REPORT OF AN INTERNATIONAL COLLOQUIUM ON ZETA-FUNCTIONS

held at the
Tata Institute of Fundamental Research, Bombay
on 14-21 February 1956

WITH THE FINANCIAL ASSISTANCE OF

THE INTERNATIONAL MATHEMATICAL UNION, THE MINISTRY OF NATURAL RESOURCES AND SCIENTIFIC RESEARCH OF THE GOVERNMENT OF INDIA, THE SIR DORABJI TATA TRUST AND THE TATA INSTITUTE OF FUNDAMENTAL RESEARCH

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INTERNATIONAL COLLOQUIUM ON ZETA-FUNCTIONS

BOMBAY, 14-21 FEBRUARY 1956

REPORT

An International Colloquium on Zeta-functions was held at the Tata Institute of Fundamental Research, Bombay, on 14-21 February, 1956. The Colloquium was the first of its kind to be held in India, and was attended by thirty-nine mathematicians, twenty members and nineteen other participants, from nine countries: France, West Germany, Hungary, India, Japan, the Netherlands, Singapore, the United Kingdom and the United States.

The Colloquium was jointly sponsored, and financially supported, by the International Mathematical Union, the Government of India in the Ministry of Natural Resources and Scientific Research, the Sir Dorabji Tata Trust, and the Tata Institute of Fundamental Research. The proposal for the Colloquium was put forward by the Tata Institute of Fundamental Research, and endorsed by the National Committee for Mathematics in India, which acted as the principal agency for executing the plan of the Colloquium, and maintained close liaison between the sponsoring institutions. The Tata Institute of Fundamental Research was the principal host institution.

An Organizing Committee consisting of Professor K. Chandrasekharan (Chairman), Professor J. F. Koksma, Professor H. Maass, Dr. K. G. Ramanathan, Professor A. Selberg, and Professor C. L. Siegel, was appointed in April, 1955, to draw up the scientific programme. Professors Koksma, Selberg and Siegel were nominated to this Committee by the International Mathematical Union.

The topics for discussion in the Colloquium were set forth as follows : (i) theory of zeta-functions for algebraic number fields, including the classical theory of the Riemann zeta-function, and general problems about the zeros ; (ii) A. Selberg's investigations ; (iii) zeta-functions in function fields and number fields, including A. Weil's methods; (iv) modular functions and Dirichlet series, Hecke's operators, and quadratic forms.

It was the policy of the Organizing Committee to invite not only acknowledged experts in these subjects, but also, wherever possible, a few other distinguished mathematicians actively interested in closely related subjects. The Committee specially welcomed the participation of younger mathematicians from inside India as well as from abroad. It was decided that English should be the language of the Colloquium.

The following sixteen mathematicians accepted invitations to speak in the Colloquium :

Professor S. Bochner (Princeton), Professor K. Chandrasekharan (Bombay), Professor S. Chowla (Boulder), Professor M. Deuring (Göttingen), Professor M. Eichler (Münster), Dr. M. Koecher (Münster), Professor H. Maass (Heidelberg), Professor A. G. Postnikov (Moscow), Professor H. Petersson (Münster), Dr. K. G. Ramanathan (Bombay), Professor R. A. Rankin (Glasgow), Dr. I. Satake (Tokyo), Professor A. Selberg (Princeton), Professor C. L. Siegel (Göttingen), Professor P. Turan (Budapest), and Professor N. G. Tchudakoff (Saratov).

Professors Postnikov and Tchudakoff could not attend the Colloquium; their papers, however, are published in the Proceedings.

Professor Y. V. Linnik (Leningrad), who was unable to accept the invitation to attend the Colloquium, sent in a paper.

The following six mathematicians accepted invitations to attend the Colloquium : Professor Y. Akizuki (Kyoto), Professor N. G. de Bruijn (Amsterdam), Dr. R. C. Gunning (Princeton), Professor S. Mandelbrojt (Paris), Professor S. Minakshisundaram (Waltair), and Professor A. Oppenheim (Singapore).

With the approval of the Organizing Committee, nineteen other research workers from India participated in the Colloquium.

In accordance with the rules framed by the International Mathematical Union, the Colloquium met in closed sessions. Abstracts of most of the papers were received in advance, mimeographed, and distributed to all the participants. Nineteen lectures were given. Each lecture lasted fifty minutes, and was followed by a discussion. Almost all the papers contained original contributions which were for the first time announced in the Colloquium. The discussions continued informally, and very fruitfully, outside the lecture room, over a period of at least two weeks, since members of the Colloquium were invited to attend the South Asian Conference on Mathematical Education held at the same place in the following week.

The Council of the Tata Institute of Fundamental Research gave a reception in honour of the members of the Colloquium on Monday, 13 February, 1956. On behalf of the Council, and on behalf of the Government of India, Dr. H. J. Bhabha, Director of the Institute, welcomed the members of the Colloquium, the names of some of whom, he said, "had already become part of the history of mathematics." He gave a brief description of the growth and development of the Institute since its foundation in 1945, and announced a tripartite agreement between the Government of India, the Government of Bombay, and the Sir D. J. Tata Trust, whereby the three parties "will jointly run the Institute in future, with, of course, the Government of India having a major voice and responsibility in running it." He also announced that the Government of India had recognized the Institute as "the national centre for advanced study and fundamental research in nuclear science and mathematics." He then read out a special message sent for the occasion by the Prime Minister, Jawaharlal Nehru. Dr. J. Matthai spoke on behalf of the Sir Dorabji Tata Trust, and expressed the interest of the Trust in the promotion of fundamental research. Professor E. Bompiani, Secretary of the International Mathematical Union, thanked the organizers for their initiative in bringing about such an international meeting of mathematicians.

The social programme during the Colloquium week included a dinner by Dr. H. J. Bhabha on 14 February; a special performance of classical Indian dances, Bharata Natyam and Manipuri, on 15 February; a special show of documentary films, produced by the Films Division of the Government of India, on 16 February; and a dinner at Juhu on 21 February.

MEMBERS

1. Professor Y. Akizuki
   Kyoto University
   Kyoto, Japan

2. Professor S. Bochner
   Fine Hall, Princeton University
   Princeton, New Jersey, U.S.A.

3. Professor N. G. de Bruijn
   Mathematisch Institut
   der Universiteit van Amsterdam
   The Netherlands

4. Professor K. Chandrasekharan
   Tata Institute of Fundamental Research
   Bombay, India

5. Professor S. Chowla
   University of Colorado
   Boulder, Colorado
   U. S. A.

6. Professor M. Deuring
   University of Göttingen
   Göttingen, West Germany

7. Professor M. EICHLER
University of Münster
Münster, West Germany

8. Dr. R. C. GUNNING
Fine Hall, Princeton University
Princeton, New Jersey
U. S. A.

9. Dr. M. KOECHER
University of Münster
Münster, West Germany

10. Professor Y. V. LINNIK*
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Leningrad, U. S. S. R.

11. Professor H. MAASS
University of Heidelberg
Heidelberg, West Germany

12. Professor S. MANDELBROJT
College de France
Paris, France

13. Professor S. MINAKSHISUNDARAM
Andhra University
Waltair, India

14. Professor A. OPPENHEIM
University of Malaya
Singapore 10

15. Professor H. PETERSSON
University of Münster
Münster, West Germany

16. Professor A. G. Postnikov*
U. S. S. R. Academy of Sciences
Moscow

17. Dr. K. G. Ramanathan
Tata Institute of Fundamental Research
Bombay, India

18. Professor R. A. Rankin
The University
Glasgow 2, Scotland

19. Dr. I. Satake
College of General Education
University of Tokyo
Tokyo, Japan

20. Professor A. Selberg
The Institute for Advanced Study
Princeton, N. J.
U. S. A.

21. Professor C. L. Siegel
University of Göttingen
Göttingen, West Germany

22. Professor P. Turan
University Matematikai Szeminariuma,
Budapest, Hungary

23. Professor N. G. Tchudakoff*
University of Saratov
Saratov, U. S. S. R.

*Did not attend the Colloquium

PARTICIPANTS

Tata Institute of Fundamental Research, Bombay

1.	Miss S. V. Adavi	Research Assistant
2.	Mr. K. Balagangadharan	Research Assistant
3.	Mr. N. S. Gopalakrishnan	Research Student
4.	Mr. M. S. Huzurbazar	Research Assistant
5.	Mr. V. C. Nanda	Research Assistant
6.	Mr. M. S. Narasimhan	Research Assistant
7.	Mr. S. Raghavan	Research Assistant
8.	Mr. M. Ramabhadran	Research Assistant
9.	Mr. P. K. Raman	Research Assistant
10.	Mr. V. V. Rao	Research Assistant
11.	Mr. C. S. Rengachari	Research Assistant
12.	Miss K. Savithri	Research Assistant
13.	Mr. C. S. Seshadri	Research Assistant
14.	Mr. B. V. Singbal	Research Assistant
15.	Mr. R. Sridharan	Research Student
16.	Mr. K. Srinivasacharyulu	Research Student
17.	Mr. M. Varada Rajan	Research Student
18.	Miss K. B. Vedak	Research Student

Punjab University, Hoshiarpur

19.	Mr. T. P. Srinivasan	Lecturer

PROGRAMME

Tuesday, February 14, 1956

10.00 A.M.— 10.50 A.M.	C. L. Siegel : A generalization of the Epstein zeta-function.
11.30 A.M. — 12.20 P.M.	N. G. Tchudakoff : Theory of characters of number semi-groups.
4.15 P.M. — 5.05 P.M.	P. Turan : On the zeros of the zeta-function of Riemann.

Wednesday, February 15, 1956

10.00 A.M. — 10.50 A.M.	Y. V. Linnik : An application of the theory of matrices and Lobatschevskian geometry to the theory of Dirichlet's real characters.
11.30 A.M. — 12.20 P.M.	A. Selberg : Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces, with applications to Dirichlet series, I.
4.15 P.M. — 5.05 P.M.	M. Deuring : The zeta-functions of algebraic curves and varieties, I.

Thursday, February 16, 1956

10.00 A.M. — 10.50 A.M.	A. Selberg : Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces, with applications to Dirichlet series, II.

11.30 A.M. — 12.20 P.M.	A. Selberg : Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces, with applications to Dirichlet series, III.
4.15 P.M. — 5.05 P.M.	R. A. Rankin : The construction of automorphic forms from the derivatives of a given form.

Friday, February 17, 1956

10.00 A.M. — 10.50 A.M.	A. Selberg : Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces, with applications to Dirichlet series, IV.
11.30 A.M. — 12.20 P.M.	H. Maass : Spherical functions and quadratic forms.
4.15 P.M. — 5.05 P.M.	M. Koecher : On the Hecke operators for modular forms of degree n.

Saturday, February 18, 1956

10.00 A.M. — 10.50 A.M.	M. Eichler : Modular correspondences and their representations, I.
11.30 A.M. — 12.10 P.M.	M. Eichler : Modular correspondences and their representations, II.
4.15 P.M. — 5.05 P.M.	A. Selberg and S. Chowla : On Epstein's zeta-function.

Sunday, February 19, 1956

11.00 A.M. — 11.50 A.M.	A. G. Postnikov : Generalization of one of the Hilbert problems.

Monday, February 20, 1956

10.00 A.M. — 10.50 A.M.	K. G. Ramanathan : Quadratic forms over involutorial division algebras.
11.30 A.M. — 12.20 P.M.	S. Bochner and K. Chandrasekharan : On Riemann's functional equation.
4.15 P.M. — 5.05 P.M.	H. Petersson : On a certain kind of zetafuchsian functions.

Tuesday, February 21, 1956

10.00 A.M. — 10.50 A.M.	S. Bochner : Gamma factors in functional equations.
11.30 A.M. — 12.20 P.M.	I. Satake : On the compactification of the Siegel space.
3.00 P.M. — 3.45 P.M.	M. Deuring : The zeta-functions of algebraic curves and varieties, II.

सत्यमेव जयते

Message

I send my greetings and good wishes to the International Colloquium on Zeta Functions and the South Asian Conference on Mathematical Education which are being organised by the Tata Institute of Fundamental Research in Bombay. This Institute has been recognised by the Government of India as the national centre for advanced study and fundamental research in mathematics and it is appropriate that it should hold this colloquium and conference.

Mathematics is supposed to be a dull subject, but it is increasingly recognised that it is of high importance in scientific developments today. Indeed, mathematical research has widened the horizon of the human mind tremendously and has helped in the understanding, to some extent, of nature and the physical world. It is a vehicle today of exact scientific thought. India has had the good fortune in the past to produce some very eminent mathematicians. I hope that the conferences that are being held in Bombay will foster this intellectual activity in the higher spheres of the mind and thus help in the progress of humanity.

Jawaharlal Nehru

New Delhi,
5th February, 1956.

A GENERALIZATION OF THE EPSTEIN ZETA FUNCTION

By CARL LUDWIG SIEGEL

[Received December 14, 1955]

1. Let $S[\underline{x}]$ be a non-degenerate even quadratic form of m variables with signature n, r and let $\underline{a}$ be a vector such that $S\underline{a}$ is integral. Put $d = (-1)^r \, |S|$, the absolute value of the determinant of S. We define

$$g_{\rho\underline{a}} = \sum_{\underline{x}(\bmod \gamma)} e^{\pi i \rho S[\underline{x}+\underline{a}]}, \quad (\rho = \alpha/\gamma, (\alpha, \gamma) = 1, \gamma > 0), \tag{1}$$

$g_{\underline{a}} = 1$ for integral $\underline{a}$ and $g_{\underline{a}} = 0$ for fractional $\underline{a}$, and we introduce the Dirichlet series

$$\phi_{\underline{a}}(s) = g_{\underline{a}} + e^{\pi i(n-r)/4} \, d^{-\frac{1}{2}} \sum_{\rho} g_{\rho\underline{a}} \, \gamma^{-1-s} (z-\rho)^{(r-1-s)/2} (\bar{z}-\rho)^{(n-1-s)/2}, \tag{2}$$

the summation carried over all rational numbers ρ, where $s = \sigma + it$, $\sigma > m/2 + 1$ and $z = \xi + i\eta$ denotes a parameter in the upper half-plane. Moreover let

$$q(s) = \frac{\pi d^{-\frac{1}{2}} \, 2^{1+m/2-s} \, \Gamma(s - m/2)}{\Gamma\{(s+1-n)/2\}\Gamma\{(s+1-r)/2\}} \sum_{0 \leqslant \rho < 1} g_{\rho 0} \, \gamma^{-1-s}, \; (\sigma > m/2).$$

THEOREM 1. *The function $\phi_{\underline{a}}(s)$ is meromorphic.*

THEOREM 2. *If $S[\underline{x}]$ is a stem-form, then*

$$\phi_{\underline{a}}(s) = \eta^{m/2-s} \, q(s) \, \phi_{\underline{a}}(m-s).$$

In the special case $n = r = 1$, $\underline{a} = 0$, $S[\underline{x}] = 2x_1 x_2$, we obtain

$$q(s) = \pi^{s-1} \frac{\Gamma(1-s/2)\,\zeta(2-s)}{\Gamma(s/2)\,\zeta(s)},$$

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and $2\,\zeta(2s)\ \phi_0(2s)$ becomes Epstein's Zeta-function corresponding to the definite binary quadratic form $(\alpha - \gamma z)\ (\alpha - \gamma \bar{z})$ of the variables α, γ.

2. Let P be a real solution of $P\,S^{-1}\,P = S$, $P = P' > 0$, and put

$$\frac{S+P}{2} = K,\ \frac{S-P}{2} = L,\ \xi\,S + i\eta\,P = z\,K + \bar{z}\,L = R,$$

$$f_{\underline{a}}(z, \underline{w}) = \sum_{\underline{x}} e^{\pi i (R[\underline{y}] + 2\underline{w}'\underline{y})}, \quad (\underline{y} = \underline{x} + \underline{a}),$$

the summation carried over all integral $\underline{x}$. Considered as a function of $\underline{a}$ this theta series depends only on the residue class of $\underline{a}$ modulo 1. Denote by $\underline{a}_1 = 0, \dots, \underline{a}_l$ a complete set of such classes.

Consider any modular substitution

$$z_M = \frac{\alpha z + \beta}{\gamma z + \delta},$$

with the matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and $\alpha\delta - \beta\gamma = 1$, and define

$$\underline{w}_M = ((\gamma z + \delta)^{-1}\,K + (\gamma \bar{z} + \delta)^{-1}\,L)\,S^{-1}\,\underline{w}.$$

In the particular case $\gamma = 0$, we have $\alpha = \delta = \pm 1$, $z_M = z + \alpha\beta$ and

$$f_{\underline{a}}(z_M, \underline{w}_M) = e^{\pi i \alpha\beta S[\underline{a}]}\,f_{\underline{a}\alpha}(z, \underline{w}). \tag{3}$$

Let now $\gamma \neq 0$, $z_M = \alpha/\gamma + \gamma^{-2} z_1$, $z_1 = -z_2^{-1}$, $z_2 = z + \delta/\gamma$, $R_1 = z_1\,K + \bar{z}_1\,L$; then

$$-R_1^{-1} = (z_2\,K + \bar{z}_2 L)[S^{-1}] = R[S^{-1}] + (\delta/\gamma)\,S^{-1},\ |R_1| = d z_1^n (-\bar{z}_1)^r.$$

Replace $\underline{x}$ by $\underline{x}\gamma + \underline{g}$, where $\underline{g}$ runs over a complete system of residues modulo γ, and define $\underline{q} = \underline{g} + \underline{a}$, $\alpha/\gamma = \rho$, such that

$$f_{\underline{a}}(z_M, \underline{w}) = \sum_{\underline{g}} e^{\pi i \rho S[\underline{q}]} \sum_{\underline{x}} e^{\pi i (R_1[\underline{x} + \underline{q}\gamma^{-1}] + 2\underline{w}'(\underline{x}\gamma + \underline{q}))}$$

$$R_1[\underline{x} + \underline{q}\gamma^{-1}] + 2\underline{w}'(\underline{x}\gamma + \underline{q}) = R_1[\underline{x} + \underline{q}\gamma^{-1} + R_1^{-1}\,\underline{w}\gamma] - R_1^{-1}[\underline{w}\gamma].$$

We obtain

$$\sum_{\underline{x}} e^{\pi i(R_1[\underline{x}+\underline{q}\gamma^{-1}]+2\underline{w}'(\underline{x}\gamma+\underline{q}))} = e^{\pi i(r-n)/4}\, d^{-\frac{1}{2}}\, z_2^{n/2}\, \bar{z}_2^{r/2}\, e^{-\pi i R_1^{-1}[\underline{w}\gamma]} \times$$
$$\times \sum_{\underline{x}} e^{-\pi i R_1^{-1}[\underline{x}]+2\pi i \underline{x}'(\underline{q}\gamma^{-1}+R_1^{-1}\underline{w}\gamma)} .$$

Let $\underline{b}$ run over the class representatives $\underline{a}_1, \dots \underline{a}_l$, and replace $\underline{x}$ by $-S(\underline{x}+\underline{b})$. Moreover we introduce the abbreviations

$$h_{\underline{a}\underline{b}} = h_{\underline{a}\underline{b}}(M) = \sum_{\underline{g}(\mathrm{mod}\,\gamma)} e^{(\pi i/\gamma)(\alpha S[\underline{g}+\underline{a}]-2\underline{b}'S(\underline{g}+\underline{a})+\delta S[\underline{b}])}\,, \tag{4}$$

$$v = v(M, z, \underline{w}) = e^{\pi i((z+\delta/\gamma)^{-1}K+(\bar{z}+\delta/\gamma)^{-1}L)[S^{-1}\underline{w}]},\ \epsilon = e^{\pi i(r-n)/4}.$$

Because of

$$-SR_1^{-1}\,\underline{w}_M\gamma = \underline{w},$$

$$-R_1^{-1}[\underline{w}_M\gamma] = -R_1[S^{-1}\underline{w}] = ((z+\delta/\gamma)^{-1}K + (\bar{z}+\delta/\gamma)^{-1}L)[S^{-1}\underline{w}],$$

$$\alpha S[\underline{q}] - 2\underline{q}'S(\underline{x}+\underline{b}) + \delta S[\underline{x}+\underline{b}] \equiv \alpha S[\underline{q}-\underline{x}\delta] - 2\underline{b}'S(\underline{q}-\underline{x}\delta) + $$
$$+\delta S[\underline{b}] \qquad (\mathrm{mod}\ 2\gamma) \tag{5}$$

we get the transformation formula

$$f_{\underline{a}}(z_M, \underline{w}_M) = \epsilon\, d^{-\frac{1}{2}}\, (z+\delta/\gamma)^{n/2}\, (\bar{z}+\delta/\gamma)^{r/2}\, v \sum_{\underline{b}} h_{\underline{a}\underline{b}}\, f_{\underline{b}}(z, \underline{w}). \tag{6}$$

Defining the l-rowed matrix

$$G = G(M, z) = \epsilon\, d^{-\frac{1}{2}}\, (z+\delta/\gamma)^{n/2}\, (\bar{z}+\delta/\gamma)^{r/2}\, (h_{\underline{a}\underline{b}}),$$

we write (6) in the form

$$\underline{f}(z_M, \underline{w}_M) = G\,\underline{f}(z, \underline{w})\, v, \tag{7}$$

where $\underline{f}$ is the column of the l functions $f_{\underline{a}}$.

It follows from (4) and (5) that

$$h_{\underline{a}\underline{b}}\,(M^{-1}) = \overline{h_{\underline{b}\underline{a}}\,(M)}. \tag{8}$$

To cover the special case $\gamma = 0$, we define $v = 1$ and

$$G = (h_{\underline{a}\underline{b}}),\ h_{\underline{a}\underline{b}} = e^{\pi i\alpha\beta S[\underline{a}]}\,\delta_{\underline{a}\underline{b}\alpha},\ \delta_{\underline{a}\underline{b}} = \begin{cases} 1, & \underline{a} \equiv \underline{b}\ (\mathrm{mod}\ 1), \\ 0, & \underline{a} \not\equiv \underline{b}\ (\mathrm{mod}\ 1). \end{cases} \tag{9}$$

Obviously (3) implies (7) and (8), in this case.

3. The l functions $f_{\underline{a}}(z, \underline{w})$ are linearly independent, considered as Fourier series in $\underline{w}$. Hence, for any two modular matrices M and M_1, we have the composition formula

$$G(M_1 M, z) = G(M_1, z_M)\, G(M, z) \tag{10}$$

and, in particular,

$$E = G(M^{-1}, z_M)\, G(M, z). \tag{11}$$

Suppose again that $\gamma \neq 0$; then

$$G(M^{-1}, z_M) = \epsilon\, d^{-\frac{1}{2}}\, (z_M - \alpha/\gamma)^{n/2}\, (\bar{z}_M - \alpha/\gamma)^{r/2}\, (h_{\underline{a}\underline{b}}(M^{-1}))$$

and

$$(z_M - \alpha/\gamma)\,(z + \delta/\gamma) = \gamma^{-2} z_1 z_2 = -\gamma^{-2}.$$

In view of (8) it follows from (11) that the matrix $d^{-\frac{1}{2}}\, \gamma^{-m/2}(h_{\underline{a}\underline{b}})$ is unitary.

Putting

$$H(M, z) = \epsilon^{-1}\, d^{-\frac{1}{2}}\, (\gamma^2)^{-m/2}\, (z - \alpha/\gamma)^{-n/2}\, (\bar{z} - \alpha/\gamma)^{-r/2}\, (h_{\underline{a}\underline{b}}(M)), \tag{12}$$

we obtain

$$H(M, z_M) = G(M, z).$$

We use this formula as a definition in case $\gamma = 0$. With this notation (10) can be replaced by

$$H(M M_1, z_M) = H(M, z_M)\, H(M_1, z).$$

Let $\underline{h}(M, z)$ be the first column of $H(M, z)$, corresponding to the subscript $\underline{b} = \underline{a}_1 = 0$; then

$$\underline{h}(M M_1, z_M) = H(M, z_M)\, \underline{h}(M_1, z). \tag{13}$$

The integral modular substitutions are characterized by $\gamma = 0$; they constitute a subgroup Δ in the modular group Γ. Because of (4) and (9) the column $\underline{h}(M_1, z)$, as a function of M_1, only depends upon the left cosets of Δ in Γ. If

$$M_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

runs over a complete set of representatives of these cosets, then MM_1 does the same, M being any fixed modular matrix. Let $\underline{\phi}(s) = \underline{\phi}(z, s)$ be the column of the l functions $\phi_{\underline{a}}(s)$ in (2), corresponding to $\underline{a} = \underline{a}_1, \dots, \underline{a}_l$, and define

$$\underline{\psi}(z, s) = \underline{\phi}(z, s)\, \eta^{(s+1-m)/2}\,; \tag{14}$$

then

$$\underline{\psi}(z, s) = \sum_{M_1} \underline{h}(M_1, z)\, (\eta^{-1}\, |\gamma z - \alpha|^2)^{(m-1-s)/2}.$$

This Dirichlet series converges absolutely in the half-plane $\sigma > m/2 + 1$. Since

$$z_{M_1^{-1}} - \bar{z}_{M_1^{-1}} = 2\, i\, \eta\, |\gamma z - \alpha|^{-2}, \tag{15}$$

it follows from (13) that

$$\underline{\psi}(z_M, s) = H(M, z_M)\, \underline{\psi}(z, s). \tag{16}$$

4. The analytic continuation of $\phi_{\underline{a}}(s)$ into the whole s-plane follows from the Fourier expansion with respect to the parameter z. Let the real parts of μ, ν, $\mu + \nu - 1$ be positive and define

$$j_u(z) = \int_{|u|}^{\infty} (w + u)^{\mu-1} (w - u)^{\nu-1}\, e^{2\pi(iu\xi - \eta w)}\, dw,\ (u \text{ real}); \tag{17}$$

then

$$\sum_{k=-\infty}^{\infty} (z - k)^{-\mu} (\bar{z} - k)^{-\nu} = \frac{2\, e^{\pi i(\nu-\mu)/2}\, \pi^{\mu+\nu}}{\Gamma(\mu)\, \Gamma(\nu)} \sum_{k=-\infty}^{\infty} j_k(z).$$

Choosing $\mu = \frac{1}{2}(s + 1 - r)$, $\nu = \frac{1}{2}(s + 1 - n)$, we write more explicitly $j_u(z) = j_u(z, s)$, and we get the expansion

$$\phi_{\underline{a}}(s) = g_{\underline{a}} + \frac{2\, d^{-\frac{1}{2}}\, \pi^{s+1-m/2}}{\Gamma\{\frac{1}{2}(s + 1 - n)\}\, \Gamma\{\frac{1}{2}(s + 1 - r)\}} \sum_u f_{u\underline{a}}(s)\, j_u(z, s), \tag{18}$$

where the summation extends over all rational numbers $u \equiv \frac{1}{2} S[\underline{a}]$ (mod 1) and

$$f_{u\underline{a}}(s) = \sum_{0 \leqslant \rho < 1} g_{\rho\underline{a}}\, e^{-2\pi i\rho u}\, \gamma^{-1-s},\ (\rho = \alpha/\gamma,\ (\alpha, \gamma) = 1,\ \gamma > 0). \tag{19}$$

The latter Dirichlet series is of the well-known " singular series " type, and it has an Euler product decomposition

$$f_{u\underline{a}}(s) = \prod_p f_{p,u\underline{a}}.$$

The factor $f_{p,u\underline{a}} = f_p$ is obtained by restricting the summation in (19) to the powers $\gamma = p^k$ ($k = 0, 1, \ldots$) of a prime number p.

Let p^κ be the p-adic denominator of S^{-1}, and suppose that h is an integer satisfying $h + \kappa \leqslant k \leqslant 2h$. Substituting

$$\underline{x} = \underline{y} + p^h \underline{z}, \quad \underline{y}(\mathrm{mod}\ p^h), \quad \underline{z}(\mathrm{mod}\ p^{k-h}),$$

we have

$$S[\underline{x} + \underline{a}] \equiv S[\underline{y} + \underline{a}] + 2\, p^h\, \underline{z}'\, S(\underline{y} + \underline{a})\ (\mathrm{mod}\ 2\, p^k).$$

Therefore the contribution of any given $\underline{y}$ to the Gaussian sum $g_{\rho\underline{a}}$ ($\rho = \alpha/\gamma$, $\gamma = p^k$) in (1) will be 0, if

$$S(\underline{y} + \underline{a}) \not\equiv 0\ (\mathrm{mod}\ p^{k-h}).$$

Hence $g_{\rho\underline{a}} = 0$, if $\underline{a}$ is not p-adically integral. It follows in this case that $f_p(s)$ is a polynomial in p^{-s} of degree $< 2\kappa$.

Now consider the remaining case that $\underline{a}$ is p-adically integral. Then the condition

$$S(\underline{y} + \underline{a}) \equiv 0\ (\mathrm{mod}\ p^{k-h})$$

implies $p^{k-h-\kappa} \mid \underline{y} + \underline{a}$, and this shows that $g_{\rho\underline{a}}$, as a function of $\underline{a}$, only depends upon the residue class of $\underline{a}$ (mod $2\, p^{2h+\kappa-k}$). Moreover we see that

$$g_{\rho\underline{a}} = p^m\, g_{p^2\rho\underline{a}}, \quad (k > 2\,\kappa + 1).$$

It follows that

$$(1 - p^{m-2s})\, f_{p,0\underline{a}}\,(s) = E_{p\underline{a}}\,(s)$$

is a polynomial in p^{-s} of degree $< 2\,\kappa + 2$. Finally let $0 \neq u \equiv \frac{1}{2} S[\underline{a}]$ (mod 1), and let p^λ be the highest power of p dividing $2u$. Suppose $k > \lambda + 2\,\kappa + 2$ and choose $h = [\frac{1}{2}(k + 1)]$; then

$$2h \geqslant k,\ k - h - \kappa \geqslant \tfrac{1}{2}(k-1) - \kappa > 0,$$

$$1 + 2h + \kappa - k + \lambda < 2 + \kappa + \lambda < k,$$

$$4\, p^{2h+\kappa-k}\, u \not\equiv 0 \pmod{p^k},$$

whence

$$\sum_{\alpha (\mathrm{mod}\, \gamma)} g_{\rho \underline{a}}\, e^{-2\pi i \rho u} = 0, \quad (\rho = \alpha/\gamma, (\alpha, \gamma) = 1, \gamma = p^k).$$

Therefore the function $f_{p,u\underline{a}}(s)$ is a polynomial in p^{-s} of degree $< \lambda + 2\kappa + 3$, in case $u \neq 0$.

5. For the primes $p \nmid d$ the Gaussian sums $g_{\rho \underline{a}}$ can be explicitly evaluated in the usual way. If m is even, we denote by

$$\chi(k) = \left(\frac{(-1)^{m/2} \, |S|}{k} \right) \quad (k = 1, 2, \ldots)$$

the Kronecker symbol; if m is odd and $u \neq 0$, we define

$$\chi_u(k) = \left(\frac{(-1)^{\frac{1}{2}(m-1)} \, |S| \, u^*}{k} \right),$$

where u^* is the discriminant of the field generated by $\sqrt{u}$. Then

$$f_{p,u\underline{a}}(s) = \begin{cases} (1 - \chi(p)\, p^{m/2-1-s}) \displaystyle\sum_{k=0}^{\lambda} \chi(p^k)\, p^{k(m/2-s)}, & (m \text{ even}), \\ \dfrac{1 - p^{m-1-2s}}{1 - \chi_u(p)\, p^{\frac{1}{2}(m-1)-s}} \times \\ \times \left(\displaystyle\sum_{k=0}^{[\lambda/2]} p^{k(m-2s)} - \chi_u(p)\, p^{\frac{1}{2}(m-1)-s} \sum_{k=0}^{[\lambda/2]-1} p^{k(m-2s)} \right), & (m \text{ odd}) \end{cases}$$

and correspondingly

$$f_{p,0\underline{a}}(s) = \begin{cases} \dfrac{1 - \chi(p)\, p^{m/2-1-s}}{1 - \chi(p)\, p^{m/2-s}} & (m \text{ even}), \\ \dfrac{1 - p^{m-1-2s}}{1 - p^{m-2s}} & (m \text{ odd}). \end{cases}$$

Introducing the Dirichlet L-series

$$L(s) = \sum_{k=1}^{\infty} \chi(k)\, k^{-s}, \quad L_u(s) = \sum_{k=1}^{\infty} \chi_u(k)\, k^{-s} \quad (\sigma > 1),$$

we obtain

$$f_{0\underline{a}}(s) = \begin{cases} \dfrac{L(s - m/2)}{L(s + 1 - m/2)} \prod\limits_{p|d} (1 - p^{m-2s})^{-1} E_{p\underline{a}}(s), & (m \text{ even}), \\[2ex] \dfrac{\zeta(2s - m)}{\zeta(2s + 1 - m)} \prod\limits_{p|d} (1 - p^{m-1-2s})^{-1} E_{p\underline{a}}(s), & (m \text{ odd}), \end{cases} \tag{20}$$

and, in case $u \neq 0$,

$$f_{u\underline{a}}(s) = \begin{cases} \dfrac{F_{u\underline{a}}(s)}{L(s + 1 - m/2)} & (m \text{ even}), \\[2ex] \dfrac{L_u(s - \frac{1}{2}(m-1))}{\zeta(2s + 1 - m)} F_{u\underline{a}}(s) \prod\limits_{p|d} (1 - p^{m-1-2s})^{-1}, & (m \text{ odd}), \end{cases} \tag{21}$$

where $F_{u\underline{a}}(s)$ is a finite Dirichlet series of the form

$$F_{u\underline{a}}(s) = \sum_{k|d^4u} c_k\, k^{-s}, \quad c_k = O(k^{m/2}).$$

The analytic continuation of $j_u(z, s)$ into the whole s-plane follows from the known properties of the confluent hypergeometric function. Define $n^* = n$, $r^* = r$ for $u > 0$ and $n^* = r$, $r^* = n$ for $u < 0$. Because of (17), the function $\sin\left\{\frac{1}{2}\pi(s + 1 - n^*)\right\} j_u(z, s)$ $(u \neq 0)$ is entire; moreover

$$j_u(z, s) = \left(\frac{\pi\eta}{|u|}\right)^{m/2-s} \frac{\Gamma\{\frac{1}{2}(s + 1 - n^*)\}}{\Gamma\{\frac{1}{2}(r^* + 1 - s)\}}\, j_u(z, m - s),$$

$$j_0(z, s) = (2\pi\eta)^{m/2-s}\, \Gamma(s - m/2).$$

By using simple estimates for the order of magnitude of $j_u(z, s)$ and $L_u(s - \frac{1}{2}(m-1))$, as $|u| \to \infty$, it follows from (18), (20), (21) that the expansion (18) is valid in the whole s-plane and that the functions $\sin \pi s\, \phi_{\underline{a}}(s)\, L(s + 1 - m/2) \prod_{p|d} (1 - p^{m-2s})$ $(m$ even$)$, $\sin \pi s\, \phi_{\underline{a}}(s)\, \zeta(2s + 1 - m) \prod_{p|d} (1 - p^{m-1-2s})$ $(m$ odd$)$ are entire.

This accomplishes the proof of Theorem 1.

6. Suppose now that $S[\underline{x}]$ is a stem form. Then $S[\underline{a}]$, $(\underline{a}=\underline{a}_1,\ldots,\underline{a}_l)$ is even only for $\underline{a}=\underline{a}_1=0$, and the term $u=0$ appears only in this case. Define

$$\frac{2d^{-\frac{1}{2}}\,\pi^{s+1-m/2}}{\Gamma\{\frac{1}{2}(s+1-n)\}\,\Gamma\{\frac{1}{2}(s+1-r)\}}\,j_u(z,s)=J_u(z,s),$$

so that

$$\phi_{\underline{a}}(s)=g_{\underline{a}}(1+\eta^{m/2-s}\,q(s))+\sum_{u\neq 0} f_{u\underline{a}}(s)\,J_u(z,s),$$

$$q(s)=\frac{\pi d^{-\frac{1}{2}}\,2^{1+m/2-s}}{\Gamma\{\frac{1}{2}(s+1-n)\}\,\Gamma\{\frac{1}{2}(s+1-r)\}}\,\Gamma(s-m/2)\,f_{0\underline{a}}(s),$$

and put

$$(\underline{\phi}(s)-\underline{\phi}(m-s)\,\eta^{m/2-s}\,q(s))\,\eta^{(s+1)/2-m/4}=\underline{\chi}(z,s)=\underline{\chi},$$

$$\underline{\chi}'\,\bar{\underline{\chi}}=\omega.$$

Because of (12), (14), (15), (16) the function $\omega=\omega(z,s)$ of z is invariant under the modular group. On the other hand,

$$\chi_{\underline{a}}\,\eta^{m/4-(s+1)/2}=g_{\underline{a}}(1-q(s)\,q(m-s))+$$

$$+\sum_{u\neq 0}\left(f_{u\underline{a}}(s)\,(\pi\,|\,u\,|)^{s-m/2}\,\frac{\Gamma\{\frac{1}{2}(n^*+1-s)\}}{\Gamma\{\frac{1}{2}(s+1-r^*)\}}-f_{u\underline{a}}(m-s)\,q(s)\right)\times$$

$$\times\,\eta^{m/2-s}\,J_u(z,m-s).$$

This function is bounded in the fundamental domain of the modular group, therefore the same holds for $\chi_{\underline{a}}$ if $\sigma<m/2-1$. Since ω is invariant, it follows that then $\chi_{\underline{a}}$ is bounded throughout the upper z-half plane.

Compute the Fourier coefficient

$$\frac{1}{2d}\int_{-d}^{d}\chi_{\underline{a}}\,e^{-\pi i u\xi}\,d\xi=c_u(\eta,s),$$

$$\eta^{m/4-(s+1)/2}\, c_u(\eta, s) = \begin{cases} g_{\underline{a}}(1 - q(s)\, q(m-s)), \quad (u = 0) \\ \left(f_{u\underline{a}}(s)\, (\pi \mid u \mid)^{s-m/2} \dfrac{\Gamma\{\frac{1}{2}(n^* + 1 - s)\}}{\Gamma\{\frac{1}{2}(s + 1 - r^*)\}} - \right. \\ \left. - f_{u\underline{a}}(m-s)\, q(s) \right) \eta^{m/2-s} J_u(i\eta, m - s), \ (u \neq 0) \end{cases}$$

and let $\eta \to 0$; then

$$\eta^{m/2-s}\, j_u(i\eta, m-s) \to (2\pi)^{s-m/2}\, \Gamma(m/2 - s) \neq 0, \ (\sigma < m/2 - 1).$$

It follows that $c_u(\eta, s) = 0$, $\underline{\chi} = 0$ and Theorem 2 is proved.

Furthermore we have the functional equations

$$\left. \begin{aligned} q(s)\, q(m-s) &= 1, \\ f_{u\underline{a}}(s) &= (\pi \mid u \mid)^{m/2-s} \frac{\Gamma\{\frac{1}{2}(s+1-r^*)\}}{\Gamma\{\frac{1}{2}(n^*+1-s)\}}\, q(s)\, f_{u\underline{a}}(m-s), \ u \neq 0. \end{aligned} \right\} \tag{22}$$

Using the expressions (20), (21) for $f_{0\underline{a}}(s)$, $f_{u\underline{a}}(s)$ we can obtain from (22) the functional equations for $\zeta(s)$, $L(s)$, $L_u(s)$. Besides, a functional equation for the finite Dirichlet series $f_{p,u\underline{a}}$ is found. It seems rather complicated to get the latter result in an elementary way, if p is a factor of d.

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THEORY OF THE CHARACTERS OF NUMBER SEMIGROUPS

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1. Characters of number semigroups. It is known that non-principal characters modulo k have a bounded sum-function.

Generalizing this terminology, we set up the following definitions. Let $\mathfrak{G}$ be a commutative semigroup, such that for every $\alpha \in \mathfrak{G}$ we have

$$\alpha = \mathfrak{p}_1^{x_1} \dots \mathfrak{p}_\lambda^{x_\lambda}, \quad \mathfrak{p}_i \in \mathfrak{G}, \quad (i = 1, \dots, \lambda).$$

Here $\mathfrak{p}_1, \dots, \mathfrak{p}_\lambda$ are said to be *basic elements*, $x_1, \dots, x_\lambda$ are non-negative integers. Every finite set of $\mathfrak{p}_i$ forms a system of independent elements, i.e. $\alpha = 0$ if and only if

$$x_1 = x_2 = \dots = x_\lambda = 0.$$

The totality of all $\mathfrak{p}_i$ is said to be a *basis* of $\mathfrak{G}$.

A non-negative number $N(\alpha)$ is said to be a *norm* of α, if

1. $N(\alpha \beta) = N(\alpha) N(\beta)$.
2. There are only a finite number of $\alpha \in \mathfrak{G}$ with $N(\alpha) \leqslant x$.

By definition, the character $\chi(\alpha)$ of $\mathfrak{G}$ is a homomorphism of $\mathfrak{G}$ into a set of complex numbers. $\chi(\alpha)$ is said to be *normalized*, if $|\chi(\alpha)| = 1$. $\chi(\alpha)$ is said to be a *finite homomorphism* if the totality of all values of $\chi(\alpha)$ is a finite set.

Let

$$H(x) = \sum_{1 \leqslant N(\alpha) \leqslant e^x} \chi(\alpha), \; \alpha \in \mathfrak{G},$$

be a sum-function of $\chi(\alpha)$.

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A character $\chi(\alpha)$ is said to be a *generalized Dirichlet's character* (*GD-character*) if $H(x) = O(1)$, as $x \to \infty$.

If a GD-character is a finite homomorphism, then it is said to be a *GD-finite homomorphism*.

There are two classical examples of GD-characters.

1. Non-principal Dirichlet's characters modulo k.
2. The non-principal characters of cyclic semigroups.

There is a GD-character which does not belong to the last two classes [7].

THEOREM 1. *No GD-character exists for* $\mathfrak{G}$ *with a finite basis under the condition that* $N(\alpha)$ *is a positive integer.*

PROOF. We write

$$L(s, \chi) = \sum_{\nu=1}^{\infty} \chi(\alpha_\nu)\,(N(\alpha_\nu))^{-s} = \prod_{k=1}^{\lambda} (1 - \chi(\mathfrak{p}_k)\,(N(\mathfrak{p}_k))^{-s})^{-1},$$

where λ is the number of basic elements; the numbers $N(\alpha_\nu)$ $(\nu = 1, 2, \ldots)$ form an increasing sequence which tends to infinity with ν.

Let $\vartheta_{i,j}$ be equal to $\lg N(\mathfrak{p}_i)/\lg N(\mathfrak{p}_j)$; $E_{i,j,\gamma}(x)$ is a set of positive integers $n \in [0, x]$ with $((\vartheta_{i,j}\; n + \gamma)) \leq 5n^{-1}$, where $((\alpha))$ is the smallest distance of α from any rational integer; $P(x, i, j, \gamma)$ is equal to the number of elements of $E(x, i, j, \gamma)$, γ is a real number.

Parseval's identity gives us

$$(2\pi)^{-1} \int_{-\infty}^{+\infty} |(\sigma + it)^{-1}\, L(\sigma + it)|^2\, dt = \int_0^{\infty} |H(x)|^2 e^{-2\sigma x}\, dx. \quad (1)$$

For some system (i, j, γ) we have

$$(2\pi)^{-1} \int_{-\infty}^{+\infty} |(\sigma + it)^{-1}\, L(\sigma + it)|^2\, dt \geqslant 4^{-\lambda}\sigma^{-1}\, P(\sigma^{-1}, i, j, \gamma). \quad (2)$$

If $H(x) = O(1)$ then

$$\int_0^{\infty} |H(x)|^2 e^{-2\sigma x}\, dx \leqslant \sigma^{-1},$$

From (1), (2) and (3) it follows that

$$4^{-\lambda} P(\sigma^{-1}, i, j, \gamma) \ll 1, \quad \text{as } \sigma \to 0;$$

this contradicts the known Chebyshev theorem on the approximation of real numbers.

For example, this theorem is applicable to the semigroup of all entire ideals of every finite algebraic number field, including the rational number field.

We state a generalization of Theorem 1 for $\mathfrak{G}$ with an infinite basis.

THEOREM 2. *Suppose that* $\mathfrak{G}$ *satisfies the following conditions* :

1. *The number of basic elements* $\mathfrak{p}$ *with* $N(\mathfrak{p}) \leqslant x$ *is equal to* $\pi(x) = O(\lg_m x)$.

2. *There are two basic elements* $\mathfrak{p}_i$ *and* $\mathfrak{p}_j$ *such that* $a_\nu = O(\exp_m \nu)$, *where* a_ν *is a partial denominator of the continued fraction*

$$\vartheta_{ij} = [a_0, a_1, \dots].$$

Then no GD-character exists for this $\mathfrak{G}$.

2. Ω-theorems for $H(x)$. We can estimate the upper bound of $H(x)$, as $x \to \infty$. I shall give in this section brief statements of the main theorems of this kind.

Put

$$\vartheta_{ij} = \lg N(\mathfrak{p}_i)/\lg N(\mathfrak{p}_j) = [a_0, a_1, \dots],$$

where a_ν is a partial denominator of the continued fraction for ϑ_{ij}.

THEOREM 3. *Let all* $\mathfrak{p}$ *be real positive numbers; if* $\alpha_\nu = O(\exp_m \nu)$, *then* $H(x) = \Omega(\sqrt{\lg_m x})$.

In this case we put $N(\mathfrak{p}) = \mathfrak{p}$.

Further, let $\mathfrak{G}$ be some semigroup of entire ideals of the algebraic number field of degree n; $\chi(\mathfrak{a})$ is the normalized character of an ideal $\mathfrak{a}$.

THEOREM 4. A. *If the basis of* $\mathfrak{G}$ *contains* N *prime ideals, where* $n < N$, *then*

$$\sum_{1 \leqslant N(\mathfrak{a}) \leqslant e^x} \chi(\mathfrak{a}) = \Omega(\sqrt{\lg_3 x}).$$

B. *If the basis of* $\mathfrak{G}$ *contains an infinity of prime ideals, and the number of prime ideals* $\mathfrak{p} \in \mathfrak{G}$ *with* $N(\mathfrak{p}) \leqslant x$ *is equal to* $\pi(x) = O(\lg_4 x)$, *then*

$$\sum_{1 \leqslant N(\mathfrak{a}) \leqslant e^x} \chi(\mathfrak{a}) = \Omega((\lg_3 x)^\mu), \quad 0 < \mu < \tfrac{1}{2}.$$

3. **Characters with dense bases.** By definition, the basis is dense in $\mathfrak{G}$, when there are only a finite number of basic elements $\mathfrak{p}$, which do not belong to $\mathfrak{G}$.

I shall state a theorem on characters of this kind. Let $\mathfrak{G}$ be the totality of positive integers. Then we can prove the following theorem [1].

THEOREM 5. *Let* $\chi(n)$ *be a primitive character modulo* k; $p_1, p_2, \ldots, p_h$ *are prime divisors of* k, *and* $h \geqslant 2$. *The complex numbers* $\alpha_1, \ldots, \alpha_h$ *are given in such a manner that* $|\alpha_i| = 1$. *The character* $\chi(p)$ *is defined to have the value* α_i *or* $\chi(p)$ *according as the prime* p *is equal to some* p_i *or not. Then*

$$H(x, \chi) = \Omega(\sqrt{x}(\lg x)^{-1}).$$

The proof of this theorem is based on Parseval's identity for $L(s, \chi)$.

RESEARCH PROBLEMS. I shall give a list of unsolved problems.

1. Let $\mathfrak{G}$ be a semigroup of positive numbers $\alpha > 1$ and satisfying the condition 1 of Theorem 2. If we turn down condition 2 of this theorem, does it remain right ?

2. Show that the totality of all positive integers does not possess a *GD*-finite homomorphism.

3. Does a *GD*-finite homomorphism exist for a given semigroup of positive integers apart from the classical Dirichlet's characters ?

4. Let $\mathfrak{G}$ be a semigroup of positive integers, $\chi(n)$ is such a GD-character of $\mathfrak{G}$ that the corresponding function $L(s, \chi)$ is an entire function. Prove that $\chi(n) = \chi_1(n)n^{\alpha i}$, where $\chi_1(n)$ is a Dirichlet's character, α is a real number.

