IDEALS OF DIFFERENTIABLE FUNCTIONS

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IDEALS OF DIFFERENTIABLE FUNCTIONS

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INTRODUCTION

This book reproduces, with some additions, the contents of a course of lectures given at the Tata Institute of Fundamental Research, Bombay in January/February 1964.

During the elaboration of the material here presented, I benefited from numerous discussions with various mathematicians, notably H. Cartan, G. Glaeser, L. Schwartz and R. Thom, and drew much profit from these discussions. In particular, I consider it my duty to state that one of the main results, "the preparation theorem for differentiable functions", was proposed to me as a conjecture by R. Thom, and that he had to make a great effort to overcome my initial scepticism.

I would like also to thank specially Raghavan Narasimhan and N. Venkateswara Rao, who wrote one part of these notes and translated the rest, and despite their own work, have undertaken all the work connected with the publication of this book.

B. Malgrange

Orsay, July 1965

I
WHITNEY'S EXTENSION THEOREM

1 Notations. 1

R denotes the set of real numbers, **N** denotes the set of natural numbers. For any open set Ω in $\mathbf{R}^n$, $\mathscr{E}^m(\Omega)$ (resp. $\mathscr{E}_c^m(\Omega)$) denotes the space of all C^m-real-valued functions in Ω (resp. with compact support in Ω). We omit m when $m = +\infty$. When $\Omega = \mathbf{R}^n$, we write $\mathscr{E}^m$, $\mathscr{E}_c^m$ instead of $\mathscr{E}^m(\mathbf{R}^n)$, $\mathscr{E}_c^m(\mathbf{R}^n)$. $k = (k_1, \ldots, k_n)$ in $\mathbf{N}^n$ is called an "n-integer". We write $|k| = k_1 + \cdots + k_n$ (order of k), $k! = (k_1!)\ldots(k_n!)$. We order $\mathbf{N}^n$ by the relation: "$k \leqslant l$ if and only if, for every j, $k_j \leqslant l_j$"; we write $\binom{l}{k} = \dfrac{l!}{k!(l-k)!}$ if $k \leqslant l$ and sometimes $\binom{l}{k} = 0$ if $k > l$. For $x \in \mathbf{R}^n$, $|x|$ denotes the euclidean norm of x.

Let K be a compact set in $\mathbf{R}^n$ and consider all $F = (f^k)_{|k| \leqslant m}$ where f^k are continuous functions on K. Any such F is called a *jet of order m*. Let $J^m(K)$ denote the space of all jets of order m provided with the natural structure of a vector space on **R**. We define $|F|_m^K = \sup\limits_{\substack{x \in K \\ |k| \leqslant m}} |f^k(x)|$; we write sometimes $|F|_m$ instead of $|F|_m^K$.

We write $F(x) = f^0(x)$ for all x in K, F in $J^m(K)$. For $|k| \leqslant m$, $D^k : J^m(K) \to J^{m-|k|}(K)$ is a linear map defined by $D^k F = (f^{k+l})_{|l| \leqslant m-|k|}$, and for any $g \in \mathscr{E}^m$, $J^m(g)$ denotes the jet $\left(\dfrac{\partial^k g}{\partial x^k}\right)_{|k| \leqslant m}$ in $J^m(K)$ where each $\dfrac{\partial^k g}{\partial x^k}$ is restricted to K. Now for $x \in \mathbf{R}^n$, $a \in K$, $F \in J^m(K)$, we define

$$T_a^m F(x) = \sum_{|k| \leqslant m} \frac{(x-a)^k}{k!} f^k(a).$$

We observe that for a fixed a in K and F in $J^m(K)$, $T_a^m F$ is a C^∞-function on $\mathbf{R}^n$. So we write $J^m(T_a^m F) = \widetilde{T}_a^m F$ and $R_a^m F = F - \widetilde{T}_a^m F$.

2 For $x \in X$, $y \in K$, one has obviously

$$T_x^m \circ \widetilde{T}_y^m = T_y^m, \text{ and then } \widetilde{T}_x^m \circ \widetilde{T}_y^m = \widetilde{T}_y^m, \tag{1.1}$$

$$R_x^m \circ R_y^m = R_x^m, \tag{1.2}$$

$$R_x^m \circ R_y^m = R_x^m, \tag{1.3}$$

$$\widetilde{T}_x^m \circ R_y^m = -\widetilde{T}_y^m \circ R_x^m = \widetilde{T}_x^m - \widetilde{T}_y^m = R_y^m - R_x^m, \tag{1.4}$$

$$D^k \circ \widetilde{T}_x^m = \widetilde{T}_x^{m-|k|} \circ D^k. \tag{1.5}$$

For any F in $J^m(K)$,

$$(R_x^m F)^k = f^k - T_x^{m-|k|} \circ D^k F. \tag{1.6}$$

From now on we omit the $\sim$ and we "identify" $T_a^m F$ and $J^m(T_a^m F)$.

2 Differentiable functions in the sense of Whitney.

Definition 2.1. *An increasing, continuous, concave function $\alpha : [0, \infty[\to [0, \infty[$ with $\alpha(0) = 0$ is called a modulus of continuity.*

Theorem 2.2. *The following statements are equivalent:*

(2.2.1) $(R_x^m F)^k(y) = o(|x-y|^{m-|k|})$ *for x, y in K and $|k| \leqslant m$, as* $|x-y| \to 0$.

(2.2.2) *There exists a modulus of continuity α such that $|(R_x^m F)^k(y)| \leqslant |x-y|^{m-|k|}\alpha(|x-y|)$ for x, y in K and $|k| \leqslant m$.*

(2.2.3) *There exists a modulus of continuity α_1 such that $|T_x^m F(z) - T_y^m F(z)| \leqslant \alpha_1(|x-y|)(|x-z|^m + |y-z|^m)$ for x, y in K, z in $\mathbf{R}^n$.*

In addition, if (2.2.2) *is true, we can choose $\alpha_1 = c\alpha$, c depending only on m and n. If* (2.2.3) *is true, we can choose $\alpha = c\alpha_1$, c depending only on m and n.*

Proof. Evidently (2.2.2) implies (2.2.1). Suppose that (2.2.1) is true. Then $\beta(t)$ defined by $\sup\limits_{\substack{x,y\in K\\ x\neq y\\ |x-y|\leqslant t\\ |k|\leqslant m}} \dfrac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}}$ is an increasing function, continuous at zero with $\beta(0) = 0$. Hence we can choose a modulus of continuity α such that $\alpha(t) \geqslant \beta(t)$ for all t (consider the convex enve- 3
lope of the positive t-axis and the graph of β). Therefore (2.2.1) implies (2.2.2). Note that α is constant for $t \geqslant \operatorname{diam} K$ and equal to $\beta(\operatorname{diam} K)$.

Now suppose that (2.2.2) is true. Using (1.5) and (1.6) we have

$$T_x^m F(z) - T_y^m F(z) = \sum_{|k|\leqslant m} \frac{(z-x)^k}{k!} (R_y^m F)^k(x).$$

Hence

$$\begin{aligned} |T_x^m F(z) - T_y^m F(z)| &\leqslant \sum_{|k|\leqslant m} \frac{|z-x|^{|k|}}{k!} |x-y|^{m-|k|} \alpha(|x-y|) \\ &\leqslant c\alpha(|x-y|)(|x-z|^m + |y-z|^m), \end{aligned}$$

where c depends only on m and n. Taking $\alpha_1 = c\alpha$, we see that (2.2.3) holds.

We shall now prove that (2.2.3) implies (2.2.2). Again by (1.6) we have for all z in $\mathbf{R}^n$, x, y in K

$$\left| \sum_{|k|\leqslant m} (R_y^m F)^k(x) \frac{(z-x)^k}{k!} \right| \leqslant \alpha_1(|x-y|)(|x-z|^m + |y-z|^m).$$

Writing $z - x = |x-y|(z'-x)$, $|x-y| = \lambda$, we have

$$\left| \sum_{|k|\leqslant m} \frac{\lambda^{|k|}}{k!} (z'-x)^k (R_y^m F)^k(x) \right| \leqslant c\alpha_1(|x-y|)\lambda^m(1 + |z'-x|^m)$$

where c is a constant depending only on m and n. Fixing x and y and treating sum on the left as polynomial in $z'-x$, (observing that the coefficients are determined linearly in terms of the values of the polynomial at a suitable finite system of points) we see that there exists a constant c_1 depending only on m and n such that

$$\left|\frac{\lambda^{|k|}}{k!}(R_y^m F)^k(x)\right| \leqslant c_1\alpha_1(|x-y|)\lambda^m$$

which implies (2.2.2).

The last assertion is evident from the proof. □

Definition 2.3. *$\mathscr{E}^m(K)$ is the space of all those F in $J^m(K)$ for which*
4 *any one of* (2.2.1), (2.2.2), (2.2.3) *is true. Each such F is called a*
Whitney function of class C^m on K. (They are not usual functions of
course, but this will not lead to any confusion.)

A modulus of continuity verifying (2.2.2) *is called a modulus of continuity for F.*

We define

$$||F||_m^K = |F|_m^K + \sup_{\substack{x,y\in K\\ x\neq y\\ |k|\leqslant m}} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-k}},$$

$$||F||_m'^K = |F|_m^K + \sup_{\substack{x,y\in K, x\neq y\\ |k|\leqslant m\\ z\in\mathbf{R}^n}} \frac{|T_x^m F(z) - T_y^m F(z)|}{|x-z|^m + |y-z|^m}.$$

(We usually omit the index K.) There exist constants c and c_1 depending only on m and n such that

$$||F||_m \leqslant c||F||_m' \leqslant c_1||F||_m.$$

(Proof is similar to the preceding one.)

Remark 2.4. We also note that we can choose α and α_1 in (2.2.2) and (2.2.3) such that

$$||F||_m = |F|_m + \alpha(\operatorname{diam} K), ||F||_m' = |F|_m + \alpha_1(\operatorname{diam} K).$$

The preceding norms being equivalent, we shall omit the prime and denote by $||F||_m$ one or the other. Under these norms, $\mathscr{E}^m(K)$ is a Banach space, the proof of which is left to the reader.

Remark 2.5. Let α denote a modulus of continuity for F. Then there exists a constant c depending only on m and n such that for any F in $J^m(K)$, x, y in K and $|k| \leqslant m$, we have

$$|D^k \circ T_x^m F(z) - D^k \circ T_y^m F(z)| \leqslant c\alpha(|x-y|)(|x-z|^{m-|k|} + |y-z|^{m-|k|}).$$

The proof is similar to the proof of the fact that (2.2.2) implies (2.2.3).

3 The extension theorem of Whitney.

We shall first prove an important

Lemma 3.1. *Given any compact set K in $\mathbf{R}^n$ there exists a family of functions $\phi_i (i \in I)$ each $\in \mathscr{E}(\mathbf{R}^n - K)$ satisfying the following properties:*

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(3.1.1) $0 \leqslant \phi_i$ for $i \in I$

(3.1.2) $\operatorname{supp} \phi_i (i \in I)$ *is a locally finite family and if $N(x)$ denotes the number of* $\operatorname{supp} \phi_i$ *to which x belongs, then* $N(x) \leqslant 4^n$

(3.1.3) $\sum_{i \in I} \phi_i(x) = 1$ *for all x in* $\mathbf{R}^n - K$*;*

(3.1.4) *for $i \in I$, one has* $2d(\operatorname{supp} \phi_i, K) \geqslant \operatorname{diam}(\operatorname{supp} \phi_i)$*;*

(3.1.5) *there exists a constant C_k depending only on k and n such that for $x \in \mathbf{R}^n - K$,*

$$|D^k \phi_i(x)| \leqslant C_k \left(1 + \frac{1}{d(x,K)^{|k|}}\right).$$

Proof. For $p \geqslant 0$, we divide $\mathbf{R}^n$ into closed cubes each with sides of length $\dfrac{1}{2^p}$ by the planes $x_v = \dfrac{j_v}{2^p}$ where $1 \leqslant v \leqslant n$ and $j_1, j_2, \ldots, j_n$ run through all integers.

Let K_0 be the family of all cubes S of the 0th division such that $d(S, K) \geqslant \sqrt{n}$. We inductively define $K_p (p > 0)$ to be the family of all cubes S of the pth division which are not contained in any cube of K_{p-1} and are such that $d(S, K) \geqslant \dfrac{\sqrt{n}}{2^p}$. Let $I = \bigcup\limits_{p \geqslant 0} K_p$. Let us consider a C^∞-function ψ such that $0 \leqslant \psi \leqslant 1$, $\psi(x) = 1$ if $|x_i| \leqslant \frac{1}{2}$ for $1 \leqslant i \leqslant n$; $\psi(x) = 0$ if $|x_i| \geqslant \frac{3}{4}$ for one i. For any $S \in I$, let ψ_S be defined as $\psi_S(x) = \psi\left(\dfrac{x - x_S}{l_S}\right)$ where x_S is the center, l_S is the length of a side of S. By the construction of I, $\operatorname{supp} \psi_S (S \in I)$ form a locally finite family. Therefore, we can define $\psi_S(x) = \dfrac{\psi_S(x)}{\sum\limits_{T \in I} \psi_T(x)}$. Now it is easy to verify that this family $\psi_s (S \in I)$ satisfies 1., 2. and 3.

As for 4., we observe that

$$d(\operatorname{supp} \psi_S, K) \geqslant \frac{3\sqrt{n}\, l_S}{4} \geqslant \frac{\operatorname{diam}(\operatorname{supp} \psi_S)}{2}$$

Now to prove 5. By definition

$$|D^k \psi_S(x)| = \left| \frac{1}{l_S^{|k|}} D^k \psi \left(\frac{x - x_S}{l_S} \right) \right| \leqslant \frac{C}{l_S^{|k|}}$$

6 where C is a constant depending only on k. But in view of 2., for x in $\mathbf{R}^n - K$,

$$1 \leqslant \sum_{S \in I} \psi_S(x) \leqslant 4^n.$$

Therefore we get, by Leibniz's formula, $|D^k \psi_S(x)| \leqslant \dfrac{C'}{l_S^{|k|}}$ where C' depends only on k and n. Therefore if $l_S = 1$, $|D^k \psi_S(x)| \leqslant C'$ And if $l_S < 1$, we find that for x in $\operatorname{supp} \psi_S$, $d(x, K) \geqslant \dfrac{\sqrt{n} l_S}{2}$ and so in any case for $x \in \mathbf{R}^n - K$,

$$\left|D^k \psi_S(x)\right| \leqslant C' \left(1 + \left(\frac{\sqrt{n}}{2d(x,K)}\right)^{|k|}\right).$$

□

Theorem 3.2 (Whitney [23].)**.** *There exists a linear mapping $W : \mathscr{E}^m(K) \to \mathscr{E}^m$ such that for every F in $\mathscr{E}^m(K)$ and every $x \in K$, $D^k W F(x) = f^k(x)$ for $|k| \leqslant m$.*

Proof. For every S in I, choose a point a_S in K such that $d(\operatorname{supp}\psi_S, K) = d(\operatorname{supp}\psi_S, a_S)$. Let $\tilde{f}$ be a function defined on $\mathbf{R}^n$ as follows

$$\tilde{f}(x) = f^0(x) \text{ for } x \in K;$$
$$\tilde{f}(x) = \sum_{S \in I} \psi_S(x) T^m_{a_S} F(x) \text{ if } x \text{ is not in } K.$$

Obviously $\tilde{f}$ is infinitely differentiable on $\mathbf{R}^n - K$. For $|k| \leqslant m$, define $\tilde{f}^k$ as follows

$$\tilde{f}^k(x) = f^k(x) \text{ if } x \text{ is in } K;$$
$$\tilde{f}^k(x) = D^k \tilde{f}(x) \text{ if } x \text{ is not in } K.$$

Let L be a cube such that $K \subset \overset{\circ}{L}$ and $\lambda = \sup_{x \in L} d(x,K)$. Then we prove the following, where α denotes a modules of continuity satisfying (2.2.3):

□

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(3.2.1) There exists a constant C depending only on m, n and λ such that for every k with $|k| \leqslant m$, for a in K, x in L, one has

$$|\tilde{f}^k(x) - D^k T^m_a F(x)| \leqslant C\alpha(|x-a|)|x-a|^{m-|k|}.$$

(3.2.2) For every k with $|k| > m$, there exists a constant C_k depending only on k, n and λ such that for x in $L - K$, one has:

$$|D^k f(x)| \leqslant \frac{C_k \alpha(d(x,K))}{d(x,K)^{|k|-m}}.$$

In fact one has, for every x in $L - K$,

$$\widetilde{f}(x) - T_a^m F(x) = \sum_{S \in I} \psi_S(x)(T_{aS}^m F(x) - T_a^m F(x)).$$

Hence applying Leibniz's formula, one has

$$D^k \widetilde{f}(x) - D^k T_a^m F(x) = \sum_{S \in I} \sum_{l \leqslant k} \binom{k}{l} D^l \psi_S(x) D^{k-l}[T_{aS}^m F(x) - T_a^m F(x)].$$

We consider those terms for which $l = 0$.

For $x \in \operatorname{supp} \psi_S$, obviously $d(x,K) \leqslant |x - a|$ and by (3.1.4), $d(x,K) \leqslant |x - a_S| \leqslant 3d(x,K)$ so that

$$\alpha(|a - a_S|) \leqslant \alpha(|x - a| + |x - a_S|) \leqslant \alpha(4|x - a|) \leqslant 4\alpha(|x - a|)$$

(due to the concavity of α). Now appealing to Remark 2.5 and Lemma 3.1, we obtain an estimate of the type (3.2.1). Now, if $l \neq 0$, the sum $\sum_{S \in I} D^l \psi_S(x) D^{k-l}[T_{aS}^m F(x) - T_a^m F(x)]$ is equal, since $\sum_{S \in I} D^l \psi_S(x) = 0$, to $\sum_{S \in I} D^l \psi_S(x) D^{k-l}[T_{aS}^m F(x) - T_b^m F(x)]$ for any $b \in K$. If b is so chosen that $|x - b| = d(x,K)$, we may argue as above and we obtain (3.2.1) for each such sum.

This gives us (3.2.1) when $x \notin K$. But for x in K, (3.2.1) results from the definition of $\mathscr{E}^m(K)$ and α.

To prove (3.2.2), proceed in the same way choosing a point a in K such that $|x - a| = d(x,K)$ and observing that

$$D^k T_a^m F(x) = 0, \text{ and } D^{k-l}[T_a^m F(x) - T_a^m F(x)] = 0$$

if $|k - l| > m$.

We assert now that $\widetilde{f}$ is of class C^m and that for $|k| \leqslant m$, $D^k\widetilde{f} = f^k$. 8
Then by defining $WF = f$, the theorem will be proved. For this, we proceed by induction and suppose that the result is true for all l with $l < k$. We can write $k = l + (j)$ where $|l| = |k| - 1$, $(j) = (0, \quad, 1, 0, \ldots, 0)$ with 1 in the jth place and all others equal to zero.

We then prove that one has for every a in K, $\dfrac{\partial \widetilde{f}^l}{\partial x_j}(x) = f^k(x)$ which will prove the result because one knows already that f is of class C^∞ outside K. For this, apply (3.2.1) replacing k by l. Retaining only in the first member, terms of degree 0 and 1 in $(x - a)$, it follows, for x in L,

$$\widetilde{f}^l(x) - f^l(a) - \sum_{i=1}^{n}(x_i - a_i)\widetilde{f}^{l+(i)}(a) = o(|x - a|)$$

which implies the desired result.

We shall now prove some complements to Whitney's theorem, due to Glaeser [4]. The $W : \mathscr{E}^m(K) \to \mathscr{E}^m$ which we have constructed induces in an obvious manner a map from $\mathscr{E}^m(K)$ to $\mathscr{E}^m(L)$. We denote by $\widetilde{F}$ the image of WF in $\mathscr{E}^m(L)$, and we denote by α a modulus of continuity of F.

Complement 3.3. *For $|k| > m$, x in $L - K$, there exists a constant C depending only on k, n and λ such that* $|D^k\widetilde{F}(x)| \leqslant \dfrac{C\alpha(d(x, K))}{d(x, K)^{|k|-m}}$.

This is a restatement of (3.2.2)

Remark 3.4. We observe that $|| \ ||_m$, $| \ |_m$ are equivalent on $\mathscr{E}^m(L)$. Also, let F be in $\mathscr{E}^m(L)$, and suppose that α is a modulus of continuity for every f^k, $|k| = m$ (i.e. $|f^k(x) - f^k(y)| \leqslant \alpha(|x - y|)$ for x, $y \in L$); then a constant (depending only on m and n) multiple of α is a modulus of continuity for F. These are easily verified by Taylor's formula. The same result is true for many other compact sets (for instance the convex ones), but it is not true in general.

Complement 3.5. *There exists a constant C depending only on m, n and λ such that* $||\widetilde{F}||_m^L \leqslant C||F||_m^K$ *for all F in* $\mathscr{E}^m(K)$.

9 *Proof.* In view of the preceding remarks, we need only prove this for $|\widetilde{F}|_m^L$ instead of $||\widetilde{F}||_m^L$. Take any x in L, a in K. By (3.2.1), we have for $|k| \leqslant m$,

$$\left| \widetilde{f}^k(x) - \sum_{|l| \leqslant m-|k|} \frac{(x-a)^l}{l!} \widetilde{f}^{k+l}(a) \right| \leqslant C\alpha(|x-a|)|x-a|^{m-|k|}$$

and hence it follows (Remark 2.4) that $|\widetilde{f}^k(x)| \leqslant C(\lambda, m, n)||F||_m$. □

Complement 3.6. *There exists a constant C depending only on m, n and λ such that if α is a modulus of continuity verifying* (2.2.3)*, then $C\alpha$ is a modulus of continuity for $\widetilde{F}$.*

Proof. In view of Remark 3.4, it suffices to find such a modulus of continuity for each $\widetilde{f}^k$, $|k| = m$. Let x, $y \in L$. If one of them belongs to K, then by (3.2.1) we see that $|\widetilde{f}^k(x) - \widetilde{f}^k(y)| \leqslant C\alpha(|x-y|)$.

Now suppose that x, y are in $L - K$.

Case (i). Suppose that $d(x, K) \geqslant 2|x-y|$. Using the mean-value theorem we have

$$\widetilde{f}^k(x) - \widetilde{f}^k(y) = \sum_{i=1}^{n} (x_i - y_i) \frac{\partial \widetilde{f}^k}{\partial x_i}(z)$$

where z belongs to the segment joining x and y. Hence in virtue of (3.2.2)

$$|\widetilde{f}^k(x) - \widetilde{f}^k(y)| \leqslant \frac{C\alpha(d(z,K))}{d(z,K)}|x-y|$$

where C depends only on m, n and λ. But by hypothesis $d(z, K) \geqslant |x-y|$ and so by the concavity of α,

$$\frac{\alpha(d(z,K))}{d(z,K)}|x-y| \leqslant \alpha(|x-y|).$$

Case (ii). Suppose that $d(x,K) < 2|x-y|$. Select a, b in K such that

$$|x-a| = d(x,K), \quad |y-b| = d(y,K).$$

Then we have

$$|y-b| \leqslant 3|x-y|, \quad |a-b| \leqslant 6|x-y|.$$

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Writing

$$\tilde{f}^k(x) - \tilde{f}^k(y) = \tilde{f}^k(x) - \tilde{f}^k(a) + \tilde{f}^k(a) - \tilde{f}^k(b) + \tilde{f}^k(b) - \tilde{f}^k(y)$$

and using (3.2.2), we obtain the result.

Note that the hypothesis of concavity on α is essential here. (For instance, one can find a compact connected K and a continuous non-constant function for K with $t^{3/2}$ as modulus of continuity; but, obviously, f cannot be extended to a cube with the same modulus of continuity for the extension! cf. Glaeser [4].) □

4 Whitney's theorem for the C^∞-case.

We adopt the same notations as before. Let $\mathscr{I}^m(K;L)$ denote the family of those jets of $\mathscr{E}^m(L)$, the restrictions of which to K are zero. Let $i_m \mathscr{I}^m(K;L) \to \mathscr{E}^m(L)$ be the canonical injection, $\psi_m : \mathscr{E}^m(L) \to \mathscr{E}^m(K)$ be the natural restriction map. Now Theorem 3.2 can be stated as

The sequence $0 \to \mathscr{I}^m(K;L) \xrightarrow{i_m} \mathscr{E}^m(L) \xrightarrow{\psi_m} \mathscr{E}^m(K) \to 0$ *is exact.*

Let $\eta_m J^{m+1}(K) \to J^m(K)$ be defined as $\eta_m(F) = (f^k)_{|k|\leqslant m}$. Obviously $\eta_m(\mathscr{E}^{m+1}(K)) \subset \mathscr{E}^m(K)$. Also if $\eta_m : J^{m+1}(L) \to J^m(L)$ is the analogue of the previous map for L, $\eta_m(\mathscr{I}^{m+1}(K;L)) \subset \mathscr{I}^m(K;L)$. Moreover η_m is injective on $\mathscr{E}^{m+1}(L)$.

We consider the projective limits, $\varprojlim J^m(K) = J(K)$, $\varprojlim \mathscr{E}^m(L) = \mathscr{E}(L)$, $\varprojlim \mathscr{E}^m(K) = \mathscr{E}(K)$, $\varprojlim \mathscr{I}^m(K;L) = \mathscr{I}(K;L)$. Elements of $J(K)$ are called C^∞-jets on K, and elements of $\mathscr{E}(K)$, C^∞ functions on K in the sense of Whitney. Obviously, $\mathscr{E}(L)$ can be identified with the usual C^∞-functions on the cube L. Defining $i = \varprojlim i_m$, $\psi = \varprojlim \psi_m$, we have the following theorem:

Theorem 4.1. *The following sequence*

$$0 \to \mathscr{I}(K;L) \xrightarrow{i} \mathscr{E}(L) \xrightarrow{\psi} \mathscr{E}(K) \to 0$$

is exact.

11 To prove this, we need the following

Lemma 4.2. *There exist constants $C_k \geqslant 0$, depending only on k in $\mathbf{N}^n$ with the following property:*

given K a compact subset of $\mathbf{R}^n$ and $\delta > 0$, there exists a C^∞-function α on $\mathbf{R}^n$ which verifies

(i) $\alpha = 0$ *on a neighbourhood of K, $\alpha(x) = 1$ when $d(x, K) \geqslant \delta$ and $\alpha \geqslant 0$ everywhere,*

(ii) *for every x in $\mathbf{R}^n$ and every k*

$$|D^k \alpha(x)| \leqslant \frac{C_k}{\delta^{|k|}}.$$

Proof. Consider a non-negative function ϕ in $\mathscr{E}(\mathbf{R}^n)$, with $\phi = 1$ if $|x| \leqslant \frac{1}{4}$, $\psi = 0$ if $|x| \geqslant \frac{3}{8}$, $\int \phi = 1$ and define $\phi_\delta(x) = \dfrac{1}{\delta^n}\phi\left(\dfrac{x}{\delta}\right)$. Take now the characteristic function α' of the set $\left\{x | d(x, K) \geqslant \dfrac{\delta}{2}\right\}$ and set $\alpha = \alpha' \, \phi_\delta$. □

Lemma 4.3. *$\mathscr{I}(K;L)$ is dense in $\mathscr{I}^m(K;L)$.*

Actually, we prove that those F in $\mathscr{I}(K;L)$ which vanish in a neighbourhood of K are dense in $\mathscr{I}^m(K;L)$.

Take any F in $\mathscr{I}^m(K;L)$ and $K_1 = \{x | d(x, K) \leqslant \delta\}$. Consider the function α (depending on K and δ) obtained in the preceding lemma. Denote $F\alpha$ by F_δ. One verifies immediately by Leibniz's formula that F_δ tends to F in $\mathscr{E}^m(L)$ as $\delta \to 0$. Therefore the class of functions in $\mathscr{E}^m(L)$ which vanish in a neighbourhood of K is dense in $\mathscr{I}^m(K;L)$. (This can also be proved using Whitney's theorem!) The result follows immediately by regularization.

Proof of Theorem 4.1. Let F be in $\mathscr{E}(K)$ and for $m \geqslant 0$, F_m denote F as an element of $\mathscr{E}^m(K)$. Let $\widetilde{F}_m = WF_m$ be a Whitney extension of F_m to L. Since $\widetilde{F}_m - \widetilde{F}_{m-1}$ is in $\mathscr{I}^{m-1}(K;L)$, by the lemma, there exists a H_{m-1} in $\mathscr{I}(K;L)$ such that

$$||\widetilde{F}_m - \widetilde{F}_{m-1} - H_{m-1}||_{m-1} \leqslant 1/2^m.$$

Now let us consider **12**

$$\widetilde{F}(x) = \widetilde{F}_0(x) + \sum_{m \geqslant 1} \left\{ \widetilde{F}_m(x) - \widetilde{F}_{m-1}(x) - H_{m-1}(x) \right\}.$$

It is easy to verify that $\widetilde{F}$ is a C^∞-function and the C^∞-jet induced by it, when restricted to K, is F.

5 Regularly situated sets.

Let X be a closed subset of an open set Ω in $\mathbf{R}^n \cdot J^m(\Omega)$, $J^m(X)$, $J(\Omega)$, $J(X)$ are defined in the usual way. Define $\mathscr{E}^m(X)$ to be the family of all jets F in $J^m(X)$ such that given any compact set K in X, the restriction $F|K$ of F to K is in $\mathscr{E}^m(K)$. Define

$$||F||_m^K = ||F|K||_m^K.$$

This $||F||_m^K$ is a seminorm on $\mathscr{E}^m(X)$ for all K in X. Now provide $\mathscr{E}^m(X)$ with the topology defined by the totality of these seminorms. Obviously $\mathscr{E}^m(X)$ is a Fréchet space.

$\mathscr{E}(X)$ is defined to be the family of all jets F in $J(X)$ such that given any compact set K in X, $F|K$ is in $\mathscr{E}(K)$. For all $m \geqslant 0$ and all K in X, $||F||_m^K$ defines a seminorm on $\mathscr{E}(X)$. $\mathscr{E}(X)$ provided with the topology defined by the totality of these seminorms is a Fréchet space.

When $X = \Omega$, $\mathscr{E}^m(\Omega)$ (resp. $\mathscr{E}(\Omega)$) defined in the preceding way is identified with the space of usual C^m (resp. C^∞) functions on Ω.

Definition 5.1. *Let X, Y be closed subsets of an open set Ω such that $X \subset Y \cdot \mathscr{I}^m(X;Y)$ is defined to be the class of all jets F in $\mathscr{E}^m(Y)$ the restrictions of which to X are zero.*

When $m = \infty$, we shall denote this space by $\mathscr{I}(X;Y)$.

In order to avoid possible confusion, we refer to an element of $\mathscr{I}^m(X;Y)$ (resp. $\mathscr{I}(X;Y)$) as a Whitney function of order m (resp. of infinite order) on Y *m-flat on X* (resp. *flat on X*).

Proposition 5.2. *Let X, Y be closed subsets of an open set Ω such that $X \subset Y$. Then $\mathscr{I}(X;Y)$ is dense in $\mathscr{I}^m(X;Y)$.*

Actually those $F \in \mathscr{I}(X;Y)$ which are flat in a neighbourhood of X are dense in $\mathscr{I}^m(X;Y)$.

13 *Proof.* Evident by the statement made in Lemma 4.3. □

Proposition 5.3. *The sequences*

$$0 \longrightarrow \mathscr{I}^m(X;\Omega) \xrightarrow{i_m} \mathscr{E}^m(\Omega) \xrightarrow{\psi_m} \mathscr{E}^m(X) \longrightarrow 0,$$

$$0 \longrightarrow \mathscr{I}(X;\Omega) \xrightarrow{i} \mathscr{E}(\Omega) \xrightarrow{\psi} \mathscr{E}(X) \longrightarrow 0$$

are exact.

(i_m and i are the canonical injections and ψ_m and ψ are the canonical restriction maps.)

Proof. Immediate by a partition of unity. □

Now, let X, Y be closed subsets of Ω, an open set in $\mathbf{R}^n$. Let δ be the 'diagonal mapping'

$$\mathscr{E}(X \cup Y) \to \mathscr{E}(X) \oplus \mathscr{E}(Y)$$

defined by

$$\delta(F) = (F|X, F|Y),$$

and π the mapping $\mathscr{E}(X) \oplus \mathscr{E}(Y) \to \mathscr{E}(X \cap Y)$ defined by

$$\pi(F,G) = (F|X \cap Y) - (G|X \cap Y).$$

Obviously δ is injective, π is surjective $\pi \circ \delta = 0$; moreover $\operatorname{im}\delta$ is dense in $\ker\pi$.

