Differential Analysis

Papers presented at the Bombay Colloquium, 1964, by

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KOHN MALGRANGE MATSUSHIMA MILNOR
MONTGOMERY MORREY MORSE MOSER
NARASIMHAN RAGHUNATHAN DE RHAM
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DIFFERENTIAL ANALYSIS

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
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REPORT

AN International Colloquium on Differential Analysis was held at the Tata Institute of Fundamental Research, Bombay, on 7-14 January 1964. The Colloquium was a closed meeting of experts and of others seriously interested in differential analysis. It was attended by 23 members, and 26 other participants, from France, India, Japan, the Netherlands, Sweden, Switzerland, the United Kingdom, and the United States.

The Colloquium was jointly sponsored, and financially supported, by the International Mathematical Union, the Sir Dorabji Tata Trust, and the Tata Institute of Fundamental Research. An Organizing Committee consisting of Professor K. Chandrasekharan (Chairman), Professor K. G. Ramanathan, Professor M. S. Narasimhan, Professor Raghavan Narasimhan, Professor G. de Rham, and Professor D. Montgomery was in charge of the scientific programme. Professor de Rham and Professor Montgomery acted as representatives of the Union on the Organizing Committee. The purpose of the Colloquium was to discuss recent developments in some aspects of (i) the theory of differential equations, (ii) analysis in the large and differential geometry, and (iii) differential topology.

The following nineteen mathematicians accepted invitations to address the Colloquium :

Professor M. F. Atiyah, Professor R. Bott, Professor L. Gårding, Professor L. Hörmander, Professor J. J. Kohn, Professor B. Malgrange, Professor Y. Matsushima, Professor J. W. Milnor, Professor D. Montgomery, Professor C. B. Morrey, Jr., Professor J. K. Moser, Professor M. S. Narasimhan, Mr. M. S. Raghunathan, Professor G. de Rham, Professor C. S. Seshadri, Professor S. Smale, Professor D. C. Spencer, Professor R. Thom and Professor A. Van de Ven.

Professor M. Morse, who was unable to accept the invitation to attend the Colloquium, sent in a paper.

The Colloquium met in closed sessions. Eighteen lectures were given. Each lecture lasted fifty minutes, followed by discussions. Informal lectures and discussions continued during the week, outside the official programme.

The social programme during the Colloquium week included a ballet and dinner on 7 January; a show of cultural films on 8 January; a performance of Indian music on the Veena, and on the Sitar, on 9 January; a performance of classical Indian dances on 10 January; an excursion to Elephanta on 12 January; and a violin recital followed by a dinner on 13 January.

CONTENTS

CONDITIONED DIFFERENTIABLE ISOTOPIES

By WILLIAM HUEBSCH† and MARSTON MORSE‡

1. Introduction. The theorems on differentiable isotopies found in recent papers, such as [6], [5] and the "Reduction Theorems" in §3 of [1] are inadequate for the purpose of proving some of the more recent theorems in differential topology. In particular the principal theorem concerning the elimination of a pair of critical points, as stated in [2], seems to require deeper Reduction Theorems and differentiable isotopies. Theorem 1.3 of this paper is one such theorem. This paper will establish Theorem 1.3 with an appropriate background.

We refer to a euclidean n-space E_n with rectangular coordinates $x_1, \ldots, x_n$. The point $x = (x_1, \ldots, x_n)$ can be considered a vector with components equal to the respective coordinates of x. Let $\|x\|$ be the magnitude of x conceived as a vector. Corresponding to a prescribed positive constant ρ set

$$D_\rho = \{x \in E_n \mid \|x\| < \rho\}. \tag{1.0}$$

Given a subset Y of E_n set $E_n - Y = {}^cY$. Let $\mathbf{0}$ denote the origin in E_n. Let R denote the axis of reals.

For simplicity all differentiable mappings used in this paper will be differentiable of class C^∞. It is clear that this condition could be greatly relaxed.

DEFINITION. *A differentiable mapping of E_n onto E_n which leaves $\mathbf{0} \cup {}^cD_\rho$ point-wise invariant, will be termed a mapping with domain of identity $\mathbf{0} \cup {}^cD_\rho$.*

DEFINITION. *Two diffeomorphisms whose domains of definition include $\mathbf{0}$, will be said to be $\mathbf{0}$-related if their restrictions to some neighborhood of the origin are identical.*

† Work of Huebsch supported in part by the National Science Foundation under NSF-G19884.

‡ Work of Morse supported by the Air Force Office of Scientific Research under AF-AFOSR-63-357.

Theorem 3.1 of [1] can be reformulated as follows. {Cf. [5] Lemma 8.1, and [6] Lemma 3.2.}

THEOREM 1.1. *Let X be an open neighborhood of* $\mathbf{0}$, *and let* $x \to f(x)$ *be a sense-preserving diffeomorphism of X into E_n which leaves* $\mathbf{0}$ *invariant.*

Corresponding to a prescribed positive constant ρ there exists a diffeomorphism of E_n onto E_n, $\mathbf{0}$*-related to f, with domain of identity* $\mathbf{0} \cup {}^cD_\rho$.

To state an extension of Theorem 1.1 we recall a definition.

DEFINITION. *An isotopy H. Let X be an open subset of E_n. A diffeomorphism h of X into E_n will be said to be differentiably isotopic to a diffeomorphism k of X into E_n if there exists a differentiable mapping:*

$$H: X \times R \to E_n; \ (x, t) \to H(x, t) \tag{1.1}$$

such that each partial mapping:

$$x \to H(x, t) = H^t(x) \ \text{(introducing } H^t\text{)} \tag{1.2}$$

is a diffeomorphism of X into E_n, and if $H^t = h$ for $t \leqslant 0$, and $H^t = k$ for $t \geqslant 1$. We then term H a differentiable isotopy of h into k, and H^t the t-section of H.

The following extension of Theorem 3.1 of [1] is a consequence of Theorem 1.3 of this paper.

THEOREM 1.2. *Let X and f be given as in Theorem 1.1. Corresponding to a prescribed positive constant ρ there exists a diffeomorphism h of E_n onto E_n,* $\mathbf{0}$*-related to f, with domain of identity* $\mathbf{0} \cup {}^cD_\rho$, *admitting a differentiable isotopy H into the identity, such that each section H^t of H is a diffeomorphism of E_n onto E_n with domain of identity* $\mathbf{0} \cup {}^cD_\rho$.

In this paper a differentiable m-manifold Σ_m, $0 < m \leqslant n$, "in E_n" is a differentiable manifold which is *regular* and *proper* in E_n. Σ_m is proper in the sense that its topology is induced by that of E_n; it is regular in E_n if each point in Σ_m has a neighborhood N, relative to Σ_m, such that rectangular coordinates in E_n of an arbitrary point $q \in N$ are functions of class C^∞ of some m of these coordinates of q.

DEFINITION. *An indicatrix of Σ_m at $\mathbf{0}$. Suppose that Σ_m meets the origin $\mathbf{0}$. An ordered set of m linearly independent vectors tangent to Σ_m at $\mathbf{0}$ will be called an indicatrix of Σ_m at $\mathbf{0}$. Two indicatrices of Σ_m at $\mathbf{0}$ are termed equivalent if one can be deformed, as a linearly independent ordered set of m vectors tangent to Σ_m at $\mathbf{0}$, into the other. Non-equivalent indicatrices are termed opposite.*

DEFINITION. *The f-image of an indicatrix. Let f be a diffeomorphism into E_n of a neighborhood X of $\mathbf{0}$. If $\Sigma_m \subset X$, $f(\Sigma_m)$ is well-defined. Let*

$$(w) = (w(1), \dots, w(m))$$

be an ordered set of m contravariant vectors which define an indicatrix of Σ_m at $\mathbf{0}$. The contravariant image under f of the vectors $w(1), \dots, w(m)$, is a set

$$(w') = (w'(1), \dots, w'(m))$$

of vectors tangent to the manifold $f(\Sigma_m)$ at $\mathbf{0}$ which serves as an indicatrix of $f(\Sigma_m)$ at $\mathbf{0}$. We term (w') the f-image of (w).

It is clear that f maps equivalent indicatrices into equivalent indicatrices.

DEFINITION. *Relative similarity of indicatrices. Let r and s be positive integers such that $r + s = n$. Let M_r, M_r^* and L_s be differentiable manifolds in E_n with dimensions r, r and s, respectively. Suppose that*

$$M_r \cap L_s = \mathbf{0}, \quad M_r^* \cap L_s = \mathbf{0},$$

and that M_r and L_s have no tangent vector in common at $\mathbf{0}$, nor M_r^* and L_s. Let

$$(w) = (w(1), \dots, w(r)) \tag{1.3}$$

$$(w^*) = (w^*(1), \dots, w^*(r)) \tag{1.4}$$

be indicatrices of M_r and M_r^*, respectively, at $\mathbf{0}$. Let

$$(\lambda) = (\lambda(1), \dots, \lambda(s)) \tag{1.5}$$

be an arbitrary indicatrix of L_s at $\mathbf{0}$.

We say that the indicatrices (w) and (w^*) at $\mathbf{0}$ are similar relative to L_s if the two ensembles of vectors

$$(\lambda(1), \ldots, \lambda(s) : w(1), \ldots, w(r)) \tag{1.6}$$

$$(\lambda(1), \ldots, \lambda(s) : w^*(1), \ldots, w^*(r)) \tag{1.7}$$

are equivalent as indicatrices of E_n at $\mathbf{0}$.

The property of (w) and (w^*) being similar relative to L_s, is independent of the choice of (λ) as an indicatrix of L_s at $\mathbf{0}$, and of the choice of (w) and (w^*) in equivalence classes of (w) and (w^*) respectively.

Theorem 1.3 is the principal theorem of this paper.

Data in Theorem 1.3. Let X be an open neighborhood of $\mathbf{0}$ in E_n, and L_s and M_r differentiable manifolds in X such that

$$M_r \cap L_s = \mathbf{0}, \tag{1.8}$$

with $r + s = n$ and $0 < s < n$. Suppose moreover that M_r and L_s have no tangent in common at $\mathbf{0}$.

THEOREM 1.3. *Let f be a sense-preserving diffeomorphism of X into E_n, leaving $\mathbf{0}$ fixed, and such that* (a_1) *and* (a_2) *are satisfied.*

(a_1) *$L_s \cap f(M_r) = \mathbf{0}$ and there is no tangent line common to L_s and $f(M_r)$ at $\mathbf{0}$.*

(a_2) *If (w) is an indicatrix of M_r at $\mathbf{0}$, and if (w^*) is the indicatrix of $f(M_r)$ at $\mathbf{0}$ which is the f-image of (w), then (w) and (w^*) are similar relative to L_s.*

Corresponding to a prescribed positive constant ρ there then exists a diffeomorphism h of E_n onto E_n, $\mathbf{0}$-related to f, with domain of identity $\mathbf{0} \cup {}^cD_\rho$, with

$$L_s \cap h(M_r) = \mathbf{0} \tag{1.9}$$

and such that there exists a differentiable isotopy H of h into the identity on E_n each section H^t of which is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$.

The proof of Theorem 1.3 will be completed in § 5.

Methods. In proving Theorem 1.3 we shall rely on two special types of diffeomorphisms of E_n onto E_n termed *ξ-diffeomorphisms* and *perispherical* diffeomorphisms. They will be sense-preserving and leave $\mathbf{0}$ invariant.

A fundamental condition on ξ-diffeomorphisms will be that they "deviate" from the identity in a measured way to be defined in §2. These ξ-diffeomorphisms have been used in [1] in proving Theorem 3.1. However they do not seem adequate in proving Theorem 1.3 of the present paper.

The major difficulty in proving Theorem 1.3 arises from the problem of choosing the diffeomorphism h so that (1.9) of Theorem 1.3 is satisfied, as well as the other conditions on h and H in Theorem 1.3. There are many choices of h such that the conditions of Theorem 1.2 are satisfied, but condition (1.9) of Theorem 1.3 is not satisfied. *Perispherical* diffeomorphisms aid in defining the diffeomorphism h and homotopy H so that *all* conditions on h in Theorem 1.3 are satisfied.

We close this section by recalling some useful definitions.

A product of two isotopies. Let P and Q be differentiable isotopies whose sections P^t and Q^t are diffeomorphisms of E_n onto E_n. If $P^1 = Q^0$, a differentiable isotopy, $W = QP$, termed the *product* of P and Q, is defined by setting

$$\begin{aligned} W^t &= P^0 && (t \leq 0) \\ W^t &= P^{2t} && (0 < t \leq \tfrac{1}{2}) \\ W^t &= Q^{2t-1} && (\tfrac{1}{2} < t \leq 1) \\ W^t &= Q^1 && (t \geq 1). \end{aligned}$$

So defined W is a differentiable isotopy of P^0 into Q^1 as one readily shows.

Deformations of indicatrices represented. For each $t \in R$ let $w^t = (w_1^t, \ldots, w_n^t)$ be a vector in E_n. The mapping $t \to w^t$ is regarded as *continuous* (*differentiable*) if each mapping $t \to w_i^t$, $i = 1, \ldots, n$ of R into R is continuous (differentiable).

(a) For each $t \in R$ and for $0 < m \leq n$ let

$$(w^t) = (w^t(1), \dots, w^t(m))$$

be an ordered set of linearly independent vectors in E_n. The mapping $t \to (w^t)$ is regarded as *continuous* (*differentiable*) if each mapping $t \to w^t(r)$, $r = 1, \dots, m$ is continuous (differentiable).

(b) The preceding mapping $t \to (w^t)$, if continuous (differentiable), will represent a *continuous* (*differentiable*) *deformation* of the indicatrix (w^0) into the indicatrix (w^1) if

$$(w^t(1), \dots, w^t(m)) = (w^0(1), \dots, w^0(m)) \qquad (t \leq 0)$$

$$(w^t(1), \dots, w^t(m)) = (w^1(1), \dots, w^{(1)}(m)) \qquad (t \geq 1).$$

2. ξ-diffeomorphisms. Let $h = (h_1, \dots, h_n)$ be a differentiable mapping of E_n into E_n. Understanding that $x = (x_1, \dots, x_n)$, set

$$d_0(h) = \sup_{x \in E_n} \| x - h(x) \|. \tag{2.1}$$

We suppose that $d_0(h)$ is finite. Assuming that the partial derivatives of the mappings h_i, $i = 1, \dots, n$, are bounded, set

$$d_1(h) = \max_{i,j} \left(\sup_{x \in E_n} \left| \delta_i^j - \frac{\partial h_i}{\partial x_j}(x) \right| \right) \tag{2.2}$$

where i and j have the range $1, \dots, n$ and δ_i^j is a "Kronecker delta". Set

$$d(h) = d_0(h) + d_1(h). \tag{2.3}$$

We term $d(h)$ the 1st-order *deviation of h from the identity.*

The constant ξ. There clearly exists a positive constant ξ so small that a C^1-mapping h of E_n onto E_n for which $d(h) < \xi$ has the property that

$$\frac{3}{2} > \frac{D(h_1, \dots, h_n)}{D(x_1, \dots, x_n)} > \frac{1}{2} \quad (x \in E_n). \tag{2.4}$$

So chosen, ξ will be invariable in this paper.

DEFINITION. *A ξ-mapping. A differentiable mapping h of E_n into E_n such that $d(h) < \xi$, and such that h leaves $0 \cup {}^cD_\rho$ point-wise invariant for some positive constant ρ, will be called a ξ-mapping.*

LEMMA 2.1. *A ξ-mapping h is a diffeomorphism of E_n onto E_n.*

We begin by proving (i).

(i) *The mapping h is onto E_n.*

It is readily seen that the set $h(E_n)$ is both open and closed relative to E_n. Since E_n is connected, $h(E_n) = E_n$. Thus (i) is true.

The mapping h is "proper" in the sense that $h^{-1}(K)$ is compact for arbitrary choice of K as a compact subset of $h(E_n) = E_n$. However a proper mapping of E_n onto E_n, which is locally a diffeomorphism, is a diffeomorphism. See Lemma 4.1 [4].

Thus h is a diffeomorphism of E_n onto E_n.

DEFINITION. *Taking account of Lemma 2.1, a ξ-mapping of E_n onto E_n will be referred to as a ξ-diffeomorphism.*

LEMMA 2.2. *A ξ-diffeomorphism k of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$ admits a differentiable isotopy K into the identity each section K^t of which is a ξ-diffeomorphism with domain of identity $\mathbf{0} \cup {}^cD_\rho$, and such that*

$$d(K^t) \leqslant d(k). \tag{2.5}$$

The mapping μ. In proving this lemma we shall make use of a differentiable mapping μ of R onto $[0, 1]$ such that

$$(0 = \mu(t) \mid t \leqslant 0) \quad (1 = \mu(t) \mid t \geqslant 1). \tag{2.6}$$

Given k as in the lemma we define a mapping K of $E_n \times R$ into E_n by setting

$$K(x, t) = (1 - \mu(t))\, k(x) + \mu(t)\, x \quad (x \in E_n, t \in R). \tag{2.7}$$

One sees that K is a differentiable mapping of $E_n \times R$ into E_n such that for each t, K^t leaves $\mathbf{0} \cup {}^cD_\rho$ point-wise invariant. Moreover for each t

$$d_0(K^t) = (1 - \mu(t))\, d_0(k), \quad d_1(K^t) = (1 - \mu(t))\, d_1(k)$$

so that (2.5) holds. For each t, $d(K^t) < \xi$, since $d(k) < \xi$ by hypothesis. By Lemma 2.1 then, for each t, K^t is a diffeomorphism of E_n onto E_n.

Finally one sees that K^t, as defined by (2.7), is an isotopy of k into the identity, thereby completing the proof of Lemma 2.2.

Lemma 2.3, below, implies Theorem 1.1 in the special case in which the linear terms at $\mathbf{0}$ in the diffeomorphism f define the identity, that is, in the case in which f is a mapping

$$x \to x + A(x) = M(x) \quad (x \in X) \tag{2.8}$$

in which A is differentiable on X and each component A_i of A has a critical point at the origin.

Lemma 2.3 contains information not conveyed by Theorem 1.1, information useful in proving Theorem 1.3.

LEMMA 2.3. *Corresponding to the above diffeomorphism, $x \to M(x)$, of X into E_n, to any positive constant ρ and any positive constant ϵ, there exists a diffeomorphism k of E_n onto E_n, $\mathbf{0}$-related to M, with domain of identity $\mathbf{0} \cup {}^cD_\rho$ and with $d(k) < \epsilon$.*

Let $t \to \lambda(t)$ be a differentiable mapping of R onto $[0, 1]$ such that

$$(1 = \lambda(t) \mid t \leqslant 1) \quad (0 = \lambda(t) \mid t \geqslant 4).$$

Let σ be a positive constant at most $\rho/2$ such that $\bar{D}_{2\sigma} \subset X$. Denote $\| x \|$ by r. In vector notation, set

$$k(x) = x + \lambda\left(\frac{r^2}{\sigma^2}\right) A(x) \quad (r \leqslant 2\sigma) \tag{2.9}$$

and $k(x) = x$ for $r \geqslant 2\sigma$. Then

$$k(x) = x + A(x) = M(x) \quad (r \leqslant \sigma). \tag{2.10}$$

It is clear that k is a differentiable mapping of E_n into E_n. For $i, j = 1, \dots, n$ and for $r \leqslant 2\sigma$

$$\left|\delta_j^i - \frac{\partial k_j}{\partial x_i}(x)\right| = \left|\lambda\left(\frac{r^2}{\sigma^2}\right)\frac{\partial A_j}{\partial x_i}(x) + 2\,\lambda'\left(\frac{r^2}{\sigma^2}\right)\frac{x_i}{\sigma}\,\frac{A_j(x)}{\sigma}\right|. \tag{2.11}$$

The right member of (2.11) is at most $\epsilon/2$ for $r \leqslant 2\sigma$ if σ is sufficiently small. The left member of (2.11) vanishes for $r \geqslant 2\sigma$. Hence $d_1(k) < \epsilon/2$. Moreover

$$d_0(k) \leqslant \max\,(\|A(x)\| \mid \|x\| \leqslant 2\sigma)$$

in accord with (2.9), so that $d_0(k) < \epsilon/2$ if σ is sufficiently small.

Hence $d(k) < \epsilon$ for σ sufficiently small.

By definition k has a domain of identity $\mathbf{0} \cup {}^c D_{2\sigma}$. By Lemma 2.1, k is then a diffeomorphism of E_n onto E_n. Since $\rho \geqslant 2\sigma$, $\mathbf{0} \cup {}^c D_\rho$ is also a domain of identity of k. By (2.10) k is $\mathbf{0}$-related to M.

This completes the proof of Lemma 2.3.

We return to a diffeomorphism $x \to f(x)$ of Theorem 1.1 and, for $i = 1, \ldots, n$, set

$$g_i(x) = \frac{\partial f_i}{\partial x_j}(\mathbf{0})\, x_j \quad (x \in E_n) \tag{2.12}$$

summing as to j on the range $1, \ldots, n$. Theorem 2.1 below is a corollary of Lemma 2.3. In it we refer to the linear diffeomorphism

$$x \to g(x) = (g_1(x), \ldots, g_n(x)). \tag{2.13}$$

THEOREM 2.1. *Let X be an open neighborhood of $\mathbf{0}$ in E_n and $x \to f(x)$ a sense-preserving diffeomorphism of X into E_n which leaves $\mathbf{0}$ invariant. Corresponding to prescribed positive constants ρ and ϵ, there exists a diffeomorphism k_ϵ of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^c D_\rho$, with $d(k_\epsilon) < \epsilon$, and such that the composite diffeomorphism $g k_\epsilon$ of E_n onto E_n is $\mathbf{0}$-related to f.*

NOTE. Theorem 2.1 is also true if f is sense-inverting as our proof shows. We have written Theorem 2.1 as above to preserve the continuity of our development.

PROOF OF THEOREM 2.1. Observe that the mapping

$$x \to (g^{-1} f)(x) = M(x) \quad (x \in X) \tag{2.14}$$

(introducing $M(x)$) has the form

$$x \to M(x) = x + A(x) \tag{2.15}$$

where A has the properties ascribed to A in (2.8). From Lemma 2.3 we then infer the following. There exists a diffeomorphism k_ϵ of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^c D_\rho$, with $d(k_\epsilon) < \epsilon$, and k_ϵ $\mathbf{0}$-related to $g^{-1} f$. It follows that $g k_\epsilon$ is $\mathbf{0}$-related to f.

This completes the proof of Theorem 2.1.

Theorem 1.2 *will follow from Theorem* 1.3 as proved in § 5. However a *proof of Theorem* 1.2 can here be sketched as follows.

Lemma 2.2 implies the following. For $0 < \epsilon < \xi$ there exists a differentiable isotopy K_ϵ of k_ϵ (of Theorem 2.1) into the identity such that for each t, K_ϵ^t is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$.

The mapping g is a linear sense-preserving diffeomorphism of E_n onto E_n leaving $\mathbf{0}$ invariant. One could readily show that there exists a diffeomorphism γ of E_n onto E_n, $\mathbf{0}$-related to g, with domain of identity $\mathbf{0} \cup {}^cD_\rho$, admitting an isotopy Γ into the identity such that for each t, Γ^t is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$.

For each $\epsilon < \xi$ Theorem 1.2 would be satisfied by the composite diffeomorphism $h_\epsilon = \gamma k_\epsilon$ and by an isotopy H_ϵ of which the section H_ϵ^t is the composite diffeomorphism,

$$H_\epsilon^t = \Gamma^t K_\epsilon^t \quad (t \in R), \tag{2.16}$$

taking h_ϵ and H_ϵ in place of h and H in Theorem 1.2. To verify this one notes that h_ϵ is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\rho$ and is $\mathbf{0}$-related to f. The isotopy H_ϵ deforms h_ϵ into the identity. Its sections H_ϵ^t are diffeomorphisms of E_n onto E_n with domains of identity $\mathbf{0} \cup {}^cD_\rho$. Theorem 1.2 will thus be satisfied by h_ϵ and H_ϵ in place of h and H for *arbitrary* choice of $\epsilon < \xi$.

However h_ϵ and H_ϵ will not in general satisfy Theorem 1.3 because (1.9) will not in general be satisfied by such an h_ϵ.

The structure of the proof of Theorem 1.3 *is similar to the above. One chooses* k_ϵ *and* K_ϵ *as above, but then chooses* γ *and* Γ *in a special way so that Theorem* 1.3, *including* (1.9), *is satisfied by* h_ϵ *and* H_ϵ, *as defined by* (3.16), *provided* ϵ *is sufficiently small.*

"Perispherical diffeomorphisms" will aid in defining Γ and γ.

3. Perispherical diffeomorphisms.

For each positive number c let S_c denote the $(n-1)$-sphere in E_n with center at the origin and radius c.

PERISPHERICAL DIFFEOMORPHISMS DEFINED. A diffeomorphism ζ of E_n onto E_n leaving $\mathbf{0}$ invariant will be termed *perispherical* if for some positive constant a, $\zeta(S_r) = S_r$ when $r > a$. Under these circumstances ${}^{e}D_a$ will be termed a *domain of sphericity* of ζ.

We shall need polar coordinates in E_n.

POLAR COORDINATES. To each point $x \neq \mathbf{0}$ we assign coordinates $(r, z) = (r, z_1, \ldots, z_n)$, termed *polar*, by setting

$$r = \| x \|, \quad z_i = \frac{x_i}{\|x\|} \quad (i = 1, \ldots, n). \tag{3.1}$$

Thus $x = rz$. The points z are on the sphere $\mathscr{S}$ on which $\| x \| = 1$. Let R_+ denote the axis of positive real numbers. The mapping $x \to (r, z)$ defined by (3.1) is a real analytic diffeomorphism of $E_n - \mathbf{0}$ onto $R_+ \times \mathscr{S}$.

THE DIFFEOMORPHISM g. Let g be a linear diffeomorphism of E_n onto E_n leaving the origin invariant. Cf. (2.12). In polar coordinates

$$\| g(rz) \| = r\phi(z) > 0 \quad (r > 0) \tag{3.2}$$

introducing ϕ. The mapping $z \to \phi(z)$ is a real analytic mapping of $\mathscr{S}$ into R_+. Set

$$m = \max\,(\phi(z) \,|\, z \in \mathscr{S}). \tag{3.3}$$

Under g a point in $E_n - \mathbf{0}$ with polar coordinates (r, z) corresponds to a point with polar coordinates (r', z') such that

$$r' = r\phi(z), \; z'_i = \frac{g_i(rz)}{\| g(rz) \|} = \psi_i(z) \quad (r > 0) \tag{3.4}$$

introducing $\psi_i(z)$, $i = 1, \ldots, n$. One sees that the mapping $z \to \psi(z)$ is an analytic diffeomorphism of $\mathscr{S}$ onto $\mathscr{S}$.

Lemma 3.1, below, is an aid in proving the fundamental Lemma 3.2. Both lemmas concern the above linear diffeomorphism g.

LEMMA 3.1. *Corresponding to the linear diffeomorphism g, to the constant m, defined in* (3.3), *and to a prescribed positive constant a, there exists a diffeomorphism p of E_n onto E_n, $\mathbf{0}$-related to g, with $p(\pi) = g(\pi)$ for each ray π emanating from $\mathbf{0}$, and such that in polar coordinates*

$$\| p(rz) \| < am + r, \quad \frac{\partial \, \| p(rz) \|}{\partial r} = \frac{\phi(z)}{m} < 1 \quad (r > a). \tag{3.5}$$

Let μ be a monotone differentiable mapping of R into R such that

$$(1 = \mu(t) \mid t \leqslant a/2) \quad (1/m = \mu(t) \mid t \geqslant a). \tag{3.6}'$$

To define p we first set $p(\mathbf{0})=\mathbf{0}$. Under p a point $x \neq \mathbf{0}$ with polar coordinates (r, z) shall go into a point in $E_n - \mathbf{0}$ with polar coordinates (r'', z'') such that

$$r'' = \phi(z) \int_0^r \mu(\alpha)\, d\alpha, \quad z''_i = \psi_i(z) \quad (i = 1, \dots, n). \tag{3.6}''$$

where ϕ and ψ_i are defined in (3.2) and (3.4) respectively. For $0 < r \leqslant a/2$, (3.6)″ takes the form

$$r'' = r\,\phi(z) \quad z''_i = \psi_i(z) \qquad \text{(Cf. (3.4))}$$

so that $p(x) = g(x)$ for $\|x\| \leqslant a/2$.

For fixed z, $\partial r''/\partial r > 0$ for $r > 0$. For $r \geqslant a$ this partial derivative equals $\phi(z)/m$, as indicated in (3.5). Under p each ray π is accordingly mapped biuniquely onto $g(\pi)$. It follows that p maps E_n biuniquely onto E_n. Since $\partial r''/\partial r > 0$ for $r > 0$, and since the mapping $z'' = \psi(z)$ of $\mathscr{S}$ onto $\mathscr{S}$ is the same as the mapping $z' = \psi(z)$ defined by g, we infer that p is locally a diffeomorphism for $r > 0$. Since p is $\mathbf{0}$-related to g, p like g, is a diffeomorphism neighboring $\mathbf{0}$. Thus p is a diffeomorphism.

The first relation in (3.5) follows readily from (3.6)″.

This completes the proof of Lemma 3.1.

Lemma 3.2. *Corresponding to the linear diffeomorphism g of Lemma 3.1, and to a prescribed positive constant c, there exists a perispherical diffeomorphism ω of E_n onto E_n, $\mathbf{0}$-related to g, with domain of sphericity ${}^cD_c$, and such that $\omega(\pi) = g(\pi)$ for each ray π issuing from $\mathbf{0}$.*

Let b be a constant such that $0 < b < c$. Let $t \to \theta(t)$ be a differentiable mapping of R onto [0, 1] such that

$$(1 = \theta(t) \mid t \leqslant b) \quad (0 = \theta(t) \mid t \geqslant c) \quad (0 > \theta'(t) \mid b < t < c).$$

With ϕ defined on $\mathscr{S}$ by (3.2), set

$$\chi = \min\left(\theta(r)\frac{\phi(z)}{m} + 1 - \theta(r) \mid r \geqslant 0, z \in \mathscr{S}\right) \tag{3.7}$$

and note that $\chi > 0$. We shall apply Lemma 3.1 taking a so small that $0 < a < b$ and

$$am\theta'(\tau) > -\chi \quad (\tau \in R). \tag{3.8}$$

Definition of ω. We begin by setting $\omega(\mathbf{0}) = \mathbf{0}$. Under the mapping $x \to \omega(x)$ the point in $E_n - \mathbf{0}$ with polar coordinates (r, z) shall go into a point in $E_n - \mathbf{0}$ with polar coordinates (r^*, z^*) such that

$$z^*_i = \psi_i(z) \quad (i = 1, \ldots, n) \tag{3.9}$$

and

$$r^* = \theta(r)\, \| p(rz) \| + (1 - \theta(r))\, r \tag{3.10}$$

where $\psi_i(z)$ is defined in (3.4).

Properties of ω. For $r \leqslant b$ (3.10) gives the relation $r^* = \| p(rz) \|$ so that $\omega(x) = p(x)$ for $\| x \| \leqslant b$. For $r \geqslant c$ (3.10) gives the relation $r^* = r$. For fixed z and $b < r < c$ one can apply the second relation in (3.5), since $b > a$. One finds that

$$\frac{\partial r^*}{\partial r} = \theta(r)\frac{\phi(z)}{m} + (1 - \theta(r)) + \theta'(r)\,(\| p(rz) \| - r) \quad (b \leqslant r \leqslant c). \tag{3.11}$$

It follows from the first relation in (3.5) and the condition (3.8) on a, that

$$\theta'(r)\,(\| p(rz) \| - r) \geqslant am\theta'(r) > -\chi \quad (b < r < c, z \in \mathscr{S}). \tag{3.12}$$

From (3.7), (3.11) and (3.12) we conclude that $\partial r^* / \partial r > 0$ for $b < r < c$ and $z \in \mathscr{S}$. Since $r^* = r$ for $r \geqslant c$ and since $r^* = \| p(rz) \|$ for $0 < r \leqslant b$ it follows that $\partial r^* / \partial r > 0$ for all positive r.

One sees that on each ray π issuing from $\mathbf{0}$, ω is biunique and, in accord with (3.9), maps π onto $p(\pi) = g(\pi)$. It follows that ω is biunique and onto E_n. Since ω is also locally a diffeomorphism ω is a diffeomorphism.

Since ω is $\mathbf{0}$-related to p and p is $\mathbf{0}$-related to g, ω is $\mathbf{0}$-related to g. The relation $r^* = r$ for $r \geqslant c$ means that ${}^cD_c$ is a domain of sphericity of ω.

This completes the proof of Lemma 3.2.

DEFINITION. *When ω is related to g as in Lemma 3.2, ω will be called a perispherical counterpart of g with domain of sphericity ${}^cD_c$.*

Lemma 3.2 admits the following easy extension.

LEMMA 3.2a. *Let g be a linear diffeomorphism of E_n onto E_n leaving $\mathbf{0}$ invariant and admitting a differentiable isotopy G into the identity such that for each t, G^t is a linear diffeomorphism of E_n onto E_n leaving $\mathbf{0}$ invariant.*

Corresponding to a prescribed constant $c > 0$ there exists a perispherical counterpart ω of g and a differentiable isotopy Ω of ω into the identity such that for each t, Ω^t is a perispherical counterpart of G^t with domain of sphericity ${}^cD_c$.

The proof of Lemma 3.2a is similar to that of Lemma 3.2. In brief it runs as follows.

For each t set

$$\| G^t(rz) \| = r\,\phi^t(z) \quad (r > 0)$$

as in (3.2). As in (3.3) set

$$m = \max\,(\phi^t(z) \mid z \in \mathscr{S},\ 0 \leqslant t \leqslant 1). \tag{3.13}$$

For each t (3.4) takes the form

$$r' = r\,\phi^t(z), \quad z'_i = \frac{G_i^t(rz)}{\| G^t(rz) \|} = \psi_i^t(z), \quad (r > 0) \tag{3.14}$$

introducing $\psi_i^t(z)$ for i on the range $1, \dots, n$.

One could extend Lemma 3.1 by a Lemma 3.1a, replacing g by its isotopy G and deriving from G a differentiable isotopy P of p into the identity in which for each t, P^t is related to G^t as p is to g in Lemma 3.1. In the definition of P^t in terms of G^t the auxiliary mapping μ, as defined in (3.6)′ is unchanged, while (3.6)″ takes the form

$$r'' = \phi^t(z) \int_0^r \mu(\alpha)\, d\alpha, \quad z''_i = \psi_i^t(z) \quad (r > 0)$$

for each t.

One uses Lemma 3.1a to prove Lemma 3.2a. The auxiliary mapping θ is unchanged. The constant χ, defined in (3.7), is replaced by the constant

$$\chi = \min \left(\theta(r) \frac{\phi^t(z)}{m} + (1 - \theta(r)) \,\middle|\, r \geqslant 0, z \in \mathcal{S}, 0 \leqslant t \leqslant 1 \right) > 0.$$

Condition (3.8) on a takes the same form. For each t one replaces (3.9) by the relation

$$z^*_i = \psi^t_i(z) \quad (i = 1, \dots, n)$$

and (3.10) by

$$r^* = \theta(r) \, \| P^t(rz) \| + (1 - \theta(r))r \quad (r > 0).$$

The proof is concluded essentially as before.

In § 4 use will be made of the following readily established lemma.

Approximation Lemma. *Let $t \to \lambda(t)$ be a continuous mapping of R into R such that*

$$(\lambda(t) = \lambda(0) \mid t \leqslant 0) \quad (\lambda(t) = \lambda(1) \mid t \geqslant 1).$$

Corresponding to a prescribed constant $e > 0$, there exists a differentiable mapping $t \to \eta(t)$ of R into R such that

$$| \lambda(t) - \eta(t) | < e \, (t \in R) \tag{3.15}$$

and

$$(\eta(t) = \lambda(0) \mid t \leqslant 0) \quad (\eta(t) = \lambda(1) \mid t \geqslant 1). \tag{3.16}$$

One method of proof is to define a differentiable mapping $t \to \eta(t)$ of R into R such that

$$| \lambda(t) - \eta(t) | < e/2 \tag{3.17}$$

but for which (3.16) does not necessarily hold. One can modify this function η so that it remains of class C^∞ and satisfies both (3.15) and (3.16).

4. Deformation of indicatrices. In this section certain implications of hypotheses (a_1) and (a_2) will be presented.

We refer to the manifolds L_s and M_r of Theorem 1.3, and to the image manifold $f(M_r)$. Let $(\lambda(1), \dots, \lambda(s)) = (\lambda)$ and $(w(1), \dots, w(m)) = (w)$ be indicatrices of L_s and M_r respectively at $\mathbf{0}$.

Let g be the sense-preserving linear diffeomorphism of E_n onto E_n determined by f at $\mathbf{0}$ and defined in (2.13). The mapping $x \to g(x) = y$ has the form

$$y_i = g_{ij}\, x_j \quad (i = 1, \ldots, n) \tag{4.1}$$

summing with respect to j on the range $1, \ldots, n$. Let

$$(w^*(1), \ldots, w^*(r)) = (w^*) \tag{4.2}$$

be the image under g of the indicatrix (w) of M_r at $\mathbf{0}$, or equivalently the contravariant image of (w) under f. Then (w^*) is an indicatrix of $f(M_r)$ at $\mathbf{0}$.

The vectors in the ensemble

$$(\lambda(1), \ldots, \lambda(s) : w(1), \ldots, w(r)) = (\lambda : w) \tag{4.3}$$

of n vectors are linearly independent, since L_s and M_r have no tangent in common at the origin. The vectors in the ensemble

$$(\lambda(1), \ldots, \lambda(s) : w^*(1), \ldots, w^*(r)) = (\lambda : w^*) \tag{4.4}$$

are similarly linearly independent, since L_s and $f(M_r)$ have no tangent in common at the origin. See Theorem 1.3.

By the *determinant* of an *indicatrix* (h) of E_n at $\mathbf{0}$ is meant the determinant, the elements of whose rows are the components of the vectors in (h), taking these vectors in the order given in (h). Recall that the indicatrices (w) and (w^*), by hypothesis, are similar relative to L_s; equivalently, the determinants of the vectors $(\lambda : w)$ and $(\lambda : w^*)$ have the same sign, or equivalently, (for proper choice of (λ)) these determinants are positive. We suppose (λ) so chosen.

Since the determinants of $(\lambda : w)$ and $(\lambda : w^*)$ have the same sign there exists a continuous deformation (Cf. § 1)

$$t \to (w^t(1), \ldots, w^t(r)) = (w^t) \quad (t \in R) \tag{4.5}$$

of $(w^0) = (w^*)$ into $(w^1) = (w)$ such that the vectors of the ensemble

$$(\lambda(1), \ldots, \lambda(s) : w^t(1), \ldots, w^t(r)) = (\lambda : w^t) \tag{4.6}$$

remain linearly independent for all t.

With the aid of the Approximation Lemma of § 3 we infer the following.

LEMMA 4.1. *There exists a differentiable deformation* $t \rightarrow (w^t)$ *of the indicatrix* (w^*) *of* $f(M_r)$ *at* $\mathbf{0}$ *into the indicatrix* (w) *of* M_r *at* $\mathbf{0}$ *such that the vectors of the corresponding ensemble* (4.6) *remain linearly independent for all* $t \in R$.

We shall state a lemma which is complementary to Lemma 4.1, introducing this lemma as follows.

The image under g of the indicatrix (λ) of L_s at $\mathbf{0}$ will be denoted by

$$(\lambda^*(1), \ldots, \lambda^*(s)) = (\lambda^*). \tag{4.7}$$

So defined (λ^*) is an indicatrix of $f(L_s)$ at $\mathbf{0}$. The ensemble of vectors

$$(\lambda^*(1), \ldots, \lambda^*(s) : w^*(1), \ldots, w^*(r)) = (\lambda^* : w^*) \tag{4.8}$$

is the image under g of the indicatrix $(\lambda : w)$, and has a positive determinant since g is sense-preserving. Since the indicatrix $(\lambda : w^*)$ also has a positive determinant there exists a continuous deformation,

$$t \rightarrow (\lambda^t(1), \ldots, \lambda^t(s)) = (\lambda^t) \quad (t \in R) \tag{4.9}$$

of (λ^*) into (λ) such that the vectors in the ensemble

$$(\lambda^t(1), \ldots, \lambda^t(s) : w^*(1), \ldots, w^*(r)) = (\lambda^t : w^*) \tag{4.10}$$

remain independent for $t \in R$.

With the aid of the Approximation Lemma of § 3 we can infer the following.

LEMMA 4.2. *There exists a differentiable deformation* $t \rightarrow (\lambda^t)$ *of the indicatrix* (λ^*) *of* $f(L_s)$ *at* $\mathbf{0}$ *into the indicatrix* (λ) *of* L_s *at* $\mathbf{0}$ *such that the vectors of the corresponding ensemble* (4.10) *remain linearly independent for all* t.

A differentiable isotopy G *of* g *into the identity.* We shall define a special differentiable isotopy G of g into the identity such that each section G^t of g is a linear diffeomorphism of E_n onto E_n leaving $\mathbf{0}$ invariant. We shall define G as a "product" QP of differentiable isotopies Q and P. See § 1.

DEFINITION. *Let* (h) *and* (h^*) *be indicatrices of* E_n *at* $\mathbf{0}$. *The linear diffeomorphism* H *carrying* (h) *into* (h^*) *is by definition the linear diffeomorphism of* E_n *onto* E_n *which maps* $\mathbf{0}$ *into* $\mathbf{0}$ *and maps the* n *points whose coordinates are the components of the respective vectors* $h(1), \ldots, h(n)$ *in* (h), *into the* n *points whose coordinates are the components of the respective vectors* $h^*(1), \ldots, h^*(n)$ *in* (h^*).

With this understood let p be the linear diffeomorphism carrying $(\lambda : w)$ into $(\lambda : w^*)$. Differentiable deformations $t \to (w^t)$ and $t \to (\lambda^t)$ are characterized in Lemmas 4.1 and 4.2 respectively. For each $t \in R$ let Q^t and P^t be the linear diffeomorphisms carrying $(\lambda : w)$ into $(\lambda : w^t)$ and $(\lambda^t : w^*)$ respectively. Recall that g is the linear diffeomorphism carrying $(\lambda : w)$ into $(\lambda^* : w^*)$. These diffeomorphisms are sense-preserving.

The product $G = QP$. Since P^0 maps $(\lambda : w)$ into $(\lambda^0 : w^*) = (\lambda^* : w^*)$, we see that $P^0 = g$. Now $P^1 = p$, since P^1 maps $(\lambda : w)$ into $(\lambda^1 : w^*) = (\lambda : w^*)$. Thus P is a differentiable isotopy of g into p. Moreover Q^0 maps $(\lambda : w)$ into $(\lambda : w^0) = (\lambda : w^*)$ so that $Q^0 = p$, while Q^1 maps $(\lambda : w)$ into $(\lambda : w^1) = (\lambda : w)$ so that Q^1 reduces to the identity I. Thus Q is a differentiable isotopy of p into I. The product $G = QP$, as defined in § 1, is a differentiable isotopy, and deforms g into I.

NOTATION. Given an indicatrix (h) of an r-manifold meeting $\mathbf{0}$, let the r-plane of vectors in (h), assumed to have initial points at $\mathbf{0}$, be denoted by $\langle h \rangle$.

We shall prove the following.

(i) *If* $\mathscr{L}_s$ *is the s-plane tangent to* L_s *at* $\mathbf{0}$, *and* $\mathscr{M}_r$ *the r-plane tangent to* M_r *at* $\mathbf{0}$, *then*

$$\mathscr{L}_s \cap G^t(\mathscr{M}_r) = \mathbf{0} \quad (t \in R). \tag{4.11}$$

PROOF OF (i). Recall that $\mathscr{L}_s = \langle \lambda \rangle$ and $\mathscr{M}_r = \langle w \rangle$. The definition of QP in § 1 implies that

$$G^t(\mathscr{M}_r) = P^{2t}(\mathscr{M}_r) = \langle w^* \rangle = \langle w^0 \rangle \quad (0 \leq t \leq \tfrac{1}{2})$$

and with $\mathrm{T} = 2t - 1$

$$G^t(\mathscr{M}_r) = Q^{\mathrm{T}}(\mathscr{M}_r) = \langle w^{\mathrm{T}} \rangle \quad (\tfrac{1}{2} \leq t \leq 1).$$

Finally

$$G^t(\mathscr{M}_r) = \langle w^* \rangle = \langle w^0 \rangle \qquad (t \leqslant 0)$$
$$G^t(\mathscr{M}_r) = \langle w \rangle = \langle w^1 \rangle \qquad (t \geqslant 1).$$

Since the vectors (4.6) are linearly independent

$$\langle \lambda \rangle \cap \langle w^t \rangle = \mathbf{0} \qquad (t \in R) \tag{4.12}$$

so that (4.11) follows.

Corresponding to each positive constant c set

$$\bar{L}_s \cap \bar{D}_c = L_s^c \qquad \bar{M}_r \cap \bar{D}_c = M_r^c. \tag{4.13}$$

Given a point $x \in E_n - \mathbf{0}$ let π_x denote the ray emanating from $\mathbf{0}$ and meeting x. Set

$$\Pi\, M_r^c = \text{Cl (Union } \pi_x \mid x \in M_r^c - \mathbf{0}). \tag{4.14}$$

This set is a "cone" in that it is the union of a set of rays emanating from $\mathbf{0}$. It is clear that $\Pi\, M_r^c$ includes M_r^c and the r-plane $\mathscr{M}_r$ tangent to M_r at $\mathbf{0}$.

Statement (i) leads to Lemma 4.3.

Lemma 4.3. *If σ is a sufficiently small positive constant*

$$L_s^\sigma \cap G^t(\Pi\, M_r^\sigma) = \mathbf{0} \quad (t \in R). \tag{4.15}$$

Were L_s and M_r suitably chosen, regular manifolds in E_n, not "properly" embedded in E_n, (4.15) would be false regardless of the choice of $\sigma > 0$.

Proof of Lemma 4.3. Since L_s and M_r are by hypothesis regularly and properly embedded in E_n, a sufficiently small positive constant e will have the following property. The intersections $D_e \cap M_r$ and $D_e \cap L_s$ are differentiable r-and s-manifolds tangent respectively to the r-plane $\mathscr{M}_r$ and s-plane $\mathscr{L}_s$ at $\mathbf{0}$, and such that their orthogonal projections into $\mathscr{M}_r$ and $\mathscr{L}_s$ are diffeomorphisms. If σ is a sufficiently small constant such that $0 < \sigma < e$ it is clear that (4.15) is a consequence of (4.12) and of the preceding statement.

5. Proof of Theorem 1.3. The constant $\rho > 0$ is given in Theorem 1.3.

Choice of σ. Of constants σ such that $0 < \sigma < \rho$ let σ be so small that (4.15) holds.

A diffeomorphism ω *and isotopy* Ω. Let g be the linear diffeomorphism defined by the terms of first order at $\mathbf{0}$ in the diffeomorphism f given in Theorem 1.3. Let a differentiable isotopy $G = QP$ of g into the identity be defined as in §4. By virtue of Lemma 3.2a there exists a perispherical counterpart ω of g and a differentiable isotopy Ω of ω into the identity, such that for each t, Ω^t is a perispherical counterpart of G^t with domain of sphericity ${}^cD_{\sigma/2}$. As a special consequence

$$\Omega^t(D_a) = D_a \quad (a = \sigma/2,\ t \in R). \tag{5.0}$$

A diffeomorphism γ *and isotopy* Γ. Let $t \to \nu(t)$ be a differentiable mapping of R into [0,1] such that

$$(0 = \nu(t) \mid t \leqslant a) \quad (1 = \nu(t) \mid t \geqslant \sigma) \quad (a = \sigma/2). \tag{5.1)$'$$$

Let $t \to \alpha(t)$ be a differentiable mapping of R onto [0,1] such that

$$(0 = \alpha(t) \mid t \leqslant 0) \quad (1 = \alpha(t) \mid t \geqslant 1). \tag{5.1)$''$$$

We shall define a differentiable mapping

$$(x, t) \to \Gamma(x, t)\,;\ E_n \times R \to E_n \tag{5.2}$$

by setting

$$\Gamma(x, t) = \Omega(x,\ \alpha(t)\,(1 - \nu(\|x\|)) + \nu(\|x\|)) \tag{5.3}$$

and establish the following lemma.

LEMMA 5.1. I. *For each* $t \in R$, Γ^t *is a diffeomorphism of* E_n *onto* E_n *with domain of identity* $\mathbf{0} \cup {}^cD_\sigma$.

II. *The mapping* Γ *is a differentiable isotopy of* Γ^0 *into the identity.*

III. *The diffeomorphism* $\gamma = \Gamma^0$ *is* $\mathbf{0}$*-related to* g.

IV. $L_s^\sigma \cap \gamma(\Pi\, M_r^\sigma) = \mathbf{0}$. (Cf. (4.15).)

Note the following consequences of the definition (5.3) of Γ.

$$\Gamma(x, t) = \Omega(x, \alpha(t)) \qquad (x \in D_a,\ a = \sigma/2\,;\ t \in R) \tag{5.4)$'$$$

$$\Gamma(x, t) = \Omega(x, 1) = x \qquad (x \in {}^cD_\sigma\,;\ t \in R) \tag{5.4)$''$$$

$$\Gamma(x, t) = \Omega(x, 1) = x \qquad (x \in E_n\,; t \geqslant 1) \tag{5.5)'$$

$$\Gamma(x, t) = \Omega(x, \nu(\|x\|)) \qquad (x \in E_n\,; t \leqslant 0) \tag{5.5)''$$

According to (5.4)″, for each t, Γ^t has the domain of identity $\mathbf{0} \cup {}^cD_\sigma$.

As previously S_r shall denote the $(n-1)$-sphere of radius r with center at $\mathbf{0}$.

We establish I by proving (Ia) and (Ib).

(Ia) *For each t, Γ^t is a biunique mapping of E_n onto E_n.*

It is trivial that

$$\Gamma^t(E_n) = \Gamma^t(D_a) \cup \Gamma^t({}^cD_a) \quad (t \in R). \tag{5.6}$$

To show that Γ^t is biunique and onto E_n it is sufficient to verify that

$$\Gamma^t(D_a) = D_a \quad \Gamma^t({}^cD_a) = {}^cD_a \tag{5.7}$$

and that the mappings

$$\Gamma^t \mid D_a \quad , \quad \Gamma^t \mid {}^cD_a \tag{5.8}$$

are biunique.

The case of D_a. It follows from (5.4)′ that

$$\Gamma^t \mid D_a = \Omega^{\alpha(t)} \mid D_a \quad (t \in R). \tag{5.9}$$

Now Ω^t is biunique for each t. Hence $\Gamma^t \mid D_a$ is biunique for each t. It follows from (5.9) and (5.0) that

$$\Gamma^t(D_a) = \Omega^{\alpha(t)}(D_a) = D_a \quad (a = \sigma/2, \ t \in R).$$

The case of ${}^cD_a$. It is trivial that

$$\Gamma^t({}^cD_a) = \underset{r \geqslant a}{\text{Union}}\ \Gamma^t(S_r) \quad (t \in R). \tag{5.10}$$

Moreover for $r \geqslant a$

$$\Gamma^t \mid S_r = \Omega^{\mathrm{T}} \mid S_r\,;\ \Omega^{\mathrm{T}}(S_r) = S_r \ \ (\mathrm{T} = \alpha(t)\,(1 - \nu(r)) + \nu(r)\,;\, t \in R) \tag{5.11}$$

as a consequence of (5.3) and the fact that Ω^t has the domain of sphericity ${}^cD_a$ for each t. By virtue of (5.11), (5.10) takes the form

$$\Gamma^t({}^cD_a) = \underset{r \geqslant a}{\text{Union}}\ S_r = {}^cD_a \tag{5.12}$$

establishing the second relation in (5.7).

When $r \geqslant a$, Ω^t maps S_r biuniquely onto S_r for each t, including T in (5.11); hence Γ^t maps S_r biuniquely onto S_r. Taking account of (5.12) we see that $\Gamma^t \mid {}^cD_a$ is biunique for arbitrary $t \in R$.

Thus Γ^t is biunique for each t.

We continue by proving (Ib).

(Ib) *The jacobian of the mapping* $x \to \Gamma^t(x)$, $t \in R$ *vanishes at no point* $x \in E_n$.

The case of D_a. (Ib) is true for $x \in D_a$ by virtue of (5.4)′ and the fact that Ω^t is a diffeomorphism for each t.

The case of ${}^cD_a$. (Ib) is true for $x \in {}^cD_a$. One verifies this most readily on referring ${}^cD_a - \mathbf{0}$ to polar coordinates (r, z) as in § 3. From (5.11) one infers that the mapping $\Gamma^t \mid {}^cD_a$ makes a point $(r, z) \in {}^cD_a - \mathbf{0}$ correspond to a point with polar coordinates (r^*, z^*) such that $r = r^*$, and that $\Gamma^t | S_r$ is a diffeomorphism of S_r onto S_r for $r \geqslant a$. It follows that the jacobian of Γ^t differs from zero at each point of ${}^cD_a$.

This establishes (Ib) and I follows.

Proof of II. That Γ is an isotopy of Γ^0 into the identity Γ^1 follows from I and relations (5.5)′ and (5.5)″.

Proof of III. From (5.5)″, the definition of ν (see (5.1)′) and the definitions of γ and ω we see that

$$\gamma(x) = \Gamma(x, 0) = \Omega(x, 0) = \omega(x) \quad (x \in D_a)$$

where ω is a perispherical diffeomorphism of E_n onto E_n. According to Lemma 3.2 ω is $\mathbf{0}$-related to g. Hence γ is $\mathbf{0}$-related to g and III is established.

Proof of IV. Since $\gamma(x) = \Gamma(x, 0)$ by definition, (5.3) shows that for $x \in E_n$

$$\gamma(x) = \Omega(x, \nu(\| x \|)) = \Omega^{t(x)}(x) \quad (t(x) = \nu(\| x \|)).$$

For $x \neq \mathbf{0}$, $x \in \pi_x$, so that

$$\gamma(x) \in \Omega^{t(x)}(\pi_x) \quad (x \in E_n - \mathbf{0}). \tag{5.13}$$

Since Ω^t is a "perispherical counterpart" of G^t (as ω is of g in Lemma 3.2)

$$\Omega^t(\pi_x) = G^t(\pi_x) \quad (x \in E_n - \mathbf{0}, t \in R) \tag{5.13$'$}$$

so that it follows from the definition (4.14) of ΠM_r^σ, from (5.13) and (5.13)$'$, that

$$\gamma(x) \in G^{t(x)}(\Pi M_r^\sigma) \quad (x \in \Pi M_r^\sigma - \mathbf{0}). \tag{5.14}$$

The constant σ has been chosen so that (4.15) is true. We infer from (4.15) and (5.14) that $\gamma(x)$ is not in L_s^σ for x conditioned as in (5.14).

It follows that IV is true, thus completing the proof of Lemma 5.1.

Completion of proof of Theorem 1.3. According to Theorem 2.1, for each positive constant ϵ there exists a diffeomorphism k_ϵ of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\sigma$, with $d(k_\epsilon) < \epsilon$, and such that the composite diffeomorphism $g\,k_\epsilon$ of E_n onto E_n is $\mathbf{0}$-related to the diffeomorphism f given in Theorem 1.3. According to Lemma 2.2 if $\epsilon < \xi$, there exists a differentiable isotopy, K_ϵ, of k_ϵ into the identity, such that for each t, K_ϵ^t is a diffeomorphism of E_n onto E_n with domain of identity $\mathbf{0} \cup {}^cD_\sigma$.

Diffeomorphism h_ϵ and isotopy H_ϵ. As chosen previously the constant σ is such that $0 < \sigma < \rho$, where ρ is given in Theorem 1.3, and so small that (4.15) holds. Let Γ be the isotopy affirmed to exist in Lemma 5.1. By Lemma 5.1 the diffeomorphism $\gamma = \Gamma^0$ is $\mathbf{0}$-related to g, the diffeomorphism defined by the terms of first order in the diffeomorphism f. For each $\epsilon < \xi$ we introduce the composite diffeomorphism $h_\epsilon = \gamma\, k_\epsilon$. The composite diffeomorphisms

$$H_\epsilon^t = \Gamma^t K_\epsilon^t \quad (t \in R) \tag{5.15}$$

serve to define an isotopy H_ϵ of h_ϵ into the identity. Cf (2.16) and paragraphs at end of §2.

Theorem 1.3 is implied by Lemma 5.2.

LEMMA 5.2. *If $\epsilon_0 < \xi$ is a sufficiently small positive constant and if $0 < \epsilon < \epsilon_0$, the diffeomorphism $h_\epsilon = \gamma k_\epsilon$ of E_n onto E_n and its isotopy H_ϵ into the identity satisfy Theorem* 1.3 *in place of h and H.*

For each constant ϵ such that $0 < \epsilon < \xi$ the diffeomorphism h_ϵ and isotopy H_ϵ have the following properties.

(c_1) The diffeomorphism $h_\epsilon = \gamma k_\epsilon$ is $\mathbf{0}$-related to f since γ is $\mathbf{0}$-related to g and since gk_ϵ is $\mathbf{0}$-related to f.

(c_2) Each section H_ϵ^t of H_ϵ has the domain of identity $\mathbf{0} \cup {}^cD_\sigma$, since Γ^t and K_ϵ^t have this domain of identity.

(c_3) $$L_s \cap h_\epsilon(M_r - M_r^\sigma) = \varnothing.$$

That (c_3) holds follows from the relation

$$h_\epsilon(M_r - M_r^\sigma) = M_r - M_r^\sigma \tag{5.16}$$

a consequence of (c_2) when $t = 0$, and from the relation $L_s \cap M_r = \mathbf{0}$ presupposed in (1.8). Cf. Theorem 1.3.

Properties (c_1), (c_2) and (c_3) hold for each ϵ such that $0 < \epsilon < \xi$. We are supposing that $\epsilon_0 < \xi$. We must now further condition ϵ_0 so that Lemma 5.2 is satisfied.

Verification of (1.9). We shall show that if ϵ_0 is sufficiently small and if $0 < \epsilon < \epsilon_0$ then

$$L_s \cap h_\epsilon(M_r) = \mathbf{0}. \tag{5.17}$$

We shall modify the definition of the closed cone $\Pi\, M_r^c$ in (4.14), and define a closed cone

$$\Pi_\epsilon\, k_\epsilon(M_r^\sigma) = \Pi_\epsilon \tag{5.18}$$

with vertex $\mathbf{0}$ as the closure of the union of the rays π_x which meet points $x \in k_\epsilon(M_r^\sigma) - \mathbf{0}$. Recall that $\mathscr{S}$ designates the unit sphere in E_n with center at $\mathbf{0}$. If ϵ_0 is sufficiently small and if $0 < \epsilon < \epsilon_0$, the condition $d(k_\epsilon) < \epsilon$ will imply that $\Pi_\epsilon \cap \mathscr{S}$ is included in so small a neighborhood, relative to $\mathscr{S}$, of $\Pi\, M_r^\sigma \cap \mathscr{S}$ that the relation

$$L_s^\sigma \cap \gamma(\Pi_\epsilon) = \mathbf{0} \tag{5.19}$$

will hold as a consequence of the relation

$$L_s^\sigma \cap \gamma(\Pi\, M_r^\sigma) = \mathbf{0} \tag{5.20}$$

of Lemma 4.3. We suppose ϵ_0 so chosen.

As a special consequence of (5.19)

$$L_s^\sigma \cap \gamma(k_\epsilon(M_r^\sigma)) = 0 \quad (0 < \epsilon < \epsilon_0)$$

or equivalently, since $h_\epsilon = \gamma k_\epsilon$,

$$L_s^\sigma \cap h_\epsilon(M_r^\sigma) = 0 \quad (0 < \epsilon < \epsilon_0). \tag{5.21}$$

Hence replacing L_s^σ by L_s

$$L_s \cap h_\epsilon(M_r^\sigma) = 0 \tag{5.22}$$

since

$$h_\epsilon(M_r^\sigma) \subset \bar{D}_\sigma; \quad L_s - L_s^\sigma \subset {}^c\bar{D}_\sigma.$$

The desired relation (5.17) is a consequence of (5.22) and (c_3).

This completes the proof of Lemma 5.2 and thereby the proof of Theorem 1.3.

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REIDEMEISTER'S TORSION INVARIANT AND ROTATIONS OF S^n

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Two transformations t_1 and t_2 of the sphere S^n of n dimensions into itself are called homeomorphic, diffeomorphic or isometric if there exists a transformation f with $t_2 f = f t_1$ which is, respectively, a homeomorphism, a diffeomorphism or an isometry. An isometry of S^n will also be called a rotation.

I propose to indicate here the main lines of a proof of the following theorem.

If two rotations of S^n are diffeomorphic, then they are isometric.

1. By introducing suitable coordinates in $R^{n+1} \supset S^n$ which are either real or pairwise conjugate, the equations of a rotation r which sends the point $z = (z_1, \ldots, z_{n+1})$ into $r(z) = z' = (z'_1, \ldots, z'_{n+1})$ can be brought to the form

$$z'_j = \zeta_j z_j \quad (j = 1, \ldots, n+1) \tag{1.1}$$

where the ζ_j are the characteristic roots of the rotation. This system of coordinates corresponds to a decomposition of R^{n+1} into a direct sum of lines and 2-planes which are pairwise orthogonal and invariant under r. It follows from this that two rotations are isometric if they have the same characteristic roots, and only in this case.

(1.2) *If two rotations of S^n are homeomorphic, they have the same number of characteristic roots which are not roots of unity, and, for any integer $h > 0$, the same number of characteristic roots which are primitive h-th roots of unity.*

To prove this theorem, we will need the following lemma.

LEMMA. *If α and β are complex numbers of absolute value 1, and $\beta^{n_k} \to 1$ for every sequence of integers n_k for which $\alpha^{n_k} \to 1$, then $\beta = \alpha^m$ (m an integer).*

If α is not a root of unity, the group G generated by α is dense in the group S^1 of all complex numbers of modulus 1, and the hypothesis implies that the mapping f of G into S^1 defined by setting $f(\alpha^k) = \beta^k$ is uniformly continuous and so extends to an endomorphism of S^1 which is necessarily of the form $\zeta \to \zeta^m$, whence $\beta = \alpha^m$. If α is a primitive h-th root of unity, we see that $\beta^h = 1$ if we take $n_k = h$, and the conclusion follows.

Let us denote by $E(\alpha, r)$ the set of points z of S^n having the following property: $r^{n_k}(z) \to z$ for every sequence of integers n_k for which $\alpha^{n_k} \to 1$. It follows from the lemma and (1.1) that $E(\alpha, r)$ is a subsphere of S^n of dimension $n(\alpha, r) - 1$, $n(\alpha, r)$ being the number of characteristic roots of r which are equal to a power of α. This implies that if r_1 and r_2 are homeomorphic, then $n(\alpha, r_1) = n(\alpha, r_2)$; in other words r_1 and r_2 have the same number of characteristic roots belonging to the group G generated by α. This being the case for every α, these rotations have also the same number of characteristic roots which belong to G and to no proper subgroup of G, which is equivalent to the assertion of the theorem.

Let us call the restriction of r to the subsphere of S^n defined by setting equal to zero the z_j which correspond, in (1.1), to the characteristic roots ζ_j which are not roots of unity, the rotation of finite order associated to r. It is clear that if r_1 and r_2 are diffeomorphic (or homeomorphic), the rotations of finite order associated to r_1 and r_2 are also diffeomorphic (or homeomorphic). Therefore, because of (1.2), to prove our theorem, we have only to consider the case of rotations of finite order.

For rotations of finite order which generate groups without fixed points, the problem becomes that of the diffeomorphism classification of lens spaces, which has been solved by means of the torsion invariant of Reidemeister and Franz. The same method can also be applied to arbitrary rotations of finite order, as I shall indicate. I shall start with the definition of torsion, by introducing the notion of an (A, G)-system.

2. Torsion of an (A,G)-system. Let A be a ring, G a multiplicative subgroup of the group of invertible elements of A, and let C be a free A-module of finite rank. From a basis $e_1,\ldots, e_m$ of C, we may obtain other bases by means of the following operations:

(a) permute $e_1,\ldots, e_m$;

(b) replace one of the e_i by $\pm\gamma e_i$, where $\gamma \in G$;

(c) replace e_i by $e_i + \lambda e_j$ where $\lambda \in A$ and $i \neq j$.

The set of all bases which can be obtained from one of them by any finite number of these operations will be called a *family of distinguished bases of C relative to G.*

We shall call an (A,G)-*system* two free A-modules C', C'' of finite rank, each provided with a family of distinguished bases relative to G, together with an endomorphism ∂ of $C' \oplus C''$ such that $\partial C' \subset C''$, $\partial C'' \subset C'$ and $\partial^2 = 0$. The quotient A-modules $B' = F'/H'$ and $B'' = F''/H''$, where $H' = \partial C''$, $H'' = \partial C'$, $F' = C' \cap \partial^{-1}(0)$, $F'' = C'' \cap \partial^{-1}(0)$, will be called the *Betti modules* of the system. If they are zero, the system is said to be acyclic.

We call *volume* in a vector space E of dimension m any element $\neq 0$, of the m^{th} exterior power of E (if $m = 0$, we shall agree that it is a scalar $\neq 0$). If F is a subspace of E, the natural isomorphism of E onto $F \oplus E/F$ makes correspond to every pair of volumes, f on F, and d on E/F, a well-determined volume $e = f\,d$ on E. Any two of these three volumes determine the third, so that we shall also write $d = e/f$.

Let us suppose now that A is a (commutative) field and that the (A,G)-system S is acyclic. Then $F' = H'$ and $F'' = H''$. Let c', c'', h', h'' be volumes on C', C'', H', H'' respectively. The isomorphism of C'/H' onto H'' induced by ∂ makes correspond to c'/h' a volume $\partial(c'/h')$ on H'', whose ratio with h'' is a well-determined number, $\neq 0$, of A. In the same way, c''/h'' is a volume on C''/H'' to which corresponds a volume $\partial(c''/h'')$ on H', whose ratio with h' is a number, $\neq 0$, of A. The quotient of these numbers

$$\frac{\partial(c''/h'')}{h'} : \frac{\partial(c'/h')}{h''} = D\left(\frac{c''}{c'}\right),$$

depends only on c' and c''. Let us agree to take for c' and c'' the volumes equal to the exterior products of the elements of the distinguished bases of C' and C''; these volumes being determined up to a factor of the form $\pm\gamma$, $\gamma \in G$, the quotient defined above is also determined upto such a factor, and we shall call it the *torsion of* S.

We can also define the torsion if S is not acyclic (see [4] and [5]), but we shall not make any use of this here. On the other hand, it is essential to consider the case where A is not a field.

If A is not a field, let us consider a homomorphism θ of A into a (commutative) field D. We associate to it in a natural way a homomorphism of any free A-module C of finite rank into a vector space E over D of the same rank, which we denote again by θ. To a family of distinguished bases of C relative to G corresponds a family of distinguished bases of E relative to $\theta(G)$. In particular, θ gives us homomorphisms of the A-modules C' and C'' of an (A,G)-system S into vector spaces E' and E'' and the endomorphism ∂ of $C' \oplus C''$ is transformed into an analogous endomorphism of $E' \oplus E''$ in such a way that E' and E'' form a $(D,\ \theta(G))$-system, which we call the image of S by θ, and denote by ${}_\theta S$. If this latter system is acyclic, we denote its torsion by $\Delta_\theta(S)$ and by the *torsion of the system* S, we mean *the set of the* $\Delta_\theta(S)$, *associated with homomorphisms* θ *of* A *into commutative fields such that* ${}_\theta S$ *is acyclic, which are determined upto a factor of the form* $\theta(\pm\gamma)$, *where* $\gamma \in G$ *and* $\pm\gamma$ *does not depend on* θ.

In what follows, we consider exclusively the case where G is a cyclic group of finite order h and A is the group algebra of G over the complex numbers D. We denote by γ a generator of G. To each h-th root ζ of unity corresponds a homomorphism θ such that $\theta(\gamma) = \zeta$. From the properties of the algebra A, one deduces the following theorem.

(2.1) *If the operation induced by* γ (*resp.* γ^2) *in the Betti modules of the system* S *is the identity,* ${}_\theta S$ *is acyclic so long as* $\theta(\gamma) \neq 1$ (*resp.* $\theta(\gamma) \neq \pm 1$).

3. Complexes with automorphisms. We now consider a finite cell complex K with an automorphism γ which generates a cyclic group G of order h, satisfying the following condition: *if an automorphism of G leaves a cell fixed, it leaves fixed every point of the cell.*

The set of cells of K which are fixed by at least one automorphism of G different from the identity forms a closed subcomplex K_f of K invariant under γ. Let L be a closed subcomplex of K, invariant under γ and containing K_f. We associate to the pair (K,L) an (A,G)-system $S(K,L)$ in the following way.

The set of chains of odd dimension of K, that is to say the set of linear combinations with coefficients in D of cells of odd dimension of K, each cell being taken with a fixed orientation, forms, in a natural way, an A-module C, for which we obtain a basis by taking a set of cells such that every cell of odd dimension is the transform of one and only one cell of this set by an element of G. This basis gives rise to a family of distinguished bases relative to G which is perfectly well determined. In the same way, the chains of even dimension form an A-module C'' with a family of distinguished bases relative to G. If we take for the endomorphism ∂ the operator which gives the boundary (mod L), C' and C'' form an (A,G)-system $S(K,L)$.

The torsion of this system will be called the *torsion of the pair* (K, L), and we will set $\Delta_\theta(K, L) = \Delta_\theta(S(K, L))$. In view of (2.1), if γ (resp. γ^2) leaves invariant the homology class of K (mod L), ${}_\theta S(K,L)$ is acyclic, so long as $\theta(\gamma) \neq 1$ (resp. $\theta(\gamma) \neq \pm 1$) and $\Delta_\theta(K,L)$ is then defined upto a factor $\theta(\pm \gamma^d)$.

One has then the following proposition (Milnor [2]).

(3.1) *If M is an invariant subcomplex of K containing the invariant subcomplex L, which in turn contains K_f, and if ${}_\theta S(K,M)$ and ${}_\theta S(M,L)$ are acyclic, then ${}_\theta S(K,L)$ is also acyclic, and*

$$\Delta_\theta(K,L) = \Delta_\theta(K,M)\, \Delta_\theta(M,L).$$

Let us suppose that the pair $(\widetilde{K},\widetilde{L})$ is obtained from the pair (K, L) by subdividing a cell of K and, correspondingly, all its transforms by the automorphisms of G, in such a way that these automorphisms can be extended to the subdivided complex. Let us call the passage from one of these pairs to the other an *elementary operation*, and let us say that two pairs are *combinatorially equivalent*, if one of them becomes isomorphic to the other by a finite number of elementary operations. One has then the following proposition.