5. Find a function $L(s, \chi)$ for which the Riemann hypothesis is right.

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ON THE ZEROS OF THE ZETA-FUNCTION OF RIEMANN

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1. More than a hundred years have passed since Riemann published his fundamental work on prime numbers. Many investigations have been published to prove its audacious unproved assertions, and still more to make clear the various questions about the distribution of the zeros which were raised mainly by number theory. These questions and investigations are contained, by and large, in the following five groups.

I. To find possibly large domains in the critical strip $0 < \sigma < 1$ (where the complex variable will be denoted by $s = \sigma + it$) which are free of zeros (which will be always denoted by $\rho = \sigma_\rho + it_\rho$).

II. Equivalent formulations of Riemann's conjecture.

III. Distribution of the zeros on the line $\sigma = \frac{1}{2}$.

IV. Estimation of the number of zeros in different domains of the critical strip.

V. The connection between prime numbers and non-trivial zeta-roots.

In this paper I shall deal exclusively with the fourth group of these questions, which seems for number theory to be one of the most important among the above groups. This started with the theorem of Bohr-Landau according to which " most " of the zeros lie " near " the line $\sigma = \frac{1}{2}$. More exactly if $N(T)$ denotes the number of zeta-zeros in the parallelogram

$$0 < \sigma < 1, \quad 0 < t \leqslant T,$$

and $N(\alpha,T)$, for $\frac{1}{2} \leqslant \alpha \leqslant 1$, those in the parallelogram

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$$\alpha \leqslant \sigma \leqslant 1, \quad 0 < t \leqslant T,$$

then for any fixed $\alpha > \frac{1}{2}$ and $T \to +\infty$, we have

$$\frac{N(\alpha, T)}{T} \to 0.$$

A decisive step was then taken by Carlson, who proved — with a slight modification of Hoheisel — that

$$N(\alpha, T) < c_1 T^{4\alpha(1-\alpha)} \log^6 T, \tag{1.1}$$

where c_1, and later c_2 ,..., denote numerical constants (except when they depend on an ϵ or η ; in this case the dependence will always be explicitly stated). The point is of course that the exponent $4\alpha(1-\alpha)$ is < 1 for $\frac{1}{2} < \alpha \leqslant 1$, and according to Riemann-Mangoldt,

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} \right| \leqslant c_2 \log T. \tag{1.2}$$

The theorem of Carlson was considered for eight years only as a probability-argument for the truth of Riemann's hypothesis. In 1930 Hoheisel discovered its significance for the upper estimation of $p_{n+1} - p_n$, the difference of consecutive primes. Before Hoheisel a proof of the estimation

$$p_{n+1} - p_n < p_n^{\vartheta} \tag{1.3}$$

with a $\vartheta < 1$ was unimaginable without a proof of the fact that $\zeta(s) \neq 0$ in the half-plane $\sigma > \vartheta$. Hoheisel found that to prove (1.3), besides (1.1), Littlewood's theorem is sufficient, according to which $\zeta(s) \neq 0$ for

$$\sigma > 1 - c_3 \frac{\log \log t}{\log t}, \quad t \geqslant c_4. \tag{1.4}$$

Hoheisel's ϑ was very small; but his analysis showed that if (1.4) is true with an arbitrarily large c_3, then (1.3) holds for $n > n_0(\epsilon)$ with

$$\vartheta = \tfrac{3}{4} + \epsilon, \tag{1.5}$$

with arbitrarily small $\epsilon > 0$. This result, incredible formerly, was proved, using Vinogradoff's estimations, by Tchudakoff in 1936.

But Hoheisel's analysis gave even more. This showed that if instead of Carlson's estimation (1.1), we have for $\frac{1}{2} \leqslant \alpha \leqslant 1$,

$$N(\alpha, T) \leqslant c_5 T^{c_6(1-\alpha)} \log^{c_7} T, \tag{1.6}$$

then also the estimation

$$p_{n+1} - p_n < p_n^{1-1/c_6+\epsilon} \tag{1.7}$$

holds with arbitrarily small $\epsilon > 0$ for $n > n_1(\epsilon)$. Carlson's theorem gave $c_6 = 4$, and therefore Tchudakoff reached the value $\frac{3}{4} + \epsilon$. Tchudakoff's world-record lived only for a year; then Ingham proved (1.6) with $c_6 = 8/3$ (and even with a little less c_6-value) so that his result, combined with Hoheisel's, gave

$$\vartheta = \tfrac{5}{8}.$$

What is the smallest ϑ-value one can hope for in this way? Owing to (1.2) and (1.6) we get easily

$$c_6 \geqslant 2,$$

that is

$$\vartheta \geqslant \tfrac{1}{2} + \epsilon.$$

Hence the sharpest inequality of this type is

$$N(\alpha, T) < c_8 T^{2(1-\alpha)} \log^{c_9} T. \tag{1.8}$$

Theorems of Carlson's type are often called in the literature "density-theorems"; the inequality (1.8), which is unproved so far, is called the "density-hypothesis". A discussion of this will be the main subject of this paper; as we see, a proof of it would imply the inequality

$$p_{n+1} - p_r < p_n^{\frac{1}{2}+\epsilon}, \quad n > n_0(\epsilon), \tag{1.9}$$

which is essentially the fourth main problem of the analytical theory of numbers.

2. It turned out in the papers of Linnik that the density-theorems, in particular if generalized to the L-functions of Dirichlet, are still more important for the theory of numbers than thought after the discovery of Hoheisel. It is well known that Hardy and Littlewood proved the Goldbach conjecture concerning the odd integers, supposing that no L-function vanishes in the half-plane

$\sigma \geqslant \frac{3}{4} - \epsilon$, and also that after the first complete proof of Vinogradoff, Linnik succeeded in obtaining a proof along the original Hardy-Littlewood lines, but working with density-theorems. In a paper dated 1936 in the *Acta Szeged*, Erdös and myself proved, by supposing the truth of the extended Riemann-hypothesis, that for all irrational ω, the sequence

$$p_n\omega, \quad n = 1,2,\ldots$$

(p_n again the nth prime) is uniformly distributed mod 1. This theorem is implicitly contained in Vinogradoff's later work, which uses different ideas and needs no supposition. Linnik succeeded again in a proof based on density-theorems, without any hypothesis. Particularly interesting, from our point of view, is the discussion of Linnik's results concerning the binary Goldbach problem. Let us call Goldbach-numbers those even numbers which can be represented as sums of two primes. Since all numbers of the form $2p$ are evidently Goldbach numbers, Ingham's above-quoted result gives that for all $N > N_0$ there are Goldbach numbers between N and $N + N^{5/8}$. Now Linnik proved, by supposing the truth of Riemann's conjecture, that for arbitrarily small $\epsilon > 0$, and for all $N > N_1(\epsilon)$, there are Goldbach-numbers between

$$N \text{ and } N + \log^{3+\epsilon} N\,;$$

if he used instead of it only the density-hypothesis (1.8) with $c_9 = 2$, he could deduce the not essentially weaker result that for all $N > N_2(\epsilon)$ there are Goldbach-numbers between

$$N \text{ and } N + \log^{7+\epsilon} N.$$

Using, however, instead of the estimation (1.8) the above-quoted result of Ingham, which gives the estimation

$$N(\alpha, T) < c_{10}\, T^{(8/3)(1-\alpha)} \quad \log^5 T \text{ for } \tfrac{7}{8} \leqslant \alpha \leqslant 1, \tag{2.1}$$

$$N(\alpha, T) < c_{10}\, T^{3(1-\alpha)/(2-\alpha)} \log^5 T \text{ for } \tfrac{1}{2} \leqslant \alpha \leqslant \tfrac{7}{8}, \tag{2.2}$$

Linnik's method gives curiously enough only the existence of a positive constant κ between 0 and $\frac{1}{2}$ so that for $N > c_{11}$ there is a Goldbach-number between

$$N \text{ and } N + N^{\kappa},$$

and the same holds in the case when we know (1.6) only with a $c_6 > 2$. The conclusion of these considerations is that the density theorems can replace in many respects the Riemann-hypothesis and the density-hypothesis (1.8) is among the density-theorems not only the deepest, but *essentially* deeper. Practically the same holds for the inequality

$$N(\alpha, T) < c_{12}(\epsilon)\, T^{2(1+\epsilon)(1-\alpha)}, \tag{2.3}$$

valid for $\frac{1}{2} \leqslant \alpha \leqslant 1$, $T \geqslant 2$; a proof of it would give besides (1.9) also the fact that for $N > N_3(\epsilon)$ there is a Goldbach-number between

$$N \text{ and } N + N^{\epsilon}. \tag{2.4}$$

3. What was the idea underlying the above theorems? This can be described, somewhat differently from the usual, shortly as follows.

Let $f(s)$ be regular in the closed domain E with a rectifiable G-boundary; suppose $f(s)$ does not vanish on G and we denote the number of its zeros (counted according to the multiplicity) by N. Then according to Cauchy we have

$$N = \frac{1}{2\pi i} \int_{(G)} \frac{f'}{f}(s)\, ds.$$

Now if $\phi(s)$ is regular in E and k an arbitrary positive integer, then Cauchy's integral theorem gives at once

$$N = \frac{1}{2\pi i} \int_{(G)} \frac{f'}{f}(s) \left\{1 - f(s)\,\phi(s)\right\}^k ds. \tag{3.1}$$

This is the basic formula. If we have on the boundary

$$\left| \frac{f'}{f}(s) \right| \leqslant M, \tag{3.2}$$

then from (3.1) with $k = 2$ follows the inequality

$$N \leqslant \frac{M}{2\pi} \int_{(G)} |1 - f(s)\,\phi(s)|^2\, |ds|. \tag{3.3}$$

We now apply (3.3) to the estimation of

$$N(\alpha, 2T) - N(\alpha, T), \quad T \geqslant 20.$$

From the well-known representation

$$\left| \frac{\zeta'}{\zeta}(s) - \sum_{\substack{\rho \\ |t-t_\rho| \leqslant 2}} \frac{1}{s-\rho} \right| \leqslant c_{13} \log(2 + |t|), \tag{3.4}$$

it follows that for each integer $n \geqslant 1$ there is a τ_n with

$$n < \tau_n < n + 1,$$

so that for

$$-1 \leqslant \sigma \leqslant 2, \quad t = \tau_n,$$

the inequality

$$\left| \frac{\zeta'}{\zeta}(s) \right| \leqslant c_{14} \log^3 \tau_n$$

holds. From this and (3.4) we get easily for each $n \geqslant 1$ the existence of a σ_n with

$$\alpha - \frac{1}{\log \tau_n} \leqslant \sigma_n \leqslant \alpha,$$

so that the inequality

$$\left| \frac{\zeta'}{\zeta}(s) \right| \leqslant c_{15} \log^3 T \tag{3.5}$$

holds on the non-intersecting closed broken-line consisting of the segments

$$\sigma = 2, \; \tau_{[T]-1} \leqslant t \leqslant \tau_{[2T]+1},$$

$$t = \tau_{[T]-1}, \; \sigma_{[T]-1} \leqslant \sigma \leqslant 2,$$

$$t = \tau_{[2T]+1}, \; \sigma_{[2T]+1} \leqslant \sigma \leqslant 2,$$

of the vertical segments

$$\sigma = \sigma_n, \; \tau_n \leqslant t \leqslant \tau_{n+1},$$

$$[T] - 1 \leqslant n \leqslant [2T] + 1,$$

and of the horizontal ones

$$t = \tau_n, \; \min(\sigma_{n-1}, \sigma_n) \leqslant \sigma \leqslant \max(\sigma_{n-1}, \sigma_n).$$

$$[T] \leqslant n \leqslant [2T].$$

We choose this line as G, and

$$\phi(s) = \sum_{n \leqslant x} \frac{\mu(n)}{n^s},$$

where $x = x(T, \alpha)$, will be determined later; this gives from (3.3) and (3.5)

$$N(\alpha, 2T) - N(\alpha, T) < c_{16} \log^3 T. \int_{(G)} \left| 1 - \zeta(s) \sum_{n \leqslant x} \frac{\mu(n)}{n^s} \right|^2 |ds|. \quad (3.6)$$

If $g(w)$ is regular for $|w - w_0| \leqslant r$, $(w = u + iv)$, then as is well known,

$$|g(w_0)|^2 \leqslant \frac{1}{r^2 \pi} \int_{|w - w_0| \leqslant r} |g(w)|^2 du\, dv.$$

Applying this at fixed s-values with

$$g(w) = 1 - \zeta(w) \sum_{n \leqslant x} \frac{\mu(n)}{n^w}$$

to the circles

$$|w - s| \leqslant \frac{1}{\log T}, \quad (3.7)$$

where s runs over the whole G-line, an easy elementary-geometrical reflection gives at once from (3.6)

$$N(\alpha, 2T) - N(\alpha, T) < c_{17} \log^6 T \int_{(H)} \left| 1 - \zeta(w) \sum_{n \leqslant x} \frac{\mu(n)}{n^w} \right|^2 du\, dv,$$

where H is the domain swept by the circles (3.7). Thus

$$N(\alpha, 2T) - N(\alpha, T) < c_{18} \log^6 T \int_{\alpha - 1/\log T}^{1} du \times$$
$$\times \int_{T-2}^{2T+2} \left| 1 - \zeta(u + iv) \sum_{n \leqslant x} \frac{\mu(n)}{n^{u+iv}} \right|^2 dv.$$

From this a direct estimation of

$$J(u, T) = \int_{T-2}^{2T+2} \left| 1 - \zeta(u + iv) \sum_{n \leqslant x} \frac{\mu(n)}{n^{u+iv}} \right|^2 dv$$

gives Carlson's estimation (1.1); estimating $J(u, T)$ by a convexity-argument furnishes Ingham's result.

4. What has been reached towards the density-hypothesis either in the stronger form (1.8) or in the weaker form (2.3)? As to the

first, nothing has been done. As to the second there is some progress. (2.3) was proved by Ingham by supposing the truth of Lindelöf's conjecture, according to which the inequality

$$|\zeta(\sigma+it)| < c_{19}(\epsilon)\, t^{\epsilon} \tag{4.1}$$

is valid for arbitrarily small $\epsilon > 0$, $\sigma \geqslant \frac{1}{2}$ and $t \geqslant 1$. I proved in 1950 without any suppositions the inequality (2.3) for the α-values near to 1, i.e. even for values for which (2.3) is the deepest. The reason why all the density-theorems are the deepest for $\alpha = 1$ is clear; for $\alpha < \frac{3}{4}$ already Carlson's inequality (1.1) gives (2.1) and (2.2) gives for $\alpha < \frac{2}{3}$ even

$$N(\alpha, T) < c_{20}\, T^{(9/4)(1-\alpha)}.$$

More exactly I proved — with an unpublished refinement — that for a (small) c_{21} for

$$1 - c_{21} \leqslant \alpha \leqslant 1, \quad T > c_{22},$$

the estimation

$$N(\alpha, T) < T^{2(1-\alpha)+(1-\alpha)^{1,14}} \tag{4.2}$$

holds. The number-theoretical consequences of (4.2) are still not quite clear to me. The methods I used in proving (4.2) were quite different from those sketched in §3, and I was willing to think their power is confined to the neighbourhood of the line $\sigma = 1$. Therefore my next aim was to deduce with my methods Ingham's above-mentioned theorem, the inequality (2.3) supposing the truth of Lindelöf's conjecture. I succeeded in doing so in 1954.

Ingham's method was essentially the same as mentioned previously. A sketch of my proof for (4.2), which will be a good preparation for more sophisticated arguments later, runs as follows. We start from the inequality

$$\left|\frac{1}{l!}\sum_{n\geqslant\xi}\frac{\Lambda(n)}{n^{s}}\log^{l}\frac{n}{\xi} - \frac{\xi^{1-s}}{(s-1)^{l+1}} + \sum_{\rho}\frac{\xi^{\rho-s}}{(s-\rho)^{l+1}}\right| \leqslant c_{23}\,\xi^{-1-\sigma}\log(2+|t|). \tag{4.3}$$

Here $\Lambda(n)$ denotes the usual Dirichlet-symbol; $\sigma > 1$, $\xi \geqslant 2$, l an integer $\geqslant 2$. Choosing $\sigma_0 = \sigma_0(\alpha) > 1$ "near" to 1, an easy upper estimation of the square-integral

$$J = \int_T^{2T} \left| \sum_{n \geqslant \xi} \frac{\Lambda(n)}{n^{\sigma_0 + it}} \log \frac{n}{\xi} \right|^2 dt$$

(T large) gives that

$$\left| \sum_{n \geqslant \xi} \frac{\Lambda(n)}{n^s} \log^l \frac{n}{\xi} \right| \tag{4.4}$$

is relatively " small " on the segment $\left\{ \begin{matrix} \sigma = \sigma_0 \\ T \leqslant t \leqslant 2T \end{matrix} \right\}$, except an R_l-set of measure $\leqslant \frac{T^{2(1-\alpha)}}{\log^{10} T}$. If l is restricted to an interval Δ of length about $\log T$, then also the union R of all R_l-sets has a measure $\leqslant T^{2(1-\alpha)} \log^{-9} T$ and if $\bar{R}$ is the complementary set of R on our segment, then the functions (4.4) are small in $\bar{R}$ for *each* permitted l-value. Dividing the segment into pieces

$$l_j : \left\{ \begin{matrix} \sigma = \sigma_0 \\ \frac{T + j}{[\log^3 T]} \leqslant t \leqslant T + \frac{j+1}{[\log^3 T]} \end{matrix} \right\} \tag{4.5}$$

we call l_j a "good" segment, or a "bad" one, according as it contains at least one point of $\bar{R}$ or not. Let us fix in each "good" l_j-segment an s_j-point belonging to $\bar{R}$; then the above reasoning gives that the number of the "bad" l_j-segments is "small" and the quantities

$$\left| \sum_{n \geqslant \xi} \frac{\Lambda(n)}{n^{s_j}} \log^l \frac{n}{\xi} \right| \tag{4.6}$$

are for the permitted l-values " small " if j is the index of a " good " segment. The inequality (4.3) gives a possibility of transition to zeta-roots ; since $s - 1$ is of order T, the estimation (4.6) gives that all

$$U_j \equiv \left| \sum \frac{\xi^{\rho - s_j}}{(s_j - \rho)^{l+1}} \right| \tag{4.7}$$

quantities are " small ". Let us call the strip L_j :

$$T + \frac{j}{[\log^3 T]} \leqslant t < T + \frac{j+1}{[\log^3 T]} \tag{4.8}$$

a "good" or a "bad" one according as the segment l_j is a "good" or a "bad" one. Since the number of the ρ's in each L_j-strip is $O(\log T)$, the contribution of the "bad" L_j-strips to

$$N(\alpha, 2T) - N(\alpha, T)$$

(and also to $N(2T) - N(T)$) is

$$\leqslant c_{24}\, T^{2(1-\alpha)} \log^{-8} T. \tag{4.9}$$

Hence if we can show that all zeros in "good" strips have a real part

$$< \alpha + \{ 2(1-\alpha) \}^{1,14} \tag{4.10}$$

if only

$$\alpha \leqslant 1 - \log^{-0,9} T,$$

then (4.2) would easily follow. Let us fix an arbitrary "good" L_j-strip. To deduce (4.10) for it, we remark first that the contribution of those ρ-zeros to U_j, for which $|\rho - s_j| \geqslant \{2(1-\alpha)\}^{6/7}$, is "small" if l lies between two multiples of $\log T$. Suppose that (4.10) were not true for our L_j-strip and α is sufficiently near to 1, i.e. there were a

$$\rho^* = \sigma^* + i\, t^*$$

zero in L_j such that

$$\sigma^* > \alpha + \{2(1-\alpha\}^{1,14}. \tag{4.11}$$

This would mean the remaining sum

$$U_j^* \equiv \Bigg| \sum_{\substack{\rho \\ |\rho - s_j| \leqslant \{2(1-\alpha)\}^{6/7}}} \frac{\xi^{\rho - s_j}}{(s_j - \rho)^{l+1}} \Bigg| \tag{4.12}$$

is "small", though *some* terms of it are very big, since $\mathrm{Re}\, s_j = \sigma_0(\alpha)$ is only a *little* greater than 1 and the strip L_j is "narrow". Choosing

$$\xi = \exp\left[\frac{200}{\{2(1-\alpha)\}^{0,99}} \cdot (l+1) \right], \tag{4.13}$$

U_j^* becomes the $(l+1)$th power-sum of complex z_ν-numbers, where neither the z_ν's, nor their number depends on l. The integer l is so far indeterminate and the only restriction upon it is to lie between two multiples of $\log T$. Now the main idea of the proof is the use of the second main theorem of my book, which asserts in a sharpened† form that having n complex $\xi_1, \xi_2, \ldots, \xi_n$ numbers with

$$|\xi_1| \geqslant |\xi_2| \geqslant \ldots \geqslant |\xi_n|$$

and a positive integer m there is an integer l with

$$m + 1 \leqslant l + 1 \leqslant m + n \tag{4.14}$$

such that

$$|\xi_1^{l+1} + \ldots + \xi_n^{l+1}| \geqslant \left(\frac{n}{24(m+n)}\right)^n |\xi_1|^{l+1}. \tag{4.15}$$

It is easy to see that this estimation holds *a fortiori* if n is replaced by N_1, an upper bound of n. This lower estimation gives—using the supposition (4.11)—already the required contradiction, if for N_1 we can choose, e.g. the quantity

$$8\{2(1-\alpha)\}^{1,11} \log T + 4 \log\log T. \tag{4.16}$$

But owing to the sharp, known estimations of $\zeta(s)$ near to the line $\sigma = 1$, Jensen's estimation gives (4.16) at once and this completes the sketch of proof of (4.2).

As mentioned it was also possible to prove Ingham's theorem using these ideas. The only really new feature of the proof was that the vertical line $\sigma = \sigma_0$ was assumed "far" from the line $\sigma = 1$. Lindelöf's conjecture was used, in a form seemingly quite different from (4.1); the form used was that for any fixed $\alpha > \frac{1}{2}$,

$$\lim_{T \to +\infty} \frac{N(\alpha, T+1) - N(\alpha, T)}{\log T} = 0. \tag{4.17}$$

The equivalence of (4.1) and (4.17) was proved by Littlewood.

5. Recently I resumed once more the question of density-hypothesis in the form (2.3). My proof for Ingham's theorem (4.1) gave me the impression that from the truth of Lindelöf's hypothesis a much stronger conclusion can be derived for $N(\alpha, T)$ than the density-hypothesis. This gave me the belief that on my way the density-hypothesis can either be proved or at least derived from a much weaker hypothesis. Time worked quicker than myself; what

† In my book I had instead of 24 the greater constant $24e^2$ only. This sharper estimation will be published in *Acta Math. Hung.* in a joint paper with Vera T. Sós.

I can do at present is a deduction of the density-hypothesis in the form (2.3) from a hypothesis (conjecture B or C below) which is certainly weaker than Lindelöf's conjecture and whose proof seems not to be hopeless. The new ideas would enable one to improve Ingham's estimation (2.1) without any supposition in a much bigger neighbourhood of $\sigma = 1$ than (4.2) did; but we did not investigate thoroughly these possibilities, playing double or quits.

6. Before turning to our theorem we analyse some assertions. First we introduce

CONJECTURE A. *There is a $g(x)$, which is positive, for $x > 0$ monotonically increasing, with*

$$\lim_{x \to +0} g(x) = 0 \tag{6.1}$$

and having the following property. Let a γ with $\frac{1}{2} < \gamma < 1$ be fixed, and denote by $M_\delta(\tau, \alpha_1)$ the number of the zeros of $\zeta(s)$ in the square

$$(\gamma \leqslant)\ \alpha_1 - \delta \leqslant \sigma \leqslant \alpha_1\ (\leqslant 1), \quad |t - \tau| \leqslant \delta/2. \tag{6.2}$$

Then for $\tau > c_{25}(\gamma, \delta)$ (> 3) the estimation

$$M_\delta(\tau, \alpha_1) < \delta\, g(\delta) \log \tau \tag{6.3}$$

holds, independently of α_1.

The truth of conjecture A would be a trivial consequence of the truth of Lindelöf's conjecture in the form (4.17). It is obvious that from the truth of conjecture A that of Lindelöf's conjecture does not follow, i.e. conjecture A is definitely weaker than Lindelöf's conjecture. As stated before, conjecture A can be proved indeed at least in the case $\alpha_1 = 1$. Nevertheless a general proof of conjecture A, even instead of (6.3), e.g. with

$$M_\delta(\tau, \alpha_1) < \delta^{1/10} \log \tau \tag{6.4}$$

seems to be very difficult, so requiring conjecture A instead of Lindelöf's conjecture I should not consider as an *essential* progress. This opinion would be certainly still better founded if to an arbitrarily small $\delta > 0$, I could construct a function $f_\delta(s)$ representable for $\sigma > \frac{1}{2}$ by the convergent Dirichlet-series

$$f_\delta(s) = \sum_n \frac{a_n}{n^s} \tag{6.5}$$

with

$$\sum_n \frac{|a_n|^2}{n^{1+\epsilon}} < \infty, \quad \epsilon > 0, \tag{6.6}$$

and such that for an infinity of squares

$$(\tfrac{2}{3} \leqslant)\ \alpha_1 - \delta \leqslant \sigma \leqslant \alpha_1\ (\leqslant 1), \quad |t - \tau| \leqslant \delta/2,$$

the number of zeros should be greater than

$$\delta \log \tau_n. \tag{6.7}$$

My few superficial attempts were so far unsuccessful.

7. There were two reasons why the proof of conjecture A succeeded at least in the case $\alpha_1 = 1$, with the aid of Jensen's estimation. The first of them is that the square (6.2) has in this case from the right a " big " zero-free neighbourhood (namely the whole half-plane $\sigma \geqslant 1$). Therefore it seems very favourable to work with

CONJECTURE B. *There is a $g(x)$ which is positive for $x > 0$, monotonically increasing, with*

$$\lim_{x \to +0} g(x) = 0, \tag{7.1}$$

and having the following property. Let γ with $\frac{1}{2} < \gamma < 1$ be fixed, and suppose that $\zeta(s)$ does not vanish in a parallelogram

$$(\gamma \leqslant)\ \alpha_2 \leqslant \sigma \leqslant 1, \quad |\tau - t| \leqslant \log \tau \tag{7.2}$$

with a $\tau \geqslant 3$. Then for $0 < \delta < \frac{1}{20}(\gamma - \frac{1}{2})$ the number $M_\delta(\tau, \alpha_2)$ of zeros of $\zeta(s)$ in the square

$$\left(\frac{\frac{1}{2} + \gamma}{2} \leqslant\right) \alpha_2 - \delta \leqslant \sigma \leqslant \alpha_2, \quad |t - \tau| \leqslant \delta/2, \tag{7.3}$$

satisfies for $\tau > c_{26}(\gamma, \delta)$ the inequality

$$M_\delta(\tau, \alpha_2) < \delta g(\delta) \log \tau, \tag{7.4}$$

independently of α_2.

That conjecture B is not stronger than conjecture A is trivial. That a proof of conjecture B is much easier than that of conjecture A will be clear on confronting the statement in (6.4) with the fact that the proof of the inequality

$$M_\delta(\tau, \alpha_2) < 1.26\,\delta\log\tau \tag{7.5}$$

instead of (7.4) is very easy; it needs only the three-circle theorem and the usual Jensen's formula. The constant 1.26 could have been diminished, in particular when we are content to prove the density-hypothesis for greater α-values only.

8. But we shall see that the truth of a still weaker conjecture would be sufficient for the proof of (2.3). This is

CONJECTURE C. *There is a $g(x)$ which is positive for $x > 0$, monotonically increasing, with*

$$\lim_{x\to+0} g(x) = 0, \tag{8.1}$$

and having the following property. Let γ with $\frac{1}{2} < \gamma < 1$ be fixed and suppose that $\zeta(s)$ does not vanish in a parallelogram

$$(\gamma\leqslant)\ \alpha_3 \leqslant \sigma \leqslant 1, \quad |t-\tau| \leqslant \log\tau \tag{8.2}$$

with a $\tau \geqslant 3$. Then for $0 < \delta < \frac{1}{20}(\gamma - \frac{1}{2})$ the number $M_\delta^*(\tau, \alpha_3)$ of the zeros of $\zeta(s)$ in the parallelogram†

$$\left(\frac{\frac{1}{2}+\gamma}{2}\leqslant\right) \alpha_3 - 5\delta\sqrt{(g(\delta))} \leqslant \sigma \leqslant \alpha_3,\ |t-\tau| \leqslant \delta/2, \tag{8.3}$$

satisfies for $\tau > c_{27}(\gamma, \delta)$ the inequality

$$M_\delta^*(\tau, \alpha_3) < \delta\, g(\delta)\log\tau \tag{8.4}$$

independently of α_3.

The weakening of conjecture C compared with conjecture B consists in replacing the square (7.3) by the thin parallelogram (8.3). At present we can obtain for $M_\delta^*(\tau, \alpha_3)$ no better estimation than (7.4); it is very probable that a "more suitable" Jensen-formula will furnish a simple proof of conjecture C. Since we shall

†(8.3) could have been still weakened.

sketch in §§10-13 a proof of the theorem that '*The truth of the density-hypothesis in its form* (2.3) *follows from hypothesis* C', this would give a complete proof of the density-hypothesis in its (2.3)-form.

9. I think the most hopeful way to prove conjecture C is via a suitable Jensen-formula. Are there any more possibilities when this way is impassable? First it is not quite impossible that by changing some details in my proof, even (7.5) will be sufficient to prove (2.3). Further possibilities can be described as follows. The usual Jensen-formula and the three-circle theorem give at once that conjecture C follows from

CONJECTURE D. *There is a* $g(x)$ *which is for* $x > 0$ *positive, monotonically increasing, with*

$$\lim_{x \to +0} g(x) = 0, \tag{9.1}$$

and having the following property. Suppose that for $\frac{1}{2} \leqslant \alpha_4 \leqslant 1$ *the zeta-function does not vanish in a parallellogram*

$$\alpha_4 \leqslant \sigma \leqslant 1, \quad | t - \tau | \leqslant \log \tau \tag{9.2}$$

with a $\tau \geqslant 3$. *Then we have for* $0 < \delta < 1/10$, $\tau > c_{28}(\delta)$, *in the parallelogram*

$$\alpha_4 - 2\,\delta \leqslant \sigma \leqslant \alpha_4, \quad |t - \tau| \leqslant \delta, \tag{9.3}$$

the inequality

$$|\zeta(s)| \leqslant \tau^{g(\delta)(\alpha_4 - \sigma)}. \tag{9.4}$$

As to this conjecture D we remark that the weaker inequality

$$|\zeta(s)| \leqslant \tau^{\alpha_4 - \sigma}$$

in (9.3) can be proved easily using properly the three-circle theorem and for $\alpha_4 = 1$ the whole conjecture D was proved by Hardy-Littlewood even in a much stronger form. How could they succeed? They started from the approximative Dirichlet-polynomial representation of $\zeta(s)$ and applied the method of Weyl-sums. Thus one way of proving conjecture D could be to find an approximative Dirichlet-polynomial-representation of $\zeta(s)$ in the domain (9.3), which moreover uses the fact that $\zeta(s) \neq 0$ in the parallelogram (9.2). This way, though not quite hopeless, seems to be very difficult.

But it is also not quite impossible that the above mentioned theorem of Hardy-Littlewood can be deduced from the general theory of functions, using about $\zeta(s)$ only the fact that it possesses a quadratic mean-value on each vertical line $\sigma = \beta, \frac{1}{2} < \beta \leqslant 1$. In terms of the well-known Lindelöf $\mu(\sigma)$-function of a function $f(s)$ regular for $\sigma > \frac{1}{2}$ this question asks, does it follow or not from the fact that this $f(s)$ possesses moreover a quadratic mean-value along each vertical line $\sigma = \beta$ and $\mu(\sigma) = 0$ for $\sigma > \kappa\ (\frac{1}{2} < \kappa \leqslant 1)$, that the graph of $y = \mu(x)$ touches the x-axis at $x = \kappa$? In the affirmative case this would give at once the proof of conjecture B. Finally a possibility is given by sharpening the three-circle-theorem among the Dirichlet-polynomials.

10. Finally we turn to the sketch of a proof of our theorem announced in §8. In all my previous investigations I started from inequalities valid for $\sigma > 1$. Now my starting inequality will be valid for $\sigma > \frac{1}{2}$, which is a big advantage. Let ϵ_1 be an arbitrarily small positive number with $\epsilon_1 \leqslant 1/1000$, let

$$\eta = \frac{\sqrt{\epsilon_1}}{20}, \quad (\text{i.e. } 12\,\eta \log 3/\eta < 1), \tag{10.1}$$

further let α with

$$\sqrt{\epsilon_1} \leqslant 2\,(1-\alpha) \leqslant 1 - \epsilon_1/4 \tag{10.2}$$

be fixed. By the requirement

$$g\,(\delta) = \eta^2 \tag{10.3}$$

δ is uniquely determined ; let then

$$N_1\,(= N_1\,(\eta)) = \frac{1}{\eta\,\delta}. \tag{10.4}$$

Let T be large, and of the integer $k \geqslant 6$ we require at this moment only

$$\log T \leqslant k\,N_1 \leqslant (1 + 12\,N_1\,\delta g(\delta)) \log T \equiv (1 + 12\eta) \log T. \tag{10.5}$$

Finally let $s = \sigma + it$ be restricted by

$$\tfrac{1}{2} + \frac{2}{[\log^2 T]} \leqslant \sigma \leqslant 1 - \frac{2}{[\log^2 T]}, \quad T \leqslant t \leqslant 2T. \tag{10.6}$$

Then our new starting inequality is

$$\Bigg| \sum_{e^{kN_1(1-\eta)} \leqslant n \leqslant e^{kN_1(1+\eta)}} \frac{\Lambda(n) R_k(n)}{n^s} + \\ + \sum_\rho \left\{ e^{N_1(\rho - s)} \frac{e^{\eta N_1(\rho - s)} - e^{-\eta N_1(\rho - s)}}{2\eta\, N_1(\rho - s)} \right\}^k \Bigg| \\ \leqslant c_{29}(\eta)\, T^{-(3/2)(1-\eta)} \log T, \tag{10.7}$$

where for the $R_k(n)$-numbers we have the estimation

$$| R_k(n) | < \frac{1}{\eta N_1}. \tag{10.8}$$

For the sum containing the primes a modification of the square-integral treatment described in §4 leads to the following result. Forming the strips

$$L_j: \left\{ \begin{array}{l} T + \dfrac{j}{[\log^3 T]} \leqslant t < T + \dfrac{j+1}{[\log^3 T]}, \\ 0 \leqslant j \leqslant [T[\log_3 T]], \end{array} \right. \tag{10.9}$$

with at most

$$T^{2(1-\alpha)} \log^{-9} T \tag{10.10}$$

exceptions of "bad" strips, all the further "good" L_j-strips have the property that there is a τ_j in L_j such that with

$$\left. \begin{array}{c} \sigma_\nu = 1 - \dfrac{\nu}{[\log^2 T]} \\ 2 \leqslant \nu \leqslant \tfrac{1}{2}[\log^2 T] - 2 \end{array} \right\} \tag{10.11}$$

and

$$s_{\nu j} = \sigma_\nu + i\, \tau_j, \tag{10.12}$$

the inequality

$$\left| \sum_\rho \left\{ e^{N_1(\rho - s_{\nu j})} \frac{e^{N_1 \eta(\rho - s_{\nu j})} - e^{-N_1 \eta(\rho - s_{\nu j})}}{2 N_1\, \eta(\rho - s_{\nu j})} \right\}^k \right| \\ \leqslant c_{30}(\eta)\, T^{\alpha + \eta^2/2 + \eta - (1+\eta)\sigma_\nu} \log^{13} T \tag{10.13}$$

holds.

11. Next comes another new step, a simple reduction-process which allows us to show that throwing away at most

$$T^{2(1-\alpha)} \log^{-6} T \tag{11.1}$$

"good" strips the remaining "very good" strips have the following property. Denoting by $\rho_j^* = \sigma_j^* + i\, t_j^*$ a zero in L_j with the greatest real part, let us call a ρ_j^* an "outstanding" zero if $\zeta(s)$ does not vanish in the domain

$$\sigma \geqslant \sigma_j^* + \frac{1}{[\log^4 T]}, \qquad |t - t_j^*| \leqslant [\log T]. \tag{11.2}$$

If λ is an upper bound of the real parts of *all* "outstanding" zeros then the property mentioned asserts that these "very good" strips do not contain zeta-zeros lying in the half-plane

$$\sigma \geqslant \lambda + \frac{1}{[\log^4 T]}. \tag{11.3}$$

The significance of this reduction lies in the fact that the investigation of the "outstanding" zeros by (10.13)—as we shall see—is greatly enlightened by the use of (11.2). Hence if we succeed in proving that

$$\lambda \leqslant \alpha + 3\,\eta, \tag{11.4}$$

then it follows from (10.10) and (11.1) that

$$N\left(\alpha + 3\eta + \frac{1}{[\log^4 T]}, 2\,T\right) - N\left(\alpha + 3\eta + \frac{1}{[\log^4 T]}, T\right) < c_{31}(\eta)\, T^{2(1-\alpha)} \log^{-5} T, \tag{11.5}$$

from which the theorem easily follows.

12. Hence we have only (11.4) to prove. Let L_μ be a "good" strip whose ρ_μ^*-root is an "outstanding" one; we fix this μ. If

$$\sigma_\mu^* \leqslant \alpha,$$

we have nothing to prove. Hence we may suppose

$$\left(1 - \frac{10}{\log T} \geqslant\right) \sigma_\mu^* > \alpha \left(\geqslant \tfrac{1}{2} + \frac{12}{N_1}\right), \tag{12.1}$$

and we have to show (10.13) leads then to a contradiction. We apply (10.13) with $j = \mu$; for ν we choose the index satisfying

$$\sigma_\nu = 1 - \frac{\nu + 1}{[\log^2 T]} < \sigma_\mu^* \leqslant - \frac{\nu}{[\log^2 T]}. \tag{12.2}$$

Trivial estimations give that the contribution of the ρ-zeros on the left of (10.13) with

$$|t_\rho - t_\mu^*| \geqslant [\log T]$$

is negligible ; easy ones assure that also those of

$$\sigma_\rho \leqslant \sigma_\mu^* - \frac{4}{N_1}, \quad |t_\rho - t_\mu^*| < [\log T]$$

and of

$$\sigma_\mu^* - \frac{4}{N_1} < \sigma_\rho \leqslant \sigma_\mu^* + \frac{1}{[\log^4 T]}$$

$$\frac{4}{\eta N_1} + \frac{1}{\eta [\log^4 T]} \leqslant |t_\rho - t_\mu^*| \leqslant [\log T]$$

contribute

$$< \log^2 T/4^k.$$

Hence

$$Z \equiv \Bigg| \sum_{\substack{\rho \\ |t_\rho - t_\mu^*| \leqslant 4/\eta N_1 + 1/\eta[\log^4 T] \\ \sigma_\mu^* - 4/N_1 \leqslant \sigma_\rho \leqslant \sigma_\mu^* + 1/[\log^4 T]}} \left\{ e^{N_1(\rho - s_{\nu\mu})} \frac{e^{\eta N_1(\rho - s_{\nu\mu})} - e^{-\eta N_1(\rho - s_{\nu\mu})}}{2\eta N_1 (\rho - s_{\nu\mu})} \right\}^k \Bigg|$$

$$\leqslant c_{33}(\eta) \left\{ T^{-1/N_1} + T^{\alpha + 2\eta - (1+\eta)\sigma_\mu^*} \right\} \log^{13} T, \tag{12.3}$$

taking in account (10.5), (12.2) and (10.13).

13. In order to obtain the required contradiction we shall estimate Z from below, applying again the second main theorem of my book quoted in (4.14) - (4.15). Z is the kth power-sum of fixed complex numbers and also the number of terms is independent of k, if the ξ's are identified with the numbers

$$e^{N_1(\rho - s_{\nu\mu})} \frac{e^{\eta N_1(\rho - s_{\nu\mu})} - e^{-\eta N_1(\rho - s_{\nu\mu})}}{2\eta N_1 (\rho - s_{\nu\mu})}. \tag{13.1}$$

We shall choose our k as the $(l + 1)$ of the second main theorem. We choose

$$m = \left[\frac{1}{N_1} \log T \right]. \tag{13.2}$$

First we have to estimate N of this theorem in our case. The domain of summation in Z is for $T > c_{34}(\eta)$ increased on taking the parallelogram

$$\sigma_\mu{}^* - \frac{4}{N_1} \leqslant \sigma \leqslant \sigma_\mu{}^* + \frac{1}{N_1}; \quad |t - t_\mu{}^*| \leqslant \frac{5}{\eta N_1}. \tag{13.3}$$

To estimate the number of zeros in the domain (13.3) we apply conjecture C with

$$\alpha_3 = \sigma_\mu{}^* + \frac{1}{N_1}, \quad \delta = \frac{10}{\eta N_1}, \quad \gamma = \tfrac{1}{2} + \frac{1}{N_1}, \quad \tau = t_\mu{}^*.$$

Since by this choice, using (10.3) and (10.4),

$$5\,\delta\,\sqrt{\{g(\delta)\}} = 5/N_1,$$

(8.3) is indeed identical with (13.3), and thus the upper bound of the number of zeros in Z is, for $T > c_{35}(\eta)$,

$$\frac{10}{\eta N_1}\,\eta^2 \log 2\,T < \frac{12\,\eta}{N_1} \log T, \tag{13.4}$$

which we may choose as N of the second main theorem. $(l + 1)$ lies between $\frac{1}{N_1}$ log T and

$$\frac{1}{N_1} \log T + N = \frac{1}{N_1}\,(1 + 12\,\eta) \log T,$$

i.e. the only requirement for k is by this choice fulfilled. Thus

$$Z \geqslant \left(\frac{12\,\eta/N_1}{24\,(1/N_1 + 12\,\eta/N_1)}\right)^{(12\eta/N_1)\log T} \max_\nu |\xi_\nu|^k$$
$$> T^{-(12\eta/N_1)\log 3/\eta} \max_\nu |\xi_\nu|^k. \tag{13.5}$$

Since, as is easy to see taking the $\xi_\mu{}^*$, corresponding to $\rho = \rho_\mu{}^*$,

$$\max_\nu |\xi_\nu|^k > \tfrac{1}{2},$$

we get from (13.5) and (12.3)

$$\tfrac{1}{2}\,T^{-(12\eta/N_1)\log 3/\eta} < c_{35}(\eta)\left\{T^{-1/N_1} + T^{\alpha+2\eta-(1+\eta)\sigma_\mu{}^*}\right\} \log^{13} T.$$

This gives, taking T sufficiently large,

$$\sigma_\mu{}^* < \frac{\alpha + 2\eta + (12\eta/N_1)\log 3/\eta}{1 + \eta},$$

from which (11.4) is an easy consequence and the sketch of the proof of the theorem is finished.

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AN APPLICATION OF THE THEORY OF MATRICES AND OF LOBATSCHEVSKIAN GEOMETRY TO THE THEORY OF DIRICHLET'S REAL CHARACTERS

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1. Many difficult problems of the analytical theory of Dirichlet characters are not resolved up to the present. Several of these problems concern the summation of the character values and the distribution of these values on narrow intervals.

A survey of these problems and some theorems on the connections between the different problems may be found in the paper [5]. One of the most difficult of such problems is the problem of the least quadratic non-residue. (I. M. Vinogradov, 1918).

Let D be an odd number, and $\chi(n)$ a real non-principal character (mod D); we shall call a number m such that $\chi(m) = -1$ a *non-residue* (mod D). Let $N_{\min}(D)$ be the least non-residue amongst the numbers $1, 2, \ldots, D-1$; I. M. Vinogradov's hypothesis consists in the relation

$$\lim_{D\to\infty} \frac{\ln N_{\min}(D)}{\ln D} = 0. \tag{1.1}$$

This hypothesis is not completely proved so far* though there are many reasons for the probable truth of it (see, for instance, [3]). The hypothesis (1.1) is an easy consequence of the Riemann hypothesis for $L(s, \chi)$; one may prove even the following conditional theorem.

This paper was communicated by title to the International Colloquium on Zeta-functions held at the Tata Institute of Fundamental Research, Bombay, on February 14-21, 1956.

* In the *Duke Math. Journal,* 21 (1954) appeared N. C. Ankeney's paper "Quadratic residues", where the author claims to prove (1.1) for primes $D \equiv -1$ (mod 4). Unluckily, his arguments are completely erroneous and give no information about least non-residues (see K. A. Rodosski's review [6]).

Conditional Theorem 1. *Let* $\psi(D) \to \infty$ *be a monotonic (as slowly increasing as we please) positive function. Form the series* $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$, $\chi(n)$ *being a real non-principal character* (mod D). *If the number of zeros of these L-series inside the corresponding semi-circles*

$$|s-1| \leqslant \frac{\psi(D)}{\ln D}, \quad \operatorname{Re} s < 1, \tag{1.2}$$

may be estimated as $o(\psi(D))$, *then hypothesis* (1.1) *is true.*

2. For the sums of the non-principal characters $\chi(n)$ (mod D) the following Vinogradov-Polya estimate(†) is known:

$$\sum_{n \leqslant x} \chi(n) = O(\sqrt{D}.\ln D) \tag{2.1}$$

for all values of x.

This estimate enables us to study the distribution of the character values on intervals of length exceeding $\sqrt{D}.\ \ln D$. The behaviour of the $\chi(n)$ values on more narrow segments escapes all efforts so far.

Very little was added to the result of I. M. Vinogradov [8]:

$$\limsup_{D \to \infty} \frac{\ln N_{\min}(D)}{\ln D} \leqslant \frac{1}{2\sqrt{e}}. \tag{2.2}$$

As is well known, Dirichlet's real characters $\chi(n)$ are connected with binary quadratic forms; therefore it is possible to study them by means of these forms.

Such an approach is considered in the present report; the theorems obtained are essentially theorems in quadratic form theory, but some of them are directly related to Dirichlet's series and real characters.

We shall consider classical integral properly primitive (*propriae primitivae*) binary quadratic forms $\phi(x, y) = ax^2 + 2bxy + cy^2$ with the determinant $-D = b^2 - ac < 0$; the class number of these

† An equivalent of this inequality as well as Polya's summation formula were found recently in G. F. Voronoi's scientific diaries for the year 1907 (see [9]).

forms will be denoted by $h(-D)$; the Lagrange reduced forms will be those under the conditions

$$c > a > 2\,|b|, \tag{2.3}$$

(the equality signs are allowed sometimes).

To each form $\phi(x, y) = ax^2 + 2bxy + cy^2$ we make correspond the integral matrix with trace zero:

$$L = \begin{bmatrix} b & -a \\ c & -b \end{bmatrix}, \tag{2.4}$$

so that

$$L^2 = -D, \tag{2.5}$$

where $D = ac - b^2 > 0$ is the determinant of the matrix.

The classical theorem of C. L. Siegel holds :

$$\ln h\,(-D) \sim \tfrac{1}{2} \ln D, \tag{2.6}$$

for $D \to \infty$.

By means of the machinery of the matrices (2.4) it is possible to prove a refinement of this theorem and to connect it with Lobatschevskian geometry.

3. To each form $\phi(x, y) = ax^2 + 2bxy + cy^2$, besides the matrix (2.4) we make correspond the point (a, b, c) on the half-hyperboloid H :

$$ac - b^2 = D\,;\, a > 0. \tag{3.1}$$

Introducing the normed coordinates

$$x_1 = \frac{a}{\sqrt{D}},\, x_2 = \frac{c}{\sqrt{D}},\, x_3 = \frac{b}{\sqrt{D}},$$

we obtain the normed hyperboloid H_0:

$$x_1 x_2 - x_3{}^2 = 1\,;\, x_1 > 0. \tag{3.2}$$

The hyperboloid H_0 may be considered as an interpretation of the Lobatschevsky plane, with the points (x_1, x_2, x_3), and the straight lines, the hyperbolic sections of H_0 by the planes $a_1x_1 + a_2x_2 + a_3x_3 = 0$. If we are given a finitely connected figure S_0 on H_0, with a piecewise smooth contour, then its Lobatschevsky area, which will be denoted by $\Lambda(S_0)$, is numerically equal to the Euclidean volume of the cone with the vertex (0, 0, 0) and base S_0.

