorphic functions $h_1, \ldots, h_r$ on V such that, on V, $g = \sum_{i=1}^{r} h_i\, g_i$.*

PROOF. Let I be the ideal of $\mathscr{H}^m$ generated by the $\mathbf{g}$, $g \in \mathscr{G}$. Let the notation be that of Theorem 7. Since the $\mathbf{f}_j \in I$, we have finitely many members $g_1, \ldots, g_r$ *of* $\mathscr{G}$, such that relations of the type $\mathbf{f}_j = \sum_{i=1}^{r} \mathbf{h}_{j_i}\, \mathbf{g}_i$, $\mathbf{h}_{j_i} \in \mathscr{H}^m$, hold ($j = 0, \ldots, q$). Hence, if n is sufficiently large, we have holomorphic functions h_{j_i} on $P_n (\subset U)$ such that $f_j = \sum_{i=1}^{r} h_{j_i}\, g_i (j = 0, \ldots, q)$. Clearly Corollary 3 is valid with any such P_n as V.

With the notation of Corollary 3, we have that, *on* V, the set of common zeros of all members of $\mathscr{G}$ is equal to the set of common zeros of finitely many members of $\mathscr{G}$. Hence the following definition of analytic sets is equivalent to the one given earlier (Definition 4 of § 6).

DEFINITION 7. *An analytic set in an open set U in C^m is a subset S of U with the following property: for every $a \in U$, there exist an open neighbourhood $V \subset U$ of a and a family $\{f_i\}$ of holomorphic functions on V such that $V \cap S = \{x \in V \mid f_i(x) = 0$ for every $i\}$.*

DEFINITION 8. *An element $\mathbf{f} \in \mathscr{H}^m$ vanishes on the germ of analytic set $\mathbf{S}$ at $\mathfrak{o} \in C^m$ if $\mathbf{S} \subset \mathbf{S}_f$, where $\mathbf{S}_f$ is the germ of principal analytic set at $\mathfrak{o}$ defined by $\mathbf{f}$ ($\mathbf{S}_f = \mathbf{C}^m$ if $\mathbf{f} = \mathbf{0}$).*

It is clear that the set of all $\mathbf{f} \in \mathscr{H}^m$ vanishing on a given germ of analytic set $\mathbf{S}$ at $\mathfrak{o}$ is an ideal in $\mathscr{H}^m$.

Definition 9. *The ideal associated to the germ of analytic set* $\mathbf{S}$ *at* $\mathfrak{o} \in C^m$ *is the ideal in* $\mathscr{H}^m$ *formed by all the elements of* $\mathscr{H}^m$ *vanishing on* $\mathbf{S}$; *it is denoted by* $I(\mathbf{S})$.

For example, $I(\mathbf{C}^m) = \{\mathbf{0}\}$, $I(\{\mathfrak{o}\}) = \mathscr{H}'^m, I(\varnothing) = \mathscr{H}^m$.

As an immediate consequence of the definition of the associated ideal, we have

Proposition 2. *Let* $\mathbf{S}, \mathbf{T}$ *be germs of analytic sets at* $\mathfrak{o} \in C^m$. *Then*: (a) $\mathbf{S} \subset \mathbf{T}$ *if and only if* $I(\mathbf{T}) \subset I(\mathbf{S})$ (*hence* $\mathbf{S} = \mathbf{T}$ *if and only if* $I(\mathbf{S}) = I(\mathbf{T})$; (b) $I(\mathbf{S} \cup \mathbf{T}) = I(\mathbf{S}) \cap I(\mathbf{T})$.

Proposition 3. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be germs of analytic sets at* $\mathfrak{o} \in C^m$. *Then*: (a) $\mathbf{S}$ *is irreducible if and only if* $I(\mathbf{S})$ *is prime*; (b) *the germ of analytic set at* $\mathfrak{o}$ *defined by* $I(\mathbf{S})$ (*as in Corollary* 2 *to Theorem* 7) *is precisely* $\mathbf{S}$, *and any ideal in* $\mathscr{H}^m$ *defining* $\mathbf{S}$ *is contained in* $I(\mathbf{S})$.

Proof. (a) Suppose $I(\mathbf{S})$ is not prime. Then there exist $\mathbf{f}, \mathbf{g} \in \mathscr{H}^m$, $\mathbf{f}, \mathbf{g} \notin I(\mathbf{S})$, such that $\mathbf{f}\,\mathbf{g} \in I(\mathbf{S})$. Let $\mathbf{S}' = \mathbf{S} \cap \mathbf{S}_f$, $\mathbf{S}'' = \mathbf{S} \cap \mathbf{S}_g$. Since $\mathbf{f}, \mathbf{g} \notin I(\mathbf{S})$, $\mathbf{S}'$ and $\mathbf{S}''$ are both distinct from $\mathbf{S}$. Further $\mathbf{S}' \cup \mathbf{S}'' = \mathbf{S} \cap (\mathbf{S}_f \cup \mathbf{S}_g) = \mathbf{S} \cap \mathbf{S}_{fg} = \mathbf{S}$, since $\mathbf{f}\,\mathbf{g} \in I(\mathbf{S})$. Thus, if $I(\mathbf{S})$ is not prime, $\mathbf{S}$ is reducible.

Conversely, suppose $\mathbf{S}$ is reducible: $\mathbf{S} = \mathbf{S}' \cup \mathbf{S}''$, where $\mathbf{S}'$ and $\mathbf{S}''$ are germs of analytic sets at $\mathfrak{o}$ both distinct from $\mathbf{S}$. There exist $\mathbf{f}_1, \ldots, \mathbf{f}_r$ (resp. $\mathbf{g}_1, \ldots, \mathbf{g}_s$) $\in \mathscr{H}^m$ such that $\mathbf{S}' = \bigcap_{i=1}^{r} \mathbf{S}_{f_i}$ (resp. $\mathbf{S}'' = \bigcap_{j=1}^{s} \mathbf{S}_{g_j}$). Since $\mathbf{S}' \subsetneq \mathbf{S}$, there exists an i_0, $1 \leqslant i_0 \leqslant r$, such that $\mathbf{f}_{i_0} \notin I(\mathbf{S})$; similarly there exists a j_0, $1 \leqslant j_0 \leqslant s$, such that $\mathbf{f}_{j_0} \notin I(\mathbf{S})$. Now $\mathbf{S} = \mathbf{S}' \cup \mathbf{S}'' = \bigcap_{i,j} \mathbf{S}_{f_i g_j}$, hence in particular $\mathbf{f}_{i_0} \mathbf{g}_{j_0} \in I(\mathbf{S})$. Thus $I(\mathbf{S})$ is not prime, and (a) is proved.

(b) Let $\mathbf{g}_1, \ldots, \mathbf{g}_p \in I(\mathbf{S})$ generate $I(\mathbf{S})$. The germ of analytic set at $\mathfrak{o}$ defined by $I(\mathbf{S})$ is $\mathbf{T} = \mathbf{S}_{g_1} \cap \ldots \cap \mathbf{S}_{g_p}$. Clearly $\mathbf{S} \subset \mathbf{T}$. But we also have $I(\mathbf{S}) \subset I(\mathbf{T})$, since $\mathbf{g}_1, \ldots, \mathbf{g}_p$ vanish on $\mathbf{T}$ by definition of $\mathbf{T}$, and generate $I(\mathbf{S})$. Hence (by Proposition 2 (a)) $\mathbf{T} \subset \mathbf{S}$, i.e., $\mathbf{S} = \mathbf{T}$. Thus $I(\mathbf{S})$ defines $\mathbf{S}$. Finally it is clear from the definitions that any ideal defining $\mathbf{S}$ is contained in $I(\mathbf{S})$.

We can now deduce another corollary to Theorem 7 (we use only the fact that $\mathscr{H}^m$ is Noetherian).

COROLLARY 4. *Every germ of analytic set (at a point of C^m) is a finite union of irreducible germs; these irreducible germs are uniquely determined by the given germ if we require that none of them be contained in the union of the remaining.*

PROOF. Let $\mathbf{S}$ be a germ of analytic set, at $\mathfrak{o} \in C^m$ say. We first show that $\mathbf{S}$ is a finite union of irreducible germs of analytic sets at $\mathfrak{o}$. Suppose this is false. Then $\mathbf{S}$ is in particular reducible, say $\mathbf{S} = \mathbf{S}_1 \cup \mathbf{S}_2$, where $\mathbf{S}_1, \mathbf{S}_2$ are germs of analytic sets at $\mathfrak{o}$ distinct from $\mathbf{S}$. At least one of $\mathbf{S}_1, \mathbf{S}_2$ is not a finite union of irreducible germs, say $\mathbf{S}_1$. Repeating the argument used with $\mathbf{S}$, we obtain a germ $\mathbf{S}_3 \underset{\neq}{\subset} \mathbf{S}_1$, which is again not a finite union of irreducible germs. If we go on repeating the construction, we obtain an infinite descending sequence of germs of analytic sets at $\mathfrak{o}$: $\mathbf{S} \underset{\neq}{\supset} \mathbf{S}_1 \underset{\neq}{\supset} \mathbf{S}_3 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} \mathbf{S}_{2n+1} \underset{\neq}{\supset} \ldots$. By Proposition 2(a), we have for the sequence of associated ideals, $I(\mathbf{S}) \underset{\neq}{\subset} I(\mathbf{S}_1) \underset{\neq}{\subset} I(\mathbf{S}_3) \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} I(\mathbf{S}_{2n+1}) \ldots$. But this contradicts the fact that $\mathscr{H}^m$ is Noetherian (§2 (d)). Hence $\mathbf{S}$ is a finite union of irreducible germs $\mathbf{S}_i$, $1 \leqslant i \leqslant p$. By discarding some of the $\mathbf{S}_i$ if necessary, we may suppose that none of the $\mathbf{S}_i$ is contained in the union of the remaining.

Now suppose $\mathbf{S}$ is also the union of the irreducible germs $\mathbf{S}'_j$, $j = 1, \ldots, q$, where no $\mathbf{S}'_j$ is contained in the union of the remaining. We shall show that each $\mathbf{S}'_j$ is one of the $\mathbf{S}_i$. In fact, we have

$$\mathbf{S}'_j = \mathbf{S}'_j \cap \mathbf{S} = \mathbf{S}'_j \cap \left(\bigcup_{i=1}^{p} \mathbf{S}_i\right) = \bigcup_{i=1}^{p} (\mathbf{S}'_j \cap \mathbf{S}_i).$$

Since $\mathbf{S}'_j$ is irreducible, we must have, for at least one i, $1 \leqslant i \leqslant p$ $\mathbf{S}'_j = \mathbf{S}'_j \cap \mathbf{S}_i$, i.e., $\mathbf{S}'_j \subset \mathbf{S}_i$. Repeating the argument with $\mathbf{S}_i$, we obtain similarly $\mathbf{S}_i \subset \mathbf{S}'_{j'}$ for some j', $1 \leqslant j' \leqslant q$. Hence $\mathbf{S}'_j \subset \mathbf{S}_i \subset \mathbf{S}'_{j'}$. But $\mathbf{S}'_j \not\subset \mathbf{S}'_{j'}$ for $j' \neq j$, hence we have $\mathbf{S}'_j = \mathbf{S}_i$. Thus every $\mathbf{S}'_j$ is one of the $\mathbf{S}_i$, and similarly every $\mathbf{S}_i$ is one of the $\mathbf{S}'_j$, q.e.d.

The uniquely determined irreducible germs of which $\mathbf{S}$ is thus the union are called the *irreducible components* of $\mathbf{S}$.

Remark. If $\mathbf{S} \subset \mathbf{T}$ and $\mathbf{S}$ is irreducible, then $\mathbf{S}$ is contained in at least one irreducible component of $\mathbf{T}$.

III

ANALYTIC SETS: A LOCAL DESCRIPTION

1. Hartogs' continuity theorem and Levi's convexity theorem. Let S be an analytic set in an open set U in C^m. It is easy to see that the interior of S in U is both open and closed in U. In particular, if U is connected, then S is nowhere dense in U if (and only if) $S \neq U$.

THEOREM 1. *Let U be a connected open set in C^m, and S an analytic set in U, $S \neq U$. Then $U - S$ is connected.*

PROOF. We may assume $m \geqslant 2$, and $S \neq \varnothing$. Suppose now that $U - S$ is not connected. Let $U - S = U_0 \cup U_1$ where U_0, U_1 are disjoint non-empty open subsets of $U - S$. Then $\bar{U}_0 \cup \bar{U}_1 = \overline{U - S} = U$ (all the closures being taken in U); for, as remarked above, $U - S$ is dense in U. Since U is connected, $\bar{U}_0 \cap \bar{U}_1 \neq \varnothing$. Let $a \in \bar{U}_0 \cap \bar{U}_1$. Then $a \in S$, since $U_0 \cap \bar{U}_1 = \bar{U}_0 \cap U_1 = \varnothing$ (U_0, U_1 are open in U). Suppose now that we have proved the following assertion: every $a \in S$ has a connected open neighbourhood $V \subset U$ such that $V \cap {}^cS$ is connected. Then, for $a \in \bar{U}_0 \cap \bar{U}_1$ this leads to a contradiction: $V \cap U_0$, $V \cap U_1$ are non-empty disjoint open sets, and $V \cap {}^cS = (V \cap U_0) \cup (V \cap U_1)$. So the theorem is proved.

It remains to prove the assertion made above. Let $a \in S$ be given. Let $P \subset U$ be an open polydisc with centre a, such that $P \cap S$ is the set of common zeros in P of finitely many holomorphic functions $f_0, f_1, \ldots, f_q$ on P. S being nowhere dense in U, we may assume that none of the f_j vanishes identically on P. We assert now that $P \cap {}^cS$ is connected. In fact let x_0 be a fixed point of $P \cap {}^cS$, such that say $f_0(x_0) \neq 0$. If x is any other point of $P \cap {}^cS$, let L be the complex line in C^m joining x_0 and x. Then $L_0 = L \cap P$ is connected and $f_0 \,|L_0 \not\equiv 0$, since $f_0(x_0) \neq 0$. Hence x_0 and x can be joined in L_0 by an arc on which f_0 does not vanish except possibly at x, q.e.d.

COROLLARY. *Let U be a connected open set in C^m and f and g holomorphic functions on U. Let $f = g$ on a subset S of U with the* *property: there exists a connected open subset V of U such that $V \cap {}^cS$ is disconnected. Then $f \equiv g$ on U.*

PROOF. Let A be the set of zeros of $f-g$ in U, and suppose $A \neq U$. Then by Theorem 1, $V \cap {}^cA$ is connected; further $V \cap {}^cA$ is dense in V. Hence $V \cap {}^cS$, which contains $V \cap {}^cA$, must also be connected. This contradiction proves that $A = U$, q.e.d.

THEOREM 2. *Let U be a connected open set in C^m, and S an analytic set in U, $S \neq U$. Then:* (a) *suppose h is a holomorphic function on $U - S$ which is bounded in a neighbourhood of each point of S (more precisely, we suppose that every $a \in S$ has a neighbourhood $V \subset U$ such that $h \mid V \cap {}^cS$ is bounded); then h has a unique holomorphic extension to U;* (b) *suppose, for every $a \in S$, that $\mathbf{S}_a$ does not contain any (non-empty) germ of principal analytic set at a; then every holomorphic function on $U - S$ has a unique holomorphic extension to U. (In case* (b)*, we must necessarily have $m \geqslant 2$, if $S \neq \varnothing$.)*

PROOF. In both the cases, the uniqueness of a holomorphic extension of h to U is trivial, because U is connected. Hence it is sufficient to prove that every $a \in S$ has an open neighbourhood $P(\subset U)$ such that $h \mid P \cap {}^cS$ has a holomorphic extension to P.

Let $a \in S$ be given. We shall assume, for convenience of notation, that $a = \mathfrak{o}$, the origin of C^m. Let $V \subset U$ be an open connected neighbourhood of $\mathfrak{o}$, such that $V \cap S = \{x \in V \mid f_j(x) = 0, j = 0, 1, \dots, q\}$, the f_j being finitely many holomorphic functions on V. Since S is nowhere dense in U, we may assume that none of the f_j vanishes identically on V. Also all the f_j vanish at $\mathfrak{o}$, since $\mathfrak{o} \in S$. Hence, as in the Weierstrass preparation theorem, we can find: (i) a basis for C^m and an open polydisc $P = \{(x', x_m) \in C^m \mid x' \in P', \ |x_m| < \rho\}$ with centre $\mathfrak{o}$, whose closure $\bar{P} \subset V$, and (ii) distinguished pseudo-polynomials $\phi_0, \dots, \phi_q$ in x_m with coefficients which are holomorphic functions of x' on P', such that, for $0 \leqslant j \leqslant q, \phi_j(x', x_m) \neq 0$ if $x' \in P'$ and $|x_m| = \rho$, and $\bar{P} \cap S = \{x \in \bar{P} \mid \phi_j(x) = 0, \ j = 0, 1, \dots, q\}$. In particular if $|x_m| = \rho$ and $x' \in P'$, $(x', x_m) \in V \cap {}^cS$. Hence if γ is the positively oriented circle $|t| = \rho$ in C^1, and $x = (x', x_m) \in P$, the integral

$$h^*(x) = \frac{1}{2\pi i} \int_\gamma \frac{h(x', t)}{t - x_m} \, dt$$

is defined, and $h^*(x)$ is a holomorphic function of x on P. We shall prove that, under the assumptions of (a) or (b), $h^* = h$ on $P \cap {}^cS$, and Theorem 2 will be proved.

(a) In this case, let x' be any given point of P'. There are only finitely many points in $|t| < \rho$ such that $(x', t) \in S$, and none on $|t| = \rho$ (since, for instance, all such points are roots of the polynomial $\phi_0(x', t)$). Hence $t \to h(x', t)$ is a holomorphic function of t in $|t| \leqslant \rho$, except for finitely many points in $|t| < \rho$, and is bounded in a neighbourhood of each of the exceptional points. By the classical theorem of Riemann, $t \to h^*(x', t)$ is the unique holomorphic extension of $t \to h(x', t)$ to $\{|t| < \rho\}$. Consequently, for every $x' \in P'$, $h^*(x', t) = h(x', t)$ whenever $|t| < \rho$ and $(x', t) \notin S$, q.e.d.

(b) In this case the argument is quite different. Let $\boldsymbol{\varphi}_0 = \mathbf{P}_1 \mathbf{P}_2 \dots \mathbf{P}_n$ be the unique factorisation of $\boldsymbol{\varphi}_0$ into distinguished irreducible polynomials in $\mathscr{H}^{m-1}[x_m]$. If P' is sufficiently small, there exist distinguished pseudo-polynomials $P_1, \dots, P_n$ in x_m, whose coefficients are holomorphic functions of x' on P', such that P_i induces the germ $\mathbf{P}_i$ at $\mathfrak{o}$, $1 \leqslant i \leqslant n$. Now, since $\mathbf{S}_0$ contains no non-empty germ of principal analytic set at $\mathfrak{o}$, none of the $\mathbf{P}_i$ can divide all the $\boldsymbol{\varphi}_j$, $j = 1, \dots, q$ (in $\mathscr{H}^m$ and hence) in $\mathscr{H}^{m-1}[x_m]$. For each i, $1 \leqslant i \leqslant n$, let j_i, $1 \leqslant j_i \leqslant q$, be such that $\mathbf{P}_i$ does not divide $\boldsymbol{\varphi}_{j_i}$. $\mathbf{P}_i$ being irreducible, this is equivalent to saying that $\mathbf{P}_i$ and $\boldsymbol{\varphi}_{j_i}$ are relatively prime in $\mathscr{H}^{m-1}[x_m]$, and, since $\mathscr{H}^{m-1}$ is a factorial ring, this means that $\rho(\mathbf{P}_i, \boldsymbol{\varphi}_{j_i}) \neq \mathbf{0}$, where $\rho(\mathbf{P}_i, \boldsymbol{\varphi}_{j_i})$ is Sylvester's resultant of $\mathbf{P}_i$ and $\boldsymbol{\varphi}_{j_i}$, $1 \leqslant i \leqslant n$.

Now let R_i be the Sylvester resultant of ϕ_{j_i} and P_j (regarded as polynomials over the ring of holomorphic functions on P'). R_i is a holomorphic function on P', and $R_i \not\equiv 0$ on P' since the germ it induces at $\mathfrak{o}'$, viz. $\rho(\mathbf{P}_i, \boldsymbol{\varphi}_{j_i})$, is not $\mathbf{0}$. Hence $W' = \{x' \in P' \mid R_i(x') \neq 0,$ $i = 1, \dots, n\}$ is a non-empty open subset of P'. Let $W = \{(x', x_m) \in P \mid x' \in W'\}$. We assert that $W \cap S = \varnothing$. Suppose on the contrary that $x = (x', x_m) \in W \cap S$. Then $\phi_j(x) = 0$, $0 \leqslant j \leqslant q$. In particular, $P_i(x) = 0$ for at least one i, $1 \leqslant i \leqslant n$, and $\phi_{j_i}(x) = 0$. But this means $R_i(x') = 0$, contradicting $x' \in W'$.

It is now obvious that $h^* \equiv h$ on W : for each $x' \in W'$, $t \to h(x', t)$ is holomorphic in $|t| \leqslant \rho$, and the Cauchy integral formula applies. But, by Theorem 1, $P \cap {}^cS$ is connected, and the holomorphic functions h^* and h on $P \cap {}^cS$ are identical on the non-empty open subset W of $P \cap {}^cS$. Hence $h^* \equiv h$ on $P \cap {}^cS$, q.e.d.

REMARKS. It is easily seen, using Theorem 6 of Chapter II, that the assumption about S made in (b) above is equivalent to the following : for every $a \in S$, if $\mathbf{f}_1, \ldots, \mathbf{f}_q \in \mathscr{H}_a^m - \{0\}$ are such that $\mathbf{S}_a = \mathbf{S}_{f_1} \cap \ldots \cap \mathbf{S}_{f_q}$, then the greatest common divisor of $\mathbf{f}_1, \ldots, \mathbf{f}_q$ is $\mathbf{1}$. As a consequence we have the

COROLLARY. *Let U be an open set in C^m, and S the set of points of U at which a meromorphic function on U is indeterminate. Then every holomorphic function on $U - S$ can be extended, in a unique way, to a holomorphic function on U.*

Theorem 2(b) is an illustration (due to Hartogs himself) of the following principle, whose proof is the same as the last part of the proof of Theorem 2 (with U instead of $U - S$).

"CONTINUITY THEOREM" OF HARTOGS. *Let*

$$P = \{(x', x_m) \in C^m \mid x' \in P', |x_m| < \rho\}$$

be an open polydisc in C^m ($m \geqslant 2$) with centre $\mathfrak{o}$, and let U be an open set with the following properties :

(i) *$x' \in P'$, $|x_m| = \rho$ imply $(x', x_m) \in U$;*

(ii) *there is a non-empty open subset W' of P' such that $x' \in W'$, $|x_m| < \rho$ imply $(x', x_m) \in U$;*

(iii) *$P \cap U$ is connected.*

Then, any holomorphic function on U has a unique holomorphic extension to $P \cup U$.

Another classical illustration of the above theorem is the following.

"CONVEXITY THEOREM" OF LEVI. *Let ϕ be a real-valued and twice continuously differentiable function on an open neighbourhood*

ω of the origin $\mathfrak{o}$ in C^m ($m \geqslant 2$); using the variables x_j, $\bar{x}_j$ instead of Re x_j, Im x_j *($j = 1, \ldots, m$), set*

$$f(x) = \sum_{j=1}^{m} \frac{\partial \phi}{\partial x_j}(\mathfrak{o})\, x_j,$$

$$g(x) = \tfrac{1}{2} \sum_{j,k=1}^{m} \frac{\partial^2 \phi}{\partial x_j\, \partial x_k}(\mathfrak{o})\, x_j x_k,$$

$$h(x) = \tfrac{1}{2} \sum_{j,k=1}^{m} \frac{\partial^2 \phi}{\partial x_j\, \partial \bar{x}_k}(\mathfrak{o})\, x_j \bar{x}_k,$$

and assume that $f(x) \not\equiv 0$, $\phi(\mathfrak{o}) = 0$.

(a) *If $f(x) = 0$ and $h(x) > 0$ for at least one x, then, for every open neighbourhood $U \subset \omega$ of $\mathfrak{o}$, there exists an open neighbourhood $V \subset U$ of $\mathfrak{o}$ such that any holomorphic function on $U_0 = \{x \in U \mid \phi(x) > 0\}$ has a unique holomorphic extension to $U_0 \cup V$.*

(b) *Conversely, if such a V exists for every U, then $f(x) = 0$ and $h(x) \geqslant 0$ for at least one $x \neq \mathfrak{o}$.*

Proof. We set $x = (x', x_m)$ and $\|x\|^2 = \sum_{j=1}^{m} |x_j|^2$. Let $\dfrac{\partial \phi}{\partial x_1}(\mathfrak{o}) \neq 0$; if the variables x_j, $\bar{x}_j$ are replaced by $x'_1 = f(x) + g(x)$, $x'_j = x_j$ for $j = 2 \ldots, m$, and the $\bar{x}'_j$, $f(x)$ is replaced by x'_1 and $g(x)$ by 0; so we may assume $f(x) \equiv x_1$, $g(x) \equiv 0$, and then

$$\phi(x) = 2 \operatorname{Re} x_1 + h(x) + o(\|x\|^2) \text{ as } x \to \mathfrak{o}. \tag{1}$$

Near $\mathfrak{o}$, $\phi(x) > 0$ if and only if Re $x_1 > \psi(x)$, where $\psi(x)$ and its first derivatives vanish at $\mathfrak{o}$. Hence for any sufficiently small polydisc P about $\mathfrak{o}$, the set $\{x \in P \mid \phi(x) > 0\}$ is connected.

(a) Let $\tfrac{1}{2} \dfrac{\partial^2 \phi}{\partial x_m\, \partial \bar{x}_m}(\mathfrak{o}) = \alpha > 0$, or $h(\mathfrak{o}', x_m) = \alpha |x_m|^2$; we assume that U is a sufficiently small neighbourhood of $\mathfrak{o}$ for (1) to imply

$$\phi(x) \geqslant \phi_1(x) = 2 \operatorname{Re} x_1 + h(x) - \frac{\alpha}{3}\|x\|^2 \text{ for any } x \in U. \tag{2}$$

Let $U_1 = \{x \in U \mid \phi_1(x) > 0\}$, $S = \{x \in U \mid 2 \operatorname{Re} x_1 + h(x) - \dfrac{\alpha}{3}\|x\|^2 \leqslant 0\}$, and P be an open polydisc with centre $\mathfrak{o}$, radii r_j, such that: (i) $\bar{P} \subset U$, $P \cap U_0$, $P \cap U_1$ are connected; (ii) $\left|\frac{\partial^2 \phi}{\partial x_1 \partial \bar{x}_m}(\mathfrak{o})\right| r_m \leqslant 1$; (iii) $x' \in P'$, $|x_m| = r_m$ imply $h(x) > \frac{2\alpha}{3}\|x\|^2$; (iv) $r_1 \leqslant \frac{\alpha}{6} r_m^2$.

Then $x' \in P'$, $|x_m| = r_m$ imply

$$2\,\mathrm{Re}\, x_1 + h(x) - \frac{\alpha}{3}\|x\|^2 > -2r_1 + \frac{\alpha}{3}\|x\|^2 \geqslant -2r_1 + \frac{\alpha}{3} r_m^2 \geqslant 0,$$

i.e., $x \notin S$, which is condition (i) in Hartogs' theorem. On the other hand, if $x_1 = t > 0$, $x_2 = \dots = x_{m-1} = 0$, $|x_m| \leqslant r_m$, then

$$2\,\mathrm{Re}\, x_1 + h(x) - \frac{\alpha}{3}\|x\|^2$$

$$= \frac{2\alpha}{3}|\bar{x}_m|^2 + \left[\tfrac{1}{2}\frac{\partial^2 \phi}{\partial x_1 \partial \bar{x}_1}(\mathfrak{o}) - \frac{\alpha}{3}\right] t^2 + 2t\,\mathrm{Re}\left[1 + \tfrac{1}{2}\frac{\partial^2 \phi}{\partial x_1 \partial \bar{x}_m}(\mathfrak{o})\,\bar{x}_m\right]$$

$$\geqslant t + \left[\tfrac{1}{2}\frac{\partial^2 \phi}{\partial x_1 \partial \bar{x}_1}(\mathfrak{o}) - \frac{\alpha}{3}\right] t^2$$

by (ii); hence we can choose t, $0 < t < r_1$, such that $x_1 = t$, $x_2 = \dots = x_{m-1} = 0$, $|x_m| \leqslant r_m$ imply $x \notin S$, and therefore there exists a non-empty open subset W' of P' such that $x' \in W'$, $|x_m| \leqslant r_m$ imply $(x', x_m) \in U_1$, so that U_1 meets the requirements of Hartogs' theorem. If f is holomorphic on U_0, $f \mid U_1$ has on extension to $P \cup U_1$ and also, since $P \cap U_0$ is connected, to $P \cup U_0$.

(b) It is sufficient to show, given $\alpha > 0$, that $x_1 = 0$ and $h(x) \geqslant -\alpha\|x\|^2$ for at least one $x \neq \mathfrak{o}$. Assume the contrary: $x_1 = 0$ implies $h(x) < -\alpha\|x\|^2$ and, by (1), there exists an open neighbourhood $U \subset \omega$ of $\mathfrak{o}$ such that $x \in U$ and $x_1 = 0$ imply $\phi(x) \leqslant 0$; then $\frac{1}{x_1}$ is holomorphic on $\{x \in U \mid \phi(x) > 0\}$ and has no holomorphic extension to any neighbourhood of $\mathfrak{o}$.

2. The regular points of an analytic set.

DEFINITION 1 (a). *Let S be an analytic set in an open set U in C^m. $a \in S$ is a regular point of S (or S is regular at a) if there exist an open neighbourhood $V \subset U$ of a and a one-one biholomorphic mapping*

f of V onto an open set V' in C^m, such that $f(V \cap S) = V' \cap L$, where L is an affine subvariety of C^m with dimension k, $0 \leq k \leq m$.

REMARK 1. With the notation as above, let a be a regular point of S. Then the dimension k of L is uniquely determined by S. In fact let f_1 be a biholomorphic mapping of the open neighbourhood $V_1(\subset U)$ of a onto an open set V'_1 in C^m, mapping $V_1 \cap S$ onto $V'_1 \cap L_1$, where L_1 is a k_1-dimensional affine subvariety of C^m. Let $W = V \cap V_1$. Then the mapping $g = f_1 \circ f^{-1} : f(W) \to f_1(W)$ is one-one, surjective, and biholomorphic. Hence its Jacobian does not vanish at $f(a)$. From $g(f(W) \cap L) \subset L_1$, we easily deduce $k \leq k_1$. A similar argument with g^{-1} shows that $k_1 \leq k$, hence $k_1 = k$. Thus, in Definition 1(a), we may add:

DEFINITION 1(a′). *The dimension of S at a is the dimension of the affine variety L.*

REMARK 2. With the notation as above, let a be a regular point of S, of dimension k. If f' is the affine mapping tangent to f at the point a, $f'^{-1}(L)$ is uniquely determined by S: it is an affine variety of dimension k, containing a. We shall refer to $f'^{-1}(L)$ as the affine variety tangent to S at the point a.

REMARK 3. Let S, S' be analytic sets in open sets U, U' of C^m respectively, and let $a \in S \cap S'$. Suppose $\mathbf{S}_a = \mathbf{S}'_a$. Then clearly a is a regular point of S of dimension k if and only if a is a regular point of S' of dimension k.

DEFINITION 1(b). *A germ of an analytic set at $a \in C^m$ is a regular germ of dimension k if a is a regular point of dimension k of some analytic set which induces the given germ at a.*

By Remark 2, if $\mathbf{S}$ is a regular germ of dimension k at a, then a is a regular point of dimension k of every analytic set which induces the germ $\mathbf{S}$ at a. Further since an affine subvariety of C^m induces an irreducible germ at every point of C^m, we see easily that a regular germ is always irreducible.

EXAMPLE. If S is defined by $m - k$ equations $x_j = g_j(x_1, \dots, x_k)$, $k + 1 \leq j \leq m$, where the g_j are holomorphic on the projection of

U on the subspace of C^m generated by the first k elements of the basis, then any point of S is a regular point of S of dimension k.

In fact the formulae

$$x'_j = x_j, \; 1 \leqslant j \leqslant k,$$
$$x'_j = x_j - g_j(x_1, \ldots, x_k), \quad k+1 \leqslant j \leqslant m,$$

define a one-one biholomorphic mapping f of U onto an open set U', such that $f(S)$ is the intersection of U' and the linear variety

$$x'_{k+1} = \ldots = x'_m = 0.$$

An immediate consequence of Definition 1,(a) and (a′), is the

Proposition 1. *Let S be an analytic set in an open set U in C^m, and F a one-one biholomorphic mapping of U onto the open set $F(U)$ in C^m. Then $a \in S$ is a regular point of S of dimension k if and only if $F(a)$ is a regular point of $F(S)$ of dimension k (plainly $F(S)$ is an analytic set in $F(U)$).*

Let S be an analytic set in the open set U in C^m, and S^* the set of regular points of S.

Proposition 2. *S^* is an open, locally connected subset of S, and the dimension of S is constant on each connected component of S^* (the topologies on S and S^* being the ones induced from U).*

Proof. Let $a \in S^*$. With the notation of Definition 1(a), it is clear that the open neighbourhood $V \cap S$ of a in S consists entirely of regular points, all of the same dimension. Since $V \cap S$, being homeomorphic to $V' \cap L$, is locally connected, the proposition is proved.

Remark. We shall see (Chapter IV, § 2) that S^* is dense in S.

Proposition 3. *Let S, S^* be as above, and $b \in S^*$. Then:*

(a) *if, for a holomorphic function h on U, $\mathbf{h}_b$ vanishes on $\mathbf{S}_b$, then h vanishes at every point of the connected component of S^* which contains b;*

(b) *if, for an analytic set S' in U, $\mathbf{S'}_b \supset \mathbf{S}_b$, then the connected component of S^* which contains b is contained in S'.*

PROOF. Suppose given a holomorphic function h on U such that $\mathbf{h}_b$ vanishes on $\mathbf{S}_b$ (respectively an analytic set S' in U such that $\mathbf{S}'_b \supset \mathbf{S}_b$). Let $X = \{x \in S^* \mid \mathbf{h}_x$ vanishes on $\mathbf{S}_x$ (resp. $\mathbf{S}'_x \supset \mathbf{S}_x$)$\}$. Obviously X is an open subset of S^*. Proposition 3 will be proved if X is closed in S^*. Hence let $a \in S^*$ be in the closure of X; we have to prove that $a \in X$, and may assume the dimension k of S at $a > 0$.