For, let (F, G) be in $\ker \pi$. We can suppose that $G = 0$. (If not, extend G to $\widetilde{G}$ in $\mathscr{E}(X \cup Y)$ by Proposition **??** and take $(F, G) - \delta\widetilde{G}$ instead of (F, G).) Therefore one has $F|X \cap Y = 0$, i.e. F is in $\mathscr{I}(X \cap Y; X)$. Then by Proposition 5.3, F is the limit of a sequence $\{F_m\}$ flat on a neighbourhood of $X \cap Y$ and one has, obviously, $(F_m, 0) \in \operatorname{im} \delta$ which proves the result.

Definition 5.4. *Two closed subsets X, Y of an open set Ω are said to be regularly situated if* $\operatorname{im} \delta = \ker \pi$ *or equivalently if the sequence*

$$0 \to \mathscr{E}(X \cup Y) \xrightarrow{\delta} \mathscr{E}(X) \oplus \mathscr{E}(Y) \xrightarrow{\pi} \mathscr{E}(X \cap Y) \to 0$$

is exact.

Theorem 5.5 (Łojasiewicz [10])**.** *Given X, Y closed in an open set Ω a* 14
necessary and sufficient condition that they are regularly situated is the following : Either $X \cap Y = \varnothing$ or

(Λ) *Given any pair of compact sets $K \subset X$, $L \subset Y$, there exists a pair of constants $C > 0$ and $\alpha > 0$ such that, for every x in K, one has $d(x, L) \geqslant C\, d(x, X \cap Y)^{\alpha}$, ($d$ denoting the euclidean distance in $\mathbf{R}^n$).*

Direct verification of the fact that the condition is symmetric with respect to X and Y is left to the reader.

Proof. (a) $(\Lambda) \Rightarrow$ "$\ker \pi = \operatorname{im} \delta$" Let $f = (f^k)$ (resp. $g = g^k$) be an element of $\mathscr{E}(X)$ (resp. $\mathscr{E}(Y)$); suppose that $f = g$ on $X \cap Y$. Define $h = (h^k)$ in $J(X \cup Y)$ by $h^k = f^k$ on X and $h^k = g^k$ on Y and it is sufficient to prove that $h \in \mathscr{E}(X \cup Y)$.

Let M be a compact subset of $X \cup Y$ and set $X \cap M = K$, $Y \cap M = L$. We should verify that for every m in $\mathbf{N}$ and every k in $\mathbf{N}^n$, there exists a $c' > 0$ such that for every x in M and every y in M:

$$\left| h^k(x) - \sum_{|l| \leqslant m} h^{k+1}(y) \frac{(x-y)^l}{l!} \right| \leqslant c' ||x - y||^m.$$

The case when x and y both belong to X, or both belong to Y, is immediate. Therefore, suppose that, for instance $x \in X$, $y \in Y$. If we extend g to $\widetilde{g} \in \mathscr{E}(X \cup Y)$, and replace f by $f - \widetilde{g}$, it reduces to the case when $g = 0$ and consequently $f|X \cap Y = 0$; in this case our inequality can be written simply as

$$|f^k(x)| \leqslant c'||x - y||^m.$$

By hypothesis, we can find a $z \in X \cap Y$ such that one has $||x-y|| \geqslant \frac{c}{2}||x - z||^\alpha$; we can suppose that when x varies in K, and y in L, z varies in a compact subset of $X \cap Y$. Let m' be an integer such that $\alpha m \leqslant m'$; as $||x-z||$ is bounded, one has $||x-z||^{m'} \leqslant c''||x-y||^m$, with $c'' \geqslant 0$. Since $f \in \mathscr{E}(X)$, we have

$$\left| f^k(x) - \sum_{|l| \leqslant m'} f^{k+l}(z) \frac{(x-z)^l}{l!} \right| \leqslant c_1 ||x - z||^{m'}$$

15 therefore, since $f = 0$ on $X \cap Y$,

$$|f^k(x)| \leqslant c_1||x - z||^{m'} \leqslant c'' c_1 ||x - y||^{m'},$$

which gives the result.

(b) "$\ker \pi = \operatorname{im} \delta$" $\Rightarrow (\Lambda)$.

By hypothesis, the image of δ is closed and therefore δ is a homomorphism; let M be a compact subset of $X \cup Y$. For every f in $\mathscr{E}(X \cup Y)$, there exists in particular a seminorm p on $\mathscr{E}(X)$ and a seminor q on $\mathscr{E}(Y)$ such that one has, for every x in M and every y in M

$$\left| f^0(x) - f^0(y) - \sum_{i=1}^{n} (x_i - y_i) f^{(i)}(y) \right| \leqslant (p(f) + q(f))||x - y||.$$

In particular, if f is zero on Y, setting $X \cap M = K$, $Y \cap M = L$, this means that for every x in K, $|f(x)| \leqslant p(f)d(x, L)$.

"Lifting" the preceding inequality to $\mathscr{E}(\Omega)$ one finds the following: there exists a compact $N \subset \Omega$, an integer $m \geqslant 0$ and a $C > 0$ such that, for every F in $\mathscr{E}(\Omega)$, flat on Y and every $x \in K$ one has

$$|F(x)| \leqslant C|F|_m^N \cdot d(x, L).$$

Let ϕ be a C^∞-function with support in the unit ball, with $\phi(0) = 1$. For any $x_0 \in K$, apply the preceding inequality to the function $x \to \phi\left(\dfrac{x - x_0}{\epsilon}\right)$ with $\epsilon = d(x_0, X \cap Y)$. That gives

$$1 \leqslant \frac{C}{\epsilon^m} d(x, L)|\phi|_m^N$$

which proves the result at once.

□

Remark 5.6. Naturally we ask whether an analogue of Theorem 5.5 is valid for $\mathscr{E}^m(X)$, $\mathscr{E}^m(Y)$, $\mathscr{E}^m(X \cup Y)$, i.e. whether the sequence

$$0 \to \mathscr{E}^m(X \cup Y) \to \mathscr{E}^m(X) \oplus \mathscr{E}^m(Y) \to \mathscr{E}^m(X \cap Y) \to 0$$

is exact for finite m. The results are the following:

If $m = 0$, the sequence is always exact (trivial).

If $0 < m$, X and Y are "m-regularly situated" if the condition (Λ) is 16
replaced by an analogous one with $\alpha = 1$.

(Proof similar to the preceding one.) This is of course much more restrictive than (Λ).

6 A theorem of composition*.

Let $\mathscr{O}$ be an open set $\subset \mathbf{R}^m$, Ω an open set $\subset \mathbf{R}^n$, and $g : \mathscr{O} \to \Omega$, $f : \Omega \to \mathbf{R}$ two mappings of class C^s. Set $A^0 = \Omega$ and $A^r = \{x \in \Omega | D^k f(x) = 0 \text{ for } 1 \leqslant |k| \leqslant r\}$ $(1 \leqslant r \leqslant s)$.

*The results of this section and the next will not be used in the rest of this book, except in Chap. V, §5, (iii).

The theorem of composition which we have in mind is based on the following remark.

Let $y \in \mathscr{O}$ be such that $x = g(y) \in A^r$. The derivatives (of order $\leqslant s$) of $f \circ g$ at y do not depend on the derivatives of g of order $> s - r$.

In fact, let $y \in \mathscr{O}$ and $x = g(y)$. The formula for the differentiation of composite functions can be written

$$T_y^s(f \circ g) = T_x^s f(T_y^s g(z)) \text{mod } (z-y)^{s+1};$$

this identity means that the two expressions above, considered as polynomials in z, are congruent modulo the ideal generated by the $(z-y)^l$, $|l| = s+1$. This formula can also be written in the form $T_y^s(f \circ g)(z) =$

$$f(x) + \sum_{|k|=1}^{s} \frac{D^k f(x)}{k!} \left[\sum_{|l|=1}^{s} \frac{D^l g(y)}{l!}(z-y)^l \right]^k \text{mod } (z-y)^{s+1}.$$

For a fixed $k = (k_1, \ldots, k_n)$, the bracket occuring on the right can be written

$$\frac{1}{l^1! \ldots l^n!}[D^{l^1} g_1(y)]^{k_1} \quad [D^{l^n} g_n(y)]^{k_n}(z_1 - y_1)^{k_1 l^1} \quad (z_n - y_n)^{k_n l^n},$$

where $g = (g_1, \ldots, g_n)$ and the sum is over systems $(l^1, \ldots, l^n)$ for which $l^i \in \mathbf{N}^n$, $|l^i| \geqslant 1$ for each i; only those (l^i) for which $k_1 l^1 + \cdots + k_n l^n \leqslant s$ occur here. However, if $|k| \geqslant r$, these conditions imply that $|l^i| \leqslant s - r$ for each i. Hence, for $x \in A^r$, we have

$$T_y^s(f \circ g)(z) = T_x^s f[T_y^{s-r} g(z)] \text{mod } (z-y)^{s+1},$$

17 and the result follows.

Let Ω, f, A^i be as above. Let r be an integer with $1 \leqslant r \leqslant s$, and let K be a compact subset of $\mathbf{R}^m$ and G a system of n elements of $\mathscr{E}^{s-r}(K)$ which we consider as a "function" with values in $\mathbf{R}^n$, and suppose that $g_0(K) = G(K) \subset A^r$. The calculations made above lead us to define a jet $\in J^s(X)$, which we denote $f \circ G$, by means of the formula

$$T_y^s(f \circ G)(z) = T_x^s f(T_y^{s-r} G(z)) \text{mod } (z-y)^{s+1},$$

where $y \in K$, $x = G(y) \in A^r$.

Theorem 6.1 (M. Kneser [9], see also G. Glaeser [4].)**.** *For* $1 \leqslant r < s$, *we have* $f \circ G \in \mathscr{E}^s(K)$.

Proof. We have to prove that for a certain modulus of continuity α, we have

$$|T^s_{y_1}(f \circ G)(z) - T^s_{y_2}(f \circ G)(z)| \leqslant \alpha(|y_1 - y_2|)\times \\ \times (|y_1 - z|^s + |y_2 - z|^s) \tag{6.2}$$

for $y_1 \in K$, $y_2 \in K$, $z \in \mathbf{R}^m$. Let B be an open ball in $\mathbf{R}^m$ for which $K \subset B$. It is enough to prove this estimate for $z \in B$ (to see this, one has only to repeat the argument given for the implication (2.2.3) $\Rightarrow$ (2.2.2), using (2.2.3) only for $z \in B$).

(i) Let us first show that it is sufficient to establish the following formula:

$$|T^s_{x_1} f(T^{s-r}_{y_1} G(z)) - T^s_{x_2} f(T^{s-r}_{y_2} G(z))| \leqslant \alpha(|y - y_2|)\times \\ \times (|y_1 - z|^s + |y_2 - z|^s) \tag{6.3}$$

for y_1, $y_2 \in K$, $x_1 = G(y_1)$, $x_2 = G(y_2)$, $z \in B$ and a suitable modulus α of continuity. For this purpose, it suffices to verify that the terms on the left in (6.2) and (6.3) differ only by terms satisfying the required inequality. Now, this difference is a sum of terms of the form

$$h(y_1)(z - y_1)^k - h(y_2)(z - y_2)^k$$

with a continuous h, and $|k| > s$. If we write this in the form

$$\{h(y_1) - h(y_2)\}(z - y_1)^k + h(y_2)\{(z - y_1)^k - (z - y_2)^k\}$$

and mojorise the two terms in the obvious way, we obtain the required result (note that the restriction $z \in B$ is essential since we have $|k| > s$). **18**

(ii) Let us write the term on the left in (6.3) in the form

$$\{T^s_{x_1} f(T^{s-r}_{y_1} G(z)) - T^s_{x_1} f(T^{s-r}_{y_2} G(z))\} + \\ + \{T^s_{x_1} f(T^{s-r}_{y_2} G(z)) - T^s_{x_2} f(T^{s-r}_{y_2} G(z))\}. \quad (6.4)$$

The second term is majorised, in absolute value, by

$$\alpha(|x_1 - x_2|)(|x_1 - x_2|^s + |T^{s-r}_{y_2} G(z) - x_2|^s).$$

Since $r < s$, we have $|x_1 - x_2| = |G(y_1) - G(y_2)| \leqslant C|y_1 - y_2|$, $(y_1, y_2 \in K)$ and

$$|T^{s-r}_{y_2} G(z) - x_2| \leqslant C|z - y_2|(x \in B, y_2 \in K), \quad (6.5)$$

which gives us the required estimate for the second term.

(iii) It remains to majorise the first term in (6.4). Let us put $T^{s-r}_{y_i} G(z) = u_i (i = 1, 2)$. We have

$$T^s_{x_1} f(u_1) - T^s_{x_1} f(u_2) = \sum_{1 \leqslant |k| \leqslant s} \frac{1}{k!} D^k_{u_1} T^s_x f(u_1)(u_2 - u_1)^k, \quad (6.6)$$

and

$$|u_2 - u_1| \leqslant \alpha(|y_2 - y_1|)(|z - y_1|^{s-r} + |z - y_2|^{s-r})(y_1, y_2 \in K, z \in \mathbf{R}^m); \quad (6.7)$$

here α is a suitable modulus of continuity.

The right hand side of (6.6) is estimated as follows.

If $1 \leqslant |k| \leqslant r$, in $D^k_{u_1} T^s_{x_1} f(u_1) = T^{s-|k|}_{x_1} D^k f(u_1)$, the terms containing $(u_1 - x_1)^l$ are zero if $|l| \leqslant r - |k|$. Hence, if $y_1 \in K, z \in B$, we have

$$|D^k_{u_1} T^s_{x_2} f(u_1)| \leqslant C|x_1 - u|^{r-|k|+1}.$$

Using (6.5) and (6.7) we easily obtain the required estimate from this. Finally, if $|k| > r$, we have $|u_2 - u_1|^k \leqslant C|u_2 - u_1|^{r+1}$, which, with (6.7) and the obvious inequality $(s - r)(r + 1) \geqslant s$, gives us the required result.

□

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7 The theorem of Sard.

Let Ω be an open set in $\mathbf{R}^n$ and f a mapping $\Omega \to \mathbf{R}^p$ of class C^s, $s \geqslant 1$. As in §6, let us set $A^0 = \Omega$ and

$$A^r = \{x \in \Omega | D^k f(x) = 0 \text{ for } 1 \leqslant |k| \leqslant r\}, \text{ where } 1 \leqslant r \leqslant s.$$

Lemma 7.1. *For $r \geqslant \frac{n}{p}$, $f(A^r)$ has measure zero.*

Proof. Let K be a closed cube $\subset \Omega$. It is obviously sufficient to prove that $f(A^r \cap K)$ has measure zero.

There exists a modulus α of continuity such that for $x \in K \cap A^r$ and any $y \in K$ we have $|f(x) - f(y)| \leqslant |x-y|^r \alpha(|x-y|)$. Let l be the length of the edge of K. Let us divide K into N^n equal cubes K_i, $1 \leqslant i \leqslant N^n$. Let J be the set of indices i for which K_i meets A^r. If $x, y \in K_j$, $j \in J$, we have $|f(x) - f(y)| \leqslant C \left(\frac{l}{N}\right)^r \alpha \left(\frac{l}{N}\right)$ with $C = 2(\sqrt{n})^{r+1}$. The volume V_j of $f(K_j)$ is therefore at most $N^n C' \left(\frac{l}{N}\right)^{pr} \alpha \left(\frac{l}{N}\right)^p$ C' being a constant depending only on n and p. The volume of $f(K \cap A^r)$ is therefore at most $C' N^n \left(\frac{l}{N}\right)^{pr} \alpha \left(\frac{l}{N}\right)^p$. Choosing N large enough and using our hypothesis that $n - pr \leqslant 0$ we obtain the required result. □

Remark 7.2. If $n < p$, the same argument shows that $f(\Omega)$ has measure zero if f is of class C^1.

Lemma 7.3. *If $s \geqslant \frac{n}{p}$, $f(A^1)$ has measure zero.*

Proof. By Lemma **??**, the result is true for $p = n$. Keeping p fixed, we shall use induction on n. Suppose therefore that the lemma has already been established for $n - 1$. We shall show that for $1 \leqslant r < s$, $f(A^r - A^{r+1})$ has measure zero, which implies our result, since, by the preceding lemma, $f(A^s)$ has measure zero.

Let $x \in A^r \ldots A^{r+1}$ and set $f = (f_1, \ldots, f_p)$. There exists i, $1 \leqslant i \leqslant n$, j, $1 \leqslant j \leqslant p$, and $k \in \mathbf{N}^n$, $|k| = r$, such that $\frac{\partial}{\partial x_i} D^k f_j(x) \neq 0$.

Near x, the set of points of Ω which satisfy $D^k f_j(x) = 0$ is therefore 20
a submanifold of class C^{s-r}. Hence there is an open set $U \subset \Omega$ with $x \in U$, an open set $\mathscr{O} \subset \mathbf{R}^{n-1}$ and a proper mapping $g : \mathscr{O} \to U$ of class C^{s-r} such that $U \cap A^r \subset g(\mathscr{O})$.

Let K be a compact set in U with $x \in \overset{\circ}{K}$. Set $L = g^{-1}(K)$ and $B^r = L \cap g^{-1}(A^r)$. By Theorem 6.1 and the extension theorem, there exists $h : \mathscr{O} \to \mathbf{R}^p$ of class C^s coinciding with $f \circ g$ on B^r and satisfying $D^k h(y) = 0$ for $y \in B^r$, $1 \leqslant |k| \leqslant r$.

By induction, $f \circ g(B^r)$, hence $f(A^r \cap K)$, is of measure zero. Since $A^r - A^{r+1}$ is a countable union of compact sets, $f(A^r - A^{r+1})$ has measure zero and the lemma follows.

The above lemma is due to A. P. Morse [15] (at least if $p = 1$). The method used is due to M. Kneser [9]. □

Theorem 7.4 (Sard [17].). *Let K be the set of critical points of f (i.e. the set of points where the differential map f' has rank $< p$). If $s \geqslant$* $\max(1, n-p+1)$, *then $f(K)$ has measure zero.*

Proof. For $n < p$, this follows from Remark 7.2. Suppose therefore that $n \geqslant p$. For $0 \leqslant r < p$, let K^r- be the set of points of Ω where f' has rank r, and let $a \in K^r$. We shall show that there exists a neighbourhood U of a such that $f(U \cap K^{r-})$ has measure zero. Since K^r is locally closed in Ω, hence a countable union of compact subsets of Ω, the theorem follows from this.

We can find a neighbourhood U of a, a neighbourhood V of $f(a)$ and changes of coordinates of class C^s in U and V such that, in the new coordinates, f is given by the system of equations

$$
\begin{aligned}
y_i &= x_i, 1 \leqslant i \leqslant r, \\
y_i &= f_i(x_1, \ldots, x_n), r+1 \leqslant i \leqslant p.
\end{aligned}
$$

The f_i are of class C^s and for a point $(x_1, \ldots, x_n) \in U$ to belong to K^r, it is necessary and sufficient that $\dfrac{\partial f_i}{\partial x_j}(x_1, \ldots, x_n) = 0$ for $i \geqslant r+1$, $j \geqslant r+1$.

Let $E(x_1, \ldots, x_r)$ (resp. $F(x_1, \ldots, x_r)$) be the set of points of U 21
(resp. V) whose first r coordinates are $x_1, \ldots, x_r$. From the inequality $s \geqslant n - p + 1$, we deduce that $s \geqslant \dfrac{n-r}{p-r}$; fixing $x_1, \ldots, x_r$ and applying (7.3) to $(f_{r+1}, \ldots, f_p)$ considered as a function of $(x_{r+1}, \ldots, x_n)$, we find that $f(K^r \cap E(x_1, \ldots, x_r))$ has measure zero in $F(x_1, \ldots, x_r)$. By the Lebesgue-Fubini theorem, $f(U \cap K^r)$ has measure zero, and the theorem is established. □

II
CLOSED IDEALS

22

1 Jets of vector-valued functions.

Let L be a closed cube in $\mathbf{R}^n$, K a closed subset of L, E a finite dimensional vector space over $\mathbf{R}$. Until now we considered jets $(f^k)_{|k|\leqslant m}$ where f^k are real-valued functions but we can also consider vector-valued functions with values in E. The spaces $J^m(K,E)$, $\mathscr{E}^m(K,E)$, $\mathscr{I}^m(K;L,E)$, $J(K,E)$, $\mathscr{E}(K,E)$, $\mathscr{I}(K;L,E)$ are all defined in the obvious way. The results of Chapter I hold for these spaces with the obvious modifications. Also it is claar that there is an identification of $\mathscr{E}^m(L,E)$ with the product space $(\mathscr{E}^m(L))^r$ where r is the rank of E over $\mathbf{R}$. So naturally we provide $\mathscr{E}^m(L,E)$ with the product topology and also the structure of an $\mathscr{E}^m(L)$-module. In the sequel, all the modules considered are $\mathscr{E}^m(L)$-modules.

Definition 1.1. *For $a \in L$, we denote by T_a^m the natural mapping*

$$\mathscr{E}^m(L,E) \to \mathscr{E}^m(L,E)/\mathscr{I}^m(\{a\},L,E).$$

Obviously, the image of an $f \in \mathscr{E}^m(L,E)$ under this mapping can be identified with the Taylor expansion of order m of f at a, which explains the notation.

For any sub-module M of $\mathscr{E}^m(L,E)$, $T_a^m M$ is a sub-module of

$$\mathscr{E}^m(L,E)/\mathscr{I}^m(\{a\},L,E);$$

and as a vector space over $\mathbf{R}$, it has finite rank because the latter does.

Definition 1.2. *An f in $\mathscr{E}^m(L,E)$ is said to be pointwise in a sub-module M of $\mathscr{E}^m(L,E)$ if $T_a^m f \in T_a^m M$ for all a in L.*

Theorem 1.3 (Whitney [24].). *If M is a sub-module of $\mathscr{E}^m(L,E)$, $\overline{M}$ is the closure of M in $\mathscr{E}^m(L,E)$, and $\widehat{M}$ is the module of all functions f pointwise in M, then $\widehat{M} = \overline{M}$.*

Lemma 1.4. *Let K be a compact subset of L such that for all $a \in K$, the rank of $T_a^m M$ over $\mathbf{R} = p$, a constant. Let $F \in M$. Then given any*
23 *$\epsilon > 0$, we can find $\phi \in \mathscr{E}^m(L)$; $\phi = 1$ in a neighbourhood of K and $f \in M$ such that $|\phi F - f|_m < \epsilon$.*

Here, $||_m$ stands for $||_m^L$ which is defined in the same way as in Chapter I, §1 if we have chosen a norm on E. Also we observe that Chapter I, (2.2) holds even for finite dimensional vector-valued jets and we define α to be a modulus of continuity for F, if it is a modulus of continuity and it verifies Chapter I, (2.2.3).

Proof of the Lemma. Let $a \in K$. By hypothesis there exists a neighbourhood V_a of a and $f_1, f_2, \ldots, f_p$ in M such that for x in $V_a \cap K$, $T_x^m f_1$, $T_x^m f_2, \ldots, T_x^m f_p$ is a basis of $T^m M$ over $\mathbf{R}$. Hence there exist continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_p$ on $V_a \cap K$ such that

$$T_x^m F = \sum_{i=1}^{p} \lambda_i(x) T_x^m f_i \text{ for all } x \in V_a \cap K.$$

Using a partition of unity we can find $f_1, f_2, \ldots, f_s \in M$, functions $\lambda_1, \lambda_2, \ldots, \lambda_s$ on L and a constant C such that for all $x \in K$,

$$T_x^m F = \sum_{i=1}^{s} \lambda_i(x) T_x^m f_i$$

and

$$\sup_{\substack{1 \leqslant i \leqslant s \\ x \in L}} |\lambda_i(x)| \leqslant C.$$

Let α be a modulus of continuity for $F, f_1, f_2, \ldots, f_s$. Define for any $a \in K$, $x \in L$, $f_a(x) = \sum_{i=1}^{s} \lambda_i(a) f_i(x)$. Obviously

$$T_a^m F(z) = T_a^m f_a(z).$$

Therefore for $a \in K$, $x \in L$, $z \in \mathbf{R}^n$,

$$|T_x^m F(z) - T_x^m f_a(z)| \leqslant |T_x^m F(z) - T_a^m F(z)| + |T_a^m f_z(z) - T_x^m f_z(z)|$$

$$\leqslant C'(|z-x|^m + |z-a|^m)\alpha(|x-a|), \qquad (1.4.2)$$

where C' is independent of a, x, z. Hence again using the same argument as in proving in Chapter I that (2.2.3) $\Rightarrow$ (2.2.2), we see that there exists a constant C'' not depending on a, x, z, such that

$$|D^k F(x) - D^k f_a(x)| \leqslant C''|x-a|^{m-|k|}\alpha(|x-a|). \qquad (1.4.2)$$

24 Let us divide $\mathbf{R}^n$ into cubes each of side d and for each such cube, consider the open cube of side $2d$ with the same centre and let I denote the family of these cubes. By a construction similar to Lemma 3.1 of Chapter I, (and even simpler), we obtain a partition of unity $\phi_i (i \in I)$ subordinate to I such that for $|k| \leqslant m$,

$$\sum_{i \in I} |D^k \phi_i(x)| \leqslant \frac{C}{d^{|k|}}, \qquad (1.4.3)$$

where C is a constant depending only on m and n. Let I' be the family of those S in I which meet K and for each such S, let a_S be a point in $S \cap K$. I' is a finite set. Define

$$\phi = \sum_{S \in I'} \phi_S, \; f = \sum_{S \in I'} \phi_S f_{a_S}.$$

Obviously $\phi(x) = 1$ in a neighbourhood of K and

$$\begin{aligned} |\phi F - f|_m &= \sum_{|k| \leqslant m} \sup_{x \in L} |D^k(\phi F - f)(x)| \\ &\leqslant \sum_{|k| \leqslant m} \sum_{S \in I'} \sup_{x \in L} |D^k(\phi_S F - \phi_S f_{a_S})(x)| \end{aligned}$$

and so by Leibniz's formula and (1.4.2), (1.4.3), it follows that

$$|\phi F - f|_m \leqslant C''' \alpha(d)$$

where C''' is independent of d.

Hence if we choose d sufficiently small the lemma follows.

Proof of Theorem 1.3. Let $B_p = \{x \in L|$ rank of $T_a^m M \leqslant p\}$. Let $A_p = B_p - B_{p-1}$ for $p \geqslant 0$. Let us make the statement H_p: Given any $F \in \widehat{M}$ and $\epsilon > 0$, there exists a function ϕ in $\mathscr{E}^m(L)$, f in M such that $\phi(x) = 1$ in a neighbourhood of B_p and $|\phi F - f|_m \leqslant \epsilon$.

H_0 is true because of Lemma 1.4 and the fact that $B_0 = A_0$ is closed. So, let us suppose that H_{p-1} is true for some $p \geqslant 1$. Therefore given any $\epsilon > 0$, $F \in M$, there exists functions $\phi_{p-1} \in \mathscr{E}^m(L)$, and $f_{p-1} \in M$ such that $\phi_{p-1}(x) = 1$ for all x in a neighbourhood of B_{p-1} and $|\phi_{p-1}F - f_{p-1}|_m \leqslant \dfrac{\epsilon}{2}$.

Let K' be a compact neighbourhood of supp$(1 - \phi_{q-1})$ such that 25
$K' \cap B_{p-1} = \varnothing$. Let $K = K' \cap B_p$. Therefore $K \subset A_p$ and so applying our lemma to K, taking $(1 - \phi_{p-1})F$ instead of F, we get a function $\psi \in \mathscr{E}^m(L)$, with $\psi = 1$ in a neighbourhood of K and an $f \in M$ such that

$$|\psi(1 - \phi_{p-1})F - f|_m \leqslant \frac{\epsilon}{2}.$$

Consider ϕ_p, f_p defined by $1 - \phi_p = (1 - \psi)(1 - \phi_{p-1})$ and $f_p = f + f_{p-1}$. One has obviously $\phi_p \in \mathscr{E}^m(L)$, $f_p \in M$, $|\phi_p F - f_p|_m \leqslant \epsilon$ and $\phi_p = 1$ in a neighbourhood of B_p. This proves the theorem.

Corollary 1.5. *Let M be a sub-module of $\mathscr{E}^m(L, E)$. Then for any $x \in L$, $T_x^m M = T_x^m \overline{M}$.*

Corollary 1.6. *Let Ω be an open set in $\mathbf{R}^n$ and M a sub-module of $\mathscr{E}^m(\Omega, E)$. Then $\widehat{M} = \overline{M}$ where $\widehat{M}$ and $\overline{M}$ are defined in the same way as in 1.3.*

Proof. Let us take a C^∞-partition of unity $\phi_i (i \in I)$ in Ω. Let $f \in \widehat{M}$. Then applying the theorem to $\phi_i f$, we get $\phi_i f \in \overline{M}$. By the definition of the topology on $\mathscr{E}^m(L, E)$, one obtains $\sum_{i \in I} \phi_i f \in \overline{M}$. □

Corollary 1.7. *Let Ω be a C^∞-manifold countable at infinity. Let M be a sub-module of $\mathscr{E}(\Omega, E)$. Then $\widehat{M} = \overline{M}$ where $\widehat{M}$ is defined as the module of all f in $\mathscr{E}(\Omega, E)$ such that $T_x^m f \in T_x^m M$ for all $x \in \Omega$ and all $m \geqslant 0$.*

Proof. Let K be any compact set in Ω, m any positive integer, ϵ any positive number. Let $f \in \hat{M}$. Then since $T_x^m f \in T_x^m M$ for all x in Ω, f is in the closure of the module generated by M over $\mathscr{E}^m(\Omega)$ and so there exist $\phi_1', \phi_2', \ldots, \phi_k'$ in $\mathscr{E}^m(\Omega)$ and $g_1, g_2, \ldots, g_k$ in M such that $\left| f - \sum_{i=1}^{k} g_i \phi_i' \right|_m^K \leqslant \epsilon$. But $\mathscr{E}(\Omega)$ is dense in $\mathscr{E}^m(\Omega)$; therefore ϕ_i' can be replaced by ϕ_i in $\mathscr{E}(\Omega)$ such that

$$\left| f - \sum_{i=1}^{k} g_i \phi_i \right|_m^K \leqslant \epsilon.$$

Therefore $f \in \overline{M}$ which proves the result. □

26 **Remark 1.8.** We know (Chapter I, §4) that given $a \in \Omega$, $\mathscr{E}(\Omega)/\mathscr{I}(\{a\}; \Omega)$ is isomorphic to the ring of formal power series in $n(= \dim \Omega)$ variables. Now define T_a as the natural mapping $\mathscr{E}(\Omega, E) \to \mathscr{E}(\Omega, E)/\mathscr{I}(\{a\}; \Omega, E)$ and let M be a sub-module of $\mathscr{E}(\Omega, E)$. From Krull's theorem (see Chapter III) it follows that "$T_a f \in T_a M$" is equivalent to "for all $m \geqslant 0$, $T_a^m f \in T_a^m M$". Namely, for all $a \in \Omega$, we have $T_a^m M = T_a^m \hat{M} = T_a^m \overline{M}$.

III
ANALYTIC AND DIFFERENTIABLE ALGEBRAS

1 Local R-algebras.

In this chapter, rings and algebras are supposed to be commutative with a unit and modules over these rings and algebras are supposed to be unitary. Further if A is a ring, we say that an A-module is "finite over A" if it is of finite type as an A-module. 27

Let A be a local ring, i.e. a ring possessing a proper ideal $\mathfrak{m}(A)$ containing all other proper ideals, which consists necessarily of all non-invertible elements of A. Let us recall the following result which we shall have frequently to use.

Proposition 1.1 (Nakayama's lemma.)**.** *Let M be an A-module of finite type and M' a submodule of M satisfying*

$$M = M' + \mathfrak{m}(A)M.$$

Then we have $\quad M' = M.$

Proof. If we set $N = M/M'$, we have $N = \mathfrak{m}(A)N$, and we have to show that N reduces to $\{0\}$. Now, let $n_1, \ldots, n_p$ be a system of generators of N. There exist elements

$$a_{ij} \in \mathfrak{m}(A) \quad (1 \leqslant i \leqslant p, 1 \leqslant j \leqslant p)$$

such that

$$n_i = \sum_{j=1}^{p} a_{ij} n_j,$$

and since $\det(\delta_{ij} - a_{ij}) \notin \mathfrak{m}(A)$ (δ_{ij} being the Kronecker symbol), we have $n_i = 0$ for every i.

Let now A be a local **R**-algebra and 1 its unit element. If $A \neq \{0\}$ (which we suppose in all that follows), the element 1 defines an injection $\epsilon : \mathbf{R} \to A$ by $\epsilon(\alpha) = \alpha 1$.

In all that follows, the following hypotheses are made (explicitly or implicitly) when we speak of local **R***-algebras.*

28 **(1.2; i)** $\mathfrak{m}(A)$ *is finite over* A.