The basic points (a, b, c) on H (with the reducibility conditions (2.3) and the condition: g.c.d. $(a, 2b, c) = 1$ (greatest common divisor $= 1$)), are projected from (0, 0, 0) on the triangle $\Delta_0 \subset H_0$. This triangle is bounded by the Lobatschevsky straight lines: $x_2 - x_1 = 0$; $x_1 - 2x_3 = 0$; $x_1 + 2x_3 = 0$, and has the angles

$$\frac{\pi}{3}, \frac{\pi}{3}, 0; \ \Lambda(\Delta_0) = \frac{2}{9}\pi.$$

The number of the projections of basic points (a, b, c) on Δ_0 equals $h(-D)$ and is governed by C. L. Siegel's theorem (2.6). One may prove a theorem on the asymptotic number of these projections inside rather arbitrary subsets of Δ_0.

Let a convex figure Σ be given on the initial hyperboloid H, with piecewise smooth contour and the projection Σ_0 on Δ_0 completely situated inside Δ_0.

Let $H(\Sigma)$ be the number of the basic integer points inside Σ. We are interested in the ratio

$$\frac{H(\Sigma)}{h(-D)}.$$

4. In what follows, $\alpha \leqslant 0.01$ is a small positive constant. D being given, we draw the Lobatschevsky straight line $x_2 - \lambda x_1 = 0$, where $\lambda = (\ln D)^2$.

It cuts off the basic triangle Δ_0, the quadrangle $G_0(\lambda)$; moreover

$$\Lambda(\Delta_0) \sim \Lambda(G_0(\lambda)), \text{ for } D \to \infty.$$

THEOREM 2. *Let $D \geqslant 0$ be odd, and let there exist a prime $p \geqslant 3$ such that $(-D/p) = +1$. Let the figure Σ_0, the projection of $\Sigma \subset H$, be situated completely inside $G_0(\lambda)$. Let, moreover, the Lobatschevsky area of Σ_0 satisfy the condition*

$$\Lambda(\Sigma_0) > (\ln D)^{-\alpha}. \tag{4.1}$$

Then the following asymptotic relation holds :

$$H(\Sigma)/h(-D) = \Lambda(\Sigma_0)/\Lambda(\Delta_0)\,(1 + \eta\,(p, D)), \tag{4.2}$$

where $\eta(p, D) \to 0$ *for a fixed* p *and* $D \to \infty$.

As we have $\Lambda(\Delta_0) = 2\pi/9$, we obtain

$$H(\Sigma) = \frac{9}{2\pi}\, h(-D)\, \Lambda(\Sigma_0)\, (1 + \eta(p, D)). \tag{4.3}$$

Thus we obtain a complement to C. L. Siegel's theorem in terms of Lobatschevskian geometry.

It should be remarked that we use an auxiliary arbitrary prime p under the condition $(-D/p) = +1$. Apparently the use of this prime is not essential, it is a technical defect of the proof.

We must remark that Theorem 2 is not a direct consequence of the Riemann hypothesis for $L(s, \chi)$ with $\chi = (-D/n)$; $-D < 0$ being a fundamental discriminant. But if $L(s, \chi)$ is known not to have zeros for $\sigma > 9/10$, $|t| \leqslant \frac{1}{2}$ $(s = \sigma + it)$, Theorem 2 can be improved : instead of $\eta(p, D)$ in (4.2) and (4.3) one can take $\eta(D) \to 0$ for $D \to \infty$; no auxiliary prime p is needed then.

5. From formula (4.2), by a simple computation of area on the Lobatschevsky plane, we can deduce the theorems immediately connected with Dirichlet's L-series.

Let $h(-D, \gamma\sqrt{D})$ be the number of properly primitive forms (a, b, c) for which $a \leqslant \gamma\sqrt{D}$. Computing the area $\Lambda(\Sigma_0)$ in (4.3) we deduce

THEOREM 3. *For* $0 \leqslant \gamma \leqslant 1$, $\gamma \geqslant (\ln D)^{-\alpha}$ *and a fixed* p, *we have*

$$h(-D, \gamma\sqrt{D}) = \frac{3\gamma}{\pi} h(-D)\,(1 + \eta\,(p, D)), \tag{5.1}$$

where $\eta(p, D) \to 0$ *for* $D \to \infty$ *uniformly on* $\gamma \geqslant (\ln D)^{-\alpha}$.

For $1 \leqslant \gamma \leqslant \sqrt{(4/3)}$ *we have*

$$h(-D, \gamma\sqrt{D}) = f(\gamma)\, h(-D)\, (1 + \eta(p, D)), \tag{5.2}$$

where

$$f(\gamma) = \frac{6}{\pi} \text{ arc sin } \sqrt{(1-\gamma^{-2})} + \frac{3\gamma}{\pi} (1 - 2\sqrt{(1-\gamma^{-2})}).$$

For $\gamma > \sqrt{(4/3)}$ *we have (trivially)*

$$h(-D, \gamma\sqrt{D}) = h(-D).$$

Here $\eta(p, D) \to 0$, *for fixed* p *and* $D \to \infty$.

A part of Theorem 3 may be formulated in terms of L-series.

Let $D < 0$ be an odd fundamental discriminant and $\chi(n) = (-D/n)$ Kronecker's character.

Let

$$(-D/p) = +1; L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s};$$

$$\zeta(s)\, L(s, \chi) = \sum_{n=1}^{\infty} a_n\, n^{-s}, \quad (s > 1).$$

THEOREM 4. *For* $\gamma \geqslant (\ln D)^{-\alpha}$ *we have*

$$\sum_{n \leqslant \gamma\sqrt{D}} a_n = \gamma\sqrt{D}.\, L(1, \chi)\, (1 + \eta(p, D)), \tag{5.3}$$

where $\eta(p, D) \to 0$ *for fixed* p, $D \to \infty$ *uniformly on* $\gamma \geqslant (\ln D)^{-\alpha}$.

6. Relation (5.3) represents but a slight advance in the theory of the real characters, but the method of the analytical arithmetic of matrices (2.4) is also fit for more refined investigations. In particular, it gives some partial results on the hypothesis (1.1) for certain progressions modulo 16.

The study of the behaviour of characters on narrow segments of the type $(1, x)$ for $x = D^{\epsilon}$, $\epsilon > 0$ small and fixed, is reduced to the study of the distribution of the projections of integer points inside the zero angle of the Lobatschevskian triangle Δ_0.

We will not formulate the results thus obtained, which are cumbersome enough, except a theorem on the zeros of $L(s, \chi)$, though rather cumbersome also.

THEOREM 5. *Let* $-D < 0$ *be an odd fundamental discriminant;* $\chi(n) = (-D/n)$; K *a large positive constant. Consider all odd primes not exceeding* K:

$$3, 5, 7, \dots, p_r \leqslant K. \tag{6.1}$$

Let $\rho > 0$ *be a small constant, and suppose for at least* $100\,\rho$ % *of primes* (6.1) *we have the condition*

$$(-D/p_i) = +1. \tag{6.2}$$

Then the series $L(s, \chi)$ *has no zeros in the domain*

$$\sigma > 1 - \frac{\psi_1(K)}{\ln D}; \quad \psi_2(K) \geqslant |t| \geqslant 1, \tag{6.3}$$

where $\psi_1(K)$ *and* $\psi_2(K) \to \infty$ *for* $K \to \infty$.

7. The methods of proof of theorems slightly feebler than those mentioned above (except Theorems 1 and 5) are exposed in detail in the paper [4].

We shall sketch here the principal ideas of these methods.

For a given odd $D > 0$, we consider the matrices (2.4) satisfying the conditions of reducibility (2.3) and primitivity: g.c.d. $(a, 2b, c) = 1$ (basic vector-matrices),

$$L_1, L_2, \dots, L_h;\ h = h(-D);\quad L_i^2 = -D. \tag{7.1}$$

They form a part of the ring of the integral matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

If A is any non-singular matrix and $L' = ALA^{-1}$, then

$$\operatorname{tr}(L') = \operatorname{tr}(L) = 0;\ L'^2 = -D.$$

If the matrix L' is integral and basic, it must be one of the matrices (7.1).

If we have the equality $b + L_i = CA$ with an integer $b = b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and integer matrices C and A, then $b + L'_i = b + ALA^{-1} = AC$, so that $L' = AL_i A^{-1}$ is an integer vector matrix. If ϵ is a unimodular matrix, then $\epsilon L'_i \epsilon^{-1}$ is also an integer vector matrix; it can be basic only for two particular values of ϵ, $\epsilon = \pm \epsilon_0$.

The connection between inequivalent matrices in this sense with the class number of the corresponding field (of any degree, not only quadratic ones) was established by A. Hurwitz some sixty years ago [1] and forms essentially the background of these investigations.

Let $(-D/p) = +1$; then it proves possible and expedient to consider the equalities

$$l + L_i = \Pi_{1i} \Pi_{2i} \dots \Pi_{si} V_i, \quad (i = 1, 2, \dots, h), \tag{7.2}$$

where Π_{ki} are integer matrices of determinant p, and l an integer under the condition

$$l^2 + D \equiv 0 \pmod{0\, p^s}.$$

The matrices Π_{ki} are so chosen in the " bundle " of matrices unimodularly associated from the right hand side, that if

$$Q_{\nu i} = \Pi_{1i} \Pi_{2i} \dots \Pi_{\nu i}, \tag{7.3}$$

then

$$Q_{\nu i}^{-1} L_i Q_{\nu i} = L'_i$$

is a basic vector matrix.

The matrices $Q_{\nu i}$ may be considered as operators generating the Lobatschevskian motions on the hyperboloid H. The application of these operators in a convenient way (see [4]) enables us to shift L_i into the given domain.

For the corresponding asymptotic computations some probability-theoretic limit theorems for Markov chains are applied (for analogous applications see the paper [2]).

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HARMONIC ANALYSIS AND DISCONTINUOUS GROUPS IN WEAKLY SYMMETRIC RIEMANNIAN SPACES WITH APPLICATIONS TO DIRICHLET SERIES

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IN THE following lectures we shall give a brief sketch of some representative parts of certain investigations that have been undertaken during the last five years. The center of these investigations is a general relation which can be considered as a generalization of the so-called Poisson summation formula (in one or more dimensions). This relation we here refer to as the " trace-formula."

1. Let S be a Riemannian space, whose points we denote by x and the (local) coordinates by $x^1, x^2, \ldots, x^n$, with a positive definite metric

$$ds^2 = \sum g_{ij}\, dx^i\, dx^j.$$

We shall assume the g_{ij} to be analytic in the coordinates. Further we assume that we have a locally compact group G of isometries of S (not necessarily the full group of isometries), whose elements we denote by m, and that G acts transitively on S so that given x and y in S, there exists an $m \in G$ such that $x = my$. We shall be concerned with the linear operators on functions $f(x)$ defined on S, which have the property that the operators are invariant under G, or otherwise expressed, linear operators that commute with the isometries m in G. We restrict ourselves here to the class of linear operators that are differential operators of finite order, integral operators of the form $\int_S k(x, y)\, f(y)\, dy$ (where dy denotes the invariant element of volume derived from the metric), or any

This is a summary of the results presented by the author to the International Colloquium on Zeta-functions held at the Tata Institute of Fundamental Research, Bombay, on February 14-21, 1956.

finite combination (by addition or multiplication) of such. This class evidently forms a ring.

Turning first to the integral operators, one observes that in order that the operator

$$\int_S k(x, y) f(y)\, dy$$

should be invariant, it is necessary and sufficient that the kernel satisfy the relation

$$k(mx, my) = k(x, y), \tag{1.1}$$

for all x, y in S and all m in G. We shall refer to such a kernel as a "point-pair invariant". If we consider such a "point-pair invariant" $k(x, y)$ as a function of one of the arguments, say x, keeping the other point y fixed, we see that $k(x, y)$ is invariant under the subgroup of G that leaves y fixed. This subgroup we denote by Ry and call it the rotation group of y. We express this property of k by saying that it has as a function of x rotational symmetry around the point y. Let x_0 be a chosen fixed point in S and R^0 with elements r^0 the rotation group of x_0. R^0 is isomorphic to a compact (or possibly finite) subgroup of the orthogonal group of n elements. Norming the bi-invariant Haar measure on R^0 so that $\int_{R^0} dr^0 = 1$, we can define for a function $f(x)$ a symmetrized function

$$f(x\,; x_0) = \int_{R^0} f(r^0 x)\, dr^0\,; \tag{1.2}$$

$f(x\,; x_0)$ clearly has rotational symmetry around the point x_0. Furthermore, if we have a function $f(x\,; x_0)$ with rotational symmetry around x_0, we can define a point-pair invariant $k(x, y)$ by the relation

$$k(x, y) = f(mx\,; x_0), \quad \text{where} \quad my = x_0,$$

this definition is seen not to depend on the particular choice of m if there is more than one m satisfying the relation $my = x_0$. Therefore the study of point pair invariants is equivalent to the study of functions with rotational symmetry around some point x_0.

We observe also the following facts, before turning to the consideration of differential operators. Because G acts transitively on S, an invariant operator, say L, of our class is completely characterized by its action at one point, say x_0. By this we mean that, introducing the notation $[Lf(x)]_{x=x_0}$ to denote the value of the function $Lf(x)$ at the point $x = x_0$, we can for an arbitrary point x_1 express $[Lf(x)]_{x=x_1}$ by means of the relation

$$[L f(x)]_{x=x_1} = [L f(mx)]_{x=x_0},$$

where m is a solution of $mx_0 = x_1$. Conversely if we have an operator $\mathscr{L}$ (not necessarily invariant), we can from its action at x_0, construct an invariant operator L by the relation

$$[L f(x)]_{x=x_0} = [\mathscr{L} f(x)]_{x=x_0},$$

provided

$$[\mathscr{L} f(r^0 x)]_{x=x_0} = [\mathscr{L} f(x)]_{x=x_0},$$

for every element r^0 in the rotation group R^0 of x_0. Finally if $\mathscr{L}$ does not have this property we may define

$$[L f(x)]_{x=x_0} = [\mathscr{L} f(x\,;x_0)]_{x=x_0},$$

where $f(x\,;x_0)$ is the symmetrized function of f around x_0 defined by (1.2), because $f(r^0x)$ and $f(x)$ have the same symmetrized function around x_0. If $\mathscr{L}$ is invariant then $L = \mathscr{L}$.

Furthermore, one observes that an invariant operator applied to a function with rotational symmetry around a point, gives a function which again is rotationally symmetric around the same point. Also an invariant operator applied to a point-pair invariant as a function of say the first point, gives as result again a point-pair invariant.

Consider now the class of invariant differential operators of finite order, and let for simplicity the local co-ordinates around x_0 be chosen such that the matrix (g_{ij}) at $x = x_0$ is the identity matrix E_n. Let D be an invariant differential operator, its action at the point $x = x_0$ is identical to that of a differential operator $D^{(0)}$ with constant coefficients,

$$D^{(0)} = \sum a_{i_1, i_2, \ldots, i_n} \left(\frac{\partial}{\partial x^1}\right)^{i_1} \left(\frac{\partial}{\partial x^2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial x^n}\right)^{i_n}.$$

By the highest homogeneous part of $D^{(0)}$ we mean the aggregate of terms in the above sum, where $a_{i_1, \ldots, i_n} \neq 0$ and $i_1 + i_2 + \ldots + i_n$ attains its maximal value; we denote this by $\overline{D}^{(0)}$, and write

$$\overline{D}^{(0)} = p_D \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right),$$

where p_D is a homogeneous polynomial. The rotation group R^0 induces on the tangent space of S at x_0 a subgroup R of the orthogonal group $\mathcal{O}_n$, and the polynomial $p_D(u_1, u_2, \ldots, u_n)$ is seen to be invariant under this group R of orthogonal transformations. Conversely, if we have a homogeneous polynomial $p(u_1 \ldots u_n)$ which is invariant under the group R, we may define an invariant differential operator D_p by the relation

$$[D_p f(x)]_{x=x_0} = \left[p \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right) f(x; x_0) \right]_{x=x_0}.$$

It should be observed that whereas $p_{(D_p)} = p$, all one can say about $D_{(p_D)} - D$ is that it is an invariant operator of lower order than D. One also easily shows that if p_1 and p_2 are two such homogeneous polynomials invariant under R, we have that $D_{p_1 p_2} - D_{p_1} D_{p_2}$ is an operator of lower order than $D_{p_1 p_2}$. Using these facts, and a well-known result by Hilbert which says that the polynomials p have a finite basis of homogeneous polynomials $p_1, p_2, \ldots, p_l$, $1 \leqslant l$, such that every homogeneous polynomial p can be written as a polynomial (not necessarily in a unique way) of $p_1, p_2, \ldots, p_l$, with constant coefficients, one obtains the result that $D_{p_1}, D_{p_2}, \ldots, D_{p_l}$ generate the ring of the invariant differential operators in the sense that any invariant differential operator D can be written as a finite expression

$$D = \sum A_{\nu_1, \nu_2, \dots, \nu l} \; D_{p_1}^{\nu_1} D_{p_2}^{\nu_2} \dots D_{pl}^{\nu l}, \tag{1.3}$$

where the A's are constants. Writing $D_{p_i} = D_i$ for $i = 1, 2, \dots, l$, we shall call $D_1, D_2, \dots, D_l$ a set of fundamental operators and we may assume that it is so chosen that l is minimal.

The fundamental operators in general do not commute, and as commutativity is essential for our later considerations, we shall make an additional assumption about G and S, which will imply commutativity (as we do not know, however, whether this assumption is necessary for commutativity, we should note that it is only the commutativity that is really necessary for the following developments).

We assume that there is a fixed isometry μ of S (possibly not in G), such that $\mu G \mu^{-1} = G$, $\mu^2 \in G$, and that for any pair of points x and y in S, there exists an m in G for which $mx = \mu y$ and $my = \mu x$. We may call a space for which there is some group of isometries G with these properties (if that is the case then the full group of all isometries will have these properties too) a " weakly symmetric " Riemannian space. This concept is more general than E. Cartan's concept of a symmetric space, as symmetric implies weakly symmetric, whereas it can be shown by examples that weakly symmetric does not imply symmetric.

Under this assumption we can prove that all the invariant operators commute. We first show that they commute when applied to point-pair invariants $k(x, y)$ considered as functions of the first point x. We first notice that if L is an invariant operator then so is also $\widetilde{L}$ defined by

$$\widetilde{L} f(x) = [L f(\mu^{-1}x)]_{x \to \mu x}.$$

Also from our assumption about G follows that for any point-pair invariant $k(x, y)$ we have

$$k(\mu y, \mu x) = k(mx, my) = k(x, y).$$

Denoting by a subscript the argument (x or y) that the operator is to act on, we have

$$L_x k(x, y) = k'(x, y),$$

where $k'(x, y)$ again is a point-pair invariant. Now we have

$$\begin{aligned} \widetilde{L}_y k(x, y) &= \widetilde{L}_y k(\mu y, \mu x) \\ &= [L_y k(y, \mu x)]_{y \to \mu y} = [k'(y, \mu x)]_{y \to \mu y} \\ &= k'(\mu y, \mu x) = k'(x, y). \end{aligned}$$

Thus

$$L_x k(x, y) = \widetilde{L}_y \, k(x, y),$$

so that we may shift the operator from the first to the second argument by replacing it with $\widetilde{L}$. If we now have two operators $L^{(1)}$ and $L^{(2)}$ we may write

$$\begin{aligned} L_x^{(1)} L_x^{(2)} k(x, y) &= L_x^{(1)} \widetilde{L}_y^{(2)} k(x, y) \\ &= \widetilde{L}_y^{(2)} L_x^{(1)} k(x, y) = L_x^{(2)} L_x^{(1)} k(x, y), \end{aligned}$$

(since the operators clearly may be interchanged when they act on different arguments). Thus we have commutativity when our operators are applied to point-pair invariants. Therefore we have also commutativity if our operators are applied to a function with rotational symmetry around a point, say x_0. For a function without rotational symmetry we notice that

$$[L^{(1)} L^{(2)} f(x)]_{x=x_0} = [L^{(1)} L^{(2)} f(x; x_0)]_{x=x_0},$$

where $f(x; x_0)$ is the function with rotational symmetry defined by (1.2). From this follows

$$[L^{(1)} L^{(2)} f(x)]_{x=x_0} = [L^{(2)} L^{(1)} f(x)]_{x=x_0},$$

or what is the same

$$L^{(1)} L^{(2)} f(x) = L^{(2)} L^{(1)} f(x),$$

that is, the operators commute.

It can be shown that the operator $\bar{\tilde{L}}$ where the bar denotes conjugation, is the formal adjoint of the operator L.

Returning to (1.3) we may now write

$$D = P\,(D_1, D_2, \ldots, D_l), \tag{1.4}$$

where P is a polynomial with constant coefficients. It should be noted that though our fundamental operators were chosen so that l was minimal, there may sometimes still be algebraic relations between them, so that the representation (1.4) may not necessarily be unique. Further it can be shown that one can always choose a set of fundamental operators with minimal l, such that each of them is self-adjoint.

Now let $f(x)$ be a function which is an eigenfunction of all our fundamental operators D_i so that

$$D_i f(x) = \lambda_i f(x), \quad i = 1, 2, \ldots, l, \tag{1.5}$$

where the λ_i are constants; because of (1.4) it will then be an eigenfunction of all the invariant differential operators, and in particular of the Laplace operator derived from the metric, therefore $f(x)$ will be analytic in the coordinates. If we take a point x_0 such that $f(x_0) \neq 0$, and form $f(x; x_0)$ defined by (1.2), this will again satisfy the equations (1.5) and will not vanish identically in x since $f(x_0\,; x_0) = f(x_0) \neq 0$. We now write

$$f(x\,; x_0) = f(x_0)\, \omega_\lambda(x, x_0), \tag{1.6}$$

where the subscript λ is an abbreviation for the l-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ so that $\omega_\lambda(x_0, x_0) = 1$. We call this the " normed " eigenfunction with rotational symmetry around x_0, and shall show that it is unique, that is to say a function with rotational symmetry around x_0 which takes the value 1 at the point x_0 and which satisfies the equations (1.5) is identical with $\omega_\lambda(x, x_0)$. To prove this we observe that for such a function $g(x)$, we have, because $g(x) = g(x\,; x_0)$,

$$\left[\left(\frac{\partial}{\partial x^1}\right)^{\nu_1}\left(\frac{\partial}{\partial x^2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n} g(x)\right]_{x=x_0}$$
$$=\left[\left(\frac{\partial}{\partial x^1}\right)^{\nu_1}\left(\frac{\partial}{\partial x^2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n} g(x;\, x_0)\right]_{x=x_0} = [D\, g(x)]_{x=x_0},$$

where D is an invariant differential operator depending only on $(\nu_1, \nu_2,\ldots,\nu_n)$. Because of (1.4) and since $g(x)$ satisfies the equations (1.5) we thus get on using $g(x_0) = 1$,

$$\left[\left(\frac{\partial}{\partial x^1}\right)^{\nu_1}\left(\frac{\partial}{\partial x^2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x^n}\right)^{\nu_n} g(x)\right]_{x=x_0} = P(\lambda_1, \lambda_2,\ldots,\lambda_l),$$

where P is a polynomial depending only on $(\nu_1, \nu_2,\ldots, \nu_n)$. This shows that all the partial derivatives of $g(x)$ at the point x_0 are uniquely determined by the l-tuple $(\lambda_1, \lambda_2,\ldots, \lambda_l)$ and so since $g(x)$ is analytic in the coordinates, $g(x)$ is unique, that is, it coincides with $\omega_\lambda(x, x_0)$. We may from $\omega_\lambda(x, x_0)$ construct the point-pair invariant $\omega_\lambda(x, y)$ which will, because of the relation

$$D_x\, \omega_\lambda(x, y) = \tilde{D}_y\, \omega_\lambda(x, y),$$

be a normed eigenfunction also in y with rotational symmetry around the point x. Therefore we must have

$$\omega_\lambda(x, y) = \omega_{\tilde{\lambda}}(y, x), \qquad (1.7)$$

where $\tilde{\lambda}$ denotes an l-tuple $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_l)$ not necessarily identical to the original one. $\omega_\lambda(x, y)$ is now easily seen to be an eigenfunction (considered as a function of x) of our whole class of invariant operators for the reason that

$$L_x\, \omega_\lambda(x, y)$$

because of the commutativity of L and the D_i, $i = 1, 2,\ldots, l$, again satisfies the equations (1.5), and furthermore it is again a function with rotational symmetry around y, and differs therefore only by a factor independent of x (and hence since the factor is a point pair invariant it is independent of y also) from $\omega_\lambda(x, y)$, that is to say

$$L_x\, \omega_\lambda(x, y) = \Lambda\, \omega_\lambda(x, y),$$

where Λ is a constant depending on L and the l-tuple λ only.

We can now show that any function which satisfies the equations (1.5) will be an eigenfunction of our class of invariant operators, namely we have

$$\begin{aligned}[L\, f(x)]_{x=x_0} &= [L\, f(x\,;\, x_0)]_{x=x_0} \\ &= [L\, f(x_0)\, \omega_\lambda\, (x, x_0)]_{x=x_0} \\ &= \Lambda\, f(x_0)\, \omega_\lambda\, (x_0, x_0) = \Lambda\, f(x_0).\end{aligned}$$

Since this holds for any point x_0, we have

$$L\, f(x) = \Lambda\, f(x),$$

and we see that the eigenvalue Λ does depend only on L and the l-tuple λ, but not on the particular function $f(x)$.

Thus for an integral operator we may write

$$\int_S k(x, y)\, f(y)\, dy = h(\lambda)\, f(x), \tag{1.8}$$

where $h(\lambda) = h(\lambda_1, \lambda_2, \ldots, \lambda_l)$ depends on k and λ only. In order to get an expression for $h(\lambda)$ it is therefore enough to produce a " representative " set of eigenfunctions, that is, one that exhausts all the possibilities for the l-tuple $(\lambda_1, \ldots, \lambda_l)$, that is, l-tuples for which there really do exist functions satisfying the equations (1.5).

In a number of cases that are of particular interest for applications, such a set can be obtained from the following lemma:

Let T with elements t be a subgroup of G which is simply transitive on S, that is, such that the equation $x = tx_0$, where x is any point in S and x_0 a chosen fixed point, always has one and only one group element t as a solution. Further suppose that we have a continuous non-vanishing function $\phi(t)$ on T that satisfies the relation

$$\phi(t_1 t_2) = \phi(t_1)\,\phi(t_2),$$

for all t_1 and t_2 in T. If we now define $f(x) = f(tx_0) = \phi(t)$, where $tx_0 = x$, then $f(x)$ is an eigenfunction of our operators, because

$$L\,f(x) = [L\,f(tx)]_{x=x_0} = \phi(t)\,[L\,f(x)]_{x=x_0} = [L\,f(x)]_{x=x_0}\,f(x).$$

If we have several such multiplicatively independent functions $\phi_1(t), \phi_2(t), \ldots, \phi_\kappa(t)$, then

$$\phi_1^{s_1}(t)\,\phi_2^{s_2}(t)\,\ldots\,\phi_\kappa^{s_\kappa}(t)$$

will also be one (where, if T or what is the same S, is not simply connected the exponents $s_1, s_2, \ldots, s_\kappa$ have to be chosen such that the resulting function is single-valued). It is of course not always so that different choices of the κ-tuple $s_1, \ldots, s_\kappa$ necessarily lead to different l-tuples $\lambda_1, \ldots, \lambda_l$. In many cases one gets all possibilities for which eigenfunctions exist covered by this construction.

The nature of the set of possible λ's may differ from the completely discrete set that would occur if S is compact,† to the situation for many non-compact spaces where the set of all l-tuples of complex numbers $\lambda_1, \ldots, \lambda_l$, which satisfy the possible algebraic relations between the D_i, $i = 1, 2, \ldots, l$, does occur. Intermediary situations can of course also occur. In the case when the set of all l-tuples λ of complex numbers satisfying the algebraic relations between the D_i's does occur, it is easily shown that $\omega_\lambda(x, y)$ as a function of λ is an analytic function on the algebraic variety defined by these relations, which is regular whenever all λ_i's are finite.

As an illustration we may for instance consider the space of n by n positive definite symmetric matrices $Y = (y_{ij})$ with the metric

$$ds^2 = \sigma\,(Y^{-1}\,dY\;Y^{-1}\,dY),‡$$

† Because we require our functions to be regular globally, if one admits " local " eigenfunctions (that cannot be continued everywhere in S, or that by such continuation would not be single-valued) the situation is different as shown by the examples of the surface of a sphere or the periphery of a circle.

‡ σ here and in the following denotes the trace.

where $dY = (dy_{ij})$, and the group G may be taken as the group of all non-singular real n by n matrices A, the isometries being

$$Y \to A\,Y\,A'$$

(A' is the transposed of A); finally the isometry μ may be taken as

$$Y \to Y^{-1}.$$

It is then easily established that all our requirements are satisfied. The point-pair invariants are easily seen to be of the form that $k(Y_1, Y_2)$ is a symmetric function of the n eigenvalues of the matrix $Y_2\,Y_1^{-1}$, or if one prefers it, $k(Y_1, Y_2)$ is a function of the n arguments $\sigma((Y_2\,Y_1^{-1})^{\nu})$, $\nu = 1, 2, \ldots, n$. Conversely any such function is a point-pair invariant.

A set of fundamental operators can be obtained as follows: let $\dfrac{\partial}{\partial Y}$ denote the matrix $\left(\dfrac{1+\delta_{i,j}}{2}\,\dfrac{\partial}{\partial y_{ij}}\right)$, where $\delta_{i,j}$ is the Kronecker symbol; then the operators

$$D_i = \sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, \ldots, n \tag{1.9}$$

are a set of fundamental operators, and they are algebraically independent.

To obtain a representative set of eigenfunctions, consider the subgroup of G formed by the "triangular" matrices $T = (t_{ij})$ with $t_{ij} = 0$ for $i < j$ and $t_{ii} > 0$ for $i = 1, 2, \ldots, n$. This group acts simply transitive on our space, and for any complex n-tuple $s = (s_1, s_2, \ldots, s_n)$ the function

$$\phi_s(T) = \prod_{i=1}^{n} t_{ii}^{\,2s_i + i - (n+1)/2} \tag{1.10}$$

is single-valued and continuous on this group and has the property

$$\phi_s(T_1)\,\phi_s(T_2) = \phi_s\,(T_1\,T_2).$$

Thus defining for $Y = TT'$

$$f_s(Y) = \phi_s(T), \tag{1.11}$$

this is an eigenfunction. One can show that

$$D_i\, f_s(Y) = \lambda_i\,(s)\, f_s(y),$$

where $\lambda_i(s) = \lambda_i(s_1, s_2, \ldots, s_n)$ is a polynomial in the s_j of degree i which is symmetric in the s_j and of the form

$$\lambda_i(s_1, s_2, \ldots, s_n) = s_1^{\,i} + s_2^{\,i} + \ldots + s_n^{\,i} + \text{terms of lower degree.}$$

From this one sees that λ_i are a basis for the symmetric polynomials of the s_j, so that the s_j are determined as roots of an algebraic equation of nth degree whose coefficients are rational in the λ_i, so they are determined up to a permutation of the s_j. From this it follows that we may by suitable choice of the s_j make the λ_i any n-tuple of finite complex numbers. One also can show that

$$\widetilde{\lambda}_i(s_1, s_2, \ldots, s_n) = \lambda_i(-s_1, -s_2, \ldots, -s_n).$$

To find an expression for the $h(\lambda)$ defined in (1.8),

$$\int_S k(Y_1, Y)\, f_s(Y)\, dY = h(\lambda)\, f_s(Y_1),$$

where dY is the invariant element of volume

$$dY = \frac{2^{\frac{1}{2}n(n-1)}}{|Y|^{\frac{1}{2}(n+1)}} \prod_{i \leqslant j} dy_{ij},$$

we may write

$$k(Y_1, Y) = k(\sigma\,(YY_1^{-1}), \sigma(\,(YY_1^{-1})^2), \ldots, \sigma(\,(YY_1^{-1})^n\,)\,),$$

and take $Y_1 = E_n$ the identity matrix so that $f_s(Y_1) = 1$; further we may introduce the t_{ij}, $i \geqslant j$, in $Y = TT'$ as new coordinates in our space, the element of volume then becomes $\dfrac{2^{n(n+3)/4}}{t_{11}\, t_{22}^2 \ldots t_{nn}^n} \prod_{i \geqslant j} dt_{ij}$ and the relation becomes

$$2^{\frac{1}{4}n(n+3)} \int k(\,\sigma(TT'),\, \sigma(\,(TT')^2\,),\, \ldots,\, \sigma(\,(TT')^n\,)\,) \times$$

$$\times \prod_{i=1}^{n} t_{ii}^{\,2s_i - \frac{1}{2}(n+1)} \prod_{i \geqslant j} dt_{ij} = h(\lambda),$$

where the integration is carried from 0 to ∞ over the t_{ii} and from $-\infty$ to ∞ over the t_{ij} with $i > j$. For special forms of k these integrations can be carried out explicitly, for instance if

$$k(Y_1, Y) = |\,Y Y_1^{-1}\,|^{\alpha}\, e^{-\beta\sigma(Y Y_1^{-1} - 1)},$$

where the real part of β is positive, and the real part of $2s_i + 2\alpha > \frac{1}{2}(n-1)$ for $i = 1, 2, \ldots, n$; the integral then becomes

$$2^{\frac{1}{4}n(n+3)} \int \left\{ \exp\left(-\beta \sum_{i \geqslant j} t_{ij}^{\,2} \right) \right\} \prod_{i=1}^{n} t_{ii}^{\,2s_i + 2\alpha - (n+1)/2} \prod_{i \geqslant j} dt_{ij},$$

and splits into a product of $\dfrac{n(n+1)}{2}$ simple integrals, each of which is expressible in terms of Gamma functions.§

2. Let now Γ be a discrete subgroup of G which acts properly discontinuous on the space S, and let there be given a representation of Γ by unitary ν by ν matrices $\chi(M)$, where we denote the elements of Γ by M. Consider function vectors $F(x)$, that are column vectors, whose ν components are scalar functions of the point x, and which furthermore satisfy the relation

$$F(M\,x) = \chi(M)\, F(x), \tag{2.1}$$

for all x in S and M in Γ. Such a function $F(x)$ is then of course fully determined by its values on a fundamental domain $\mathscr{D}$ of Γ in S. Applying one of the invariant integral operators to such a function $F(x)$ one sees that

$$\int_S k(x, y)\, F(y)\, dy = \int_{\mathscr{D}} K(x, y\,;\, \chi)\; F(y)\, dy,$$

§ For this special choice of k, the resulting form of formula (1.8) has in the meantime been derived by different means by H. Maass, *Journal of the Indian Math. Soc.* 19 (1955), 1-24.

where the kernel K is a matrix given by

$$K(x, y; \chi) = \sum_{M \in \Gamma} \chi(M)\, k(x, My). \tag{2.2}$$

Considering now the Hilbert space defined by the inner product

$$(F_1, F_2) = \int_{\mathscr{D}} \bar{F}_1'(x)\, F_2(x)\, dx,$$

where $\bar{F}_1'$ is the conjugate transposed of F_1, one sees easily that the operator

$$\int_{\mathscr{D}} K(x, y; \chi)\; F(y)\, dy \tag{2.3}$$

is normal since the adjoint operator has a kernel that is derived from the right-hand side of (2.2) by replacing $k(x, y)$ by $\tilde{k}(x, y) = \overline{k(y, x)}$, and thus it commutes with the operator (2.3). The invariant differential operators are also seen to be normal.

We have not up to now put any restrictions on our point-pair invariants $k(x, y)$, but always only assumed that the kernel and the function that the operator acted on were such that the integral also existed if absolute values were taken of the integrands.

It is now time to impose conditions that will enable us to make definite statements about the absolute convergence of the series on the right-hand side of (2.2) and also about the behavior of $K(x, y; \chi)$.

We make the following assumptions :

$k(x, y)$ should have a majorant,

$k_1(x, y)$ such that (a) $\int_S k_1(x, y)\, dy < \infty$, (b) $k_1(x, y)$ is of regular growth; that is to say, there should exist positive constants δ and A such that for all x and y,†

† One can relax this, and permit kernels with, for instance, a singularity at $x = y$ by requiring (b′) to be fulfilled only if the smallest geodesic distance $d(x, y)$ exceeds some fixed number.

$$k_1(x, y) \leqslant A \int\limits_{d(y,y')<\delta} k_1(x, y')\, dy', \tag{b'}$$

where $d(y, y')$ denotes the smallest geodesic distance between y and y'. Under these assumptions the above series for $K(x, y; \chi)$ converges absolutely for x and y in S, and uniformly if x and y are in some compact subregion of S.‡

We also make the assumption that the fundamental domain $\mathscr{D}$ of Γ in S is compact. Then $K(x, y; \chi)$ will be uniformly bounded for x and y in $\mathscr{D}$ (and therefore also for all x and y in S). Therefore also the expression

$$\int\limits_{\mathscr{D}} \int\limits_{\mathscr{D}} \sigma\,(K(x, y\,;\chi)\,\overline{K(x, y\,;\chi)}')\, dx\, dy$$

is finite ($\overline{K}'$ denotes the conjugate transposed of the matrix K) so that the integral operator is of the Hilbert-Schmidt class, and the classical methods from the theory of integral equations can be applied.

Consider now the functions $F(x)$ satisfying (2.1) which are eigenfunctions of our fundamental operators D_i for $i = 1, 2, \ldots, l$. We can then show from the preceding results about our integral and differential operators, that there exist an orthonormal system of eigenfunctions $F_i(x)$, which is complete in our Hilbert space, and such that if we write

$$D_j\, F_i(x) = \lambda^i_j\, F_i(x) \tag{2.4}$$

for $j = 1, 2,\ldots, l$; the l-tuples $\lambda^i = (\lambda^i_1, \lambda^i_2,\ldots, \lambda^i_l)$ have no finite point of accumulation in l-dimensional space. The completeness in particular follows from the easily established fact that the system of all admissible kernels $K(x, y; \chi)$ is complete.

About the eigenvalues, the l-tuples λ^i, one could at once make statements based upon the fact that if the kernel of an integral

‡ Thus in particular $K(x, y\,; \chi)$ is continuous if $k(x, y)$ is.

operator is Hermitian (which is the case if $k(x, y) = \overline{k(y, x)}$), the eigenvalues $h(\lambda^i)$ must be real; also, by looking at the differential operators, if we have chosen the fundamental operators self-adjoint, as one always can, the λ^i_j for $j = 1, 2, \ldots, l$ have to be real, and for the elliptic ones the sign of the eigenvalue is also given. In terms of the corresponding normed rotationally symmetric eigenfunctions, it follows that $\omega_{\lambda^i}(x, y) = \overline{\omega_{\lambda^i}(y, x)}$, and $|\omega_{\lambda^i}(x, y)| \leqslant 1$ for all x and y in S.

Formally we have the expansion of $K(x, y; \chi)$ in terms of the eigenfunctions F_i,

$$\sum_{M \in \Gamma} \chi(M)\, k(x, My) = \sum_i h(\lambda^i)\, F_i(x)\, \overline{F_i}'(y).^{\S} \tag{2.5}$$

The absolute convergence of the right-hand side and the equality of the two sides could be proved under suitable additional assumptions about $k(x, y)$. However, since the eigenfunctions themselves occur in (2.5), our attention here will instead centre on the trace of the integral operators, where the eigenfunctions do not anymore occur.

We may formally compute the trace of the integral operator in two ways, namely on the one hand as

$$\sum_i h(\lambda^i), \tag{2.6}$$

and on the other hand as

$$\int_{\mathscr{D}} \sigma(K(x, x; \chi))\, dx = \sum_{M \in \Gamma} \sigma(\chi(M)) \int_{\mathscr{D}} k(x, Mx)\, dx. \tag{2.7}$$

We leave aside for the moment the question whether the series (2.6) is convergent or only summable in some sense and also the

§ This formula in the case $\chi(M)$ identically 1, can be used for estimation of the number of points Mx in large regions with rotational symmetry about the point y.

question whether the sum is actually equal to the expression (2.7), and turn our attention first to the latter expression. Under our assumption on $\mathscr{D}$ and $k(x, y)$ the series on the right-hand side of (2.7) actually is absolutely convergent, even when we take absolute values under the integral signs.

We shall rearrange the series on the right-hand side of (2.7) by combining the terms in a suitable way. For this purpose we introduce some notations.

Two elements M_1 and M_2 in Γ are said to be conjugate within Γ if there exists $M_3 \in \Gamma$ such that $M_1 = M_3 M_2 M_3^{-1}$; we call the class of all elements in Γ which are conjugate to a given M the conjugate class of M in Γ, and denote it by the symbol $\{M\}_\Gamma$. The subgroup of Γ formed by the elements which commute with M we call Γ_M and denote its elements by N_M. Similarly we define conjugacy within G, and denote by $\{m\}_G$ the class of all elements in G conjugate to an m in G. Clearly $\{M\}_\Gamma$ is contained in $\{M\}_G$. Also the subgroup of G formed by the elements of G which commute with m we call G_m and denote its elements by n_m; clearly Γ_M is contained in G_M.

We now group together the terms on the right-hand side of (2.7), where M belongs to the same conjugacy class in Γ. The factor $\sigma(\chi(M))$ has the same value for all elements M belonging to the same conjugacy class in Γ. Therefore we consider the sum

$$\sum_{M \in \{M_0\}_\Gamma} \int_{\mathscr{D}} k(x, Mx)\, dx. \tag{2.8}$$

The terms here are of the form

$$\int_{\mathscr{D}} k(x, M_1^{-1} M_0 M_1 x)\, dx = \int_{\mathscr{D}} k(M_1 x, M_0 M_1 x)\, dx$$

$$= \int_{M_1 \mathscr{D}} k(x, M_0 x)\, dx,$$

with $M_1\mathscr{D}$ denoting the image of $\mathscr{D}$ under the transformation M_1. Two M_1 give the same $M_1^{-1}M_0M_1$, if and only if they differ on the left by an element of Γ_{M_0}. Thus the expression (2.8) becomes

$$\int_{\mathscr{D}_{M_0}} k(x, M_0 x)\, dx,$$

where the domain of integration is given by $\mathscr{D}_{M_0} = \sum^*_{M\in\Gamma} M\mathscr{D}$, Σ^* indicating that the summation is carried over a complete set of elements M such that no two differ on the left by an element of Γ_{M_0}. It is easily seen that $\mathscr{D}_{M_0}$ is actually a fundamental domain of the discontinuous group Γ_{M_0} in S. Thus we may rewrite the right-hand side of (2.7) as

$$\sum_{\{M\}_\Gamma} \sigma(\chi\,(M)\,) \int_{\mathscr{D}_M} k(x, Mx)\, dx, \tag{2.9}$$

where the summation is extended over one representative for each conjugacy class in Γ. We shall transform the expression

$$\int_{\mathscr{D}_M} k(x, M\, x)\, dx$$

still further. We introduce on G_M with elements n_M the Haar measure dn_M which is invariant with respect to multiplication on the right. We construct some function $p(x)$ which is everywhere on S real and non-negative, and for which

$$\int_{G_M} p(n_M x)\, dn_M = 1, \quad \text{for all } x \text{ in } S.$$

This can be done by constructing first a function $q(x) \geqslant 0$, everywhere on S, for which the integral

$$\int_{GM} q(n_M x)\, dn_M = q_1(x)$$

exists and is positive for every x in S. This can be done for instance by defining

$$q(x) = \begin{cases} 1 + \rho(x) - d(x), & \text{for } d(x) < 1 + \rho(x), \\ 0, & \text{for } d(x) \geqslant 1 + \rho(x), \end{cases}$$

where $d(x)$ denotes the smallest geodesic distance from x to some fixed point x_0, and $\rho(x) = \min_{n_M \in G_M} d(n_M\, x)$. Then $p(x) = q(x)/q_1(x)$ is seen to satisfy the above requirements. The group Γ_M acting on the right of G_M is discontinuous and we may denote by G_M/Γ_M a fundamental domain of Γ_M in G_M; we then get

$$\begin{aligned}
&\int_{\mathscr{D}_M} k(x, Mx)\, dx \\
&= \int_{G_M} \int_{\mathscr{D}_M} k(x, Mx)\, p(n_M x)\, dx\, dn_M \\
&= \sum_{N_M \in \Gamma_M} \int_{G_M/\Gamma_M} \int_{\mathscr{D}_M} k(x, Mx)\, p(n_M\, N_M\, x)\, dx\, dn_M \\
&= \sum_{N_M \in \Gamma_M} \int_{G_M/\Gamma_M} \int_{\mathscr{D}_M} k(N_M\, x, M\, N_M\, x)\, p\, (n_M\, N_M x)\, dx\, dn_M \\
&= \sum_{N_M \in \Gamma_M} \int_{G_M/\Gamma_M} \int_{N_M \mathscr{D}_M} k(x, Mx)\, p(n_M x)\, dx\, dn_M \\
&= \int_{G_M/\Gamma_M} \int_S k(x, M\, x)\, p\, (n_M x)\, dx\, dn_M \\
&= \int_{G_M/\Gamma_M} \int_S k(n_M x, M n_M x)\, p(n_M x)\, dx\, dn_M \\
&= \int_{G_M/\Gamma_M} \int_S k(x, Mx)\, p(x)\, dx\, dn_M \\
&= \int_{G_M/\Gamma_M} dn_M \int_S k(x, Mx)\, p(x)\, dx,
\end{aligned}$$

where we repeatedly have used the fact that $k(x, Mx)$ is invariant under the group G_M, that the measure dn_M is right-invariant, that the measure dx is invariant, and also that

$$G_M = \sum_{N_M \in \Gamma_M} (G_M/\Gamma_M).\, N_M \text{ and } S = \sum_{N_M \in \Gamma_M} N_M\, D_M.$$

Writing now

$$\int_{G_M/\Gamma_M} dn_M = \mu(G_M/\Gamma_M),$$

this factor measures the volume of the fundamental domain of Γ_M in G_M, and does not in any way depend on $k(x, y)$. For the other factor we write

$$\int_S k(x, Mx)\, p(x)\, dx = g(\{M\}_G),$$

and observe that this factor only depends on $k(x, y)$ and on the conjugacy class $\{M\}_G$ of M in G. Combining our results we may now write

$$\int_{\mathscr{D}} \sigma(K(x, x; \chi))\, dx = \sum_{\{M\}_\Gamma} \sigma(\chi(M))\, \mu(G_M/\Gamma_M)\, g(\{M\}_G). \quad (2.10)$$

We now turn to the question 'when and in what sense are the two expressions (2.6) and (2.10) equal?' We can at first say that the series (2.6) converges absolutely and is equal to (2.10) if $k(x, y)$ can be written in the form

$$\int_S k_1(x, z)\, \overline{k_1(y, z)}\, dz, \quad (2.11)$$

where k_1 is a point-pair invariant satisfying our conditions (a) and (b). From this we get next that the same conclusion holds if k can be written in the form

$$\int_S k_1(x, z)\, k_2(z, y)\, dz, \quad (2.11')$$

if k_1 and k_2 both satisfy the conditions (a) and (b), since (2.11′) can be written as a linear combination of expressions of the form (2.11).

Introducing the notation, for $\epsilon > 0$,

$$\kappa_\epsilon(x, y) = \begin{cases} C_\epsilon, & \text{for } d(x, y) < \epsilon, \\ 0, & \text{for } d(x, y) \geqslant \epsilon, \end{cases}$$

where $d(x, y)$ is the smallest geodesic distance between x and y, and where C_ϵ is a constant depending on ϵ, chosen such that

$$\int_S \kappa_\epsilon(x, y)\, dy = 1,$$

(the integral clearly is independent of x), $\kappa_\epsilon(x, y)$ is a point-pair invariant satisfying (a) and (b). Writing

$$\theta_\epsilon(\lambda) = \int_S \kappa_\epsilon(x, y)\, \omega_\lambda(y, x)\, dy,$$

we have

$$\lim_{\epsilon \to 0} \theta_\epsilon(\lambda) = 1,$$

and that for the λ^i, in addition $|\theta(\lambda^i)| \leqslant 1$.†

Now let $k(x. y)$ satisfy (a) and (b) and in addition be continuous; considering the class

$$k_\epsilon(x, y) = \int_S \kappa_\epsilon(x, z)\, k(z, y)\, dz,$$

for $0 < \epsilon < 1$, we get that the class k_ϵ satisfy our conditions (a) and (b) uniformly, and that $\lim_{\epsilon \to 0} k_\epsilon(x, y) = k(x, y)$, uniformly for x and y in any compact subregion of S. Using this we can show that the " trace formula "

$$\sum_i h(\lambda^i) = \sum_{\{M\}_\Gamma} \sigma(\chi\,(M))\, \mu(G_M/\Gamma_M)\, g(\{\,M\,\}_G) \tag{2.12}$$

† With equality only if $\omega_{\lambda^i}(x, y) = 1$.

is valid if we give the left-hand side the interpretation

$$\lim_{\epsilon \to 0} \sum_i h(\lambda^i)\, \theta_\epsilon(\lambda^i).$$

In particular (2.12) holds whenever k satisfies (a) and (b) and is continuous and the left-hand side of (2.12) converges absolutely.