(3.2) *Two combinatorially equivalent pairs have the same torsion.*

With a terminology which is a bit different, this proposition has been proved in [4]. Another proof is sketched, incompletely, in [5] (there is a gap in the proof of Lemma 3, p. 56). See also J.H.C. Whitehead [7] and Milnor [2].

4. Complex with automorphisms associated to a rotation of finite order of S^n. Let r be a rotation of S^n, defined by equations of the form (1.1), which generates a cyclic group G of order h. On the lines of the invariant coordinates, corresponding to the real coordinates z_j and to the characteristic roots $\zeta_j = \pm 1$, let us mark the two points of its intersection with S^n. In every 2-plane of the invariant coordinates, which correspond to two complex conjugate coordinates z_j and $\bar{z}_j$ and to the non real characteristic roots ζ_j and $\bar{\zeta}_j$, let us mark on its circle of intersection witb S^n the point where $\arg z_j = 0$ and all its transforms by G, which form the vertices of a regular polygon of centre 0. The points thus marked out on S^n are the vertices of a convex polyhedron inscribed in S^n which is invariant under r, and each of whose faces is a simplex. Projecting this from 0, we obtain a simplicial subdivision of S^n, invariant under r. We shall denote by $P(r)$ this complex with the automorphism γ induced by r. The simplices of $P(r)$ which are left invariant by at least one automorphism of G different from the identity form an invariant subcomplex $P_f(r)$. We have the following proposition.

(4.1) *If the rotations r and r' are diffeomorphic, the pairs $(P(r), P_f(r))$ and $(P(r'), P_f(r'))$ have the same torsion.*

This is a special case of a more general theorem about discontinuous groups of diffeomorphisms which can be proved using the Whitehead theory of differentiable complexes [6] and (3.2), or with the help of the notion of convex coverings (see [5] for the case where G operates without fixed points, i.e. when $P_f(r)$ is empty). I hope to come back to this point elsewhere.

To establish that two diffeomorphic rotations r' and r'' have the same characteristic roots, we proceed by induction on n. The theorem is obvious for $n = 0$ and immediate for $n = 1$.

Let d be a proper divisor of h (i.e. a divisor such that $1 \leqslant d < h$). The points of S^n which are left fixed by r^d form a subsphere of S^n defined by $z_j = 0$ for every j with $\zeta_j^d \neq 1$, which is of dimension $< n$. Let $r_{(d)}$ be the restriction of r to this subsphere, and $r'_{(d)}$ the analogous restriction of r'. If r and r' are diffeomorphic, so are $r_{(d)}$ and $r'_{(d)}$, and by induction hypothesis they have the same characteristic roots. This being true for every proper divisor of h, *r and r' have the same characteristic roots which are not primitive h-th roots of unity.*

Let us denote by r_0 the restriction of r to the subsphere of S^n defined by equating to zero the coordinates z_j associated to the ζ_j which are primitive h-th roots of unity. The characteristic roots of r_0 are those of r which are not primitive h-th roots of unity. $P_f(r)$ is an invariant subcomplex of $P(r_0)$ which is itself an invariant subcomplex of $P(r)$.

(4.2) *If $\theta(\gamma) \neq \pm 1$, the systems ${}_\theta S(P(r), P_f(r))$, ${}_\theta S(P(r_0), P_f(r))$ and ${}_\theta S(P(r), P(r_0))$ are acyclic.*

This follows, if we take account of (2.1), from the fact that r^2 belongs always to a one parameter group of rotations leaving invariant the topological spaces $P(r_0)$ and $P_f(r)$ (subsphere of S^n or union of subspheres); consequently, the topological transformation r^2 of each pair $(P(r), P_f(r))$, $(P(r_0), P_f(r))$ and $(P(r), P(r_0))$ is homotopic to the identity, and so leaves homology classes invariant, i.e. induces the identity on the Betti modules. If r preserves orientation, we may take r instead of r^2 and it suffices to suppose that $\theta(\gamma) \neq 1$.

The restriction r'_0 of r', defined in the same way as the restriction r_0 of r, has the same characteristic roots as r_0, so that $P(r_0)$ and $P(r'_0)$ are isomorphic, as are also $P_f(r)$ and $P_f(r')$, so that the pairs

$(P(r_0), P_f(r))$ and $(P(r'_0), P_f(r'))$ have the same torsion. But, because of (3.1), we have

$$\Delta_\theta(P(r), P_f(r)) = \Delta_\theta(P(r), P(r_0)) \,.\, \Delta_\theta(P(r_0), P_f(r)),$$

and the same relation with r' instead of r, so that, taking into account what we have just said, and (4.1), we have

(4.3) $\Delta_\theta(P(r), P(r_0)) = \Delta_\theta(P(r'), P(r'_0))$ *for any* θ *for which* $\theta(\gamma) \neq \pm 1$.

Let us suppose that among the characteristic roots of r, there are $2m$ primitive h-th roots of unity, say $\zeta_1, \ldots, \zeta_m$ and their conjugates. Let us denote, as always, by z_j the complex coordinate associated to ζ_j in (1.1). Let r_k be the restriction of r to the subsphere of S^n defined by $z_{k+1} = z_{k+2} = \ldots = z_m = 0$, so that for $k = 0$ we obtain the rotation r_0 already considered and $r_m = r$. Because of (3.1) we have

$$\Delta_\theta(P(r), P(r_0)) = \prod_{k=1}^{m} \Delta_\theta(P(r_k), P(r_{k-1})). \tag{4.4}$$

The complex $P(r_k)$ on the sphere S^{n_k} of dimension $n_k = n - 2(m-k)$ is a subdivision of the complex $Q(r_k)$ which we obtain by adjoining to $P(r_{k-1})$ the cell $\mathfrak{a}$ of dimension n_k defined in S^{n_k} by $0 < \arg z_k < \frac{2\pi}{h}$, the cell $\mathfrak{b}$ of dimension $n_k - 1$ defined by $\arg z_k = 0$ and the transforms of $\mathfrak{a}$ and $\mathfrak{b}$ by G. Let μ_k be an integer (determined mod h) such that $\zeta_k^{\mu_k} = \exp(2\pi i/h)$. Under the action of γ^{μ_k} (or r^{μ_k}), z_k is multiplied by $\exp(2\pi i/h)$, and so the cell $\mathfrak{b}$ goes into the cell $\gamma^{\mu_k}\mathfrak{b}$ defined by $\arg z_k = \frac{2\pi}{h}$, and consequently, $\mathfrak{a}$, $\mathfrak{b}$ being suitably oriented, we have, if r preserves the orientation, $\partial\mathfrak{a} = (\gamma^{\mu_k} - 1)\,\mathfrak{b}$, while $\partial\mathfrak{b} = 0 \pmod{P(r_{k-1})}$. The A-modules C' and C'' of the system $S(Q(r_k), P(r_{k-1}))$ are of rank 1, $\mathfrak{a}$ is a distinguished basis of one (C' if n is odd and C'' if n is even) and $\mathfrak{b}$ is a distinguished basis of the other. If we set $\theta(\gamma) = \zeta$ and $\epsilon = (-1)^n$, we obtain $\Delta_\theta(Q(r_k), P(r_{k-1})) = (\zeta^{\mu_k} - 1)^\epsilon$. This number is only determined upto a factor of the form $\pm \zeta^d$; let us remark that, if instead of z_k and ζ_k

we had taken their conjugates, we would have obtained the above result with $-\mu_k$ instead of μ_k, which is the same upto the above mentioned factor.

By virtue of (3.2.), $\Delta_\theta(P(r_k), P(r_{k-1})) = \Delta_\theta(Q(r_k), P(r_{k-1}))$, and (4.4) implies that

$$\Delta_\theta(P(r), P(r_0)) = \prod_{k=1}^{m} (\zeta^{\mu_k} - 1)^{\epsilon}.$$

By (4.3), this number is equal, upto a factor $\pm \zeta^d$, to the number formed in the same way with the numbers μ_k replaced by the analogous numbers μ'_k defined relative to the rotation r'. We deduce from this that

$$\prod_{k=1}^{m} (\zeta^{\mu_k} - 1)(\zeta^{-\mu_k} - 1) = \prod_{k=1}^{m} (\zeta^{\mu'_k} - 1)(\zeta^{-\mu'_k} - 1)$$

for every h-th root ζ of unity, and the same relations hold if r does not preserve the orientation. Now, by a theorem of W. Franz [1], this implies that the set of residues (mod h) $\mu_k, -\mu_k$ is identical (upto order) with $\mu'_k, -\mu'_k$, and this implies that r and r' have the same characteristic roots which are primitive h-th roots of unity, which completes the proof of our theorem.

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SOME FREE ACTIONS OF CYCLIC GROUPS ON SPHERES

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LET $p \geqslant 5$ be an integer different from 6 and let $n \geqslant 5$ be odd. *This note will show that the cyclic group* Π *of order p can act differentiably on the n-sphere, without fixed points, in infinitely many different ways.* These actions are "different" in the sense that the corresponding quotient manifolds $M = S^n/\Pi$ can be distinguished by their Reidemeister-Franz-de Rham torsion invariants. Hence two such "different" manifolds M, M' cannot have the same simple homotopy type, cannot be piecewise-linearly homeomorphic, and cannot be diffeomorphic. (It is not known whether or not M and M' can be homeomorphic.)

First let me review the basic properties of the torsion invariant, following [3], [4]. Let K be a finite, connected† CW-complex and let Π denote the fundamental group of K. Let

$$f : Z[\Pi] \to \mathbf{C}$$

be a ring homomorphism from the integral group ring to the complex numbers. If the homology groups $H_i(K\,;\mathbf{C}_f)$ are all zero (homology with local coefficients twisted by f) then the torsion invariant $\Delta_f \widetilde{K} \in \mathbf{C}_0/\pm f\Pi$ is defined. (Here $\widetilde{K}$ denotes the universal covering complex, $\mathbf{C}_0$ the multiplicative group of non-zero complex numbers, and $\pm f\Pi$ the subgroup generated by $f(\Pi)$ and ± 1.) To simplify the notation we will henceforth leave off the tilde, and write simply $\Delta_f K$.

Similarly, given a pair K, L with $H_*(K, L\,;\mathbf{C}_f) = 0$ the torsion $\Delta_f(K, L)$ is defined. This satisfies the identity

$$\Delta_f(K, L) = \Delta_f K/\Delta_f L, \tag{1}$$

providing that the three terms are defined. (If two out of three are defined, then the third is automatically defined.)

† For a complex with several components the torsion can be defined as the product of the torsions of the components.

If W is a triangulated orientable manifold of dimension n with boundary bW, then the following duality theorem holds. We must assume that $|f(t)| = 1$ for $t \in \Pi = \pi_1(W)$. Then

$$\Delta_f(bW) = (\Delta_f W)(\bar{\Delta}_f W)^{\epsilon(n)} \tag{2}$$

where $\bar{\Delta}$ denotes the complex conjugate and $\epsilon(n) = (-1)^n$. We will also need the following variant form. If M is a triangulated manifold without boundary of dimension $n-1$ then

$$\Delta_f M = (\bar{\Delta}_f M)^{\epsilon(n)}. \tag{3}$$

Now consider an h-cobordism $(W;\ M,\ M')$. That is, assume that W is a smooth manifold with boundary $M + M'$, and that both M and M' are deformation retracts of W. Choosing a C^1-triangulation of $(W;\ M,\ M')$ we will assume that the torsion

$$\Delta_f M \in \mathbf{C}_0 / \pm f\,\Pi$$

is defined.

LEMMA 1. *With the above assumptions,* $\Delta_f M'$ *is defined and equal to*

$$(\Delta_f M)\ \Delta_f(W, M)\ (\bar{\Delta}_f(W, M))^{\epsilon(n)}.$$

PROOF. Since M is a deformation retract of W it is clear that $\Delta_f(W, M)$ is defined. Thus $\Delta_f W$ is defined, and similarly $\Delta_f M'$ is defined. Consider the duality statement

$$\Delta_f(bW) = (\Delta_f W)\ (\bar{\Delta}_f W)^{\epsilon(n)}.$$

Since $\Delta_f(bW) = (\Delta_f M)\ (\Delta_f M')$ and since $\Delta_f W = (\Delta_f M)\ \Delta_f(W, M)$, this can be rewritten as

$$(\Delta_f M)\ (\Delta_f M') = (\Delta_f M)\ \Delta_f(W, M)\ (\bar{\Delta}_f M)^{\epsilon(n)}\ (\bar{\Delta}_f(W, M))^{\epsilon(n)}.$$

Now dividing through by

$$\Delta_f M = (\bar{\Delta}_f M)^{\epsilon(n)}$$

we obtain the required formula

$$\Delta_f M' = (\Delta_f M)\ \Delta_f(W, M)\ (\bar{\Delta}_f(W, M))^{\epsilon(n)}.$$

Henceforth we will assume that the dimension n of W is even. Thus Lemma 1 can be rewritten in the form

$$\Delta_f M' = (\Delta_f M) \, | \Delta_f (W, M) |^2. \tag{4}$$

Suppose that we are given the manifold M with fundamental group Π, and wish to construct the h-cobordism $(W; M, M')$.

LEMMA 2 (Stallings). *If* $\dim (M) \geqslant 5$ *then the* h*-cobordism* $(W: M, M')$ *can be constructed so that* $\Delta_f(W, M)$ *is equal to the image, in* $\mathbf{C}_0/\pm f\,\Pi$, *of any unit of the group ring* $Z[\Pi]$.

PROOF. Stallings actually observes that the h-cobordism can be constructed so that the Whitehead torsion invariant $\tau(W, M)$ is any desired element of the Whitehead group

$$\mathrm{Wh}(\Pi) = GL(\infty, Z[\Pi])/(\text{Commutators}, \pm \Pi).$$

(See Stallings [6, § 2]. The manifold W is constructed by adjoining handles of index 2 and 3 to $M \times [0, 1]$ along one boundary, in such a way that the matrix of "incidence numbers" between the two types of handles is equal to a given invertible matrix over $Z[\Pi]$.) In particular if u is a unit of $Z[\Pi]$ then W can be chosen so that $\tau(W, M)$ is the element of Wh (Π) corresponding to the matrix

$$\begin{bmatrix} u & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix} \in GL(\infty, \; Z[\Pi]).$$

It is then clear that $\Delta_f(W, M)$ is equal to the image of u in $\mathbf{C}_0/\pm f\,\Pi$. (Compare Cockcroft [1], or [3, p. 589].) This completes the proof.

Thus in order to construct examples of h-cobordisms, we need only look for units in $Z[\Pi]$. To be more specific, let us now assume that Π is cyclic of order p with generator t. Define $f\colon Z[\Pi] \to \mathbf{C}$ by $f(t) = \exp(2\pi i/p)$. The following case is particularly easy.

LEMMA 3 (Higman). *If* $p \geqslant 5$ *is an integer of the form* $6k \pm 1$ *then* $Z[\Pi]$ *contains a unit* u *with* $|f(u)| \neq 1$.

PROOF. This follows from Higman [2]. Alternatively, here is a direct proof. Let

$$u = t + t^{-1} - 1$$

so that $f(u) = 2\cos(2\pi/p) - 1 \neq \pm 1$. To see that u is a unit it is only necessary for the reader to verify the identity

$$u(1 + t - t^3 - t^4 + t^6 + t^7 - - + + \ldots - - + t^{p-1}) = 1$$

for $p \equiv 1 \pmod 6$; or

$$u(-1 + t^2 + t^3 - t^5 - t^6 + + - - \ldots - - + t^{p-3} + t^{p-2}) = 1$$

for $p \equiv -1 \pmod 6$. This completes the proof. (Some further discussion of this lemma is included as an appendix.)

Now combining the three lemmas we have the following.

THEOREM. *Let M be a smooth manifold of odd dimension $\geqslant 5$ whose fundamental group is cyclic of order $p = 6k \pm 1$, $p \geqslant 5$. Suppose that the torsion $\Delta_f M$ is defined. Then there exist infinitely many manifolds $M_1, M_2, M_3, \ldots$ which are h-cobordant to M, but such that no two have the same simple homotopy type.*

PROOF. For each integer m we can choose the h-cobordism $(W_m; M, M_m)$ so that

$$|\Delta_f(W_m, M)| = |f(u^m)|.$$

Then

$$\Delta_f M_m = (\Delta_f M)\, |f(u)|^{2m}.$$

Since $|f(u)| \neq 0, 1$ the real numbers $|\Delta_f M_m|$ are all distinct. This does not yet prove that the M_m all have distinct simple homotopy types, since the invariant $|\Delta_f M_m|$ depends on the choice of f. But there are only finitely many homomorphisms from $Z[\Pi]$ to $\mathbf{C}$, so out of the infinite sequence $M_1, M_2, \ldots$ one can certainly extract an infinite subsequence consisting of pairwise distinct manifolds. This completes the proof.

In particular let us apply this theorem to a lens space

$$L = S^n/\Pi,\ n \text{ odd}.$$

The resulting h-cobordant manifolds $L_1, L_2, \ldots$ will all have universal covering spaces diffeomorphic to the sphere. (See Smale [5].) Thus we have infinitely many distinct free actions of the cyclic group

Π on S^n. But there are only finitely many orthogonal actions of Π on S^n. Thus we have:

COROLLARY. *For n odd $\geqslant 5$ and $p = 6k \pm 1 \geqslant 5$ there exist infinitely many smooth fixed point free actions of the cyclic group of order p on S^n which are not smoothly equivalent to orthogonal actions, and are not smoothly equivalent to each other.*

It would be interesting to know whether any corresponding phenomenon occurs in dimension 3.

APPENDIX: FURTHER DISCUSSION OF LEMMA 3.

Higman's theorem actually applies more generally to any finite abelian group Π which does not have exponent 1, 2, 3, 4 or 6. Hence the theorem also applies in this generality. In fact suppose that $t \in \Pi$ is an element whose order p is different from 1, 2, 3, 4, 6. Then the Euler ϕ-function satisfies $\phi(p) > 2$. Hence there exists an integer a, with $1 < a < p/2$, which is relatively prime to p. Choose b so that $ab \equiv 1 \pmod p$, and set

$$x = (t^a - 1)/(t - 1) \quad = 1 + t + t^2 + \dots + t^{a-1},$$

$$y = (t^{ab} - 1)/(t^a - 1) = 1 + t^a + t^{2a} + \dots + t^{(b-1)a}.$$

Then $(t - 1)xy = t - 1$, from which it follows easily that $xy - 1$ is a multiple of the element $s = 1 + t + t^2 + \dots + t^{p-1}$. Thus x is a unit modulo s. To obtain an actual unit, choose integers k, l, m so that $a^k = lp + 1$, $b^k = mp + 1$. Then $(x^k - ls)(y^k - ms) = 1$; so that $u = x^k - ls$ is the required unit.

As before we can choose $f: Z[\Pi] \to \mathbf{C}$ so that $f(t) = \exp(2\pi i/p)$. Then $f(s) = 0$, hence $|f(u)| = |f(x)|^k > 1$.

For any integer $p \geqslant 5$, $p \neq 6$, it follows that the cyclic group of order p can act freely on a sphere in infinitely many different ways. *Problem.* Can a cyclic group of order 2, 3, 4 or 6 act freely on a sphere in infinitely many different ways?

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COMPACT GROUPS OF TRANSFORMATIONS

By DEANE MONTGOMERY

1. Introduction. This paper gives an account of some of the results recently obtained on the topology of groups of transformations, particularly in the differentiable case. No attempt has been made to be complete, and the topics covered are mainly those of special interest to the author. Probably the most glaring omission is the important work of Conner and Floyd on the cobordism theory of groups of transformations. This constitutes a new and very interesting subject, much of which can be found in the book by Conner and Floyd [11].

A topological transformation group $\phi(G, X)$ consists of a topological group G, a space X, and a map

$$\phi : G \times X \to X$$

satisfying

(a) $\phi[g_1, \phi(g_2, x)] = \phi[g_1 g_2, x]$;

(b) for g fixed, $\phi(g, x)$ is a homeomorphism of X onto X.

A topological transformation group is often called an action and X is often called a G-space. If X is a differentiable manifold and G is a Lie group, and if ϕ is a differentiable map, then ϕ is called a differentiable transformation group or a differentiable action. When ϕ is fixed $\phi(g, x)$ is often denoted by $g(x)$. Here the main concern will be with differentiable actions of a compact Lie group G. This of course includes the case where G is a finite group of diffeomorphisms; this is the case most thoroughly explored so far. An action is called effective if the identity is the only element leaving every point fixed.

If G acts on X and Y, then a map $f : X \to Y$ is called equivariant if

$$fg(x) = gf(x) \quad \text{for all} \quad g \in G,\ x \in X.$$

If G and X are fixed, two actions ϕ_1 and ϕ_2 are called equivalent (differentiably equivalent) if there is a homeomorphism (diffeomorphism) $H : X \to X$ such that

$$\phi_1(g, Hx) = H\,\phi_2(g, x).$$

Thus two actions are equivalent if there is an equivariant homeomorphism from one action to the other. Some of the problems of topological transformation groups are concerned with finding whether actions are equivalent to certain simple well known actions or the extent to which they do or do not resemble them.

2. Preliminary remarks. If G (compact Lie) acts on M, a fixed point or a stationary point p is one which is fixed under every element of the group, that is $g(p) = p$, $g \in G$. The orbit $G(p)$ of a point p is the set of all $g(p)$, $g \in G$; the isotropy group G_p is the closed subgroup of those elements g which leave p fixed. Two orbits, say of x and y, are called equivalent or of the same type if G_x and G_y are conjugate in G. The set of orbits M/G can be given a topology in a natural way and there is a natural map p

$$p : M \to M/G$$

which is open. The space M/G (also denoted by M^*) is not necessarily a manifold even though M is.

Assume that $x \in M$ and let G_x be the isotropy group. There is a neighborhood U of x such that if $y \in U$ then G_y is conjugate to a subgroup of G_x. Hence for $y \in U$, dim $G_y \leqslant$ dim G_x, and this implies dim $G(y) \geqslant$ dim $G(x)$.

Let G (compact Lie) act differentiably on M and let p be a fixed point. Bochner has shown that in a neighborhood of p, coordinates may be chosen in which the action of G is linear. It follows that the set of fixed points, denoted by $F(G)$, is a manifold. This manifold need not be connected and there are easy examples to show that different components may have different dimensions.

C. T. Yang has proved

THEOREM 2.1. *Let G be a compact Lie group acting differentiably on a manifold M. Then the orbit space M/G is triangulable.*

A compact Lie group G can act on a manifold in a manner which is not equivalent to a differentiable action and the remainder of this section is devoted to a few remarks on this more general case. In fact, Bing has given an example of such an action of Z_2 on S^3. In his action $F(Z_2)$ is S^2 but neither of the components of $S^3-F(Z_2)$ is simply connected. It is not known whether or not every compact G acting effectively on a manifold without the hypothesis of differentiability is a Lie group. It is known for this case that G must be finite dimensional. The question of whether a compact non-Lie group G can act effectively on a manifold reduces to the case where G is infinite, compact, and zero-dimensional. Several interesting contributions to this case have recently been made by Yang, Bredon, Raymond, and Williams.

Assume for the moment that G is a compact Lie group acting on a compact manifold (no differentiability assumed). Then Floyd and Mostow have proved that there are only a finite number of orbit types and Mostow has shown that an action may be imbedded equivariantly in an orthogonal action. He also shows the existence of a slice and this existence was also proved by Yang and the author. Intuitively a slice is something like a cell orthogonal to $G(p)$ at p and invariant under G_p. Mann and Su have obtained information on the number of orbit types.

3. Number of inequivalent actions. The following has been proved by Palais.

THEOREM 3.1. *If G and M are compact and differentiable then the number of inequivalent differentiable actions of G on M is at most countable.*

Palais has obtained a direct proof of this and it is also a consequence of another of his theorems (as yet unpublished) which may be stated as follows.

THEOREM 3.2. *Let* Diff(M) *be the group of all* C^∞ *diffeomorphisms of a compact* C^∞ *manifold* M *in the* C^1 *topology. Let* G *be any compact subgroup and let* U *be a neighborhood of the identity. Then there is a neighborhood* V *of* G *such that if* H *is a subgroup in* V, *then there is an element* $g \in U$ *satisfying* $g^{-1}Hg \subset G$.

Theorem 3.1 would not be true under weaker hypotheses. Palais and Richardson have shown that if the M is not compact then there can be non-countably many mutually non-equivalent differentiable (even real analytic) actions of a compact Lie group G.

4. Examples. In this section G will be a compact Lie group acting differentiably on M which is to be R^n respectively S^n. Long ago the question raised for this case was whether such an action was equivalent to a linear action. If G is Z_p or more generally a finite p-group, Smith showed (he did not need differentiability) that the set of fixed points has the homology properties (mod p) of an R^k respectively S^k, $-1 \leqslant k \leqslant n$. Since differentiability is assumed, $F(G)$ is always a manifold so that if G is a p-group, $F(G)$ is a manifold with the mod p homology or cohomology properties of R^k respectively S^k. However, as we shall see below, $F(G)$ may not be homeomorphic to R^k or to S^k as it would be if the action were equivalent to a linear action.

Many examples have been given by Conner, Floyd, Whitehead, Kister, Rosen, and others which have given a great deal of information. A few of these will be mentioned below and we begin with a simple example suggested by Rosen.

Let J be a closed arc imbedded in R^3 where R^3 is imbedded naturally in R^4. It has been proved by Klee that J is tamely imbedded in R^4, that is, there is a homeomorphism of R^4 onto itself which takes J onto a closed interval on a straight line. It is known however that J need not be tame in R^3 and that the fundamental group of $R^3 - J$ may be non-trivial.

By the remarks just made $R^4 - J$ is homeomorphic to $R^4 -$ (a point). Now $R^4 - J$ admits an involution

$$T : (x, t) \to (x, -t)$$

and $F(T) = R^3 - J$. Adding a point at infinity and extending T to leave it fixed gives an involution T' of $S^4 - J$ whose set of fixed points is $S^3 - J$. This shows

THEOREM 4.1. *There is a diffeomorphic involution of R^4 whose set of fixed points is not homeomorphic to R^3.*

Another example of this kind can be given using a space W constructed by Whitehead. The space W is a simply connected 3-dimensional manifold which is not homeomorphic to R^3. Shapiro (unpublished) showed that $W \times R^1 = R^4$. A differentiable involution of $W \times R^1$ is given by

$$(x, t) \to (x, -t)$$

and the set of fixed points is W. By using $W \times R^2 = R^5$ and the action of the circle group $SO(2)$ on R^2 we obtain a differentiable action of $SO(2)$ on R^5 whose set of fixed points is not homeomorphic to a euclidean space.

McMillan has found a non-countable set of 3-manifolds analogous to W and these were used by Palais and Richardson to construct the example mentioned earlier. Bing has shown that there is a non-manifold B whose product with R^1 is R^4. This gives an action of Z_2 on R^4 (not, of course, a differentiable one) whose set of fixed points is not a manifold. Rosen showed that the set of fixed points may fail to be a manifold at every point.

Whitehead found actions of Z_2 on S^n whose fixed point sets are not simply connected. Following his technique Samelson and the author showed

THEOREM 4.2. *Let G be a compact Lie group containing more than one element. Then there is a positive integer k such that G has an infinite number of differentiable actions on S^k in its usual structure, no two of which are equivalent.*

By the theorem of Palais the number cannot be more than countable.

It follows from the theorem of Smith that a prime power cyclic homeomorphism of R^n must have a fixed point. However, it has been shown by Kister, using earlier ideas of Conner and Floyd, that there is a differentiable action of Z_6 on R^8 without a fixed point. Also by using ideas of Conner and of Floyd, it has been shown by Conner and the author that there exists an action of $SO(3)$ on R^{12} without a fixed point. As constructed this action was not differentiable, but it can be made differentiable by using more care.

It follows from this that there is an action of $SO(3)$ on S^{12} with precisely one fixed point. This action, however, cannot be differentiable at the fixed point. At present it is not known whether or not there exists a differentiable action of $SO(3)$ on S^n (for any n) with precisely one fixed point.

Bredon has constructed a set of interesting examples which show, in the simplest case, that there is an action of Z_2 on S^5 whose fixed point set is the lens space $L(2k+1, 1)$.

Milnor has found a smooth free involution $f : S^7 \to S^7$ so that the orbit manifold is not diffeomorphic to P^7. Hirsch and Milnor have found a smooth free involution of S^7 whose base space is neither diffeomorphic nor piecewise linearly homeomorphic to P^7. They have analogous examples for S^6 and S^5. It is not known whether or not these respective base spaces are homeomorphic to the corresponding projective spaces. Equivalently it is not known whether or not there is a topological free involution (even a smooth one) which is topologically distinct from the antipodal map. Floyd and Richardson have given an example of a finite group acting on a closed cell without a fixed point.

5. Distribution of orbits. We continue to assume that G is a compact Lie group acting differentiably on a manifold M. The number of components in the isotropy group G_x is denoted by $m(x)$. There is a highest dimension for the orbits in M and this is denoted by r. The points of M which lie on orbits of dimension less than r will be denoted by B which is a closed set. Let $u \geqslant 0$ and $v > 1$ be integers and let

$$M_{u,v} = \{x \mid x \in M, \ \dim G(x) = u, \ m(x) = v\}$$
$$M_u = \bigcup_v M_{u,v}$$
$$Z_{u,v} = M_r \cup M_{r-1} \cup \ldots \cup M_{u+1} \cup M_{u,1} \cup \ldots \cup M_{u,v}.$$

For $x \in M$ and y near to x it is known that G_y is conjugate to a subgroup of G_x so that $Z_{u,v}$ is open. Each set $M_{u,v}$ is locally a product of an orbit and a cross-section. Any component of $M_{u,v}$ contains orbits of only one type.

THEOREM 5.1. *The set B satisfies* $\dim B \leqslant n - 2$. *If M is connected, then $M - B = M_r$ is connected.*

THEOREM 5.2. *Let t be any integer $0 \leqslant t < r$. The union of all orbits of dimension $\leqslant t$ is of dimension $\leqslant n - r + t - 1$.*

We shall now define principal orbits and orbits of highest dimension. Let

$$s = \min m(x), \ x \in M_r$$
$$D = \bigcup_{v>s} M_{r,v}.$$

Then orbits in D are called *exceptional orbits of highest dimension* and orbits in $M_{r,s}$ are called *principal orbits*. Let $M_{r,s} = U$. Then M is divided into mutually exclusive sets as follows :

$$M = B \cup D \cup U.$$

If M is connected then $M_{r,s}/G$ is connected and if G is also connected then $M_{r,s}$ is connected. Hence for a connected G, the principal orbits form a connected open set which is fibered by the principal orbits. We always have $\dim D \leqslant n - 1$.

THEOREM 5.3. *If $M = R^n$ or S^n and G is connected, then* $\dim D \leqslant n - 2$. *The same is true more generally for an M satisfying* $H_c^{n-1}(M; Z_2) = 0$.

THEOREM 5.4. *Let a compact connected group act differentiably on R^n or S^n. Then $(U \cup D)/G$ is simply connected.*

A principal orbit in an oriented manifold is orientable and it is a corollary of the theorem above that under the hypothesis of Theorem 5.3, any highest dimensional orbit is orientable.

6. Actions of $SO(3)$. The actions about which the most information is available are for finite cyclic groups and for toral groups. It is natural to study $SO(3)$ partly for itself and partly because it is one of the most familiar groups beyond those just mentioned. There are serious difficulties even in this case, but a few results have been found. Samelson and the author have shown that $SO(3)$ cannot act differentiably on S^7 with every orbit three-dimensional. It has been proved that if a compact Lie group G acts on S^n with all orbits of the same dimension, then rank $G = 1$, that is, G is the circle or $SO(3)$ or $SU(1)$. There are familiar examples to show that the circle acts freely on S^{2n-1} and that $SU(1)$ acts freely on S^{4n-1}. The group $SO(3)$ cannot act freely on any sphere, but it is not known in general whether it can act on S^{4n-1} (the only possibility) with all orbits of dimension 3.

THEOREM 6.1. *If $SO(3)$ acts differentiably on S^n with 3-dimensional principal orbits and if* $\dim B < n - 2$ *then the principal isotropy group is the identity.*

The case when $SO(3)$ acts on S^n with $\dim B = n - 2$ has been considered by Yang and the author. Below are three examples for actions of this kind.

1. $SO(3)$ acts trivially on S^1.

2. $SO(3)$ acts on $R^{n+1} = R^5 \times R^{n-4} (n \geqslant 4)$ by the definition

$$g(x, y) = (gx, y)$$

where the action of $SO(3)$ on R^5 is the irreducible orthogonal action. Then $SO(3)$ acts on S^n regarded as the unit sphere in R^{n+1}. In this action, the two-dimensional orbits are all projective planes, $F(G)$ is an $(n - 5)$-sphere, and for every $x \in U$, G_x is a dihedral group of order 4.

3. $SO(3)$ acts on $R^{n+1} = R^3 \times R^3 \times R^{n-5} (n \geqslant 5)$ by the definition

$$g(x, y, z) = (gx, gy, z)$$

where the action on R^3 is the familiar one. Then $SO(3)$ acts on S^n (the unit sphere) and in this action the two-dimensional orbits are all 2-spheres, $F(G)$ is an $(n-6)$-sphere and for every $x \in U$, $G_x = e$.

In each of these three examples $\dim B = n-2$ and $D = \emptyset$. The orbit space in cases 2 and 3 is a closed $(n-3)$-cell with boundary $B/SO(3)$. The next theorem shows that every action of $SO(3)$ on S^n with $\dim B = n-2$ resembles one of these examples rather closely.

THEOREM 6.2. *Let $SO(3)$ act on $M = S^n$ with* $\dim B = n-2$. *Then $D = \emptyset$ and one of the following holds:*

1. *$n = 1$ and $SO(3)$ acts trivially;*

2. *$n \geqslant 4$ and for every $x \in U$, G_x is a dihedral group of order 4, the two-dimensional orbits are all projective planes and*

$$H^*(F(G); Z_2) = H^*(S^{n-5}; Z_2);$$

3. *$n \geqslant 5$ and for every $x \in U$, $G_x = e$; the two-dimensional orbits are all 2-spheres and $H^*(F(G); Z_2) = H^*(S^{n-6}; Z_2)$.*

In the last two cases $H^*(B^*; Z) = H^*(S^{n-4}; Z)$ and M^* has trivial cohomology.

7. Principal orbits of dimension $n-2$ or $n-3$.

THEOREM 7.1. *Let G be a compact connected group which acts differentiably on R^n in such a way that the highest dimension of any orbit is either $(n-1)$ or $(n-2)$. Then the action is equivalent to a linear action.*

Bredon has recently studied actions on S^n with $(n-2)$-dimensional orbits and has classified the possibilities.

THEOREM 7.2. *Let G be a compact connected Lie group acting differentiably on S^n, $n > 4$, with principal orbits of dimension $n-3$ and a stationary point. Then S^n/G is a simply connected 3-manifold with boundary $B^* = S^2$ and $D^* = \emptyset$. There exists a second stationary point, and the set $F(G)$ is S^0, S^1, S^2, or S^3. In case $F(G) = S^2$, then $F(G) = B$ and principal orbits are $n-3$ spheres.*

8. Conditions implying linearity. Stewart has proved

THEOREM 8.1. *Let the circle group $SO(2)$ act differentiably on S^n with precisely two fixed points and freely otherwise. Then the action is equivalent to a linear action.*

By using the engulfing theorem, Connell, Yang and the author have proved

THEOREM 8.2. *Let G be a compact group which acts differentiably on R^n with a fixed point set F which is assumed to be diffeomorphic to R^k, $0 \leqslant k \leqslant n-3$, and with all other orbits of the same type and dimension r. Then the action of G is differentiably equivalent to a linear action of G on R^n if $n-r \geqslant 5$.*

The case where G acts freely outside of F is perhaps the main case covered by the theorem, and to specialize further for concreteness one may think of the case $G = Z_p$ or $G = SO(2)$.

COROLLARY. *Let G be a compact group which acts differentiably on S^n with a fixed point set which is diffeomorphic to S^k, $0 \leqslant k \leqslant n-3$, and with all other orbits of the same type and dimension r. Then the action of G is topologically equivalent to a linear action on S^n if $n-r \geqslant 5$.*

Hsiang has found several cases of actions of $SO(n)$ on S^m to be linear when n and m are suitably related.

When Z_2 acts on S^3 it is known that $F(Z_2)$ is homeomorphic to S^i, $-1 \leqslant i \leqslant 3$.

Livesay has proved

THEOREM 8.3. *If Z_2 acts differentiably on S^3, then the action is equivalent to a linear action.*

In the cases $i = -1$, $i = 0$, it is not necessary to assume differentiability. In the case $i = 1$, the main problem is to show that S^1 is unknotted. In the case $i = 0$, Hirsch and Smale gave a proof which was completed by Livesay.

Wang (Amer. Jour. Math. 82 (1960), 698-748) has studied the case of a differentiable action of a compact connected G on S^n with an $(n-1)$-dimensional orbit, and has given a classification of such actions.

Milnor (Some free actions of cyclic groups on spheres, these Proceedings, pp. 37-42) has shown that if $p \geqslant 5$ is a prime and n is an odd integer $\geqslant 5$, then Z_p can act differentiably and freely on S^n in infinitely many different ways (in the sense that the quotient manifolds S^n/Z_p have different simple homotopy types).

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DIFFERENTIAL OPERATORS ON MANIFOLDS WITH BOUNDARY*

By J. J. KOHN

LET M be a domain in $\mathbf{R}^n$ which is bounded and whose boundary bM is a C^∞ manifold (of dimension $n-1$). Let $\mathscr{F}$ be the space of C^∞ complex-valued functions on $\overline{M}$. We shall consider a differential operator $A: \mathscr{F}^m \to \mathscr{F}^p$. The results presented here can easily be generalized to the case where M is a finite differentiable manifold and A is a differential operator on C^∞ sections of a fibre bundle. If $u \in \mathscr{F}^m$, $u = (u_1, \ldots, u_m)$ we write:

$$(Au)_i = \sum_j A_{ij}(D)\, u_j,\ i = 1, \ldots, p, \tag{1}$$

where $D = (D_1, \ldots, D_n)$, $D_k = -(-1)^{\frac{1}{2}} \dfrac{\partial}{\partial x^k}$ and the A_{ij} are polynomials in the D_k with C^∞ coefficients. We shall suppose that A is homogeneous and of *first order*, so that we have

$$A_{ij}(D) = \Sigma\, a_{ij}^k\, D_k, \tag{2}$$

where the a_{ij}^k are in $\mathscr{F}$.

First we consider the inhomogeneous equation

$$Au = \phi\,; \tag{3}$$

the problem is, given $\phi \in \mathscr{F}^p$, to find $u \in \mathscr{F}^m$, satisfying (3).

We introduce the inner products:

$$(u, v) = \sum_{i=1}^{m} \int_M u_i\, \bar{v}_i\, dx \text{ and } (\phi, \psi) = \sum_{j=1}^{p} \int_M \phi_j\, \bar{\psi}_j\, dx, \tag{4}$$

the corresponding norms are denoted by $\|\ \|$ and $dx = dx^1 \ldots dx^n$; we complete $\mathscr{F}^m$ and $\mathscr{F}^p$ under these norms and obtain hilbert spaces which we denote by $\tilde{\mathscr{F}}^m$ and $\tilde{\mathscr{F}}^p$ respectively. We let A also

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denote the hilbert space closure of A with domain $\mathscr{D}_A \subset \mathscr{F}^m$ and we denote by A^* the hilbert space adjoint of A, with domain $\mathscr{D}_{A^*} \subset \tilde{\mathscr{F}}^p$.

Now it is clear that a necessary condition for the existence of u satisfying (3) is

$$(\phi, \psi) = 0 \text{ for all } \psi \in \mathscr{D}_{A^*} \text{ with } A^* \psi = 0. \tag{5}$$

If $B : \mathscr{F}^p \to \mathscr{F}^q$ is such that

$$BA = 0, \tag{6}$$

then another necessary condition is that

$$B\phi = 0, \tag{7}$$

or, equivalently, that $(\phi, B^* \alpha) = 0$, for all $\alpha \in \mathscr{D}_{B^*} \subset \tilde{\mathscr{F}}^q$. Let $\mathscr{D} = \mathscr{D}_{A^*} \cap \mathscr{D}_B$. We define the following inner product on $\mathscr{D}$:

$$\mathbf{D}(\phi, \psi) = (A^* \phi, A^* \psi) + (B\phi, B\psi) + (\phi, \psi) ; \tag{8}$$

it is clear that $\mathscr{D}$ is a hilbert space under the inner product $\mathbf{D}$. Now we define the space $\mathscr{H}$ by

$$\mathscr{H} = \{\phi \in \mathscr{D} \mid A^* \phi = 0 \text{ and } B\phi = 0\}. \tag{9}$$

Then we have

10. Proposition. *If* $\mathbf{D}$ *is completely continuous, in the sense that a bounded sequence in the* $\mathbf{D}$*-norm has a convergent subsequence in the* $\| \; \|$*-norm, then there exists* $u \in \mathscr{D}_A$ *satisfying* (3) *if and only if* $B\phi = 0$ *and* $\phi \perp \mathscr{H}$. *Furthermore* $\mathscr{H}$ *is finite dimensional.*

The above proposition is proven by showing that the complete continuity of $\mathbf{D}$ implies that the operator L has a closed range where L is defined by:

$$L = AA^* + B^*B. \tag{11}$$

In fact we obtain the orthogonal decomposition:

$$\tilde{\mathscr{F}}^p = AA^* \mathscr{D}_L \oplus B^*B\mathscr{D}_L \oplus \mathscr{H}, \tag{12}$$

it then follows that $A\mathscr{D}_A = AA^* \mathscr{D}_L$.

For each $x \in M$ and $\xi \in \mathbf{R}^n$ we define $\sigma A(x, \xi) : \mathbf{C}^m \to \mathbf{C}^p$ by:

$$((\sigma A(x,\xi))(\gamma))_i = \sum a_{ij}^k(x)\,\xi_k\,\gamma_j; \tag{13}$$

the map $\sigma A : M \times \mathbf{R}^m \times \mathbf{C}^m \to \mathbf{C}^p$ is called the *symbol* of A.

In what follows we will assume that B satisfies (6), that B *is of the first order* and that for *each* $x \in M$ *and* $\xi \in \mathbf{R}^n$ *with* $\xi \neq 0$ *the sequence*

$$\mathbf{C}^m \xrightarrow{\sigma A(x,\xi)} \mathbf{C}^p \xrightarrow{\sigma B(x,\xi)} \mathbf{C}^q \tag{14}$$

is exact.

15. Proposition. *If B satisfies the above conditions then* $\mathbf{D}$ *is elliptic in the interior; that is, there exists a constant $C > 0$ such that:*

$$\|\phi\|_1^2 \leqslant C\,\mathbf{D}(\phi,\phi)$$

for all $\phi \in \mathscr{F}^p$ whose support does not meet bM, where $\|\ \ \|_1$ is defined by:

$$\|\phi\|_1^2 = \sum_{k,j} \|D_k\phi_j\|^2 + \|\phi\|^2. \tag{16}$$

If the above inequality holds for all $\phi \in \mathscr{D} \cap \mathscr{F}^p$ then we say that $\mathbf{D}$ is *coercive*; in that case the complete continuity of $\mathbf{D}$ and the existence and regularity of solutions of (3) follow by standard methods. However, it happens often that $\mathbf{D}$ is not coercive but that, nevertheless these conclusions can be established. In fact in certain situations in which $\mathbf{D}$ is not coercive the following estimate can be established:

$$\sum \int_{bM} |\phi_j|^2\,dS \leqslant C\,\mathbf{D}(\phi,\phi) \tag{17}$$

for all $\phi \in \mathscr{D} \cap \mathscr{F}^p$, were dS is the volume element on bM and $C > 0$ is independent of ϕ. It then follows that:

$$|||\phi|||_{1/2}^2 \leqslant C\,\mathbf{D}(\phi,\phi) \tag{18}$$

for all $\phi \in \mathscr{D} \cap \mathscr{F}^p$. The norms $|||\ \ |||_s$ are defined with the aid of a partition of unity with support in coordinate neighborhoods which "flatten" the boundary. If s is an integer then $|||\ \ |||_s$ is the L^2-norm of the tangential derivatives of order s. The inequality (18) implies the complete continuity of $\mathbf{D}$ on $\mathscr{D} \cap \mathscr{F}^p$. Furthermore, if $\phi \in \mathscr{D}_L \cap \mathscr{F}^p$ and if we set $\beta = L\phi + \phi$ we can establish the following *a priori* estimates:

$$\| \phi \|_s^2 + ||| \phi |||_{s+1/2}^2 \leqslant C_s (\| \beta \|_{s-2}^2 + ||| \beta |||_{s-1/2}^2), \tag{19}$$

where s is any integer. These estimates imply the complete continuity of $\mathbf{D}$ on $\mathscr{D}$ and the existence and regularity of solutions of (3) and of $L\phi = \gamma$.

Now we suppose that $m = 1$ and that the following sequence is exact

$$0 \longrightarrow \mathbf{C} \xrightarrow{\sigma A(x,\xi)} \mathbf{C}^p \xrightarrow{\sigma B(x,\xi)} \mathbf{C}^q. \tag{20}$$

We will then write

$$(Au)_j (x) = (P_j (x, D) u) (x), j = 1,..., p \tag{21}$$

and we can suppose that σB is given by $\sigma P_i - \sigma P_j$.

For $x \in \mathbf{R}^n$ we denote by $\mathscr{T}_x$ the complex tangent space at x; if $x \in bM$ then $\mathscr{T}_x(bM)$ denotes the subspace of $\mathscr{T}_x$ consisting of those vectors which are tangent to bM. We denote by $\mathscr{P}_x$ the subspace of $\mathscr{T}_x$ spanned by the $P_j (x, D)$. The exactness of (20) then implies that

$$\mathscr{T}_x = \mathscr{P}_x + \overline{\mathscr{P}}_x. \tag{22}$$

We define the subspace $\mathscr{R}_x$ of $\mathscr{T}_x$ to be the space spanned by $\mathscr{R}'_x$ where $\mathscr{R}'_x$ is defined by :

$$\mathscr{R}'_x = \{ \tau \in \mathscr{P}_x \mid \tau = \bar{\tau} \} \tag{23}$$

and we define the subset $(bM)_0$ of bM by :

$$(bM)_0 = \{ x \in bM \mid \mathscr{R}_x \subset \mathscr{T}_x (bM) \}. \tag{24}$$

Observe that $\mathscr{R}_x \subset \mathscr{P}_x \cap \overline{\mathscr{P}}_x$. Let $\mathscr{S}_x$ be a subspace of $\mathscr{P}_x$ such that $\mathscr{P}_x = \mathscr{R}_x \oplus \mathscr{S}_x$ and, for $x \in (bM)_0$ let $\mathscr{Q}_x = \mathscr{S}_x \cap \mathscr{T}_x (bM)$. We now have $\dim \mathscr{Q}_x = \dim \mathscr{S}_x - 1$ and since $2 \dim \mathscr{S}_x + \dim \mathscr{R}_x = n$, we have:

$$2 \dim \mathscr{Q}_x + \dim \mathscr{R}_x = n - 2. \tag{25}$$

Thus, for $x \in (bM)_0$ there exists $\tau_x \in \mathscr{T}_x (bM)$, $\tau_x \neq 0$ such that:

$$\mathscr{T}_x (bM) = \mathscr{Q}_x \oplus \overline{\mathscr{Q}}_x \oplus \mathscr{R}_x \oplus \{\tau_x \}. \tag{26}$$

Let f be a C^∞ non-singular real-valued function defined in a neighborhood of bM and suppose that $f > 0$ outside of $\overline{M}$ and $f < 0$ in M. Let $\sigma^1_x,..., \sigma^s_x$ be a basis of $\mathscr{Q}_x$ and $\rho^1_x,..., \rho^t_x$ be a basis of $\mathscr{R}_x$.

Then we have the hermitian matrices $A^{ij}(x) = (\sigma_x^i \bar{\sigma}_x^j f)(x)$ and $B^{ij}(x) = (\rho_x^i \rho_x^j f)(x)$.

27. THEOREM. *The estimates* (17) *and* (19) *are satisfied if and only if for each $x \in (bM)_0$ the matrix $A^{ij}(x)$ has either s-positive eigen-values or two negative eigen-values and the matrix $B^{ij}(x)$ has either t-positive eigen-values or two negative eigen-values.*

Now let $\mathscr{F}_b = C^\infty(bM)$ and $\mathscr{F}_0 = \{u \in \mathscr{F} \mid u(x) = 0 \text{ if } x \in bM\}$. Then we have the exact sequence

$$0 \to \mathscr{F}_0 \to \mathscr{F} \to \mathscr{F}_b \to 0. \tag{28}$$

Further we define the subspace $\mathscr{C}$ of $\mathscr{F}^p$ by: $\mathscr{C} = \{\phi \in \mathscr{F}^p \mid$ there exists $\lambda \in \mathscr{F}_b$ such that for $x \in bM$ we have $\phi_x = \lambda(x) A(f)_x\}$, (29)

where f is the function defined above. Let $\mathscr{D}$ be the quotient space $\mathscr{F}^p/\mathscr{C}$. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{C} & \longrightarrow & \mathscr{F}^p & \longrightarrow & \mathscr{D} & \longrightarrow & 0 \\ & & \uparrow A & & \uparrow A & & \uparrow A_b & & \\ 0 & \longrightarrow & \mathscr{F}_0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}_b & \longrightarrow & 0 \end{array} \tag{30}$$

which defines the map $A_b : \mathscr{F}_b \to \mathscr{D}$. Observe that the space $\mathscr{D}$ depends only on bM (locally it can be identified with a subspace of $\mathscr{F}_b^p$) and that A_b is a first order differential operator. In a similar way we can define a first order operator B_b on bM (corresponding to B) such that $B_b A_b = 0$, but the symbol sequence will no longer be exact and the operator $L_b : \mathscr{D} \to \mathscr{D}$ defined by $L_b = A_b A_b^* + B_b^* B_b$ is not elliptic.

THEOREM. *If each of the matrices $A^{ij}(x)$ and $B^{ij}(x)$ have at least one positive eigen-value for each $x \in (bM)_0$, then if $\psi \in \mathscr{D}$, every weak solution of the equation $L_b \phi = \psi$ is in C^∞ and the operator $(L_b + I)^{-1}$ is completely continuous.* (31)

Thus we obtain an example of a determined self-adjoint system of partial differential equations on a compact manifold which is not elliptic but nevertheless has the usual regularity, existence and complete continuity properties.

REMARKS. (A) A system of the type we discuss here can be associated with a very general system by means of the "Spencer resolution", see [10] and [11].

(B) When $\mathscr{R}_x = \mathscr{T}_x$ for all x, the problem discussed here reduces to the classical Neumann problem.

(C) When $\mathscr{R}_x = 0$ for all x, then the P_j define an integrable almost-complex structure on M and the problem becomes the so called "$\bar{\partial}$-Neumann" problem for forms of type (0, 1).

(D) The $\bar{\partial}$-Neumann problem motivates the developments described here. This problem was first formulated in [2]. C. B. Morrey discovered and established the basic estimate (17) for a special case of the problem, see [8]. The problem was solved on strongly pseudo-convex manifolds by the author, see [4]. M. E. Ash generalized this solution to pseudo-groups, see [1]. L. Hörmander, in [3], found necessary and sufficient conditions on M for (17) to hold.

(E) There are three different proofs of the regularity of solutions of the $\bar{\partial}$-Neumann problem, see [4], [5], [6] and [8].

(F) The works quoted above contain several applications of the solution of the $\bar{\partial}$-Neumann problem. H. Rossi and the author have used the problem to characterize boundary values of holomorphic functions, see [7]. The boundary operators A_b, L_b, etc. arise from that work.

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L^2 ESTIMATES AND EXISTENCE THEOREMS FOR THE $\bar\partial$ OPERATOR†

By LARS HÖRMANDER

1. Introduction. Let Ω be an open subset of a paracompact complex analytic manifold M of complex dimension n. If p and q are integers ($p \geqslant 0$, $q > 0$), and u is a differential form of type $(p, q-1)$ in Ω (with distribution coefficients), the exterior differential du can be written in a unique way as a sum

$$du = \partial u + \bar\partial u$$

where ∂u is of type $(p+1, q-1)$ and $\bar\partial u$ is of type (p, q). (For definitions see Weil [11] and also Sections 3 and 4.) The purpose of this lecture is to discuss the existence of solutions of the system of differential equations

$$\bar\partial u = f \tag{1.1}$$

where f is a given form of type (p, q). Since $\bar\partial\,\bar\partial = 0$, a solution of (1.1) can only exist if

$$\bar\partial f = 0. \tag{1.2}$$

The problem is to decide when (1.2) is sufficient to guarantee that (1.1) has a solution. By the Dolbeault theorem, a positive answer to this question is equivalent to the vanishing of the cohomology groups of Ω with values in the sheaf of germs of holomorphic p-forms; in particular, the case $p = 0$, $q = 1$, means that the additive Cousin problem can be solved. (See e.g. Malgrange [8].) The existence theorems in this paper have therefore been obtained before with methods of sheaf theory (Cartan [3], Andreotti and Grauert [1]), but we shall give direct proofs with Hilbert space methods. These are developed from those of Morrey [9], Kohn [6] and Ash [2], combined with techniques used in the study of a single differential equation, particularly the Carleman method for proving uniqueness

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theorems. The interest of such an approach depends of course largely on the extent to which it is applicable to general overdetermined systems of differential equations. As yet the author has only verified that the methods used in this paper give a rather simple proof of the complex Frobenius theorem of Nirenberg [10] (see also Kohn [6]).

During the colloquium the author was informed that Andreotti and Vesentini, in a manuscript entitled "Carleman estimates for the Laplace-Beltrami equation on complex manifolds", have used similar methods to prove the results of Andreotti and Grauert [1].

2. Basic facts from functional analysis. Let H_1 and H_2 be two Hilbert spaces, T a closed densely defined linear operator from H_1 to H_2, and F a closed subspace of H_2 containing the range R_T of T.

LEMMA 2.1. *If the range of T is equal to F, it follows that*

$$\|f\|_2 \leq C \, \|T^* f\|_1, f \in F \cap D_{T^*}, \tag{2.1}$$

where C is a constant. Conversely, if (2.1) is valid, the equation $Tu = g$ with $g \in F$ has a solution u such that $\|u\|_1 \leq C \, \|g\|_2$.

PROOF. Assume that $R_T = F$. We must prove that the set

$$B = \{f \mid f \in F \cap D_{T^*}, \|T^* f\|_1 \leq 1\}$$

is bounded. To do so it is sufficient to prove that B is weakly bounded in F, that is, that $|(f, g)_2|$ is bounded when $f \in B$ for every fixed $g \in F$. But by hypothesis we can choose $u \in D_T$ so that $Tu = g$, which gives

$$|(f, g)_2| = |(T^* f, u)_1| \leq \|u\|_1, f \in B.$$

This proves the first part of the lemma.

Now assume that (2.1) is valid, and let $g \in F$. Since $T^{**} = T$, the equation $Tu = g$ is equivalent to the identity

$$(u, T^* f)_1 = (g, f)_2, f \in D_{T^*}.$$

If we prove the inequality

$$|(g, f)_2| \leq C \, \|g\|_2 \, \|T^* f\|_1, f \in D_{T^*}, g \in F, \tag{2.2}$$

the Hahn-Banach theorem will prove that the equation $Tu = g$ has a solution u with $\|u\|_1 \leqslant C \|g\|_2$. If f is orthogonal to F, we have $(g, f)_2 = 0$, and $T^*f = 0$ since $R_T \subset F$. Hence it is sufficient to prove (2.2) when $f \in F$, and then it follows immediately from (2.1).

LEMMA 2.2. *The range of T is closed and has finite codimension in F if and only if from every sequence $f_n \in F \cap D_{T^*}$ with $\|f_n\|_2 = 1$ and $T^*f_n \to 0$ one can select a strongly convergent subsequence.*

PROOF. (a) *Sufficiency.* The hypothesis implies that the vector space

$$N = \{f \mid f \in F \cap D_{T^*},\ T^*f = 0\}$$

is finite dimensional, for it is locally compact. We have $R_T \subset F \ominus N$ since N is orthogonal to R_T, and we claim that (2.1) is valid if F is replaced by $F \ominus N$. In fact, otherwise we can choose a sequence $f_n \in (F \ominus N) \cap D_{T^*}$ so that $\|f_n\|_2 = 1$ and $T^*f_n \to 0$. By hypothesis there exists a strong limit f of this sequence, and since $f \in (F \ominus N) \cap D_{T^*}$, $\|f\|_2 = 1$ and $T^*f = 0$, we have a contradiction. Hence it follows from Lemma 2.1 that $R_T = F \ominus N$.

(b) *Necessity.* Let N be the orthogonal complement of R_T in F. Then we have $T^*f = 0$ for every $f \in N$, and N is finite dimensional. If f_n is a sequence with the properties described in the lemma and we set $f_n = f_n' + f_n''$ where $f_n' \in F \ominus N$ and $f_n'' \in N$, it follows that $T^*f_n' = T^*f_n \to 0$. But since $R_T = F \ominus N$ by assumption, Lemma 2.1 shows that (2.1) is valid with F replaced by $F \ominus N$. Hence $f_n' \to 0$. Since the sequence f_n'' is bounded and lies in a finite dimensional space, the lemma is proved.

The lemmas will be applied to Hilbert spaces H_1 and H_2 of forms of type $(p, q-1)$ and (p, q) respectively, with T defined by the $\bar{\partial}$ operator. We shall choose F as the space of forms in H_2 satisfying (1.2). To prove estimates such as (2.1) it is convenient to introduce a third Hilbert space H_3 of forms of type $(p, q+1)$ and the operator S from H_2 to H_3 defined by the $\bar{\partial}$ operator. Then we have $ST = 0$, so that $R_T \subset N_S$, the null space of S. We wish to prove that R_T is equal to N_S or at least that R_T has finite codimension in N_S. To do so we note that the inequality (2.1) with $F = N_S$ is implied by

$$\|f\|_2^2 < C^2(\|T^*f\|_1^2 + \|Sf\|_3^2), \ f \in D_{T^*} \cap D_S, \tag{2.1)'$$

for when $f \in F = N_S$, the last term drops out. Similarly, the hypotheses of Lemma 2.2 are fulfilled if from every sequence $f_n \in D_{T^*} \cap D_S$ with $\|f_n\|_2 = 1$ and $\|T^*f_n\|_1 + \|Sf_n\|_3 \to 0$ we can extract a convergent subsequence. In what follows we shall therefore study estimates in terms of $\|T^*f\|_1^2 + \|Sf\|_3^2$, valid for all $f \in D_{T^*} \cap D_S$, instead of estimates of the form (2.1) where f is restricted to lie in a subspace F. This implies a considerable simplification since we shall see that the study of (2.1)′ can to some extent be made locally. Incidentally, (2.1)′ also contains information about the operator S, but we shall not make use of that.

3. Pseudo-convex domains in $\mathbf{C}^n$. We denote the real coordinates in $\mathbf{C}^n$ by x_j, $1 \leqslant j \leqslant 2n$, and the complex coordinates by $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$. A differential form f is said to be of type (p, q) if it can be written in the form

$$f = \sum_{|I|=p}' \sum_{|J|=q}' f_{I,J}\, dz^I \wedge d\bar{z}^J$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multi-indices, that is, sequences of indices between 1 and n, of length $|I| = p$ and $|J| = q$. The notation Σ' means that the summation shall only be extended over strictly increasing multi-indices, and we have written

$$dz^I \wedge d\bar{z}^J = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

The coefficients f_{IJ} may be distributions in an open set Ω, and are supposed to be defined for arbitrary I and J so that they are antisymmetric in the indices of I as well as in those of J. With $\partial/\partial\bar{z}_k = \frac{1}{2}(\partial/\partial x_{2k-1} + i\,\partial/\partial x_{2k})$ we have

$$\bar{\partial} f = \sum_{I,J}' \sum_{k} \frac{\partial f_{I,J}}{\partial \bar{z}_k}\, d\bar{z}_k \wedge dz^I \wedge d\bar{z}^J. \tag{3.1}$$

If $\mathscr{F}$ is a space of distributions, we denote by $\mathscr{F}_{(p,q)}$ the space of forms of type (p, q) with coefficients belonging to $\mathscr{F}$. In particular, we shall use this notation with $\mathscr{F} = C^k(\Omega)$, where Ω is an open set in $\mathbf{C}^n$, or with $\mathscr{F} = C^k(\bar{\Omega})$, the space of restrictions to Ω of functions belonging to C^k in the whole space. We shall also use the space $\dot{C}^k(\bar\Omega)$ consisting of elements in $C^k(\bar\Omega)$ vanishing outside a large sphere. If ϕ is a measurable function in Ω, locally bounded from above, we denote by $L^2(\Omega, \phi)$ the space of functions in Ω which are square integrable with respect to the density $e^{-\phi}$; the norm in $L^2_{(p,q)}(\Omega, \phi)$ is defined by

$$(f, f)_\phi = \|f\|_\phi^2 = \int_\Omega |f(z)|^2 e^{-\phi}\, dV,\ f \in L^2_{(p,q)}(\Omega, \phi), \tag{3.2}$$

where dV is the Lebesgue measure and

$$|f(z)|^2 = \langle f(z), f(z)\rangle = \sum_{I,J}{}' |f_{I,J}(z)|^2. \tag{3.3}$$

Finally, we write $L^2(\Omega, \mathrm{loc})$ for the space of functions which are in L^2 on all compact subsets of Ω.

We shall now illustrate our methods by studying the case when $p = 0$, $q = 1$, and the set Ω is an open set in $\mathbf{C}^n$ with a C^2 boundary. With $\phi \in C^2(\bar\Omega)$ we choose the Hilbert spaces H_j, $j = 1, 2, 3$, as $L^2_{(0,j-1)}(\Omega, \phi)$ and let T and S be the maximal differential operators between these spaces defined by $\bar\partial$. Thus, for example, $u \in D_T$ if $u \in L^2_{(0,0)}(\Omega, \phi)$ and $\bar\partial u$, defined in the sense of distribution theory, belongs to $L^2_{(0,1)}(\Omega, \phi)$; then we have $Tu = \bar\partial u$.

It follows from a theorem of Lax and Phillips [7] that $\dot{C}^1_{(0,1)}(\bar\Omega) \cap D_{T^*}$ is dense in $D_{T^*} \cap D_S$ with respect to the graph norm $f \to \|f\|_2 + \|Sf\|_3 + \|T^* f\|_1$. The essential step in the proof is a regularization by convolution along the boundary of Ω in a local coordinate system where the boundary is a hyperplane. The operators $\partial/\partial z_j$ and $\partial/\partial \bar z_j$ do not necessarily have constant coefficients in such coordinates, so the techniques of Friedrichs [4] have to be used.

We shall now describe the space $\dot{C}^1_{(0,1)}(\bar\Omega) \cap D_{T^*}$. To do so we note that Green's formula can be written in the form

$$\int_\Omega \frac{\partial v}{\partial x_j} \cdot \bar w \cdot e^{-\phi}\, dV = -\int_\Omega v \left(\overline{\frac{\partial w}{\partial x_j} - w \frac{\partial \phi}{\partial x_j}}\right) e^{-\phi}\, dV + \int_{\partial\Omega} \frac{\partial \rho}{\partial x_j} v \bar w e^{-\phi}\, dS,$$

where v and $w \in \dot{C}^1(\overline{\Omega})$, dS is the Euclidean surface element on $\partial\Omega$, and ρ denotes the distance to $\partial\Omega$, defined to be > 0 outside $\overline{\Omega}$ and < 0 in Ω. In fact, grad ρ is the exterior unit normal on $\partial\Omega$. If we introduce the notation

$$\delta_j w = \frac{\partial w}{\partial z_j} - w \frac{\partial \phi}{\partial z_j} = e^{\phi} \frac{\partial (we^{-\phi})}{\partial z_j}, \tag{3.4}$$

we obtain

$$\int_\Omega \frac{\partial v}{\partial \bar{z}_j} \overline{w} e^{-\phi}\, dV = - \int_\Omega v \,\overline{\delta_j w}\, e^{-\phi}\, dV + \int_{\partial\Omega} \frac{\partial \rho}{\partial \bar{z}_j} v \overline{w} e^{-\phi}\, dS. \tag{3.5}$$

Hence

$$(f, \bar{\partial} u)_\phi = \int_\Omega \langle f, \bar{\partial} u \rangle e^{-\phi}\, dV = - \int_\Omega \left(\sum_1^n \delta_j f_j \right) \overline{u}\, e^{-\phi}\, dV +$$
$$+ \int_{\partial\Omega} \left(\sum_1^n \frac{\partial \rho}{\partial z_j} \right) \overline{u} e^{-\phi}\, dS,$$

if $f \in \dot{C}^1_{(0,1)}(\overline{\Omega})$ and $u \in \dot{C}^1(\overline{\Omega})$. This proves that every $f \in \dot{C}^1_{(0,1)}(\overline{\Omega}) \cap D_{T^*}$ satisfies the boundary condition

$$\sum_1^n f_j \frac{\partial \rho}{\partial z_j} = 0 \text{ on } \partial\Omega, \tag{3.6}$$

and that

$$T^* f = - \sum_1^n \delta_j f_j. \tag{3.7}$$

Before proceeding to the proof of estimates, we note the commutation relations

$$\left(\delta_k \frac{\partial}{\partial \bar{z}_j} - \frac{\partial}{\partial \bar{z}_j} \delta_k \right) w = w \frac{\partial^2 \phi}{\partial \bar{z}_j\, \partial z_k}, \; w \in C^2. \tag{3.8}$$

These imply the identities

$$\int_\Omega \delta_j v\, \overline{\delta_k w}\, e^{-\phi}\, dV - \int_\Omega \frac{\partial v}{\partial \bar{z}_k} \overline{\left(\frac{\partial w}{\partial \bar{z}_j} \right)} e^{-\phi}\, dV$$
$$= \int_\Omega v\, \overline{w} \frac{\partial^2 \phi}{\partial z_j\, \partial \bar{z}_k} e^{-\phi}\, dV + \int_{\partial\Omega} v\, \overline{w} \frac{\partial^2 \rho}{\partial z_j\, \partial \bar{z}_k} e^{-\phi}\, dS +$$
$$+ \int_{\partial\Omega} \frac{\partial \rho}{\partial z_j} v\, \overline{\delta_k w}\, e^{-\phi}\, dS - \int_{\partial\Omega} v \left(\overline{\partial \left(w \frac{\partial \rho}{\partial z_k} \right) \Big/ \partial \bar{z}_j} \right) e^{-\phi}\, dS \tag{3.8'}$$

when $v, w \in \dot{C}^1(\overline{\Omega})$. In fact, (3.8)′ follows immediately from (3.8) and (3.5) if $w \in \dot{C}^2(\overline{\Omega})$, and this is a dense subset of $\dot{C}^1(\overline{\Omega})$.

With $f \in \dot{C}^1_{(0,1)}(\overline{\Omega}) \cap D_{T^*}$ we now form

$$\| T^* f \|_\phi^2 + \| Sf \|_\phi^2 = \int_\Omega \sum_{j,k=1}^n \left(\delta_j f_j \overline{\delta_k f_k} - \frac{\partial f_j}{\partial \bar{z}_k} \left(\overline{\frac{\partial f_k}{\partial \bar{z}_j}} \right) + \left| \frac{\partial f_j}{\partial \bar{z}_k} \right|^2 \right) e^{-\phi}\, dV. \tag{3.9}$$

If we drop the last term, which is positive, and apply (3.8)′ to the others, we obtain in view of (3.6)

$$\| T^* f \|_\phi^2 + \| Sf \|_\phi^2 \geqslant \int_\Omega \sum_{j,k=1}^n f_j \bar{f}_k \frac{\partial^2 \phi}{\partial z_j\, \partial \bar{z}_k} e^{-\phi}\, dV + \int_{\partial\Omega} \sum_{j,k=1}^n f_j \bar{f}_k \frac{\partial^2 \rho}{\partial z_j\, \partial \bar{z}_k} e^{-\phi}\, dS,\ f \in \dot{C}^1_{(0,1)}(\overline{\Omega}) \cap D_{T^*}. \tag{3.10}$$

Now recall that $\partial\Omega$ is called pseudo-convex if on $\partial\Omega$

$$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j\, \partial \bar{z}_k} f_j \bar{f}_k \geqslant 0 \text{ when } \sum_1^n \frac{\partial \rho}{\partial z_j} f_j = 0. \tag{3.11}$$

Also recall that a function $\phi \in C^2$ is called plurisubharmonic if the quadratic form

$$\sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_j\, \partial \bar{z}_k} f_j \bar{f}_k$$

is positive semi-definite. From (3.10) we then obtain

THEOREM 3.1. *Let $\phi \in C^2(\overline{\Omega})$, $\partial\Omega \in C^2$, and assume that $\partial\Omega$ is pseudo-convex and that $\phi(z) - \epsilon |z|^2$ is plurisubharmonic for some $\epsilon > 0$. Then we have*

$$\epsilon \| f \|_\phi^2 \leqslant \| T^* f \|_\phi^2 + \| Sf \|_\phi^2,\ f \in D_{T^*} \cap D_S. \tag{3.12}$$

Hence it follows from Lemma 2.1 that the range of T is equal to the null space of S.

By a straightforward regularization procedure one can remove from Theorem 3.1 the hypotheses that $\partial\Omega \in C^2$ and that $\phi \in C^2$. Instead of assuming the Levi condition (3.11) one then requires that

Ω is pseudo-convex in the sense that there is a plurisubharmonic function π in Ω such that $\{z \mid z \in \Omega, \ \pi(z) < M\}$ is relatively compact in Ω for every M. Choosing ϕ as the sum of $|z|^2$ and a convex, sufficiently rapidly increasing function of π, one can achieve that any given form $f \in L^2_{(0,1)}(\Omega, \text{loc})$ belongs to the space $L^2_{(0,1)}(\Omega, \phi)$. If Ω is pseudo-convex, the equation (1.1) has therefore a solution $u \in L^2_{(0,0)}(\Omega, \text{loc})$ for every $f \in L^2_{(0,1)}(\Omega, \text{loc})$ satisfying (1.2). This implies that the Cousin problem can be solved in Ω. If the above is extended to forms of type $(0, q)$, which only requires somewhat longer computations, we also obtain the Cartan-Oka-Serre theorem that all the cohomology groups of Ω with values in the sheaf of germs of holomorphic functions are equal to 0 if Ω is pseudo-convex. This implies that Ω is a domain of holomorphy, so the Levi problem is solved at the same time.