(1.2; ii) *The composite* $\mathbf{R} \xrightarrow{\epsilon} A \to A/\mathfrak{m}(A)$ *is bijective.*

Let us recall that if $\mathfrak{p}$ is an ideal of A, we may put on A a structure of topological algebra (called the $\mathfrak{p}$-adic topology) by requiring that the powers $\mathfrak{p}^k$ (k an integer > 0) constitute a fundamental system of neighbourhoods of 0. For this topology to be Hausdorff, it is necessary and sufficient that

$$\bigcap_k \mathfrak{p}^k = \{0\}.$$

If $\mathfrak{p} = \mathfrak{m}(A)$, we call this the "Krull topology of A" (or simply "topology of A" if no confusion is possible). The $\mathfrak{p}$-adic topology on A coincides with the Krull topology if and only if there is an integer k such that $\mathfrak{m}^k(A) \subset \mathfrak{p}$; in this case, we shall say that $\mathfrak{p}$ is *an ideal of definition* of A. □

Proposition 1.3. *For* $\mathfrak{p}$ *to be an ideal of definition, it is necessary and sufficient that* $A/\mathfrak{p}$ *be finite over* **R**.

In fact, since $\mathfrak{m}(A)$ is finite over A, $\mathfrak{m}^k(A)$ is finite over A for every k, so that $\mathfrak{m}^k(A)/\mathfrak{m}^{k+1}(A)$ is finite over $A/\mathfrak{m}(A) \simeq \mathbf{R}$. Hence, for each k, $A/\mathfrak{m}^k(A)$ is finite over $\mathbf{R}$. If $\mathfrak{p}$ is an ideal of definition, $A/\mathfrak{p}$ is therefore finite over $\mathbf{R}$.

Conversely, suppose that $A/\mathfrak{p}$ is finite over $\mathbf{R}$. The $\mathfrak{m}^k(A/\mathfrak{p})$ form a decreasing sequence of finite modules over $\mathbf{R}$, and the sequence is therefore stationary. By Nakayama's lemma, we have, for a certain k, $\mathfrak{m}^k(A/\mathfrak{p}) = \{0\}$, whence $\mathfrak{m}^k(A) \subset \mathfrak{p}$.

Let $\widehat{A}$ be the algebra obtained by making Hausdorff the completion of A for the Krull topology*. It is obvious that $\widehat{A}$ can be identified with

*In what follows, we shall say "completion" for this Hausdorff completion, and "complete" for rings which are Hausdorff and complete.

the projective limit $\varprojlim A/\mathfrak{m}^k(A)$, that $\widehat{A}$ is again a local $\mathbf{R}$-algebra (satisfying (i) and (ii)), and that the natural mappings $A/\mathfrak{m}^k(A) \to \widehat{A}/\mathfrak{m}^k(\widehat{A})$ are isomorphisms. (We leave the details to the reader.)

Let $x_1, \ldots, x_p$ be elements of $\mathfrak{m}(A)$. We define, in an obvious way, **29**
a mapping of the ring $\mathbf{R}[[X_1, \ldots, X_p]]$ of formal power series into $\widehat{A}$. This mapping will be surjective if (and only if) $x_1, \ldots, x_p$ are generators of $\mathfrak{m}(A)$ over A. Consequently, $\widehat{A}$ is a quotient of an algebra of formal power series. It follows from this that $\widehat{A}$ is noetherian. (We shall, furthermore, recall the proof of this fact later on.)

Let now A and B be two local $\mathbf{R}$-algebras, and u a homomorphism (which, in what follows, will always be supposed unitary) $A \to B$. We have $u^{-1}(\mathfrak{m}(B)) = \mathfrak{m}(A)$: in fact, the $\mathbf{R}$-linear mapping $A/u^{-1}(\mathfrak{m}(B)) \to B/\mathfrak{m}(B)$ is not zero, since $u(1) = 1$, so that the mapping is surjective; hence $u^{-1}(\mathfrak{m}(B))$ is maximal and thus equal to $\mathfrak{m}(A)$. A fortiori, we have $u(\mathfrak{m}(A)) \subset \mathfrak{m}(B)$; in other words, u is local, that is, continuous with respect to the Krull topology. It follows that u induces a homomorphism $\widehat{u} : \widehat{A} \to \widehat{B}$, which is again local, and a homomorphism

$$\overline{u} : A/\mathfrak{m}(A) \to B/B\, u(\mathfrak{m}(A)).$$

This last mapping coincides with the canonical injection

$$\epsilon : \mathbf{R} \to B/B\, u(\mathfrak{m}(A)).$$

In what follows, we shall equip B with the structure of A-module defined by u. We shall write therefore

$$ab (a \in A, b \in B) \text{ for } u(a)b,$$
$$\mathfrak{m}(A)B \text{ for } B\, u(\mathfrak{m}(A)),$$

and so on.

Definition 1.4. *(i) We say that u is finite if B is finite over A.*

(ii) We say that u is quasi-finite if $\overline{u}$ is finite, that is, if $B/\mathfrak{m}(A)B$ is finite over $\mathbf{R}$.

By Proposition 1.3, u is quasi-finite if and only if $\mathfrak{m}(A)B$ is an ideal of definition of B. It is clear that every finite homomorphism is quasi-finite; but in general, the converse is false (counter example: A = ring of convergent power series in $n \geqslant 1$ variables, B its completion). One
30 of the main objects of this course is to prove that this converse (called the "preparation theorem") is true in a certain number of cases. Let us note at once the following

Proposition 1.5 (The formal preparation theorem.)**.** *If A and B are complete (and Hausdorff) and if $u : A \to B$ is quasi-finite, then u is finite.*

We shall utilise this proposition here, but postpone its proof to §3.

Let us go back to the general case : the map $u : A \to B$ being continuous defines, by passage to completions, a mapping $\hat{u} : \hat{A} \to \hat{B}$ (and, by composition, a mapping $A \to \hat{B}$ which we shall use incidentally).

Proposition 1.6. *The properties "u quasi-finite", "$\hat{u}$ quasi-finite" and "$\hat{u}$ finite" are equivalent. If they are satisfied, the canonical mapping*

$$B/\mathfrak{m}(A)B \to \hat{B}/\mathfrak{m}(\hat{A})\hat{B}$$

is bijective.

Proof. By Proposition 1.5, "$\hat{u}$ finite" and "$\hat{u}$ quasi-finite" are equivalent. Let us prove the equivalence of "u quasi-finite" and "$\hat{u}$ quasi-finite". For this, it is sufficient to prove that $\mathfrak{m}(A)B$ is an ideal of definition of B if and only if $\mathfrak{m}(\hat{A})\hat{B}$ is an ideal of definition of $\hat{B}$.

Let $\mathfrak{p}$ be an ideal in B. For each r, the canonical mapping

$$(\mathfrak{p} + \mathfrak{m}^r(B))/\mathfrak{m}^r(B) \to (\mathfrak{p}\hat{B} + \mathfrak{m}^r(\hat{B}))/\mathfrak{m}^r(\hat{B})$$

is evidently bijective. Put $\mathfrak{p} = \mathfrak{m}(A)B$, and remark that $\mathfrak{m}(A)\hat{B} + \mathfrak{m}^r(\hat{B})$ is closed (since it contains $\mathfrak{m}^r(\hat{B})$), hence is equal to $\mathfrak{m}(\hat{A})\hat{B} + \mathfrak{m}^r(\hat{B})$. We obtain thus an isomorphism

$$(\mathfrak{m}(A)B + \mathfrak{m}^r(B))/\mathfrak{m}^r(B) \xrightarrow{\sim} (\mathfrak{m}(\hat{A})\hat{B} + \mathfrak{m}^r(\hat{B}))/\mathfrak{m}^r(\hat{B}). \tag{1.7}$$

Suppose now that $\mathfrak{m}(A)B \supset \mathfrak{m}^k(B)$. Using this isomorphism for
31 $r = k+1$, we obtain $\mathfrak{m}(\hat{A})\hat{B} + \mathfrak{m}^{k+1}(\hat{B}) \supset \mathfrak{m}^k(\hat{B})$. Applying Nakayama's

lemma to the couple $\mathfrak{m}^k(\widehat{B})$, $\mathfrak{m}(\widehat{A})\widehat{B} \cap \mathfrak{m}^k(B)$, we find that $\mathfrak{m}(\widehat{A})\widehat{B} \supset \mathfrak{m}^k(\widehat{B})$. Conversely, the same argument shows that $\mathfrak{m}(\widehat{A})\widehat{B} \supset \mathfrak{m}^k(\widehat{B})$ implies that $\mathfrak{m}(A)B \supset \mathfrak{m}^k(B)$, whence the result.

Finally, suppose that $\mathfrak{m}(A)B \supset \mathfrak{m}^k(B)$. The preceding result, together with the isomorphism (1.7) for $r = k$ gives an isomorphism

$$\mathfrak{m}(A)B/\mathfrak{m}^k(B) \xrightarrow{\sim} \mathfrak{m}(\widehat{A})\widehat{B}/\mathfrak{m}^k(B).$$

The isomorphism stated in Proposition 1.6 follows from this, the isomorphism

$$B/\mathfrak{m}(A)B \simeq (B/\mathfrak{m}^k(B))/(\mathfrak{m}(A)B/\mathfrak{m}^k(B)),$$

and the corresponding isomorphism for the completions. □

Proposition 1.6 has the following corollary which is useful for applications.

Corollary 1.7. *Let $u : A \to B$ be a homomorphism of local* **R***-algebras and let $b_1, \dots, b_p$ be a finite family of elements of B. Let us denote by $\widehat{b}_i$ their images in $\widehat{B}$ and by $\overline{b}_i$ their images in $B/\mathfrak{m}(A)B$. The following properties are equivalent.*

(i) *$\widehat{b}_1, \dots, \widehat{b}_p$ generate $\widehat{B}$ over $\widehat{A}$.*

(ii) *$\overline{b}_1, \dots, \overline{b}_p$ generate $B/\mathfrak{m}(A)B$ over* **R**.

(iii) *$\overline{\widehat{b}}_1, \dots, \overline{\widehat{b}}_p$ generate $\widehat{B}/\mathfrak{m}(\widehat{A})\widehat{B}$ over* **R**.

Furthermore, if u is finite, they are equivalent to

(iv) *$b_1, \dots, b_p$ generate B over A.*

The equivalence of (ii) and (iii) follows from the isomorphism (1.6). On the other hand, it is obvious that (iv) implies (ii). If u is finite, (ii) implies (iv) by Nakayama's lemma. Taking into account Proposition 1.5. the equivalence of (i) and (iii) is proved in the same way.

2 Analytic and differentiable algebras.

In what follows, we denote respectively by $\mathscr{O}_n$, $\mathscr{E}_n$ the rings of germs at 0 in $\mathbf{R}^n$ of real analytic and C^∞ functions with real values, and by $\mathscr{F}_n$ the ring
32 of formal power series in n indeterminates over $\mathbf{R}$. One has obvious mappings $\mathscr{O}_n \to \mathscr{E}_n$ (an injection), $\mathscr{O}_n \to \mathscr{F}_n$ and $\mathscr{E}_n \to \mathscr{F}_n$ (Taylor expansion at 0). These rings are local $\mathbf{R}$-algebras satisfying (1.2). The only point which is not entirely obvious is the fact that $\mathscr{E}_n$ satisfies (1.2; ii), which fact results from the following lemma in which $x_1, \ldots, x_n$ stand for coordinates in $\mathbf{R}^n$.

Lemma 2.1. *Let $f \in \mathscr{E}_n$ and k be an integer $\leqslant n$. Suppose that*

$$f(0, \ldots, 0, x_{k+1}, \ldots, x_n) = 0.$$

There exist then $h_i \in \mathscr{E}_n$, $i = 1, \ldots, k$ with

$$f = \sum_{i=1}^{k} x_i h_i.$$

Proof. We may, in fact, take

$$h_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_k, x_{k+1}, \ldots, x_n)dt.$$

It follows from this lemma that $x_1, \ldots, x_n$ form a system of generators of $\mathfrak{m}(\mathscr{E}_n)$ over $\mathscr{E}_n$. One also deduces from it at once that $\mathscr{F}_n$ is the completion of $\mathscr{E}_n$ for the Krull topology, the corresponding fact for $\mathscr{O}_n$ instead of $\mathscr{E}_n$ being obvious. We note also an important difference between the two cases: while the mapping $\mathscr{O}_n \to \mathscr{F}_n$ is *injective*, the mapping $\mathscr{E}_n \to \mathscr{F}_n$ is surjective (Chapter I, §4), so that $\mathscr{E}_n$ is, in some sense, "complete but not Hausdorff". □

Definition 2.2. *By a differentiable algebra, we mean a local $\mathbf{R}$-algebra together with a surjective homomorphism $\mathscr{E}_n \xrightarrow{\pi} A$ (which is assumed unitary). Replacing $\mathscr{E}_n$ by $\mathscr{O}_n$ (resp. $\mathscr{F}_n$), we define in the same way an analytic (resp. formal) algebra.*

We will now define the morphisms of differentiable algebras. First, if $A = \mathscr{E}_n$, $B = \mathscr{E}_m$, a homomorphism $u : A \to B$ is called a morphism if there exists a germ ϕ (at 0) of C^∞-mapping from $\mathbf{R}^m$ into $\mathbf{R}^n$, $\phi(0) = 0$, such that for any $f \in \mathscr{E}_n$, we have $u(f) = f \circ \phi$ (ϕ if it exists, is obviously unique). In the general case, let $\mathscr{E}_n \xrightarrow{\pi} A$, $\mathscr{E}_m \xrightarrow{\psi} B$ be two differentiable algebras and u a homomorphism $A \to B$. We say that u is a morphism 33
if there exists a morphism $\tilde{u} : \mathscr{E}_n \to \mathscr{E}_m$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{E}_n & \xrightarrow{\tilde{u}} & \mathscr{E}_m \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow[u]{} & B \end{array}$$

It is evident that the composite of two morphisms is a morphism. In accordance with general definitions in a category, we say that a morphism u is an isomorphism if there exists a morphism $v : B \to A$ such that $v \circ u =$ identity, $u \circ v =$ identity (it is in fact sufficient that u be bijective; this results easily from the considerations that follow).

Proposition 2.3. *Given a differentiable algebra $\mathscr{E}_m \xrightarrow{\pi} B$ and n elements $b_i \in \mathfrak{m}(B)$, there is one and only one morphism $u : \mathscr{E}_n \to B$ such that $u(x_i) = b_i$ (x_i standing for the coordinates in $\mathbf{R}^n$).*

Proof. For each i, let us choose an $f_i \in \mathscr{E}_m$ satisfying $\pi(f_i) = b_i$, and let v be the morphism $\mathscr{E}_n \to \mathscr{E}_m$ defined by $v(x_i) = f_i$, i.e. the morphism induced by the mapping $(y_1, \ldots, y_m) \to (f_1, \ldots, f_n)$ from $\mathbf{R}^m \to \mathbf{R}^n$. Then $u = \pi \circ v$ has the required property. To prove the uniqueness, let us choose $f_i' \in \mathscr{E}_m$ with $\pi(f_i') = b_i$ and let us denote by $\mathfrak{I}$ the ideal $\pi^{-1}(0)$. It is sufficient to prove that for any $g \in \mathscr{E}_n$, we have

$$g(f_1, \ldots, f_n) - g(f_1', \ldots, f_n') \in \mathfrak{I}.$$

Now, Lemma 2.1 shows that there exist functions $h_i \in \mathscr{E}_{2n}$ satisfying

$$g(f_1, \ldots, f_n) - g(f_1', \ldots, f_n') = \sum_{i=1}^{n} (f_i - f_i') h_i(f_1, \ldots, f_n, f_1', \ldots, f_n'),$$

and the result follows.

It will be convenient to denote the element $u(g)$ constructed in the above proposition by $g(b_1, \ldots, b_n)$. The reader will verify easily that
34 a homomorphism $u : A \to B$ is a morphism if and only if it has the following property: for any $a_1, \ldots, a_p \in \mathfrak{m}(A)$ and $f \in \mathscr{E}_p$, we have

$$u(f(a_1, \ldots, a_p)) = f(u(a_1), \ldots, u(a_p))$$

(in other words: u is compatible with composition by differentiable functions). □

Remark 2.4. As far as I am aware, one does not know at present whether every homomorphism $A \to B$ (as **R**-algebras) is a morphism. One does not even know whether two differentiable algebras which are isomorphic as **R**-algebras are also isomorphic as differentiable algebras. It is precisely this fact which has forced us to adopt the preceding definitions rather than the "naive" definitions, with which we will not be able to work.

We adopt analogous definitions also in the case of formal and analytic rings, leaving to the reader the task of formulating them explicitly. This is only provisional, till the preparation theorem is proved: we shall see in §3, as a consequence of this theorem, that any homomorphism of analytic (resp. formal) **R**-algebras is a morphism.

3 The preparation theorem for formal and analytic algebras.

Theorem 3.1. *Let u be a morphism of analytic (resp. formal) algebras. Then u is quasi-finite if and only if u is finite.*

Proof. We shall give the proof in the case of analytic algebras, leaving to the reader the task of treating that of formal algebras.

Let, then $\mathscr{O}_n \xrightarrow{\pi} A$, $\mathscr{O}_m \xrightarrow{\psi} B$ be two analytic algebras, and u a morphism $A \to B$, which we suppose quasi-finite. The problem reduces to proving that u is finite.

(A) *Reduction to the regular case.* $(A = \mathscr{O}_n, B = \mathscr{O}_m)$.

In the first place, $u \circ \pi$ is again quasi-finite, and if it is finite, then so is u; we may, therefore, suppose that $A = \mathscr{O}_n$ (and π = identity).

Let us now put $\mathfrak{p} = \ker \psi$; the second reduction consists in reducing to the case when $\mathfrak{p}$ is finitely generated (property which is, a posteriori, true of all ideals since $\mathscr{O}_m$ is noetherian, but which we cannot use here!). For this, let $\tilde{u}$ be a morphism $\mathscr{O}_n \to \mathscr{O}_m$ 35
such that $u = \psi \circ \tilde{u}$. Since $\mathfrak{m}(\mathscr{O}_n)B$ is an ideal of definition of B, $\mathfrak{p} + \mathfrak{m}(\mathscr{O}_n)\mathscr{O}_m$ is an ideal of definition of $\mathscr{O}_m$, hence contains $\mathfrak{m}^k(\mathscr{O}_m)$ for a certain k. It follows that there exists an ideal $\mathfrak{p}' \subset \mathfrak{p}$ *finitely generated over $\mathscr{O}_m$,* such that

$$\mathfrak{p}' + \mathfrak{m}(\mathscr{O}_n)\mathscr{O}_m + \mathfrak{m}^{k+1}(\mathscr{O}_m) \supset \mathfrak{m}^k(\mathscr{O}_m).$$

By Nakayama's lemma, we deduce from this that $\mathfrak{p}' + \mathfrak{m}(\mathscr{O}_n)\mathscr{O}_m \supset \mathfrak{m}^k(\mathscr{O}_m)$. Let us now put $B' = \mathscr{O}_m/\mathfrak{p}'$, $\psi' : \mathscr{O}_m \to B'$ the natural projection and let $u' = \psi' \circ u$. Then u' is again quasi-finite, and if it is finite, then so is u.

Let us, therefore, suppose that $\mathfrak{p}$ is finitely generated, and let $g_1, \dots, g_p$ be a system of generators. Let us denote by $y_i (1 \leqslant i \leqslant n)$ resp. $z_j (n + 1 \leqslant j \leqslant n + p)$ generators of the maximal ideal of $\mathscr{O}_n$ (resp. $\mathscr{O}_{n+p}$), and let v be the morphism $\mathscr{O}_{n+p} \to \mathscr{O}_m$ defined by the formulas

$$\begin{aligned} v(z_i) &= \tilde{u}(y_i),\ 1 \leqslant i \leqslant n \\ v(z_{n+j}) &= g_j,\ 1 \leqslant j \leqslant p. \end{aligned}$$

The morphism v is still quasi-finite; if v is finite, then so is u. We are thus reduced, after a change of notation, to the case $A = \mathscr{O}_n$, $B = \mathscr{O}_m$.

(B) Proof in the regular case. (cf. Houzel [8]).

Let us denote by $x_1, \dots, x_n$ (resp. $y_1, \dots, y_m$) coordinates in $\mathscr{O}^n$ (resp. $\mathscr{O}^m$). Let us put $\phi_i = u(x_i)$, and let ϕ be the mappings $(\phi_1, \dots, \phi_n)$ of $\mathbf{R}^m \to \mathbf{R}^n$. By hypothesis, we have $u(f) = f \circ$

ϕ for $f \in \mathscr{O}_n$. Since u is quasi-finite, there is an r such that $\mathfrak{m}(\mathscr{O}_n)\mathscr{O}_m \supset \mathfrak{m}^r(\mathscr{O}_m)$. If for an m-tuple $k = (k_1, \ldots, k_m) \in \mathbf{N}^m$, we set as usual: $|k| = k_1 + \cdots + k_m$, $y^k = y_1^{k_1} \ldots y_m^{k_m}$, we have, for $|k| = r$, the formula

$$y^k = \sum_{i=1}^{n} \lambda_{k^i}\phi_i,\ \lambda_{k^i} \in \mathscr{O}_m. \tag{3.2}$$

We shall prove that the y^k with $|k| \leqslant r$ generate $\mathscr{O}_m$ over $\mathscr{O}_n$. For this, let $f \in \mathscr{O}_m$. If we denote by $\pi(f)$ the terms of degree $< r$ in f we can write f in the form

$$f = \pi(f) + \sum_{|k|=r} y^k \sigma_k(f),\ \text{where } \sigma_k(f) \in \mathscr{O}_m. \tag{3.3}$$

36 Using (3.2), we obtain from this an expression of f in the form

$$f = \tau(f) + \sum_{i=1}^{n} \phi_i \rho_i(f),\ \text{where } \rho_i(f) \in \mathscr{O}_m. \tag{3.4}$$

[The σ and the ρ are not, in general, unique, but this causes no trouble.]

Applying the same formula to the ρ_i and iterating this procedure, one obtains for $p \in \mathbf{N}$,

$$\begin{aligned} f = \tau(f) + \quad &+ \sum_{1 \leqslant i_1, \ldots, i_p \leqslant n} \phi_{i_1} \ldots \phi_{i_p} \tau_{i_1 \ldots i_p}(f) + \\ &+ \sum_{1 \leqslant i_1, \ldots, i_p + 1 \leqslant n} \phi_{i_1} \ldots \phi_{i_{p+1}} \rho_{i_1 \ldots i_{p+1}}(f). \end{aligned} \tag{3.4p}$$

Here the τ are polynomials of degree $\leqslant r - 1$ in y and the ρ are functions in $\mathscr{O}_m$. Since the last term belongs to $\mathfrak{m}^{p+1}(\mathscr{O}_n)$, we see already that this defines a series converging formally to f. We have thus only to prove that for a suitable choice of the τ, the family

$$x_{i_1} \ldots x_{i_p} \tau_{i_1 \ldots i_p}(f) \tag{3.5}$$

(which has values in the space of polynomials of degree $\leqslant r - 1$ in y) is summable in a neighbourhood of the origin.

For this, let $R > 0$ and $f = \sum a_k y^k$. Let us put $|f|_R = \sum |a_k| R^{|k|}$. The mapping $f \to |f|_R$ has all the usual properties of norms, except that it may take the value $+\infty$. For f fixed, the function $R \to |f|_R$ is increasing and finite for R small enough. Finally, a straight-forward calculation shows that $|fg|_R \leqslant |f|_R |g|_R$.

In (3.3), we have $|\tau(f)|_R \leqslant |f|_R$, and, if we choose the σ suitably, we will have, for any $R > 0$,

$$|\sigma_k(f)|_R \leqslant \frac{1}{R^r} |f|_R.$$

Let us choose R_0 such that $|\lambda_{ki}|_{R_0} < \infty$ for all (k, i). Substituting in (3.3) the expressions for the y^k given by (3.2), we obtain, for $R < R_0$, the following estimate in (3.4):

$$|\rho_i(f)|_R \leqslant \frac{C}{R^r} |f|_R \text{ with } C \text{ independent of } R.$$

Iterating this, we find that we can choose the $\tau_{i_1 \ldots i_p}$ in such a way that, 37
for $R < R_0$, we have

$$|\tau_{i_1 \ldots i_p}(f)|_R \leqslant \frac{C^p}{R^{rp}} |f|_R.$$

Finally, choosing $R < R_0$ such that $|f|_R < \infty$, we conclude from the above inequality that the family (3.5) is summable in $|x_i| \leqslant \rho$ for $\rho \leqslant R^r/nC$, which proves the theorem.

In the following corollaries, we consider $\mathscr{O}_{n-1}$(resp. $\mathscr{F}_{n-1}$) as imbedded in $\mathscr{O}_n$ (resp. $\mathscr{F}_n$) by means of the morphism π^* induced by the projection $\pi : (x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1})$. □

Corollary 3.6 (Division algorithm)**.** *Let $\Phi \in \mathscr{O}_n$ be such that*

$$\Phi(0, \quad, 0, x_n) = x_n^p g(x_n), g(0) \neq 0.$$

For any $f \in \mathscr{O}_n$, there exist $Q \in \mathscr{O}_n$ and $R \in \mathscr{O}_{n-1}[x_n]$ with degree $R < p$, such that $f = \Phi Q + R$. Moreover Q and R are uniquely determined by these conditions.

Also, the statement remains valid if we replace $\mathscr{O}_n$, $\mathscr{O}_{n-1}$ by $\mathscr{F}_n$, $\mathscr{F}_{n-1}$.

Let $A = \mathcal{O}_{n-1}$, $B = \mathcal{O}_n/(\Phi)$ and let u be the composite of π^* and the canonical mapping $\mathcal{O}_n \to \mathcal{O}_n/(\Phi)$. It is obvious that the images of $1, x_n, \ldots, x_n^{p-1}$ in $B/\mathfrak{m}(A)B$ form a basis of the latter space over **R**. By our theorem, and Nakayama's lemma $(1, x_n, \ldots, x_n^{p-1})$ is a system of generators of B over A, whence the existence of Q and R.

To prove the uniqueness, let us write $\Phi = \Sigma\Phi_k$, Φ_k being a convergent series in x_n with coefficients which are homogeneous polynomials of degree k in $(x_1, \ldots, x_{n-1})$. Suppose further that there exist $Q \in \mathcal{O}_n$, $R \in \mathcal{O}_{n-1}[x_n]$, $\deg R < p$, for which $0 = \Phi Q + R$ and $Q \neq 0, R \neq 0$. Let us write $Q = \Sigma Q_k$, $R = \Sigma R_k$ in the same way as above, and let l be the smallest integer such that $Q_l \neq 0$ or $R_l \neq 0$. We have $\Phi_0 Q_l + R_l = 0$; but $\Phi_0 Q_l$ contains x_n^p as factor, so that $Q_l = R_l = 0$, a contradiction.

38 **Corollary 3.7** (Weierstrass). *With the same hypotheses as in the preceding corollary, there exists $Q \in \mathcal{O}_n$ with $Q(0) \neq 0$ and a distinguished polynomial $P \in \mathcal{O}_{n-1}[x_n]$ (i.e. a monic polynomial, all of whose other coefficients are zero at the origin) such that $P = \Phi Q$. Further, P and Q are completely determined by these conditions. The statement remains valid with $\mathcal{F}_n$, $\mathcal{F}_{n-1}$ instead of $\mathcal{O}_n$, $\mathcal{O}_{n-1}$.*

We have only to apply (3.6) to $f = x_n^p$ and take $P = x_n^p - R$ (it is easy to verify that P is distinguished and that $Q(0) \neq 0$).

Theorem 3.8. *Analytic (resp. formal) algebras are noetherian.*

Proof. It is sufficient to prove that $\mathcal{O}_n$ (resp. $\mathcal{F}_n$) is noetherian. Let $\mathfrak{p} \neq \{0\}$ be an ideal in $\mathcal{O}_n$ and let $f \in \mathfrak{p}$, $f \neq 0$. By a linear change of coordinates, we may suppose that $f(0, \ldots, 0, x_n) \neq 0$, and it is enough to prove that the image $\overline{\mathfrak{p}}$ of $\mathfrak{p}$ in $\mathcal{O}_n/(f)$ is finite over $\mathcal{O}_n$. A fortiori, it is sufficient to prove that $\overline{\mathfrak{p}}$ is finite over $\mathcal{O}_{n-1}$. This follows from the induction hypothesis and the fact that $\mathcal{O}_n/(f)$ is finite over $\mathcal{O}_{n-1}$. □

Theorem 3.9. *The ring $\mathcal{O}_n$ (resp. $\mathcal{F}_n$) is factorial.*

Proof. The ring $\mathcal{O}_n$ is obviously an integral domain. In view of the theorem above, it is sufficient to prove the following: if $f \in \mathcal{O}_n$ is irreducible, then f is prime. We proceed by induction on n, and suppose that $\mathcal{O}_{n-1}$ is factorial, so that (Gauss' theorem) $\mathscr{O}_{n-1}[x_n]$ is factorial. By a change of coordinates and multiplication by an invertible factor, we may suppose that f is a distinguished polynomial in x_n. The theorem results from the next lemma, which is a little stronger. □

Lemma 3.10. *Let $P \in \mathscr{O}_{n-1}[x_n]$ be a distinguished polynomial, irreducible in $\mathscr{O}_{n-1}[x_n]$. Then P is prime in $\mathscr{O}_n$.*

Proof. Let $gh \in \mathscr{O}_n$ be such that P divides gh. Let $\overline{g}$ and $\overline{h}$ be the remainders of g and h after division by P. P divides $\overline{g}\,\overline{h}$ in $\mathscr{O}_n$. Because of our inductive hypothesis, P is prime in $\mathscr{O}_{n-1}[x_n]$, so that it suffices to prove that P divides $\overline{g}\,\overline{h}$ in $\mathscr{O}_{n-1}[x_n]$.

Now, we have on the one hand $\overline{g}\overline{h} = PQ$, $Q \in \mathscr{O}_n$, and on the other 39
(euclidean division) $\overline{g}\overline{h} = PQ' + R'$, $Q', R' \in \mathscr{O}_{n-1}[x_n]$, $\deg R' < \deg P$. Because of the uniqueness in (3.6), we have necessarily $Q = Q', R' = 0$, whence the lemma. □

Remark 3.11. Let $P \in \mathscr{O}_{n-1}[x_n]$ be distinguished. One verifies easily that P admits a decomposition into irreducible factors which are all distinguished polynomials in x_n.

Theorem 3.8 will enable us to apply some of the theorems of local algebra to analytic and formal rings. We recall some of these results. Let A be a local ring, E and A-module. We define the structure of a topological group on E, "the Krull topology", by taking, for a fundamental system of neighbourhoods of 0, the sets $\mathfrak{m}^k(A)E$. [If $E = A$, this coincides with the definition given in §1; further, we could also consider the $\mathfrak{p}$-adic topology of E for an arbitrary ideal $\mathfrak{p}$ of A, but we will not need this.]

Let F be a submodule of E. It is obvious that the Krull topology of E/F is the quotient topology of that on E. To study the topology of F, we use the following result.

Theorem 3.12 (Artin-Rees)**.** *Suppose that A is noetherian, and that E is finite over A. There is an integer $p > 0$ such that, for $n > p$, we have,* *(writing* $\mathfrak{m}$ *for* $\mathfrak{m}(A)$*)*

$$F \cap \mathfrak{m}^n E = \mathfrak{m}^{n-p}.(F \cap \mathfrak{m}^p(E)).$$

For the proof, see e.g. Bourbaki [2].

Corollary 3.13 (Krull)**.** *With the same hypothesis*

(i) *the Krull topology of F coincides with the topology induced on F by the Krull topology of E;*

(ii) *E is hausdorff;*

(iii) *F is closed.*

(i) follows trivially from (3.12). To prove (ii), we apply (i) with $F =$
40 closure of 0, $\overline{0} = \cap \mathfrak{m}^n E$: we have $\mathfrak{m}\overline{0} = \overline{0}$, whence (Proposition 1.1), $\overline{0} = 0$. Finally (iii) follows from (ii) applied to E/F.

From this and Theorem 3.8, we deduce that *formal and analytic algebras are hausdorff.* We will deduce from this a result stated at the end of §2.

Proposition 3.14. *Every homomorphism of analytic (resp. formal) algebras is a morphism.*

Proof. Let $\mathscr{O}_n \xrightarrow{\pi} A$, $\mathscr{O}_m \xrightarrow{\psi} B$ and let u be a (unitary $\mathbf{R}$-algebra) homomorphism $A \to B$. Let $x_1, \ldots, x_n$ be coordinates in $\mathbf{R}^n$ and let $\widetilde{u}$ be the morphism $\mathscr{O}_n \to B$ defined by $\widetilde{u}(x_i) = u \circ \pi(x_i)$ (Proposition 2.3). It is sufficient to prove that, for any $f \in \mathscr{O}_n$, we have $\widetilde{u}(f) = u \circ \pi(f)$. Now this formula is true if f is a polynomial in $x_1, \ldots, x_n$. Since B is hausdorff and the two sides of our formula depend continuously on f, the result is obtained by passage to the limit. □

Remark 3.15. The same argument proves the following: let $\mathscr{E}_n \to A$, $\mathscr{E}_m \to B$ be two differentiable algebras. If B is hausdorff every homomorphism $A \to B$ is a morphism. We shall see that, in general, B is not hausdorff, so that this does not answer the question raised in 2.4.