Various types of sufficient conditions for absolute convergence can be given,‡ for instance that $K(x, y; \chi)$ have partial derivatives up to the order $[n/2] + 1$, which is the case if $k(x, y)$ has partial derivatives up to this order which are such that (2.2) can be differentiated term by term and the resulting series converges absolutely.

The trace formula (2.12) may be used on the one hand to investigate the distribution of the l-tuples λ^i and on the other hand also to investigate the distribution of the conjugate classes $\{M\}_\Gamma$, the latter in the following sense : The conjugate classes in G can be characterized by a certain number of numerical parameters and so with each $\{M\}_\Gamma$ can be associated the numerical parameters that characterize $\{M\}_G$; it is the distribution of these numerical parameters that can be investigated by means of (2.12).

We shall mention briefly a certain generalization of (2.12) which is of interest in connection with the so-called Hecke-operator for the classical modular group and their analogues.

Let us have given in connection with our group Γ and the representation $\chi(M)$, a subset Γ^* of elements M^* of G with the following properties : The set Γ^* (it does not need to be a group) and the

‡ Actually in the case of a particular G and S, the more convenient such conditions are those that can be expressed in terms of $h(\lambda)$ only. This involves expressing $k(x, y)$ in the form $\int h(\lambda)\, \omega_\lambda(x, y) d\lambda$ where $d\lambda$ is a certain measure, and seeing what properties of $h(\lambda)$ are sufficient to ensure that $k(x, y)$ is continuous and satisfies (a) and (b), then determining enough about the asymptotic distribution of the $\lambda^{(i)}$ to see what additional condition should be imposed to ensure the absolute convergence of $\sum_i h(\lambda^i)$.

elements M^*, are such that with M^* the inverse M^{*-1} is also in Γ^*, further for M^* in Γ^* and M in Γ the element M^*M is also in Γ^*, and there should be a finite set of "left-representatives" M_1^*, M_2^*, $\ldots$, M_κ^* such that $\Gamma^* = \sum_{i=1}^{\kappa} M_i^* \Gamma$, or otherwise expressed, every M^* can in a unique way be represented as $M_i^* M$ with $M \in \Gamma$. Further let there be associated with each M^* a ν by ν matrix $\chi(M^*)$ (not necessarily unitary) such that $\chi(M^*M) = \chi(M^*)\chi(M)$ for M^* in Γ^* and M in Γ, and such that

$$\chi(M^{*-1}) = \overline{\chi(M^*)}'.$$

Defining now the operator T^* by

$$T^*F(x) = \sum_{i=1}^{\kappa} \chi(M_i^*)\ F(M_i^{*-1}x), \tag{2.13}$$

one establishes that $T^*F(x)$ again satisfies (2.1). T^* is seen to be self-adjoint in our Hilbert space and further to commute with our invariant integral operators (2.3) and with the fundamental differential operators. Therefore our complete orthonormal system of eigenfunctions $F_i(x)$ may be chosen such that they are also eigenfunctions of T^*; writing then

$$T^*F_i(x) = \lambda_*^i F_i(x),$$

it can be shown by multiplying the T^* with an operator of the form (2.3), which gives us an integral operator with the kernel

$$K^*(x, y; \chi) = \sum_{M^* \in \Gamma^*} \chi(M^*)\ k(x, M^*y),$$

and computing the trace of this integral operator in a similar way that

$$\sum_i h(\lambda^i)\ \lambda_*^i = \sum_{\{M^*\}_\Gamma} \sigma(\chi(M^*))\ \mu(G_{M^*}/\Gamma_{M^*})\ g(\{M^*\}_G), \tag{2.14}$$

where the conjugacy classes $\{M^*\}_\Gamma$ are defined by conjugacy with respect to Γ (that is M_1^* and M_2^* belong to the same conjugacy class, if and only if there exists $M \in \Gamma$ such that $M_1^* = MM_2^* M^{-1}$), and Γ_{M^*} is the subgroup of Γ that commute with M^*. What was said about the validity of (2.12) also holds for (2.14).

If for some l-tuple Λ it happens that $\omega_\Lambda(x, y)$ satisfies the condition

$$\int_S |\omega_\Lambda(x, y)| \, dy < \infty,$$

then one can show that $\omega_\Lambda(x, y)$ satisfies both our conditions (a) and (b), and it can therefore be used as a $k(x, y)$ in our trace formula. Since it is seen that for the $h(\lambda)$ corresponding to $\omega_\Lambda(x, y)$ one has $h(\lambda) = 0$ for $\lambda \neq \Lambda$, and

$$h(\Lambda) = \int_S |\omega_\Lambda(x, y)|^2 \, dy,$$

we get on the left-hand side of the formula (2.12) simply $N(\Lambda)\, h(\Lambda)$, where $N(\Lambda)$ is the number of the l-tuples λ^i that are equal to Λ. I conjecture, but have only so far been able to verify this conjecture for special types of spaces, that in this case $g\,\{M\}_G$† $= 0$ for all M which do not belong to some compact subgroup of G so that (as one easily establishes) the number of terms on the right-hand side which are not zero is finite. This would imply that one gets a finite expression for $N(\Lambda)$. As will be indicated later this has interesting applications to the problem of determining the number of linearly independent regular analytic automorphic forms of a given dimension, in one or more complex variables.‡

We have so far assumed that the fundamental domain $\mathscr{D}$ of our group Γ is compact. If we relax this condition and only require that D have finite volume, the situation changes somewhat. While the kernel $K(x, y; \chi)$ will behave as before as long as at least one

† Of course the special g that is derived from $\omega_\Lambda(x, y)$.

‡ Similar remarks apply to formula (2.14), which is of interest for the theory of Hecke-operators, as applied to the analytic modular forms.

of the points x and y is restricted to a compact subregion of $\mathscr{D}$ (or of S for that matter), the kernel may exhibit a singular behavior as both points tend simultaneously towards the " non-compact boundary " of $\mathscr{D}$, such that the integral

$$\int_{\mathscr{D}} \int_{\mathscr{D}} \sigma(K(x, y; \chi) \overline{K(x, y; \chi)}') \, dx \, dy \tag{2.15}$$

does not exist. If, as it may happen for some χ, the kernels K behave well enough at the " non-compact boundary " for (2.15) to exist, the situation is not significantly changed, the spectrum of l-tuples λ for which there are eigenfunctions F is still discrete and the eigenfunctions are in our Hilbert space, and one may in specific cases by showing special care with the transformations M that leave some " part " of the " non-compact boundary " fixed (namely by grouping together those that have the same Γ_M), prove a trace formula that is not essentially different in form from (2.12), only that some terms on the right-hand side will no longer correspond to a single conjugacy class $\{M\}_\Gamma$, but to an aggregate of conjugacy classes.

If however χ is such that (2.15) does not exist, there are in general continuous spectra (which may even be multi-dimensional) besides the discrete spectrum. In some of the simpler cases, where these continuous spectra have been studied, it is possible to remove them by replacing the kernel $K(x, y; \chi)$ with a modified kernel which retains only the eigenfunctions from the discrete spectrum and with unchanged eigenvalues $h(\lambda^i)$, the computation of the trace of this modified integral operator leads then to a trace formula, which however besides terms of the type occurring on the righthand side of (2.12) will contain terms of a radically new nature.

3. We shall in the following give some explicit illustrations of the formulas in the case of some simpler spaces S and groups G satisfying our conditions.

First we consider the case when S is the hyperbolic plane for which we use the model represented by the upper complex half-plane $z = x + iy$, $y > 0$, with the metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$. Our group G may be taken as the group formed by all motions $mz = \dfrac{az + b}{cz + d}$ where $ad - bc = 1$, a, d, b, and c real.[†] The Laplacian corresponding to the metric $y^2\Delta = y^2\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$ is the only fundamental operator, and the point-pair invariants are seen to be all of the form

$$k(z, z') = k\left(\frac{|z - z'|^2}{yy'}\right).$$

A representative set of eigenfunctions is given by y^s since

$$y^2\Delta\, y^s = \lambda\, y^s$$

with $\lambda = -s(1 - s)$. Writing $s = \frac{1}{2} + ir$, we shall use for convenience the r instead of the λ as parameter. The connection between $k(z, z')$ and $h(r)$ is given by the relations

$$\left.\begin{aligned} &\int_w^\infty \frac{k(t)}{\sqrt{(t - w)}}\, dt = Q(w), \quad k(t) = -\frac{1}{\pi}\int_t^\infty \frac{dQ(w)}{\sqrt{(w - t)}}, \\ &Q(e^u + e^{-u} - 2) = g(u), \\ &h(r) = \int_{-\infty}^{\infty} e^{iru}\, g(u)\, du, \quad g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru}\, h(r)\, dr. \end{aligned}\right\} \quad (3.1)$$

Regarding now $h(r)$ as the primary function, we see that if $h(r)$ satisfies the conditions[§]:

(1) $h(r) = h(-r)$,

(2) $h(r)$ is regular analytic in a strip $|\operatorname{Im} r| < \frac{1}{2} + \epsilon$, where $\epsilon > 0$,

and

§ This is not the full group of isometries, since this also contains the elements $\dfrac{a\bar{z} + b}{c\bar{z} + d}$, with $ad - bc = -1$. However, we shall for simplicity assume that our discontinuous group Γ has only true motions as elements.

§ The conditions (2) and (3) could be somewhat weakened.

(3) $h(r) = O((1 + |r|^2)^{-1-\epsilon}$ in this strip;

then k will exist and satisfy our conditions (a) and (b).

The elements of G can, as it is well known, be divided into four types, of which the first consists only of the identity element, while the others are respectively the hyperbolic, the elliptic and the parabolic elements. For a hyperbolic element m there is always a representative of the conjugacy class $\{m\}_G$ of the form $z \to \rho z$, where ρ is real and > 1. We call ρ the norm of m, and also the norm of the hyperbolic conjugacy class $\{m\}_G$ and denote it by $N\{m\}$, leaving the subscript G out. An elliptic element has always one (and only one) fixed point in the space and represents a rotation of the plane around this point, by an angle which we may count positive in the counter-clockwise direction; we call this the rotation angle of the elliptic element and also of the elliptic conjugacy class in G represented by the element. Finally if an element is parabolic it belongs to one of the two parabolic conjugacy classes represented by $z \to z + 1$ and $z \to z - 1$ respectively.

In Γ we shall call a hyperbolic element P primitive, if it is not a power with exponent > 1 of any other element in the group Γ, correspondingly we say that the conjugacy class $\{P\}_\Gamma$ is primitive. For the elliptic elements of Γ, those with the same fixed point form a finite group generated by a single element, and the one that has the smallest positive rotation angle we call primitive and denote it by R and call the corresponding class a primitive elliptic conjugacy class $\{R\}_\Gamma$ in Γ. Finally a parabolic element of Γ which is not a power with exponent > 1 of any other element in Γ, and which belongs to the first of the two parabolic conjugacy classes in G, we call a primitive parabolic element of Γ and, denoting it by S, the corresponding class $\{S\}_\Gamma$ a primitive parabolic class. It should be mentioned that if the area of the fundamental domain $\mathscr{D}$ of Γ is finite, that is to say

$$A(\mathscr{D}) = \int_{\mathscr{D}} \frac{dx\,dy}{y^2} < \infty,$$

there are only a finite number of elliptic and primitive parabolic conjugacy classes in Γ, and if $\mathscr{D}$ is compact there are no parabolic ones. The primitive hyperbolic classes $\{P\}_\Gamma$ on the other hand are always present in infinite number.

Assuming first that $\mathscr{D}$ is compact, the trace formula takes the form

$$\sum_i h(r_i) = \frac{A(\mathscr{D})\,\nu}{2\pi} \int_{-\infty}^{\infty} r\,\frac{e^{\pi r} - e^{-\pi r}}{e^{\pi r} + e^{-\pi r}}\, h(r)\, dr +$$

$$+ \sum_{\{R\}_\Gamma} \sum_{k=1}^{m-1} \frac{\sigma(\chi^k(R))}{M \sin k\pi/m} \int_{-\infty}^{\infty} \frac{e^{-2\pi rk/m}}{1 + e^{2-\pi r}}\, h(r)\, dr +$$

$$+ 2 \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \frac{\sigma(\chi^k(P)) \log N\{P\}}{(N\{P\})^{k/2} - (N\{P\})^{-k/2}}\, g(k \log N\{P\}). \quad (3.2)$$

Here the r_i are the values for which there is a solution of the equation

$$y^2 \Delta F(z) = \lambda F(z), \quad \lambda = -\left(\tfrac{1}{4} + r^2\right)$$

with $F(z)$ in our Hilbert space ; since we count both values of r that give the same λ (and if $\lambda = -\tfrac{1}{4}$, $r = 0$ with double multiplicity) our formula actually represents twice the trace of the integral operator. $A(\mathscr{D})$ is the area of the fundamental domain. $m = m(R)$ represents the order of the primitive elliptic element R, and the summations $\sum_R$ and $\sum_P$ are taken over one representative from each primitive elliptic and each primitive hyperbolic class respectively. The r_i have to be such that $\tfrac{1}{4} + r^2$ is real and non-negative, so that the r_i are either real, or they are purely imaginary with absolute value $\leqslant \tfrac{1}{2}$.† The formula (3.2) can now on the one hand be used for determining the asymptotic distribution of the r_i, and on the other hand the asymptotic distribution of the norms of the primitive hyperbolic classes in Γ. Under our assumptions on $h(r)$ all infinite series occurring in (3.2) converge absolutely.

† These latter could of course only occur in finite number, but one can show that their number for suitable Γ and χ may become arbitrarily large, although it can be shown to be less than a certain constant times $\nu\, A(\mathscr{D})$.

(3.2) has a rather striking analogy to certain formulas that arise in analytic number theory from the zeta - and L-functions of algebraic number fields. This leads us to introduce the function defined by

$$Z_\Gamma(s;\chi) = \prod_{\{P\}_\Gamma} \prod_{k=0}^{\infty} | E_\nu - \chi(P)\,(N\{P\})^{-s-k} | \, , \qquad (3.3)$$

for real part of $s > 1$, when the product converges absolutely. E_ν is here the ν by ν identity matrix, and $| \dots |$ denotes the determinant.

From (3.2) one can derive the following facts about this analytic function of s :

(A). $Z_\Gamma(s;\chi)$ is an integral function of s of order 2, except in the case when the genus of the fundamental domain $\mathscr{D}$ is zero, in this case there may be a pole at $s = 0$ of order at most ν.‡

(B). $Z_\Gamma(s;\chi)$ has "trivial" zeros at the integers $-k$ for $k \geqslant 0$, whose multiplicity can be explicitly given in terms of k, ν, $A(\mathscr{D})$ (or the genus of the fundamental domain if one prefers), and the $m(R)$, the orders of the primitive elliptic classes, and the traces $\sigma(\chi^i(R))$ for $i = 1, 2, \dots m(R) - 1$. In the particular case that there are no elliptic classes in Γ one has that the multiplicity of the trivial zero at $-k$ is $(2k+1)(2p-2)$, where p is the genus§ (which is in this case always > 1).

(C). $Z_\Gamma(s;\chi)$ satisfies a functional equation which relates the value of $Z_\Gamma(1-s;\chi)$ to that of $Z_\Gamma(s;\chi)$. The form of this functional equation depends on the quantities ν, $A(\mathscr{D})$, and the orders $m(R)$ of the primitive elliptic classes and the traces $\sigma(\chi^i(R))$ for $i = 1, 2, \dots, m(R) - 1$. In the particular case that there are no elliptic classes in Γ this functional equation has the form

$$Z_\Gamma(1-s;\chi) = Z_\Gamma(s;\chi) \exp \left\{ -\nu A(\mathscr{D}) \int_0^{s-\frac{1}{2}} v \operatorname{tg} \pi v \, dv \right\} . \qquad (3.4)$$

‡ If one assumes the representation $\chi(M)$ to be irreducible, this pole only occurs for $\chi(M)$ identically equal to 1, and is then a simple pole.

§ In this particular case $p-1 = \frac{A(D)}{4\pi}$.

(D). The zeros of $Z_\Gamma(s, \chi)$ which are not mentioned under (B), are the numbers $\frac{1}{2} + ir_i$, and have thus real part equal to $\frac{1}{2}$, with the possible exception of a finite number of zeros that are real and lie in the interval $0 \leqslant s \leqslant 1$.

As one sees from (D) the analog of the Riemann hypothesis is true for our $Z_\Gamma(s; \chi)$ with the slight modification that real zeros may occur in the interval $0 \leqslant s \leqslant 1$.

If we only require $A(\mathscr{D})$ to be finite, there will, if $\mathscr{D}$ is not compact, always be at least one primitive parabolic class $\{S\}_\Gamma$. If $\{S_i\}_\Gamma$ for $i = 1, 2, \ldots, \kappa$, are the different primitive parabolic classes in Γ, the situation will depend on the matrices $\chi(S_i)$; if $\chi(S_i)$ has μ_i eigenvalues equal to 1, we say that χ is singular of degree μ_i with respect to the class $\{S_i\}_\Gamma$, and singular of degree $\mu = \sum_{i=1}^{\kappa} \mu_i$ with respect to Γ. If $\mu = 0$, that is if χ is non-singular with respect to Γ, the situation is only slightly altered from the compact case. The spectrum is still discrete and in our trace-formula (3.2), will occur on the right-hand side the new term

$$- 2\, g(0) \sum_{i=1}^{\kappa} \log \| E_\nu - \chi(S_i) \| . \tag{3.5}$$

This new term does not essentially alter the statements (A), (B), (C) and (D) about $Z_\Gamma(s; \chi)$. If $\mu \geqslant 1$ however the situation is very much altered, in that we have then for our eigenvalue problem, besides the discrete spectrum, also a continuous spectrum of multiplicity μ. As mentioned in the previous section we have then first to investigate the eigenfunctions in the continuous spectra and then to remove their contribution to the kernel K and develop a trace formula for the modified kernel. As a description of the general case is rather complicated, we shall here only briefly indicate the results in the simplest case when there is only one parabolic class $\{S\}_\Gamma$ with respect to which χ is singular, and further that χ is one dimensional, so that $\chi(S) = 1$.

We may assume for simplicity that one representative S of the class $\{S\}_\Gamma$ is $Sz = z + 1$. Forming for real part of s greater than 1 the function

$$\left.\begin{aligned} E(z, s; \chi) = \sum_{M \in S/\Gamma} \overline{\chi(M)}\, (\operatorname{Im} M z)^s &= \sum_{M \in S/\Gamma} \overline{\chi(M)}\, \frac{y^s}{|cz + d|^{2s}} \\ M z &= \frac{a z + b}{c z + b}, \end{aligned}\right\} \tag{3.6}$$

where $M \in S/\Gamma$ means that M runs over a complete set of elements of Γ that do not differ by a power of S on the left, one establishes that this series is absolutely convergent for $\sigma > 1$, $s = \sigma + it$. Further one has

$$E(Mz, s; \chi) = \chi(M)\, E(z, s; \chi),$$

for M in Γ, and

$$y^2 \Delta\, E(z, s; \chi) = -\, s(1 - s)\, E(z, s; \chi).$$

It can then be proved that $E(z, s; \chi)$ is a meromorphic function of s in the whole s-plane, and that the poles are all in the region $\sigma < \frac{1}{2}$, with the possible exception of a finite number of simple poles which are real and lie in the interval $\frac{1}{2} < s \leqslant 1$; these poles are independent of z, and $E(z, s; \chi)$ may be written as a quotient of two integral functions in s, each of which is at most of order 2 and where the denominator is independent of z. Further $E(z, s; \chi)$ satisfies a functional equation, which may be described as follows :

We write for $\sigma > 1$,

$$\phi(s, \chi) = \frac{\pi^{\frac{1}{2}}\, \Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c \neq 0} \sum_{0 \leqslant d < |c|} \frac{\overline{\chi(M)}}{|c|^{2s}}; \tag{3.7}$$

then one can show that $\phi(s, \chi)$ is meromorphic in the whole s-plane and regular for $\sigma \geqslant \frac{1}{2}$ with the possible exception of a finite number of simple poles in the interval $\frac{1}{2} < s \leqslant 1$, and can be written as a quotient of two integral functions at most of order 2. Further one has the functional equation

$$\phi(s,\ \chi)\ \phi(1-s,\ \chi) = 1.\dagger \tag{3.8}$$

Then the functional equation of $E(z, s; \chi)$ is

$$E(z, s; \chi) = \phi(s, \chi)\ E(z, 1-s; \chi). \tag{3.9}$$

Forming now the kernel

$$H(z, z'; \chi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\, E(z, \tfrac{1}{2} + ir; \chi)\ \overline{E(z', \tfrac{1}{2} + ir; \chi)}\, dr, \tag{3.10}$$

one can show that the kernel

$$K^*(z, z'; \chi) = K(z, z'; \chi) - H(z, z'; \chi), \tag{3.11}$$

where

$$K(z, z'; \chi) = \sum_{M\in\Gamma} \chi(M)\, k(z, Mz'),$$

has the property that it retains only the discrete spectrum (that is all eigenfunctions which are not in our Hilbert space are annihilated by the integral operator with kernel K^*), and this is retained with unchanged eigenvalues $h(r_i)$. The evaluation of the trace of this modified integral operator then gives us a trace formula which differs from the earlier in that on the right-hand side we have the new terms

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)\, \frac{\phi'}{\phi}\, (\tfrac{1}{2} + ir, \chi)\, dr - \frac{1}{\pi} \int_{-\infty}^{\infty} h(r)\, \frac{\Gamma'}{\Gamma}\, (1 + ir)\, dr -$$

$$- 2 \log 2.\ g(0) + \tfrac{1}{2}\, (1 - \phi(\tfrac{1}{2}, \chi))\, h(0). \tag{3.12}$$

These new terms make a rather drastic change in the nature of $Z_\Gamma(s, \chi)$, in particular $Z_\Gamma(s; \chi)$ will have simple poles at $s = -1/2$, $-3/2, -5/2, \ldots$; because of the second term in (3.12), the last term produces a simple pole at $s = \frac{1}{2}$ if $\phi(\frac{1}{2}, \chi) = -1$ (this pole may however be cancelled by a zero if one or more of the r_i equals zero), so that $Z_\Gamma(s; \chi)$ is no longer an integral function. Furthermore in addition to the non-trivial zeros at the points $\frac{1}{2} + ir_i$, namely wherever $\phi(s, \chi)$ has a pole in the region $\sigma < \frac{1}{2}$, $Z_\Gamma(s, \chi)$ will have a

† Since the coefficients of the Dirichlet series of (3.7) are actually real, this implies $|\phi(\frac{1}{2} + ir, \chi)| = 1$.

a zero of the same multiplicity. The functional equation is also correspondingly modified in that besides simple factors also the function $\phi(s, \chi)$ occurs in it.

In the general case one has a system of μ series like (3.6), with similar properties : when s is replaced by $1 - s$, this system transforms by a matrix $\phi(s, \chi)$ whose elements are of a similar nature as (3.7) ; the determinant of this matrix will essentially then play the role that $\phi(s, \chi)$ does in the former case.

In the 3-dimensional hyperbolic space, the situation is similar but in some respects simpler. One can introduce also there a $Z_\Gamma(s; \chi)$‡ which although it will be a function of order 3, has a functional equation which is essentially simpler than in the case of the hyperbolic plane. For general n-dimensional hyperbolic space the explicit computations are somewhat complicated by the fact that the groups Γ_M and G_M now may not always be abelian when M is different from the identity element; this complicates the form of the trace formula, which however is always in a certain sense simpler when n is odd than when n is even. The non-compact case with finite volume of $\mathscr{D}$ can in all these cases be treated satisfactorily.

For groups acting simultaneously on the product of a finite number of such spaces,§ the situation can also be handled even in the non-compact case as long as the " non-compact boundaries " of $\mathscr{D}$ are point-like.

For other higher dimensional spaces, as for instance the space of positive definite, n by n symmetric matrices with determinant 1, the situation, for $n > 2$, is not so simple. The continuous spectra that may occur in the non-compact case at present cannot be handled properly. One will also here try to obtain them by analytic continuation of certain Dirichlet series, like we did for the hyperbolic plane; only these Dirichlet series are more complicated and in the case of spectra that have a dimension > 1, they are Dirichlet series in several

‡ Defined by a somewhat more complicated product than (3.3).

§ Like the so-called Hilbert group acting on a product of hyperbolic planes.

variables. This problem of analytic continuation cannot be handled at present, except for special groups that arise from arithmetic, where one may be able to utilize this to effect the continuation. As an example could be mentioned the case when $n = 3$ for the above space, and the group Γ is the group of 3 by 3 matrices with determinant 1 and integral rational elements, and χ identical to 1; when one is led to consider the series

$$\zeta_Y(s, s') = \sum_{X'Z=0} (X' \, Y \, X)^{-s} \, (Z' \, Y^{-1} \, Z)^{-s'},$$

where the summation is carried over all pairs of column vectors X and Z, with integral rational components which satisfy the conditions $X'X > 0$, $Z'Z > 0$, $X'Z = 0$. The series converges absolutely for $\sigma > 1, \sigma' > 1$, where $s = \sigma + it$, $s' = \sigma' + it'$. One can in this case show that

$$(s - 1) \; (s' - 1) \; (s + s' - 3/2) \; \zeta(2s + 2s' - 1) \; \zeta_Y(s, s'),$$

where $\zeta(2s + 2s' - 1)$ is the ordinary Riemann zeta-function, is an integral function in the two complex arguments s and s'. Further if one writes

$$\xi_Y(s, s') = \pi^{-2s-2s'} \, \Gamma(s) \, \Gamma(s') \, \Gamma(s + s' - \tfrac{1}{2}) \; \zeta(2s + 2s' - 1) \; \zeta_Y(s, s'),$$

then the function $\xi_Y(s, s')$ remains invariant by replacing (s, s') by any of the following pairs of complex arguments $(s + s' - 1/2, 1 - s')$, $(1 - s, s + s' - 1/2)$, $(3/2 - s - s', s)$, $(s', 3/2 - s - s')$ and $(1 - s', 1 - s)$, so that it has a larger number of functional equations than the zeta-functions in one variable. It should be noted that the group under which $\xi_Y(s, s')$ is invariant is isomorphic to the permutation group of 3 elements, as the three quantities $4s + 2s' - 3$, $2s' - 2s$, $-4s' - 2s + 3$, undergo permutations. $\xi_Y(1/2 + it, 1/2 + it')$ is here connected with the two-dimensional continuous spectrum. Besides this there is a denumerably infinite sequence of Dirichlet series in one complex variable that are connected with one-dimensional continuous spectra.

Similar Dirichlet series in up to $(n - 1)$ complex variables, can be defined for general n, by looking at the definite forms in $(n - 1)$

variables that can be represented by the quadratic form with matrix Y, then the $(n-2)$ forms that can be represented by the $(n-1)$ form, and so on down to a form in one variable, and forming a product of the determinants of the $(n-1)$, $(n-2)$, ..., 1 form raised to complex exponents $-s_{n-1}$, $-s_{n-2}$, ..., $-s_1$ respectively, and summing over all such " descending " series of forms that are inequivalent in a certain sense. In the case $n = 3$ this would lead to a function which differs only by a simple factor (which is independent of Y) from $\zeta_Y(s_1, s_2)$ as it was defined above. The general study of these series has not yet been undertaken, but it is conceivable that it may prove of value for the theory of quadratic forms.

4. We shall finally give some applications to more classical problems. We go back to the hyperbolic plane $z = x + iy$, $y > 0$, and add a third coordinate ϕ, where we will identify ϕ and $\phi + 2\pi$. On this space consisting now of points (z, ϕ), we take the following group G with elements m_α, where m is a real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant 1, and α a real number, and let it act on the space (z, ϕ) in the way that

$$m_\alpha(z, \phi) = \left(\frac{a\,z + b}{c\,z + d},\ \phi + \arg\,(cz + d) + \alpha\right).$$

Further we define μ such that

$$\mu(z, \phi) = (-\,\bar{z}, -\,\phi).$$

One then establishes that the two differential forms

$$\frac{dx^2 + dy^2}{y^2} \text{ and } d\phi - \frac{dx}{2y},$$

have the property that they both are invariant under G, the first one is also invariant under μ whereas the second only changes sign. We may therefore take for instance

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + \left(d\phi - \frac{dx}{2y}\right)^2$$

as our invariant metric. We have two fundamental differential operators in this case, namely $\frac{\partial}{\partial \phi}$ and the Laplacian derived from the metric. The point-pair invariants are seen to be all functions of the two real arguments

$$\frac{|z-\bar{z}'|^2}{4\,yy'} \text{ and } \phi - \phi' + \arg \frac{z-\bar{z}'}{2i},$$

where (z, ϕ) and (z', ϕ') are the two points.

If we now have a group Γ which is discontinuous in the hyperbolic plane, it is seen that the group $\bar{\Gamma}$ obtained from Γ by, for each transformation $Mz = \frac{az+b}{cz+d}$ in Γ, counting both $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $-M = \begin{pmatrix} -a, & -b \\ -c, & -d \end{pmatrix}$ as different elements of $\bar{\Gamma}$, has the property that when $\bar{\Gamma}$ acts on our space (z, ϕ) in the way

$$M(z, \phi) = \left(\frac{a\,z+b}{c\,z+d},\ \phi + \arg\,(c\,z+d)\right),$$

the group $\bar{\Gamma}$ is discontinuous in this space, and if the fundamental domain $\mathscr{D}$ in the hyperbolic plane of Γ is compact, then so is the fundamental domain $\bar{\mathscr{D}}$ of $\bar{\Gamma}$, and if $\mathscr{D}$ has finite area $\bar{\mathscr{D}}$ has finite volume. The converse is also true.

Similarly a representation χ of Γ can be extended to $\bar{\Gamma}$ by letting both M and $-M$ correspond to the same χ as the transformation Mz in Γ. There may however also be other representations χ of $\bar{\Gamma}$ where the two elements M and $-M$ correspond to different χ.

If we have such a representation $\chi(M)$ of $\bar{\Gamma}$ one now sees that the eigenfunctions of our operators, because of the presence of the fundamental operator $\frac{\partial}{\partial \phi}$ and the identification of (z, ϕ) and $(z, \phi + 2\pi)$ must be of the form $e^{-ik\phi}$ times a function of the point z, where k is an integer. The eigenfunctions $F(z, \phi)$ which satisfy the relation

$$F(M(z, \phi)) = \chi(M)\ F(z, \phi)$$

can therefore be written in the form

$$F(z, \phi) = y^{k/2}\ F(z)\ e^{-ik\phi}, \tag{4.1}$$

and the former relation takes the form

$$F(Mz) = \chi(M)\ (cz + d)^k\ F(z). \tag{4.2}$$

Thus we see that we have the same type of transformation law as (in the case of one-dimensional χ) is known from the theory of the analytic automorphic forms.†

Instead of studying the general eigenfunction, and the general form of the trace-formula, which can be carried out without serious difficulties, we shall here only study a particular type that is associated with certain eigenfunctions with rotational symmetry which have the property that they satisfy our conditions (a) and (b) and so can be used as point-pair invariants in forming our kernels K.

It can be established that

$$\omega_k\ (z, \phi\,; z', \phi') = \frac{(y\ y')^{k/2}}{\{(z - \bar{z}')/2i\}^k}\, e^{-ik(\phi - \phi')}, \tag{4.3}$$

for any integer k is an eigenfunction in (z, ϕ) which is a point-pair invariant in the two points (z, ϕ) and (z', ϕ'); further that for $k > 2$ the conditions (a) and (b) are satisfied. As a consequence the integral operator with the kernel

$$K_k\ (z, \phi\,; z', \phi'\,; \chi) = \sum_{M \in \bar{\Gamma}} \chi(M)\ \omega_k\ \{z, \phi\,; M(z', \phi')\}, \tag{4.4}$$

can be shown to have only eigenfunctions§ of the form (4.1) where $F(z)$ is an analytic function of z regular in the interior of the upper half plane and satisfying the condition that

† We get here only integral dimensions, k; if one wants to study arbitrary real dimension, one has to give up the identification $(z, \phi) = (z, \phi + 2\pi)$, also $\bar{\Gamma}$ has to be defined in a different way.

§ That is corresponding to an eigenvalue different from zero.

$$y^{k/2} F(z)$$

is uniformly bounded throughout this region, and every such eigenfunction corresponds to the same eigenvalue given by

$$\int_S |\omega_k (z, \phi ; z', \phi')|^2 \, d(z, \phi)$$

where the integral is taken over our whole (z, ϕ)-space and $d(z, \phi)$ is the invariant element of volume. The trace-formula for this particular kernel gives us then the number N_k of regular analytic forms $F(z)$ satisfying (4.2)‡ as a finite expression depending on k, ν, the area $A(\mathscr{D})$ of the fundamental domain, the elliptic primitive classes $\{R\}_\Gamma$ and the eigenvalues of the χ's that correspond to them, and the primitive parabolic classes and the eigenvalues of the χ's that correspond to them. The hyperbolic classes give no contribution at all. For $k = 2$ it is possible to obtain a similar result by replacing ω_2 with $\omega_2 \left(\frac{(y\, y')^{1/2}}{|(z - \bar{z}')/2i\,|} \right)^\delta$ where $\delta > 0$, and in the trace formula for this kernel letting δ tend to zero.

If we consider the classical modular group Γ with elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc = 1$, and a, d, b, c are rational integers, and the representation χ identical to 1, Hecke has introduced certain operators T_n for each positive integer n and studied their action on the regular modular forms, in connection with his theory about Dirichlet series with functional equations (of a certain type) and Euler products.

These T_n are of the type (2.13), associated with the set of transformations $M^{(n)} = \frac{1}{\sqrt{n}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are rational integers with $ad - bc = n$, in the way that T^* was associated with the set M^*. The generalized trace formula (2.14) gives then applied to the

‡ If D is not compact, but has finite area, the condition that $y^{k/2} F(z)$ is uniformly bounded, implies that we are only counting the so-called cusp-forms here.

point-pair invariant eigenfunction ω_k for $k > 2$,[§] the following formula for the trace of the Hecke operator T_n acting on the space of cusp-forms of dimension $-k$.

$$\sigma_k(T_n) = -\frac{1}{2} \sum_{-2\sqrt{n}<m<2\sqrt{n}} H(4n - m^2) \frac{\eta_m^{k-1} - \bar{\eta}_m^{k-1}}{\eta_m - \bar{\eta}_m} - \sum_{\substack{d/n \\ d \leqslant \sqrt{n}}}' d^{k-1} + \delta(\sqrt{n}) \frac{k-1}{12} n^{k/2-1}. \quad (4.5)$$

Here the $H(d)$ denotes the number of inequivalent positive definite forms $ax^2 + bxy + cy^2$ with $4ac - b^2 = d$, counted in the usual way that a form equivalent to $a(x^2 + y^2)$ is counted with the weight $\frac{1}{2}$ and one equivalent to $a(x^2 + xy + y^2)$ with weight $1/3$. Further

$$\eta_m = \frac{m + i\,(4n - m^2)^{\frac{1}{2}}}{2} .$$

$\delta(x)$ is defined as 1 if x is an integer and zero otherwise, and Σ' means that if $d = \sqrt{n}$ the corresponding term is counted with weight $\frac{1}{2}$. For $k = 2$ one can again by a limit process arrive at a similar formula which however will contain one new term, and turns out (since there are no cusp forms of dimension -2 for the modular group, so that $\sigma_2(T_n) = 0$), to be identical with the so-called Kronecker class number relation. For $k = 4$, 6, 8, 10 and 14 there are again no cusp forms, so that the left-hand side of (4.5) is zero, which gives five new class number relations, while for $k = 12$, for instance the left-hand side is identical to the number theoretical function $\tau(n)$ of Ramanujan, so that one gets an explicit (admittedly rather complicated) formula for this. While the results about the number of regular analytic forms of a given dimension $-k$ and representation χ of $\bar{\Gamma}$ are classical[†] and previously were derived from the Riemann-Roch formula, the evaluation of the trace of

§ k will here be even, since with χ identical to one there are no non-vanishing functions satisfying (4.2), for k odd, since the left-hand side remains the same by replacing M by $-M$ whereas the right-hand side changes sign.

† Although as far as I know only the case of one-dimensional χ occurs in the literature.

the Hecke operator has not yet been accomplished by other means. From our point of view these expressions are finite elementary cases of the general trace formulas (2.12) and (2.14).

It is of interest to note that the method sketched above carries immediately over to the analytic automorphic forms in higher dimensional spaces, as for instance a product space formed by a finite number of hyperbolic planes or the general symplectic space, which can all be handled in a similar way without any essential difficulties occurring as long as the discontinuous group Γ has compact fundamental domain. For the symplectic space for instance, one can introduce in a similar way as before a space (Z, ϕ) and define the group G acting on the space with elements M_α, where the symplectic matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $MZ = (AZ + B)\,(CZ + D)^{-1}$, and we define

$$M_\alpha(Z, \phi) = (MZ,\ \phi + \arg |CZ + D| + \alpha),$$

and as before

$$\mu(Z, \phi) = (-\bar{Z}, -\phi).$$

In this space again the point-pair invariant of the two points (Z, ϕ) and (Z^*, ϕ^*) which has the form

$$\omega_k(Z, \phi;\ Z^*, \phi^*) = \frac{|Y|^{k/2}\,|Y^*|^{k/2}}{\left|\dfrac{Z - \bar{Z}^*}{2i}\right|^{k}}\, e^{-ik(\phi - \phi^*)},$$

where $Z = X + iY$, is an eigenfunction for every integer k and for k positive and large enough‡ it will again satisfy our requirements (a) and (b).

We shall finally briefly indicate the most general result that we at present can obtain along these lines. Let there be in our space S a sequence of l-tuples $\lambda^{(k)}$, $k = 1, 2, 3 \ldots$, with the property that we have the relation

‡ If we consider the symplectic space of dimension $n^2 + n$, this takes place for $k > 2n$.

$$\omega_{\lambda(k)}(x, y) = (\omega_{\lambda(1)}(x, y))^k$$

for all positive integers k, where as before $\omega_\lambda(x, y)$ denotes the eigenfunction in x that corresponds to the l-tuple λ, and has rotational symmetry around the point y and is normed so as to take the value 1 for $x = y$. Further assume that for k sufficiently large and positive,

$$\int_S |\omega_{\lambda(k)}(x, y)| \, dy < \infty;$$

then $\omega_{\lambda(k)}(x, y)$ can be seen to satisfy both conditions (a) and (b). If we now have a discontinuous group Γ whose fundamental domain is compact, with a representation by unitary matrices χ, in our space, and denote the number of eigenfunctions corresponding to the eigenvalue $\lambda^{(k)}$ by N_k, then one can show that for k sufficiently large, N_k is given by a finite expression,

$$N_k = P_0(k) + \Sigma\, \epsilon_i^k P_i(k), \qquad (4.6)$$

where P_0 is a polynomial and the P_i certain polynomials in general of lower degree† and the ϵ_i are certain roots of unity, such that if q_i is the smallest positive integer for which $\epsilon_i^{q_i} = 1$, the number q_i divides the order of some element§ in Γ which is of finite order.

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† The only case when some of them can be of the same degree as P_0 is when Γ contains other elements than the identity which commute with the whole group G.

§ Different from the identity.

THE ZETA-FUNCTIONS OF ALGEBRAIC CURVES AND VARIETIES

By M. DEURING

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1. Let k be a field with a finite number q of elements and K a field of algebraic functions of one variable over k as field of constants. The zeta function $\zeta(s,K)$ of K is defined by

$$\zeta(s, K) = \sum_{\mathfrak{a}} N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}, \tag{1}$$

where $\mathfrak{a}$ runs over all integral divisors, and $\mathfrak{p}$ over all prime divisors of K; the norm $N\mathfrak{a}$ denotes the number of residue classes modulo $\mathfrak{a}$, i.e.

$$N\mathfrak{a} = q^{\deg \mathfrak{a}}.$$

The series and the product in (1) converge absolutely for Re $s > 1$. $\zeta(s, K)$ is meromorphic; more precisely

$$\zeta(s, K) = \frac{1}{1 - q^{-s}} \frac{1}{1 - q^{1-s}} \prod_{\nu=1}^{2g} (1 - \pi_\nu q^{-s}), \tag{2}$$

where g denotes the genus of K,

$$P(X) = \prod_{\nu=1}^{2g} (1 - \pi_\nu X) \tag{3}$$

is a polynomial with rational integral coefficients, so that the π_ν's are integral algebraic numbers. $\zeta(s,K)$ satisfies the functional equation

$$q^{(1-g)s} \zeta(s, K) = q^{(1-g)(1-s)} \zeta(1 - s, K). \tag{4}$$

(4), together with (2) and (3), is a consequence of the Riemann-Roch theorem for the field K.

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If we assume an expression of the type (2) for $\zeta(s, K)$, the functional equation (4) reduces to

$$\pi_\nu \pi_{2g+1-\nu} = q; \quad \nu = 1,2,\ldots, 2g, \tag{5}$$

(if the π_ν's are arranged in a suitable order).

$\zeta(s,K)$ has poles of order 1 at

$$s_\nu = \frac{2\pi i\nu}{\log q}, \quad \nu = 0, \pm 1, \pm 2,\ldots$$

and

$$s_\nu' = 1 + \frac{2\pi i\nu}{\log q}, \quad \nu = 0, \pm 1, \pm 2,\ldots.$$

The zeros of $\zeta(s, K)$ are

$$s_{\mu,\nu} = -\frac{\log \pi_\mu}{\log q} + \frac{2\pi i\nu}{\log q}, \quad \nu = 0, \pm 1, \pm 2,\ldots. \tag{6}$$

All zeros have the real part equal to $\frac{1}{2}$, or

$$|\pi_\nu| = \sqrt{q}. \tag{7}$$

This analogue of the Riemann hypothesis has been proved by Hasse for $g = 1$ and by Weil for general g.

2. We introduce $U = q^{-s}$ as an independent variable instead of s and write $Z(U, K) = \zeta(s, K)$, so that (2) becomes

$$Z(U,K) = \frac{\prod_{\nu=1}^{2g} (1 - \pi_\nu U)}{(1 - U)(1 - qU)} \tag{2'}$$

and (4) becomes

$$Z\left(\frac{1}{qU}, K\right) = q^{1-g}\, U^{2(1-g)}\, Z(U,K). \tag{4'}$$

From the definition (1) of $\zeta(s, K)$ one derives easily the following power series for the logarithmic derivative of $Z(U, K)$:

$$\frac{Z'}{Z}(U,K) = \sum_{f=1}^{\infty} N_f\, U^{f-1}, \tag{8}$$

where N_f is the number of prime divisors of degree f in the extended field $K_f = Kk_f$, k_f denoting the extension of degree f of k. The radius of convergence of (8) is q^{-1}.

Introducing (2′) into (8), we obtain formulae for the numbers N_f:

$$N_f = q^f + 1 - \sum_{\nu=1}^{2g} \pi_\nu^f. \tag{9}$$

3. Let C be a curve without singularities, defining K over k. Then N_f is the number of points of C which are rational over k_f. This remark leads to Weil's definition of the zeta function $\zeta(s, V) = Z(U, V)$ of an absolutely irreducible algebraic variety V without singular points over k.

Let N_f denote the number of points of V which are rational over k_f. Then $\zeta(s, V) = Z(U, V)$ is defined up to a constant factor by

$$\frac{Z'}{Z}(U, V) = \sum_{f=1}^{\infty} N_f U^{f-1}, \ U = q^{-s}. \tag{10}$$

Weil stated the following conjectures, which are generalizations of the theorems on curves outlined in § 1.

Let d be the dimension of V. Then the power series defining $Z(U, V)$ converges for $|U| < q^{-d}$, so that $Z(U, V)$ is holomorphic and nowhere zero in $|U| < q^{-d}$. The constant factor in $Z(U, V)$ can be fixed by taking $Z(0, V) = 1$.

$Z(U, V)$ is a rational function of U, which we write as the quotient of two coprime polynomials in the form

$$Z(U, V) = \frac{\prod_{i=1}^{N} (1 - \alpha_i U)}{\prod_{j=1}^{M} (1 - \beta_j U)}.$$

$Z(U, V)$ satisfies a functional equation of the type

$$Z\left(\frac{1}{q^d U}, V\right) = c\, U^{\varkappa} Z(U, V), \tag{11}$$

with certain constants c and χ. This is equivalent to the following set of equations

$$\left.\begin{aligned} \alpha_i\,\alpha_{N+1-i} &= q^d, \quad i = 1, 2, \ldots, N, \\ \beta_j\,\beta_{M+1-j} &= q^d, \quad j = 1, 2, \ldots, M, \end{aligned}\right\} \tag{12}$$

if the α's and the β's are arranged in a suitable order. Furthermore, we must have of necessity

$$\chi = M - N \quad \text{and} \quad c = \pm q^{\frac{1}{2}d\chi}, \tag{13}$$

so that the functional equation can be written more precisely in the form

$$Z\left(\frac{1}{q^d U}, V\right) = \pm q^{\frac{1}{2}d\chi}\, U^\chi Z(U, V), \tag{14}$$

the $-$ sign being possible only if at least one of the numbers N and M is odd.

$Z(U, V)$ has poles of order 1 at $U = 1$ and $U = q^{-d}$, which implies that the radius of convergence of the power series (10) is equal to q^{-d}. The other poles of $Z(U, V)$, if there are any, are distributed on the circles

$$|U| = q^{-1},\ |U| = q^{-2}, \ldots,\ |U| = q^{-(d-1)}.$$

The zeros of $Z(U, V)$, if there are any, are distributed on the circles

$$|U| = q^{-1/2},\ |U| = q^{-3/2}, \ldots,\ |U| = q^{-(2d-1)/2}.$$

In other words, we must have

$$Z(U, V) = \frac{P_1(U)\, P_3(U) \ldots P_{2d-1}(U)}{P_0(U)\, P_2(U) \ldots P_{2d}(U)}, \tag{15}$$

where

$$P_\mu(U) = \prod_{\nu=1}^{B_\mu} (1 - \pi_{\mu,\nu} U)$$

is a polynomial with constant term equal to 1, whose inverse roots have absolute values equal to $q^{\mu/2}$,

$$|\pi_{\mu,\nu}| = q^{\mu/2}, \tag{16}$$

and in particular

$$P_0(U) = 1 - U,\ P_{2d}(U) = 1 - q^d U.$$

The equations (12) are equivalent to

$$B_\mu = B_{2d-\mu},\ \mu = 1, 2, \dots, 2d \tag{18}$$

and

$$\pi_{\mu,\nu}\, \pi_{2d-\mu, B_\mu+1-\nu} = q^d;\ \mu = 0, 1, \dots, 2d;\ \nu = 1, 2, \dots, B_\mu, \tag{19}$$

where the inverse roots $\pi_{\mu,\nu}$ of each P_μ are arranged in a suitable order. (15)–(19) comprise all statements on $Z(U, V)$ made hitherto.

For χ we get

$$\chi = \sum_{\mu=0}^{2d} (-1)^\mu B_\mu. \tag{20}$$

A further conjecture of Weil is that χ is equal to the Euler-Poincaré characteristic of V, i. e. the intersection number of the diagonal of $V \times V$ with itself.

For the number N_f we deduce easily the relations

$$N_f = \sum_{\mu=0}^{2d} (-1)^\mu \sum_{\nu=1}^{B_\mu} \pi_{\mu\nu}^f. \tag{21}$$

(21) is equivalent to (15).