We shall now show how approximation theorems of the Runge type can be proved with our methods. Let $A(\Omega)$ be the space of analytic functions in Ω with the topology of uniform convergence on compact subsets, or, equivalently, L^2 convergence on compact subsets.

THEOREM 3.2. *Let the hypotheses of Theorem 3.1 be fulfilled and let c be a constant such that* $\Omega_c = \{z | z \in \Omega, \ \phi(z) < c\}$ *is relatively compact in* Ω. *Then the restrictions to* Ω_c *of the functions in* $A(\Omega)$ *are dense in* $A(\Omega_c)$.

PROOF. Let K be a compact subset of Ω_c. In view of the Hahn-Banach theorem we only have to prove that if v is a function in $L^2(\Omega)$, vanishing outside K, such that

$$\int v \bar{u} \, dV = 0 \tag{3.13}$$

for every $u \in A(\Omega)$, then (3.13) is valid for every $u \in A(\Omega_c)$. This will follow if we prove that there exist functions g_j with compact supports in Ω_c such that $v = -\sum_1^n \frac{\partial g_j}{\partial z_j}$, for this implies that

$$\int v \bar{u} \, dV = \sum_1^n \int g_j \overline{\left(\frac{\partial u}{\partial \bar{z}_j}\right)} \, dV = 0, \ u \in A(\Omega_c).$$

That (3.13) holds for every $u \in A(\Omega)$ implies that $v e^{\phi}$ is orthogonal to the null space of T in $L^2_{(0,0)}(\Omega, \phi)$. Hence $v e^{\phi}$ belongs to the closure of R_{T^*}. But the range of T^* is equal to the range of its restriction to the orthogonal complement of N_{T^*}, which is the closure of R_T and therefore contained in N_S. Hence it follows from (3.12) that R_{T^*} is closed. If we choose f so that $Sf = 0$ and $T^*f = v e^{\phi}$, the estimate (3.10) gives

$$\int_\Omega \sum_{j,k=1}^{n} f_j \bar{f}_k \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} e^{-\phi} \, dV \leqslant \int_\Omega |v|^2 e^{\phi} \, dV.$$

The equation $T^*f = v e^{\phi}$ means, besides boundary conditions on f, that

$$\sum_1^n e^{\phi} \frac{\partial (f_j e^{-\phi})}{\partial z_j} = - v e^{\phi},$$

so if $g = f e^{-\phi}$, we have

$$\sum_1^n \frac{\partial g_j}{\partial z_j} = -v ; \int_\Omega \sum_{j,k=1}^{n} g_j \bar{g}_k \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} e^{\phi} \, dV \leqslant \int_\Omega |v|^2 e^{\phi} \, dV.$$

We shall apply this result with ϕ replaced by suitable functions ϕ_λ depending on a parameter λ. Choose $\gamma < c$ so that $\phi \leqslant \gamma$ in K, and let χ be a convex function in $C^2(\mathbf{R})$ such that $\chi(t) = 0$ when $t \leqslant \gamma$, and $0 < \chi'(t)$ when $t > \gamma$. Then $\chi(\phi)$ is plurisubharmonic, and if we set $\phi_\lambda = \phi + \lambda\chi(\phi)$ where λ is a positive parameter, we have $\phi \leqslant \phi_\lambda$ with equality in K. If we now replace ϕ by ϕ_λ in the first part of the proof, it follows that one can find g^λ so that

$$\sum_1^n \frac{\partial g_j^\lambda}{\partial z_j} = -v ; \int_\Omega \sum_{j,k=1}^{n} g_j^\lambda \overline{g_k^\lambda} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} e^{\phi_\lambda} \, dV \leqslant \int_\Omega |v|^2 e^{\phi} \, dV.$$

If g is a weak limit of g^λ in $L^2_{(0,1)}(\Omega, -\phi)$ when $\lambda \to +\infty$, it follows that

$$\sum_1^n \frac{\partial g_j}{\partial z_j} = - v,$$

and that g_j is in L^2 and vanishes when $\phi(z) > \gamma$. This completes the proof.

In this theorem, as in Theorem 3.1, it is of course easy to remove the regularity assumptions on $\partial\Omega$ and on ϕ.

4. The $\bar\partial$ operator on a manifold. Let M be a paracompact complex analytic manifold of complex dimension n. The decomposition of differential forms into forms of type (p, q), the definition of the $\bar\partial$ operator, and the definition of plurisubharmonic functions, which we have introduced in Section 3 for domains in $\mathbf{C}^n$, can immediately be extended to forms and functions on the manifold M. In fact, all these definitions are invariant for analytic changes of coordinates.

In order to study the operator $\bar\partial$ with the Hilbert space techniques of Section 3, we must introduce hermitian norms on differential forms in M. Thus we choose a hermitian metric on M, that is, a Riemannian metric such that in any analytic coordinate system with coordinates $z_1, \dots, z_n$ we have

$$ds^2 = \sum_{j,k=1}^{n} h_{jk}\, dz_j\, d\bar z_k$$

where (h_{jk}) is a positive definite hermitian matrix with C^∞ coefficients. The existence of a hermitian structure is trivial locally and is immediately proved in the large by means of a partition of unity. We keep the hermitian structure on M fixed in all that follows. The element of volume defined by the structure we denote by dV, and the element of area on a smooth hypersurface we denote by dS. (For these and the following definitions see also Weil [11].)

If f is a form of type $(1, 0)$ and $f = \sum_1^n f_j\, dz_j$ in a local coordinate system, we set

$$\langle f, f\rangle = \sup_{dz} \Big|\sum_1^n f_j\, dz_j\Big|^2 \Big/ \sum_{j,k=1}^{n} h_{jk}\, dz_j\, d\bar z_k .$$

This definition is of course independent of the choice of coordinates. The Gram-Schmidt orthogonalization process shows that every point in M has a neighborhood U where there are n forms $\omega^1, \dots, \omega^n$ of type $(1, 0)$ with C^∞ coefficients such that $\langle \omega^j, \omega^k\rangle = \delta_{jk}$

$(j, k = 1, \ldots, n)$, where δ is the Kronecker delta. If we set $f = \sum_1^n f_j \omega^j$, it follows that $\langle f, f \rangle = \sum_1^n |f_j|^2$. Now an arbitrary differential form f of type (p, q) can be written in a unique way as a sum

$$f = \sum_{|I|=p}{}' \sum_{|J|=q}{}' f_{I,J}\, \omega^I \wedge \bar\omega^J$$

where $f_{I,J}$ are antisymmetric in I and in J, the notation Σ' is explained in Section 3, and

$$\omega^I \wedge \bar\omega^J = \omega^{i_1} \wedge \ldots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \ldots \wedge \bar\omega^{j_q}.$$

We can define $\langle f, f \rangle$ by

$$\langle f, f \rangle = |f|^2 = \Sigma' \, |f_{I,J}|^2,$$

for this is independent of the choice of orthonormal basis $\omega^1, \ldots, \omega^n$ for forms of type $(1, 0)$.

Let Ω be an open subset of M and ϕ a continuous function in Ω. We then define $L^2_{(p,q)}(\Omega, \phi)$ as the space of all forms f in Ω of type (p, q) with measurable coefficients, such that

$$\|f\|_\phi^2 = \int_\Omega |f|^2 e^{-\phi}\, dV < \infty,$$

forms which are equal almost everywhere being identified. If $p \geqslant 0$, $q > 0$, the operator $\bar\partial$ defines in the weak sense a closed densely defined operator

$$T : L^2_{(p,q-1)}(\Omega, \phi) \to L^2_{(p,q)}(\Omega, \phi)$$

and another

$$S : L^2_{(p,q)}(\Omega, \phi) \to L^2_{(p,q+1)}(\Omega, \phi).$$

(See also Section 3.) We have $ST = 0$. If Ω is relatively compact in M, if $\partial\Omega \in C^3$, and if $\phi \in C^1(\bar\Omega)$, which we assume from now on, it follows from the results of Lax and Phillips [7] that $C^2_{(p,q)}(\bar\Omega) \cap D_{T^*}$ is dense in $D_{T^*} \cap D_S$ for the graph norm $f \to \|f\|_\phi + \|T^* f\|_\phi + \|Sf\|_\phi$.

As noted by Ash [2], the operators S and T^* have very simple expressions in terms of the "moving frame" $\omega^1, \ldots, \omega^n$, so the arguments of Section 3 can be reproduced with rather small modifications. However, at one point in the proof of (3.10) the argument was much too crude. Namely, when passing from (3.9) to (3.10) we dropped the square of the norm of $\partial f_j/\partial \bar{z}_k$ for all j and k. Now it turns out that these terms become very important when ϕ is not plurisubharmonic or $\partial\Omega$ is not pseudo-convex. Another integration by parts should then be performed in some of these terms. A careful analysis along the lines of Hörmander [5], Chapter VIII, gives the following results.

THEOREM 4.1. *In order that, for a fixed* ϕ, *there shall exist a constant* C *such that*

$$\int_{\partial\Omega} |f|^2 e^{-\phi}\, dS \leqslant C(\|T^*f\|_\phi^2 + \|Sf\|_\phi^2 + \|f\|_\phi^2),\ f \in C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}, \tag{4.1}$$

it is necessary and sufficient that the Levi form has either at least $n-q$ *positive eigenvalues or at least* $q+1$ *negative eigenvalues at every point on* $\partial\Omega$.

We recall that if ρ denotes the distance to $\partial\Omega$, defined to be positive outside $\overline{\Omega}$ and negative in Ω, then the Levi form is the restriction of the quadratic form $\langle \partial\bar{\partial}\rho, t \wedge \bar{t}\rangle$ to the plane $\langle \partial\rho, t\rangle = 0$. Here t denotes a form of type $(1, 0)$. It is easily seen that the condition on the Levi form found in Theorem 4.1 is independent of the choice of hermitian metric. When the Levi form is positive definite, the estimate (4.1) is due to Ash [2] and Kohn [6].

Combining the estimate (4.1) with results from the theory of elliptic systems of differential operators one can prove that the hypotheses of Lemma 2.2 are fulfilled. (See Kohn [6]. It is very important in the proof that $C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}$ is dense in $D_{T^*} \cap D_S$ for the graph norm as we observed above.) Hence the range of T is closed and has finite codimension in the null space of S when the conditions on the Levi form in Theorem 4.1 are fulfilled.

To proceed further we must be able to vary the weight function ϕ as we did in proving Theorem 3.2, and this also leads to a simpler proof of the fact that the hypotheses of Lemma 2.2 are implied by those of Theorem 4.1. In the next two theorems we shall therefore study estimates where ϕ is replaced by a function of ϕ.

THEOREM 4.2. *Let $z_0 \in \Omega$. In order that for some neighborhood $U \subset \Omega$ of z_0 there shall exist constants C and τ_0 such that*

$$\tau \, \|f\|_{\tau\phi}^2 \leqslant C(\|T^* f\|_{\tau\phi}^2 + \|Sf\|_{\tau\phi}^2), \quad \tau > \tau_0, \tag{4.2}$$

for all $f \in C^2_{(p,q)}(\overline{\Omega})$ with support contained in U, it is necessary and sufficient that the hermitian form $\langle \partial \bar\partial \phi, t \wedge \bar t \rangle$, where t is of type$(1, 0)$, has either at least $n - q + 1$ positive or at least $q + 1$ negative eigenvalues.

Here T^* denotes the adjoint of the operator $T = \bar\partial$ with respect to the norms $\| \quad \|_{\tau\phi}$, so that the coefficients of the differential operator T^* involve τ.

For neighborhoods of boundary points and for non-linear functions $\chi(\phi)$ of ϕ instead of linear ones as in Theorem 4.2, our results are not quite complete but still adequate for the applications.

DEFINITION 4.3. *We shall say that a real valued function $\phi \in C^2$ satisfies the condition A_q at a point z_0 if grad $\phi(z_0) \neq 0$ and*

$$\lambda_1 + \dots + \lambda_q + \sum_1^{n-1} \max(-\mu_j, 0) > 0,$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_n$ are the eigenvalues of the quadratic form $\langle \partial \bar\partial \phi, t \wedge \bar t \rangle$ with respect to the form $\langle t, t \rangle$, and $\mu_1, \dots, \mu_{n-1}$ are the eigenvalues of the same form restricted to the plane $\langle \partial\phi, t \rangle = 0$. If at least $n - p$ of the eigenvalues μ_j are positive or at least $q + 1$ of them are negative, we say that ϕ satisfies the condition a_q.

It is easy to see that A_q implies a_q. The condition a_q is independent of the choice of hermitian structure, which is not true for A_q. The difference between the two conditions is rather small, however, for if ϕ satisfies a_q it follows that $\exp(\gamma\phi)$ satisfies A_q if the constant γ is chosen sufficiently large.

THEOREM 4.4. *Let $z_0 \in \overline{\Omega}$ and let ϕ satisfy condition A_q at z_0. If $z_0 \in \partial\Omega$ we also assume that ϕ is constant on $\partial\Omega$ and that $\phi < \phi(z_0)$*

in Ω. Then there is a neighborhood U of z_0 and a constant C such that for all convex increasing functions $\chi \in C^2(\mathbf{R})$ and all $f \in C^2_{(p,q)}(\overline{\Omega}) \cap D_{T^*}$ with support in $U \cap \overline{\Omega}$ we have

$$\int_{\Omega} \chi'(\phi)\,|f|^2\,e^{-\chi(\phi)}\,dV \leqslant C(\|T^*f\|^2_{\chi(\phi)} + \|Sf\|^2_{\chi(\phi)} + \|f\|^2_{\chi(\phi)}). \tag{4.3}$$

Without giving the details of the proof we shall now state an existence and approximation theorem which follows from Theorem 4.4. In doing so, we use the following notation.

DEFINITION 4.5. *If Ω is relatively compact in the manifold M and ϕ is continuous in $\overline{\Omega}$, we denote the quotient space N_S/R_T by $\overline{H}_{(p,q)}(\Omega)$. (Recall that $N_S = \{f \mid f \in L^2_{(p,q)}(\Omega, \phi), \bar\partial f = 0\}$, and that R_T is the range of the weak maximal $\bar\partial$ operator from $L^2_{(p,q-1)}(\Omega, \phi)$ to $L^2_{(p,q)}(\Omega, \phi)$. This quotient space is of course independent of ϕ and the hermitian structure.) We also denote by $H_{(p,q)}(\Omega)$ the quotient space of*

$$\{f \mid f \in L^2_{(p,q)}(\Omega, \text{loc}), \bar\partial f = 0\}$$

with respect to

$$L^2_{(p,q)}(\Omega, \text{loc}) \cap \{\bar\partial f \mid f \in L^2_{(p,q-1)}(\Omega, \text{loc})\};$$

here Ω may be any paracompact complex analytic manifold.

As mentioned in the introduction, the Dolbeault theorem (see Malgrange [8]) gives a natural isomorphism between the space $H_{(p,q)}(\Omega)$ and the q^{th} cohomology group of Ω with values in the sheaf of germs of holomorphic p-forms.

THEOREM 4.6. *Let Ω be a complex analytic manifold of complex dimension n, and let ϕ be a real valued function in $C^3(\Omega)$ such that the open sets*

$$\Omega_c = \{z \mid z \in \Omega,\ \phi(z) < c\}$$

are relatively compact in Ω for every real number c. Further assume that ϕ satisfies the condition a_q in the complement of Ω_c for some c. Then the restriction homomorphism $H_{(p,q)}(\Omega) \to \overline{H}_{(p,q)}(\Omega_c)$ is injective for the same c, and $\overline{H}_{(p,q)}(\Omega_c)$ has finite dimension. Further, every solution $u \in L^2_{(p,q-1)}(\Omega_c, \phi)$ of the equation $\bar\partial u = 0$ belongs to the closure in

$L^2_{(p,q-1)}(\Omega_c, \phi)$ of the restrictions to Ω_c of forms $u_1 \in L^2_{(p,q-1)}(\Omega, \text{loc})$ such that $\bar{\partial} u_1 = 0$. If ϕ satisfies both conditions a_q and a_{q+1} outside Ω_c, the homomorphism $H_{(p,q)}(\Omega) \to \bar{H}_{(p,q)}(\Omega_c)$ is an isomorphism.

This is essentially a result of Andreotti and Grauert [1].

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THE $\bar{\partial}$-NEUMANN PROBLEM ON STRONGLY PSEUDO-CONVEX MANIFOLDS

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1. Introduction. In this paper we present a simplification of the recent solution due to J. J. Kohn ([7], [8]) of the so-called $\bar{\partial}$-Neumann problem introduced by Garabedian and Spencer [5] for complex exterior differential forms on a compact complex-analytic manifold with strongly pseudo-convex boundary. The problem in its present form was investigated by D. C. Spencer and J. J. Kohn [9] by means of integral equations. The present author [13] solved this problem for the special cases of 0-forms and $\bar{z}$-1-forms (i.e. forms of the types (0,0) and (0,1) in our current notation) on certain "tubular" manifolds and used those results to prove that any compact real-analytic manifold can be analytically embedded in a Euclidean space of sufficiently high dimension. Unfortunately there is an error in that paper which is corrected in § 2 by using the results of Kohn presented in this paper. These results apply to forms of arbitrary type (p, q) and the solution forms are shown to be of class C^∞ on the closed manifold provided the metric, boundary, and non-homogeneous term $\in C^\infty$ there. Recently Hörmander has extended some of these results but they are not published as yet.

In his recent papers, Kohn sketched applications of his results (a) to the study of the $\bar{\partial}$-cohomology theory, (b) to the study of deformations of complex structures, and (c) to obtain a new proof of the result of Nirenberg and Newlander [15] which showed that a complex analytic structure could be introduced on an integrable almost-complex manifold. However, part of the interest in this problem to those working in partial differential equations lies in the fact that the problem is not a regular boundary value problem in the sense of Agmon-Douglis-Nirenberg ([1] and a forthcoming paper on systems), Browder [4], Lopatinsky [11], etc. We shall give an example below after we have introduced the notations and sketched the results; we shall also show the connection with the $\bar{\partial}$-cohomology.

We assume that $\bar{M} = M \cup bM$ is a compact complex-analytic manifold having boundary bM of class C^∞. We assume that we are given a hermitian metric

$$ds^2 = g_{\alpha\beta}\, dz^\alpha\, d\bar{z}^\beta\dagger \quad (g_{\beta\alpha} = \bar{g}_{\alpha\beta},\ \alpha,\ \beta = 1, \ldots, \nu) \tag{1.1}$$

which is of class C^∞ on $\bar{M}$. We suppose that the function $r \in C^\infty(\bar{M})$ and equals the negative of the geodesic distance to bM for points within a distance $-s_0$ of bM, $s_0 < 0$. It is clear that there exists a slightly larger such manifold M' such that $\bar{M} \subset M'$ and that the metric and r can be extended to $\in C^\infty(M')$ so that r is the geodesic distance from bM on $M' - \bar{M}$. The strong pseudo-convexity of the boundary bM implies that there is a constant $c_0 > 0$ such that at any point P_0 on bM (where $r = 0$) we have

$$r_{z^\beta \bar{z}^\gamma}\, T^\beta \bar{T}^\gamma \geqslant c_0\, g_{\beta\gamma} T^\beta \bar{T}^\gamma \tag{1.2}$$

for all complex vectors $(T^1, \ldots, T^\nu)$ such that

$$r_{z^\beta}\, T^\beta = 0. \tag{1.3}$$

If f is any other real function of class C^2 near bM such that $\nabla f \neq 0$ and $f = 0$ on bM, and $f < 0$ on M near bM, then the positiveness of the form $f_{z^\beta \bar{z}^\gamma}\, T^\beta\, \bar{T}^\gamma$ for T such that $f_{z^\beta}\, T^\beta = 0$ follows. In the above and throughout this chapter we assume that the operators $\partial/\partial z^\alpha$ and $\partial/\partial \bar{z}^\alpha$ are defined by

$$\frac{\partial}{\partial z^\alpha} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\,\frac{\partial}{\partial y^\alpha}\right),\ \frac{\partial}{\partial \bar{z}^\alpha} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\,\frac{\partial}{\partial y^\alpha}\right),$$

$$z^\alpha = x^\alpha + iy^\alpha,\ \bar{z}^\alpha = x^\alpha - iy^\alpha. \tag{1.4}$$

We let $\mathfrak{A}$ denote the set of all exterior differential forms of class $C^\infty(\bar{M})$ (i.e. C^∞ on $\bar{M}$) and denote by $\mathfrak{A}^{p,q}$ the set of all those which are of type (p, q), i.e. which can be expressed in any local analytic coordinate system in the form

† Repeated Greek indices are summed from 1 to ν.

$$\phi = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_q}} \phi_{i_1 \dots i_p j_1 \dots j_q} \, dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} . \tag{1.5}$$

We abbreviate the notation to

$$\phi = \Sigma \, \phi_{IJ} \, dz^I \wedge d\bar{z}^J ; \tag{1.6}$$

when we use this notation I and J will always stand for increasing sequences as in (1.5). However, we shall often wish to have the $\phi_{I,j_1 \dots j_q}$ defined for *all* sequences of indices $j_1 \dots j_q$; in this case, we assume that the ϕ's are defined so as to be antisymmetric in the j-indices. We shall at times wish to do the same with the I indices and shall sometimes write $\phi_{I,\alpha R}$ where $R = (r_1, \dots, r_{q-1})$ with $r_1 < \dots < r_{q-1}$ and α runs 1 to ν independently of R.

We shall wish to consider M (or $\bar{M}$) as a real manifold with metric given in $(x, y) = (x^1, \dots, x^\nu, y^1, \dots, y^\nu)$ coordinates by (1.1) which becomes, on setting $dz^\alpha = dx^\alpha + i dy^\alpha$ and $d\bar{z}^\beta = dx^\beta - i dy^\beta$,

$$g_{1\alpha\beta}(dx^\alpha \, dx^\beta + dy^\alpha \, dy^\beta) + 2 g_{2\alpha\beta} \, dx^\alpha \, dy^\beta,$$

$$g_{\alpha\beta} = g_{1\alpha\beta} + i g_{2\alpha\beta}, \, g_{1\beta\alpha} = g_{1\alpha\beta}, \, g_{2\beta\alpha} = - g_{2\alpha\beta} . \tag{1.7}$$

Then the dual $*\phi$ would be defined by first expressing ϕ in terms of real differentials dx^α and dy^α and then taking the ordinary real dual of the real and imaginary parts. This procedure introduces a factor 2^{p+q} into the customary inner product

$$(\phi, \psi) = \int_M \phi \wedge * \bar{\psi} \tag{1.8}$$

of two forms of the same type. Along with most workers in this field we omit these factors. The space $\mathfrak{L}^{p,q}$ is the completion of the space $\mathfrak{A}^{pq}$ using the inner product (1.8) and $\mathfrak{L}$ is just the Hilbert space sum of all the $\mathfrak{L}^{pq}$. For two forms of the same type, it is convenient to define the point function $\langle \phi, \psi \rangle$ by

$$\langle \phi, \psi \rangle \, dM = \phi \wedge * \bar{\psi}, \; \langle \phi, \phi \rangle = |\phi|^2$$

where dM is the element of volume on M. The formulas for (ϕ, ψ), dM, and $\langle \phi, \psi \rangle$ are

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle \, dM \tag{1.9}$$

where in any analytic coordinate system

$$dM = \Gamma(x, y)\, dxdy$$

$$\langle \phi, \psi \rangle = \Sigma g^{KI} g^{JL} \phi_{IJ} \bar{\psi}_{KL} \tag{1.10}$$

if ϕ is given by (1.6) and ψ is correspondingly defined. Here Γ is the $\nu \times \nu$ determinant of the $g_{\alpha\beta}$, g^{KI} is the $p \times p$ determinant of the $g^{k_\gamma i_\delta}$ and g^{JL} is the $q \times q$ determinant of the $g^{j_\alpha l_\beta}$. If we use the antisymmetry of the ϕ's and ψ's in all their indices, we may write

$$\langle \phi, \psi \rangle = \frac{1}{p!\,q!} g^{k_1 i_1} \ldots g^{k_p i_p} g^{j_1 l_1} \ldots g^{j_q l_q} \phi_{i_1 \ldots i_p j_1 \ldots j_q} \bar{\psi}_{k_1 \ldots k_p l_1 \ldots l_q}.$$

For forms in $\mathfrak{A}$ we define the operator $\bar{\partial}$ as follows:

If ϕ is of type (p, q) with $q = \nu$, we define $\bar{\partial}\phi = 0$; if $q < \nu$ and ϕ is given by (1.6), we define

$$\bar{\partial}\phi = \Sigma \phi_{IJ\bar{z}^\alpha} \, d\bar{z}^\alpha \wedge dz^I \wedge d\bar{z}^J. \tag{1.11}$$

For forms in $\mathfrak{A}$ we define $\mathfrak{b}\,\phi$ as follows: If $\phi \in \mathfrak{A}^{p,q}$ and $q = 0$, we define $\mathfrak{b}\,\phi = 0$; otherwise we define $\mathfrak{b}\,\phi$ as that form of type $(p, q-1)$ such that

$$(\mathfrak{b}\,\phi, \psi) = (\phi, \bar{\partial}\psi) \tag{1.12}$$

for every ψ in $\mathfrak{A}^{p,q-1}$ with compact support in M. As is seen by integrating by parts (see § 3), this leads to a formula of the form

$$(\mathfrak{b}\,\phi)_{IR} = (-1)^{p+1} g^{\alpha\beta} (\phi_{I,\alpha R z^\beta} + \Sigma A^{ST}_{IR\beta} \phi_{S,\alpha T}), \tag{1.13}$$

for suitable functions $A^{ST}_{IR\beta}$, and to the general formula

$$(\phi, \bar{\partial}\,\psi) = (\mathfrak{b}\,\phi, \psi) + \int_{bM} \langle \omega, \psi \rangle \, dS \tag{1.14}$$

where dS is the invariant surface element on bM and

$$\omega = \nu\,\phi, \ \omega_{IR} = (-1)^p g^{\alpha\beta} \phi_{I,\alpha R} r_{z^\beta}. \tag{1.15}$$

From (1.14), we may also derive the formula

$$(\bar{\partial}\phi, \psi) = (\phi, \mathfrak{d}\psi) + \int_{bM} \langle \phi, \nu\psi \rangle \, dS. \tag{1.16}$$

We let $\mathfrak{A}_0$ denote the subset of ϕ in $\mathfrak{A}$ for which $\nu\phi = 0$ on bM.

It follows immediately from (1.11) and the antisymmetry of the exterior product that

$$\bar{\partial}\bar{\partial}\phi = 0, \ \phi \in \mathfrak{A}. \tag{1.17}$$

From (1.14) and (1.16) it follows that if ϕ and $\psi \in \mathfrak{A}$, then

$$\begin{aligned}(\mathfrak{d}\phi, \bar{\partial}\psi) &= (\phi, \bar{\partial}\cdot\bar{\partial}\psi) - \int_{bM} \langle \nu\phi, \bar{\partial}\psi \rangle \, dS \\ &= (\mathfrak{d}\mathfrak{d}\phi, \psi) + \int_{bM} \langle \nu\mathfrak{d}\phi, \psi \rangle \, dS.\end{aligned} \tag{1.18}$$

By first letting ψ be arbitrary with compact support in M and then letting it be arbitrary we find that

$$\mathfrak{d}\mathfrak{d}\phi = 0 \text{ on } M \text{ and } \nu\mathfrak{d}\phi = 0 \text{ on } bM \text{ if } \phi \in \mathfrak{A}_0. \tag{1.19}$$

The $\bar{\partial}$-Neumann problem is to show the existence and regularity of the solutions of the complex Poisson equation

$$\Box\phi \equiv \mathfrak{d}\bar{\partial}\phi + \bar{\partial}\mathfrak{d}\phi = \omega \tag{1.20}$$

subject to the boundary conditions

$$\nu\phi = \nu\bar{\partial}\phi = 0 \text{ on } bM. \tag{1.21}$$

This boundary value problem is seen to arise formally from the variational problem of minimizing the integral

$$d(\phi, \phi) - 2\mathrm{Re}(\omega, \phi), \ \ d(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\mathfrak{d}\phi, \mathfrak{d}\psi) \tag{1.22}$$

among all $\phi \in \mathfrak{A}_0 (\nu\phi = 0$ on $bM)$. If ω and $\phi \in C^\infty(\overline{M})$, we see that ϕ satisfies

$$\begin{aligned}&(\bar{\partial}\phi, \bar{\partial}\psi) + (\mathfrak{d}\phi, \mathfrak{d}\psi) - (\omega, \psi) = 0 \\ &= (\Box\phi - \nu\omega, \psi) + \int_{bM} \{\langle \nu\bar{\partial}\phi, \psi \rangle - \langle \mathfrak{d}\phi, \nu\psi \rangle\} \, dS\end{aligned} \tag{1.23}$$

for all $\psi \in \mathfrak{A}_0$; the second line follows from (1.14) and (1.16). Since $\nu\phi = 0$, we see from (1.23) (and known formulas) that the condition $\nu\bar{\partial}\phi = 0$ on bM is a natural one.

2. Results. Examples. The analytic embedding theorem. In order to get a complete picture of the results we let T be the closure (with respect to the composite Hilbert space $\mathfrak{L}$) of the operator $\bar{\partial}$ as restricted to $\mathfrak{A}$ and let T^* be its adjoint. It is easy to see (using (1.14)) that $\mathfrak{A} \cap \mathfrak{D}(T^*) = \mathfrak{A}_0$. We then define

$$L = T^*T + TT^*. \tag{2.1}$$

We also define the domain $\mathfrak{D} = \mathfrak{D}(T) \cap \mathfrak{D}(T^*)$ and $\mathfrak{H}$ as that subset of $\mathfrak{D}$ for which $T\phi = T^*\phi = 0$; we define $\mathfrak{D}^{pq} = \mathfrak{D} \cap \mathfrak{L}^{pq}$ and $\mathfrak{H}^{pq} = \mathfrak{H} \cap \mathfrak{L}^{pq}$. In §5, we first prove the following simple preliminary results:

THEOREM. *$\mathfrak{R}(T) \subset \mathfrak{D}(T)$, $\mathfrak{R}(T^*) \subset \mathfrak{D}(T^*)$, and $T^2 = (T^*)^2 = 0$. L is self-adjoint and $\phi \in \mathfrak{D}(L) \Longleftrightarrow \phi$, $T\phi$, and $T^*\phi$ all $\in \mathfrak{D}$.*

$\phi \in \mathfrak{L} \ominus \mathfrak{R}(L) \Longleftrightarrow L\phi = 0 \Longleftrightarrow \phi \in \mathfrak{H}$.

Then in Sections 5 and 6, we prove the following principal results.

THEOREM. (i) *$\mathfrak{R}(L) = \mathfrak{L} \ominus \mathfrak{H}$ and $\mathfrak{H}$ is closed.*

(ii) *If $\omega \in \mathfrak{L} \ominus \mathfrak{H}$, there exists a unique*

$$\phi \in \mathfrak{D}(L) \cap (\mathfrak{L} \ominus \mathfrak{H}) \text{ such that } L\phi = \omega.$$

(iii) *If we define $N\omega = 0$ for $\omega \in \mathfrak{H}$ and $N\omega$ as the solution ϕ in* (ii) *if $\omega \in \mathfrak{L} \ominus \mathfrak{H}$, then N is completely continuous.*

(iv) *If $q \geqslant 1$, $\mathfrak{H}^{pq}$ is finite-dimensional.*

(v) *If $\omega \in \mathfrak{A}$, then $N\omega \in \mathfrak{A}$.*

(vi) *If $q \geqslant 1$, $\mathfrak{H}^{pq} \subset \mathfrak{A}_0^{pq}$.*

Parts (i) through (iv) are proved in §5 and require only the Hilbert space technique used there and the $\mathfrak{L}_2$, H_{20}^1 regularity theorem 4.4. This theorem is closely related to some recent results of Lions and Magenes [10]. The smoothness results in (v) and (vi) are proved in §6. One of the principal tools in the proofs is the important formula (3.15) of integration by parts.

Before proceeding, we introduce some additional notations. The manifold $M_s = M(s)$ for $s < 0$ consists of all points P on M

for which $r(P) \leqslant s$. An analytic coordinate patch with domain G and range $\mathfrak{R}$ is said to be *tangential* at some point P_0 of bM if a part g of bG contains the origin and corresponds, under the mapping from $\bar{G}$ to $\bar{\mathfrak{R}}$, to $\bar{\mathfrak{R}} \cap bM$, the origin corresponding to P_0, and at the origin $g_{\alpha\beta}(0) = \delta_{\alpha\beta}$ and the exterior normal to M at P_0 corresponds to the positive y^ν axis (i.e. $r_{y^\nu}(0, 0) = 1$). In case $\tau = (\tau^1, \ldots, \tau^r)$ or $(\tau_1, \ldots, \tau_r)$ is a set of indices, τ'_γ denotes the set

$$(\tau^1, \ldots, \tau^{\gamma-1}, \tau^{\gamma+1}, \ldots, \tau^r).$$

If $\alpha = (\alpha_1, \ldots, \alpha_\nu)$ is a sequence of non-negative integers, then D^α means $D^{\alpha_1}_{x^1} \ldots D^{\alpha_\nu}_{x^\nu}$. If ϕ is a vector function, $\nabla\phi$ denotes its gradient.

Next, we give an example to illustrate the fact that the $\bar{\partial}$-Neumann problem is not regular, except in the case where $q = \nu$ when it reduces to the Dirichlet problem since $\bar{\partial}\phi \equiv 0$ and $\nu\phi = 0$ on bM if and only if $\phi = 0$ on bM in that case. In the case $q = 0$, the problem is obviously not regular since the null space is just the space of holomorphic forms $\mathfrak{H}^{p0}$. To show that the problem is not regular for $1 \leqslant q \leqslant \nu - 1$, we take, as an example, $\nu = 2$, M the unit ball in R_4, the metric Euclidean, and set

$$\phi = \phi_1\, d\bar{z}^1 + \phi_2\, d\bar{z}^2,\ \phi_1 = -\bar{z}^2\Phi,\ \phi_2 = \bar{z}^1\Phi,$$

$$\Phi = \frac{3 - 2r^2}{6} A(z),\ r^2 = z^1\bar{z}^1 + z^2\bar{z}^2;$$

where $A(z) \in H^1_2(M)$ and is holomorphic on $M^{(0)}$ but is not in $H^2_2(M)$. Then

$$r.\,\nu\phi = \bar{z}^1\phi_1 + \bar{z}^2\phi_2 \equiv 0,\ \bar{\partial}\phi = \omega d\bar{z}^1 \wedge d\bar{z}^2$$

$$\omega = \phi_{2\bar{z}^1} - \phi_{1\bar{z}^2} = \bar{z}^1\Phi_{\bar{z}^1} + \bar{z}^2\Phi_{\bar{z}^2} + 2\Phi = (1 - r^2)\, A(z) \in H^1_{20}(M)$$

$$\Delta\phi_1 = -\bar{z}^2\Delta\Phi - \Phi_{z^2},\ \Delta\phi_2 = \bar{z}^1\Delta\Phi + \Phi_{z^1}$$

$$\Delta\Phi = -\tfrac{1}{3}(z^1 A_{z^1} + z^2 A_{z^2} + 2A) \in L_2(M).$$

It follows easily that $\phi \in \mathfrak{D}(L)$ but ϕ does not $\in H^2_2(M)$ as it would be if the problem were regular.

We now prove a theorem indicating the connection with the $\bar{\partial}$-cohomology theory.

Theorem. *If $\phi \in \mathfrak{A}^{pq}$ with $q \geqslant 1$ and if $\bar{\partial}\phi = 0$, there is a harmonic field $\phi_0 (\in \mathfrak{H}^{pq})$ such that $\phi - \phi_0 = \bar{\partial}\Phi$ for some $\Phi \in \mathfrak{A}^{p,q-1}$.*

Proof. First of all, suppose $f \in C^\infty$ on $\mathbf{R}^1$, and we define $\Phi_1 = f(r)(\nu\phi)$, $f(0) = 0$, $f'(0) = 4$, $f(r) = 0$ for $r \leqslant s_0 < 0$ for some such s_0. Then, from Lemma 6.2, it follows that

$$\nu(\phi - \bar{\partial}\Phi_1) = 0 \text{ on } bM.$$

Hence we may assume that $\phi \in \mathfrak{A}_0^{pq}$. Then, let $\Phi = N\mathfrak{d}\phi$. Then $\Phi \in C^\infty(\bar{M})$. Also, since $\phi \in \mathfrak{A}_0^{pq}$, we see from (1.16) that

$$(\bar{\partial}\mathfrak{d}\Phi, \mathfrak{d}\phi) = -\int_{bM} \langle \bar{\partial}\mathfrak{d}\Phi, \nu\phi \rangle \, dS + (\bar{\partial}\bar{\partial}\mathfrak{d}\Phi, \phi) = 0.$$

Hence, since $\bar{\partial}\phi = 0$ and $\nu\bar{\partial}\Phi = 0$ (so that $\nu\mathfrak{d}\bar{\partial}\Phi = 0$ by (1.19)),

$$\mathfrak{d}(\phi - \bar{\partial}\Phi) = \bar{\partial}(\phi - \bar{\partial}\Phi) = 0 \text{ and } \phi - \bar{\partial}\Phi \in \mathfrak{A}_0$$

so $\phi - \bar{\partial}\Phi \in \mathfrak{H}^{pq}$.

The following analog of the Kodaira decomposition theorem is of some interest.

Theorem. *$\mathfrak{L} = \mathfrak{H} \oplus D \oplus C$ where $\mathfrak{H}$ has its usual significance, D is the totality of forms of the form $\bar{\partial}\phi$ for some $\phi \in \mathfrak{D}$, and C is that of forms of the form $\mathfrak{d}\psi$ for some ψ in $\mathfrak{D}$.*

Proof. It is clear that if $h \in \mathfrak{H}$ and ϕ and $\psi \in \mathfrak{D}$, then the forms h, $\bar{\partial}\phi$, and $\mathfrak{d}\psi$(i.e. $T\phi$ and $T^*\psi$; see Theorem 5.6) are mutually orthogonal. Since $\mathfrak{H}^{p0}$ just consists of the holomorphic forms of degree p and the $\mathfrak{H}^{pq}$ with $q > 0$ are finite-dimensional, it follows that $\mathfrak{H}$ is closed. If $\omega \in \mathfrak{L} \ominus \mathfrak{H}$, let $\Phi = N\omega$, $\phi = \mathfrak{d}\Phi$, and $\psi = \bar{\partial}\Phi$. From our principal results, it follows that ϕ and $\psi \in \mathfrak{D} \cap (\mathfrak{L} \ominus \mathfrak{H})$ and that

$$\bar{\partial}\phi + \mathfrak{d}\psi = \omega \text{ and } \mathfrak{d}\phi = \bar{\partial}\psi = 0. \tag{2.2}$$

It is clear from the first statement above that the decomposition is unique. That C and D are closed follows from our principal results and the particular choice of ϕ and ψ in (2.2).

We can now present a simplification and correction to the proof given in [13] of the possibility of embedding analytically a real-analytic abstract manifold in Euclidean space. The error in that paper was in the proof of Theorem C of that paper which was given in §§8-11. The embedding theorem was proved by Bochner for compact manifolds in 1937 [3] assuming the existence of an analytic metric; this result was extended by Malgrange [12] in 1957 to the case of non-compact manifolds. The result of [13] was generalized to manifolds with a countable topology by Grauert [6] using methods of the theory of functions of several complex variables and some results of Remmert which had not been published at that time.

We now outline our method of proof. First of all, on account of Bochner's result, it is sufficient to show the following :

THEOREM A. *With each point P_0 of the given real-analytic compact manifold M_0 there are associated ν functions w_γ which are analytic over the whole of M_0 and have linearly independent gradients at P_0.*

For the gradients will remain linearly independent in some neighborhood of P_0 and thus M_0 can be covered by neighborhoods $\mathfrak{N}_q$, $q = 1, \ldots, Q$, where the functions $w_{q\gamma}$, $\gamma = 1, \ldots, \nu$ are analytic over M_0 and have linearly independent gradients over $\mathfrak{N}_q$, $q = 1, \ldots, Q$. The mapping $w_{q\gamma} = w_{q\gamma}(P)$ maps M_0 analytically into Euclidean space of $Q\nu$ dimensions; the mapping may not be $1 - 1$ in the large but is locally $1 - 1$ and non-singular and the Euclidean metric induces an analytic metric on M_0.

To prove Theorem A, we first embed M_0 in an open complex extension M (see [13], §2 or [17] and [18] where this embedding is discussed for manifolds with countable topology). Let P_0 be any point on M_0, let τ_0 be a complex-analytic coordinate patch with domain G_0 containing the origin and range $\mathfrak{N}_0$ containing P_0, in which P_0 and the origin correspond and the part of G_0 in $\mathbf{R}^\nu$ (i.e. for which $y = 0$) corresponds to $\mathfrak{N}_0 \cap M_0$, and choose a

Hermitian metric (1.1) which is of class $C^\infty(M)$, which is real on M_0 (i.e. $g_{\alpha\beta}$ is real on M_0), and for which we have

$$g_{\alpha\beta}(x, y) = \delta_{\alpha\beta}, \quad (x, y) \text{ on } G_0, \tag{2.3}$$

with respect to the coordinate system τ_0.

For points P on M near M_0, we define $r'(P)$ to be the geodesic distance from P to M_0. It is easily shown (see [13], §3) that the function $K(P) = [r'(P)]^2/2$ is of class C^∞ in a neighborhood of M_0 including all points where $r'(P) \leqslant R_0$. We define M_R as the complex analytic manifold $r'(P) \leqslant R$. It is easy to show that ($g_{\alpha\beta}(x, 0)$ is real)

$$4K_{z^\alpha \bar{z}^\beta}(x, 0)\, T^\alpha \overline{T}^\beta = K_{y^\alpha y^\beta}(x, 0)\, T^\alpha \overline{T}^\beta = g_{\alpha\beta}(x, 0)\, T^\alpha \overline{T}^\beta \tag{2.4}$$

for any complex-analytic patch which carries the points $(x, 0)$ into M_0. Thus if $0 < R \leqslant R_1$, bM_R is regular and of class C^∞ and M_R is strongly pseudo-convex; the function $r(P)$ used in this paper reduces to $r'(P) - R$ on M_R and the pseudo-convexity follows from (2.4). In fact, since $r(P) = r'(P) - R$ and $K(P) = [r'(P)]^2/2$ on M_R, we see by following through the proof of Theorem 5.8 as far as equation (5.18), that we obtain

$$\begin{aligned} d_R(\phi) &= I_R(\phi) + \int_{bM(R)} \sum g^{KI} g^{ST} g^{\alpha\beta} g^{\gamma\delta} r_{z^\beta \bar{z}^\gamma} \phi_{I\alpha S} \overline{\phi}_{K\delta T}\, dS(P) \\ &\geqslant -C_1 (\phi, \phi)_R + C_2 R^{-1} \int_{bM(R)} |\phi|^2\, dS, \quad \phi \in \mathfrak{D}(M_R). \end{aligned}$$

Thus we obtain

$$\int_{bM(R)} |\phi|^2\, dS \leqslant CR[d_R(\phi) + (\phi, \phi)_R],\ 0 < R \leqslant R_1 \leqslant R_0,\ \phi \in \mathfrak{D}^{pq}(M_R), \tag{2.5}$$

where C is independent of R.

We now sketch the proof, given in §7 of the cited paper [13], of the important inequality

$$(\phi, \phi)_R \leqslant CR^2\, d_R(\phi),\ 0 < R \leqslant R_2 \leqslant R_1,\ \phi \in \mathfrak{D}^{pq}(M_R),\ q \geqslant 1. \tag{2.6}$$

Incidentally, this shows that *$\mathfrak{H}^{pq}(M_R)$ consists only of the zero element if $q \geqslant 1$ and R is small enough.* We conclude first that there is an R_3, $0 < R_3 \leqslant R_1$, such that $\overline{M(R_3)}$ can be covered by a finite number of neighborhoods $\mathfrak{N}_t^*$ each of which $\subset \mathfrak{N}_t$, the range of a complex analytic coordinate patch τ_t of the type of τ_0 (i.e. real on M_0) with domain G_t, and is the range of a "quasi-geodesic" non-analytic (but C^∞) coordinate system τ_t^* with domain of the form $\bar{G}_{0t} \times \overline{B(0, R_3)}$, where G_{0t} is a domain of class C^∞ in real ν-space $\mathbf{R}^\nu$ and $\bar{G}_{0t} \subset G_t \cap \mathbf{R}^\nu$. If $P \in \mathfrak{N}_t^*$, its quasi-geodesic coordinates (ξ, η) with respect to τ_t^* are determined as follows: There is a unique geodesic through P which is orthogonal to M_0 at some point of M_0. Let its equations be

$$x^\alpha = x^\alpha(r), \ y^\alpha = y^\alpha(r), \ 0 \leqslant r \leqslant r'(P),$$

in the (x, y) coordinates of τ_t. We define

$$\xi^\alpha(P) = x^\alpha(0), \ \zeta^\alpha(P) = {}'y^\alpha(0) \ ({}'y = dy/dr).$$

Since the metric is real along M_0, we see that

$$\begin{aligned} ds^2 &= g_{1\alpha\beta}(x, 0)\,(dx^\alpha\, dx^\beta + dy^\alpha\, dy^\beta), \text{ along } M_0, \\ &= g_{1\alpha\beta}(x, 0)\,(d\xi^\alpha\, d\xi^\beta + d\zeta^\alpha\, d\zeta^\beta). \end{aligned}$$

By using the Lagrange method of reducing a quadratic form to a sum of squares, we introduce the η^α by

$$\zeta^\alpha = d_\gamma^\alpha(\xi)\,\eta^\gamma$$

where the $d_\gamma^\alpha \in C^\infty$ and the matrix is non-singular so that

$$ds^2 = g_{1\alpha\beta}(\xi, 0)\, d\xi^\alpha\, d\xi^\beta + \delta_{\alpha\beta}\, d\eta^\alpha\, d\eta^\beta \text{ along } M_0. \tag{2.7}$$

From the construction, it follows that

$$K(P) = |\eta(P)|^2/2, \ r'(P) = |\eta(P)|. \tag{2.8}$$

It is shown in [13], §3, that the coordinates are C^∞.

With each t and each R, $0 < R \leqslant R_3$, we define an analytic manifold M_{tR} as follows: Let $\bar{\mathfrak{N}}_{tR} = \tau_t^*[\bar{G}_{0t} \times \overline{B(0, R)}]$, and let $\bar{G}_{tR} = \tau_t^{-1}(\bar{\mathfrak{N}}_{tR})$; clearly $\bar{G}_{0t} = \bar{G}_{tR} \cap \mathbf{R}^\nu$. Choose positive numbers a_t and A_t such that if we define

$$F_t = \overline{F_t^{(0)}}, \ F_t^{(0)} : |x^\alpha| < A_t, \ \alpha = 1, \ldots, \nu,$$

then

$$G_{tR_3} \subset F_t^{(0)} \times B(0, a_t/2), \ \bar{G}_{0t} \subset F_t^{(0)}.$$

Extend the metric $g_{t\alpha\beta}$ to be of class C^∞ for all (x, y), to be periodic of period $2A_t$ in each x^α to be real if $y = 0$, and so that $g_{t\alpha\beta}(x, y) = \delta_{\alpha\beta}$ for all x on and near ∂F_t and all y with $|y| \geqslant 3a_t/4$. Then, if R_3 is small enough, the quasi-geodesic (ξ, η) coordinates can be extended to all (ξ, η) with $|\eta| \leqslant a_t$ to be periodic of period A_t in each ξ^α. We then let M_{tR} be the set of all (x, y) corresponding to the (ξ, η) with $|\eta| \leqslant R$, any two points (x_1, y) and (x_2, y) where each $x_2^\alpha - x_1^\alpha = 2A_t\, n^\alpha$, n^α an integer, being identified.

Now, suppose $\phi \in \mathfrak{D}^{pq}(M_R)$, $0 < R \leqslant R_3$. Let $\{\zeta_s\}$, $s = 1, \ldots S$, be a partition of unity such that each $\zeta_s \in C^\infty(\overline{M}_{R_3})$ and has support in $\overline{M}_{R_3} \cap \overline{\mathfrak{N}}_t^*$ for some t (ζ_s need not vanish on bM_{R_3}) and let $\phi_s = \zeta_s\, \phi$. Then, by approximating ϕ by smooth forms, as we may on account of our principal results, we see that each $\phi_s \in \mathfrak{D}^{pq}(M_R)$. But now, we may associate each ϕ_s with a form ϕ_{st} on M_{tR} by defining the components of ϕ_{st} on $\overline{G}_{tR}$ to agree with those of ϕ_s there and to vanish elsewhere on M_{tR}. Then, clearly,

$$d_R(\phi_s) = d_{tR}(\phi_{st}),\ (\phi_s, \phi_s)_R = (\phi_{st}, \phi_{st})_{tR},\ d_R(\phi_s) \leqslant C\,[d_R(\phi) + (\phi, \phi)_R]$$

since the ζ_s do not depend on R. Accordingly, it is sufficient to prove (2.6) for forms $\in \mathfrak{D}^{pq}(M_{tR})$, since, if this is done, we would have

$$(\phi, \phi)_R^{\frac{1}{2}} \leqslant \sum_{s=1}^{S} (\phi_s, \phi_s)_R^{\frac{1}{2}} \leqslant CSR[d_R(\phi) + (\phi, \phi)_R]^{\frac{1}{2}}$$

from which the result follows easily if $0 < R \leqslant R_2 \leqslant R_3$.

So we consider some M_{tR} and drop the t. From Theorems 5.6 and 5.4 it follows that we may write

$$\phi = \phi_0 + H,\ d_R(\phi) = d_R(\phi_0) + d_R(H),\ \phi_0 \in H_{20}(M_R) \cap (\mathfrak{L}^{pq} \ominus \mathfrak{H}^{pq}). \tag{2.9}$$

We shall hereafter denote the components ϕ_{IJ} simply by ϕ^j. Then

$$(\phi_0, \phi_0)_R = \int_0^R \int_{bM(r)} \langle \phi_0, \phi_0 \rangle\, dSdr$$

$$\leqslant C \int_0^R r^{\nu-1}\, dr \int_F d\xi \int_\Sigma \sum_j |\phi_0^j(r, \xi, \theta)|^2\, d\Sigma(\theta), \tag{2.10}$$

where $\Sigma = bB(0, 1)$ and θ denotes coordinates on $bB(0, 1)$, (ξ, η) being the quasi-geodesic coordinates, and (r, θ) being polar coordinates in the η-space. Since $\phi_0(R, \xi, \theta) = 0$, we have

$$\left.\begin{aligned} \phi_0^j(r, \xi, \theta) &= -\int_r^R \phi_{0r}^j(s, \xi, \theta)\, ds \\ |\phi_0^j(r, \xi, \theta)|^2 &\leqslant (R-r)\int_r^R |\phi_{0r}^j(s, \xi, \theta)|^2\, ds. \end{aligned}\right\} \tag{2.11}$$

Substituting (2.11) into (2.10) and using the condition $s \geqslant r$, we obtain

$$\begin{aligned}(\phi_0, \phi_0)_R \leqslant CR^2\,((\phi_0, \phi_0))_R = CR^2((\phi_0, \phi_0))_{R_3} &\leqslant CR^2\, d_{R_3}(\phi_0) \\ &= CR^2\, d_R(\phi_0)\end{aligned}$$

(for the definition of the strong norm $((\phi, \phi))^{\frac{1}{2}}$ in H_2^1, see §5) if we define $\phi_0 = 0$ for $R \leqslant r \leqslant R_3$ and use Theorem 5.4 for $M(R_3)$. This is (2.6) for ϕ_0.

To prove (2.6) for a harmonic H, we shall prove for any harmonic H, whether in $\mathfrak{D}^{pq}(M_R)$ or not, that

$$(H, H)_R \leqslant CR \int_{bM(R)} \langle H, H\rangle\, dS \quad \text{(for } R \text{ small enough)} \tag{2.12}$$

from which (2.6) follows, using (2.5). We may assume $H \in C^\infty(\overline{M}_R)$. To prove (2.12), we see as in §5, that the real and imaginary parts H^j of the components satisfy a system of differential equations of the form

$$a^{\alpha\beta} H^j_{\xi^\alpha\xi^\beta} + 2b^{\alpha\beta} H^j_{\xi^\alpha\eta^\beta} + c^{\alpha\beta} H^j_{\eta^\alpha\eta^\beta} + 2d_k^{j\alpha} H^k_{\xi^\alpha} + 2e_k^{j\alpha} H^k_{\eta^\alpha} + f_k^j H^k = 0 \tag{2.13}$$

in terms of the quasi-geodesic coordinates, where

$$a^{\alpha\beta}(\xi, 0) = g_1^{\alpha\beta}(\xi, 0),\ b^{\alpha\beta}(\xi, 0) = 0,\ c^{\alpha\beta}(\xi, 0) = \delta^{\alpha\beta}. \tag{2.14}$$

Let us now take spherical coordinates in the η-space as above. Then the equations (2.13) are seen to be equivalent to

$$H^j_{rr} + (\nu - 1)\, r^{-1} H^j_r + r^{-2} \Delta_{2\theta} H^j + 2C^\gamma H^j_{r\theta^\gamma} + r^{-1} C^{\gamma\delta} H^j_{\theta^\gamma\theta^\delta} +$$
$$+ 2rB^\alpha H^j_{\alpha_r} + 2B^{\alpha\gamma} H^j_{\xi^\alpha\theta^\gamma} + A^{\alpha\beta} H^j_{\xi^\alpha\xi^\beta} + 2D^{j\alpha}_k H^k_{\xi^\alpha} +$$
$$+ E^j_k H^k_r + 2r^{-1} E^{j\gamma}_k H^k_{\theta^\gamma} + F^j_k H^k = 0 \tag{2.15}$$

where $\Delta_{2\theta}$ denotes the Beltrami operator on the unit sphere and all the coefficients $\in C^\infty$ in (r, ξ, θ). We define the positive form Q by

$$Q = \sum_{j=1}^{N} [(H^j_r)^2 + r^{-2} |\nabla_\theta H^j|^2 + 2C^\gamma H_r H_{\theta^\gamma} + r^{-1} C^{\gamma\delta} H^j_{\theta^\gamma} H^j_{\theta^\delta} +$$
$$+ 2rB^\alpha H^j_{\xi^\alpha} H^j_r + 2B^{\alpha\gamma} H^j_{\xi^\alpha} H^j_{\theta^\gamma} + A^{\alpha\beta} H^j_{\xi^\alpha} H^j_{\xi^\beta}], \tag{2.16}$$

$$F(s) = \int_{F\times\Sigma} \sum_j [H^j(s, \xi, \theta)]^2 \, d\xi \, d\Sigma(\theta). \tag{2.17}$$

Then we see easily that

$$\left.\begin{aligned} F'(s) &= 2 \int_{F\times\Sigma} \sum_j H^j(s, \xi, \theta)\, H^j_r(s, \xi, \theta)\, d\xi\, d\Sigma(\theta) \\ F''(s) &= 2 \int_{F\times\Sigma} \sum_j [H^j H^j_{rr} + (H^j_r)^2]\, d\xi\, d\Sigma \\ F'(0) &= 2 \int_{F\times\Sigma} \sum_j H^j(\xi, 0)\, H^j_{\eta^\alpha}(\xi, 0)\, \eta^\alpha \, d\xi\, d\Sigma = 0. \end{aligned}\right\} \tag{2.18}$$

Using (2.15) to eliminate the H^j_{rr}, integrating by parts with respect to the ξ^α and θ^γ, and using the fact that

$$\int_\Sigma u \Delta_{2\theta} u d\Sigma = - \int_\Sigma |\nabla_\theta u|^2 \, d\Sigma$$

we find that

$$\left.\begin{aligned} F''(s) = 2 \int_{F\times\Sigma} \Big\{ Q &+ \sum_j [-(\nu - 1)\, r^{-1} H^j H^j_r + 2C^\gamma_{\theta^\gamma} H^j H^j_r + \\ &+ r^{-1} C^{\gamma\delta}_{\theta^\delta} H^j H^j_{\theta^\gamma} + 2rB^\alpha_{\xi^\alpha} H^j H^j_r + 2B^{\alpha\gamma}_{\theta^\gamma} H^j H^j_{\xi^\alpha} + \\ &+ A^{\alpha\beta}_{\xi^\alpha} H^j H^j_{\xi^\beta}] - 2\, D^{j\alpha}_k H^j H^k_{\xi^\alpha} - \text{etc.} \Big\}\, d\xi\, d\Sigma(\theta). \end{aligned}\right\} \tag{2.19}$$

Using the positivity of Q and the simple device $|2ab| \leqslant \epsilon a^2 + \epsilon^{-1} b^2$, we conclude from (2.19) that

$$F''(s) \geqslant -(\nu - 1)\, s^{-1} F'(s) - \lambda^2 F(s),$$

where λ depends only on bounds for the coefficients and their derivatives (i.e. on the metric). Thus if R is small enough

$$F(s) \leqslant 2F(R),\ 0 \leqslant s \leqslant R. \tag{2.20}$$

Thus

$$(H, H)_R \leqslant C \int_0^R r^{\nu-1} F(r)\, dr \leqslant CR^\nu F(R) \leqslant CR \int_{bM(R)} \langle H, H \rangle\, dS$$

as desired.

The inequality (2.6) states that the constant C_3 in Theorem 5.9 can be replaced by CR^2 for the manifolds M_R. Thus, from Theorem 5.10 we conclude the following theorem :

THEOREM B′. *If $w \in \mathfrak{D}(L_R^{0,0})$ on M_R and $R \leqslant R_2$, then*

$$\| N_R L_R w \| \leqslant C_5 R^2 \| L_R w \|, \tag{2.21}$$

where $\|\ \ \|$ denotes the norm in $\mathfrak{L}^{(0,0)}$.

We now show how to prove Theorem A using Theorem B′. We first construct, for each R, functions $w_{2R\gamma}$, $\gamma = 1, \ldots, \nu$ which $\in \mathfrak{D}(L_R^{0,0})$, are of class $C^\infty(\overline{M}_R)$, which are analytic at least in the ball $B(P_0, R)$ with

$$\left.\begin{aligned} w_{2R\gamma z\beta}(0) &= \delta_{\gamma\beta} \quad \text{(with respect to } \tau_0) \\ | L_R w_{2R\gamma} | &\leqslant Z_1 R^h \quad (\text{on } \overline{M}_R) \\ h = [\nu/2],\ 0 < R \leqslant R_4 &\leqslant R_2,\ Z_1 \text{ independent of } R. \end{aligned}\right\} \tag{2.22}$$

To do this we first define $w_{1\gamma} = z^\gamma$ in $\overline{B(P_0, R_2)}$ with respect to τ_0, extend $w_{1\gamma}$ to be of class C^∞ on $M_0 \cup \overline{B(P_0, R_2)}$. We then extend $w_{1\gamma}$ into some M_{R_2} using Whitney's extension theorem, assigning the various derivatives of order $\leqslant h + 1$ with respect to the y^α in each complex-analytic patch, real on M_0, in such a way that the Cauchy-Riemann equations and all their derivatives of order $\leqslant h$ hold along M_0. Thus the second condition in (2.22) is satisfied. The functions $w_{2R\gamma}$ are constructed to $\in \mathfrak{D}\,(L_R^{0,0})$ by a method like that of Lemma 6.2 which retains the second condition in (2.22). Then if we set $w_{3R\gamma} = N_R L_R w_{2R\gamma}$, we see from Theorem B′ that

$$\| w_{3R\gamma} \| \leqslant CR^2 \| L_R w_{2R\gamma} \| \leqslant Z_2 R^{h+2+\nu/2}.$$

But also $w_{4R\gamma} = w_{2R\gamma} - w_{3R\gamma}$ is analytic on M_R, so $w_{3R\gamma}$ is analytic on $B(P_0, R)$. From the inequalities of (2.22), it follows that

$$|\nabla w_{3R\gamma}(0)| \leqslant Z_3 R^k, \; k = 1 + (\nu/2) + [\nu/2] - \nu = \tfrac{1}{2} \text{ or } 1.$$

Hence if R is small enough, the gradients of the $w_{3R\gamma}$ are so small that those of the $w_{4R\gamma}(0)$ are linearly independent at 0.

3. Some important formulas. In case $\phi \in \mathfrak{A}^{pq}$, $\bar{\partial}\phi$ was defined to be 0 if $q = \nu$ and was defined in (1.11) otherwise. Starting from that definition, we obtain

$$\begin{aligned}
\bar{\partial}\phi &= \frac{(-1)^p}{q!} \sum_I \sum_{j_1 \dots j_q} \sum_\alpha \phi_{I, j_1 \dots j_q \bar{z}^\alpha}\, dz^I \wedge d\bar{z}^\alpha \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} \\
&= \frac{(-1)^p}{(q+1)!} \sum_I \sum_{(j)\alpha} \phi_{I, j_1 \dots j_q \bar{z}^\alpha} \Big[dz^I \wedge d\bar{z}^\alpha \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} + \\
&\quad + \sum_{\delta=1}^{q} (-1)^\delta\, dz^I \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_\delta} \wedge d\bar{z}^\alpha \wedge d\bar{z}^{j_{\delta+1}} \wedge \dots \wedge d\bar{z}^{j_q} \Big] \\
&= (-1)^p \sum_{I,M} \Big[\sum_{\gamma=1}^{q+1} (-1)^{\gamma-1} \phi_{I, M_\gamma \bar{z}^{m_\gamma}} \Big] dz^I \wedge d\bar{z}^M \;\; (|M| = q+1, q < \nu).
\end{aligned} \tag{3.1}$$

The form (3.1) has the advantage that the coefficients are antisymmetric in all the indices $m_1, \dots, m_{q+1}$.

We shall be concerned with boundary integrals over the manifolds bM_s. We note that the function r defined in §1 is a real-valued function of class $C^\infty(\overline{M})$ such that $|\nabla r|$ (as measured on M) $= 1$ near bM, $r = 0$ on bM, and $r < 0$ on M near bM. If we let $dM(P)$ denote the volume element and $dS(P)$ denote the surface element along some bM_s at P, then

$$\left.\begin{aligned}
dM(P) &= \Gamma(x, y)\, dx dy, \; dM(P) = dS(P)\, dr(P) \\
dS(P) &= |\nabla r(x, y)|^{-1}\, \Gamma(x, y)\, dS(x, y),
\end{aligned}\right\} \tag{3.2}$$

(x, y) corresponding to P in some coordinate patch; here $|\nabla r(x, y)|$ denotes the gradient of r with respect to the coordinates (x, y) and $dS(x, y)$ denotes the surface element of the surface of integration in the (x, y)-space. Let G be the domain of an analytic coordinate patch having range $\mathfrak{N}$ such that $\mathfrak{N} \cap bM$ is not empty and let g be the part of bG which corresponds to $\mathfrak{N} \cap bM$. Then, if ϕ or ψ vanishes on and near $bG - g$, we note that

$$\int_G \phi\psi_{z\beta}\Gamma(x, y)dxdy = \int_g \phi\psi \,.\, r_{z\beta} \,.\, |\nabla r(x, y)|^{-1}\, \Gamma(x, y)\, dS(x, y) - \\ - \int_G \psi[\Gamma^{-1} \frac{\partial}{\partial z^\beta} \Gamma\phi] \,.\, \Gamma\, dxdy$$

and the integral over g can be expressed in the form

$$\int_{\mathfrak{N} \cap bM} \phi\psi\, r_{z\beta}\, dS(P). \tag{3.4}$$

Next, suppose that $\phi \in \mathfrak{A}^{pq}$, $\psi \in \mathfrak{A}^{p,q-1}$, $q \geqslant 1$, ϕ is given by (1.6), and ψ is given by a similar formula. Then, using (3.1), (1.10), and the antisymmetry in the indices, we obtain

$$\left.\begin{aligned} \langle \phi, \bar{\partial}\psi \rangle &= \frac{1}{q!} \sum_{I,K} \sum_{(j),(m)} g^{KI} g^{j_1 m_1} \ldots g^{j_q m_q} \times \\ &\quad \times \sum_{\gamma=1}^{q} (-1)^{p+\gamma-1} \phi_{I,j_1\ldots j_q} \bar{\psi}_{K,m_\gamma z m_\gamma} \\ &= \sum_{I,K} \sum_{R,L} \sum_{\alpha,\beta} (-1)^p \bar{\psi}_{KLz\beta}\, g^{KI} g^{RL} g^{\alpha\beta} \phi_{I,\alpha R}. \end{aligned}\right\} \tag{3.5}$$

If we cover M with coordinate patches and use a partition of unity $\{\zeta_s\}$, each with support in one patch, let $\phi_s = \zeta_s \phi$, integrate by parts using (3.3), (3.4), and (3.5), and add up, we obtain

$$(\phi, \bar{\partial}\psi) = \int_{bM} \sum_{I,K} \sum_{R,L} g^{KI} g^{RL}\, \omega_{IR}\, \bar{\psi}_{KL}\, dS(P) + (\mathfrak{d}\phi, \psi) \tag{3.6}$$

where $\omega = \nu\phi$ is given by (1.15) and $\mathfrak{d}\phi$ is given by (1.13) where the $A^{ST}_{IR\beta}$ are determined so that

$$(\mathfrak{d}\phi, \psi) = \int_M \Sigma g^{KI}\, g^{RL}\, (\mathfrak{d}\phi)_{IR}\, \overline{\psi}_{KL}\, dM(P)$$

$$= (-1)^{p+1} \int_M \overline{\psi}_{KL} \left\{ \Gamma^{-1} \frac{\partial}{\partial z^\beta} (\Gamma g^{KI}\, g^{RL}\, g^{\alpha\beta}\, \phi_{I,\alpha R}) \right\} dM(P) \tag{3.7}$$

for all $\psi \in \mathfrak{A}^{p,q-1}$.

Now, let ϕ and $\psi \in \mathfrak{A}^{pq}$; we wish to develop an important formula for

$$d(\phi, \psi) = (\mathfrak{d}\phi, \mathfrak{d}\psi) + (\overline{\partial}\phi, \overline{\partial}\psi). \tag{3.8}$$

Clearly, we may write ϕ and ψ each as sums of forms each having compact support and such that if the supports of $\phi^{(u)}$ and $\psi^{(v)}$ intersect, their union lies in one coordinate patch. So we assume ϕ and ψ have supports in one patch. Suppose ϕ and ψ are given by (1.6), $q < \nu$, and $\rho = \overline{\partial}\phi$, $\sigma = \overline{\partial}\psi$, then ρ and σ are given by formulas like (3.1), so

$$(\overline{\partial}\phi, \overline{\partial}\psi) = \frac{1}{(q+1)!} \int_M \sum_{K,I} \sum_{\substack{m_1 \dots m_{q+1} = 1 \\ r_1 \dots r_{q+1} = 1}}^{\nu} \sum_{\gamma,\delta=1}^{q+1} (-1)^{\gamma+\delta} g^{KI} \times$$

$$\times\, g^{m_1 r_1} \dots g^{m_{q+1} r_{q+1}}\, \phi_{Im'_\gamma \bar{z}^{m_\gamma}}\, \overline{\psi}_{K,r'_\delta z^{r_\delta}}\, dM = I_1 + I_2 \tag{3.9}$$

where I_1 is the part of the sum where $\delta = \gamma$ and I_2 is the remainder. We obtain†

$$I_1 = \int \Sigma g^{KI}\, g^{JL}\, g^{\alpha\beta}\, \phi_{IJ\bar{z}^\alpha}\, \overline{\psi}_{KLz^\beta}\, dM. \tag{3.10}$$

Using the antisymmetry of the indices we see that

$$\phi_{I,m'_\gamma \bar{z}^{m_\gamma}}\, \overline{\psi}_{K,r'_\delta z^{r_\delta}} = (-1)^{\gamma+\delta-1}\, \phi_{I,m_\delta m'_{\gamma\delta} \bar{z}^{m_\gamma}}\, \overline{\psi}_{K,r_\gamma r'_{\gamma\delta} z^{r_\delta}}$$

where $m'_{\gamma\delta}$ denotes the m sequence with m_γ and m_δ both omitted, etc. Thus, we obtain

$$I_2 = -\int_M \Sigma g^{KI}\, g^{ST}\, g^{\alpha\beta}\, g^{\gamma\delta}\, \phi_{I\gamma S\bar{z}^\alpha}\, \overline{\psi}_{K,\beta T z^\delta}\, dM. \tag{3.11}$$

Using (3.7), we thus obtain (interchanging (α, γ) and (β, δ) in (3.11))

† Strictly speaking, the integrand in I_1 is not invariant under changes of coordinates ; however the final result in (3.15) is invariant.

$$
\begin{aligned}
d(\phi, \psi) &= I_1 + \int \Sigma g^{KI} g^{ST} g^{\alpha\beta} g^{\gamma\delta}[- \phi_{I,\alpha s\bar{z}^\gamma} \bar{\psi}_{K,\delta T z^\beta} + \\
&\qquad + (\phi_{I,\alpha s z^\beta} + \Sigma A^{uv}_{Is\beta} \phi_{U,\alpha V}) (\bar{\psi}_{K,\delta T \bar{z}^\gamma \delta} + \Sigma \bar{A}^{WX}_{KT\gamma} \bar{\psi}_{w,\delta X})] \Gamma \, dx dy \\
&= I_1 + I_3. \qquad (3.12)
\end{aligned}
$$

Now we define the forms $'\chi$ and $'\omega \in \mathfrak{A}^{p,q-1}$ by

$$
'\chi = \Sigma' \chi_{Is} \, dz^I \wedge d\bar{z}^s = (-1)^p \nu \phi, \; '\omega = (-1)^p \nu\psi = \Sigma' \omega_{KT} \, dz^K \wedge d\bar{z}^T \qquad (3.13)
$$
$$
'\chi_{Is} = g^{\alpha\beta} r_{z\beta} \phi_{I,\alpha s}, \; '\omega_{KT} = g^{\delta\gamma} r_{z\gamma} \psi_{K\delta T}.
$$

Next, we note that

$$
\begin{aligned}
&2(\phi_{I,\alpha s z^\beta} \bar{\psi}_{K,\delta T \bar{z}^\gamma} - \phi_{I,\alpha s \bar{z}^\gamma} \bar{\psi}_{K,\delta T z^\beta}) \qquad (3.14) \\
&= \frac{\partial}{\partial z^\beta} (\phi_{I,\alpha s} \bar{\psi}_{K,\delta T \bar{z}^\gamma} - \phi_{I,\alpha s \bar{z}^\gamma} \bar{\psi}_{K,\delta T}) + \frac{\partial}{\partial \bar{z}^\gamma} (\phi_{I,\alpha s z^\beta} \bar{\psi}_{K,\delta T} - \phi_{I,\alpha s} \bar{\psi}_{K,\delta T z^\beta}) \\
&g^{\alpha\beta} r_{z\beta} \phi_{I,\alpha s \bar{z}^\gamma} = {'\chi}_{Is\bar{z}^\gamma} - (g^{\alpha\beta} r_{z\beta\bar{z}^\gamma} + g^{\alpha\beta}_{\bar{z}^\gamma} r_{z\beta}) \phi_{I,\alpha s} \\
&g^{\gamma\delta} r_{\bar{z}\gamma} \psi_{K,\delta T z^\beta} = {'\bar{\omega}}_{KTz^\beta} - (g^{\gamma\delta} r_{z\beta\bar{z}^\gamma} + g^{\gamma\delta}_{z^\beta} r_{\bar{z}\gamma}) \bar{\psi}_{K,\delta T}.
\end{aligned}
$$

We then integrate I_3 by parts until there are no terms like $\phi_{U,\alpha V z^\beta}$ and $\psi_{w,\delta X \bar{z}^\gamma}$ in the remaining integral over M. The result is of the form

$$
\left.
\begin{aligned}
d(\phi, \psi) = &\int_M \Sigma \{ g^{KI} g^{JL} g^{\alpha\beta} \phi_{IJ\bar{z}^\alpha} \bar{\psi}_{KLz^\beta} + C^{IJ,KL\beta} \phi_{IJ} \bar{\psi}_{KLz^\beta} + \\
&+ \bar{C}^{KL,IJ\alpha} \phi_{IJ\bar{z}^\alpha} \bar{\psi}_{KL} + D^{IJ,KL} \phi_{IJ} \bar{\psi}_{KL} \} \, dM + \\
&+ \tfrac{1}{2} \int_{bM} \Sigma g^{KI} g^{ST} \{ '\chi_{Is} [\epsilon(\mathfrak{d}\psi)_{KT} + \bar{B}^{UV\delta}_{KT} \bar{\psi}_{U,\delta V}] + \\
&+ {'\bar{\omega}}_{KT} [\epsilon(\mathfrak{d}\phi)_{Is} + B^{uv\alpha}_{Is} \phi_{U,\alpha V}] - \\
&- g^{\gamma\delta} \, '\chi_{Is\bar{z}^\gamma} \bar{\psi}_{K,\delta T} - g^{\alpha\beta} \, '\bar{\omega}_{KTz^\beta} \phi_{I,\alpha s} + \\
&+ 2 r_{z\beta\bar{z}^\gamma} g^{\alpha\beta} g^{\gamma\delta} \phi_{I,\alpha s} \bar{\psi}_{K,\delta T} \} \, ds
\end{aligned}
\right\} \qquad (3.15)
$$

$$
(\epsilon = (-1)^{p+1}).
$$

In the case $q = \nu$, $\bar{\partial}\phi = \bar{\partial}\psi = 0$; a special computation leads to the result (3.15) in this case. In the case $q = 0$, $\mathfrak{d}\,\phi = \mathfrak{d}\,\psi = 0 = {'\chi} = {'\omega}$

and (3.15) holds without a boundary integral. In the case $q = \nu$ it follows that $\nu\phi = 0$ on $bM \Longleftrightarrow$ the components of $\phi = 0$ on bM.