(a) Let the notation be as in Definition 1(a). Let $P' \subset V'$ be an open polydisc containing $f(a)$, and let $L_0 = P' \cap L$. Then $f^{-1}(L_0)$ is an open neighbourhood of a in S^*, hence contains a point $x \in X$. Now, L_0 can be regarded as a connected open set in C^k, and $g = h \circ f^{-1}$ is a holomorphic function on L_0 such that $\mathbf{g}_{f(x)} = \mathbf{0}$. Hence $g \equiv 0$ on L_0, i.e., $h \mid f^{-1}(L_0) \equiv 0$. Hence $\mathbf{h}_a$ vanishes on $\mathbf{S}_a$, q.e.d.

(b) Let V be an open neighbourhood of a in U, such that $V \cap S'$ is the set of common zeros in V of the holomorphic functions $h^{(1)}, \ldots, h^{(q)}$ on V. S^* being a locally connected open subset of S, we may assume that $V \cap S \subset S^*$, and that $V \cap S$ is connected. $V \cap S$ must contain a point $x \in X$. This means that all the $\mathbf{h}_x^{(j)}$ vanish on $\mathbf{S}_x$, $1 \leqslant j \leqslant q$. By (a), all the $h^{(j)}$ must then vanish identically on $V \cap S$, i.e., $V \cap S \subset V \cap S'$, q.e.d.

PROPOSITION 4. *Let S be an analytic set in the open set U in C^m, s an open connected subset of S^*, S' another analytic set in U such that $s \cap S' \neq s$. Then $s \cap {}^cS'$ is connected.*

PROOF. The proposition is a consequence of Theorem 1 of § 1, and the proof begins in the same way, by assuming that $s \cap {}^cS'$ is not connected: then $s \cap {}^cS' = s_0 \cup s_1$, where s_0, s_1 are disjoint non-empty open subsets of s; $\bar{s}_0 \cup \bar{s}_1 = \overline{s \cap {}^cS'} = s$ (all the closures being taken in s), for, by Prop. 3 (b), the assumption $s \cap S' \neq s$ implies that $s \cap {}^cS'$ is dense in s. Since s is connected, $\bar{s}_0 \cap \bar{s}_1 \neq \emptyset$. Let $a \in \bar{s}_0 \cap \bar{s}_1$; the dimension of S at a is $k > 0$. Since $a \in s \subset S^*$, there exist an open set $V \subset U$, $V \ni a$, and a one-one biholomorphic mapping f of V onto an open set V' in C^m, such that $V \cap S$ is connected, $V \cap S \subset s$ and $f(V \cap S) = V' \cap L$, where L is an affine variety of dimension k. $V \cap S \cap S'$ is an analytic set in V, which is not $V \cap S$, so $f(V \cap S \cap S')$ is an analytic set in V', which is contained in

$V' \cap L$, but is not $V' \cap L$; since it may be considered as an analytic set in the open connected set $V' \cap L$ in C^k, $(V' \cap L) - f(V \cap S \cap S')$ is connected (§ 1, Theorem 1), and so is $(V \cap S) - (V \cap S \cap S') = V \cap (S \cap {}^c S')$; this connected subset of $s \cap {}^c S'$ meets s_0 and s_1, which is absurd.

REMARK. The assumption of regularity in Propositions 3 and 4 can be weakened to irreducibility: Proposition 3 (resp. 4) still holds with an open connected subset s of S such that $\mathbf{S}_x$ is irreducible for every $x \in s$, instead of a connected component of S^* (resp. an open connected subset of S^*). This will be proved in Chap. IV, § 3.

3. Local description of an analytic set: choice of basis of $\mathbf{C}^m$. Given a germ of an analytic set $\mathbf{S}$ at say the origin $\mathfrak{o}$ of C^m, we wish to describe an analytic set S in an open neighbourhood of $\mathfrak{o}$ which induces the germ $\mathbf{S}$ at $\mathfrak{o}$, and is adapted to a detailed study of $\mathbf{S}$. Our treatment of this subject is different from the ones available in the literature, which are due to H. Cartan [3] and Remmert-Stein [4].

Any germ of an analytic set being a finite union of irreducible germs, we may consider irreducible germs; their associated ideals are prime ideals in $\mathscr{H}^m_{\mathfrak{o}} = \mathscr{H}^m$ (Chap. II, § 8).

In what follows, I is always an ideal in $\mathscr{H}^m (m \geqslant 2)$; we exclude the trivial ideals $I = \{0\}$, $I = \mathscr{H}'^m$, $I = \mathscr{H}^m$ which are associated respectively to the germs $\mathbf{C}^m, \{\mathfrak{o}\}, \varnothing$. Given an ordered basis of C^m, we adopt the following notation: for $1 \leqslant r \leqslant m$,

$C^r =$ the subspace of C^m generated by the first r elements of the basis,

$\mathfrak{o}_r =$ the origin of C^r,

$\mathscr{H}^r =$ ring of germs of holomorphic functions of the first r coordinates at $\mathfrak{o}_r \in C^r$ (regarded as a subring of $\mathscr{H}^m$),

$I_r = I \cap \mathscr{H}^r$. (Thus I prime in $\mathscr{H}^m$ implies I_r prime in $\mathscr{H}^r$ for each r.)

PROPOSITION 1. *Let $1 \leqslant k \leqslant m - 1$, and let $x_1, \dots, x_k$ be k independent linear forms on C^m. Then, for any ideal I in $\mathscr{H}^m$, $I \neq \mathscr{H}^m$, the following statements are equivalent.*

A. *The ideal generated in* $\mathscr{H}^m$ *by* $\mathbf{x}_1, \dots, \mathbf{x}_k$ *and* I *defines the germ* $\{\mathfrak{o}\}$ *at* $\mathfrak{o}$; *i.e.,* $\mathbf{S}_I \cap \mathbf{S}_{x_1} \cap \dots \cap \mathbf{S}_{x_k} = \{\mathfrak{o}\}$.

A′. *For any (or one) basis of* C^m *for which* $x_1, \dots, x_k$ *are the first* k *coordinates, and any* r, $k < r \leqslant m$,

A'_r : *the ideal in* $\mathscr{H}^r$ *generated by* I_r *and* $\mathbf{x}_1, \dots, \mathbf{x}_k$ *defines the germ* $\{\mathfrak{o}_r\}$ *at* $\mathfrak{o}_r$.

A″. *For any (or one) basis of* C^m *for which* $x_1, \dots, x_k$ *are the first* k *coordinates, and any* r, $k < r \leqslant m$,

A''_r : $\mathscr{H}^{r-1}[x_r] \cap I_r$ *contains a distinguished polynomial.*

Proof. Trivially A and A'_m (for an arbitrary basis of C^m) are identical. We shall prove that A″ (for a certain basis of C^m) implies A, and that A'_r (for a certain basis of C^m for which $x_1, \dots, x_k$ are the first k coordinates, and a certain r, $k < r \leqslant m$) implies A''_r (for the same basis of C^m), and implies A'_{r-1}(for the same basis) if $r > k + 1$. This will prove the proposition.

(1) A'_r *implies* A''_r. If there exists a holomorphic function f on an open neighbourhood of $\mathfrak{o}_r$ in C^r such that $\mathbf{f} \in I_r$ while $f(\mathfrak{o}_{r-1}, x_r) \not\equiv 0$ in any neighbourhood of $x_r = 0$, the Weierstrass preparation theorem shows that A''_r is valid. Suppose no such f exists. Then every $\mathbf{f} \in I_r$ vanishes on the germ of analytic set $\mathbf{L}_r$ at $\mathfrak{o}_r$ induced by the subspace $\{x_1 = \dots = x_{r-1} = 0\}$ of C^r. Trivially $\mathbf{x}_1, \dots, \mathbf{x}_k$ vanish on $\mathbf{L}_r$. Hence the germ at $\mathfrak{o}_r$ defined by I_r and $\mathbf{x}_1, \dots, \mathbf{x}_k$ contains the germ $\mathbf{L}_r \underset{\neq}{\supset} \{\mathfrak{o}_r\}$, so that A'_r is not valid.

(2) A'_r *implies* $A'_{r-1}(r > k + 1)$. Suppose A'_r holds (for a certain basis), and let $r > k + 1$. By (1), A''_r holds. Let $\mathbf{Q}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$ be a distinguished polynomial in x_r, of degree q say. Let $\mathbf{f}_1, \dots, \mathbf{f}_n \in I_r$ generate I_r in $\mathscr{H}^r$. We can find an open polydisc $\pi_r = \pi$ in C^r with centre $\mathfrak{o}_r$, such that : (i) each $\mathbf{f}_j$ is induced at $\mathfrak{o}_r$ by a holomorphic function f_j on π, $1 \leqslant j \leqslant n$; (ii) $\mathbf{Q}_r$ is induced at $\mathfrak{o}_r$ by a distinguished pseudo-polynomial Q_r in x_r, whose coefficients are holomorphic functions of $x' = (x_1, \dots, x_{r-1})$ on the projection $\pi' = \pi_{r-1}$ of π on C^{r-1} ; (iii) $x' \in \pi'$, $Q_r(x', x_r) = 0$ imply $(x', x_r) \in \pi$; (iv) $\pi \cap \{f_1 = \dots = f_n = 0,\ x_1 = \dots = x_k = 0\} = \{\mathfrak{o}_r\}$; (v) by the Späth-Cartan

preparation theorem, there exist holomorphic functions h_j on π such that the $f_j - h_j Q_r = u_j$ are pseudo-polynomials (of degree $< q$) in x_r, with coefficients which are holomorphic functions of x' on π'.

Let $x' \neq \mathfrak{o}_{r-1}$ be any given point of π' whose first k coordinates vanish (recall that $r-1 > k$). Let $t_1, \dots, t_q$ be the roots of the polynomial $Q_r(x', t)$ in t. By (iv) above, we have for each s, $1 \leqslant s \leqslant q$, a $j_s (1 \leqslant j_s \leqslant n)$ such that $f_{j_s}(x', t) \neq 0$, hence $u_{j_s}(x', t) \neq 0$. Hence there exists a $\lambda = (\lambda_1, \dots, \lambda_n) \in C^n$, such that the polynomials $Q_r(x', t)$ and $u_\lambda(x', t) = \sum_{j=1}^{n} \lambda_j u_j(x', t)$ in t have no common root; in fact such λ's form a dense open subset of C^n. Thus if $\rho_\lambda = \rho(Q_r, u_\lambda)$ is the Sylvester resultant of Q_r and u_λ (regarded as polynomials over the ring of holomorphic functions on π'), then ρ_λ is a holomorphic function on π' for every $\lambda \in C^n$, and $\rho_\lambda(x') \neq 0$ for suitable λ.

Now, since $\mathbf{Q}_r$ and the $\mathbf{u}_j \in I_r$, we have, for every $\lambda \in C^n$, $\boldsymbol{\rho}_\lambda \in \mathscr{H}^{r-1} \cap I_r = I_{r-1}$. By Theorem 7 (Chapter II), there exist holomorphic functions $g_1, \dots, g_{n'}$ on an open neighbourhood $U' \subset \pi'$ of $\mathfrak{o}_{r-1}$ in C^{r-1} such that $\mathbf{g}_1, \dots, \mathbf{g}_{n'} \in I_{r-1}$ and generate I_{r-1}, and such that each ρ_λ is, on U', a linear combination of $g_1, \dots, g_{n'}$ with coefficients which are holomorphic functions on U'. Then any common zero in U' of all the g_i, $1 \leqslant i \leqslant n'$, is a common zero of all the ρ_λ, $\lambda \in C^n$. But we have seen already that there is no $x' \in \pi'$, $x' \neq \mathfrak{o}_{r-1}$, whose first k coordinates vanish, such that $\rho_\lambda(x') = 0$ for every $\lambda \in C^n$. Hence the $\mathbf{g}_i$, $1 \leqslant i \leqslant n'$, and the $\mathbf{x}_j$, $1 \leqslant j \leqslant k$, define the germ $\{\mathfrak{o}_{r-1}\}$ at $\mathfrak{o}_{r-1}$. Since the g_i generate I_{r-1}, A'_{r-1} is proved.

(3) A″ *implies* A. For a certain basis of C^m for which $x_1, \dots, x_k$ are the first k coordinates, let $\mathbf{Q}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$ be distinguished. Let π be an open polydisc in C^m with centre $\mathfrak{o}$, such that $\mathbf{Q}_r$ is the germ induced at $\mathfrak{o}_r$ by a distinguished pseudo-polynomial Q_r in x_r with coefficients which are holomorphic functions on the projection π_{r-1} of π on C^{r-1}, $r = k+1, \dots, m$. Let the first k coordinates of $\xi \in \pi$ be zero. Then $Q_{k+1}(\xi) = 0$ implies $\xi_{k+1} = 0$, and inductively we deduce, from $Q_{k+1}(\xi) = \dots = Q_m(\xi) = 0$, that $\xi_{k+1} = \dots = \xi_m = 0$, i.e., $\xi = 0$. Hence $\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{Q}_{k+1}, \dots, \mathbf{Q}_m$ define the germ $\{\mathfrak{o}\}$ at $\mathfrak{o}$.

A fortiori the ideal generated by I and $\mathbf{x}_1, \ldots, \mathbf{x}_k$ defines the germ $\{\mathfrak{o}\}$ at $\mathfrak{o}$, q.e.d.

PROPOSITION 2. *Given an ideal* I *in* $\mathscr{H}^m$, *such that* $I \neq \{\mathbf{0}\}$ *and* rad $I \subsetneq \mathscr{H}'^m$, *there exist* k *independent forms* $x_1, \ldots, x_k$ *on* C^m, $0 < k < m$, *such that* A *of Proposition* 1 *holds, and in addition,*

B. *For any basis of* C^m *for which* $x_1, \ldots, x_k$ *are the first* k *coordinates,* $I_k = \{\mathbf{0}\}$.

If conversely, there are k independent forms on C^m, $0 < k < m$, such that A(resp. B) holds, then $I \neq \{\mathbf{0}\}$ (resp. rad $I \subsetneq \mathscr{H}'^m$).

PROOF. We shall find a basis for C^m such that A″ and B hold when $x_1, \ldots, x_k$ are chosen to be the first k coordinates for the basis, $0 < k < m$; this will prove the proposition.

Since $\{\mathbf{0}\} \neq I \subset \mathscr{H}'^m$, there exists an $\mathbf{f} \in I, \mathbf{f} \neq \mathbf{0}$; and $\mathbf{f}$ is non-invertible. By the Weierstrass preparation theorem, we can find a basis of C^m such that $\mathbf{f} \sim \mathbf{Q}_m$, where $\mathbf{Q}_m$ is a distinguished polynomial in the last coordinate x_m of the basis. Clearly, $\mathbf{Q}_m \in I$. With respect to such a basis, we consider I_{m-1}. Plainly, $I_{m-1} \subset \mathscr{H}'^{m-1}$. We assert that rad $I_{m-1} \subsetneq \mathscr{H}'^{m-1}$. Suppose in fact that rad $I_{m-1} = \mathscr{H}'^{m-1}$. Then, if $q_m = d^0 Q_m$, we see easily that $x_m{}^{q_m} \in$ rad I, and this implies $x_m \in$ rad I. But if rad $I_{m-1} = \mathscr{H}'^{m-1}$ and $x_m \in$ rad I, then rad $I = \mathscr{H}'^m$, contrary to the hypothesis.

Hence either $I_{m-1} = \{\mathbf{0}\}$, in which case the proposition is proved with $k = m - 1$ (and $x_1, \ldots, x_{m-1}$ as the first $m-1$ coordinates for the basis), or $\{\mathbf{0}\} \neq I_{m-1}$, rad $I_{m-1} \subsetneq \mathscr{H}'^{m-1}$. In the latter case, $m - 1 \geqslant 2$. For, it is easily seen that the radical of any ideal in $\mathscr{H}^1$ is $\{\mathbf{0}\}$, $\mathscr{H}'^1$ or $\mathscr{H}^1$. Repeating the earlier argument, we can find a basis of C^{m-1} such that I_{m-1} contains a distinguished polynomial in the last coordinate x_{m-1} of this basis. This basis of C^{m-1} and the last element of the previous basis of C^m form a new basis of C^m, with respect to which x_m continues to be the last coordinate, and $\mathbf{Q}_m$ remains a distinguished polynomial in x_m.

We now consider I_{m-2} with respect to the new basis of C^m. Using rad $I_{m-1} \subsetneq \mathscr{H}'^{m-1}$ we conclude as before that rad $I_{m-2} \subsetneq \mathscr{H}'^{m-2}$. Proceeding as above, we arrive at the conclusion of the proposition at the end of $m-k$ steps, where $0 < k < m$, — for the k forms $x_1, \ldots, x_k$ of the proposition we may choose any which, together with the $m-k$ forms $x_m, \ldots, x_{k+1}$ obtained above, form a coordinate system for C^m.

DEFINITION 2. *An ordered basis of* C^m *is* k*-proper for the ideal* I $(0 < k < m)$ *if the first* k *coordinates satisfy condition* A *of Proposition* 1 *and condition* B *of Proposition* 2.

Such a basis can exist only if $\{0\} \neq I$, rad $I \subsetneq \mathscr{H}'^m$.

REMARKS. (1) We shall see (Chap. IV, § 2) that, if a basis of C^m is k-proper for an ideal I, then $k =$ the dimension of the germ $\mathbf{S}_I$ defined by I: hence k depends only on I. We shall see also that, if k independent linear forms on C^m satisfy condition A only, then $k \geqslant$ the dimension of $\mathbf{S}_I$; but, if the forms satisfy condition B only, then nothing similar can be said, even if I is prime.

(2) Let I be a *principal* ideal, $\{0\} \neq I \subset \mathscr{H}'^m$: I is generated by a single element $\mathbf{f} \in \mathscr{H}'^m$, $\mathbf{f} \neq \mathbf{0}$, which is determined up to an invertible factor $\in \mathscr{H}^m$. If $\mathscr{H}^{m-1}[x_m] \cap I$ contains a distinguished polynomial, then $\mathbf{f}$ satisfies the assumptions of the Späth-Cartan preparation theorem and, by the uniqueness assertion of this theorem, $\mathbf{f}$ cannot divide in $\mathscr{H}^m$ a non-zero element of $\mathscr{H}^{m-1}$. Therefore: (a) if a basis of C^m is k-proper for I, then $k = m-1$ (for the converse statement, see Remark 2 on Lemma 1); (b) a basis of C^m is $(m-1)$-proper for I if and only if $f(0_{m-1}, x_m) \not\equiv 0$ in any neighbourhood of $x_m = 0$.

(3) Given two ideals I, I' in $\mathscr{H}^m$ such that $\{0\} \neq I \subsetneq I' \subset \text{rad } I' \subsetneq \mathscr{H}'^m$, and an ordered basis of C^m which is k-proper for I, $0 < k < m$, then the first k coordinates satisfy condition A for I'; if they also satisfy condition B for I', the basis is k-proper for I' too; if not, according to the procedure of Proposition 2 applied

to I'_k, I'_{k-1}, ..., by altering the basis of the subspace C^k only, we can find a new basis of C^m which is still k-proper for I and also is k'-proper for I', $0 < k' < k$.

Consequently, given two ideals I, I' in $\mathscr{H}^m$ with the above properties, then there exist integers k, k', $0 < k' \leqslant k < m$, and a basis of C^m which is k-proper for I and k'-proper for I'.

From now on we shall suppose that a basis of C^m has been chosen which is k-proper for the *prime* ideal I in $\mathscr{H}^m$. We recall that $\{\mathbf{0}\} \neq I \subsetneq \mathscr{H}'^m$, $m \geqslant 2$, and $0 < k < m$.

LEMMA 1. *For* $k+1 \leqslant r \leqslant m$, *we can find* (1) $\mathbf{P}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$, $p_r = d^0\, \mathbf{P}_r \geqslant 1$, *(coefficient of* $x_r^{p_r}$ *in* $\mathbf{P}_r) \notin I_{r-1}$, (2) $\boldsymbol{\varphi}_{r-1} \in \mathscr{H}^{r-1} - I_{r-1}$, $\boldsymbol{\varphi}_k = \mathbf{1}$, *such that:*

(a) *for any* $\mathbf{U} \in \mathscr{H}^{r-1}[x_r] \cap I_r$, $d^0\, \mathbf{U} < p_r$ *implies* $\mathbf{U} \in I_{r-1}[x_r]$;

(b) *for any* $\mathbf{f} \in I_r$ *(resp.* $\mathbf{U} \in \mathscr{H}^{r-1}[x_r] \cap I_r$*),* $\boldsymbol{\varphi}_{r-1}\,\mathbf{f}$ *(resp.* $\boldsymbol{\varphi}_{r-1}\,\mathbf{U}$*) belongs to the ideal generated in* $\mathscr{H}^r$ *(resp.* $\mathscr{H}^{r-1}[x_r]$*) by* I_{r-1} *and* $\mathbf{P}_r$ *(in particular* I_{k+1} *is generated by* $\mathbf{P}_{k+1}$*);*

(c) *for any* $\mathbf{U} \in \mathscr{H}^{r-1}[x_r]$, $\mathbf{U} \in I_r$ *if and only if* $\rho(\mathbf{P}_r, \mathbf{U}) \in I_{r-1}$ *(in order to form the Sylvester resultant* $\rho(\mathbf{P}_r, \mathbf{U})$*, we must assume that* $\mathbf{U}$ *has "formal" degree* > 0*).*

PROOF. Let us fix an r, $k < r \leqslant m$, and let A be the residue class ring $\mathscr{H}^{r-1}/I_{r-1}$. Since I_{r-1} is a proper prime ideal in $\mathscr{H}^{r-1}$, A is a Noetherian integral domain (§ 2, (c) and (d), Chapter II). Let μ denote the natural epimorphism $\mathscr{H}^{r-1} \to A$, and the induced epimorphism $\mathscr{H}^{r-1}[x_r] \to A[x_r]$. Let $I^* = \{U^* = \mu(\mathbf{U}) \in A[x_r] \mid \mathbf{U} \in \mathscr{H}^{r-1}[x_r] \cap I_r\}$. Clearly I^* is an ideal in $A[x_r]$. There exist $P^*_r \in I^*$, and $\phi^*_{r-1} \in A$, $\phi^*_{r-1} \neq 0^*$, such that P^*_r divides $\phi^*_{r-1}\, U^*$ for every $U^* \in I^*$ (§ 2, (d), Chapter II). Let $\mathbf{P}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$ and $\boldsymbol{\varphi}_{r-1} \in \mathscr{H}^{r-1}$ be such that $\mu(\mathbf{P}_r) = P^*_r$ and $\mu(\boldsymbol{\varphi}_{r-1}) = \phi^*_{r-1}$, while $d^0\, \mathbf{P}_r = d^0 P^*_r$. We claim that $p_r = d^0\, \mathbf{P}_r > 0$. In fact $p_r = 0$ implies $\mathbf{P}_r \in \mathscr{H}^{r-1} \cap I_r = I_{r-1}$, i.e., $P^*_r = 0^*$, which in turn implies $I^* = \{0^*\}$. But $I_r \cap \mathscr{H}^{r-1}[x_r]$ contains a distinguished polynomial

$\mathbf{Q}_r$ and $Q^*_r = \mu(\mathbf{Q}_r) \in I^*$ is different from zero. This contradiction proves that $p_r > 0$. Further $\phi^*_{r-1} \neq 0^*$ means precisely that $\boldsymbol{\varphi}_{r-1} \notin I_{r-1}$. We shall show that $\mathbf{P}_r$ and $\boldsymbol{\varphi}_{r-1}$ satisfy (a), (b) and (c), indicating how we may choose $\boldsymbol{\varphi}_k = \mathbf{1}$.

(a) For any $\mathbf{U} \in \mathscr{H}^{r-1}[x_r] \cap I_r$, P^*_r divides $\phi^*_{r-1}U^*$, where $U^* = \mu(\mathbf{U})$, and $\phi^*_{r-1} \neq 0^*$. Hence either $U^* = 0^*$, which means precisely that $\mathbf{U} \in I_{r-1}[x_r]$, or $d^0\mathbf{U} \geqslant d^0U^* \geqslant d^0P^*_r = d^0\mathbf{P}_r$, and (a) is proved.

(b) Again let $\mathbf{U} \in \mathscr{H}^{r-1}[x_r] \cap I_r$. With the notation as before, there exists $V^* \in A[x_r]$ such that $\phi^*_{r-1}U^* = P^*_rV^*$. If $\mathbf{V} \in \mathscr{H}^{r-1}[x_r]$ is such that $\mu(V) = V^*$, then $\mu(\boldsymbol{\varphi}_{r-1}\mathbf{U} - \mathbf{P}_r\mathbf{V}) = 0^*$, i.e., $\boldsymbol{\varphi}_{r-1}\mathbf{U} - \mathbf{P}_r\mathbf{V} \in I_{r-1}[x_r]$.

In particular, let $r = k+1$. Then, for every $\mathbf{U} \in \mathscr{H}^k[x_{k+1}] \cap I_{k+1}$, $\mathbf{P}_{k+1}$ divides $\boldsymbol{\varphi}_k\mathbf{U}$ in $\mathscr{H}^k[x_{k+1}]$. But we may assume that $\mathbf{P}_{k+1}$ is a primitive polynomial (by dividing it out by the greatest common divisor of all its coefficients). Then $\mathbf{P}_{k+1}$ must divide $\mathbf{U}$ itself. Hence in the part of (b) which we have already proved, we may take $\boldsymbol{\varphi}_k = \mathbf{1}$.

Now let $\mathbf{f} \in I_r$ be arbitrary. By assumption $\mathscr{H}^{r-1}[x_r] \cap I_r$ contains a distinguished polynomial in x_r, say $\mathbf{Q}_r$. By the preparation theorem of Späth-Cartan, there exists $\mathbf{h} \in \mathscr{H}^r$ such that $\mathbf{u} = \mathbf{f} - \mathbf{h}\,\mathbf{Q}_r \in \mathscr{H}^{r-1}[x_r]$. Plainly $\mathbf{u} \in I_r$. Applying the part of (b) already proved to $\mathbf{u}$ and $\mathbf{Q}_r$, we conclude that $\boldsymbol{\varphi}_{r-1}\mathbf{f}$ belongs to the ideal generated in $\mathscr{H}^r$ by $\mathbf{P}_r$ and I_{r-1}, and (b) is proved.

(c) If $\mathbf{U} \in \mathscr{H}^{r-1}[x_r] \cap I_r$, then $\rho(\mathbf{P}_r, \mathbf{U}) \in I_r \cap \mathscr{H}^{r-1} = I_{r-1}$. Conversely, let $\rho(\mathbf{P}_r, \mathbf{U}) \in I_{r-1}$. Then with the notation as before, $\rho(P^*_r, U^*) = 0^*$. Further P^*_r has actual degree $p_r = d^0\,\mathbf{P}_r$. Hence (§ 2 (a), Chapter II) there exist polynomials X^*, $Y^* \in A[x_r]$ not both identically zero, with degrees $< d^0U^*$, p_r respectively, such that $X^*P^*_r + Y^*U^* = 0^*$. Since $P^*_r \neq 0^*, Y^* \neq 0^*$. Let $\mathbf{X}, \mathbf{Y} \in \mathscr{H}^{r-1}[x_r]$ be such that $\mu(\mathbf{X}) = X^*$, $d^0\,\mathbf{X} = d^0X^*$, $\mu(\mathbf{Y}) = Y^*$, $d^0\,\mathbf{Y} = d^0Y^*$. Then $\mu(\mathbf{XP}_r + \mathbf{YU}) = 0^*$, i.e., $\mathbf{XP}_r + \mathbf{YU} \in I_{r-1}[x_r]$. Since $\mathbf{P}_r \in I_r$, this implies that $\mathbf{Y}\mathbf{U} \in I_r$. Now $\mathbf{Y} \notin I_r$. For $d^0\,\mathbf{Y} = d^0\,Y^* < p_r$, so that by (a), $\mathbf{Y} \in I_r$ implies $\mathbf{Y} \in I_{r-1}[x_r]$, i.e., $Y^* = 0^*$, which is impossible. Finally, I_r being prime, $\mathbf{Y}\mathbf{U} \in I_r$, $\mathbf{Y} \notin I_r$ imply $\mathbf{U} \in I_r$, q.e.d.

REMARKS 1. It is useful to note that, with the notation of Lemma 1, $\dfrac{\partial \mathbf{P}_r}{\partial x_r} \notin I_r$, $r = k+1, \ldots, m$. In fact if $\dfrac{\partial \mathbf{P}_r}{\partial x_r} \in I_r$, then by (a) $\dfrac{\partial \mathbf{P}_r}{\partial x_r} \in I_{r-1}[x_r]$, i.e., $d^0 P_r{}^* = 0$, which is impossible.

2. If $k = m-1$, I is necessarily a principal ideal, generated by $\mathbf{P}_m$.

3. Each polynomial $\mathbf{P}_r$ may be multiplied by any invertible factor $\in \mathscr{H}^{r-1}$; in particular, since $\mathbf{P}_{k+1}$ divides the distinguished polynomial $\boldsymbol{\varphi}_{k+1}$ in $\mathscr{H}^k[x_{k+1}]$, $\mathbf{P}_{k+1}$ may be chosen distinguished (Chap. II, § 3, Lemma 2).

LEMMA 2. *For every r, $k+1 \leqslant r \leqslant m$, and every $\mathbf{f} \in \mathscr{H}^r - I_r$, there exists an $\mathbf{f}' \in \mathscr{H}^{r-1} - I_{r-1}$ which belongs to the ideal generated by $\mathbf{f}$ and I_r in $\mathscr{H}^r$.*

PROOF. Let $\mathbf{Q}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$ be a distinguished polynomial in x_r. By the Späth-Cartan preparation theorem, there exists an $\mathbf{h} \in \mathscr{H}^r$ such that $\mathbf{U} = \mathbf{f} - \mathbf{h}\,\mathbf{Q}_r \in \mathscr{H}^{r-1}[x_r]$. $\mathbf{f} \notin I_r$ implies that $\mathbf{U} \notin I_r$. Hence, if $\mathbf{P}_r$ is as in Lemma 1, then $\mathbf{f}' = \rho(\mathbf{P}_r, \mathbf{U}) \notin I_{r-1}$, by Lemma 1 (c). Also, it is clear that $\mathbf{f}'$ is in the ideal generated by $\mathbf{f}$ and I_r in $\mathscr{H}^r$.

PROPOSITION 3. *Let I, I' be two ideals in $\mathscr{H}^m$ such that I is prime and $\{0\} \neq I \subset I' \subsetneq \mathscr{H}'^m$, and suppose the basis of C^m is k-proper for I and k'-proper for I'. Then $I = I'$ if and only if $k = k'$.*

PROOF. Obviously, $k' \leqslant k$, and $k' < k$ implies $I \neq I'$. Now let $I \subsetneq I'$, and let $\mathbf{f} = \mathbf{f}_m \in I' - I$. By Lemma 2, there exists an $\mathbf{f}_{m-1} \in \mathscr{H}^{m-1} - I_{m-1}$, belonging to the ideal generated in $\mathscr{H}^m$ by I and $\mathbf{f}_m$. Then $\mathbf{f}_{m-1} \in \mathscr{H}^{m-1} \cap I' = I'_{m-1}$. Hence we may repeat the argument, until we obtain an $\mathbf{f}_k \in I'_k - I_k = I'_k - \{0\}$. Thus $I'_k \neq \{0\}$, i.e., $k' < k$, q.e.d.

4. Local description of an analytic set: special dense subsets. Given a *prime* ideal I in $\mathscr{H}^m$, $\{0\} \neq I \subsetneq \mathscr{H}'^m$, we choose a basis of

C^m which is k-proper for I. For $k+1 \leqslant r \leqslant m$, let $\mathbf{P}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$ and $\boldsymbol{\varphi}_{r-1} \in \mathscr{H}^{r-1} - I_{r-1} (\boldsymbol{\varphi}_k = \mathbf{1})$ be as in Lemma 1 of § 3, and let $\mathbf{Q}_r \in \mathscr{H}^{r-1}[x_r] \cap I_r$ be a distinguished polynomial in x_r. As already noted, $\dfrac{\partial \mathbf{P}_r}{\partial x_r} \notin I_r$. Hence, by Lemma 1(c), the Sylvester resultant

$\boldsymbol{\delta}_{r-1} = \rho \left(\mathbf{P}_r, \dfrac{\partial \mathbf{P}_r}{\partial x_r} \right) \notin I_{r-1}$. I_{r-1} being a prime ideal, we also have $\boldsymbol{\varphi}_{r-1}\, \boldsymbol{\delta}_{r-1} \notin I_{r-1}$. Hence, if $k \leqslant m-2$, we can, by Lemma 2 of § 3, find a $\boldsymbol{\gamma}_{m-2} \in \mathscr{H}^{m-2} - I_{m-2}$, which belongs to the ideal generated in $\mathscr{H}^{m-1}$ by $\boldsymbol{\varphi}_{m-1}\, \boldsymbol{\delta}_{m-1}$ and I_{m-1}. Further, if $k \leqslant m-3$, we can find inductively a $\boldsymbol{\gamma}_{r-1} \in \mathscr{H}^{r-1} - I_{r-1}$ belonging to the ideal generated in $\mathscr{H}^r$ by $\boldsymbol{\gamma}_r\, \boldsymbol{\delta}_r\, \boldsymbol{\varphi}_r$ and I_r, $k+1 \leqslant r \leqslant m-2$.

For $k \leqslant r \leqslant m$, let $\mathbf{f}_r^{(1)}, \ldots, \mathbf{f}_r^{(i_r)} \in I_r$ generate I_r. (I_{k+1} is generated by the single element $\mathbf{P}_{k+1}$, which for uniformity is denoted by $\mathbf{f}_{k+1}^{(1)}$, and similarly I_k is generated by the single element $\mathbf{0}$, denoted by $\mathbf{f}_k^{(1)}$). We have the following relations.

(1) $\mathbf{P}_r$ is a linear combination, over $\mathscr{H}^r$, of the $\mathbf{f}_r^{(i)}$, $1 \leqslant i \leqslant i_r$. $(k+1 \leqslant r \leqslant m)$.

(2) Each $\boldsymbol{\varphi}_{r-1}\, \mathbf{f}_r^{(i)}$, $1 \leqslant i \leqslant i_r$, (resp. $\boldsymbol{\varphi}_{r-1} \mathbf{Q}_r$) is a linear combination, over $\mathscr{H}^r$(resp. $\mathscr{H}^{r-1}[x_r]$), of $\mathbf{P}_r$ and the $\mathbf{f}_{r-1}^{(i)}$, $1 \leqslant i \leqslant i_{r-1}$. $(k+1 \leqslant r \leqslant m)$.

(3) $\boldsymbol{\gamma}_{r-1}$ is a linear combination, over $\mathscr{H}^r$, of $\boldsymbol{\gamma}_r\, \boldsymbol{\delta}_r\, \boldsymbol{\varphi}_r$ and the $\mathbf{f}_r^{(i)}$, $1 \leqslant i \leqslant i_r (k+1 \leqslant r \leqslant m-1, \boldsymbol{\gamma}_{m-1} = \mathbf{1})$.