From (21) it follows that the numbers $\pi_{\mu,\nu}$ are algebraic. We state the conjecture that the polynomials $P_\mu(U)$ have rational integral coefficients, in particular, that the numbers $\pi_{\mu,\nu}$ are algebraic integers.

All these conjectures are true for $d = 1$. For general d, Weil and Lang proved the convergence of (10) for $|U| < q^{-d}$, so that $Z(U, V)$ exists and is holomorphic and different from zero in $|U| < q^{-d}$. The proof is based on an estimation formula for the numbers N_f: $N_f = q^{fd} + A_f q^{f(d-\frac{1}{2})}$, where A_f is bounded in absolute value by a positive constant A. For $d = 1$, this estimation reduces essentially to (7). The general proof is by induction on d, using (7) at each step. Introducing $N_f = q^{df} + A_f q^{f(d-\frac{1}{2})}$ in (10), we get

$$\frac{d\log[Z(U,\ V)\ (1-q^dU)]}{dU} = \sum_{f=1}^{\infty} A_f(d^{d-\frac{1}{2}}U)^f,$$

and since this series, on account of $|A_f| \leqslant A$, is convergent for $|U| < q^{-(d-\frac{1}{2})}$, we see that $Z(U, V)$ has exactly one pole, of order 1, inside the circle $|U| < q^{-(d-\frac{1}{2})}$, which confirms one point in the above-mentioned conjectures. We mention two examples of higher dimension treated by Weil.

Let $V = G_{m,r}$ be the Grassmann variety of the r-dimensional linear subspaces of the m-dimensional projective space over k. V has the dimension $d = (r+1)(m-r)$. The numbers N_f can be computed;

$$N_f = F_{m+1,r+1}(q^f),$$

where $F_{a,b}(X)$, $a \geqslant b$, denotes the polynomial

$$F^{a,b}(X) = \frac{X^a - 1}{X^b - 1}\cdot\frac{X^{a-1} - 1}{X^{b-1} - 1}\cdots\frac{X^{a-b+1} - 1}{X - 1}$$

of degree d. This leads to

$$Z(U, G_{m,r}) = \prod_{\mu=0}^{d} \frac{1}{(1 - q^{\mu}\,U)^{\beta_{2\mu}}}, \tag{22}$$

where

$$F_{m+1,r+1}(X) = \sum_{\mu=0}^{d} B_{2\mu}\,X^{\mu},$$

and since obviously $B_{d-2\mu} = B_{\mu}$, $B_0 = 1$, the conjectures are true for $G_{m,r}$.

The other example is the variety V of dimension d, defined over k by a single equation

$$a_0 x_0^n + a_1 x_1^n + \ldots + a_d x_d^n = 0,\ a_\nu \neq 0,\ a_\nu \text{ in } k. \tag{23}$$

In this case the numbers N_ν can be expressed by Gauss sums; the result is

$$Z(U, V) = \frac{[\prod'_{\alpha}(1 - C(\alpha)\,U^{\mu(\alpha)})]^{(-1)^{d-1}}}{(1-U)\,(1-qU)\ldots(1-q^{d-1}U)}. \tag{24}$$

Here α denotes a system $\alpha_0, \ldots, \alpha_d$ of $d+1$ residue classes mod 1, satisfying

$$\alpha_\nu \not\equiv 0 \bmod 1, \; n\alpha_\nu \equiv 0 \bmod 1, \; \alpha_0 + \ldots + \alpha_d \equiv 0 \bmod 1,$$

$\mu(\alpha)$ is the smallest integer for which

$$(q^{\mu(\alpha)} - 1)\,\alpha_\nu \equiv 0 \bmod 1, \nu = 0, 1, \ldots, d$$

holds, $C(\alpha)$ is the product of certain Gauss sums and

$$|\,C(\alpha)\,| = q^{\mu(\alpha)(d-1)/2}.$$

Two systems α, α' are called equivalent, if there exists an integer t for which

$$\alpha' \equiv \alpha.\, q^t \bmod 1, \quad \nu = 0, 1, \ldots, d$$

holds. The product in (24) is to be extended over a complete set of inequivalent systems α, the number of which will be called A.

This proves Weil's conjecture for V, the numbers B_ν having the following values

$$\left.\begin{array}{l} B_{2h} = 1,\; B_{2h+1} = 0 \text{ for } 2h+1 \neq d-1,\; B_{d-1} = A \text{ if } d \text{ is even}, \\ B_{2h} = 1 \text{ for } 2h \neq d-1,\; B_{d-1} = A+1,\; B_{2h+1} = 0, \text{ if } d \text{ is odd}. \end{array}\right\} \quad (25)$$

Another case has been treated successfully by Taniyama, that of certain singular Abelian varieties. We shall come back to this later.

4. We now proceed to the investigation of an absolutely irreducible algebraic variety V without singular points, of dimension d, defined over a finite algebraic number field k.

For a prime ideal $\mathfrak{p}$ of k we denote by $k_\mathfrak{p}$ the residue class field of k modulo $\mathfrak{p}$.

The defining equations of V, considered as congruences modulo $\mathfrak{p}$, define a variety $V_\mathfrak{p}$ over $k_\mathfrak{p}$. It is probably true, and it can be proved for curves (Deuring) and for Abelian varieties (Shimura) that $V_\mathfrak{p}$ is absolutely irreducible, without singular points and of dimension d for almost all $\mathfrak{p}$.

The zeta function $Z(U, V_\mathfrak{p})$ of such a $V_\mathfrak{p}$ defines a set

$$B_{\mathfrak{p},0}, \ldots, B_{\mathfrak{p},2d}$$

of integral numbers, the degrees of the polynomials P_ν comprising $Z(U, V_\mathfrak{p})$. Now the variety V, defined over the field of complex numbers, is a topological manifold of dimension $2d$. Weil states the conjecture, that for almost all $\mathfrak{p}$, each $B_{\mathfrak{p},\nu}$, $\nu = 0, 1, \dots, 2d$, is equal to the ν-dimensional Betti number B_ν of V.

This is true for curves ($d = 1$), for it can be proved (Deuring) that for a curve C the genus of the reduced curve $C_\mathfrak{p}$ is equal to the genus of C for almost all $\mathfrak{p}$.

It is true also for the Grassmann variety $G_{m,r}$ defined over k according to the results of Ehresman on the Betti numbers of Grassmann varieties.

Weil verified his conjecture for a variety V, defined over the number field k by an equation

$$a_0x_0^n + a_1x_1^n + \dots + a_dx_d^n = 0, \quad a_\nu \neq 0.$$

For this purpose it is necessary to show that the Betti numbers of the variety V have the values given in (25). This has been done by Dolbeault.

A general proof of the conjecture in question (granted that the conjectures in §3 are correct) would require an algebraic definition of the Betti numbers of an algebraic variety, which is applicable also to a variety defined over a field of prime characteristic. Such a definition is still lacking, but we have an algebraic definition for the Euler-Poincaré characteristic

$$\chi(V) = \sum_{\nu=0}^{2d} (-1)^\nu B_\nu,$$

$\chi(V)$ is defined as the intersection number with itself of the diagonal of the product variety $V \times V$. Now it should be possible to prove, that for a variety V over a finite number field, $\chi(V)$ is equal to $\chi(V_\mathfrak{p})$ for almost all $\mathfrak{p}$, using Shimura's reduction theory for algebraic varieties. The conjectures in §3 granted, this would prove the conjecture on the Betti numbers at least partly, namely for their alternating sum.

5. From now on, we shall assume the conjectures in §3 and §4 to be correct. In $Z(U, V_{\mathfrak{p}})$ we introduce the variable s by means of

$$U = N\mathfrak{p}^{-s},$$

i.e.

$$\zeta(s, V_{\mathfrak{p}}) = Z(N\mathfrak{p}^{-s}, V_{\mathfrak{p}}).$$

The product

$$\prod_{\mathfrak{p}} \zeta(s, V_{\mathfrak{p}}),$$

extended over all $\mathfrak{p}$, for which $B_{\mathfrak{p},\nu} = B_\nu$, is absolutely convergent for Re $s > 1$, it therefore defines an analytic function $\zeta(s, V)$ of s, which is holomorphic and different from zero in Re $s > 1$. We call $\zeta(s, V)$ the zeta function of the variety V over k. Several examples suggest that $\zeta(s, V)$ is meromorphic and that

$$\frac{\zeta(d - s, V)}{\zeta(s, V)}$$

can be expressed as a product of a finite number of "elementary" factors, built up of exponentials and Γ-functions.

For example, if V is the Grassmann variety $G_{m,r}$ over k, we have

$$\zeta(s, V) = \prod_{\mu=0}^{2d} \zeta_k(s - \mu/2)^{(-1)^\mu B_\mu},$$

where $\zeta_k(s)$ denotes the zeta function of the field k, the B_ν's being the Betti numbers of $G_{m,r}$. In particular, we have for the projective space P^d of dimension d over k,

$$\zeta(s, P^d) = \prod_{\mu=0}^{d} \zeta_k(s - \mu).$$

Apart from this simple case, this conjecture (which, for curves, is called Hasse's conjecture) has been confirmed in the following cases:

I. (A. Weil): V is a curve defined over k by an equation

$$ax^f + by^e + c = 0, \quad a, b, c \neq 0, \text{ numbers in } k.$$

In the sequel we shall use the following abbreviation: if $f(s)$ is the L-function belonging to a Hecke character in a number field (in particular, if $f(s)$ is the zeta function of a number field), $f^*(s)$ shall denote the function which is obtained from $f(s)$ by depriving its Euler product of a finite number of its factors.

Now, if g is the genus of V, we have

$$\zeta(s, V) = \zeta^*(s)\, \zeta^*(s-1) \prod_{i=1}^{2g} L^*(s - \tfrac{1}{2}, \chi_i)^{-1}.$$

Here the χ_i's are certain Hecke characters in cyclotomic extensions of k.

II. V is a curve of genus 1 with complex multiplication (singular elliptic curve). We then have (Deuring)

$$\zeta(s, V) = \zeta^*_k(s)\, \zeta^*_k(s-1)\, [L^*(s - \tfrac{1}{2}, \chi)\, L^*(s - \tfrac{1}{2}, \bar{\chi})]^{-1}, \quad (26)$$

where χ is a Hecke character of k.

This result has been generalized in two ways by Taniyama:

III. V is a curve of genus g. The endomorphism algebra of the Jacobian J of C (which is an Abelian variety of dimension g) is supposed to contain a number field of degree $2g$, which then is necessarily contained in the base field k. In this case one gets

$$\zeta(s, V) = \zeta^*_k(s)\, \zeta^*_k(s-1) \prod_{i=1}^{2g} L^*(s - \tfrac{1}{2}, \chi_i)^{-1}, \quad (27)$$

where the χ_i's are Hecke characters of k.

IV. V is an Abelian variety of dimension g whose endomorphism algebra contains a number field of degree $2g$ (a subfield of the base field k). Taniyama proved

$$\zeta(s, V) = \zeta^*_k(s)\, \zeta^*_k(s-g) \prod_{\nu=1}^{2g} \prod_{i_1,\ldots,i_\nu} L^*\left(s - \frac{\nu}{2}, \chi_{i_1}, \ldots, \chi_{i_\nu}\right)^{(-1)^{\nu-1}}, \quad (28)$$

where the $\chi_1, \ldots, \chi_{2g}$ are Hecke characters of k.

Obviously, III and IV reduce to II if $g = 1$.

The equations (26), (27), (28) suggest considering as the "true" zeta function in each case the function obtained by dropping the *'s, that is, by introducing the full Euler product for each L-series occurring, because it is the resulting functions which satisfy functional equations of the familiar type. In connection with this, one may ask the question, whether the conductors of the Hecke characters occurring in (26), (27), (28) have a simple algebraic meaning for the variety V.

In Cases I and II an answer is known. Consider first Case II (Deuring):

Let K be the field of rational functions on the singular elliptic curve V over k. If V reduces modulo $\mathfrak{p}$ to an *elliptic* curve $V_{\mathfrak{p}}$, this reduction defines an extension of $\mathfrak{p}$ to K, such that the residue class field $K_{\mathfrak{p}}$ of K modulo $\mathfrak{p}$ is elliptic with $k_{\mathfrak{p}}$ as its field of constants. For a given $\mathfrak{p}$, there may exist among all curves birationally equivalent to V over k, one, V', which reduces to an elliptic $V'_{\mathfrak{p}}$.

The extension of $\mathfrak{p}$ to K obtained in this way does not depend upon the choice of the curve V'. Consequently, $\zeta(s, V'_{\mathfrak{p}})$ is defined independently of V' as an invariant of the field K and it shall be denoted by $\zeta(s, K, \mathfrak{p})$. A prime $\mathfrak{p}$, for which no curve V' birationally equivalent to V over k reduces to an elliptic $V'_{\mathfrak{p}}$, shall be called irregular; in this case we put

$$\zeta(s, K, \mathfrak{p}) = \frac{1}{1 - N\mathfrak{p}^{-s}} \frac{1}{1 - N\mathfrak{p}^{1-s}}.$$

If we then define the zeta function $\zeta(s, K)$ of the *field* K by

$$\zeta(s, K) = \prod_{\mathfrak{p}} \zeta(s, K, \mathfrak{p})$$

the product being extended over *all* primes $\mathfrak{p}$ of k, it turns out that we have

$$\zeta(s, K) = \zeta_k(s)\, \zeta_k(s-1)\, [L(s-\tfrac{1}{2}, \chi)\; L(s-\tfrac{1}{2}, \bar{\chi})]^{-1},$$

with the same character χ as in (26). In particular, the irregular primes $\mathfrak{p}$ are exactly the primes dividing the conductor f_{χ} of the character χ. The functional equation for $\zeta(s, K)$ is

$$(Nf_{\chi})^{-s}\, \zeta(s, K) = (-1)^{n/2}\, (Nf_{\chi})^{-(2-s)}\, \zeta(2-s, k),$$
$$n = \text{degree of } k.$$

Case I has been treated by Hasse; the result is quite similar to that in Case II.

In all cases considered so far, $\zeta(s, V)$ has been expressed by means of L- series. Of a quite different type are the results of Eichler, who proved Hasse's conjecture for certain curves (or function fields of one variable) over an algebraic number field, which arise from the theory of modular functions. In particular, his results include elliptic curves, which are not singular.

6. One may replace the number field k by an algebraic function field of one variable over a finite field of constants, the arithmetic of such fields being very similar to that of number fields. Zeta functions of curves over such fields have been investigated by Lamprecht, with results similar to those in § 5. These zeta functions may of course be considered as zeta functions of surfaces over finite fields.

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THE CONSTRUCTION OF AUTOMORPHIC FORMS FROM THE DERIVATIVES OF A GIVEN FORM

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1. Let $f(z)$ be a meromorphic automorphic form of real or complex dimension $-k$ belonging to the horocyclic [2] group Γ (Fuchsian group of the first kind, *Grenzkreisgruppe*), and furnished with a multiplier system v. We write this $f(z) \in \{\Gamma, k, v\}$, and denote by v^r, for integral r, the multiplier system which is the rth power of v. It is probably well known that

$$kf(z)\, f''(z) - (k+1)\,\{f'(z)\}^2 \in \{\Gamma, 2k+4, v^2\}, \tag{1}$$

and the object of this paper is to generalize this result by finding all polynomials in the derivatives of $f(z)$ which are automorphic forms for Γ. The results which we obtain hold also for more general groups (see end of §4).

We denote by T the bilinear transformation

$$w = \frac{az+b}{cz+d} = Tz,$$

and by $\mathscr{H}$ the half-plane $\Im z > 0$. If $f(z) \in \{\Gamma, k, v\}$, where Γ is a given horocyclic group, then

$$f(Tz) = v(T)\,(cz+d)^k\, f(z), \tag{2}$$

for all $T \in \Gamma$ and $z \in \mathscr{H}$. For non-integral k, $(cz+d)^k$ denotes a certain uniquely determined root (see, for example, [1]) of $cz+d$.

Throughout the paper we shall suppose that $f(z)$ is a fixed automorphic form belonging to $\{\Gamma, k, v\}$ and that $z \in \mathscr{H}$. Also, for any $T \in \Gamma$, we write

$$S = S_T = cz+d, \quad \lambda = c/S, \quad f_r = \left(\frac{d}{dz}\right)^r f(z). \tag{3}$$

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Then, by (2), for $w = Tz$,

$$f(w) = v(T)S^k f_0,$$
$$f'(w) = v(T)S^{k+2}(f_1 + k\lambda f_0),$$
$$f''(w) = v(T)S^{k+4}\{f_2 + 2(k+1)\lambda f_1 + (k+1)k\lambda^2 f_0\},$$

and, in general,

$$f^{(r)}(w) = v(T)S^{k+2r}\sum_{m=0}^{r}\binom{r}{m}(k+r-1)(k+r-2)\times\ldots\times (k+r-m)\lambda^m f_{r-m}. \tag{4}$$

We wish to find those polynomials $P(f_0, f_1, \ldots, f_n) = P^*(z)$ in $f(z)$ and its first n derivatives, which, for any Γ, k and v, are automorphic forms belonging to $\{\Gamma, k', v'\}$ for some dimension $-k'$ and multiplier system v' depending on k and v respectively. A term $A\, f_0^{\alpha_0} f_1^{\alpha_1} \ldots f_n^{\alpha_n}$ in a polynomial is said to be of degree r and weight s if the non-negative integers $\alpha_0, \alpha_1, \ldots, \alpha_n$ satisfy

$$\alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_n = r,\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n\alpha_n = s.$$

When we transform from z to w by means of (4), each term

$$A\{f(w)\}^{\alpha_0}\{f'(w)\}^{\alpha_1}\ldots\{f^{(n)}(w)\}^{\alpha_n}$$

of $P^*(w)$ will transform into an expression which is a polynomial in λ. If $P^*(z) \in \{\Gamma, k', v'\}$, the coefficients of positive powers of λ will cancel out between the different terms of $P^*(w)$, and so it suffices to consider the contribution from the part of the expression which is independent of λ, namely

$$A\{v(T)\}^r S^{kr+2s} f_0^{\alpha_0} f_1^{\alpha_1}\ldots f_n^{\alpha_n}.$$

Accordingly $k' = kr + 2s$, $v' = v^r$, and each term of $P^*(z)$ must be of the same degree r and weight s. For this reason it suffices to consider polynomials in $f(z)$ and its derivatives, for which each term is of the same degree r and weight s; we denote such a polynomial by $P_{r,s}(\ldots)$.

We define, for each non-negative integer r,

$$h_r = h_r(z) = \frac{f_r}{\Gamma(k+r)\,r!},\ H_r = H_r(w) = \frac{S^{-k-2r} f^{(r)}(w)}{v(T)\,\Gamma(k+r)\,r!}, \tag{5}$$

where $w = Tz$. Note that $h_0 = 0$ if $k = 0$, for example.

Also, for each integer $m \geqslant 2$, let

$$\psi_m = \psi_m(z) = (-1)^{m-1} \begin{vmatrix} h_1 & 2h_2 & 3h_3 & \dots & (m-1)h_{m-1} & mh_m \\ h_0 & h_1 & h_2 & \dots & h_{m-2} & h_{m-1} \\ 0 & h_0 & h_1 & \dots & h_{m-3} & h_{m-2} \\ 0 & 0 & h_0 & \dots & h_{m-4} & h_{m-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & h_1 & h_2 \\ 0 & 0 & 0 & \dots & h_0 & h_1 \end{vmatrix} . \quad (6)$$

THEOREM 1. *Suppose that $f(z) \in \{\Gamma, k, v\}$ and that k is neither zero nor a negative integer.*

(i) *If $P_{r,s}(f_0, f_1, \dots, f_n) \in \{\Gamma, rk + 2s, v^r\}$, then*

$$P_{r,s}(f_0, f_1, \dots, f_n) = f_0^{r-s}\, Q_s(\psi_2, \psi_3, \dots, \psi_n),$$

where Q_s is a polynomial of weight s in $\psi_2, \psi_3, \dots, \psi_n$.

(ii) *Conversely, if Q_s is any such polynomial, then*

$$Q_s \in \{\Gamma, (k+2)s, v^s\},$$

and an integer $r \leqslant s$ can be chosen so that $f_0^{r-s} Q_s$ is a polynomial $P_{r,s}$ in the h_i $(0 \leqslant i \leqslant n)$ and belongs to $\{\Gamma, rk + 2s, v^r\}$; this remains true for any $r > s$, but then each term in $P_{r,s}$ is divisible by the form f.

This theorem shows that the functions ψ_m, together with f, form a basis for all automorphic forms which are polynomials, or rational functions, in f and its derivatives.

2. Proof of Theorem 1. We suppose that z is not a zero or a pole of $f(z)$. When k is not a non-positive integer, any polynomial in $f_0, f_1, \dots, f_n$ is a polynomial in $h_0, h_1, \dots, h_n$ and conversely; it is more convenient to consider polynomials in the h_i.

Consider the formal power series

$$h(x) = \sum_{r=0}^{r} h_r x^r, \quad H(x) = \sum_{r=0}^{\infty} H_r x^r, \tag{7}$$

where h_r and H_r are defined by (5). The equation (4) can be written as

$$H_r = \sum_{m=0}^{r} h_m \frac{\lambda^{r-m}}{(r-m)!}, \tag{8}$$

so that we have

$$H(x) = \sum_{r=0}^{\infty} x^r \sum_{m=0}^{r} h_m \frac{\lambda^{r-m}}{(r-m)!} = \sum_{m=0}^{\infty} h_m x^m \sum_{r=m}^{\infty} \frac{(\lambda x)^{r-m}}{(r-m)!}.$$
$$= e^{\lambda x} h(x).$$

Accordingly, if we write

$$\frac{h'(x)}{h(x)} = \sum_{m=1}^{\infty} c_m x^{m-1}, \quad \frac{H'(x)}{H(x)} = \sum_{m=1}^{\infty} C_m x^{m-1}$$

we obtain

$$\frac{H'(x)}{H(x)} = \lambda + \frac{h'(x)}{h(x)},$$

so that

$$C_m = c_m \quad (m \geqslant 2). \tag{9}$$

These processes are justified whether the series converge or not, as they can be replaced by arguments involving finite systems of equations. Further, we have

$$\frac{h'(x)}{h(x)} h(x) = h'(x),$$

so that

$$\sum_{m=1}^{\infty} c_m x^{m-1} \sum_{r=0}^{\infty} h_r x^r = \sum_{n=1}^{\infty} n h_n x^{n-1}.$$

Comparing coefficients of powers of x^{n-1}, we obtain

$$\sum_{m=1}^{n} c_m h_{n-m} = n h_n \quad (n \geqslant 1),$$

and deduce that, for $n \geqslant 1$,

$$c_n = (-1)^{n-1} h_0^{-n} \begin{vmatrix} h_1 & 2h_2 & 3h_3 & \dots & nh_n \\ h_0 & h_1 & h_2 & \dots & h_{n-1} \\ 0 & h_0 & h_1 & \dots & h_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & h_2 \\ 0 & 0 & 0 & \dots & h_1 \end{vmatrix}. \quad (10)$$

Thus, by (6),

$$c_1 = h_1/h_0, \quad h_0^n\, c_n(z) = \psi_n(z) \quad (n \geqslant 2).$$

Further, the coefficients C_m are similar functions of the H_r, and it follows from (9) that

$$\psi_n(z) \in \{\Gamma,\, n(k+2),\, v^n\} \quad (n \geqslant 2),$$

since $\psi_n(z)$ is of degree n and weight n in the h_r. From this part (ii) of the theorem follows.

To prove part (i) we suppose that

$$P_{r,s}(f_0, f_1, \dots, f_n) \in \{\Gamma,\, rk+2s,\, v^r\}.$$

Write

$$P_s(z) = P_{r,s}\, h_0^{-r},$$

so that $P_s(z) \in \{\Gamma, 2s, 1\}$. Now $c_n(z) \in \{\Gamma, 2n, 1\}$ for $n \geqslant 2$ and both $P_s(z)$ and $c_n(z)$ are polynomials in the h_m/h_0 for $m \geqslant 1$. Further h_n/h_0 occurs only to the first power and with coefficient n in $c_n(z)$. It follows that we can eliminate successively h_n/h_0, h_{n-1}/h_0, $\dots$, h_2/h_0 from $P_s(z)$ by using c_n, $c_{n-1}, \dots, c_2$ and obtain finally

$$P_s(z) = \sum_{m=0}^{s} \left(\frac{h_1}{h_0}\right)^m p_m,$$

where $p_m \in \{\Gamma, 2s-2m, 1\}$ and is a polynomial in the $c_k(z)$ for $2 \leqslant k \leqslant n$.

We now show that p_m vanishes identically for $m > 0$. For we have, if $w = Tz$, where $T \in \Gamma$, that

$$P_s(w) = \sum_{m=0}^{s} \left\{ S^2\left(\lambda + \frac{f_1}{kf_0}\right) \right\}^m S^{2(s-m)} p_m.$$

Thus

$$\sum_{m=1}^{s} p_m \left\{ \left(\lambda + \frac{f_1}{kf_0}\right)^m - \left(\frac{f_1}{kf_0}\right)^m \right\} = 0. \tag{11}$$

This holds for all $\lambda = c/(cz + d)$. For fixed $z \in \mathscr{H}$ there are infinitely many different values of λ, corresponding to different $T \in \Gamma$, and hence the coefficients of $\lambda, \lambda^2, \dots, \lambda^s$ on the left of (11) must vanish identically. We thus have s linear equations in the s quantities $p_1, p_2, \dots, p_s$ with determinant 1, which is impossible unless $p_1 = p_2 = \dots = p_s = 0$.

Hence $P_s(z) = p_0$ and so is a polynomial in the $c_k(z)$ for $2 \leqslant k \leqslant n$. It consists of terms of the form

$$B\, c_2^{\beta_2} c_3^{\beta_3} \dots c_n^{\beta_n},$$

where

$$2\beta_2 + 3\beta_3 + \dots + n\beta_n = s,$$

and so $P_{r,s}$ consists of terms of the form

$$B\, h_0^{r-s}\, \psi_2^{\beta_2}\, \psi_3^{\beta_3} \dots \psi_n^{\beta_n},$$

from which the first part of the theorem follows.

If $m > 3$, ψ_m is an algebraic function of ψ_2 and ψ_3. For there is an algebraic relation connecting the automorphic functions ψ_2^3/ψ_3^2 and $\psi_m/\psi_2^\alpha\, \psi_3^\beta$ $(2\alpha + 3\beta = m)$.

3. We now suppose that k is zero or a negative integer, and write

$$k = 1 - N,$$

so that N is a positive integer. We assume further that z is not a zero or pole of $f(z)$ or $f^{(N)}(z)$. Then $h_0 = h_1 = \dots = h_{N-1} = 0$, and we put

$$g_r = g_r(z) = \frac{f_r}{r!}\,\Gamma(N-r) \quad (0 \leqslant r < N), \quad g_r = 0 \; (r \geqslant N), \tag{12}$$

$$G_r = G_r(w) = \frac{\Gamma(N-r)\, f^{(r)}(w)}{v(T)\, S^{k+2r}\, r!} \quad (0 \leqslant r < N).$$

Then we have, by (4),

$$G_r = \sum_{m=0}^{r} g_m \frac{(-\lambda)^{r-m}}{(r-m)!} \quad (0 \leqslant r < N), \tag{13}$$

and we define G_r for $r \geqslant N$ by this relation. If

$$g(x) = \sum_{r=0}^{\infty} g_r x^r, \quad G(x) = \sum_{r=0}^{\infty} G_r x^r,$$

we then deduce from (13), exactly as in § 2, that

$$G(x) = e^{-\lambda x} g(x).$$

If we put

$$\frac{g'(x)}{g(x)} = \sum_{n=1}^{\infty} b_n x^{n-1}, \quad \frac{G'(x)}{G(x)} = \sum_{n=1}^{\infty} B_n x^{n-1},$$

we obtain

$$B_m = b_m \quad (m \geqslant 2), \tag{14}$$

and

$$b_m = (-1)^{m-1} g_0^{-m} \begin{vmatrix} g_1 & 2g_2 & \dots & mg_m \\ g_0 & g_1 & \dots & g_{m-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g_1 \end{vmatrix} = g_0^{-m} \phi_m(z), \tag{15}$$

say.

We now suppose that $N \geqslant 3$. Since the coefficients B_m are similar functions of the G_m, we deduce from (14) that

$$\phi_m(z) \in \{\Gamma, m(k+2), v^m\} \quad (2 \leqslant m < N; N \geqslant 3).$$

For values of $r \geqslant N$, where $N \geqslant 1$, we employ the functions h_r and proceed as in §2. We note, in the first place, that, by (8),

$$f_N/(N!) = h_N = h_N(z) \in \{\Gamma, k + 2N, v\} = \{\Gamma, 2-k, v\}.$$

If we put

$$\frac{h'(x)}{h(x)} = \sum_{m=0}^{\infty} c_m x^{m-1}, \quad \frac{H'(x)}{H(x)} = \sum_{m=0}^{\infty} C_m x^{m-1},$$

then (9) holds and we have

$$\sum_{m=0}^{\infty} c_m x^{m-1} \sum_{r=N}^{\infty} h_r x^r = \sum_{n=N}^{\infty} nh_n x^{n-1}.$$

From this we obtain

$$\sum_{m=0}^{n} c_m h_{N+n-m} = (N+n) h_{N+n},$$

which gives, in particular, $c_0 = N$, so that, for $n \geqslant 1$,

$$\sum_{m=1}^{n} c_m h_{N+n-m} = nh_{N+n}.$$

Thus

$$c_m = h_N^{-m} \chi_m(z), \tag{16}$$

where

$$\chi_m(z) = (-1)^{m-1} \begin{vmatrix} h_{N+1} & 2h_{N+2} & \cdots & mh_{N+m} \\ h_N & h_{N+1} & \cdots & h_{N+m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & h_{N+2} \\ 0 & 0 & \cdots & h_{N+1} \end{vmatrix} \tag{17}$$

and is, in fact, the ψ_m function of §2 formed from h_N instead of h_0.

We have

$$\chi_m \in \{\Gamma, m(4-k), v^m\} \quad (m \geqslant 2).$$

We also define, for $N \geqslant 2$,

$$d_1 = b_1 + c_1 = \frac{g_1}{g_0} + \frac{h_{N+1}}{h_N},$$

$$\delta_2 = d_1\, g_0\, h_N = g_1\, h_N + g_0\, h_{N+1}$$
$$= \frac{1}{N(N^2-1)} \left\{ (N+1) f_1 f_N + (N-1)\, f_0\, f_{N+1} \right\}, \qquad (18)$$

so that, by (8) and (13),

$$d_1 \in \{\Gamma, 2, 1\}, \quad \delta_2 \in \{\Gamma, 4, v^2\}.$$

We accordingly have

THEOREM 2. *The following polynomials in $f(z)$ and its derivatives are automorphic forms for* Γ.

(i) $f, f^{(N)}$ *and* χ_m *for* $N \geqslant 1, m \geqslant 2$.

(ii) δ_2 *for* $N \geqslant 2$.

(iii) ϕ_m *for* $N \geqslant 3$ *and* $2 \leqslant m < N$.

We now show that the functions of Theorem 2 form a basis for automorphic forms which are polynomials in f and its derivatives.

Suppose that $P_{r,s}$ is a polynomial in the derivatives of $f(z)$ and that $P_{r,s} \in \{\Gamma, rk + 2s, v^r\}$. Then $P_{r,s}$ is a polynomial in the functions $g_0, g_1, \dots, g_{N-1}, h_N, h_{N+1}, \dots, h_{N+n}$, say, and is a sum of terms of the form

$$\tau_m = A_m\, g_0^{\alpha_0}\, g_1^{\alpha_1} \dots g_{N-1}^{\alpha_{N-1}}\, h_N^{\beta_0}\, h_{N+1}^{\beta_1} \dots h_{N+n}^{\beta_n},$$

for $1 \leqslant m \leqslant M$, say, where the non-negative integers α_μ, β_ν satisfy

$$\sum_{\mu=0}^{N-1} \alpha_\mu = r_1, \quad \sum_{\nu=0}^{n} \beta_\nu = r_2, \quad r = r_1 + r_2,$$

$$\sum_{\mu=0}^{N-1} \mu\alpha_\mu = s_1, \quad \sum_{\nu=1}^{n} \nu\, \beta_\nu = s_2, \quad s = s_1 + s_3 + Nr_2.$$

For different values of m, the integers r_1, r_2, s_1, s_2 may differ. By means of (15) and (17) we can express τ_m in terms of the functions

$$g_0, h_N, h_{N+1}, \chi_m \quad (N \geqslant 1, m \geqslant 2),$$
$$g_1\, (N \geqslant 2), \quad \phi_m \quad (N \geqslant 3, 2 \leqslant m < N),$$

and obtain

$$\tau_m = g_0^{r_1 - s_1} h_N^{r_2 - s_2} R_m(g_1, h_{N+1}),$$

where R_m is a polynomial in g_1 and h_{N+1} of degree s_1 in g_1 and s_2 in h_{N+1}, its coefficients being polynomials in g_0, h_N, ϕ_m and χ_m $(m \geqslant 2)$. Since $P_{r,s} \in \{\Gamma, rk + 2s, v^r\}$ we have, for all $T \in \Gamma$, that

$$P_{r,s} = \sum_{m=1}^{M} \tau_m = \sum_{m=1}^{M} g_0^{r_1 - s_1} h_N^{r_2 - s_2} R_m(g_1 - \lambda g_0, h_{N+1} + \lambda h_N),$$

where λ is given by (3). Since λ can take an infinity of different values for each fixed z, this relation is an identity in λ and so we may substitute $\lambda = -h_{N+1}/h_N$ and obtain

$$P_{r,s} = \sum_{m=1}^{M} g_0^{r_1 - s_1} h_N^{r_2 - s_2} R_m(\delta_2/h_N, 0)$$
$$= \sum_{m=1}^{M} g_0^{r_1 - s_1} h_N^{r_2 - s_2 - s_1} R_m^*(\delta_2),$$

where R_m^* is a polynomial in δ_2, g_0, h_N, ϕ_m and χ_m. Put

$$p = \max(s_1 - r_1), \quad q = \max(s_1 + s_2 - r_2),$$

so that

$$p \leqslant s - r, \quad q \leqslant s. \tag{19}$$

We deduce

Theorem 3. *Suppose that $f(z) \in \{\Gamma, k, v\}$ and that $k = 1 - N$, where N is a positive integer. If $P_{r,s}(f_0, f_1, \ldots, f_{n+N}) \in \{\Gamma, rk + 2s, v^r\}$, then for some integers p, q satisfying* (19),

$$P_{r,s} = g_0^{-p} h_N^{-q} Q\{g_0, h_N; \delta_2; \phi_2, \phi_3, \ldots, \phi_{N-1}; \chi_2, \chi_3, \ldots, \chi_n\},$$

where Q is a polynomial in the variables indicated, not all of which need be present, and is of degree $r + p + q$ and weight $qN + s$ in the f_m.

We note, in conclusion, that an alternative method of considering the case $k = 1 - N$, for $N \geqslant 2$, is to apply Theorem 1 to $1/f$ which is of dimension $N - 1 \geqslant 1$.

4. Other determinantal expressions for automorphic forms can be obtained. For example, if

$$u_r = f_r/\Gamma(k+r), \tag{20}$$

then it is easily verified from (4) that

$$U_n = \begin{vmatrix} u_0 & u_1 & u_2 & \cdots & u_n \\ u_1 & u_2 & u_3 & \cdots & u_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_n & u_{n+1} & u_{n+2} & \cdots & u_{2n} \end{vmatrix} \tag{21}$$

belongs to $\{\Gamma, (n+1)(k+2n), v^{n+1}\}$ for $n = 0, 1, 2, \ldots$. In particular

$$U_1 = \psi_2,\ U_2 = h_0^{-3}\{\psi_2^3 - 2\psi_3^2 + 3\psi_2\psi_4\}.$$

However the functions U_n, unlike the ψ_n, do not form a basis together with $f(z)$, for all automorphic forms which are polynomials in $f(z)$ and its derivatives. For example, ψ_3 cannot be expressed in terms of the U_n and h_0.

Also, if D_n denotes the discriminant of the equation

$$u_0 x^n + \binom{n}{1} u_1 x^{n-1} + \binom{n}{2} u_2 x^{n-2} + \ldots + u_n = 0, \tag{22}$$

then $D_n \in \{\Gamma,\ 2(n-1)(k+n),\ v^{2n-2}\}$. For example,

$$D_3 = \begin{vmatrix} 3u_0 & 6u_1 & 3u_2 & 0 \\ 0 & 3u_0 & 6u_1 & 3u_2 \\ 3u_1 & 6u_2 & 3u_3 & 0 \\ 0 & 3u_1 & 6u_2 & 3u_3 \end{vmatrix}$$

$$= 81(u_0^2 u_3^2 + 4u_0 u_2^3 - 6u_0 u_1 u_2 u_3 + 4u_1^3 u_3 - 3u_1^2 u_2^2)$$

$$= 324\, h_0^{-2}(\psi_2^3 + \psi_3^2).$$

These are particular cases of the result that any symmetric function of the roots α_i $(i = 1, 2, \ldots, n)$ of (22), which is invariant when α_i is replaced by $\alpha_i + v$, for any v, is an automorphic form for Γ, We can also consider automorphic forms constructed from systems of functions $f^{(i)}(z)$, where $f^{(i)} \in \{\Gamma, k_i, v_i\}$ for $i = 0, 1, 2, \ldots, n$.

Write

$$u_{ir} = \frac{1}{\Gamma(k_i + r)} \left(\frac{d}{dz}\right)^r f^{(i)}(z),$$

and put

$$W_n(f^{(i)}|z) = \begin{vmatrix} u_{00} & u_{01} & u_{02} & \cdots & u_{0n} \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{n0} & u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix}. \tag{23}$$

Then it is easily verified that $W_n \in \{\Gamma, K, V\}$, where

$$K = n(n+1) + \sum_{i=0}^{n} k_i, \quad V = v_0 v_1 \ldots v_n.$$

For example,

$$W_1(f, \psi_2) = \frac{2}{\Gamma(2k+4)} \psi_3,$$

where $f^{(0)} = f$, $f^{(1)} = \psi_2$.

We note, in conclusion, that the results which we have obtained hold more generally for any group for which λ takes an infinity of different values for each fixed z, i.e. for any group containing infinitely many distinct points congruent to the point at infinity. Even when there are only finitely many such distinct points, partial results can be obtained; for instance, if the weight of the polynomial $P_{r,s}$ is sufficiently small.

5. We conclude by giving some examples, where $\Gamma = \Gamma(1)$, the full modular group, and $v = 1$. We denote by $\Delta(z)$ the modular discriminant

$$\Delta(z) = e^{2\pi i z} \left\{ \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) \right\}^{24}$$

and by $G_k(z)$ for even $k \geqslant 4$, the Eisenstein series

$$G_k(z) = 1 + \alpha_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i n z},$$

where

$$\alpha_k = \frac{2\,(-1)^{\frac{1}{2}k}\, k}{B_{\frac{1}{2}k}}, \quad \sigma_r(n) = \sum_{\substack{d|n \\ d>0}} d^r.$$

We then find, from (1), or by considering ψ_2, that

$$4G_4G_4'' - 5G_4'^2 = -3840\pi^2\Delta,$$

$$6G_6G_6' - 7G_6'^2 = 12{,}096\pi^2 G_4\Delta.$$

Also, by taking $f = \Delta$ we find that

$$12\,!\ \ 13\,!\ \psi_2 = 12\Delta\Delta'' - 13\Delta'^2 = 4\pi^2\, G_4\, \Delta^2, \tag{24}$$

$$\tfrac{1}{2}\,(12\,!)^2\, 14\,!\ \psi_3 = 91\,(\Delta')^3 + 36\Delta^2\Delta''' - 126\,\Delta\Delta'\Delta'' = (2\,\pi\, i)^3\, G_6\, \Delta^3. \tag{25}$$

It follows, in particular, that every meromorphic modular form for $\Gamma(1)$ is a rational function of Δ, Δ', Δ'' and Δ'''. We also find that

$$(12\,!)^2\ 13\,!\ 15\,!\ \psi_4 = -\,66\,(2\pi)^4\,\Delta^4\, G_4^{\,2}, \tag{26}$$

and from this and the result for ψ_2 we find the following differential equation satisfied by Δ, namely

$$13\,\Delta_1^{\,4} + 10\,\Delta^2\,\Delta_1\,\Delta_2 - 24\,\Delta\,\Delta_1^{\,2}\,\Delta_2 - 2\Delta^3\,\Delta_4 + 3\,\Delta^2\,\Delta_2^{\,2} = 0, \tag{27}$$

where Δ_r denotes the rth derivative of Δ.

This differential equation is homogeneous. By expressing Δ in terms of G_4 and G_6, we can obtain from (24) and (25) an inhomogeneous differential equation satisfied by Δ which involves only Δ, Δ_1, Δ_2 and Δ_3.

A more curious result is obtained by taking $f = 1/\Delta$ and evaluating $h_N = h_{13}$. We find that

$$\Delta \left(\frac{1}{2\pi i}\frac{d}{dz}\right)^{13} \frac{1}{\Delta} = G_{14}\left\{\frac{65520}{691}\Delta - G_{12}\right\}. \tag{28}$$

Similarly,

$$\Delta \left(\frac{1}{2\pi i}\frac{d}{dz}\right)^{s-1} \frac{G_{14-s}}{\Delta} = G_s\left\{(24 + \alpha_{12} + \alpha_s)\Delta - G_{12}\right\}, \tag{29}$$

for $s = 4, 6, 8$ and 10.

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SPHERICAL FUNCTIONS AND QUADRATIC FORMS

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INTRODUCTION. An analytical treatment of the problem of representation of quadratic forms $T[\mathfrak{x}]$ by a given positive form $S[\mathfrak{x}]$ seems to be possible in the following general shape: Let $S = S^{(m)}$ and $T = T^{(n)}$ with $m > n$ be positive real matrices of m and n rows respectively. In the set of all real matrices $X = X^{(m,n)}$, having m rows and n columns, we denote by $\mathfrak{B}$ a domain of homogeneity, i.e. a subset which contains with X also XV, $V = V^{(n)}$ being an arbitrary non-singular real matrix of n rows. Further let $\mathfrak{C}$ be a subset of the set of all reduced positive real matrices $Y = Y^{(n)}$ in the sense of Minkowski, such that with Y, $\mathfrak{C}$ also contains λY, λ being an arbitrary positive real number. Then the number $a_t(\mathfrak{B}, \mathfrak{C})$ of all integral matrices $G = G^{(m,n)}$ which yield a representation

$$S[G] = G' S G = T, \tag{1}$$

with $G \in \mathfrak{B}$, $T \in \mathfrak{C}$, $|T| = t$ or at least the mean value

$$A_t(\mathfrak{B}, \mathfrak{C}) = \frac{1}{t} \sum_{\tau \leqslant t} a_\tau (\mathfrak{B}, \mathfrak{C}) \tag{2}$$

allows an asymptotic computation provided that $\mathfrak{B}$ and $\mathfrak{C}$ are measurable in a certain sense.

A method which is fitted for an analogous problem in algebraic number fields was developed by E. Hecke [2]. This method will probably work also in our case. It is based on the approximation of

$$\phi(s\,; \mathfrak{B}, \mathfrak{C}) = \sum_{t=1}^{\infty} a_t (\mathfrak{B}, \mathfrak{C})\, t^{-s} \tag{3}$$

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by a finite or infinite linear combination of certain zeta functions, i.e. by functions having a well-known behaviour on the strength of a Dirichlet series development and a functional equation of Riemannian type. We introduce $S = Q'Q$, $Q' = Q > 0$,

$$f(X) = \begin{cases} 1, & \text{for } Q^{-1}X \in \mathfrak{B} \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

and

$$g(Y) = \begin{cases} 1, & \text{for } Y \in \mathfrak{C}, \\ 0, & \text{for } Y \notin \mathfrak{C},\ Y \text{ reduced}; \end{cases} \tag{5}$$

$$g(Y[U]) = g(Y) \text{ for unimodular } U.$$

Then we have obviously

$$\phi(s; \mathfrak{B}, \mathfrak{C}) = \sum_{G} f(QG)\, g(S[G])\, |S[G]|^{-s}, \tag{6}$$

the summation taken over a complete set of integral matrices $G = G^{(m,n)}$ of rank n, such that each two do not differ by a unimodular right factor. The approximation of $\phi(s; \mathfrak{B}, \mathfrak{C})$ amounts to one of the functions $f(X)$ and $g(Y)$. Here we have to make use of the angular characters of quadratic forms [5] in so far as it concerns the function $g(Y)$. The theory of these angular characters is at the present sufficiently developed [6] only in the case $n = 2$ so that number-theoretical investigations of the desired kind are possible. Provided that $\mathfrak{B}$ is the full space of all real matrices $X = X^{(m,n)}$ of rank n, an asymptotic computation of $A_t(\mathfrak{B}, \mathfrak{C})$ with the method I have in mind could be carried out indeed in the case $n = 2$ [7]. For the approximation of $f(X)$ we need in the case $n = 1$ the spherical harmonics of m variables [1]. Apparently nobody has so far observed the significance of the spherical harmonics for this number theoretical problem.

The aim of this paper is to introduce a generalized class of spherical functions which are useful for the approximation of $f(X)$ for arbitrary n. One obtains a reasonable theory if one replaces the special but discontinuous functions $f(X)$ by the class of all functions $g(X)$ continuous in $X'X > 0$ which satisfy, just as $f(X)$, the relation of homogeneity

$$g(XV) = g(X) \text{ for non-singular } V = V^{(n)}. \qquad (7)$$

Then we can ask for the uniform approximation of these functions by elementary functions with a certain typical behaviour. Applying Weierstrass's well-known approximation theorem for continuous functions to $g(X)$, and using a certain positive hermitian metric in the space of the functions $g(X)$, we obtain by a straight-forward conclusion the following result: Let $X = (x_{\mu\nu})$, $\frac{\partial}{\partial X} = \left(\frac{\partial}{\partial x_{\mu\nu}}\right)$, $\Lambda = X\frac{\partial}{\partial X'} - \left(X\frac{\partial}{\partial X'}\right)'$ and denote by $\sigma(W)$ the trace of the square matrix W. Then we can find a finite set of polynomials $u_{ij}(X)$ with the properties

$$\left.\begin{array}{l} 1.\ u_{ij}(XV) = |V|^{2i}\, u_{ij}(X) \quad \text{for non-singular } V = V^{(n)}, \\ 2.\ \sigma(\Lambda^{2h})\, u_{ij}(X) = \lambda_{ij}^{(h)}\, u_{ij}(X) \text{ with constant eigenvalues} \\ \quad \lambda_{ij}^{(h)} \text{ for } h = 1, 2, \ldots, n, \\ 3.\ \left|\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right| u_{ij}(X) = 0, \end{array}\right\} \qquad (8)$$

such that

$$\left| g(X) - \sum_{i,j} |X'X|^{-i}\, u_{ij}(X) \right| < \epsilon \qquad (9)$$

for all X of rank n, where ϵ denotes a given positive real number.

All functions of X we are taking into consideration depend only upon the equivalence class $\tilde{X}$ of X which consists of all matrices XV with arbitrary real $V = V^{(n)}$ of determinant $|V| = 1$. Thus it is obvious to introduce the Plücker coordinates

$$\xi_\alpha = \xi_{\alpha_1\alpha_2\ldots\alpha_n} = |x_{\alpha_\mu\nu}|, \quad (\mu, \nu = 1, 2, \ldots, n) \qquad (10)$$

of $\tilde{X}$. For brevity we shall call these coordinates also Plücker co-ordinates of X. We denote by ξ the set of all ξ_α's. The first of the characteristic properties (8) says that $u_{ij}(x)$ is representable as an algebraic form in ξ of degree $2i$:

$$u_{ij}(X) = v_{ij}(\xi).$$

In particular we have

$$| X'X | = \sum_{\alpha} \xi_{\alpha}^2, \tag{11}$$

where the sum must be extended over all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Thus $| X'X |^{-1} u_{ij}(X)$ defines a function on the sphere

$$\sum_{\alpha} \xi_{\alpha}^2 = 1,$$

and it seems to be justified to speak of $u_{ij}(x)$ as a generalized spherical function. However we have to observe that the Plücker coordinates are not independent so that $u_{ij}(X)$ *de facto* is only defined on the Grassmannian manifold represented by the ξ_{α}'s.