From the result (3.6) it follows that $\phi \in \mathfrak{A} \cap \mathfrak{D} \Longleftrightarrow \nu\phi = 0$ on bM. But now if ϕ and $\psi \in \mathfrak{A} \cap \mathfrak{D}$, the boundary integral in (3.15) is seen to reduce to that of the last term since $'\chi$ and $'\omega$ vanish on bM and hence

$$'\chi_{Is\bar{z}\gamma} = \lambda_{Is}\, r_{\bar{z}\gamma} \text{ and } '\overline{\omega}_{KTz\beta} = \overline{\mu}_{KT}\, r_{z\beta} \text{ on } bM$$

for suitable functions λ_{Is} and μ_{KT}. From our hypothesis of the pseudo-convexity of bM, it follows that

$$r_{z\beta\bar{z}\gamma}\, \tau^\beta\, \bar{\tau}^\gamma \geqslant C\, g_{\beta\gamma}\, \tau^\beta \bar{\tau}^\gamma,\ C > 0, \text{ whenever} \qquad (3.16)$$

$$r_{z\beta}\, \tau^\beta = 0.$$

Consequently, if $\psi = \phi$ and $\phi \in \mathfrak{A} \cap \mathfrak{D}$, then

$$\int_{bM} g^{KI}\, g^{ST}\, r_{z\beta\bar{z}\gamma}\, g^{\alpha\beta}\, g^{\gamma\delta}\, \phi_{I,\alpha s}\, \overline{\phi}_{K,\delta T}\, ds$$

$$\geqslant c \int_{bM} g^{KI}\, g^{ST}\, g^{\alpha\delta}\, \phi_{I,\alpha s}\, \overline{\phi}_{K,\delta T}\, ds = C \int_{bM} |\phi|^2\, ds. \qquad (3.17)$$

Now, for forms ϕ having support in the range $\mathfrak{R}$ of some coordinate patch, it is sometimes desirable to introduce a new basis

$$\zeta^1, \ldots, \zeta^\nu, \bar{\zeta}^1, \ldots, \bar{\zeta}^\nu$$

for the 1-forms, given by

$$\left.\begin{aligned} \zeta^\alpha &= c^\alpha_\gamma\, dz^\gamma,\ dz^\gamma = d^\gamma_\alpha\, \zeta^\alpha \\ \bar{\zeta}^\alpha &= \bar{c}^\alpha_\gamma\, d\bar{z}^\gamma,\ d\bar{z}^\gamma = \bar{d}^\gamma_\alpha\, \bar{\zeta}^\alpha \end{aligned}\right\} \qquad (3.18)$$

and to introduce the corresponding differential operators

$$\left.\begin{aligned} u_{,\gamma} &= d^\alpha_\gamma\, u_{z\alpha},\ u_{z\alpha} = c^\gamma_\alpha\, u_{,\gamma}, \\ u_{,\bar{\gamma}} &= \bar{d}^\alpha_\gamma\, u_{\bar{z}\alpha},\ u_{\bar{z}\alpha} = \bar{c}^\gamma_\alpha\, u_{,\bar{\gamma}}, \end{aligned}\right\} \qquad (3.19)$$

where of course the matrices (c^α_γ) and (d^γ_α) are inverses of one another as are the matrices $(\bar{c}^\alpha_\gamma)$ and (d^γ_α). The exterior multiplication allows us to express any form in terms of the ζ^γ and $\bar{\zeta}^\gamma$:

$$\left.\begin{aligned}\phi &= \frac{1}{p!\,q!}\sum_{\substack{i_1\ldots i_p\\ j_1\ldots j_q}} \tilde{\phi}_{i_1\ldots i_p j_1\ldots j_q}\, dz^{i_1}\wedge\ldots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\ldots\wedge d\bar{z}^{j_q}\\ &= \frac{1}{p!\,q!}\sum \tilde{\phi}_{i_1\ldots i_p j_1\ldots j_q}\, d^{i_1}_{k_1}\ldots d^{i_p}_{k_p}\,\bar{d}^{j_1}_{l_1}\ldots\bar{d}^{j_q}_{l_q}\,\zeta^{k_1}\wedge\ldots\wedge\zeta^{k_p}\wedge\\ &\qquad\qquad\wedge\bar{\zeta}^{l_1}\wedge\ldots\wedge\bar{\zeta}^{l_q}.\end{aligned}\right\}\quad(3.20)$$

In case the bases are introduced so that

$$g^{\alpha\beta}\,\bar{c}^{\gamma}_{\alpha}\,c^{\delta}_{\beta} = \Delta^{\gamma\delta} \text{ and } g_{\alpha\beta}\,d^{\alpha}_{\gamma}\,\bar{d}^{\beta}_{\delta} = \Delta_{\gamma\delta},\quad \gamma,\delta = 1,\ldots,\nu, \qquad (3.21)$$

we see that

$$\left.\begin{aligned}\langle\phi,\psi\rangle &= \sum_{I,J}\phi_{IJ}\,\bar{\psi}_{IJ}, \text{ if}\\ \phi = \sum_{I,J}\phi_{IJ}\,\zeta^I\wedge\bar{\zeta}^J,\ \psi &= \sum_{I,J}\psi_{IJ}\,\zeta^I\wedge\bar{\zeta}^J.\end{aligned}\right\}\quad(3.22)$$

We call a basis in which the c^{γ}_{α} satisfy (3.21) an *orthogonal basis*. Such bases were used by Kohn [8].

In terms of such a basis, we see that

$$\left.\begin{aligned}\bar{\partial}\phi &= \sum\chi_{IM}\,\zeta^I\wedge\bar{\zeta}^M,\ \mathfrak{d}\phi = \sum\rho_{IR}\,\zeta^I\wedge\bar{\zeta}^R\\ \chi_{IM} &= \sum_{\gamma=1}^{q+1}(-1)^{p+\gamma-1}\,\phi_{I,M_\gamma,\bar{m}_\gamma} + \sum B^{KT}_{IM}\,\phi_{KT}\\ \rho_{IR} &= (-1)^{p+1}\sum_{\alpha}\phi_{I,\alpha R,\alpha} + \sum A^{UV\alpha}_{IR}\,\phi_{U,\alpha V}\end{aligned}\right\}\quad(3.23)$$

where A's and B's are suitable C^∞ functions.

Such bases are more useful in boundary neighborhoods $\mathfrak{N}$ (in which $\mathfrak{N}\cap bM$ is not empty). In case σ is a tangential analytic coordinate patch with domain $G\cup g$ and range $\mathfrak{N}$, we may, by taking a smaller $\mathfrak{N}$ if necessary, choose an orthogonal basis ζ such that

$$2ig^{\alpha\beta}\,r_{z^\beta} = \bar{d}^{\alpha}_{\nu}. \qquad (3.24)$$

It is also possible, choosing $\mathfrak{N}$ smaller if need be, to introduce non-analytic boundary coordinates (t, r), $t = (t^1,\ldots,t^{2\nu-1})$ of class C^∞ which range over some $G_R\cup\sigma_R$ and which are such that the metric takes the form

$$ds^2 = \sum_{\lambda,\mu=1}^{2\nu-1} a_{\lambda\mu}(t, r)\, dt^\lambda\, dt^\mu + dr^2,\ a_{\lambda\mu}(0, 0) = \delta_{\lambda\mu}. \tag{3.25}$$

Now, since the basis is complex-orthogonal, each

$$u_{,\gamma} = ('D^\gamma u - i\,''D^\gamma u)/2 \text{ and } u_{,\bar\gamma} = ('D^\gamma u + i\,''D^\gamma u)/2,$$

where $'D^\gamma$ and $''D^\gamma$ are real operators which are, in fact, directional derivatives along real unit vectors e'_γ and e''_γ in which all 2ν are mutually orthogonal and $e''_\nu = \nabla r$. Thus, in terms of the (t, r) coordinates

$$\left.\begin{aligned} u_{,\gamma} &= \sum_\lambda e^\lambda_\gamma\, u_{t\lambda},\ \gamma < \nu\ (e^\lambda_\gamma \text{ complex});\\ u_{,\nu} &= \sum_\lambda e^\lambda_\nu\, u_{t\lambda} - (i/2)\, u_r\ (e^\lambda_\nu \text{ real});\\ u_{,\bar\gamma} &= \sum_\lambda \bar e^\lambda_\gamma\, u_{t\lambda},\ \gamma < \nu;\\ u_{,\bar\nu} &= \sum_\lambda e^\lambda_\nu\, u_{t\lambda} + (i/2)\, u_r. \end{aligned}\right\} \tag{3.26}$$

Finally, by using the relations implied by (3.20) between the components with respect to the $(dz^\alpha, d\bar z^\alpha)$ basis and the orthogonal $(\zeta^\alpha, \bar\zeta^\alpha)$ basis, we find easily that

$$\nu\phi = \sum \omega_{IR}\, \zeta_I \wedge \bar\zeta_R,\ \omega_{IR} = (-1)^p\, (-i/2)\, \phi_{I,\nu R} \tag{3.27}$$

provided the $(\zeta^\alpha, \bar\zeta^\alpha)$ basis satisfies (3.24) on the boundary neighborhood $\mathfrak{N}$.

4. Some regularity theorems. In this section we shall consider an equation of the form

$$\left.\begin{aligned} &u_{yy}(x, y) + \sum_{\alpha,\beta=1}^{\nu} a^{\alpha\beta}(x, y)\, u_{,\alpha\beta}(x, y)\\ &= \sum_{\alpha=1}^{\nu} f^\alpha_{.\alpha}(x, y) + f_y(x, y) + g(x, y)\ (x = x^1, \ldots, x^\nu,\ y = y) \end{aligned}\right\} \tag{4.1}$$

in which the coefficients $a^{\alpha\beta}$ are real and of class C^∞ for all (x, y) with $-R < y \leqslant 0$ and are periodic of period $2R$ in each x^α and we are looking for solutions u which are periodic of period $2R$ in each x^α and we assume this to be true of the f^α, f and g. Actually, we assume that the $a^{\alpha\beta}$ depend on R, as do the other functions, and satisfy

$$\left.\begin{aligned} &|\nabla^p \epsilon^{\alpha\beta}(x, y)| \leqslant K_p R^{1-p}, p = 0, 1, 2, \ldots \\ &\epsilon^{\alpha\beta}(0, 0) = 0, \epsilon^{\alpha\beta}(x, y) = \delta^{\alpha\beta} - a^{\alpha\beta}(x, y). \end{aligned}\right\} \tag{4.2}$$

NOTATIONS. In this chapter σ_R denotes the hypercube $|x^\alpha| < R$, $\alpha = 1, \ldots, \nu$, $G_{R,\epsilon} \equiv G(R, \epsilon)$ denotes the cell $x \in \sigma_R$, $-R < y < -\epsilon$, and $G_R = G_{R,0}$.

We shall be interested in functions f^α, f, and g and solutions u of (4.1) which vanish near $y = -R$ and $x^\alpha = \pm R$, are of class C^∞ for $-R \leqslant y < 0$, and $\in L_2(G_R)$ and we wish to prove further smoothness properties near $\mu = 0$. In order to do this, we shall use many Hilbert spaces of functions which are periodic of period $2R$ in each x^α. The spaces $H_2^m(G_R)$ shall have their usual significance, except that we shall assume that the functions are periodic and $\in H_2^m$ in any cell $-R < y < 0$ (or $-\epsilon$) and $|x^\alpha| <$ any A. The space $H_{20}^m(G_R)$ will denote the subspace of $H_2^m(G_R)$ of functions u such that

$$\lim_{y\to(-R)+} \int_{\sigma_R} u(x, y)\, \bar{v}(x. y)\, dx = 0, \lim_{R\to 0-} \int_{\sigma_R} u(x, y)\, \bar{v}(x, y)\, dx = 0 \tag{4.3}$$

for every $v \in C^\infty$ for $-R \leqslant y \leqslant 0$ (and periodic).

We also define $\mathfrak{H}^s(\sigma_R)$ as the set of u in $L_2(\sigma_R)$ for which

$$(\|u\|^s_{\sigma(R)})^2 = \sum_m (1 + |m|^2)^s\, |u_m|^2 < \infty \tag{4.4}$$

where here and below

$$u_m = (2R)^{-\nu} \int_{\sigma_R} u(x)\, e^{-i\pi m\cdot x/R}\, dx, \left(u(x) = \sum_m u_m\, e^{i\pi m\cdot x/R} \right). \tag{4.5}$$

We also define the space $\mathfrak{H}^s(G_R)$ as the set of u in $L_2(G_R)$ for which

$$(\|u\|^s_{G(R)})^2 \equiv R^{-1} \int_{-R}^{0} \sum_m (1 + |m|^2)^s\, |u_m(y)|^2\, dy < \infty. \tag{4.6}$$

The space $\mathfrak{H}_0^s(G_R)$ is the subspace of $\mathfrak{H}^s(G_R)$ of functions u satisfying (4.3). The spaces $\mathfrak{H}^s(G_{R,\epsilon})$ and $\mathfrak{H}_0^s(G_{R,\epsilon})$ can be defined in the corresponding way.

LEMMA 4.1. *If* $u \in \mathfrak{H}^s(G_R)$ $(\mathfrak{H}^s(\sigma_R))$ *and* $s \geqslant k$, *where* k *is an integer* $\geqslant 0$. *then* $\nabla_x^k u \in \mathfrak{H}^{s-k}(G_R)$ $(\mathfrak{H}^{s-k}(\sigma_R))$ *and*

$$\| \nabla_x^k u \|^{s-k} \leqslant \pi^k R^{-k} \| u \|^s$$

where the norm is on G_R *or* σ_R, *respectively.*

PROOF. This is evident, since

$$D_x^\alpha u(x, y) = \sum_m (i\pi)^{|\alpha|} R^{-|\alpha|} m^\alpha u_m(y) e^{i\pi m \cdot x/R}.$$

THEOREM 4.1. *Suppose* H *is harmonic and periodic of period* $2R$ *in each* x^α *for* $-R < y < 0$ *and suppose* $H \in \mathfrak{H}^0(G_R)$.

(a) *If the first limit in* (4.3) *holds, then* H, *as extended across* $y = -R$ *by reflection, is harmonic for* $-2R < y < 0$; *if the second limit holds,* H *can be extended similarly to be harmonic for* $|y| < R$. *If both limits hold,* $H \equiv 0$.

(b) *If* $H(x, -R) = 0$ *and* $H(x, y) \in \mathfrak{H}^s(\sigma_R)$ *with* $\| H(.\,, y) \|_{\sigma_R}^s \leqslant L$ *uniformly for some* $s \geqslant 0$, *then* $H \in \mathfrak{H}^{s+1/2}(G_R)$ *and* $H(.\,, y)$ *tends to a limit* $H(.\,, 0)$ *in* $\mathfrak{H}^s(\sigma_R)$ *as* $y \to 0^-$. *Moreover*

$$\| H \|_{G(R)}^{s+1/2} \leqslant CL \text{ if } s \geqslant 0,\; R.\, \| H \|_{G(R)}^{s-1/2} \leqslant CL \text{ if } s \geqslant 1/2, \quad (4.7)$$

where C *is an absolute constant. The corresponding results hold for* $G_{R,\epsilon}$.

PROOF. Expand H into a Fourier series (4.5) with coefficients $H_m(y)$. Then the H_m are each analytic in y for $-R < y < 0$ and satisfy

$$H''_m(y) - \pi^2 |m|^2 R^{-2} H_m(y) = 0. \quad (4.8)$$

Thus if the first limit in (4.3) holds, we see that

$$H_0(y) = c_0 R^{-1}(y + R)$$

$$H_m(y) = c_m(\sinh \pi |m|)^{-1} \sinh \pi |m| (y + R)/R, \; m \neq 0. \quad (4.9)$$

The results in (a) follow easily. Now

$$(\|H(\,.\,,y)\|^s_{\sigma_R})^2 = \sum_m (1+|m|^2)^s\,|H_m(y)|^2. \tag{4.10}$$

Evidently (4.10) is $\leqslant L^2$ for all $y<0$ if and only if

$$(\|H(\,.\,,0)\|^s_{\sigma_R})^2 = \sum_m (1+|m|^2)^s\,|c_m|^2 \leqslant L^2. \tag{4.11}$$

It follows easily from (4.9), (4.10), and (4.11) that $H(.,y) \to H(.,0)$ in $\mathfrak{H}^s(\sigma_R)$ and that (4.7) holds.

THEOREM 4.2. *If $g \in H^0(G_R)$, there exists a unique solution $u \in H^2_{20}(G_R)$ of $\Delta u = g$. If $g \in \mathfrak{H}^s(G)$ $(s \geqslant 0)$ then $u \in \mathfrak{H}^{s+2}_0(G_R)$, $\nabla u \in \mathfrak{H}^{s+1}$,*

$$\|u\|^{s+2}_{G(R)} \leqslant C_1 R^2 \|g\|,\quad \|\nabla u\|^{s+1}_{G(R)} \leqslant C_2 R \|g\|,\quad \|\nabla^2 u\|^s_{G(R)} \leqslant \|g\|,$$
$$\|g\| = \|g\|^s_{G(R)},$$

C_1 and C_2 being absolute constants.

If $s \geqslant 1$, then $u_{,\alpha}$ is the solution in $\dot{H}^{s+1}_0(G)$ of $\Delta u_{,\alpha} = g_{,\alpha}$. Corresponding results hold for $G_{R,\epsilon}$.

PROOF. If g is given by a finite Fourier series (4.5) with coefficients g_m continuous in y for $-R \leqslant y \leqslant 0$, then the finite series for u in which $u_m(y)$ is given by

$$\left.\begin{aligned} u_m(y) &= R^2\, u^*_m(y/R),\ g_m(y) = g^*_m(y/R) \\ u^*_m(\eta) &= \int_{-1}^{0} K_m(\eta, t)\, g^*_m(t)\, dt \end{aligned}\right\} \tag{4.12}$$

where

$$\left.\begin{aligned} K_0(\eta,t) &= \begin{cases} \eta(t+1),\ -1 < t < \eta, \\ (\eta+1)\,t,\ \eta < t < 0, \end{cases} \\ (k\sinh k)\,K_m(\eta,t) &= \begin{cases} \sinh k(t+1)\sinh k\eta,\ -1 < t < \eta \\ \sinh k(\eta+1)\sinh kt,\ \eta \leqslant t \leqslant 0 \end{cases} \\ & (k = \pi\,|m|), \end{aligned}\right\} \tag{4.13}$$

is seen to be a solution. We also have

$$u_m^{*''}(\eta) - \pi^2 |m|^2 u_m^*(\eta) = g_m^*(\eta).$$

To show that $\nabla^2 u \in \mathfrak{H}^s(G_R)$ we observe that

$$u_{,\alpha\beta} = \sum - \pi^2 m_\alpha m_\beta u_m^*(\eta)\, e^{i\pi m.x/R},$$

$$u_{y,\alpha} = \sum i\pi m_\alpha u_m^{*\prime}(\eta)\, e^{i\pi m.x/R},$$

$$u_{yy} = \sum u^{*''}(\eta)\, e^{i\pi m.x/R}.$$

Accordingly we conclude that

$$\sum_{\alpha,\beta} (\|u_{,\alpha\beta}\|^s)^2 + 2\sum (\|u_{y,\alpha}\|^s)^2 + (\|u_{yy}\|^s)^2$$

$$= \int_{-1}^{0} \sum_m (1+|m|^2)^s [|u_m^{*''}(\eta)|^2 +$$

$$+ 2\pi^2 |m|^2 |u_m^{*\prime}(\eta)|^2 + \pi^4 |m|^4 |u_m^*(\eta)|^2]\, d\eta$$

$$= \int_{-1}^{0} \sum_m (1+|m|^2)^s |u_m^{*''} - \pi^2 |m|^2 u_m^*|^2\, d\eta$$

$$= R^{-1} \int_{-R}^{0} \sum_m (1+|m|^2)^s |g_m(y)|^2\, dy = (\|g\|^s_{G(R)})^2. \qquad (4.14)$$

The equality between the first and second integrals uses the fact that $d_m^*(\eta) = 0$ for $\eta = -1$ and $\eta = 0$.

The fact that $u \in H^{s+2}(G_R)$, etc. follows from (4.14) and the fact that

$$R^{-1} \int_{-R}^{0} |u_0(y)|^2\, dy$$

$$= \int_{-1}^{0} |u_0^*(\eta)|^2\, d\eta < \left[\int_{-1}^{0} \int_{-1}^{0} K_0^2(\eta, t)\, d\eta\, dt \right] \cdot \int_{-1}^{0} |g_0^*(t)|^2\, dt.$$

The other inequalities are proved similarly. One may pass to the limit if the series for g is not finite.

The solution is unique since the difference between two solutions would be a harmonic function satisfying (4.3).

THEOREM 4.3. *If $f \in \mathfrak{H}^0(G_R)$, there is a unique solution $u \in H^1_{20}(G_R)$ of the equation*

$$\int_{G_R} [\bar v_{,\alpha}\, u_{,\alpha} + \bar v_y(u_y - f)]\, dx\, dy = 0,\ v \in H^1_{20}(G_R). \tag{4.15}$$

In case $f \in C^\infty$ for $-R < y < 0$, then $\Delta u = f_y$ there. If $f \in \mathfrak{H}^s(G_R)$, then $u \in \mathfrak{H}^{s+1}_0(G_R)$ and $\nabla u \in \mathfrak{H}^s_0(G_R)$ and

$$\| u \|^{s+1}_{G(R)} \leqslant C_1 R\, \| f \|^s_{G(R)},\ \| \nabla u \|^s_{G(R)} \leqslant C_2\, \| f \|^s_{G(R)}, \tag{4.16}$$

C_1 and C_2 being absolute constants. Corresponding results hold on $G_{R,\epsilon}$.

PROOF. The solution is clearly unique. If f is smooth for $-R \leqslant y \leqslant 0$ and is given by a finite series, the equation $\Delta u = f_y$ may be solved as in the proof of Theorem 4.2. We obtain

$$\left.\begin{aligned} u_m(y) &= R u^*_m(\eta),\ f_m(y) = f^*_m(\eta),\ \eta = R^{-1} y \\ u^*_m(\eta) &= \int_{-1}^{0} K_m(\eta, t)\, f^{*\prime}_m(t)\, dt = - \int_{-1}^{0} K_{mt}(\eta, t)\, f^*_m(t)\, dt \end{aligned}\right\} \tag{4.17}$$

since K_m satisfies a Lipschitz condition in t for each η. Since $K_m(-1, t) = K_m(0, t) = 0$, we see that $K_{mt}(-1, t) = K_{mt}(0, t) = 0$. In the last form in (4.17), we may pass to the limit.

In the case of the smooth f with a finite series, we obtain

$$\int_{-1}^{0} u^{*\prime}_m(\eta) . \bar u^{*\prime}_m(\eta)\, d\eta = - \operatorname{Re} \int_{-1}^{0} \bar u^*_m(\eta) u^{*\prime\prime}_m(\eta)\ d\eta$$

$$= - \operatorname{Re} \int_{-1}^{0} \bar u^*_m(\eta)\, [f^{*\prime}_m(\eta) + \pi^2\, | m |^2\, u^*_m(\eta)]\, d\eta \leqslant \operatorname{Re} \int_{-1}^{0} f^*_m\, \bar u^{*\prime}_m\, d\eta. \tag{4.18}$$

From (4.18), we obtain the second result in (4.16) by using the Schwarz inequality, squaring, multiplying by $(1 + | m |^2)^s$, and summing. The first result in (4.16) may also be obtained from (4.18) and the first result except for the term $m = 0$ which can be obtained from Poincaré's inequality.

THEOREM 4.4. *Suppose the $a^{\alpha\beta}$ satisfy* (4.2) *and suppose $u, f^\alpha, f,$ and $g \in C^\infty$ for $-R < y < 0$, are periodic of period $2R$ in each x^α, $\in L_2(G_R)$, and suppose u satisfies* (4.3).

Then $u \in H^1_{20}(G_R)$ (i.e. $u \in H_2$, vanishes for $y = 0$ and $-R$ and is periodic of period $2R$ in each x^α).

PROOF. We begin by writing the equation (4.1) in the form

$$\Delta u = \epsilon^{\alpha\beta}\, u_{,\alpha\beta} + \sum_\alpha f^\alpha_{,\alpha} + f_y + g, \quad (\epsilon^{\alpha\beta} = \delta^{\alpha\beta} - a^{\alpha\beta}). \tag{4.19}$$

We first assume that $f^\alpha, f,$ and g are given by finite Fourier series. Then we can proceed as in the proof of Theorem 4.2 to find the solution v of

$$-\Delta v + f^\alpha_{,\alpha} + f_y + g = 0. \tag{4.20}$$

Multiplying (4.20) by $\bar{v}$ and integrating by parts, we obtain

$$\int_{G_R} |\nabla v|^2\, dxdy = \int_{G_R} (f^\alpha\, \bar{v}_{,\alpha} + f\bar{v}_y - g\bar{v})\, dxdy$$

from which we conclude that

$$\int_{G_R} |\nabla v|^2\, dxdy \leqslant 2 \int_{G_R} \left[\sum_\alpha |f^\alpha|^2 + |f|^2 + R^2\, |g|^2 \right] dxdy. \tag{4.21}$$

Under our assumption that f^α, f and $g \in C^\infty$ for $-R < y < 0$, we see that the Fourier series for them and for v converge nicely so we may pass to the limit and conclude that the limiting v satisfies (4.20) and (4.21) and hence that $v \in H^1_{20}(G_R)$.

Next we notice that we may write

$$\epsilon^{\alpha\beta} u_{,\alpha\beta} = \sum_{\alpha,\beta} [\, (\epsilon^{\alpha\beta}\, u)_{,\alpha\beta} - 2(a^{\alpha\beta}_{,\beta}\, u)_{,\alpha} + (a^{\alpha\beta}_{,\alpha\beta})\, u]. \tag{4.22}$$

For each $u \in L_2(G_R)$ we define the operator T_R by

$$T_R\, u = U = \sum_{\alpha,\beta} V^{\alpha\beta}_{,\alpha\beta} - 2 \sum_\alpha V^\alpha_{,\alpha} + V \tag{4.23}$$

where $V^{\alpha\beta}$, V^α and V are the respective solutions of Theorem 4.2 for $\epsilon^{\alpha\beta} u$, $\sum_\beta a^{\alpha\beta}_{,\beta}\, u$, and $\sum_{\alpha,\beta} (a^{\alpha\beta}_{,\alpha\beta}\, u)$. From Theorem 4.2 and our assumptions on the $a^{\alpha\beta}$, we see that

$$\| U \|_2^0 \leqslant C_1 K_0 R \| u \|_2^0,$$

so that $\| T \| \leqslant 1/2$ if $R \leqslant R_1(K_0)$. From our assumptions and interior regularity theorems, it follows that

$$\Delta U = \epsilon^{\alpha\beta} u_{,\alpha\beta} \text{ for } -R < y < 0. \tag{4.24}$$

Moreover, by integrating by parts in the tangential directions we see that U (as well as v) satisfies (4.3). From (4.19), (4.20), (4.22), (4.23), and (4.24), it follows that the function w defined by

$$w = u - U - v = u - T_R^\alpha - v$$

is harmonic for $-R < y < 0$ and satisfies (4.3), so that $w \equiv 0$, and u satisfies

$$u - T_R^\alpha = v. \tag{4.25}$$

Now, let us suppose that $u \in H_{20}^1$ (and so satisfies (4.3)). Using Theorem 4.2 we see that

$$T_R u = U = \sum_\alpha W^\alpha_{,\alpha} - W,\ W^\alpha = \sum_\beta V^{\alpha\beta}_{,\beta} - V^\alpha,\ W = \sum_\alpha V^\alpha_{,\alpha} - V,$$

$$\Delta W^\alpha = \epsilon^{\alpha\beta} u_{,\beta},\ \Delta W = \epsilon^{\alpha\beta}_{,\alpha} u_{,\beta}. \tag{4.26}$$

It follows as in the case of v that $U \in H_{20}^1(G_R)$ and that

$$\int_{G_R} |\nabla U|^2\, dxdy \leqslant C_2^2 K^2 R^2 \int_{G_R} |\nabla u|^2\, dx \tag{4.27}$$

so that $\| T \| \leqslant C_2 K R \leqslant 1/2$ if $R \leqslant R_2$ where we now consider T_R as an operator in $H_{20}^1(G_R)$. Again u satisfies (4.25) considered as an equation in H_{20}^1, v being in H_{20}^1. If $R \leqslant$ smaller of R_1 and R_2, then (4.25) has a unique solution in $L_2(G_R)$ and in $H_{20}^1(G_R)$. Hence the solutions coincide and $u \in H_{20}^1(G_R)$ as required.

We now prove some useful theorems concerning the s-norms. The following theorem and its proof are essentially due to Peetre [16], although a similar theorem has been proved, for instance, by Nirenberg.

THEOREM 4.5 *Suppose* $a \in C^\infty(G_R)$ *and is periodic as usual and* $u \in \mathfrak{H}^s(G_R)$, $s \geqslant 0$. *Then* $a u \in \mathfrak{H}^s(G_R)$ *and*

$$\| au \|^s_{G_R} \leqslant \left(\sum_m |a_m| \right) . \| u \|^s_{G_R} + C(s, a, R). \| u \|^{s-1}_{G_R}, s > 0.$$

$$C = \sum_n 2^{|s-1|/2} |s| . |n| . [1 + (1 + |n|^2)^{(s-1)/2}] |a_n|.$$

PROOF. Let us first assume that a and u are given by finite Fourier series. It evidently suffices to prove the inequality where the s norm is defined as in (4.4) and the dependence on y neglected, since one can integrate with respect to y. We have

$$(au)_m = \sum_n a_n u_{m-n}.$$

The s-norm of (au) is then the l_2-norm of the multiple series $(1 + |m|^2)^{s/2}(au)_m$. By the triangle inequality, we obtain

$$\left\| (1 + |m|^2)^{s/2} \sum_n a_n u_{m-n} \right\| < \left\| \sum_n a_n (1 + |m - n|^2)^{s/2} u_{m-n} \right\| +$$

$$+ \left\| \sum_n a_n [(1 + |m|^2)^{s/2} - (1 + |m - n|^2)^{s/2}] u_{m-n} \right\|; \quad (4.28)$$

here the norm denotes the l_2-norm. Now, if $u > 0$ and $v > 0$, $|u^s - v^s| \leqslant |s| (u^{s-1} + v^{s-1}) |u - v|$ so that

$$|(1 + |m|^2)^{s/2} - (1 + |m - n|^2)^{s/2}| < |s|.|n| [(1 + |m|^2)^{(s-1)/2} + + (1 + |m - n|^2)^{(s-1)/2}]$$

$$< 2^{|s-1|/2} |s| |n| [1 + (1 + |n|^2)^{(s-1)/2}]. [1 + |m - n|^2]^{(s-1)/2} \quad (4.29)$$

since $1 + |m|^2 \leqslant 2(1 + |n|^2). (1 + |m - n|^2)$. Clearly the first term in (4.28)

$$\leqslant \|u\|^s . \sum_n |a_n|. \quad (4.30)$$

Using (4.28) and (4.29), we see that the second term in (4.28)

$$< \|u\|^{s-1} . \sum_n 2^{|s-1|/2} |s|.|n| [1 + (1 + |n|^2)^{(s-1)/2}].|a_n|. \quad (4.31)$$

The general result follows by a passage to the limit.

LEMMA 4.2. *Suppose* $\chi \in C_c^\infty(\mathbf{R}^\nu)$ *and its Fourier transform* $\hat{\chi}(\lambda) \in C^\infty$ *for all real* λ *and satisfies*

$$|\hat{\chi}(\lambda)| \leqslant C_1 |\lambda|^k \text{ for } |\lambda| \leqslant 1 \text{ and } |\hat{\chi}(\lambda)| \leqslant C_2|\lambda|^{-l}, |\lambda| \geqslant 1, l > 1, k \geqslant 0. \tag{4.32}$$

Then, for all λ,

$$\int_0^1 \epsilon^{-2s-1}|\hat{\chi}(\epsilon\lambda)|^2 d\epsilon + (1+|\lambda|^2)^{s-1} \leqslant C_3(C_1, C_2, k, l, s)(1+|\lambda|^2)^s, \ s<k. \tag{4.33}$$

If, also, χ *satisfies*

$$|\hat{\chi}(\lambda)| \geqslant C_4 |\lambda|^k \text{ for } |\lambda| \leqslant h, \ 0 < h < 1 \tag{4.34}$$

then, for all λ

$$\int_0^1 \epsilon^{-2s-1}|\hat{\chi}(\epsilon\lambda)|^2 d\epsilon + (1+|\lambda|^2)^{s-1} \geqslant C_5(C_4, h, k, s)\ (1+|\lambda|^2)^s, \ s<k. \tag{4.35}$$

PROOF. If $|\lambda| \leqslant 1$, the result (4.33) is evident. If $|\lambda| > 1$, then

$$\int_0^1 \epsilon^{-2s-1}|\hat{\chi}(\epsilon\lambda)|^2 d\epsilon \leqslant C_1^2 |\lambda|^{2k} \int_0^{1/|\lambda|} \epsilon^{2k-2s-1} d\epsilon + C_2^2|\lambda|^{-2l} \int_{1/|\lambda|}^1 \epsilon^{-2l-2s-1} d\epsilon \leqslant \left[\frac{C_1^2}{(2k-2s)} + \frac{C_2^2}{2l+2s}\right] |\lambda|^{2s}.$$

The result (4.35) is evident if $|\lambda| < h$ because of the addition of the term $(1+|\lambda|^2)^{s-1}$. If $|\lambda| > h$, then

$$\int_0^1 \epsilon^{-2s-1}|\hat{\chi}(\epsilon\lambda)|^2 d\epsilon \geqslant \int_0^{h/|\lambda|} C_4^2 |\lambda|^{2k}\epsilon^{2k-2s-1} d\epsilon = C_4^2 h^{2k-2s} |\lambda|^{2s}/(2k-2s)$$

and the second result follows in general.

THEOREM 4.6. *Suppose* $\chi \in C_c^\infty[B(0,1)]$ *and suppose its Fourier transform* $\hat{\chi}$ *satisfies* (4.32). *Let us define* $S_\epsilon u$ *by the formula*

$$(S_\epsilon u)(x) = \rho^{-\nu} \int_{B(x,\rho)} \chi[(x-x')/\rho]\, u(x')\, dx', \rho = R\epsilon/\pi. \quad (4.36)$$

Then, if $u \in \mathfrak{H}^s(G_R)$ *(or* $\mathfrak{H}^s(\sigma_R)$ *),*

$$\int_0^1 \|S_\epsilon u\|^2 \epsilon^{-2s-1}\, d\epsilon \leqslant C_3(C_1, C_2, k, l)\, (\|u\|^s)^2. \quad (4.37)$$

If, also, $\hat{\chi}$ *satisfies* (4.34), *then*

$$\int_0^1 \|S_\epsilon u\|^2 \epsilon^{-2s-1}\, d\epsilon + (\|u\|^{s-1})^2 \geqslant C_5(C_4, h, k, s)\,.\,(\|u\|^s)^2.$$

PROOF. Letting $\rho = R\epsilon/\pi$ as in (4.36) above, we obtain

$$(S_\epsilon u)_m = (2R)^{-\nu}\rho^{-\nu} \int_{\sigma_R} \left[\int_{B(x,\rho)} \chi\left(\frac{x-x'}{\rho}\right) u(x')\, dx' \right] e^{-i\pi m.x/R}\, dx$$

$$= (2R)^{-\nu}\, \rho^{-\nu} \int_{\sigma_R} \left[\int_{B(0,\rho)} \chi\left(\frac{\eta}{\rho}\right) u(x-\eta)\, e^{-i\pi m.x/R}\, d\eta \right] dx$$

$$= (2R)^{-\nu}\, \rho^{-\nu} \int_{B(0,\rho)} \chi\left(\frac{\eta}{\rho}\right) e^{-i\pi m.\eta/R} \left[\int_{\sigma_R} u(y)\, e^{-i\pi m.y/R}\, dy \right] d\eta$$

$$= u_m\, \hat{\chi}\left(\frac{\pi m \rho}{R}\right) = u_m\, \hat{\chi}(\epsilon m).$$

The theorem follows immediately from Lemma 4.2.

REMARK. If we take $\chi(\lambda) = \omega(|\lambda|^2)$ where $\omega \in C_c^\infty(\mathbf{R}^1)$ and has support in $[-1, 1]$, then χ satisfies all the hypotheses above with $k = 2q$ if

$$\int_0^1 r^{\nu-1+2j}\, \omega(r^2)\, dr = 0,\ 0 \leqslant j < q,$$

$$\Delta^q \hat{\chi}(0) = (-1)^q \int_0^1 r^{\nu-1+2q}\, \omega(r^2)\, dr > 0.$$

THEOREM 4.7. *Suppose* χ *satisfies the first hypothesis of the preceding theorem,* $u \in \mathfrak{H}^s(G_R)$ *and* a *is periodic and* $\in C^\infty(G_R)$. *Suppose also that there is a positive integer* p *such that* $s + 1 < p \leqslant k$ *and suppose* $s \geqslant 0$. *Then*

$$\int_0^1 [\|aS_\epsilon u - S_\epsilon au\|^0]^2 \epsilon^{-2s-3} d\epsilon \leqslant C(s, k, a, \chi) (\|u\|^s)^2,$$

$$\int_0^1 [\|aS_\epsilon u_{,\lambda} - S_\epsilon au_{,\lambda}\|^0]^2 \epsilon^{-2s-1} d\epsilon \leqslant C(s, k, a, \chi) R^{-2} (\|u\|^s)^2 ;$$

here $u_{,\lambda}$ *denotes* $\partial u/\partial x^\lambda$.

PROOF. Using the formula above for $(S_\epsilon u)_m$ and for the coefficient in the series for a product we find that

$$(S_\epsilon au)_m - (aS_\epsilon u)_m = \sum_n a_n \{\hat{\chi}(\epsilon m) - \hat{\chi}[\epsilon(m-n)]\} u_{m-n}. \quad (4.38)$$

It is easy to see that the j-th derivatives of $\hat{\chi}$ satisfy those same hypotheses with k replaced by $k-j$ (since $\hat{\chi}$ is analytic at the origin) if $j \leqslant k$; the same l may be used. It is also clear that every derivative of $\hat{\chi}$ is uniformly bounded. Thus, we may expand $\hat{\chi}(\lambda)$ about $\lambda_0 = \pi(m-n)\epsilon/R$ out to the terms in $(\lambda-\lambda_0)^{p-1}$, using the boundedness of the derivatives to estimate the remainder; here we take $p = 2 + [s]$. Thus

$$|(aS_\epsilon u)_m - (S_\epsilon au)_m| \leqslant [\epsilon^p/p!] \sum_n |a_n| . |n|^p. C. |u_{m-n}| +$$
$$+ \sum_{j=1}^{p-1} [\epsilon^j/j!] \sum_n |a_n| . |n|^j. |\nabla^j \chi[\epsilon(m-n)]| . |u_{m-n}| \quad (4.39)$$

where C is a bound for $|\nabla^p \chi(\lambda)|$. Now, let us write

$$v_m = |(aS_\epsilon u)_m - (S_\epsilon au)_m|, \; u_m^j = |\nabla^j \chi(\epsilon m)| . |u_m|, \; 1 \leqslant j \leqslant p-1$$

$$u_m^p = C|u_m|, \; v_m^j = \frac{\epsilon^j}{j!} \sum_n |a_n| . |n|^j u_{m-n}^j.$$

Letting $\| \ \|$ denote the l_2 norm, we see that

$$\left.\begin{aligned} &\|v^j\| \leqslant \frac{\epsilon^j}{j!} \left(\sum_n |a_n| . |n|^j\right) \|u^j\| \leqslant C \frac{\epsilon^j}{j!} \|u^j\|, j = 1, \ldots, p, \\ &\|v\| \leqslant C \sum_{j=1}^p \frac{\epsilon^j}{j!} \|u^j\|, \|v\|^2 \leqslant \sum_{j=1}^p C_j \epsilon^{2j} \|u^j\|^2. \end{aligned}\right\} \quad (4.40)$$

Multiplying the last inequality in (4.40) by ϵ^{-2s-3}, integrating, and using Lemma 4.2, we obtain

$$\int_0^1 [\| a S_\epsilon u - S_\epsilon au \|^0]^2 \epsilon^{2s-3} d\epsilon \leq C_p \int_0^1 \epsilon^{2p-2s-3} (\| u \|^0)^2 d\epsilon +$$

$$+ \sum_{j=1}^{p-1} C_j \int_0^1 \epsilon^{2j-2s-3} \sum_m | \nabla^j \hat{\chi}(\epsilon m) |^2 | u_m |^2 d\epsilon$$

$$\leq \sum_{j=1}^{p-1} C_j^* (\| u \|^{s+1-j})^2 + C_p (2p - 2s - 2)^{-1} (\| u \|^0)^2$$

from which the result follows, since $\| u \|^s$ is a non-decreasing function of s.

To prove the second statement we notice that

$$(aS_\epsilon u_{,\lambda} - S_\epsilon au_{,\lambda})(x, y)$$

$$= \rho^{-\nu} \int_{B(x,\rho)} \chi\left(\frac{x - x'}{\rho}\right) [a(x, y) - a(x', y)] u_{,\lambda}(x', y) \, dx'$$

$$= (S_\epsilon a_{,\lambda} u)(x, y) +$$

$$+ \rho^{-\nu-1} \int_{B(x,\rho)} \chi_{,\lambda}\left(\frac{x - x'}{\rho}\right) [a(x, y) - a(x', y)] u(x', y) \, dx'.$$

The desired inequality holds for the first term. The analysis of the first part of the theorem now applies with the $\hat{\chi}$ difference of (4.38) replaced by ρ^{-1} times the $\hat{\chi}^{(\lambda)}$ difference, where

$$\hat{\chi}^{(\lambda)}(\gamma) = i\gamma^\lambda \hat{\chi}(\gamma), \; \rho = \pi\epsilon/R.$$

THEOREM 4.8. *Suppose the* $a^{\alpha\beta}$ *satisfy the conditions in and near equation* (4.2) *for each* $R \leq$ *some* $A > 0$ *and suppose the functions* u, f^α, f, *and* $g \in C^\infty$ *for* $-R \leq y < 0$ *and are periodic of period* $2R$ *in each* x^α. *Suppose that* f^α, f, *and* $g \in \mathfrak{H}^s(G_R)$, *that* $u \in \mathfrak{H}^s(G_R)$, *that* $u(x, -R) = 0$, *that* u *satisfies* (4.1), *and also that* $\| u(\,.\,, y) \|^s_{\sigma R} \leq L$ *for* $-R \leq y < 0$. *Then, if* $0 < R \leq R(\nu, s, K_0, K_1, \ldots)$,

$$u \in \mathfrak{H}^{s+1/2}(G_R) \text{ and } u_y \in \mathfrak{H}^{s-1/2}(G_R),$$

the latter holding if $s \geq 1/2$.

PROOF. For each η with $0 < \eta < R/2$, let H_η be the harmonic function coinciding with u when $y = -R$ and $y = -\eta$ and let $u_\eta = u - H_\eta$. Then $u_\eta \in C^\infty$ for $-R \leqslant y \leqslant -\eta$ and satisfies (4.1) with the right side replaced by

$$(\epsilon^{\alpha\beta} H_\eta)_{,\alpha\beta} + (f^\alpha - 2\epsilon^{\alpha\beta}_{,\beta} H_\eta)_{,\alpha} + f_y + (g + \epsilon^{\alpha\beta}_{,\alpha\beta} H_\eta) \tag{4.41}$$

as is seen by employing the device in (4.22). From Theorem 4.1 and its proof, we conclude that $H_\eta \in \mathfrak{H}^{s+1/2}(G_{R,\eta})$ with $\|H_\eta\|^{s+1/2}_{G(R,\eta)} \leqslant CL$, independently of η. Moreover, if $s \geqslant 1/2$, then $H_{\eta y} \in \mathfrak{H}^{s-1/2}(G_{R,\eta})$ with $\|H_{\eta y}\|^{s-1/2}_{G(R,\eta)} \leqslant CL$ independently of η.

Now, if we expand $\epsilon^{\alpha\beta}(x, y)$ in a Fourier series,

$$\epsilon^{\alpha\beta}(x, y) = \sum_m \epsilon^{\alpha\beta}_m(y)\, e^{i\pi m\cdot x/R}$$

we find by differentiation and using (4.2) that

$$D^\lambda \epsilon^{\alpha\beta}(x, y) = \sum_m (i\pi)^{|\lambda|} \,.\, R^{-|\lambda|}\, m^\lambda\, \epsilon^{\alpha\beta}_m(y)\, e^{i\pi m\cdot x/R} \quad (\lambda = \lambda_1, \ldots, \lambda_\nu),$$

$$\sum_m |m|^{2p}\, |\epsilon^{\alpha\beta}_m(y)|^2 < C_p^2 K_p^2 R^2,\ p = 0, 1, 2, \ldots \tag{4.42}$$

Consequently it follows from Theorem 4.5 that

$$\left.\begin{aligned} &\epsilon^{\alpha\beta} H_\eta \in \mathfrak{H}^{s+1/2}(G_{R,\eta}) \text{ and} \\ &\|\epsilon^{\alpha\beta} H_\eta\|^{s+1/2}_{G(R,\eta)} < CLR,\ 0 < \eta < R/2. \end{aligned}\right\} \tag{4.43}$$

In like manner, we find that $\epsilon^{\alpha\beta}_{,\beta} H_\eta$ and $\epsilon^{\alpha\beta}_{,\alpha\beta} H_\eta \in \mathfrak{H}^{s+1/2}$ and

$$\|\epsilon^{\alpha\beta}_{,\beta} H_\eta\|^{s+1/2}_{G(R,\eta)} < CL,\ \|\epsilon^{\alpha\beta}_{,\alpha\beta} H_\eta\|^{s+1/2}_{G(R,\eta)} < CLR^{-1},\ 0 < \eta < R/2. \tag{4.44}$$

Next, let $V^{\alpha\beta}_\eta$, V^α_η, and V_η be the solution of Theorem 4.2 with g replaced by $\epsilon^{\alpha\beta} H_\eta$, $f^\alpha - 2\epsilon^{\alpha\beta}_{,\beta} H_\eta$, and $g + \epsilon^{\alpha\beta}_{,\alpha\beta} H_\eta$, respectively, on $G(R, \eta)$, let W_η be the solution of Theorem 4.3 for f on $G(R, \eta)$, and let

$$v_\eta = V^{\alpha\beta}_{\eta,\alpha\beta} + V^\alpha_{\eta,\alpha} + V_\eta + W_\eta. \tag{4.45}$$

Then $v_\eta \in \mathfrak{H}^{s+1/2}(G_{R,\eta})$, vanishes on $y = -R$ and $-\eta$ and satisfies

$$\Delta v_\eta = \epsilon^{\alpha\beta} H_{\eta,\alpha\beta} + f^\alpha_{,\alpha} + f_y + g \text{ on } G(R, \eta). \tag{4.46}$$

Now we write (4.1) for u_η in the form

$$\Delta u_\eta = \epsilon^{\alpha\beta} u_{\eta,\alpha\beta} + \Delta v_\eta$$
$$= (\epsilon^{\alpha\beta} u_\eta)_{,\alpha\beta} - 2(\epsilon^{\alpha\beta}_{,\beta} u_\eta)_{,\alpha} + \epsilon^{\alpha\beta}_{,\alpha\beta} u_\eta + \Delta v_\eta .$$

Suppose we let $X^{\alpha\beta}_\eta$, X^α_η, and X_η be the solutions of Theorem 4.2 for $\epsilon^{\alpha\beta} u_\eta$, $-2\epsilon^{\alpha\beta}_{,\beta} u_\eta$, and $\epsilon^{\alpha\beta}_{,\alpha\beta} u_\eta$, respectively, and define

$$T_R u_\eta = X^{\alpha\beta}_{\eta,\alpha\beta} + X^\alpha_{\eta,\alpha} + X_\eta .$$

Using the estimates on the $\epsilon^{\alpha\beta}$ in (4.42)-(4.44), we see that

$$\left.\begin{array}{l} \| T_R u_\eta \|^s_{G(R,\eta)} < C_1 R . \| u \|^s_{G(R,\eta)}, \\ \| T_R u_\eta \|^{s+1/2}_{G(R,\eta)} < C_2 R . \| u \|^{s+1/2}_{G(R,\eta)}, \end{array}\right\} 0 < \eta < R/2, \tag{4.47}$$

and, moreover, we see as in the proof of Theorem 4.4 that

$$u_\eta - T_R u_\eta = v_\eta, \; v_\eta \in \mathfrak{H}^{s+1/2} (G_{R,\eta}), \tag{4.48}$$

$\|v_\eta\|^{s+1/2}_{G(R,\eta)} < C_3 + C_4 RL$ independently of η. From (4.47), it follows that if $R < R(\nu, s, K_0, K_1, \ldots)$, then (4.48) has a unique solution u_η in $\mathfrak{H}^s(G_{R,\eta})$ and in $\mathfrak{H}^{s+1/2}(G_{R,\eta})$, so that $u_\eta \in \mathfrak{H}^{s+1/2}(G_{R,\eta})$ with norm uniformly bounded, independently of η. The theorem follows by letting $\eta \to 0$.

5. The domain $\mathfrak{D}$; the Hilbert space results.

In this section we prove the first four principal results stated in § 2.

THEOREM 5.1. ([8, I], *Theorem* 2.3). *The transformation L is self-adjoint.*

PROOF. Since T is a closed operator, it follows that TT^* and T^*T are self-adjoint and that $(I + TT^*)^{-1}$ and $(I + T^*T)^{-1}$ are bounded and self-adjoint and are defined everywhere ([14], p. 307). Define

$$S = (I + T^*T)^{-1} + (I + TT^*)^{-1} - I.$$

We shall show that $(L + I)S = I$. First we note that

$$(I + TT^*)^{-1} - I = [I - (I + TT^*)]\,(I + TT^*)^{-1}$$
$$= -TT^*(I + TT^*)^{-1}.$$

Thus $\mathfrak{R}(S) \subset \mathfrak{D}(T^*T)$ (since $\mathfrak{R}(T) \subset \mathfrak{D}(T)$ and $T^2 = 0$) and

$$T^*TS = T^*T(I + TT^*)^{-1}.$$

Similarly $\mathfrak{R}(S) \subset \mathfrak{D}(TT^*)$ and

$$TT^*S = TT^*(I + TT^*)^{-1}.$$

Hence $(L + I)S = I$. The operator S is self-adjoint since it is the sum of bounded self-adjoint operators. Thus $L + I$ and hence L is self-adjoint.

For a given covering $\mathscr{U}$ of $\overline{M}$ by the ranges $\mathfrak{R}_s$ of coordinate patches τ_s with domains G_s, we define

$$((\phi, \psi))_{\mathscr{U}} = (\phi, \psi) + \sum_s \int_{G_s} \sum_{I,J,\alpha} \left(\phi^{(s)}_{IJx^\alpha} \overline{\psi}^{(s)}_{IJx^\alpha} + \phi^{(s)}_{IJy^\alpha} \overline{\psi}^{(s)}_{IJy^\alpha} \right) dxdy$$

$$((\phi, \psi))_{\bar{z}\mathscr{U}} = (\phi, \psi) + \sum_s \int_{G_s} \sum_{I,J,\alpha} \left(\phi^{(s)}_{IJ\bar{z}^\alpha} \overline{\psi}^{(s)}_{IJ\bar{z}^\alpha} \right) dxdy \tag{5.1}$$

the $\phi^{(s)}_{IJ}$ and $\psi^{(s)}_{IJ}$ being the components of ϕ and ψ with respect to τ_s. It is clear that any two norms defined by different coverings are topologically equivalent; actually non-analytic coordinates may be used in the first norm. So we shall omit the subscript. The space H^1_2 is the closure of $\mathfrak{A}$ with respect to the first norm and H^1_{20} is the closure of those forms of $\mathfrak{A} \in C^\infty_c(M^0)$.

THEOREM 5.2 (a). *$\mathfrak{R}(T) \subset \mathfrak{D}(T)$ and $T^2\phi = 0$ if $\phi \in \mathfrak{D}(T)$.*

(b) *$\mathfrak{R}(T^*) \subset \mathfrak{D}(T^*)$ and $(T^*)^2\phi = 0$ if $\phi \in \mathfrak{D}(T^*)$.*

(c) *$\phi \in \mathfrak{L} \ominus \mathfrak{R}(L) \Longleftrightarrow \phi \in \mathfrak{D}(L)$ and $L(\phi) = 0 \Longleftrightarrow \phi \in \mathfrak{H}$.*

(d) *$H^1_{20} \subset \mathfrak{D}$ and if $\phi \in H^1_{20}$, $T\phi = \bar{\partial}\phi$ and $T^*\phi = \mathfrak{d}\phi$, it being understood that $\bar{\partial}\phi$ and $\mathfrak{d}\phi$ are formed using the distribution derivatives of the components of ϕ.*

PROOF. (a) Let $\psi \in \mathfrak{R}(T)$. Then $\exists$ a $\phi \in \mathfrak{D}(T)$ *such that* $T\phi = \psi$. Hence $\exists \{\phi_n\} \subset \mathfrak{A}$ *such that* $\phi_n \to \phi$ and $\bar{\partial}\phi_n \to \psi$. But, for each n,

$$\bar{\partial}\phi_n \in \mathfrak{D}(T) \text{ and } \bar{\partial}\phi_n \to \psi \text{ and } \bar{\partial}(\bar{\partial}\phi_n) = 0.$$

(b) Let $\psi \in \mathfrak{R}(T^*)$. Then $\exists\ \phi \in \mathfrak{D}(T^*)$ *such that* $T^*\phi = \psi$.

Thus, for each $\omega \in \mathfrak{A}$,

$$(\psi, \bar{\partial}\omega) = (T^*\phi, \bar{\partial}\omega) = (\phi, \bar{\partial}\bar{\partial}\omega) = 0 = (0, \omega).$$

Hence $\psi \in \mathfrak{D}(T^*)$ and $T^*\psi = 0$.

(c) The first statement follows from the self-adjointness of L and the second follows since

$$(L\phi, \phi) = 0 = (T^*T\phi + TT^*\phi, \phi) = (T\phi, T\phi) + (T^*\phi, T^*\phi).$$

(d) is evident.

THEOREM 5.3. *If $\phi \in H^1_{20}(M)$, then*

$$d(\phi) \geqslant c((\phi, \phi))_z - C(\phi, \phi), \; c > 0,$$
$$((\phi, \phi)) \leqslant C((\phi, \phi))_{\bar{z}}.$$

PROOF. From the definition of H^1_{20}, it follows that it is sufficient to prove these inequalities for forms $\phi \in C^\infty_c(M)$. By taking $\psi = \phi$ in (3.15), we find that

$$d(\phi) = I(\phi) = \int_M \Sigma \{g^{KI} g^{JL} g^{\alpha\beta} \phi_{IJ\bar{z}^\alpha} \overline{\phi}_{KL\bar{z}^\beta} + C^{IJ,KL\beta} \phi_{IJ} \overline{\phi}_{KL\bar{z}^\beta} +$$
$$+ \overline{C}^{KL,IJ\alpha} \phi_{IJ\bar{z}^\alpha} \overline{\phi}_{KL} + D^{IJ,KL} \phi_{IJ} \overline{\phi}_{KL}\} \, dM \tag{5.2}$$

from which the first inequality follows. By using a partition of unity, the proof of the second is reduced to proving it for ϕ having compact support in a single sufficiently small coordinate patch with domain G, say. For such a ϕ,

$$((\phi, \phi)) \leqslant C \int_G \sum_{I,J,\alpha} (|\phi_{IJx^\alpha}|^2 + |\phi_{IJy^\alpha}|^2) \, dxdy,$$

$$\int_G \sum_{I,J,\alpha} (|\phi_{IJx^\alpha}|^2 + |\phi_{IJy^\alpha}|^2) \, dxdy$$
$$= 4 \int_G \sum_{I,J,\alpha} |\phi_{IJ\bar{z}^\alpha}|^2 \, dxdy - 2 \int_G \sum_{I,J,\alpha} (\phi_{1IJx^\alpha} \phi_{2IJy^\alpha} - \phi_{2IJx^\alpha} \phi_{1IJy^\alpha}) \, dxdy$$
$$\leqslant C((\phi, \phi))_{\bar{z}}$$

since the second integral vanishes.

We define $\mathfrak{H}_0 = \mathfrak{H} \cap H^1_{20}$. It can be shown, using the Unique Continuation Theorem of Aronszajn et al [2] for forms that $\mathfrak{H}_0$ is just the zero element. However, all we need is the following easily proved theorem :

THEOREM 5.4. *$\mathfrak{H}_0$ has finite dimensionality. If $\phi \in H^1_{20} \cap (\mathfrak{L} \ominus \mathfrak{H}_0)$, then*

$$d(\phi) \geqslant c((\phi, \phi)), \quad c > 0.$$

PROOF. From Theorem 5.3, we conclude that

$$((\phi, \phi)) \leqslant C_1 d(\phi) + C_2(\phi, \phi), \quad \phi \in H^1_{20}. \tag{5.3}$$

Since (ϕ, ϕ) is completely continuous with respect to $((\phi, \phi))$, we see from (5.3) that there is a $\phi_1 \in H^1_{20}$ which minimizes $d(\phi)$ among all ϕ in H^1_{20} for which $(\phi, \phi) = 1$. Next, there is a ϕ_2 which minimizes $d(\phi)$ among all ϕ in H^1_{20} with $(\phi, \phi) = 1$ and $(\phi, \phi_1) = 0$ and then a ϕ_3 which minimizes $d(\phi)$ among all ϕ in H^1_{20} with $(\phi, \phi) = 1, (\phi, \phi_1) = 0$, and $(\phi, \phi_2) = 0$. This may be continued. Obviously

$$0 \leqslant d(\phi_1) \leqslant d(\phi_2) \leqslant \cdots. \tag{5.4}$$

If all the $d(\phi_i) = 0$, then $((\phi_i, \phi_i))$ is uniformly bounded and one could extract a subsequence converging strongly in $\mathfrak{L}$ to some limit. But this is clearly impossible. From (5.4), we conclude that

$$d(\phi) \geqslant d(\phi_k) . (\phi, \phi) \text{ if } \phi \in H^1_{20} \cap (\mathfrak{L} \ominus \mathfrak{H}_0),$$

where k is the first integer for which $d(\phi_k) > 0$ and $\mathfrak{H}_0$ is the space spanned by $\phi_1, \ldots, \phi_{k-1}$.

THEOREM 5.5. *Suppose $\phi \in C^\infty(M^{(0)})$. Then a necessary and sufficient condition for $\phi \in \mathfrak{D}$ is that $\bar{\partial}\phi$ and $\mathfrak{b}\phi \in \mathfrak{L}$ and that*

$$(\mathfrak{b}\phi, \psi) = (\phi, \bar{\partial}\psi) \text{ for all } \psi \in \mathfrak{A}. \tag{5.5}$$

In this case $T\phi = \bar{\partial}\phi$ and $T^*\phi = \mathfrak{b}\phi$.

PROOF. Suppose $\phi \in \mathfrak{D}$. Then $\phi \in \mathfrak{D}(T)$ and $\exists$ a sequence $\{\phi_n\}$ with ϕ_n in $\mathfrak{A}$ such that $\phi_n \to \phi$ and $\bar{\partial}\phi_n \to T\phi$ in $\mathfrak{L}$. Let ζ be a 0-form $\in C^\infty_c(M)$ with support in some coordinate patch, and let $\phi' = \zeta\phi$, $\phi'_n = \zeta\phi_n$. Then $\phi'_n \to \phi'$ and $\bar{\partial}\phi'_n \to \psi' = \zeta T\phi + \bar{\partial}\zeta \wedge \phi$, which must be $T\phi'$, in $\mathfrak{L}$. If we let π be a mollifier, then $\phi'_{n\rho} \to \phi'_\rho$ and $\bar{\partial}\phi'_{n\rho} \to \psi'_\rho = \bar{\partial}\phi'_\rho$, where ψ'_ρ, etc., denote the π-mollified forms (all defined on the coordinate patch containing the support of ζ). Letting $\rho \to 0$, we see that $\psi' = \bar{\partial}\phi'$. Since this is true for every such ζ, we conclude that $T\phi = \bar{\partial}\phi \in \mathfrak{L}$. Also $\phi \in \mathfrak{D}(T^*)$. Hence $\exists$ a $\chi \in \mathfrak{L}$ such that $(\phi, \bar{\partial}\omega) = (\chi, \omega)$ for all $\omega \in \mathfrak{A}$. But if $\omega \in C^\infty_c(M)$, we see, using a manifold M_s, that $(\phi, \bar{\partial}\omega) = (\mathfrak{d}\phi, \omega)$ so that $(\mathfrak{d}\phi - \chi, \omega) = 0$ for all such ω. Thus $T^*\phi = \mathfrak{d}\phi \in \mathfrak{L}$ and (5.5) holds.

Now, suppose the conditions are satisfied. Let $\mathfrak{R}$ be the range of a coordinate patch with domain G which is tangential at some point P_0 on bM. Let ζ be a 0-form $\in C^\infty(\bar{M})$ which vanishes outside $\mathfrak{R}$ and on and near the part of $\partial\mathfrak{R}$ interior to G, and let $\phi' = \zeta\phi$; we assume that G is bounded above by the surface $y^\nu = f(x, y'_\nu)$ of class C^∞ if $\mathfrak{R} \cap bM$ is not empty. Then ϕ' can be approximated by ϕ'_n, where the component in $\mathfrak{A}^{pq}$ is

$$\phi'_{nIJ}(x, y) = \phi'_{IJ}(x, y'_\nu, y^\nu - n^{-1}) \text{ if } \phi' = \Sigma\, \phi'_{IJ}\, dz^I \wedge d\bar{z}^J.$$

Since this can be done for each patch we see that $\phi \in \mathfrak{D}(T)$ with $T\phi = \bar{\partial}\phi$. From (5.5), it follows that $\phi \in \mathfrak{D}(T^*)$ with $T^*\phi = \mathfrak{d}\phi$.

THEOREM 5.6. *If $\phi \in \mathfrak{D}$, there exists a unique $\phi \in H^1_{20} \cap (\mathfrak{L} \ominus \mathfrak{H}_0)$ and a harmonic form H in $\mathfrak{D}$ such that $\phi = \phi_0 + H$. It follows that*

$$d(\phi) = d(\phi_0) + d(H). \tag{5.6}$$

PROOF. We first note that

$$(\bar{\partial}\omega - T\phi, \bar{\partial}\omega - T\phi) + (\mathfrak{d}\omega - T^*\phi, \mathfrak{d}\omega - T^*\phi)$$
$$= d(\omega) - 2\,\mathrm{Re}(\bar{\partial}\omega, T\phi) - 2\,\mathrm{Re}(\mathfrak{d}\omega, T^*\phi) + (T\phi, T\phi) + (T^*\phi, T^*\phi)$$
$$\geqslant c((\omega, \omega)) - 2\,C((\omega, \omega))^{1/2} D + D^2, \tag{5.7}$$
$$D^2 = (T\phi, T\phi) + (T^*\phi, T^*\phi), \ \omega \in H^1_{20} \cap (\mathfrak{L} \ominus \mathfrak{H}_0).$$

Thus there is a form ϕ_0 which minimizes the expression in (5.7) among all ω of the class indicated. Then ϕ_0 satisfies

$$(\bar{\partial}\phi_0 - T\phi, \bar{\partial}\zeta) + (\mathfrak{d}\phi_0 - T^*\phi, \mathfrak{d}\zeta) = 0, \ \zeta \in H^1_{20}. \tag{5.8}$$

Since $\bar{\partial}\phi_0 = T\phi_0$ and $\mathfrak{d}\phi_0 = T^*\phi_0$, it follows that

$$(\phi_0 - \phi, \Box\,\zeta) = 0, \ \zeta \in C^\infty_c(M).$$

Thus, if we define $H = \phi - \phi_0$, we see that $H \in C^\infty(M^{(0)})$ and $\Box\, H = 0$ there. From Theorem 5.5, it follows that $TH = \bar{\partial}H$ and $T^*H = \mathfrak{d}H$ so we conclude from (5.8) with $\zeta = \phi_0$ that

$$(\bar{\partial}H, \bar{\partial}\phi_0) + (\mathfrak{d}H, \mathfrak{d}\phi_0) = 0$$

from which (5.6) follows easily.

THEOREM 5.7. *If H is harmonic and $H \in \mathfrak{D}$, then $\nu H \in H^1_{20}$.*

PROOF. If $H \in \mathfrak{D}^{p0}$, then $\nu H = 0$. So we assume that $H \in \mathfrak{D}^{pq}$ with $q \geqslant 1$. Choose an analytic coordinate patch which is tangential at some $P_0 \in bM$ and suppose that

$$H = \sum H_{IJ}\, dz^I \wedge d\bar{z}^J,\ \omega = \nu H = \sum \omega_{IM}\, dz^I \wedge d\bar{z}^M,$$
$$\omega_{IM} = (-1)^p \sum g^{\alpha\beta}\, r_{z\beta}\, H_{I,\alpha M}.$$

Since H is harmonic, $d(H, \zeta) = 0$ for all $\zeta \in C_c^\infty(M)$, so one sees, by using the formula (3.15), that the components of H satisfy equations of the form

$$g^{\alpha\beta}\, H_{IJ\bar{z}^\alpha z^\beta} + (\text{lower order terms in all the } H_{ST}) = 0. \quad (5.9)$$

Since the components of ω are just linear combinations of those of H, they satisfy similar equations which may be adjoined to (5.9). If we multiply by 4, write $g^{\alpha\beta} = g_1^{\alpha\beta} + i g_2^{\alpha\beta}$, and express the derivatives in terms of those with respect to the x^α and y^α, the totality of the equations for H and ω take the form

$$g_1^{\alpha\beta}\, u^j_{x^\alpha x^\beta} + g_2^{\alpha\beta}\, u^j_{x^\alpha y^\beta} + g_2^{\beta\alpha}\, u^j_{y^\alpha x^\beta} + g_1^{\alpha\beta}\, u^j_{y^\alpha y^\beta} + \dots = 0, \quad (5.10)$$

where we have denoted all the various components by u^j. The condition $H \in \mathfrak{D}$ is seen to imply, as one sees from (5.5) with $\phi = H$ and (1.14) by integrating over M_s and letting $s \to 0^-$, that

$$\lim_{s\to 0^-} \int_{bM_s} \sum g^{KI} g^{RL}\, \omega_{IR}\, \bar{\psi}_{KL}\, dS(P) = 0, \psi \in C^\infty(\bar{M}). \quad (5.11)$$

Let us now choose non-analytic, but C^∞, boundary coordinates $\xi^1, \dots, \xi^{2\nu}$ so $x = y = 0$ corresponds to $\xi = 0$, bM corresponds to $\xi^{2\nu} = 0$, $r \equiv \xi^{2\nu}$, and the metric is of the form

$$ds^2 = G_{\gamma\delta}\, \xi^\gamma\, \bar{\xi}^\delta,\ G_{2\nu,2\nu} \equiv 1,\ G_{2\nu,\delta} \equiv 0\ (\delta < 2\nu)$$
$$G_{\gamma\delta}(0) = \Delta_{\gamma\delta}.$$

Then the equations (5.10) take the form

$$a^{\alpha\beta}\, u^j_{,\alpha\beta} + b_k^{j\alpha}\, u^k_{,\alpha} + c_k^j\, u^k = 0,$$
$$a^{\alpha\beta}(0) = \delta^{\alpha\beta},\ a^{2\nu,2\nu} \equiv 1,\ a^{2\nu,\beta} \equiv 0,\ \beta < 2\nu, \quad (5.12)$$

and the conditions (5.11) reduce to

$$\left.\begin{aligned}&\lim_{s\to 0-}\int_{\sigma_R} u^j(\xi_{2\nu}, s)\,\bar{v}^j(\xi_{2\nu}, s)\,d\xi_{2\nu} = 0,\ v \in C^\infty(\overline{G}_R),\\ &\sigma_R : |\xi^\alpha| < R,\ \alpha < 2\nu,\ G_R : \xi_{2\nu} \in \sigma_R, -R < \xi^{2\nu} < 0,\end{aligned}\right\} \quad (5.13)$$

for those indices j for which $u^j = \omega_{IM}$ for some (I, M).