Let π be an open polydisc in C^m with centre $\mathfrak{o}$, having the following properties:

(0′) For $k+1 \leqslant r \leqslant m$, the elements $\mathbf{f}_r^{(i)}$, $1 \leqslant i \leqslant i_r$, $\mathbf{P}_r$, $\mathbf{Q}_r$ of $\mathscr{H}^r$ are induced at $\mathfrak{o}_r$ by holomorphic functions $f_r^{(i)}$, $1 \leqslant i \leqslant i_r$, P_r, Q_r respectively on the projection π_r of π on C^r. Similarly, for $k \leqslant r \leqslant m-1$, $\boldsymbol{\varphi}_r$, $\boldsymbol{\delta}_r$ and $\boldsymbol{\gamma}_r$ are induced at $\mathfrak{o}_r$ by holomorphic functions ϕ_r, δ_r and γ_r on π_r respectively.

(1′), (2′), (3′). The relations (1), (2), (3) among the $\mathbf{f}_r^{(i)}$, $\mathbf{P}_r$, $\mathbf{Q}_r$ etc. over $\mathscr{H}^r$(or $\mathscr{H}^{r-1}[x_r]$) are "induced" by corresponding relations among the $f_r^{(i)}$, P_r, Q_r etc. over the ring $\mathscr{H}_{\pi_r}$ of holomorphic functions on π_r (or over $\mathscr{H}_{\pi_{r-1}}[x_r]$). For instance part of (2′) would read: $\phi_{r-1}Q_r$ is a linear combination, over $\mathscr{H}_{\pi_{r-1}}[x_r]$, of P_r and the $f_{r-1}^{(i)}$, $1 \leqslant i \leqslant i_{r-1}$.

(4′) For $k+1 \leqslant r \leqslant m$, and $\xi \in \pi_{r-1}$, $Q_r(\xi, t) = 0$ (for a $t \in C$) implies $(\xi, t) \in \pi_r$. (This condition can be secured since the $\mathbf{Q}_r$ are distinguished.)

Let us write $\pi_k = \pi'$. Since $\gamma_k\ \delta_k \neq 0$, $U' = \{x' \in \pi' \mid \gamma_k(x')\ \delta_k(x') \neq 0\}$ is a dense connected open subset of π'(cf. § 1). Let $U = \{(x_1, \dots, x_m) \in \pi \mid (x_1, \dots, x_k) \in U'\}$.

Finally, let S be the set of common zeros in π of all the $f_r^{(i)}$, $1 \leqslant i \leqslant i_r$, $k+1 \leqslant r \leqslant m$; thus $\mathbf{S}_I$ is the germ of analytic set at $\mathfrak{o}$ induced by S.

Lemma 1. (a) $S \cap U = \{x \in U \mid P_{k+1}(x) = \dots = P_m(x) = 0\}$. (b) *For each $x' \in U'$, there exist precisely* $p = \prod_{r=k+1}^{m} (d^0\ \mathbf{P}_r)$ *points $x^{(j)}(x')$ of $S \cap U$ which project on x'. $x^{(j)}(x') \to \mathfrak{o}$, $1 \leqslant j \leqslant p$, as $x' \to \mathfrak{o}_k = \mathfrak{o}'(x' \in U')$.* (c) *On a sufficiently small neighbourhood of any given point of U', all the mappings $x' \to x^{(j)}(x')$ are holomorphic.*

Proof. Let $s = \{x \in U \mid P_{k+1}(x) = \dots = P_m(x) = 0\}$. Condition (1′) implies $S \cap U \subset s$. Now suppose that $x' \in U'$ is given. Since $\delta_k(x') \neq 0$, the polynomial $P_{k+1}(x', t)$ (in t) has $p_{k+1} = (d^0\ \mathbf{P}_{k+1})$ distinct roots, say $x_{k+1}^{(j)}(x') = x_{k+1}^{(j)}$, $1 \leqslant j \leqslant p_{k+1}$. (2′) implies that the polynomial $Q_{k+1}(x', t)$ also vanishes at all the $x_{k+1}^{(j)}$. Hence, by (4′), $(x', x_{k+1}^{(j)}(x')) \in \pi_{k+1}$, $1 \leqslant j \leqslant p_{k+1}$. Further, since Q_{k+1} is distinguished, we see easily that $x_{k+1}^{(j)}(x') \to 0$, $1 \leqslant j \leqslant p_{k+1}$ as $x' \to \mathfrak{o}'$.

If $k+1 = m$, (a) and (b) are proved. Let $k+1 < m$, and let $\xi = (x', x_{k+1}^{(j)}(x'))$ be any one of the p_{k+1} zeros of $P_{k+1}(x', t)$, $x' \in U'$. Then first $\delta_{k+1}\ \phi_{k+1}\ \gamma_{k+1}$ does not vanish at ξ; for otherwise $\gamma_k(x') = 0$ by (3′), contradicting $x' \in U'$. In particular $\delta_{k+1}(\xi) \neq 0$, hence the polynomial $P_{k+2}(\xi, t)$ (in t) has $p_{k+2} = (d^0\ \mathbf{P}_{k+2})$ distinct roots $x_{k+2}^{(j)}(\xi)$, $1 \leqslant j \leqslant p_{k+2}$. From (2′) and $\phi_{k+1}(\xi) \neq 0$, we see that $Q_{k+2}(\xi, t)$ also vanishes at all the $x^{(j)}_{k+2}(\xi)$. It follows that $(\xi, x^{(j)}_{k+2}(\xi)) \in \pi_{k+2}$, and $x^{(j)}_{k+1}(\xi) \to 0$ as $\xi \to \mathfrak{o}_{k+1}$ $(1 \leqslant j \leqslant p_{k+2})$. If we now use (2′) (and $\phi_{k+2}(\xi) \neq 0$) a second time, we see that all the $f^{(j)}_{k+2}$, $1 \leqslant i \leqslant i_{k+2}$, vanish at all the $(\xi, x^{(j)}_{k+2}(\xi))$. If $k+2 = m$, (a) and (b) are proved. If $k+2 < m$, we can use $\gamma_{k+1}(\xi) \neq 0$, and repeat the above argument, until we obtain (a) and (b). (c) can be secured, for a suitable ordering of the $x^{(j)}(x')$, for the following reason: if $(x_1, \ldots, x_m) \in S$, then for $k+1 \leqslant r \leqslant m$, x_r is a *simple* root of the polynomial $P_r(x_1, \ldots, x_{r-1}, t)$. Thus Lemma 1 is proved.

REMARKS. If I $(\{0\} \neq I \subsetneq \mathscr{H}'^m)$ is a *principal* ideal in $\mathscr{H}^m$, then by Remark 2 on Definition 2 (§ 3) and the Weierstrass preparation theorem, we have bases of C^m which are $(m-1)$-proper for I, and for any such basis a distinguished polynomial $\mathbf{Q} \in \mathscr{H}^{m-1}[x_m]$ such that $\mathbf{S}_I = \mathbf{S}_Q$. However, the situation is different in the general case: if I is a prime ideal in $\mathscr{H}^m (m \geqslant 3, \{0\} \neq I \subsetneq \mathscr{H}'^m)$, and the basis of C^m k-proper for I, $0 < k < m-1$, we cannot always find distinguished polynomials $\mathbf{Q}_r \in I_r \cap \mathscr{H}^{r-1}[x_r]$, $k+1 \leqslant r \leqslant m$, such that $\mathbf{S}_I = \mathbf{S}_{Q_{k+1}} \cap \ldots \cap \mathbf{S}_{Q_m}$. The following is an example.

Let $\mathbf{S}$ be the germ at $\mathfrak{o} \in C^3$ induced by the analytic set $S = \{(x_1, x_2, x_3) \in C^3 \mid x_2^2 + x_3^2 - x_1 = x_2^2 - x_1 x_3 = 0\}$, and let $I = I(\mathbf{S})$. Then first, I is prime. In fact, each $x \in S$ can be written uniquely in the form $x = \left(\dfrac{t^4}{1+t^2}, \dfrac{t^3}{1+t^2}, \dfrac{t^2}{1+t^2}\right)$, where $t \in C$, $t \neq \pm i$. Hence, with the usual notation, $\mathbf{f} \in \mathscr{H}^3$ belongs to I if and only if $f\left(\dfrac{t^4}{1+t^2}, \dfrac{t^3}{1+t^2}, \dfrac{t^2}{1+t^2}\right) \equiv 0$ in a neighbourhood of $t = 0$, and it follows that I is prime. Further the basis for C^3 is 1-proper for I: $(0, x_2, x_3) \in S$ implies $x_2 = x_3 = 0$, and this is condition A of Definition 2 for I, while condition B is obvious. Suppose now that there exist distinguished polynomials $\mathbf{Q}_2 \in \mathscr{H}^1[x_2]$, $\mathbf{Q}_3 \in \mathscr{H}^2[x_3]$ such that $\mathbf{S}_I = \mathbf{S} = \mathbf{S}_{Q_2} \cap \mathbf{S}_{Q_3}$. Given $x_1, x_2 \in C$, there exists at the most one $x_3 \in C$ such that $(x_1, x_2, x_3) \in S$; then, for $(x_1, x_2, x_3) \in S$ and x_1, x_2 sufficiently near 0, x_3 is the only root of the polynomial

$Q_3(x_1, x_2, t)$ in t, i.e., is a root of order $q_3 = d^0\, \mathbf{Q}_3$: in particular

$$\frac{\partial^{q_3-1}\mathbf{Q}_3}{\partial x_3^{q_3-1}} = q_3!\left(x_3 - \sum_{r+s\geqslant 1} a_{r,s}\, x_1^r x_2^s\right) \in I.$$

This means that

$$\frac{t^2}{1+t^2} = \sum_{r+s\geqslant 1} a_{r,s}\left(\frac{t^4}{1+t^2}\right)^r\left(\frac{t^3}{1+t^2}\right)^s$$

in a neighbourhood of $t = 0$, and plainly this is impossible.

We now deduce some immediate consequences of Lemma 1.

PROPOSITION 1. *Let I be a prime ideal in $\mathscr{H}^m$, $I \subsetneq \mathscr{H}'^m$. Then* $\mathbf{S}_I \supsetneq \{\mathfrak{o}\}$.

PROOF. We may suppose $I \neq \{0\}$. Then, with the notation of Lemma 1, U' is a dense subset of π'. Let $\{x'_n\}$ be a sequence of points of U' converging to $\mathfrak{o}'$, but distinct from $\mathfrak{o}'$. Then, by Lemma 1, $x^{(j)}(x'_n)$ is a sequence of points of S converging to $\mathfrak{o}$ (for any j, $1 \leqslant j \leqslant p$). Thus $\mathfrak{o}$ is not an isolated point of S.

LEMMA 2. *Let the notation be that of Lemma 1. Let h be a holomorphic function on an open neighbourhood of $\mathfrak{o}$, such that, for every $x' \in U'$ sufficiently near $\mathfrak{o}'$, $h(x^{(j)}(x')) = 0$ for at least one $j = j(x')$, $1 \leqslant j \leqslant p$. Then $\mathbf{h} = \mathbf{h}_{\mathfrak{o}} \in I$.*

PROOF. Suppose $\mathbf{h} \in \mathscr{H}^m - I$ satisfies the hypothesis of the lemma. By Lemma 2 of §3, there exists an $\mathbf{h}_{m-1} \in \mathscr{H}^{m-1} - I_{m-1}$, which is a linear combination of $\mathbf{h}$ and the $\mathbf{f}_m^{(i)}$, $1 \leqslant i \leqslant i_m$. Clearly $\mathbf{h}_{m-1}$ also satisfies the hypothesis of the lemma. Hence, if $k < m-1$, we may proceed as above, until we finally obtain an $\mathbf{h}_k \in \mathscr{H}^k - I_k = \mathscr{H}^k - \{0\}$ which vanishes at all $x' \in U'$ sufficiently near $\mathfrak{o}'$. Since U' is dense in π', this is impossible. Hence $\mathbf{h} \in I$, and Lemma 2 is proved.

PROPOSITION 2. *Let I be a prime ideal in $\mathscr{H}^m$, $\{0\} \neq I \subsetneq \mathscr{H}'^m$ and let the basis of C^m be k-proper for I, where $1 \leqslant k \leqslant m-1$. Suppose given an open neighbourhood W of $\mathfrak{o}$, and let $\mathscr{F}$ be the set of all holomorphic functions on W vanishing at $\mathfrak{o}$. Then, there exist*: (1) *an analytic set S in an open polydisc π with centre $\mathfrak{o}$, inducing at $\mathfrak{o}$ the germ $\mathbf{S}_I$ (π and S depending only on I and the basis)*; (2) *an open* *polydisc* π_0 $(\subset \pi \cap W)$ *in* C^m *with centre* $\mathfrak{o}$ *(which can be chosen to have arbitrarily small radii)*; *and* (3) *for each* $h \in \mathscr{F}$, *a distinguished pseudo-polynomial* $R_h(x', u)$ *of degree* p *in a complex variable* u, *with coefficients which are holomorphic functions of* x' *on* π'_0 (= *projection of* π_0 *on* C^k), *such that*:

(a) *the germ* $\mathbf{R}_h(\mathbf{h})$ *at* $\mathfrak{o}$ *induced by the holomorphic function* $x \to R_h(x', h(x))$ *on* π_0 *belongs to* I;

(b) *for every* $x'_0 \in \pi'_0$, *and every* $h \in \mathscr{F}$,

$$\{u \in C \mid R_h(x'_0, u) = 0\}$$
$$= \{h(x) \mid x \in \pi_0 \cap S, \text{ and projection of } x \text{ on } C^k = x'_0\};$$

(c) *for any* $h \in \mathscr{F}$, *either* $R_h(x', u)$ *contains the factor* u, *and then* $\mathbf{h} = \mathbf{h}_0 \in I$, *or* $R_h(x', u)$ *does not contain the factor* u, *and then* $h(\pi_0 \cap S)$ *is a neighbourhood of* 0 *in* C;

(d) *for each function* h_0 *holomorphic on* π_0, *there exists a pseudo-polynomial* $X_0(x', u)$ *of degree* $\leqslant p-1$ *in* u, *with coefficients holomorphic on* π'_0 *(and vanishing at* $\mathfrak{o}'$ *if* h_0 *vanishes at* $\mathfrak{o}$*), such that the germ* $\frac{\partial}{\partial u}\mathbf{R}_h(\mathbf{h})\mathbf{h}_0 - \mathbf{X}_0(\mathbf{h})$ *at* $\mathfrak{o}$, *induced by the holomorphic function* $x \to \frac{\partial}{\partial u} R_h(x', h(x))\, h_0(x) - X_0(x', h(x))$ *on* π_0, *belongs to* I.

Proof. Let π and S be as in Lemma 1. Let $\tilde{\pi}$ be an open polydisc with centre $\mathfrak{o}$, $\tilde{\pi} \subset \pi$, such that $\bar{\tilde{\pi}} \subset W$, and (with the notation of Lemma 1), for every $x' \in \tilde{\pi}' \cap U'$, each $x^{(j)}(x') \in \tilde{\pi}$, $1 \leqslant j \leqslant p$. For any $x' \in \tilde{\pi}' \cap U'$, $h \in \mathscr{F}$, and any j, $1 \leqslant j \leqslant p$, let

$$c_j(x') = c_{j,h}(x') = (-1)^j \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant p} h(x^{(i_1)}(x')) \ldots h(x^{(i_j)}(x')).$$

One sees easily that the c_j are *bounded* holomorphic functions on $\tilde{\pi}' \cap U'$. Since $\sigma' = \pi' - U'$ is a nowhere-dense analytic set in π', each c_j has, by Theorem 2 (§ 1) a unique (bounded) holomorphic extension to $\tilde{\pi}'$, which we shall again denote by c_j. Now each c_j vanishes at $\mathfrak{o}'$. This is because $h(\mathfrak{o}) = 0$ and, for $1 \leqslant j \leqslant p$, $x^{(j)}(x') \to \mathfrak{o}$ as $x'(\in U') \to \mathfrak{o}'$. Hence $R_h(x', u) = u^p + \sum_{j=1}^{p} c_j(x')\, u^{p-j}$ is a distinguished pseudo-polynomial. Since, by definition, $R_h(x', h(x)) = 0$ for

$x \in \pi \cap (S \cap U)$, Lemma 2 shows that $\mathbf{R}_h(\mathbf{h}) \in I$, hence (a) is proved. Using Theorem 7 of Chapter II, we can now find an open polydisc $\pi_0 \subset \tilde{\pi}$ with centre $\mathfrak{o}$ and radii ρ_j, such that $R_h(x', h(x)) = 0$ for every $x \in \pi_0 \cap S$ and every $h \in \mathscr{F}$, while

$$\sup_{\substack{x' \in \pi'_0 \cap U' \\ 1 \leqslant j \leqslant p}} |x_r^{(j)}(x')| = \rho'_r < \rho_r, \; r = k+1, \dots, m \qquad (*)$$

We shall show that (b), (c), (d) hold for this π_0, $R_h(x', u)$ being as above.

(b) By (a) and by our choice of π_0, we already have, for every $x'_0 \in \pi'_0$ and every $h \in \mathscr{F}$,

$$\{h(x) \mid x \in \pi_0 \cap S, \text{ projection of } x \text{ on } C^k = x'_0\} \subset \{u \in C \mid R_h(x'_0, u) = 0\}.$$

The converse inclusion is also valid for every $x'_0 \in \pi'_0 \cap U'$, by the definition of R_h. Hence suppose $x'_0 \in \pi'_0 \cap {}^cU'$, and let u_0 be a root of $R_h(x'_0, u)$. Since U' is dense in π', we can find a sequence $\{x'_n\}$ of points of $\pi'_0 \cap U'$ converging to x'_0. Then, for each n, there is a root u_n of the polynomial $R_h(x'_n, u)$, such that $u_n \to u_0$. As already remarked, we have, for each n, a j_n, $1 \leqslant j_n \leqslant p$, such that $u_n = h(x^{(j_n)}(x'_n))$. On account of the condition (*), we may assume, by passing to a subsequence if necessary, that $\{x^{(j_n)}(x'_n)\}$ converges to a point $x_0 \in \pi_0$. Since $\pi_0 \cap S$ is closed in π_0, we have $x_0 \in S \cap \pi_0$. And $h(x_0) = \lim_{n\to\infty} h(x^{(j_n)}(x'_n)) = \lim_{n\to\infty} u_n = u_0$. Since plainly the projection of x_0 on C^k is x'_0, (b) is proved.

(c) For any $h \in \mathscr{F}$, the coefficients $c_j = c_{j,h}$ of R_h are *bounded* holomorphic functions on π_0'. Suppose first that $c_p \equiv 0$. By the definition of c_p, this means that for every $x' \in \pi'_0 \cap U'$, h vanishes at at least one of the $x^{(j)}(x')$, $1 \leqslant j \leqslant p$. By Lemma 2, we then have $\mathbf{h} \in I$.

Suppose now that $c_p \not\equiv 0$. If now (c) were false, we would have a sequence $\{\lambda_n\}$ of complex numbers tending to 0, such that h takes the value λ_n nowhere in $\pi_0 \cap S$. By (b), this means that the holomorphic function $g_n(x') = R_h(x', \lambda_n)$ of x' on π'_0 vanishes nowhere in π'_0. However, the sequence $\{g_n(x')\}$ converges *uniformly* on π'_0

to $R_h(x', 0) = c_p(x')$, since the c_j are bounded on π'_0. This contradicts $c_p \not\equiv 0$, $c_p(\mathfrak{o}') = 0$. In fact let $a' \in \pi'_0$, $c_p(a') \neq 0$. Then, on the domain $D = \{t \in C \mid ta' \in \pi'_0\}$ in C, the $g_n(ta')$ are holomorphic functions converging uniformly on D to $c_p(ta') \not\equiv 0$ ($c_p(1) \neq 0$). Hence, by a well-known argument, $\mathfrak{o}'$ is a limit of zeros $x'_n = t_n a'$ of the g_n respectively, and we have a contradiction to the assumption that the $g_n(x')$ vanish nowhere in π'_0. This proves (c).

(d) For $x' \in \pi'_0 \cap U'$,

$$X_0(x', u) = \sum_{1 \leqslant j \leqslant p} \left\{ h_0(x^{(j)}(x')) \prod_{\substack{1 \leqslant j' \leqslant p \\ j' \neq j}} [u - h(x^{(j')}(x'))] \right\} \qquad (**)$$

is a pseudo-polynomial of degree $\leqslant p-1$ in u, with coefficients which are holomorphic functions of x' on $\pi'_0 \cap U'$, bounded in a neighbourhood of each point of $\pi'_0 \cap \sigma'$; by Theorem 2 (§ 1), each of them has a unique holomorphic extension to π'_0, which vanishes at $\mathfrak{o}'$ if h_0 vanishes at $\mathfrak{o}$.

Putting $u = h\,(x^{(j)}(x'))$, $x' \in \pi'_0 \cap U'$, $1 \leqslant j \leqslant p$, in the relation (**), we get

$$X_0(x', h(x^{(j)}(x'))) = h_0(x^{(j)}(x')) \prod_{\substack{1 \leqslant j' \leqslant p \\ j' \neq j}} [h(x^{(j)}(x')) - h(x^{(j')}(x'))]$$

$$= h_0\,(x^{(j)}(x'))\,\frac{\partial}{\partial u}\, R_h(x', h(x^{(j)}(x')));$$

thus the function $\dfrac{\partial}{\partial u} R_h(x', h(x))\, h_0(x) - X_0\,(x', h(x))$ vanishes for any $x \in \pi_0 \cap (S \cap U)$, and therefore, by Lemma 2, induces at $\mathfrak{o}$ a germ belonging to I.

REMARK 1. By (b), $R_h(x', h(x)) = 0$ for any $x \in \pi_0 \cap S$, while, in (d), we have only proved $\dfrac{\partial}{\partial u} R_h(x', h(x))\, h_0(x) - X_0(x', h(x)) = 0$ for any $x \in \pi_0 \cap (S \cap U)$; but Theorem 3(B) will show that $\pi_0(S \cap U)$ is dense in $\pi_0 \cap S$ and, therefore,

$$\frac{\partial}{\partial u} R_h(x', h(x))\, h_0(x) - X_0(x', h(x)) = 0 \text{ for any } x \in \pi_0 \cap S.$$

Remark 2. Proposition 2(c) has the following consequence. Given an analytic set S in an open set U, a point $a \in S$, and a function h holomorphic on U, either the restriction $h \mid S$ is constant in some neighbourhood of a, or the set of values of $h \mid S$ in any neighbourhood of a is a neighbourhood of $h(a)$ in C. This result, in which $h \mid S$ may be replaced by a function holomorphic on S (Chap. V, § 6), generalizes the classical result: a non-constant holomorphic mapping of a connected open set in C^m into C is an open mapping.

Lemma 3. *With the notation of Lemma 1 and Proposition 2, suppose that, for at least one point $x' \in \pi'_0 \cap U'$, the p values $h(x^{(j)}(x'))$, $1 \leqslant j \leqslant p$, are all distinct; then:*

(a) *the germ $\mathbf{R}_h(u) \in \mathscr{H}^k[u]$, induced by $R_h(x', u)$ at the origin of C^{k+1}, is irreducible in $\mathscr{H}^k[u]$; the discriminant of $R_h(x', u)$, which is a holomorphic function of x' on π'_0, is $\not\equiv 0$;*

(b) *for any function $\phi(x', u)$, holomorphic on an open neighbourhood of the origin in C^{k+1}, such that $\phi(x', h(x))$ induces at $\mathfrak{o}$ a germ $\boldsymbol{\varphi}(\mathbf{h}) \in I$, $\mathbf{R}_h(u)$ divides, in $\mathscr{H}^{k+1}$, the germ $\boldsymbol{\varphi}(u) \in \mathscr{H}^{k+1}$ induced by ϕ at the origin of C^{k+1}.*

Proof. (a) Assume that $\mathbf{R}_h(u)$ is reducible in $\mathscr{H}^k[u]$: then (cf. Lemma 2 of Chap. II, §3), there exist two distinguished pseudopolynomials $R_1(x', u)$, $R_2(x', u)$ in u, of degrees p_1, $p_2 \geqslant 1$, with coefficients holomorphic on an open neighbourhood $V' \subset \pi'_0$ of $\mathfrak{o}'$, such that $R_h(x', u) \equiv R_1(x', u)\, R_2(x', u)$ for $x' \in V'$.

By the definition of R_h: for every $x' \in V' \cap U'$, $R_1(x', h(x))$ vanishes at exactly p_1 points $x^{(j)}(x')$, and $R_2(x', h(x))$ at the remaining p_2 points $x^{(j)}(x')$, $1 \leqslant j \leqslant p$. Then, by Lemma 2, each of these two functions induces at $\mathfrak{o}$ a germ belonging to I and, therefore, vanishes at each point $x^{(j)}(x')$, $1 \leqslant j \leqslant p$, for every $x' \in V' \cap U'$ sufficiently near $\mathfrak{o}'$. Hence at least two values $h(x^{(j)}(x'))$ are equal, i.e., the discriminant of R_h vanishes for every $x' \in \pi'_0 \cap U'$ sufficiently near $\mathfrak{o}'$; since this discriminant is a holomorphic function of x' on π'_0, it vanishes identically on π'_0, and at least two values $h(x^{(j)}(x'))$ are equal for *any* $x' \in \pi'_0 \cap U'$, which is contrary to the assumption.

(b) Proposition 2(b), with $h(x) = x_{k+1}, \dots, x_m$, shows that $x \in \pi_0 \cap S$ and $x' \to \mathfrak{o}'$ imply $x \to \mathfrak{o}$. Now suppose $\boldsymbol{\varphi}(\mathbf{h}) \in I$: $\phi(x', h(x)) = 0$ for any $x \in \pi_0 \cap S$ sufficiently near $\mathfrak{o}$, i.e., for any $x \in \pi_0 \cap S$ such that x' is sufficiently near $\mathfrak{o}'$, or, by Proposition 2(b), $\phi(x', u) = 0$ whenever $R_h(x', u) = 0$ and x' is sufficiently near $\mathfrak{o}'$. In other words, $\boldsymbol{\varphi}(u)$ vanishes on the germ of principal analytic set, at the origin of C^{k+1}, defined by $\mathbf{R}_h(u)$; since $\mathbf{R}_h(u)$ is irreducible in $\mathscr{H}^{k+1}$ (cf. Lemma 3 of Chap. II, §3), by Theorem 6 of Chap. II, $\mathbf{R}_h(u)$ divides $\boldsymbol{\varphi}(u)$ in $\mathscr{H}^{k+1}$.

REMARK. It is useful to note that, under the assumptions of Lemma 3, $\dfrac{\partial}{\partial u}\mathbf{R}_h(\mathbf{h}) \notin I$. In fact, $\mathbf{R}_h(u)$ cannot divide $\dfrac{\partial}{\partial u}\mathbf{R}_h(u)$ in $\mathscr{H}^{k+1}$ by the uniqueness assertion in the Späth-Cartan preparation theorem.

THEOREM 3. (Local description theorem.) *Let a prime ideal I in $\mathscr{H}^m (m \geqslant 2)$ be given, $\{0\} \neq I \subsetneq \mathscr{H}'^m$, and suppose the basis of C^m is k-proper for I. Then there exist an open polydisc π_0 with centre $\mathfrak{o}$ (which can be chosen with arbitrarily small radii), and an analytic set S_0 in π_0 inducing the germ $\mathbf{S}_I$ at $\mathfrak{o}$, with the following properties.*

(A) *The sets $S_0(x') = \{x \in S_0 \mid = \text{projection of } x \text{ on } C^k = x'\}$, $x' \in \pi'_0$ ($=$ projection of π_0 on C^k) satisfy:*

(a) *for any $x' \in \pi'_0$, $S_0(x')$ is a finite non-empty set; the maximum number of points $\in S_0(x')$ is a finite integer p (which depends only on I); $S_0(\mathfrak{o}') = \{\mathfrak{o}\}$;*

(b) *$S_0(x')$ depends continuously on x', i.e., given a sequence of points $x'_n \in \pi'_0$ converging to $x'_0 \in \pi'_0$, $x_0 \in S_0(x'_0)$ if and only if, for each n, we can choose $x_n \in S_0(x'_n)$ so that $x_n \to x_0$;*

(c) *for any dense subset E' of π'_0, $E = \bigcup_{x' \in E'} S_0(x')$ is a dense subset of S_0.*

(B) A local description of regular points of S_0. *For every linear form l on C^m, there exists a distinguished pseudo-polynomial $R_l(x', u)$, of degree p in u, with coefficients holomorphic on π'_0, such that $l(S_0(x')) = \{u \in C \mid R_l(x', u) = 0\}$ for any $x \in \pi'_0$.*

(a) *Given* $x_0 \in S_0$, *the following statements are equivalent*: (a_1) *for at least one linear form* l *on* C^m, $l(x_0)$ *is a simple root of* $R_l(x'_0, u)$ (x'_0 = *projection of* x_0 *on* C^k); (a_2) x_0 *is a regular point of* S_0 *of dimension* k, *and the affine variety tangent to* S_0 *at* x_0 *has* C^k *itself as its projection on* C^k.

(b) *There exists a principal analytic set* σ'_0 *in* π'_0 *such that, for any* $x' \in U'_0 = \pi'_0 - \sigma'_0$, $S_0(x')$ *has precisely* p *points* $x^{(j)}(x')$, $1 \leqslant j \leqslant p$, *which are regular points of* S_0 *of dimension* k; $s_0 = \bigcup_{x' \in U'_0} S_0(x')$ *is a dense subset of* S_0.

(c) *For every connected open neighbourhood* $\omega'_1 \subset \pi'_0$ *of* $\mathfrak{o}'$, *the sets* $s_1 = \{x \in s_0 \mid x' \in \omega'_1\}$ *and* $S_1 = \{x \in S_0 \mid x' \in \omega'_1\}$ *are connected.*

(C) The classical local description. *Suppose that, for at least one point* $x' \in U'_0$, *the* $(k+1)$*-th coordinates of the* p *points* $x^{(j)}(x')$ *are all distinct. Then there exist a distinguished pseudo-polynomial* $R(x', u)$, *of degree* p *in* u, *and, for each* $r = k+2, \dots, m$, *a pseudo-polynomial* $X_r(x', u)$, *of degree* $\leqslant p-1$ *in* u, *such that*:

(a) *the coefficients of* R *and the* X_r *are holomorphic on* π'_0 *and, but for the leading coefficient of* R, *vanish at* $\mathfrak{o}'$;

(b) *the germs* $\mathbf{R}(x_{k+1}) \in \mathscr{H}^k[x_{k+1}]$ *induced by* $R(x', x_{k+1})$ *at* $\mathfrak{o}_{k+1}$, *and the germs* $\frac{\partial}{\partial x_{k+1}} \mathbf{R}(x_{k+1})\, x_r - \mathbf{X}_r(x_{k+1})$ *induced at* $\mathfrak{o}$ *by the functions* $\frac{\partial}{\partial x_{k+1}} R(x', x_{k+1})\, x_r - X_r(x', x_{k+1})$, $k+2 \leqslant r \leqslant m$, *belong to* I;

(c) $\mathbf{R}(x_{k+1})$ *is irreducible in* $\mathscr{H}^k[x_{k+1}]$ *and generates, in* $\mathscr{H}^{k+1}$, *the ideal* $I_{k+1} = \mathscr{H}^{k+1} \cap I$; *the discriminant* $\delta(x')$ *of* $R(x', x_{k+1})$ *is* $\not\equiv 0$;

(d) *the dense subset* $\bigcup_{\substack{x' \in \pi'_0 \\ \delta(x') \neq 0}} S_0(x')$ *of* S_0 *is the set of points* x *satisfying the conditions*:

$$\left.\begin{aligned} & x' \in \pi'_0,\ \delta(x') \neq 0,\ R(x', x_{k+1}) = 0, \\ & \frac{\partial}{\partial x_{k+1}} R(x', x_{k+1})\, x_r - X_r(x', x_{k+1}) = 0,\ r = k+2, \dots, m. \end{aligned}\right\} \quad (1)$$

PROOF. Let π, S, U' (and $\sigma' = \pi' - U'$) be as in Lemma 1. Let $\pi_0 \subset \pi$ be an open polydisc with centre $\mathfrak{o}$ which serves in Proposition 2 for the family $\mathscr{F}$ of all linear forms on C^m, and let $S_0 = S \cap \pi_0$, $\sigma'_0 = \sigma' \cap \pi'_0$, $U'_0 = U' \cap \pi'_0$. The distinguished pseudo-polynomials $R_l(x', u)$, of degree p in u, with coefficients holomorphic on π'_0, constructed in Proposition 2, are such that

$$l(S_0(x')) = \{u \in C \mid R_l(x', u) = 0\} \text{ for every } x' \in \pi'_0 \text{ and every } l \in \mathscr{F}. \quad (\dagger)$$

(A) By condition (†), each set $S_0(x')$ has at least one and at most p distinct points; by Lemma 1, if $x' \in U'_0$, $S_0(x')$ actually has p distinct points $x^{(j)}(x')$, $1 \leqslant j \leqslant p$; but $S_0(\mathfrak{o}') = \{\mathfrak{o}\}$. Further, by condition (*) in the proof of Proposition 2, for $r = k+1, \dots, m$, each root of $R_{x_r}(x', u)$ has a modulus $\leqslant \rho'_r$ for any $x' \in U'_0$; this remains true for any $x' \in \pi'_0$, i.e., by condition (†), the r-th coordinate of each point $\in S_0(x')$ has a modulus $\leqslant \rho'_r$.

Now consider a sequence of points $x'_n \in \pi_0$ converging to $x'_0 \in \pi'_0$: if, for each n_ν in an arbitrary subsequence, we can choose $x_{n_\nu} \in S_0(x'_{n_\nu})$ so that $x_{n_\nu} \to x_0$, then $x_0 \in \pi_0$, and $x_0 \in S_0(x'_0)$ since S_0 is closed in π_0. If we denote by X_0 the set of limits of convergent subsequences x_{n_ν} such that $x_{n_\nu} \in S_0(x'_{n_\nu})$ for each ν, we have proved that $X_0 \subset S_0(x'_0)$. Then suppose $X_0 \subsetneq S_0(x'_0)$: since $S_0(x'_0)$ is finite, there exist a form $l \in \mathscr{F}$ and a value $l_0 \in l(S_0(x'_0)) - l(X_0)$; by condition (†), l_0 is a root of $R_l(x'_0, u)$, and each root of $R_l(x'_n, u)$ is the value of l at a point $\in S_0(x'_n)$; for each n, we can choose a root $l(x_n)$ of $R_l(x'_n, u)$, $x_n \in S_0(x'_n)$, so that $l(x_n) \to l_0$, and a suitable subsequence x_{n_ν} converges to a point $x_0 \in X_0$ such that $l(x_0) = l_0$.

This contradiction proves that, given $x_0 \in S_0(x'_0)$ and the sequence $\{x'_n\}$, there exist a subsequence x'_{n_ν} and, for each ν, $x_{n_\nu} \in S_0(x'_{n_\nu})$ such that $x_{n_\nu} \to x_0$; this being true for any subsequence of the given sequence $\{x'_n\}$, there exists, for each n, $x_n \in S_0(x'_n)$ such that $x_n \to x_0$.