Let A be a local ring, $\mathfrak{p}$ an ideal of A with $\mathfrak{p} \neq A$. Then $A/\mathfrak{p}$ is a local ring and we have $\mathfrak{m}^k(A/\mathfrak{p}) = \mathfrak{m}^k(A)(A/\mathfrak{p})$. Consequently, the topology of the local ring $A/\mathfrak{p}$ and the A-module $A/\mathfrak{p}$ coincide.

Let us take $A = \mathscr{F}_n$ which is complete; since $\mathfrak{p}$ is closed in $\mathscr{F}_n$, $\mathscr{F}_n/\mathfrak{p}$ is again a complete $\mathscr{F}_n$-module, and hence a complete local ring. Consequently, we obtain

Proposition 3.16. *Every formal algebra is complete.*

Take now $A = \mathscr{E}_n$. The mapping of "Taylor expansion" $\mathscr{E}_n \to \mathscr{F}_n$ is surjective, and its kernel is the ideal $\mathfrak{m}^\infty(\mathscr{E}_n) = \cap \mathfrak{m}^k(\mathscr{E}_n)$ of functions flat at 0. $\mathscr{F}_n$ can therefore be identified with the completion $\widehat{\mathscr{E}_n}$ of $\mathscr{E}_n$.

Let $\mathfrak{p}$ be an ideal of $\mathscr{E}_n$, $\widetilde{\mathfrak{p}} = (\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n))/\mathfrak{m}^\infty(\mathscr{E}_n)$ its image in $\mathscr{F}_n$. 41
We have an isomorphism $\mathscr{E}_n/(\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n)) \to \mathscr{F}_n/\widetilde{\mathfrak{p}}$; in particular, the first space is hausdorff (and even complete) and $\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n)$ is therefore closed. Put $B = \mathscr{E}_n/\mathfrak{p}$, and $\mathfrak{m}^\infty(B) = \cap \mathfrak{m}^k(B)$. We have obviously an injection $i : (\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n))/\mathfrak{p} \to \mathfrak{m}^\infty(B)$, and since $\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n)$ is closed in $\mathscr{E}_n$, $(\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n))/\mathfrak{p}$ is closed in B, and hence i is an isomorphism. Denoting by $\widehat{B}$ the completion of B, and using the fact that $\mathscr{F}_n/\widetilde{\mathfrak{p}}$ is complete, we deduce from this the

Proposition 3.17. *The canonical mappings*

$$\mathscr{F}_n/\widetilde{\mathfrak{p}} \simeq \mathscr{E}_n/(\mathfrak{p} + \mathfrak{m}^\infty(\mathscr{E}_n)) \to B/\mathfrak{m}^\infty(B) \to \widehat{B}$$

are isomorphisms.

In particular, the completion of a differentiable algebra is simply the largest hausdorff quotient; and for a differentiable algebra to be complete, it is necessary and sufficient that it be hausdorff, or again, that it be isomorphic (as a local algebra) to a formal algebra.

4 Analytic algebras: completion and coherence.

A. Flat modules. We recall here a certain number of definitions and elementary properties, and refer to Serre [21] or Bourbaki [1] for the proofs. [These proofs are, moreover, almost all immediate so that it would be a good exercise for the reader to reconstruct them.]

Let us also state that the concept of flatness finds its natural interpretation in homological algebra. We will not develop this point of view here.

Definition 4.1. *Let A be a ring, E an A-module. We say that E is flat if the following equivalent conditions are satisfied.*

(i) *For every exact sequence $M' \to M \to M''$ of A-modules, the sequence $E \otimes_A M' \to E \otimes_A M \to E \otimes_A M''$ is exact.*

(ii) *For any ideal $\mathfrak{I}$ of A, the natural map $\mathfrak{I} \otimes_A E \to E$ is injective.*

The property (ii) can be interpreted in the following way. Let $f =$
42 $(f_1, \ldots, f_n) \in A^n$. Let us denote by $R(f, E)$ (the "relations of f in E") the submodule of E^n consisting of the $(e_1, \ldots, e_n)$ verifying $\Sigma f_i e_i = 0$. Then

Proposition 4.2. *E is flat if and only if for every n and $f \in A^n$, we have $R(f, E) = R(f, A)E$.*

Remark 4.3. Let $f = (f_1, \quad , f_n)$ be a system of n elements of A^m. We may again define $R(f, E)$ as the submodule of E^n consisting of the $(e_1, \quad e_n)$ for which $\Sigma f_i e_i = 0$. Using induction and Proposition 4.2, we see that if E is flat, we have again $R(f, E) = R(f, A)E$.

Proposition 4.4. (i) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are flat, then so is M. If M and M'' are flat, so is M'*

(ii) *Let $A \to B$ be a homomorphism of rings such that B is flat as an A-module. If M is a flat B-module, then M, considered as an A-module, is again flat.*

(iii) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules and suppose that M'' is flat. For any A-module E, the sequence*

$$0 \to M' \otimes_A E \to M \otimes_A E \to M'' \otimes_A E \to 0$$

is again exact.

Proposition 4.5. *Let E be a flat A-module, M' and M'' two submodules of M. Consider $M' \otimes_A E$ (resp. $M'' \otimes_A E$) as a submodule of $M \otimes_A E$ by means of the natural injection $M' \otimes_A E \to M \otimes_A E$ (resp. etc...). Then, we have*

$$(M' \cap M'') \otimes_A E = (M' \otimes_A E) \cap (M'' \otimes_A E).$$

[One uses the exact sequence

$$0 \to M/(M' \cap M'') \xrightarrow{i} M/M' \oplus M/M'' \xrightarrow{\delta} M/(M' + M'') \to 0$$

where

$$i(x) = (x \text{mod } M', x \text{mod } M'')$$

and

$$\delta(x', x'') = x' \text{mod } (M' + M'') - x'' \text{mod } (M' + M''),$$

and the exact sequence obtained by tensoring with E.]

Definition 4.6. *Let A be a ring, E an A-module. We say that E is faithfully flat if it has the following properties* 43

(i) *E is flat.*

(ii) *For any A-module M, $E \otimes_A M = \{0\}$ implies that $M = \{0\}$.*

[It is sufficient to require (ii) for modules of finite type.]

Let B be a ring containing A. For B to be a faithfully flat A-module, it is necessary and sufficient that it verify one of the following equivalent conditions.

(i) B/A is a flat A-module.

(ii) B is flat and, for any ideal $\mathfrak{I}$ of A, we have $(\mathfrak{I}B) \cap A = \mathfrak{I}$.

We shall have occasion to study the following more general situation.

Proposition 4.7. *Let $A \subset B \subset C$ be three rings having the following properties:*

(i) *C is faithfully flat over A.*

(ii) *For any ideal $\mathfrak{I}$ of A, we have $(\mathfrak{I}C) \cap B = \mathfrak{I}B$.*

Then B is faithfully flat over A.

Proof. In fact, let us consider the following exact sequence of A-modules

$$0 \to B/A \to C/A \to C/B \to 0.$$

We wish to show that B/A is flat, and we know that C/A is flat. It is, therefore, enough to show that C/B is flat (Proposition 4.4). Consider the following commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{I} \otimes B & \longrightarrow & \mathfrak{I} \otimes C & \longrightarrow & \mathfrak{I} \otimes (C/B) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{I}B & \longrightarrow & \mathfrak{I}C & \longrightarrow & \mathfrak{I}(C/B) & \longrightarrow & 0.
\end{array}$$

The first row is exact, since the tensor product is right exact, and the second row is exact because of the hypothesis (ii). Now, the first and third vertical arrows are surjective and the second is bijective; hence the third is bijective, whence the result. □

44 **Remark 4.8.** Let us consider again the situation of the preceding proposition, and let $F \subset E$ be two A-modules. To simplify the notation, let us put $FB = F \otimes_A B$, and let us define FC, $EB, \ldots$ in the same way. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & FB & \longrightarrow & FC & \longrightarrow & F(C/B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & EB & \longrightarrow & EC & \longrightarrow & E(C/B) & \longrightarrow & 0
\end{array}$$

Since B, C, C/B are flat, the columns in this diagram are exact, and the rows are also exact because of Proposition 4.4, (iii) and the fact that

C/B is flat. We deduce immediately that if we consider FB, FC, EB as submodules of EC, we have $FC \cap EB = FB$. [This formula, with $E = A$, $F = \mathfrak{I}$ an ideal in A, is nothing but condition (ii) of Proposition 4.7.]

B. Completion of analytic algebras. Let A be a noetherian local ring, E an A-module of finite type furnished with the Krull topology, and $\widehat{E}$ the completion of E. One has the following result (Serre [21]; see also Bourbaki [1].)

Theorem 4.9. (i) *The natural map $\widehat{A} \otimes_A E \to \widehat{E}$ is an isomorphism.*

(ii) *$\widehat{A}$ is faithfully flat over A.*

Let us recall rapidly the proof. In the first place, if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of A-modules of finite type, the topology of E' is induced by that of E, and the topology of E'', which is trivially the quotient of that of E are hausdorff (Corollary 3.13). From properties of the completion of topological groups, we deduce that the sequence $0 \to \widehat{E}' \to \widehat{E} \to \widehat{E}'' \to 0$ is exact. From this we deduce, by a well-known method, that for any exact sequence $E' \to E \to E''$ of A-modules of finite type, the sequence $\widehat{E}' \to \widehat{E} \to \widehat{E}''$ is exact.

Let us apply this to a presentation of E, i.e. to an exact sequence 45
$A^p \to A^q \to E \to 0$. We obtain (i); the fact that $\widehat{A}$ is flat over A is then immediate.

Finally, any E of finite type is hausdorff, so that the mapping $E \to \widehat{E}$ is injective; in particular, $\widehat{E} = \{0\}$ implies that $E = \{0\}$, which completes the proof of (ii).

Let $\mathfrak{p}$ be an ideal in A, $B = A/\mathfrak{p}$. The "intrinsic" topology of B and its topology as A-module coincide. Consequently

$$\widehat{B} \simeq \widehat{A}/\widehat{\mathfrak{p}} \simeq (A/\mathfrak{p}) \otimes_A \widehat{A} \text{ and } \widehat{\mathfrak{p}} \simeq \widehat{A} \otimes_A \mathfrak{p};$$

so that $\widehat{\mathfrak{p}}$ is the closure of $\mathfrak{p}$ in $\widehat{A}$. If we take $A = \mathscr{O}_n$, we have, obviously, $\widehat{A} = \mathscr{F}_n$. The preceding results give then a description of the completions of analytic algebras.

Theorem 4.9, together with Proposition 4.7, has the following consequence.

Proposition 4.10. *Let $A \subset B$ be two noetherian local rings such that the mapping $\widehat{A} \to \widehat{B}$ is an isomorphism. Then B is faithfully flat over A.*

Example 4.11. Let R_n be the field $\mathbf{R}(x_1, \dots, x_n)$ of rational functions and S_n the subring of fractions whose denominator does not vanish at the origin. We have an obvious injection $S_n \to \mathscr{O}_n$ and the two completions are equal to $\mathscr{F}_n$. Hence $\mathscr{O}_n$ is faithfully flat over S_n. Since S_n is trivially flat over $\mathbf{R}[x_1, \dots, x_n]$, we deduce that $\mathscr{O}_n$ is flat over $\mathbf{R}[x_1, \dots, x_n]$.

One could show, in the same way, that $\mathscr{O}_n$ is flat over $\mathscr{O}_{n-1}[x_n]$ (imbedded in $\mathscr{O}_n$ in the obvious way).

C. Coherence. Let $\mathscr{U}$ be the set of open neighbourhoods of 0 in $\mathbf{R}^n$; for $V \in \mathscr{U}$, let $\widetilde{\mathscr{O}}_n(V) = \prod_{x \in V} \widetilde{\mathscr{O}}_{n,x}$ and let us denote by $\mathscr{O}_n(V)$ the space of real valued analytic functions on V. We obtain a mapping
46 $\mathscr{O}_n(V) \to \widetilde{\mathscr{O}}_n(V)$ by associating to $f \in \mathscr{O}_n(V)$ the collection of the Taylor series of f at the various points of V. Let $\widetilde{\mathscr{O}}_n$ be the inductive limit $\widetilde{\mathscr{O}}_n(V)$ following the filtered set $\mathscr{U}$; the above mapping defines an injection $\mathscr{O}_n \to \widetilde{\mathscr{O}}_n$ (which we shall refer to as the canonical injection).

Starting with $\mathscr{F}_n$ instead of $\mathscr{O}_n$, we can define a ring $\widetilde{\mathscr{F}}_n$ and a canonical injection $\mathscr{E}_n \to \widetilde{\mathscr{F}}_n$ (obtained by associating to each $f \in \mathscr{E}_n$ the germ at 0 of the collection of the Taylor series of f at points near 0). Finally, from the injection $\mathscr{O}_n \to \mathscr{F}_n$, we obtain in the obvious way an injection $\widetilde{\mathscr{O}}_n \to \widetilde{\mathscr{F}}_n$ and the following diagram is commutative

$$\begin{array}{ccc} \mathscr{O}_n & \longrightarrow & \widetilde{\mathscr{O}}_n \\ \downarrow & & \downarrow \\ \mathscr{F}_n & \longrightarrow & \widetilde{\mathscr{F}}_n \end{array}$$

One of the main results of this course is to establish that the triple $(\mathscr{O}_n, \mathscr{E}_n, \widetilde{\mathscr{F}}_n)$ satisfies the hypotheses of Proposition 4.7. We shall take up in Chapter VI the condition (ii) which is more difficult, and we shall establish (i) here. For this we need a definition and some results, which we do not number formally since they will not be used outside this article.

Let A be a ring and E an A-module. We say that E is *quasi-flat* if, for any ideal $\mathfrak{I} \subset A$ of *finite presentation* (i.e. for which there is an exact sequence $A^p \to A^q \to \mathfrak{I} \to 0$), the mapping $\mathfrak{I} \otimes_A E \to E$ is injective. This is equivalent to saying that for any $f \in A^q$ such that $R(f, A)$ is of finite type, we have $R(f, E) = R(f, A)E$.

If A is noetherian, "quasi-flat" is equivalent with "flat" since any ideal is of finite presentation. On the other hand, if we have a ring homomorphism $A \to B$ and a B-module, E, "E quasi-flat over B" and "B quasi-flat over A" imply "E quasi-flat over A". (Proof left to the reader.)

Examples. From the fact that $\mathscr{F}_n$ is flat over $\mathscr{O}_n$, we deduce at once that $\widetilde{\mathscr{F}}_n$ is quasi-flat over $\widetilde{\mathscr{O}}_n$. In the same way, $\widetilde{\mathscr{O}}_n$ is quasi-flat over $\widetilde{\mathscr{O}}_{n-1}[x_n]$ **47**
(cf. Example 4.11). This being the case, the theorem we have in view is the following.

Theorem 4.12 (Oka). *$\widetilde{\mathscr{O}}_n$ is faithfully flat over $\mathscr{O}_n$.*

Proof. It is obviously sufficient to show that $\widetilde{\mathscr{O}}_n$ is flat over $\mathscr{O}_n$. The proof is by induction on n. The result being trivial for $n = 0$, suppose the theorem proved for $n - 1$. Let $\mathfrak{I}$ be an ideal in $\mathscr{O}_n$. Let us prove that the map $\mathfrak{I} \otimes_{\mathscr{O}_n} \widetilde{\mathscr{O}}_n \to \widetilde{\mathscr{O}}_n$ is injective. If $\mathfrak{I} = 0$, this is trivial. If $\mathfrak{I} \neq 0$, we may suppose, by an application of 3.7, that after a linear change of coordinates, $\mathfrak{I}$ contains a distinguished polynomial in x_n, say f. Applying 3.6, we find that there is an ideal $\mathfrak{I}' \subset \mathscr{O}_{n-1}[x_n]$ such that

$$\mathfrak{I} = \mathfrak{I}' \mathscr{O}_n \simeq \mathfrak{I}' \otimes_{\mathscr{O}_{n-1}[x_n]} \mathscr{O}_n.$$

Hence $\mathfrak{I} \otimes_{\mathscr{O}_n} \widetilde{\mathscr{O}}_n \simeq \otimes_{\mathscr{O}_{n-1}[x_n]} \mathscr{O}_n$, and it is sufficient to verify that $\widetilde{\mathscr{O}}_n$ is flat over $\mathscr{O}_{n-1}[x_n]$. Now, by induction, $\widetilde{\mathscr{O}}_{n-1}$ is flat over $\mathscr{O}_{n-1}$. One deduces at once that $\widetilde{\mathscr{O}}_{n-1}[x_n]$ is flat over $\mathscr{O}_{n-1}[x_n]$. Since $\widetilde{\mathscr{O}}_n$ is quasi-flat over $\widetilde{\mathscr{O}}_{n-1}[x_n]$, it is quasi-flat, hence flat over the noetherian ring $\mathscr{O}_{n-1}[x_n]$. The theorem follows. □

Corollary 4.13. *$\widetilde{F}_n$ is faithfully flat over $\mathscr{O}_n$.*

In fact, $\widetilde{\mathscr{F}}_n$ is quasi-flat over $\widetilde{\mathscr{O}}_n$, and $\widetilde{\mathscr{O}}_n$ is flat over $\mathscr{O}_n$.

Remark 4.14. The theorem of Oka is usually stated somewhat differently.

(C) Let $f = (f_1, \ldots, f_n)$ be analytic functions in a neighbourhood of 0, and g_i, $1 \leqslant i \leqslant q$, be p-tuples of analytic functions in a neighbourhood of 0 such that their germs g_i^0 at 0 generate $R(f^0, \mathscr{O}_n)$. Then, for any point a sufficiently near 0, the g_i^a generate $R(f^a, \mathscr{O}_n)$.

Using the interpretation of flatness in terms of relations, it is clear that (C) implies (4.12). Conversely, let us prove (C) using (4.12). If
48 (C) were false, there would exist a sequence a_k of points tending to 0 and $\gamma_k \in R(f^{a_k}, \mathscr{O}_n)$ such that γ_k is not a linear combination of the $g_i^{a_k}$. Consider in $\widetilde{\mathscr{O}}_n^p$ the germ defined by $\gamma = \gamma_k$ at a_k, $\gamma = 0$ otherwise. This germ does not belong to $R(f^0, \mathscr{O}_n)\widetilde{\mathscr{O}}_n$, which is absurd.

5 Dimension of analytic algebras and analytic germs.

A. The concept of dimension. Let us recall the following definition.

Definition 5.1. *Let A be a noetherian local ring. By the dimension of A (written* $\dim A$*) we mean the largest integer n for which there exists a strictly decreasing sequence $\mathfrak{p}_0 = \mathfrak{m}(A), \mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of prime ideals of A.*

One proves the following results (see Zariski-Samuel [26]).

Proposition 5.2. *For any noetherian local ring A,*

(i) $\dim A$ *is finite and equal to the minimum number of generators of an ideal of definition of A;*

(ii) we have $\dim A = \dim \widehat{A}$.

It follows at once from (i) that $\dim \mathscr{O}_n = \dim \mathscr{F}_n = n$.

Theorem 5.3 (Cohen-Seidenberg)**.** *Given two noetherian local rings A and B with $A \subset B$ and B finite over A, we have* $\dim A = \dim A$.

The inequality $\dim B \leqslant \dim A$ results immediately from 5.2 (i) since any ideal of definition of A generates an ideal of definition of B (cf. §1).

It would also result from the following proposition, which we shall need in any case.

Proposition 5.4. *Let A and B be two rings with $A \subset B$, B finite over A, and let $\mathfrak{p}$ and $\mathfrak{q}$ be two ideals of B, $\mathfrak{p}$ being prime, such that $\mathfrak{p} \subset \mathfrak{q}$, $\mathfrak{p} \neq \mathfrak{q}$. Then $\mathfrak{p} \cap A \neq \mathfrak{q} \cap A$.*

Proof. Passing to the quotient by $\mathfrak{p}$, we are reduced to the case when $\mathfrak{p} = \{0\}$ and A and B are integral domains. Let $\overline{A}$ (resp. $\overline{B}$) be the 49
quotient field of A (resp. B). Since $\overline{A}[B]$ is a finite $\overline{A}$ algebra which is an integral domain, it is a field, so that $\overline{A}[B] = \overline{B}$. Let $f \in \mathfrak{q}$, $f \neq 0$. We have $\dfrac{1}{f} = \dfrac{b}{a}$ with $b \in B$, $a \in A$, $a \neq 0$. We have $a = fb \in \mathfrak{q} \cap A$, whence the proposition. □

B. Integral analytic algebras. Let $\mathfrak{p}$ be an ideal in $\mathscr{O}_n$, and $A = \mathscr{O}_n/\mathfrak{p}$. Let $g_1, \ldots, g_p \in \mathfrak{m}(\mathscr{O}_n)$, and $\overline{g}_1, \ \overline{g}_p$ their canonical images in A. Recall that there is a unique morphism $u : \mathscr{O}_p \to A$ with $u(y_i) = \overline{g}_i$ ($y_1, \ldots, y_p$ are the coordinates of $\mathbf{R}^p$). If u is injective, we say that $g_1, \ldots, g_p$ are analytically independent modulo $\mathfrak{p}$ (or that $\overline{g}_1, \ldots, \overline{g}_p$ are analytically independent).

Put $k = \dim A$; clearly we have $0 \leqslant k \leqslant n$.

Theorem 5.5. *Under the above hypotheses, we may make a linear change of coordinates in $\mathbf{R}^n$ such that, $x_1, \ldots, x_n$ being the new coordinates, we have*

(i) *$x_1, \ldots, x_k$ are analytically independent modulo $\mathfrak{p}$,*

(ii) *the morphism $\mathscr{O}_k \to A$ defined by $\overline{x}_1, \ldots, \overline{x}_k$ is finite.*

Consider the set (E) of couples $(\underline{f}, S)$ where $\underline{f} = (f_1, \ldots, f_l)$ is a family of elements of $\mathfrak{p}$, all $\neq 0$, and S is a system $(x_1, \ \ , x_n)$ of coordinates obtained by linear change from the canonical system having the following properties for $1 \leqslant p \leqslant l$.

(a) $f_p \in \mathscr{O}_{n-p+1}$, i.e. f_p depends only on $x_1, \ldots, x_{n-p+1}$;

(b) $f_p(0, \ldots, 0, x_{n-p+1}) \neq 0$.

We say that such a couple is maximal if there is no couple $(\underline{f}', S') \in (E)$ with $\underline{f}' = (f_1, \dots, f_l, g)$ and S' being obtained from S by linear change of the $n - 1$ first coordinates *among themselves*. Let us take such a maximal couple (which evidently exists). By applying the preparation theorem, we see that A is finite over $\mathscr{O}_{n-l}$. On the other hand, the mapping $\mathscr{O}_{n-l} \to A$ is injective: if it were not, there would be a $g \in \mathscr{O}_{n-l} \cap \mathfrak{p}$,
50 $g \neq 0$, and by a linear change of the first $n - l$ coordinates among themselves, we could ensure that $g(0, \dots, 0, x_{n-l}) \neq 0$.

By (5.3), we have $\dim A = n - l$, and the theorem follows. From now onwards, we suppose that $\mathfrak{p}$ is prime, and we retain the preceding notation. Let $\overline{\mathscr{O}}_k$ (resp. $\overline{A}$) be the quotient field of $\mathscr{O}_k$ (resp. A). $\overline{A}$ is a finite algebraic extension of $\overline{\mathscr{O}}_k$.

Proposition 5.6. *For any $f \in A$ (resp. $\mathfrak{m}(A)$), the minimal polynomial of f over $\overline{\mathscr{O}}_k$ has its coefficients in $\mathscr{O}_k$ (resp. $\mathfrak{m}(\mathscr{O}_k)$ and is distinguished).*

Proof. $\mathscr{O}_k$ is factorial, hence integrally closed. Since any $f \in A$ is integral over $\mathscr{O}_k$, its minimal polynomial, P, has its coefficients in $\mathscr{O}_k$. Let us show that if $f \in \mathfrak{m}(A)$, F is distinguished. If it were not, we would have $P = P'P''$ with $P' \in \mathscr{O}_k[t]$ distinguished and $P'' \in \mathscr{O}_{k+1}$ (and in fact $P'' \in \mathscr{O}_k[t]$, but this has no importance) is invertible in $\mathscr{O}_{k+1}$. Hence $P''(x_1, \quad , x_k, f)$ is invertible in $\mathscr{O}_k$, and $P'(x_1, \dots, x_k, f) = 0$, and P is not the minimal polynomial of f (if P'' is not constant). The proposition follows.

In the same situation, the preparation theorem shows that the classes $\overline{x}_{k+1}, \dots, \overline{x}_n$ of $x_{k+1}, \dots, x_n$ modulo $\mathfrak{p}$ generate A as an $\mathscr{O}_k$-algebra. A fortiori, they generate $\overline{A}$ over $\overline{\mathscr{O}}_k$. It follows from the theorem of the primitive element that, by a linear change of coordinates of $x_{k+1}, \dots, x_n$ among themselves, we may suppose that $\overline{A} = \overline{\mathscr{O}}_k[\overline{x}_{k+1}]$. Let, then, P be the minimal polynomial of $\overline{x}_{k+1}$, Δ its discriminant, and let $p = \deg P = [\overline{A} : \overline{\mathscr{O}}_k]$. □

Proposition 5.7. *For any $f \in \overline{A}$, integral over A, there exists a unique $Q \in \mathscr{O}_k[t]$, $\deg Q < p$, such that $\Delta f = Q(\overline{x}_{k+1})$.*

Proof. Let $\sigma_1, \ldots, \sigma_p$ be the $\overline{\mathscr{O}}_k$-isomorphisms of $\overline{A}$ in an algebraic closure of $\overline{\mathscr{O}}_k$. For $0 \leqslant i \leqslant p-1$, we have

$$\sigma_1(\overline{x}^i_{k+1} f) + \cdots + \sigma_p(\overline{x}^i_{k+1} f) = a^{(i)} \in \mathscr{O}_k$$

(since $\mathscr{O}_k$ is integrally closed). If we identify A with $\sigma_1(A)$ and solve these equations for $\sigma_1(f)$, we obtain a Q with the required property. The uniqueness is obvious. □

C. Real analytic germs. To conclude this chapter, we shall recall rapidly 51
some results which we require. Let Ω be an open set in $\mathbf{R}^n$. A closed set $F \subset \Omega$ is called analytic if, in a neighbourhood of each of its points, F is the set of common zeros of a finite family of analytic functions. Let us take a point of $\mathbf{R}^n$, say 0. If we identify two analytic sets defined in two neighbourhoods of 0 if they coincide in a third neighbourhood of 0, we define the notion of a (real) analytic germ at 0. To any such germ, E, we make correspond the ideal $\mathfrak{I}(E) \subset \mathscr{O}_n$ of germs of analytic functions which are zero on E. Conversely, to any ideal $\mathfrak{p} \subset \mathscr{O}_n$, we make correspond the germ $V(\mathfrak{p})$ defined by the vanishing of a finite system of generators of the ideal. (It is obvious that $V(\mathfrak{p})$ does not depend on the system chosen.) We always have $V(\mathfrak{I}(E)) = E$ and $\mathfrak{I}(V(\mathfrak{p})) \supset \mathfrak{p}$, but, in general $\mathfrak{I}(V(\mathfrak{p})) \neq \mathfrak{p}$.

A finite union (resp. intersection) of analytic germs is defined in the obvious way, and is again an analytic germ.

We say that a germ E is reducible if we have $E = E' \cup E''$ with $E \neq E'$, $E \neq E''$, and it is irreducible if this is not the case. One verifies at once that E is irreducible if and only if $\mathfrak{I}(E)$ is prime. Any decreasing sequence of analytic germs is stationary (since $\mathscr{O}_n$ is noetherian). We deduce that every analytic germ can be decomposed in a unique way into a finite set of irreducible germs, none of which is contained in the union of the others. We call these the irreducible components.

Let E be an analytic germ. The dimension of E (written $\dim E$) is the dimension of $\mathscr{O}_n/\mathfrak{I}(E)$. If $E = \bigcup_{i=1}^{p} E_i$, we have obviously $\mathfrak{I}(E) = \bigcap_{i=1}^{p} \mathfrak{I}(E_i)$. We deduce from this and Definition 5.1 that the dimension of

E is equal to the largest dimension of its irreducible components.

Let E be irreducible, and $\dim E = k$. Let $\mathfrak{p} = \mathfrak{I}(E)$, and apply to $\mathfrak{p}$ the considerations of §5.B. Using the notation of Proposition 5.7, let us denote by $Q_{k+2}, \dots Q_n$, the polynomials associated to $\overline{x}_{k+2}, \dots, \overline{x}_n$.

52 **Proposition 5.8.** *Let $\overline{E}$ be an analytic set in a neighbourhood of* 0 *whose germ E at* 0 *is irreducible. In a neighbourhood of* 0*, the set of points $x = (x_1, \dots, x_n)$ satisfying $x \in \overline{E}$, $\Delta(x_1, \dots, x_k) \neq 0$ coincides with the set*

$$\begin{aligned} &P(x_1, \dots, x_k, x_{k+1}) = 0 \\ &\Delta(x_1, \dots, x_k)x_{k+j} - Q_{k+j}(x_1, \dots, x_k, x_{k+1}) = 0, 2 \leqslant j \leqslant n-k \\ &\Delta(x_1, \dots, x_k) \neq 0. \end{aligned}$$

Proof. Let $\mathfrak{q}$ be the ideal in $\mathscr{O}_n$ generated by P and the $\Delta x_{k+j} - Q_{k+j}$. Clearly $\mathfrak{q} \subset \mathfrak{p}$, and it is sufficient to show that for any $f \in \mathfrak{p}$, there is an integer p such that $\Delta^p f \in \mathfrak{q}$. For this, denote by $\widetilde{\mathscr{O}}_k$ (resp. $\widetilde{\mathscr{O}}_n, \widetilde{\mathfrak{p}}, \widetilde{\mathfrak{q}}$) the localisation of $\mathscr{O}_k$ (resp. $\mathscr{O}_n$, $\mathfrak{p}$, $\mathfrak{q}$) with respect to Δ, i.e. the set of fractions f/Δ^p with $f \in \mathscr{O}_k$ (resp. $\mathscr{O}_n$, $\mathfrak{p}$, $\mathfrak{q}$). We have to show that $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{q}}$, or that the natural surjection $\widetilde{\mathscr{O}}_n/\widetilde{\mathfrak{q}} \to \widetilde{\mathscr{O}}_n/\widetilde{\mathfrak{p}}$ is bijective.

The ring $\widetilde{\mathscr{O}}_k$ can be considered, in a natural way, as a subring of the two preceding rings, and, if we denote by x'_{k+j} (resp. x''_{k+j}) the image of x_{k+j} in $\widetilde{\mathscr{O}}_n/\widetilde{\mathfrak{p}}$ (resp. $\widetilde{\mathscr{O}}_n/\widetilde{\mathfrak{q}}$), we have the following isomorphisms:

$$\widetilde{\mathscr{O}}_n/\widetilde{\mathfrak{p}} \simeq \widetilde{\mathscr{O}}_k[x'_{k+1}] \simeq \widetilde{\mathscr{O}}_k[x''_{k+1}] \simeq \widetilde{\mathscr{O}}_k[t]/(P)$$

(the first follows from 5.7). Thus we have only to show that x''_{k+1} generates $\widetilde{\mathscr{O}}_n/\widetilde{\mathfrak{q}}$. Now, because of the equations $x''_{k+j} = \dfrac{Q_{k+j}}{\Delta}$, we have $x''_{k+j} \in \widetilde{\mathscr{O}}_k[x''_{k+1}]$. Consequently, the result will be proved if we show that modulo $\mathfrak{q}$, every element of $\widetilde{\mathscr{O}}_n$ is equivalent to an element of $\widetilde{\mathscr{O}}_k[x_{k+1}, \dots, x_n]$. Now, if P_{k+j} is the minimal polynomial of $\overline{x}_{k+j}$ over $\mathscr{O}_k$, $(2 \leqslant j \leqslant n-k)$, the above isomorphisms show that $P_{k+j}(x''_{k+j}) = 0$, so that $P_{k+j} \in \widetilde{\mathfrak{q}}$. This, together with the preparation theorem (more precisely, formula (3.6) applied successively to P, $P_{k+2}, \dots, P_n$) implies the requied result. ☐

Remark 5.9. One must pay attention to the fact that, contrary to the complex case, a prime ideal $\mathfrak{p} \subset \mathscr{O}_n$ is not necessarily of the form $\mathfrak{I}(E)$: in other words, the “Nullstellensatz” is *false* in the real domain. (Counter-example: $n = 2$, $\mathfrak{p}$ = principal ideal generated by $x_1^2 + x_2^2$!) 53

IV
METRIC AND DIFFERENTIAL PROPERTIES OF ANALYTIC SETS

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1 Multipliers.