Now, suppose $\zeta \in C^\infty$, $\zeta = 1$ in $C_{R/2}$ (the cube $|\xi^\alpha| < R/2$ for $1 \leqslant \alpha \leqslant 2\nu$) and has support in $C_{3R/4}$, let $U^j = \zeta u^j$, extend the coefficients from $G_{3R/4}$ so they are periodic of period $2R$ in each ξ^α with $\alpha < 2\nu$. It is easy to see that for each R the coefficients may be modified so they also $\in C^\infty$ for $-R \leqslant \xi^{2\nu} \leqslant 0$, satisfy a Lipschitz condition with constant K_1 (independent of R) and satisfy (4.2). Then the U^j are periodic in the ξ^α with $\alpha < 2\nu$, (5.13) holds with u^j replaced by U^j and v^j by any C^∞ function, periodic in these ξ^α, and the U^j satisfy

$$a^{\alpha\beta}\, U^j_{,\alpha\beta} = G^{j\alpha}_{,\alpha} + G^j$$

where the $G^{j\alpha}$ and G^j are C^∞ for $\xi^{2\nu} < 0$ and in $L_2(G_R)$. Thus from Theorem 4.4, we conclude that the $U^j \in H^1_{20}(G_R)$ if j is one of the indices in (5.13) and R is small enough. Thus, we see that $\omega = \nu H \in H^1_2$ near P_0 and vanishes along a part of bM near P_0. Since this is true for each P_0, the result follows.

THEOREM 5.8. *Suppose $\phi \in \mathfrak{D}^{p,q}$ and $q \geqslant 1$. Then*

$$\int_{bM(s)} |\phi|^2\, dS \leqslant C_2\,[d(\phi) + (\phi, \phi)],\ s_0 \leqslant s \leqslant 0.$$

PROOF. Using Theorem 5.6, we write $\phi = \phi_0 + H$ where $\phi_0 \in H^1_{20}$, H is harmonic and $d(\phi) = d(\phi_0) + d(H)$. By approximating to ϕ_0 by forms in $\mathfrak{A}$, we see easily that

$$\int_{bM(s)} |\phi_0|^2\, dS(P) \leqslant 2 \int_{bM(s)} |\phi_0|^2\, d\,\Sigma(P)$$

$$\leqslant C\,|s| \int_{M-M(s)} |\phi_{0r}|^2\, dM \leqslant C\,|s|\, d(\phi), \quad s_1 \leqslant s < 0, \qquad (5.14)$$

$d\Sigma(P)$ being the area element on bM at the foot of the geodesic through P. Thus it is sufficient to prove the theorem for ϕ harmonic.

Thus, in formula (3.15), let us set $\phi=\psi=H$, let $\chi'=\omega'=(-1)^p\nu H$ and apply the formula to $M(s)$. Let $d_s(\phi)$ and $I_s(\phi)$ denote the integrals over $M(s)$ on the left and right, respectively, in (3.15). Let us now integrate with respect to s for $s_1 \leqslant s \leqslant s_2 < 0$ and then integrate by parts in the integral over $M(s_2)-M(s_1)$ which arises from the integration of the boundary integral. The result is

$$\int_{s_1}^{s_2} d_s(H)\,ds = \int_{s_1}^{s_2} I_s(H)\,ds - \int_{bM(s_2)} |\omega|^2\,dS(P) + \int_{bM(s_1)} |\omega|^2\,dS(P) +$$

$$+ \int_{M(s_2)-M(s_1)} \operatorname{Re} \sum g^{KI} g^{ST} \omega'_{IS}\,[2\epsilon(\overline{\mathfrak{d} H})_{KT} + {}'\overline{B}{}^{UV\delta}_{KT}\overline{H}_{U,\delta V}]\,dM(P) +$$

$$+ \int_{M(s_2)-M(s_1)} \sum g^{KI} g^{ST} r_{z\beta\bar{z}\gamma}\, g^{\alpha\beta} g^{\gamma\delta} H_{I,\alpha S}\overline{H}_{K,\delta T}\,dM(P) \quad (\epsilon=(-1)^{p+1}) \tag{5.15}$$

for suitable functions ${}'B^{UV\delta}_{KT}$. Since $\omega \in H^1_{20}$ and H and $\mathfrak{d}\,H \in \mathfrak{L}$, we may let $s_2 \to 0^-$. From (3.17), it follows that the last integral in (5.15) is positive. Now if we choose coordinates (P, s) in $M - M(s_1)$, where $P \in bM$ and P is constant along a gradient curve for r, we see as in (5.14) that

$$\int_{bM(s)} |\omega(s,P)|^2\,dS(P) \leqslant 2C\,|s| \int_{M-M(s)} |\nabla\omega|^2\,dM,\ s_1 < s \leqslant 0. \tag{5.16}$$

Integrating, we obtain

$$\int_{M-M(s_1)} |\omega(s,P)|^2\,dM \leqslant Cs_1^2 \int_{M-M(s_1)} |\nabla\omega|^2\,dM. \tag{5.17}$$

If we now divide (5.15) (with $s_2=0$) by $-s_1$, use the Schwarz inequality and (5.16) and (5.17), and let $s_1 \to 0^-$, we see that

$$I(H) \leqslant d(H). \tag{5.18}$$

Since it is obvious from (3.15) that

$$I(\phi) \geqslant c_2((\phi,\phi))_{\bar{z}} - Z_1(\phi,\phi)$$

for any ϕ we see that

$$((H,H))_{\bar{z}} \leqslant C[d(H) + (H,H)]. \tag{5.19}$$

Now, we first note that

$$((\omega, \omega))_{\bar{z}} \leqslant C((H, H))_z$$

so that

$$((\omega, \omega)) \leqslant C((H, H))_z \tag{5.20}$$

since $\omega \in H^1_{20}$. Next, let us define

$$\phi = H - 4\rho, \ \rho_{I,j_1 \ldots j_q} = \sum_{\gamma=1}^{q} (-1)^{\gamma-1} r_{\bar{z}j_\gamma} \, \omega_{I,j'_\gamma} \tag{5.21}$$

Then

$$g^{\alpha\beta} \, r_{z\beta} \, \phi_{I,\alpha M} = \omega_{IM} - 4g^{\alpha\beta} \, r_{z\beta} \, [r_{\bar{z}\alpha} \, \omega_{IM} + \\ + \sum_{\gamma=1}^{q-1} (-1)^{\gamma} \, r_{\bar{z}m_\gamma} \, \omega_{I,\alpha m'_\gamma}] = 0 \quad (\text{near } bM), \tag{5.22}$$

since

$$(-1)^p \, g^{\alpha\beta} \, r_{z\beta} \, \omega_{I,\alpha m'_\gamma} = g^{\alpha\beta} \, g^{\epsilon\theta} \, r_{z\beta} \, r_{z\theta} \, H_{I,\epsilon\alpha m'_\gamma} = 0$$

on account of the antisymmetry of H in the indices ϵ and α. Thus $\gamma\phi \equiv 0$ near bM. Hence, for s sufficiently small, (3.15) with $\psi = \phi$, yields

$$d_s(\phi) = I_s(\phi) + \int_{bM(s)} \Sigma \, g^{KI} \, g^{ST} \, r_{z\beta\bar{z}\gamma} \, g^{\alpha\beta} \, g^{\gamma\delta} \, \phi_{I,\alpha S} \, \overline{\phi_{K,\delta T}} \, dS$$

$$\geqslant c_2((\phi, \phi))_{\bar{z},s} - C(\phi, \phi)_s + c \int_{bM(s)} |\phi|^2 \, dS, \quad c > 0, c_2 > 0. \tag{5.23}$$

Thus

$$\int_{bM(s)} |\phi|^2 \, dS(P) \leqslant C[d_s(\phi) + (\phi, \phi)] \leqslant C[d(\phi) + (\phi, \phi)_s]. \tag{5.24}$$

The result follows from (5.18), (5.20), (5.21) and (5.24).

THEOREM 5.9. *Suppose each* $\phi_n \in \mathfrak{D}^{pq}$ *with* $q \geqslant 1$ *and suppose* $\phi_n \to \phi$, $\bar{\partial}\phi_n \to \psi$, *and* $\mathfrak{d}\phi_n \to \chi$ *in* $\mathfrak{L}$. *Then* $\phi_n \to \phi$, $\phi \in \mathfrak{D}$, $\psi = \bar{\partial}\phi$ *and* $\chi = \mathfrak{d}\phi$. *Also* $\mathfrak{H}^{pq}$ *has finite dimensionality. If* ϕ *is* $\mathfrak{L}$*-orthogonal to* $\mathfrak{H}^{pq}$, *then*

$$(\phi, \phi) \leqslant C_3 d(\phi).$$

PROOF. For each n, let $\phi_n = \phi_{0n} + H_n$. From Theorem 5.6 it follows that

$$d(\phi_{0n}) \leqslant d(\phi_n) \leqslant M, \ d(H_n) \leqslant d(\phi_n) \leqslant M$$

for all n. Let $\{p\}$ be a subsequence of $\{n\}$. Using Theorem 5.4 and its proof for ϕ_{0n} and for ζH_n, ζ being any function of class C_c^∞, we see that there is a subsequence $\{q\}$ of $\{p\}$ such that $\phi_{0q} \to$ some ϕ_0 in $\mathfrak{L}(M)$ and $H_q \to$ some H in $\mathfrak{L}(M_s)$ for each $s < 0$ and $H_q \to H$ in $\mathfrak{L}(M)$. But, now, let $\epsilon > 0$. From Theorem 5.8 it follows that there is a sufficiently small $s_2 < 0$ such that

$$\int_{M-M(s_2)} |H_q|^2\, dM \leqslant C[M + (H_q, H_q)].\, |s_2| < (\epsilon/3)^2.$$

From the weak convergence of H_q to H, we see that this holds for H. Finally, since $H_q \to H$ in $\mathfrak{L}[M(s_2)]$, we see that

$$\int_{M(s_2)} |H - H_q|^2\, dM < (\epsilon/3)^2 \text{ for } q > q_0.$$

Thus $H_q \to H$ in $\mathfrak{L}(M)$. Since this is true for any subsequence, the first result follows. From the interior regularity theorems it follows that H is harmonic. It is easy to see that ϕ, and hence ϕ_0 and $H \in \mathfrak{D}$ and that $\psi = T\phi = \bar\partial\phi$ and $\chi = T^*\phi = \mathfrak{d}\phi$.

From this, it follows easily that if $\mathfrak{M}$ is any closed linear manifold in $\mathfrak{L}^{pq}$, there is a form $\phi \in \mathfrak{D}^{pq}$ which minimizes $d(\phi)$ among all $\phi \in \mathfrak{D}^{pq} \cap \mathfrak{M}$ for which $(\phi, \phi) = 1$. We may let $\mathfrak{M}_1 = \mathfrak{L}^{pq}$ and ϕ_1 a minimizing form in $\mathfrak{M}_1$, $\mathfrak{M}_2$ be those forms $\mathfrak{L}$-orthogonal to ϕ_1 and ϕ_2 a minimizing form in $\mathfrak{M}_2$, etc. Clearly $0 \leqslant d(\phi_1) \leqslant d(\phi_2) \leqslant \ldots$. Now, suppose all the $d(\phi_k) = 0$. Since each $(\phi_k, \phi_k) = 1$, we may extract a subsequence $\{\phi_n\}$ such that $\phi_n \rightharpoonup \phi$, $\bar\partial\phi_n \rightharpoonup \psi$ and $\mathfrak{d}\phi_n \rightharpoonup \chi$. We conclude that $\phi_n \to \phi$. But this is impossible since the ϕ_n form a normal orthogonal set. Hence $\mathfrak{H}^{pq}$ has finite dimensionality and

$$d(\phi) \geqslant C(\phi, \phi),\ C > 0, \text{ if } \phi \in \mathfrak{D}^{pq} \cap (\mathfrak{L}^{pq} \ominus \mathfrak{H}^{pq})$$

from which the last result follows.

We can now prove the remainder of our first four principal results.

Theorem 5.10. (a) *$\phi \in \mathfrak{D}(L) \Longleftrightarrow \phi$, $T\phi$, and $T^*\phi \in \mathfrak{D}$ and*

$$L\phi = \Box\phi = \mathfrak{d}\,\bar\partial\phi + \bar\partial\,\mathfrak{d}\phi.$$

(b) $d(\phi) \leqslant C_3(\Box\phi, \Box\phi)$ *if* $\phi \in \mathfrak{D}(L)$,

$(\phi, \phi) \leqslant C_3\, d(\phi)$ *if* $\phi \in \mathfrak{D}(L) \cap (\mathfrak{L} \ominus \mathfrak{H})$,

C_3 being the constant in Theorem 5.9.

(c) $\mathfrak{N}(L) = \mathfrak{L} \ominus \mathfrak{H}$.

(d) *If we define the operator N on $\mathfrak{L}$ by setting $N\psi = 0$ if $\psi \in \mathfrak{H}$ and $N\psi$ equal to the unique solution ϕ in $\mathfrak{D}(L) \cap (\mathfrak{L} \ominus \mathfrak{H})$ of $L\phi = \psi$, then N is bounded and completely continuous.*

PROOF. (a) follows from Theorems 5.2 and 5.6.

(b) To prove (b), we first assume that $\phi \in \mathfrak{D}(L^{pq}) \cap (L^{pq} \ominus \mathfrak{H}^{pq})$ with $q \geqslant 1$. Then, from Theorem 5.9, we conclude that

$$d(\phi) = (T\phi, T\phi) + (T^*\phi, T^*\phi) = (L\phi, \phi) \leqslant \|\phi\| \, \|\square \phi\|,$$

$$(\phi, \phi) \leqslant C_3 \, d(\phi) \leqslant C_3 \, \|\phi\| \, \|\square \phi\|,$$

from which both results follow. Clearly the inequality for $d(\phi)$ holds whether $\phi \in \mathfrak{L}^{pq} \ominus \mathfrak{H}^{pq}$ or not. In the cases where $q = 0$, let $\psi = \bar{\partial}\phi$. Then $\psi \in \mathfrak{D}^{p,1}$, $\bar{\partial}\psi = 0$, and $\mathfrak{d}\psi = \square\phi$. Thus $(T^2 = T^{*2} = 0)$ $\psi \in \mathfrak{D}^{p,1} \cap (\mathfrak{L}^{p,1} \ominus \mathfrak{H}^{p,1})$ so that

$$d(\phi) = (\psi, \psi) \leqslant C_3 \, d(\psi) = C_3 \, (\square\phi, \square\phi).$$

Next, let Ψ be the unique form in $\mathfrak{D}(L^{p,1}) \cap (\mathfrak{L}^{p,1} \ominus \mathfrak{H}^{p,1})$ such that

$$\square \Psi \equiv \mathfrak{d}\bar{\partial}\Psi + \bar{\partial}\mathfrak{d}\Psi = \psi = \bar{\partial}\phi.$$

Since $\mathfrak{d}\bar{\partial}\Psi$ and $\bar{\partial}\mathfrak{d}\Psi$ are orthogonal (Theorem 5.2) and since $(T^*T\Psi, T\phi) = (T\Psi, T^2\phi) = 0$, we see that $\bar{\partial}\mathfrak{d}\Psi = \bar{\partial}\phi$ and $\mathfrak{d}\bar{\partial}\Psi = 0$. Since $\mathfrak{d}\Psi$ and ϕ are both orthogonal to $\mathfrak{H}^{p,0}$, it follows that $\mathfrak{d}\Psi = \phi$. Thus, we obtain

$$(\phi, \phi) \leqslant d(\Psi) \leqslant C_3 \, (\square\Psi, \square\Psi) = C_3 \, (\bar{\partial}\phi, \bar{\partial}\phi) = C_3 \, d(\phi).$$

(c) follows immediately from (b) and Theorem 5.2.

(d) In view of (b), it is necessary only to prove the complete continuity of N on $\mathfrak{L} \ominus \mathfrak{H}$. To that end, let us assume that $\omega_n \in \mathfrak{L}^{pq} \ominus \mathfrak{H}^{pq}$ with $\|\omega_n\|$ uniformly bounded, and let $\phi_n = N\omega_n$. From part (b), it follows that a subsequence, still called $\{n\}$, is such that $\omega_n \rightharpoonup \omega$, $\bar{\partial}\phi_n \rightharpoonup \bar{\partial}\phi$, $\mathfrak{d}\phi_n \rightharpoonup \mathfrak{d}\phi$ and $\phi_n \rightharpoonup \phi$. If $q \geqslant 1$, it follows from Theorem 5.9 that $\phi_n \to \phi$. In case $q = 0$, let $\psi_n = \bar{\partial}\phi_n$ and $\psi = \bar{\partial}\phi$ as in the proof of (b) for $q = 0$. Then $\mathfrak{d}\psi_n = \omega_n$, $\bar{\partial}\psi_n = 0$, $\mathfrak{d}\psi = \omega$, $\bar{\partial}\psi = 0$

so that we conclude from Theorem 5.9 that $\psi_n - \psi \to 0$ from which it follows (using (b) with $q = 0$) that $\phi_n \to \phi$.

6. The smoothness of the solutions. In this section we prove the smoothness results stated in Section 2.

LEMMA 6.1. *Suppose* $\phi \in \mathfrak{D}$, $\psi \in \mathfrak{D}(T^*)$, $\chi \in \mathfrak{D}(T)$, $\omega \in \mathfrak{L}$, *and*

$$(T\phi - \psi, T\zeta) + (T^*\phi - \chi, T^*\zeta) = (\omega, \zeta) \textit{ for all } \zeta \in \mathfrak{D}. \quad (6.1)$$

Then $\phi \in \mathfrak{D}(L)$ *and* $L\phi = \omega + T^*\psi + T\chi$. *Also* ω, $T^*\psi$, *and* $T\chi \in \mathfrak{L} \ominus \mathfrak{H}$.

PROOF. If ϕ satisfies (6.1), then it satisfies (6.1) with $\psi = \chi = 0$ and ω replaced by $\omega + T^*\psi + T\chi$. Hence it is sufficient to prove this for the case where $\psi = \chi = 0$, since $T^*\psi$ and $T\chi \in \mathfrak{L} \ominus \mathfrak{H}$.

Then, let $\omega = \omega_0 + \omega_1$ where $\omega_0 \in \mathfrak{L} \ominus \mathfrak{H}$ and $\omega_1 \in \mathfrak{H}$. Let $\phi_0 = N\omega_0$ and let $\phi_1 = \phi - \phi_0$. Then $\phi_1 \in \mathfrak{D}$ and satisfies

$$(T\phi_1, T\zeta) + (T^*\phi_1, T^*\zeta) = (\omega_1, \zeta) \text{ for all } \zeta \in \mathfrak{D}.$$

By setting $\zeta = \omega_1 (\in \mathfrak{H})$, we conclude that $\omega_1 = 0$. By setting $\zeta = \phi_1$, we then see that $\phi_1 \in \mathfrak{H}$. The result follows since ϕ_1 and $\phi_0 \in \mathfrak{D}(L)$.

LEMMA 6.2. *Suppose* $\phi \in C^\infty(\overline{M})$ *and define*

$$\chi = f(r).\nu\phi,\ f(0) = 0, f'(0) = 4,\ f \in C^\infty(\overline{M}) \textit{ and } f(r) = 0 \textit{ in } M_{r_0}. \quad (6.2)$$

Then $\chi = 0$ *and* $\nu(\bar\partial\chi - \phi) = 0$ *on* bM.

PROOF. Obviously $\chi = 0$ on bM. In a boundary coordinate patch, let

$$\phi = \sum_{I,J} \phi_{IJ}\, dz^I \wedge d\bar z^J,\ \rho = \nu\phi = \Sigma\, \rho_{IR}\, dz^I \wedge d\bar z^R.$$

Then

$$\rho_{IR} = (-1)^p g^{\epsilon 0} r_{z^\epsilon} \phi_{I,0R},\ \chi_{IR} = f(r)\rho_{IR}.$$

Using (3.1) and (6.2) we see that on bM

$$(\bar\partial\chi)_{IJ} = 4 \sum_{\gamma=1}^{q} (-1)^{\gamma-1} (\partial r/\partial \bar z^{j_\gamma}).\ \rho_{IJ'_\gamma}.$$

Then, on bM

$$
\begin{aligned}
(\nu\bar{\partial}\chi)_{IR} &= g^{\alpha\beta}\, r_{z\beta}(\bar{\partial}\chi)_{I,\alpha R} \\
&= 4g^{\alpha\beta}\, r_{z\beta}\left\{ r_{\bar{z}\alpha}\ \rho_{IR} + \sum_{\gamma=1}^{q-1} (-1)^{\gamma}\, r_{\bar{z}r_{\gamma}}\, \rho_{I,\alpha R'_{\gamma}} \right\} \\
&= 4\left\{ (g^{\alpha\beta}\, r_{\bar{z}\alpha}\, r_{z\beta})\, \rho_{IR} + \sum_{\gamma=1}^{q-1} (-1)^{\gamma}\, r_{\bar{z}r_{\gamma}}\, g^{\alpha\beta} g^{\epsilon\theta}\, r_{z\beta}\, r_{z\theta}\, \phi_{I,\epsilon\alpha R'_{\gamma}} \right\} \\
&= \rho_{IR}, \ .
\end{aligned}
$$

since the second term is zero because $\phi_{I,\epsilon\alpha R'_{\gamma}}$ is antisymmetric in the indices ϵ and α. This is the result.

LEMMA 6.3. *Suppose that $\phi \in \mathfrak{D}(L)$, $L\phi = \omega$, and $\omega \in C^{\infty}(\bar{M})$. Then $\phi \in C^{\infty}(M^{0})$ and $\square^{n}\phi$, $\square^{n}\bar{\partial}\phi$, $\square^{n}\mathfrak{d}\phi$, $\square^{n}\mathfrak{d}\bar{\partial}\phi$, and $\square^{n}\bar{\partial}\mathfrak{d}\phi$ are each the sum of a form in $\mathfrak{A}$ and one in $\mathfrak{D}$.*

PROOF. That $\phi \in C^{\infty}(M^{0})$ follows from the interior regularity theorem. Let $\psi_1 = \bar{\partial}\phi$. Then $\psi_1 \in \mathfrak{D}$ and satisfies

$T\psi_1 = 0$, $(T\psi_1, T\zeta) + (T^{*}\psi_1 - \omega, T^{*}\zeta) = -(TT^{*}\phi, T^{*}\zeta) = 0$, $\zeta \in \mathfrak{D}$, since $T^{*}\zeta \in \mathfrak{D}(T^{*})$. It follows from Lemma 6.1 that $\psi_1 \in \mathfrak{D}(L)$, $\square\psi_1 = \omega_1 = \bar{\partial}\omega$.

Next, choose $\chi_2 = -f(r)\,(\nu\bar{\partial}\omega)$, as in Lemma 6.2, let $\psi_2 = \mathfrak{d}\psi_1 + \chi_2$. Then $\chi_2 \in \mathfrak{D}$, $\chi_2 \in C^{\infty}(\bar{M})$, $\nu(\bar{\partial}\chi_2 + \bar{\partial}\omega) = 0$ on bM. So $\psi_2 \in \mathfrak{D}$ and satisfies

$$(T\psi_2 - \bar{\partial}\omega - \bar{\partial}\chi_2, T\zeta) + (T^{*}\psi_2 - \mathfrak{d}\chi_2, T^{*}\zeta) = 0$$

in which $\bar{\partial}\omega + \bar{\partial}\chi_2 \in \mathfrak{A}_0 \subset \mathfrak{D}(T^{*})$ and $\mathfrak{d}\chi_2 \in \mathfrak{D}(T)$. Thus

$$\psi_2 \in \mathfrak{D}(L) \text{ and } L\,\psi_2 = \omega_2 = \mathfrak{d}\bar{\partial}\omega + \square\chi_2 \in C^{\infty}(\bar{M}).$$

Evidently the cycle can be repeated indefinitely to obtain the theorem, using the fact that $\mathfrak{d}\bar{\partial}\phi = \psi_2 - \chi_2$ and $\bar{\partial}\mathfrak{d}\phi = \omega - \mathfrak{d}\bar{\partial}\phi$, etc.

We now wish to introduce the spaces H^{s} of *forms* ϕ on M; we shall consider only forms $\in C^{\infty}(M^{(0)})$. As was seen in § 3, each point $P_0 \in bM$ is in a neighborhood $\mathfrak{N}$ on M which is simultaneously the range of a tangential analytic coordinate patch σ and a non-analytic boundary coordinate system $\tau : (t, r)$ in which the metric takes the form (3.25) and (t, r) range over $G_R \cup \sigma_R$ for some $R > 0$.

By choosing $\mathfrak{N}$ and R smaller if need be, we may choose a $(\zeta^\alpha, \bar{\zeta}^\alpha)$ basis for the 1-forms which is orthogonal and satisfies (3.24). A *form* $\phi \in C^\infty(M^{(0)})$ also $\in \mathfrak{H}^s(M) \Longleftrightarrow$ each point P_0 is in a neighborhood $\mathfrak{N}$ of the type just described such that if $\mu \in C^\infty(\bar{M})$ and has support in $\mathfrak{N}$ (perhaps $\neq 0$ along $\mathfrak{N} \cap bM$) then the components of $\mu\phi$, with respect to σ and any orthogonal basis $(\zeta^\alpha, \bar{\zeta}^\alpha)$ which satisfies (3.24), $\in \mathfrak{H}^s(G_R)$ when considered as functions of (t, r).

Now suppose $\phi \in C^\infty(M^{(0)}) \cap \mathfrak{L}^{pq}$ and satisfies $L\phi = \omega$, and suppose $\mathfrak{N}$, σ, τ, and $(\zeta^\alpha, \bar{\zeta}^\alpha)$ satisfy the conditions above. Then, since the components of ϕ with respect to σ and the ζ-basis are linear combinations of those with respect to σ and the $(dz, d\bar{z})$ basis, it follows that the σ-ζ-components satisfy (cf (5.12))

$$\left.\begin{aligned} u^j_{rr} + a^{\alpha\beta} u^j_{,\alpha\beta} + b^j_k u^k_r + b^{j\alpha}_k u^k_{,\alpha} + c^j_k u^k &= w^j \\ a^{\alpha\beta}(0, 0) &= \delta^{\alpha\beta} \end{aligned}\right\} \quad (6.3)$$

in the (t, r) system ($u^j_{,\alpha}$ means $\partial u^j/\partial t^\alpha$, etc.) the coefficients being in $C^\infty(\bar{G}_R)$ for R small. It is clear that there exist positive numbers $A, K_0, K_1, \ldots$, independent of P_0, such that each P_0 is in the range of coordinate systems σ and τ above in which an orthogonal ζ-basis exists which satisfies (3.24) with the property that for each $R \leqslant A$ the coefficients in (6.3) can be altered outside $\bar{G}_{3R/4}$ so as to $\in C^\infty$ for $-R \leqslant r \leqslant 0$, to be periodic of period $2R$ in each t^α and to satisfy

$$\left.\begin{aligned} |\nabla^p(a^{\alpha\beta} - \delta^{\alpha\beta})| \leqslant K_p R^{1-p}, \ |\nabla^p(b^j_k - b^j_{k0})| &\leqslant K_p R^{1-p} \\ |\nabla^p(b^{j\alpha}_k - b^{j\alpha}_{k0})| \leqslant K_p R^{1-p}, \ |\nabla^p(c^j_k - c^j_{k0})| &\leqslant K_p R^{1-p}, \\ p = 0, 1, \ldots, \ b^j_{k0} = b^j_k(0, 0), \text{ etc.} & \end{aligned}\right\} \quad (6.4)$$

Moreover, we have seen in § 5 that if $\phi \in \mathfrak{D}^{pq}$ with $q \geqslant 1$, then $\nu\phi \in H^1_{20}(M)$ and if

$$\phi = \sum_{I,J} \phi_{IJ}\, \zeta^I \wedge \bar{\zeta}^J \quad (6.5)$$

then (see (3.27))

$$\nu\phi = \sum_{IR} \omega_{IR}\, \zeta^I \wedge \bar{\zeta}^R, \ \omega_{IR} = (-1)^p(-i/2)\, \phi_{I,\nu R} \quad (6.6)$$

LEMMA 6.4. *Suppose* $\phi \in \mathfrak{D}(L^{pq})$ *with* $q \geqslant 1$ *and* $L\phi = \omega$ *where* $\omega \in \mathfrak{A}^{pq}$ *and suppose* ϕ, $\bar{\partial}\phi$, *and* $\mathfrak{d}\phi \in \mathfrak{H}^s$. *Then* $\phi \in \mathfrak{H}^{s+1/2}$ *and its derivative* $\phi_r \in \mathfrak{H}^{s-1/2}$, *the latter in case* $s \geqslant 1/2$.

PROOF. $\phi \in C^\infty(M^{(0)})$ from interior regularity. Let $P_0 \in bM$ and choose $\mathfrak{R}$ and ζ as above and $R < R(\nu, s, K_0, K_1, \ldots)$, the constant of Theorem 4.8. Let μ have support in $G_{\lambda R} \cup \sigma_{\lambda R}$, $\lambda = (3/4) - (1/\pi)$, with $\mu \in C^\infty(\overline{M})$ and let $\psi = \mu\phi$. Using the formulas (3.23) it is easily seen that ψ, $\bar{\partial}\psi$, and $\mathfrak{d}\,\psi \in \mathfrak{H}^s$. Extend the coefficients in (6.3) to satisfy all the conditions above. For $0 < \epsilon \leqslant 1$, we define the form $S_\epsilon\,\psi = 0$ outside $\mathfrak{R}$ and we define

$$S_\epsilon \psi = \sum_{I,J} (S_\epsilon\, \psi_{IJ})\ \zeta^I \wedge \bar{\zeta}^J \text{ on } \mathfrak{R}, \tag{6.7}$$

the functions $S_\epsilon\,\psi_{IJ}$ being defined in (4.36). For the ϵ considered, $S_\epsilon\,\psi$ has support in the part of $\mathfrak{R}$ corresponding to $G_{3R/4} \cup \sigma_{3R/4}$. From (6.6) and (4.36), we note that $S_\epsilon\,\psi \in \mathfrak{D}$ for these ϵ. From Theorem 5.8, it follows that

$$\int_{\sigma_R} |S_\epsilon\,\psi(t, r)|^2\, dt \leqslant C[d(S_\epsilon \psi) + (S_\epsilon\,\psi, S_\epsilon\,\psi)], \ -R \leqslant r \leqslant 0,\ 0 < \epsilon \leqslant 1. \tag{6.8}$$

We shall prove our result using Theorems 4.6—4.8.

From (3.22), it follows that

$$\begin{aligned}(\bar{\partial}S_\epsilon\,\psi, \bar{\partial}S_\epsilon \psi) &= \sum_{IM} (\Gamma\bar{\partial}S_\epsilon\,\psi_{IM}, \bar{\partial}S_\epsilon\,\psi_{IM}) \\ &\leqslant C \sum_{IM} [\|S_\epsilon(\bar{\partial}\psi)_{IM}\|^2 + \|(\bar{\partial}S_\epsilon\,\psi - S_\epsilon\,\bar{\partial}\psi)_{IM}\|^2],\end{aligned} \tag{6.9}$$

the inner products and norms being those for $L_2(G_R)$. We assume that the function χ (see (4.36)) satisfies the hypotheses of Theorem 4.7 with s replaced by $s + 1/2$. Using (3.23) and (3.26) and the fact that the operators S_ϵ and $\partial/\partial^\lambda$ and $\partial/\partial r$ commute, we see that

$$\begin{aligned}(\bar{\partial}S_\epsilon \psi - S_\epsilon\,\bar{\partial}\psi)_{IM} &= \sum_{\gamma=1}^{q+1} (-1)^{p+\gamma-1} \sum_{\lambda} (\bar{e}^\lambda_{m_\gamma} S_\epsilon\,\psi_{IM'_\gamma t^\lambda} - S_\epsilon\,\bar{e}^\lambda_{m_\gamma}\,\psi_{IM'_\gamma t^\lambda}) + \\ &+ \sum (B^{KT}_{IM}\, S_\epsilon\,\psi_{KT} - S_\epsilon\, B^{KT}_{IM}\,\psi_{KT}).\end{aligned} \tag{6.10}$$

The derivatives with respect to r do not occur since their coefficients are constants. We conclude from Theorem 4.7 that

$$\int_0^1 \|\bar{\partial} S_\epsilon \psi - S_\epsilon \bar{\partial}\psi\|_G^2 \, \epsilon^{-2s-1} \, d\epsilon \leqslant C(\|\psi\|_G^s)^2 \quad (G = G_R).$$

From our hypothesis and Theorem 4.6, we conclude that

$$\int_0^1 \|S_\epsilon \bar{\partial}\psi\|_G^2 \, \epsilon^{-2s-1} \, d\epsilon \leqslant C(\|\bar{\partial}\psi\|_G^s)^2 \quad (G = G_R).$$

A similar analysis holds for $\mathfrak{d}\, S_\epsilon \psi$ and, of course for $S_\epsilon \psi$. Thus, we conclude using (6.8) that

$$\int_0^1 \|S_\epsilon \psi(\,.\,, y)\|_\sigma^2 \, \epsilon^{-2s-1} \, d\epsilon \quad (\sigma = \sigma_R)$$

is uniformly bounded. Consequently the result follows from Theorems 4.6 and 4.8.

THEOREM 6.1. *Suppose $\phi \in \mathfrak{D}(L^{pq})$ with $q \geqslant 1$ and $L\phi = \omega$, where $\omega \in C^\infty(\bar{M})$. Then $\phi \in C^\infty(\bar{M})$. The same result holds if $q = 0$, provided that $\phi \in \mathfrak{L}^{p0} \ominus \mathfrak{H}^{p0}$.*

PROOF. We assume first that $q \geqslant 2$. Then all the forms $\square^n\phi$, $\square^n\bar{\partial}\phi$, $\square^n\mathfrak{d}\phi$, $\square^n\bar{\partial}\mathfrak{d}\phi$, and $\square^n\mathfrak{d}\,\bar{\partial}\phi$ satisfy the conclusions of Lemma 6.3 and are of type (p, q) with $q' \geqslant 1$. Thus since $\mathfrak{d}\,\phi$ and $\bar{\partial}\phi \in \mathfrak{H}^0$, we conclude that $\phi \in \mathfrak{H}^{1/2}$. Then, since $\bar{\partial}\mathfrak{d}\,\phi$ and $\mathfrak{d}\,\bar{\partial}\phi \in \mathfrak{H}^0$, we conclude that $\bar{\partial}\phi$ and $\mathfrak{d}\,\phi \in \mathfrak{H}^{1/2}$ and hence that $\phi \in \mathfrak{H}^1$. This process may be continued to show that $\phi \in$ every $\mathfrak{H}^s$. But then, from Theorem 4.8 or Lemma 6.4, we first conclude that $\phi_r \in$ every $\mathfrak{H}^s$. But then by repeated use of the differential equations (6.3), we conclude that all derivatives $\in$ every $\mathfrak{H}^s$. That $\phi \in C^\infty(\bar{M})$ then follows from the Sobolev lemma.

In case $q = 1$, we first conclude that $\bar{\partial}\,\phi$ and $\mathfrak{d}\,\phi \in \mathfrak{H}^0$ so that $\phi \in \mathfrak{H}^{1/2}$. Then we conclude that $\mathfrak{d}\,\bar{\partial}\,\mathfrak{d}\,\phi$ (and $\bar{\partial}\,\bar{\partial}\,\mathfrak{d}\,\phi$) $\in \mathfrak{H}^0$ so $\bar{\partial}\,\mathfrak{d}\,\phi \in \mathfrak{H}^{1/2}$. Using the inequality $(\mathfrak{d}\,\phi, \mathfrak{d}\,\phi) \leqslant C(\bar{\partial}\,\mathfrak{d}\,\phi, \bar{\partial}\,\mathfrak{d}\,\phi)$ with ϕ replaced by $S_\epsilon(\mu\,\phi)$ in patches, we conclude that $\mathfrak{d}\,\phi \in \mathfrak{H}^1$; that

$\bar{\partial}\phi \in \mathfrak{H}^{1/2}$ follows from the fact that $\mathfrak{d}\,\bar{\partial}\phi \in \mathfrak{H}^0$. Thus $\phi \in \mathfrak{H}^1$. The process can be continued indefinitely.

In case $q = 0$, we conclude that $\bar{\partial}\phi \in C^\infty(\bar{M})$, since $\bar{\partial}\phi \in \mathfrak{D}(L)$. If we choose $\Phi \in \mathfrak{D}(L)$ so that $\square\,\Phi = \bar{\partial}\phi$, we see that $\Phi \in C^\infty(\bar{M})$ and, as in the proof of Theorem 5.10(b), that $\phi = \mathfrak{d}\,\Phi$.

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EXISTENCE OF LOCAL COORDINATES FOR STRUCTURES DEFINED BY ELLIPTIC PSEUDOGROUPS

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1. Introduction. By "differentiable" we shall always mean "differentiable of class C^∞."

Let Γ be a transitive, continuous pseudogroup of order μ_0 of local bidifferentiable transformations of real n-space $\mathbf{R}^n$. We say that a differentiable manifold is a Γ-manifold if it is covered by local differentiable coordinates, called Γ-coordinates, which are transformed into one another by elements of Γ. Let M be a Γ-manifold of (real) dimension n. Then the pseudogroup Γ operating locally on $\mathbf{R}^n$ induces a pseudogroup of local bidifferentiable transformations of M which, for simplicity, we also denote by Γ. The infinitesimal transformations of the pseudogroup Γ operating (locally) on M or, as we shall say, the Γ-vector fields, have a structure of Lie pseudo-algebra and are defined by a system of linear partial differential equations of order μ_0 on M which we denote by $\mathscr{S}^{\mu_0} = \mathscr{S}^{\mu_0}(\Gamma)$.

For each non-negative integer μ, let P^μ be the bundle over the Γ-manifold M of jets of order μ of bidifferentiable maps of $\mathbf{R}^n$ into M with source at the origin of $\mathbf{R}^n$. Then P^μ, the bundle over M of frames of order μ, is a principal bundle with fibre and group G^μ, where G^μ is a Lie group, namely a subgroup of the prolongation $GL^\mu(n, \mathbf{R})$ of order μ of the general linear group $GL(n, \mathbf{R}) = GL^1(n, \mathbf{R})$.

Now let M be an arbitrary differentiable manifold of (real) dimension n. Then the bundle Q^μ over M of frames of order μ is a principal bundle with fibre and group $GL^\mu(n, \mathbf{R})$.

DEFINITION 1.1. *Suppose that Γ is a transitive, continuous pseudogroup of order μ_0 of local bidifferentiable transformations. An almost Γ-structure on M is a differentiable reduction of the structure group $GL^{\mu_0}(n, R)$ of Q^{μ_0} to the subgroup $G^{\mu_0} = G^{\mu_0}(\Gamma)$ associated with Γ.*

We call a differentiable reduction of the structure group $GL^{\mu}(n, \mathbf{R})$ of Q^{μ} to G^{μ} a reduction of order μ, or μ-reduction, and we denote by P^{μ} the reduced principal bundle over M with fibre and group G^{μ}. We observe that a reduction of order μ defines by projection, for each ν, $0 < \nu < \mu$, a reduction of order ν—namely its ν-projection. A μ-reduction will be called a prolongation of a ν-reduction, $0 < \nu < \mu$, if the μ-reduction projects onto the ν-reduction.

Suppose that a μ-reduction, with reduced bundle P^{μ}, is given. The group G^{μ} operates on the right on P^{μ}, hence it operates on the right on the tangent bundle $T(P^{\mu})$ of P^{μ}, and we let $S^{\mu} = T(P^{\mu})/G^{\mu}$ be the vector bundle over M with fibre $\mathbf{R}^n \oplus \mathfrak{g}^{\mu}$, where $\mathfrak{g}^{\mu}$ is the Lie algebra of G^{μ} and $T(P^{\mu})/G^{\mu}$ is obtained by identifying conjugate points of $T(P^{\mu})$ under the right action of G^{μ}. We remark that S^{μ} can be identified with the bundle over M of jets of order μ of the Γ-vector fields. Let $S^0 = T(M)$, the reduced tangent bundle of M, and let $\Sigma^{\mu} = \oplus_i \Sigma^{\mu,i}$, where $\Sigma^{\mu,i}$ is the sheaf of germs of differential forms on M with values in S^{μ}. We set $\Sigma^{\mu} = 0$ if $\mu < 0$, and denote by $\Sigma^{\mu}_{\mu-1} = \oplus_i \Sigma^{\mu,i}_{\mu-1}$ the kernel of the projection of Σ^{μ} onto $\Sigma^{\mu-1}$, i.e.

$$0 \longrightarrow \Sigma^{\mu}_{\mu-1} \longrightarrow \Sigma^{\mu} \longrightarrow \Sigma^{\mu-1} \longrightarrow 0.$$

Next, a reduction of order $\mu + 1$ defines, by projection, reductions of orders μ, $\mu - 1$, with bundles S^{μ}, $S^{\mu-1}$, and it is represented locally by a connection from S^{μ} to $S^{\mu-1}$, that is by an $\mathbf{R}$-linear map of degree 1,

$$D^{\mu-1} : \Sigma^{\mu} \longrightarrow \Sigma^{\mu-1}. \tag{1.1}$$

We call the (local) connection from S^{μ} to $S^{\mu-1}$ a (local) connection of order μ or μ-connection. The negative of the (local) map obtained by restriction of $D^{\mu-1}$ to $\Sigma^{\mu}_{\mu-1}$ is linear over the local rings of differentiable functions and will be denoted by δ, namely

$$\delta : \Sigma^{\mu}_{\mu-1} \longrightarrow \Sigma^{\mu-1}_{\mu-2}. \tag{1.2}$$

We have $\delta^2 = \delta \circ \delta = 0$. Let $\Lambda^{\mu}_{\mu-1} = \oplus_i \Lambda^{\mu,i}_{\mu-1}$ be the kernel of the map (1.2), and let $H^{\mu}_{\mu-1} = \oplus_i H^{\mu,i}_{\mu-1}$ where

$$H^{\mu,i}_{\mu-1} = \Lambda^{\mu,i}_{\mu-1}/\delta\,(\Sigma^{\mu+1,i-1}_{\mu})$$

is the cohomology of order μ and degree i. The following theorem is a consequence of Theorem 5.1 of the paper [13 (a)] :

THEOREM 1.1. *There is a positive integer $\mu_1 = \mu_1(\mu_0, n)$, depending only on the order μ_0 and n, such that $H^{\mu}_{\mu-1} = 0$ for $\mu \geqslant \mu_1$.*

For formal reasons, we define the order μ_0 such that it is always positive, and we let ν_1 be the smallest positive integer satisfying $\nu_1 \geqslant \mu_0$ and $H^{\mu}_{\mu-1} = 0$ for $\mu \geqslant \nu_1$. Then $\mu_0 \leqslant \nu_1 \leqslant \mu_1$.

A (local) $(\mu + 1)$-connection, or (local) map $D^{\mu} : \Sigma^{\mu+1} \to \Sigma^{\mu}$, defines by projection a (local) μ-connection, or (local) map $D^{\mu-1} : \Sigma^{\mu} \to \Sigma^{\mu-1}$, and the composition $D^{\mu-1} \circ D^{\mu}$ is a (local) map of degree 2, linear over the local rings of differentiable functions, namely

$$D^{\mu-1} \circ D^{\mu} : \Sigma^{\mu+1} \longrightarrow \Sigma^{\mu-1}. \tag{1.3}$$

DEFINITION 1.2. *We say that a (local) μ-connection is torsionless if the map $D^{\mu-1} \circ D^{\mu}$, defined by any prolongation of it to a $(\mu + 1)$-connection, vanishes (locally) on the subsheaf $\Sigma_0^{\mu+1}$ of $\Sigma^{\mu+1}$* (see Goldschmidt [5], Singer-Sternberg [12], Guillemin [6]).

DEFINITION 1.3. *An almost Γ-structure is torsionless, or integrable, if each point has a neighborhood on which the (local) connection of order $\mu_0 - 1$ can be prolonged to a torsionless connection of order ν_1.*

We remark that, if an almost Γ-structure is integrable, its (local) connection of order $\mu_0 - 1$ can be prolonged to a torsionless connection of arbitrarily high order (see, for example, Goldschmidt [5]).

Finally, we say that a transitive, continuous pseudogroup Γ is elliptic if the system $\mathscr{S}^{\mu_0}(\Gamma)$ of linear partial differential equations defining its Γ-vector fields (infinitesimal transformations) is elliptic. The purpose here is to outline a proof of the following theorem : *if Γ is elliptic, an integrable almost Γ-structure on a manifold M is a Γ-structure, i.e. M is covered by local Γ-coordinates transforming into one another by elements of Γ.*

This result is a generalization of the well-known complex Frobenius theorem of Newlander-Nirenberg [10] and Nirenberg [11].

Sections 2—5 of this paper concern arbitrary elliptic systems of linear partial differential equations and do not involve the notion of a pseudogroup. In these sections the theory of harmonic differential forms associated with an arbitrary elliptic system of equations is described (see [13 (b), (c)]). In the remaining sections of the paper the existence of coordinates compatible with an integrable almost Γ-structure is reduced to a Poincaré lemma by means of a theorem proved in [13 (b)], and the Poincaré lemma is then established, in the elliptic case, by solving a local Neumann problem.

2. The resolution associated with a system of linear partial differential equations. Let $p=(p_1, \dots, p_n)$ denote an ordered set of n non-negative integers $p_1, \dots, p_n$, and write $|p| = p_1 + p_2 + \dots + p_n$. Moreover, if $x = (x^1, \dots, x^j, \dots, x^n)$ is a coordinate, we define $\partial_p = (\partial/\partial x^1)^{p_1}(\partial/\partial x^2)^{p_2} \dots (\partial/\partial x^n)^{p_n}$. Finally let $O^\nu(\mathbf{R}^n)$, $\nu \geqslant 0$, be the ν-tuple symmetric product of (real) n-space $\mathbf{R}^n$, and write

$$F^\mu = \mathrm{Hom}\ (\bigoplus_{0\leqslant\nu\leqslant\mu} O^\nu(\mathbf{R}^n),\ \mathbf{R}^m) = \bigoplus_{0\leqslant\nu\leqslant\mu} F^\nu_{\nu-1}$$

where

$$F^\nu_{\nu-1} = \mathrm{Hom}\ (O^\nu(\mathbf{R}^n), \mathbf{R}^m)$$

and $F^0_{-1} = F^0 = \mathbf{R}^m$. Let $\sigma^\mu = \{\sigma^j_p \mid 1 \leqslant j \leqslant m, 0 \leqslant |p| \leqslant \mu\}$ be the coordinate of F^μ where $\sigma = \sigma^0 = (\sigma^1, \dots, \sigma^j, \dots, \sigma^m)$ is the coordinate of $\mathbf{R}^m$.

Now let M be a differentiable manifold of dimension n, and let Q be a differentiable vector bundle over M with fibre $\mathbf{R}^m$. For each non-negative integer μ, we denote by $S^\mu_d = S^\mu_d(Q)$ the differentiable vector bundle over M, with fibre F^μ, of all jets of order μ of differentiable sections of Q over M. Let U be a neighborhood of M, covered by a differentiable coordinate $x = (x^1, \dots, x^j, \dots, x^n)$, such that $Q \mid U \simeq U \times \mathbf{R}^m$. Then $S^\mu_d \mid U \simeq U \times F^\mu$ is covered by the coordinate $(x, \sigma) = (x, \sigma^\mu)$.

A differentiable section $f\colon M \to Q$ induces, for each μ, a differentiable section $\iota^\mu(f)\colon M \to S^\mu_d$ which, expressed in terms of a local coordinate (x, σ), sends x into $\iota^\mu(f)\,(x) = (x, (\partial f)\,(x))$ where

$$\partial f = \partial^\mu f = \{\partial_p f^j \mid 1 \leqslant j \leqslant m,\ 0 \leqslant |p| \leqslant \mu\}.$$

DEFINITION 2.1. *A linear partial differential equation on M of order μ_0 is the kernel $E^{\mu_0} = E^{\mu_0}(Q, R)$ of a differentiable map $a^{\mu_0}: S_d^{\mu_0}(Q) \to R$, where Q, R are differentiable vector bundles over M, and a^{μ_0} maps each fibre of $S_d^{\mu_0} = S_d^{\mu_0}(Q) \to M$ linearly onto a fibre of $R \to M$ and induces the identity map on the base space M. A solution of the equation $E^{\mu_0} = E^{\mu_0}(Q, R)$ is a differentiable section $f: M \to Q$ which induces a section $\iota^{\mu_0}(f): M \to E^{\mu_0}$.*

A linear partial differential equation $E^{\mu_0} = E^{\mu_0}(Q, R)$ is defined locally, in terms of a local coordinate (x, σ) for $S_d^{\mu_0}$, by a finite number of equations

$$\mathfrak{f}^k(x, \sigma) = 0, \quad k = 1, 2, \ldots, h,$$

each of which is linear in σ. Therefore, a solution f of E^{μ_0}, expressed locally in terms of a coordinate (x, σ), satisfies the equations

$$\mathfrak{f}^k(x, (\partial f)(x)) = 0, \quad k = 1, 2, \ldots, h.$$

Let $E^{\mu_0} = E^{\mu_0}(Q, R)$ be a partial differential equation (of order μ_0). Then E^{μ_0} can be prolonged, by differentiation, to a linear partial differential equation $E^{\mu_0+1} = \iota^1(E^{\mu_0})$ of order $\mu_0 + 1$ where E^{μ_0+1} is defined locally, in terms of a coordinate, by the equations

$$\begin{cases} \mathfrak{f}^k(x, \sigma^\mu) = 0, \\ \partial \mathfrak{f}^k(x, \sigma^{\mu+1}) = 0, \end{cases} \quad k = 1, 2, \ldots, h,$$

where

$$\partial \mathfrak{f}^k(x, \sigma^{\mu+1}) = \partial^1 \mathfrak{f}^k(x, \sigma^{\mu+1}) = \{\partial_j \mathfrak{f}^k(x, \sigma^{\mu+1}) \mid j = 1, 2, \ldots, n\}$$

and

$$\partial_j \mathfrak{f}^k = \frac{\partial \mathfrak{f}^k}{\partial x^j} + \sum_{l=1}^{m} \sum_{0 \leqslant |p| \leqslant \mu} \frac{\partial \mathfrak{f}^k}{\partial \sigma_p^l} \cdot \sigma_{p+1_j}^l,$$

where 1_j is the set of n integers all of which are zero except the j-th, which is equal to 1. For each non-negative integer μ, we define $E^\mu = E^\mu(E^{\mu_0})$ as follows :

$$E^\mu = \begin{cases} E^{\mu_0} = \iota^0(E^{\mu_0}), & \mu = \mu_0, \\ \iota^{\mu-\mu_0}(E^{\mu_0}) = \iota^1(\iota^{\mu-\mu_0-1}(E^{\mu_0})), & \mu > \mu_0, \\ pr^\mu(E^{\mu_0}), & 0 \leq \mu < \mu_0, \end{cases}$$

where $pr^\mu(E^{\mu_0})$ denotes the projection of E^{μ_0} in S_d^μ. Set

$$S^\mu = \bigcap_{\nu \geq \mu} pr^\mu(E^\nu).$$

For each non-negative integer μ, we then have the projection $S^{\mu+1} \to S^\mu$ whose kernel we denote by $S_\mu^{\mu+1}$, i.e.,

$$0 \longrightarrow S_\mu^{\mu+1} \longrightarrow S^{\mu+1} \longrightarrow S^\mu \longrightarrow 0 \tag{2.1}$$

where

$$S^\mu \subseteq E^\mu \subseteq S_d^\mu. \tag{2.2}$$

DEFINITION 2.2. *We say that the equation is regular if, for each μ, E^μ coincides with S^μ and is a differentiable vector bundle over M.*

If the equation is regular, we denote it by $\mathscr{S}^{\mu_0}$ and we say that the manifold M, over which it is defined, is an $\mathscr{S}^{\mu_0}$-manifold.

Let M be an $\mathscr{S}^{\mu_0}$-manifold, denote by $T^*(M)$ the dual of the tangent bundle $T(M)$ of M, and let $\wedge^i T^*(M)$ be the i-tuple exterior product of $T^*(M)$. We let

$$S_d^{\mu,i} = S_d^\mu \otimes \wedge^i T^*(M)$$

for $\mu \geq 0$, and we define $S_d^{\mu,i} = 0$ for $\mu < 0$. Moreover, we let $S_{d,\mu-1}^{\mu,i}$ be the kernel of the projection $S_d^{\mu,i} \to S_d^{\mu-1,i}$. Let U be a sufficiently small neighborhood of an arbitrary point of M; then

$$S_{d,\mu-1}^{\mu,i} \mid U \simeq U \times \{F_{\mu-1}^\mu \otimes \text{Hom}\,(\wedge^i(\mathbf{R}^n), \mathbf{R})\}$$

and $S_{d,\mu-1}^{\mu,i}$ is covered by the coordinate (x, σ) where $\sigma = \{\sigma_q \mid |q| = \mu\}$, $\sigma_q = \{\sigma_q^j \mid i \leq j \leq m\}$ and

$$\sigma_q^j = \sum \sigma_{q,k_1 \ldots k_i}^j \, dx^{k_1} \wedge \ldots \wedge dx^{k_i},$$

where the summation is over $k_1, \ldots, k_i$ satisfying $1 \leq k_1 < \ldots < k_i \leq n$. We have the map

$$\delta : S_{d,\mu}^{\mu+1,i} \longrightarrow S_{d,\mu-1}^{\mu,i+1} \tag{2.3}$$

sending σ into $\delta\sigma$ where $\delta\sigma$ has the components $\{(\delta\sigma)_p \mid |p| = \mu\}$ and

$$(\delta\sigma)_p = \sum_{j=1}^{n} dx^j \wedge \sigma_{p+1_j} \tag{2.4}$$

where $p+1_j = (p_1, \ldots, p_j + 1, \ldots, p_n)$. Clearly we have $\delta^2 = \delta \circ \delta = 0$.

Define $S^{\mu,i}$ in the same way as $S_d^{\mu,i}$, with S^μ replacing S_d^μ. Then $S^{\mu,i} \subset S_d^{\mu,i}$ and, in particular, $S^{\mu,i} = 0$ for $\mu < 0$. If $\mu \geqslant \mu_0$, the restriction of δ to $S_\mu^{\mu+1,i}$, the kernel of the projection $S^{\mu+1,i} \to S^{\mu,i}$, defines a map

$$\delta : S_\mu^{\mu+1,i} \longrightarrow S_{\mu-1}^{\mu,i+1}, \tag{2.5}$$

and we denote the kernel of (2.5) by $L_\mu^{\mu+1,i}$.

The following theorem is essentially a restatement, in terms of vector bundles, of Theorem 1.1.

THEOREM 2.1. *If M is an $\mathscr{S}^{\mu_0}$-manifold, there is an integer $\mu_1 = \mu_1(\mu_0, n)$, depending only on the order μ_0 of $\mathscr{S}^{\mu_0}$ and the dimension n of M, where $\mu_1 > \mu_0$, such that the sequence*

$$0 \longrightarrow L_\mu^{\mu+1,i} \longrightarrow S_\mu^{\mu+1,i} \xrightarrow{\ \delta\ } L_{\mu-1}^{\mu,i+1} \longrightarrow 0 \tag{2.6}$$

is exact for $\mu \geqslant \mu_1$ (and all i, $0 \leqslant i \leqslant n$).

Suppose that M is an $\mathscr{S}^{\mu_0}$-manifold, and let $\Sigma^\mu = \oplus_i \Sigma^{\mu,i}$ where $\Sigma^{\mu,i}$ is the sheaf over M of germs of (differentiable) sections of $S^{\mu,i}$. Moreover, let $J^{\mu,i}$ be the sheaf of pairs $u = (\sigma, \xi)$ where, for some element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1,i}$, $\sigma = \sigma^\mu$ is the projection of $\sigma^{\mu+1}$ in $\Sigma^{\mu,i}$ and $\xi = \delta\sigma^{\mu+1}$ has components defined by (2.4) for $0 \leqslant |p| < \mu$. For each i, $0 \leqslant i < n$, we have the surjective map

$$\#^\mu : \Sigma^{\mu+1,i} \longrightarrow J^{\mu,i}$$

sending $\sigma^{\mu+1}$ into $u = (\sigma, \xi)$, and its kernel is the sheaf $\Lambda_\mu^{\mu+1,i}$ of germs of sections of $L_\mu^{\mu+1,i}$, i.e.

$$0 \longrightarrow \Lambda_\mu^{\mu+1,i} \longrightarrow \Sigma^{\mu+1,i} \xrightarrow{\ \#^\mu\ } J^{\mu,i} \longrightarrow 0.$$

We have the map (see [13])

$$D : J^{\mu,i} \longrightarrow J^{\mu,i+1}$$

sending $u = (\sigma, \xi)$ into $Du = D(\sigma, \xi) = (d\sigma - \xi, -d\xi)$. Clearly we have $D^2 = D \circ D = 0$.

DEFINITION 2.3. *We call $J^{\mu} = \oplus J^{\mu,i}$ the sheaf (over M) of jet forms of order μ belonging to $\mathscr{S}^{\mu_0}$.*

The regularity implies that J^{μ} is a sheaf of germs of differentiable sections of a (differentiable) vector bundle over M and therefore, in particular, J^{μ} is a fine sheaf.

Finally, let $D^{\mu} : \Sigma^{\mu+1,i} \longrightarrow \Sigma^{\mu,i+1}$ be the map sending $\sigma^{\mu+1}$ into $D^{\mu}(\sigma^{\mu+1}) = d\sigma^{\mu} - \delta\sigma^{\mu+1}$. For $\nu \leqslant \mu$, let $\Sigma^{\mu+1,i}_{\nu}$ be the kernel of the projection $\Sigma^{\mu+1,i} \longrightarrow \Sigma^{\nu,i}$. The restriction of D^{μ} to $\Sigma^{\mu+1}_{\nu}$ maps $\Sigma^{\mu+1}_{\nu}$ into $\Sigma^{\mu}_{\nu-1}$, where $\Sigma^{\mu}_{-1} = \Sigma^{\mu}$, and we denote by $\Xi^{\mu+1}_{\nu} = \oplus\, \Xi^{\mu+1,i}_{\nu}$ the maximal subsheaf of $\Sigma^{\mu+1}_{\nu}$ which is mapped by D^{μ} into Ξ^{μ}_{ν}. Then (see [13(a)], page 383)

$$\Xi^{\mu}_{\mu-1} = \Lambda^{\mu}_{\mu-1},$$

where $\Lambda^{\mu}_{\mu-1} = \oplus\, \Lambda^{\mu,i}_{\mu-1}$, and we have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{\mu+1}_{\mu} & \longrightarrow & \Xi^{\mu+1}_{\mu-1} & \longrightarrow & \Lambda^{\mu}_{\mu-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Sigma^{\mu+1}_{\mu} & \longrightarrow & \Sigma^{\mu+1} & \longrightarrow & \Sigma^{\mu} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & J^{\mu-1} & = & J^{\mu-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Moreover, denoting by $J^{\mu}_{\mu-1} = \oplus J^{\mu,i}_{\mu-1}$ the kernel of the projection $J^{\mu} \to J^{\mu-1}$, we have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{\mu+1}_{\mu} & \longrightarrow & \Xi^{\mu+1}_{\mu-1} & \xrightarrow{\#^{\mu}} & J^{\mu}_{\mu-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^{\mu+1}_{\mu} & \longrightarrow & \Sigma^{\mu+1} & \xrightarrow{\#^{\mu}} & J^{\mu} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & J^{\mu-1} & = & J^{\mu-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now let M be an $\mathscr{S}^{\mu_0}$-manifold of dimension n, and let Θ be the sheaf over M of germs of solutions of the regular partial differential equation of order μ_0. Moreover, let

$$i = i^{\mu} : \Theta \longrightarrow J^{\mu,0}$$

be the injection sending f into

$$i(f) = i^{\mu}(f) = (\iota^{\mu}(f),\ \delta\iota^{\mu+1}(f)) = (\iota^{\mu}(f),\ d\iota^{\mu}(f)).$$

If $\mu \geqslant \mu_0 - 1$, it is easily seen that the sequence

$$0 \longrightarrow \Theta \xrightarrow{\ i\ } J^{\mu,0} \xrightarrow{\ D\ } J^{\mu,1} \xrightarrow{\ D\ } J^{\mu,2} \xrightarrow{\ D\ } \cdots \xrightarrow{\ D\ } J^{\mu,n} \longrightarrow 0 \tag{2.7}$$

is exact at $J^{\mu,0}$.

DEFINITION 2.4. *We call* (2.7) *the resolution* (*by jet forms*) *of order* μ *of the sheaf* Θ *of solutions of* $\mathscr{S}^{\mu_0}$.

EXAMPLES. (1) (de Rham's theorem). Let M be an $\mathscr{S}^1$-manifold, where $\mathscr{S}^1$ is the equation $df = 0$ for the real-valued function f. Then $\Theta = \mathbf{R}$ (real numbers) and $J^{\mu,i} = A^i$ for $\mu \geqslant 0$ and $0 \leqslant i \leqslant n$, where A^i is the sheaf of germs of (real-valued) differential forms of degree i.

In this case (2.7) coincides (for arbitrary $\mu \geqslant 0$) with the classical (exact) resolution of de Rham, namely

$$0 \longrightarrow \mathbf{R} \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \dots \xrightarrow{d} A^n \longrightarrow 0.$$

(2) Let M be a differentiable manifold with a foliate structure whose sheets are real m-dimensional manifolds. This structure is represented by a covering $\mathfrak{U} = \{U_\alpha\}$, where U_α is a domain of the local coordinates $(x_\alpha, y_\alpha) = (x_\alpha^1, \dots, x_\alpha^m, y_\alpha^1, \dots, y_\alpha^{n-m})$, $n > m$, and the transition functions for these coordinates have the form

$$\left.\begin{aligned} x_\alpha &= f_{\alpha\beta}(x_\beta, y_\beta) \\ y_\alpha &= g_{\alpha\beta}(y_\beta), \end{aligned}\right\} \tag{2.8}$$

where $f_{\alpha\beta}$ is differentiable in x_β, y_β, the jacobian matrix $\partial(x_\alpha)/\partial(x_\beta)$ is non-singular, and $g_{\alpha\beta}$ is differentiable in y_β and the jacobian matrix $\partial(y_\alpha)/\partial(y_\beta)$ is non-singular. Then M is an $\mathscr{S}^1$-manifold, where $\mathscr{S}^1$ is represented, in terms of the coordinates (x_α, y_α), by the equations

$$\frac{\partial f}{\partial x_\alpha^j} = 0, \quad j = 1, 2, \dots, m, \tag{2.9}$$

for the real-valued function f. It is now convenient to write $x^{m+1} = y^1, \dots, x^n = y^{n-m}$. The equations (2.9) (which remain unchanged) imply that an element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1} = \oplus_i \Sigma^{\mu+1,i}$ has the components σ_p, where $\sigma_p = 0$ unless $p = (p_1, \dots, p_n)$ where $p_1 = 0, \dots, p_m = 0$. In this case $J^{0,i}$ is composed of the pairs $u = (\sigma, \xi)$, where σ is a (real-valued) differential form of degree i and ξ is locally equal to a (real-valued) differential form of degree $i+1$ which belongs to the ideal generated by $dx^{m+1}, \dots, dx^n$. The sequence (2.7) is exact for $\mu \geqslant 0$ (see [7], [13(a)]).

(3) (Cauchy-Riemann equations). Let M be a complex analytic manifold of (complex) dimension m, and let $z = (z^1, \dots, z^j, \dots, z^m)$ be a local holomorphic coordinate on M. Write $z^j = x^{2j-1} + (-1)^{\frac{1}{2}} x^{2j}$, $j = 1, 2, \dots, m$, where $x = (x^1, \dots, x^j, \dots, x^n)$, $n = 2m$, and define

$$\frac{\partial}{\partial \bar{z}^j} = \tfrac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} + (-1)^{\frac{1}{2}} \frac{\partial}{\partial x^{2j}}\right), \quad j = 1, 2, \dots, m.$$

The equations

$$\frac{\partial f}{\partial \bar{z}^j} = 0, \quad j = 1, 2, \dots, m, \tag{2.10}$$

for the complex-valued function f, have as solutions the functions holomorphic in $z = (z^1, \dots, z^j, \dots, z^n)$. Introduce the self-conjugate coordinate $(z, \bar{z}) = (z^1, \dots, z^j, \dots, z^m, \bar{z}^1, \dots, \bar{z}^j, \dots, \bar{z}^m)$, and write $r = p + \bar{p} = (p^1, \dots, p^j, \dots, p^n, \bar{p}^1, \dots, \bar{p}^j, \dots, \bar{p}^n)$ where p^j and $\bar{p}^j$ are non-negative integers. An element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1} = \oplus_i \Sigma^{\mu+1,i}$ has the components $\sigma_{p+\bar{p}}$, where $\sigma_{p+\bar{p}} = 0$ unless $p = 0$. In this case $J^{0,i}$ is the sheaf of germs of pairs $u = (\sigma, \xi)$, where σ is a (complex-valued) differential form of degree i and ξ is a (complex-valued) differential form of degree $i + 1$ which belongs to the ideal generated over the (differentiable) functions by $dz^1, \dots, dz^j, \dots, dz^n$. The sequence (2.7) is exact for $\mu \geqslant 0$ (see [7], [13(a)]). Finally, let $A^{0,i}$ denote the sheaf over M of germs of (complex-valued) differential forms of type $(0, i)$, and let $\pi : J^{0,i} \to A^{0,i}$ be the projection sending $u = (\sigma, \xi)$ into the component of σ of type $(0, i)$. The differential operator D on J^0 splits into the sum of two operators D', D'', where

$$D'(\sigma, \xi) = (\partial\sigma - \xi, -\partial\xi), \; D''(\sigma, \xi) = (\bar{\partial}\sigma, -\bar{\partial}\xi)$$

and $d = \partial + \bar{\partial}$ is the usual splitting of the exterior differential operator d into operators ∂, $\bar{\partial}$ of types $(1, 0)$, $(0, 1)$, respectively. The following diagram is exact and commutative :

$$\begin{array}{ccccccccccccccccc}
0 & \to & \Theta & \to & J^{0,0} & \xrightarrow{D} & J^{0,1} & \xrightarrow{D} & J^{0,2} & \xrightarrow{D} & \dots & \to & J^{0,m} & \xrightarrow{D} & \dots \xrightarrow{D} & J^{0,n} & \to 0 \\
 & & \| & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & & & \downarrow{\scriptstyle\pi} & & & & \\
0 & \to & \Theta & \to & A^{0,0} & \xrightarrow{\bar{\partial}} & A^{0,1} & \xrightarrow{\bar{\partial}} & A^{0,2} & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & A^{0,m} & \to & 0 & &
\end{array}$$

The second line of this diagram is the classical Dolbeault resolution of the sheaf Θ of germs of holomorphic functions on M.

3. Elliptic systems of equations. Let M be an $\mathscr{S}^{\mu_0}$-manifold, choose a metric, let $S^*(M)$ be the corresponding unit cotangent sphere bundle and let $\pi : S^*(M) \to M$ be the projection. Denote by $\pi^* S^{\nu,i}_{\nu-1}$ the bundle over $S^*(M)$ which is induced from the bundle $S^{\nu,i}_{\nu-1}$ over M by the map π.

If $\mu \geqslant \mu_0$, we have the map

$$d\delta : \Sigma_{\mu}^{\mu+1,0} \longrightarrow \Sigma_{\mu-1}^{\mu,2} \tag{3.1}$$

where $d\delta = d \circ \delta$ is the composition of formal and actual exterior differentiation and $d\delta = -\delta d$. The symbol $s(d\delta)$ of the differential operator $d\delta$ then defines a homomorphism of vector bundles, namely

$$s(d\delta) : \pi^* S_{\mu}^{\mu+1,0} \longrightarrow \pi^* S_{\mu-1}^{\mu,2}. \tag{3.2}$$

The map $s(d\delta)$ is described in terms of a local coordinate as follows. Let σ be a vector belonging to the fibre of $\pi^* S_{\mu}^{\mu+1,0}$, and let the point of $S^*(M)$ over which σ lies be (x, ξ), where $x = (x^1, \ldots, x^k, \ldots, x^n)$ and

$$\xi = \sum_{k=1}^{n} \xi_k \, dx^k.$$

Denote by $\delta_k \sigma$ the vector of $\pi^* S_{\mu-1}^{\mu,0}$ lying over the same point of $S^*(M)$, which has the components $(\delta_k \sigma)_p = \{\sigma_{p+1_k}^j \mid 1 < j < m\}$. Then the map (3.2) sends σ into $s(d\delta)\sigma$ where

$$s(d\delta)\sigma = (-1)^{\frac{1}{2}} \sum_{j<k} (\xi_j \, \delta_k \, \sigma - \xi_k \, \delta_j \, \sigma) \, dx^j \wedge dx^k. \tag{3.3}$$

The following definition of ellipticity is a natural one in the present context.