So (b) of part (A) is proved, and it obviously implies (c).

(B) In order to prove (a), we first suppose $\frac{\partial}{\partial u} R_l(x'_0, l(x_0)) \neq 0$; we can find an open polydisc $\omega_0 \subset \pi_0$, with centre x_0, such that: (i) $x \in \omega_0$ implies $\frac{\partial}{\partial u} R_l(x', l(x)) \neq 0$; given $x' \in \omega'_0$ ($=$ projection of ω_0 on C^k), $x \in \omega_0$ and $R_l(x', l(x)) = 0$ determine $l(x)$ uniquely; (ii) any point $\in \omega'_0$ is the projection of at least one point $\in \omega_0 \cap S_0$ (this by using (b) of part (A)). By Proposition 2 (d), for $r = k + 1, \ldots,$ m, there exists a pseudo-polynomial $X_r(x', u)$ in u, with coefficients holomorphic on π'_0, such that $\frac{\partial}{\partial u} R_l(x', l(x))\, x_r = X_r(x', l(x))$ for any $x \in \bigcup_{x' \in U'_0} S_0(x')$, hence, by (c) of part (A), for any $x \in S_0$.

Then, by (i) and (ii), any point $x' \in \omega'_0$ is the projection of one and only one point $x(x') \in \omega_0 \cap S_0$; this point depends continuously on x' by (b) of part (A), then $l(x(x'))$ is a holomorphic function of x' on ω'_0, and hence the same is true for each coordinate of $x(x')$: $x_r = g_r(x')$, $r = k + 1, \ldots, m$. Since $\omega_0 \cap S_0$ is defined by the $m-k$ equations $x_r = g_r(x')$, x_0 is a regular point of $\omega_0 \cap S_0$, or S_0, of dimension k, and the affine variety tangent to S_0 at x_0 has C^k itself as its projection on C^k (cf. the example in §2).

Conversely, if these conditions are fulfilled, we can find an open polydisc $\omega_0 \subset \pi_0$, with centre x_0, such that $\omega_0 \cap S_0$ is defined by the $m - k$ equations $x_r = g_r(x')$, $r = k + 1, \ldots, m$, where each g_r is holomorphic on ω'_0. Then, for $x' \in \omega'_0 \cap U'_0$, $x^{(j)}(x') \in \omega_0 \cap S_0$ for one and only one j; the other $p - 1$ points $x^{(j)}(x')$ are in ${}^c\omega_0$ and hence, by (b) of part (A), for $x' \in \omega'_0 \cap U'_0$ and sufficiently near x'_0, they lie in a given neighbourhood of $S_0(x'_0) - \{x_0\}$. Now choose $l \in \mathscr{F}$ such that $l(x_0) \notin l(S_0(x'_0) - \{x_0\})$: for $x' \in \omega'_0 \cap U'_0$, the roots of $R_l(x', u)$ are the $l(x^{(j)}(x'))$ by the construction of R_l; therefore, as $x' \to x'_0$, one of them tends to $l(x_0)$ and the other $p - 1$ to values $\in l(S_0(x'_0) - \{x_0\})$; then $l(x_0)$ is a simple root of $R_l(x'_0, u)$.

Thus (a) is proved, and (b) too, since (a_1) is fulfilled for $x_0 = x^{(j)}(x')$. Finally, since s_0 is dense in S_0, (c) will be proved if we show that $s_1 = s_0 \cap S_1$ is connected.

Let $\tilde{s}_1$ be a connected component of s_1. Let $x'_0 \in \omega'_1 \cap U'_0$. Suppose q of the $x^{(j)}(x'_0)$ lie in s_1, $0 \leqslant q \leqslant p$: $x^{(1)}(x'_0), \ldots, x^{(q)}(x'_0)$ say. Then if $W' \subset \omega_1' \cap U_0'$ is an open *connected* neighbourhood of x'_0 such that all the p mappings $x' \to x^{(j)}(x')$ are holomorphic on W', we see easily that $x^{(j)}(W') \subset \tilde{s}_1$ for $1 \leqslant j \leqslant q$, while $x^{(j)}(W') \cap \tilde{s}_1 = \emptyset$ for $q+1 \leqslant j \leqslant p$. Thus the number $q(x')$ of points of a given connected component of s lying over $x' \in \omega' \cap U'_0 = \omega' \cap {}^c\sigma'_0$ is a locally constant function of x' on $\omega' \cap U'_0$. Since, by Theorem 1, $\omega'_1 \cap U'_0$ is connected, the same number of points of a given connected component of s_1 lie over any point of $\omega'_1 \cap U'_0$.

Let $\tilde{s}_1$ now be any connected component of s_1, and suppose $q(>0)$ points of s_1 lie over any point of $\omega'_1 \cap U'_0$. Imitating the procedure by which the $R_h(x', u)$ were constructed in Proposition 2, we obtain, for each linear form l on C^m, a distinguished pseudo-polynomial $\tilde{R}_l(x', u)$ in u of degree q, with coefficients which are holomorphic functions of x' on ω'_1. By construction of $\tilde{R}_l$, for each $x' \in \omega'_1 \cap U'_0$, the holomorphic function $x \to \tilde{R}_l(x', l(x))$ vanishes at $q\ (>0)$ of the $x^{(j)}(x')$. Hence, by Lemma 2, if $x' \in \omega'_1 \cap U'_0$ is sufficiently close to $0'$, $\tilde{R}_l(x', u)$ vanishes for $u = l(x^{(j)}(x'))$, $j=1,\ldots,p$. But we can, for any $x' \in \omega'_1 \cap U'_0$, choose l such that the $l(x^{(j)}((x'))$ are all distinct, $1 \leqslant j \leqslant p$. Hence $q =$ degree of $\tilde{R}_l$ cannot be less than p. Thus $\tilde{s}_1 = s_1$, and s_1 is connected, q.e.d.

(C) R is the pseudo-polynomial R_l corresponding to $l(x) = x_{k+1}$; the X_r are the pseudo-polynomials X_0 of Proposition 2(d) corresponding to $h_0(x) = x_{k+2}, \ldots, x_m$: thus (a) and (b) are included in Proposition 2, (c) in Lemma 3, and only (d) remains to be proved.

By condition (†), $R(x', x_{k+1}) = 0$ for any $x \in S_0$; by Remark 1 on Proposition 2, $\frac{\partial}{\partial x_{k+1}} R(x', x_{k+1})\, x_r - X_r(x', x_{k+1}) = 0, r = k+2, \ldots, m,$ for any $x \in S_0$. Thus any point $x \in \bigcup_{\substack{x' \in \pi'_0 \\ \delta(x') \neq 0}} S_0(x')$ satisfies conditions (1). Conversely, if x satisfies conditions (1), then, by (†), there exists

$\tilde{x} \in S_0(x')$ such that x and $\tilde{x}$ have the same first $k+1$ coordinates; since each function $\frac{\partial}{\partial x_{k+1}} R(x', x_{k+1})\, x_r - X_r(x', x_{k+1})$ vanishes at x and $\tilde{x}$, and $\frac{\partial}{\partial x_{k+1}} R(x', x_{k+1}) \neq 0, x = \tilde{x}$.

REMARK 1. By (a) of part (B), the set of points $x_0 \in S_0$ which do not satisfy (a_2) is the set of points $x \in S_0$ such that $\frac{\partial}{\partial u} R_l(x', l(x)) = 0$ for every $l \in \mathscr{F}$. This will allow us to show (Chap. IV, § 2) that the set of non-regular points of an analytic set is again an analytic set.

In the same field of ideas: for each $r = k+1, \ldots, m$, a point $(x', x_{k+1}, \ldots, x_r) \in C^r$ is the projection on C^r of at least one point $\in S_0$ if and only if $x' \in \pi'_0$ and $R_l(x', l(x', x_{k+1}, \ldots, x_r)) = 0$ for every linear form l on C^r. This too will be used in Chap. IV, §1.

REMARK 2. The last lines in the proof of Theorem 3 actually prove that the conditions

$$\left.\begin{aligned} & x' \in \pi'_0, \quad R(x', x_{k+1}) = 0, \quad \frac{\partial}{\partial x_{k+1}} R(x', x_{k+1}) \neq 0, \\ & \frac{\partial}{\partial x_{k+1}} R(x', x_{k+1})\, x_r - X_r(x', x_{k+1}) = 0, r = k+2, \ldots, m, \end{aligned}\right\} \quad (2)$$

imply $x \in S_0$; since the conditions (1) imply (2), the classical local description can be (and is usually) stated as follows: S_0 is the closure in π_0 of the set of points satisfying (2).

This classical local description requires the hypothesis of part (C), which can be formulated in the following equivalent ways:

(i) for at least one point $x' \in U'_0$, the $(k+1)$-th coordinates of the p points $x^{(j)}(x')$ are all distinct;

(ii) there exists a dense open subset V' of π'_0 such that, for any $x' \in V'$, two distinct points $\in S_0(x')$ have distinct $(k+1)$-th coordinates.

In fact, if (i) holds, $\delta(x') \not\equiv 0$ by (c) of part (C), and (ii) holds with $V' = \{x' \in U'_0 \mid \delta(x') \neq 0\}$. Thus we see that the validity of the hypothesis of part (C) does not depend on the choice of π_0. Finally, if this hypothesis is not fulfilled for the given basis of C^m, since the p points $x^{(j)}(x')$ themselves are all distinct for every $x' \in U'_0$, we can find a new basis of C^m, for which $x_1, \ldots, x_k$ remain the first k coordinates (so that this new basis remains k-proper for I), and the hypothesis of part (C) is fulfilled.

REMARK 3. If $p = 1$: by (a) of part (B), every point of S_0 is a regular point of S_0 of dimension k.

If $k = m - 1$, i.e., if I is principal (cf. the remarks in § 3) : by condition (†), S_0 is the set of zeros of $R(x', x_m)$ in π_0, and, by (c) of part (C), the germ induced by $R(x', x_m)$ at $\mathfrak{o}$ generates the principal ideal I.

If $k = 1$, $p > 2$, and π_0 is chosen small enough : $\delta(x_1) = 0$ if and only if $x_1 = 0$, i.e., the subset of S_0 defined by the conditions (1) or (2) is $S_0 - \{\mathfrak{o}\}$; hence $S_0 - \{\mathfrak{o}\}$, or S_0, can be parametrized, with $t = x_1^{1/p}$ as the parameter :

$$x_1 = t^p,\ x_j = g_j(t),\ j = 2, \ldots, m,\ |t| < \rho_1^{1/p},$$

where ρ_1 is the first radius of π_0, the g_j are holomorphic for $|t| < \rho_1^{1/p}$ and vanish for $t = 0$.

COROLLARY 1. *Given an analytic set S in an open set U, any compact subset of U contains only a finite number of isolated points of S.*

PROOF. If the contrary is true, we can find a point $\in S$, which we choose as the origin $\mathfrak{o}$, an analytic set S' in an open neighbourhood of $\mathfrak{o}$ which induces at $\mathfrak{o}$ an irreducible germ $\mathbf{S}'$, and a sequence of isolated points of S' converging to $\mathfrak{o}$; the ideal $I = I(\mathbf{S}')$ is prime, $\{0\} \neq I \subsetneq \mathscr{H}'^m$; then Theorem 3 gives a set S_0 which coincides with S' in some neighbourhood of $\mathfrak{o}$, and has no isolated point by (b) of part (A).

COROLLARY 2. *If the basis of C^m is k-proper for a prime ideal I in $\mathscr{H}^m$, $\{0\} \neq I \subsetneq \mathscr{H}'^m$, and an analytic set S in an open neighbourhood of $\mathfrak{o}$ induces at $\mathfrak{o}$ the germ $\mathbf{S}_I$, then, for any $a \in S$ sufficiently* *near* $\mathfrak{o}$, *the basis of* C^m *is also* k*-proper for the ideal* $I(\mathbf{S}_a)$ *in* $\mathscr{H}_a^m$ *i.e.,* $x_1 - a_1, \dots, x_k - a_k$ *have the properties of Definition* 2).

PROOF. We shall prove that, for any a in the set S_0 of Theorem 3, the basis is k-proper for the ideal I_a in $\mathscr{H}_a^m$ associated to the germ at a induced by S_0. Let a' = projection of a on C^k : the ideal in $\mathscr{H}_a^m$ generated by I_a and $(x_1 - a_1), \dots, (x_k - a_k)$ defines the germ at a induced by $S_0(a')$, which is $\{\mathbf{a}\}$ since $S_0(a')$ is finite ; $(I_a)_k = \{0\}$ since, by (b) of part (A), any neighbourhood of a in S_0 has a neighbourhood of a' in C^k as its projection on C^k.

REMARK. In particular, the basis of C^m is k-proper for $I' = I(\mathbf{S}_I)$. But $I' \supset I$, so that, by §3, Proposition 3, $I' = I$, i.e. $I = I(S_I)$. A more direct proof of this important fact will be given in Chap. IV (Theorem 2 (a)).

IV

LOCAL PROPERTIES OF ANALYTIC SETS

1. Direct consequences of the local description.

THEOREM 1 (H. Cartan). *Given an analytic set S in an open set U, for any $a_0 \in U$, there exists a finite family $\mathscr{F}_0$ of holomorphic functions on an open neighbourhood $V \subset U$ of a_0 such that, for any $a \in V$, the germs induced at a by the functions $\in \mathscr{F}_0$ generate, in $\mathscr{H}_a^m$, the ideal I_a associated to the germ $\mathbf{S}_a$. In other words : the sheaf of an analytic set is coherent.*

(See H. Cartan, *Bull. Soc. Math. France*, vol. 78, 1950, pp. 29-64.)

Proof. By putting aside a few trivial cases and using the fact that a finite intersection of coherent sheaves is again a coherent sheaf (see the paper by H. Cartan mentioned above, p. 41), it is enough to prove that, with the notation of Theorem 3 (Chap. III), there exists a finite family $\mathscr{F}_0$ of holomorphic functions on π_0 such that, for any $a \in \pi_0$ sufficiently near 0, the germs induced at a by the functions $\in \mathscr{F}_0$ generate, in $\mathscr{H}_a^m$, the ideal I_a associated to the germ induced by S_0 at a.

We can find $1 + (p-1)(m-k-1)$ linear forms l_i on C^m, $i = 1, \dots, 1 + (p-1)(m-k-1)$, with these two properties :

(α) $x_1, \dots, x_k$ and any $m - k$ forms among the l_i are independent;

(β) there exists an $x' \in U'_0$ such that, for each i, the p values $l_i(x^{(j)}(x'))$, $1 \leqslant j \leqslant p$, are all distinct.

For each $i = 1, \dots, 1 + (p-1)(m-k-1)$ and each $r = k+1, \dots, m$, we denote by $X_{i,r}$ the pseudo-polynomial constructed in Proposition 2 (d) (Chap. III) with $h = l_i$ and $h_0(x) = x_r$. The functions $R_{x_r}(x', x_r)$, $R_{l_i}(x', l_i(x))$ and $\dfrac{\partial}{\partial u} R_{l_i}(x', l_i(x))\, x_r - X_{i,r}(x', l_i(x))$, $i = 1, \dots, 1 + (p-1)(m-k-1)$, $r = k+1, \dots, m$, form a finite family $\mathscr{F}_1$ of holomorphic functions on π_0, each of which vanishes for any $x \in S_0$ (cf. Remark 1 on Proposition 2, Chap. III), and hence induces at a a germ $\in I_a$; omitting the subscript a, we shall denote these germs by $\mathbf{R}_{x_r}(x_r)$, $\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)$, $\frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\, x_r - \mathbf{X}_{\iota,r}(\mathbf{l}_\iota)$ respectively.

Given $a \in S_0$, with a' = projection of a on C^k, $S_0(a') - \{a\}$ consists of at most $p-1$ points; since any one of these points, say $\tilde{a}$, is distinct from a, but has the same projection a' on C^k, we have $l_\iota(\tilde{a}) = l_\iota(a)$ for at most $m-k-1$ indices i, on account of (α); we can find an i, depending only on a, such that $l_\iota(\tilde{a}) \neq l_\iota(a)$ for any $\tilde{a} \in S_0(a') - \{a\}$; the integer i has this value in what follows.

For any $\mathbf{f} \in I_a$, by Theorem 2 of Chap. II, there exists a polynomial $\mathbf{X}(x_{k+1}, \dots, x_m) \in \mathscr{H}^k_{a'}[x_{k+1}, \dots, x_m]$, of degree $< p$ in each x_r, such that $\mathbf{f} - \mathbf{X}(x_{k+1}, \dots, x_m)$ is a linear combination, over $\mathscr{H}^m_a$, of the germs $\mathbf{R}_{x_r}(x_r)$, $r = k+1, \dots, m$; since these germs belong to I_a, so does $\mathbf{X}(x_{k+1}, \dots, x_m)$. We may set

$$\left[\frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\right]^{(p-1)(m-k)} \mathbf{X}(x_{k+1}, \dots, x_m)$$
$$\equiv \mathbf{Y}\left(\frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\, x_{k+1}, \dots, \frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\, x_m, \mathbf{l}_\iota\right),$$

where

$$\mathbf{Y}(y_{k+1}, \dots, y_m, u) \in \mathscr{H}^k_{a'}[y_{k+1}, \dots, y_m, u],$$

and

$$\mathbf{Y}\left(\frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\, x_{k+1}, \dots, \frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\, x_m, \mathbf{l}_\iota\right) - \mathbf{Y}\left(\mathbf{X}_{\iota,k+1}(\mathbf{l}_\iota), \dots, \mathbf{X}_{\iota,m}(\mathbf{l}_\iota), \mathbf{l}_\iota\right)$$

is a linear combination, over $\mathscr{H}^m_a$, of the germs $\frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\, x_r - \mathbf{X}_{\iota,r}(\mathbf{l}_\iota)$. Let $\mathbf{Y}(\mathbf{X}_{\iota,k+1}(\mathbf{l}_\iota), \dots, \mathbf{X}_{\iota,m}(\mathbf{l}_\iota), \mathbf{l}_\iota) = \boldsymbol{\varphi}(\mathbf{l}_\iota)$, where $\boldsymbol{\varphi}(u) \in \mathscr{H}^k_{a'}[u]$; thus $\boldsymbol{\varphi}(\mathbf{l}_\iota) \in I_a$ and $\left[\frac{\partial}{\partial u}\mathbf{R}_{l_\iota}(\mathbf{l}_\iota)\right]^{(p-1)(m-k)} \mathbf{f} - \boldsymbol{\varphi}(\mathbf{l}_\iota)$ is a linear combination over $\mathscr{H}^m_a$ of the germs induced at a by the functions $\in \mathscr{F}_1$.

Now, the germ $\boldsymbol{\varphi}(\mathbf{l}_\iota)$ is induced at a by the function $\phi(x', l_\iota(x))$, where $\phi(x', u)$ is a pseudo-polynomial in u with coefficients holomorphic on some open neighbourhood of a'; $\boldsymbol{\varphi}(\mathbf{l}_\iota) \in I_a$ means that

$\phi(x', l_i(x)) = 0$ for any $x \in S_0$ sufficiently near a. By Theorem 3 of Chap. III, (b) of part (A), and the choice of $\mathbf{l}_i$, this is equivalent to saying that $\phi(x', u) = 0$ whenever $R_{l_i}(x', u) = 0$ and (x', u) is sufficiently near $(a', l_i(a))$; then, by Theorem 6 of Chap. II, every irreducible factor, in $\mathscr{H}^{k+1}_{a', l_i(a)}$, of the germ $\mathbf{R}_{l_i}(u)$ induced by $R_{l_i}(x', u)$ at the point $(a', l_i(a)) \in C^{k+1}$, divides, in $\mathscr{H}^{k+1}_{a', l_i(a)}$, the germ $\boldsymbol{\varphi}(u)$ induced by $\phi(x', u)$ at the same point.

But $\mathbf{R}_{l_i}(u)$ is equivalent, in $\mathscr{H}^{k+1}_{a', l_i(a)}$, to a distinguished polynomial $\mathbf{A}$ in $u - l_i(a)$, which, by the uniqueness assertion in Theorem 2 of Chap. II, divides $\mathbf{R}_{l_i}(u)$ in $\mathscr{H}^k_{a'}[u]$; on account of (β), $\mathbf{R}_{l_i}(u)$ has a discriminant $\neq \mathbf{0}$, and therefore the non-invertible irreducible factors of $\mathbf{A}$, in $\mathscr{H}^{k+1}_{a', l_i(a)}$ or $\mathscr{H}^k_{a'}[u - l_i(a)]$ (cf. Chap. II, § 3), are mutually non-equivalent. So $\mathbf{R}_{l_i}(u)$ divides $\boldsymbol{\varphi}(u)$ in $\mathscr{H}^{k+1}_{a', l_i(a)}$, hence $\mathbf{R}_{l_i}(\mathbf{l}_i)$ divides $\boldsymbol{\varphi}(\mathbf{l}_i)$ in $\mathscr{H}^m_a$, and $\left[\dfrac{\partial}{\partial u}\mathbf{R}_{l_i}(\mathbf{l}_i)\right]^{(p-1)(m-k)} \mathbf{f}$ is a linear combination, over $\mathscr{H}^m_a$, of the germs induced at a by the functions $\in \mathscr{F}_1$.

(From now on, all germs are taken at a.) As i may depend on a, let $\rho(x) = \left[\prod_{i=1}^{1+(p-1)(m-k-1)} \dfrac{\partial}{\partial u} R_{l_i}(x', l_i(x))\right]^{(p-1)(m-k)}$: we have proved that, for any $a \in S_0$ and any $\mathbf{f} \in I_a$, $\mathbf{f}\,\boldsymbol{\rho}$ is a linear combination, over $\mathscr{H}^m_a$, of the germs induced at a by the functions $\in \mathscr{F}_1$. Now, by (β) and the remark on Lemma 3 of Chap. III, § 4, for each i, the germ induced at $\mathfrak{o}$ by $\dfrac{\partial}{\partial u} R_{l_i}(x', l_i(x))$ does not belong to I ; since I is prime, the germ induced at $\mathfrak{o}$ by ρ does not belong to I.

The polydisc π_0 defined by Theorem 3 of Chap. III, is contained in the polydisc π defined at the beginning of Chap. III, § 4 : so I is generated by the germs induced at $\mathfrak{o}$ by a finite family of functions holomorphic on π_0, which have S_0 as their set of common zeros ; let $\mathscr{F}_0 = \{\rho_\nu \mid 1 \leqslant \nu \leqslant n\}$ be the union of this family and $\mathscr{F}_1$. Since the sheaf of relations between the ρ_ν and ρ is coherent (see the same paper by H. Cartan, p. 37), there exist a finite number, say s, of systems of holomorphic functions $\phi_1^{(\sigma)}, \ldots, \phi_n^{(\sigma)}, \phi^{(\sigma)}$, $1 \leqslant \sigma \leqslant s$, on an open neighbourhood $V \subset \pi_0$ of $\mathfrak{o}$, such that :

(i) $\phi_1^{(\sigma)} \rho_1 + \ldots + \phi_n^{(\sigma)} \rho_n + \phi^{(\sigma)} \rho \equiv 0$ on V for each σ ;

(ii) for any $a \in V$, any system of germs $\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{f} \in \mathscr{H}_a^m$ such that $\mathbf{f}_1 \boldsymbol{\rho}_1 + \ldots + \mathbf{f}_n \boldsymbol{\rho}_n + \mathbf{f} \boldsymbol{\rho} = \mathbf{0}$ is a linear combination, over $\mathscr{H}_a^m$, of the s systems $\boldsymbol{\varphi}_1^{(\sigma)}, \ldots, \boldsymbol{\varphi}_n^{(\sigma)}, \boldsymbol{\varphi}^{(\sigma)}$, $1 \leqslant \sigma \leqslant s$.

Then, by (ii), for any $a \in V \cap S_0$ and any $\mathbf{f} \in I_a$, $\mathbf{f}$ is a linear combination, over $\mathscr{H}_a^m$, of the $\boldsymbol{\varphi}^{(\sigma)}$, $1 \leqslant \sigma \leqslant s$; by (i), the germ induced at $\mathfrak{o}$ by each $\phi^{(\sigma)} \rho$ belongs to I ; so does the germ induced by each $\phi^{(\sigma)}$, and hence, on a suitably chosen open neighbourhood $W \subset V$ of $\mathfrak{o}$, each $\phi^{(\sigma)}$ is a linear combination, with coefficients holomorphic on W, of the ρ_ν, $1 \leqslant \nu \leqslant n$. Finally, for any $a \in W \cap S_0$ and any $\mathbf{f} \in I_a$, $\mathbf{f}$ is a linear combination, over $\mathscr{H}_a^m$, of the germs induced at a by the functions $\in \mathscr{F}_0$; this remains true for $a \in W \cap {}^cS_0$, since at least one function $\in \mathscr{F}_0$ does not vanish at a, and the theorem is proved.

THEOREM 2. (a) *If I is any prime ideal in $\mathscr{H}^m$, then the ideal associated to the germ $\mathbf{S}_I$ is I itself, and therefore $\mathbf{S}_I$ is irreducible.*

(b) *If I is any ideal in $\mathscr{H}^m$, then the ideal associated to the germ $\mathbf{S}_I$ is* rad I.

PROOF. (a) Obviously it is sufficient to consider the case $\{\mathbf{0}\} \neq I \underset{\neq}{\subset} \mathscr{H}'^m$. In this case let the notation be as in Lemma 1 (Chap. III, § 4), and suppose $\mathbf{h} \in \mathscr{H}^m$ vanishes on $\mathbf{S}_0 = \mathbf{S}_I$. Then $\mathbf{h}$ in particular satisfies the hypothesis of Lemma 2, hence $\mathbf{h} \in I$. Thus $I(S_I) \subset I$. Since $I \subset I(\mathbf{S}_I)$ is always valid for any ideal I in $\mathscr{H}^m$, (a) is proved.

(b) Let I be an arbitrary ideal in $\mathscr{H}^m$. We may again assume that $\{\mathbf{0}\} \neq I \underset{\neq}{\subset} \mathscr{H}'^m$. We first remark that $\mathbf{S}_I = \mathbf{S}_{\mathrm{rad}\, I}$. In fact $\mathbf{S}_{\mathrm{rad}\, I} \subset \mathbf{S}_I$, since $I \subset \mathrm{rad}\, I$. Now, for any $\mathbf{f} \in \mathrm{rad}\, I$, there exists an integer $n \geqslant 1$ such that $\mathbf{F} = \mathbf{f}^n \in I$. Hence $\mathbf{S}_I \subset \mathbf{S}_F = \mathbf{S}_f$. We have therefore $\mathbf{S}_I \subset \mathbf{S}_f$, for every $\mathbf{f} \in \mathrm{rad}\, I$. Hence $\mathbf{S}_I \subset \mathbf{S}_{\mathrm{rad}\, I}$. It follows that $\mathbf{S}_{\mathrm{rad}\, I} = \mathbf{S}_I$.

Suppose first that I is primary. Then rad I is prime, and by (a) the ideal associated to $\mathbf{S}_I = \mathbf{S}_{\mathrm{rad}\, I}$ is rad I; hence (b) is proved if I is primary.

Now let I be an arbitrary ideal in $\mathscr{H}^m$. Then I can be written as a finite intersection of primary ideals in $\mathscr{H}^m$, say $I = \bigcap_{i=1}^{n} J_i$ (Chap. II, § 2 (d)). Then $\mathbf{S}_I = \bigcup_{i=1}^{n} \mathbf{S}_{J_i}$ (Chap. II, § 8, Proposition 1). Hence

$$I(\mathbf{S}_I) = I(\mathbf{S}_{J_1}) \cap \dots \cap I(\mathbf{S}_{J_n}) \text{ (Chap. II, § 8, Proposition 2),}$$

$$= \mathrm{rad}\, J_1 \cap \dots \cap \mathrm{rad}\, J_n = \mathrm{rad}\, I,$$

since the J_i are primary, and since the radical of the intersection of finitely many ideals (in any ring) is the intersection of their radicals, q.e.d.

Corollary 1. *For any ideal I in $\mathscr{H}^m$, the following statements are equivalent*: (1) $I = \mathrm{rad}\, I$; (2) $I = I(\mathbf{S}_I)$; (3) $I = I(\mathbf{S})$ *for some germ of analytic set* $\mathbf{S}$ *at* $\mathbf{0}$.

Corollary 2. *If I, J are ideals in $\mathscr{H}^m$*: $\mathbf{S}_I \subset \mathbf{S}_J$ *and* rad $J \subset$ rad I *are equivalent statements.*

This result includes Proposition 1 of Chap. III, §4.

Since $\mathscr{H}^m$ is Noetherian, Theorem 2 (b) easily implies

Corollary 3 (Hilbert's Nullstellensatz). *For any ideal I in $\mathscr{H}^m$, there exists an integer $n(I) = n > 0$ such that, if $\mathbf{f} \in \mathscr{H}^m$ vanishes on $\mathbf{S}_I$, then $\mathbf{f}^n \in I$.*

Theorem 3. *Given an analytic set S in an open set U in C^m and a point $a \in S$, we can find an open neighbourhood $V \subset U$ of a with the property : for any analytic set S' in U, $\mathbf{S}'_a \supset \mathbf{S}_a$ implies $S' \supset V \cap S$.*

Proof. Plainly we may assume $\{\mathbf{a}\} \subsetneq \mathbf{S}_a = \mathbf{S} \neq \mathbf{C}^m_a$. Then, if $\mathbf{T}_1, \dots, \mathbf{T}_n$ are the irreducible components of $\mathbf{S}$, we also have $\{\mathbf{a}\} \subsetneq \mathbf{T}_j \neq \mathbf{C}^m_a$, $1 < j < n$. Hence, by Theorem 3 (B) of Chap. III, we have, for each j $(1 \leqslant j \leqslant n)$, an open neighbourhood $V_j \subset U$ of a, and an analytic set T_j in V_j such that : (i) T_j induces the germ $\mathbf{T}_j$ at a ; (ii) T_j contains a dense connected open subset t_j, all of whose points are regular points of T_j. Now let S' be any analytic set in U, and suppose $\mathbf{S}'_a \supset \mathbf{S}_a$. Then, for any j $(1 \leqslant j \leqslant p)$, $\mathbf{S}'_a \supset \mathbf{T}_j$, hence also $\mathbf{S}'_b \supset (T_j)_b$ for all $b \in t_j$ close enough to a. Proposition 3 of § 2 (Chap. III) implies therefore that $S' \cap V_j \supset t_j$. Since $S' \cap V_j$ is closed in V_j, and t_j is dense in T_j, we have $S' \supset T_j$. Finally, since $\mathbf{S} = \bigcup_{j=1}^{n} \mathbf{T}_j$, there exists an open neighbourhood $V \subset \bigcap_{j=1}^{n} V$ of a, such that $S \cap V = V \cap (\bigcup_{j=1}^{n} T_j)$. Then clearly $S' \supset S \cap V$, q.e.d.

Corollary 1. *The intersection of an arbitrary family $\mathscr{F}$ of analytic sets in an open set U in C^m is again an analytic set in U.*

Proof. Since the family of finite intersections of members of $\mathscr{F}$ has the same intersection as $\mathscr{F}$, and is decreasingly filtered, we may assume that $\mathscr{F}$ is decreasingly filtered. For any $a \in U$, the family $\{I(\mathbf{S}_a)\}_{S \in F}$ of ideals in $\mathscr{H}_a^m$ is then increasingly filtered. Since $\mathscr{H}_a^m$ is Noetherian, $I = \bigcup_{S \in F} I(\mathbf{S}_a)$ is again an ideal in $\mathscr{H}_a^m$ which belongs to the family $\{I(\mathbf{S}_a)\}_{S \in F}$. Let $I = I(\mathbf{T}_a)$, $T \in \mathscr{F}$. For every $S \in \mathscr{F}$, we have $I \supset I(\mathbf{S}_a)$, hence $\mathbf{T}_a \subset \mathbf{S}_a$. Hence, by Theorem 3, we have a neighbourhood $V \subset U$ of a, such that $S \supset T \cap V$ for every $S \in \mathscr{F}$. Hence, if $S_0 = \bigcap_{S \in F} S$, we have $S_0 \cap V \supset T \cap V$. Since $T \in \mathscr{F}$, we must actually have $S_0 \cap V = T \cap V$. $T \cap V$ being an analytic set in V, it follows from the local character of the definition of an analytic set in U that S_0 is an analytic set in U, q.e.d.

Remark. Note that we have actually proved the stronger result : if $\mathscr{F}$ is a decreasingly filtered family of analytic sets in U, then, given any compact subset K of U, there exists a $T \in \mathscr{F}$ such that

$$\left(\bigcap_{S \in F} S\right) \cap K = T \cap K.$$

Regarding arbitrary *unions* of analytic sets in an open set U of C^m, one can only assert : the union of a locally finite family of analytic sets in U is an analytic set in U.

COROLLARY 2 (Remmert). *Let S be an analytic set in an open set U, and let the basis of C^m be such that the origin $\mathfrak{o}$ is an isolated point of the analytic set $S \cap \{x_1 = ... = x_r = 0\}$ for an integer r, $1 \leqslant r \leqslant m-1$. Then there exsits a sequence P_n of open polydiscs, with centre $\mathfrak{o}$ and radii decreasing to 0, such that, for each n, the projection of $S \cap P_n$ on C^r is an analytic set in the projection of P_n.*

PROOF. With the notation of Theorem 3 (Chap. III), the projection of S_0 on C^k is the polydisc π'_0 itself and, for $r = k+1,...,m$, by Remark 1 on Theorem 3, the projection of S_0 on C^r is the set of common zeros in $(\pi_0)_r$ ($=$ projection of π_0 on C^r) of a family of functions holomorphic on $(\pi_0)_r$, i.e., by Corollary 1, an analytic set $(S_0)_r$ in $(\pi_0)_r$.

Let $k \leqslant r \leqslant m-1$: given positive numbers ρ_i, $i = r+1,..., m$, there exists a neighbourhood $W \subset (\pi_0)_r$ of $\mathfrak{o}_r$ such that (projection of x on C^r) $\in W$ and $x \in S_0$ imply $|x_i| < \rho_i$, $i = r+1, ..., m$; then, for any open set $\omega \subset W$ in C^r, the projection on C^r of the set

$$\{x \in S_0 \mid (\text{projection of } x \text{ on } C^r) \in \omega, |x_{r+1}| < \rho_{r+1}, \dots, |x_m| < \rho_m\}$$

is an analytic set in ω, namely ω itself if $r = k$, $\omega \cap (S_0)_r$ if $r = k+1, ..., m-1$.