Let Ω be an open set in $\mathbf{R}^n$ and X a closed subset of Ω. We shall denote by $\mathscr{M}(X;\Omega)$ the set of C^∞-functions f of $\Omega - X$ which satisfy the following condition

(1.1) For any compact set $K \subset \Omega$ and any n-tuple of positive integers $k \in \mathbf{N}^n$, there exist constants $C > 0$, $m > 0$ such that

$$|D^k f(x)| \leqslant C/(d(x,X))^m \text{ for } x \in K - X.$$

We start with the following

Lemma 1.2. *If $g \in \mathscr{M}(X;\Omega)$ and $g \neq 0$ everywhere in $\Omega - X$, then $1/g \in \mathscr{M}(X;\Omega)$ if and only if for any compact $K \subset \Omega$, there are constants $c > 0$, $\alpha > 0$ such that*

$$|g(x)| \geqslant c(d(x,X))^\alpha \text{ for } x \in K - X. \tag{1.3}$$

Proof. If $g^{-1} \in \mathscr{M}(X;\Omega)$, then (1.1) applied to $f = g^{-1}$ with $k = 0$ on K gives (1.3). Conversely, if (1.3) holds, then the condition (1.1) for $f = g^{-1}$ follows from (1.1) for g and the relation

$$D^k f = g^{-|k|-1} P_k(g, \ldots, D^k g),$$

where P_k is a polynomial in the derivatives $D^l g$ with $l \leqslant k$./ □

Proposition 1.4. *If $\mathscr{I}(X;\Omega)$ is the space of C^∞-functions in Ω which are flat on X, then $\mathscr{M}(X;\Omega)$ is a space of multipliers for $\mathscr{I}(X;\Omega)$. More precisely, if $F \in \mathscr{I}(X;\Omega)$, $g \in \mathscr{M}(X;\Omega)$, then the C^∞-function gF on $\Omega - X$ has a unique extension to a C^∞-function on Ω which is flat on X.*

Proof. Since the space $\mathscr{I}_*$ of C^∞-functions in Ω vanishing in a neighbourhood of X are dense in $\mathscr{I}(X;\Omega)$, we have only to prove that multiplication by g is a continuous mapping of $\mathscr{I}_*$ into itself, in the topology induced from $\mathscr{I}(X;\Omega)$, i.e. given $m > 0$, and $K \subset \Omega$ compact, there exists $m' > 0$ and a compact $K' \subset \Omega$, such that for $F \in \mathscr{I}_*$, we have

$$\|gF\|_m^K \leqslant C\|F\|_{m'}^{K'},$$

where $C > 0$ is independent of F. But since for $F \in \mathscr{I}(X;\Omega)$, compact 55
$K \subset \Omega$, and $k \in \mathbf{N}^n$, there is a compact set $K' \subset \Omega$ such that

$$|D^k F(x)| \leqslant C_N(d(x,X))^N \|F\|_{N+k}^{K'} \text{ for } x \in K \text{ and any } N > 0,$$

this follows at once from the condition (1.1) applied to g and Leibniz's formula. □

Proposition 1.5. *If X and Y are closed subsets of Ω which are regularly situated, and $\mathscr{I}(X \cap Y;Y)$ is the space of Whitney C^∞-functions on Y which are flat on $X \cap Y$, then $\mathscr{M}(X;\Omega)$ is a space of multipliers for $\mathscr{I}(X \cap Y;Y)$ (in a sense analogous to that in Proposition 1.4).*

Proof. Since X and Y are regularly situated, if $F \in \mathscr{I}(X \cap Y;Y)$, then the function $\tilde{F}$ defined to be F on Y, 0 on X is induced by a function $f \in \mathscr{I}(X;\Omega)$. Proposition 1.5 thus follows at once from Proposition 1.4. □

2 Quasi-Hölderian functions.

Let Ω be a bounded open set in $\mathbf{R}^n$ and f a real valued function in Ω.

Definition 2.1. *We say that f is quasi-hölderian of order α $0 < \alpha \leqslant 1$, if there exists $C > 0$ such that for any pair of points x, y such that the closed segment $[x,y]$ joining x and y belongs to Ω, we have*

$$|f(x) - f(y)| \leqslant C|x-y|^\alpha.$$

(Note that the condition need not be satisfied for all x, $y \in \Omega$.)

Proposition 2.2. *Let Ω be a bounded open set in $\mathbf{R}^n$ and $a_i, i = 1, \ldots, p$, bounded functions which are quasi-hölderian of order α. Let f be a continuous function on Ω satisfying*

$$f^p + \sum_{i=1}^{p} a_i f^{p-i} = 0.$$

Then f is bounded and quasi-hölderian of order α/p.

The proof is based on three very elementary lemmas.

56 **Lemma 2.3.** *If $c_1, \ldots, c_p, z$ are complex numbers and*

$$z^p + \sum_{i=1}^{p} c_i z^{p-i} = 0,$$

then $|z| \leqslant 2 \sup |c_i|^{1/i}$.

Proof. For reasons of homogeneity, we may suppose $|c_i| \leqslant 1$. Then

$$|z|^p \leqslant 1 + |z| + \quad + |z|^{p-1};$$

a fortiori

$$\sum_{k=1}^{\infty} |z|^{-k} \geqslant 1,$$

whence $|z| \leqslant 2$. □

Lemma 2.4. *Let $z_j (resp.\ z'_k)(j, k = 1, \ldots, p)$ be the roots of the equation*

$$z^p + \sum_{i=1}^{p} c_i z^{p-i} = 0 \left(resp.\ z^p + \sum_{i=1}^{p} c'_i z^{p-i} = 0 \right)$$

where the c_i, c'_i are complex numbers. Suppose that

$$|c_i| \leqslant K^i, |c_i - c'_i| \leqslant K^i \delta, \text{ where } K, \delta > 0.$$

Then for any j, there exists k such that $|z_j - z'_k| \leqslant 2K.\delta^{1/p}$

Proof. Since $z_j^p + \sum\limits_{i=1}^{p} c_i z_j^{p-i} = 0$, we have

$$\prod_k |z_j - z'_k| = \left| z_j^p + \sum_{i=1}^{p} c'_i z_j^{p-i} \right| = \left| \sum_{i=1}^{p} (c'_i - c_i) z_j^{p-l} \right|.$$

By Lemma 2.3, $|z_j| \leqslant 2K$, so that

$$\prod_k |z_j - z'_k| \leqslant 2^p K^p \delta;$$

Lemma 2.4 follows at once.

Proposition 2.2 obviously follows from the next lemma. □

Lemma 2.5. *Let $K > 0$, $0 < \alpha \leqslant 1$ and let $b_1, \ldots, b_p$ be complex-valued functions defined on the closed interval $t_1 \leqslant t \leqslant t_2$ such that if $t_1 \leqslant t, t' \leqslant t_2$, we have*

$$|b_i(t)| \leqslant K^i, \ |b_i(t) - b_i(t')| \leqslant K^i |t - t'|^\alpha.$$

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Let f be a continuous functions on $[t_1, t_2]$ such that

$$f^p + \sum_{i=1}^{p} b_i f^{p-i} = 0.$$

Then, we have

$$|f(t_2) - f(t_1)| \leqslant 4p.K.|t_2 - t_1|^{\alpha/p}.$$

Proof. Let $z_1 = f(t_1), \ldots, z_p$ be the roots of the equation

$$z^p + \sum_{i=1}^{p} b_i(t_1) z^{p-i} = 0$$

and let Ω be the union of the closed discs of radius $2K|t_2 - t_1|^{\alpha/p}$ and center z_j. Then, by Lemma 2.4, $f([t_1, t_2]) \subset \Omega$, and so, since f is continuous, is contained in the connected component of Ω containing z_1. Lemma 2.5 follows at once. □

3 Notations.

In the rest of the chapter, and the following ones we will need to appeal several times to the local description of a real analytic set which was given in Chapter III, §5. For this reason, we shall fix the conventions and notations to which we shall adhere.

Let X be an analytic set in an open set $\Omega \subset \mathbf{R}^n$, let $0 \in X$ and suppose that the germ X_0 of X at 0 is irreducible. We suppose that $\mathfrak{I} = \mathfrak{I}(X)$ is the ideal in $\mathcal{O}_n$ of germs of analytic functions vanishing on X_0. Suppose $\dim X_0 = k$ and that the coordinates $x_1, \ldots, x_n$ of $\mathbf{R}^n$ satisfy the following conditions; as we have seen these can always be achieved by a linear change of coordinates in $\mathbf{R}^n$.

(a) $x_1, \ldots, x_k$ are analytically independent mod $\mathfrak{I}$, i.e. the natural mapping $\mathcal{O}_k \to \mathcal{O}_n/\mathfrak{I}$ is injective ; further $\mathcal{O}_n/\mathfrak{I}$ is a finite $\mathcal{O}_k$ module.

(b) The image in $\mathcal{O}_n/\mathfrak{I}$ of x_{k+1} generates the quotient field of $\mathcal{O}_n/\mathfrak{I}$ over the quotient field of $\mathcal{O}_k$.

(c) The images $\bar{x}_{k+j}$, $j = 1, \ldots, n-k$ of x_{k+j} in $\mathcal{O}_n/\mathfrak{I}$ satisfy the monic polynomial equations

$$P_j(\bar{x}_{k+j}, x') = 0, (x' = (x_1, \ldots, x_k))$$

58 over $\mathcal{O}_k$, (i.e. with coefficients in $\mathcal{O}_k$). Further, we may suppose that these are the minimal equations for $\bar{x}_{k+j}$; therefore the P_j are *distinguished*. We shall denote P_1 by P. Let $\Delta(x_1, \ldots, x_k)$ be the discriminant of the polynomial P in $\bar{x}_{k+1}$. Then, there exist polynomials $Q_j(\bar{x}_{k+j}; x')$ over $\mathcal{O}_k$ such that

$$\Delta(x').\bar{x}_{k+j} = Q_j(\bar{x}_{k+1}; x').$$

In what follows, we write $x = (x_1, \ldots, x_n) = (x', x'')$ where $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_n)$. We denote $n - k$ by l. We choose a neighbourhood $\Omega_1 \subset \Omega$ of $0, \Omega_1 = \Omega' \times \Omega''$ where $\Omega' \subset \mathbf{R}^k$, $\Omega'' \subset \mathbf{R}^l$, such that there are polynomials $P_j(x_{k+j}; x')$, $Q_j(x_{k+1}; x')$, with coefficients analytic on Ω' such that the image of these in $\mathcal{O}_n/\mathfrak{I}$ are the polynomials considered above and which have the same degree in x_{k+j}. We denote again by Δ the discriminant of $P_1 = P$. Δ is analytic on Ω' and its germ at 0 is $\neq 0$. P being distinguished the roots of the equation $P(t, 0) = 0$ are all zero, so that given any neighbourhood V'' of 0 in $\mathbf{R}^l$, there exists a neighbourhood V' of 0 in $\mathbf{R}^k$ such that if $x' \in V'$, $x'' \in \mathbf{R}^l$, $x'' = (x_{k+1}, \ldots, x_n)$ and $P_j(x_{k+j}; x') = 0$, then $x'' \in V''$. We may choose V'' and V' such that $V = V' \times V''$ is relatively compact in Ω_1. We also suppose that V' and V'' are cubes in $\mathbf{R}^k$, $\mathbf{R}^l$ respectively.

Let $\Delta = \{x' \in V' | \Delta(x') = 0\}$. If V is sufficiently small, the set $X \cap ((V' - \delta) \times V'')$ coincides with the set defined by the relations

$$x' \in V' - \delta,\ P(x_{k+1}; x') = 0,$$
$$\Delta(x')x_{k+j} - Q_j(x_{k+1}; x') = 0,\ 2 \leqslant j \leqslant l.$$

Clearly, for $x' \in V' - \delta$, all the roots of $P(x_{k+1}; x') = 0$ are distinct. Let $V_s (1 \leqslant s \leqslant p)$ be the set of points $x' \in V' - \delta$ for which the polynomial $P(x_{k+1}; x')$ has at least s real roots. Then V_s is open and its boundary in V' is contained in δ. Let $F^1(x') < \ldots < F^s(x')$ be the s smallest real roots of $P(x_{k+1}; x')$ on V_s. Clearly, F^r is defined, continuous and bounded on $V_r (1 \leqslant r \leqslant p)$. For $x' \in V_r$, put

$$F_1^r = F^r,\ F_j^r(x') = \frac{Q_j(F^r(x'); x')}{\Delta(x')}$$

The F_j^r are again defined and continuous on V_r, and, being roots of **59**
the equation $P_j(t; x') = 0$, are bounded on V_r. Set $\Phi^r = (F_1^r, \ldots, F_l^r)$ on V_r and

$$X_r' = \{x = (x', x'') \in V | x' \in V_r, x'' = \Phi^r(x')\}.$$

Let $D = X \cap (\delta \times V'')$, $X_r = X_r' \cup D$. Then X_r is closed in V and we have $\bigcup_{1 \leqslant r \leqslant p} X_r = X \cap V$.

4 The inequality of Łojasiewicz.

The aim of this section is to prove the following important theorem of Łojasiewicz [10].

Theorem 4.1. *Let Ω be an open set in $\mathbf{R}^n$ and let f be real analytic in Ω. Let $E = \{x \in \Omega | f(x) = 0\}$. Then for any compact set $K \subset \Omega$, there exist constants c, $\alpha > 0$ such that, for all $x \in K$, we have*

$$|f(x)| \geqslant c(d(x, E))^\alpha.$$

(in other words, $1/f \in \mathscr{M}(E; \Omega)$).

We shall suppose that the theorem is proved for all analytic functions in all open sets in $\mathbf{R}^m$ for $m < n$. The proof consists of two steps.

Step 1. (L). *With the above hypothesis of induction, given an analytic set S in Ω of dimension $< n$ at every point, if f, K, E are as in Theorem 1, then exist constants c, $\alpha > 0$ such that, for $x \in S \cap K$, we have*

$$|f(x)| \geqslant c(d(x, E))^\alpha.$$

Step 2. *Deduction of Theorem 1.1 for $\Omega \subset \mathbf{R}^n$ from (L).*

Proof of (L). It is clearly sufficient to prove that for $a \in S \cap E$, there is a neighbourhood W such that for $x \in W \cap S$, we have

$$|f(x)| \geqslant c(d(x, E))^\alpha$$

for suitable constants c, $\alpha > 0$. We may suppose that $a = 0$. Clearly if X is an analytic subset of $S \cap W$ such that the germ X_0 of X at 0 is irreducible, it is sufficient to prove the above inequality for all $x \in X$. Let $k = \dim X_0$; we may clearly suppose that $X \not\subset E$. We shall proceed by induction on k, and suppose that (L) is proved for all sets S of dimension $< k$. We begin by reducing (L) to the following.

60 (L′). *There is an analytic set $Y \subset X$ in a neighbourhood of 0, $Y \neq X$, and constants $c > 0$, $\alpha > 0$ such that for $x \in X$ near enough to 0, we have*

$$|f(x)| \geqslant c(d(x, Y))^\alpha. \tag{4.2}$$

Proof that (L′) Implies (L). By induction hypothesis, there are constants $B, \beta > 0$ such that for $y \in Y$ sufficiently near 0, we have

$$B|f(y)|^\beta \geqslant d(y, E).$$

Let $x \in X$ and $y \in Y$ be such that $|x - y| = d(x, Y)$. Such a y exists if x is sufficiently near 0. Now, $|f(x) - f(y)| \leqslant B_1|x - y|$ (mean value theorem), so that

$$d(y, E) \leqslant B|f(y)|^{\beta} \leqslant B_2\{|x - y|^{\beta} + |f(x)|^{\beta}\},$$

so that

$$d(x, E) \leqslant |x - y| + B_2\{|x - y|^{\beta} + |f(x)|^{\beta}\}.$$

The result now follows from the fact that $|x - y| = d(x, Y) \leqslant 1/c.|f(x)|^{1/\alpha}$ (by (4.2)).

We will now prove L$'$. Since the ideal $\mathfrak{I}$ is prime, there is $h \in \mathcal{O}_n$ and $f_1 \in \mathcal{O}_k$, $f_1 \neq 0$ such that $hf - f_1 \in \mathfrak{I}$. Obviously, in (L$'$), we may replace f by f_1 and E by the set E_1 of zeros of f_1 in some neighbourhood of 0. We therefore suppose that $f \in \mathcal{O}_k$.

We take now for Y the set $D \cup (E \cap X)$. Since $f \not\equiv 0$ on X, $Y \neq X$ it suffices to prove that on X_s, (notation as in §3), near 0, we have

$$|f(x)| \geqslant c(d(x, (E \cap X_s) \cup D))^{\alpha}.$$

Now f is a function of $x_1, \ldots, x_k$. If E' denotes its zeros in a small neighbourhood of 0, then Theorem 1 applied to $\mathbf{R}^k$ (induction hypothesis) shows that we have an inequality of the form

$$|f(x')| \geqslant c(d(x', E'))^{\alpha}.$$

To complete the proof, we have only to obtain an inequality of the form

$$d(x; (E \cap X_s) \cup D) \leqslant B_3(d(x', E'))^{\gamma} \text{ if } x \in X_s. \tag{4.3}$$

Now, if $x \in D$, there is nothing to prove. Suppose then that $x' \in V_s$ $x = (x', \phi^s(x'))$ and let $y' \in E'$ satisfy $d(x', E') = |x' - y'|$.

If the half-open segment $[x', y'[$ meets δ (4.3) is obvious. If not, the **61**
segment $[x'y'[\subset V_s$ and if y' belongs to this segment and $y'_\nu \to y'$ as $\nu \to \infty$, then any limit point y'' of $\Phi^s(y'_\nu)$ has the property that $(y', y'') \in D \cup (E \cap X_s)$. Hence

$$|\Phi^s(x') - y''| \leqslant \limsup_{\nu \to \infty} |\Phi^s(x') - \Phi^s(y'_\nu)|$$

$$\leqslant B_4|x' - y'|^\gamma = B_4(d(x', E'))^\gamma,$$

the second inequality being valid by Proposition 2.2, since the F_j^s satisfy the monic equations $P_j(F_j^s(x); x') = 0$. This completes the proof of (4.2), and with it the proof of (L′), and thus of (L).

To prove Theorem 1, we have now only to complete Step 2, i.e. to prove that Theorem 1 follows from (L). It suffices to find an analytic set S near $0 \in \mathbf{R}^n$, $\dim_0 S < n$, and constants c, $\alpha > 0$ for which we have, near 0,

$$|f(x)| \geqslant c(d(x, E \cup S))^\alpha.$$

This is because, we have by (L), for $y \in S$ an inequality of the form

$$|f(y)| \geqslant c_1(d(y, E))^{\alpha_1} \quad (c_1, \alpha_1 > 0)$$

and we may repeat the argument used to prove that (L′) implies (L) to obtain the desired inequality. Now, by the Weierstrass preparation theorem, we may suppose that f is a distinguished polynomial in x_n and further, that f is irreducible. [In fact, if the Łojasiewicz inequality is true for two functions it is trivially true for their product.] Thus the discriminant $\Delta_f(x_1, \ldots, x_{n-1}) \neq 0$. Suppose that the coefficients of f and Δ_f are defined in a neighbourhood U of 0. We may then take $S = \{x \in U | \Delta_f(x_1, \ldots, x_{n-1}) = 0\}$. Let $\lambda_1, \ldots, \lambda_r$, be the real roots of the equation $f(x_n; x_1, \ldots, x_{n-1}) = 0$ and $\mu_1, \ldots, \mu_s$, the other roots. Then

$$|f(x)| = \prod_{i=1}^{r} |x_n - \lambda_i| \prod_{j=1}^{s} |x_n - \mu_j|.$$

The first product $\prod_{i=1}^{r}$ is trivially $\geqslant d(x, E)^r$. Now,

$$\prod_{j=1}^{s} |x_n - \mu_j| \geqslant \prod_{j=1}^{s} |\operatorname{im} \mu_j| \geqslant 2^{-s} \prod_{j=1}^{s} |\mu_j - \bar{\mu}_j|.$$

Now the λ_i, μ_j are all bounded and $\Delta_f(x_1, \ldots, x_{n-1})$ is the product of **62**
the squares of the differences of all roots of $f(x_n; x_1, \ldots, x_{n-1}) = 0$. Hence

$$\prod_{j=1}^{s} |\mu_j - \bar{\mu}_j| \geqslant c_2 \Delta_f(x_1, \ldots, x_{n-1}).$$

Thus,

$$|f(x)| \geqslant c_3(d(x, E))^r \Delta_f(x_1, \ldots, x_{n-1}).$$

By induction hypothesis, $\Delta_f(x_1, \ldots, x_{n-1}) \geqslant c_4(d(x; S))^\beta$, and it follows that

$$|f(x)| \geqslant c(d(x, E \cup S))^\alpha.$$

This proves Theorem 4.1.

Corollary 4.4. *Let Ω be an open set in $\mathbf{R}^n$ and let X and Y be two analytic sets in Ω. Then X and Y are regularly situated.*

Proof. Clearly, it is enough to prove that for any $a \in X \cap Y$, there exists a neighbourhood U of a such that $X \cap U$ and $Y \cap U$ are regularly situated in U. Hence we may suppose that there exist analytic functions f, g in Ω such that $\{x \in \Omega | f(x) = 0\} = X$, $\{x \in \Omega | g(x) = 0\} = Y$. Let K be any compact set in Ω. Then, there exists a constant $B > 0$ such that for $x \in K$,

$$|g(x)| \leqslant Bd(x, Y). \tag{4.5}$$

By Theorem 4.1, applied to the function $f^2 + g^2$ there are constants c, $\alpha > 0$ so that for $x \in K$, we have

$$f^s(x) + g^2(x) \geqslant c(d(x, X \cap Y))^\alpha,$$

since $X \cap Y = \{x \in \Omega | f^2(x) + g^2(x) = 0\}$. Combining this with (4.5), we obtain,

$$d(x, Y) \geqslant c_1(d(x, X \cap Y))^{\alpha/2} \text{ for } x \in X \cap K, \text{ q.e.d.}$$

□

5 Further properties of analytic sets.

The above corollary gives us information on the metric properties of two analytic sets. We now go back to the notation of §3, and prove some metric properties of different "sheets" of the same irreducible analytic set, due also to Łojasiewicz [10].

63 Let X be an analytic set in the open set $\Omega \subset \mathbf{R}^n$, irreducible at the origin. suppose $k = \dim_0 X$ and let $V = V' \times V''$ be a neighbourhood of 0 as in §3. We have defined closed sets X_r in V, $1 \leqslant r \leqslant p$ in §3. We have

Proposition 5.1. *For any pair of integers r, s, the sets X_r, X_s are regularly situated.*

Proof. We may obviously suppose $r < s$, so that $V_r \supset V_s$. It is clear that for any compact set $K' \subset V'$, there exists a compact set $K'' \subset V''$ such that $(K' \times V'') \cap X = (K' \times K'') \cap X$. Let $K = K' \times K''$ We have to prove that there exist constants $c, \alpha > 0$ so that for $x \in K \cap X_s$, $y \in K \cap X_r$, we have $|x-y| \geqslant c(d(x, D))^\alpha$ (since $X_s \cap X_r = D$). We have already seen (in the proof of Theorem 4.1) that for $x \in X$, we obtain an inequality of the form

$$d(x', \delta) \geqslant B(d(x, D))^\beta$$

from the fact that the functions F^r_j are quasi-hölderian. Hence we have only to prove an inequality of the form

$$|x - y| \geqslant c(d(x', \delta))^\alpha. \tag{5.2}$$

Let $x = (x', x'')$, $y = (y', y'')$, where $x' \in V_s$, $y' \in V_r$. If the closed segment $[x', y'] \not\subset V_s$, then it meets δ and (5.2) is trivial. Suppose therefore that $[x', y'] \subset V_s$. Let $\eta = (x', \Phi^r(x'))$. Now, $F^r(x')$ and $F^s(x')$ are two distinct roots of the equation $P(t; x') = 0$. Hence there is a constant $A > 0$ so that

$$|F^r(x') - F^s(x')| \geqslant A|\Delta(x')|.$$

Hence, by Theorem 1 applied to Δ, we have

$$|x - \eta| \geqslant |F^r(x') - F^s(x')| \geqslant B_1(d(x', \delta))^{\beta_1}.$$

On the other hand,

$$\begin{aligned} |y - n| &\leqslant |y' - x'| + \Phi^r(x') - \Phi^r(y')| \\ &\leqslant B_2 |x' - y'|^{\beta_2} \end{aligned} \tag{5.3}$$

(since the functions F^r_j are quasi-hölderian). If now $B_2|x' - y'|^{\beta_2} \geqslant \frac{1}{2}B_1(d(x', \delta))^{\beta_1}$, (5.2) is trivial. Otherwise, we have $|y - \eta| \leqslant \frac{1}{2}|x - \eta|$, so that $|x - y| \geqslant \frac{1}{2}|x - \eta| \geqslant \frac{1}{2}B_1(d(x', \delta))^{\beta_1}$, and is (5.2) proved. □

Proposition 5.3. *For $1 \leqslant j \leqslant l(= n - k)$, $1 \leqslant r \leqslant p$, the functions F^r_j* 64
belong to the space $M(V' - V_r; V')$.

Proof. By Lemma 1.2 and Theorem 4.1, $\dfrac{1}{\Delta} \in \mathscr{M}(V' - V_r; V')$. Hence, we have only to prove the proposition for $j = 1$, i.e. for the function F^r. We prove by induction on $|q|(q \in \mathbf{N}^k)$ an estimate of the form

$$|D^q F^r(x')| \leqslant \frac{C_q}{d(x', \delta)^{m_q}} \tag{5.4}$$

for $x' \in K' - \delta$, K' being a compact subset of V'. Suppose $q \in \mathbf{N}^n$ and suppose (5.4) proved for all q' with $|q'| < |q|$. Since $P(F^r(x'); x') = 0$, we have a relationship

$$\left(\frac{\partial P}{\partial x_{k+1}}(F^r; x')\right)^{\lambda_q} \quad D^q F^r = R_q(F^r, D^{q'} F^r; x'),$$

where λ_q is an integer > 0 and R_q is a polynomial in F^r and its derivatives $D^{q'} F^r$ of order $< |q|$ (differentiation of composite functions). After our induction hypothesis and Theorem 4.1, we have only to prove an inequality of the form

$$\left|\frac{\partial P}{\partial x_{k+1}}(F^r; x')\right| \geqslant c.|\Delta(x')|.$$

But this is immediate, and the proposition follows. □

We shall end this section by giving a description of the space $\mathscr{I}(D; X_r)$ of Whitney functions on X_r which are flat on D.

Let $\lambda \in \mathbf{N}^n = \mathbf{N}^k \times \mathbf{N}^l$, and let

$$F = \{f^\lambda\} \in \mathscr{I}(D; X_r).$$

We remark that F is determined uniquely by the collection $\{g^\mu\}_{\mu \in \mathbf{N}^l}$ where

$$g^\mu = f^\lambda \text{ with } \lambda = 0 \times \mu,\ 0 \in \mathbf{N}^k.$$

In fact, if $\lambda = \nu \times \mu$, $\nu \in \mathbf{N}^k$ then f^λ is a linear combination of derivatives $D_{x'}^{\nu'} g^{\mu'}(x', \Phi^r(x'))$ with $\mu' \leqslant \mu$. (See also proof of (5.5b) given below.)

65 Given (g^μ) which determines an element of $\mathscr{I}(D; X_r)$, let us set

$$h^u(x') - g^\mu(x'; \Phi^r(x')) \in \mathscr{E}(V_r).$$

This gives us a mapping

$$\pi : \mathscr{I}(D; X_r) \to [\mathscr{E}(V_r)]^{\mathbf{N}^l}$$

Proposition 5.5. *π maps $\mathscr{I}(D; X_r)$ bijectively onto $[\mathscr{I}(V' - V_r; V')]^{\mathbf{N}^l}$.*

Proof. As remarked above, π is injective. We have only to prove the following two facts;

$$\pi(\mathscr{I}(D; X_r)) \subset [\mathscr{I}(V' - V_r; V')]^{\mathscr{N}^l}, \tag{5.5a}$$

$$\pi(\mathscr{I}(D; X_r)) \supset [\mathscr{I}(V' - V_r' V')]^{\mathscr{N}^l} \tag{5.5b}$$

□

Proof of (a). We remark that any derivative of $h^\mu(x')$ can be expressed as a finite linear combination of the functions $f^\lambda(x', \Phi^r(x'))$ with coefficients which are polynomials in the derivatives of F_j^r. (This can be proved, for example, by choosing a C^∞-function in V inducing (f^λ) and applying the rule for differentiation of composite functions.) To prove (a), we have only to prove : given a compact subset K' of V',

$f^\lambda(x', \Phi^r(x'))$ tends to zero faster than any positive power of $d(x', \delta)$ when $x' \in V_r \cap K'$ tends to δ. But this follows from the definition of $\mathscr{I}(V' - V_r, V')$ and the fact that Φ^r is quasi-hölderian.

Proof of (b). Let $h = (h^\mu)_{\mu \in \mathbf{N}^l}$, $h^\mu \in \mathscr{I}(V' - V_r; V')$ be given. It is enough to prove that for any integer $m > 0$, there is a C^m functions H on V, m-flat on $(V' - V_r) \times V''$, such that for $\mu \in \mathbf{N}^l$, $|\mu| \leqslant m$, we have

$$D^\mu_{x''} H(x'; \Phi^r(x')) = h^\mu(x') (D^\mu_{x''} = D^{\mu_1}_{x_{k+1}} \ldots D^{\mu_l}_{x_n}).$$

We take $H = 0$ on $(V' - V_r) \times V''$ and

$$H(x) = \sum_{|\mu| \leqslant m} h^\mu(x') \frac{(x'' - \Phi^r(x'))^\mu}{\mu!} \text{ for } x = (x', x'') \in V_r \times V''.$$

By Proposition 5.3 since $h^\mu(x') \in \mathscr{I}(V' - V_r; V')$, H is C^∞ on V. Clearly this has the required properties.

V
THE PREPARATION THEOREM FOR DIFFERENTIABLE FUNCTIONS

66

1 The special preparation theorem.

The aim of this chapter is to prove the preparation theorem for differentiable functions. We begin by stating the theorem in a special case.

(1.1) The special Preparation Theorem. Let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $t \in \mathbf{R}^n$, $t \in \mathbf{R}$, and let

$$\Pi(x,t) = t^p + \sum_{i=1}^{p} a_i(x) t^{p-i}$$

be a distinguished polynomial in t with coefficients which are analytic functions of x in a neighbourhood of $x = 0$ in $\mathbf{R}^n$. Then, for any $f \in \mathscr{E}_{n+1}$, there exists $g \in \mathscr{E}_{n+1}$ and $\rho_i \in \mathscr{E}_n$, $0 \leqslant i \leqslant p-1$, such that

$$f(x,t) = \Pi(x,t) g(x,t) + \sum_{i=0}^{p-1} \rho_i(x) t^i. \tag{W}$$

We first state a more general theorem which is more convenient to handle. To state this, we introduce some notations.

Let X be an analytic set in a neighbourhood of $0 \in \mathbf{R}^n$, X_0 its germ at $x = 0$. For any set $A \supset \mathbf{R}^n$, denote by A_0 the germ of the set A at $x = 0$ and $\widehat{A}_0$ the germ of the set $A \times \mathbf{R}$ at $(x,t) = (0,0)$, with a similar convention for germs of sets in $\mathbf{R}^n$. Let Y be an analytic subset of X, and let $\mathscr{I}(Y_0; X_0)$ denote the space of germs of Whitney functions on X_0 which are flat on Y_0, and define $\mathscr{I}(\widehat{Y}_0, \widehat{X}_0)$ in a similar way. Then we have

Theorem 1.2. *Let* Π, X_0, Y_0 *be as above. Then for any* $f \in \mathscr{I}(\widehat{Y}_0; \widehat{X}_0)$, *there exist germs* $g \in \mathscr{I}(\widehat{Y}_0; \widehat{X}_0)$ *and* $\rho_i \in \mathscr{I}(Y_0, X_0)$, $0 \leqslant i \leqslant p-1$ *such that* (W) *holds.*

The special preparation theorem follows on taking $X = \mathbf{R}^n$, $Y = \varnothing$.