DEFINITION 3.1. *We say that* $\mathscr{S}^{\mu_0}$ *is elliptic if and only if the map* (3.2) *is injective for* $\mu \geqslant \mu_1 = \mu_1 (\mu_0, n)$.

EXAMPLES. (1) (Cauchy-Riemann equations). Let M be a complex analytic manifold of (complex) dimension m, and let $\mathscr{S}^1$ be the system of Cauchy-Riemann equations (2.10). We suppose that $\mu \geqslant 0$. Let σ be a vector of $\pi^* S_{0,\mu}^{\mu+1,0}$ lying over the point (z, ξ) of the *real* unit cotangent sphere, where $z = (z^1, \ldots, z^k, \ldots, z^m)$,

$$\xi = \sum_{k=1}^{n} (\xi_k dz^k + \xi^{\bar{k}} d\bar{z}^k).$$

Then (see Example (3), § 2) we have $\delta_{\bar{k}} \, \sigma = 0$, and hence

$$s(d\delta)\sigma = (-1)^{\frac{1}{2}} \left\{ \sum_{j<k} (\xi_j \delta_k \, \sigma - \xi_k \delta_j \, \sigma) \, dz^j \wedge dz^k - \sum_{j,k} \bar{\xi}_k \delta_j \, \sigma \, dz^j \wedge d\bar{z}^k \right\}$$

Therefore, the vanishing of $s(d\delta)\sigma$ implies that

$$\bar{\xi}_k \delta_j \sigma = 0, \quad j, k = 1, 2, \ldots, m.$$

Since $\bar{\xi}_k \neq 0$ for one value of k, at least, we conclude that $\delta_j \sigma = 0$, $j = 1, 2, \ldots, m$, i.e. $\sigma = 0$. Thus the map (3.2) is injective and hence the Cauchy-Riemann equations are elliptic.

(2) Let M be a subdomain of euclidean n-space with coordinates $x = (x^1, \ldots, x^k, \ldots, x^n)$, and let $\mathscr{S}^2$ be represented by the usual laplacian

$$\sum_{k=1}^{n} \left(\frac{\partial}{\partial x^k}\right)^2 f = 0.$$

We suppose that $\mu \geqslant 1$, and let σ be a vector of $\pi^* S_{0,\mu}^{\mu+1,0}$ over (x, ξ). Let $\delta_k^2 \sigma = \delta_k(\delta_k \sigma)$ be the element with the components $(\delta_k^2 \sigma)_p = \sigma_{p+2_k}$, where $2_k = 1_k + 1_k$. Then

$$\sum_{k=1}^{n} \delta_k^2 \sigma = 0. \tag{3.4}$$

Now suppose that $s(d\delta)\sigma = 0$, i.e.

$$\xi_j \, \delta_k \, \sigma = \xi_k \, \delta_j \, \sigma, \quad j, k = 1, 2, \ldots, n. \tag{3.5}$$

Operating on (3.5) with $\xi_k \, \delta_j$, we obtain

$$\xi_j \, \xi_k \, \delta_j \, \delta_k \, \sigma = \xi_k^2 \, \delta_j^2 \, \sigma$$

and hence, by symmetry,

$$\xi_j^2 \, \delta_k^2 \sigma = \xi_k^2 \, \delta_j^2 \, \sigma.$$

Summing on k from 1 to n, we have by (3.4)

$$\left(\sum_k \xi_k^2\right) . \, \delta_j^2 \, \sigma = \xi_j^2 \, . \sum_k \delta_k^2 \, \sigma = 0$$

and, since $\Sigma \, \xi_k^2 \neq 0$, we conclude that $\delta_j^2 \sigma = 0$, $j = 1, 2, \ldots, n$. Applying δ_j to (3.5), we therefore have

$$\xi_j \, \delta_j \, \delta_k \, \sigma = \xi_k \, . \, \delta_j^2 \, \sigma = 0. \tag{3.6}$$

Choose j such that $\xi_j \neq 0$. Then we infer from (3.5) that $\delta_k \sigma = 0$ if $\xi_k = 0$, and hence $\delta_j \, \delta_k \sigma = 0$ if either ξ_j or ξ_k is equal to zero. If $\xi_j \neq 0$, we infer from (3.6) that $\delta_j \, \delta_k \sigma = 0$. Thus $\delta_j \, \delta_k \sigma = 0$ for all j, k, i.e. $\sigma = 0$ and the map (3.2) is injective. We have thus verified that the laplacian is elliptic !

Let D^* be the (formal) adjoint operator defined in terms of a metric, and let $\square = DD^* + D^*D$ be the corresponding laplacian. We have the following theorem (see [13(b)]), which justifies Definition 3.1.

THEOREM 3.1. *The system $\mathscr{S}^{\mu_0}$ is elliptic (in the sense of Definition 3.1) if and only if there is an integer $\mu_2 = \mu_2(\mathscr{S}^{\mu_0})$ such that the laplacian $\square = DD^* + D^*D$ is an elliptic operator (in the "interior" sense) on the sections of $J^\mu = \oplus_i J^{\mu,i}$ for $\mu \geqslant \mu_2$.*

4. Neumann decompositions. We say that a manifold M is *finite* if it is a subdomain of a differentiable manifold M' where M has compact closure in M' and a boundary ∂M which is a regularly imbedded differentiable submanifold of M' of codimension 1. We say that M is a finite $\mathscr{S}^{\mu_0}$-manifold if it is a finite subdomain of an $\mathscr{S}^{\mu_0}$-manifold M'.

Suppose that $\mathscr{S}^{\mu_0}$ is elliptic, and let M be a finite $\mathscr{S}^{\mu_0}$-manifold, i.e. M is a finite subdomain of an $\mathscr{S}^{\mu_0}$-manifold M'. Let μ be a fixed integer and suppose that $\mu \geqslant \mu_2$, where $\mu_2 = \mu_2 (\mathscr{S}^{\mu_0})$. We have over M' the sheaf $J^\mu = \oplus_i J^{\mu,i}$, and we denote by $\mathbf{A} = \oplus_i \mathbf{A}^i$ the restriction to M of the space of sections over M' of $J^\mu = \oplus_i J^{\mu,i}$. Thus $\mathbf{A}$ is the space of sections of J^μ over M which are differentiable up to and including the boundary of M.

Choose a metric on M', which fits the structure as closely as possible, denote by (u, v) the scalar product, defined in terms of the metric, of the elements u, v of $\mathbf{A}$, and let D^* be the formal adjoint of the differential operator D, i.e. if u has compact support on M, D^* is the operator satisfying $(Du, v) = (u, D^*v)$ for all elements v of $\mathbf{A}$. Let $\mathbf{N} = \oplus_i \mathbf{N}^i$ (Neumann space) be the (graded) subspace of $\mathbf{A}$ composed of the forms u which satisfy the following pair of boundary conditions:

$$\left.\begin{aligned} (D^*u, v) &= (u, Dv), \\ (D^*Du, v) &= (Du, Dv), \end{aligned}\right\} \tag{4.1}$$

for all v of $\mathbf{A}$. Denote by $\mathbf{H} = \oplus_i \mathbf{H}^i$ the (graded) subspace of $\mathbf{N}$ composed of the forms which are annihilated by the laplacian

$DD^* + D^*D$ or, equivalently (in view of (4.1)), $\mathbf{H}$ is the subspace of $\mathbf{N}$ composed of the elements u satisfying $Du = 0$, $D^*u = 0$. If $\mathbf{H}$ is finite dimensional, we denote by $H: \mathbf{A} \to \mathbf{H}$ the orthogonal projection of $\mathbf{A}$ onto $\mathbf{H}$. If $\mathbf{H}$ is infinite dimensional, let $\overline{\mathbf{A}}$, $\overline{\mathbf{H}}$ be the completions of $\mathbf{A}, \mathbf{H}$, respectively, and let $\overline{H} : \overline{\mathbf{A}} \to \overline{\mathbf{H}}$ be the orthogonal projection of $\overline{\mathbf{A}}$ onto $\overline{\mathbf{H}}$.

Definition 4.1. *We say that the Neumann problem is solvable for a finite $\mathscr{S}^{\mu_0}$-manifold M if the following assertions are true.*

(I) *The restriction of $\overline{H}$ to $\mathbf{A}$ is a projection*

$$H : \mathbf{A} \to \mathbf{H} \tag{4.2}$$

of $\mathbf{A}$ onto $\mathbf{H}$.

(II) *The Neumann operator N exists, i.e. there is the surjective map, of degree* 0,

$$N : \mathbf{A} \to \mathbf{N} \tag{4.3}$$

which is characterized by the following conditions:

(i) $HN = NH = 0$.

(ii) $DN = ND$.

(iii) (Neumann decomposition). *For $u \in \mathbf{A}$, we have the orthogonal decomposition*

$$u = DD^*Nu + D^*DNu + Hu, \tag{4.4}$$

which, in view of (ii), *can be written in the form*

$$u = D(D^*N)u + (D^*N)Du + Hu. \tag{4.5}$$

The Neumann decomposition therefore has the form of a cochain homotopy. In fact, let $Z(\mathbf{A}) = \oplus_i Z(\mathbf{A}^i)$ be the kernel of the map $D : \mathbf{A} \to \mathbf{A}$; then

$$Z(\mathbf{A})/D(\mathbf{A}) = Z(\mathbf{A}^0) \oplus Z(\mathbf{A}^1)/D(\mathbf{A}^0) \oplus \ldots \oplus Z(\mathbf{A}^n)/D(\mathbf{A}^{n-1}) \tag{4.6}$$

is the D-cohomology of $\mathbf{A}$, where $Z(\mathbf{A}^0)$ is the space of sections of Θ over M which are differentiable up to and including the boundary of M. The Neumann decomposition (if it exists) provides a representation of the D-cohomology of $\mathbf{A}$ by the space $\mathbf{H} = \oplus_i \mathbf{H}^i$ of harmonic forms, i.e. it gives a linear isomorphism (of graded vector spaces)

$$\mathbf{H} \simeq Z(\mathbf{A})/D(\mathbf{A}) \tag{4.7}$$

The solvability of the Neumann problem for a given finite manifold M depends only on $\mathscr{S}^{\mu_0}$, i.e. it is independent of the choice of metric. We denote by $\mathscr{C}(\mathscr{S}^{\mu_0})$ the set of finite $\mathscr{S}^{\mu_0}$-manifolds for which the Neumann problem is solvable. Our program is to solve the following problem :

PROBLEM. *Determine* $\mathscr{C} = \mathscr{C}(\mathscr{S}^{\mu_0})$ *for each elliptic* $\mathscr{S}^{\mu_0}$.

EXAMPLES (1) $\mathscr{S}^1$ is the system of equations $df = 0$ (see the first set of Examples, § 3). Then $\mathscr{C}(\mathscr{S}^1)$ is the set of all finite manifolds (see Duff and Spencer [3], Conner [2], Morrey [9]).

(2) $\mathscr{S}^1$ is the system of Cauchy-Riemann equations in m variables, i.e. the system of equations (2.10). Then $\mathscr{C}(\mathscr{S}^1)$ is the class of all strongly pseudoconvex (finite) manifolds (see Kohn [8]).

(3) Let $x = (x^1, \dots, x^j, \dots, x^m), z = (z^1, \dots, z^k, \dots, z^n)$, where x^j is real, z^k complex, and write $\bar{z} = (\bar{z}^1, \dots, \bar{z}^k, \dots, \bar{z}^n)$, where $\bar{z}^k$ is the complex conjugate of z^k. Let $\mathscr{S}^1$ be the system of equations

$$\left.\begin{aligned} \frac{\partial f}{\partial x^j} &= 0, \qquad j = 1, 2, \dots, m, \\ \frac{\partial f}{\partial \bar{z}^k} &= 0, \qquad k = 1, 2, \dots, n, \end{aligned}\right\} \tag{4.8}$$

for the complex-valued function f (compare (2.10)).

Now let M' be a differentiable manifold with a foliate structure whose sheets are real m-dimensional manifolds with a complex analytic structure transverse to them. This mixed structure is represented by a locally finite covering $\mathfrak{V} = \{V_\alpha\}$, where V_α is an open set covered by the coordinates (x_α, z_α), and the transition functions have the form

$$\begin{cases} x_\alpha = f_{\alpha\beta}(x_\beta, z_\beta, \bar{z}_\beta), \\ z_\alpha = g_{\alpha\beta}(z_\beta), \end{cases}$$

where $f_{\alpha\beta}$ is differentiable in x_β, z_β, $\bar{z}_\beta$, the jacobian matrix $\partial(x_\alpha)/\partial(x_\beta) = \partial(f_{\alpha\beta})/\partial(x_\beta)$ is non-singular, and $g_{\alpha\beta}$ is a biholomorphic transformation. Through each point of M' there passes a real

m-dimensional sheet, which is defined in V_α by setting the z_α^k equal to (complex) constants, and the local differentiable coordinate along the sheet is $x_\alpha = (x_\alpha^1, \ldots, x_\alpha^j, \ldots, x_\alpha^m)$. Let $T(M')$ be the tangent bundle of M', and denote by $T_s(M')$ the sub-bundle of $T(M')$ of tangent vectors along the sheets. Then the restriction $T_s(V_\alpha)$ of $T_s(M')$ to V_α is covered by the coordinates $(x_\alpha, z_\alpha, \partial/\partial x_\alpha)$, where $\partial/\partial x_\alpha = (\partial/\partial x_\alpha^1, \ldots, \partial/\partial x_\alpha^j, \ldots, \partial/\partial x_\alpha^m)$. The equations (4.8) are defined on M', i.e., M' is an $\mathscr{S}^1$-manifold where $\mathscr{S}^1$ is represented, in terms of local coordinates (x_α, z_α), by a system of the form (4.8). Let M be a finite subdomain of M', and denote by ∂M the boundary of M. The boundary ∂M is tangent to a sheet, at the point x_0, if $T_s(M')|_{x_0}$ is contained in the tangent space of ∂M at x_0. We denote by $\partial_s M$ the (closed) set of boundary points of M at which ∂M is tangent to the sheets. The work of Ash [1] and Kohn [8], together with an observation of L. Nirenberg, yields the following result:

$\mathscr{C}(\mathscr{S}^1)$ is the class of finite $\mathscr{S}^1$-manifolds M such that, at each point of $\partial_s M$, the boundary is strongly pseudoconvex in the sense of the complex structure transverse to the sheet and strongly convex along the sheet through the point.

In the following section we shall indicate how the method of Kohn [8], in the form applied by Ash [1], can be used to establish the following result:

PROPOSITION 4.1. *Suppose that $\mathscr{S}^{\mu_0}$ is elliptic. Then $\mathscr{C}(\mathscr{S}^{\mu_0})$ contains all sufficiently small spherical subdomains of euclidean n-space and, for these domains,* $\mathrm{H}^i = 0$ *for* $i > 0$.

Suppose that $\mathscr{S}^{\mu_0}$ is elliptic, and let M be an $\mathscr{S}^{\mu_0}$-manifold of dimension n. The exactness of the sequence (2.7), for $\mu \geqslant \mu_2 = \mu_2(\mathscr{S}^{\mu_0})$, follows at once from Proposition 4.1. In fact, suppose that μ is a fixed integer, $\mu \geqslant \mu_2$, and let $\tilde{u}$ be a germ of $J^{\mu,i}$, where $i > 0$, which satisfies $D\tilde{u} = 0$. Then $\tilde{u}$ is represented by a section u of $J^{\mu,i}$, which is defined over a neighborhood containing the closure of a sufficiently small coordinate ball and satisfies $Du = 0$. By Proposition 4.1, the Neumann problem is solvable on the coordinate ball and $Hu = 0$. Hence, by formula (4.5), $u = Dw$ where $w = D^*Nu$, i.e. the Poincaré lemma for D is valid and the sequence (2.7) is exact.

Now let $\mathbf{L}(J^\mu) = \oplus_i \mathbf{L}(J^{\mu,i})$ be the graded vector space of sections of J^μ over M, and let $\mathbf{Z}(J^\mu) = \oplus_i \mathbf{Z}(J^{\mu,i})$ be the kernel of the map $D : \mathbf{L}(J^\mu) \to \mathbf{L}(J^\mu)$. Then

$$\mathbf{Z}(J^\mu)/D\mathbf{L}(J^\mu) = \mathbf{Z}(J^{\mu,0}) \oplus \mathbf{Z}(J^{\mu,1})/D\mathbf{L}(J^{\mu,1}) \oplus \dots \oplus \mathbf{Z}(J^{\mu,n})/D\mathbf{L}(J^{\mu,n-1}) \tag{4.9}$$

is the (graded) D-cohomology of sections of J^μ over M. Moreover, let

$$H^*(M, \Theta) = H^0(M, \Theta) \oplus H^1(M, \Theta) \oplus \dots \oplus H^n(M, \Theta) \tag{4.10}$$

be the (graded) cohomology of M with values in the sheaf Θ of germs of solutions of the system $\mathscr{S}^{\mu_0}$ of linear partial differential equations on M.

We denote by $\nu_2 = \nu_2 (\mathscr{S}^{\mu_0})$ the smallest positive integer for which the sequence (2.6) is exact for $\mu \geqslant \nu_2$ and $0 \leqslant i \leqslant n-1$, and we denote by $\nu_1 = \nu_1(\mathscr{S}^{\mu_0})$ the larger of the two integers μ_0, ν_2 (see Section 1).

The following theorem is an immediate consequence of Proposition 4.1.

THEOREM 4.1. (Theorem of de Rham for elliptic systems.) *Let M be an $\mathscr{S}^{\mu_0}$-manifold of dimension n, and suppose that $\mathscr{S}^{\mu_0}$ is elliptic. Then, for $\mu \geqslant \nu_1 - 1$, a fortiori for $\mu \geqslant \mu_1 - 1$, the sequence*

$$0 \longrightarrow \Theta \xrightarrow{\ i\ } J^{\mu,0} \xrightarrow{\ D\ } J^{\mu,1} \xrightarrow{\ D\ } \dots \xrightarrow{\ D\ } J^{\mu,n} \longrightarrow 0$$

is an exact sequence of fine sheaves, and we have the isomorphism of graded vector spaces

$$H^* (M, \Theta) \simeq \mathbf{Z}(J^\mu)/D\mathbf{L}\,(J^\mu) \tag{4.11}$$

which is derived from the exact sequence of sheaves in a canonical manner.

In fact, suppose that $\mu \geqslant \nu_1 - 1, i > 0$, and let u be a local section of $J^{\mu,i}$ satisfying $Du = 0$. If $\mu < \mu_2$, then u can be lifted up to a section v of $J^{\mu_2,i}$ satisfying $Dv = 0$ (see [13(a)], § 5). By Proposition 4.1,

$v = D(D^*Nv)$, and it follows that $u = Dw$, where w is the projection of D^*Nv into a (local) section of $J^{\mu, i-1}$.

5. The (formal) adjoint operator defined in terms of a metric. Suppose that $\mu \geqslant \nu \geqslant \mu_1$. Then we have the decomposition (over **R**)

$$J^{\mu,i}_{\nu-1} = (D \circ \#)\,(\Sigma^{\mu+1,i-1}_{\nu}) \oplus \#\,(\Sigma^{\mu+1,i}_{\nu}). \tag{5.1}$$

Moreover, the following sequence is exact (see [13 (a)], Proposition 5.1)

$$0 \longrightarrow J^{\mu,0}_{\nu-1} \xrightarrow{D} J^{\mu,1}_{\nu-1} \xrightarrow{D} \cdots \xrightarrow{D} J^{\mu,n}_{\nu-1} \longrightarrow 0. \tag{5.2}$$

Take $\mu = \nu \geqslant \mu_1$; then (5.1) reduces to the following decomposition:

$$J^{\mu,i}_{\mu-1} = D(J^{\mu,i-1}_{\mu-1}) \oplus \delta(\Sigma^{\mu+1,i}_{\mu}). \tag{5.3}$$

From (5.3) we obtain the decomposition (over the differentiable functions)

$$J^{\mu,i}_{\mu-1} = \delta(\Sigma^{\mu+1,i-1}_{\mu}) \oplus \delta(\Sigma^{\mu+1,i}_{\mu}). \tag{5.4}$$

Next, suppose that a riemannian metric has been chosen along the fibres of the bundle $S^{\mu+1}_{d}$, where μ is a fixed integer, $\mu \geqslant \mu_1$. Then, at each point x of M, we have a scalar product $(\sigma, \tau)_x$ of vectors σ, τ belonging to the fibre of S^{μ}_{d} over x, and write $\| \sigma \|_x = [(\sigma, \sigma)_x]^{1/2}$. This scalar product induces a scalar product in the sheaf Σ^{μ}_{d}, and we let

$$a : \Sigma^{\mu}_{d} \to \Sigma^{\mu}$$

be the orthogonal projection, in the sense of the scalar product$(\ldots)_x$, of Σ^{μ}_{d} onto Σ^{μ}. Setting $b = 1 - a$, we have the orthogonal decomposition

$$\Sigma^{\mu}_{d} = a(\Sigma^{\mu}_{d}) \oplus b(\Sigma^{\mu}_{d})$$

where $a(\Sigma^{\mu}_{d}) = \Sigma^{\mu}$. Moreover, let

$$\alpha : \Sigma^{\mu,i}_{d} \to \delta(\Sigma^{\mu+1,i-1}_{\mu})$$

be the orthogonal projection, and let β be the orthogonal projection of $\Sigma^{\mu,i}_{d}$ onto the orthogonal complement of $\delta(\Sigma^{\mu+1,i-1}_{\mu})$ in $\Sigma^{\mu,i}$, i.e. we have the orthogonal decomposition

$$\Sigma^{\mu,i} = \beta(\Sigma^{\mu,i}) \oplus \alpha(\Sigma^{\mu,i}), \tag{5.5}$$

where $\beta(\Sigma^{\mu,i}) = \beta(\Sigma^{\mu,i}_{d})$, $\alpha(\Sigma^{\mu,i}) = \alpha(\Sigma^{\mu,i}_{d})$, and $a = \beta + \alpha$.

The decomposition (5.4) can be written

$$J^{\mu,i}_{\mu-1} = \alpha(\Sigma^{\mu,i}) \oplus \alpha(\Sigma^{\mu,i+1}). \tag{5.6}$$

Moreover, we have

$$J^{\mu,i} = \Sigma^{\mu,i} \oplus \alpha(\Sigma^{\mu,i+1}) \tag{5.7}$$

where $J^{\mu-1,i}$ is isomorphic to $\beta(\Sigma^{\mu,i})$. In fact, we have the exact sequence (see § 2)

$$0 \longrightarrow \Xi^{\mu+1}_{\mu-1} \longrightarrow \Sigma^{\mu+1} \xrightarrow{\varpi} J^{\mu-1} \longrightarrow 0, \tag{5.8}$$

and we let

$$s : J^{\mu-1} \to \Sigma^{\mu+1}$$

be the injective map sending the element $u^{\mu-1}$ of $J^{\mu-1}$ into the unique element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1}$ such that $\varpi(\sigma^{\mu+1}) = u^{\mu-1}$ and $\sigma^{\mu+1}$ is orthogonal (in the sense of the scalar product $(\ldots)_x$) to $\Xi^{\mu+1}_{\mu-1}$. Then we have the following splitting of (5.8):

$$\Sigma^{\mu+1} = (s \circ \varpi)\,(\Sigma^{\mu+1}) \oplus \Xi^{\mu+1}_{\mu-1}. \tag{5.9}$$

We obtain an induced splitting of the sequence

$$0 \longrightarrow J^{\mu}_{\mu-1} \longrightarrow J^{\mu} \longrightarrow J^{\mu-1} \longrightarrow 0,$$

namely

$$J^{\mu} = (\# \circ s)\,(J^{\mu-1}) \oplus J^{\mu}_{\mu-1}, \tag{5.10}$$

where $\# : \Sigma^{\mu+1} \to J^{\mu}$. Finally, let $\pi : J^{\mu} \to \Sigma^{\mu}$ be the projection of J^{μ} onto Σ^{μ}. Then

$$\pi \circ \# \circ s : J^{\mu-1} \to \beta(\Sigma^{\mu})$$

is plainly bijective, and we therefore have

$$J^{\mu,i} = \beta(\Sigma^{\mu,i}) \oplus J^{\mu,i}_{\mu-1}. \tag{5.11}$$

The decompositions (5.5), (5.6) and (5.11) imply (5.7).

Let $q = s \circ \# \circ \beta$, where $\# : \Sigma^{\mu} \to J^{\mu-1}$. Then

$$q : \Sigma^{\mu,i} \to \Sigma^{\mu+1,i},$$

and we have the orthogonal decomposition

$$\Sigma^{\mu+1,i} = q(\Sigma^{\mu,i}) \oplus \Xi^{\mu+1,i}_{\mu-1}, \tag{5.12}$$

where $q(\Sigma^{\mu,i})$ is isomorphic to $J^{\mu-1,i}$, therefore isomorphic to $\beta(\Sigma^{\mu,i})$. Moreover, we have the exact sequence

$$0 \longrightarrow K_{\mu-1}^{\mu+1,i} \longrightarrow \Xi_{\mu-1}^{\mu+1,i} \xrightarrow{\delta} \delta(\Sigma_{\mu}^{\mu+1,i}) \longrightarrow 0 \qquad (5.13)$$

where $K_{\mu-1}^{\mu+1,i}$ is the kernel of the map $\delta : \Xi_{\mu-1}^{\mu+1,i} \to \delta(\Sigma_{\mu}^{\mu+1,i})$, and we have the orthogonal decomposition

$$K_{\mu-1}^{\mu+1,i} = p(\Lambda_{\mu-1}^{\mu,i}) \oplus \Lambda_{\mu}^{\mu+1,i}$$

where $p : \Lambda_{\mu-1}^{\mu,i} \to K_{\mu-1}^{\mu+1,i}$ is the prolongation of $\Lambda_{\mu-1}^{\mu,i}$ into a subsheaf of the kernel $K_{\mu-1}^{\mu+1,i}$. Thus we have the decomposition

$$\Sigma^{\mu+1,i} = q(\Sigma^{\mu,i}) \oplus p(\Lambda_{\mu-1}^{\mu,i}) \oplus \Sigma_{\mu}^{\mu+1,i} \qquad (5.14)$$

where $\Lambda_{\mu-1}^{\mu,i} = \delta(\Sigma_{\mu}^{\mu+1,i-1}) = \alpha(\Sigma^{\mu,i})$ (since $\mu \geqslant \mu_1$).

Now let $u = (\sigma, \xi) = \# (\sigma^{\mu+1})$ be an element of $J^{\mu,i}$, where $\sigma = \pi(\sigma^{\mu+1})$ is the projection of the element $\sigma^{\mu+1}$ in $\Sigma^{\mu,i}$ and $\xi = \delta\, \sigma^{\mu+1}$. In accordance with (5.7) we write $u = (\sigma, \zeta)$ where $\zeta = \delta(\sigma^{\mu+1} - q(\sigma))$, and we henceforth denote an element u of $J^{\mu,i}$ by the pair (σ, ζ) where $\sigma \in \Sigma^{\mu,i}$ and $\zeta \in \alpha(\Sigma^{\mu,i+1}) = \delta(\Sigma_{\mu}^{\mu+1,i})$.

Let $u = (\sigma, \zeta)$ be a section of $J^{\mu,i}$, and set

$$\left.\begin{aligned} \|u\|_x^2 &= \|\sigma\|_x^2 + \|\zeta\|_x^2, \\ \|u\|^2 &= \int_M \|u\|^2\, dM, \end{aligned}\right\} \qquad (5.15)$$

where dM denotes the volume element determined by the riemannian metric on M. If u, v are sections of $J^{\mu,i}$, we denote by $(u, v)_x$, (u, v) the scalar products corresponding to the above norms.

For $u = (\sigma, \zeta)$, the operator D, expressed in terms of the decomposition (5.7), has the form

$$Du = D(\sigma, \zeta) = ((d - \delta_0)\, \sigma - \zeta,\ -(d - \delta_0)^2\, \sigma - d\zeta) \qquad (5.16)$$

where $\delta_0 \sigma = (\delta \circ q)\, \sigma$. Let u be a section of $J^{\mu,i}$; then the (formal) adjoint D^* of D satisfies the equation $(D^*u, v) = (u, Dv)$ for every section v of $J^{\mu,i-1}$ with compact support, and we verify that

$$D^*u = D(\sigma, \zeta) = (a.((d^* - \delta_0^*)\, \sigma - (d^* - \delta_0^*)^2 \zeta),\ -\alpha(\sigma + d^*\zeta)) \qquad (5.17)$$

where δ_0^* is the adjoint of δ_0 and d^* the (formal) adjoint of the exterior differential operator d. For simplicity, let $D_0 = d - \delta_0$. Formulas (5.16) and (5.17) then become

$$Du = D(\sigma, \zeta) = (D_0\sigma - \zeta, -D_0^2\sigma - d\zeta), \tag{5.16$'$}$$

$$D^*u = D^*(\sigma, \zeta) = (D_0^*\sigma - D_0^{*2}\zeta, -\alpha(\sigma + d^*\zeta)), \tag{5.17$'$}$$

where

$$\begin{cases} D_0 : \Sigma^{\mu,i} \longrightarrow \Sigma^{\mu,i+1}, \\ D_0^* : \Sigma^{\mu,i+1} \longrightarrow \Sigma^{\mu,i}, \end{cases}$$

and (as is easily seen)

$$\begin{cases} D_0^2 : \Sigma^{\mu,i} \longrightarrow \alpha(\Sigma^{\mu,i+2}), \\ D_0^{*2} : \alpha(\Sigma^{\mu,i+2}) \longrightarrow \Sigma^{\mu,i}, \end{cases}$$

are linear over the differentiable functions.

The formulas (5.16)′, (5.17)′ are used in proving Proposition 4.1 (where it is assumed that $\mu \geqslant \mu_2 = \mu_2(\mathscr{S}^{\mu_0})$).

6. Elliptic pseudogroups. Let M be a Γ-manifold, where Γ is a transitive, continuous pseudogroup of order μ_0. Then the Γ-vector fields (infinitesimal transformations of Γ) are defined by a system $\mathscr{S}^{\mu_0} = \mathscr{S}^{\mu_0}(\Gamma)$ of linear partial differential equations which is regular in the sense of Definition 2.2, and the considerations of the preceding sections are therefore applicable.

We remark that the sheaf Θ of germs of Γ-vector fields, i.e. the sheaf of germs of solutions of $\mathscr{S}^{\mu_0}(\Gamma)$, has a structure of Lie algebra over $\mathbf{R}$, and the cohomology $H^*(M, \Theta) = \oplus_i H^i(M, \Theta)$ has an induced structure of graded Lie algebra (over $\mathbf{R}$). Moreover (see [13(a)], Proposition 3.4) the sheaf $J^\mu = \oplus_i J^{\mu,i}$ has a natural structure of graded Lie algebra (over $\mathbf{R}$) with bracket $[u, v]$, $u, v \in J^\mu$, and we have the formula (formula (3.37) of the paper [13(a)])

$$D[u, v] = [Du, v] + (-1)^i [u, Dv], \quad u \in J^{\mu,i},\ v \in J^\mu.$$

Therefore (see (4.9)) we see that $\mathbf{Z}(J^\mu)$ is a graded subalgebra of $\mathbf{L}(J^\mu)$ and the image $D\mathbf{L}(J^\mu)$ of $\mathbf{L}(J^\mu)$, under the map $D : \mathbf{L}(J^\mu) \to \mathbf{L}(J^\mu)$, is an ideal of the algebra $\mathbf{Z}(J^\mu)$. It follows that the D-cohomology of sections of J^μ, namely (4.9), inherits a structure of graded Lie algebra (over $\mathbf{R}$).

DEFINITION 6.1. *An elliptic pseudogroup is a transitive, continuous pseudogroup* Γ *for which* $\mathscr{S}^{\mu_0}(\Gamma)$ *is elliptic (in the sense of Definition* 3.1).

Theorem 4.1 is valid for elliptic pseudogroups, with the additional assertion that (4.11) is an anti-isomorphism of graded Lie algebras over $\mathbf{R}$ (see [13(a)], Theorem 7.1, and [13(b)]).

Let G be an arbitrary linear Lie group, $G \subset GL(n, \mathbf{R})$, with Lie algebra $\mathfrak{g}$, $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbf{R})$, and denote by Γ_G the pseudogroup of all local bidifferentiable transformations of $\mathbf{R}^n$ whose jacobian matrices, defined in terms of the coordinate $(x^1, \ldots, x^j, \ldots, x^n)$ of $\mathbf{R}^n$, belong to G. The Γ_G-vector fields $\theta = (\theta^1, \ldots, \theta^j, \ldots, \theta^n)$ are the solutions of the system $\mathscr{S}^1$ of equations which assert that the $(n \times n)$-matrices $(\partial\theta^j/\partial x^k)$ belong to $\mathfrak{g}$, and we say that G is elliptic if and only if $\mathscr{S}^1$ is elliptic. Professor I. M. Singer has remarked to the author that G is elliptic if and only if its Lie algebra $\mathfrak{g}$ contains no real subalgebra generated by matrices with precisely one non-vanishing (real) coefficient.

EXAMPLES. (1) A complex Lie group G, $G \subset GL(n, \mathbf{C})$, is elliptic.

(2) A complex pseudogroup, i.e. a pseudo-subgroup of the general complex pseudogroup of all local biholomorphic transformations, is elliptic.

7. Almost Γ-structure referred to an osculating Γ-structure. Suppose that M has an almost Γ-structure, and let O be an arbitrary point of M. Then there is a neighborhood U of O in M with a Γ-structure osculating to the given almost Γ-structure at the point O. Let $J^\mu = \oplus_i J^{\mu,i}$ be the sheaf over U of jet forms of order μ belonging to the osculating Γ-structure. Then (see [13(a), (b)]) the almost Γ-structure on U is represented by a section v over U of $J^{\mu,1}$, where $\mu \geqslant \mu_1$, and v vanishes at the point O. We introduce the differential operator

$$\widetilde{D} = \widetilde{D}_v : J^\mu \to J^\mu \tag{7.1}$$

which sends the element u of $J^{\mu,i}$ into the element $\widetilde{D}u$ of $J_{\mu,i+1}$ where

$$\tilde{D}u = Du - [v, u]. \tag{7.2}$$

The following proposition is proved in the paper [13(a)] :

PROPOSITION 7.1. *The almost Γ-structure on U is integrable if and only if*

$$\tilde{D}^2 = Dv - \tfrac{1}{2}[v, v] = 0. \tag{7.3}$$

Next, suppose that the section v of $J^{\mu,1}$, $\mu \geqslant \mu_1$, depends differentiably on a real parameter t for all sufficiently small t, i.e. $v = v(t)$ and suppose also that $v(0) = 0$. As in [13(a)], we let

$$D(t) : J^{\mu} \to J^{\mu}$$

be the operator defined by (7.2) with $v = v(t)$, i.e.

$$D(t)\, u = Du - [v(t), u]. \tag{7.4}$$

Then $v(t)$ defines over U a one-parameter family of almost Γ-structures depending differentiably on t and tending, as t approaches zero, to the osculating Γ-structure on U. By Proposition 7.1 the family of almost Γ-structures is integrable if and only if

$$D(t)^2 = Dv(t) - \tfrac{1}{2}[v(t), v(t)] = 0. \tag{7.5}$$

Let $J^{\mu}_{(1)} = \oplus_i J^{\mu,i}_{(1)}$ be the sheaf over U of germs of jet forms of order μ depending differentiably on the real parameter t for all sufficiently small t. A germ of $J^{\mu}_{(1)}$ over a point of U is represented by a family of sections $u(t)$ of J^{μ}, defined on a neighborhood of the point in question which is independent of t, and depending differentiably on t for all sufficiently small t.

DEFINITION 7.1. *We say that the Poincaré lemma for the operator $D(t)$ or, shortly, the $D(t)$-Poincaré lemma, is true in degree $i+1$, where $0 \leqslant i \leqslant n-1$, if, for each section $v = v(t)$ of $J^{\mu,1}_{(1)}$ over U satisfying* (7.5) *and vainshing at $t = 0$, the sequence*

$$J^{\mu,i}_{(1)} \xrightarrow{D(t)} J^{\mu,i+1}_{(1)} \xrightarrow{D(t)} J^{\mu,i+2}_{(1)}$$

is exact at $J^{\mu,i+1}_{(1)}$.

The following theorem is proved in [13(b)] :

THEOREM 7.1. *The existence of local coordinates, compatible with a family of almost* Γ*-structures defined by a section* $v(t)$ *of* $J^{\mu,1}_{(1)}$ *satisfying* (7.5) *and vanishing at* $t = 0$, *is equivalent to the validity of the* $D(t)$*-Poincaré lemma in degree* 1.

The validity of the $D(t)$-Poincaré lemma in all degrees, in particular in degree 1, for elliptic pseudogroups is established by the method described in Section 5. In fact, if the pseudogroup Γ is elliptic, then the basic estimates which hold for D and D^*, also hold for $D(t)$ and $D^*(t)$, provided that t is sufficiently small. We therefore have the following result (Theorem 5.2 of [13(b)]) :

THEOREM 7.2. *The* $D(t)$*-Poincaré lemma is valid for elliptic pseudogroups.*

We remark (see [13(a), (b)]) that (7.5) is the integrability condition for the equation

$$\mathscr{D} h(t) = v(t) \tag{7.6}$$

where $h(t) = (f(t), g(t))$ and $f(t)$ is a local bidifferentiable transformation of a neighborhood of the point 0, $g(t)$ is a differentiable map of a neighborhood of 0 into the group $G^{\mu+1}$ associated with Γ. If $h(t) = (f(t), g(t))$ satisfies (7.6), the map $f(t)$ transforms a Γ-coordinate $x = (x^1, \ldots, x^j, \ldots, x^n)$ for the osculating structure into a Γ-coordinate $f(x, t) = (f^1(x, t), \ldots, f^j(x, t), \ldots, f^n(x, t))$ for the structure defined by $v(t)$. At $t = 0$, $f(x, 0) = x$ and $g(0)(x) = g(x, 0)$ is the unit element of $G^{\mu+1}$.

Finally, a transformation $h(t) = (f(t), g(t))$ of the type described above is a section of the sheaf $\mathscr{H}^{\mu+1}_{(1)}$ of groups defined in the paper [13 (a)]. Let $h(t)$ be a section of $\mathscr{H}^{\mu+1}_{(1)}$ over a neighborhood of the point 0, and set

$$v_1(t) = \text{Ad}\, h(t) \,.\, (v(t) - \mathscr{D} h(t)) \tag{7.7}$$

where the operation $\text{Ad}\, h(t)$ is that described in [13(a)]. Then (see [13(a), (b)]) the element $v_1(t)$ is a local section of $J^{\mu,1}_{(1)}$ which also satisfies the integrability condition (7.5).

8. Existence of local coordinates compatible with an integrable almost Γ-structure. We are now able to prove the following generalization of the theorem of Newlander-Nirenberg [10] (see also Nirenberg [11]) :

THEOREM 8.1. *If Γ is elliptic, an integrable almost Γ-structure is a Γ-structure.*

In fact, suppose that M has an almost Γ-structure, let O be an arbitrary point of M, and let U be a neighborhood of O in M with a Γ-structure osculating to the given almost Γ-structure at the point O. Choosing U smaller if necessary, we can suppose that U is covered by a Γ-coordinate $x = (x^1, \ldots, x^j, \ldots, x^n)$ for the osculating structure which is centered at the point O. Let v be a section of $J^{\mu,1}$ over U, $\mu \geqslant \mu_1 = \mu_1(\mu_0, n)$, which represents the given almost Γ-structure on U and vanishes at the point O, where $J^{\mu} = \oplus_i J^{\mu,i}$ is the sheaf over U of jet forms of order μ belonging to the osculating Γ-structure. Then we have the map (7.1), where $\widetilde{D}$ is defined by (7.2). We suppose that the almost Γ-structure is integrable, i.e. v satisfies (7.3) and $\widetilde{D}^2 = 0$.

For simplicity, we suppose that $\Gamma = \Gamma_G$, where G is a linear Lie group (see Section 6). Let t be a sufficiently small real parameter, and let $y = (y^1, \ldots, y^j, \ldots, y^n)$ where $y^j = x^j/t$ or, shortly, $y = x/t$. The small spherical neighborhood $|x| < t$ of the point O is transformed into the unit coordinate ball $|y| < 1$. However, we remark that $y = (y^1, \ldots, y^j, \ldots, y^n)$ will not generally be a Γ-coordinate for the osculating Γ-structure. The transformation $y = x/t$ induces a transformation of v into a section $v(t)$ of the sheaf $J^{\mu,1}_{(1)}$ of jet forms associated with the osculating structure, and $v(t)$ satisfies (7.5) and vanishes at $t = 0$.

Suppose now that the pseudogroup is elliptic (i.e. G is elliptic), and let $D(t)$ be the operator defined by (7.4) in terms of $v(t)$. By Theorem 7.2, the Poincaré lemma is valid for $D(t)$ and hence, for all sufficiently small t, there exists on a neighborhood of the point O a solution $h(t)$ of the equation (7.6). Transforming back to the Γ-coordinate $x = t.y$, we obtain a solution h of the equation $\mathscr{D}h = v$ on a neighborhood of O, where $h = (f, g)$, and

$$f(x) = (f^1(x), \ldots, f^j(x), \ldots, f^n(x))$$

is the desired Γ-coordinate for the given almost Γ-structure. We have thus proved Theorem 8.1 in the special case where $\Gamma = \Gamma_G$, G a linear Lie group. The proof in the general case can be carried out along somewhat similar lines.

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SOME REMARKS ON THE NOTION OF CONVEXITY FOR DIFFERENTIAL OPERATORS

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I. Complexes of differential operators. Let $A = C[X_1, \ldots, X_n]$ be the ring of polynomials in n variables over C. This ring operates on the functions (and distributions) in R^n by the formula

$$X_j f = -i \frac{\partial f}{\partial x_j} \qquad (i = \sqrt{-1}).$$

Let M_* be a complex of A-modules, with differential of degree -1

$$\cdots \longrightarrow M_k \xrightarrow{d_k} M_{k-1} \longrightarrow \cdots$$

Throughout this paper, we make the following assumptions:

1. The M_k are free and of finite type.
2. All but a finite number are zero.

We denote by M^* the complex $\mathrm{Hom}\,(M_*, A)$ (we write Hom, Ext^k, ... instead of Hom_A, Ext^k_A, etc. ...). It is known that one has a spectral sequence connecting $H^*(M^*)$ and $H_*(M_*)$:

$$H^*(M^*) \Leftarrow E_2^{p,q} = \mathrm{Ext}^p(H_q(M_*), A). \tag{I. 1}$$

We denote by $\mathscr{E}$ (resp. $\mathscr{D}'$, $\mathscr{A}$) the sheaf of germs of $\mathscr{C}^\infty$ functions (resp. distributions, analytic functions) on R^n; if Ω is an open set in R^n, we write $\mathscr{E}(\Omega) = \Gamma(\Omega; \mathscr{E})$, $\mathscr{E}_c(\Omega) = \Gamma_c(\Omega; \mathscr{E})$ (here, c stands for the family of compact subsets of Ω), and we define $\mathscr{D}'(\Omega)$, $\mathscr{D}'_c(\Omega)$ etc. ... in the same way.

DEFINITION (I. 2). *The complex M_* is said to be "hypoelliptic" (resp. "elliptic") if, for every Ω, the natural mapping*

$$H^*(\mathrm{Hom}(M_*, \mathscr{E}(\Omega))) \to H^*(\mathrm{Hom}(M_*, \mathscr{D}'(\Omega)))$$

[resp. $H^*(\mathrm{Hom}(M_*, \mathscr{A}(\Omega))) \to H^*(\mathrm{Hom}(M_*, \mathscr{E}(\Omega)))$]

is bijective.

Let M be an A-module of finite type, and suppose that M_* is a free resolution of M

$$\text{(i.e. } M_k = 0 \text{ for } k < 0,\ H_k(M_*) = 0 \text{ for } k > 0,\ H_0(M_*) = M).$$

Then, M_* is hypoelliptic (resp. elliptic) if and only if M is hypoelliptic (resp. elliptic), which is equivalent with the following property: For every $\Omega \subset R^n$, the natural mapping:

$$\operatorname{Hom}(M, \mathscr{E}(\Omega)) \to \operatorname{Hom}(M, \mathscr{D}'(\Omega))$$

$$[\text{resp. } \operatorname{Hom}(M, \mathscr{A}(\Omega)) \to \operatorname{Hom}(M, \mathscr{E}(\Omega))]$$

is bijective.

[Denote by $\mathscr{E}^M$ the sheaf $\Omega \to \operatorname{Hom}(M, \mathscr{E}(\Omega))$ and define similarly $\mathscr{D}'^M$ and $\mathscr{A}^M$; one has isomorphisms

$$H^k(\operatorname{Hom}(M_*, \mathscr{E}(\Omega)) \simeq \operatorname{Ext}^k(M, \mathscr{E}(\Omega)) \simeq H^k(\Omega; \mathscr{E}^M)$$

and similar isomorphisms with $\mathscr{E}$ replaced by $\mathscr{D}'$ or $\mathscr{A}$ [10]; the result is an immediate consequence of these isomorphisms.]

We recall the following theorem (due to Hörmander-Lech [7] in the hypoelliptic case, and—essentially—to Petrowsky in the elliptic case).

THEOREM (I. 3). *The A-module M is elliptic (resp. hypoelliptic) if and only if* supp(M) *(in the sense of* Bourbaki [3]) *has no real points at infinity (resp. the projection $C^n \to R^n : z \to \operatorname{Im} z$ induces on* supp(M)*, considered as a subset of C^n, a proper mapping).*

PROPOSITION (I. 4). *The complex M_* is elliptic (resp. hypoelliptic) if and only if the modules $H_k(M_*)$ are elliptic (resp. hypoelliptic).*

PROOF. Let $a \in R^n$. It is known that the spaces of germs at a $\mathscr{E}_a$, $\mathscr{D}'_a$, $\mathscr{A}_a$ are injective A-modules [10]; then, one has

$$H^k(\operatorname{Hom}(M_*, \mathscr{E}_a)) = \operatorname{Hom}(H_k(M_*), \mathscr{E}_a)$$

and the same result is true with $\mathscr{E}$ replaced by $\mathscr{D}'$ or $\mathscr{A}$. Then, if M_* is elliptic (resp. hypoelliptic), the $H_k(M_*)$ are elliptic (resp. hypoelliptic).

Conversely, suppose the $H_k(M_*)$ to be elliptic (the hypoelliptic case is similar); one has a homomorphism of convergent spectral sequences

$$\begin{array}{ccc} H^*(\mathrm{Hom}(M_*, \mathscr{E}(\Omega))) & \Leftarrow & E_2^{p,q} = \mathrm{Ext}^p(H_q(M_*), \mathscr{E}(\Omega)) \\ \uparrow & & \uparrow \\ H^*(\mathrm{Hom}(M_*, \mathscr{A}(\Omega))) & \Leftarrow & E_2^{p,q} = \mathrm{Ext}^p(H_q(M_*), \mathscr{A}(\Omega)) \end{array}$$

and the mapping in the second column is bijective ; then the mapping in the first column is also bijective. q. e. d.

In paragraph 4, we use the following result, which can be proved in a similar manner.

If M_* is hypoelliptic, then, for every $\Omega \subset R^n$, and for every k the natural mapping

(I. 5) $$H^k(\mathrm{Hom}(M_*, \mathscr{E}_c(\Omega))) \to H^k(\mathrm{Hom}(M_*, \mathscr{D}'_c(\Omega)))$$

is bijective.

(But, we note that it happens very often that for given M_* and k (I. 5) is bijective although M_* is not hypoelliptic ; and, actually, we shall need just this property for given M_* and Ω, and that, only for some values of k.)

We consider now the complex M'_* obtained from

$$M^* = \mathrm{Hom}\,(M_*, A)$$

by replacing the degrees by their negatives :

$$M'_k = M^{-k} (= \mathrm{Hom}\,(M_{-k}, A)),$$

and keeping the same differential ; we write here d'_k for the component $M'_k \to M'_{k-1}$ of the differential of M'_* (that is : $d'_{-k} = \mathrm{Hom}\ (d_{k+1}, A)$). We consider M'_* as a complex of A-modules, not in the usual way, but as follows :

if $u \in M'_{-k} = \mathrm{Hom}\,(M_k, A)$ and $m \in M_k$, we put $(X_j u)(m) = u(-X_j m)$.

The reason is that $\dfrac{\partial}{\partial x_j}$ is the transpose of $-\dfrac{\partial}{\partial x_j}$!

Using (I. 1), (I. 3) and (I. 4), one proves easily the following result.

Proposition (I. 5). *M'_* is elliptic (resp. hypoelliptic) if and only if M_* is elliptic (resp. hypoelliptic).*

II. Duality. Let Ω be an open set in R^n, and M_* a complex (satisfying the conditions 1 and 2 of paragraph I). We obtain a topological and algebraic duality between Hom $(M_*, \mathscr{E}(\Omega))$ and Hom $(M'_*, \mathscr{D}'_c(\Omega))$, in the following way:

1. M_k being free of finite type, we have an isomorphism $M_k \simeq A^{l_k}$; then we have an isomorphism Hom $(M_k, \mathscr{E}(\Omega)) \simeq \mathscr{E}(\Omega)^{l_k}$; here the second term is, with its natural topology, a space of type (F) (and even of type (FS) [5]); we transfer this topology to the first member; one verifies easily that this topology is independent of the isomorphism chosen.

2. In the same way, Hom $(M'_{-k}, \mathscr{D}'_c(\Omega))$ is a space of type (DF), and even (DFS) [5]; using the "canonical" pairing $\mathscr{E}(\Omega)\times\mathscr{D}'_c(\Omega) \to C$ and the isomorphism

$$\text{Hom}\,(\text{Hom}\,(M_k, A), \mathscr{D}'_c(\Omega)) \simeq M_k \otimes \mathscr{D}'_c(\Omega)$$

(M_k is free of finite type !) one gets a pairing

$$\text{Hom}\,(M_k, \mathscr{E}(\Omega)) \times \text{Hom}(M'_{-k}, \mathscr{D}'_c(\Omega)) \to C.$$

One verifies immediately that this pairing is separately continuous, and that the first term is the topological dual of the second.

3. Denote by P_k the differential operator Hom $(d_k, \mathscr{E}(\Omega))$ and define P'_k in the same way. Then, from the preceding duality we see that P'_{-k+1} is the *transpose* of P_k.

Now, the cohomology group H^k(Hom $(M_*, \mathscr{E}(\Omega))$ is isomorphic to ker P_{k+1}/im P_k. The space ker P_{k+1} is a closed subspace of Hom $(M_k, \mathscr{E}(\Omega))$, and is, with the induced topology, a space (F). We put on H^k(Hom $(M_*, \mathscr{E}(\Omega)))$ the quotient topology, *which is not necessarily* a Hausdorff topology (it is if and only if im P_k is closed); we call this type of topology "quotient of an (F)-space", or "$(q\text{-}F)$".

Similarly, H^{-k}(Hom $(M_*, \mathscr{D}'_c(\Omega)))$ has a topology of "quotient of a (DF)-space", or "$(q\text{-}DF)$" (according to a theorem of Grothendieck, a closed subspace of a (DFS)-space is also (DFS) [5]). H^{-k}(Hom $(M'_*, \mathscr{D}'_c(\Omega)))$ is Hausdorff if and only if im P'_{-k} is closed;

this is equivalent to saying that im P_{k+1} is closed, or that $H^{k+1}(\text{Hom}\,(M_*, \mathscr{E}(\Omega)))$ is Hausdorff.

Now, the preceding pairing induces a pairing

$$\alpha_k : H^k(\text{Hom}\,(M_*, \mathscr{E}(\Omega))) \times H^{-k}(\text{Hom}\,(M'_*, \mathscr{D}'_c(\Omega))) \to C.$$

THEOREM (II. 1). 1. *The pairing* α_k *is separately continuous, and induces a duality between the associated Hausdorff spaces* (*i. e. the largest Hausdorff quotients*).

2. $H^k(\text{Hom}\,(M_*, \mathscr{E}(\Omega)))$ *is Hausdorff if and only if*

$$H^{-k+1}(\text{Hom}\,(M'_*, \mathscr{D}'_c(\Omega)))$$

is Hausdorff.

(2 has been proved, and 1 is trivial).

This theorem is of course not new. (As far as I know, results of this kind were studied systematically for the first time by Serre [13]. The form given here was suggested to me by Martineau).

III. Convexity. The main purpose of this paper is an attempt to define the "convexity" of an open set with respect to a given complex. We justify this definition by giving "abstract" theorems connecting this definition with the "existence theory" (the fact that some H^k are Hausdorff, finite-dimensional, etc....). We shall not discuss examples here, but note that it generalises naturally both the "p-convexity" of Rothstein—Grauert—Andreotti [1] for complex variables and the so-called "uniqueness for Cauchy problem" in the theory of *one* linear equation (for the results of this theory, we refer to Hörmander [8]). I must say that, for the moment, I have no examples, except the examples existing in the literature (or which can be easily deduced from known results; in this direction, we mention, without giving precise statements, the ' universal" convexity of ordinary convex sets, and the fact that the problem of continuation of holomorphic solutions of some equations, studied by Leray, is more or less connected with this convexity).

First, we give a condition for $H^k(\text{Hom}\,(M_*, \mathscr{D}'_c(\Omega)))$ to be Hausdorff. Here, if K is compact $\subset R^n$, we denote by $\mathscr{E}_K$ the space $\Gamma_K(R^n, \mathscr{E})$ of sections of $\mathscr{E}$ with support in K (of course, one must not confuse this with $\mathscr{E}(K)$, the space of germs of sections of $\mathscr{E}$ near K); we define $\mathscr{D}'_K$ similarly.

THEOREM (III. 1). *Consider a matrix* $P \in \mathrm{Hom}\,(A^p, A^q)$, *and an open set* $\Omega \subset R^n$. *Suppose the following property is satisfied*:
(III. 2). *For every compact* $K \subset \Omega$, *there exists a compact* $K' \subset \Omega$ *which verifies*

$$P\,\mathscr{D}'_c\,(\Omega)^p \cap \mathscr{E}^q_K \subset P\,\mathscr{E}^p_{K'}.$$

Then, $P\,\mathscr{D}'_c(\Omega)^p$ *is closed in* $\mathscr{D}'_c\,(\Omega)^q$.

PROOF. (a) We prove first that for K compact $\subset \Omega$, the subspace E of $\mathscr{E}^q_K$, defined by

$$E = \mathscr{E}^q_K \cap P\,\mathscr{D}'_c\,(\Omega)^p$$

is closed.

We denote by $\mathscr{C}$ the sheaf of germs of continuous functions in Ω; by hypothesis, we have

(III. 3) $E = \mathscr{E}^q_K \cap P\,\mathscr{E}^p_{K'}$, therefore $E = \mathscr{E}^q_K \cap P\,\mathscr{C}^p_{K'}$.

Let N be the kernel of the mapping $P : \mathscr{D}'_c\,(\Omega)^p \to \mathscr{D}'_c\,(\Omega)^q$ and put $N_1 = N \cap \mathscr{C}^p_{K'}$, $N_2 = N \cap \mathscr{E}^p_{K'}$. Denote by V_1 (resp. V_2) the subspace of

$$(\mathscr{C}^p_{K'}/N_1) \times \mathscr{E}^q_K \text{ (resp. } (\mathscr{E}^p_{K'}/N_2) \times \mathscr{E}^q_K)$$

consisting of the pairs (f, g) verifying $Pf = g$. The natural injection $i : V_2 \to V_1$ is a bijection (and then is bicontinuous), according to (III. 3); denoting the canonical projection of V_1 into $\mathscr{C}^p_{K'}/N_1$ (resp. $\mathscr{E}^q_K$) by pr_1 (resp. pr_2), we have

$$i = (pr_1 \circ i,\ 0) + (0, pr_2 \circ i).$$

The injection $\mathscr{E}_{K'} \to \mathscr{C}_{K'}$ is compact (Ascoli's theorem), then $pr_1 \circ i$ is compact (all the spaces introduced here being equipped with their natural topologies). Therefore, $pr_2 \circ i$ is a strict morphism, and has a closed image (see [12], Theorem 1).

This argument is essentially due to F. Treves (unpublished).

(b) Using this result, one proves easily, by an argument of regularisation, the following result:

Let K_1 and K_2 be compact subsets of Ω, with $K_1 \subset \mathring{K}$, $K' \subset \mathring{K}_2$. Then, $\mathscr{D}'^q_{K_1} \cap P\,\mathscr{D}'_c(\Omega)^p$ is a closed subspace of $\mathscr{D}'^q_{K_1}$, and is equal to $\mathscr{D}'^q_{K_1} \cap P\,\mathscr{D}'^p_{K_2}$ (see [9], proof of Lemma 3.1).

(c) To prove now that $P\,\mathscr{D}'_c(\Omega)^p$ is closed, it suffices, according to a theorem of Banach, to prove that its intersection with any closed bounded convex set is closed; and this results immediately from (b) and the fact that, in $\mathscr{D}'_c(\Omega)$, all elements of a bounded set have their supports in a fixed compact.

REMARK (III. 4). I do not know if the converse of the preceding theorem is true. Actually, one can obtain a necessary and sufficient condition for the closure of $P\,\mathscr{D}'_c(\Omega)^p$, very close to (III. 2):

for every compact $K \subset \Omega$, there exists a compact $K' \subset \Omega$ and an integer $k > 0$, with the following property:

if $f \in P\,\mathscr{D}'_c(\Omega)^p \cap \mathscr{D}'^q_K$ has continuous derivatives up to order $s + k$ (s, any integer $\geqslant 0$), there exists g having continuous derivatives up to order s verifying $g \in \mathscr{D}'^p_{K'}$ and $Pg = f$.

REMARK (III. 5). Using well-known arguments of duality, one can deduce from Theorem (III. 1) some properties of density for the solutions of the transposed equation (see for instance [8], chap. 3).

Now, we note that property (III. 2) can be decomposed in two parts.

(i) *A property of regularity.*

(III. 6). $P\,\mathscr{D}'_c(\Omega)^p \cap \mathscr{E}_c(\Omega)^q = P\,\mathscr{E}_c(\Omega)^q$.

(ii) *A property of the supports.*

(III. 7). *For every compact $K \subset \Omega$, there exists a compact $K' \subset \Omega$ verifying*

$$P\,\mathscr{E}_c(\Omega)^p \cap \mathscr{E}^q_K \subset P\,\mathscr{E}^p_{K'}.$$

Now we are interested only in this last property.

DEFINITION (III. 8). *Let M_* be a complex (in the sense of § 1), and p an integer. We say that Ω is p-convex with respect to M_* if, for $k \leqslant -p+1$, the mapping*

P'_k: Hom $(M'_{k-1}, \mathscr{E}_c(\Omega)) \to$ Hom $(M'_k, \mathscr{E}_c(\Omega))$ *satisfies* (III.7).

REMARK (III.9). If M_* (and then M'_*) is hypoelliptic, P'_k verifies (III.6) for all k (see § 1); then, for $k \leqslant -p+1$, the hypotheses of Theorem (III.1) are satisfied.

We shall give now a sufficient condition, *of local character on* $\partial\Omega$ (the boundary of Ω), for p-convexity. (We follow essentially here an idea due to Ehrenpreis [4], in the case of complex variables; it is slightly different from the method of Andreotti-Grauert [1], which uses global properties of $\partial\Omega$. A recent, and yet unpublished, approach due to Andreotti avoids the use of global properties, using the local cohomology of $\overline{\Omega}$; but simple examples show that, for general differential systems, local cohomology gives satisfactory results only in the elliptic case [11]. This is the reason for our use of the family Φ of supports defined below, rather than local cohomology).

Let Ω be a fixed open set in R^n, and $\mathcal{O}$ a variable subset of $\overline{\Omega}$, *open in* $\overline{\Omega}$; we denote by $\Phi(\mathcal{O})$ (or simply Φ) the family of subsets of $\mathcal{O} \cap \Omega$ which are closed in $\mathcal{O}$, and we put $\mathscr{E}_\Phi(\mathcal{O}) = \Gamma_{\Phi(\mathcal{O})}(\mathcal{O}; \mathscr{E})$.

DEFINITION (III.10). *Let $a \in \partial\Omega$; we say that Ω is strictly p-convex at a with respect to M_* if the following property is satisfied.*

For every $\mathcal{O} \subset \overline{\Omega}$ with $a \in \mathcal{O}$, $\mathcal{O}$ open in $\overline{\Omega}$, there exists $\mathcal{O}' \subset \mathcal{O}$, with $a \in \mathcal{O}'$, $\mathcal{O}'$ open in $\overline{\Omega}$, such that the natural mapping

$$H^k(\text{Hom}\,(M'_*, \mathscr{E}_\Phi(\mathcal{O}))) \to H^k(\text{Hom}\,(M'_*, \mathscr{E}_\Phi(\mathcal{O}')))$$

reduces to zero for $k \leqslant -p$.

THEOREM (III.11). *Suppose that Ω is bounded, and strictly p-convex with respect to M_* at every point of $\partial\Omega$. Then*

(i) *Ω is p-convex with respect to M_*;*

(ii) *there exists $\Omega' \subset\subset \Omega$ such that the natural mapping*

$$H^k(\text{Hom}\,(M'_*, \mathscr{E}_c(\Omega'))) \to H^k(\text{Hom}\,(M'_*, \mathscr{E}_c(\Omega)))$$

is surjective for $k \leqslant -p$.

It is obviously sufficient to prove

(III. 12). *Let $k \leqslant -p$, and let K be a compact $\subset \Omega$; there exists a compact K' with $K \subset K' \subset \Omega$, verifying the following property:*

For any $\phi \in \text{Hom}(M'_k, \mathscr{E}_\Phi(\bar{\Omega} - K))$ with $P'_{k+1}\phi = 0$, there exists $\psi \in \text{Hom}(M'_{k-1}, \mathscr{E}_\Phi(\bar{\Omega} - K'))$ with $P'_k \psi = \phi$ in $\bar{\Omega} - K'$.

Proof of (III.12). Denote by $\mathscr{E}^r_\Phi$ the sheaf $\mathscr{O} \to \text{Hom}(M'_r, \mathscr{E}_\Phi(\mathscr{O}))$, and consider the mapping $P'_{r+1} : \mathscr{E}^r_\Phi \to \mathscr{E}^{r+1}_\Phi$; we denote by Z^r_Φ its kernel; its image is obviously contained in Z^{r+1}_Φ. The hypothesis on p-convexity can be expressed in the following manner: let $r \leqslant -p$, $a \in \partial\Omega$, and let $\mathscr{O}$ be an open set in $\bar{\Omega}$, $a \in \mathscr{O}$; there exists $\mathscr{O}'$, open in $\bar{\Omega}$, $a \in \mathscr{O}'$, such that $P'_r\, \mathscr{E}^{r-1}_\Phi(\mathscr{O})$, *restricted to* $\mathscr{O}'$, is equal to $Z^r_\Phi(\mathscr{O}')$. We express this fact by saying that the sequence

$$0 \to Z^{r-1}_\Phi \to \mathscr{E}^{r-1}_\Phi \to Z^r_\Phi \to 0 \tag{III.13}$$

is "exact near $\partial\Omega$". (This is of course stronger than saying that the restriction to $\partial\Omega$ is exact ; actually, this last statement is trivial, everything being zero on $\partial\Omega$!)

The idea of the proof is to derive from (III.13) a variant of the "exact sequence of cohomology". Let $\mathscr{U} = (U_i)_{i \in I}$ be a finite collection of open sets in $\bar{\Omega}$, covering $\partial\Omega$ (we call this a covering of $\bar{\Omega}$ near $\partial\Omega$); we consider the groups of cohomology of this covering $H^s(\mathscr{U}; Z^r_\Phi)$; if $\mathscr{U}'$ is a refinement of $\mathscr{U}$ (every set of $\mathscr{U}'$ is contained in some set of $\mathscr{U}$; note that $\mathscr{U}'$ need not cover $\bigcup U_i$), we have the usual restriction mapping

$$\rho_{r,s}(\mathscr{U}, \mathscr{U}') : H^s(\mathscr{U}; Z^r_\Phi) \to H^s(\mathscr{U}'; Z^r_\Phi).$$

For $r \leqslant -p$, $s \geqslant 1$, we prove : given r, s, $\mathscr{U}$, there exists $\mathscr{U}'$ with $\rho_{r,s}(\mathscr{U}, \mathscr{U}') = 0$; we express this fact by saying "the direct system $H^{s,r}_\Phi : \mathscr{U} \to H^s(\mathscr{U}; Z^r_\Phi)$ is zero near $\partial\Omega$".

To do this, using (III.13), given $r \leqslant -p$, $s \geqslant 0$ and $\mathscr{U}$, we construct a refinement $\mathscr{U}'$ of $\mathscr{U}$ and a connecting homomorphism :

$$\delta_{r,s}(\mathscr{U}, \mathscr{U}') : H^s(\mathscr{U}; Z^r_\Phi) \to H^{s+1}(\mathscr{U}'; Z^{r-1}_\Phi)$$

with the following property for $s \geqslant 1$:

If $\mathscr{U}''$ is a refinement of $\mathscr{U}'$ and if

$$\rho_{r-1,s+1}(\mathscr{U}', \mathscr{U}'')\ \delta_{r,s}(\mathscr{U}, \mathscr{U}') = 0, \quad \text{then} \quad \rho_{r,s}(\mathscr{U}, \mathscr{U}') = 0$$

(we omit the details, which are the standard ones).

Now, for $r \leqslant -p$, $s \geqslant 1$, if $H_\Phi^{s+1,r-1}$ is zero near $\partial\Omega$, $H_\Phi^{s,r}$ has the same property; but, for large negative values of r, M'_r is zero and then $H_\Phi^{s,r}$ is zero; by induction on r, we obtain the desired result.

Now, to prove (III.12), we consider a covering $\mathscr{U} = (U_i)_{i \in I}$ of $\overline{\Omega}$ near $\partial\Omega$, with $U_i \subset \overline{\Omega} - K$, such that in each U_i, one has $\phi = P'_k \psi_i$, with $\psi_i \in \mathscr{E}_\Phi^{k-1}(U_i)$; in $U_i \cap U_j$, we put $\chi_{ij} = \psi_i - \psi_j$; the collection (χ_{ij}) is a 1-cocycle of $\mathscr{U}$ with values in Z_Φ^{k-1} (note that this construction is just that of the connecting homomorphism $\delta_{k,0}$). Using the preceding assumption, we can get a refinement $\mathscr{U}'$ of $\mathscr{U}$ (depending on K, but not on ϕ) such that the image of this cocycle in $H^1(\mathscr{U}'; Z_\Phi^{k-1})$ is zero; if we write, for simplicity, $\mathscr{U}$ instead of $\mathscr{U}'$, we have then: $\chi_{ij} = \chi_i - \chi_j$, with $\chi_i \in Z_\Phi^{k-1}(U_i)$. Defining $\psi \in \mathscr{E}_\Phi^{k-1}(\bigcup U_i)$ by: $\phi = \phi_i - \chi_i$ in U_i, we get a solution of (III.12).

REMARK (III.14). Suppose M_* to be hypoelliptic, and let Ω verify the hypotheses of Theorem (III.11). Using (III.1) and (III.11.i), one obtains the result that $H^k(\mathrm{Hom}\,(M'_*, \mathscr{D}'_c(\Omega)))$ *is Hausdorff for* $k \leqslant -p+1$.

Actually, we obtain more, using (III.11.ii): *for* $k \leqslant -p$, *this space is finite-dimensional* (we follow here Andreotti-Vesentini [2]). For, the injection $i : \mathscr{E}_K \to \mathscr{D}'_c(\Omega)$ (we put here $K = \overline{\Omega}'$) induces a surjective (and obviously continuous) mapping

$$\tilde{i} : H^k(\mathrm{Hom}\,(M'_*, \mathscr{E}_K)) \to H^k(\mathrm{Hom}\,(M'_*, \mathscr{D}'_c(\Omega))).$$

The first space is (q-F), and the second (DFS) (note we have already proved it is Hausdorff); the "theorem of strict morphism" is then valid here [6]; i being a compact mapping, it proves that the identity mapping of $H^k(\mathrm{Hom}\,(M'_*, \mathscr{D}'_c(\Omega)))$ into itself is compact; therefore this space is finite-dimensional.

With the same hypotheses, one gets now, using Theorem (II.1) the following result:

$H^k(\text{Hom}\ (M_*, \mathscr{E}(\Omega)))$ *is finite-dimensional for* $k \geqslant p$.

REMARK (III.15) We note that the preceding method does not use the hypoellipticity, of M_* (or M'_*, which is equivalent), but only the fact that the natural mapping

$$H^k(\text{Hom}\ (M'_*, \mathscr{E}_c(\Omega))) \to H^k(\text{Hom}\ (M'_*, \mathscr{D}'_c(\Omega)))$$

is bijective for $k \leqslant -p$, and injective for $k = -p + 1$. We do not discuss this question here. We wish only to mention that this question is more or less related (but not equivalent) to the question of the "convexity modulo $\mathscr{C}^\infty$" for which a theory could probably be developed on the same lines as those developed here for the theory of convexity. But, at the moment, I have no examples, except of course in the case of "one equation" where very satisfactory results have been obtained by Hörmander [8] (see chapters 3 and 8).

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THE INDEX PROBLEM FOR MANIFOLDS WITH BOUNDARY

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1. Introduction. The aim of these lectures is to report on the progress of the index problem in the last year. We will describe an extension of the index formula for closed manifolds, (see Atiyah and Singer, [3]) to manifolds with boundary. The work of Section 4, i.e. the proof of the general index theorem from Theorem 1 was done in collaboration with Singer.

The first question which one encounters when seeking such an extension is how to measure the topological implications of elliptic boundary conditions. The boundary conditions of course have a definite effect on the index, as the following example shows.

Let X denote the unit disc in the plane. let $Y = \partial X$ be its boundary, and let b be a vector-field defined and *never zero* on Y. Let D be the Laplacian on X, and consider the operator

$$(D, b): C^\infty(X) \to C^\infty(X) \oplus C^\infty(Y)$$

which sends f into $Df \oplus (bf \mid Y)$, where bf denotes the directional derivative of f along b. Because D is elliptic and b is non-vanishing, the kernel and cokernel of this operator are finite dimensional and the difference of their dimensions is by definition the index of the boundary value problem, (D, b). Our problem is now to describe this integer in terms of the topological data which are implicitly given by D and the boundary conditions.

In the present instance this problem is completely solved by the following formula due to Vekua (see Hörmander [4]):

$$\text{index } (D, b) = 2\,(1 - \text{winding number of } b).$$

To proceed further we need to recall the form which the formula for the index on a closed manifold takes; and for simplicity we will describe this formula only in the case of systems—i.e. trivial vector bundles.