The set of all differential operators $\sigma(\Lambda^{2h})$, $(h = 1, 2, \ldots, n)$ completed by $\sigma(X'\partial/\partial X)$ has a remarkable basic property which can be described in the following way. We define a linear differential operator Ω as a polynomial in the elements of $\partial/\partial X$ with functions of X as coefficients which have derivatives of arbitrary high order. We call Ω simply 'invariant' if Ω is invariant relative to the group of substitutions $X \to UXV$ where $U = U^{(m)}$ is an arbitrary orthogonal matrix and $|V = V^{(n)}$ an arbitrary non-singular one. Two invariant linear operators are said to be equal if they are of the same effect on all functions $f(X)$ which are invariant relative to $X \to XV$, $|V| = 1$. The invariant linear differential operators form obviously a ring $\mathfrak{R}$. We shall prove that $\mathfrak{R}$ is generated by the operators $\sigma\left(X' \frac{\partial}{\partial X}\right)$, $\sigma(\Lambda^{2h})$, $(h = 1, 2, \ldots, n)$. Thus the first two of the conditions (8) say that $u_{ij}(X)$ is a polynomial in the Plücker coordinates and also an eigenfunction of the ring $\mathfrak{R}$.

Our further interest is now concentrated on the series

$$\phi(s, S; u, v) = \sum_G u(QG)\, v(S[G])\, |S[G]|^{-s}, \tag{12}$$

$u(X)$ being an arbitrary spherical function of degree $2kn$ and $v(Y)$ an arbitrary angular character. The sum must be extended over the same set of matrices G as in (6). Now the question arises whether the functions (12) are zeta functions in the described sense, i.e. whether these functions satisfy a functional equation which expresses a simple transformation property relative to the substitution $s \to k' - s$ with a suitable $k' > 0$. Let $v(Y)$ run over all angular characters, then we obtain in $\phi(s, S; u, v)$ a set of functions which is supposed to be linear equivalent with the single series

$$\vartheta(Y, S; u) = \sum_G u(Q\,G)\, e^{-\pi\sigma(YS[G])},\ (Y = Y^{(n)} > 0), \tag{13}$$

(see [5]). At the present, this fact is provable only for $n = 2$. W. Roelcke investigated this case by using Mellin's integral-transformation [6]. In (13) G runs over all integral matrices of the type $G^{(m,n)}$.

We can probably expect that $\vartheta(Y, S; u)$ has a simple transformation property relative to the substitution $Y \to Y^{-1}$ if the functions $\phi(s, S; u, v)$ satisfy a functional equation of Riemannian type at all. In this respect we meet the following situation. Applying Poisson summation method to the theta-series

$$\vartheta(X, Y, S; u) = \sum_G u(Q(G + X))\, e^{-\pi\sigma(YS[G+X])},$$

we obtain with regard to $u(X\,V) = |\,V\,|^{2k}\, u(X)$ for

$$\vartheta(Y, S; u) = \vartheta(0,\ Y, S; u),$$

the representation

$$\vartheta(Y, S; u) = |\,S\,|^{-n/2}\,|\,Y\,|^{-m/2-k} \sum_G u^*(-\,i\,Q^{-1}G\,R^{-1})\, e^{-\pi\sigma(Y^{-1}S^{-1}[G])} \tag{14}$$

with $Y = R'R$, $R = R' > 0$ and

$$u^*(X) = \int_{\mathfrak{T}} u(X + T)\, e^{-\pi\sigma(T'T)}\, [dT], \tag{15}$$

where $\mathfrak{T}$ denotes the full space of all real matrices $T = T^{(m,n)} = (t_{\mu\nu})$ and $[dT]$ the product of all differentials $dt_{\mu\nu}$ [1]. According to

$$\int_{\mathfrak{T}} e^{-\pi\sigma(T'T)}\,[d\,T] = 1$$

one can state that $u^*(X) - u(X)$ is a polynomial in the elements of X with a degree less than that of $u(X)$ provided $u(X) \neq 0$. In general however it is

$$u^*(X) - u(X) \neq 0,$$

as examples show, and even $u^*(X)$ no algebraic form in the Plücker coordinates of X. Therefore it is also impossible to split off the factor R^{-1} in $u^*(-\,i\,Q^{-1}\,G\,R^{-1})$. If we assume however

$$\sigma\left(\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\right) u(X) = 0, \tag{16}$$

it follows, as it was proved recently also by C. S. Herz [4],

$$u^*(X) = u(X),$$

i.e. $u(X)$ is an eigenfunction of Gauss integral-transformation (15). Moreover it can be shown that (16) is not only sufficient but also necessary for $u(X)$ being an eigenfunction of this kind. Assuming (16) we now obtain

$$u^*(-\,i\,Q^{-1}\,G\,R^{-1}) = (-\,1)^{kn}\,|\,Y\,|^{-k}\,u(Q^{-1}\,G),$$

and thus we see that (14) can be rewritten in the form

$$\vartheta(Y, S; u) = (-\,1)^{kn}\,|\,S\,|^{-n/2}\,|\,Y\,|^{-m/2-2k}\,\vartheta(Y^{-1}, S^{-1}; u). \tag{17}$$

The relation of homogeneity $u(X\,V) = |\,V\,|^{2k}\,u(X)$ effects a decomposition of the differential equation (16) into the system

$$\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\,u(X) = 0. \tag{18}$$

A consequence of this is $\sigma(\Lambda^{2h})\,u(X) = \lambda^{(h)}\,u(X)$ for $h = 1, 2, \ldots, n$ with certain constant eigenvalues $\lambda^{(h)}$. In the case $n = 1$ our supposition (16) does not go beyond (8). Thus the general transformation formula (17) corresponds with the results of

Schoeneberg [8]. Maybe it is sufficient also in the case $n > 1$ to take into consideration only those spherical functions $u(x)$ which are solutions of the Laplacian differential equation (16) in order to approximate the functions defined by (4). Solutions of (18) which also satisfy the relation of homogeneity are given by

$$u(x) = |L'X|^{2k}, \tag{19}$$

$L = L^{(m,\, n)}$ being an arbitrary complex solution of $L'L = 0$, [3].

Now we assume the generalized spherical function $u(X)$ to be a non-constant eigenfunction of Gauss integral-transformation. Further let $v(Y)$ be a bounded angular character, i.e. we have

$$\left.\begin{aligned} \left(\sigma\left(Y\frac{\partial}{\partial Y}\right)^h + \lambda_h\right) v(Y) &= 0, \text{ for } h = 1, 2, \ldots, n, \\ v(Y[U]) &= v(Y) \text{ for unimodular } U, \end{aligned}\right\} \tag{20}$$

with the notation

$$Y = (y_{\mu\nu}), \frac{\partial}{\partial Y} = \left(e_{\mu\nu}\frac{\partial}{\partial y_{\mu\nu}}\right), \; e_{\mu\nu} = \begin{cases} 1, \text{ for } \mu = \nu, \\ \frac{1}{2}, \text{ for } \mu \neq \nu. \end{cases} \tag{21}$$

$\lambda_1, \lambda_2, \ldots, \lambda_\mu$ are constant eigenvalues; in particular we have $\lambda_1 = 0$. It is easy to show that $v^*(Y) = v(Y^{-1})$ also defines an angular character which in general however belongs to another system of eigenvalues λ_h. Since now rank $X < n$ implies $u(X) = 0$, it is sufficient to extend the sum in

$$\vartheta(Y, S; u) = \sum_G u(Q\ G)\, e^{-\pi\sigma(YS[G])}, \tag{22}$$

over all integral matrices G of rank n so that always $S[G] > 0$. This is important because at present we can prove a functional equation for $\phi(s, S; u, v)$ only if the theta-series shows the behaviour of a so-called cusp form. By means obtained in [5] we shall prove that the function defined by the Dirichlet series (12) is an entire function of s which satisfies the functional equation

$$\xi(\tfrac{1}{2}m + 2k - s, S; u, v) = (-1)^{kn}\, |S|^{-n/2}\, \xi(s, S^{-1}; u, v^*), \tag{23}$$

where

$$\xi(s, S; u, v) = \pi^{-ns}\,\Gamma(s-\beta_1)\,\Gamma(s-\beta_2)\ldots\Gamma(s-\beta_n)\,\phi(s, S; u, v), \quad (24)$$

with certain constants $\beta_1, \beta_2, \ldots, \beta_n$ which depend upon the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ only. This is the main result of the present paper.

1. Plücker coordinates. We denote by $\delta_{\mu\nu}$ the Kronecker symbol and introduce

$$I_\alpha = I(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\delta_{\alpha_\nu \mu}), \quad (\mu = 1, 2, \ldots, m;\ \nu = 1, 2, \ldots, n), \quad (25)$$

$\alpha_1, \alpha_2, \ldots, \alpha_n$ being an arbitrary system of integers in the interval from 1 to m, so that

$$I'_\alpha X = (x_{\alpha_\mu \nu}), \quad (\mu, \nu = 1, 2, \ldots, n).$$

The Plücker coordinates ξ_α of X are given by

$$\xi_\alpha = \xi_{\alpha_1\alpha_2\ldots\alpha_n} = |I'_\alpha X|. \quad (26)$$

Any summation over α is to extend always over the full system of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n \leqslant m$. If λ_α are the Plücker coordinates of $L = L^{(m,n)}$ then we have as is well known

$$|L' X| = \sum_\alpha \lambda_\alpha \xi_\alpha, \text{ particularly } |X' X| = \sum_\alpha \xi_\alpha^2.$$

We compute the effect of some differential operators on functions of the type $f(\xi) = f(\ldots, \xi_\alpha, \ldots)$. First we state

$$\frac{\partial}{\partial X}\xi_\alpha = I_\alpha (X' I_\alpha)^{-1} \xi_\alpha. \quad (27)$$

Denoting by $A^\alpha_{\mu\nu}$ the algebraic complement of $x_{\alpha_\mu \nu}$ in $|I'_\alpha X|$ we obtain indeed

$$\frac{\partial}{\partial X}\xi_\alpha = \left(\frac{\partial}{\partial x_{\mu\nu}}\xi_\alpha\right) = \left(\sum_\sigma \delta_{\alpha_\sigma \mu} A^\alpha_{\sigma\nu}\right) = (\delta_{\alpha_\nu \mu})(A^\alpha_{\mu\nu}) = I_\alpha (X' I_\alpha)^{-1} \xi_\alpha.$$

(27) yields in particular

$$X' \frac{\partial}{\partial X}\xi_\alpha = \xi_\alpha E, \ E = \text{unit matrix}. \quad (28)$$

Consequently

$$X' \frac{\partial}{\partial X} f(\xi) = \sum_{\alpha} \frac{\partial f(\xi)}{\partial \xi_\alpha} X' \frac{\partial}{\partial X} \xi_\alpha = \sum_{\alpha} \xi_\alpha \frac{\partial}{\partial \xi_\alpha} f(\xi) E. \tag{29}$$

Let $L = L^{(m,n)} = (l_{\mu\nu})$ be a constant matrix of rank n and $(\zeta_{\mu\nu}) = (X'L)^{-1}$, then it is

$$\sum_{\rho,\sigma} \zeta_{\mu\rho}\, x_{\sigma\rho}\, l_{\sigma\nu} = \delta_{\mu\nu}.$$

Differentiation yields

$$\sum_{\rho,\sigma} \frac{\partial \zeta_{\mu\rho}}{\partial x_{\kappa\lambda}} x_{\sigma\rho}\, l_{\sigma\nu} + \zeta_{\mu\lambda}\, l_{\kappa\nu} = 0,$$

from which

$$\frac{\partial \zeta_{\sigma\nu}}{\partial x_{\rho\mu}} = - \sum_{\kappa} \zeta_{\sigma\mu}\, l_{\rho\kappa}\, \xi_{\kappa\nu}$$

follows. Thus we find

$$\frac{\partial}{\partial X'} L(X'L)^{-1} = \Big(\sum_{\rho,\sigma} \frac{\partial}{\partial x_{\rho\mu}} l_{\rho\sigma}\, \zeta_{\sigma\nu}\Big) = -\Big(\sum_{\rho,\sigma} l_{\rho\sigma}\, \zeta_{\sigma\mu}\, l_{\rho\kappa}\, \zeta_{\kappa\nu}\Big)$$
$$= -(\zeta_{\mu\nu})' L' L(\zeta_{\mu\nu}),$$

particularly for $L = I_\alpha$

$$\frac{\partial}{\partial X'} I_\alpha (X' I_\alpha)^{-1} = -(I'_\alpha X)^{-1} (X' I_\alpha)^{-1}, \tag{30}$$

according to $I'_\alpha I_\alpha = E$. Applying the operator $\partial/\partial X'$ to

$$\frac{\partial}{\partial X} f(\xi) = \sum_{\alpha} \frac{\partial f(\xi)}{\partial \xi_\alpha} I_\alpha (X' I_\alpha)^{-1} \xi_\alpha,$$

we obtain

$$\frac{\partial}{\partial X'} \frac{\partial}{\partial X} f(\xi) = \sum_{\alpha,\beta} \xi_\beta \frac{\partial}{\partial \xi_\beta} \Big(\xi_\alpha \frac{\partial f(\xi)}{\partial \xi_\alpha}\Big) (I'_\beta X)^{-1} I'_\beta I_\alpha (X' I_\alpha)^{-1} -$$
$$- \sum_{\alpha} \xi_\alpha \frac{\partial f(\xi)}{\partial \xi_\alpha} (I'_\alpha X)^{-1} (X' I_\alpha)^{-1}$$
$$= \sum_{\alpha,\beta} (I'_\beta X)^{-1} I'_\beta I_\alpha (X' I_\alpha)^{-1} \xi_\beta\, \xi_\alpha \frac{\partial^2 f(\xi)}{\partial \xi_\beta\, \partial \xi_\alpha}.$$

Therefore

$$X\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)' f(\xi) = \sum_{\alpha,\beta} \Xi_{\alpha\beta}\frac{\partial}{\partial \xi_\alpha}\frac{\partial}{\partial \xi_\beta} f(\xi), \tag{31}$$

with

$$\Xi_{\alpha\beta} = \xi_\alpha \xi_\beta\, X(I'_\alpha X)^{-1} I'_\alpha I_\beta (X' I_\beta)^{-1} X' = \Xi'_{\beta\alpha} \tag{32}$$

holds. The operators

$$\frac{\partial}{\partial X'}\frac{\partial}{\partial X} \text{ and } X\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)'$$

annihilate the same functions. From

$$X\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)' g(X) = X\left[\frac{\partial}{\partial X'}\frac{\partial}{\partial X} g(X)\right] X' = 0,$$

follows, by left and right-hand multiplication with X' and X respectively, since $X'X > 0$, indeed that

$$\frac{\partial}{\partial X'}\frac{\partial}{\partial X} g(X) = 0.$$

We compute the elements of the matrix

$$\Xi_{\alpha\beta} = (\xi_{\mu\nu}^{\alpha\beta}),\ (\mu, \nu = 1, 2, \ldots, m), \tag{33}$$

as functions of the Plücker coordinates. Since $I'_\alpha I_\beta = (\delta_{\alpha_\mu \beta_\nu})$ we obtain

$$\xi_{\mu\nu}^{\alpha\beta} = \sum_{\rho,\sigma,\tau,\kappa} x_{\mu\rho}\, A_{\sigma\rho}^{\alpha}\, \delta_{\alpha_\sigma \beta_\tau}\, A_{\tau\kappa}^{\beta}\, x_{\nu\kappa}, \tag{34}$$

with $A_{\sigma\rho}^{\alpha}$ in the significance already introduced. The replacement of the σth row in $|I'_\alpha X|$ by $(x_{\mu 1}, x_{\mu 2}, \ldots x_{\mu n})$ leads obviously to $\sum\limits_{\rho} x_{\mu\rho}\, A_{\sigma\rho}^{\alpha}$.

Thus we obtain

$$\begin{aligned}\sum_{\rho} x_{\mu\rho}\, A_{\sigma\rho}^{\alpha} &= |I'(\alpha_1, \ldots, \alpha_{\sigma-1}, \mu, \alpha_{\sigma+1}, \ldots, \alpha_n)\, X| \\ &= \xi_{\alpha_1 \ldots \alpha_{\sigma-1} \mu \alpha_{\sigma+1} \ldots \alpha_n}.\end{aligned}$$

Now we introduce the notation

$$\xi_{\alpha}^{\mu\to\nu} = \xi_{\alpha_1 \ldots \alpha_{\sigma-1} \nu \alpha_{\sigma+1} \ldots \alpha_n} \text{ or } 0, \tag{35}$$

according as $\mu = \alpha_\sigma$ or $\mu \neq \alpha_1, \alpha_2, \ldots \alpha_n$. So we can rewrite (34) as

$$\xi^{\alpha\beta}_{\mu\nu} = \sum_{\sigma,\tau} \xi^{\alpha_\sigma \to \mu}_{\alpha} \, \delta_{\alpha_\sigma \beta_\tau} \, \xi^{\beta_\tau \to \nu}_{\beta} = \sum_{\rho} \xi^{\rho \to \mu}_{\alpha} \, \xi^{\rho \to \nu}_{\beta}. \tag{36}$$

According to the signification of the symbol $\xi^{\mu \to \nu}_{\alpha}$ the sum in (36) can be extended over all integers ρ from 1 to m. A special consequence of (32) is also

$$\sigma\Big(X' X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big) f(\xi) = \sigma\Big(X \Big(X \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \Big)' \Big) f(\xi)$$

$$= \sum_{\alpha,\beta} \xi^{\alpha\beta} \frac{\partial}{\partial \xi_\alpha} \frac{\partial}{\partial \xi_\beta} f(\xi), \tag{37}$$

with

$$\xi^{\alpha\beta} = \sigma(\Xi_{\alpha\beta}) = \sum_{\rho,\mu} \xi^{\rho \to \mu}_{\alpha} \, \xi^{\rho \to \mu}_{\beta}. \tag{38}$$

The remaining formulae of this section apply to the special case $m = n + 1$. Now we note

$$\xi_\alpha = \eta_\kappa, \quad \xi^{\alpha\beta}_{\mu\nu} = \eta^{\kappa\lambda}_{\mu\nu}, \quad \xi^{\alpha\beta} = \eta^{\kappa\lambda}$$

for

$$\alpha = (1, \ldots, \kappa - 1, \kappa + 1, \ldots, m), \beta = (1, \ldots, \lambda - 1, \lambda + 1, \ldots, m).$$

It is easy to see that

$$\xi^{\rho \to \mu}_{1 \ldots \kappa - 1 \kappa + 1 \ldots m} = \delta_{\mu\kappa} (-1)^{\kappa + \rho + 1} \eta_\rho + \delta_{\mu\rho} \eta_\kappa.$$

Thus we obtain

$$\eta^{\kappa\lambda}_{\mu\nu} = \sum_{\rho} \xi^{\rho \to \mu}_{1 \ldots \kappa - 1 \kappa + 1 \ldots m} \, \xi^{\rho \to \nu}_{1 \ldots \lambda - 1 \lambda + 1 \ldots m}$$

$$= \sum_{\rho} \Big(\delta_{\mu\kappa} (-1)^{\kappa + \rho + 1} \eta_\rho + \delta_{\mu\rho} \eta_\kappa) \cdot (\delta_{\nu\lambda} (-1)^{\lambda + \rho + 1} \eta_\rho + \delta_{\nu\rho} \eta_\lambda \Big)$$

$$= \delta_{\mu\kappa} \delta_{\nu\lambda} (-1)^{\kappa + \lambda} \sum_{\rho} \eta^2_\rho + \delta_{\mu\nu} \eta_\kappa \eta_\lambda +$$

$$+ \delta_{\mu\kappa} (-1)^{\kappa + \nu + 1} \eta_\nu \eta_\lambda + \delta_{\nu\lambda} (-1)^{\lambda + \mu + 1} \eta_\mu \eta_\kappa$$

and

$$\eta^{\kappa\lambda} = \sum_{\mu} \eta^{\kappa\lambda}_{\mu\mu} = \delta_{\kappa\lambda} \sum_{\rho} \eta^2_\rho + (m - 2) \eta_\kappa \eta_\lambda.$$

Now we find for (37) the expression

$$\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)f(\eta)$$
$$= \left\{\left(\sum_\rho \eta_\rho^2\right)\left(\sum_\kappa \frac{\partial^2}{\partial \eta_\kappa^2}\right) + (m-2)\left(\sum_\kappa \eta_\kappa \frac{\partial}{\partial \eta_\kappa}\right)^2 - \right.$$
$$\left. - (m-2)\left(\sum_\kappa \eta_\kappa \frac{\partial}{\partial \eta_\kappa}\right)\right\} f(\eta). \quad (39)$$

2. Invariant differential operators. Let Ω be a linear differential operator, i.e. a polynomial in the elements of X, and let us assume that the coefficients which are functions of X have derivatives of arbitrary high order. Ω as a polynomial in the elements of $\partial/\partial X$ has a certain degree; this we call plainly the degree of Ω. All linear operators Ω of degree $\leqslant h$ constitute a module which we denote by $\mathfrak{M}_h$. Obviously $\mathfrak{M}_h \subset \mathfrak{M}_{h+1}$ for all h. The module of all linear differential operators which is indentical with $\mathfrak{M} = \bigcup_h \mathfrak{M}_h$ defines a non-commutative ring. It is easy to see that

$$\Omega_1\,\Omega_2 \equiv \Omega_2\,\Omega_1 \pmod{\mathfrak{M}_{h-1}}, \quad (40)$$

provided that the product $\Omega_1\,\Omega_2$ lies in $\mathfrak{M}_h$. The aim of the following considerations is the determination of a basis for the subring $\mathfrak{R}$ of $\mathfrak{M}$ consisting of all linear operators which are invariant relative to the substitution

$$X \to U X V,\ \frac{\partial}{\partial X} \to U\frac{\partial}{\partial X}V'^{-1} \text{ with } U'U = E,\ |V| \neq 0. \quad (41)$$

In the sequel we use the notation $\mathfrak{R}_h$ for the intersection $\mathfrak{R} \cap \mathfrak{M}_h$.

Let $\Omega = F(X, \partial/\partial X)$ be a given operator in $\mathfrak{R}_h$. We choose a matrix $T = T^{(m,n)}$ with variable elements which are commutable with those of X. Then we have in particular

$$F(UX, UT) = F(X, T) \text{ for } U'U = E.$$

Thus, according to well-known theorems of the theory of algebraic invariants, we see that $F(X, T)$ is a polynomial in the elements of $X'T$ and $T'T$ with functions of $X'X$ as coefficients. Then there exists also a representation

$$F(X, T) = \sum_\nu G_\nu(X'X,\ T'T)\ H_\nu(X'T),$$

G_ν being a polynomial in the elements of $T'T$ and H_ν a polynomial in the elements of $X'T$. Now we observe the invariance of $F(X, T)$ relative to the substitutions $X \to XV$, $T \to TV'^{-1}$. We set $V = V_0 V_1$ with V_0 determined by $(X'X)[V_0] = E$ and an arbitrary orthogonal matrix V_1. Using the notation $W = (T'T)\ [V'^{-1}_0]$ we obtain

$$F(X, T) = \sum_\nu G_\nu(E, W[V_1])\ H_\nu(V'_1(V_0^{-1}(V'_0 X' T)')' V_1).$$

The argument of H_ν is of course, since we are still moving in commutative domains, with $V'_1 V'_0 X'T V_0'^{-1} V_1 = V'X'T\ V'^{-1}$ identical. Because of (40) all products performed in $F\left(X, \frac{\partial}{\partial X}\right)$ admit commutations if we carry out the computations modulo $\mathfrak{M}_{h-1}$. So it turns out

$$\Omega \equiv \sum_\nu G_\nu\left(E, \left(\left(\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right)[V_0'^{-1}]\right)[V_1]\right) \times$$

$$\times H_\nu\left(V_1'\left(V_0^{-1}\left(V_0' X' \frac{\partial}{\partial X'}\right)'\right)' V_1\right) \pmod{\mathfrak{M}_{h-1}}.$$

Applying (29) we see that

$$V_1'\left(V_0^{-1}\left(V_0' X' \frac{\partial}{\partial X}\right)'\right)' V_1 f(\xi) = \sum_\alpha \xi_\alpha \frac{\partial}{\partial \xi_\alpha} f(\xi)\ E$$

$$= \frac{1}{n}\,\sigma\left(X' \frac{\partial}{\partial X}\right) f(\xi)\ E$$

holds for an arbitrary function $f(\xi)$. Thus we obtain

$$H_\nu\left(V_1'\left(V_0^{-1}\left(V_0' X' \frac{\partial}{\partial X}\right)'\right)' V_1\right) f(\xi) = H_\nu\left(\frac{1}{n}\,\sigma\left(X' \frac{\partial}{\partial X}\right) E\right) f(\xi)$$

$$= h_\nu\left(\sigma\left(X' \frac{\partial}{\partial X}\right)\right) f(\xi),$$

where $h_\nu(z)$ denotes a polynomial of z. This leads to

$$\Omega f(\xi) \equiv \sum_{\nu} G_\nu\Big(E, \Big(\Big(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\Big)[V_0'^{-1}]\Big)[V_1]\Big) \times$$

$$\times\, h_\nu\Big(\sigma\Big(X'\frac{\partial}{\partial X}\Big)\Big) f(\xi) \; \Big(\bmod \mathfrak{M}_{h-1} f(\xi)\Big). \quad (42)$$

$G_\nu(E, W[V_1])$ represents a continuous function on the compact group of all orthogonal matrices V_1. Thus the mean value

$$M_{V_1}\{G_\nu(E, W[V_1])\} = g_\nu(W)$$

(in the sense of the theory of almost periodic functions) exists. It is a polynomial in the elements of W which is invariant relative to orthogonal substitutions :

$$g_\nu(W[V_1]) = g_\nu(W), \; (V_1' V_1 = E).$$

Accordingly $g_\nu(W)$ is a symmetric polynomial in the characteristic roots of W, thus a polynomial in $\sigma(W^h)$, $(h = 1, 2, \dots, n)$:

$$g_\nu(W) = p_\nu(\sigma(W), \sigma(W^2), \dots, \sigma(W^n)).$$

With regard to the signification of W and V_1 we state easily

$$\sigma(W^h) = \sigma(X'XT'T)^h.$$

If we compute the mean value with respect to V_1 on the right hand side of (42) we obtain by means of the deduced relations

$$\Omega f(\xi) \equiv \Omega^* f(\xi) \;\; (\bmod \mathfrak{M}_{h-1} f(\xi)),$$

with

$$\Omega^* = \sum_{\nu} p_\nu\Big(\dots, \sigma\Big(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\Big)^h, \dots\Big) h_\nu\Big(\sigma\Big(X'\frac{\partial}{\partial X}\Big)\Big). \quad (43)$$

It is obvious that this operation is invariant relative to the substitutions (41).

In the sequel we shall identify invariant operations which have the same effect on all functions of the kind $f(\xi)$. Then we can state the following facts: To a given operator $\Omega \in \mathfrak{R}_h$ there exists an operator $\Omega^* \in \mathfrak{R}_h$ of the special form (43) such that $\Omega - \Omega^* \in \mathfrak{R}_{h-1}$. Induction on h yields at once

THEOREM 1. *The invariant operators* $\sigma\left(X'\frac{\partial}{\partial X}\right)$, $\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^h$ $(h=1,2,\ldots,n)$ *form a basis for the ring* $\mathfrak{R}$ *of all invariant linear operators.*

Our argument shows moreover that we only need the basis elements $\sigma\left(X'\frac{\partial}{\partial X}\right)$, $\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^k$ $(k=1,2,\ldots,[h/2])$ for the representation of an invariant operator of degree $h \leqslant 2n$. Now it is easy to see that the invariant operators

$$\sigma\left(X'\frac{\partial}{\partial X}\right), \sigma\left(X\frac{\partial}{\partial X'}\left(X\frac{\partial}{\partial X'}\right)'\right)^h, (h=1,2,\ldots,n) \tag{44}$$

also generate $\mathfrak{R}$. We have

$$\sigma\left(X\frac{\partial}{\partial X'}\left(X\frac{\partial}{\partial X'}\right)'\right)^h \equiv \sigma\left(X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}X'\right)^h \equiv \sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^h$$

$$(\operatorname{mod} \mathfrak{M}_{2h-1}).$$

Thus by induction on h we obtain

$$\sigma\left(X\frac{\partial}{\partial X'}\left(X\frac{\partial}{\partial X'}\right)'\right)^h$$

$$= \sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^h +$$

$$+ q_h\left(\sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right), \ldots, \sigma\left(X'X\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right)^{h-1}, \sigma\left(X'\frac{\partial}{\partial X}\right)\right)$$

where q_h denotes a certain polynomial. This proves the basis property for the system (44). We determine yet a third basis for the ring $\mathfrak{R}$.

THEOREM 2. *If* $\Lambda = X\frac{\partial}{\partial X} - \left(X\frac{\partial}{\partial X'}\right)'$ *then the invariant operators* $\sigma\left(X'\frac{\partial}{\partial X}\right)$, $\sigma(\Lambda^{2h})$ $(h=1,2,\ldots,n)$ *form a basis for the ring* $\mathfrak{R}$ *of all invariant linear operators.*

In order to prove this we compute

$$\begin{aligned}\sigma(\Lambda^{2h}) &= \sigma\Big(X\frac{\partial}{\partial X'} - \Big(X\frac{\partial}{\partial X'}\Big)'\Big)^{2h} \\ &= (-1)^h\,\sigma\Big(X\frac{\partial}{\partial X'}\Big(X\frac{\partial}{\partial X'}\Big)'\Big)^h + \\ &\quad + (-1)^h\,\sigma\Big(\Big(X\frac{\partial}{\partial X'}\Big)' X\frac{\partial}{\partial X'}\Big)^h + \sum_\nu \pm\,\sigma(P_\nu).\end{aligned}$$

Here P_ν are products of the form

$$\Big(X\frac{\partial}{\partial X'}\Big)^{\mu_1}\Big(\Big(X\frac{\partial}{\partial X'}\Big)'\Big)^{\nu_1}\cdots\Big(X\frac{\partial}{\partial X'}\Big)^{\mu_r}\Big(\Big(X\frac{\partial}{\partial X'}\Big)'\Big)^{\nu_r},$$

and it happens at least once that one of the exponents μ_i, ν_i is greater than 1. Hence it is

$$P_\nu = Q_\nu\Big(X\frac{\partial}{\partial X'}\Big)^2 R_\nu \quad\text{or}\quad P_\nu = Q_\nu\Big(\Big(X\frac{\partial}{\partial X'}\Big)'\Big)^2 R_\nu,$$

with certain products Q_ν and R_ν which are also of the given form. Now it follows in the first case (the second one can be treated analogously)

$$\begin{aligned}\sigma(P_\nu) &= \sigma(P_\nu') \\ &\equiv \sigma\Big(R_\nu'\frac{\partial}{\partial X}X'\frac{\partial}{\partial X}X'Q_\nu'\Big) \\ &\equiv \sigma\Big(X'Q_\nu'R_\nu'\frac{\partial}{\partial X}\cdot X'\frac{\partial}{\partial X}\Big) \quad (\mathrm{mod}\ \mathfrak{M}_{2h-1}),\end{aligned}$$

and therefore by means of (29)

$$\begin{aligned}\sigma(P_\nu)f(\xi) &\equiv \sigma\Big(X'Q_\nu'R_\nu'\frac{\partial}{\partial X}\Big)\frac{1}{n}\sigma\Big(X'\frac{\partial}{\partial X}\Big)f(\xi) \\ &\equiv \sigma\Big(\frac{\partial}{\partial X'}R_\nu Q_\nu X\Big)\frac{1}{n}\sigma\Big(X'\frac{\partial}{\partial X}\Big)f(\xi) \\ &\equiv \sigma\Big(X\frac{\partial}{\partial X'}R_\nu Q_\nu\Big)\frac{1}{n}\sigma\Big(X'\frac{\partial}{\partial X}\Big)f(\xi) \quad (\mathrm{mod}\ \mathfrak{M}_{2h-1}\,f(\xi)).\end{aligned}$$

So we obtain

$$\sigma(\Lambda^{2h})\, f(\xi) \equiv (-1)^h\, 2\, \sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^h f(\xi) +$$

$$+ \Omega_h\, \sigma\Big(X' \frac{\partial}{\partial X} \Big) f(\xi) \quad (\mathrm{mod}\ \mathfrak{M}_{2h-1} f(\xi)),$$

with a certain invariant operator Ω_h of degree at most $2h - 1$ which can be represented as a polynomial of the form

$$\Omega_h = p_h \Big(\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big), \dots,$$

$$\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^{h-1}, \sigma\Big(X' \frac{\partial}{\partial X} \Big) \Big).$$

The same is true for the operator

$$\sigma(\Lambda^{2h}) - (-1)^h\, 2\, \sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^h - \Omega_h\, \sigma\Big(X' \frac{\partial}{\partial X} \Big).$$

So we see that

$$\sigma(\Lambda^{2h}) = (-1)^h\, 2\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^h +$$

$$+ q_h \Big(\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big), \dots,$$

$$\sigma\Big(X \frac{\partial}{\partial X'} \Big(X \frac{\partial}{\partial X'} \Big)' \Big)^{h-1}, \sigma\Big(X' \frac{\partial}{\partial X} \Big) \Big)$$

holds with a certain polynomial q_h. Theorem 2 now is an easy consequence of this.

3. Spherical functions. A polynomial $u(X)$ shall be called a spherical function of type (m, n) if the following conditions are satisfied:

$$\left.\begin{array}{ll} 1. & u(XV) = u(X) \text{ for } |V| = 1, \\ 2. & u(X) \text{ is an eigenfunction of all invariant linear differential operators,} \\ 3. & \left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right| u(X) = 0, \end{array}\right\} \quad (45)$$

The first condition says that $u(X)$ is a polynomial in the Plücker coordinates ξ_α of X:

$$u(X) = f(\xi).$$

The second condition is, according to Theorem 2, equivalent to

$$\sigma\left(X' \frac{\partial}{\partial X}\right) u(X) = n \sum_\alpha \xi_\alpha \frac{\partial}{\partial \xi_\alpha} f(\xi) = k\, n\, f(\xi) = k\, n\, u(X) \quad (46)$$

and

$$\sigma(\Lambda^{2h})\, u(X) = \lambda^{(h)}\, u(X) \; (h = 1, 2, \dots, n), \quad (47)$$

$k, \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(n)}$ being certain constants. Thus $f(\xi)$ is an algebraic form of degree k so k is a non-negative integer. In order to understand the third condition we observe that $|X'X| \left|\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right|$ is an invariant linear operator. So we have also

$$|X'X| \left|\frac{\partial}{\partial X'} \frac{\partial}{\partial X}\right| \; u(X) = \lambda\, u(X) \quad (48)$$

with a certain constant λ. The third condition is obviously equivalent to $\lambda = 0$.

Let us assume that the polynomial $u(X)$ satisfies only the first two but not the third of the conditions (45). Then it is obvious that $|X'X|$ is a divisor of $u(X)$. We prove the existence of an integer $j \geqslant 1$ such that $u_j(X) = |X'X|^{-j}\, u(X)$ is still a polynomial which satisfies also the third of the conditions (45); in other words $u_j(X)$ represents a spherical function. First of all we observe that the elements $y_{\kappa\lambda}$ of the matrix $Y = X'X$ can be considered as constants with respect to the operator Λ. It is indeed

$$\begin{aligned}\Lambda\, y_{\kappa\lambda} &= \left(\sum \left(x_{\mu\rho} \frac{\partial}{\partial x_{\nu\rho}} - x_{\nu\rho} \frac{\partial}{\partial x_{\mu\rho}}\right)\right) \sum_\sigma x_{\sigma\kappa}\, x_{\sigma\lambda} \\ &= \left(\sum_{\rho,\sigma} x_{\mu\rho}(\delta_{\nu\sigma}\, \delta_{\rho\kappa}\, x_{\sigma\lambda} + x_{\sigma\kappa}\, \delta_{\nu\sigma}\, \delta_{\rho\lambda})\right) - \\ &\quad - \left(\sum_{\rho,\sigma} x_{\nu\rho}(\delta_{\mu\sigma}\, \delta_{\rho\kappa}\, x_{\sigma\lambda} + x_{\sigma\kappa}\, \delta_{\mu\sigma}\, \delta_{\rho\lambda})\right) \\ &= (x_{\mu\kappa}\, x_{\nu\lambda} + x_{\mu\lambda}\, x_{\nu\kappa}) - (x_{\nu\kappa}\, x_{\mu\lambda} + x_{\nu\lambda}\, x_{\mu\kappa}) = 0,\end{aligned}$$

so that each function $\phi(Y)$, in particular $|X'X|^j$, is commutable with the operators $\sigma(\Lambda^{2h})$. Therefore (46) and (47) imply

$$\sigma\left(X'\frac{\partial}{\partial X}\right) u_j(X) = (k-2j)\, n\, u_j(X),$$

$$\sigma(\Lambda^{2h})\, u_j(X) = \lambda^{(h)}\, u_j(X) \quad (h = 1, 2, \dots, n),$$

consequently also

$$|X'X| \left|\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right| u_j(X) = \lambda_j\, u_j(X) \quad (j = 1, 2, 3, \dots)$$

with certain constants λ_j. Now we deduce: If $\lambda_1 \neq 0$ then $|X'X|$ divides $u_1(X)$, i.e. $u_2(X)$ is a polynomial. If $\lambda_2 \neq 0$ the same conclusion shows that $u_3(X)$ is a polynomial. So it turns out that an integer $j \geqslant 1$ exists such that $u_j(X)$ is a polynomial but $\lambda_j = 0$.

We assert that a polynomial $u(X)$ with the three properties

$$\left.\begin{array}{ll} 1. & u(XV) = |V|^k\, u(X) \;\text{ for } |V| \neq 0, \\ 2. & \sigma\left(\dfrac{\partial}{\partial X'}\dfrac{\partial}{\partial X}\right) u(X) = 0, \\ 3. & |X'X| \text{ is no divisor of } u(X), \end{array}\right\} \tag{49}$$

is already a spherical function. First we state that

$$\sigma\left(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right) u(XV) = |V|^k\, \sigma\left(\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\right) u(X) = 0$$

is true for arbitrary non-singular V. Replacing $X \to XV^{-1}$, we obtain

$$\sigma\left(V\frac{\partial}{\partial X'}\frac{\partial}{\partial X}V'\right) u(X) = 0.$$

This implies

$$\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\, u(X) = 0. \tag{50}$$

According to Theorem 1 it turns out that $u(X)$ is an eigenfunction of all invariant linear operators. The third of the conditions (45) is a consequence of the fact that $|X'X|$ does not divide $u(X)$. This proves our assertion.

A special class of spherical functions is given by

$$u(X) = |L'X|^k \text{ with } L = L^{(m,n)},\ L'L = 0. \tag{51}$$

We assume that L is of rank n. Then at least one of the Plücker coordinates λ_α of L differs from 0. According to a well-known formula we have

$$u(X) = \Big(\sum_\alpha \lambda_\alpha \xi_\alpha\Big)^k,$$

an expression which is obviously not divisible by $|X'X| = \sum_\alpha \xi_\alpha^2$.

So it is sufficient to prove that the algebraic forms (51) are solutions of (50). This can be done in the following way. We set

$$X = (x_{\mu\nu}) = (\mathfrak{x}_1\, \mathfrak{x}_2 \cdots \mathfrak{x}_n),\ L = (l_{\mu\nu}) = (\mathfrak{l}_1\, \mathfrak{l}_2 \cdots \mathfrak{l}_n)$$

and denote by $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_m$ the columns of the m-rowed unit matrix. Then we have

$$\frac{\partial}{\partial X} |L'X|^k = k\, |L'X|^{k-1} M$$

with

$$M = \frac{\partial}{\partial X} |L'X| = \Big(\frac{\partial}{\partial x_{\mu\nu}} |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_n)|\Big)$$
$$= (|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\mu\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|).$$

For $\mu \neq \nu$ we have

$$\frac{\partial}{\partial x_{\rho\mu}} |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|$$
$$= |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\mu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\mu+1} \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| = 0,$$

since two columns of this determinant are equal. For $\mu = \nu$ we have also

$$\frac{\partial}{\partial x_{\rho\mu}} |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| = 0$$

since the elements of this determinant do not depend upon $x_{\rho\nu}$ at all. So we obtain

$$\frac{\partial}{\partial X'} M = \Big(\sum_{\rho} \frac{\partial}{\partial x_{\rho\mu}} |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| \Big) = 0,$$

and it follows that

$$\begin{aligned} \frac{\partial}{\partial X'} \frac{\partial}{\partial X} |L' X|^k &= k \frac{\partial}{\partial X'} |L' X|^{k-1} M \\ &= k(k-1) |L' X|^{k-2} M' M + k |L' X|^{k-1} \frac{\partial}{\partial X'} M \\ &= k(k-1) |L'X|^{k-2} M'M. \end{aligned}$$

Now it remains to show that $L'L = (\mathfrak{l}'_\mu\, \mathfrak{l}_\nu) = 0$ implies $M'M = 0$. First we form with a variable matrix $Z = Z^{(m,n)} = (z_{\mu\nu})$ the product

$$\begin{aligned} Z'M &= \Big(\sum_{\rho} z_{\rho\mu}\, |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| \Big) \\ &= (\, |L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{z}_\mu\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|\,). \end{aligned}$$

Here $\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_n$ denote the columns of Z. We choose $Z = M$ and prove that in this case $\mathfrak{z}_\mu$ is a linear combination of the columns $\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_n$ of L. Then it turns out that $L'\mathfrak{z}_\mu = 0$ and finally $M'M = 0$. We introduce $L'\mathfrak{x}_\nu = \mathfrak{a}_\nu$ and denote by $\mathfrak{e}^*_1, \mathfrak{e}^*_2, \ldots, \mathfrak{e}^*_n$ the columns of the n-rowed unit matrix. Because of $L'\, \mathfrak{e}_\mu = \sum l_{\mu\rho}\, \mathfrak{e}^*_\rho$ we now obtain indeed

$$\begin{aligned} \mathfrak{Z}_\nu &= (\,|L'(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{e}_\mu\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)|\,) \\ &= \Big(\Big| \Big(\mathfrak{a}_1 \cdots \mathfrak{a}_{\nu-1} \sum_{\rho} l_{\mu\rho}\, \mathfrak{e}^*_\rho\, \mathfrak{a}_{\nu+1} \cdots \mathfrak{a}_n \Big) \Big| \Big) \\ &= \Big(\sum l_{\mu\rho}\, |(\mathfrak{a}_1 \cdots \mathfrak{a}_{\nu-1}\, \mathfrak{e}^*_\rho\, \mathfrak{a}_{\nu+1} \cdots \mathfrak{a}_n)| \Big) \\ &= \sum_{\rho} |(\mathfrak{a}_1 \cdots \mathfrak{a}_{\nu-1}\, \mathfrak{n}^*_\rho\, \mathfrak{a}_{\nu+1} \cdots \mathfrak{a}_n)|\, \mathfrak{l}_\rho. \end{aligned}$$

Vice versa we shall prove that the conditions $M'M = 0$, which is an identity in X, and rank $L = n$ imply also $L'L = 0$. Based on the deduced formulae we have

$$M'M = \Big(\sum_{\rho} |L(\mathfrak{x}_1 \cdots \mathfrak{x}_{\nu-1}\, \mathfrak{l}_\rho\, \mathfrak{x}_{\nu+1} \cdots \mathfrak{x}_n)| \times$$
$$\times |(\mathfrak{a}_1 \cdots \mathfrak{a}_{\mu-1}\, \mathfrak{e}^*_\rho\, \mathfrak{a}_{\mu+1} \cdots \mathfrak{a}_n)|\Big)$$

Since rank $L = n$, a column $\mathfrak{C}_\kappa$ exists which solves $L'\mathfrak{C}_\kappa = \mathfrak{Q}_\kappa$, if $\mathfrak{Q}_\kappa$ is given. So we can choose in particular $\mathfrak{Q}_\kappa = \mathfrak{u}^*_\kappa$. The identical vanishing of $M'M$ now implies

$$|(\mathfrak{e}^*_1 \cdots \mathfrak{e}^*_{\nu-1}\, L'\, \mathfrak{l}_\mu\, \mathfrak{e}^*_{\nu+1} \cdots \mathfrak{e}^*_n)| = 0,$$

thus $\mathfrak{l}'_\nu\, \mathfrak{l}_\mu = 0$ or $L'L = 0$ as we asserted.

The above argument also shows that $|L'X|$ is a solution of the system

$$\frac{\partial}{\partial X'}\frac{\partial}{\partial X}\, |L'X|^k = 0, \tag{52}$$

if and only if either

$$k = 0,\ 1 \text{ or rank } L < n \text{ or } L'L = 0. \tag{53}$$

Without proof it may be mentioned that

$$\left|L'\,\frac{\partial}{\partial X}\right|^k e^{-\sigma(X'X)} = (-2)^{nk}\, |L'X|^k\, e^{-\sigma(X'X)} \text{ for } L'L = 0.$$

By means of (31), (33), (36) it can be proved in the particular case $m = 3$, $n = 2$ that the special function

$$u(X) = \xi_{23}\,\xi_{13} = \eta_1\,\eta_2 \tag{54}$$

satisfies the differential equations

$$\sigma\Big(X'\,\frac{\partial}{\partial X}\Big)\, u(X) = 4\, u(X),\ \sigma\Big(X\Big(X\,\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\Big)'\Big)\, u(X) = 2\, u(X),$$
$$\sigma\Big(X\Big(X\,\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\Big)'\Big)^2 u(X) = 8\, u(X),\ \left|\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\right| u(X) = 0. \tag{55}$$

Observing that in general the operators

$$\sigma\Big(X'\,\frac{\partial}{\partial X}\Big),\ \sigma\Big(X\Big(X\,\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\Big)'\Big)^h,\ (h = 1, 2, \ldots, n)$$

generate the ring of all invariant linear operators, this can be proved easily with the given methods, we see that $u(X)$ is a spherical function of type (3, 2). However, we have

$$\sigma\left(\frac{\partial}{\partial X'}\,\frac{\partial}{\partial X}\right) u(X) = 2\,(x_{11}\,x_{21} + x_{12}\,x_{22}) \neq 0. \tag{56}$$

4. The approximation theorem. Let $F(X)$ be a complex-valued function defined and continuous in the domain $X'X > 0$. Let $M_U\{F(UX)\}$ denote the mean value of $F(UX)$ relative to the compact group of all orthogonal matrices U in the sense of the theory of almost periodic functions. This mean value is a function of X which is invariant relative to the substitutions $X \to UX$ $(U'U = E)$, therefore it depends only upon $X'X$. If $F(X)$ is invariant relative to the substitutions $X \to XV (|V| \neq 0)$ then the mean value is obviously independent of X and therefore is a constant.

For two complex-valued functions $\phi(X)$ and $\psi(X)$, defined and continuous in $X'X > 0$ with the transformation invariance

$$\phi(XV) = |V|^k\,\phi(X), \quad \psi(XV) = |V|^k\,\psi(X) \text{ for } |V| \neq 0, \tag{57}$$

we define a scalar product by

$$(\phi(X), \psi(X))_k = M_U\{\phi(UX)\,\overline{\psi(UX)}\,|X'X|^{-k}\}. \tag{58}$$

It has the property of the translation invariance:

$$(\phi(UX), \psi(UX))_k = (\phi(X), \psi(X))_k \text{ for } U'U = E \tag{59}$$

and determines a positive hermitian metric, i.e.

$$(\phi(X), \phi(X))_k = 0 \text{ implies } \phi(X) = 0.$$

This metric and Weierstrass's approximation theorem for continuous functions are the essential means for the proof of the following approximation theorem.