We begin by reducing Theorem 1.2 to a weaker statement. For this **67**
purpose, n, Π being given, let us denote by $Th(Y_0, X_0)$ the statement of Theorem 1.2. It is clear that if $Z_0 \subset Y_0 \subset X_0$, and $Th(Z_0, Y_0)$ and $Th(Y_0, X_0)$ are true, then $Th(Z_0, X_0)$ is true. The weaker statement referred to is

$P(X_0)$. *Given the germ X_0 of an analytic set in $\mathbf{R}^n$, for any analytic germ $Y_0 \subset X_0$, $Y_0 \neq X_0$, there exists an analytic germ Y'_0, $Y_0 \subset Y'_0 \underset{\neq}{\subset} X_0$ such that $Th(Y'_0, X_0)$ is true.*

We prove that $P(X_0)$ for any X_0 implies $Th(Y_0, X_0)$. Let $\mathfrak{H}$ be the set of analytic germs Z_0, $Y_0 \subset Z_0 \subset X_0$ such that $Th(Z_0, X_0)$ is true. Clearly $\mathfrak{H}$ is nonempty $(X_0 \in \mathfrak{H})$, so that since any decreasingly filtered family of analytic germs is stationary, $\mathfrak{H}$ contains a minimal element, which we again denote Z_0. We prove that $Y_0 = Z_0$. If this were not so, then, by $P(Z_0)$, there is Z'_0, $Y_0 \subset Z'_0 \underset{\neq}{\subset} Z_0$ such that $Th(Z'_0, Z_0)$ is true. But since $Th(Z_0, X_0)$ is true, it follows that $Th(Z'_0, X_0)$ is true, and Z_0 is not minimal.

Thus, we have only to prove $P(X_0)$. We do this in two stages; first, when $X = \mathbf{R}^n$, we will prove a stronger result (which proves an analogue of $P(X_0)$ for a fixed neighbourhood of 0) and then we will reduce the general case to this.

2 The case $X = \mathbf{R}^n$.

Let V be a neighbourhood of 0, Y an analytic subset of V, Y_0 its germ at the origin; we suppose that the coefficients of Π are analytic in a neighbourhood of $\overline{V}$.

Let I be a bounded open interval in $\mathbf{R}$ such that every real root of $\Pi(x, t) = 0$ lies in I if $x \in V$; for any subset A of V, we set $\widehat{A} = A \times I$.

We may suppose in Theorem 1.2 that the polynomial Π is irreducible in $\mathscr{O}_n[t]$. Hence its discriminant $\Delta \not\equiv 0$ in V. Let $\delta = \{x \in V | \Delta(x) = 0\}$. We shall prove that $Th(Y'_0, \mathbf{R}^n)$ is true, where $Y'_0 = Y_0 \cup \delta_0$. More precisely, we have

Proposition 2.1. *Let $Y' = Y \cup \delta$. For any $f \in \mathscr{I}(\widehat{Y}'; \widehat{V})$, there exists $g \in \mathscr{I}(\widehat{Y}'_0, \widehat{V})$ and $\rho_i \in \mathscr{I}(Y', V)$, $0 \leqslant i \leqslant p-1$ such that $f = \Pi g + \sum \rho_i t^i$ on $\widehat{V}$.*

68 *Proof.* Let U_s be the set of points of $V - \delta$ where Π has exactly s real roots. Then U_s is open and its boundary is contained in δ. Let $U'_s = U_s - Y$, $F_s = V - U_s$. Let $f^{(s)} = f$ in $\widehat{U}_s = 0$ in $\widehat{F}_s$. Then $f = \sum_{s=0}^{p} f^{(s)}$ and it is enough to prove the proposition for each $f^{(s)}$. Hence we may suppose that $f = 0$ outside U_s for some s. □

Let $\tau_1(x) < \ldots < \tau_s(x)$ be the real roots of $\Pi(x,t)$ for $x \in U_s$. Then, $\tau_i(x)$ are quasi-hölderian, and belong to the space $\mathscr{M}(F_s; V)$ [Proposition 5.3.]

Let $f_1(x,t)$ be the functions defined by

$$f(x,t) = (t - \tau_1(x)) f_1(x,t) + f(x, \tau_1(x))$$

in $\widehat{U}_s$, $f_1 = 0$ in $\widehat{F}_s$. We assert that $f_1(x,t)$, $f(x,\tau_1(x))$ belong to the spaces $\mathscr{I}(\widehat{F}_s \cup \widehat{Y}; \widehat{V})$ and $\mathscr{I}(F_s \cup Y; V)$ respectively. It is clearly sufficient to prove that they belong to the spaces $\mathscr{I}(\widehat{F}_s; \widehat{V})$ and $\mathscr{I}(F_s; V)$ respectively since they are clearly flat on $\widehat{Y} \cap \widehat{U}_s$. We have seen in the proof of Proposition 5.3 that $f(x,\tau_1(x)) \in \mathscr{I}(F_s; V)$. For f_1, we write, for $(x,t,\tau) \in V \times I \times I$,

$$f(x,t) - f(x,\tau) = (t - \tau) h(x,t,\tau),$$

with $h \in \mathscr{I}(\widehat{F}_s \times I; \widehat{V} \times I)$. Clearly

$$f_1(x,t) = h(x,t,\tau_1(x)),$$

and it follows, as before, that $f_1 \in \mathscr{I}(\widehat{F}_s; \widehat{V})$. We apply the same procedure with f replaced by f_1 and $\tau_1(x)$ by $\tau_2(x)$ and obtain

$$f(x,t) = (t - \tau_1(x))(t - \tau_2(x)) f_2(x,t) + (t - \tau_1(x)) f_1(x, \tau_1(x)) + f(x, \tau_1(x)),$$

where $f_2 \in \mathscr{I}(\widehat{F}_s \cup \widehat{Y}; \widehat{V})$. Furthermore we have

$$f_1(x, \tau_2(x)), \tau_1(x) f_1(x, \tau_2(x)) \in \mathscr{I}(F_s \cup Y; V).$$

Repeating this process s times, we find

$$f(x,t) = (t - \tau_1(x)) \ldots (t - \tau_s(x)) f_s(x,t) + \sum_{i=0}^{s-1} \rho_i(x) t^i,$$

where $f_s \in \mathscr{I}(\widehat{F}_s \cup \widehat{Y}; \widehat{V})$ and $\rho_i \in \mathscr{I}(F_s \cup Y; V)$.

Now we have **69**

$$\Pi(x,t) = \prod_{i=1}^{s}(t - \tau_i(x))\Pi'(x,t), x \in U_s.$$

The proposition would obviously be proved if we show that the function

$$g = \begin{cases} f_s/\Pi' & \text{in } \widehat{U}_s, \\ 0 & \text{in } \widehat{F}_s \end{cases}$$

belongs to $\mathscr{I}(\widehat{F}_s \cup \widehat{Y}; \widehat{V})$. Clearly, since Π' does not vanish at any point of U_s, we have only to prove that $1/\Pi' \in \mathscr{M}(\widehat{F}_s; \widehat{V})$. By (IV, 4.1) it suffices to prove that

(a) $\Pi' \in \mathscr{M}(\widehat{F}_s; \widehat{V})$;

(b) for any compact $K \subset \widehat{V}$, $|\Pi'(x,t)| \geqslant c(d(x,\delta))^{\alpha}$ for suitable c, $\alpha > 0$ and $(x,t) \in K - \widehat{\delta}$.

Proof of (a). Let $\lambda_1, \ldots \lambda_s$ be new variables and let us divide the polynomial $\Pi(x,t)$ by $(t-\lambda_1)\ldots(t-\lambda_s)$. This gives us, with $\lambda = (\lambda_1, \ldots, \lambda_s)$,

$$\Pi(x,t) = (t-\lambda_1)\ldots(t-\lambda_s)\Psi(x;t;\lambda) + \Psi'(x;t;\lambda)$$

where Ψ, Ψ' are polynomials in t, λ with coefficients which are analytic functions on V. Clearly $\Psi'(x;t;\tau) = 0$, where $\tau = (\tau_1(x), \ldots, \tau_s(x))$, so that

$$\Pi'(x,t) = \Psi(s;t;\tau).$$

Since $\tau_1(x) \in \mathscr{M}(F_s, V)$, (a) is proved.

Proof of (b). If σ_j are the complex roots of $\Pi(x,t)$ we have for $(x,t) \in K$,

$$|\Pi'(x,t)| = \prod_j |t - \sigma_j| \geqslant \prod_j |\operatorname{im}\sigma_j| \geqslant c_1 \prod_j |\overline{\sigma}_j - \sigma_j| \geqslant c_2|\Delta(x)|^2,$$

where c_1, c_2, > 0. The result follows from the Łojasiewicz inequality (Theorem IV, 4. 1.).

Remark. We remark here that the uniqueness of the remainder is not 70
guaranteed. The remainder we obtained is of degree $\leqslant s-1$ on U_s, and it is not difficult to see that we could obtain any function R of degree $\leqslant p-1$ as remainder so long as $R \in \mathscr{I}(\widehat{F}_s \cup \widehat{Y}; \widehat{V})$ and $R(x;\tau_i(x)) = f(x;\tau_i(x))$ for $i = 1,\ldots,s$. This shows that we have uniqueness in Proposition 2.1 only in the case when *all the roots of* Π *are real*.

3 The proof of Theorem 1.2 in the general case.

We begin with the following remark. Suppose X_0 is as in §§1, 2, and $X_0 = X_0' \cup X_0''$. Suppose the statements $P(X_0')$ and $P(x_0'')$ are true. Then $P(X_0)$ is true. In fact, we may suppose that $S_0 = X_0' \cap X_0''$ is properly contained in both X_0' and X_0''; let now $Y_0 \subset X_0$, and $Z_0' = X_0' \cap (Y_0 \cup S_0)$, $Z_0'' = X_0'' \cap (Y_0 \cup S_0)$. If $Z_0' \subset W_0' \subset X_0'$, $Z_0'' \subset W_0'' \subset X_0''$ are such that $Th(W_0', X_0')$ and $Th(W_0'', X_0'')$ are true, then since X_0', X_0'' are regularly situated, we conclude that $Th(W_0' \cup W_0'', X_0)$ is true.

Hence we may suppose that X_0 is irreducible. We now go back to the notations of Chapter IV, §3.

Let the neighbourhood V of 0 be chosen sufficiently small and so as to have all the properties stated in Chapter IV, §3. Let Y be an analytic subset of X, (we regard X as a closed subset of V in what follows).

We may suppose that Y is the inverse image under the projection $X \to V' \subset \mathbf{R}^k$ of an analytic set $S \subset V'$, $S \neq V'$. To see this, we have only to use the fact that if $\phi \in \mathscr{O}_n$, $\phi = 0$ on Y, $\phi \notin \mathfrak{J} = \mathfrak{J}(X_0)$, then there is $h \in \mathscr{O}_n - \mathfrak{J}$ such that $h\phi - \phi_1 \in \mathfrak{J}$, where $\phi_1 \in \mathscr{O}_k$. We assert that there exists a polynomial $\Psi \in mathscrO_n[t]$ such that Ψ is a multiple of Π in $\mathscr{O}_n[t]$ and $\Psi = \Psi' + \Psi''$, where $\Psi'' \in \mathfrak{J}[t]$, and $\Psi' \in \mathscr{O}_k[t]$ is a distinguished polynomial. To prove this, let Ψ' be the product of the conjugates of the image $\overline{\Pi}$ of Π in $(\mathscr{O}_n/\mathfrak{J})[t]$ over $\mathscr{O}_k[t]$, say $\Psi' = \overline{\Lambda}\,\overline{\Pi}$. Let $\Lambda \in \mathscr{O}_n[t]$ induce $\overline{\Lambda}$, and $\Psi = \Lambda\Pi$. Clearly $\Psi' = \overline{\Lambda}\,\overline{\Pi}$ is a distinguished polynomial in $\mathscr{O}_k[t]$, and $\Psi'' = \Lambda\Pi - \Psi' \in \mathfrak{J}[t]$.

71 We remark that it is enough to prove $P(X_0)$ with Π replaced by Ψ; in fact if $R(x,t) = \sum\limits_{i=0}^{p} r_i(x)t^i$ is the remainder of the division of f by Ψ, we have only to carry out the standard polynomial division of R by Π

to prove $P(X_0)$ for Π, since clearly the coefficients in this polynomial division will be flat wherever the $r_i(x)$ are flat.

We suppose that Ψ' and Ψ'' have coefficients analytic on V and that I is a bounded open interval in $\mathbf{R}$ such that any real root of $\Psi(x';t)$ lies in I for $x' \in V'$. For any subset A of V', we shall write $\widehat{A}_k$ for $A \times I \subset \mathbf{R}^{k+1}$, $\tilde{A}$ for $A \times V'' \subset \mathbf{R}^n$, $\widehat{A}_n$ for $A \times V'' \times I = \tilde{A} \times I \subset \mathbf{R}^{n+1}$.

Applying Proposition 2.1 to the irreducible factors of Ψ', we find that if V is sufficiently small, there is an analytic set S' of V', $S \cup \delta \subset S'$, such that $\dim S' < k$ for which any $f \in \mathscr{I}(\widehat{S}'_k; \widehat{V}'_k)$ can be written $f = \Psi' g + \sum\limits_{i=0}^{p'} \rho_i t^i$, $p' = \deg . \Psi'$, where $\rho_i \in \mathscr{I}(S'; V')$ and $g \in \mathscr{I}(\widehat{S}'_k; \widehat{V}'_k)$. Let Y' be the set $\tilde{S}' \cap X$. We shall prove the following result, which clearly implies $P(X_0)$ for Ψ.

Proposition 3.4. *If $f \in \mathscr{I}(\widehat{Y}'; \widehat{X})$, then there exist $g \in \mathscr{I}(\widehat{Y}'; \widehat{X})$ and $\rho_i \in \mathscr{I}(Y'; X)$, $0 \leqslant i \leqslant p' - 1$, such that*

$$f = \Psi_g + \sum \rho_i t^i.$$

Proof. Since the sets X_r are regularly situated and $X_r \cap X_s \subset Y'$, it is enough to prove the proposition with X replaced by X_r. To do this, we remark that if π is the isomorphism of $\mathscr{I}(D; X_r)$ onto $[\mathscr{I}(V' - V_r; V')]^{\mathbf{N}^l}$ given by Chapter IV, Proposition 5.5, then π induces an isomorphism of $\mathscr{I}(Y'; X_r)$ onto $[\mathscr{I}(C; V')]^{\mathbf{N}^l}$, where $C = (V' - V_r) \cup S'$ There is further a similar isomorphism $\widehat{\pi}$ of $\mathscr{I}(\widehat{Y}'; \widehat{X}_r)$ onto $[\mathscr{I}(\widehat{C}; \widehat{V}')]^{\mathbf{N}^l}$, defined by $\widehat{\pi}(F) = (G^\lambda)$, $\lambda \in \mathbf{N}^l$, where $G^\lambda(x,t) = (\pi F_t)^{(\lambda)}(x)$, F_t standing for the function $x \to F(x,t)$. We have therefore only to prove that if $f \in \mathscr{I}(\widehat{Y}'; \widehat{X}_r)$, then there are $\rho_i \in \mathscr{I}(Y'; X_r)$, $t \in \mathscr{I}(\widehat{Y}'; \widehat{X}_r)$ with

$$\widehat{\pi}(f) = \Psi \widehat{\pi}(g) + \sum \pi(\rho_i) t^i. \tag{3.1}$$

If we write 72

$$\widehat{\pi}(f) = (f^\lambda), \text{ and } \Psi^\lambda = D^\lambda_{x''} \Psi(x', \Phi^r(x'), t),$$

then $\Psi^\lambda \in \mathscr{M}(\widehat{V}' - \widehat{V}_r; \widehat{V}')$, so that the equation (3.1) is equivalent with the infinite system

$$f^\lambda = \Psi' g^\lambda + \sum_{\mu<\lambda} \binom{\lambda}{\mu} \Psi^{\lambda-\mu} g^\mu + \sum_{i=0}^{p'-1} \rho_i^\lambda t^i, \tag{3.2}$$

for the functions $\rho_i^\lambda \in \mathscr{I}(C; V)$, $g \in \mathscr{I}(\widehat{C}, \widehat{V})$. The existence of the g^λ, ρ_i^λ is an immediate consequence of Proposition 5.3. In fact suppose g^μ, ρ_i^μ are constructed for $\mu < \lambda$; we have only to solve the equation

$$h^\lambda = \Psi' g^\lambda + \sum_{i=1}^{p'-1} \rho_i^\lambda t^i$$

where $h^\lambda = f^\lambda - \sum_{\mu<\lambda} \binom{\lambda}{\mu} \Psi^{\lambda-\mu} g^\mu$ which belongs to $\mathscr{I}(\widehat{C}; \widehat{V})$ by IV, Proposition 1.4. Note that since the system (3.2) is infinite, we need Proposition 2.1; the statement $P(X_0)$ would not suffice. □

Remark. Suppose that in the special preparation theorem we add the following condition:

At any point near 0, the Taylor expansion of Π divides that of f in the ring of formal power series.

The above proof then shows that we may take the $\rho_i = 0$ in the theorem. [The only point is that in §3, we must apply our considerations to Λf since we replace Π by $\Lambda\Pi$.] This gives us the following theorem.

Let Ω be open in $\mathbf{R}^n$, Π an analytic function in Ω. A functions $f \in \mathscr{E}(\Omega)$ is of the form Π_g, $g \in \mathscr{E}(\Omega)$, if and only if the Taylor expansion of Π divides that of f at any point of ω.

(Łojasiewicz [10], Hörmander [6].)

In view of the results of Chapter II, this may be formulated as asserting that *a principal ideal in $\mathscr{E}(\Omega)$ generated by an analytic function is closed.* We shall prove a generalization of this theorem to arbitrary ideals generated by analytic functions in the next chapter.

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4 The general preparation theorem.

Theorem 4.1 (Malgrange [12])**.** *Let A and B be differentiable algebras and $u; A \to B$ a morphism. If u is quasi-finite, then u is finite.*

Before beginning the proof, we remark that we have already proved that the quasi-finiteness of u if equivalent with $\hat{u}$ being finite, as also with $\hat{u}$ being quasi-finite. Moreover, we may assume that $A = \mathscr{E}_n$, $B = \mathscr{E}_m$ [$\mathscr{E}_k$ being the ring of germs of differentiable functions at $0 \in \mathbf{R}^k$]; this is proved in the same way as in the analytic case.

We have the following

Lemma 4.2. *Let u be a surjective morphism of a differentiable algebra A onto a differentiable algebra B. Let $\hat{u}$ be the induced morphism of the completions : $\hat{u} : \hat{A} \to \hat{B}$. Then* $\ker \hat{u} = (\ker u + \mathfrak{m}^{\infty}(A))/\mathfrak{m}^{\infty}(A)$.

Proof. Since clearly $\ker \hat{u} = u^{-1}(\mathfrak{m}^{\infty}(B))/\mathfrak{m}^{\infty}(A)$, we have only to prove that $u^{-1}(\mathfrak{m}^{\infty}(B)) = \ker u + \mathfrak{m}^{\infty}(A)$. Now, since u is surjective, we have $u(\mathfrak{m}(A)) = \mathfrak{m}(B)$. Hence, for any k, $u(\mathfrak{m}^k(A)) = \mathfrak{m}^k(B)$, so that $u^{-1}(\mathfrak{m}^k(B)) = \mathfrak{m}^k(A) + \ker u$. Hence $u^{-1}(\mathfrak{m}^{\infty}(B)) = \bigcap_{k \geqslant 0} (\ker u + \mathfrak{m}^k(A))$. Since the completion $\hat{A}$ of A is noetherian, it follows from Krull's theorem that $\bigcap_{k \geqslant 1} (\ker u + \mathfrak{m}^k(A)) = \ker u + \mathfrak{m}^{\infty}(A)$, and the lemma is proved. □

Let $u : \mathscr{E}_n \to \mathscr{E}_m$ be a quasi-finite morphism, and let $\phi : \mathbf{R}^m \to \mathbf{R}^n$ be a differentiable mapping with $\phi^* = u$. We denote the coordinates in $\mathbf{R}^m$ by $x = (x_1, \ldots, x_m)$, those in $\mathbf{R}^n$ by $y = (y_1, \ldots, y_n)$ and we write also $\mathscr{E}_n = \mathscr{E}(y)$, $\mathscr{E}_m = \mathscr{E}(x)$. Now u can be factored

$$\mathscr{E}(y) \to \mathscr{E}(x, y) \to \mathscr{E}(x),$$

where the first mapping is the canonical injection (associating to each $f(y) \in \mathscr{E}(y)$ the same function considered as a function of x and y) and the second is the mapping

$$f(x, y) \to f(x, \phi(x))$$

where $\phi(x) = (\phi_1(x), \phi_n(x)) = (u(y_1)(x), \ldots, u(y_n)(x))$. Let N be the kernel of the mapping $\mathscr{E}(x,y) \to \mathscr{E}(x)$. To prove that u is finite, it suffices to find an ideal $P \subset N$ such that the composite

$$\mathscr{E}(y) \to \mathscr{E}(x,y) \to \mathscr{E}(x,y)/P$$

74 is finite (since u is the composite of this mapping and a surjection of $\mathscr{E}(x,y)/P$ onto $\mathscr{E}(x)$). We denote this composite by i.

Let $\widehat{N}$ be the kernel of the map $\widehat{u} : \widehat{\mathscr{E}}(y) \to \widehat{\mathscr{E}}(x)$. As remarked above, $\widehat{u}$ is finite; hence there exists a finite number of monomials $x^\alpha = x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ which generate $\widehat{\mathscr{E}}(x)$ over $\widehat{\mathscr{E}}(y)$. Clearly, since if certain of these monomials generate $\widehat{\mathscr{E}}(x)/\widehat{\mathscr{E}}(x)\mathfrak{m}(\widehat{\mathscr{E}}(y))$ over $\mathbf{R}$, they generated $\widehat{\mathscr{E}}(x)$ over $\widehat{\mathscr{E}}(y)$, we may suppose that they are linearly independent in $\widehat{\mathscr{E}}(x)/\widehat{\mathscr{E}}(x)\mathfrak{m}(\widehat{\mathscr{E}}(y))$. Let r be a sufficiently large integer. Then, since the x^α are generators of $\widehat{\mathscr{E}}(x)$ over $\widehat{\mathscr{E}}(y)$, there exist elements $c_{i\alpha}(y) \in \widehat{\mathscr{E}}(y)$ such that

$$x_i^r = \sum_\alpha c_{i\alpha}(\phi(x))x^\alpha.$$

Since this equation holds in $\widehat{\mathscr{E}}(x)$, we conclude that if r is sufficiently large, then $c_{i\alpha}(0) = 0$. By our definition of $\widehat{N}$, we conclude that

$$Q_i(x,y) = x_i^r - \sum_\alpha c_{i\alpha}(y)x^\alpha \in \widehat{N}.$$

Let us write $[\alpha] = \max_j |\alpha_j|$ if $\alpha = (\alpha_1, \ldots, \alpha_m)$. By introducing series $c_{i\alpha} \equiv 0$ if necessary, we suppose that

$$Q_i(x,y) = x_i^r - \sum_{[\alpha]<r} c_{i\alpha}(y)x^\alpha \in \widehat{N}, c_{i\alpha}(0) = 0.$$

Because of Lemma 4.2, there exist functions $P_i(x,y) \in N$ whose Taylor expansion at 0 coincides with Q_i. Let P be the ideal in $\mathscr{E}(x,y)$ generated by the $P_i (1 \leqslant i \leqslant m)$. We shall prove that the morphism

$$i : \mathscr{E}(y) \to \mathscr{E}(x,y)/P$$

is finite. This, as we have already remarked, will terminate the proof.

We now introduce the new variables $t = (t_{i\alpha})$, $1 \leqslant i \leqslant m$, $[\alpha] < r$, and the "generic polynomials"

$$\Pi_i(x,t) = x_i^r - \sum_{[\alpha]<r} t_{i\alpha} x^\alpha,$$

considered as elements of $\mathscr{E}(x,y,t)$. Let Π be the ideal generated by Π_i in $\mathscr{E}(x,y,t)$. Our next object is to prove the following **75**

Lemma 4.3. *If $f \in \mathscr{E}(x,y,t)$, then there exist functions $g_i \in \mathscr{E}(x,y,t)$, $h_\alpha \in \mathscr{E}(y,t)$ ($1 \leqslant i \leqslant m$, $[\alpha] < r$) such that*

$$f(x,y,t) = \sum_{i=1}^{m} \Pi_i(x,t) g_i(x,y,t) + \sum_{[\alpha]<r} h_\alpha(y,t) x^\alpha. \tag{4.1}$$

Moreover, if f is flat at the origin, the g_i and h can be chosen flat at the origin.

Proof. It is clear that every polynomial in x, t is congruent to a sum of monomials x^α, $[\alpha] < r$ modulo the ideal $\mathfrak{p}$ generated by the Π_i in the ring of polynomials in x, t. Hence the composite

$$\mathbf{R}[t] \to \mathbf{R}[x,t] \to \mathbf{R}[x,t]/\mathfrak{p}$$

is a finite mapping, so that, fro each i, x_i is integral over $\mathbf{R}[x,t]/\mathfrak{p}$, so that there exists a monic polynomial in x_i, $\psi_i(x_i,t) \in \mathfrak{p}$. Since clearly $\psi_i(0,0) = 0$, there exists, by the Weierstrass preparation theorem (or better, the henselian properties of analytic rings), distinguished polynomials

$$R_i(x_i,t) = x_i^s + \sum_{j=0}^{s-1} \phi_{ij}(t) x_i^j,$$

where the $\phi_{ij}(t)$ are analytic functions of t, which belong to the ideal generated by $\mathfrak{p}$ in the ring of analytic functions of x, t. We may obviously suppose that s is independent of i .

For any $f(x,y,t) \in \mathscr{E}(x,y,t)$, we now apply the special preparation theorem and conclude that

$$f(x,y,t) = G_1(x,y,t) R_1(x_1,t) + \sum_{\alpha_1<s} H_{\alpha_1}(x_2,\dots,x_m;y;t) x_1^{\alpha_1}$$

where G_1, $H_{\alpha_1} \in \mathscr{E}(x, y, t)$. If now f is flat at the origin, the uniqueness of the division algorithm in the ring of formal power series assures us that G_1, H_{α_1} are automatically flat.

76 We repeat the process with G_1, H_{α_1} and divide them by $R_2(x_2; t)$ and so on. This gives us an identity

$$f(x, y, t) = \sum_{i=1}^{m} G_i(x, y, t) R_i(x_i, t) + \sum_{[\beta]<s} H_\beta(y, t) x^\beta,$$

where, if f is flat at 0, so are G_i, H_β. Since the $R_i \in \Pi$, and the x^β are congruent, modulo $\mathfrak{p}$, to a linear combination of the x^α, $[\alpha] < r$, this gives us an identity

$$f(x, y, t) = \sum_{i=1}^{m} \Pi_i(x, t) g_i(x, y, t) + \sum_{[\alpha]<r} h_\alpha(y, t) x^\alpha,$$

in which the g_i, $h_\alpha \in \mathscr{E}(x, y, t)$, and are flat at 0 if f is. This proves the lemma.

Now, we have

$$Q_i(x, y) = \Pi_i(x, t) + \sum_{[\alpha]<r} (t_{i\alpha} - c_{i\ alpha}(y)) x^\alpha.$$

If $\gamma_{i\alpha} \in \mathscr{E}(y)$ has the Taylor expansion $c_{i\alpha}(y)$, then the difference

$$P_i(x, y) - \Pi_i(x, t) - \sum_{[\alpha]<r} (t_{i\alpha} - \gamma_{i\alpha}(y)) x^\alpha$$

is flat at 0; from Lemma 4.3, it follows that we have a relation

$$P_i(x, y) = \Pi_i(x, t) + \sum_{j=1}^{n} \Pi_j(x, t) g_{ij}(x, y, t) + \sum_{[\alpha]<r} k_{i\alpha}(y, t) x^\alpha, \quad (4.2)$$

where the g_{ij} are flat and $k_{i\alpha}$ has, at 0, the Taylor expansion $t_{i\alpha} - c_{i\alpha}(y)$. Consequently the matrix

$$\left(\frac{\partial k_{i\alpha}}{\partial t_{j\beta}}(0, 0) \right)$$

is the unit matrix (the number of $k_{i\alpha}$, and that of $t_{i\alpha}$ is the same). By the implicit function theorem, there exist differentiable functions $\theta_{i\alpha}(y)$ such that

$$k_{i\alpha}(y,t) = 0 \text{ for all } i,$$

is equivalent with

$$t_{i\alpha} = \theta_{i\alpha}(y).$$

If we set $t_{i\alpha} = \theta_{i\alpha}(y)$ in (4.2), we obtain 77

$$P_i(x,y) = \Pi_i(x;\theta(y)) + \sum_{=1}^{n} \Pi_j(x,\theta(y))g_{ij}(x,y), \tag{4.3}$$

where the g_{ij} are flat. Consequently, the equations (4.3) can be inverted, so that the functions $\Pi_i(x,\theta(y))$ generate the same ideal P as the functions $P_i(x,y)$. If now $f \in \mathscr{E}(x,y)$, we apply (4.1), and then substitute $\theta_{i\alpha}(y)$ for $t_{i\alpha}$. We obtain

$$f(x,y) = \sum_{i=1}^{n} \Pi_i(x,\theta(y))g_i(x,y,\theta(y)) + \sum_{[\alpha]<r} h_\alpha(y,\theta(y))x^\alpha$$

This proves that the mapping

$$i : \mathscr{E}(y) \to \mathscr{E}(x,y)x/P$$

is a finite mapping; in fact the x^α with $[\alpha] < r$ generate $\mathscr{E}(x,y)/P$ over $\mathscr{E}(y)$. This proves the preparation theorem. □

Corollary 4.4. *Let $u : A \to B$ be a morphism of the differentiable algebra A and B. Let $b_1, \ldots, b_p \in B$ and let $\widehat{b}_1, \ldots, \widehat{b}_p$ be their images in $\widehat{B}$. Then the following conditions are equivalent:*

(i) the images of $\widehat{b}_i$ generate $\widehat{B}/\widehat{B}\mathfrak{m}(\widehat{A})$ over **R***;*

(ii) the $\widehat{b}_i$ generate $\widehat{B}$ over $\widehat{A}$;

(iii) the images of b_i generate $B/B\mathfrak{m}(A)$ over **R***;*

(iv) the b_i generate B over A.

(The deduction of this corollary from Theorem 4.1 has already been given; see Chapter III, Corollary 1.7.)

5 Examples.

We give now three examples to illustrate how the preparation theorem, (or rather the corollary above) can be applied.

I. Symmetric Functions. Let $\sigma_i(x)$ be the ith elementary symmetric function of $x_1, \ldots, x_n$, the coordinate functions in $\mathbf{R}^n$. Let $\phi : \mathbf{R}^n \to \mathbf{R}^n$ denote the map

$$\phi(x) = (\sigma_1(x), \ldots, \sigma_n(x))$$

78 and $u : \mathscr{E}_n \to \mathscr{E}_n$ the induced morphism. It follows at once from the elementary theorem on the representation of symmetric polynomials as polynomials in $\sigma_1, \ldots, \sigma_n$ that in the ring $\widehat{\mathscr{E}}_n$ of formal power series, the monomials

$$x_1^{\alpha_1} \ldots x_{n-1}^{\alpha_n - 1},\ 0 \leqslant \alpha_i \leqslant n - i,$$

generate $\widehat{E}_n$ over the subalgebra generated by the images of $\sigma_1, \ldots, \sigma_n$. Hence, by the above corollary, *these monomials generate $\mathscr{E}_n$ over the subalgebra of the differentiable functions of $\sigma_1, \ldots, \sigma_n$.* In particular, if $f \in \mathscr{E}_n$ is symmetric (i.e. invariant under permutations of $x_1, \ldots, x_n$) we see, by averaging over the permutation group, that there exists $g \in \mathscr{E}_n$ such that

$$f(x_1, \ldots, x_n) = g(\sigma_1, \ldots, \sigma_n).$$

Thus, every germ of a differentiable function which is symmetric can be expressed as a differentiable function of the elementary symmetric functions.

This result is due to G. Glaeser [5].

II. The Weierstrass preparation Theorem. Let $F(x_1, \ldots, x_n) \in \mathscr{E}_n$ be *regular* in x_n of order p, i.e. $F(0, \ldots, 0, x_n)$ has zero of order exactly p at $x_n = 0$. Let B be the differentiable algebra $\mathscr{E}_n/(F)$, A the algebra $\mathscr{E}_{n-1}$. Let $u : A \to B$ be the composite of the injection $\mathscr{E}_{n-1} \to \mathscr{E}_n$, and the projection $\mathscr{E}_n \to B$. It is clear that the images of $1, x_n, \ldots, x_n^{p-1}$ in $\widehat{B}/\widehat{B}\mathfrak{m}(\widehat{A})$ generate this module over $\mathbf{R}$. By the corollary above, $1, x_n, \ldots, x_n^{p-1}$ generate $\mathscr{E}_n/(F)$ over $\mathscr{E}_{n-1}$; this means that for any $f \in \mathscr{E}_n$, there exist functions $Q \in \mathscr{E}_n$ and $r_i \in \mathscr{E}_{n-1}$ such that

$$f = QF + \sum_{i=0}^{p-1} r_i x_n^i.$$

If we apply this to $f = x_n^p$, we see that since $F(0, \ldots, 0, x_n)$ has a zero of order exactly p at $x_n = 0$, we must have $r_i(0) = 0$, $Q(0) \neq 0$, so that

$$F = gP;$$

where $g = \dfrac{1}{Q}$ and $P = x_n^p - \sum_{i=0}^{p-1} r_i x_n^i$ is a distinguished polynomial. Thus, *any function, regular in x_n of order p, is equivalent to a dis-* 79
tinguished polynomial in x_n of degree p with coefficients differentiable functions of $x_1, \quad , x_{n-1}$.