Now let S and the basis of C^m meet the requirements of Corollary 2; all germs being taken at $\mathfrak{o}$, we have $\mathbf{S} \neq \mathbf{C}^m$ and may assume $\{\mathfrak{o}\} \subsetneq \mathbf{S}$; $x_1, ..., x_r$ satisfy condition A of Chap. III (§3, Proposition 1) for the ideal $I(\mathbf{S})$, *a fortiori* for each $I(\mathbf{T}_j)$, where the $\mathbf{T}_j$, $j = 1,...,n$, are the irreducible components of $\mathbf{S}$. Since $I(\mathbf{T}_j)$ is prime and $\{0\} \neq I(\mathbf{T}_j) \subsetneq \mathscr{H}'^m$, according to the procedure of Proposition 2, Chap. III, § 3, *by altering the basis of the subspace C^r only*, for each $j = 1, ..., n$, we can find a basis of C^m which is k_j-proper for $I(\mathbf{T}_j)$, $1 \leqslant k_j \leqslant r$. Then, given the positive numbers $\rho_{r+1}, \dots, \rho_m$, for each $j = 1, ..., n$, we have: (i) an open neighbourhood $V_j \subset U$ of $\mathfrak{o}$, and an analytic set T in V inducing the germ $\mathbf{T}_j$ at $\mathfrak{o}$;

(ii) a neighbourhood W_j of $\mathfrak{o}_r$ in C^r such that, for any open set $\omega \subset W_j$ in C^r, the projection on C^r of the set $\{x \in T_j |$ (projection of x on $C^r) \in \omega$, $| x_{r+1} | < \rho_{r+1}, \ldots, |x_m| < \rho_m\}$ is an analytic set in ω; finally, there exists an open neighbourhood $V \subset \bigcap_{j=1}^{n} V_j$ of $\mathfrak{o}$, such that

$$S \cap V = (\bigcup_{j=1}^{n} T_j) \cap V.$$

Let ω be an open polydisc with centre $\mathfrak{o}_r$ in C^r (for the given basis, of C^m), contained in $\bigcap_{j=1}^{n} W_j$; if $\rho_{r+1}, \ldots, \rho_m, \omega$ are small enough the polydisc

$P = \{x \mid$ (projection of x on $C^r) \in \omega, | x_{r+1} | < \rho_{r+1}, \ldots, | x_m | < \rho_m\}$ is contained in V and has the desired property.

Remark. If the basis of C^m is r-proper for $I(\mathbf{S})$, then, for each n, the projection of $S \cap P_n$ on C^r is the projection of P_n itself. If, conversely, this holds for arbitarily great n, then the basis of C^m is r-proper for $I(\mathbf{S})$.

Theorem 4 (Remmert-Stein). *Let U be an open set in C^m, and $L^{(d)}$ a d-dimensional affine subvariety of C^m, $-1 \leq d \leq m-1$ ($L^{(-1)} = \emptyset$ by definition). Then any analytic set S in $U \cap {}^c L^{(d)}$ is either discrete, or contains points arbitrarily close to the boundary ∂U of U in C^m(if U is inbounded, ∂U is to include the point at infinity of C^m).*

Proof. The proof is by induction on m. The theorem is trivial if $m = 1$. Now let $m > 1$.

(i) It is sufficient to prove the theorem for $d = m - 1$. In fact, suppose this done, and suppose, with the notation as in the theorem, that $d < m - 1$ and $a \in S$ is not isolated. If $L^{(m-1)}$ is any $(m-1)$-dimensional affine subvariety of C^m such that $L^{(d)} \subset L^{(m-1)}$ and $a \notin L^{(m-1)}$, then $S' = S \cap {}^c L^{(m-1)}$ is an analytic set in $U \cap {}^c L^{(d)} \cap {}^c L^{(m-1)} = U \cap {}^c L^{(m-1)}$, which contains a non-isolated point a. Hence S' already contains points arbitrarily near ∂U.

(ii) Clearly we may suppose that S is bounded. We may also suppose that the interior $\underset{-}{S}$ of S is empty. For if $S \neq \emptyset$, S contains an entire connected component of $U \cap {}^cL^{(d)}$, which obviously contains points arbitrarily near ∂U.

Finally we may suppose that S has non-isolated points only. For, by Corollary 1 to Theorem 3 of Chap. III, the set of isolated points of S is locally finite, and hence the set of non-isolated points of S is again an analytic set in $U \cap {}^cL^{(d)}$, which has non-isolated points only.

Thus let S be a bounded, nowhere-dense, non-empty analytic set in $U \cap {}^cL^{(m-1)}$, $m > 1$, and suppose the set of non-isolated points of S is S itself. Let $L^{(m-1)} = L = \{x \in C^m \mid A(x) = 0\}$, and let $\alpha = \sup_{x \in S} |A(x)|$. Then $0 < \alpha < \infty$. Let $\{x_n\}$ be a sequence of points of S such that $|A(x_n)| \to \alpha$. Since S is bounded, we may suppose that $\{x_n\}$ converges, to $x_0 \in \bar{U}$ say.

If $x_0 \in \partial U$, the theorem is proved. Hence let $x_0 \in U$. Since $x_0 \notin {}^cL$, we have $x_0 \in S$. Let $L' = \{x \in C^m \mid A(x) = A(x_0)\}$. Then in $U' = L' \cap U$, considered as an open set in C^{m-1}, $S' = S \cap L'$ is an analytic set; for, $L \cap L' = \emptyset$, hence $U' = (U \cap {}^cL) \cap L'$. We shall show below that x_0 is a non-isolated point of S'. This will prove the theorem; for by the induction hypothesis, S' contains points artbitrarily close to the boundary $\partial U'$ of U' in L', and $\partial U' \subset \partial U$.

Finally, x_0 is a non-isolated point of S'. In fact, we assert that the germ at x_0 induced by $A' = A - A(x_0)$ vanishes on $\mathbf{S}_{x_0}$. For, if $T \subset S$ is any analytic set in a neighbourhood of x_0 whose germ at x_0 is an irreducible component of $\mathbf{S}_{x_0}$, it follows from Proposition 2 (c) Chap. III, § 4, that either $\mathbf{A}'_{x_0}$ vanishes on $\mathbf{T}_{x_0}$ or $A'(T)$ is a neighbourhood of 0 in C (since x_0 is a non-isolated point of S, we have $\{\mathbf{x}_0\} \subsetneq \mathbf{T}_{x_0}$), hence $A(S)$ is a neighbourhood of $A(x_0)$. But this is impossible, since $|A(x_0)| = \sup_{x \in S} |A(x)|$. Hence $\mathbf{A}'_{x_0}$ vanishes on $\mathbf{S}_{x_0}$, i.e., $\mathbf{S}_{x_0} = \mathbf{S}'_{x_0}$. Since x_0 is a non-isolated point of S, our assertion is proved, and with it the theorem.

We state separately the following special case of Theorem 4.

COROLLARY 1. *A compact analytic set in an open set in C^m is finite.*

Using Corollary 1, we can also prove the following result, due in the case $m = 1$ to Ritt.

COROLLARY 2. *Let U be a connected open set in C^m, and F a holomorphic mapping of U into a compact subset K of itself. Then F has a unique fixed point.*

PROOF. Since $F(U) \subset U$, we can form the iterates F_n of F; $F_1 = F$, $F_n = F \circ F_{n-1}$ for $n > 1$. Clearly, all the F_n are holomorphic mappings of U into K. Since K is compact, all the coordinates of all the F_n are uniformly bounded on U. Hence we can find a strictly increasing sequence of integers $n_k > 0$, such that the sequences $\{F_{n_k}\}$ and $\{F_{n_{2k+1}-n_{2k}}\}$ of holomorphic mappings converge (coordinate wise) on U, uniformly on compact subsets of U (§ 3, Chapter I). Let F': $U \to U$, F'': $U \to U$ be the respective limit mappings. F' and F'' are holomorphic mappings and $F'(U)$, $F''(U) \subset K$. Now the relations

$$F_{n_{2k+1}} \equiv F_{n_{2k+1}-n_{2k}} \circ F_{n_{2k}}, \; k = 1, 2, 3, \dots,$$

imply $F' = F'' \circ F'$. Hence

$$F'(U) \subset \{x \in U \mid x = F''(x)\} = S_{F''}$$

say. $S_{F''}$ is obviously an analytic set in U, and $S_{F''} \subset F''(U) \subset K$, hence $S_{F''}$ is compact. By Corollary 1, $S_{F''}$ is a finite set, hence so is $F'(U)$. But $F'(U)$ is connected, hence must consist of just one point, say x_0, and F' is the constant mapping $x \to x_0$. We shall show that x_0 is the unique fixed point of F.

First we have

$$\begin{aligned} F(x_0) &= F(F'(x_0)) \\ &= \lim_{k \to \infty} F(F_{n_k}(x_0)) \\ &= \lim_{k \to \infty} F_{n_k}(F(x_0)) \\ &= F'(F(x_0)) = x_0, \end{aligned}$$

hence x_0 is a fixed point of F. On the other hand, for any $x \in U$, $x = F(x)$ implies, trivially, $x = F'(x) = x_0$, q.e.d.

COROLLARY 3. *Let S be an analytic set in an open set U, and let $A_1, \dots, A_k$ $(1 \leqslant k \leqslant m-1)$ be k independent linear forms on C^m such that the origin $\mathfrak{o}$ is an isolated point of the analytic set*

$$\{x \in S \mid A_1(x) = \dots = A_k(x) = 0\}.$$

Then there exist a neighbourhood $V \subset U$ of $\mathfrak{o}$ and, for each $j = 1, \dots, k$, a neighbourhood W of A_j (in the dual space C^m), such that, for any $a \in S \cap V$, $A_1' \in W_1, \dots, A_k' \in W_k$, a is an isolated point of the analytic set

$$\{x \in S \mid A_1'(x) - A_1'(a) = \dots = A_k'(x) - A_k'(a) = 0\}.$$

PROOF. Let the vectors $e_{k+1}, \dots, e_m \in C^m$ form a basis of the $(m-k)$-dimensional variety $L = \{x \in C^m \mid A_1(x) = \dots = A_k(x) = 0\}$; the hypothesis consists in the existence of a positive number α such that $y_{k+1}, \dots, y_m \in C$ and $0 < |y_{k+1}|^2 + \dots + |y_m|^2 \leqslant \alpha^2$ imply $y_{k+1} e_{k+1} + \dots + y_m e_m \in U - S$. Since the set

$$\{y_{k+1}e_{k+1} + \dots + y_m e_m \mid |y_{k+1}|^2 + \dots + |y_m|^2 = \alpha^2 \text{ (resp. } \leqslant \alpha^2)\}$$

is compact, there also exist a neighbourhood $V \subset U$ of $\mathfrak{o}$ and, for each $j = k+1, \dots, m$, a neighbourhood V_j of e_j, such that : $a \in V$, $e_{k+1}' \in V_{k+1}, \dots, e_m' \in V_m$ and $|y_{k+1}|^2 + \dots + |y_m|^2 = \alpha^2$ (resp. $\leqslant \alpha^2$) imply $a + y_{k+1}e_{k+1}' + \dots + y_m e_m' \in U - S$ (resp. $\in U$). Finally, each A_j, $j = 1, \dots, k$, has a neighbourhood W_j in the dual space such that for any $A_1' \in W_1, \dots, A_k' \in W_k$, we can find $e_{k+1}' \in V_{k+1}, \dots, e_m' \in V_m$ forming a basis of the variety $L' = \{x \in C^m \mid A_1'(x) = \dots = A_k'(x) = 0\}$.

Given $a \in S \cap V$, $A_1' \in W_1, \dots, A_k' \in W_k$, let $e_{k+1}' \in V_{k+1}, \dots, e_m' \in V_m$ form a basis of L' : if we set $y = y_{k+1}e_{k+1}' + \dots + y_m e_m'$, the set of points y such that $|y_{k+1}|^2 + \dots + |y_m|^2 < \alpha^2$ is an open set Y in C^{m-k}, and $\{y \in Y \mid a + y \in S\}$ is an analytic set in Y, which contains the origin of C^{m-k} and does not contain any sequence converging to a point $\in \partial Y$. Then, by Theorem 4, the origin of C^{m-k} is an isolated point of this analytic set, i.e. a is an isolated point of the set $\{x \in S \mid x - a \in L'\}$.

2. Regular points and dimension.

THEOREM 5. *The set S^* of regular points of an analytic set S (in an open set U in C^m) is dense in S.*

PROOF. Let a be a non-regular point of S. Then we have $\{\mathbf{a}\} \neq \mathbf{S}_a = \mathbf{S} \neq \mathbf{C}_a^m$. Let $\mathbf{T}_1, \ldots, \mathbf{T}_n$ be the irreducible components of $\mathbf{S}$. We then have, for $1 \leqslant j \leqslant n, \{\mathbf{a}\} \subsetneq \mathbf{T}_j \neq \mathbf{C}_a^m$. By Theorem 3(B) of Chap. III, we can find an open neighbourhood $V \subset U$ of a, and analytic sets T_j in V, $j = 1, \ldots, n$, such that: (i) for $1 \leqslant j \leqslant n$, T induces the germ $\mathbf{T}_j$ at a ; (ii) each T_j contains a dense open subset t_j consisting entirely of regular points of T_j of the same dimension, say $k_j (1 \leqslant k_j \leqslant m-1, 1 \leqslant j \leqslant n)$; and (iii) $\bigcup_{j=1}^{n} T_j = S \cap V$. If $n = 1$, we are already through. Hence let $n > 1$, and $T_j' = \bigcup_{j' \neq j} T_{j'}$. By the definition of the irreducible components, we have for every j, and an arbitrary open neighbourhood $W \subset V$ of a, $T_j \cap W \not\subset T_j' \cap W$. Since $t_j \cap W$ is dense in $T_j \cap W$, and $T_j' \cap W$ is closed in W, this implies $t_j \cap W \not\subset T_j' \cap W$. For every $x \in (t_j \cap W) \cap {}^c T_j'$, $\mathbf{S}_x = (\mathbf{T}_j)_x$, so that x is a regular point of S of dimension k_j. Thus, in every neighbourhood of a there exist regular points of S, and Theorem 5 is proved.

REMARK. We have actually seen that in every neighbourhood of a there exist regular points of each one of the dimensions k_j, $1 \leqslant j \leqslant n$.

DEFINITION 1(a). *Let S be an analytic set (in an open set U in C^m), S^* the set of regular points of S. The dimension of S at $a \in S$ is*

$$\dim_a S = \limsup_{x \in S^*, x \to a} \dim_x S.$$

(After Theorem 5, $\dim_a S$ is then defined for every $a \in S$, and we have $0 \leqslant \dim_a S \leqslant m$.)

REMARKS. (1) At points of S^*, the new definition gives the same dimension for S as the earlier one.

(2) If the analytic sets S and T induce the same germ at $a \in S \cap T$, then $\dim_a S = \dim_a T$. Hence :

DEFINITION 1(b). *If $\mathbf{S}$ is a non-empty germ of analytic set at $a \in C^m$, then the dimension of $\mathbf{S}$ is the dimension at a of any analytic set inducing the germ $\mathbf{S}$ at a.*

(3) $\dim \mathbf{S} = m$ if and only if $\mathbf{S} = \mathbf{C}^m$.

(4) $\dim \mathbf{S}_a = 0$ if and only if $\mathbf{S}_a = \{\mathbf{a}\}$. This is a consequence of Corollary 1 to Theorem 3 of Chap. III.

Since the dimension of an analytic set at any point of it has been defined in terms of its dimensions at its regular points, Proposition 1 of Chap. III, § 2 implies

PROPOSITION 1. *If F is a one-one biholomorphic mapping of an open set U onto an open set V in C^m, then for any analytic set S in U, and any $a \in S$, $\dim_a S = \dim_{F(a)} F(S)$.*

We now consider irreducible germs.

PROPOSITION 2(a). *If I is a prime ideal in $\mathscr{H}^m$, $\{0\} \neq I \subsetneq \mathscr{H}'^m$, and if the basis of C^m is k-proper for I, then $k = \dim \mathbf{S}_I$ (in particular, k is uniquely determined by I).*

PROOF. By Theorem 3(B) of Chap. III, there exists an analytic set S_0 in an open neighbourhood π_0 of $\mathfrak{o}$ inducing the germ $\mathbf{S}_I$ at $\mathfrak{o}$, and a dense subset s_0 of S_0 consisting entirely of regular points of S_0 of dimension k. Hence $\dim_x S_0 = k$ for every $x \in S_0$, in particular $\dim_{\mathfrak{o}} S_0 = k$, q.e.d.

COROLLARY. *An irreducible (non-empty) germ of analytic set has dimension $m - 1$ if and only if it is the germ of a principal analytic set* (cf. remarks in Chap. III, § 3).

PROPOSITION 2 (b). *If an analytic set S in an open set U in C^m induces an irreducible germ at $\mathfrak{o} \in S$, then there exists a neighbourhood $V(\subset U)$ of $\mathfrak{o}$ and a basis $\{V_n\}$ for the system of open neighbourhoods of $\mathfrak{o}$, such that S has the same dimension at every point of $V \cap S$ and $V_n \cap S^*$ is connected, $n = 1, 2, 3, \ldots$.*

PROOF. If $\mathbf{S} = \{\mathfrak{o}\}$ or $\mathbf{C}^m$, the proposition is trivially valid. Otherwise, we have $\{\mathbf{0}\} \neq I(\mathbf{S}) \subsetneq \mathscr{H}'^m$, and Theorem 3(B) of Chap. III applies, as in the proof of Proposition 2 (a).

COROLLARY. *Under the conditions of Proposition 2(b), all sufficiently small neighbourhoods of $\mathfrak{o}$ meet just one connected component of S^*.*

PROPOSITION 2 (c). *Let* $\mathbf{S}$, $\mathbf{S}'$ *be irreducible non-empty germs of analytic sets at* $\mathfrak{o}$: *if* $\mathbf{S}' \subset \mathbf{S}$, *then* $\dim \mathbf{S}' \leqslant \dim \mathbf{S}$; *if* $\mathbf{S}' \underset{\neq}{\subset} \mathbf{S}$, $\dim \mathbf{S}' < \dim \mathbf{S}$.

PROOF. If $\mathbf{S} = \mathbf{C}^m$ or $\mathbf{S}' = \{\mathfrak{o}\}$, the proposition is obvious. If $\{\mathfrak{o}\} \underset{\neq}{\subset} \mathbf{S}' \subset \mathbf{S} \neq \mathbf{C}^m$ we have $\{\mathbf{0}\} \neq I(\mathbf{S}) \subset I(\mathbf{S}') \underset{\neq}{\subset} \mathscr{H}'^m$. Hence (Remark 3 on Definition 2, Chap. III) we have a basis for C^m which is k-proper for I, and k'-proper for I', $k' \leqslant k$. If $\mathbf{S}' \underset{\neq}{\subset} \mathbf{S}$, then $I(\mathbf{S}) \underset{\neq}{\subset} I(\mathbf{S}')$, hence $k' < k$ (Proposition 3 of Chap. III, § 3). By Proposition 2 (a) $\dim \mathbf{S} = k$, $\dim \mathbf{S}' = k'$, hence the proposition.

Now to reducible germs.

PROPOSITION 3. *Let S be an analytic set in an open set U in C^m, and suppose* $\mathbf{S} = \mathbf{S}_a$ *is reducible. Let* $\mathbf{T}_1, \ldots, \mathbf{T}_n$ $(n \geqslant 2)$ *be the irreducible components of* $\mathbf{S}$. *Then there exist an open neighbourhood V ($\subset U$) of a, and analytic sets $T_1, \ldots, T_n$ in V inducing respectively the germs* $\mathbf{T}_1, \ldots, \mathbf{T}_n$ *at a, such that the following hold.*

(a) $V \cap S = \bigcup_{j=1}^{n} T_j$; $\dim_x T_j = \dim \mathbf{T}_j = k_j$ *say, for every* $x \in T_j$, $j = 1, \ldots, n$.

(b) *Let T_j^* be the set of regular points of T_j, and let* $T'_j = \bigcup_{j' \neq j} T_{j'}$. *There exists an open neighbourhood $W \subset V$ of a such that, for any $x \in W$, we have $x \in S^*$ if and only if $x \in T_j^* \cap {}^cT'_j$ for (exactly) one j, $1 \leqslant j \leqslant n$, and then* $\dim_x S = k_j$. (Each $T_j^* \cap {}^cT'_j$ meets every neighbourhood of a, by the definition of the irreducible components of a germ.)

(c) *For $1 \leqslant j \leqslant n$, there exist arbitrarily small open neighbourhoods $W_j \subset V$ of a such that $W_j \cap T_j^* \cap {}^cT'_j$ is connected, and dense in $W_j \cap T_j$.*

PROOF. By Proposition 2 (b), we can find V and the T_j such that (a) holds, $1 \leqslant j \leqslant n$. Further, by the same proposition, there exist arbitrarily small open neighbourhoods $W_j \subset V$ of a such that $W_j \cap T_j^*$ is connected.

(b) Let $W \subset \bigcap_{j=1}^{n} W_j$ be an open neighbourhood of a, and $x \in W \cap S^*$. Then, in particular, $\mathbf{S}_x$ is irreducible. It follows, from $V \cap S = \bigcup_{j=1}^{n} T_j$, that for some j, $1 \leqslant j \leqslant n$, $\mathbf{S}_x = (\mathbf{T}_j)_x$ (= germ of T_j at x). Hence $x \in T_j^*$. Now suppose we also have $x \in T'_j$. Let $x \in T_{j'}$, $j' \neq j$. Since $(\mathbf{T}_{j'})_x \subset \mathbf{S}_x = (T_j)_x$, and $T_{j'}^*$ is dense in $T_{j'}$, we have $y \in T_{j'}^* \cap W$ such that $(\mathbf{T}_{j'})_y \subset (\mathbf{T}_j)_y$. Since $T_{j'}^* \cap W_{j'}$ is *connected*, we have, by Proposition 3 of Chap. III, § 2, $T_{j'}^* \cap W_{j'} \subset T_j \cap W_{j'}$. Finally, $T_{j'}^* \cap W_{j'}$ is dense in $T_j \cap W_{j'}$, and $T_j \cap W_{j'}$ is closed in $W_{j'}$, hence $T_{j'} \cap W_{j'} \subset T_j \cap W_{j'}$. But this is impossible, since $\mathbf{T}_j$ and $\mathbf{T}_{j'}$ are distinct irreducible components of $\mathbf{S}$. Hence we have proved that $x \in W \cap S^*$ implies $x \in T_j^* \cap {}^cT'_j$ for one j ($1 \leqslant j \leqslant n$.) The converse implication is trivial (for $x \in W$), hence (b) is proved.

(c) To prove that $W_j \cap T_j^* \cap {}^cT'_j$ is dense in $W_j \cap T_j$, it is sufficient to prove that it is dense in $W_j \cap T_j^*$. Now suppose $W_j \cap T_j^* \cap {}^cT'_j$ is not dense in $W_j \cap T_j^*$. Then there exists $x \in W_j \cap T_j^*$ such that $(\mathbf{T}_j)_x \subset (\mathbf{T}'_j)_x$. Hence, as in the proof of (b), we obtain, by using Proposition 3 of Chap. III, § 2, and the connectedness of $W_j \cap T_j^*$, the contradiction $W_j \cap T_j \subset T'_j \cap W_j$.

Finally $W_j \cap T_j^* \cap {}^cT'_j$ is connected by Proposition 4 of Chap. III, § 2 : $W_j \cap T_j^* = (W_j \cap T_j)^*$ is connected, $T'_j \cap W_j \not\supset W_j \cap T_j^*$ as we have just proved. Hence Proposition 3 is proved.

We now list some immediate consequences of Proposition 3. The notation and the assumptions are as in Proposition 3.

Consequences of Proposition 3(b). 1. For any $x \in W$, x is a non-regular point of S if and only if x is a non-regular point of one T_j, or belongs to at least two of the T_j.

2. For any $x \in W$, $\dim_x S$ is one of the integers k_j, and $\dim_a S = \sup_{1 \leqslant j \leqslant n} k_j$: the dimension of a germ of analytic set is the highest of the dimensions of its irreducible components. In particular :

3. Any non-empty germ of principal analytic set has dimension $m-1$. A germ has dimension $m-1$ if and only if at least one of its irreducible components is the germ of a principal analytic set.

CONSEQUENCES OF PROPOSITION 3(b) AND 3(c). 4. $x \in W \cap S^*$ implies $x \in W_j \cap T_j^* \cap {}^cT_j'$ for precisely one j, $1 \leqslant j \leqslant n$. Since the $W_j \cap T_j^* \cap {}^cT_j'$ are connected subsets of S^*, we obtain: the neighbourhood W of x meets at the most n connected components of S^*. It follows that the family of all connected components of S^* is locally finite in S (hence in U). This result includes Corollary 1 to Theorem 3 of Chap. III.

COROLLARY 1. (a) *If S', S are analytic sets in an open set U in C^m, $S' \subset S$, then* $\dim_a S' \leqslant \dim_a S$ *for every $a \in S'$.* (b) *If* $\mathbf{S}$, $\mathbf{S}'$ *are non-empty germs of analytic sets at $a \in C^m$,* $\mathbf{S}' \subset \mathbf{S}$ *implies* $\dim \mathbf{S}' \leqslant \dim \mathbf{S}$. (c) *If, in* (b), $\mathbf{S}$ *is irreducible, then* $\mathbf{S}' \underset{\neq}{\subset} \mathbf{S}$ *implies* $\dim \mathbf{S}' < \dim \mathbf{S}$. (d) *If, in* (b), $\dim \mathbf{S}' = \dim \mathbf{S}$, *then* $\mathbf{S}$ *and* $\mathbf{S}'$ *have at least one irreducible component in common.* (e) *If* $\mathbf{S}_1, \ldots, \mathbf{S}_n$ *are non-empty germs of analytic sets at $a \in C^m$, and* $\mathbf{S} = \bigcup_{j=1}^{n} \mathbf{S}_j$, *then* $\dim \mathbf{S} = \max_{1 \leqslant j \leqslant n} \dim \mathbf{S}_j$.

PROOF. (b) implies (a). To prove (b), let $\mathbf{T}'$ be any one of the irreducible compenents of $\mathbf{S}'$. Then $\mathbf{S}' \subset \mathbf{S}$ implies $\mathbf{T}' \subset \mathbf{T}$ for one of the irreducible components $\mathbf{T}$ of $\mathbf{S}$. By Proposition 2(c) $\dim \mathbf{T}' \leqslant \dim \mathbf{T}'$. Since $\dim \mathbf{S}'$ is the greatest of the dimensions of its irreducible components, and similarly $\dim \mathbf{S}$, (b) is proved.

(c) is proved similarly. Since $\mathbf{S}' \underset{\neq}{\subset} \mathbf{S}, \mathbf{T}' \underset{\neq}{\subset} \mathbf{S}$ for every irreducible component $\mathbf{T}'$ of $\mathbf{S}'$. Hence, by Proposition 2(c), $\dim \mathbf{T}' < \dim \mathbf{S}$. (c) follows.

(d) follows from (b); we again use the fact that the dimension of a germ of an analytic set is the highest of the dimensions of its irreducible components.

For the proof of (e), let $\mathbf{S}^{(i)}$, $q_{j-1} + 1 \leqslant i \leqslant q_j$ ($q_0 = 0$), be the irreducible components of $\mathbf{S}_j$, $1 \leqslant j \leqslant n$. Then it is easily seen that the irreducible components of $\mathbf{S} = \bigcup_{j=1}^{n} \mathbf{S}_j$ are precisely those $\mathbf{S}^{(i)}$ such that

$\mathbf{S}^{(i)} \underset{\neq}{\subset} \mathbf{S}^{(i')}$ does not hold for any i'. Hence $\dim \mathbf{S} = \max_{1 \leqslant i \leqslant q_n} \dim \mathbf{S}^{(i)}$ $= \max_{1 \leqslant j \leqslant n} \dim \mathbf{S}_j$, q.e.d.

COROLLARY 2. *Let S, S' be analytic sets in an open set U in C^m, $S' \subset S$, and suppose, for some $a \in S'$, that $\mathbf{S}_a$ is irreducible and $\mathbf{S}'_a \neq \mathbf{S}_a$. Then there exists an open neighbourhood W of a, $W \subset U$, such that $\mathbf{S}'_x \neq \mathbf{S}_x$ for every $x \in W \cap S'$. (Consequently $W \cap (S - S')$ is dense in $W \cap S$.)*

PROOF. By Propositions 2(b) and 3(b), there exists an open neighbourhood W of a, $W \subset U$, such that for $x \in W \cap S$, $\dim \mathbf{S}_x = \dim \mathbf{S}_a$, while, for $x \in W \cap S'$, $\dim \mathbf{S}'_x \leqslant \dim \mathbf{S}'_a$. Since, by Corollary 1(c), $\dim \mathbf{S}'_a < \dim \mathbf{S}_a$, we then have $\dim \mathbf{S}'_x < \dim \mathbf{S}_x$ for every $x \in W \cap S'$. Hence $\mathbf{S}'_x \neq \mathbf{S}_x$, and the corollary is proved.

COROLLARY 3(a). *Let $\mathbf{S}$ be a non-empty germ of analytic set at $a \in C^m$. Then* $\dim \mathbf{S} \leqslant m - 2$ *if and only if* $\mathbf{S}$ *does not contain any non-empty germ of principal analytic set at a.*

PROOF. Any non-empty germ of principal analytic set has dimension $m - 1$. Hence, by Corollary 1(b), $\dim \mathbf{S} \leqslant m - 2$ implies that $\mathbf{S}$ does not contain any non-empty germ of principal analytic set.

To show that, if $\dim \mathbf{S} \geqslant m - 1$, $\mathbf{S}$ contains a non-empty germ of principal analytic set, we may clearly assume $\dim \mathbf{S} = m - 1$. Since one at least of the irreducible components of $\mathbf{S}$ then has dimension $m - 1$, it is sufficient to use the corollary to Proposition 2(a).

Hence Theorem 2(b) of Chap. III, § 1 may be restated as follows.

THEOREM. *Let S be an analytic set in an open set U in C^m, $m \geqslant 2$, and suppose* $\dim_x S \leqslant m - 2$ *for every $x \in S$. Then every holomorphic function on $U - S$ has a unique holomorphic extension to the whole of U.*

COROLLARY 3(b). *An analytic set S in an open set U in C^m is a principal analytic set if and only if* $\dim_x S = m - 1$ *for every $x \in S$.*

PROOF. We have already seen that, if S is a non-empty principal analytic set, $\dim_x S = m - 1$ for every $x \in S$. For the converse, it is sufficient, after the corollary to Proposition 2(a), to show that for any $x \in S$, and any irreducible component $\mathbf{T}$ of $\mathbf{S}_x$, dim $\mathbf{T} = m - 1$ (for, if every irreducible component of $\mathbf{S}_x$ is principal, $\mathbf{S}_x$ is principal). But this is clear from Proposition 3(b) : if $k = \dim \mathbf{T}$, then there are, arbitrarily near x, (regular) points of S of dimension k.

PROPOSITION 4. *If I is an arbitrary ideal in $\mathscr{H}^m$, and the ideal generated in $\mathscr{H}^m$ by I, $\mathbf{x}_1, \dots, \mathbf{x}_k$ $(1 \leqslant k \leqslant m-1)$ defines the germ $\{\mathfrak{o}\}$ at $\mathfrak{o}$* (Condition A of Chap. III), *then* :

(a) $\dim \mathbf{S}_I \leqslant k$; (b) $\dim \mathbf{S}_I = k$ *if and only if the basis of C^m is k-proper for I.*

PROOF. The hypothesis implies $\{\mathfrak{o}\} \subset \mathbf{S}_I \neq \mathbf{C}^m$; if $\mathbf{S}_I = \{\mathfrak{o}\}$, (a) is trivial: rad $I = \mathscr{H}'^m$ by Theorem 2 (b), hence, with the notation of Chap. III, §3, any germ $\in \mathscr{H}'^k$ has a power $\in I_k$ and the basis is not k-proper for I.

Now we assume $\{\mathfrak{o}\} \subsetneq \mathbf{S}_I \neq \mathbf{C}^m$; this holds also for the irreducible components $\mathbf{T}_1, \dots, \mathbf{T}_n$ of $\mathbf{S}_I$. For each $j = 1, \dots, n$, $I^{(j)} = I(\mathbf{T}_j) \subset I$, hence $x_1, \dots, x_k$ satisfy condition A for each $I^{(j)}$; since $I^{(j)}$ is prime and $\{\mathfrak{o}\} \neq I^{(j)} \subsetneq \mathscr{H}'^m$, the procedure of Proposition 2 (Chap. III, § 3) gives a basis of C^m which is k_j-proper for $I^{(j)}$, with $k_j \leqslant k$, and $k_j = k$ if and only if $I_k^{(j)} = \{0\}$. By Proposition 2 (a), $\dim \mathbf{T}_j = k_j$ and, by consequence 2 of Proposition 3, $\dim \mathbf{S}_I = \sup_{1 \leqslant j \leqslant n} k_j$; thus (a) is proved, and (b) will be proved if we show that $I_k = \{0\}$ if and only if $I_k^{(j)} = \{0\}$ for at least one j.

$I_k^{(j)} \supset I_k$, hence $I_k^{(j)} = \{0\}$ for one j obviously implies $I_k = \{0\}$. If, conversely, $I_k^{(j)} \neq \{0\}$ for each j, let $\mathbf{f}^{(j)} \in I_k^{(j)} - \{0\}$ for each j and $\mathbf{f} = \prod_{j=1}^{n} \mathbf{f}^{(j)} : \mathbf{f} \neq 0, \mathbf{f} \in \bigcap_{j=1}^{n} I^{(j)} = I(\mathbf{S}_I) = \text{rad } I$ by Theorem 2 (b), hence, for a suitable integer $\alpha > 0$, $\mathbf{f}^\alpha \in I$ or I_k, which implies $I_k \neq \{0\}$.

REMARK. We might expect the following analogue of Proposition 4 to hold : if k independent forms on C^m satisfy the condition B of Chap. III (Proposition 2, §3) for the ideal I in $\mathscr{H}^m$, $1 \leqslant k \leqslant m-1$, then $\dim \mathbf{S}_I \geqslant k$. But this is not the case, as the following example shows.

In $C^5 = \{(x_1, \ldots, x_5)\}$, let S be the analytic set $\{x_1 - x_4 = x_2 - x_4 x_5 = x_3 - x_4 x_5 e^{x_5} = 0\}$. Then, any point of S is a regular point of S of dimension 2 (cf. Chap. III, § 2, example). We now assert that x_1, x_2, x_3 satisfy the condition B of Chap. III for $I = I(\mathbf{S}_0)$. To prove this, we must show that $\mathscr{H}^3 \cap I = \{\mathbf{0}\}$. Let $\mathbf{f} \in \mathscr{H}^3 \cap I$. Let the Taylor expansion of f at $\mathfrak{o}$ be $\sum_{n=1}^{\infty} P_n$, where P_n is a homogeneous polynomial of degree n in x_1, x_2, x_3, $n = 1, 2, \ldots$. $\mathbf{f} \in I$ implies $f(x_4, x_4 x_5, x_4, x_5 e^{x_5}) = \sum_{n=1}^{\infty} x_4^n P_n(1, x_5, x_5 e^{x_5}) \equiv 0$ in some neighbourhood of $x_4 = x_5 = 0$. Since $P_n(1, x_5, x_5 e^{x_5})$ is an entire function of x_5, we must then have $P_n(1, x_5, x_5 e^{x_5}) \equiv 0$ (for all $n \geqslant 1$). It follows easily that $P_n \equiv 0$, $n = 1, 2, \ldots$. Hence $\mathbf{f} = \mathbf{0}$, and our assertion is proved.