THEOREM 3. *Let $g(X)$ be a complex-valued function, defined and continuous in $X'X > 0$, which is invariant relative to the substitutions $X \to XV$ with $|V| \neq 0$. Then there exists a finite set of spherical functions $u_{ij}(X)$ of degree $2in$ such that*

$$\left| g(X) - \sum_{i,j} | X'X |^{-i} u_{ij}(X) \right| < \epsilon \tag{60}$$

holds in the whole domain $X'X > 0$ *where* ϵ *denotes a given positive number.*

According to the Weierstrass approximation theorem there exist algebraic forms $p_h(X)$ of degree $h (h = 1, 2, \ldots, 2k)$ such that

$$\left| g(X) - \sum_{h=0}^{2k} p_h(X) \right| < \epsilon \tag{61}$$

for all X of the compact domain $X'X = E$. We introduce the mean values

$$q_h(X) = M_V \{p_h(XV)\}$$

relative to the compact group of all orthogonal matrices V. Observing that the compact domain defined by $X'X = E$ is mapped onto itself by the substitutions $X \to XV (V'V = E)$ and besides also $g(XV) = g(X)$ is valid, we obtain from (61), by computing the mean values,

$$\left| g(X) - \sum_{h=0}^{2k} q_h(X) \right| < \epsilon \text{ for } X'X = E. \tag{62}$$

According to well-known theorems of the theory of algebraic invariants, the algebraic form $q_h(X)$, being invariant relative to the substitutions $X \to XV (V'V = E)$, is representable as an algebraic form in the elements of the matrix XX':

$$q_h(X) = q^*_h(XX').$$

This shows in particular that h is even if $q_h(X) \neq 0$.

Let X be an arbitrary matrix of rank n. Then we can determine a non-singular matrix $R = R^{(h)}$ such that

$$X'X = R'R.$$

Replacing X in (62) by XR^{-1} and observing that $g(XR^{-1}) = g(X)$ we obtain

$$\left| g(X) - \sum_{h=0}^{k} q_{2h}(XR^{-1}) \right| < \epsilon, \text{ for } X'X > 0. \tag{63}$$

It is obvious that

$$q_{2h}(XR^{-1}) = q^*_{2h}(XR^{-1}R'^{-1}X') = q^*_{2h}(X(X'X)^{-1}X')$$

is independent of the choice of R, and thus represents a one-valued function of X. It is easy to see that

$$u_h(X) = |X'X|^h q_{2h}(XR^{-1}) = |X'X|^h q^*_{2h}(X(X'X)^{-1}X')$$

is an algebraic form with the invariance property

$$u_h(XV) = |V|^{2h} u_h(X) \text{ for } |V| \neq 0. \tag{64}$$

In place of (63) we obtain now

$$\left| g(X) - \sum_{h=0}^{k} |X'X|^{-h} u_h(X) \right| < \epsilon, \text{ for } X'X > 0. \tag{65}$$

The following considerations apply to the linear space consisting of all algebraic forms of degree $2hn$ with the invariance property (64). With $u(X)$ also $u(UX)$ belongs to this space, U being an arbitrary orthogonal matrix. For an arbitrary subspace $\mathfrak{L}$ which also has these two properties we prove a lemma of which the approximation theorem is an easy consequence.

Lemma. *Let $\mathfrak{L}$ be a linear space of algebraic forms $v(X)$ which satisfy the transformation formula*

$$v(XV) = |V|^{2h} v(X) \textit{ for } |V| \neq 0. \tag{66}$$

Assume that with $v(X)$, $\mathfrak{L}$ also contains $v(UX)$, U being an arbitrary orthogonal matrix. Further let k be a given integer $\geqslant 0$. Then there exists a basis $v_1(X), v_2(X), \ldots, v_s(X)$ in $\mathfrak{L}$ such that

$$\sigma(\Lambda^{2j}) v_i(X) = \lambda_{ij} v_i(X) \textit{ for } i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, k, \tag{67}$$

with non-negative real eigenvalues λ_{ij}.

The proof will be based on induction on k. For $k = 0$ our assertion is only that $\mathfrak{L}$ has a finite dimension. This is trivial. So we can assume that the lemma is valid for a given value of k ($\geqslant 0$). Then we prove it for $k+1$ in place of k. First of all we distribute the basis functions $v_i(X)$ into classes $\mathfrak{K}_\nu$ $(\nu = 1, 2, \ldots, l)$ so that two functions $v_i(X)$ have the same system of eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik}$ if and only if they belong to the same class. Let us assume perhaps $v_\nu(X) \in \mathfrak{K}_\nu$ $(\nu = 1, 2, \ldots, l)$. $\mathfrak{L}_\nu$ denote the linear space generated by all $v_i(X) \in \mathfrak{K}_\nu$. Then we have

$$\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_2 + \ldots + \mathfrak{L}_l.$$

Let $v(X) \in \mathfrak{L}$ be an arbitrary eigenfunction of the operators $\sigma(\Lambda^{2j})$ $(j = 1, 2, \ldots, k)$:

$$\sigma(\Lambda^{2j})\, v(X) = \lambda_j\, v(X), \ (j = 1, 2, \ldots, k).$$

Then there exists a unique decomposition

$$v(X) = \sum_{\nu=1}^{l} w_\nu(X) \quad \text{with } w_\nu(X) \in \mathfrak{L}_\nu.$$

On account of

$$\sigma(\Lambda^{2j})\, w_\nu(X) = \lambda_{\nu j}\, w_\nu(X), \ (\nu = 1, 2, \ldots, l)$$

it follows that

$$\sigma(\Lambda^{2j})\, v(X) = \sum_{\nu=1}^{l} \lambda_{\nu j}\, w_\nu(X) = \sum_{\nu=1}^{l} \lambda_j\, w_\nu(X),$$

therefore

$$\lambda_{\nu j}\, w_\nu(X) = \lambda_j\, w_\nu(X)$$

for all ν and j. $w_\nu(X) \neq 0$ implies

$$\lambda_{\nu j} = \lambda_j \ \text{ for } j = 1, 2, \ldots, k.$$

This of course is impossible for two different $\nu \leqslant l$. Thus only one $w_\nu(X)$ differs from 0 and $v(X) \in \mathfrak{L}_\nu$ is proved. Because of the invariance properties of the operators $\sigma(\Lambda^{2j})$ it is obvious that with $v(X)$ also $v(UX)$ is an eigenfunction of the operators $\sigma(\Lambda^{2j})$ $(j = 1, 2, \ldots, k)$ if U denotes an orthogonal matrix. $v(X)$ and $v(UX)$ even belong to the same system of eigenvalues. This proves that, with $v(X)$, $\mathfrak{L}_\nu$ contains also $v(UX)$. In other words, each subspace $\mathfrak{L}_\nu$ itself has the characteristic properties of $\mathfrak{L}$. So it suffices to consider the subspaces $\mathfrak{L}_\nu$ individually. Without loss of generality we can identify such $\mathfrak{L}_\nu$ with $\mathfrak{L}$, i.e. we can assume

$$\lambda_{ij} = \lambda_j, \ (j = 1, 2, \dots k), \tag{68}$$

for all i.

For a given orthogonal matrix U the mapping $v(X) \to v(UX)$ defines a linear transformation of $\mathfrak{L}$ into itself. Thus we have

$$v_\mu(UX) = \sum_{\nu=1}^{s} D_{\mu\nu}(U)\, v_\nu(X), \ (\mu = 1, 2, \dots s),$$

with certain coefficients $D_{\mu\nu}(U)$. We assume that the $v_\nu(X)$'s form an orthogonal and normalized basis, i.e.

$$(v_\mu(X),\ v_\nu(X))_{2h} = \delta_{\mu\nu}.$$

Because of the translation invariance of the scalar product the $v_\nu(UX)$'s are also orthogonal and normalized. This proves that

$$D(U) = (D_{\mu\nu}(U))$$

is a unitary matrix and it turns out that the function

$$F(X, X^*) = \sum_{\nu=1}^{s} v_\nu(X)\, \overline{v_\nu(X^*)} \tag{69}$$

is invariant relative to the simultaneous substitutions

$$X \to UX,\ X^* \to UX^*,\ (U'U = E).$$

Here a general conclusion applied already by E. Hecke [8] again gets importance. On the strength of the invariance property of $F(X, X^*)$ we deduce now the differential equations for the basis functions $v_\nu(X)$. It is well known that

$$U = (E + S)\ (E - S)^{-1} \text{ with } S' + S = 0,$$

defines a parameter representation of the orthogonal matrices. Since $F(UX, UX^*)$ is independent from U and so from S, the partial derivatives of this function relative to the elements $s_{\rho\sigma}(\rho < \sigma)$ of the matrix S vanish necessarily, i.e. we have

$$\frac{\partial}{\partial s_{\rho\sigma}} F(UX,\ UX^*) = 0.$$

It suffices to discuss these conditions for $S = 0$. A development into powers of S yields

$$U = E + 2S + \text{higher powers of } S,$$

therefore

$$\frac{\partial}{\partial s_{\rho\sigma}} UX = 2\frac{\partial}{\partial s_{\rho\sigma}} SX = 2\left(\sum_{\alpha} \frac{\partial}{\partial s_{\rho\sigma}} s_{\mu\alpha}\, x_{\alpha\nu}\right)$$
$$= 2(\delta_{\rho\mu}\, x_{\sigma\nu} - \delta_{\sigma\mu}\, x_{\rho\nu}), \text{ for } S = 0.$$

This leads in the case $S = 0$ to

$$\begin{aligned}\frac{\partial}{\partial s_{\rho\sigma}} F(UX,\ UX^*) &= 2\sum_{\mu,\nu} (\delta_{\rho\mu}\, x_{\sigma\nu} - \delta_{\sigma\mu}\, x_{\rho\nu})\, \frac{\partial F(X,X^*)}{\partial x_{\mu\nu}} \\ &\quad + 2\sum_{\mu,\nu} (\delta_{\rho\mu}\, x^*_{\sigma\nu} - \delta_{\sigma\mu}\, x^*_{\rho\nu})\, \frac{\partial F(X,X^*)}{\partial x^*_{\mu\nu}} \\ &= 2\sum_{\nu} x_{\sigma\nu}\, \frac{\partial F(X,\ X^*)}{\partial x_{\rho\nu}} - 2\sum_{\nu} x_{\rho\nu}\, \frac{\partial F(X,X^*)}{\partial x_{\sigma\nu}} \\ &\quad + 2\sum_{\nu} x^*_{\sigma\nu}\, \frac{\partial F(X,\ X^*)}{\partial x^*_{\rho\nu}} - 2\sum_{\nu} x^*_{\rho\nu}\, \frac{\partial F(X,X^*)}{\partial x^*_{\sigma\nu}} \\ &= 0,\end{aligned}$$

or, rewritten by means of the matrix calculus,

$$\Lambda\, F(X,\ X^*) + \Lambda^*\, F(X,X^*) = 0. \tag{70}$$

Here Λ^* denotes the operator which arises from Λ if we replace X by X^*. According to (69) we obtain

$$\sum_{\nu} \Lambda\, v_\nu(X)\, \overline{v_\nu(X^*)} + \sum_{\nu} v_\nu(X)\, \overline{\Lambda^*\, v_\nu(X^*)} = 0. \tag{71}$$

This shows, since the functions $v_\nu(X)$ are independent, that relations of the kind

$$\Lambda\, v_\nu(X) = \sum_{\mu} v_\mu(X)\, C_{\mu\nu}, \quad (\nu = 1, 2, \ldots, s), \tag{72}$$

with certain constant matrices $C_{\mu\nu}$ are valid. Λ and so all $C_{\mu\nu}$'s are skew-symmetric:

$$C'_{\mu\nu} = -C_{\mu\nu} \tag{73}$$

In (72) now we replace X by X^* and also turn over to conjugate complex expressions. By means of the relations, besides (72), which we obtain in this way, (71) can be rewritten as follows:

$$\sum_{\mu,\nu} \overline{v_\nu(X^*)}\, v_\mu(X)\, C_{\mu\nu} + \sum_{\mu,\nu} v_\nu(X)\, \overline{v_\mu(X^*)}\, \bar{C}_{\mu\nu} = 0.$$

This implies, according to the independence of the functions $v_\nu(X)$,

$$\bar{C}_{\mu\nu} = -C_{\nu\mu} = C'_{\nu\mu}. \tag{74}$$

Thus particularly the elements of $C_{\mu\mu}$ are pure imaginary numbers.

The repeated application of Λ to the basis functions $v_\nu(X)$ yields

$$\Lambda^j\, v_\nu(X) = \sum_\mu v_\mu(X)\, C^{(j)}_{\mu\nu}, \tag{75}$$

with certain constant matrices $C^{(j)}_{\mu\nu}$ which obviously satisfy the relations

$$C^{(i+j)}_{\mu\nu} = \sum_\rho C^{(i)}_{\mu\rho}\, C^{(j)}_{\rho\nu}, \quad C^{(1)}_{\mu\nu} = C_{\mu\nu}.$$

By means of induction on j it is easy to see that

$$\overline{C^{(j)}_{\mu\nu}} = C^{(j)\prime}_{\nu\mu}$$

holds for all $j \geqslant 1$. This shows

$$C^{(2j)}_{\mu\mu} = \sum_\rho C^{(j)}_{\mu\rho}\, C^{(j)}_{\rho\mu} = \sum_\rho C^{(j)}_{\mu\rho}\, \overline{C^{(j)}_{\mu\rho}}' \geqslant 0. \tag{76}$$

Taking the trace we obtain from (75) with $2j$ in place of j

$$\sigma(\Lambda^{2j})\, v_\nu(X) = \sum_\mu v_\mu(X)\, c^{(j)}_{\mu\nu}, \tag{77}$$

where

$$c^{(j)}_{\mu\nu} = \sigma(C^{(2j)}_{\mu\nu}) = \sigma(\overline{C^{(2j)}_{\nu\mu}}) = \overline{c^{(j)}_{\nu\mu}}.$$

This is to say $(c^{(j)}_{\mu\nu})$ is a hermitian matrix. So we can find a unitary matrix U which transforms $(c^{(k+1)}_{\mu\nu})$ into a diagonal matrix. We express this fact in the form

$$(c_{\mu\nu}^{(k+1)})\, U = U(\delta_{\mu\nu}\, \lambda_{k+1}^{(\nu)}). \tag{78}$$

The functions of the new basis

$$(w_1(X), w_2(X), \ldots, w_s(X)) = (v_1(X), v_2(X), \ldots, v_s(X))\, U$$

now turn out to be eigenfunctions of the operator $\sigma(\Lambda^{2k+2})$:

$$\sigma(\Lambda^{2k+2})\, w_\nu(X) = \lambda_{k+1}^{(\nu)}\, w_\nu(X), \; (v = 1, 2, \ldots, s). \tag{79}$$

The relations

$$\sigma(\Lambda^{2j})\, v_\nu(X) = \lambda_j\, v_\nu(X), \; (j = 1, 2, \ldots, k;\, \nu = 1, 2, \ldots, s)$$

of course can be carried over at once to the functions $w_\nu(X)$:

$$\sigma(\Lambda^{2j})\, w_\nu(X) = \lambda_j\, w_\nu(X), \; (j = 1, 2, \ldots, k;\, \nu = 1, 2, \ldots, s). \tag{80}$$

Finally we have to observe that for an arbitrary system of complex numbers $z_1, z_2, \ldots, z_s$ the hermitian form

$$\begin{aligned}\sum_{\mu,\nu} c_{\mu\nu}^{(j)}\, z_\mu\, \bar{z}_\nu &= \sum_{\mu,\nu} \sigma(C_{\mu\nu}^{(2j)}\, z_\mu\, \bar{z}_\nu) = \sum_{\mu,\nu,\rho} \sigma(C_{\mu\rho}^{(j)}\, C_{\rho\nu}^{(j)}\, z_\mu\, \bar{z}_\nu) \\ &= \sum_{\mu,\nu,\rho} \sigma(C_{\mu\rho}^{(j)}\, \overline{C_{\nu\rho}^{(j)\prime}}\, z_\mu\, \bar{z}) = \sum_{\rho} \sigma\Big(\sum_{\mu,\nu} C_{\mu\rho}^{(j)}\, \overline{C_{\nu\rho}^{(j)\prime}}\, z_\mu\, \bar{z}_\nu\Big) \\ &= \sum_{\rho} \sigma\Big(\sum_{\mu} C_{\mu\rho}^{(j)}\, z_\mu\Big)\Big(\sum_{\nu} C_{\nu\rho}^{(j)}\, z_\nu\Big)' \geqslant 0,\end{aligned}$$

so that in particular for $j = k + 1$

$$\lambda_{k+1}^{(\nu)} \geqslant 0, \; (\nu = 1, 2, \ldots, s). \tag{81}$$

By this our lemma is proved.

On the strength of the lemma, applied to the special case $k = n$, one can see that a decomposition of the algebraic form $u_h(X)$, appearing in (65), into eigenfunctions $u_{hj}(X)$ of the operators $\sigma(\Lambda^{2\nu})$ $(\nu = 1, 2, \ldots, n)$ is possible :

$$u_h\,(X) = \sum_j u_{hj}\,(X).$$

As we have seen before, $|X'X|^{-a}\, u_{hj}\,(X)$ represents a spherical function of the type (m, n) for a suitable exponent $a \geqslant 0$ which may depend upon h and j. This completes the proof of Theorem 3.

5. Theta series. The Gauss-transform $u^*(X)$ of a function $u(X)$ is defined by

$$u^*(X) = \int_{\mathfrak{T}} u(X+T)\, e^{-\pi\sigma(T'T)}\,[dT]. \tag{82}$$

Here $\mathfrak{T}$ denotes the full space of all real matrices $T = T^{(m,n)} = (t_{\mu\nu})$ and $[dT]$ the product of all differentials $dt_{\mu\nu}$. We call $u(X)$ an eigenfunction of the Gauss-transformation if $u(X) \neq 0$ and $u^*(X) = \lambda\, u(X)$ holds. Assume that $u(X)$ is a polynomial different from 0, then $u^*(X)$ is also one and according to

$$\int_{\mathfrak{T}} e^{-\pi\sigma(T'T)}\,[dT] = 1,$$

we have

$$\text{degree}\,(u^*(X) - u(X)) < \text{degree}\, u(X), \tag{83}$$

so that necessarily $\lambda = 1$ if $u(X)$ is an eigenfunction.

In the sequel we denote by $Y = Y^{(n)}$ a positive real matrix having variable elements and by $S = S^{(m)}$ a positive matrix having arbitrary but fixed chosen elements.

THEOREM 4. *Let $u(X)$ be a polynomial with the invariance property*

$$u(XV) = |V|^{2k}\, u(X) \textit{ for } |V| \neq 0, \tag{84}$$

assume also that $u(X)$ is an eigenfunction of the Gauss-transformation. Further on we introduce Q by $S = Q'Q$, $Q' = Q > 0$. Then the theta series

$$\vartheta(Y, S; u) = \sum_G u(QG)\, e^{-\pi\sigma(YS[G])}, \tag{85}$$

where G has to run over all integral matrices of the type $G^{(m,n)}$, satisfies the transformation formula

$$\vartheta(Y^{-1}, S; u) = (-1)^{kn}\, |S|^{-n/2}\, |Y|^{m/2+2k}\, \vartheta(Y, S^{-1}; u). \tag{86}$$

In order to prove this we develop the theta series

$$\vartheta(X, Y, S; u) = \sum_G u(Q(G+X))\, e^{-\pi\sigma(YS[G+X])}, \tag{87}$$

which is a periodic function of X, into a Fourier series. We obtain

$$\vartheta(X, Y, S; u) = \sum_{G} \alpha(G, Y, S; u)\, e^{2\pi i \sigma(G'X)} \tag{88}$$

with

$$\alpha(G, Y, S; u) = \int_{\mathfrak{X}} u(QX)\, e^{-\pi\sigma(YS[X]) - 2\pi i \sigma(G'X)}\, [dX],$$

$\mathfrak{X}$ denoting the full X-space. In this integral we substitute $X_1 = QXR$ for X, R being determined by $Y = R'R$, $R' = R > 0$. Using (84) we find by a simple computation

$$\alpha(G, Y, S; u) = |S|^{-n/2}\, |Y|^{-m/2-k}\, u^*(-i\, Q^{-1}\, G\, R^{-1})\, e^{-\pi\sigma(Y^{-1}S^{-1}[G])},$$

$u^*(X)$ denoting the Gauss-transform of $u(X)$. We can split off the factor R^{-1} from $u^*(-i\, Q^{-1}\, G\, R^{-1})$ if and only if $u^*(X) = u(X)$. In the case $u^*(X) \neq u(X)$, $u^*(X)$ is not even homogeneous as (83) shows. In order to obtain a reasonable transformation formula for our theta series it is necessary to assume $u^*(X) = u(X)$. Now it is easy to state that

$$\alpha(G, Y, S; u) = (-1)^{kn}\, |S|^{-n/2}\, |Y|^{-m/2-2k}\, u(Q^{-1}\, G)\, e^{-\pi\sigma(Y^{-1}S^{-1}[G])}.$$

Thus we can carry over (88) with $X = 0$ and Y^{-1} instead of Y immediately into (86), the asserted formula.

The question whether for spherical functions $u(X)$, $u^*(X) = u(X)$ always holds is to be answered in the negative. For instance in the special case $m = 3$, $n = 2$ the spherical function

$$\begin{aligned} u(X) &= \xi_{23}\, \xi_{13} \\ &= x_{11}\, x_{21}\, x_{32}^2 + x_{12}\, x_{22}\, x_{31}^2 - x_{12}\, x_{21}\, x_{31}\, x_{32} - x_{11}\, x_{22}\, x_{31}\, x_{32}, \end{aligned}$$

we considered already, differs from $u^*(X)$. A simple computation yields

$$u^*(X) = u(X) + \frac{1}{2\pi}\,(x_{11}\, x_{21} + x_{12}\, x_{22}).$$

It is however remarkable that generally with $u(X)$ also $u^*(X)$ is an eigenfunction of the operators $\sigma(\Lambda^{2\nu})$, $(\nu = 1, 2, \ldots, n)$, and that with $u(X)$, $\left|\frac{\partial}{\partial X'}\, \frac{\partial}{\partial X}\right|$ also annihilates $u^*(X)$. We formulate these facts more precisely in

THEOREM 5. *If a polynomial $u(X)$ satisfies the relations*

$$\left| \frac{\partial}{\partial X'} \frac{\partial}{\partial X} \right| u(X) = 0,\ \sigma(\Lambda^{2\nu})\, u(X) = \lambda_\nu\, u(X),\ (\nu = 1, 2, \ldots, n), \quad (89)$$

then they are valid also for the Gauss-transform $u^*(X)$ of $u(X)$.

The proof is based on a general integral transformation which can be considered as a generalization of the method of partial integration. Now we denote by $[dX]$ the exterior product of all differentials $dx_{\mu\nu}$. Let $\omega_{\rho\sigma}$ be the exterior product of all $dx_{\mu\nu}$ with the exception of $dx_{\rho\sigma}$. We intend to choose the order of the factors such that

$$[dX] = dx_{\rho\sigma} \cdot \omega_{\rho\sigma}$$

holds. We set $\Omega = (\omega_{\mu\nu})$. Further on let $\mathfrak{G}$ be an oriented compact and measurable domain in the X-space with a measurable boundary $\mathfrak{R}$. We carry over the orientation from $\mathfrak{G}$ *to* $\mathfrak{R}$. On these premisses we prove the following

LEMMA. *Let $A = A^{(m)} = (a_{\mu\nu})$ and $B = B^{(m)} = (b_{\mu\nu})$ be matrices with functions of X as elements which have derivatives of sufficiently high order. Then*

$$\int_{\mathfrak{G}} A\Lambda^k B[dX] = \int_{\mathfrak{G}} (B'\Lambda^k A')'[dX] + \sum_{\nu=0}^{k-1} \int_{\mathfrak{R}} (\Lambda^\nu A')'(X\Omega' - \Omega X')\,\Lambda^{k-1-\nu} B \quad (90)$$

holds for an arbitrary natural number k.

First of all we establish by means of Stokes' integral-formula

$$\int_{\mathfrak{G}} AX \frac{\partial}{\partial X'} B[dX] = \int_{\mathfrak{R}} AX\Omega' B - \int_{\mathfrak{G}} \left(B'\left(X \frac{\partial}{\partial X'}\right)' A' \right)' [dX] - n \int_{\mathfrak{G}} AB[dX]. \quad (91)$$

A straightforward computation yields indeed

$$\int_{\mathfrak{G}} AX \frac{\partial}{\partial X'} B[dX]$$

$$= \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} a_{\mu\rho}\, x_{\rho\sigma} \frac{\partial}{\partial x_{\tau\sigma}} b_{\tau\nu} \Big) [dX]$$

$$= \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} \frac{\partial}{\partial x_{\tau\sigma}} a_{\mu\rho}\, x_{\rho\sigma}\, b_{\tau\nu} \Big) [dX] - \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\nu} \frac{\partial}{\partial x_{\tau\sigma}} a_{\mu\rho}\, x_{\rho\sigma} \Big) [dX]$$

$$= \int_{\mathfrak{G}} \Big(d \sum_{\rho,\sigma,\tau} a_{\mu\rho}\, x_{\rho\sigma}\, b_{\tau\nu}\, \omega_{\tau\sigma} \Big) -$$

$$- \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\nu}\, x_{\rho\sigma} \frac{\partial}{\partial x_{\tau\sigma}} a_{\mu\sigma} \Big) [dX] - \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\nu}\, a_{\mu\rho}\, \delta_{\tau\rho} \Big) [dX]$$

$$= \int_{\mathfrak{R}} \Big(\sum_{\rho,\sigma,\tau} a_{\mu\rho}\, x_{\rho\sigma}\, \omega_{\tau\sigma}\, b_{\tau\nu} \Big) -$$

$$- \int_{\mathfrak{G}} \Big(\sum_{\rho,\sigma,\tau} b_{\tau\mu}\, x_{\rho\sigma} \frac{\partial}{\partial x_{\tau\sigma}} a_{\nu\rho} \Big)' [dX] - n \int_{\mathfrak{G}} \Big(\sum_{\rho} a_{\mu\rho}\, b_{\rho\nu} \Big) [dX]$$

$$= \int_{\mathfrak{R}} AX\Omega' B - \int_{\mathfrak{G}} \Big(B' \Big(X \frac{\partial}{\partial X'} \Big)' A' \Big)' [dX] - n \int_{\mathfrak{G}} AB[dX].$$

The analogous formula

$$\int_{\mathfrak{G}} A \Big(X \frac{\partial}{\partial X'} \Big)' B[dX]$$

$$= \int_{\mathfrak{R}} A\Omega X' B - \int_{\mathfrak{G}} \Big(B' X \frac{\partial}{\partial X'} A' \Big)' [dX] - n \int_{\mathfrak{G}} AB[dX] \tag{92}$$

can be obtained from (91) in a simple manner by transposition. Subtraction of (92) from (91) yields (90) in the special case $k = 1$. A general proof of (90) is now possible without difficulty by induction on k.

Now we choose $A = \phi(X)E$, $B = \psi(X)E$ and form the trace of (90). At the same time we extend $\mathfrak{G}$ to the full X-space which was designed with $\mathfrak{X}$. By a suitable choice of $\phi(X)$ we shall take care later that all limit processes remain legitimate, in particular also that all boundary integrals vanish. Our lemma then yields

$$\int_{\mathfrak{X}} \phi(X)\, \sigma(\Lambda^k)\, \psi(X)\, [dX] = \int_{\mathfrak{X}} \psi(X)\, \sigma(\Lambda^k)\, \phi(X)\, [X]. \tag{93}$$

In

$$u^*(T) = \int_{\mathfrak{X}} u(T+X)\, e^{-\pi\sigma(X'X)}\, [dX]$$

we substitute $X_1 = X - T$ and then replace T by $-iT$. So we obtain for

$$w(T) = u^*(-iT)\, e^{-\pi\sigma(T'T)}$$

the representation

$$w(T) = \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X) - 2\pi i\sigma(X'T)} [dX].$$

Since $\sigma(X'X)$ can be considered as a constant with respect to the operator Λ it is sufficient to prove that

$$\sigma(\Lambda^{2\nu})\, w(X) = \lambda_\nu\, w(X), \; (\nu = 1, 2, \ldots, n).$$

Then these differential equations are also valid for $u^*(X)$ in place of $w(X)$. We set more precisely $\Lambda = \Lambda_x$, and denote by Λ_t the operator which arises from Λ_x by the substitution $X \to T$. It is easy to see that

$$\Lambda_x\, e^{-2\pi i\sigma(X'T)} = \Lambda'_t\, e^{-2\pi i\sigma(X'T)},$$

and by induction on k

$$\Lambda_x^k\, e^{-2\pi i\sigma(X'T)} = (\Lambda_t^k)'\, e^{-2\pi i\sigma(X'T)}.$$

This leads indeed to

$$\begin{aligned}(\Lambda_t^{k+1})'\, e^{-2\pi i\sigma(X'T)} &= (\Lambda_t^k\, \Lambda'_x)'\, e^{-2\pi i\sigma(X'T)} \\ &= \Lambda_x(\Lambda_t^k)'\, e^{-2\pi i\sigma(X'T)} \\ &= \Lambda_x^{k+1}\, e^{-2\pi i\sigma(X'T)}.\end{aligned}$$

Taking the trace, we obtain

$$\sigma(\Lambda_x^k)\, e^{-2\pi i\sigma(X'T)} = \sigma(\Lambda_t^k)\, e^{-2\pi i\sigma(X'T)}.$$

In order to determine the effect of $\sigma(\Lambda_t^{2\nu})$ on $w(T)$ we apply (93) to $k = 2\nu$, $\phi(X) = u(X)\, e^{-\pi\sigma(X'X)}$, $\psi(X) = e^{-2\pi i\sigma(X'T)}$. So we get finally

$$\begin{aligned}\sigma(\Lambda_t^{2\nu})\, w(T) &= \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X)},\ \sigma(\Lambda_t^{2\nu})\ e^{-2\pi i\sigma(X'T)}[dX] \\ &= \int_{\mathfrak{X}} u(X)\, e^{-\pi\sigma(X'X)}\ \sigma(\Lambda_x^{2\nu})\ e^{-2\pi i\sigma(X'T)}[dX\,] \\ &= \int_{\mathfrak{X}} e^{-2\pi i\sigma(X'T)}\ \sigma(\Lambda_x^{2\nu})\ u(X)\ e^{-\pi\sigma(X'X)}[dX] \\ &= \lambda_\nu \int_{\mathfrak{X}} u(X)\ e^{-\pi\sigma(X'X)-2\pi i\sigma(X'T)}[dX] = \lambda_\nu w(T),\end{aligned}$$

the asserted relations. It is trivial that also the first of the differential equations (89) can be carried over from $u(X)$ to $u^*(X)$. So Theorem 5 is proved.

A characterization of the algebraic forms which are eigenfunctions of the Gauss-transformation yields

THEOREM 6. *The Gauss-transform $u^*(X)$ of an algebraic form $u(X)$ is identical with $u(X)$ if and only if $u(X)$ is a solution of the Laplacian differential equation* $\sigma\left(\frac{\partial}{\partial X'}, \frac{\partial}{\partial X}\right) u(X) = 0$.

In this connection the fact that X is a rectangular matrix of an arbitrary number of columns does not come into appearance. So we can assume without loss of generality that $n = 1$. Let $X = (x_\mu)$, then $\Lambda = \left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}\right)$. In the sequel we denote by $u(X)$ an algebraic form of degree k which does not vanish identically.

1. Assume $\sigma\left(\frac{\partial}{\partial X'}, \frac{\partial}{\partial X}\right) u(X) = 0$. $u(X)$ then represents an ordinary spherical harmonic of m variables. Thus we have

$$\sigma(\Lambda^2)\ u(X) = \lambda_1 u(X),$$

and also, according to Theorem 5,

$$\sigma(\Lambda^2)\, u^*(X) = \lambda_1 u^*(X),$$

with a constant eigenvalue λ_1. We apply the well-known operator identity

$$\tfrac{1}{2}\sigma(\Lambda^2) = \Big(\sum_\nu x_\nu \frac{\partial}{\partial x_\nu}\Big)^2 + (m-2)\Big(\sum_\nu x_\nu \frac{\partial}{\partial x_\nu}\Big) - \\ - \Big(\sum_\nu x_\nu^2\Big)\Big(\sum_\nu \frac{\partial^2}{\partial x_\nu^2}\Big) \tag{94}$$

to $u^*(X)$. Let

$$u^*(X) = \sum_{\nu=0}^{k} h_\nu(X) \text{ with } h_k(X) = u(X),$$

be the decomposition of $u^*(X)$ into homogeneous terms such that $h_\nu(X)$ has the degree ν. Then we obtain

$$\tfrac{1}{2}\lambda_1 u^*(X) = \sum_{\nu=0}^{k} \nu(\nu+m-2)\, h_\nu(X),$$

therefore

$$\tfrac{1}{2}\lambda_1 = \nu(\nu+m-2) \text{ if } h_\nu(X) \neq 0.$$

This happens for $\nu = k$, so that $\frac{1}{2}\lambda_1 > \nu(\nu+m-2)$ for $\nu < k$, which proves $u^*(X) = u(X)$.

2. We apply the Gauss-transformation to the polynomial

$$u(X) = \sum a_{\nu_1\nu_2\ldots\nu_m} x_1^{\nu_1} x_2^{\nu_2} \ldots x_m^{\nu_m}.$$

A first computation yields

$$(x_1^{\nu_1}\ldots x_m^{\nu_m})^* = \int_{-\infty}^{\infty}\cdots\int e^{-\pi(t_1^2+\ldots+t_m^2)}(x_1+t_1)^{\nu_1}\ldots(x_m+t_m)^{\nu_m}\, dt_1\ldots dt_m$$

$$= \prod_{i=1}^{m}\int_{-\infty}^{\infty} e^{-\pi t^2}(x_i+t)^{\nu_i}\, dt$$

$$= \prod_{i=1}^{m}\Bigg(\sum_{\substack{\alpha_i,\beta_i\geqslant 0\\ \alpha_i+\beta_i=\nu_i}} \binom{\nu_i}{\alpha_i} x_i^{\alpha_i}\int_{-\infty}^{\infty}\Big(e^{-\pi t^2} t^{\beta_i}\, dt\Big)\Bigg).$$

Obviously it suffices to define the summation on even β_i. For such β_i

$$\int_{-\infty}^{\infty} e^{-\pi t^2}\, t^{\beta_i}\, dt = \pi^{-(\beta_i+1)/2}\, \Gamma\left(\frac{\beta_i+1}{2}\right)$$

holds. Consequently

$$(x_1^{\nu_1}\dots x_m^{\nu_m})^* = \prod_{i=1}^{m} \left(\sum_{\substack{\alpha_i,\beta_i\geqslant 0\\ \alpha_i+2\beta_i=\nu_i}} \binom{\nu_i}{\alpha_i} \Gamma(\beta_i+\tfrac{1}{2})\, \pi^{-\beta_i-\frac{1}{2}}\, x_i^{\alpha_i}\right)$$

$$= \sum_{i=1}^{m} \sum_{\substack{\alpha_i,\beta_i\geqslant 0\\ \alpha_i+2\beta_i=\nu_i}} \binom{\nu_1}{\alpha_1}\dots\binom{\nu_m}{\alpha_m} \Gamma(\beta_i+\tfrac{1}{2})\dots\Gamma(\beta_m+\tfrac{1}{2})\times$$

$$\times\, \pi^{-(\beta_1+\dots+\beta_m)-\frac{1}{2}m}\, x_1^{\alpha_1}\dots x_m^{\alpha_m},$$

thus

$$u^*(X) = \sum a_{\nu_1\nu_2\dots\nu_m}\,(x_1^{\nu_1}\, x_2^{\nu_2}\dots x_m^{\nu_m})^*$$

$$= \sum b_{\alpha_1\alpha_2\dots\alpha_m}\, x_1^{\alpha_1}\, x_2^{\alpha_2}\dots x_m^{\alpha_m},$$

with

$$b_{\alpha_1\dots\alpha_m} = \sum_{i=1}^{m} \sum_{\beta_i\geqslant 0} a_{\alpha_1+2\beta_1\dots\alpha_m+2\beta_m} \binom{\alpha_1+2\,\beta_1}{\alpha_1}\dots\binom{\alpha_m+2\,\beta_m}{\alpha_m}\times$$

$$\times\, \Gamma(\beta_1+\tfrac{1}{2})\dots\Gamma(\beta_m+\tfrac{1}{2})\, \pi^{-(\beta_1+\dots+\beta_m)-m/2}$$

Since $u(X)$ is assumed to be an algebraic form of degree k we need only to consider those systems of numbers $\beta_1, \beta_2, \dots \beta_m$ for which

$$\alpha_1+\dots+\alpha_m+2(\beta_1+\dots+\beta_m)=k.$$

In particular we obtain, for $\alpha_1+\dots+\alpha_m=k-2$,

$$b_{\alpha_1\dots\alpha_m} = \frac{1}{4\pi}\sum_{i=1}^{m}(\alpha_i+2)(\alpha_i+1)\, a_{\alpha_1\dots\alpha_{i-1}\alpha_i+2\,\alpha_{i+1}\dots\alpha_m}.$$

Now $u^*(X)=u(X)$ implies

$$b_{\alpha_1\alpha_2\dots\alpha_m}=0, \text{ for } \alpha_1+\alpha_2+\dots+\alpha_m=k-2.$$

These relations first mean that

$$\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} u(X)$$

$$= \sum \left(\sum_{i=1}^{m} (\alpha_i + 2)(\alpha_i + 1)\, a_{\alpha_1 \ldots \alpha_{i-1} \alpha_i + 2 \alpha_{i+1} \ldots \alpha_m} \right) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m{}^{\alpha_m} = 0.$$

So Theorem 6 is proved.

6. Angular characters. In the space of all positive real matrices $Y = Y^{(n)} = (y_{\mu\nu})$ we can develop a theory of invariant linear differential operators analogously to that of the X-space. Since we meet a rather simpler situation in the Y-space we need now only brief considerations. Let $\frac{\partial}{\partial Y} = \left(e_{\mu\nu} \frac{\partial}{\partial y_{\mu\nu}} \right)$ with $e_{\mu\nu} = 1$ or $\frac{1}{2}$ according as $\mu = \nu$ or $\mu \neq \nu$. Ω denotes a linear differential operator, i.e. a polynomial in the elements of $\frac{\partial}{\partial Y}$ with functions of Y as coefficients. We call $\Omega = F\left(Y, \frac{\partial}{\partial Y} \right)$ invariant if

$$F\left(Y[U], \frac{\partial}{\partial Y} [U'^{-1}] \right) = F\left(Y, \frac{\partial}{\partial Y} \right) \text{ for } |U| \neq 0 \tag{95}$$

holds. Let $\mathfrak{M}$ be again the module of all linear operators, $\mathfrak{M}_h$ the module of all operators of degree $\leqslant h$ and $\mathfrak{R}$ the ring of all invariant linear operators. (40) is still valid now.

THEOREM 7. *The invariant operators* $\sigma\left(Y \frac{\partial}{\partial Y} \right)^h$ $(h = 1, 2 \ldots, n)$ *and likewise* $\sigma\left(\left(Y \frac{\partial}{\partial Y} \right)' \right)^h$, $(h = 1, 2, \ldots, n)$ *form a basis for the ring* $\mathfrak{R}$ *of all invariant linear operators.*

In order to prove this theorem, which was first announced by A. Selberg, we choose a symmetric matrix $W = W^{(n)}$ with variable elements. We assume that they are commutable with those of Y. Let $\Omega = F\left(Y, \frac{\partial}{\partial Y} \right)$ be invariant. Then we have with regard to (95)

$$F(Y, W) = F(E, (W[U_0'^{-1}])[V]), \tag{96}$$

if $Y[U_0] = E$ and V an arbitrary orthogonal matrix. The right side of (96) is a symmetric polynomial in the characteristic roots of $W[U_0'^{-1}]$, consequently also representable as a polynomial in

$$\sigma(W[U_0'^{-1}])^h = \sigma(YW)^h = \sigma((YW)')^h \ (h = 1, 2, \dots, n).$$

Let

$$F(Y, W) = p(\sigma(YW), \dots, \sigma(YW)^n) = p(\sigma(YW)', \dots, \sigma((YW)')^n)$$

be such a representation. If Ω is of degree k, we obtain on the strength of (40)

$$\Omega \equiv p\left(\sigma\left(Y\frac{\partial}{\partial Y}\right), \dots, \sigma\left(Y\frac{\partial}{\partial Y}\right)^n\right)$$

$$\equiv p\left(\sigma\left(Y\frac{\partial}{\partial Y}\right)', \dots, \sigma\left(\left(Y\frac{\partial}{\partial Y}\right)'\right)^n\right) \ (\mathrm{mod}\ \mathfrak{M}_{k-}\ ).$$

Our assertion now follows in the usual manner by induction on k.

Let $Y_1 = Y^{-1}$. Introducing $dY_1 = -Y^{-1}\,dY\,Y^{-1}$ in

$$d\phi = \sigma\left(dY\frac{\partial}{\partial Y}\right)\phi = \sigma\left(dY_1\frac{\partial}{\partial Y_1}\right)\phi,$$

where ϕ denotes an arbitrary function, we obtain

$$Y\frac{\partial}{\partial Y} = -\left(Y_1\frac{\partial}{\partial Y_1}\right)'.$$

Thus the substitution $Y \to Y^{-1}$ maps the ring $\mathfrak{R}$ of all invariant linear operators into itself according to Theorem 7.

The space defined by $Y > 0$ can be considered as a Riemannian space relative to the metric introduced by the differential form

$$ds^2 = \sigma(Y^{-1}\,dY)^2.$$

Let ω denote the invariant volume element on the determinant surface $|Y| = 1$. Further let $\mathfrak{F}_n$ be a fundamental domain in $Y > 0$ relative to the group of transformations $Y \to Y[U]$ with unimodular U. For instance we can take for $\mathfrak{F}_n$ the domain of all reduced positive Y in the sense of Minkowski. Let $\mathfrak{B}_n$ denote the intersection of $\mathfrak{F}_n$ with the determinant surface $|Y| = 1$.

A function $v(Y)$ shall be called an *angular character* if

$$\left.\begin{array}{ll} 1. & v(Y) \text{ is holomorphic in } Y > 0 \text{ and homogeneous of degree } 0, \\ 2. & v(Y) \text{ is an eigenfunction of the ring } \mathfrak{R} \text{ of all invariant linear operators}, \\ 3. & v(Y[u]) = v(Y) \text{ is valid for unimodular } U, \\ 4. & v(Y) \text{ is square integrable over } \mathfrak{B}_n, \text{ so that} \\ & \displaystyle\int_{\mathfrak{B}_n} v(Y)\,\overline{v(Y)}\,\omega \text{ exists.} \end{array}\right\} \quad (97)$$

In applications we replace the fourth condition first of all by the sharper one saying that $v(Y)$ is bounded. Since the metric introduced in $Y > 0$ and also the ring $\mathfrak{R}$ are invariant relative to the substitution $Y \to Y^{-1}$, it is easy to see, that with $v(Y)$ also $v^*(Y) = v(Y^{-1})$ represents an angular character. But in general $v(Y)$ and $v(Y^*)$ belong to different eigenvalues. This may be mentioned without proof.

Here we note yet an integral formula, a generalization of the Euler gamma-integral, which was proved already elsewhere [5]:

$$\int_{Y>0} e^{-2\pi\sigma(TY)}\, v(Y)\, |Y|^{s-(n+1)/2}\, [dY]$$
$$= (2\pi)^{-ns}\, \Gamma(s-\alpha_1)\, \Gamma(s-\alpha_2) \ldots \Gamma(s-\alpha_n)\, \pi^{n(n-1)/4}\, v(T^{-1})\, |T|^{-s}. \quad (98)$$

Here it is $T = T^{(n)} > 0$, $v(Y)$ a bounded angular character, $[dY] = \prod_{\mu \leqslant \nu} dy_{\mu\nu}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ a set of constants which are uniquely determined up to the order by the eigenvalues of $v(Y)$. We have to assume $\operatorname{Re} s > (n-1)/2$.

7. Zeta-functions. In the sequel $u(X)$ denotes a spherical function of type (m, n) and degree $2kn$ and $v(Y)$ an angular character.

We assume that $v(Y)$ is bounded and $u(X)$ a non-constant eigenfunction of the Gauss-transformation so that in particular $k > 0$. The theta series (85) can be rewritten as

$$\vartheta(Y, S; u) = \sum_{T>0} a(T, S; u)\, e^{-\pi\sigma(YT)}, \tag{99}$$

with

$$a(T, S; u) = \sum_{\substack{G \\ S[G]=T}} u(QG), \tag{100}$$

where the finite sum must be extended over all integral G with $S[G] = T$. We introduce y and Y_1 by $Y = yY_1$, $y > 0$, $|Y_1| = 1$ and apply the Mellin transformation to the theta series, this being considered as a function of y, i.e. we form the function

$$\eta(s;\, Y_1, S; u) = \int_0^\infty \vartheta(y Y_1, S; u)\, y^{ns-1}\, dy.$$

For brevity we set $c_n = n^{-\frac{1}{2}}\, 2^{n(n-1)/4}$, denote by $\{T\}$ the class of all with $T(> 0)$ equivalent matrices $T[U]$, where U denotes an arbitrary unimodular matrix, and by $\epsilon(T)$ the number of units of T. We use $\mathfrak{F}_n$, $\mathfrak{B}_n$, ω in the introduced meaning. In anology with a computation carried out in [5] we obtain now

$$\begin{aligned}
&\xi_0(s;\, S;\, u, v)\\
&= \int_{\mathfrak{B}_n} \eta(s;\, Y_1, S;\, u)\, v(Y_1)\, \omega\\
&= \int_{\mathfrak{B}_n} \int_0^\infty \vartheta(yY_1, S; u)\, v(Y_1)\, y^{ns-1}\, dy\, \omega\\
&= c_n \int_{\mathfrak{F}_n} \vartheta(Y, S; u)\, v(Y)\, |Y|^{s-(n+1)/2}\, [dY]\\
&= c_n \sum_{T>0} a(T, S; u) \int_{\mathfrak{F}_n} e^{-\pi\sigma(YT)}\, v(Y)\, |Y|^{s-(n+1)/2}\, [dY]
\end{aligned}$$

$$= c_n \sum_{\{T\}} \frac{a(T, S; u)}{\epsilon(T)} \sum_U \int\limits_{\mathfrak{F}_n} e^{-\pi\sigma(YT[U])} v(Y) \mid Y \mid^{s-(n+1)/2} [dY], \tag{101}$$

where U has to run over all unimodular matrices. The addition of all integrals and the application of the integral-formula (98) yield

$$\xi_0(s;\ S;\ u, v)$$

$$= 2\, c_n \sum_{\{T\}} \frac{a(T, S; u)}{\epsilon(T)} \int\limits_{Y>0} e^{-\pi\sigma(YT)} v(Y) \,|\, Y \,|^{s-(n+1)/2} [dY]$$

$$= \frac{2}{\sqrt{n}} (2\pi)^{n(n-1)/4} \pi^{-ns} \Gamma(s-\alpha_1)\, \Gamma(s-\alpha_2) \ldots \Gamma(s-\alpha_n)\, \phi(s, S; u, v^*)$$

with

$$\phi(s, S; u, v^*) = \sum_{\{T\}} \frac{a(T, S; u)\, v^*(T)}{\epsilon(T) \,|\, T\,|^s}, \quad v^*(T) = v(T^{-1}). \tag{102}$$

The summation over the classes $\{T\}$ can be replaced by a summation over $T \in \mathfrak{F}_n$. According to the signification of $a(T, S; u)$ we get

$$\phi(s, S; u, v^*) = \sum_{\substack{G \\ S[G] \in \mathfrak{F}_n}} \frac{u(Q\, G)\, v^*(S[G])}{\epsilon(S[G]) \,|\, S[G] \,|^s}, \tag{103}$$

where G has to run over all integral matrices such that $S[G] \in \mathfrak{F}_n$. A set of matrices of this kind can be obtained by forming the products $G = G^* U$, where G^* must run over a full set of integral matrices of rank n, such that each two are not right-associated, and U over a full set of units of $S[G^*]$ provided that G^* is given. Two matrices are called right-associated if they differ by a unimodular right factor only. Writing G^*U in place of G we see that the general term in (103) does not depend upon U, thus we obtain finally, after writing again G instead of G^*,

$$\phi(s, S; u, v^*) = \sum_G \frac{u(QG)\, v^*(S[G])}{|\, S[G] \,|^s}, \tag{104}$$

where the sum must be extended over a full set of integral matrices G of rank n, such that each two are not right-associated.