The notation is as follows. We use X for the basic manifold and $\partial X = Y$ for its boundary. $T = T(X)$ denotes the cotangent bundle of X; we write $B = B(X)$ for the ball-bundle consisting of vectors in $T(X)$ of length $\leqslant 1$ (in some fixed Riemannian structure) and set $S = S(X)$ equal to the sphere-bundle of unit vectors in T.

Observe here first of all that if X is a closed manifold (i.e. $\partial X = \emptyset$), then $S(X)$ is just the boundary of $B(X)$:

$$\partial B(X) = S(X).$$

On the other hand if the manifold X has a boundary Y then:

$$\partial B(X) = S(X) \cup B(X) \mid Y$$

where $B(X) \mid Y$ denotes the subspace of $B(X)$ lying over Y under the natural map π: $B(X) \to X$. (We denote the projection $\partial B(X) \to X$ also by π.)

With the notation and this elementary fact out of the way, it is easy to indicate the general form of the index formula for elliptic systems on closed manifolds.

Let then $Df_i \equiv \sum_{j=1}^{k} A_{ij} f_j$ be a system of k linear partial differential operators, defined on the manifold X of dimension n. The symbol of D is then a function $\sigma(D)$ on $T(X)$ which attaches to each cotangent vector λ of X, the matrix $\sigma(D\colon \lambda)$ obtained from the highest terms of A_{ij} by replacing $\dfrac{\partial^\alpha}{\partial x^\alpha}$, by $(i\,\lambda)^\alpha$. Further, the system D is *elliptic* if and only if the function $\sigma(D)$ maps $S(X)$ into the group $GL(k, C)$ of nonsingular $k \times k$ matrices with complex coefficients. Hence, for such systems, the symbol defines a map

$$\sigma(D) : S(X) \to GL$$

where we have set $GL = \lim_{m \to \infty} GL(m, C)$, and it is a basic consequence of the invariance of the index of an elliptic system under deformations that on a closed manifold the index of D depends only on the homotopy class of the map $\sigma(D)$ defined above.

The index formula we are after now takes the following form. One constructs a definite differential form $ch = \Sigma\, ch^i$ (with components in every dimension) on GL†. Further, using a universal expression in the curvature of X, one constructs a definite form $\mathscr{T}(X) = \Sigma\, \mathscr{T}^i(X)$ on X. The index formula for a closed manifold then expresses the index of D as an integral‡:

$$\text{index }(D) = \int_{S(X)} \sigma(D)^*\, ch \wedge \pi^*\, \mathscr{T}(X)$$

where of course $\sigma(D)^* ch$ denotes the form on GL pulled back to $S(X)$ via $\sigma(D)$. Now, in view of our earlier remark, one may rewrite this integral as

$$\text{index }(D) = \int_{\partial B(X)} \sigma(D)^*\, ch \wedge \pi^* \mathscr{T}(X),$$

and in this form the formula would be meaningful even for a manifold with boundary, provided only that $\sigma(D)$, which is defined only on $S(X)$, is extended in some definite way to $\partial B(X)$.

It is in this extension that the topological data of a set of elliptic boundary conditions manifest themselves. In fact our first and main aim will be the proof of the following theorem.

THEOREM 1. *A set of elliptic boundary conditions, B, on the elliptic system D, defines a definite map $\sigma(D, B) : \partial B(X) \to GL$, which extends the map $\sigma(D) : S(X) \to GL$ to all of $\partial B(X)$.*

Note that in particular then, $\sigma(D)$ *restricted to a fibre-sphere of* $S(X)$ *at points over Y is homotopic to* 0; *expressed differently, there are topological obstructions to imposing elliptic boundary conditions on elliptic systems.*

The final index theorem is then given by the same formula as the original one.

THEOREM 2. *The index of D subject to the elliptic boundary condition B, is given by*:

† Strictly speaking we define a differential form $ch\,(m)$ on each $GL(m, C)$, these being compatible with the inclusions $GL(m, C) \to GL(m+1, C)$.

‡ This formula differs slightly from that given in the note of Atiyah-Singer[3]. Here X does not have to be orientable. We orient $B(X)$ by its almost complex structure.

$$\text{index } (D,B) = \int_{\partial B(X)} \sigma(D,B)^* \, ch \wedge \pi^* \mathscr{T}(X).$$

We will give the definition of $\sigma(D, B)$ in rather complete detail in Sections 2 and 3. The proof of Theorem 2 is then sketched in Section 4.

Our original proof of Theorem 1 was not elementary. We had to use the periodicity theorem $\pi_k(GL) \simeq \pi_{k+2}(GL)$. Thereafter we noticed that our argument can in a certain sense be turned about so that the considerations needed to extend $\sigma(D)$ to $\sigma(D,B)$ can be used to give a new and in a sense completely elementary proof of the periodicity theorem (see Atiyah-Bott [2]).

In this note we will apply this method, which is suggested by the linearisation procedure in differential equations, directly to the construction of $\sigma(D, B)$.

2. Elliptic boundary conditions. Let D be a $k \times k$ elliptic system of differential operators on X and let r denote the order of D. Then, as we have remarked, the symbol $\sigma(D)$ is a function on the co-tangent vector bundle $T(X)$ whose values are $(k \times k)$-matrices, and its restriction to the unit sphere-bundle $S(X)$ takes non-singular values.

We consider a system B of boundary operators given by an $l \times k$ matrix with rows $b_1, \ldots, b_l$ of orders $r_1, \ldots, r_l$, and we suppose that they are elliptic in the sense of Lopatinski (see Hörmander : loc. cit.). This means the following. Let $\sigma(b_i)$ denote the symbol of b_i and let $\sigma(B)$ be the matrix with $\sigma(b_i)$ as i-th row. At a point of the boundary Y of X let ν be the unit inward normal and let y denote any unit tangent vector to Y. Put

$$\sigma_y(D)(t) = \sigma(D)(y + t\nu) \; ; \; \sigma_y(B)(t) = \sigma(B)(y + t\nu)$$

so that $\sigma_y(D)$ and $\sigma_y(B)$ are polynomials in t. Consider the system of ordinary linear equations

$$\sigma_y(D)\left(-i\frac{d}{dt}\right) u = 0 \tag{2.1}$$

and let $\mathscr{M}_y$ denote its space of solutions. The ellipticity of D gives rise to a decomposition

$$\mathscr{M}_y = \mathscr{M}_y^+ \oplus \mathscr{M}_y^-$$

where $\mathscr{M}_y^+$ consists only of exponential polynomials involving $\exp(i\lambda t)$ with $\mathrm{Im}(\lambda) > 0$ and $\mathscr{M}_y^-$ involves then those with $\mathrm{Im}(\lambda) < 0$. The ellipticity condition for B (relative to D) is that the equations (2.1) have a unique solution $u \in \mathscr{M}_y^+$ satisfying the boundary condition

$$\sigma_y(B)\left(-i\frac{d}{dt}\right) u\bigg|_{t=0} = V \tag{2.2}$$

for any given $V \in C^l$.

We proceed next to put this condition into a purely algebraic form. Let $\Lambda \subset C(t)$ be the ring of all rational functions of t with no poles in the half-plane $\mathrm{Im}\, t > 0$. We may then regard $\sigma_y(D)\,(t)$ as defining a homomorphism of free Λ-modules of rank k, and we let M_y^+ denote its cokernel. Thus we have the exact sequence of Λ-modules

$$0 \longrightarrow \Lambda^k \xrightarrow{\sigma_y} \Lambda^k \longrightarrow M_y^+ \longrightarrow 0. \tag{2.3}$$

Then we have the following lemma.

LEMMA. *There is a natural isomorphism of vector spaces*

$$M_y^+ \simeq \mathscr{M}_y^+. \tag{2.4}$$

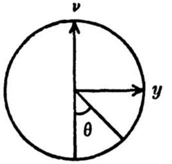
In view of (2.4) we see that the elliptic boundary condition gives an isomorphism

$$\beta_y^+ : M_y^+ \longrightarrow C^l. \tag{2.5}$$

Now the set of all M_y^+ for $y \in S(Y)$ forms a vector bundle M^+ over $S(Y)$ and (2.5) defines an isomorphism β^+ of M^+ with the trivial bundle $S(Y) \times C^l$.

If we deal only with differential boundary conditions then the map β_y^+, regarded as a function of y, cannot be an arbitrary continuous function. In order to obtain all continuous functions we need to enlarge our problem and consider integro-differential boundary conditions in the sense of Agranovic—Dynin [1]. This causes no essential analytical difficulties and is an important topological simplification. Thus from now on an elliptic problem (D, B) will have associated with it $\sigma(D)$, M^+, β^+, where β^+ can now be any vector bundle isomorphism of M^+ with the trivial bundle.

3. Extension of $\sigma(D)$ over Y. The proof of Theorem 1. In this section we shall show how a trivialization β^+ of M^+ defines a definite extension of $\sigma(D) : S(X) \to GL$ to a map $\partial B(X) \to GL$. What we will do is to perform a sequence of homotopies of $\sigma(D)|Y$ so that finally it can be extended in a trivial fashion over $B(X) \mid Y$.



Our first step is to parametrize $S(X)$ in the form

$$y \sin \theta - \nu \cos \theta \quad (0 \leqslant \theta \leqslant \pi)$$

and then put $z = \exp 2i\theta$. In this way, by assigning (y, z) to each point of $S(X) \mid Y$, we really define a continuous map

$$f : S(X) \mid Y \to S(Y) \times S^1/S(Y) \times \{1\},$$

where S^1 is the unit circle $|z| = 1$. We shall then transfer our maps from $S(X) \mid Y$ to $S(Y) \times S^1$.

Let us write $\sigma = \sigma(D)$ and put

$$\sigma(y \sin \theta - \nu \cos \theta) = \exp ir(\pi - \theta)\, \sigma(\nu)\, p_y(z)$$

so that $p_y(z)$ is a $k \times k$ matrix of polynomials in z with $p_y(1) = 1$, and non singular for $|z| = 1$. Then our first homotopy is as follows.

First Homotopy.

$$\sigma_u = \exp iru\,(\pi - \theta)\, \sigma(\nu)\, p_y(z), \quad 0 \leqslant u \leqslant 1. \qquad (3.1)$$

Then $\sigma_1 = \sigma(\nu)\, p_y(z)$, and $p_y(z)$ is a map which factors through f.

Now we shall deform $p_y(z)$. For this purpose we can work on $S(Y) \times S^1$ instead of $S(Y) \times S^1/S(Y) \times \{1\}$. The reason is that if $g(y,z)$ is a map $S(Y) \times S^1 \to GL$ then $h(y,z) = g(y,1)^{-1} g(y,z)$ is a map $S(Y) \times S^1 / S(Y) \times \{1\} \to GL$, and similarly for homotopies.

Now we recall briefly the homotopies used in the paper [2] for the proof of the periodicity theorem. Write

$$p_y(z) = \sum_{i=0}^{r} a_i(y) z^i,$$

$$L^r p_y(z) = \begin{bmatrix} a_0(y) & a_1(y) & \cdots & a_r(y) \\ -z & 1 & \cdots & 0 \\ 0 & -z & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -z & 1 \end{bmatrix}$$

Then we have the matrix identity

$$\begin{bmatrix} a_0 & a_1 & \cdots & \cdots & a_r \\ -z & 1 & \cdots & \cdots & 0 \\ 0 & -z & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -z & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ z & 1 & & & \\ z^2 & z & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \\ z^r & & & z & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & p_1 & p_2 & \cdots & p_r \\ & 1 & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} p & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{bmatrix} \qquad (3.2)$$

where $p = p_y(z)$ and the p_i are polynomials in z defined by $p_{i+1}(z) = \dfrac{p_i(z) - p_i(0)}{z}$, $p_0 = p$. This can be written briefly as

$$(L^r p)(1+N_1) = (1+N_2)\ (p \oplus 1)$$

where N_1, N_2 are nilpotent matrices. Then our second homotopy is given by

Second Homotopy.

$$(1+uN_2)^{-1} L^r(p)\ (1+uN_1), \quad 0 \leqslant u \leqslant 1. \tag{3.3}$$

For the next stage we define maps Q^1, $Q^2 : S(Y) \to GL((r+1)\,k, C)$ by

$$Q^1_y = \frac{1}{2\pi i} \int_{|z|=1} q^{-1}\, dq,$$

$$Q^2_y = \frac{1}{2\pi i} \int_{|z|=1} dq \,.\, q^{-1},$$

where for brevity we write

$$L^r(p) = q = az + b.$$

Then Q^1, Q^2 are idempotents and

$$q\, Q^1 = Q^2\, q. \tag{3.4}$$

Moreover we have

$$\left.\begin{aligned} \text{rank } q(z)\, Q^1 &= \text{rank } Q^1 \text{ for } |z| \geqslant 1 \\ \text{rank } q(z)\ (1-Q^1) &= \text{rank } (1-Q^1) \text{ for } |z| \leqslant 1. \end{aligned}\right\} \tag{3.5}$$

Our third and fourth homotopies are then given by

Third Homotopy.

$$(az+ub)\ Q^1 + (uaz+b)\,(1-Q^1), \quad 0 \leqslant u \leqslant 1. \tag{3.6}$$

Fourth homotopy.

$$(a+ub)\,z\,Q^1 + (au+b)\ (1-Q^1), \quad 0 \leqslant u \leqslant 1. \tag{3.7}$$

In view of (3.4) we may replace the expression in (3.6) by

$$Q^2(az+ub)\,Q^1 + (1-Q^2)\,(uaz+b)\,(1-Q^1)$$

which shows that it is non-singular for $|z| = 1$. A similar argument with (3.5) shows that the expression in (3.7) is non-singular for $|z| = 1$.

So far we have not used the boundary operator at all. In order to do this we now need to identify the space M^+_y with the image H^+_y of the projection operator Q^1_y. This identification arises as follows.

Let $\Lambda' \subset C(z)$ be the ring of rational functions of z with no poles in the disc $|z| < 1$. The substitution $t = \dfrac{i(1+z)}{1-z}$ then induces an isomorphism $\alpha : \Lambda \to \Lambda'$. Since $\sigma_y(t) = \left(\dfrac{z-1}{2i}\right)^{-r} . \, \sigma(-\nu) . \, p_y(z)$ it follows that α induces an isomorphism of vector spaces

$$M_y^+ \simeq \text{Coker } p_y \tag{3.8}$$

where p_y is regarded as a homomorphism of Λ'-modules. Trivially we have an isomorphism

$$\text{Coker } p_y \simeq \text{Coker } (p_y \oplus 1). \tag{3.9}$$

Also from (3.2) we get a commutative diagram in which the vertical arrows represent isomorphisms.

$$\begin{array}{ccccccc} \Lambda'^k & \xrightarrow{p_y \oplus 1} & \Lambda'^k & \longrightarrow & \text{Coker } (p_y \oplus 1) & \longrightarrow & 0 \\ \downarrow{\scriptstyle 1+N_1} & & \downarrow{\scriptstyle 1+N_2} & & \downarrow & & \\ \Lambda'^k & \xrightarrow{q_y} & \Lambda'^k & \longrightarrow & \text{Coker } q_y & \longrightarrow & 0 \end{array} \tag{3.10}$$

where as before $q_y = L^r(p_y)$. Finally from (3.5) we obtain an isomorphism

$$H_y^+ \simeq \text{Coker } q_y \tag{3.11}$$

where as before H_y^+ is the image space of the projection operator Q_y^1.

Now using (3.8) — (3.11) and the isomorphism (2.5) given by the boundary operator we obtain an isomorphism

$$C^l \to H_y^+. \tag{3.12}$$

Since H_y^+ is by definition a subspace of C^s where $s = k(r+1)$ we can regard (3.12) as a monomorphism

$$\Phi_y : C^l \to C^s.$$

Consider then the homotopy of monomorphisms

$$\psi_y(u) : C^l \oplus C^l \to C^s \oplus C^l$$

given by

$$\psi_y(u) = \begin{pmatrix} \Phi_y & 0 \\ 0 & 1_l \end{pmatrix} \begin{pmatrix} 1_l \cos u & 1_l \sin u \\ -1_l \sin u & 1_l \cos u \end{pmatrix}, \; 0 \leqslant u \leqslant \frac{\pi}{2} \tag{3.13}$$

where 1_l is the identity $l \times l$ matrix. The image of $\psi_y(u)$ is independent of u so that for each u we can define a projection operator $P_y(u)$ on $C^s \oplus C^l$ by taking

$$\ker P_y(u) = \ker Q_y \oplus \psi_y(0 \oplus C^l)$$
$$\operatorname{Im} P_y(u) = \psi_y (C^l \oplus 0).$$

We note that

$$P_y(\pi/2) = \begin{pmatrix} 0 & 0 \\ 0 & 1_l \end{pmatrix}$$

is independent of y. Then our fifth homotopy is

Fifth Homotopy.

$$zP_y(u) + (1 - P_y(u)), \quad 0 \leqslant u \leqslant \pi/2. \tag{3.14}$$

Combining our five homotopies we end up with a composite homotopy connecting

$$\sigma(\nu)^{-1} \sigma \oplus 1_{kr+l} \quad \text{with } z\, 1_l \oplus 1_s.$$

We are now essentially finished. To complete the argument observe that $z = \exp(2i\,\theta)$ so that just as in our first homotopy we can eliminate z by a homotopy $\exp(2i\,u\,\theta)$, $0 \leqslant u \leqslant 1$. Hence we get a homotopy of $\sigma(\nu)^{-1} \sigma \oplus 1_{kr+l}$ to the constant map 1_{l+s}. Multiplication by

$$\begin{pmatrix} \sigma(\nu) & 0 \\ 0 & 1 \end{pmatrix}$$

then gives a homotopy connecting $\sigma \oplus 1$ and $\sigma(\nu) \oplus 1$. But $\sigma(\nu)$ is a function on Y and so it has a natural extension to $B(X) \mid Y$, constant on the fibres. Thus we have extended the map

$$\sigma : S(X) \to GL$$

to a map of $\partial B(X) \to GL$, so that Theorem 1 is now proved.

4. Proof of the Index Theorem. We shall very briefly indicate how Theorem 2 is proved. The idea is to reduce it to the case of manifolds without boundary. If X is a manifold with boundary Y then we can associate with X two manifolds without boundary namely Y and $\widetilde{X}$ the "double" of X (obtained by glueing two copies of X along their boundaries). In order to carry out this reduction we need to establish a number of properties both for the index and for the right hand side of the index formula. Let us denote by $\alpha_X(D, B)$ either one of these two expressions. If X has no boundary, B does not occur, and D may be replaced by a singular integral operator T. We shall also write $\alpha_X(\sigma)$ instead of $\alpha_X(\sigma(T))$. Then we need the following properties of $\alpha_X(D, B)$.

$$\alpha_X(D, B_1) - \alpha_X(D, B_2) = \alpha_Y(\beta_1^+ (\beta_2^+)^{-1}). \tag{4.1}$$

$$\alpha_X((D_1, B_1) \circ (D_2, B_2)) = \alpha_X(D_1, B_1) + \alpha_X(D_2, B_2). \tag{4.2}$$

If (a) $D = \Delta^m \, 1_k$ near Y,

(b) the j-th row of B_1 is the $(2j-1)$-th normal derivative $(1 \leqslant j \leqslant k)$,

(c) the j-th row of B_2 is the $(2j-2)$-th normal derivative $(1 \leqslant j \leqslant k)$,

then

$$\alpha_X(D, B_1) + \alpha_X(D, B_2) = \alpha_{\tilde{X}}(\tilde{D}), \tag{4.3}$$

where $\tilde{D}$ is the double or reflection of D.

Remarks. Formula (4.1) for the index is due to Agranovic and Dynin. It is proved by composing B_2 with T where $\sigma(T) = \beta_1^+ (\beta_2^+)^{-1}$. The composition in (4.2) is defined by

$$\{(D_1, B_1) \circ (D_2, B_2)\} u = D_1 D_2 u \oplus B_1 D_2 u|_Y \oplus B_2 u|_Y.$$

Thus (4.1) and (4.2) use the fact that the index is additive under composition of operators. In (4.3) Δ denotes the Laplace operator and we assume that the Riemannian metric on X is chosen so that, near Y, X is isomorphic with $Y \times I$, I being the unit interval. Normal derivatives then make invariant sense. To prove (4.3) when α is the index, one uses the involution on $\tilde{X}$ obtained by reflection.

Of course one has to verify, directly from the construction of $\sigma(D, B)$, i.e. Section 3, that the right hand side of Theorem 2 satisfies (4.1) — (4.3).

Once the above properties of $\alpha_X(D, B)$ have been established one proceeds as follows. First, using (4.2) with $(D_1, B_1) = (D_2, B_2)$ we see that it is sufficient to suppose D of even order. Next, by composing with the operator (Laplace operator, Dirichlet boundary conditions) which has $\alpha = 0$, we can suppose D of arbitrarily high order. Then a topological argument based on Section 3 shows that we can suppose we are in case (4.3) (a). Finally we use (4.1) and (4.3) together with the index formula for Y and $\widetilde{X}$, to deduce Theorem 2 for X.

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ON THE CALCULUS OF VARIATIONS

By S. SMALE

We give a brief account of an approach to the calculus of variations using differential calculus on an infinite dimensional linear space and some theorems coming out of this approach which give the Morse inequalities for a general non-linear Dirichlet problem. Much of this work was done with R. Palais (see [2], [3], [4]).

We suppose M is a compact C^∞ manifold (perhaps with boundary) equipped with a smooth measure and ξ is a real finite dimensional vector space bundle over M. Let $C^k(\xi)$ be the Banach space of C^k sections of ξ with C^k norm and $J^k(\xi)$ the vector space bundle of k-jets of sections of ξ. Then differentiation defines a continuous linear map $j_k : C^k(\xi) \to C^0(J^k(\xi))$. A Riemannian metric on $J^k(\xi)$ defines an inner product on $C^k(\xi)$ by $(f, g) = \int_M (j_k f(x), j_k g(x))$. Denote the Hilbert space completion of $C^k(\xi)$ with respect to this inner product by $H^k(\xi)$.

An integrand in a calculus of variations problem can be represented by a function $F : J^k(\xi) \to R$ which we suppose C^∞. Then it can be shown that $J : C^k(\xi) \to R$ defined by $J(f) = \int_M F(j_k f)$ is C^∞. If M is a disk, then J takes the familiar form that one sees in the calculus of variations literature,

$$J(f) = \int_M F(x, f(x), Df(x), \ldots, D^k f(x))\, dx$$

where D^i denotes the i^{th} derivative.

The "first variation" of this calculus of variations problem can be concisely described as the first derivative of J (in the sense of differentiation on Banach spaces, see e.g. [1]).

In usual problems one has in addition boundary conditions imposed on the functions of the space $C^k(\xi)$, for example given by an affine subspace of $C^k(\xi)$. The case of "Dirichlet boundary conditions" is defined as follows. If $f_0 \in C^k(\xi)$, let $C_0^k(\xi)$ consist of elements f of $C^k(\xi)$ with the property $j_{k-1}f = j_{k-1}f_0$ on ∂M. Thus $C_0^k(\xi) = C^k(\xi)$ if M has no boundary.

In case M is a domain in E^n with smooth boundary, $k = 1$, and

$$F(j_k\, f) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2,$$

f_0 an arbitrary element of $C^1(\xi)$, the usual Dirichlet principle states that $J : C_0^1(\xi) \to R$ takes on a minimum value.

The critical points of J generally satisfy the Euler equations (possibly in integrated form), as well as any boundary conditions, and in fact may be thought of as solutions of the calculus of variations problem.

Suppose f is a critical point of $J : C_0^k(\xi) \to R$, i.e. $J'(f) = 0$. Then $J''(f)$, the second derivative of J at f, defines a symmetric bilinear form on $C_{00}^k(\xi)$ called the Hessian (or "second variation") where $C_{00}^k(\xi)$ is the linear subspace of $C^k(\xi)$ defined by zero Dirichlet boundary conditions. Using a Riemannian structure on ξ, it can be shown that there exists a linear differential operator, the Jacobi operator, L on ξ with the property that

$$J''(f)\,(\eta,\ \tau) = \int_M (L\eta,\ \tau), \quad \eta,\ \tau \in C_{00}(\xi),$$

which can be used to study the nature of f relative to nearby g in $C_0^k(\xi)$. For example if L has positive spectrum, then f is a local minimum of J on $C_0^k(\xi) \to R$.

To obtain a reasonable existence theory for critical points of J, it is expedient to use $H^k(\xi)$ and the subspace $H_0^k(\xi)$ defined as the closure of $C_0^k(\xi)$ in $H^k(\xi)$. One difficulty that arises is that in general J will not extend even to a continuous map $J : H^k(\xi) \to R$. Let $F_{pp} : J^k(\xi) \to L_s^2(J^k(\xi), R)$ be the second derivative of F along the fiber with range the vector space bundle over M associated to $J^k(\xi)$ whose fiber over $x \in M$ is the space of the bilinear symmetric forms on $J^k(\xi)_x$ ($J^k(\xi)_x$ is the fiber of $J^k(\xi)$ over x). Then we have the

THEOREM. *If $F : J^k(\xi) \to R$ is such that $F_{pp} : J^k(\xi) \to L_s^2(J^k(\xi),\ R)$ has a relatively compact image, the associated map $J : C^k(\xi) \to R$, $J(f) = \int_M F(j_k f)$, extends to a C^2 map, $J : H^k(\xi) \to R$.*

The actual existence theorem requires in addition to the condition on F of this theorem, something like what is called strong ellipticity.

(*) Suppose that in a neighborhood of each point of M, there exists a trivialization of $J^k(\xi)$ of the form

$$U \times E \times L_s(V, E) \times \dots \times L_s^k(V, E) = \{(x, p^0, \dots, p^k)\}$$

where p^i represents the i^{th} derivative, with the property that

$$c_1 \| p^k \|^2 - c_2 \leqslant \int_U F(x, p^0, \dots, p^k)\, dx \quad \text{and}$$

$$c_3 \| \beta \|^2 \leqslant F_{p^k p^k}(x, p)\, (\beta, \beta), \quad \beta \in L_s^k(V, E).$$

Here the c_i are positive constants and $F_{p^k p^k}$ is the 2nd partial derivative of F with respect to p^k.

THEOREM. *Let $F : J^k(\xi) \to R$ satisfy (*) in addition to the hypothesis of the previous theorem, and $f_0 \in C^k(\xi)$ represent a Dirichlet boundary condition. Then the C^2 extension J: $H_0^k(\xi) \to R$ given by the previous theorem has a minimum and further, we may apply Morse theory to it.*

For example if J has 2 non-degenerate local minima, there is some other critical point.

For more details see the references cited.

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LOCAL TOPOLOGICAL PROPERTIES OF DIFFERENTIABLE MAPPINGS

By R. THOM

THERE are admittedly two basic theorems in Local Differential Analysis : namely Taylor's expansion formula and the implicit function theorem. Very important (at least in the author's opinion) are also the now classical theorem of A. Sard on the measure of critical values of a differentiable map, and the quite recent generalization due to B. Malgrange of the Weierstrass Preparation theorem to C^∞ functions.

Roughly speaking, the most general problem in Local Differential Analysis may be stated as follows :

Given a set of equations in $\mathbf{R}^n$:

$$g_1(x_i) = g_2(x_i) = \dots = g_k(x_i) = 0,$$

where the g_j are C^m-functions, what can be said of the set A of zeros, and of its topological structure ?

It is well known that any closed set in $\mathbf{R}^n$ is the set of zeros of a differentiable function. Hence no more precise results can be obtained without new assumptions on the equations. The simplest example of such a situation is given by the implicit function theorem. If at a point $a \in A$, the differentials $dg_1, dg_2, \dots, dg_k$ are linearly independent, then, locally, the set A is a differentiably imbedded submanifold; more precisely, by a change of coordinates, the given set of equations can be transformed into the system:

$$x_1 = x_2 \dots = x_k = 0 ;$$

therefore the set A is locally, in this new chart, a linear sub-variety of $\mathbf{R}^n$.

In this simple (but very essential) case, the topological structure of the set A at a is completely determined by the knowledge of the first partial derivatives of the $g_j(x_i)$. There are other cases where this situation occurs—outside the scope of the implicit function theorem. As an example, consider a function with a critical point at a, and non vanishing Hessian $\left| \frac{\partial^2 f}{\partial x_i \, \partial x_j} \right| \neq 0$. Then, according to Marston Morse's theorem, it is possible to find local coordinates (of class C^{m-2}) such that, with respect to these new coordinates f is locally a non degenerate quadratic form; as a result, the topological nature of the set A is completely determined (it is namely a quadratic cone). In this case the knowledge of the partial derivatives up to the second order is sufficient to determine the topological nature of the set A. Conversely, there are situations (like for instance the "flat" function $f = \exp(-1/x^2)$ at $x = 0$) where the knowledge of *all the derivatives* is not sufficient to determine locally the set of zeros.

This suggests the use of C. Ehresmann's terminology of "jets". Consider two mappings $F, G : \mathbf{R}^n \to \mathbf{R}^p$, which have the same value at the origin, as also all their derivatives up to order r. This defines among local mappings an equivalence class, the *jet* of order r of the map F (denoted $j^r(F)$); the totality of all such jets is provided with a natural structure of vector space (denoted $J^r(n, p)$). If, r, s are positive integers, there is a canonical mapping:

$$h_s : J^{r+s}(n, p) \to J^r(n, p),$$

defined by omitting in the $(r+s)$-jet all derivatives of order larger than r. The inverse image h_s^{-1} of any jet $z \in J^r(n, p)$ is obviously a vector space $\mathbf{R}^m$. We shall need the following essential definitions.

Stratification of a real algebraic variety. Let A be a real algebraic variety in Euclidean n-space $\mathbf{R}^n$. A *stratification* of A is a decomposition of A into a finite union of (differentiably imbedded), singularity-free, connected, manifolds U^j, the *strata* of A, such that:

1. the adherence of a stratum U is a "semi-algebraic" set;
2. the boundary $\overline{U} - U$ of a stratum is a finite union of lower dimensional strata;

3. (regular incidence property) any differentiable mapping $g : \mathbf{R}^k \to \mathbf{R}^n$ which is transversal on a stratum U, is also transversal locally on any stratum V of the "star" of U (the star of U consists of the strata W such that $\overline{W} \supset U$.)

A *semi-algebraic set* is here a set locally defined as a finite union of sets K_j, each K_j being defined by a set of polynomial equations $P_j = 0$ and inequalities $Q_k \geqslant 0$. That such a stratification exists has been proved by H. Whitney [1]. A similar decomposition exists also for semi-algebraic sets. (See in particular the triangulation theorem of S. Lojasiewicz.)

DEFINITION. *A differentiable mapping* $g : \mathbf{R}^k \to \mathbf{R}^n$ *is said to be transversal on a (semi)-algebraic set* $A \subset \mathbf{R}^n$, *if* g *is transversal on all strata of a stratification of* A.

As a straightforward consequence of the transversality lemma, the set of all mappings transversal on a given stratification of a compact semi-algebraic set A is an open and dense set in the function space $L(\mathbf{R}^k, \mathbf{R}^n)$, provided with the C^m-topology, where m is large enough.

STRATIFIED SETS. We shall use two definitions of stratified sets, one stronger than the other. The weaker one was given in [3].

A *weakly stratified set* E is a Hausdorff space having the following properties.

1. E is a finite union of (C^∞) differentiable manifolds, $E = \bigcup U_i$. (The U_i are the strata of E.)

2. The boundary $\partial U_i = \overline{U}_i - U_i$ of a stratum is a union of lower dimensional strata.

3. To any pair (X, Y) of strata with $X \subset \partial Y$ are associated a "tubular neighbourhood" T_{XY} of X in Y and a family of differentiable retractions $k_{XY} : T_{XY} \to X$ such that if $X \subset \partial Y$, $Y \subset \partial Z$, and T_{XY}, T_{YZ}, T_{XZ} are the associated tubes, then for any system of the given retractions k_{XY}, k_{YZ}, there is a retraction k_{XZ} for which $k_{XZ} = k_{XY} \circ k_{YZ}$ in $T_{XZ} \cap T_{YZ}$.

4. *Carpeting functions.* For any stratum U, there is a function C^∞ *on* U, C^0 *on* $\bar{U}$: ϕ: $\bar{U} \to \mathbf{R}$, with $\phi \geqslant 0$, $\phi^{-1}(0) = \partial U$, $d\phi \neq 0$ in a neighbourhood of ∂U in U, which has the property that for any stratum $V \subset \partial U$, the restriction $k_{VU} \mid \{x \in T_{VU} \mid \phi(x) = \epsilon\}$ is of maximal rank.

A *strongly stratified set* E is a Hausdorff space such that for any $x \in E$, there is a local presentation of E as the transversal intersection of a semi-algebraic set $A \subset \mathbf{R}^n$ by a diffeomorphism $g : \mathbf{R}^k \to \mathbf{R}^n$.

It is easily seen that a strongly stratified set is weakly stratified, a real (or complex) analytic set has a natural weak stratification, but I do not know whether it is strongly stratified (more precisely, whether an analytic set is locally the transverse intersection of a semi-algebraic set).

Definition. Stratified Mapping. Let A be a compact semi-algebraic set in $\mathbf{R}^{m+n}$; if p denotes the canonical projection $p : \mathbf{R}^{m+n} \to \mathbf{R}^n$, then the set $B = p(A)$ is a compact semi-algebraic set (Tarski-Seidenberg theorem). It is then possible to find stratifications of A and B in such a way that: (i) the image by p of any stratum $U \subset A$ is a stratum V of B, and the rank of $p \mid U$ is equal to the dimension of V; (ii) the inverse image $p^{-1}(V)$ of any stratum of B is a finite union of strata of A.

These local projections may be used as local models to define stratified mappings; here are the definitions.

Weakly Stratified Mappings. *Let A, B be weakly stratified sets, and $f : A \to B$ a map. f is called weakly stratified if it maps any stratum of A onto a stratum of B with maximal rank, and is compatible with the local retractions.*

Strongly Stratified Mappings. *Let A, B be strongly stratified sets. A map g: $A \to B$ is called strongly stratified if, to any $a \in A$, $b = g(a)$, there are local presentations of neighbourhoods A_a, B_b respectively as transversal intersections of semi-algebraic sets S_A, S_B,*

$$h_A : \mathbf{R}^{k_1} \to \mathbf{R}^{n+p} \supset S_A$$

$$h_B : \mathbf{R}^{k_2} \to \mathbf{R}^{p} \supset S_B$$

such that $h_A^{-1}(S_A) = A_a$, $h_B^{-1}(S_B) = B_b$, *and if* ϖ *denotes the natural projection* $\mathbf{R}^{n+p} \to \mathbf{R}^p$, *we have* $\varpi(S_A) = S_B$ *and* $h_B g = \varpi h_A$.

DEFINITION. *A stratified mapping* $h: A \to B$ *is said to present "blowing down" or* σ*-degeneracy, if there exist in* A *at least two strata* U, V *with* $U \subset \partial V$, *such that* $\operatorname{corank} h \mid U > \operatorname{corank} h \mid V$.

See [3], for examples.

I recall here the following theorems on weakly stratified mappings stated without proofs in [3]. We shall deduce some consequences of these theorems relating to the local topological properties of differentiable mappings.

THEOREM 1. *If a mapping* $g: A \to I$ *is stratified over the segment* I *(provided with its trivial stratification* $0, I$*), then, over any open interval contained in* I*, the mapping* g *is a (trivial) fibration.*

THEOREM 2. *If the two mappings* $A \xrightarrow{F} B \xrightarrow{G} I$ *are stratified (and the composed map* $G \circ F$*) over* I*, provided with the trivial stratification* $(0, I)$*, and, if* F *has no* σ*-degeneracy, then for any two values* s, t *in* I*, the sectional mappings* $F_t: A_t \to B_t$, $F_s: A_s \to B_s$ *are of the same topological type.*

We want now to deduce from these the

THEOREM 3. *Let* A *be a real algebraic set containing the origin* $0 \in \mathbf{R}^n$ *; let* $z \in J^r(p, n)$ *be a fixed jet, and* $h: J^{r+1}(p, n) \to J^r(p,n)$ *be the restriction mapping; in the vector space* $h^{-1}(z)$*, there exists a proper algebraic variety* Σ *with the following property: given any jet* α *in* $h^{-1}(z) - \Sigma$*, any map* g *in this jet is such that* $g^{-1}(A)$ *is locally strongly stratified, and for any other map* g_1 *of the jet* α*, the set* $g_1^{-1}(A)$ *is locally isomorphic to* $g^{-1}(A)$.

PROOF. Let $G_j(x_i) = 0$ be the equations of the set A ($G_j(0) = 0$). Denote by $x_i = p_i(u_k)$ the polynomial representative of the jet z; then the vector space $h^{-1}(z)$ has coordinates a_i^ω, where

$$x_i = p_i(u_k) + \sum_\omega a_i^\omega u_\omega,$$

ω being a multi-index $(i_1, \ldots, i_k)$ and $u_\omega = (u_1)^{i_1} \ldots (u_k)^{i_k}$. In the auxiliary space Ω of coordinates (u_k, a_i^ω), we consider the algebraic variety (G) defined by

$$0 = G_j\left(p_i + \sum_\omega a_i^\omega u_\omega\right).$$

This variety (G) contains the a-axis $Q(u_\omega = 0)$ in Ω; call Σ the Zariski-closure of the set of points of Q where the property of regular incidence of Q with respect to (G) does not hold. Let $(\alpha) = (a_i^\omega = \alpha_i^\omega)$ be a point of Q outside of Σ. Any map g realizing the jet (α) is given by equations of the type

$$x_i = p_i(u_j) + \Sigma\, b_i^\omega(u_j)\, u_\omega$$

where the differentiable functions b_i^ω satisfy $b_i^\omega(0) = \alpha_i^\omega$. The mapping $j(g) = j^{r+1}(g) : \mathbf{R}^p \to \Omega$ defined by $a_i = b_i(u_j)$ is obviously transversal to (G) around α, because the graph of $j(g)$ is transversal to the axis Q at α. Hence, the set $g^{-1}(A) \simeq j(g)^{-1}(G)$ has a local presentation as a transversal intersection of the real algebraic set (G); hence by definition it is locally strongly stratified. Moreover if g' is another mapping belonging to the same jet there exists a homotopy F_t (defined by $tg + (1-t)g'$) between g and g', and the corresponding mapping of $F_t^{-1}(A)$ onto I is locally stratified; hence, by application of Theorem 2, the two sets $g^{-1}(A)$, $g'^{-1}(A)$ are locally isomorphic and homeomorphic.

Corollary. *Let $f_1 = \ldots = f_n = 0$ be a set of n equations such that $f_1(0) = \ldots = f_n(0) = 0$, and $F : \mathbf{R}^p \to \mathbf{R}^n$ the associated map. If $z \in J^r(p, n)$ is the jet defined by F, and $z' \in h^{-1}(z)$ lies outside a proper algebraic variety Σ, then for any mapping $F' : \mathbf{R}^p \to \mathbf{R}^n$ realizing z', the set of zeros $F'^{-1}(0)$ is locally strongly stratified, and is locally isotopic to $F^{-1}(0)$.*

We want now to study the extent to which the local topological properties of differentiable mappings depend only on their jets of sufficiently high order.

Definition. Sufficient jets. *A jet $z \in J^r(n, p)$ is sufficient if any two local mappings F and G belonging to this jet are of the same* *topological type, i.e. there exist local homeomorphisms h, h' in $\mathbf{R}^n$ and $\mathbf{R}^p$ such that the diagram*

$$\begin{array}{ccc} \mathbf{R}^n & \xrightarrow{F} & \mathbf{R}^p \\ h \big\downarrow & & \big\downarrow h' \\ \mathbf{R}^n & \xrightarrow[G]{} & \mathbf{R}^p \end{array}$$

is commutative.

The main theorem is the following.

THEOREM 4. *Let $z \in J^r(n, p)$ be a given jet. Then there exists a positive integer s, depending only on r, n, p, such that if $h_s : J^{r+s} \to J^r$ is the canonical projection map, there exists in $h_s^{-1}(z)$ a proper algebraic variety Σ (the bifurcation variety) such that any jet $\in h_s^{-1}(z)$ lying outside Σ is sufficient. Further, any two mappings realizing such a jet are locally weakly stratified and isotopic.*

In the special case $n = p + 1$, we may replace "weakly stratified" in the above theorem by "strongly stratified". I am unable to prove the stronger theorem in the general case. Further, an analogue of Theorem 3 is also valid in this special case.

PROOF OF THEOREM 4. Let $y_j = P_j(x_i) + \Sigma\, u_\omega\, x_\omega^{r+1} + \Sigma\, u_{\omega'}\, x_{\omega'}$, $r + 1 < |\omega'| < r + s$, be the equations of a polynomial mapping f of degree $r + s$ extending the given jet. We suppose s so large that all the critical loci $S(f), \ldots, S_{i_1} S_{i_2} \ldots (f)$ (for definition see [2]) of the generic f are strongly stratified. This is, according to Theorem 3, the case outside a subvariety Σ_1 in the space (u_ω). We then eliminate formally the coordinates (x_i) and obtain a semi-algebraic set K which is contained in the resultant variety

$$\Delta(y, u) = 0.$$

The set $f^{-1}(K)$ contains the u-axis U (in x, y, u space) as a sub-stratified set, and we denote by Σ the Zariski-closure of the locus of singular incidence of U with respect to the intersection set

$p^{-1}(\Delta) \cap S(f)$. We choose a point $z'(u_\omega = \alpha_\omega)$ outside the two varieties Σ and Σ_1, and a differentiable mapping F realizing z'. We stratify the set K and consider any stratum V_y which is an injective image of a stratum $\widetilde{V}$ of the critical locus $S(f)$ of the polynomial mapping f. Such a stratum has an isotopic image in the critical locus $S(F)$ of F (Theorem 3). We denote this image by V_F. The restriction $F|V_F$ is, by assumption, of maximal rank, and we may obtain local equations G_{V_F} for the image V_F. This can be done, for instance, as follows. Let

$$y_j = P_j(x_i) + \sum a_\omega(x_i)\, x_\omega^{r+1} + \sum \alpha_{\omega'}\, x_{\omega'}^{r+1},\ a_\omega(0) = \alpha_\omega,\ r+1 < |\omega'| < r+s, \tag{1}$$

be the equations of F ; then the local equations of any stratum W_x of $S(F)$ can be obtained by adding to (1) a system of equations

$$\phi_j(x_i) \equiv A_j(x_i) + R_j\left(\alpha,\ x_i,\ \frac{\partial a_\omega(x_i)}{\partial x_i}\right) = 0 \tag{2}$$

which arise from the polynomial equations associated with the variety in the space of jets attached to the stratum W_x, by substituting the functions $\partial a_\omega / \partial x_i$ for certain of the u_ω. Note that any such stratum has dimension at most $k - 1$ since $S(F)$ is strongly stratified. These lead only to finitely many equations since we have only to consider the critical varieties of codimension $\leqslant n$. We eliminate from equations (1) and (2) successively the coordinates $x_n, x_{n-1}, \ldots, x_1$ by repeated application of the Malgrange preparation theorem. This gives rise to equations

$$G_{W_y}\,(y_j, \alpha) = 0$$

for the image W_y of W_x. There is an increasing function $\tau = \tau(\sigma)$ such that the jet of order σ of the functions G_{W_y} is determined by the jet of order τ of the mapping F. The same procedure applied to the polynomial mapping f gives us local equations (which are analytic)

$$g_{\widetilde{V}_y}\,(y_j, \alpha) = 0$$

for the image of the stratum $\widetilde{V}$ of $S(f)$. The existence of $\tau(\sigma)$ implies that if s is large enough ($s > \tau(\rho)$) we have an inequality

$$|g_{\widetilde{V}_y}(y_j, \alpha) - G_{\widetilde{V}_y}(y_j, \alpha)| \leqslant K|y|^\rho, \quad K > 0. \tag{3}$$

The Lojasiewicz inequality shows that there is $\nu > 0$ such that

$$|\operatorname{grad} g_{\widetilde{V}_y}(\widetilde{y}_j, \alpha)| > c\,[\operatorname{dist}(y, \partial \widetilde{V}_y)]^\nu$$

(with obvious notation). In particular, if $\partial \widetilde{V}_y \subset U$, we have

$$|\operatorname{grad} g_{\widetilde{V}_y}(y_j, \alpha)| \geqslant c\,|y|^\nu.$$

This, and (3), show that if $\partial \widetilde{V}_y \subset U$ and $\rho > \nu$, we have

$$|\operatorname{grad} G_{\widetilde{V}_y}(y_j, \alpha)| \geqslant c'\,|y|^\nu$$

so that $\widetilde{V}_y$ is singularity free in a neighbourhood of $y = 0$, and may be taken as a stratum for the image of $S(F)$.

We wish to prove that if $W = W_x$ is a stratum of $S(F)$, then $F(W_x)$ is stratified in such a way that $F : W_x \to F(W_x)$ is an immersion. We shall do this by induction on the dimension of W_x; in particular we assume this proved for every stratum $\subset \partial W_x$. Let Z be a stratum of $f(W_f)$ (where W_f is the stratum corresponding to W in $S(f)$), and Y any stratum of ∂Z. There are three possibilities.

1. Y belongs to an injective image of a stratum of ∂W.

2. Y is an intersection (or self-intersection) of some immersed variety $f(V)$, where V is a stratum of $S(f)$.

3. Y belongs to the singular incidence locus of a stratum of $f(\partial W)$ with respect to $f(V)$, where V is a stratum of $S(f)$.

In any case, it is possible to obtain local analytic equations $\gamma_i(y) = 0$ for Y, the γ_i being obtained starting from the equations (1) of f by a finite number of the following operations (in any order).

(Π)
- (1) Adding partial derivative equations, or jacobian equations, or, more generally, canonical polynomial equations in the space of jets.
- (2) Applying Malgrange's preparation theorem with respect to a coordinate x_i or y_j and eliminating this coordinate.

Thus, we may associate with Y a tubular neighbourhood $T_{Y,f}$, defined locally by analytic inequalities of the form

$$\psi_f(y) \leqslant 0,$$

where ψ_f is obtained from f by a finite sequence of the operations (Π).

If the integer s is large enough, then Z is transversally defined, which means that the defining analytic equations $g_Z(y, \alpha)$ of Z have the property that $|\operatorname{grad} g_Z(y, \alpha)|$ does not vanish in a neighbourhood of ∂Z. Hence, for instance, in $Z \cap T_{Y,f}$, we have an inequality

$$|\operatorname{grad} g_Z(y, \alpha)| \geqslant c(h(y))^r, \sum \gamma_i^2 = h.$$

For the differentiable mapping F, we may define an analogous stratum Y_F and a tubular neighbourhood $T_{Y,F}$: $\psi_F(y) \leqslant 0$, and equations $\Gamma_i = 0$ for Y_F by the same set of operations (Π) as that which we used to construct ψ_f, $T_{Y,f}$ and γ_i. Hence, in the set

$$\psi_F(y) \leqslant 0$$

we have an inequality

$$|\operatorname{grad} G_{Z_F}(y, \alpha)| \geqslant c(H(y))^r, \quad H = \sum \Gamma_i^2,$$

provided that the jets of the functions

$$(g, G), (\psi_f, \psi_F), (h, H)$$

coincide up to a sufficiently high order, which is the case if s is large enough.

This proves that in a neighbourhood of the set ∂Z_F, Z_F which is defined by $G_{Z_F}(y, \alpha) = 0$ is an imbedded manifold. Outside a neighbourhood $T(\partial Z) = \bigcup_{Y_F \subset \partial Z} T_Y$, we have an inequality

$$|\operatorname{grad} G_{Z_F}(y, \alpha)| \geqslant k\, |y|^r$$

and the above proof applies. Hence Z_F is an imbedded manifold, and can be taken as a stratum.

This shows that there is a weak stratification of $\mathbf{R}^k$ which contains the image $F(S(F))$ as a substratified set. This is isomorphic to the corresponding stratification of the semi-algebraic set $f(\mathbf{R}^n)$. The inverse images of these stratifications by F, f respectively are then also isomorphic. [The proof of this statement is based on a step-wise identification of the inverse image of strata of $F^{-1}(S(F))$ with the corresponding set in $f^{-1}(S(f))$ which admits in (x, y, u) space a semi-algebraic stratification. This identification is done by induction on the dimension, using inequalities of the Lojasiewicz type relative to $| x |$ as above.]

Thus f, F are, locally, isomorphically stratified mappings. They do not present blowing down. The corank on a stratum is 0 if this stratum is contained in the critical locus and $n - k$ otherwise. The critical locus being a closed substratified set, there is no blowing down. Hence by Theorem 2, f and F are of the same local topological type. This proves the main Theorem 4.

CANONICAL STRATIFICATION OF THE SPACE OF JETS.

If we consider any mapping $h_s : J^{r+s} \to J^r$, we may construct the corresponding bifurcation variety Σ_z for $z \in J^r$. From the construction of Σ_z, it follows that the union $\bigcup_z \Sigma_z$ is itself an algebraic variety in J^{r+s}. The totality of these varieties for all $r \geqslant 1$, $s > 1$ form the canonical stratification in J^{r+s}.

Let f be a C^∞ map at 0. If for some q, $j^q(f)$ is sufficient, we say that f has a singularity of "finite codimension" at 0. If $j^q(f)$ is sufficient at all points of the source space, we say that f is "almost correct". [It is likely that an almost correct mapping $f : \mathbf{R}^m \to \mathbf{R}^n$ with $m < n$ is a "finite" mapping in the sense of B. Malgrange.]

Given a sufficient jet $z \in J^s(n, p)$, consider its restriction $h_{s,\lambda}(z) \in J^\lambda(n, p)$. Let λ be the largest integer for which $h_{s,\lambda}(z)$ is not sufficient. Then $h_{s,\lambda}(z) \in \Sigma_z \subset J^\lambda(\Sigma_z$ the bifurcation variety). The codimension of Σ_z in J^λ is called the codimension of the singularity of z.

A map $f : \mathbf{R}^n \to \mathbf{R}^p$ is "correct" ("generic at the source") if all its jets are sufficient and of codimension $\leqslant n$. The map $j^r(f)$ corresponding to a correct map is transversal on all bifurcation varieties of $J^r(n, p)$ for $r > 0$. A correct map is locally structurally stable, and the correct mappings form an open dense set in $L(\mathbf{R}^n, \mathbf{R}^p)$.

The maps h^s induce mappings :

$$\begin{array}{ccccccccc} \cdots \to & J^s & \to & J^{s-1} & \to & \cdots & \to & J^1 \\ & \uparrow & & \uparrow & & & & \uparrow \\ \cdots \to & \Sigma^s & \to & \Sigma^{s-1} & \to & \cdots & \to & \Sigma^1 \end{array}$$

of the canonical stratifications $\Sigma^s \subset J^s$.

I do not know whether this infinite sequence of algebraic mappings can be given a common finite stratification.

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THE PREPARATION THEOREM FOR DIFFERENTIABLE FUNCTIONS

By BERNARD MALGRANGE

LET u be the germ of an infinitely differentiable mapping of R^m into R^n, at the origin 0, with $u(0)=0$. Let us denote by $x=(x_1,\ldots,x_m)$ the coordinates in R^m and by $y=(y_1,\ldots,y_n)$ those in R^n. Let $\mathscr{E}_m$ (resp. $\mathscr{E}_n$) be the space of germs of infinitely differentiable real valued functions at 0 in R^m (resp. R^n). We also write $\mathscr{E}(x)$ for $\mathscr{E}_m$ and $\mathscr{E}(y)$ for $\mathscr{E}_n$. Let finally u^* be the mapping $\mathscr{E}(y)\to\mathscr{E}(x)$ induced by u (i.e. $u^*f=f\circ u$).

DEFINITION 1. *We say that u is finite if $\mathscr{E}(x)$, considered as an $\mathscr{E}(y)$ module by means of u^*, is of finite type.*

In other words, u is finite if there are finitely many functions $\phi_1,\ldots,\phi_p\in\mathscr{E}(x)$ such that every $f\in\mathscr{E}(x)$ can be written in the form $f=\sum_{i=1}^{p}(u^*g_i)\,\phi_i$ where $g_i\in\mathscr{E}(y)$.

Let us suppose that u is finite and let Ω be a neighbourhood of $0\in R^m$ and $\tilde{u}$ a C^∞ mapping $\Omega\to R^n$ whose germ at 0 is u. It is easy to show that there is an open set Ω', $0\in\Omega'\subset\Omega$ such that the restriction $\tilde{u}'$ of $\tilde{u}$ to Ω' is "set wise finite" (i.e. the inverse image of any point is finite). However, the converse of this is false. (Counter-example: $m=n=1$, $\tilde{u}(x)=\exp(-1/x^2)$.)

We will, in spite of this, show that the converse becomes correct, *if we count the points of the inverse image with suitable multiplicity.* In fact, it is even enough to look at the multiplicity of 0 in $u^{-1}(0)$. For this, let us introduce the set $\mathfrak{m}(\mathscr{E}(x))\subset\mathscr{E}(x)$ of functions which vanish at the origin; $\mathfrak{m}(\mathscr{E}(x))$ is the maximal ideal of $\mathscr{E}(x)$, (which is thus a local ring), and it is generated by the coordinate functions $x_1,\ldots,x_m$. Let us define $\mathfrak{m}(\mathscr{E}(y))$ similarly, and let $\mathscr{E}(x)\,u^*\,\mathfrak{m}(\mathscr{E}(y))$ be the ideal generated in $\mathscr{E}(x)$ by $u^*\,\mathfrak{m}(\mathscr{E}(y))$, i.e. by the coordinate functions $u_1,\ldots,u_n$ of u. In accordance with terminology which is usual in algebraic geometry, we should supply the point 0 of $u^{-1}(0)$

with the local ring $\mathscr{E}(x)/\mathscr{E}(x)u^*\mathfrak{m}(\mathscr{E}(y))$ and consider its multiplicity to be equal to the dimension of this ring, considered as vector space over $\mathscr{E}(x)/\mathfrak{m}(\mathscr{E}(x)) \simeq R$. This leads us to the following definition (cf. [3]).

DEFINITION 2. *We say that u is quasi-finite if*

$$\mathscr{E}(x)/\mathscr{E}(x)\, u^*\, \mathfrak{m}(\mathscr{E}(y))$$

is a finite dimensional R-vector-space.

It is clear that every finite mapping is quasi-finite. Our result is that the converse is true. Before stating this as a separate theorem, we shall indicate two conditions equivalent with "u quasi-finite" which are easier to handle.

For this purpose, let us introduce the space $\hat{\mathscr{E}}(x) \simeq R[[X_1, \ldots, X_m]]$ of formal power series in m variables over R and the mapping $\mathscr{E}(x) \to \hat{\mathscr{E}}(x)$ which associates with each germ its Taylor expansion of infinite order. It is well known that this mapping is surjective, and consequently $\hat{\mathscr{E}}(x)$ can be identified with the quotient of $\mathscr{E}(x)$ by the ideal $\mathfrak{m}^\infty(\mathscr{E}(x))$ of functions "flat" at 0 (i.e. vanish at 0 together with all their derivatives). Defining $\hat{\mathscr{E}}(y)$ in the same way, we obtain a mapping $\hat{u}^*\colon \hat{\mathscr{E}}(y) \to \hat{\mathscr{E}}(x)$.

DEFINITION 3. *u is called formally finite if $\hat{u}^*$ makes $\hat{\mathscr{E}}(x)$ an $\hat{\mathscr{E}}(y)$ module of finite type; it is called formally quasi-finite if*

$$\hat{\mathscr{E}}(x)/\hat{\mathscr{E}}(x)\, \hat{u}^*\, \mathfrak{m}(\hat{\mathscr{E}}(y))$$

is a finite dimensional R-vector space.

[We leave it to the reader to interpret the notation $\mathfrak{m}(\hat{\mathscr{E}}(y))$.]

It is elementary to verify that u is quasi-finite if and only if it is formally quasi-finite (Nakayama's lemma and the fact that $\mathfrak{m}(\mathscr{E}(x))$ is finite over $\mathscr{E}(x)$). On the other hand, the "formal preparation theorem" asserts that u is formally finite if and only if it is formally quasi-finite [3].

This being the case, the preparation theorem may be formulated as follows (we were inspired by the formulation given in [3] in the analytic case).

THEOREM 1. *The following properties are equivalent.*

(a) *u is finite.*

$(\widehat{\text{a}})$ *u is formally finite.*

(b) *u is quasi-finite.*

$(\widehat{\text{b}})$ *u is formally quasi-finite.*

This theorem and Nakayama's lemma give at once the following corollary.

COROLLARY. *Let $\phi_1, \dots, \phi_p \in \mathscr{E}(x)$. The following properties are equivalent.*

(a)′ $\phi_1, \dots, \phi_p$ *$\mathscr{E}(y)$-generate* $\mathscr{E}(x)$.

$(\widehat{\text{a}})'$ $\widehat{\phi}_1, \dots, \widehat{\phi}_p$ *$\widehat{\mathscr{E}}(y)$-generate* $\widehat{\mathscr{E}}(x)$.

(b)′ $\phi_1, \dots, \phi_p$ mod $\mathscr{E}(x)\, u^* \mathfrak{m}(\mathscr{E}(y))$ *R-generate*

$$\mathscr{E}(x)/\mathscr{E}(x)\, u^* \mathfrak{m}(\mathscr{E}(y)).$$

$(\widehat{\text{b}})'$ $\widehat{\phi}_1, \dots, \widehat{\phi}_p$ mod $\widehat{\mathscr{E}}(x)\, \widehat{u}^* \mathfrak{m}(\widehat{\mathscr{E}}(y))$ *R-generate*

$$\widehat{\mathscr{E}}(x)/\widehat{\mathscr{E}}(x)\, \widehat{u}^* \mathfrak{m}(\widehat{\mathscr{E}}(y)).$$

REMARK. Note that the hypotheses of Theorem 1 imply that $m \leqslant n$. On the other hand, if we can take $p=1$ in the corollary, our result reduces to the implicit function theorem. The latter theorem, in the C^∞ case, is therefore a particular case of Theorem 1.

We now give two examples where this theorem can be applied.

EXAMPLE 1. *Symmetric Functions.* Let us take $n = m$, and for u the mapping $u(x) = (\sigma_1(x_1, \dots, x_n), \dots, \sigma_n(x_1, \dots, x_n))$ where σ_i is the i^{th} elementary symmetric function of $x_1, \dots, x_n$. One verifies easily that the images in $\widehat{\mathscr{E}}(x)$ of the monomials

$$x_1^{\alpha_1}, \dots, x_{n-1}^{\alpha_{n-1}} \text{ with } 0 \leqslant \alpha_i \leqslant n - i$$

generate $\widehat{\mathscr{E}}(x)$ as a module over $\widehat{\mathscr{E}}(y)$, i.e. over the subalgebra of $\widehat{\mathscr{E}}(x)$ generated by $\sigma_1, \ldots, \sigma_n$. By the corollary above, these monomials generate $\mathscr{E}(x)$ over the subalgebra of germs of differentiable functions of $\sigma_1, \ldots, \sigma_n$.

In particular, if f is symmetric, we see (by an averaging argument) that there is a differentiable germ $g \in \mathscr{E}_n$ with

$$f(x_1, \ldots, x_n) = g(\sigma_1, \ldots, \sigma_n).$$

This result has been proved by a quite different method by Glaeser [1].

Example 2. *The Weierstrass preparation theorem.* Let $F(x_1, \ldots, x_n) \in \mathscr{E}_n$ be regular of order p in x_n (i.e. $F(0, \ldots, 0, x_n)$ has a zero of order exactly p at 0). Let us again take $m = n$, and for u the mapping

$$(x_1, \ldots, x_{n-1}, x_n) \to (x_1, \ldots, x_{n-1}, F(x_1, \ldots, x_n)).$$

It is immediate that the ideal generated by $x_1, \ldots, x_{n-1}, \widehat{F}$ in $\widehat{\mathscr{E}}_n$ coincides with the ideal generated by $x_1, \ldots, x_{n-1}, x_n^p$. We are therefore in a position to apply Corollary 1 (equivalence of (a)′ and $(\widehat{\text{b}})'$) if we take $\phi_i = x_n^{p-i}$ $(1 \leqslant i \leqslant p)$. In other words, every $f \in \mathscr{E}_n$ can be written

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} g_i(x_1, \ldots, x_{n-1}, F)\, x_n^{p-i},$$

where $g_i \in \mathscr{E}_n$. If we put $h_i(x_1, \ldots, x_{n-1}) = g_i(x_1, \ldots, x_{n-1}, 0)$ and remark that we have $g_i - h_i = x_n k_i$, $k_i \in \mathscr{E}_n$, we obtain the following result on substituting F for x_n.

(W) *Let $F \in \mathscr{E}_n$ be regular of order p in x_n. For every $f \in \mathscr{E}_n$, there is $Q \in \mathscr{E}_n$ and $h_i \in \mathscr{E}_{n-1}$ $(1 \leqslant i \leqslant p)$, such that*

$$f(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)\, Q(x_1, \ldots, x_n) + \sum_{i=1}^{p} h_i(x_1, \ldots, x_{n-1})\; x_n^{p-i}.$$

The analogous assertion for analytic functions is precisely the Weierstrass preparation theorem in the form given by Rückert. The form of Weierstrass himself follows easily on applying (W) to $f = x_n^p$; in fact one then finds

$$x_n^p - \sum_{i=1}^{p} h_i(x_1, \ldots, x_{n-1})\, x_n^{p-i} = F(x_1, \ldots, x_n)\, Q(x_1, \ldots, x_n)$$

and one verifies at once that $h_i(0) = 0$, $Q(0) \neq 0$. In other words, up to an invertible factor, F is a *distinguished polynomial* in x_n (i.e. a unitary polynomial, all of whose coefficients of lower order are zero at the origin).

We remark that in the analytic case, Q and h_i are unique. This property is not true in general in the differentiable case, since the equation $F(x_1, \ldots, x_n) = 0$ can have less than p real roots for fixed $x_1, \ldots, x_{n-1}$.

COUNTER-EXAMPLE. $n = 2$, $F = x_1^2 + x_2^2$. Let f be a function depending only on x_1 and *flat* at 0. We may take $Q = 0$, $h_2 = f$, $h_1 = 0$ and also $Q = \dfrac{f}{F}$, $h_1 = h_2 = 0$.

We shall conclude by giving some very brief indications of the proof. The complete proof is given in [5]. It consists of two steps.

STEP I. *One proves* (W) *in the case when F is a polynomial* (*or an analytic function*) *with respect to all the variables.*

The proof is rather long and technical. It is done by adapting the arguments developed by Hörmander [2] and Lojasiewicz [4] in the study of " division of distributions ".

STEP II. *One deduces Theorem* 1 *from Step* I.

We shall content ourselves with giving here the idea of the proof by showing how one proves (W) in the general case, given Step I. The proof of Theorem 1 does not differ from this except in technical detail.

Let us take $F \in \mathscr{E}_n$ which is regular of order p in x_n, introduce new variables $t_1, \ldots, t_p$ and consider the "generic polynomial of degree p"

$$\Pi(x_n, t) = x_n^p + \sum_{i=1}^{p} t_i\, x_n^{p-i}.$$

If Step I is given, we apply (W) to $\Pi(x_n, t)$ in $\mathscr{E}_{n+p}$; in particular for any $f \in \mathscr{E}_n$, we have

$$f(x) = \Pi(x_n, t)\, Q(x, t) + \sum_{i=1}^{p} h_i\,(x_1, \ldots, x_{n-1}\,; t)\, x_n^{p-i}, \qquad \text{(A)}$$

with $Q \in \mathscr{E}_{m+p}$, $h_i \in \mathscr{E}_{m+p-1}$ (we write t for $(t_1, \ldots, t_p)$, x for $(x_1, \ldots, x_n)$). Let us apply this to F. We obtain

$$F(x) = \Pi(x_n, t)\, G(x, t) + \sum_{i=1}^{p} R_i\,(x_1, \ldots, x_{n-1}; t)\, x_n^{p-i}. \qquad \text{(B)}$$

From the hypothesis that F is regular in x_n of order p, we deduce immediately that $R_i(0) = 0$ for each i, $G(0) \neq 0$, and that the determinant $\left| \dfrac{\partial R_i}{\partial t_j}(0) \right| \neq 0$. By the implicit function theorem, there are $\theta_j \in \mathscr{E}_{n-1}$, $\theta_j\,(0) = 0$, $1 \leqslant j \leqslant p$, such that the equations $R_i = 0$ are equivalent with $t_j = \theta_j\,(x_1, \ldots, x_{n-1})$. If we make the substitution $t_j \to \theta_j$ in the formula (B), we obtain

$$F(x) = \Pi(x_n;\ \theta(x_1, \ldots, x_{n-1}))\ H(x_1, \ldots, x_n), \qquad \text{(C)}$$

where $H(x) = G(x\,; \theta\,(x_1, \ldots, x_{n-1}))$, so that $H \in \mathscr{E}_n$, $H(0) \neq 0$. This proves the theorem in the "form of Weierstrass". We pass at once from that to the "form of Rückert" by substituting θ for t in the formula (A).

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ENERGY INEQUALITIES FOR HYPERBOLIC SYSTEMS

By LARS GÅRDING

INTRODUCTION. In [2] I generalized the energy inequality for hyperbolic differential operators to a kind of differential operator called a partial adjoint. These inequalities led to new Cauchy problems. The theory had no immediate non-classical application. I present here without proofs a new and sharper version of this theory including also hyperbolic systems. The non-classical energy inequalities for one operator are used to get energy inequalities (also classical ones) for systems. These inequalities (Theorem 9.1. below) are good in the sense that they require comparatively little differentiability of the coefficients. Cruder versions were first obtained by Petrovsky [6] and later by Leray [5].

The presentation is condensed but self-contained. A more complete version will be published in a forthcoming book by Leray and myself. The idea of using Soboleff spaces to describe the differentiability of the coefficients stems from Schauder and was perfected by Leray [5] and Dionne [1].

Good energy estimates can be used to derive existence theorems in the nonlinear case. This was done by Dionne [1] for one operator, and by myself [3] for a first-order system. Theorem 9.1 below leads to new results of this kind, which will not be given here.

We shall work in an infinite band. This masks the local character of Cauchy's problem for hyperbolic operators, but has technical advantages. It is possible to give a local version of the theory.

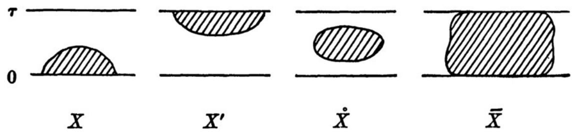
1. Distributions in a band with a boundary. Let $x = (x_1, \ldots, x_l)$ be coordinates in R^l. Later, the first coordinate will serve as a time coordinate, the others as coordinates of space. The hyperplane $x_1 = t$ will be denoted by S_t. Let

$$X : 0 \leqslant x_1 < \tau, \ X' : \ 0 < x_1 \leqslant \tau \qquad (\ )$$

be half-open bands in time with interior and closure

$$\mathring{X} : 0 < x_1 < \tau, \ \bar{X} : 0 \leqslant x_1 \leqslant \tau. \tag{2}$$

When Y is one of these four bands, let $\mathscr{C}(Y)$ be all infinitely differentiable functions with compact supports in Y, equipped with the Schwartz topology. Typical supports in the four spaces are as follows.



Limiting ourselves to, e.g. $Y = X$ and $\mathring{X}$ we see that the injection

$$\mathscr{C}(\mathring{X}) \to \mathscr{C}(X) \tag{3}$$

is continuous.

Let $\mathscr{C}'(Y)$ be the space of distributions on Y, i.e. all continuous antilinear functionals

$$f : g \to (f, g)$$

on $\mathscr{C}(Y)$. A function $f(x)$, locally integrable in $\bar{X}$, will be identified with the distribution

$$(f, g) = \int_Y f(x) \, \overline{g(x)} \, dx,$$

where $dx = dx_1 \dots dx_l$ and the integral will be used as a notation for (f, g). Corresponding to (3) there is a restriction mapping

$$\mathscr{C}'(X) \to \mathscr{C}'(\mathring{X}); \tag{4}$$

but, since $\mathscr{C}(\mathring{X})$ is not dense in $\mathscr{C}(X)$, there is no inclusion between the two spaces of distributions. The kernel of (4) consists of all $f \in \mathscr{C}'(X)$ with supports on the boundary S_0 of X.

Let

$$D^\alpha = (\partial / \partial x_1)^{\alpha_1} \dots (\partial / \partial x_l)^{\alpha_l}$$

denote a derivative of order $|\alpha| = \alpha_1 + \dots + \alpha_l$. We define the derivative $\bar{D}^\alpha f$ of a distribution f on Y by

$$(\bar{D}^\alpha f, g) = (f, (-D)^\alpha g), \ g \in \mathscr{C}(Y).$$

The reason for the $\bar{D}$ is that if $\alpha_1 > 0$, $Y = X$ and f is a smooth function, then the equality

$$(D^\alpha f, g) = (f, (-D)^\alpha g)$$

holds only modulo integrals over the boundary S_0 of X. We have for instance

$$(D_1 f, g) = -(f, D_1 g) - \int\limits_{S_0} f(x)\, \overline{g(x)}\, dx_2 \dots dx_l.$$

Hence $\bar{D}_1$ is an extension of the classical derivative D_1 in $\mathring{X}$, but it is not an extension to X. When these derivatives operate on smooth functions, imbedded as distributions on X, we have

$$\bar{D}_1 = D_1 + \delta(x_1),\ \bar{D}_k = D_k$$

where $k > 1$. J. Leray has proposed to call D and $\bar{D}$ the interior and adhesive derivatives respectively. It is clear that

$$\bar{D}^\alpha : \mathscr{C}'(X) \to \mathscr{C}'(X)$$

is a continuous mapping which reduces to the usual distribution derivative

$$D^\alpha : \mathscr{C}'(\mathring{X}) \to \mathscr{C}'(\mathring{X})$$

on the interior of X. Interior derivatives D^α shall be applied only to smooth functions. Our object of study will be mixed differential operators

$$a = \sum \bar{D}^\gamma a_{\beta\gamma}(x)\, D^\beta \tag{6}$$

with locally integrable coefficients $a_{\beta\gamma}(x)$, containing both adhesive and interior derivatives. For $f \in \mathscr{C}(\bar{X})$ we define $af \in \mathscr{C}'(X)$ by

$$(af, g) = \sum (a_{\beta\gamma}(x)\, D^\beta f, (-D)^\gamma g),\ g \in \mathscr{C}(X).$$

The product ba of two operators (6) will be performed by moving all adhesive derivatives to the left and all interior derivatives to the right, using the commutation rules

$$[D_k, h] = [\bar{D}_k, h] = \partial h(x) / \partial x_k,\ [D_k, \bar{D}_j] = 0$$

where h and $\partial h / \partial x_k$ are supposed to be locally integrable. This product of operators is associative, but if $f \in \mathscr{C}(X)$, we have

$$(ba) f = b(af)$$

only when the second member is defined, e.g. when a is free of adhesive derivatives or b free of interior derivatives.