Corollary 1. *Let S be an analytic set in an open set U in C^m. Then for any $a \in S$, $m - \dim_a S$ is the highest of the dimensions of affine subvarieties L of C^m with the property : a is an isolated point of $L \cap S$.*

Proof. We may suppose $a = \mathfrak{o}$. Further, our assertion is evident if $\mathbf{S} = \mathbf{S}_0 = \{\mathfrak{o}\}$ or $\mathbf{C}^m$. Hence let $\{\mathfrak{o}\} \subsetneq \mathbf{S} \neq \mathbf{C}^m$, i.e. $\{\mathbf{0}\} \neq I = I(S) \subsetneq \mathscr{H}'^m$: by Corollary 1 to Theorem 2, $I = \text{rad}\ I$, therefore, by Proposition 2 of Chap. III, § 3, there exists a basis of C^m which is k-proper for I, $1 \leqslant k \leqslant m-1$. Then $k = \dim S$ (Proposition 4(b)), and condition A of Chap. III means precisely this : the $(m-k)$-dimensional subspace $L_1 = \{x_1 = \ldots = x_k = 0\}$ of C^m is such that $\mathfrak{o}$ is an isolated point of $L_1 \cap S$.

Now we have to show that, for any subspace L of C^m such that $\mathfrak{o}$ is an isolated point of $L \cap S$, $\dim L \leqslant m-k$: we may suppose $1 \leqslant \dim L \leqslant m-1$, and find $(m - \dim L)$ independent linear forms on C^m having L as their set of common zeros ; these forms satisfy condition A of Chap. III for I, hence (Proposition 4(a)) $k = \dim \mathbf{S} \leqslant m - \dim L$.

Remark. This corollary establishes the identity of our definition of the dimension of an analytic set with that given by Remmert-Stein [4].

COROLLARY 2. *Let S be an analytic set in an open set U in C^m, and the basis of C^m be such that $\mathfrak{o}$ is an isolated point of $S \cap \{x_1 = \dots = x_r = 0\}$ for an integer r, $1 \leqslant r \leqslant m-1$. Suppose there exists a sequence P_n of open polydiscs, with centre $\mathfrak{o}$ and radii decreasing to 0, such that, for each n, the projection of $S \cap P_n$ on C^r is an analytic set S'_n in the projection P'_n of P_n; then, for sufficiently great n :* $\dim_{\mathfrak{o}} S'_n = \dim_{\mathfrak{o}} S$ *($\mathfrak{o}'$ = origin in C^r).*

PROOF. Consider the increasing sequence of ideals $I(\mathbf{S}'_n)$ in $\mathscr{H}^r_{\mathfrak{o}'}$: since $\mathscr{H}^r_{\mathfrak{o}'}$ is a Noetherian ring, for $n \geqslant$ a suitable n_0, they all coincide with an ideal I' such that $I' = \operatorname{rad} I'$ (cf. Corollary 1 to Theorem 2). $I' \subset \mathscr{H}'^r_{\mathfrak{o}'}$; if $I' = \mathscr{H}'^r_{\mathfrak{o}'}$, we have $\mathbf{S}'_n = \{\mathfrak{o}'\}$ for $n \geqslant n_0$, $\mathbf{S} = \{\mathfrak{o}\}$, and the corollary is proved ; if $I' = \{\mathbf{0}\}$, we have $S'_n = P'_n$, for each n, the basis of C^m is r-proper for $I(S)$ (cf. remark on Corollary 2 to Theorem 3), and the corollary is proved with the help of Proposition 4 (b).

Now let $\{\mathbf{0}\} \neq I' \subsetneq \mathscr{H}'^r_{\mathfrak{o}'}$: by Proposition 2 of § 3, Chap. III, we can find a basis of C^r which is k-proper for I', $0 < k < r$; then, by Proposition 4 (b), it is enough to know that this basis of C^r and the last $m - r$ elements of the given basis of C^m constitute a new basis of C^m which is k-proper for $I(\mathbf{S})$, and this is obvious.

REMARK. This corollary includes the remark on Corollary 2 to Theorem 3, which was referred to in the above proof.

THEOREM 6 (H. CARTAN). *The set $S - S^*$ of non-regular points of an analytic set S (in an open set U in C^m) is again an analytic set in U, and* $\dim_a (S - S^*) < \dim_a S$ *for every $a \in S - S^*$.*

FIRST PROOF. (a) We first make the following assumptions: (i) S has the same dimension k, $1 \leqslant k \leqslant m-1$, at every point of S; (ii) there exist k non-empty open sets $W_1, \dots, W_k$ in the dual space C^m such that, if $A_1 \in W_1, \dots, A_k \in W_k$, the k linear forms $A_1, \dots, A_k$ are independent and any $a \in S$ is an isolated point of the analytic set $\{x \in S \mid A_1(x) - A_1(a) = \dots = A_k(x) - A_k(a) = 0\}$.

For $x \in S^*$, let L_x be the affine variety, of dimension k, tangent to S at x: given $x \in S^*$ and $m-k$ independent linear forms $A_{k+1}, \dots, A_m$

assuming constant values on L_x, we can find $A_1 \in W_1, \ldots, A_k \in W_k$ such that the m forms $A_1, \ldots, A_m$ are independent ; then there exists a point $\in L_x$ where $A_1, \ldots, A_k$ assume any given system of k values, i.e., if we choose a basis of C^m for which $A_1, \ldots, A_k$ are the first k coordinates, the projection of L_x on the subspace generated by the first k elements of the basis is this subspace itself. We have just proved this : if $S^*(A_1, \ldots, A_k)$ is the set of points $x \in S^*$ such that there exists a point $\in L_x$ where $A_1, \ldots, A_k$ assume any given system of k values, then $S^* = \bigcup_{A_i \in W_i} S^*(A_1, \ldots, A_k)$. By Corollary 1 to Theorem 3, $S - S^*$ will be an analytic set in U if we show that, for any $A_1 \in W_1, \ldots, A_k \in W_k$, $S - S^*(A_1, \ldots, A_k)$ is an analytic set in U, i.e. : any point $a \in S$ has an open neighbourhood $V \subset U$ such that $V \cap [S - S^*(A_1, \ldots, A_k)]$ is an analytic set in V. In order to show this, we choose a as the origin and a basis of C^m for which $A_1, \ldots, A_k$ are the first k coordinates.

Let $\mathbf{T}_1, \ldots, \mathbf{T}_n$ be the irreducible components of the germ $\mathbf{S} = \mathbf{S}_a$: by (ii), $A_1, \ldots, A_k$ satisfy condition A of Chap. III (§3, Proposition 1) for the ideal $I(\mathbf{S})$, *a fortiori* for each $I(\mathbf{T}_j)$, $j = 1, \ldots, n$. By (i) and Proposition 3, $\dim \mathbf{T}_j = k$ and therefore, by Proposition 4(b), the basis of C^m which we have chosen is k-proper for $I(\mathbf{T}_j)$; then, for each $j = 1, \ldots, n$, there exists an open polydisc π_j with centre $\mathfrak{o}$ and an analytic set T_j in π_j, inducing $\mathbf{T}_j$ at $\mathfrak{o}$, with the properties listed in Theorem 3 of Chap. III : in particular, by part (B) of this theorem, there exists a dense subset of T_j consisting entirely of regular points of T_j of dimension k, hence $\dim_x T_j = k$ for every $x \in T_j$; and, by Remark 1, $T_j - T_j^*(A_1, \ldots, A_k)$ is the set of points of T_j which are common zeros of a family of functions holomorphic on π_j, i.e., by Corollary 1 to Theorem 3 again, an analytic set in π_j. Now, by Proposition 3(b), there exists an open neighbourhood $V \subset \bigcap_{j=1}^{n} \pi_j$ of $\mathfrak{o}$ such that $V \cap S = \bigcup_{j=1}^{n} (V \cap T_j)$ and $V \cap (S - S^*)$ is the union of the sets $V \cap T_j \cap T_{j'}$, $1 \leqslant j < j' \leqslant n$, and $V \cap (T_j - T_j^*)$, $1 < j < n$; then $V \cap [S - S^*(A_1, \ldots, A_k)]$ is the union of the sets $V \cap T_j \cap T_{j'}$, $1 < j < j' < n$, and $V \cap [T_j - T_j^*(A_1, \ldots, A_k)]$, $1 < j < n$, each of which is an analytic set in V. Thus we have proved that $S-S^*$ is an analytic set in U, under the assumptions (i) and (ii).

(b) In the general case, since $S-S^*$ is closed in U, we have to show that any point $x_0 \in S-S^*$ has an open neighbourhood $V \subset U$ such that $V \cap (S-S^*)$ is an analytic set in V; since $x_0 \in S-S^*$, the non-empty germ $\mathbf{S}=\mathbf{S}_{x_0}$ is *neither* $\{\mathbf{x}_0\}$ *nor* $\mathbf{C}^m$, and this holds also for the irreducible components of $\mathbf{S}$.

If $\mathbf{S}$ is irreducible : then $I(\mathbf{S})$ is prime, $\{0\} \neq I(\mathbf{S}) \subsetneq \mathcal{H}'^m_{x_0}$; we may choose x_0 as the origin $\mathfrak{o}$ and a basis of C^m which is k-proper for $I(\mathbf{S})$, $1 \leqslant k \leqslant m-1$ (Chap. III, § 3, Proposition 2). By Proposition 2(a), $k=\dim \mathbf{S}=\dim_{\mathfrak{o}} S$; by Proposition 2(b), the origin has an open neighbourhood $V_1 \subset U$ such that S has dimension k at every point of $S \cap V_1$; by Corollary 3 to Theorem 4, there exist another open neighbourhood $V_2 \subset U$ of $\mathfrak{o}$ and, for each $j=1, \ldots, k$, an open neighbourhood W_j of the linear form $x \rightarrow x_j$ in the dual space, such that, for any $a \in S \cap V_2$, $A_1 \in W_1, \ldots, A_k \in W_k$, a is an isolated point of the analytic set $\{x \in S \mid A_1(x)-A_1(a)=\ldots=A_k(x)-A_k(a)=0\}$. If the W_j are small enough, $A_j \in W_j$, $j=1, \ldots, k$, also implies the independence of the A_j : then the assumptions (i) and (ii) of part (a) are fulfilled for the analytic set $S \cap V_1 \cap V_2$ in $V_1 \cap V_2$, hence $(S \cap V_1 \cap V_2)-(S \cap V_1 \cap V_2)^*=(S-S^*) \cap (V_1 \cap V_2)$ is an analytic set in $V_1 \cap V_2$.

If $\mathbf{S}$ is reducible, let $\mathbf{T}_1, \ldots, \mathbf{T}_n$ be its irreducible components : by Proposition 3(b), there exist an open neighbourhood $W \subset U$ of x_0, and analytic sets $T_1, \ldots, T_n$ in W, inducing respectively the germs $\mathbf{T}_1, \ldots, \mathbf{T}_n$ at x_0, such that $W \cap S=\bigcup_{j=1}^{n} T_j$ and $W \cap (S-S^*)$ is the union of the sets

$$W \cap T_j \cap T_{j'},\ 1 \leqslant j<j' \leqslant n, \text{ and } W \cap (T_j-T_j^*),\ 1 \leqslant j \leqslant n.$$

We have just shown the existence, for each $j=1, \ldots, n$, of an open neighbourhood $V_j \subset W$ of x_0 such that $V_j \cap (T_j-T_j^*)$ is an analytic set in V_j; then, if $V=\bigcap_{j=1}^{n} V_j$, $V \cap (S-S^*)$ is an analytic set in V.

Finally, by Corollary 1 to Proposition 3, $S-S^* \subset S$ implies $\dim_a(S-S^*) \leqslant \dim S$ for every $a \in S-S^*$, and $\dim_a(S-S^*)=\dim_a S$

would have the following consequence : if $\mathbf{S}_a$ is irreducible, some neighbourhood of a would not meet S^*, which contradicts Theorem 5; if $\mathbf{S}_a$ is reducible, with the notation of Proposition 3(b), $S - S^*$ would include one T_j in some neighbourhood of a, which is impossible since any neighbourhood of a meets $T_j^* \cap {}^cT'_j$. Theorem 6 is proved.

SECOND PROOF. On account of Proposition 3, we may assume that $\dim_x S$ is the same, say k, $1 \leqslant k \leqslant m-1$, for all $x \in S$, and it is enough to show that every $a \in S$ has an open neighbourhood $V \subset U$ such that $V \cap (S - S^*)$ is an analytic set in V. This can be done very simply with the help of Theorem 1 : in fact, Theorem 1 means that, for every $a \in S$, there exist an open neighbourhood $V \subset U$ of a and a finite family of functions $f_1, \dots, f_n$ holomorphic on V, such that the germs induced by $f_1, f_2, \dots, f_n$ at any $x \in V$ generate the ideal $I(\mathbf{S}_x)$, which implies that $V \cap S$ is the set of points $\in V$ where $f_1 = \dots = f_n = 0$. Then Theorem 6 will be proved if we show that $V \cap (S - S^*)$ is the set of points $x \in V \cap S$ such that the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, is $< m-k$, and this is a consequence of the following two remarks.

(A) If $x \in V \cap S^*$, since $\dim_x S = k$, we can find an open neighbourhood $W \subset V$ of x and $m-k$ functions $g_1, \dots, g_{m-k}$, holomorphic on W and vanishing on $W \cap S$, such that the matrix $\left(\frac{\partial g_i}{\partial x_j}(x)\right)$, $1 \leqslant i \leqslant m-k$, $1 \leqslant j \leqslant m$, has rank $m-k$. Since the germs induced at x by $g_1, \dots, g_{m-k}$ are linear combinations, over $\mathscr{H}_x^m$, of the germs induced by $f_1, \dots, f_n$, the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, must have a rank $\geqslant m-k$.

(B) If $x \in V \cap S$, and the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant m$, is r, then the analytic set T, in V, of points $\in V$ where $f_1 = \dots = f_r = 0$, induces at x a regular germ $\mathbf{T}_x$ of dimension $m-r$ (cf. Theorem 1 of Chap. I and Definition 1 of Chap. III) ; since $T \supset V \cap S$ and $\dim \mathbf{S}_x = k$, we must have $m - r \geqslant k$ or $r \leqslant m-k$, and equality implies, by Corollary 1(c) to Proposition 3, $\mathbf{T}_x = \mathbf{S}_x$, hence $x \in S^*$.

3. Irreducible analytic sets. The irreducible components of an analytic set. We shall make use of some properties which were proved before for the set of regular points of an analytic set S (Chap. III, § 2, Propositions 3 and 4), but actually hold for the set of points $\in S$ where S induces an irreducible germ. In fact, Propositions 3 and 4 of Chap. III, § 2, are included in the following theorem.

THEOREM 7. *Let S be an analytic set in an open set U in C^m, and s an open connected subset of S such that for every $x \in s$, $\mathbf{S}_x$ is irreducible. Then, for every analytic set S' in U:* (a) *if $\mathbf{S}'_{x_0} \supset \mathbf{S}_{x_0}$ for some $x_0 \in s$, we have $S' \supset s$;* (b) *$S' \not\supset s$ implies that $s \cap {}^cS'$ is connected.*

PROOF (a). The set $\tilde{s} = \{x \in s \mid \mathbf{S}'_x \supset \mathbf{S}_x\} = \{x \in s \mid (\mathbf{S}' \cap \mathbf{S})_x = \mathbf{S}_x\}$ is open in s by definition, and is closed in s by Corollary 2 to Proposition 3 of § 2. Hence either $\tilde{s} = \varnothing$ or $\tilde{s} = s$, and (a) is proved.

(b) Suppose $s \cap {}^cS' = s_0 \cup s_1$, s_0 and s_1 being non-empty, open and disjoint subsets of s. By (a), $s \cap {}^cS'$ is dense in s. Hence there exists an $a \in \bar{s}_0 \cap \bar{s}_1 \cap s$: s being connected, $\bar{s}_0 \cap s$ and $\bar{s}_1 \cap s$ cannot be disjoint. Now, by Proposition 2(b) of § 2, there exists an open neighbourhood W of a, $W \subset U$, such that $W \cap S \subset s$, and $W \cap S^*$ is connected. By Proposition 4 of § 2, Chap. III, $W \cap S^* \cap {}^cS'$ is also connected. Hence $W \cap s \cap {}^cS'$, in which this set is dense, is also connected. But since $W \cap s \cap {}^cS' = (W \cap s_0) \cup (W \cap s_1)$, and $W \cap s_0$ $W \cap s_1$ are non-empty disjoint open subsets of $W \cap s \cap {}^cS'$ (non-empty because $a \in \bar{s}_0 \cap \bar{s}_1$), we have a contradiction, hence (b) is proved.

PROPOSITION 1. *Let S be an analytic set in an open set U in C^m, S^* the set of its regular points, and s_1 a connected component of S^*. Then the closure S_1 (in U, or equivalently, in S) of s_1 is an irreducible analytic set in U.*

Proof. S_1 is closed in U, and $S_1 \subset S$. To show that S_1 is an analytic set in U, it is sufficient to show that every $a \in S_1$ has an open neighbourhood $W \subset U$ such that $S_1 \cap W$ is an analytic set in W.

Thus let $a \in S_1$. Excluding trivial cases, we may assume that $\{\mathbf{a}\} \neq \mathbf{S} = \mathbf{S}_a \neq \mathbf{C}^m$. Then (Proposition 3 of § 2) there exist an open neighbourhood $V \subset U$ of a, analytic sets $T_1, \ldots, T_n$ in V and open neighbourhoods $W_j \subset V$ of a ($j = 1, \ldots, n$) such that : (i) the irreducible components of $\mathbf{S}$ are precisely the $\mathbf{T}_j$, and $\bigcup_{j=1}^{n} T_j = S \cap V$; (ii) each $T_j^* \cap {}^c T_j' \cap W_j$ is connected, $\subset S^*$, and dense in $T_j \cap \dot{W}_j$; (iii) if $W = \bigcap_{j=1}^{n} W_j$, $S^* \cap W \subset \bigcap_{j=1}^{n} (T_j^* \cap {}^c T_j' \cap W_j)$ (here we have used the notation of Proposition 3 of § 2, with $T_1' = \varnothing$ if $n = 1$). Hence, if the T_j are suitably indexed, there exists an integer p $(1 \leqslant p \leqslant n)$ such that $T_j^* \cap {}^c T_j' \cap W_j$ is contained in s_1 for $1 \leqslant j \leqslant p$, and disjoint with s_1 for $p + 1 \leqslant j \leqslant n$; then $s_1 \cap W = W \cap \bigcup_{j=1}^{p} (T_j^* \cap {}^c T_j' \cap W_j)$, hence, by taking the closures in W of both members, $S_1 \cap W = W \cap \bigcup_{j=1}^{p} T_j$, and our assertion is proved.

To prove that S_1 is irreducible, suppose $S_1 = S_{11} \cup S_{12}$, where S_{11} and S_{12} are analytic sets in U, and $S_{11} \neq S_1$. Then, s_1 being dense in S_1, there exists an $a \in s_1 \cap {}^c S_{11}$: at this point we have $\mathbf{S}_{12} = \mathbf{S}_1$. $s_1 \subset S_1^*$ being connected, by Proposition 3 of § 2 (Chap. III): $S_{12} \supset s_1$. Finally s_1 is dense in S_1, hence $S_{12} \supset S_1$, i.e., $S_{12} = S_1$, q.e.d.

Corollary 1. *Let S be an analytic set in an open set U in C^m. Then S is irreducible if and only if S^*, the set of its regular points, is connected.*

Proof. By Proposition 1, S is irreducible if S^* is connected. Now suppose S^* is not connected. Let $\{s_n\}$ be the countable (i.e. finite or countably infinite : cf. the consequences of Proposition 3, § 2) family of connected components of S^*, and $S_n =$ closure of s_n in U, $n = 1, 2, \ldots$. Since $\{s_n\}$ is locally finite (in U), so is $\{S_n\}$.

By Proposition 1, each S_n is an (irreducible) analytic set in U; hence $S_1' = \bigcup_{n \geqslant 2} S_n$ is also an analytic set in U, being the union of a locally finite family of analytic sets in U. Clearly $S = S_1 \cup S_1'$, and $S_1 \neq S$ by assumption. If we show that $S_1' \neq S$, then S will be reducible, and the corollary will be proved.

Suppose $S_1' = S$. Then $S_1 \subset S_1'$. In particular, at any point of s_1, $\mathbf{S} \subset \mathbf{S}_1'$. Now, since the family $\{S_n\}$ is locally finite, $\mathbf{S}_1'$ is the union of finitely many $\mathbf{S}_n$, $n \geqslant 2$. Since $\mathbf{S}_1$ is irreducible (in fact regular), we must have, for an $n \geqslant 2$, $\mathbf{S}_1 \subset \mathbf{S}_n$. Since s_1 is a connected subset of S_1^*, this implies (Proposition 3 of § 2, Chap. III) $s_1 \subset S_n$. But this is impossible since $s_1 \cap s_n = \emptyset$. This contradiction proves that $S_1' \neq S$, and the proof of Corollary 1 is complete.

Remark. With the notation as above, we have also proved that, for every n, $S_n \not\subset \bigcup_{n' \neq n} S_{n'}$.

Corollary 2. *Given an analytic set S in an open set U, and $a \in S$: the germ $\mathbf{S}_a$ is irreducible if and only if there exist arbitrarily small open neighbourhoods $V \subset U$ of a such that the analytic set $S \cap V$ in V is irreducible.*

Proof. If the germ $\mathbf{S}_a$ is reducible, the set $S \cap V$ is reducible, for any sufficiently small V, by the definition of a reducible germ. If the germ $\mathbf{S}_a$ is irreducible, by Proposition 2 (b) of § 2, there exist arbitarily small open neighbourhoods $V \subset U$ of a such that $V \cap S^* = (V \cap S)^*$ is connected. By Corollary 1, this implies that $V \cap S$ is irreducible.

Corollary 3. *Let S be an irreducible analytic set in an open set U in C^m. Then*

(a) *S is connected.*

(b) *$\dim_x S$ is the same for all $x \in S$.*

(c) (Ritt's lemma) *If S' is an analytic set in U, and $\mathbf{S}'_a \supset \mathbf{T}_a$ for some $a \in S$ and some irreducible component $\mathbf{T}_a$ of $\mathbf{S}_a$, then $S' \supset S$. In particular, if h is a holomorphic function on U, and if, for some $a \in S$, $\mathbf{h}_a$ vanishes on some irreducible component of $\mathbf{S}_a$, then $h \equiv 0$ on S.*

(c′) *For any function h holomorphic on U, the restriction* $h \mid S$ *is either constant or is an open mapping of S into C, in the following strong sense : for any* $a \in S$ *and any analytic set T in an open neighbourhood* $V \subset U$ *of a such that* $\mathbf{T}_a$ *is an irreducible component of* $\mathbf{S}_a$, *the set h(T) is a neighbourhood of h(a) in C.*

(d) *If S′ is an analytic set in U,* $S' \subsetneq S$ *implies* $\dim_x S' < \dim_x S$ *for every* $x \in S'$.

(e) *For any analytic set* S_0 *in U,* $S \cap (U - S_0)$ *is an irreducible analytic set in* $U - S_0$, *and is either empty or dense in S.*

(f) *If* $\{S_n\}$ *is any countable family of analytic sets in U, and* $S = \bigcup_n S_n$, *then* $S = S_n$ *for some n.*

Proof. (a) By Corollary 1, S^* is connected. Since S^* is dense in S, S is also connected.

(b) Since S^* is connected, $\dim_x S$ is the same for all $x \in S^*$ (Proposition 2 of Chap. III, § 2). (b) follows.

(c) Under the given conditions, there also exists (Proposition 3 of § 2) a point $b \in S^*$ such that $\mathbf{S}'_b \supset \mathbf{S}_b$. Since S^* is connected, Proposition 3 of Chap. III, § 2, shows that $S' \supset S^*$, hence $S' \supset S$.

(c′) is a consequence of (c) and Remark 2 on Proposition 2 (Chap. III, § 4).

(d) If $S' \subset S$, and $\dim_x S' = \dim_x S$ for some $x \in S'$, then, by Corollary 1(d) to Proposition 3 (§ 2), $\mathbf{S}'_x$ and $\mathbf{S}_x$ have an irreducible component in common. Hence, by (c), $S' = S$.

(e) Suppose that $S \cap (U - S_0) = S - (S \cap S_0) \neq \varnothing$: by (d), with $S' = S \cap S_0$, $S - (S \cap S_0)$ is dense in S. Further, $S^* \not\subset S_0$ and S^* connected imply (Proposition 4 of Chap. III, § 2) $S^* \cap (U - S_0) = [S \cap (U - S_0)]^*$ connected, i.e. $S \cap (U - S_0)$ irreducible.

(f) If $S_n \neq S$ for every n, then, by (e), the $S - S_n$ are all open dense subsets of S. Hence, by Baire's theorem, $S - \bigcup_n S_n$ is actually dense in S.

THEOREM 8. *Let S be an analytic set in an open set U in C^m, S^* the set of regular points of S. Let $\{s_n\}$ be the (countable) family of connected components of S^*, $\{S_n\}$ the family of their closures in U. Then :* (a) *$\{S_n\}$ is a locally finite family of irreducible analytic sets in U, with union S ; and $S_n \not\subset \bigcup_{n' \neq n} S_{n'}$ for every n.* (b) *If $\{T_i\}$ is any countable family of irreducible analytic sets in U, with $\bigcup_i T_i = S$ and $T_i \not\subset \bigcup_{i' \neq i} T_{i'}$ for every i, then $\{T_i\} = \{S_n\}$.*

PROOF. We have only to prove (b). We have, for every i, $T_i = \bigcup_n (T_i \cap S_n)$. T_i being irreducible, we have, by Corollary 3(f) of Proposition 1, $T_i = T_i \cap S_n$ for some n, say $n(i)$. Similarly, for every n, $S_n \subset T_{i(n)}$. Since, for every n and i, $S_n \not\subset \bigcup_{n' \neq n} S_{n'}$ and $T_i \not\subset \bigcup_{i' \neq i} T_{i'}$, (b) follows.

DEFINITION 1. *The irreducible components of an analytic set S in an open set U in C^m are the closures in U of the connected components of S^* (= set of regular points of S).*

As immediate consequences of this definition, we have :

(i) For any $x \in S$, $\dim_x S$ is the highest of the dimensions of the (finitely many) irreducible components of S which contain x (cf. Corollary 1(e) to Proposition 3 of §2).

(ii) For $0 \leqslant k \leqslant m$, let $S^{(k)}$ = union of the k-dimensional irreducible components of S ($S^{(k)} = \varnothing$ if S has no irreducible component of dimension k). Then $S^{(k)}$ is an analytic set in U (since the family of irreducible components of S is locally finite) which has dimension k at each of its points (or is empty), and $S = \bigcup_{k=0}^{m} S^{(k)}$.

(iii) For $0 \leqslant k \leqslant m$, the set $\{x \in S \mid \dim_x S \geqslant k\}$ is the union of $S^{(k)}, S^{(k+1)}, \dots, S^{(m)}$, hence is an analytic set in U. In particular, given two functions f, g holomorphic on U, the set $\{x \in U \mid \mathbf{f}_x$ and $\mathbf{g}_x$ not relatively prime in $\mathscr{H}^m_x\}$ is an analytic set in U : in fact, by Theorem 6 of Chap. II and Corollary 3(a) to Proposition 3 of §2, this set is $\{x \in S \mid \dim_x S \geqslant m-1\}$ for $S = \{x \in U \mid f(x) = g(x) = 0\}$. This includes Theorem 4 of Chap. II.

(iv) Given another analytic set T in U, if $S \subset T$ and S is irreducible, then S is contained in at least one irreducible component of T.

PROPOSITION 2. *Let S be an analytic set in an open set U in C^m, $\{S_i\}$ the family of its irreducible components. For an $a \in S$, suppose $S_1, \ldots, S_n$ are the S_i containing a. Then (all the germs being taken at a), the sets of irreducible components of the* $\mathbf{S}_i$, $i = 1, \ldots, n$, *are disjoint, and their union is precisely the set of all irreducible components of* $\mathbf{S}$.

PROOF. It is enough to show that, for $1 \leqslant i \leqslant i' < n$, an irreducible component of $\mathbf{S}_{i'}$ cannot contain an irreducible component of $\mathbf{S}_i$: in fact this, by Corollary 3(c) to Proposition 1, would imply $S_{i'} \supset S_i$, contradicting Theorem 8(a).

CONSEQUENCES OF PROPOSITION 2. 1. If $\mathbf{S}_a$ is irreducible, then only one irreducible component of S contains a.

2. (Identitätssatz.) Given two analytic sets S, S' in U, and $a \in S \cap S'$, let $\mathbf{T}$ (resp. $\mathbf{T}'$) be an irreducible component of the germ $\mathbf{S}$ (resp. $\mathbf{S}'$) (all the germs being taken at a), and S_1 (resp. S_1') the uniquely determined irreducible component of the set S (resp. S') such that $\mathbf{T}$ (resp. $\mathbf{T}'$) is an irreducible component of the germ $\mathbf{S}_1$ (resp. $\mathbf{S}_1'$). Then $\mathbf{T} \subset \mathbf{T}'$ implies $S_1 \subset S_1'$, hence $\mathbf{T} = \mathbf{T}'$ implies $S_1 = S_1'$.

3. If S is connected and $\mathbf{S}_a$ irreducible for every $a \in S$, then S is irreducible.

PROOF. Suppose S is connected and reducible : S has at least two irreducible components S_n ; since S_1 and $S_1' = \bigcup_{n \geqslant 2} S_n$ are closed non-empty subsets of S which have S as their union, there exists $a \in S_1 \cap S_1'$, and (consequence 1) $\mathbf{S}_a$ is reducible.

PROPOSITION 3. *Let $S_0 \neq U$ be an analytic set in an open set U in C^m, and S an analytic set in $U - S_0$, such that* $\dim_a S$ *is the same, say k, for all $a \in S$. Then the following two statements are equivalent*: (i) *the closure $\bar{S}$ of S in U is an analytic set in U* ; (ii) *there exists an* *analytic set S' in U such that $S' \supset S$ and* $\dim_a S' \leqslant k$ *for any* $a \in S'$. *If they hold,* $\dim_a \bar{S} = k$ *for every* $a \in \bar{S}$.

PROOF. If (i) holds, S is dense in $\bar{S}$; since $\dim_a \bar{S} = k$ for any $a \in S$, we also have $\dim_a \bar{S} = k$ for any $a \in \bar{S}$, and (ii) holds with $S' = \bar{S}$. Conversely, let (ii) hold: given $a \in S$, the germs $\mathbf{S}, \mathbf{S}'$ induced by S, S' at a are such that $\mathbf{S} \subset \mathbf{S}'$ and $\dim \mathbf{S} = \dim \mathbf{S}'$, hence they have at least one irreducible component, say $\mathbf{T}$, in common (Corollary 1 (d) to Proposition 3 of § 2). Let S'_1 be the irreducible component of the set S' which is such that $\mathbf{T}$ is an irreducible component of $\mathbf{S}'_1$: $S'_1 \cap (U - S_0)$ is an irreducible analytic set in $U - S_0$ (Corollary 3(e) to Proposition 1) containing a, and S an analytic set in $U - S_0$ such that the germ induced at a by S contains an irreducible component of the germ induced by $S'_1 \cap (U - S_0)$; then (Corollary 3(e) to Proposition 1) $S \supset S'_1 \cap (U - S_0)$, hence $\bar{S} \supset S'_1$, $S'_1 \cap (U - S_0)$ is dense in S'_1 (Corollary 3(e) to Proposition 1).

Repeating this argument for every $a \in S$, we get a family of irreducible components of the set S', the union of which contains S and is contained in $\bar{S}$; this union is an analytic set in U, hence is closed in U and coincides with $\bar{S}$, q.e.d.

REMARK. This proposition is a useful tool for the continuation of analytic sets (see [4]).

4. Relations between the dimensions of analytic sets and subsets.

THEOREM 9. *Let S be an analytic set in an open set U in C^m, h a holomorphic function on U, and* $S' = \{x \in S \mid h(x) = 0\}$ *(S' is an analytic set in U). Then for every $a \in S'$:*

$$\dim_a S' = \dim_a S \text{ or } \dim_a S - 1;$$

when $\mathbf{S}_a$ is irreducible, $\dim_a S' = \dim_a S$ *if and only if $\mathbf{h}_a$ vanishes on $\mathbf{S}_a$.*

PROOF. The case $\mathbf{S}_a = \{\mathbf{a}\}$ is trivial, and, if $\mathbf{S}_a = \mathbf{C}^m$, with $\mathbf{h}_a \neq \mathbf{0}$, we have $\dim \mathbf{S}'_a = m - 1 = \dim \mathbf{S}_a - 1$, by Proposition 3 (b) (consequence 3) of § 2. Thus let $\{\mathbf{a}\} \neq \mathbf{S} = \mathbf{S}_a \neq \mathbf{C}^m$. We may assume that $\mathbf{S}$ is irreducible. In fact, if $\mathbf{T}_1, \dots, \mathbf{T}_n$ are the irreducible components of $\mathbf{S}$, and we know for every j $(1 \leqslant j \leqslant n)$ that $\dim (\mathbf{T}_j \cap \mathbf{S}_h) \geqslant \dim \mathbf{T}_j - 1$, we have

$$\dim \mathbf{S}'_a = \sup_j \dim (\mathbf{T}_j \cap \mathbf{S}_h) \text{ (Corollary 1 (e) to Proposition 3, § 2)}$$

$$\geqslant \sup_j (\dim \mathbf{T}_j - 1) = \dim \mathbf{S} - 1,$$

and obviously $\dim \mathbf{S}'_a \leqslant \dim \mathbf{S}$.

Thus let $\mathbf{S}$ be an irreducible germ at $\mathfrak{o} \in C^m$, $\{\mathfrak{o}\} \subsetneq \mathbf{S} \neq \mathbf{C}^m$ and let $\mathbf{h} \in \mathscr{H}'^m$. Let $\mathbf{S}' = \mathbf{S} \cap \mathbf{S}_h$. If $\mathbf{h} \in I(\mathbf{S}) = I$, $\mathbf{S}' = \mathbf{S}$, and the theorem is trivially valid. Hence the proof of the theorem will be complete if we prove that $\mathbf{h} \notin I$ implies $\dim \mathbf{S}' = \dim \mathbf{S} - 1$; by Corollary 1 (c) to Proposition 3 of § 2, we know that $\dim \mathbf{S}' < \dim \mathbf{S}$.