III. Generic mappings $\mathbf{R}^2 \to \mathbf{R}^2$.

Let X and Y be two copies of $\mathbf{R}^2$ with coordinates (x_1, x_2) and (y_1, y_2) respectively. Let Ω be an open set $\subset X$ and let $F = (f_1, f_2)$ be a C^∞ mapping $\Omega \to Y$

(a) There exists F' as near as we like to F in $\mathscr{E}(\Omega; Y)$ and having the following property: at any point $(x_1, x_2) \in \Omega$, the rank of the mapping F' (i.e. the rank of its jacobian matrix) is $\geqslant 1$.

In fact, consider the mapping

$$\left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}\right) \ \Omega \to \mathbf{R}^4.$$

By Sard's theorem I, 7.4 its image has measure 0. Let then

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

be a point not belonging to the image (which we may choose arbitrarily small). We may take

$$f_1' = f_1 - \lambda_1 x_1 - \lambda_2 x_2, f_2' = f_2 - \lambda_3 x_1 - \lambda_4 x_2.$$

(b) Suppose that the rank of F is everywhere $\geqslant 1$. By making Ω small and making suitable changes of variables in X and Y, we may suppose that $f_1 = x_1$. We then set $f_2 = f$ for simplicity. Let us show that there exists f' arbitrarily close to f in $\mathscr{E}(\Omega; \mathbf{R})$ having the following property.

(G). At any point $a \in \Omega$ where $\dfrac{\partial f'}{\partial x_2}(a) = \dfrac{\partial^2 f'}{\partial x_2^2}(a) = 0$, we have

$$\frac{\partial^2 f'}{\partial x_1 \partial x_2}(a) \neq 0, \frac{\partial^3 f'}{\partial x_2^3}(a) \neq 0.$$

The proof is as before: using Sard's theorem, one shows that the set of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbf{R}^4$ for which

$$f' = f - \lambda_1 x_2 - \lambda_2 x_1 x_2 - \lambda_3 x_2^2 - \lambda_4 x_2^3$$

does not satisfy ((G)) is of measure zero.

Using (a) and (b) one proves the following (details are left to the reader).

80 Let M and N be two C^∞ manifolds of dimension 2 which are countable at infinity, and let K be a compact set in M. Let $\mathscr{E}(M, N)$ be the space of C^∞ mappings of M into N with the topology of uniform convergence on any compact set of functions and their derivatives of all orders (in an obvious sense). Then, *the set of mappings in* $\mathscr{E}(M, N)$ *all of whose critical points on K satisfy* ((G)) *(in a suitable coordinate system) is open and dense.*

We shall now look more closely at these critical points. We place ourselves at 0 in X and Y for simplicity. There are two types which cannot be reduced to one another.

Type 1.

$$F = (x_1, f), \frac{\partial f}{\partial x_2}(0) = 0, \frac{\partial^2 f}{\partial x_2^2} \neq 0. \tag{5.1}$$

Let us apply Corollary 4.4 to the mapping $\mathscr{E}_2 \to \mathscr{E}_2$ defined by F. We find, in particular, that there exist Φ, $\Psi \in \mathscr{E}_2$ such that

$$x_2^2 = \Phi(x_1, f) + 2\Psi(x_1, f)x_2. \tag{5.2}$$

We obviously have $\Phi(0) = \Psi(0) = 0$. Put $x_2' = x_2 - \Psi(x_1, f)$, $y_2' = \Phi(y_1, y_2) + \Psi^2(y_1, y_2)$. We deduce from (5.1) and (5.2) that (x_1, x_2'), (y_1, y_2') are local coordinates at 0. In this coordinate *system, our mapping takes the canonical form of Type 1 :* $f_1 = x_1$, $f_2 = x_2^2$.

TYPE 2.

$$F = (x_1, f), \frac{\partial f}{\partial x_2}(0) = \frac{\partial^2 f}{\partial x_2^2}(0) = 0; \tag{5.3}$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(0) \neq 0, \frac{\partial^3 f}{\partial x_2^3}(0) \neq 0.$$

Applying again Corollary 4.4, we can find functions Φ, Ψ, $\Theta \in \mathscr{E}_2$ such that

$$x_2^3 = \Phi(x_1, f) + \Psi(x_1, f)x_2 + 3\Theta(x_1, f)x_2^2. \tag{5.4}$$

Clearly, $\Phi(0) = \Psi(0) = \Theta(0) = 0$. Replacing x_2 by $x_2 - \Theta(x_1, f)$, we see that we have again local coordinates on X for which (5.3) is satisfied, so that we may suppose that $\Theta = 0$.

This being so, the conditions (5.3) and (5.4) show that we may take **81**
as coordinates

$$\text{on } X: \ x_1' = \Psi(x_1, f), \ x_2' = x_2$$
$$\text{on } Y: \ y_1' = \Psi(y_1, y_2), y_2' = \Phi(y_1, y_2)$$

as is easily verified. We obtain finally the *canonical form of Type 2:*

$$f_1 = x_1, f_2 = -x_1 x_2 + x_2^3.$$

The preceding results are due to *H*. Whitney [25]. The idea of proving them using the preparation theorem is due to *R*. Thom. A generalization is to be found in B. Morin [14].

VI
IDEALS DEFINED BY ANALYTIC FUNCTIONS

82

1 The main theorem.

The main theorem of this chapter is the following.

Theorem 1.1. *Let $\mathscr{O}_n$, $\mathscr{E}_n$ denote the rings of germs of analytic and differentiable functions respectively and $\widetilde{\mathscr{F}_n}$ the ring of germs at the origin of collections of formal power series at each point near* 0 (see Chapter III, §4). *Let $\mathfrak{a}$ be an ideal in $\mathscr{O}_n$. Then we have*

$$(\mathfrak{a}\widetilde{\mathscr{F}_n}) \cap \mathscr{E}_n = \mathfrak{a} \cdot \mathscr{E}_n.$$

This theorem is obviously equivalent with the following (partition of unity).

Theorem 1.1′. *Let Ω be an open set in $\mathbf{R}^n$ and $f_1, \ldots, f_p$ analytic functions in Ω. Let $\phi \in \mathscr{E}(\Omega)$. Then ϕ can be written in the form*

$$\psi = \sum_{i=1}^{p} f_i \psi_i, \text{ where } \psi_i \in \mathscr{E}(\Omega),$$

if and only if for any $a \in \Omega$, the Taylor expansion $T_a\phi$ belongs to the ideal generated by the $T_a f_i$ in $T_a\mathscr{E}(\Omega) = \mathscr{D}_a$ (formal power series at a). For $p = 1$, see Hörmander [6], Łojasiewicz [10]; for the general cases, Malgrange [11]: see also Palamodov [16].

For the proof of the theorem, we shall use certain reductions which are very similar to those used in the proof of the preparation theorem. We start by stating a more general form of Theorem 1.1.

If $Y_0 \subset X_0$ are germs of analytic sets at 0 in $\mathbf{R}^n$, let $\widetilde{\mathscr{F}_n}(X_0)$ denote the ring of germs at 0 of collections, at points of X_0, of formal power series. Clearly we have an inclusion $\mathscr{E}(X_0) \subset \widetilde{\mathscr{F}_n}(X_0)$ where $\mathscr{E}(X_0)$ is the ring of germs at 0 of Whitney functions on X_0. Let $\mathscr{F}(Y_0; X_0)$ denote the subring of $\mathscr{E}(X_0)$ of functions flat on Y_0. Then we have

83 **Theorem 1.2.** *If $\mathfrak{a}$ is an ideal in $\mathscr{O}_n$, we have*

$$\mathfrak{a} \cdot \widetilde{\mathscr{F}_n}(X_0) \cap \mathscr{F}(Y_0; X_0) = \mathfrak{a} \cdot \mathscr{F}(Y_0; X_0).$$

We shall call Theorem 1.2 for the germs Y_0 and X_0, $\mathrm{Th}(Y_0, X_0)$ *(we suppose $\mathfrak{a}$ given). As in the case of the preparation theorem, we may reduce Theorem 1.2 to the proof of the following statement:*

$P(X_0)$. Given the analytic germ X_0 at 0, for any analytic germ $Y_0 \subset X_0$, $Y_0 \neq X_0$ there is an analytic germ $Z_0 \neq X_0$, $Y_0 \subset Z_0 \subset X_0$ such that $\mathrm{Th}(Z_0, X_0)$ *is true.*

We remark that Theorem 1.2 implies

Theorem 1.2′. *If X is an analytic set in an open set $\Omega \subset \mathbf{R}^n$, if $f_1, \dots, f_p$ are analytic in Ω and Y is an analytic subset of X, then for any $\phi \in \mathscr{F}(Y; X)$, there exist functions $\psi_1, \dots, \psi_p \in \mathscr{F}(Y; X)$ such that $\phi = \Sigma f_i \psi_i$ if and only if $T_a\phi$ belongs to the ideal generated by the $T_a f_i$ in $\mathscr{F}_a$ for any $a \in X$.*

We shall prove $P(X_0)$ by induction on $k = \dim X_0$; we may therefore suppose Theorem 1.2′ true for any analytic set $X \subset Q$ whose dimension at any point is $< k$.

Now we shall show that it suffices to prove $P(X_0)$ when X_0 is irreducible and $\mathfrak{a}$ is contained in the ideal $\mathfrak{p} \subset \mathscr{O}_n$ of functions vanishing on X_0. The proof that we may suppose X_0 irreducible is the same as in the case of the preparation theorem and we do not repeat the argument. Suppose that X_0 is irreducible and let $\mathfrak{a} \not\subset \mathfrak{p}$; let $f \in \mathfrak{a}$, $f \notin \mathfrak{p}$, and let $Z_0 = Y_0 \cup [X \cap \{x | f(x) = 0\}]$. Then $\mathrm{Th}(Z_0, X_0)$ is true as follows from the next lemma.

Lemma 1.3. *Let Ω be an open set in $\mathbf{R}^n$, and f analytic in Ω. Let $S = \{x \in \Omega | f(x) = 0\}$. Let ϕ be a function $\in \mathscr{F}(S; \Omega)$. Then there exists $\psi \in \mathscr{F}(S; \Omega)$ such that $\phi = \psi f$.*

Proof. By the inequality of Lojasiewicz and Chapter IV, Lemma 1.2, $1/f \in \mathscr{M}(S; \Omega)$. Since $\phi \in \mathscr{F}(S; \Omega)$, $\psi = (1/f)\phi \in \mathscr{F}(S; \Omega)$ by Chapter IV, Proposition 1.4. This gives Lemma 1.3.

Before going to the proof of $P(X_0)$, we need two lemmas. □

Lemma 1.4. *Let Ω be an open set in $\mathbf{R}^n$ containing 0, $\mathfrak{a}_0$, $\mathfrak{b}_0$ two ideals* **84**
in $\mathscr{O}_n$. Let $f = (f_1, \ldots, f_p)$; $g = (g_1, \ldots, g_q)$ be generators of $\mathfrak{a}_0$, $\mathfrak{b}_0$ respectively and suppose that they are analytic in Ω. Let $\mathfrak{a}_x$, $\mathfrak{b}_x$ be the ideals generated by $f_1, \ldots, f_p$; $g_1, \ldots, g_q$ at $x \in \Omega$. Then, for any compact subset K of Ω and any integer $m \geqslant 0$, there is an integer m' such that for any $x \in K$, we have

$$\mathfrak{a}_x^{m'} \cap \mathfrak{b}_x \subset \mathfrak{a}_x^m \cdot \mathfrak{b}_x.$$

Proof. Let, for $x \in \Omega$, $m'(x)$ be the smallest integer m' such that

$$\mathfrak{a}_x^{m'} \cap \mathfrak{b}_x \subset \mathfrak{a}_x^m \cdot \mathfrak{b}_x;$$

(such an m' exists by the Artin-Rees lemma). Now there exist $h_1, \ldots, h_r$; $k_1, \ldots, k_s$ in a neighbourhood U of x, such that the h belong to $\mathfrak{a}_y^{m'(x)} \cap \mathfrak{b}_y$ and generate it for any $y \in U$, while the k belong to $\mathfrak{a}_y^{m'} \cdot \mathfrak{b}_y$ and generate it for any $y \in U$. Since $\mathfrak{a}_x^{m'(x)} \cap \mathfrak{b}_x \subset \mathfrak{a}_x^m \cdot \mathfrak{b}_x$, if U is small enough, there exist analytic functions a_{ij} in U such that $h_i = \sum_{j=1}^{s} a_{ij} k_j$, $i = 1, \ldots, r$. Then clearly, since the h_i generate $\mathfrak{a}_y^{m'(x)} \cap \mathfrak{b}_y$, we have $\mathfrak{a}_y^{m'(x)} \cap \mathfrak{b}_y \subset \mathfrak{a}_y^m \cdot \mathfrak{b}_y$, so that $m'(y) \leqslant m'(x)$ for $y \in U$. Hence $m'(x)$ is bounded on K, and the lemma follows. □

Lemma 1.5. *Let Ω be an open set in $\mathbf{R}^n$, f an analytic function on Ω and let $X = \{x \in \Omega | f(x) = 0\}$. Then any point a of Ω has a fundamental system of open neighbourhoods Ω_p such that $\Omega_p - X$ has only finitely many connected components, each of which contains a in its closure in Ω_p.*

Proof. We may clearly suppose that $a = 0$ and that f is a distinguished pseudopolynomial in x_n which is irreducible at 0. Let $y = (x_1, \ldots, x_{n-1})$. The discriminant of f has a germ at $0 \in \mathbf{R}^{n-1}$ which is not zero. If $\Omega = \Omega' \times \Omega''$, $\Omega' \subset \mathbf{R}^{n-1}$, $\Omega'' \subset \mathbf{R}$, suppose that the lemma is already proved for the set $Y \subset \Omega'$, $Y = \{y \in \Omega' | \Delta(y) = 0\}$. Let Ω'_p be a fundamental system of open neighbourhoods of $0 \in \mathbf{R}^{n-1}$ such that $\Omega'_p - Y$ has k_p components $U_{p,\nu}$ which contain 0 in their closures. Let I_p be

85 an open interval whose length $\to 0$ as $p \to \infty$ such that $f(y, x_p) = 0$, $y \in \Omega'_p$ imply $x_n \in I_p$, and let $\Omega_p = \Omega'_p \times I_p$. To show that $\Omega_p - X$ has only finitely many components each adherent to 0, it is enough to prove the same of $\Omega_p - X - (Y \times I_p)$. Now, the number of real roots of $f(y, x_n) = 0$ is constant $= s$, say, on $U_{p,\nu}$; let $\tau_1(y) < \ldots < \tau_s(y)$ be these roots. Then the connected components of $\Omega_p - X - (Y \times I_p)$ are the sets

$$\{(y, x_n) | y \in U_{p,\nu}, \tau_i(y) < x_n < \tau_{i+1}(y)\},$$

where we have set $\tau_0 = -\infty$, $\tau_{s+1} = +\infty$. Since $\tau_i(y) \to 0$ as $y \to 0$, $(1 \leqslant i \leqslant s)$ the lemma follows.

We now go to the proof of $P(X_0)$. We use the notations of Chapter IV, §3, and we may suppose, as in the preparation theorem, that there is an analytic set $Y' \subset V'$ such that $Y = (Y' \times V'') \cap X$. Let δ be the set $\{x' \in V' | \Delta(x') = 0\}$. Let $Z' = Y' \cup \delta$, and suppose that V' is so chosen that $V' - Z'$ has only finitely many connected components, each adherent to zero. Let $Z = X \cap (Z' \times V'')$. Then the same is true of $X - Z$, in fact, any component U' of $V' - Z'$ is contained in a set V_r, and the components of $X - Z$ are the sets $\{(x', x'') | x' \in U', x'' = \Phi^s(x')\}$, for any $s \leqslant r$.

We suppose that Ω is an open set containing $\overline{V}$ and that $\mathfrak{p}$ is generated by functions $f_1, \ldots, f_p$ analytic on Ω. Let, for $x \in \Omega$, $\mathfrak{p}_x$ denote the ideal at x generated in the ring of analytic functions at x by the f_i. Finally let $\mathscr{F}_x$ denote the ring of formal power series at x. □

Now we make the following remark:

(1.6) *If ϕ is a germ of C^∞-functions on X, at $a \in X - Z$ and the "normal derivatives" of ϕ vanish upto order m (i.e. $D^\lambda_{x''}\phi = 0$ for $\lambda \in \mathbf{N}^l$, $|\lambda| \leqslant m$), then the Taylor expansion of ϕ at a belongs to $\mathfrak{p}_a^{m+1} \cdot \mathscr{F}_a$.*

This is a trivial consequence of the fact that, in $\mathscr{O}_a$ the ideal generated by $P(x_{k+1}; x')$ and $\Delta x_{k+j} - Q_j(x_{k+1}; x')$ coincides with the ideal of germs vanishing on X. Suppose now that $\mathfrak{q}$ is any ideal in $\mathscr{O}_n$ generated by functions $g_1, \ldots, g_q$ analytic in Ω, $\mathfrak{q} \subset \mathfrak{p}$. We identify $\mathscr{I}(Z; X_r)$ with $[\mathscr{I}((V' - V_r) \cup Z'; V'))]^{N^l}$ (by Chapter IV, Proposition 5.5). Let $\lambda \in \mathbf{N}^l$, 86 and $g_j^\lambda = (D_{x''}^\lambda g_j)(x', \Phi^r(x'))$. We prove first the following

Lemma 1.7. *Suppose $\phi = (\phi^\lambda) \in [\mathscr{I}((V' - V_r) \cup Z'; V')]^{N^l}$, and suppose that the Taylor expansion of ϕ^λ at any point a' of V_r belongs to the ideal generated in $\mathscr{F}_{a'}$ by the g_j^λ. Then there exist functions ϕ_j^λ, $\mu < \lambda$ such that we have*

$$\phi^{\lambda'} = \sum_{j=1}^{q} \sum_{\lambda < \lambda'} \binom{\lambda'}{\mu} g_j^{\lambda' - \mu} \psi_j^\lambda \text{ for } \lambda' \leqslant \lambda.$$

Proof. If all the g_j^λ, $\mu \leqslant \lambda$ vanish on V, we have nothing to prove. Otherwise, let x' be a point at which the matrix $(g_j^{\lambda' - \mu'})$ has maximal rank (indices being λ' and the pairs (μ', j) say ρ and let A' be a $\rho \times \rho$ submatrix of $(g_j^{\lambda' - \mu'})$ whose determinant at x' if non-zero. Let A denote the corresponding $\rho \times \rho$ submatrix of $(D_{x''}^{\lambda' - \mu'} g_j)$. Then, clearly $\det A \neq 0$ at the point $(x', \Phi^r(x'))$. Let S be the set of points of X where $\det A$ is zero. We assert that $\dim S$ is $< k$ at every point. In fact, since every component of $V' - Z'$ is adherent to 0, the projection of S contains no neighbourhood of 0 in V'. Hence the germ of S at 0 is $\subset X_0$, $\neq X_0$, so that the dimension of S is $< k$ at every point of $X - Z$ since every component of $X - Z$ is adherent to 0, so that S can contain no such component.

To prove Lemma 1.7, we use now the following simple generalization of Lemma 1.3. □

Lemma 1.8. *Let $h_1, \ldots, h_\rho$ be ρ-tuples of analytic functions on the connected open set $\Omega \subset \mathbf{R}^n$, which are linearly independent at some point of Ω. Let M be the set of points of Ω where they are not linearly independent. Then for any ρ-tuple ϕ of C^∞-functions flat on M, there exist functions ψ_i, $1 \leqslant i \leqslant \rho$, flat on M, such that*

$$\phi = \sum_{i=1}^{\rho} \psi_i h_i.$$

Moreover, the ψ_i are flat at any point of $\Omega - M$ where ϕ is.

Lemma 1.7 is an immediate consequence of Lemma 1.8 if the function ϕ is flat on S.

Since, by assumption, the system under consideration is soluble at 87
every point, and A' is a submatrix of maximal rank outside the projection of $S \cap X_r$ on V_r, it is sufficient to solve the square system

$$\phi^{\lambda'} = \sum_{j,\mu} \binom{\lambda'}{\mu} g_j^{\lambda'-\mu} \psi_j^{\mu} \text{ where } (g_j^{\lambda'-\mu'}) = A',$$

with the ψ_j^{μ} flat on the projection of $S \cap X_r$ on V'; the other equations in the system are then automatically satisfied.

To prove Lemma 1.7, we proceed as follows. Let $\phi_1 \in \mathscr{I}(Z \cap S; S)$ be the restriction of ϕ to S. By the inductive hypothesis and Theorem 1.2$'$, there exist $\psi_i \in \mathscr{I}(Z \cap S; S)$ such that

$$\phi_1 = \Sigma \psi_i g_i \text{ in } \mathscr{E}(S).$$

Let $\psi_i' \in \mathscr{I}(Z; X)$ be such that their restrictions to S are the ψ_i (this is possible because any two analytic sets are regularly situated); and let $\phi' = \Sigma g_i \psi_i'$. Then $\phi - \phi' \in \mathscr{I}(Z \cap S; X)$ and we may apply the above result to $\phi - \phi'$. Since Lemma 1.7 is true for $\phi - \phi'$ and for ϕ', it is clearly true for ϕ.

We now go back to our ideal $\mathfrak{a} \subset \mathfrak{p}$, and suppose that it is generated by functions analytic on Ω; then clearly Lemma 1.7 is true for the ideal $\mathfrak{q}_m = \mathfrak{p}^m \cdot \mathfrak{a}$ for every $m \geqslant 0$. Suppose $m' = m'(m)$ so chosen that

$$\mathfrak{p}_x^{m'} \cap \mathfrak{a}_x \subset \mathfrak{p}_x^{m} \cdot \mathfrak{a}_x \text{ for } x \in V \quad \text{(Lemma 1.4).} \tag{1.9}$$

Lemma 1.7 and the assertion (1.6) show that the following holds

(1.10) *If $\phi \in \mathscr{I}(Z; X_r)$ and ϕ is m'-flat on X_r, then there exist, for any $\lambda \in \mathbf{N}^l$, functions $\psi_j^{\lambda} \in \mathscr{I}(Z; X_r)$ which are m-flat on X_r such that*

$$\phi^{\lambda'} = \sum_{j=1}^{p} \sum_{\mu<\lambda'} \binom{\lambda'}{\mu} f_j^{\lambda'-\mu} \psi_j^{\mu} \text{ for } \lambda' \leqslant \lambda. \tag{1.11}$$

It is now easy to complete the proof of $P(X_0)$. Given ϕ, it suffices to find $\psi_j \in \mathscr{I}(Z'; X_r)$ with $\phi = \sum f_i\psi_i$, since the X_r are regularly situated.

We write ϕ in the form $\phi = \phi_1 + \phi_2 + \cdots$ where $\phi_k \in \mathscr{I}(Z; X_r)$, and the component $\phi_k^\lambda \neq 0$ only if $m'(k) \leqslant |\lambda| \leqslant m'(k+1)$ [where $m'(k)$ is defined by (1.9)]. There exist functions $\psi^\mu_{j,1}, \in \mathscr{I}((V_1 - V_r) \cup Z'; V')$,
88 $|\mu| \leqslant m'(1)$ such that $\psi^\mu_{j,1}$ are 0-flat on X_r, and $\phi_1 - \sum\limits_{j=1}^{p} f_j\psi_{j,1}$ is $m'(1)$-flat. Let $\phi_2^1 = \phi_1 + \phi_2 - \sum\limits_{j=1}^{p} \psi_{j,1}f_j$. We can, as before, find functions $\psi_{j,2} \in \mathscr{I}(Z; X_r)$ which are 1-flat on X_r, such that

$$\phi_1 + \phi_2 - \sum_{j=1}^{p} f_j(\psi_{j,1} + \psi_{j,2}) \text{ is } m'(2)\text{-flat.}$$

By induction, we find $\psi_{j,k} \in \mathscr{I}(Z, X_r)$, $\psi_{j,k}$ being $(k-1)$-flat on X_r, such that

$$\phi_1 + \phi_2 + \quad +\phi_k - \sum_{j=1}^{p} f_j(\psi_{j,1} + \quad +\psi_{j,k}) \text{ is } m'(k)\text{-flat.}$$

Clearly $\psi_j = \sum\limits_{k=1}^{\infty} \psi_{j,k} \in \mathscr{I}(Z; X_r)$ (since $\psi_{j,k}$ is $(k-1)$-flat) and $\phi = \sum\limits_{j=1}^{p} f_j\psi_j$.

This proves $P(X_0)$ and hence the main theorem.

Corollary 1.12. *$\mathscr{E}_n$ is a faithfully flat $\mathscr{O}_n$-module.*

We have seen already that $\widetilde{\mathscr{F}_n}$ is a faithfully flat $\mathscr{O}_n$-module (Theorem III, 4.12). Therefore. the corollary results from Theorem 1.1 and Proposition III, 4.7.

2 A remark concerning the Łojasiewicz inequality.

Let $\Omega \in \mathbf{R}^n$ and $f \in \mathscr{E}(\Omega)$. Let $X = \{x \in \Omega | f(x) = 0\}$. We assert that if $f\mathscr{E}(\Omega)$ is closed, then for any compact $K \subset \Omega$, there exist constants C, $\alpha > 0$ such that

$$|f(x)| \geqslant C\{d(x, X)\}^{\alpha} \text{ for } x \in K. \tag{2.1}$$

In fact, suppose $f\mathscr{E}(\Omega)$ closed. Then, by Banach's theorem, to every compact set $K \subset \Omega$ and $m > 0$, there exists a compact set $K' \subset \Omega$ and $m' > 0$ such that if $g \in f\mathscr{E}(\Omega)$, there exists a $\psi \in \mathscr{E}(\Omega)$ with $\psi f = g$ such that

$$|\psi|_m^K \leqslant C|g|_{m'}^{K'}, C \text{ independent of } g. \tag{2.2}$$

If $x_0 \in K$, we may find $g \in \mathscr{E}(\Omega)$, $g(x_0) = 1$, $g = 0$ in a neighbourhood of X such that 89

$$|g|_{m'}^{K'} \leqslant \frac{A}{\{d(x_0, X)\}^p},$$

where $A > 0$ and $p > 0$ are independent of x_0, but depend only on K, K'. (2.2) clearly implies that

$$\sup_K \left|\frac{g}{f}\right| \leqslant \frac{AC}{\{d(x_0, X)\}^p};$$

in particular

$$|f(x_0)| \geqslant \frac{\{d(x_0, X)\}^p}{AC}.$$

Next we give an example to show that the situation for non-analytic functions is rather complicated.

Let $f^{\pm} = y^2 \pm e^{-1/x^2} \in \mathscr{E}(bfR^2) = \mathscr{E}$. Then $f^{+}\mathscr{E}$ is not closed, but $f^{-}\mathscr{E}$ is. In fact f^{+} does not satisfy (2.1) in any neighbourhood of 0. Since $f^{-} = (y + e^{-1/2x^2})(y - e^{-1/2x^2}) = f_1^{-} f_2^{-}$, we have only to prove the theorem for f_1^{-}, f_2^{-} separately. But, by a change of coordinates, these functions can be made linear.

3 Differentiable functions vanishing on an analytic set.

The results of this paragraph are based on the following theorem:

Theorem 3.1 (Zariski-Nagata)**.** *If the analytic algebra A is an integral domain, so is its completion $\widehat{A}$.*

For the proof, see e.g. Houzel [8] or Malgrange [13].

Here are some immediate consequences (in the statements A is an analytic algebra and $\widehat{A}$ its completion).

(3.2) *If $\mathfrak{p}$ is a prime ideal of A, $\widehat{\mathfrak{p}} = \widehat{A}\mathfrak{p}$ is prime.* (Apply (3.1) to $A/\mathfrak{p}$).

(3.3) *Let $\mathfrak{q}$ be an ideal of A and $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ the minimal prime ideals in the decomposition of $\mathfrak{q}$. Then $\widehat{\mathfrak{p}}_1, \dots, \widehat{\mathfrak{p}}_s$ are the minimal prime ideals in the decomposition of $\widehat{\mathfrak{q}}$.*

90 In fact, one is reduced at once to the case $\mathfrak{q} = \{0\}$; $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are then the minimal prime ideals of A. Let us put $\mathfrak{r} = \mathfrak{p}_1 \cap \quad \cap \mathfrak{p}_s$. It is well known that $\mathfrak{r}$ is the set of nilpotent elements of A and that for a certain n, one has $\mathfrak{r}^n = \{0\}$.

By Proposition III, 4.5 and Theorem III, 4.9, we have $\widehat{\mathfrak{r}} = \widehat{\mathfrak{p}}_1 \cap \cap \widehat{\mathfrak{p}}_s$. On the other hand, we have obviously $\widehat{\mathfrak{r}}^n = \{0\}$. Suppose that $\mathfrak{I}$ is a prime ideal of $\widehat{A}$, and let us suppose, for example, that $\widehat{\mathfrak{p}}_1 \not\subset \mathfrak{I}, \dots, \widehat{\mathfrak{p}}_{s-1} \not\subset \mathfrak{I}$. Let $a_i \in \widehat{\mathfrak{p}}_i, a_i \notin \mathfrak{I}$; $(1 \leqslant i \leqslant s-1)$. For any $x \in \widehat{\mathfrak{p}}_s$, we have

$$(a_1 \dots a_{s-1} x)^n = 0 \in \mathfrak{I},$$

whence $x \in \mathfrak{I}$. Hence $\mathfrak{p}_s \subset \mathfrak{I}$, which proves (3.3)

(3.3) shows in particular that if A is *reduced* (i.e. has no nilpotent elements), then $\widehat{A}$ is reduced.

Definition 3.4. *Let X be a subset of $\mathbf{R}^n$ adherent to 0, and let g be a function of class C^∞ in a neighbourhood of 0. We say that g has a zero of infinite order on X at 0 if, for any $p \in \mathbf{N}$, there is a neighbourhood U_p of 0 and a number $C_p > 0$ such that, on $X \cap U_p$, we have $|g(x)| \leqslant C_p |x|^p$.*

The above property depends only on the germ X_0 of X and on the Taylor expansion of g at 0. The set of these Taylor series forms an ideal in $\mathscr{F}_n$ which we call the "formal ideal defined by X (or X_0)" and denote $J(X)$.

Theorem 3.5. *Let X be an analytic set in a neighbourhood of 0 in $\mathbf{R}^n$, and let $I(X)$ be the ideal in $\mathscr{O}_n$ of germs of analytic functions vanishing on X_0. We have $I(X)\mathscr{F}_n = \widehat{I(X)} = J(X)$.*

It is sufficient to prove this theorem when X_0 is irreducible. In fact, if $X = X' \cup X''$, we have

$$I(X) = I(X') \cap I(X'').$$

By Proposition III, 4.5 and Theorem III, 4.9, we deduce that

$$\widehat{I(X)} = \widehat{I(X')} \cap \widehat{I(X'')}.$$

On the other hand, we obviously have $J(X) = J(X') \cap J(X'')$; hence if **91**
the theorem is true of X', X'', it is true of X.

Suppose then that X_0 is irreducible. Set $\dim X = k$, and let us go back to the notation of Chapter III, §3. The mapping $\mathscr{O}_k \to \mathscr{O}_n/I(X)$ defined by $\overline{x}_1, \ldots, \overline{x}_k$ is finite and injective, and hence the "intrinsic" topology of $\mathscr{O}_n/I(X)$ coincides with its topology as $\mathscr{O}_k$-module. From the exactness properties of the completion, we deduce from this that the mapping $\mathscr{F}_k \to \mathscr{F}_n/\widehat{I(X)}$ defined in the same way as above is still injective; this mapping is finite by (III, 1.6). On the other hand, $\widehat{I(X)}$ is prime (by (3.2)). Let us apply Proposition III, 5.4 to

$$A = \mathscr{F}_k, B - \mathscr{F}_n/\widehat{I(X)}, \mathfrak{p} = \{0\}, \mathfrak{q} = J(X)/\widehat{I(X)}.$$

We find that, to prove the theorem, it is sufficient to verify that one has

$$J(X) \cap \mathscr{F}_k = \{0\}.$$

($\mathscr{F}_k$ is considered as imbedded in $\mathscr{F}_n$). This amounts to proving the following:

Any function $f(x_1, \ldots, x_k)$ of class C^∞ having a zero of infinite order at 0 *on X has a Taylor series which is identically zero.*

Let U be the (germ at 0 of the) set of points of $\mathbf{R}^k - \delta$ which are images of points of X under the projection $(x', x'') \to x'$ (i.e. $U = \bigcup_{s \geqslant 1} V_s$; the notation is that of Chapter IV). There is a $C > 0$, and $p > 0$ such that, on X we have $|x''| \leqslant C|x'|^p$. Hence f, considered now as a function on $\mathbf{R}^k$, has a zero of infinite order at 0 on U. Changing the notation, we are led to prove the following proposition.