Our reason for considering distributions on the half-open band $X: 0 \leqslant x_1 < \tau$ is that we are going to estimate f in X in terms of $af \in \mathscr{C}'(X)$ and the Cauchy data of f on the hyperplane S_0. This makes it necessary to have test functions which do not vanish on S_0.

2. Soboleff spaces of functions and distributions on hyperplanes. Consider hyperplanes of constant time contained in X

$$S_t:\ x_1 = t,\ 0 \leqslant t < \tau.$$

Put

$$|f, K|_q = \left(\int_K |f(x)|^q \, dx_2 \ldots dx_l \right)^{1/q},\ 1 \leqslant q \leqslant \infty \tag{1}$$

when K is an open subset of S_t. In particular, $|f, S_t|_q$ is the usual q-norm on S_t. We shall also use the norm

$$|f, S_t|_{[q]} = \sup |f, K|_q \tag{2}$$

where K ranges over all unit cubes in S_t.

Introduce a double order of derivatives as follows

$$(\alpha) = (\alpha_1, \alpha_2 + \ldots + \alpha_l).$$

Let $r \geqslant 0$, $s \geqslant 0$ be integers. We say that α is of order $\leqslant r, s$, or briefly that

$$(\alpha) \leqslant (r, s),$$

if $\alpha_1 \leqslant r$, $|\alpha| \leqslant r + s$ and define a gradient of order (r, s) by

$$D^{r,s} = \{D^\alpha\},\ (\alpha) \leqslant (r, s).$$

We extend the norms (1) and (2) to gradients of f, putting

$$|D^{r,s} f, S_t|_q = \sum_\alpha |D^\alpha f, S_t|_q \tag{3}$$

and

$$|D^{r,s} f, S_t|_{[q]} = \sum_\alpha |D^\alpha f, S_t|_{[q]} \tag{4}$$

where $(\alpha) \leqslant (r, s)$. It is clear that these norms increase with r and s. We want to extend (3) to negative values of s, assuming for simplicity that $q = 2$. First, let us put

$$|D^{0,-s} f, S_t|_2 = \sup |(f, g, S_t)| / |D^{0,s} g, S_t|_2,\ s \geqslant 0, \tag{5}$$

where $(f, g, S_t) = \int f(x)\,\overline{g(x)}\,dx_2 \dots dx_l$ with $x_1 = t$ and g ranges over the space $\mathscr{C}(S_t)$ of all infinitely differentiable functions with compact supports in S_t. The norm (5) increases with s, it is finite if f is square integrable over S_t, but also in other cases. If, e.g., $s > (l-1)/2$, then (5) is finite when f is a δ-function in S_t. When $r = s = 0$, the two definitions (3) and (5) agree. The definition that extends (3) is

$$| D^{r,s} f, S_t |_2 = \sum_j | D^{0,r+s-j} D_1^j f, S_t |_2 \tag{6}$$

where $0 < j \leqslant r$, $s \gtrless 0$. If $s \geqslant 0$, this formula follows from (3), if $s < 0$, the right-hand side is defined by (5). If f is a function in $\mathscr{C}(X)$, (6) is a norm of the restriction of

$$f, D_1 f, \dots, D_1^r f \tag{7}$$

to S_t. In connection with hyperbolic operators we shall use (6) as a measure of the energy of f at time t.

DEFINITION 2.1. *Let*

$$L_2^{r,s}(S_t),\ 0 < t < \tau$$

be the closure with respect to the norm (6) *of the restriction of* (7) *to* S *when* $f \in \mathscr{C}(\overline{X})$.

NOTE. This space does not depend on t. It is a direct sum of the spaces

$$L_2^{0,r+s-j}(S_t)$$

for $j = 0, \dots, r$.

3. Soboleff spaces for the coefficients.

Let us put

$$| D^{r,s} f, X |_{p,[q]} = \left(\int_0^\tau | D^{r,s} f, S_t |_{[q]}^p \, dt \right)^{1/p} \tag{1}$$

where $r \geqslant 0$, $s \geqslant 0$ are integers. The following spaces are convenient coefficient spaces for our differential operators.

DEFINITION 3.1. *Let* $L_{p,[q]}^{r,s}(X)$ *be the set of all* $f \in \mathscr{C}'(\dot{X})$ *with locally integrable derivatives of order* $<(r, s)$ *and with finite norm* (1).

NOTE. This is a space of functions containing the space $\mathscr{B}(\dot{X})$ of functions in $\dot{X}$ whose derivatives are all bounded. The space decreases as r and $r+s$ increase. Since the norm (1) is an integral, the choice of band X is immaterial. We could replace it by any other band (1.1) and (1.2).

4. Soboleff spaces of functions and distributions in the band X. Let us put

$$|D^{r,s}f, X|_{p,2} = \left(\int_0^T |D^{r,s}f, S_t|_2^p\, dt\right)^{1/p} \tag{1}$$

$$|D^{-r,s}f, X|_{p,2} = \sup_{g\in\mathscr{C}(X)} |(f,g)|/|D^{r,-s}g, X|_{p',2}, \tag{2}$$

where $r \geqslant 0$, $s \gtrless 0$ are integers, $1 < p < \infty$ and

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

It is clear that $|D^{r,s}f, X|_{p,2}$ increases with r and $r+s$ and it is not difficult to see that (1) and (2) are equal when $r = 0$ and $f \in \mathscr{C}(\bar{X})$. Since (1) is defined by an integral, the choice of band X in the norm is immaterial, but in (2) it is all important. In fact, since $g \in \mathscr{C}(X)$, the right side of (2) is defined (possibly infinite) for $f \in \mathscr{C}'(X)$. Changing X to Y then changes the domain of definition of the norm and a restriction to $Y = \dot{X}$ makes the norm vanish if the support of f is contained in S_0.

Using these norms we can now define some useful spaces of distributions.

DEFINITION 4.1. *Put*

$$L_{1,2}^{r,s}(X) = \textit{the closure of } \mathscr{C}(\overline{X}) \textit{ with respect to } |D^{r,s}f, X|_{1,2} \tag{3}$$

$$C_{\infty,2}^{r,s}(X) = \textit{the closure of } \mathscr{C}(\overline{X}) \textit{ with respect to } |D^{r,s}f, X|_{\infty,2}. \tag{4}$$

NOTE. These spaces increase as r and $r+s$ decrease. When $r \geqslant 0$, (3) and (4) consist, roughly speaking, of all distribution-valued mappings

$$t \to f(t, x_2, \ldots, x_l), \ldots, D_1^r f(t, x_2, \ldots, x_l) \in L_2^{r,s}(S_0)$$

which are integrable and continuous respectively. When $r < 0$, these mappings become distributions also in time and can have supports in a hyperplane S_t.

The notations in this section have straight-forward extensions to vector-valued functions

$$f(x) = f_1(x), \dots, f_N(x).$$

We say that $f \in \mathscr{C}(Y)$ if all $f_\mu \in \mathscr{C}(Y)$ and put $(f, g) = \Sigma(f_\mu, g_\mu)$. Letting

$$r = (r_1, \dots, r_N) \text{ and } s = (s_1, \dots, s_N)$$

be vectors with integral components, we put

$$| D^{r,s} f, X |_{p,q} = \Sigma | D^{r_\mu, s_\mu} f_\mu, X |_{p,q}$$

and interpret $f \in L^{r,s}_{1,2}(X)$ and $f \in C^{r,s}_{\infty,2}(X)$ as

$$f_\mu \in L^{r_\mu, s_\mu}_{1,2}(X) \text{ and } f_\mu \in C^{r_\mu, s_\mu}_{\infty,2}(X).$$

5. Differential operators. Hyperbolic operators.

Let

$$c_+ = \max(0, c), \; c_- = \min(0, c)$$

denote the positive and negative parts of a real number c.

DEFINITION 5.1. *Let*

$$m \gtrless 0, \; n \gtrless 0$$

be integers and let

$$V(n, X, m) \tag{1}$$

be the set of all differentiable operators of the form

$$a = a(\bar{D}, x, D) = \Sigma \bar{D}^\gamma a_{\beta\gamma} D^\beta$$

with locally integrable coefficients in X such that

$$| \beta | \leqslant m_+, | \gamma | \leqslant n_+, | \beta + \gamma | \leqslant m + n \tag{2}$$

for every $a_{\beta\gamma} \neq 0$.

NOTE. (2) is equivalent to $| \beta | \leqslant m_+ + n_-$, $| \gamma | \leqslant n_+ + m_-$.

If $m + n < 0$, V contains only the zero operator.

If $n \leqslant 0$, $a \in V$ contains only derivatives D, i.e. a is a classical differential operator, and, by definition,

$$(af, g) = \sum_{|\alpha| \leqslant m+n} (a_\alpha D^\alpha f, g),$$

where $f \in \mathscr{C}(\bar{X})$, $g \in \mathscr{C}(X)$. When $m \leqslant 0$, a contains only derivatives $\bar{D}$ and

$$(af, g) = \sum_{|\alpha| \leqslant m+n} (a_\alpha(x) f, (-D)^\alpha g).$$

If $n > 0$, $m > 0$, a contains both kinds of derivatives and

$$(af, g) = \sum_{|\beta| \leqslant m, |\gamma| \leqslant n} (a_{\beta\gamma}(x) D^\beta f, (-D)^\gamma g).$$

We shall refer to this as the mixed case. In all cases, we denote by af the distribution on X defined by the right sides. For simplicity we shall also write

$$(af\bar{g})(x) = \Sigma\, a_{\beta\gamma}(x)\, D^\beta f(x)\, (-D)^\gamma\, \overline{g(x)}. \tag{3}$$

A coefficient of $a \in V$ with $|\beta + \gamma| = m + n$ will be called principal, the others secondary. The principal part $\hat{a}$ of a is defined by

$$\hat{a}\,(\bar{D}, x, D) = \Sigma\, \bar{D}^\gamma\, a_{\beta\gamma}\,(x) D^\beta$$

where $|\beta + \gamma| = m + n$. The difference $a - \hat{a}$ is called the secondary part. We say that a is homogeneous if $a = \hat{a}$.

The characteristic polynomial of $\hat{a}$ is defined by

$$\hat{a}\,(\bar{\xi}, x, \xi) = \Sigma\, a_{\beta\gamma}(x)\, \xi^\beta\, \bar{\xi}^\gamma \tag{4}$$

where $|\beta + \gamma| = m + n$, $\xi = \xi_1, \ldots, \xi_l$ has complex components and the bar denotes complex conjugation. If

$$\xi \text{ real} \Rightarrow \hat{a}(\bar{\xi}, x, \xi) = 0$$

then $\hat{a}$ vanishes unless we have the mixed case $n > 0$, $m > 0$. But then $\hat{a}$ is almost a divergence in the following sense (compare Hörmander [4] p. 188) : there exist homogeneous operators

$$a_j \in V(n, X, m-1), \quad j = 0, \ldots, l,$$

such that

$$a = \Sigma\, (D_j - \bar{D}_j)\, a_j(\bar{D}, x, D) + a_0(\bar{D}, x, D). \tag{5}$$

This requires the coefficients of a to be once differentiable. If a has constant coefficients, a_0 vanishes. Notice that (3) and (5) give

$$(af, g) = \sum \int_X \frac{\partial}{\partial x_j} (a_j f \bar{g})(x)\, dx + (a_0 f, g),$$

where the right side only contains derivatives of f of order $< m$ and derivatives of g of order $< n$, integrated over X and S_0. This formula explains the term approximative divergence.

Let $\tilde{a}(x)$ be the leading coefficient of a, i.e. the coefficient $a_{\beta\gamma}(x)$ for which $\beta_1 + \gamma_1 = m + n$. It is unique. Let us factorize the characteristic polynomial (4) for ξ real,

$$\hat{a}(\xi, x, \xi) = \tilde{a}(x) \prod_1^{m+n} (\xi_1 - \lambda_k(x, \xi_2, \dots, \xi_l)). \tag{6}$$

Since $\hat{a}$ is homogeneous, the λ_k are homogeneous of degree 1 in ξ.

DEFINITION 5.2. *An operator*

$$a \in V(n, X, m)$$

is said to be (uniformly strongly) hyperbolic in X if

(1) *the principal coefficients are bounded and uniformly continuous in X,*

(2) $1/\tilde{a}(x)$ *is bounded in X,*

(3) *the zeros λ_k of (6) are real for $\xi_2, \dots, \xi_l$ real and uniformly separated in X : there is a constant $c > 0$ such that*

$$|\lambda_j - \lambda_k| > c(|\xi_2| + \dots + |\xi_l|)$$

when $x \in X$ and $j \neq k$.

The set of hyperbolic operators in $V(n, X, m)$ will be denoted by $\mathrm{Hyp}(n, X, m)$.

We shall also need a scale of regularity for the coefficients of a. Let

$$\rho = \{\rho_{\beta\gamma}\}, \ \sigma = \{\sigma_{\beta\gamma}\}$$

be vectors whose coefficients are non-negative integers (differentiability indices) and let

$$p = \{p_{\beta\gamma}\}$$

be a vector whose components are summability indices, $1 \leqslant p_{\beta\gamma} \leqslant \infty$.

DEFINITION 5.3. *Let*

$$V^{\rho,\sigma}_{1,[p]}(n, X, m)$$

be the set of operators $a \in V(n, X, m)$ such that

$$\iota_{\beta\gamma}(x) \in L^{\rho_{\beta\gamma},\sigma_{\beta\gamma}}_{1,[p_{\beta\gamma}]}(X) \tag{7}$$

where the indices satisfy the following inequalities

$$p_{\beta\gamma} \geqslant 2, \ \lambda/p_{\beta\gamma} \leqslant \rho_{\beta\gamma} + \sigma_{\beta\gamma} + m + n - 1 - |\beta + \gamma|$$

where λ is some number $> l - 1$.

NOTE 1. An element of ρ, σ or p corresponding to a principal coefficient will be called a principal index and denoted by $\hat{\rho}$, $\hat{\sigma}$ and $\hat{p}$ respectively. The other indices are called secondary.

NOTE 2. The lower index 1 in (7) means that the derivatives of the coefficients, although they are at least locally square integrable in space, are only required to be locally integrable in time.

6. A general energy inequality for hyperbolic operators.

Let c denote a constant, not always the same. Let X_t be the band $0 \leqslant x_1 < t \leqslant \tau$. The following result sharpens Theorem 16.2 of [2].

THEOREM 6.1. *Let*

$$a \in V^{\rho,\sigma}_{1,[p]}(-n, X, m), \quad m > n \tag{1}$$

be hyperbolic and let $s \gtrless 0$ be an integer. Then the inequality

$$|D^{m-1,s} f, X_t|_{\infty,2} \leqslant c\, |D^{m-1,s} f, S_0|_2 + c\, |D^{n,s}\, a f, X_t|_{1,2} \tag{2}$$

holds for all $f \in \mathscr{C}(\overline{X})$ and all t provided

$$\left.\begin{aligned} \rho_{\beta\gamma} &\geqslant (|\beta| + 1 - m)_+ + n_+ \\ \rho_{\beta\gamma} + \sigma_{\beta\gamma} &\geqslant (\beta + 1 - m)_+ + |\gamma| + n + s \\ \rho_{\beta\gamma} + \sigma_{\beta\gamma} &\geqslant |\beta| + 1 - m + n_+ - s. \end{aligned}\right\} \tag{3}$$

NOTE 1. To avoid a minus sign in (2) we have changed the sign of n. The last two inequalities (3) account for the differentiability imposed by the parameter s. They follow from the first one when $s = 0$. When $n \geqslant 0$, a is a classical differential operator and we shall refer to this as the classical case.

NOTE 2. If $m \leqslant 0$, the first term on the right side in (2) has no sense and should be dropped.

NOTE 3. We say that a couple

$$v \in L^{n,s}_{1,2}(X),\ \ w \in L^{m-1,s}_2(S_0) \tag{4}$$

is a set of Cauchy data for $a \in V$ if

$$v - aw = 0 \text{ of order } n \text{ on } S_0, \tag{5}$$

i.e. if the restrictions of the derivatives $D^j_1(aw - v)$ to S_0 vanish for $0 \leqslant j < n$. This condition, which is necessary when $u = w$, $v = au$, is empty if $n \leqslant 0$. When a satisfies the requirements of the theorem and $m > n > 0$, the time derivatives

$$w, \ldots, D^{m-1}_1 w \ \text{ and } \ w, \ldots, D^{m-n-1}_1 w,\ v, \ldots, D^{n-1}_1 v$$

on S_0 are expressible in terms of each other and the derivatives of w on the right (the classical Cauchy data on S_0) can be chosen arbitrarily. In particular, there is a constant c such that

$$| D^{m-1,s} w, S_0 |_2 \leqslant c \,| D^{m-n-1,s} w, S_0 |_2 + c \,| D^{n-1,s} v, S_0 |_2.$$

Applying this to (2) with $w = f$, $v = af$, we can make the corresponding change on the right in (2).

NOTE 4. The Cauchy problem associated with (2) is the following : given a couple v, w of Cauchy data, find u such that

$$au = v \text{ in } \mathring{X},\ u - w = 0 \text{ of order } m \text{ on } S_0.$$

If a satisfies the assumptions of the theorem and if

$$a - \hat{a} \in V(-n, X, m-1) \tag{6}$$

this problem can be shown to have a unique solution $u \in C^{m-1,s}_{\infty,2}(X)$ satisfying the inequality (3). The condition (6) only concerns the secondary part of a. It is empty unless we have the mixed case $m > 0 > n$.

NOTE 5. The principal interest of the mixed case $m > 0 > n$ is that it requires very little of the coefficients. In fact, the maximal values of the right sides in (4) are in this case

$$1,\ 1+s,\ 1-s$$

respectively. On the other hand, if n is large positive, they are dominated by the term n and if m is large negative, they are dominated by the term $1-m$. In the last section we shall use the mixed case of (3) to deduce similar inequalities for hyperbolic systems (also in the classical case) which require comparatively little differentiability of the coefficients.

7. Systems. We shall now deal with vector-valued functions

$$f = f_1, \dots, f_N, \quad N > 1$$

and extend the notions of section 4 as described at the end of that section. A square matrix of differential operators

$$A = A(\bar{D}, x, D) = (a_{\nu\mu}(\bar{D}, x, D))\,;\ \nu, \mu = 1, \dots, N \tag{1}$$

will be called a system. We put

$$(Af)_\nu = \sum a_{\nu\mu} f_\mu .$$

DEFINITION 7.1. *Let*

$$m = m_1, \dots, m_N,\ n = n_1, \dots, n_N$$

be vectors with integral components. Let $V(n, X, m)$ *be the set of systems* (1) *such that*

$$a_{\nu\mu} \in V(n_\nu, X, m_\mu). \tag{2}$$

NOTE. For classical systems, the space V was introduced by Leray [5].

The principal part $\hat{A}$ of $A \in V$ is defined by

$$\hat{A} = \hat{A}(\bar{D}, x, D) = (\hat{a}_{\nu\mu}).$$

The secondary part is $A - \hat{A}$. The characteristic polynomial of $\hat{A}$ is

$$\hat{A}(\bar{\xi}, x, \xi) = (\hat{a}_{\nu\mu}(\bar{\xi}, x, \xi)).$$

NOTE. Given A, there are in general many m and n such that (2) holds. In fact, given e.g. n, we only have to choose m sufficiently large. But this will have the result that most principal parts vanish.

If all n_ν are $\leqslant 0$, A is a classical system. If all n_ν vanish, and the matrix $(a_{\nu\mu})$ of leading coefficients is non-singular, A is a normal system in the sense of Cauchy and Kowalevski.

Corresponding to the definition 5.3 we have

DEFINITION 7.2. *Let*

$$\rho_{\nu\mu} = \{\rho_{\nu\mu\beta\gamma}\}, \quad \sigma_{\nu\mu} = \{\sigma_{\nu\mu\beta\gamma}\}$$

be vectors with non-negative integral components and put

$$\rho = \{\rho_{\nu\mu}\}, \quad \sigma = \{\sigma_{\nu\mu}\}.$$

Let

$$p_{\nu\mu} = \{p_{\nu\mu\beta\gamma}\}$$

be vectors whose components are summability indices and put

$$p = \{p_{\nu\mu}\}.$$

Let

$$V^{\rho,\sigma}_{1,[p]}(n, X, m)$$

be the space of systems A such that

$$a_{\nu\mu} \in V^{\rho_{\nu\mu},\sigma_{\nu\mu}}_{1,[p_{\nu\mu}]}(n_\nu, X, m_\mu).$$

NOTE. A differentiability index $\rho_{\nu\mu\beta\gamma}, \sigma_{\nu\mu\beta\gamma}$ with $|\beta+\gamma| = m_\mu + n$ will be called a principal index and will be denoted by $\hat\rho_{\nu\mu}, \hat\sigma_{\nu\mu}$. The other indices are called secondary.

8. An energy inequality for almost diagonal systems with a hyperbolic diagonal.

We are going to extend Theorem 6.1 to a special class of systems.

DEFINITION 8.1. *A system*

$$A \in V(n, X, m)$$

is said to be almost diagonal if

$$\xi \text{ real} \Rightarrow \hat A(\bar\xi, x, \xi) \text{ diagonal.} \tag{1}$$

Its diagonal is said to be hyperbolic if

$$\hat{a}_{\mu\mu} \in \mathrm{Hyp}(n_\mu, X, m_\mu)$$

for all μ.

NOTE. (1) does not mean that $\hat{a}_{\nu\mu}(\bar{D}, x, D)$ vanishes when $\nu \neq \mu$. This happens only when $n_\nu \leqslant 0$ or $m_\mu \leqslant 0$. In the mixed case $m_\mu > 0$, $n_\nu > 0$, $\hat{a}_{\nu\mu}$ is an approximative divergence as described in Section 5.

THEOREM 8.1. *Let*

$$A \in V^{\rho,\sigma}_{1,[p]}(-n, X, m) \tag{2}$$

be almost diagonal with a hyperbolic diagonal, let $m_\mu > n_\mu$ *for all* μ *and let* $s \gtrless 0$ *be an integer. Then the inequality*

$$|D^{m-1,s} f, X_t|_{\infty,2} \leqslant c\,|D^{m-1,s} f, S_0|_2 + c\,|D^{n,s} A f, X_t|_{1,2} \tag{3}$$

holds for all $f \in \mathscr{C}(\bar{X})$ *and all sufficiently small* t *provided*

$$\left.\begin{aligned} \rho_{\nu\mu\beta\gamma} &\geqslant (|\beta| + 1 - m_\mu)_+ + (n_\nu)_+ \\ \rho_{\nu\mu\beta\gamma} + \sigma_{\nu\mu\beta\gamma} &\geqslant (|\beta| + 1 - m_\mu)_+ + |\gamma| + n_\nu + s \\ \rho_{\nu\mu\beta\gamma} + \sigma_{\nu\mu\beta\gamma} &\geqslant |\beta| + 1 - m_\mu + (n_\nu)_+ - s. \end{aligned}\right\} \tag{4}$$

NOTE 1. If all $m_\mu > 0$, and maybe also in the general case, (3) holds for $0 < t \leqslant \tau$. For the principal indices, (4) reads

$$\left.\begin{aligned} \hat{\rho}_{\nu\mu} &\geqslant \max(1, n_\nu, 1 - m_\mu) \\ \hat{\rho}_{\nu\mu} + \hat{\sigma}_{\nu\mu} &\geqslant \max(1, n_\nu) + s \\ \hat{\rho}_{\nu\mu} + \hat{\sigma}_{\nu\mu} &\geqslant \max(1, 1 - m_\mu) - s. \end{aligned}\right\} \tag{5}$$

NOTE 2. The note on Theorem 6.1 about Cauchy's problem applies with appropriate formal changes due to the fact that we are now dealing with a system. In particular, that $Av - w$ vanishes at order n on S_0, has to be interpreted as saying that $(Av)_\nu - w_\nu$ vanishes at order n_ν on S_0 for every ν.

9. An energy inequality for hyperbolic systems.

DEFINITION 9.1. *A system*

$$A \in V(n, X, m)$$

is said to be hyperbolic if

$$\det \hat{A}(\bar{\xi}, x, \xi)$$

is hyperbolic.

We say that $m > n$ if there exists a permutation $\mu \to \mu'$ such that $m_\mu > n_{\mu'}$ for all μ.

THEOREM 9.1. *Let*

$$A \in V^{\rho,\sigma}_{1,[p]}(-n, X, m), m > n, \tag{1}$$

be hyperbolic and let $s \gtrless 0$ *be an integer. Then the inequality*

$$| D^{m-1,s} f, X_t |_{\infty,2} \leqslant c\,|D^{m-1,s} f, S_0|_2 + c\,| D^{n,s} Af, X_t |_{1,2} \tag{2}$$

holds for all $f \in \mathscr{C}(\overline{X})$ *and all sufficiently small* t *provided* (8.4) *of the preceding theorem holds and provided the principal indices satisfy*

$$\left.\begin{aligned} \rho_{\nu\mu} &\geqslant \max_{\kappa,\lambda} \max\,(1, n_\kappa, 1 - n_\lambda) \\ \hat{\rho}_{\nu\mu} + \hat{\sigma}_{\nu\mu} &\geqslant \max_{\kappa} \max\,(1, n_\kappa) + s \\ \hat{\rho}_{\nu\mu} + \hat{\sigma}_{\nu\mu} &\geqslant \max_{\kappa} \max\,(1, 1 - n_\kappa) - s. \end{aligned}\right\} \tag{3}$$

NOTE 1. Since $m_\mu \geqslant n_\nu \Rightarrow 1 - m_\mu \leqslant 1 - n_\nu$, (3) is stronger than (8.5). If all m_μ are positive, and maybe also in the general case, (2) holds for all $t \leqslant \tau$.

NOTE 2. The conditions (3) are particularly favorable when all n_ν are equal and non-negative. In fact, then all m_μ are positive so that (3) follows from (8.5) and hence also from (8.4). In other words, for classical normal systems, our theorem is as sharp as Theorem 8.1.

NOTE 3. At least in the classical case, the Note 4 of Theorem 6.1 about Cauchy's problem applies with the appropriate formal changes.

The theorem can be reduced to the preceding one by a diagonalization which runs roughly as follows.

For simplicity we limit ourselves to homogeneous A. For ξ real, the degree of every non-vanishing term in

$$a\,(\bar{\xi}, x, \xi) = \det A\,(\xi, x, \xi)$$

then is

$$M = \Sigma m_\mu - \Sigma n_\nu = \Sigma(m_\mu - n_{\mu'}) \geqslant N > 1.$$

Since $a \neq 0$, this is also the degree of a. We shall construct a system

$$B \in V(M - m, X, n) \tag{5}$$

such that

$$C = BA \in V(M - m, X, m) \tag{6}$$

is hyperbolic and almost diagonal.

To do this, let $b_{\kappa\nu}(x, \xi)$ be the cofactor of $a_{\kappa\nu}$ in $A(\bar{\xi}, x, \xi)$ for ξ real. The degree of $b_{\kappa\nu}$ is $M - m_\kappa + n_\nu$ and changing $(M - m_\kappa)_+$ derivatives D in $b_{\kappa\nu}(x, D)$ to $\bar{D}$ and putting them in front of the coefficients we get an operator

$$b_{\kappa\nu}(\bar{D}, x, D) \in V\,(M - m_\kappa, X, n_\nu) \tag{7}$$

and hence a system B satisfying (5). Using (1) and (5) we see that the product

$$c_{\kappa\mu}(\bar{D}, x, D) = \sum_\nu b_{\kappa\nu}(\bar{D}, x, D)\; a_{\nu\mu}(\bar{D}, x, D) \tag{8}$$

has the property (6) and that we have

$$\xi \text{ real} \Rightarrow \hat{c}_{\kappa\mu}(\bar{\xi}, x, \xi) = \Sigma b_{\kappa\nu}(\bar{\xi}, x, \xi)\;\; a_{\nu\mu}(\bar{\xi}, x, \xi) = \delta_{\kappa\mu}\, a\,(\bar{\xi}, x, \xi)$$

so that C is hyperbolic and almost diagonal. Hence, provided the coefficients of A and B are sufficiently differentiable, Theorem 8.1 applies to C and gives the inequality

$$|\, D^{m-1,s} f, X_t\,|_{\infty,2} \leqslant c\;|\, D^{m-1,s} f, S_0\,|_2 + c\;|\, D^{m-M,s}\, C f, X_t|_{1,2}. \tag{9}$$

On the other hand, it follows from (1) and (5) that

$$(b_{\kappa\nu}\, a_{\nu\mu})\, f = b_{\kappa\nu}(a_{\nu\mu}\, f).$$

In fact, if $n_\nu \geqslant 0$, $a_{\nu\mu}$ is free of adhesive derivatives and if $n_\nu \leqslant 0$, $b_{\kappa\nu}$ is free of interior derivatives. In particular,

$$(Cf)_\kappa = \sum_\nu b_{\kappa\nu}(Af)_\nu,$$

so that $Cf = B(Af)$. Using this it is not difficult to show that

$$|\, D^{m-M,s}\, C f, X_t|_{1,2} \leqslant c\;|\, D^{n,s}\, A f, X_t\,|_{1,2}.$$

Combining this with (9) we get the desired inequality (2). A close check of the differentiability requirements gives the theorem. From (8) it is possible to obtain an idea of the result. In fact, the terms with $n_\nu \geqslant 0$ require n_ν derivatives of the coefficients of A and the terms with $n_\nu < 0$ require $-n_\nu$ derivatives of the coefficients of B which are polynomials in the principal coefficients of A. Hence we have to require at least $\max|n_\lambda|$ derivatives of the principal coefficients of A. The first inequality (3) requires one more derivative than this estimate unless all n_λ are positive. The two others account for the differentiability imposed by the parameter s and follow from the first one when $s = 0$.

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ON INVARIANT MANIFOLDS OF VECTOR FIELDS AND SYMMETRIC PARTIAL DIFFERENTIAL EQUATIONS†

By JÜRGEN MOSER

1. We are concerned with the question of existence and construction of solutions of some classes of nonlinear partial differential equations. There are a number of approaches to solving partial differential equations and we mention here the theorem of Cauchy-Kowalewski which is based on power series expansion and Cauchy's majorant method. For hyperbolic differential equations the theory of characteristics allows the construction of the solution in some cases and finally we mention Schauder's methods to solve quasilinear partial differential equations using the Schauder-Leray fixed point theorem. Nonlinear elliptic equations have been treated by L. Nirenberg and others.‡

All these methods have in common that one provides sufficiently good estimates for the solutions or the coefficients (in the Cauchy majorant method). These estimates are deduced from the non-characteristic nature of the initial surfaces, the hyperbolic or the elliptic character of the differential equations considered.

However, in the theory of ordinary differential equations and in several fields of applications one encounters nonlinear partial differential equations which cannot be classified as elliptic or hyperbolic. In elasticity theory and fluid dynamics one is led to partial differential equations which are hyperbolic in some domain and elliptic in another one. Recently Friedrichs [8] worked out a theory for systems of arbitrary type in the *linear* case. His theory is based on appropriate a priori estimates from which one can

† This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, with support from the Office of Naval Research, Contract Nonr-285(46).

‡ For an exposition of this subject we refer to the address [1] of L. Nirenberg at the International Congress, 1962.

establish the existence of weak solutions. To establish the differentiability of these weak solutions is a separate task. Another approach to linear partial differential equations—due to Hörmander—also is not restricted to elliptic or hyperbolic type, but assumes to a larger extent constant coefficients.

We discuss in this talk a class of symmetric systems in the *nonlinear* case and apply a method of construction of the solution which circumvents weak—or generalized—solutions entirely. In fact, for nonlinear problems weak solutions seem of doubtful value for several reasons : In general one cannot form a "function of" a distribution, an operation needed in a nonlinear theory. In the linear theory the concept of weak solutions is usually introduced for systematic reasons since it is easier to establish the existence of weak than classical solutions. The differentiability of these solutions can then be studied separately. In the nonlinear theory this separation of the problem seems not possible any more. In the nonlinear problems we are going to discuss, a priori estimates—the essence of weak solutions—do not seem available. On the contrary the estimates are supplied only—a posteriori—with the procedure. This method (published previously) reduces the construction of the solution to an iteration where at each step a finite system of equations has to be solved. The approximate solutions converge rapidly to *classical* solutions. Therefore one may expect that this approach could be used for numerical purposes.

It would be desirable to apply this method also to boundary value problems. We shall restrict ourselves to differential equations on compact manifolds where no boundaries occur. For simplicity we treat the case of a torus only.

We start with a discussion of a problem from the theory of ordinary differential equations which leads to symmetric partial differential equations of a rather special type.

2. Stable Invariant Surfaces. A closed differentiable manifold σ is called "invariant" under a vector field if the vector field is tangential at each point of σ, i.e. σ consists of orbits of the vector field. It is called stable—or asymptotically stable—if all orbits near σ approach σ as the time parameter increases to infinity.

The concept of invariant manifold occurs naturally when one considers slightly coupled oscillations, i.e. systems of the form

$$\ddot{x}_\nu = f_\nu(x_\nu, \dot{x}_\nu) + \mu\phi_\nu(x, \dot{x}) \quad (\nu = 1, 2, \ldots, n). \tag{1}$$

For $\mu = 0$ these systems decompose into n second order equations which—as we assume now—have an asymptotically stable periodic solution (i.e. one dimensional invariant manifold) :

$$x_\nu = p_\nu(s_\nu) \; ; \quad \dot{x}_\nu = q_\nu(s_\nu) \tag{2}$$

where

$$\dot{s}_\nu = 1.$$

Then (2) represents an n dimensional torus $\sigma(0)$ invariant under (1) for $\mu = 0$ *which is asymptotically stable. The question arises* whether there exists an invariant manifold $\sigma = \sigma(\mu)$ for small values of μ. This is indeed so and this problem requires solving a system of partial differential equations.

We assume, then, that the unperturbed surface $\sigma(0)$ is a torus of n dimensions which is nearly invariant. Introducing n angular variables $(x_1, \ldots, x_n)$ on this torus we describe the vector field

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

in terms of $n + m$ variables (of a Euclidean space)

$$x = (x_1, \ldots, x_n), \; y = (y_1, \ldots, y_m)$$

where y is normal to σ_0. Since σ_0, given by $y = 0$, is approximately invariant $g(x, 0)$ is small. We suppress the parameter μ. The desired invariant torus can be represented in the form

$$y_k = u_k(x), \quad (k = 1, \ldots, m)$$

where $u = (u_1, \ldots, u_n)$ satisfies

$$\sum_{\nu=1}^{n} f_\nu(x, u) \frac{\partial u}{\partial x_\nu} - g(x, u) = 0. \tag{3}$$

The system is asymptotically stable if the quadratic form

$$\left(\eta, \frac{\partial g}{\partial u} \eta \right)$$

is negative definite.

We do not exclude here that the functions f_ν vanish in some points which represent singularities of the differential equations. In fact, such singularities correspond to equilibrium points of the original differential equations.

Problems of this type have been studied by different authors—Diliberto [2], [3], Kyner [4], Bogoliubov and Mitropolski [5], [6] in their book on nonlinear oscillations. However, singularities of the differential equations are usually not admitted and extensive use of characteristics is made. Such an approach is inadequate for symmetric systems as we shall discuss them now.

3. Symmetric Systems. We consider systems of the form

$$F_k(x, u, u_x) = 0, \quad (k = 1, 2, \dots, m), \tag{4}$$

where the $F_k(x, y, p)$ are of period 2π in x_ν $(\nu = 1, \dots, n)$ and admit sufficiently many derivatives in $|y| + |p| \leqslant 1$; here p has nm components $p_{\mu\nu}$.

We introduce the matrices $a^{(\nu)}$, b with the elements

$$\left. \begin{aligned} a_{kl}^{(\nu)}(x, y, p) &= \frac{\partial F_k}{\partial p_{l\nu}} = \frac{\partial F_l}{\partial p_{k\nu}}, \quad & k, l &= 1, 2, \dots, m, \\ b_{kl}(x, y, p) &= \frac{\partial F_k}{\partial u_l}. & \nu &= 1, 2, \dots, n, \end{aligned} \right\} \tag{5}$$

and call (4) symmetric if the $a^{(\nu)}$ are symmetric. Note that these matrices $a^{(\nu)}$ determine the type of the system, which will however not be restricted.

In the linear case this reduces to systems of the form

$$\sum_\nu a^{(\nu)} \frac{\partial u}{\partial x_\nu} + b(x)\, u = f(x)$$

as they were discussed by Friedrichs, who even treated boundary value problems.

The object is to construct a solution which is of period 2π in the x-variables under appropriate assumptions. For elliptic systems such results are known. While ellipticity of the system is a condition for the $a^{(\nu)}$ we shall require instead that $b + b^T$† is positive definite, and so not restrict the type.

We formulate our result. With some $l = l(d)$ depending only on d, we assume that the derivatives of order $\leq l$ of F are bounded by a constant K in $|y| + |p| < 1$. Moreover, with a β in $0 < \beta < 1$ let

$$(\eta, b(x, 0, 0)\,\eta) > 2\beta\,|\eta|^2 \tag{6}$$

and

$$\left|\frac{\partial}{\partial x}\,a^{(\nu)}\right| < \frac{\beta}{l+1}. \tag{6'}$$

THEOREM. *There exists an* $\epsilon_0 = \epsilon_0(d, K) > 0$ *such that for*

$$\max_x |F(x, 0, 0)| < \epsilon_0\,\beta,$$

there exists a periodic solution $u(x)$ *which is twice continuously differentiable. The integer* l *can be chosen as any integer* $> (3d/2 + 6.)$

The equations (3) are obviously a special case of this problem. They are quasilinear, i.e. linear in p and, moreover, the matrices $a^{(\nu)}$ are scalar multiples of the identity matrix. In that case one can get away with milder differentiability assumptions; in fact, $l = 2$ would suffice. However, it seems unimportant to reduce the smoothness assumption since in most applications the equations are infinitely often differentiable if not analytic in their arguments. It is worth mentioning that even for analytic equations the solutions need not be analytic, as one can see from simple examples of differential equations for which F_p vanishes in some points.

This theorem can also be used to establish the existence of stable manifolds through equilibrium points.

REMARK. Assumptions (6) and (6′) can be replaced by the following weaker one: For any set of integers $\lambda_\nu \geq 0$ with $\sum \lambda_\nu = l$ replace b in (6) by

† b^T denotes the transpose of b.

$$\tilde{b} = b + \sum_{\nu=1}^{n} \lambda_\nu \frac{\partial a^{(\nu)}}{\partial x_\nu}$$

and replace (6′) by

$$\sum_{\mu \neq \nu} \left| \frac{\partial}{\partial x_\mu} a^{(\nu)} \right| < \frac{\beta}{l+1} \text{ for } y = p = 0.$$

4. Method. The construction is established by a modification of a method presented in [3]. The idea is the following: To solve equation (4) we construct approximate solutions $u_n(x)$ starting with $u_0 = 0$. We obtain u_{n+1} by use of Newton's method; i.e. in $F(x, u, u_x)$ we replace the function u by $u_n + v$ and linearize with respect to v:

$$F_p(x, u_n, u_{nx})\, v_x + F_u\,(x, u_n, u_{nx})\, v + F\,(x, u_n, u_{nx}) = 0 \tag{7}$$

and set

$$u_{n+1} = u_n + v.$$

For these linear equations one has a priori estimates on account of the assumptions (6) and (6′). Such estimates are also at the basis of Friedrichs' approach and have the following form: If $\|u\|_0$ denotes the square integral of u over $0 \leqslant x_\nu \leqslant 2\pi$ one has

$$\|v\|_0 \leqslant c\, \|F\,(x, u_n, u_{nx})\|_0$$

where c depends on β only.

One sees clearly the difficulty: the estimates for the square integral of $u_{n+1} - u_n$ require estimates of the first derivatives of u_n. Thus at each step one derivative is lost—not to speak of the necessity to have pointwise estimates of u_x.

For this reason the strategy is to *solve the linearized equation* (7) *only approximately* but by functions which are sufficiently smooth. We abbreviate the differential equation (7) as

$$Lv = f \tag{8}$$

where L denotes the linear differential operator and

$$f(x) = -\,F(x, u_n, u_{nx}).$$

Moreover, we denote by $\|f\|_l$ the sum of the square integrals of all derivatives of f up to order l. Then one obtains from (6), (6′)—under appropriate assumptions on u_n —

$$\|v\|_l \leqslant c\|f\|_l.$$

But we construct, with a small parameter h in $0 < h < 1$, an approximate solution w of (8) satisfying

$$\| Lw - f \|_0 \leqslant ch^l \|f\|_l \tag{9}$$

for which the $l + 1$ derivative can be estimated by

$$\|w\|_{l+1} \leqslant ch^{-1} \|f\|_l \tag{9'}$$

with a constant $c > 0$ independent of h. This way one gains the lost derivative, but the estimate (9′) contains the large factor h^{-1}.

Choosing $h = h_{n+1}$ at the n-th step appropriately and setting $u_{n+1} = u_n + w$ one can establish convergence of the procedure. In fact, one can establish inequalities of the type

$$\| u_{n+1} - u_n \|_0 < \epsilon_n \to 0 ; \| u_{n+1} - u_n \|_{l+1} < \frac{\epsilon_n}{h_n^{l+1}} \to \infty$$

where

$$\epsilon_n = \epsilon_{n-1}^{\kappa} \to 0 ; h_n = h_{n-1}^{\kappa} \to 0 \text{ with } 1 < \kappa < 2 \qquad .(10)$$

provided $\| F(x, 0, 0)\|_0$ is sufficiently small.

For the details of the convergence proof we refer to the paper [9]. We first mention that the method succeeds because of the quadratic convergence of Newton's method. The recursive estimate for the error yields the estimate

$$\| u_{n+2} - u_{n+1} \| < c \left(\frac{\epsilon_n^2}{h_{n+1}^{d/2+1}} + \frac{h_{n+1}^l}{h_n^{l+1}} \epsilon_n \right) \tag{11}$$

and the right hand side is required to be less than ϵ_{n+1}. The first term stems from the quadratic error term in Newton's method which is of the form $w.w_x$ which can be estimated by Sobolev's Lemma. The second term corresponds to the error in solving the linearized equation approximately.

Notice that there is an optimal choice for the parameter h_{n+1} which minimizes the expression in (11). It is easy to establish (10) and the convergence (for details see [9]).

It remains to solve the linearized equation approximately, i.e. to find w satisfying (9) and (9′). This can be done in several ways. We mention two approaches. If one adds to (8) a dissipation term $-\mu\Delta v$ with a small positive μ the system becomes linear elliptic, for which the theory is developed. Choosing μ appropriately one can achieve (9) and (9′). This has been worked out for the quasilinear case by R. Sacker, a student at New York University.

Another method is to replace the linear equation (8) by a difference equation of "mesh width" h^{l-1} and replacing $\frac{\partial}{\partial x_\nu}$ by symmetric differences. All estimates are preserved then and the finite system has a unique solution. One can devise an interpolation method which interpolates the discrete function by a smooth one satisfying (9), (9′). In fact, this is relatively elementary. The advantage of this method is that it reduces the problem even to a finite one.

Difference approximations have frequently been used for partial differential equations. The new feature here is that these discrete functions converge *with derivatives* to smooth functions.

5. This procedure has its origin in a paper of Kolmogorov [7] which concerns the constructions of invariant manifolds for conservative systems, a much more difficult problem than that of Section 2. There the use of Newton's method was suggested to overcome the small divisor difficulty of celestical mechanics.†

It was our aim to show the wide applicability of this method in conjunction with an approximation scheme. One can summarize: To achieve fast convergence one should not solve the linearized equation too precisely. High *precision* and high *convergence* compete and have to be balanced against each other (by choice of h_{n+1} in (11)). All this requires, of course, many derivatives for the original problem.

It can be expected that the Cauchy problem for nonlinear hyperbolic systems can also be attacked by this method since the

† The proofs for Kolmogorov's statement were supplied just recently in a paper of Arnold [11].

principal difficulty is of the same nature. The basic result of Schauder [10] permits one to treat quasilinear equations only, in which case one can use a trick to circumvent the loss of derivatives. Namely one solves iteratively

$$F_p(x, u, u_x)\, v_x + F_u(x, 0, 0)\, v + F(x, u, u_x) = 0.$$

The iteration is given by $u = u_n$; $v = u_{n+1} - u_n$. It turns out that the procedure converges geometrically only (like ϵ_0^n) and if one tries to achieve fast convergence one again finds a loss of derivatives.

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ON THE COHOMOLOGY GROUPS OF LOCALLY SYMMETRIC, COMPACT RIEMANNIAN MANIFOLDS

By YOZÔ MATSUSHIMA

1. LET X be a simply connected symmetric Riemannian manifold of non-compact type. Let G be the identity component of the group of all isometries of X. Then G is a semi-simple Lie group of non-compact type with center reduced to the identity element. Let Γ be a discrete subgroup of G with compact quotient space G/Γ and without element of finite order different from the identity element. The group Γ acts on X discontinuously and freely and the quotient space $M = X/\Gamma$ of X by the action of Γ is a compact Riemannian manifold, which is locally symmetric.

Let u_0 be a point of X and let K be the isotropy subgroup of G at the point u_0. Then K is a maximal compact subgroup of G and X is identified with the coset space G/K and M is identified with the double coset space $\Gamma\backslash G/K$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ the subalgebra corresponding to the subgroup K. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $\phi(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \,\mathrm{ad}\, Y)$ $(X, Y \in \mathfrak{g})$ of $\mathfrak{g}$. The restriction of ϕ to $\mathfrak{m}$(resp. $\mathfrak{k}$) is positive (resp. negative) definite and we have

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{k}, \ [\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}, \ [\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}.$$

The vector space $\mathfrak{m}$ is identified with the tangent vector space of X at the point u_0. We take a basis $X_1, \ldots, X_r$ of $\mathfrak{m}$ such that $\phi(X_i, X_j) = \delta_{ij}$ and we normalize the G-invariant Riemannian metric g so that $g_{u_0}(X_i, X_j) = \delta_{ij}$. Let R_{ijkh} be the components of the curvature tensor with respect to the orthonormal basis $\{X_1, \ldots, X_r\}$ of the tangent space $\mathfrak{m}$ at the point u_0.

We suppose now that $\mathfrak{g}$ is simple and define a quadratic form $H_0(\xi)$ on the vector space $\mathfrak{S}$ of twice contravariant symmetric tensors $\xi = (\xi^{ij})$ at the point u_0 by putting

$$H_{\mathfrak{g}}(\xi) = b(\mathfrak{g}) \sum_{i,j=1}^{r} (\xi^{ij})^2 + \sum_{i,j,k,h=1}^{r} R_{ikhj}\, \xi^{ij}\, \xi^{kh},$$

where $b(\mathfrak{g})$ is a positive constant depending on $\mathfrak{g}$ (cf. [8]).

Then we can prove the following theorem.

THEOREM 1. *Any harmonic* 1*-form on the space* X/Γ *vanishes identically if, for each simple ideal* $\mathfrak{g}_i$ *of* $\mathfrak{g}$, *the quadratic form* $H_{\mathfrak{g}_i}(\xi)$ *is positive definite.*

2. Let $(\xi, \eta) = \sum_{i,j=1}^{r} \xi^{ij}\, \eta^{ij}$ be the inner product on the tensor space $\mathfrak{S}$ and P the linear endomorphism of $\mathfrak{S}$ defined by $P(\xi)^{ij} = \sum_{k,h} R_{ikhj}\, \xi^{kh}$. Then $(P(\xi), \eta) = (\xi, P(\eta))$ and $H_{\mathfrak{g}}(\xi) = b(\mathfrak{g})\ (\xi, \xi) + (P(\xi), \xi)$. The quadratic form $H_{\mathfrak{g}}(\xi)$ is positive definite (resp. non-negative) if and only if $b(\mathfrak{g})$ is $>$ (resp. $\geqslant$) the absolute value of the minimal eigen-value of the symmetric linear endomorphism P of $\mathfrak{S}$.

The eigen-values of P are calculated by Calabi-Vesentini [3] and Borel [1] in the case where X is a bounded symmetric domain in $\mathbf{C}^n$ and by Kaneyuki-Nagano [6, 7] for the other types of irreducible symmetric spaces. From their results we conclude that:

(1) $H_{\mathfrak{g}}(\xi)$ is always non-negative;

(2) $H_{\mathfrak{g}}(\xi)$ is positive definite if and only if the rank of the corresponding symmetric space G/K is greater than 1.

From this and Theorem 1 we get

THEOREM 2. *Let* X *be a simply connected symmetric Riemannian manifold all of whose irreducible factors are of rank greater than* 1. *Let* Γ *be a discontinuous group of isometries of* X *acting freely on* X *and with compact quotient space* X/Γ. *Then the first Betti number of* X/Γ *is equal to* 0.

Combined with a lemma of Selberg [12] we get the following corollary.

COROLLARY. *Let* X *be as in Theorem* 2 *and let* Γ *be a discontinuous group of isometries of* X *with compact fundamental domain. Then* *the factor group* $\Gamma/[\Gamma, \Gamma]$ *of* Γ *by its commutator subgroup* $[\Gamma, \Gamma]$ *is a finite group.*

REMARK. L. Greenberg informed me that he has constructed an example of a discontinuous group Γ of isometries of the 3-dimensional hyperbolic space such that $\Gamma/[\Gamma, \Gamma]$ is not a finite group.

3. Let $X = X_1 \times \dots \times X_p$ be the decomposition of X into the product of irreducible factors and let G (resp. G_i) be the identity component of the group of all isometries of X (resp. X_i). Then G is identified canonically with the direct product $G_1 \times \dots \times G_p$; $G = G_1 \times \dots \times G_p$. Suppose that the rank of X_i is equal to 1 for $i = 1, \dots, s$ and greater than 1 for $i = s+1, \dots, p$ and that $s < p$. Then we can prove the following theorem ([8], Theorem 3).

THEOREM 3. *The notation and the assumptions being as above, let* Γ *be an irreducible*† *discrete subgroup of* $G = G_1 \times \dots \times G_p$ *with compact fundamental domain. Then* $\Gamma/[\Gamma, \Gamma]$ *is a finite group.*

REMARK. The case where all of the X_i are of rank 1 is excluded in Theorem 3. But the case where all of the X_i are isomorphic to the hyperbolic plane is treated in the paper [11]. We shall refer to this in § 5.

4. Concerning the p-th Betti numbers ($p \geqslant 1$) of X/Γ we can prove the following theorem ([9]).

THEOREM 4. *Let* X *be irreducible and of rank greater than* 1. *Let* ω *be a harmonic* p*-form on the Riemannian manifold* X *which is invariant by* Γ. *Then* ω *is invariant by* G *if the ratio* $p : \dim X$ *is small.*

We conclude from this theorem that the p-th Betti number $b^p(X/\Gamma)$ of X/Γ is equal to the p-th Betti number $b^p(X_u)$ of the compact form X_u of X, if the ratio $p : \dim X$ is small.

EXAMPLE. Let X be the Siegel upper half space consisting of all complex symmetric matrices $x + iy$ of degree n with y positive

†For the definition of irreducible discrete subgroups, see, for example, [10].

definite. In this case the assertion of Theorem 4 holds for $p < \frac{n+2}{2}$ in the case $n > 2$, and for $p = 1$ in the case $n = 2$.

Refining the arguments in [9] we can prove that a holomorphic p-form on X invariant by Γ vanishes identically if $p < \frac{n+1}{2}$.

5. The results mentioned in the preceding sections are obtained by the method based on the theory of harmonic forms, some integral formulas and some of the detailed results on the curvature tensor of symmetric spaces. It seems to be rather difficult to get more precise results along these lines. One of the difficulties comes from the fact that the tangential representation of the isotropy group K is rather difficult to treat. In the special case where X is the product of upper half-planes the isotropy group K is abelian and we can obtain satisfactory results in this case.

Let H be the upper half-plane consisting of all complex numbers z with $\mathrm{Im}(z)>0$ and let $X = H \times \dots \times H$ (n factors). Let

$$G = SL(2, R) \times \dots \times SL(2, R) \ (n \text{ factors}).$$

Then G acts in a well-known way transitively on X and $X = G/K$, where $K = SO(2) \times \dots \times SO(2)$. Let Γ be an irreducible discrete subgroup of G and suppose that Γ acts freely on X and that X/Γ is compact. Then X/Γ is a compact Kähler manifold, in fact a Hodge manifold. We denote by $h^{p,q}$ the dimension of the complex vector space of all harmonic forms of type (p, q) on X/Γ. We can prove the following result [11].

THEOREM 5. $h^{p,q} = 0$, if $p \neq q$, $p + q \neq n$,

$$h^{p,p} = \binom{n}{p}, \text{ if } 2p \neq n,$$

$$h^{n-q,q} = \binom{n}{q} (\delta_{n-q,q} + h^{n,0}).$$

COROLLARY. *Let X_u be the product of n copies of the Riemann sphere. Then*

$$b^r(X/\Gamma) = b^r(X_u) \ (r \neq n),$$

$$b^n(X/\Gamma) = b^n(X_u) + 2^n \, h^{n,0}(X/\Gamma).$$

In the proof of Theorem 5 given in [11], the following results are utilized: (1) a vanishing theorem for automorphic forms, (2) the Chern decomposition theorem for harmonic forms [4], and (3) the Hirzebruch proportionality relation [5].

6. In the above considerations, we used the theory of real or complex valued harmonic forms on the space X which are invariant under the action of Γ. Recently, in connection with the theory of deformations of discrete subgroups by Selberg, Calabi-Vesentini and Weil and the theory of automorphic forms by Eichler and Shimura, it turns out that we should consider vector-valued differential forms on the space X which are transformed under the action of Γ according to a given representation of Γ. A harmonic theory for such forms has been treated in the paper [10] and in the paper [11] detailed results on these harmonic forms have been obtained in the case where X is a product of the upper half-plane.

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DEFORMATIONS OF LINEAR CONNECTIONS AND RIEMANNIAN METRICS

By M. S. RAGHUNATHAN

1. Introduction. The aim of this talk is to outline a theory of deformation of linear connections and Riemannian metrics.† The results are analogous to those in the theory of deformations of complex structures as developed by Kodaira and Spencer (See [2]). Our techniques, however, are different and enable us to treat also a class of non-compact manifolds. We indicate an application of the results to the problem of deformations of discrete subgroups of Lie groups considered by A. Weil ([4] and [5]).

2. Notation and Definitions. All manifolds, linear connections, vectorfields, etc. which occur are assumed to be differentiable of class C^∞. Also, all manifolds are assumed to be paracompact and connected.

Let $(W, B, p : W \to B)$ be a locally trivial differentiable fibre space, with total space W, base B, projection p and typical fibre M. Let, for each $b \in B$, w_b be a linear connection on $p^{-1}(b)$. Let Θ (resp. Π) denote the sheaf of germs of vertical (resp. projectable) vectorfields on W which leave the connections along the fibres invariant (a projectable vectorfield generates a local one parameter group of diffeomorphisms which map fibres into each other). We denote by T the inverse image by p of the sheaf of germs of C^∞ vectorfields on B. Also Θ_b (resp. Π_b) shall denote the sheaf of germs of vectorfields on $p^{-1}(b)$ (resp. vectorfields on W defined along $p^{-1}(b)$) which are restrictions to $p^{-1}(b)$ of sections of Θ (resp. Π). Again, T_b denotes the constant sheaf on $p^{-1}(b)$ with stalk the tangent space at b to B. Clearly, Θ_b is the sheaf of germs of Killing vectorfields on $p^{-1}(b)$.

† Detailed proofs will be given in a paper entitled "Deformations of linear connections and Riemannian manifolds" to appear in Journal of Mathematics and Mechanics.

We have then the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Theta & \xrightarrow{i} & \Pi & \xrightarrow{p} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r_b} & & \downarrow{\scriptstyle r_b} & & \downarrow{\scriptstyle r_b} & & \\ 0 & \longrightarrow & \Theta_b & \xrightarrow{i_b} & \Pi_b & \xrightarrow{p_b} & T_b & \longrightarrow & 0 \end{array} \qquad \text{(I)}$$

In the diagram (I), i (resp. i_b) is an inclusion and p(resp. p_b) is the map induced by $p : W \to B$ and r_b is the 'restriction' homomorphism.

Definition 1. *A family $(w_b)_{b \in B}$ of linear connections, w_b being defined on $p^{-1}(b)$, is a family of deformations of a linear connection w on M if*

(i) *there exists $b_0 \in B$ and a connection-preserving diffeomorphism $\phi : p^{-1}(b_0) \to M$, M being provided with the connection w;*

(ii) *in the associated diagram* (I), *the rows are exact.*

Definition 2. *A family of deformations $p : W \to B$ is locally trivial at $b \in B$, if there is an open set $U \subset B$, $b \in U$, and a homomorphism Φ of fibre spaces, $\Phi : p^{-1}(U) \to M \times U$ which induces a connection-preserving diffeomorphism of each $p^{-1}(b')$, $b' \in U$, (provided with $w_{b'}$) onto M (with the connection w_b).*

Associated to the exact sequence (I), we have an exact sequence of direct images of sheaves :

$$\begin{array}{ccccccccc} 0 \to & R^0p(\Theta) & \xrightarrow{i} & R^0p(\Pi) & \xrightarrow{p} & R^0p(T) & \xrightarrow{\delta} & R^1p(\Theta) \to \ldots \\ & \downarrow{\scriptstyle r_b} & & \downarrow{\scriptstyle r_b} & & \downarrow{\scriptstyle r_b} & & \downarrow{\scriptstyle r_b} \\ 0 \to & H^0(p^{-1}(b),\Theta_b) & \xrightarrow{i_b} & H^0(p^{-1}(b),\Pi_b) & \xrightarrow{p_b} & H^0(p^{-1}(b),T_b) & \xrightarrow{\delta_b} & H^1(p^{-1}(b),\Theta_b) \to \ldots \end{array}$$

Definition 3. *The homomorphism δ(resp. δ_b) in the second diagram above will be called the infinitesimal deformation map (resp. punctual deformation map at b).*

We have then

Proposition 1. *If a family $p : W \to B$ is locally trivial at $b \in B$, then δ and δ_b are zero at b. Conversely, if each connection w_b is complete, and δ is zero at b, then the family is locally trivial at b.*

The proposition is proved by lifting vectorfields on the base to sections of Π and, using completeness, integrating them to obtain global 1-parameter groups.

3. Punctual maps. In this paragraph, we give a stronger result than Proposition 1, by making further assumptions on the connections.

DEFINITION 4. *A connection w on a manifold M is regular if the sheaf Θ_0 of germs of Killing vectorfields is a local system.*

Examples of regular connections: (i) analytic connections; (ii) locally homogeneous connections.

The regularity of a connection is equivalent to the following: the function $d(x)$ = the dimension of the stalk at x of Θ_0 (which is necessarily finite, in fact, less than n^2+n+1, n dimension of M) is independent of x.

We can now state our main results.

THEOREM 1. *Let M be a connected paracompact manifold with a finitely presentable fundamental group. Let $p: W \to B$ be a family of deformations of complete regular linear connections on M. If then the punctual deformation maps δ_b are zero for every $b \in B$, and further $d_0(b) = \dim H^0(p^{-1}(b), \Theta_b)$ is a function independent of b, then the family is locally trivial.*

The proof uses the following lemma on finite dimensional vector spaces.

Let $\sum_{i=0}^{\infty} (K_i) = K$, be a graded vector space, each K_i being finite dimensional. Let $(d_b)_{b\in B}$ be a family of endomorphisms of degree 1 of K such that $d_b^2 = 0$, B being a differentiable manifold. Then (K, d_b) is a complex. If then, the maps $b \to d_b \in \mathrm{Hom}\,(K_p, K_{p+1})$ are C^∞, the functions $d_i(b) = \dim H^i(K, d_b)$ are upper semi-continuous functions of b. Further, if $d_i(b)$ and $d_{i+2}(b)$ are constants, then so is $d_{i+1}(b)$. Also if $d_0(b)$ is independent of b and $f: B \to K_1$ is a

C^∞ function such that each $f(b)$ is a coboundary in (K, d_b), then there is a C^∞ function $g: B \to K_0$ such that $d_b(g(b)) = f(b)$.

THEOREM 2. *If $p: W \to B$ is a family of deformations of regular connections on M, where M is a manifold with a finitely presentable fundamental group, then $d_i(b) = \dim H^i(p^{-1}(b), \Theta_b)$ are upper semi-continuous functions of b for $i = 0, 1$. If $d_1(b)$ is independent of b so is $d_0(b)$.*

The proof utilises, besides the lemma given above, the well-known isomorphisms $H^i(p^{-1}(b), \Theta_b) \simeq H^i(\Gamma, \rho_b)$, where Γ is the Poincaré group of M and ρ_b a suitable representation of Γ [1].

Combining Theorems 1 and 2, we get

THEOREM 3. *Let $p: W \to B$ be a family of deformations on a manifold M with a finitely presentable fundamental group. If for some $b \in B$, $H^1(p^{-1}(b), \Theta_b) = 0$, and if each connection is complete, then the family is locally trivial at the point b.*

Analogous results can be proved for the deformation of Riemannian manifolds by similar methods.

4. Discrete subgroups of Lie groups. Let G be a simply connected Lie group and $\rho_t: \Gamma \to G$ a differentiable 1-parameter family of isomorphisms of a group Γ in G. The following result is due to A. Weil ([5]).

There exists $\delta > 0$ such that if $I = \{t \mid |t| < \delta\}$ then the action of Γ on $G \times I$, defined by $(g, t)\,\gamma = (g.\,\rho_t(\gamma), t)$, $\gamma \in \Gamma$, is properly discontinuous.

We deduce from the above that the right invariant Maurer-Cartan forms w_t on $G/\rho_t(\Gamma)$ (left coset space) define on $G/\rho_0(\Gamma)$ a deformation family. Applying Theorem 3 to this situation, we deduce

PROPOSITION 2. *If $H^1(\rho_0(\Gamma), \mathfrak{g}) = 0$, $\mathfrak{g}$ the Lie algebra of G, then $\rho_0(\Gamma)$ and $\rho_t(\Gamma)$ are inner conjugate for t sufficiently near zero.*

The results in § 7-10 of [5] and the theory of harmonic forms with coefficients in a local system (see [3]) imply the following:

If $\mathfrak{g}$ *is semisimple and without compact or* 3-*dimensional components, then* $H^1(\rho_0(\Gamma), \mathfrak{g}) = 0$.

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HOLOMORPHIC VECTOR BUNDLES ON A COMPACT RIEMANN SURFACE

By M. S. NARASIMHAN and C. S. SESHADRI†

Let X be a compact Riemann surface of genus $g(\geqslant 2)$ and let π be the fundamental group of X. We prove that *the set M of equivalence classes of holomorphic vector bundles on X arising from n-dimensional irreducible unitary representations of π has a natural structure of a complex analytic manifold of complex dimension $n^2(g-1)+1$.* For $n=1$, this complex manifold coincides with the Picard variety of X. The number $n^2(g-1)+1$ has been calculated heuristically by A. Weil as the dimension of the "field of hyperabelian functions" [J. Math Pures. App. (1938)].

The details of the proof of the theorem are given in a paper which is due to appear shortly in the Mathematische Annalen. We give below the main steps.

Two holomorphic vector bundles on X arising from unitary representations of π are isomorphic if and only if the representations are equivalent. This fact enables us to introduce the structure of a real analytic manifold on M in the following way. Let $\Omega = U(n) \times \ldots \times U(n)$ ($2g$ times) and $f: \Omega \to SU(n)$ the map defined by

$$(A_1, B_1, \ldots, A_g, B_g) \to A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1}.$$

The set of homomorphisms of π into the unitary group $U(n)$ is identified with $f^{-1}(I)$, where I is the identity matrix. One can prove that the tangent vectors to Ω at $\rho \in f^{-1}(I)$ which are mapped into zero by the differential of f at ρ can be identified with the 1-cocycles of π with respect to the representation $\text{ad}\,\rho$ of π in $\mathfrak{U}(n)$, the Lie algebra of skew-Hermitian matrices ($\text{ad}\,\rho$ is the composite of ρ and the adjoint representation of $U(n)$ in $\mathfrak{U}(n)$). The dimension of the space of 1-cocycles is then computed by using the isomorphism between the cohomology groups of π (with respect to $\text{ad}\,\rho$) and the

† Presented by M. S. Narasimhan at the International Colloquium on Differential Analysis, Bombay, 1964.

cohomology groups of X with coefficients in the local coefficient system determined by the representation ad ρ. It follows that f is of maximal rank at a point $\rho \in f^{-1}(I)$ if and only if ρ is an irreducible representation and that the set of irreducible unitary representations has a natural structure of a real analytic manifold. As a consequence, one sees that the equivalence classes of irreducible unitary representations (of a given dimension) form a real analytic manifold M, and that the tangent space at a point $m \in M$ can be identified with the cohomology space $H^1(\pi, \text{ad}\,\rho)$, where ρ is a unitary representation in the class m.

To introduce the complex structure on M, let $\text{Ad}\,P(\rho)$ be the holomorphic vector bundle adjoint to the holomorphic $GL(n; C)$-principal bundle $P(\rho)$ determined by a unitary representation ρ of π and let $H^1(X, \text{Ad}\,P(\rho))$ denote the first cohomology space of X with coefficients in the sheaf of germs of holomorphic sections of $\text{Ad}\,P(\rho)$. We prove, using harmonic forms with coefficients in a local system of vector spaces, that the natural map

$$J : H^1(\pi, \text{ad}\,\rho) \to H^1(X, \text{Ad}\,P(\rho))$$

is an isomorphism of real vector spaces. Since $H^1(X, \text{Ad}\,P(\rho))$ has a natural structure of a complex vector space, M thus acquires an almost complex structure. Now the map J turns out, locally, to be the infinitesimal deformation map of a differentiable family of holomorphic bundles on X and hence, by a theorem of Kodaira-Spencer-Nakano, this almost complex structure is in fact a complex structure on M [Mem. Coll. Sci. Univ. Kyoto. Ser A. Math. (1961)].

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HOLOMORPHIC FIELDS OF COMPLEX LINE ELEMENTS WITH ISOLATED SINGULARITIES

By A. VAN DE VEN

LET V_d be a compact, connected, complex manifold of complex dimension d, $d \geqslant 2$. If to each singular point (zero point) of a continuous field of tangent vectors with isolated zeros on V_d we attach in a standard way an index of singularity, the index sum is always the same, namely equal to the Euler-Poincaré characteristic of V_d. An analogous result is no longer true for a continuous field of complex line elements with isolated singularities (by this I mean a continuous 1-dimensional subbundle of the restriction to $V_d - A$ of the complex tangent bundle of V_d, where A is a discrete subset of V). This can be seen already in the projective plane: a field obtained by joining all points with a fixed point has index sum 1, a field obtained from a "general" collineation, the index sum 3. The question as to what numbers occur as index sums was answered by E. Kundert, who proved (as a special case of a general theorem on 1-dimensional subbundles of complex vector bundles) that the integers occuring as index sums of fields with isolated singularities are precisely the Chern numbers $c_d(\Theta \otimes \xi)$, where Θ is the (contravariant) tangent bundle of V_d and ξ runs through all continuous complex line bundles on V_d.

It is clear that one can also define holomorphic fields of complex line elements (the two examples given above are such fields). It seems natural (also from the point of view of holomorphic foliations) to ask questions of the following type.

(i) What are the index sums $c_d(\Theta \otimes \xi)$ which can be represented by holomorphic fields with isolated singularities?

(ii) Can the holomorphic fields with isolated singularities, as in the topological case, be classified in some way by means of holomorphic sections in vector bundles $\Theta \otimes \xi$, where ξ is now a holomorphic line bundle on V?

The second question is answered by

THEOREM I. *Let V_d $(d \geqslant 2)$ be a complex manifold, X a discrete subset of V_d, α a holomorphic vector bundle of dimension d on V_d, β a holomorphic 1-dimensional subbundle of $\alpha \mid V_d - X$. Then there exists a holomorphic line bundle ξ on V_d and a holomorphic section S of $\alpha \otimes \xi$, $S \neq 0$ on $V_d - X$, such that on $V_d - X$, β is the bundle determined by S in a canonical way.*

It follows that the holomorphic fields of line elements with isolated singularities on a compact complex manifold can be parametrised in a natural way by (possibly empty) Zariski open subsets of the projective spaces of the complex vector spaces $H^0(V_d, \Theta \otimes \xi)$, where ξ runs through all holomorphic line bundles on V_d.

However, the answer is not completely satisfactory in as far as we have no *necessary and sufficient* criterion for a bundle $\Theta \otimes \xi$, having holomorphic sections, to have also holomorphic sections *with isolated singularities.*

Nevertheless, our methods provide complete answers in many familiar cases. For example, we have

THEOREM II. *The product $V = V^{(1)} \times \ldots \times V^{(k)}$ of the rational homogeneous manifolds $V^{(1)}, \ldots, V^{(k)}$, all with second Betti number 1, admits a holomorphic field of complex line elements without singularities if and only if at least one $V^{(i)}$ is a projective line P_1; and in that case V admits only the obvious fields i.e. the fields attaching to the point $(x_1, \ldots, x_k)$ the line element tangent to $(x_1, \ldots, x_{i-1}, P_1, x_{i+1}, \ldots, x_k)$.*

In particular, the odd dimensional projective spaces and quadrics of dimension $\geqslant 3$ have no holomorphic field of line elements without singularities though they have continuous fields of such type. As another example, the 2-dimensional quadric $P_1 \times P_1$ has only two such fields.

By a Hirzebruch surface we mean the bundle space of a holomorphic P_1-bundle on P_1.

THEOREM III. *The only holomorphic field of complex line elements without singularities on a Hirzebruch surface which is different from* $P_1 \times P_1$, *is the field along the fibres.*

On a Hirzebruch surface there are in general many homotopy classes of continuous fields.

For the manifolds appearing in Theorems II and III, question (i) can be answered completely.

REMARK. Holomorphic fields of line elements which have as singular set an analytic subset of V of codimension $\geqslant 2$ can also be treated with our methods.

Full details will appear in a forthcoming paper in the *Annales de Fourier.*

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