$I = I(\mathbf{S})$ is a prime ideal in $\mathscr{H}^m$, $\{0\} \neq I \subsetneq \mathscr{H}'^m$. Let the basis of C^m be k-proper for I $(1 \leqslant k \leqslant m - 1)$. Let $I' = I(\mathbf{S}')$. We claim that (with the usual notation) $I'_k \neq \{0\}$. In fact, by Proposition 2 of Chap. III, § 4, there exists a distinguished polynomial $R_h(u) \in \mathscr{H}^k[u]$ such that

$$\mathbf{R}_h(\mathbf{h}) = \mathbf{h}^p + \mathbf{c}_1 \mathbf{h}^{p-1} + \dots + \mathbf{c}_p \in I,$$

and, by the same proposition, $\mathbf{h} \notin I$ implies $\mathbf{c}_p \neq \mathbf{0}$. Since $\mathbf{R}_h(\mathbf{h}) \in I \subset I'$, and $\mathbf{h} \in I'$, we have $\mathbf{c}_p \in I'_k$. Hence $I'_k \neq \{0\}$.

Now if $k = 1$, Proposition 4 of § 2 gives $\dim \mathbf{S} = 1$, which implies $\dim \mathbf{S}' = 0 = \dim \mathbf{S} - 1$. Hence let $k > 1$. By suitably altering the basis in C^k, we may assume that $\mathbf{c}_p \sim \mathbf{Q}_k$, where $\mathbf{Q}_k \in \mathscr{H}^{k-1}[x_k]$ is distinguished (we recall that, since $R_h(u)$ is distinguished, $\mathbf{c}_p \in \mathscr{H}'^k$). Then $\mathbf{Q}_k \in I'_k$. Hence condition A″ of Proposition 1 (Chap. III, § 3) is satisfied by $x_1, \dots, x_{k-1}$ for I'. We assert that $I'_{k-1} = \{0\}$. This assertion will be proved if we show that any neighbourhood of $\mathfrak{o}$ in S' has a neighbourhood of $\mathfrak{o}_{k-1}$ as its projection on C^{k-1} ; in fact, by Proposition 2(b) of Chap. III, § 4, there exist arbitrarily small open polydiscs $\pi_0 \subset U$, with centre $\mathfrak{o}$, such that $\pi_0 \cap S'$ has the set of zeros of c_p in π'_0 as its projection on C^k ; c_p and the distinguished pseudo-polynomial Q_k have the same set of zeros in some neighbourhood of $\mathfrak{o}_k$, hence the set of zeros of c_p in π'_0 has a neighbourhood of $\mathfrak{o}_{k-1}$ as its projection on C^{k-1}.

Thus our assertion is proved : we have obtained a basis of C^m which is k-proper for I and $(k-1)$-proper for I'. Hence, by Proposition 4 of § 2, dim $\mathbf{S} = k$, dim $\mathbf{S}' = k - 1$, and the proof of Theorem 9 is complete.

COROLLARY 1. *Let S be any analytic set in U, and S' a principal analytic set in U. Then, either* $\dim_a (S \cap S') = \dim_a S - 1$ *for every $a \in S \cap S'$, or S' contains an irreducible component of S. (It is not asserted that $S \cap S' \neq \varnothing$.)*

PROOF. Given $a \in S \cap S'$, let $\mathbf{T}_1, \ldots, \mathbf{T}_n$ be the irreducible components of $\mathbf{S}$ (all the germs being taken at a), and suppose dim $(\mathbf{S} \cap \mathbf{S}')$ $=$ dim $\mathbf{S}$: by Corollary 1 (e) to Proposition 3 of § 2, dim $(\mathbf{T}_j \cap \mathbf{S}')$ $=$ dim $\mathbf{T}_j$ for at least one j ; then (Theorem 9) $\mathbf{S}' \supset \mathbf{T}_j$, and finally, by Corollary 3 (c) to Proposition 1 of § 3, S' contains the irreducible component S_j of S such that $\mathbf{T}_j$ is an irreducible component of $\mathbf{S}_j$.

COROLLARY 2. *Let S be an analytic set in $U, h_1, \ldots, h_p$ holomorphic functions on U, and $S' = \{x \in S \mid h_1(x) = \ldots = h_p(x) = 0\}$. Then, for every $a \in S'$,* $\dim_a S' \geqslant \dim_a S - p$.

REMARK. If F is a meromorphic function on an open set U in C^m, it follows that the set of points at which F is indeterminate has dimension $m - 2$ at every one of its points (cf. Corollary 3 (a) to Proposition 3 of § 2).

COROLLARY 3. *If S_1, S_2 are any analytic sets in U:* $\dim_a (S_1 \cap S_2)$ $\geqslant \dim_a S_1 + \dim_a S_2 - m$ *for any $a \in S_1 \cap S_2$.*

PROOF. This is already known (Corollary 2) if S_1 or S_2 is the intersection of U with an affine subvariety of C^m. In order to prove it in the general case, we first consider the analytic set $S_1 \times S_2$ in $U \times U$: $S_1{}^* \times S_2{}^*$ is a dense subset of $S_1 \times S_2$, and any point $(x, y) \in S_1{}^* \times S_2{}^*$ is a regular point of $S_1 \times S_2$ of dimension $\dim_x S_1 + \dim_y S_2$; consequently, for any point $(a, b) \in S_1 \times S_2$,

$$\dim_{(a,b)} (S_1 \times S_2) = \dim_a S_1 + \dim_b S_2.$$

Now let $a \in S_1 \cap S_2$ be given, and $k = \dim_a (S_1 \cap S_2)$: by Corollary 1 to Proposition 4 of § 2, there exists an $(m-k)$-dimensional affine subvariety L of C^m such that a is an isolated point of $L \cap (S_1 \cap S_2)$. Then, if D is the linear subvariety $\{(x, y) \mid y = x\}$ of $C^m \times C^m$, (a, a) is an isolated point of $D \cap (L \times C^m) \cap (S_1 \times S_2)$; since $D \cap (L \times C^m)$ is a $(m-k)$-dimensional affine subvariety of $C^m \times C^m$, this, by the same corollary as above, implies

$$m - k \leqslant 2m - \dim_{(a,a)} (S_1 \times S_2) = 2m - \dim_a S_1 - \dim_a S_2, \text{ q.e.d.}$$

5. Holomorphic mappings.

PROPOSITION 1. *Let f map an open set $U \subset C^m$ into C^n, and $S = \{(x, f(x)) \mid x \in U\}$ be the graph of f in $U \times C^n$.*

(a) *If f is holomorphic, then S is an analytic set in $U \times C^n$, and any point of S is a regular point of S of dimension m.*

(b) *Let U be connected : if f is holomorphic, then S is an irreducible analytic set in $U \times C^n$, and conversely, if S is a connected analytic set in $U \times C^n$, then f is holomorphic.* ("S irreducible" implies "S connected" by Corollary 3 (a) to Proposition 1 of § 3).

PROOF. (a) is an immediate consequence of the example in Chap. III, §2. The first part of (b) then follows from Proposition 2 of § 3 (consequence 3). It remains only to prove the second part of (b). As an ordered basis of $C^m \times C^n$, we choose the given basis of C^m, with respect to which the coordinates are $x_1, \ldots, x_m$, followed by the given basis of C^n, with respect to which the coordinates are $x'_1, \ldots, x'_n$.

Since S is the graph of f, for any $a \in U$, the set $S \cap \{x_1 - a_1 = \ldots = x_m - a_m = 0\}$ reduces to the point $(a, f(a))$, hence, by Proposition 4 of § 2, $\dim_{(a,f(a))} S \leqslant m$, with equality if and only if the chosen basis of $C^m \times C^n$ is m-proper for $I(\mathrm{S}_{(a,f(a))})$. By Corollary 2 to Theorem 3, and the remark on this corollary, if $\dim_{(a,f(a))} S < m$, there exists an open polydisc $P \subset U \times C^n$, with centre $(a, f(a))$, such that the projection of $S \cap P$ on C^m is an analytic set in the projection of P, but not this projection itself, and therefore the closure in U of the projection of $S \cap P$ on C^m has an empty interior. By Baire's theorem, this cannot happen for every $a \in U$, since the projection of S on C^m is U : so we must have $\dim_{(a,f(a))} S = m$ for some $a \in U$, i.e., S has at least one m-dimensional irreducible component, S_1 say.

Let $a \in U$ be such that $(a, f(a)) \in S_1$, and T_1 be an analytic set, in an open neighbourhood $V \subset U \times C^n$ of $(a, f(a))$, such that $T_1 \subset S_1$ and $\mathbf{T}_1$ is an m-dimensional irreducible component of $\mathbf{S}_1$, all germs being taken at the point $(a, f(a))$: by Proposition 4 of § 2, the chosen basis of $C^m \times C^n$ is m-proper for $I(\mathbf{T}_1)$; by Corollary 2 to Theorem 3, and the remark on this corollary, there exists an open polydisc $P \subset V$, with centre $(a, f(a))$, such that $T_1 \cap P$ and P have the same projection on C^m. Then $T_1 \cap P = S \cap P = S_1 \cap P$, which implies : (i) $\mathbf{T}_1 = \mathbf{S}_1$, or $\mathbf{S}_1$ is irreducible ; and (ii) S_1 and the union of the remaining irreducible components of S are disjoint, by Corollary 3(c) to Proposition 1 of § 3. Since S is connected, we must have $S = S_1$. Hence for any $a \in U$, the germ induced by S at the point $(a, f(a))$ is irreducible and m-dimensional, and the chosen basis of $C^m \times C^n$ is m-proper for $I(\mathbf{S})$. Then we may use the local description theorem (Theorem 3 of Chap. III), with (x, x') instead of x, and x instead of x' : since S is the graph of f, the integer p in this theorem is 1, and therefore (Remark 3 on the theorem) f is holomorphic on some open neighbourhood of a. This being true for any $a \in U$, the proof of Proposition 1 is complete.

Remark. In Proposition 1(b), to conclude the holomorphy of f from the fact that its graph S is an analytic set, the connectedness of S is necessary. For example, $f : C \to C$, defined by $f(x) = x^{-1}$ for $x \neq 0$, $f(0) = 0$, is not holomorphic in C, although its graph in C^2, namely $\{\mathfrak{o}\} \cup \{(x_1, x_2) \in C_2 \mid x_1 x_2 = 1\}$, is an analytic set in C^2.

Theorem 10. *Let $f : x \to x' = (f_1(x), \ldots, f_n(x))$ be a holomorphic mapping of an open set U in C^m into C^n, and let $a \in U$ be an isolated point of $f^{-1}(a')$, $a' = f(a)$. Then :* (a) $n \geqslant m$; (b) $n > m$ *if and only if there exists a holomorphic function g on an open neighbourhood of a', such that $\mathbf{g}_{a'} \neq \mathbf{0}$ and $g(f_1, \ldots, f_n) \equiv 0$ in a neighbourhood of a.*

Proof. Let the notation be as in the proof of Proposition 1 : S is the graph of f, $a = \mathfrak{o}$, $a' = \mathfrak{o}'$, $\mathbf{S}$ is the germ induced by S at

$(0, 0')$; we know that $\dim S = m$. Now our assumption is that the n coordinates in C^n satisfy condition A of Chap. III for the ideal $I(S)$ in $\mathscr{H}^{m+n}$: then, by Proposition 4 of § 2, $m = \dim S \leqslant n$, which proves (a), and $m < n$ if and only if the n coordinates in C^n do not satisfy condition B for $I(S)$, which is equivalent to (b).

REMARK. Theorem 10 is false without the assumption that a be an isolated point of $f^{-1}(f(a))$. For instance, consider the holomorphic mapping $f : (x_1, x_2) \to (x_1', x_2', x_3')$ of C^2 into C^3 defined by $x_1' = x_1$, $x_2' = x_1 x_2$, $x_3' = x_1 x_2 e^{x_2}$. Then, as in the remark following Proposition 4 of § 2, we see that there is no non-trivial holomorphic relation among x_1', x_2', x_3' at $0 \in C^2$. In this case,

$$f^{-1}(f(0)) = \{(0, x_2) \mid x_2 \in C\}.$$

COROLLARY. *Let f be a holomorphic mapping of an open set U in C^m into C^n: if every point $a \in U$ is an isolated point of $f^{-1}(f(a))$ and $f(U)$ is an open set in C^n, then $n = m$.*

PROOF. By Theorem 10 (a), $n \geqslant m$. Suppose $n > m$: by Theorem 10 (b), any $a \in U$ has a compact neighbourhood $A \subset U$ such that $f(A)$ has an empty interior, i.e., $f(U) - f(A)$ is an open dense subset of $f(U)$; U is the union of a suitably chosen countable family of sets A_n, hence $f(U)$ is the union of the $f(A_n)$, while, by Baire's theorem, $f(U) - \bigcup_n f(A_n)$ is dense in $f(U)$.

THEOREM 11. *Let $f : x \to x' = (f_1(x), \ldots, f_m(x))$ be a holomorphic mapping of an open set $U \subset C^m$ into C^m, and let $a \in U$ be an isolated point of $f^{-1}(a')$, $a' = f(a)$. Then :*

(a) *$f(U)$ is a neighbourhood of a' in C^m;*

(b) *there exist open polydiscs $\pi \subset U$ and π', with centres a and a' respectively (which can be chosen with arbitrarily small radii), such that, for any $x' \in \pi'$, $\pi \cap f^{-1}(x')$ is a finite non-empty set ; the maximum number of points $\in \pi \cap f^{-1}(x')$ is a finite integer p (which does not depend on the choice of π and π') ; $\pi \cap f^{-1}(a') = \{a\}$;*

(c) *$J_f(a) \neq 0$ is a necessary and sufficient condition for the existence of a neighbourhood $V \subset U$ of a such that the restriction $f \mid V$ is one-one (i.e., injective).*

PROOF. Let the notation be as in the proofs of Proposition 1 and Theorem 10 : S is the graph of f, $a = a' = \mathfrak{o}$, $\mathbf{S}$ is the germ induced by S at $(\mathfrak{o}, \mathfrak{o})$; let (x, x') be any point of $C^m \times C^m$. By our assumption and the proof of Theorem 10, the m coordinates $x_1', \ldots, x_m'$ of x' satisfy conditions A and B of Chap. III for the ideal $I(\mathbf{S})$ in $\mathscr{H}^{2m}$, which is prime since $\mathbf{S}$ is regular (Proposition 1).

Then, by the local description theorem (Theorem 3 of Chap. III), there exist an open polydisc $\pi_0 \times \pi'_0$, with centre $(\mathfrak{o}, \mathfrak{o})$, and an analytic set S_0 in $\pi_0 \times \pi'_0$ with the properties listed in the theorem ; as to the notation, x' remains x', but x becomes (x, x'), and $S_0(x')$ has the same meaning as in the theorem.

We choose the polydiscs π and π', with centre $\mathfrak{o}$, $\pi \subset \pi_0 \cap U$, $\pi' \subset \pi'_0$, such that $(\pi \times \pi') \cap S = (\pi \times \pi') \cap S_0$, and $x' \in \pi'$ implies $S_0(x') \subset \pi \times \pi'$. Since $S_0(x') = [\pi \cap f^{-1}(x')] \times \{x'\}$ for any $x' \in \pi'$, part (A) of the local description theorem implies (b) and $\pi' \subset f(U)$, hence (a).

(c) The condition is sufficient by Theorem 1 of Chap. I ; now suppose $f \mid V$ injective : if $\pi \subset V$, this implies $p = 1$. Then, for any $x' \in \pi'$, $\pi \cap f^{-1}(x')$ consists of a single point $g(x')$ and, by the classical local description, the mapping g is holomorphic on π' ; $f \circ g$ is the identical mapping of π' onto π' and $g(\mathfrak{o}) = \mathfrak{o}$, hence $J_f(\mathfrak{o})\, J_g(\mathfrak{o}) = 1$.

REMARK. Theorem 11 is false without the assumption that a be an isolated point of $f^{-1}(f(a))$. For instance consider the holomorphic mapping $f : (x_1, x_2) \to (x_1', x_2')$ of C^2 into C^2 defined by $x_1' = x_1$, $x_2' = x_1 x_2$: if $U = \{(x_1, x_2) \mid |x_1| \text{ and } |x_2| < 1\}$, $f(U)$ is not a neighbourhood of the origin.

COROLLARY 1. *Let f be a holomorphic mapping of an open set $U \subset C^m$ into C^m : if every point $a \in U$ is an isolated point of $f^{-1}(f(a))$, then $f(U)$ is an open set in C^m.*

COROLLARY 2. *If f is a one-one holomorphic mapping of an open set $U \subset C^m$ into C^m, then : $f(U) = V$ is an open set in C^m, the Jacobian J_f vanishes nowhere in U, and f^{-1} is a holomorphic mapping of V onto U.*

PROOF. The first two statements are included in Theorem 11; that f^{-1} is holomorphic in some neighbourhood of any point $\in V$ was proved once with Theorem 1 of Chap. I, and a second time (by quite different methods) with Theorem 11.

REMARK 1. The corollary to Theorem 10 and Corollary 2 to Theorem 11 yield a proof of what was asserted in the remark at the end of Chap. I.

REMARK 2. H. Cartan (*Bull. Soc. Math.*, vol. 57, 1933) proved the following remarkable fact: given a holomorphic mapping f of an open set U in C^m into C^n, the set $\{a \in U \mid a \text{ is not an isolated point of } f^{-1}(f(a))\}$ is an analytic set in U. See also R. Remmert, "Holomorphe und meromorphe Abbildungen komplexer Räume", *Math. Annalen*, vol. 133 (1957), pp. 328-370.

6. Holomorphic functions on an analytic set, according to H. Cartan [3]. In [3], given an analytic set S in an open set U, an abstract topological space $\tilde{S}$ is defined, and holomorphic functions on $\tilde{S}$ are introduced. This definition is equivalent to the following one; we use simply the words "holomorphic on S" for these functions.

DEFINITION 3. *Given an analytic set S in an open set U in C^m, a complex-valued function h, defined on S^*, is holomorphic on S if h is holomorphic on S^*, and is bounded in a neighbourhood of each point of S (more precisely, if every $a \in S$ has a neighbourhood V such that $h \mid V \cap S^*$ is bounded).*

REMARK. A complex-valued function h on S^* is holomorphic at $a \in S^*$ if it satisfies either of the following conditions, which are obviously equivalent.

(i) There exist an open neighbourhood V of a in C^m, and a holomorphic function g in V such that $h = g \mid V \cap S^*$.

(ii) Let f be a biholomorphic mapping of an open neighbourhood V of a in C^m onto an open set V' in C^m, such that $f(S \cap V) = V' \cap L$, L being an affine sub-variety of C^m, of dimension k say.

Then there exists an open neighbourhood $W \subset V$ of a, such that $h \circ f^{-1}$ is holomorphic on $f(W \cap S)$ (which may be regarded as an open set in C^k).

SOME IMMEDIATE CONSEQUENCES OF DEFINITION 3.

1. If h and h' are holomorphic on S, $h + h'$ and $h . h'$ are again holomorphic on S.

2. If h is holomorphic on the analytic set S in U, then, for any open set $U' \subset U$, $h \mid S \cap U'$ is holomorphic on the analytic set $S \cap U'$ in U'.

3. If h is holomorphic on the analytic set S in U, and f is a one-one biholomorphic mapping of U onto another open set U', then $h \circ f^{-1}$ is holomorphic on the analytic set $f(S)$ in U'.

4. (a) Germs of holomorphic functions on analytic sets are defined in the following natural way : given an analytic set S in an open set U and a point $a \in S$, consider the couples (V, h), where $V \subset U$ is an open neighbourhood of a and h a holomorphic function on $V \cap S$, and write $(V, h) \sim (V', h')$ if there exists a neighbourhood $W \subset V \cap V'$ of a such that $h \equiv h'$ on $W \cap S^*$; then a germ of a holomorphic function on S at a is an equivalence class, for this relation, in the set of all couples (V, h).

(b) If $\mathbf{h}_a$ is a germ of a holomorphic function on S at a, the words "$\mathbf{h}_a$ vanishes on $\mathbf{T}_a$" have an obvious meaning if $\mathbf{T}_a$ is a union of irreducible components of $\mathbf{S}_a$.

5. (Principle of analytic continuation.) If h is holomorphic on an irreducible analytic set S in an open set U and, for some $a \in S$, the germ $\mathbf{h}_a$ induced at a by h vanishes on some irreducible component of $\mathbf{S}_a$, then $h \equiv 0$ on S.

This is a consequence of the connectedness of S^* (Corollary 1 to Proposition 1 of § 3) and generalizes Ritt's lemma (Corollary 3(c) to the same proposition).

6. (a) Let S, S' be analytic sets in an open set U, with $S' \subset S$ and $\dim_a S' < \dim_a S$ for every $a \in S'$: if h is holomorphic on $S^* \cap {}^cS'$ and bounded in a neighbourhood of each point of S, then

h is the restriction to $S^* \cap {}^c S'$ of a uniquely determined holomorphic function on S.

(b) Given an analytic set S in an open set U, suppose $\mathbf{S} = \mathbf{S}_a$ is reducible, and let $T_1, \ldots, T_n$ $(n \geqslant 2)$ be analytic sets in an open neighbourhood $V \subset U$ of a, inducing at a the irreducible components of $\mathbf{S}$, with the following properties (cf. Proposition 3 of § 2) : $V \cap S = \bigcup_{j=1}^{n} T_j$; if $T_j' = \bigcup_{j' \neq j} T_{j'}, j=1, \ldots, n$, $\dim_x(T_j \cap T_j') < \dim_x T$ for any $x \in T_j \cap T_j'$, $j = 1, \ldots, n$; $V \cap S^* = \bigcup_{j=1}^{n} (T_j^* \cap {}^c T_j')$. Then, for any h holomorphic on S each $h \mid S^* \cap T_j^*$ is the restriction to $S^* \cap T_j^* = T_j^* \cap {}^c T_j'$ of a uniquely determined holomorphic function on T_j, $j = 1, \ldots, n$.

(c) Given an analytic set S in an open set U and an irreducible component S_1 of S, for any h holomorphic on S, $h \mid S^* \cap S_1^*$ is the restriction to $S^* \cap S_1^*$ of a uniquely determined holomorphic function on S_1.

Proposition 1. *Let S be an analytic set in an open set U in C^m ; suppose the origin $\mathfrak{o}$ belongs to S, S induces an irreducible germ $\mathbf{S}$ at $\mathfrak{o}$, and the basis of C^m is k-proper for $I(\mathbf{S})$, $1 \leqslant k \leqslant m-1$. Then there exist :* (a) *an open polydisc $\pi_0 \subset U$ with centre $\mathfrak{o}$ (which can be chosen with arbitrarily small radii) and a principal analytic set σ_0' in π_0' ($=$ projection of π_0 on C^k), such that $S_0 = \pi_0 \cap S$ is an irreducible analytic set in π_0, and $x \in S_0$, $x' = (x_1, \ldots, x_k) \in \pi'_0 - \sigma'_0$ imply $x \in S_0^*$, and* (b) *for any function h holomorphic on S_0, a monic pseudo-polynomial $R_h(x', u)$ in u, with coefficients holomorphic on π_0', such that $\{u \in C \mid R_h(x'_0, u) = 0\} = \{h(x) \mid x \in S_0, x' = x_0'\}$ for any $x_0' \in \pi_0' - \sigma_0'$, and $R_h(x', h(x)) = 0$ for any $x \in S_0^*$.* (c) *If $R_h(x', u)$, contains the factor u, then $h \equiv 0$ on S_0.*

Proof. By the local description theorem (Theorem 3 of Chap. III), we can choose an arbitrarily small π_0 such that $S_0 = \pi_0 \cap S$ has the properties listed in (a); we need only remark that S_0 is irreducible since it contains a dense connected set of regular points, namely $\{x \in S_0 \mid x' \in \pi_0' - \sigma_0'\}$. With the same notation, each function

$$c_j(x') = (-1)^j \sum_{1 \leqslant i_1 < \ldots < i_j \leqslant p} h(x^{(i_1)}(x')) \ldots h(x^{(i_j)}(x')),\ j = 1, \ldots, p$$

is holomorphic for $x' \in U'_0 = \pi'_0 - \sigma'_0$, bounded in a neighbourhood of each point of σ'_0, and therefore, by Theorem 2 of Chap. III, has a unique holomorphic extension to π'_0, which vanishes at the origin $0'$ in C^k. If we denote this extension by c_j and set

$$R_h(x', u) = u^p + \sum_{j=1}^{p} c_j(x')\, u^{p-j},$$

then R_h obviously serves in (b).

To prove (c), suppose $c_p \equiv 0$. Given $x'_0 \in U'_0$, for a suitable choice of the indices j, the p functions $h(x^{(j)}(x'))$, $j = 1, \ldots, p$, are holomorphic on some open connected neighbourhood $V'_0 \subset U'_0$ of x'_0, and their product is $\equiv 0$ on V'_0; hence one of them is $\equiv 0$ on V'_0, i.e., h vanishes identically on S_0 in some neighbourhood of one of the points $x^{(j)}(x'_0)$. By consequence 5 of Definition 3, this implies $h \equiv 0$ on S_0.

Lemma 1. *Let h be a continuous complex-valued function on an open set U in C^m, $b_1, \ldots, b_p$ holomorphic functions on U such that $P(x, h) = h^p + b_1 h^{p-1} + \ldots + b_p \equiv 0$ on U; then h is holomorphic on U.*

Proof. Given $a \in U$, we consider only germs of holomorphic functions at a. By Gauss' Theorem (Chap. II, § 2), $\mathscr{H}^m_a[u]$ is a factorial ring: Let $\mathbf{P}(u) = u^p + \mathbf{b}_1 u^{p-1} + \ldots + \mathbf{b}_p$ and $\mathbf{P} = \mathbf{P}_1^{\alpha_1} \ldots \mathbf{P}_n^{\alpha_n}$, where the α_j are positive integers, the $\mathbf{P}_j \in \mathscr{H}^m_a[u]$ are irreducible and mutually non-equivalent in $\mathscr{H}^m_a[u]$, have degrees $\geqslant 1$ and $\mathbf{1}$ as their leading coefficient.

There exist an open connected neighbourhood $V \subset U$ of a and, for each $j = 1, \ldots, n$, a monic pseudo-polynomial $P_j(x, u)$ in u, the coefficients of which are holomorphic functions on V inducing at a the respective coefficients of $\mathbf{P}_j$; $P(x, u) = P_1(x, u)^{\alpha_1} \ldots P_n(x, u)^{\alpha_n}$ for $x \in V$ implies $P_1(x, h) \ldots P_n(x, h) \equiv 0$ on V. But the discriminant of $\mathbf{P}_1 \ldots \mathbf{P}_n$ is not $\mathbf{0}$, hence the discriminant of $P_1(x, u) \ldots P_n(x, u)$, a holomorphic function on V, is not $\equiv 0$, i.e., the set S of zeros of this discriminant is a principal analytic set in V ; any continuous root of $P_1(x, u) \dots P_n(x, u)$, e.g. $h(x)$, is holomorphic on $V - S$; since $h \mid V$ is continuous on V, $h \mid V$ is holomorphic on V by Theorem 2 of Chap. III.

THEOREM 12 (H. Cartan). *Given an analytic set S in an open set U and a complex-valued function h defined on S^*,* (i), (ii), (iii) *below are equivalent:*

(i) *h is holomorphic on S^* and bounded in a neighbourhood of each point of S (i.e., by Definition 3, h is holomorphic on S)*;

(ii) *h is holomorphic on S^* and has the following property*: *given $a \in S$ and analytic sets $T_1, \dots, T_n (n \geqslant 1)$ in an open neighbourhood of a, inducing at a the irreducible components of $\mathbf{S}_a$, then, for each $j = 1, \dots, n$, $h(x)$ has a finite limit as $x \to a$, $x \in S^* \cap T_j$*;

(iii) *h is continuous on S^* and, for any $a \in S$, there exist an open neighbourhood $V \subset U$ of a and holomorphic functions $b_1, \dots, b_p$ on V such that $h^p + b_1 h^{p-1} + \dots + b_p \equiv 0$ on $V \cap S^*$.*

PROOF. Obviously (ii) implies (i); so we have to prove that (i) implies (iii) and (iii) implies (ii).

(1) Assuming (i), by consequence 6(b) of Definition 3, it is enough to consider a point $a \in S$ such that $\mathbf{S} = \mathbf{S}_a$ is irreducible. Let a be the origin $\mathfrak{o}$ and the basis of C^m be k-proper for $I(\mathbf{S})$, $1 \leqslant k \leqslant m-1$: by Proposition 1, there exist an open polydisc $\pi_0 \subset U$ with centre $\mathfrak{o}$ and holomorphic functions $c_1, \dots, c_p$ on π'_0 (= projection of π_0 on C^k) such that $[h(x)]^p + \sum_{j=1}^{p} c_j(x')[h(x)]^{p-j} = 0$ for every $x \in \pi_0 \cap S^*$.

(2) Now we assume (iii). Then h is holomorphic on S^*, by the lemma above. Given $a \in S$, the set of limits of $h(x)$ as $x \to a$, $x \in S^*$, is contained in $\{u \in C \mid u^p + b_1(a)u^{p-1} + \dots + b_p(a) = 0\}$, hence finite. If $\mathbf{S}_a$ is irreducible, by Proposition 2 (b) of § 2, there exist arbitrarily small open neighbourhoods W of a such that each $W \cap S^*$ is connected : then the set of limits of $h(x)$ as $x \to a$, $x \in S^*$, is connected, therefore consists of only one point. If $\mathbf{S}_a$ is reducible, let $T_1, \dots, T_n$ $(n \geqslant 2)$ have the properties listed in Proposition 3 of § 2 : by part (c)

of this proposition, for each $j = 1, \ldots, n$, there exist arbitrarily small open neighbourhoods W_j of a such that each $W_j \cap (T_j{}^* \cap {}^cT'_j)$ is connected ; then the set of limits of $h(x)$ as $x \to a$, $x \in S^* \cap T_j$, is connected, therefore consists of only one point.

COROLLARY. *Let S be an analytic set in an open set U and S_1 an analytic set in an open set $U_1 \subset U$, $S_1 \subset S$.*

(a) *Suppose that, for any $a \in S_1$, each irreducible component of $\mathbf{S}_1$ is contained in only one irreducible component of $\mathbf{S}$ ($\mathbf{S}_1$ and $\mathbf{S}$ being taken at a); then any holomorphic function h on S induces a holomorphic function h_1 on S_1 as follows : if $a \in S_1{}^* \cap S^*$, $h_1(a) = h(a)$; if $a \in S_1{}^* \cap (S - S^*)$, and T_1 is an analytic set in an open neighbourhood of a inducing at a the irreducible component of $\mathbf{S}$ which contains $\mathbf{S}_1$,*

$$h_1(a) = \lim_{\substack{x \to a \\ x \in S^* \cap T_1}} h(x).$$

(b) *Suppose that* $\dim_a S_1 \cap (S - S^*) < \dim_a S_1$ *for any* $a \in S_1 \cap (S - S^*)$: *then the hypothesis of* (a), *and therefore the conclusion of* (a), *remains valid.*

PROOF. (a) h satisfies (iii) of Theorem 12 with respect to S : therefore, given $a \in S_1$, there exist an open neighbourhood $V \subset U$ of a and holomorphic functions $b_1, \ldots, b_p$ on V such that $h^p + b_1 h^{p-1} + \ldots + b_p \equiv 0$ on $V \cap S^*$; then $h_1^p + b_1 h_1^{p-1} + \ldots + b_p \equiv 0$ on $V \cap S_1{}^*$ and, by Theorem 12, we only have to prove the continuity of h_1 on $S_1{}^*$. This continuity is obvious at a point $a \in S_1{}^* \cap S^*$, and also at a point $a \in S_1{}^* \cap (S - S^*)$ where S induces an irreducible germ, since we have $h_1(a) = \lim_{\substack{x \to a \\ x \in S^*}} h(x)$ in the latter case.

So we consider an $a \in S_1{}^* \cap (S - S^*)$ with $\mathbf{S} = \mathbf{S}_a$ reducible. By Proposition 3 of § 2, we can find an open neighbourhood $W \subset U_1$ of a and analytic sets $T_1, \ldots, T_n$ ($n \geqslant 2$) in W, inducing at a the irreducible components of $\mathbf{S}_a$, with the following properties : (1) $W \cap S = \bigcup_{j=0}^{n} T_j$; (2) $W \cap S_1 \subset T_1$; (3) $|h(x) - h_1(a)| <$ a given

$\epsilon > 0$, for any $x \in S^* \cap T_1$; (4) if $T_1' = \bigcup_{j=2}^{n} T_j$, $T_1^* \cap {}^cT_1'$ is connected and dense in T_1^*.

Property (4) implies that T_1^* is connected, i.e., T_1 is an irreducible analytic set in W (Corollary 1 to Proposition 1 of § 3). Properties (2) and (3) imply $|h_1(x)-h_1(a)| < \epsilon$ for any $x \in W \cap S_1^* \cap S^*$. Now let $x \in W \cap S^* \cap (S - S^*)$: by property (1), any irreducible component of $\mathbf{S}_x$ is either an irreducible component of $(\mathbf{T}_1)_x$ or an irreducible component of $(\mathbf{T}_1')_x$; by property (2), $(\mathbf{S}_1)_x$ is contained in an irreducible component of $(\mathbf{T}_1)$, which in turn is contained in an irreducible component of $\mathbf{S}_x$; if this one were an irreducible component of $(\mathbf{T}_1')_x$, since T_1 is an irreducible analytic set in W, we should have $T_1 \subset T_1'$, by Corollary 3 (c) to Proposition 1 of § 3. So the irreducible component of $\mathbf{S}_x$ which contains $(\mathbf{S}_1)_x$ is an irreducible component of $(\mathbf{T}_1)_x$, and property (3) implies $|h_1(x) - h_1(a)| < \epsilon$. This proves (a).

(b) Consider a given $a \in S_1$ such that $\mathbf{S}$ is reducible. Let the notation be as above, and further let $(T_1)_1, \ldots, (T_1)_{n_1}$, $n_1 \geqslant 1$, be analytic sets in W inducing at a the irreducible components of $\mathbf{S}_1$, with $W \cap S_1 = \bigcup_{j=1}^{n_1} (T_1)_j$. If (a) does not hold for a, there exists an open neighbourhood $W_1 \subset W$ of a such that (for example)

$$W_1 \cap (T_1)_1 \subset W_1 \cap (T_1 \cap T_2) \subset S - S^*$$

(cf. consequence 1 of Proposition 3 of § 2); then S_1 and $S_1 \cap (S - S^*)$ induce the same non-empty germ at a if $n_1 = 1$, any point $b \in W_1 \cap (T_1)_1$ such that $b \notin \bigcup_{j=2}^{n_1} (T_1)_j$ if $n_1 \geqslant 2$. Hence (b) is proved.