Proposition 3.6. *Let Ω be an open set in $\mathbf{R}^k$, $0 \in \Omega$, and Φ be an analytic function in Ω with $\Phi(0) = 0$, $\Phi \not\equiv 0$, and let D be the set of zeros of Φ. Let Γ be an open and closed subset of $\Omega - D$ which is adherent to* 0. *Then we have $J(\Gamma) = \{0\}$.*

92 To prove this proposition, we shall proceed as follows. We shall suppose that $\Phi(0, \quad , 0, x_k)$ is not identically zero near 0, and shall show that, under this condition, any f having a zero of infinite order on Γ at 0 satisfies

$$\frac{\partial^q f}{\partial x_k^q}(0, \ldots, 0) = 0 \forall q \in \mathbf{N}.$$

This implies the required result: in fact, the set of lines through 0 on which Φ is not identically zero near 0 is an open dense set in the set of lines through 0. Since, by a linear change of coordinates, we can take any one of these lines as $0x_k$ axis, it follows, by an elementary argument, that all the derivatives of f are zero at the origin. One has thus $J(\Gamma) = 0$.

Suppose, then, that $\Phi(0, \ldots, 0, x_k)$ is not identically zero in a neighbourhood of 0. By making Ω smaller, we may suppose that Φ is a distinguished polynomial in x_k, whose germ Φ_0 has no multiple factors. We have then

$$\Phi = x_k^p + \sum_{i=1}^{p} a_i(x_1, \ldots, x_{k-1}) x_k^{p-i},$$

the a_i being analytic in Ω with $a_i(0) = 0$, $1 \leqslant i \leqslant p$, and the discriminant Δ of Φ is not identically zero near 0.

For $x = (x_1, \ldots, x_k)$, set $x = (x', x_k)$ and $\mathrm{pr}(x) = x'$. Let Ω' be a neighbourhood of 0 in $\mathbf{R}^{k-1}$ such that the conditions

$$x' \in \Omega', \Phi(x', z) = 0, z \in \mathbf{C},$$

imply that $|z| \leqslant \frac{1}{2}$, and, if z is real, that $(x', z) \in \Omega$. Let δ be the set of zeros of Δ in Ω' and $V' \subset \Omega'$ an open neighbourhood of 0 in $\mathbf{R}^{k-1}$ which is relatively compact in Ω'. By the inequality (IV, 4.1) of Łojasiewicz, there exists $C > 0$ and $\alpha > 0$ such that

$$\forall x' \in V', |\Delta(x')| \geqslant C d(x', \delta)^{\alpha}.$$

If $z^1, \ldots, z^p$ are the roots of the equation $\Phi(x', z) = 0$, we always have

$$|z^i - z^j| \leqslant 1, \text{ hence } |z^i - z^j| \geqslant C\, d(x', \delta)^\alpha \text{ if } i \neq j.$$

We may suppose that, n addition to the conditions imposed above, 93
we have $\Omega = \Omega' \times (-a, a)$, $a > 0$. For $x' \in \Omega$, the interval $\{x'\} \times (-a, a)$ is decomposed into at most $p + 1$ intervals by the zeros of $\Phi(x', z)$, and we always have

$$\Phi(x', \pm a) \neq 0.$$

This implies that $\gamma = \mathrm{pr}(\Gamma') - \delta$ is open and closed in $\Omega' - \delta$; further γ is clearly adherent to 0. For any $x' \in \gamma$, the set $\mathrm{pr}^{-1}(x') \cap \Gamma$ contains at least one of the preceding intervals; we denote the origin of this interval by $b(x')$, its extremity by $c(x')$. If $-a \neq b(x')$, $a \neq c(x')$, $b(x')$, $c(x')$ are distinct (consecutive) zeros of Φ, hence, for $x' \in V'$ we have

$$c(x') - b(x') \geqslant C d(x', \delta)^\alpha. \tag{3.6.i}$$

If we have $b(x') = -a$, $c(x') \neq a$, we replace $b(x')$ by $c(x') - C d(x', \delta)^\alpha$ (which $\to 0$ as $x' \to 0$, so that, if V' is small enough, this is $> -a$); we proceed in a similar way if $b(x') \neq -a$, $c(x') = a$. If $b(x') = -a$, $c(x') = a$, we replace $b(x')$ by 0 and $c(x')$ by $C d(x', \delta)^\alpha$.

After these modifications, the inequality (3.6.i) is valid at any point of V', and there exist constants $C' > 0$, $\alpha' > 0$, such that $\forall x' \in V'$, we have

$$|b(x')|, |c(x')| \leqslant C' |x'|^{\alpha'}. \tag{3.6.ii}$$

Lemma 3.7. *With the hypotheses of the preceding proposition, there exists a sequence* $\{x^1\}$ *of points of* Γ, $x^l \to 0$, *and numbers* $C'' > 0$, $\alpha'' > 0$ *such that*

$$|x^l| \leqslant C'' d(x^l, D)^{\alpha''} \forall l.$$

Proof. This lemma is obvious if $k = 1$. Suppose the lemma verified for $k - 1$. It is sufficient to find a sequence x^l of points of Γ, tending to zero, such that

$$|x^l| \leqslant C'' |\Phi(x^l)|^{\alpha''}$$

By induction, there is a sequence x'^l of points of γ satisfying

$$|x'^l| \leqslant C''' d(x'^l, \delta)^{\alpha'''} \tag{3.7$'$}$$

One verifies easily that the sequence

$$x^l = \left(x'^l = \frac{b(x'^l) + c(x'^l)}{2} \right)$$

94 has the required properties: it is sufficient to estimate from below the distance of x^l from the roots of $\Phi(x'^l, z) = 0$. For the real roots, this follows from (3.6.i), for the imaginary roots from the estimate from below of the imaginary part of a root in terms of $\Delta(x')$. The lemma follows.

We apply this lemma to Δ and γ (instead of Φ and Γ as in the statement). There is a sequence of points x'^l of points of γ, $x'^l \to 0$, satisfying (3.7′). Divide the interval $[b(x'^l), c(x'^l)]$ into q equal intervals with extremities

$$b_0(x'^l) = b(x'^l), b_1(x'^l), \ldots, b_q(x'^l) = c(x'^l)$$

and consider the expression

$$\frac{1}{(b_1 - b_0)^q} \left\{ f(x'^l, b_0) - \binom{q}{1} f(x'^l, b_1) + \cdots + (-1)^q f(x'^l, b_q) \right\}.$$

As $l \to \infty$, this expression tends to $\dfrac{\partial^q f}{\partial x_k^q}(0)$. On the other hand, the inequalities (3.6.i), (3.6.ii), (3.7′) and the fact that f has a zero of infinite order at 0 on Γ show that this limit is 0. Thus we have

$$\frac{\partial^q f}{\partial x_k^q}(0) = 0 \quad \forall q \in \mathbf{N},$$

which proves Proposition 3.6 and hence Theorem 3.5. □

Remark 3.8. Let X be a subset of $\mathbf{R}^n$, adherent to 0. Besides $J(X)$, we may consider also the ideal $J'(X) \subset \mathscr{F}_n$ of Taylor series at 0 of functions $f \in \mathscr{E}_n$ vanishing on X. We have $J'(X) \subset J(X)$. If X is an analytic set, we have $\widehat{I(X)} \subset J'(X)$. Hence, by (3.5), $J(X) = J'(X)$ in this case.

We shall examine now what one can say about differentiable functions vanishing on an analytic set, and not just about their Taylor series.

Definition 3.9. *Let Ω be an open set in $\mathbf{R}^n$ and X an analytic set in Ω,*
$a \in X$. We say that X is coherent at a if there exists a neighbourhood 95
Ω' of a and a finite number of analytic functions $f_i (1 \leqslant i \leqslant p)$ in Ω', vanishing on X and having the following property:

For any $b \in \Omega'$, the images of $f_1, \ldots, f_p$ in $\mathscr{O}_b$ (the ring of germs of analytic functions at b) generate $I(X_b)$.

Contrary to what happens in the complex case, this property is not verified for all analytic sets. The simplest counter-example is the "umbrella" $x_3(x_1^2 + x_2^2) = x_1^3$ which has the line $x_1 = x_2 = 0$ as isolated generator, and so is not coherent at 0.

Theorem 3.10. *Let X_0 be a real analytic germ at 0 in $\mathbf{R}^n$, $I(X_0)$ its analytic ideal, and let $K(X_0)$ its analytic ideal, and let $K(X_0)$ be the ideal in $\mathscr{E}_n$ of C^∞ functions vanishing on X_0. Then the following properties are equivalent.*

(i) $K(X_0) = I(X_0)\mathscr{E}_n$.

(ii) X_0 *is coherent at* 0.

Proof. (ii) $\Rightarrow$ (i). Let X_0 be coherent at 0 and let X be a representative of X_0 in a neighbourhood Ω' of 0 with the property given in (3.8). Let $\phi \in \mathscr{E}(\Omega')$, $\phi = 0$ on X. By (3.5), for any $b \in \Omega'$, $T_b\phi$ is a linear combination of the $T_b f_i$. Hence, by (1.1′), ϕ is a linear combination of the f_i in $\mathscr{E}(\Omega')$.

(i) $\Rightarrow$ (ii) (Tougeron [22]). Suppose that X_0 is not coherent. Let $f_1, \ldots, f_p$ be generators of $I(X_0)$, Ω a neighbourhood of 0 in which the f_i are defined, and set

$$X = \{x \in \Omega | f_1(x) = \quad = f_p(x) = 0\}.$$

Since X_0 is not coherent, there is a sequence $\{x^l\}$ of distinct points of X, $x^l \to 0$, and a sequence of functions g^l defined near x^l, such that, for each l, g^l is not a linear combination of the f_i. Let $\{\phi^l\}$ be a sequence

of functions $\in \mathscr{E}(\Omega)$, $\phi^l = 1$ near x^l, having compact support in Ω and in the set where g^l is defined such that the supports of ϕ^l, $\phi^{l'}$ do not meet if $l \neq l'$. Let $h^l = \phi^l g^l$, extended to Ω by 0. By an argument which is well known in the theory of Fréchet spaces (which we leave to the reader) we can find a sequence $\{\lambda^l\}$ of real numbers $\neq 0$ such that the series $\Sigma\lambda^l h^l$
96 converges, in $\mathscr{E}(\Omega)$, to a function g. The germ $g_0 \in \mathscr{E}_n$ of g at 0 is not a linear combination of the f_i in $\mathscr{E}_n$, whence the theorem.

We refer to Malgrange [13] for applications of Theorems 3.5 and 3.10 to complex analytic sets. In conclusion, let us note another application of Theorem 3.5. □

Proposition 3.11. *Let X_0 be an analytic germ at* 0 *in* $\mathbf{R}^n$ *with* $\dim X_0 = k$. *Suppose that X_0 contains the germ V_0 of a C^∞ manifold of dimension k. Then V_0 is the germ of an analytic manifold (which is then an irreducible component of X_0).*

Before giving the proof, we give two examples.

Example 3.11.1. If X_0 is a C^∞ manifold, it is an analytic manifold. However, one sees easily that even for $n = 2$, if we replace C^∞ by $C^r (r \in \mathbf{N})$, the statement is no longer true.

Example 3.11.2. Let $\Phi \in \mathscr{O}_{n+1}$, $\Phi \neq 0$, and let $f \in \mathscr{E}_n$, $f(0) = 0$ satisfy

$$\Phi(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = 0.$$

Then f is analytic [take for X the set defined by $\Phi(x_1, \ldots, x_{n+1}) = 0$ and for V that defined by $x_{n+1} = f(x_1, \ldots, x_n)$].

Proof of the Proposition. Denote by $I(X_0)$ the analytic ideal of X and by $J(V_0)$ the formal ideal of V_0. The structure of $J(V_0)$ is obvious because of our hypothesis that V_0 is non-singular. On the other hand, $I(X_0) \subset J(V_0)$, hence $\widehat{I(X_0)} \subset J(V_0)$. Since

$$\dim(\mathscr{F}_n/\widehat{I(X_0)}) = \dim(\mathscr{F}_n/J(V_0)) = k,$$

$J(V_0)$ is a minimal prime ideal in the decomposition of $\widehat{I(X_0)}$. By (3.3), there exists a prime ideal $\mathfrak{p} \subset \mathscr{O}_n$ with

$$\mathfrak{p} \supset I(X_0), \widehat{\mathfrak{p}} = J(V_0).$$

There remain two things to be proved.

(i) *The germ W_0 defined by $\mathfrak{p}$ is an analytic manifold of dimension k.* (This is an easy consequence of the Jacobian criterion for regular points; we leave the details to the reader.)

(ii) *We have $V_0 = W_0$.* **97**

By an analytic change of coordinates, we may suppose that W_0 is defined by equations $x_{k+1} = \ldots = x_n = 0$. On the other hand, V_0 is obviously tangent to W_0 of infinite order at 0, hence defined by equations

$$x_{k+j} = \phi_{k+j}(x_1, \ldots, x_k), \phi_{k+j} \in \mathscr{E}_k, \phi_{k+j} \text{ flat at } 0.$$

Suppose that $W_0 \neq V_0$. Let X_0' be the union of the irreducible components of X_0 different from W_0, and let $g(x_1, \ldots, x_k) \in \mathscr{O}_k$ be a function not identically zero, which vanishes on $X_0' \cap W_0$. Let D be the set of zeros of g, and U be the set of points of $\mathbf{R}^n - D$ near 0, for which we do not have $\phi_{k+1}(x_1, \quad , x_k) = \ldots = \phi_n(x_1, \ldots, x_k) = 0$. U is clearly open and closed in $\mathbf{R}^n - D$ near 0 and is adherent to 0. Let f be a function $\in \mathscr{O}_n$ vanishing on X_0'. In particular, f vanishes on $V_0 - W_0$. Hence $f(x_1, \ldots, x_k, 0, \ldots, 0)$ has a zero of infinite order at 0 on U. By Proposition 3.6, f has a Taylor series which is zero at 0, hence is itself 0. Hence f vanishes on W_0, contradicting the fact that $W_0 \not\subset X_0'$. The proposition follows.

VII
APPLICATIONS TO THE THEORY OF DISTRIBUTIONS

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1 Support of a distribution. Continuable distributions.

Let Ω be an open set in $\mathbf{R}^n$. We denote by $\mathscr{D}'(\Omega)$ [resp. $\mathscr{D}'_c(\Omega)$, $\mathscr{D}'^m(\Omega)$, $\mathscr{D}'^m_c(\Omega)$] the space of distributions [resp. with compact support, of order m, of order m with compact support] in Ω (L. Schwartz [18]). It is known that $\mathscr{D}'_c(\Omega)$ [resp. $\mathscr{D}'^m_c(\Omega)$] is the dual of $\mathscr{E}(\Omega)$ [resp. $\mathscr{E}^m(\Omega)$] with its topology of Fréchet space that we have considered in Chapter I.

Let X be a closed subset of Ω. We denote by $\mathscr{D}'(X)$ [resp. $\mathscr{D}'_c(X)$, $\mathscr{D}'^m(X)$, $\mathscr{D}'^m_c(X)$] the subspace of the corresponding space of distributions in Ω having support in X. Let us show that $\mathscr{D}'_c(X)$ [resp. $\mathscr{D}'^m_c(X)$] is the orthogonal of $\mathscr{I}(X;\Omega)$ (resp. $\mathscr{I}^m(X;\Omega)$). In fact, by the definition of support, $\mathscr{D}'_c(X)$ is orthogonal to the set of $f \in \mathscr{E}(\Omega)$ which are zero in a neighbourhood of X; on the other hand $\mathscr{I}(X;\Omega)$ is the closure in $\mathscr{E}(\Omega)$ of this set (Proposition I, 5.2). For $\mathscr{D}'^m_c(X)$, the same argument applies.

It follows from this that $\mathscr{D}'_c(X)$ [resp. $\mathscr{D}'^m_c(X)$] can be identified naturally with the dual of $\mathscr{D}(X) = \mathscr{E}(\Omega)/\mathscr{I}(X;\Omega)$ [resp. $\mathscr{E}^m(X) = \mathscr{E}^m(\Omega)/\mathscr{I}^m(X;\Omega)$].

Let now Y be another closed set in Ω with $Y \subset X$. Set $\mathscr{P}'(Y;X) = \mathscr{D}'(X)/\mathscr{D}'(Y)$, $\mathscr{P}'_0(Y;X) = \mathscr{D}'_c(X)/\mathscr{D}'_c(Y)$. The space $\mathscr{P}'(Y;X)$ can be interpreted as the space of distributions on $\Omega - Y$, with support in $X - Y$, which can be continued to a distribution on Ω (which, then, necessarily has support in X) and $\mathscr{P}'_c(Y;X)$, can be interpreted analogously. If we consider $\mathscr{D}'_c(X)$ as the dual of $\mathscr{E}(X)$, then $\mathscr{D}'_c(Y) \subset \mathscr{D}'_c(X)$ is the orthogonal complement of $\mathscr{I}(Y;X)$ (same reasoning as above). Hence $\mathscr{P}'_c(Y;X)$ is the dual of $\mathscr{I}(Y;X)$, the latter space being equipped with the topology induced from $\mathscr{E}(X)$.

Let now X and Y be two arbitrary closed sets in Ω. Consider the sequence introduced in (I, 5.4): **99**

$$0 \to \mathscr{E}(X \cup Y) \xrightarrow{\delta} \mathscr{E}(X) \oplus \mathscr{E}(Y) \xrightarrow{\pi} \mathscr{E}(X \cap Y) \to 0. \tag{1.1}$$

The transposed sequence is

$$0 \to \mathscr{D}'_c(X \cap Y) \xrightarrow{\pi^*} \mathscr{D}'_c(X) \oplus \mathscr{D}'_c(Y) \xrightarrow{\delta^*} \mathscr{D}'_c(X \cup Y) \to 0 \tag{1.2}$$

where π^* is (up to sign) the diagonal mapping $\pi^*(T) = (T, -T)$ and where $\delta^*(T, S) = T + S$. From the properties of (1.1) and the theory of duality in Fréchet spaces, we deduce at once that π^* is injective, that $\ker \delta^* = \operatorname{im} \pi^*$ and that $\operatorname{im} \delta^*$ is dense in $\mathscr{D}'_c(X \cup Y)$. Moreover, for δ^* to be surjective (i.e. for (1.2) to be exact) it is necessary and sufficient that $\operatorname{im} \delta$ be closed, i.e. that (1.1) be exact. Finally, by a partition of unity, we see that the exactness of (1.2) is equivalent to that of the sequence:

$$0 \to \mathscr{D}'(X \cap Y) \xrightarrow{\pi'} \mathscr{D}'(X) \oplus \mathscr{D}'(Y) \xrightarrow{\delta'} \mathscr{D}'(X \cup Y) \to 0, \tag{1.3}$$

π' and δ' being defined in the same way as π^* and δ^*, and this is equivalent to the surjectivity of δ'. Consequently

Proposition 1.4 (Łojasiewicz [10].)**.** *Under the above hypotheses the following properties are equivalent.*

(i) *X and Y are regularly situated.*

(ii) *The sequence* (1.3) *is exact.*

(iii) *The mapping δ' is surjective: in other words, every distribution $T \in \mathscr{D}'(X \cup Y)$ can be written $T = S_1 + S_2$ with support $(S_1) \subset X$, support $(S_2) \subset Y$.*

2 Division of distributions.

The statement dual to Theorem VI, 1.1 is the following.

Theorem 2.1. *Let Ω be an open set in $\mathbf{R}^n$, $Y \subset X \subset \Omega$ two analytic subsets of Ω, and let $f_1, \ldots, f_p$ be analytic functions on Ω. Let*

$T_1, \ldots, T_p \in \mathscr{P}'(Y;X)$. Then a necessary and sufficient condition that there exist $S \in \mathscr{P}'(Y;X)$ satisfying $f_1 S = T_1, \ldots, f_p S = T_p$ is the following.

100 (R). *For any $a \in X - Y$, analytic relations between the f_i at a are relations between the T_i, i.e. if $g_1, \ldots, g_p$ are germs of analytic functions at a, then $g_1 f_1 + \cdots + g_p f_p = 0$ implies that $g_1 T_1 + \cdots + g_p T_p = 0$ near a.*

Remark 2.2. Using the global theory of coherent analytic sheaves on a real analytic manifold, this condition can be replaced by the following : if $g_1, \ldots, g_p$ are analytic in Ω, then $g_1 f_1 + \cdots + g_p f_p = 0$ implies that $g_1 T_1 + \cdots + g_p T_p = 0$.

Example 2.3. Take $Y = \emptyset$, $X = \Omega$. Given $T \in \mathscr{D}'(\Omega)$ there exists $S \in \mathscr{D}'(\Omega)$ with $f_1 S = T$. In other words, "the division of a distribution by an analytic function is always possible". This theorem was proved for $p = 1$ (before the general case) by Hörmander [6] when f_1 is a polynomial, and Łojasiewicz [10].

Proof of Theorem 2.1. It suffices to prove the theorem for $\mathscr{P}'_c(Y;X)$ instead of $\mathscr{P}'(Y;X)$, as one sees using a partition of unity. Consider the mapping

$$F : \mathscr{P}'_c(Y;X) \to [\mathscr{P}'_c(Y;X)]^p$$

defined by $F(S) = (f_1 S, \ldots, f_p S)$. We shall prove that the image of F is closed and that it is dense in the set E of $(T_1, \ldots, T_p)$ satisfying ((R)); one would then have $\operatorname{im}(F) = E$. The transpose of F is the mapping

$$F^* : [\mathscr{I}(Y;X)]^p \to \mathscr{I}(Y;X)$$

defined by $F^*(\phi_1, \ldots, \phi_p) = f_1\phi_1 + \cdots + f_p\phi_p$. Theorem VI, 1.2 implies that the ideal generated in $\mathscr{I}(Y;X)$ by the f_i is closed; hence $\operatorname{im}(F^*)$ is closed. By transposition it follows that $\operatorname{im}(F)$ is closed.

We now prove that $\operatorname{im}(F)$ is dense in E. It suffices to show that for any $\phi = (\phi_1, \ldots, \phi_p) \in [\mathscr{I}(Y;X)]^p$ which is orthogonal to $\operatorname{im}(F)$, we have $f_1\phi_1 + \cdots + f_p\phi_p = 0$. By a partition of unity, it suffices

to examine the ϕ with compact support in a given neighbourhood of a (a being any point of Ω). Now, Corollary VI, 1.12 shows that we can find analytic relations $g^{(1)}, \ldots, g^{(r)}$ between the f in a neighbourhood of a and functions $\psi_1, \ldots, \psi_r \in \mathscr{E}(\Omega)$ with compact support in the given 101
neighbourhood of a such that $\phi = \Sigma \psi_j g^{(j)}$. One deduces at once that ϕ is orthogonal to E, and the theorem follows.

The preceding theorem can be interpreted in terms of the concept of injective modules.

Let A be a (unitary, commutative) ring, and M a unitary A-module. M is called injective if, for any ideal $\mathfrak{I} \subset A$, the natural mapping $M \simeq \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(\mathfrak{I}, M)$ is surjective. We take a system $(f_i)_{i \in I}$ of generators of $\mathfrak{I}$ and a family $(T_i)_{i \in I}$ of elements of M such that every relation between the f_i with coefficients in A is also a relation between the T_i. Then $u : f_i \to T_i$ defines an element of $\mathrm{Hom}_A(\mathfrak{I}, M)$ and conversely. To say that M is injective amounts therefore to saying that in this situation, there is an $S \in M$ such that $f_i S = T_i$ for each i. This being the case, Theorem 2.1, the noetherian nature of $\mathscr{O}_n$ and Oka's theorem III, 4.12 [in the form (III, 4.14)] give us the

Theorem 2.4. *Let X_0, Y_0 be germs of analytic sets at $0 \in \mathbf{R}^n$ with $Y_0 \subset X_0$, and let $\mathscr{P}'(Y_0; X_0)$ be the space of germs induced at 0 by $\mathscr{P}'(Y : X)$ (Y, X being representatives of Y_0, X_0 near 0 with $Y \subset X$). Then $\mathscr{P}'(Y_0; X_0)$ is an injective $\mathscr{O}_n$-module. In particular, the space $\mathscr{D}'_n$ of germs at 0 of distributions is an injective $\mathscr{O}_n$-module.*

Remark 2.5. With the hypotheses of Theorem 2.1, let $\mathscr{O}(\Omega)$ be the ring of real valued analytic functions in Ω. Then, using Remark 2.2, one shows easily that $\mathscr{P}'(Y; X)$ (and in particular $\mathscr{D}'(\Omega)$) is an injective $\mathscr{O}(\Omega)$-module.

3 Harmonic synthesis in $\mathscr{S}'$.

We being by giving the statements dual to those given in Chapter II.

Proposition 3.1. *Let Ω be an open set in $\mathbf{R}^n$ and V a sub-$\mathscr{E}^m(\Omega)$-module of $\mathscr{D}'^m(\Omega)$ which is weakly closed. Then, in V (with the weak topology* *induced from $\mathscr{D}'^m(\Omega)$) distributions with point support form a total system.*

102 *The same statement is true with $\mathscr{D}'^m(\Omega)$ and $\mathscr{E}^m(\Omega)$ respectively replaced by $\mathscr{D}'(\Omega)$ and $\mathscr{E}(\Omega)$.*

The proof, which is immediate by transposition and partition of unity, is left to the reader.

In the case of $\mathscr{E}(\Omega)$ and $\mathscr{D}'(\Omega)$, the result is true even with the strong topology, since these spaces are reflexive. We remark that, using a partition of unity [or directly, using II, 1.7], we see that these results are true if Ω is any C^∞ manifold countable at infinity.

This being the case, let $\mathscr{S}$ be the space of C^∞ functions on $\mathbf{R}^n$ which, together with derivatives of all orders, tend to zero faster than any negative power of $x_1^2 + \quad + x_n^2$. Let $\mathbf{R}^n \to S^n$ be the natural mapping of $\mathbf{R}^n$ into the n-dimensional sphere (S^n being obtained from $\mathbf{R}^n$ by adding a point ∞ at infinity). This mapping identifies $\mathscr{S}$ with $\mathscr{I}(\{\infty\}; S^n)$, and the usual topology of $\mathscr{S}$ is compatible with this isomorphism. The dual $\mathscr{S}'$ of $\mathscr{S}$ can be identified then with $\mathscr{P}'(\{\infty\}; S^n) = \mathscr{P}'_c(\{\infty\}; S^n)$. We look upon this space as imbedded in $\mathscr{D}'(S^n - \{\infty\}) = \mathscr{D}'(\mathbf{R}^n)$.

Let V be a (weakly or strongly) closed sub-$\mathscr{S}$-module of $\mathscr{S}'$ (the two being equivalent since $\mathscr{S}$ is reflexive). We show that distributions with point support form a total set in V. Let $\widetilde{V}$ be the inverse image of V in $\mathscr{D}'(S^n)$. It is sufficient to show that $\widetilde{V}$ is closed (which is obvious) and that it is invariant under multiplication by any $f \in \mathscr{E}(S^n)$. Now, if $f \in \mathscr{I}(\{\infty\}; S^n)$ this is true by hypothesis. If f is arbitrary, we show that any $\phi \in \mathscr{E}(S^n)$ orthogonal to V is orthogonal to $f\widetilde{V}$: given such a ϕ, it is orthogonal to $\mathscr{D}'(\{\infty\})$, hence $\phi \in \mathscr{I}(\{\infty\}; S^n)$. Hence there is a sequence $\{\alpha_k\}$ of functions in $\mathscr{E}(S^n)$, zero in a neighbourhood of ∞, such that $f\phi = \lim \alpha_k f\phi$ (cf. proof of Lemma I, 4.3). Hence, for $T \in \widetilde{V}$, we have

$$\langle fT, \phi\rangle = \langle T, f\phi\rangle = \lim\langle T, \alpha_k f\phi\rangle = \lim\langle(\alpha_k f)T, \phi\rangle = 0$$

and the result follows. [The same reasoning would apply to $\mathscr{P}'_c(Y; X)$, $Y \subset X$ being any closed sets of a manifold.]

By the Fourier transformation, one knows that $\mathscr{S}$ is transformed **103**
into $\mathscr{S}$, $\mathscr{S}'$ into $\mathscr{S}'$, and that multiplication transforms into convolution. One deduces easily the following : If V is a closed sub-$\mathscr{S}$-module of $\mathscr{S}'$, its Fourier transform $\widehat{V}$ is a vector $\mathbf{R}$-subspace of $\mathscr{S}'$ which is closed and invariant by translation, and conversely. Further, the Fourier transforms of distributions with point support are the "exponential polynomials", i.e. the functions $x \to P(x)e^{i\langle\lambda,x\rangle}$, where P is a polynomial and $\lambda \in \mathbf{R}^n$. Thus one has the following result.

Theorem 3.2 (Whitney-Schwartz; cf. Schwartz [19])**.** *In any vector subspace of $\mathscr{S}'$ which is closed and translation invariant, exponential polynomials form a total system.*

One knows, on the other hand, that this statement is false in $L^{\infty}(\mathbf{R}^n)$ with the weak topology (Schwartz for $n \geqslant 3$: Malliavin for $n = 1, 2$). One conjectures that it is true in $\mathscr{E}(\mathbf{R}^n)$ [it is then necessary to take "complex" exponential polynomials, i.e. $\lambda \in \mathbf{C}^n$], but, at present, this has only been proved for $n = 1$ (Schwartz [20]).

4 Partial differential equations with constant coefficients.

Let $P_n = \mathbf{R}[X_1, \ldots, X_n]$ be the polynomial ring in n indeterminates. We shall consider it, at least at the beginning of this section, as imbedded in the ring of analytic functions on $\mathbf{R}^n$ by the mapping $X_j \to x_j$, the x_j being the coordinates in $\mathbf{R}^n$. Let $f_1, \ldots, f_p \in P_n$ and $T_1, \ldots, T_p \in \mathscr{S}'(\mathbf{R}^n)$. We first prove the following result.

There exists $S \in \mathscr{S}'$ with $f_j S = T_j$, $1 \leqslant j \leqslant p$, if and only if the following condition is verified.

(R): *at any point $a \in \mathbf{R}^n$, the analytic relations at a between the f_j are relations between the T_j in a neighbourhood of a.*

For this, consider $\mathbf{R}^n$ imbedded in S^n as in §3, and let us identify $\mathscr{S}'$ with $\mathscr{D}'(\{\infty\}; S^n)$. It is enough to prove the result in the neighbourhood of any point a of S^n (partition of unity). If $a \neq \infty$ this follows from Theorem 2.1. If $a = \infty$, we make the change of variable $y_j = x_j/\Sigma x_j^2$, and
remark that, if m is large enough, $(\Sigma y_i^2)^m f_j$ is a polynomial in $y_1, \ldots, y_n$; **104**
the result follows then again from Theorem 2.1.

Let us remark that the condition (R) is equivalent to the following.

(R′). *Relations between the f_i with coefficients in P_n are relations between the T_i (*i.e. $\Sigma g_i f_j = 0$, $g_j \in P_n$ *implies that* $\Sigma g_j T_j = 0$*).*

In fact, if we denote by $\mathscr{O}_a(a \in \mathbf{R}^n)$ the ring of germs of functions analytic at a, we know that $\mathscr{O}_a$ is flat over P_n [III, (4.11)]. Interpreting flatness in terms of relations, we see at once that (R′) ⇒ (R). From this and the fact that P_n is noetherian, we deduce (arguing as in the proof of Theorem 2.4)

Theorem 4.1. *P operating on $\mathscr{S}'$ by $X_j T = x_j T$ makes of $\mathscr{S}'$ an injective P_n-module.*

By the Fourier transformation, we deduce

Theorem 4.1. *If P_n operates on $\mathscr{S}'$ by $X_j T = \dfrac{\partial T}{\partial x_j}$, $\mathscr{S}'$ is an injective P_n-module.*

Example 4.2. Let $f \in P_n$ and $\delta \in \mathscr{S}'$ be defined by $\langle \delta, \phi \rangle = \phi(0)$. Then there exists $E \in \mathscr{S}'$ with $f\left(\dfrac{\partial}{\partial x_j}\right) E = \delta$. In other words, *every linear differential operator with constant coefficients has a temporate fundamental solution* (i.e. one in $\mathscr{S}'$). This is mainly of historical interest (the condition $E \in \mathscr{S}'$ is artificial; see Hörmander [7] for a discussion of this question). We have, however, given this here because it was the origin of a large part of the results contained in this book.

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