Proposition 2. *Given analytic sets S, S' in an open set U, in C^m, with $S' \subset S$, and a point $a \in S'$, let $\mathbf{S}$, $\mathbf{S}'$ be the germs induced at a by S, S'; let $\mathbf{S}$ be irreducible and* $\dim \mathbf{S}' < \dim \mathbf{S}$ *(i.e., by Corollary 1 (c) to Proposition 3 of § 2, $\mathbf{S}' \subsetneq \mathbf{S}$): for any h holomorphic on S such that $h(x) \to 0$ as $x \to a$, $x \in S^*$, either $\mathbf{h}$ vanishes on $\mathbf{S}$, or $h(V \cap S^* \cap {}^cS') \cup \{0\}$ is a neighbourhood of 0 in C, for any neighbourhood V of a.*

PROOF. Let a be the origin $\mathfrak{o}$. We first suppose $\dim \mathbf{S} = 1$: given a basis of C^m which is 1-proper for $I(\mathbf{S})$, if the polydisc π_0 of Proposition 1 is chosen small enough, $\sigma'_0 = \{0\}$ and $\pi_0 \cap S' = \{\mathfrak{o}\}$. The pseudo-polynomial $R_h(x_1, u) = u^p + \sum_{j=1}^{p} c_j(x_1)\, u^{p-j}$ of Proposition 1 is distinguished since $h(x) \to 0$ as $x \to \mathfrak{o}$, $x \in S^*$, and, by Proposition 1, $c_p \equiv 0$ implies $h \equiv 0$ on $\pi_0 \cap S^*$.

If $c_p \not\equiv 0$, let $0 < |x_1| < r$ imply $c_p(x_1) \neq 0$: then, for any complex number $\lambda \neq 0$ with a small enough $|\lambda|$, $R_h(x_1, \lambda) = 0$ for some x_1, $0 < |x_1| < r$, i.e., $h = \lambda$ at some point $\in \pi_0 \cap S^* \cap {}^cS'$. So Proposition 2 is true if $\dim \mathbf{S} = 1$.

We now suppose Proposition 2 is true if $\dim \mathbf{S} < k$, $2 \leqslant k \leqslant m-1$. Let $\dim \mathbf{S} = k$: choosing a basis of C^m which is k-proper for $I(\mathbf{S})$, we can again find an open polydisc $\pi_0 \subset U$, with centre $\mathfrak{o}$ and arbitrarily small radii, such that $S_0 = \pi_0 \cap S$ has the properties listed in Proposition 1; so there exists a holomorphic function c_p on the projection π'_0 of π_0 on C^k, vanishing at the origin $\mathfrak{o}'$ in C^k, such that $h(x) = 0$, $x \in \pi_0 \cap S^*$ imply $c_p(x') = 0$, and $c_p \equiv 0$ implies $h \equiv 0$ on $\pi_0 \cap S^*$.

Let $c_p \not\equiv 0$; we may assume that $S' \supset S - S^*$. Since $\dim \mathbf{S}' < k$, by Proposition 4 of § 2, the basis of C^m is not k-proper for $I(\mathbf{S}')$; then, if π_0 is chosen small enough, there exist: (1) a holomorphic function c' on π'_0, $c'(\mathfrak{o}') = 0$, $c' \not\equiv 0$, such that $x \in \pi_0 \cap S'$ implies $c'(x') = 0$; (2) a holomorphic function ϕ on π'_0, $\phi(\mathfrak{o}') = 0$, $\phi \not\equiv 0$, such that the germs induced by $c_p\, c'$ and ϕ at $\mathfrak{o}'$ are relatively prime in $\mathscr{H}^k$: in fact, we may choose the basis of C^k in such a way that $(c_p\, c')\,(0, \dots, 0, x_k) \not\equiv 0$ in any neighbourhood of $x_k = 0$ in C, and take any $\phi \in \mathscr{H}'^{k-1} - \{0\}$. Since S_0 is an irreducible analytic set in π_0, by Corollary 1 to Theorem 9, the analytic set $\Sigma = \{x \in S_0 \mid \phi(x') = 0\}$ in π_0 has dimension $k-1$ at any point $\in \Sigma$, and there also exists an analytic set $S_1 \subset \Sigma$, in an open neighbourhood $U_1 \subset \pi_0$ of $\mathfrak{o}$, containing $\mathfrak{o}$, inducing at $\mathfrak{o}$ an irreducible germ $\mathbf{S}_1$ and having dimension $k-1$ at any point $\in S_1$ (cf. Proposition 3 of § 2).

By our choice of ϕ, the set of common zeros of $c_p c'$ and ϕ in π_0' has dimension $k-2$ at $\mathfrak{o}'$(cf. the remark on Corollary 2 to Theorem 9); by Corollary 1 to Proposition 4 of § 2, there exists a 2-dimensional linear subvariety L' of C^k such that $\mathfrak{o}'$ is an isolated point of the intersection of L' and the set of common zeros to $c_p\, c'$ and ϕ in π'. L' is the projection on C^k of an $(m-k+2)$-dimensional linear subvariety L of C^k such that $\mathfrak{o}$ is an isolated point of $L \cap S_1 \cap S'$ and therefore, by the same corollary as above, $\dim_{\mathfrak{o}} S_1 \cap S' \leqslant k-2$. If $\mathfrak{o} \in S - S^*$, this inequality implies $\dim_{\mathfrak{o}} S_1 \cap (S - S^*) \leqslant k-2$, hence $\dim_x S_1 \cap (S-S^*) < \dim_x S_1$ for any $x \in S_1 \cap (S-S^*)$ if U_1 is chosen sufficiently small ; then, by corollary to Theorem 12, h induces a holomorphic function h_1 on S_1, such that $h_1(x) \to 0$ as $x \to \mathfrak{o}$, $x \in S_1{}^*$. If $\mathbf{h}_1$ vanishes on $\mathbf{S}_1$, there exists an open neighbourhood $V_1 \subset U_1$ of $\mathfrak{o}$ such that $h = 0$ on $V_1 \cap S_1{}^* \cap S^*$, or $V_1 \cap S_1 \cap S^*$; in other words, $x \in V_1 \cap S_1$ implies $x \in S - S^* \subset S'$ or $h(x) = 0$, in both cases $(c_p\, c')\,(x') = \phi\,(x') = 0$; then $\mathfrak{o}$ is an isolated point of $L \cap S_1$, which contradicts $\dim S_1 = k-1$. Thus $\mathbf{h}_1$ does not vanish on $\mathbf{S}_1$, and, since $\dim_{\mathfrak{o}} S_1 \cap S' < \dim_{\mathfrak{o}} S_1 < k$, for any neighbourhood V of $\mathfrak{o}$, $h_1(V \cap S_1{}^* \cap {}^cS') \cup \{0\}$ or $h(V \cap S_1{}^* \cap {}^cS') \cup \{0\}$, *a fortiori* $h(V \cap S^* \cap {}^cS') \cup \{0\}$, is a neighbourhood of 0 in C.

Finally the case $\dim \mathbf{S} = m$ is treated by considering S as a subset of C^{m+1}.

Corollary 1. *If S is an irreducible analytic set in an open set U, any function h holomorphic on S is either constant or is an open mapping of S^* into C, in the following sense : for any $a \in S$ and any analytic set T in an open neighbourhood of a inducing at a an irreducible component $\mathbf{T}_a$ of $\mathbf{S}_a$, $h(S^* \cap T) \cup \{\lambda\}$ is a neighbourhood of λ in C, where* $\lambda = \lim\limits_{\substack{x \to a \\ x \in S^* \cap T}} h(x)$.

Proof. For a point a such that $\mathbf{S}_a$ is reducible, we use consequences 5 and 6 (b) of Definition 3, and Proposition 2 with T_j instead of S, $T_j \cap T_j'$ instead of S'.

Corollary 2. (Maximum principle.) *Given an analytic set S in an open set U, and a holomorphic function h on S, if there exist a point $a \in S$, and an analytic set T in an open neighbourhood of a inducing at* *a an irreducible component* $\mathbf{T}_a$ *of* $\mathbf{S}_a$, *such that* $|\lim_{\substack{x\to a\\ x\in S^*\cap T}} h(x)| = \sup_{S^*} |h|$, *then* $h = \text{const.}$ *on* $S^* \cap S_1$, *where* S_1 *is the irreducible component of* S *such that* $\mathbf{T}_a$ *is an irreducible component of* $(\mathbf{S}_1)_a$ (cf. *Proposition* 2 *of* § 3).

Proof. By consequence 6 (c) of Definition 3, $h \mid S^* \cap S_1^*$ is the restriction to $S^* \cap S_1^*$ of a holomorphic function h_1 on S_1 ; since $S^* \cap S_1^* \cap T$ meets any neighbourhood of a, $\lim_{\substack{x\to a\\ x\in S_1^*\cap T}} h_1(x) = \lim_{\substack{x\to a\\ x\in S^*\cap T}} h(x)$, hence $|\lim_{\substack{x\to a\\ x\in S_1^*\cap T}} h_1(x)| = \sup_{S_1^*} |h_1|$ and, by Corollary 1, h_1 is constant.

Theorem 13. *Let* S *be an analytic set in an open set* U *in* C^m, h *a holomorphic function on* S, *and* S' *the closure in* U(*or in* S) *of the set* $\{x \in S^* \mid h(x) = 0\}$. *Then*

(a) S' *is again an analytic set in* U ;

(b) *for any* $a \in S'$, $\dim_a S' \geqslant -1 +$ *the greatest of the dimensions of the irreducible components* S_n *of* S *such that* a *is in the closure of* $\{x \in S^* \cap S_n^* \mid h(x) = 0\}$;

(c) *if* S *is irreducible, either* $h \equiv 0$ *or* $\dim_a S' = \dim_a S - 1$ *for every* $a \in S'$.

Proof. Let $\{s_n\}$ be the countable family of connected components of S^*, and $\{S_n\}$ the family of irreducible components of S : for each n, S_n is the closure in U of s_n, and $s_n = S^* \cap S_n^*$. Since these families are locally finite in U, we have to prove that, for each n, the closure S_n' in U of the set $\{x \in s_n \mid h(x) = 0\}$ is an analytic set in U and, for any $a \in S_n'$, $\dim_a S_n' = k_n$ or $k_n - 1$, where k_n is the constant dimension of S_n.

This being obvious if $k_n = 0$ or m, or if $h \mid s_n \equiv 0$, let $1 \leqslant k_n \leqslant m - 1$ and $h \mid s_n \not\equiv 0$; s_n is an irreducible k_n-dimensional analytic set in $U - (S - S^*)$ consisting only of regular points, and therefore, $\{x \in s_n \mid h(x) = 0\}$ is a $(k_n - 1)$-dimensional analytic set in $U - (S - S^*)$. Hence by Proposition 3 of § 3, Theorem 13 will be proved if we show that, for any $a \in S_n \cap (S - S^*)$, there exists a $(k_n - 1)$-dimensional analytic set in an open neighbourhood $\pi_0 \subset U$ of a, containing $\{x \in \pi_0 \cap s_n \mid h(x) = 0\}$.

If the germ induced at a by S_n is irreducible, let $T = S_n$, $V = U$; if not, let T be one of the analytic sets T_j in an open neighbourhood $V \subset U$ of a, having the properties listed in consequence 6(b) of Definition 3, with S_n instead of S: it is enough to find a $(k_n - 1)$-dimensional analytic set, in an open neighbourhood $\pi_0 \subset V$ of a, containing $\{x \in \pi_0 \cap s_n \cap T^* \mid h(x) = 0\}$. If the basis of C^m is k_n-proper for $I(\mathbf{T}_a)$, by Proposition 1 and consequences 6(b) and 6(c) of Definition 3, there exist an open polydisc $\pi_0 \subset V$ with centre a and a holomorphic function c_p on π'_0 (= projection of π_0 on C^{k_n}), such that: (i) $T_0 = \pi_0 \cap T$ is an irreducible analytic set in π_0; (ii) $h(x) = 0$ implies $c_p(x') = 0$, $x' = (x_1, \ldots, x_{k_n})$, for any $x \in \pi_0 \cap s_n \cap T^*$; (iii) $c_p \equiv 0$ implies $h \mid \pi_0 \cap s_n \cap T^* \equiv 0$, hence $h \mid s_n \equiv 0$ by the principle of analytic continuation. So $c_p \not\equiv 0$ and, since T_0 is an irreducible analytic set in π_0, by Corollary 1 to Theorem 9, $\{x \in T_0 \mid c_p(x') = 0\}$ is a $(k_n - 1)$-dimensional analytic set in π_0, which contains $\{x \in \pi_0 \cap s_n \cap T^* \mid h(x) = 0\}$, q.e.d.

Lemma 2. *Let S be an analytic set in an open set U in C^m; suppose the origin $\mathfrak{o} \in S$, S induces an irreducible germ* $\mathbf{S}$ *at* $\mathfrak{o}$, $k = \dim \mathbf{S}$, $1 < k \leqslant m - 1$. *Then $\mathfrak{o}$ has an open neighbourhood $V \subset U$, such that $V \cap (S - S^*) = \{x \in V \cap S \mid \rho_j(x) = 0$ for each $j\}$, where $\{\rho_j\}$ is a finite family of holomorphic functions on V, built as follows: for each j, there exist a basis $\mathscr{B}_j$ of C^m which is k-proper for $I(\mathbf{S})$, an open polydisc π_j (with respect to $\mathscr{B}_j$) with centre $\mathfrak{o}$, $V \subset \pi_j \subset U$, such that $S_j = \pi_j \cap S$ has the properties listed in the local description theorem with respect to $\mathscr{B}_j$ (let the first k coordinates with respect to $\mathscr{B}_j$ be denoted by a single letter x_j', let $\pi_j' = \{x_j' \mid x \in \pi_j\}$, and let p_j be the maximum number of points $\in S_j$ for a given $x_j' \in \pi_j'$), and finally a linear form l_j on C^m; then* $\rho_j(x) = \dfrac{\partial}{\partial u} R_j(x_j', l_j(x))$, *where $R_j(x_j', u)$ is a distinguished pseudo-polynomial of degree p in u, with coefficients holomorphic on π_j', such that*

$\{u \in C \mid R_j(x_0', u) = 0\} = \{l_j(x) \mid x \in S_j,\ x_j' = x_0'\}$ *for any* $x_0' \in \pi_j'$

PROOF. This is a consequence of the first proof of Theorem 6, the remark on Corollary 1 to Theorem 3, and part (B, a) of the local description theorem.

DEFINITION 4. *Given an analytic set S in an open set U and a point $a \in S$, a function v, holomorphic on an open neighbourhood $V \subset U$ of a, is a universal denominator for S at a if* : (i) *all germs being taken at a, $\mathbf{v}$ does not vanish on any irreducible component of $\mathbf{S}$* ; (ii) *for any open neighbourhood $V' \subset V$ of a and any holomorphic function h on $V' \cap S$, there exists a holomorphic function on an open neighbourhood $W \subset V'$ of a which has the same restriction to $W \cap S^*$ as vh.*

THEOREM 14 (K. OKA). *Let S be an analytic set in an open set U ; suppose the origin $\mathfrak{o} \in S$.*

(a) *If S induces at $\mathfrak{o}$ an irreducible germ, there exists an integer $n > 0$ such that, for any function v holomorphic on an open neighbourhood of $\mathfrak{o}$ and vanishing on $S - S^*$ in some neighbourhood of $\mathfrak{o}$, either v vanishes on S in some neighbourhood of $\mathfrak{o}$, or v^n is a universal denominator for S at any point $a \in S$ sufficiently near $\mathfrak{o}$.*

(b) *In any case, there exists a function v, holomorphic on an open neighbourhood of $\mathfrak{o}$, which is a universal denominator for S at any point $a \in S$ sufficiently near $\mathfrak{o}$.*

PROOF. (a) Let $\mathbf{S}$ be the germ induced by S at $\mathfrak{o}$ and $k = \dim \mathbf{S}$; we may assume $1 \leqslant k \leqslant m - 1$. Let the basis of C^m be k-proper for $I(\mathbf{S})$, and $\pi_0 \subset U$ be an open polydisc with centre $\mathfrak{o}$, such that $S_0 = \pi_0 \cap S$ has the properties listed in the local description theorem ; with the notation of that theorem, given a linear form l on C^m, let $R_l(x', u)$ be a distinguished pseudo-polynomial in u, with coefficients holomorphic on π'_0, the roots of which, for $x' \in U'_0 = \pi'_0 - \sigma'_0$, are the p values $l(x^{(j)}(x'))$, $j = 1, \dots, p$.

For any holomorphic function h on S_0, and $x' \in U'_0$, consider

$$X(x', u) = \sum_{1 \leqslant j \leqslant p} \left\{ h(x^{(j)}(x')) \prod_{\substack{1 \leqslant j' \leqslant p \\ j' \neq j}} [u - l(x^{(j')}(x'))] \right\} \qquad (*)$$

which is a pseudo-polynomial of degree $< p-1$ in u; since its coefficients are holomorphic on U'_0 and bounded in a neighbourhood of each point of σ'_0, they have a holomorphic extension to π'_0, and then $X(x', l(x))$ is a holomorphic function on π_0 such that

$$\frac{\partial}{\partial u} R_l(x', l(x))\, h(x) = X(x', l(x)) \text{ for any } x \in S_0^*. \qquad (**)$$

Thus $\frac{\partial}{\partial u} R_l(x', l(x))$ either induces at $\mathfrak{o}$ a germ vanishing on **S**, or is a universal denominator for S at $\mathfrak{o}$, since π_0 can be chosen with arbitrarily small radii.

Given $a \in S_0$, let a' be the projection of a on C^k; we can find open polydiscs $\pi, \pi_1, \pi_2, \ldots$ (with arbitrarily small radii) such that: (i) the centre of π is a, the centres of $\pi_1, \pi_2, \ldots$ are the other points $\in S_0(a')$ (with the notation of the local description theorem); (ii) $\pi, \pi_1, \pi_2, \ldots$ are contained in π_0, mutually disjoint, and have the same projection π' on C^k; (iii) the union of $\pi, \pi_1, \pi_2, \ldots$ contains $S_0(x')$ for any $x' \in \pi'$.

Given a holomorphic function h on $\pi \cap S$, we may set $h \equiv 0$ on $\pi_1 \cap S^*, \pi_2 \cap S^*, \ldots$, and consider (*) for $x' \in \pi' \cap U'_0$: then $X(x', l(x))$ is a holomorphic function on π, such that (**) holds for any $x \in \pi \cap S^*$. Thus, for any $a \in S_0$, the same function $\frac{\partial}{\partial u} R_l(x', l(x))$ is a universal denominator for S at a, unless it induces at a a germ vanishing on some irreducible component of $\mathbf{S}_a$. Finally, since S_0 is an irreducible analytic set in π_0, by Corollary 3(c) to Proposition 1 of §3, if $\frac{\partial}{\partial u} R(x', l(x))$ induces, at some $a \in S_0$, a germ vanishing on some irreducible component of $\mathbf{S}_a$, then it vanishes identically on S_0: in other words, $\frac{\partial}{\partial u} R(x', l(x))$ either vanishes identically on S_0, or is a universal denominator for S at any point $a \in S_0$.

By Corollary 2 to Theorem 7, Lemma 2 and the results just obtained, there exist an open neighbourhood $V \subset U$ of $\mathfrak{o}$ and a finite family of holomorphic functions ρ_j on V, $j = 1, \ldots, q$, such that

(i) $V \cap S$ is an irreducible analytic set in V,

(ii) $V \cap (S-S^*)=\{x \in V \cap S \mid \rho_j(x) = 0 \text{ for each } j\}$,

(iii) for each j, ρ_j is a universal denominator for S at any point $a \in V \cap S$.

Let v satisfy the hypotheses of (a): by (ii), the ideal generated in $\mathscr{H}^m$ by $I(\mathbf{S})$ and the $\boldsymbol{\rho}_j$ defines the germ induced at $\mathfrak{o}$ by $S - S^*$; by the Nullstellensatz (Corollary 3 to Theorem 2), this ideal contains $\mathbf{v}^n$ for a suitable integer $n > 0$, which depends only on S; if V is small enough, v is holomorphic on V and, for each $j = 1, \dots, q$, there exists a holomorphic function ϕ_j on V such that

(iv) $v^n - \sum_{j=1} \rho_j\phi_j \equiv 0$ on $V \cap S$.

Given $a \in V \cap S$, an open neighbourhood $V' \subset V$ of a and a holomorphic function h on $V' \cap S$, by (iii), there exist an open neighbourhood $W \cap V'$ of a and, for each $j = 1, \dots, q$, a holomorphic function u_j on W such that u_j and $\rho_j h$ have the same restriction to $W \cap S^*$; then, by (iv), $\sum_{j=1}^{q} u_j \phi_j$ and $v^n h$ have the same restriction to $W \cap S^*$, i.e., v^n is a universal denominator for S at a, unless v induces at a a germ vanishing on some irreducible component of $\mathbf{S}_a$. Finally, by (i) and Corollary 3(c) to Proposition 1 of § 3, either $v \equiv 0$ on $V \cap S$, or v^n is a universal denominator for S at any point $a \in v \cap S$.

(b) If S is irreducible, (b) is an immediate consequence of (a). So let $\mathbf{T}_1, \dots, \mathbf{T}_n$ ($n \geqslant 2$) be the irreducible components of $\mathbf{S}$; if $V \subset U$ is a sufficiently small open neighbourhood of $\mathfrak{o}$, there exist analytic sets T_j in V and holomorphic functions f_j, v_j on $V, j = 1, \dots, n$, with the properties listed in consequences 6(b) of Definition 3, and further: (i) f_j vanishes identically on $T'_j = \bigcup_{j' \neq j} T_{j'}$, but $\mathbf{f}_j$ does not vanish on $\mathbf{T}_j$, $j = 1, \dots, n$; (ii) v_j is a universal denominator for T_j at any point $\in T_j$, $j = 1, \dots, n$; (iii) there exists an open neighbourhood $W_j \subset V$ of $\mathfrak{o}$ such that $W_j \cap T$ is an irreducible analytic set in W_j (cf. Proposition 3(c) of § 2), $j = 1, \ldots, n$. By (i) and (iii), and Corollary 3(c) to Proposition 1 of § 3, the germ induced by f_j at any point $a \in W_j \cap T_j$ does not vanish on any irreducible component of the germ induced by T at a.

We claim that $v = \sum_{j=1}^{n} f_j v$ is a universal denominator for S at any point $a \in \left(\bigcap_{j=1}^{n} W_j\right) \cap S$. In fact, given an open neighbourhood $V' \subset \bigcap_{j=1}^{n} W_j$ of a and a holomorphic function h on $V' \cap S$, by (ii), there exist an open neighbourhood $W \subset V'$ of a and, for each $j = 1, \ldots, n$, a holomorphic function u_j on W such that : either $a \notin T_j$, and then $W \cap T_j = \varnothing$; or $a \in T_j$, and then u_j and $v_j h$ have the same restriction to $W \cap T^* \cap {}^cT'_j$ (cf. consequence 6(a) of Definition 3, with $V' \cap T_j$ instead of S and $V' \cap T_j \cap T'_j$ instead of S') ; therefore $\sum_{j=1}^{n} f_j u_j$ and vh have the same restriction to $W \cap S^*$; on the other hand, for each j such that $a \in T_j$, the germ induced at a by $f_j v_j$ does not vanish on any irreducible component of the germ induced by T_j, hence the same is true for the germ $\mathbf{v}_a$ induced at a by v. Since each irreducible component of the germ $\mathbf{S}_a$ induced by S is an irreducible component of a germ induced by some $T_j \ni a$, $\mathbf{v}_a$ does not vanish on any irreducible component of $\mathbf{S}_a$, and the proof is complete.

PROPOSITION 3. *Given an analytic set S in an open set U in C^m and a point $a \in S$, the following two statements are equivalent :*

(i) *the constant* 1 *is a universal denominator for S at a ;*

(ii) *the germ $\mathbf{S}_a$ induced by S at a is irreducible, and the integral domain $\mathscr{H}^m_a/I(\mathbf{S}_a)$ is integrally closed in its field of quotients.*

PROOF. Let a be the origin $\mathfrak{o}$, $\mathbf{S}_a = \mathbf{S}$.

1.. Let (i) hold : for any open neighbourhood $V \subset U$ of $\mathfrak{o}$ and any holomorphic function h_0 on $V \cap S$, there exists a holomorphic function on an open neighbourhood $W \subset V$ of $\mathfrak{o}$ which has the same restriction to $W \cap S^*$ as h_0. This is impossible if h_0 is defined on $V \cap S^*$ by $h_0 \mid T^*_j \cap {}^cT'_j \equiv j$ for each $j = 1, \dots, n$, with the notation of consequence 6 (b) of Definition 3 ; hence $\mathbf{S}$ is irreducible.

Now let f^*/g^* be an element of the field of quotients of the ring $\mathscr{H}^m/I(\mathbf{S})$ which belongs to the integral closure of this ring : f^* and g^* are the images, under the canonical mapping $\mathscr{H}^m \to \mathscr{H}^m/I(\mathbf{S})$ of $\mathbf{f} \in \mathscr{H}^m$ and $\mathbf{g} \in \mathscr{H}^m - I(\mathbf{S})$, and $\mathbf{f}^p + \sum_{j=1}^{p} \mathbf{b}_j \mathbf{f}^{p-j} \mathbf{g}^j \in I(\mathbf{S})$, where each $\mathbf{b}_j \in \mathscr{H}^m$. Let $V \subset U$ be an open neighbourhood of $\mathfrak{o}$ such that $V \cap S$ is irreducible (cf. Corollary 2 to Proposition 1 of § 3), and f, g, b_j, $j = 1, \dots, p$, holomorphic functions on V, inducing at $\mathfrak{o}$ the germs $\mathbf{f}, \mathbf{g}, \mathbf{b}_j$: since $\mathbf{g} \notin I(\mathbf{S})$, by Corollary 1 to Theorem 9, the set $S' = \{x \in V \cap S \mid g(x) = 0\}$ is such that $\dim_x S' < \dim_x (V \cap S)$ for any $x \in S'$; $f/g \mid V \cap S^* \cap {}^cS'$ is a holomorphic function on $V \cap S^* \cap {}^cS'$, which is bounded in a neighbourhood of each point $\in V \cap S$: in fact $(f/g)^p + \sum_{j=1}^{p} b_j (f/g)^{p-j} \equiv 0$ on $V \cap S^* \cap {}^cS'$ (cf. Corollary 3(c) to Proposition 1 of § 3). Hence, by consequence 6(a) of Definition 3, $f/g \mid V \cap S^* \cap {}^cS'$ is the restriction to $V \cap S^* \cap {}^cS'$ of a holomorphic function h_0 on $V \cap S$; f and gh_0 have the same restriction to $V \cap S^*$, and there exists a holomorphic function h, on an open neighbourhood $W \subset V$ of $\mathfrak{o}$, which has the same restriction to $W \cap S^*$ as h_0. Then $f - gh \equiv 0$ on $W \cap S^*$, therefore on $W \cap S$, i.e., $f^* = g^* h^*$, q.e.d.

2. Let (ii) hold ; consider an open neighbourhood $V \subset U$ of $\mathfrak{o}$ and a holomorphic function h_0 on $V \cap S$. By Theorems 12 and 14, there exist holomorphic functions f, g, b_j, $j = 1, \dots, p$, on an open neighbourhood $V' \subset V$ of $\mathfrak{o}$, with $\mathbf{g} \notin I(\mathbf{S})$, such that $h_0^p + \sum_{j=1}^{p} b_j h_0^{p-j} \equiv 0$ on $V' \cap S^*$, and f and gh_0 have the same restriction to $V' \cap S^*$. Then $\mathbf{f}^p + \sum_{j=1}^{p} \mathbf{b}_j \mathbf{f}^{p-j} \mathbf{g}^j \in I(\mathbf{S})$, i.e., with the same notation as above, f^*/g^* belongs to the integral closure of $\mathscr{H}^m/I(\mathbf{S})$; hence there exist an open neighbourhood $W \subset V'$ of $\mathfrak{o}$, such that $W \cap S$ is irreducible, and a holomorphic function h on W, such that $\mathbf{f} - \mathbf{gh} \in I(\mathbf{S})$, and therefore $f - gh \equiv 0$ on $W \cap S$ by Corollary 3(c) to Proposition 1

of § 3. By the same corollary, the set $S' = \{x \in W \cap S \mid g(x) = 0\}$ is such that $W \cap S^* \cap {}^cS'$ is dense in $W \cap S^*$; h and h_0 have the same restriction to $W \cap S^* \cap {}^cS'$, hence to $W \cap S^*$, q.e.d.

DEFINITION 5. *Given an analytic set S in an open set U in C^m and a point $a \in S$, the germ $\mathbf{S}_a$ induced by S at a is normal if* (i) *or* (ii) *of Proposition* 3 *holds.*

REMARK. By a deep theorem of Oka, the set of points $a \in S$ such that $\mathbf{S}_a$ is not normal is again an analytic set in U (cf. [3]), with this consequence: if $\mathbf{S}_a$ is normal, then $\mathbf{S}_x$ is normal, hence irreducible, for any $x \in S$ sufficiently near a.

7. Holomorphic functions on an analytic set, according to R. Remmert. We consider here a class of functions on an analytic set S more special than the ones considered in § 6. This class of functions is the one considered by R. Remmert [4a] and is more natural in that it consists of functions defined everywhere on S.

DEFINITION 6. *Given an analytic set S in an open set U in C^m, a complex-valued function h defined on S is C-holomorphic on S if h is continuous on S and $h \mid S^*$ is holomorphic on S^*.*

Immediate consequences of this definition are properties 1 to 3 of holomorphic functions on S also for C-holomorphic functions on S. We may define, as for holomorphic functions on S, the germ $\mathbf{h}_a$ of a C-holomorphic function h on S at a point $a \in S$.

REMARK. If h is C-holomorphic on S, then $h \mid S^*$ is holomorphic on S. As a counter-example to the converse statement, consider the analytic set $S = \{(x_1, x_2) \mid x_1^3 + x_1^2 - x_2^2 = 0\}$ in C^2. Here $S^* = S - \{0\}$ and the function $(x_1, x_2) \to x_2/x_1$ is holomorphic but not C-holomorphic on S. However Theorem 12 shows that the converse is true if S induces an irreducible germ at every point of S.

Naturally the theorems of § 6 are also true for C-holomorphic functions: in particular the principle of analytic continuation and the dimension theorem (Theorem 13) remain valid without change if we replace "holomorphic on S" by "C-holomorphic on S". Also Proposition 2 of § 6 gives us

(a) If S is an irreducible analytic set in an open set U, any function h C-holomorphic on S is either constant or is an open mapping of S into C, in the sense that if $a \in S$, and T is an analytic set in an open neighbourhood of a such that $\mathbf{T}_a$ is an irreducible component of $\mathbf{S}_a$, then $h(T)$ is a neighbourhood of $h(a)$ in C.

(b) (Maximum principle). If h is C-holomorphic on S, and $|h|$ is maximum at some point $a \in S$, then h is constant on the connected component of S containing a.

Theorem 12 gives us

THEOREM 15. *Given an analytic set S in an open set U, a continuous complex-valued function h defined on S is C-holomorphic on S if and only if, for any $a \in S$, there exist an open neighbourhood $V \subset U$ of a and holomorphic functions $b_1 \dots, b_p$ on V such that*

$$h^p + b_1 h^{p-1} + \dots + b_p \equiv 0 \text{ on } V \cap S.$$

COROLLARY (Remmert). *If h is C-holomorphic on an analytic set S in an open set U, then for any analytic set $S_1 \subset S$ in an open set $U_1 \subset U$, $h \mid S_1$ is C-holomorphic on S_1.*

We conclude with

PROPOSITION 1. *Let S be an analytic set in an open set U in C^m, and g a homeomorphism of S onto an open set X in C^k, $k > 1$, such that g^{-1} is a holomorphic mapping of X into C^m. Then:*

(a) *for any $a \in S$, $\dim_a S = k$, and $\mathbf{S}_a$ is irreducible;*

(b) *a function h, defined on S, is C-holomorphic on S if and only if $h \circ g^{-1}$ is holomorphic on X.*

PROOF. (a) Given $a \in S^*$, let $V \subset U$ be an open neighbourhood of a and f a one-one biholomorphic mapping of V onto an open set V' in C^m, such that $f(V \cap S) = V' \cap L$, where L is an affine subvariety of C^m: $f \circ g^{-1} \mid g(V \cap S)$ is a one-one holomorphic mapping of $g(V \cap S)$, which is an open set in C^k, onto $V' \cap L$, which is an open set in $C^{\dim_a S}$. Then $\dim_a S = k$ by the corollary to Theorem 10. This result implies $\dim_a S = k$ for any $a \in S$.

On the other hand, for any analytic set $T \subset S$ in an open subset V of U, $g(T)$ is an analytic set in the open subset $g(V \cap S)$ of X. Now, given $a \in S$, let $V \subset U$ be an open neighbourhood of a and T_1, T_2 two analytic sets in V such that $V \cap S = T_1 \cup T_2$: then $g(V \cap S) = g(T_1) \cup g(T_2)$ implies that either $g(T_1)$ or $g(T_2)$ coincides with $g(V \cap S)$ in some neighbourhood of the point $g(a)$, therefore either T_1 or T_2 coincides with S in some neighbourhood of a : this proves that $\mathbf{S}_a$ is irreducible.

(b) Let $h \circ g^{-1}$ be holomorphic on X; then h is continuous on S. Given $a \in S^*$, let the notation be as in the proof of (a): by Corollary 2 to Theorem 11, since $f \circ g^{-1} \mid g(V \cap S)$ is a one-one holomorphic mapping of $g(V \cap S)$ onto $V' \cap L$, $g \circ f^{-1} \mid V' \cap L$ is a holomorphic mapping of $V' \cap L$ onto $g(V \cap S)$, and $h \circ f^{-1} \mid V' \cap L = (h \circ g^{-1}) \circ (g \circ f^{-1}) \mid V' \cap L$ is holomorphic on $V' \cap L$. Thus h is C-holomorphic on S.

Now let h be C-holomorphic on S; then $h \circ g^{-1}$ is continuous on X. Given $a \in S^*$, with the same notation again, by Definition 3 (a), there exists an open neighbourhood $W \subset V$ of a such that $h \circ f^{-1} \mid f(W \cap S)$ is holomorphic on $f(W \cap S)$; then $h \circ g^{-1} \mid g(W \cap S) = \{(h \circ f) \circ (f \circ g^{-1})\} \mid g(W \cap S)$ is holomorphic on $g(W \cap S)$, which is an open subset of X containing $g(a)$. This proves that $h \circ g^{-1} \mid g(S^*)$ is holomorphic on $g(S^*)$, which is a dense open subset of X; since $S - S^*$ is an analytic set in V (Theorem 6), $g(S - S^*) = X - g(S^*)$ is an analytic set in X; then $h \circ g^{-1}$ is holomorphic on X, by Theorem 2 of Chap. III. This proves (b).

Remark. The corollary and proposition given above show that C-holomorphic functions are precisely the holomorphic functions in the sense of Remmert [4a].

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LIST OF MAIN DEFINITIONS AND RESULTS

CHAPTER I

1. DEFINITION 1. Holomorphic functions.

   COROLLARY 2 to PROPOSITION 2. Principle of analytic continuation.

2. DEFINITION 2. Germs of holomorphic functions.

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