A functional equation for the functions $\phi(s, S; u, v)$ and $\phi(s, S; u, v^*)$ can be obtained by decomposing the integral over $\mathfrak{F}_n$ in

$$\xi_0(s, S; u, v) = c_n \int_{\mathfrak{F}_n} \vartheta(Y, S; u)\, v(Y)\, |Y|^{s-(n+1)/2} [dY] \tag{105}$$

into two parts corresponding to the decomposition of $\mathfrak{F}_n$ by the determinant surface $|Y| = 1$. We assume that $\mathfrak{F}_n$ is invariant relative to the substitution $Y \to Y^{-1}$. In that part of the integral which must be extended over the intersection of $\mathfrak{F}_n$ with $|Y| < 1$ we substitute $Y \to Y^{-1}$ and apply the transformation formula. Observing that $|Y|^{-(n+1)/2} [dY]$ is invariant relative to the substitution $Y \to Y^{-1}$ we obtain the following representations

$$\begin{aligned}
&\xi_0(s, S; u, v) \\
&= c_n \int_{\substack{Y \in \mathfrak{F}_n \\ |Y| \geq 1}} \vartheta(Y, S; u)\, v(Y)\, |Y|^{s-(n+1)/2} [dY] + \\
&\quad + c_n \int_{\substack{Y \in \mathfrak{F}_n \\ |Y| \geq 1}} \vartheta(Y^{-1}, S; u)\, v^*(Y)\, |Y|^{-s-(n+1)/2} [dY] \\
&= c_n \int_{\substack{Y \in \mathfrak{F}_n \\ |Y| \geq 1}} \{\vartheta(Y, S; u)\, v(Y)\, |Y|^s + \\
&\quad + (-1)^{kn} |S|^{-n/2} \vartheta(Y, S^{-1}; u)\, v^*(Y)\, |Y|^{m/2+2k-s}\} |Y|^{-(n+1)/2} [dY].
\end{aligned}$$

All these expressions have first of all a meaning only if the real part of s is sufficiently large. The last integral however represents, as can be seen easily, an entire function of s. So the analytical continuation of $\phi(s, S; u, v^*)$ into the whole s-plane is performed. It is obvious that $\phi(s, S; u, v^*)$ is an entire function of s. The integral-representation of $\xi_0(s, S; u, v)$ yields directly

$$\xi_0(m/2 + 2k - s, S; u, v) = (-1)^{kn} |S|^{-n/2} \xi_0(s, S^{-1}; u, v^*). \tag{106}$$

The replacement of v by v^* and consequently v^* by v may carry over $\alpha_1, \alpha_2, \dots \alpha_n$ into $\beta_1, \beta_2, \dots \beta_n$. Introducing

$$\xi(s,\, S;\, u,\, v) = \frac{\sqrt{n}}{2}\,(2\,\pi)^{-n(n-1)/4}\,\xi_0(s,\, S;\, u,\, v^*)$$

we obtain finally the following result.

THEOREM 8. *Let $u(X)$ be a spherical function of type $(m,\, n)$ and degree $2kn$, $v(Y)$ an angular character. Assume that $u(X)$ is a non-constant eigenfunction of the Gauss-transformation and that $v(Y)$ is bounded. Then the Dirichlet series*

$$\phi(s,\, S;\, u,\, v) = \sum_G \frac{u(QG)\, v\,(S[G])}{|\,S[G]\,|^s},$$

where the sum must be extended over a full set of integral matrices G of rank n such that each two are not right-associated, represents an entire function of s. It satisfies the functional equation

$$\xi(m/2 + 2k - s,\, S;\, u,\, v) = (-1)^{kn}\,|\,S\,|^{-n/2}\,\xi(s,\, S^{-1};\, u,\, v^*)$$

where

$$\xi(s,\, S;\, u,\, v) = \pi^{-ns}\,\Gamma(s - \beta_1)\,\Gamma(s - \beta_2)\ldots\Gamma(s - \beta_n)\,\phi(s,\, S;\, u,\, v)$$

with certain constants $\beta_1, \beta_2, \ldots, \beta_n$ which depend only upon the eigenvalues of v.

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MODULAR CORRESPONDENCES AND THEIR REPRESENTATIONS

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INTRODUCTION. In the works of Hecke, Petersson, Maass, and other modern writers certain linear operators T_n have been used to find relations between the coefficients of modular forms and of corresponding Dirichlet series. Although this idea has already brought rich success we have to bear in mind that the operators T_n are nothing but special representations of correspondences of special algebraic varieties. So the theory of Hecke's T_n is only one section of a vast branch of number theory whose limits can scarcely be surveyed today.

The correspondences which are represented by Hecke's T_n are the so-called modular correspondences. The latter are the principal tool in the theory of complex multiplication. Moreover they have been used in proving almost innumerably many class number relations of definite binary quadratic forms.

Common to all these theories is the fact that the general concept of a correspondence assumes definite shape as a connection between certain subgroups of the modular group. This observation leads at once to a vast generalization of modular correspondences. One has only to replace the modular group by other appropriate groups. Examples of such groups are the groups of units of an order of a central simple algebra over an algebraic number field or of certain quadratic forms. Thus the theory of modular correspondences becomes in the first line a part of pure algebra and number theory, while function theoretic aspects are shifted to the second.

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In this paper we wish to put forward this idea of modular correspondences although we shall restrict ourselves to the very special case of units in a quaternion algebra. The possibility of generalizations and certain inherent difficulties will be pointed out at the end. Even under these restricting assumptions there remain enough of open questions, both of an elementary and a deeper nature, the solutions of which promise many results and applications in number theory.

Our chief endeavour will be put to the task of determining the traces of the representations of the T_n, and in some of the cases considered we have been successful. In these cases the traces turn out as certain sums containing the class numbers of definite binary quadratic forms. This fact gives, at the same time, an explanation of the class number relations which were, up to now, barely connected with the rest of number theory. Knowing the traces of, and the multiplicative relations discovered by Hecke between, the T_n, they are now fully determined, and so are the Fourier developments of some modular forms (see the example in [6]).

1. The arithmetical and geometrical background. Let Q be an indefinite quaternion algebra over the field of rational numbers and $\mathfrak{T}$ an order of rank 4 in Q. A left ideal $\mathfrak{m}$ with respect to $\mathfrak{T}$ is defined by means of its p-adic extensions:

$$\mathfrak{m} = Q \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \cap \mathfrak{m}_5 \cap \ldots, \mathfrak{m}_p = \mathfrak{T}_p \alpha_p,$$

where, for each rational prime p, $\mathfrak{T}_p$ is the p-adic extension of $\mathfrak{T}$, and α_p is a non-singular element of Q_p. Almost all α_p have to be units. All left ideals belong to a finite number h of classes. In many cases which are the most interesting with respect to applications, the class number $h = 1$.

We shall always make the assumption that the p-adic extensions $\mathfrak{T}_p$ are either maximal orders of Q_p or isomorphic to the order of matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with rational p-adic integers a_{ik} and $a_{21} \equiv 0 \bmod p$. Orders of this kind have been called by the author *orders of square-free level* (Stufe), their arithmetical properties have been sufficiently investigated as to working out the theory in all details [6]. Especially the class number is $h = 1$. We shall call q_1 the product of all p for which Q_p does not contain divisors of zero, and q_2 the product of all the other primes dividing the discriminant of $\mathfrak{T}$. The discriminant of $\mathfrak{T}$ is then $D = q_1^2 q_2^2$.

Let K be real quadratic splitting field of k. The elements of Q can be represented by two-rowed matrices with elements in K. A special case of Q is the algebra of all two-rowed matrices with rational coefficients, and a special case of $\mathfrak{T}$ is the order of all two-rowed matrices with integral rational coefficients. The group of units of norm 1 of this $\mathfrak{T}$ is the modular group.

Now let τ be a complex variable with positive imaginary part. To each unit $\epsilon = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$ of $\mathfrak{T}$ of norm 1 represented by a matrix in K (we shall always identify elements of Q and their matrix representation) there corresponds a transformation of the complex upper half-plane

$$\tau \rightarrow \frac{e_{11}\tau + e_{12}}{e_{21}\tau + e_{22}} = \epsilon \bigcirc \tau. \tag{1}$$

These transformations form a representation of the group $\mathfrak{U}$ of units of $\mathfrak{T}$ of norm 1 which is a faithful representation of the factor group $\Gamma_{\mathfrak{T}} = \mathfrak{U}/\{1, -1\}$ of the invariant subgroup formed by the two elements 1 and -1 in $\mathfrak{U}$. The starting point of all our considerations is the fact that this group of transformations is properly discontinuous in the upper half-plane. There exists a fundamental domain bounded by a finite number of arcs of circles perpendicular to the real axis. In the following the half plane will be considered as the Poincaré model of the hyperbolic plane, circles perpendicular to the real axis are the straight lines. Now the point sets (Σ means the set theoretical union)

$$P = \sum_{\epsilon \in \Gamma_{\mathfrak{T}}} \epsilon \bigcirc \tau = \Gamma_{\mathfrak{T}} \bigcirc \tau$$

form a closed surface $S_{\mathfrak{T}}$ which has even a hyperbolic metric apart from the elliptic and parabolic vertices. The latter have to be added in order to make $S_{\mathfrak{T}}$ a closed surface, in case there are any (only if Q contains divisors of zero).

The topological genus g of $S_{\mathfrak{T}}$ can be calculated by means of the residue at $s = 1$ of the zeta function $\zeta(s)$ of $\mathfrak{T}$ [11, 3]. Because

$$\zeta(s) = \zeta_0(2s)\,\zeta_0(2s-1)\prod_{p|q_1}(1-p^{1-2s})\prod_{p|q_2}(1+p^{1-2s}),$$

where $\zeta_0(s)$ is the zeta function of Riemann, this residue is

$$R = \frac{\pi^2}{12}\prod_{p|q_1}(1-p^{-1})\prod_{p|q_2}(1+p^{-1}) = \frac{\pi^2}{12\sqrt{D}}\prod_{p|q_1}(p-1)\prod_{p|q_2}(p+1)$$

(D is the discriminant of $\mathfrak{T}$). On the other hand one can show that R is $\frac{\pi}{4\sqrt{D}}$ times the hyperbolic area F of $S_{\mathfrak{T}}$, therefore

$$F = \frac{\pi}{3}\prod_{p|q_1}(p-1)\prod_{p|q_2}(p+1).$$

According to a well-known theorem of Gauss and Bonnet this area is

$$F = 4\pi(g-1) + 2\pi\sum\left(1-\frac{1}{n}\right),$$

the sum to be extended over the vertices of a fundamental domain, where n is the order of the vertex ($n = \infty$ for parabolic cusps). There occur only vertices of orders 2, 3, and ∞, the latter if and only if Q contains divisors of zero. The numbers of elliptic vertices of orders 2 and 3 have been calculated by R. Hull [12] in the case $q_2 = 1$; his calculations can immediately be generalized by means of a former paper of the author [7]. These numbers are

$$\prod_{p|q_1}\left(1-\left(\frac{-4}{p}\right)\right)\prod_{p|q_2}\left(1+\left(\frac{-4}{p}\right)\right),$$

and

$$\prod_{p|q_1}\left(1-\left(\frac{-3}{p}\right)\right)\prod_{p|q_2}\left(1+\left(\frac{-3}{p}\right)\right) \text{ respectively.}$$

The number of parabolic cusps is 0 if $q_1 > 1$, otherwise 2^κ, where κ is the number of prime divisors of q_2. Comparison of both expressions of F yields now

$$g = 1 - \tfrac{1}{4}\prod_{p|q_1}\left(1 - \left(\frac{-4}{p}\right)\right)\prod_{p|q_2}\left(1 + \left(\frac{-4}{p}\right)\right) -$$
$$- \tfrac{1}{3}\prod_{p|q_1}\left(1 - \left(\frac{-3}{p}\right)\right)\prod_{p|q_2}\left(1 + \left(\frac{-3}{p}\right)\right) +$$
$$+ \frac{1}{12}\prod_{p|q_1}(p-1)\prod_{p|q_2}(p+1) - \begin{cases} 0 \text{ for } q_1 > 1, \\ 2^{\kappa-1} \text{ for } q_1 = 1. \end{cases} \tag{2}$$

In the case $q_1 = 1$ this formula has been proved by Hecke.

2. The modular correspondences. Let $\mathfrak{T}$, $\mathfrak{T}'$ be two orders of rank 4 in Q, and $\mathfrak{T}^* = \mathfrak{T} \cap \mathfrak{T}'$ their intersection. The groups of units of norm 1 of these orders will be denoted by $\mathfrak{U}, \mathfrak{U}', \mathfrak{U}^*$. $\mathfrak{U}^*$ is a subgroup of finite index in both $\mathfrak{U}$, $\mathfrak{U}'$. The same is true for the transformation groups $\Gamma_{\mathfrak{T}}, \Gamma_{\mathfrak{T}'}, \Gamma_{\mathfrak{T}^*}$. Let

$$\Gamma_{\mathfrak{T}} = \sum_{i=1}^{d'} \Gamma_{\mathfrak{T}^*}\epsilon_i', \quad \Gamma_{\mathfrak{T}'} = \sum_{i=1}^{d} \Gamma_{\mathfrak{T}^*}\epsilon$$

be the developments into left cosets.

Now $S_{\mathfrak{T}^*}$ can be considered as a covering surface of $S_{\mathfrak{T}'}$ of d sheets by the definition that each point $\Gamma_{\mathfrak{T}'} \circ \tau$ of $S_{\mathfrak{T}'}$ is covered by the points $\Gamma_{\mathfrak{T}^*} \circ (\epsilon_i \circ \tau) = \Gamma_{\mathfrak{T}^*} \circ \tau_i (i = 1, \ldots, d)$ of $S_{\mathfrak{T}^*}$. One easily checks that the topological requirements of a covering surface are met. Ramifications are not excluded. Two points $\Gamma_{\mathfrak{T}^*} \circ (\epsilon_i \circ \tau)$, $\Gamma_{\mathfrak{T}^*} \circ (\epsilon_k \circ \tau)$ are equal if and only if there exists an $\epsilon^* \in \Gamma_{\mathfrak{T}^*}$ such that $\epsilon^*\, \epsilon_i\, \epsilon_k^{-1} \circ \tau = \tau$. Thus τ is a fixed point of the substitution $\epsilon = \epsilon^*\, \epsilon_i\, \epsilon_k^{-1}$. If τ lies in the interior of the upper half-plane, ϵ has order 2 or 3, and $S_{\mathfrak{T}^*}$ is ramified in $\Gamma_{\mathfrak{T}^*} \circ \tau$ of order 2 or 3. If τ is a parabolic cusp, ϵ has order ∞, and $S_{\mathfrak{T}^*}$ is ramified of order n, where n is the least exponent for which $\epsilon^n \in \Gamma_{\mathfrak{T}^*}$. We make the convention once and for all that ramification points of $S_{\mathfrak{T}^*}$ are to be counted with the multiplicity of the ramifications.

Thus there lie exactly $d = [\Gamma_{\mathfrak{T}'}: \Gamma_{\mathfrak{T}^*}]$ points of $S_{\mathfrak{T}^*}$ over each point of $S_{\mathfrak{T}'}$.

Conversely $\Gamma_{\mathfrak{T}'} \circ \tau$ is called the *trace in* $S_{\mathfrak{T}'}$ of each of the points $\Gamma_{\mathfrak{T}^*} \circ \epsilon^i$ in $S_{\mathfrak{T}^*}$.

Now the set-theoretical union of the traces P_i in $S_{\mathfrak{T}}$ of all points P_i^* of $S_{\mathfrak{T}^*}$ covering a given point P' in $S_{\mathfrak{T}'}$ is called the *geometrical correspondence* of $S_{\mathfrak{T}'}$ to $S_{\mathfrak{T}}$. We shall denote it by

$$C_{\mathfrak{T}'/\mathfrak{T}}(P') = \sum_{i=1}^{d} P_i. \tag{3}$$

This is a function whose arguments are points P' of $S_{\mathfrak{T}'}$ and whose values are finite sets of points in $S_{\mathfrak{T}}$.

The only interesting case is $\mathfrak{T}' = \nu \mathfrak{T} \nu^{-1}$ with an element $\nu \in Q$. Now $\Gamma_{\mathfrak{T}'} = \nu \Gamma_{\mathfrak{T}} \nu^{-1}$, and the application of the substitution representing ν maps $S_{\mathfrak{T}}$ onto $S_{\mathfrak{T}'}$:

$$\nu \circ P = \nu \circ \Gamma_{\mathfrak{T}} \circ \tau = \Gamma_{\mathfrak{T}'} \circ \nu \circ \tau = P'. \tag{4}$$

The multiplication of (3) and (4) yields a function

$$C(P) = C_{\mathfrak{T}'/\mathfrak{T}}(\nu \circ P) = \sum_{i=1}^{d} P_i, \tag{5}$$

which is a (geometrical) *correspondence of* $S_{\mathfrak{T}}$ *to itself.*

Functions of points of $S_{\mathfrak{T}}$ whose values are finite point sets in $S_{\mathfrak{T}}$, howsoever they may be defined, can be added and multiplied by the following definitions : for

$$C_k(P) = \sum_{i=1}^{d_k} p_i^k, \quad (k = 1, 2),$$

put

$$C_1(P) + C_2(P) = \sum_{i=1}^{d_1} p_i^1 + \sum_{i=1}^{d_2} p_i^2,$$

$$C_1(P) . C_2(P) = \sum_{i=1}^{d_1} C_1(p_i^2).$$

As readily verified such functions form an associative ring with an unity element, the latter being the identity mapping $C(P) = P$.

Especially the correspondences of $S_{\mathfrak{T}}$ to itself defined above generate an associative ring with unity element.

Obviously every point P' of $S_{\mathfrak{T}}$ occurs among the P_i on the right hand side of (5) for some P on the left. Let $P'_1, \ldots, P'_{d'}$ be all P for which a given point P' occurs in the point sets $C(P)$. Then

$$C^*(P') = \sum_{i=1}^{d'} P'_i \tag{6}$$

is also a correspondence, the *conjugate* of $C(P)$. $C^*(P')$ can be derived from $C_{\mathfrak{T}/\mathfrak{T}'}(P)$ as $C(P)$ had been derived from $C_{\mathfrak{T}'/\mathfrak{T}}(P)$; therefore the number d' is finite. The following equations

$$\left.\begin{aligned} (C_1(P) + (C_2(P))^* &= C^*_1(P) + C^*_2(P) \\ (C_1(P).\, C_2(P))^* &= C^*_2(P).\, C^*_1(P) \end{aligned}\right\} \tag{7}$$

show that forming the conjugate is an anti-automorphism of the ring correspondences; it is called the *anti-automorphism of Rosati.* Equation (7) holds whenever $C(P)$ and $C^*(P)$ are finite point sets defined for all P; the proof is elementary.

Let $\mathfrak{T}$ be an order of square-free level (§ 1) and n a natural number. Furthermore let $\nu\mathfrak{T}$ be an integral right ideal for $\mathfrak{T}$ of norm n which is not divisible by a divisor of q. Such ideals are called *primitive.* The left order of $\nu\,\mathfrak{T}$ is $\mathfrak{T}' = \nu\mathfrak{T}\nu^{-1}$. We now consider the groups $\mathfrak{U}, \mathfrak{U}' = \nu\,\mathfrak{U}\,\nu^{-1}$, $\mathfrak{U}^* = \mathfrak{U} \cap \mathfrak{U}'$ of units of norm 1 of $\mathfrak{T}, \mathfrak{T}'$, $\mathfrak{T}^* = \mathfrak{T} \cap \mathfrak{T}'$ and the corresponding transformation groups $\Gamma_{\mathfrak{T}}$, $\Gamma_{\mathfrak{T}'} = \nu\,\Gamma_{\mathfrak{T}}\,\nu^{-1}$, $\Gamma_{\mathfrak{T}^*}$. If

$$\Gamma_{\mathfrak{T}'} = \sum_{i=1}^{d} \Gamma_{\mathfrak{T}^*}\,\epsilon_i, \tag{8}$$

the correspondence of $S_{\mathfrak{T}}$ to itself defined above is

$$C(\Gamma_{\mathfrak{T}} \circ \tau) = C_{\mathfrak{T}'/\mathfrak{T}}\,(\nu\,\Gamma_{\mathfrak{T}} \circ \tau) = \sum_{i=1}^{d} \Gamma_{\mathfrak{T}}\,\epsilon_i\,\nu \circ \tau. \tag{9}$$

We now want to show that $\mathfrak{T}\,\epsilon_i\,\nu$ $(i = 1,\ldots, d)$ are all integral primitive left ideals of norm n, and that they are all different from each other. If $\mathfrak{T}\,\epsilon_i\,\nu = \mathfrak{T}\,\epsilon_\kappa\,\nu$, $\epsilon_i\,\epsilon_k^{-1} \in \Gamma_{\mathfrak{T}}$, and therefore $\epsilon_i\,\epsilon_k^{-1} \in \Gamma_{\mathfrak{T}^*}$, contrary to (8), unless $i = k$. All ideals $\mathfrak{T}\,\epsilon_i\,\nu$ correspond in a unique manner to all ideals $\nu(\mathfrak{T}\,\epsilon_i\,\nu)\,\nu^{-1} = \nu\,\mathfrak{T}\,\epsilon_i = \mathfrak{T}'\,\nu\,\epsilon_i$, both are integral, primitive, and of norm n. It will be shown that the latter exhaust all these ideals, if it is proved that each integral primitive quaternion $\nu' \in \mathfrak{T}'$ of norm n can be written in the form $\nu' = \epsilon'\,\nu\,\epsilon$, ϵ and ϵ' units of $\mathfrak{T}'$. This fact is well known in connection with the theory of elementary divisors if $\mathfrak{T}'$ is the order of all two-rowed matrices with rational integral coefficients. Under our more general assumptions the proof can implicitly be found in the proof that the ideal class number $h = 1$ [4,5].

For the sake of brevity the reader is requested to content himself with this remark.

Thus (9) can also be written as follows :

$$C(\Gamma_{\mathfrak{T}} \circ \tau) = C_n(\Gamma_{\mathfrak{T}} \circ \tau) \sum_{i=1}^{d} \Gamma_{\mathfrak{T}}\,\nu_i \circ (\Gamma_{\mathfrak{T}} \circ \tau), \tag{10}$$

where the ν_i represent a full system of integral primitive left ideals $\mathfrak{T}\nu_i$ of norm n. Therefore C_n depends only on the number n. (10) is the *primitive modular correspondence* associated with n. If the ν_i represent all integral right ideals of norm n, the sum on the right in (10) is the (general) *modular correspondence* associated with n; it will be written as T_n and can be derived from C_n by means of $T_n = \sum_{t^2/n} C_{nt^{-2}}$. In the following we shall only refer to the definition (10) of modular correspondences which can even be abbreviated

$$T_n = \sum_{i=1}^{n} \Gamma_{\mathfrak{T}}\,\nu_i, \tag{11}$$

the sum to be extended over the system ν_i representing all integral left ideals $\mathfrak{T}\,\nu_i$ of norm n. The object of the preceding considerations was to connect the modular correspondences with geometric and more general ideas.

The T_n are conjugate to themselves. Indeed, according to (10), to a point τ of the upper half-plane there correspond all points $\nu \circ \tau$ of the upper half-plane with $\nu \in \mathfrak{I}$, $n(\nu) = n$. Because $\nu \bar{\nu} = n(\nu) = n$, to $\bar{\nu} \circ \tau$ there corresponds especially $\nu \circ (\bar{\nu} \circ \tau) = (\nu\, \bar{\nu}) \circ \tau = \tau$. Since $\bar{\nu} \in \mathfrak{I}$, $n(\bar{\nu}) = n$,

$$T_n = T^*_n. \tag{12}$$

An essential feature of the theory of modular correspondences is the equation

$$T_m\, T_n = T_{mn} \quad (\text{for } (m, n) = 1), \tag{13}$$

and similar formulas which will be discussed later. If μ_i, ν_k represent all integral ideals of norms m, n respectively,

$$T_m(\Gamma_{\mathfrak{I}} \circ \tau)\,.\, T_n(\Gamma_{\mathfrak{I}} \circ \tau) = \sum_{i,k} \Gamma_{\mathfrak{I}}\, \mu_i\, \nu_k \circ (\Gamma_{\mathfrak{I}} \circ \nu_k);$$

here the products $\mu_i\, \nu_k$ represent all integral ideals of norm mn, which proves (13).

Let p be a prime not dividing $q_1\, q_2$ (or the discriminant of $\mathfrak{I}$) and $m = p^r$, $n = p$. Now

$$T_p(\Gamma_{\mathfrak{I}} \circ \tau) \;\; T_{p^r}(\Gamma_{\mathfrak{I}} \circ \tau) = \sum_{i,k} \Gamma_{\mathfrak{I}}\, \mu_i\, \nu_k \circ (\Gamma_{\mathfrak{I}} \circ \tau),$$

where the μ_i, ν_k have the same meaning as above. $\mathfrak{I}\mu_i\nu_k$ are integral ideals of norm p^{r+1}. Because an integral ideal of prime power norm is uniquely decomposable into prime factors if it is primitive, there occur, among the $\mathfrak{I}\mu_i\nu_k$, all integral primitive right ideals of norm p^{r+1} exactly once. But $\mathfrak{I}\mu_i\nu_k$ may also be imprimitive, which is the case if $\nu_k = \bar{\mu}_i\, \nu'_k$, $\nu'_k \in \mathfrak{I}$, and then $\mu_i\, \nu_k = p\, \nu'_k$. Since the number of integral left ideals $\mathfrak{I}\, \mu_i$ of norm p is $p + 1$, there occur, among the $\mathfrak{I}\, \mu_i\, \nu_k$, all ideals $\mathfrak{I}_p\, \nu'_k$ of norm p^{r+1}, $(p + 1)$ times each. Therefore

$$T_{p^r}\,.\, T_p = C_{p^{r+1}} + (p+1)\, T_{p^{r-1}},$$

$(p + 1)\, T_{p^{r-1}}$ means $T_{p^{r-1}} + T_p^{\,r-1} + \dots + T_{p^{r-1}}$ (p + 1 times).

Evidently

$$T_{p^{r+1}} = C_{p^{r+1}} + T_{p^{r-1}},$$

therefore

$$T_{p^r} T_p = T_{p^{r+1}} + p\, T_{p^{r-1}}. \tag{14}$$

From (14) we can derive by induction on s [8],

$$T_{p^r} T_{p^s} = \sum_{t=0}^{\min(r,s)} p^t T_{p^{r+s-2t}}, \quad (p \nmid q_1 q_2). \tag{15}$$

For primes p dividing q_1 there exists exactly one integral ideal of norm p^r, therefore

$$T_{p^r} T_{p^s} = T_{p^{r+s}}, \quad (p | q_1). \tag{16}$$

For primes $p | q_2$ multiplication rules analoguous to (15), (16) are more complicated. However, for a given r there exists exactly one integral ambiguous ideal $\pi \mathfrak{T} = \mathfrak{T} \pi$ of norm p^r. The modified correspondence T_p^0 defined by

$$T^0_{p^r}(\Gamma_{\mathfrak{T}} \circ \tau) = \Gamma_{\mathfrak{T}}\, \pi \circ (\Gamma_{\mathfrak{T}} \circ \tau), \quad n(\pi) = p^r \tag{17}$$

satisfies

$$T^0_{p^r} T^0_{p^s} = T^0_{p^{r+s}}, \quad (p | q_2). \tag{18}$$

In his papers on the representation of modular correspondences by modular forms, Hecke employed chiefly these modified correspondences.

In the next paragraphs we shall study various representations of the ring $\mathfrak{M}$ generated by the modular correspondences. Every such representation $R(\mathfrak{M})$ gives rise to a special zeta function

$$\zeta_R(s) = \sum_{n=1}^{\infty} R(T_n)\, n^{-s}$$

which is, more precisely expressed, a matrix whose elements are certain Dirichlet series. The most simple representation is a one-rowed one: $R(T_n) = d_n$ (see (11)), the number of integral right ideals of norm n. In this case $\zeta_R(s)$ is the zeta function of $\mathfrak{T}$.

In this connection the meaning of (13) is that $\zeta_R(s)$ can be written as an Euler product

$$\zeta_R(s) = \prod_p \sum_{r=0}^{\infty} R(T_{p^r})\, p^{-rs} = \prod_p \zeta_{R,p}(s),$$

while (15), (16), (18) make certain statements on the nature of the factors $\zeta_{R,p}(s)$.

The chief task in the following will be the determination of the traces of the representations $R(\mathfrak{m})$, because the traces determine the representations uniquely. For some $R(\mathfrak{m})$ the calculation of the trace leads into topological considerations, for others the traces are not yet known.

3. The representation of the modular correspondences as endomorphisms of the Betti groups. The geometrical origin of the concept of a correspondence makes it clear that the property of an r-dimensional chain τ^r of being contained in the boundary of an $(r+1)$ dimensional chain τ^{r+1} is preserved by the modular correspondences. In other words, the incidence relations

$$\mathrm{Bd}(\tau_i^{r+1}) = \sum_k \beta_{ik}^r\, \tau_k^r$$

for a simplicial dissection of $S_{\mathfrak{T}}$ entail

$$\mathrm{Bd}(T_n(\tau_i^{r+1})) = \sum_k \beta_{ik}^r\, T_n(\tau_k^r).$$

Consequently T_n maps r-cycles onto r-cycles, and boundaries of r-chains onto boundaries of r-chains. So the T_n are representable by certain endomorphisms $R^r(T_n)$ of the rth homology groups (which, in our case, coincide with the Betti groups), $r = 0, 1, 2$.

Let

$$t^r(T_n) = \mathrm{trace}\,(R^r(T_n)) \tag{19}$$

be the traces. Evidently

$$t^0(T_n) = d_n, \tag{20}$$

which is equal to the number of integral left ideals of $\mathfrak{T}$ of norm n. Equally

$$t^2(T_n) = d_n. \tag{21}$$

$t^1(T_n)$ can be calculated by means of the Lefschetz fixed point theorem [13]

$$t^0(T_n) - t^1(T_n) + t^2(T_n) = f(T_n), \tag{22}$$

where the right hand side is the number of fixed points of T_n, that is the number of points P of $S_{\mathfrak{T}}$ which have the property that at least one point of the set $T_n(P)$ is equal to P. These fixed points have to be counted with a certain multiplicity which is defined as follows. The mapping $T_n(P) = T_n(\Gamma_{\mathfrak{T}} \circ \tau) = \sum_i \Gamma_{\mathfrak{T}} \circ \tau_i$ is in general a conformal mapping. In the neighbourhood of a fixed point $\Gamma_{\mathfrak{T}} \circ \tau$ we shall see in §4 that each of the mappings $\Gamma_{\mathfrak{T}} \circ \tau \to \Gamma_{\mathfrak{T}} \circ \tau_i$ can be written in the form

$$\tau_i = \tau^0 + c_1(\tau - \tau^0)^{a/b} + c_2(\tau - \tau^0)^{(a+1)/b} + \dots, \; c_1 \neq 0, \tag{23}$$

where τ and τ_i mean local uniformizing parameters, and where b is the common denominator of all exponents occurring in (23). Now the multiplicity of P as a fixed point is min (a, b).

With the multiplicities of fixed points defined in this way (22) is a theorem on mappings of $S_{\mathfrak{T}}$ onto itself which are conformal at least in the vicinities of fixed points (the points themselves being excepted). We shall give a simple proof under these assumptions. However, the multiplicities can also be defined by purely topological means, and (22) is a special case of a well-known topological theorem of Lefschetz. As the rather lengthy original proof has not been reproduced in elementary text books on topology the following proof may be appreciated though it makes a superfluous assumption.

At first we cut up $S_{\mathfrak{T}}$ into a finite number of triangles τ_i^2, the fixed points being among the vertices. The τ_i^2 can be assumed to have the following three properties.

1. The image $T_n(\tau_i^2)$ consists of a number of simply connected domains $\bar{\tau}_{ik}^2$ bounded by Jordan curves without double points. Consequently the images of the sides τ_j^1 of the τ_i^2 are open continuous curves. This property is ensured by placing vertices in each ramification point with respect to $S_{\mathfrak{X}}$ of the surface $S_{\mathfrak{X}^*}$, and by making the τ_i^2 sufficiently small.

2. All the images of each vertex are situated in the interior of some other triangle, except for the fixed points.

3. All images of all points of a τ_i^2 lie in such triangles τ_k^2 as have no point in common with τ_i^2, except for those τ_i^2 which contain a fixed point.

These three properties remain valid if the τ_i^2 are cut up into smaller triangles, which will be necessary under certain conditions.

We start the proof of (22) by defining a linear mapping ϕ_n of the r-chains τ_i^r having the same effect on the r-cycles as T_n. If

$$T_n(\tau_i^r) = \sum_k \bar{\tau}_{ik}^r, \tag{24}$$

where the $\bar{\tau}_{ik}^r$ are simply connected domains ($r = 2$), arcs of curves ($r = 1$), or points ($r = 0$), we put

$$\phi_n(\tau_i^r) = \phi(T_n(\tau_i^r)) = \sum_k \phi(\bar{\tau}_{ik}^r), \tag{25}$$

with an auxiliary function ϕ of arbitrary simply connected domain σ^2 on $S_{\mathfrak{X}}$, of curves σ^1, or of points σ^0. The values of ϕ will be r-chains on $S_{\mathfrak{X}}$ consisting of linear combinations of the τ_i^2 or their sides or vertices.

The definition of ϕ differs according as σ^r contains a fixed point or not. At first we assume the second. ϕ is an additive function:

$$\phi(\sigma_1^r + \sigma_2^r) = \phi(\sigma_1^r) + \phi(\sigma_2^r). \tag{26}$$

In detail

$$\phi(\sigma^2) = \tfrac{1}{3} \sum_j \epsilon_j \, \tau_j^2, \tag{27}$$

where ϵ_j is the number of vertices of τ_j^2 lying in σ^2.

$$\phi(\sigma^1)=\begin{cases}0, \text{ if } \sigma^1 \text{ lies entirely inside of a } \tau_j^2, & (28)\\ \frac{1}{6}(\tau_1'+\ldots+\tau_4'), \text{ if } \sigma^1 \text{ crosses one boundary once between two } \tau_i^2;\end{cases}$$

the meaning of $\tau_1',\ldots,\tau_4'$ is shown in Figure 1.

$$\phi(\sigma^0) = \tfrac{1}{3}(\tau_{k,1}^0 + \tau_{k,2}^0 + \tau_{k,3}^0), \tag{29}$$

where $\tau_{k,j}^0$ are the vertices of the triangle τ_k^2 in which σ^0 lies. Of course, it has to be verified that (27)-(29) are consistent with (26), which is almost trivial. The definition of ϕ is not applicable if the boundary of σ^r passes through a vertex of a τ_i^2; our definition will prove sufficient in later applications.

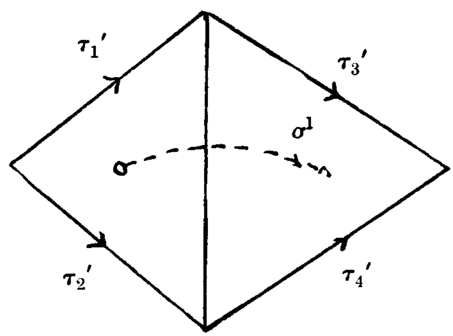

FIG. 1

The function ϕ is commutative with the formation of the boundary:

$$\phi(\mathrm{Bd}(\sigma^r)) = \mathrm{Bd}\,(\phi(\sigma^r)). \tag{30}$$

Because of (26) it is sufficient to prove (30) in the following cases: (a) $r=2$, σ^2 does not contain a vertex, and is contained in one or the union of two τ_i^2; (b) $r=2$, σ^2 contains exactly one vertex; (c) $r=1$, and conditions of Figure 1 are satisfied. In all three cases (30) is readily verified.

We now prove the following lemma under the additional assumption that σ^1 does not meet a fixed point. Later we shall show that this assumption can be dropped.

Lemma 1. For a closed curve σ^1 without double points, $\phi(\sigma^1)$ is a cycle homotopic with σ^1, if the τ_i^2 are sufficiently small.

Proof. We make the τ_i^2 so small that all τ_i^2 through which σ passes form a stripe without double points. The description of $\phi(\sigma^1)$ is more easily achieved by an example shown in Figure 2:

$$\phi(\sigma^1) = \tfrac{1}{3}\,[(P_1P_2) + (P_2P_3) + \dots + (P'_1P'_2) + (P'_2P'_3) + \dots \\ + (P'_1P_1) + (P_1P'_4) + (P'_4P_2) + (P_2P'_5) + (P'_5P_4) + \dots],$$

and the lemma becomes evident. The proof under the most general conditions can be left to the reader.

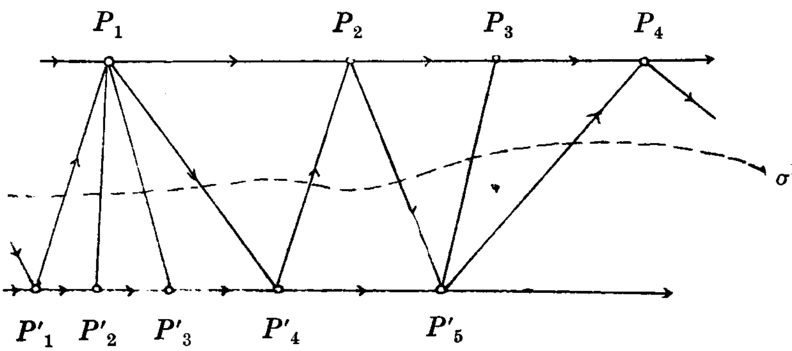

Fig. 2

We now supplement the definition of ϕ under the condition that the boundary of σ^r contains a fixed point P; this will be general enough for our purposes. With respect to the additivity relation (26) it will suffice to define ϕ for such σ^r as are contained in the union of all the τ_i^2 having P as a vertex; this union is called *the star of* P. Furthermore, with respect to (26), we can assume without loss of generality, that σ^r lies inside the star of P and that a σ^1 has no other points in common with a τ_j^1 except P. Under these conditions we put

$$\phi(\sigma^2) = \frac{1}{6} \sum_j \epsilon_j\, \tau_j^2, \tag{31}$$

where ϵ_j is the number of sides of τ_j^2 originating from P which pass through σ^2. Furthermore,

$$\phi(\sigma^1) = \tfrac{1}{3}(\tau^1_{i,1} + \tau^1_{i,2}), \tag{32}$$

where $\tau^1_{i,j}$ are the sides originating from P of the τ^2_i through which σ^1 passes. Lastly

$$\phi(P) = P. \tag{33}$$

Again we have to prove that (31) - (33) are consistent with (26) which is almost trivial. Moreover we have to verify (30) and to ascertain that Lemma 1 remains valid if σ^1 passes through P. Both are quite easy and can be left to the reader.

The function ϕ being defined in all cases in which we want to apply it, we now proceed to the proof of (22). From Lemma 1 and (24), (25) follows immediately

$$T_n(\sigma^r) \sim \phi_n(\sigma^r) \tag{34}$$

($\sim$ means homologuous) for a 1-cycle $\sigma^r (r = 1)$ consisting of some sides of the τ^2_i. The assumption of Lemma 1 that σ^1 be without double points can be dropped because σ^1 could otherwise be pieced together of curves without double points. (34) holds as well for 0-cycles ($r = 0$) which follows from (29), (33). Lastly (34) holds for $r = 2$ in consequence of (27) and (31).

Because of (34) we may replace T_n by ϕ_n in (22). Now ϕ_n is a linear function operating in the spaces of linear combinations of the τ^2_i, of linear combinations of the sides of the τ^2_i, and of linear combinations of the vertices of the τ^2_i. From (24), (25) and (30) follows that it has the property

$$\phi_n(\mathrm{Bd}(\tau^r_i)) = \mathrm{Bd}(\phi_n(\tau^r_i)). \tag{35}$$

(35) is the condition under which the Euler-Poincaré-Hopf formula is applicable, stating that the left hand side of (22) is equal to that of

$$s^0(\phi_n) - s^1(\phi_n) + s^2(\phi_n) = f(T_n), \tag{36}$$

where $s^r(\phi_n)$ is the trace of the linear transformation $\tau^r_i \to \phi_n(\tau^r_i)$. The final proof of (22) or of equation (36), equivalent to (22), will now consist of the calculation of the traces $s^r(\phi_n)$. Because of the assumption 3, made at the beginning, we need only consider the effect of ϕ_n on the simplices τ_i^r belonging to the stars of the fixed points.

In the neighbourhood of a fixed point $P = \tau^0$ the mapping $\tau_i^r \to \bar{\tau}_{ik}^r$ is effected by the function (23). This entails that the star of P is mapped b times on a Riemann surface of a sheets lying over the neighbourhood of P, the ramification (of order a) being at P. If $a > b$, or $a = b$ and $|c| < 1$, the image of the star of P lies entirely inside of the star of P (case 1). If $a > b$ or $a = b$ and $|c| > 1$, the image of the star of P lies entirely outside the star of P (case 2).

We treat the third case $a = b$, $|c| = 1$ first. The mapping is approximately a rotation about P. We now subdivide the star of P into sectors (the sides τ_i^1 of which need not be straight lines) which are so fine that the images of each side τ_i^1 falls into a sector whose both sides differ from τ_i^1. So we get $s^1(\phi_n) = b$, $s^1(\phi_n) = 0$, $s^2(\phi_n) = 0$ what concerns the star of P. The first term is explained by the fact that P is mapped b times on P.

In both cases when the image of the boundary of the star of P lies either completely inside or completely outside the star of P we deform the mapping homotopically. A homotopic deformation leaves the left hand side of (22) unchanged and therefore also the left hand side of (36). If the τ_i^2 are small enough, the assumption 3. made at the beginning remains valid after the deformation. We chose it in such a way that almost all images $\bar{\tau}_{ik}^2$ of all τ_i^2 are sectors with an angle nearly 0 at P, and that they lie inside of τ_1^2 as far as the immediate neighbourhood of P is concerned (see Figures 3 and 4). Only a of the $\bar{\tau}_{ik}^2$ with $k \neq 1$ are sectors of an angle nearly 2π. This is always possible if there are enough τ_i^2 in the star of P.

We now consider the first case (Figure 3). Because of (31) the coefficient of τ_i^2 in $\phi_n(\bar{\tau}_{ik}^2)$ is 0 or $\frac{1}{3}$ according as $\bar{\tau}_{ik}^2$ is a small or a large sector. Hence the contribution of the neighbourhood of P to $s^2(\phi_n)$ is $a/3$. Because of (32) the coefficient of a side $\tau_{i,j}^1 (j = 1, 2)$ of τ_i^2 originating from P in the function $\phi_n(\bar{\tau}_{ik,j}^1)$ is 0 or $\frac{1}{3}$ according

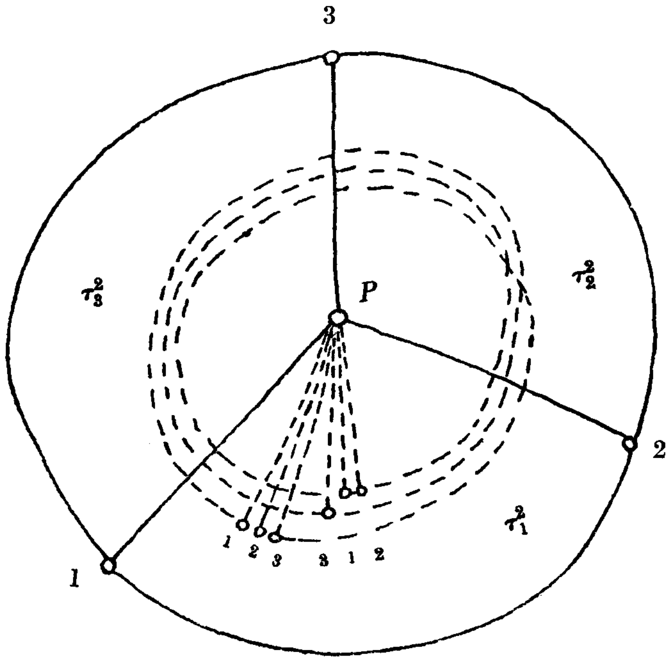

FIG. 3 : $a = 3$, $b = 2$.
(*The broken lines show the* $\bar{\tau}_{ik}^r$.)

as $i \neq 1$ or $i = 1$. Furthermore because of (28) for each side τ_i^1 of τ_i^2 opposite to P, $\phi(\bar{\tau}_{ik}^1) = 0$ or $\frac{1}{3}$ times the boundary of the star of P minus the side τ_1^1 of τ_1^2 opposite to P according as $\bar{\tau}_{ik}^1$ is small or large. Hence the contribution of the neighbourhood of P to $s^1(\phi_n)$ is $(a + 2b)/3$. Lastly $\phi(P) = b.P$, and for the other vertices $\bar{\tau}_{ik,j}^0$ of $\bar{\tau}_{ik}^2 : \phi_n(\bar{\tau}_{ik,j}^0) = \frac{1}{3}$ times the sum of the vertices of τ_1^2. Hence the contribution of the neighbourhood of P to $s^0(\phi_n)$ is $b + \frac{2}{3}b$. Consequently the contribution of the neighbourhood of P to the left hand side of (36) is $b = \min(a, b)$.

There remains now the second case to be treated (Figure 4). In this case the large $\bar{\tau}_{ik}^2$ are cut up into three parts as shown in Figure 5. Using (26) we find the coefficient of τ_i^2 in $\phi_n(\bar{\tau}_{ik}^2)$ for a large $\bar{\tau}_{ik}^2$ to be 1. For a small $\bar{\tau}_{ik}^2$ it is again 0. So the contribution of the neighbourhood of P to $s^2(\phi_n)$ is a.

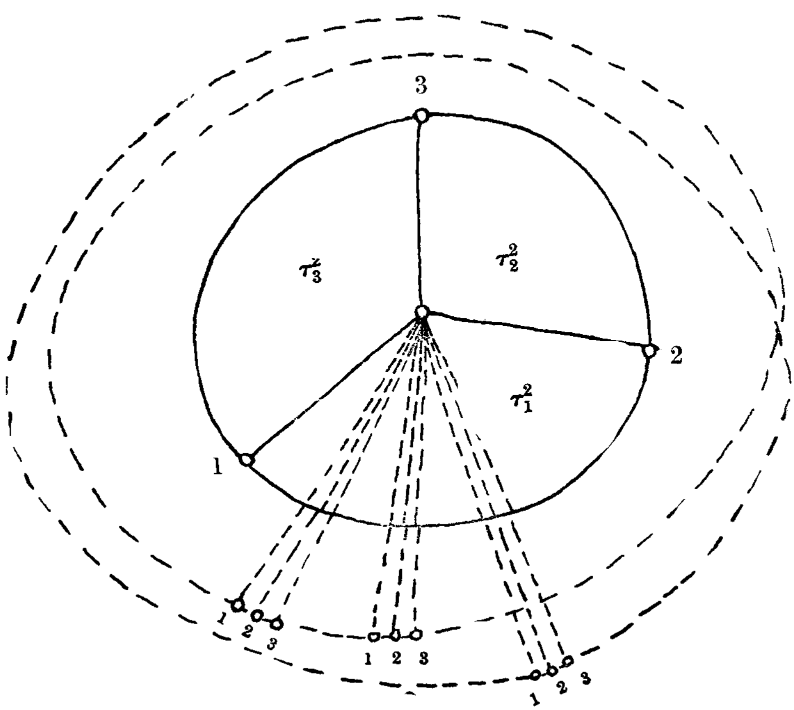

FIG. 4: $a = 2$, $b = 3$.
(*The broken lines show the* $\bar{\tau}^r_{ik}$.)

The sides $\tau^1_{i,j}$ ($j = 1, 2$) of τ^2_i originating in P are divided correspondingly into two parts. Now (26), (32) and (28) show that the coefficient of such a $\tau^1_{i,j}$ in $\phi_n(\bar{\tau}^1_{ik,j})$ is 0 or $\frac{1}{2}$ according as $i \neq 1$ or $i = 1$. The other sides of the τ^2_i do not contribute to the trace. The contribution of the former to $s^1(\phi_n)$ is b. Lastly the only contribution to $s^0(\phi_n)$ is yielded by P, it is equal to b. So the contribution of the neighbourhood of P to the left hand side of (36) is $a = \min(a, b)$.

In all three cases the contribution of the neighbourhood of P to the left hand side is equal to the multiplicity of P as a fixed point. Taking the sum over all P completes the proof.

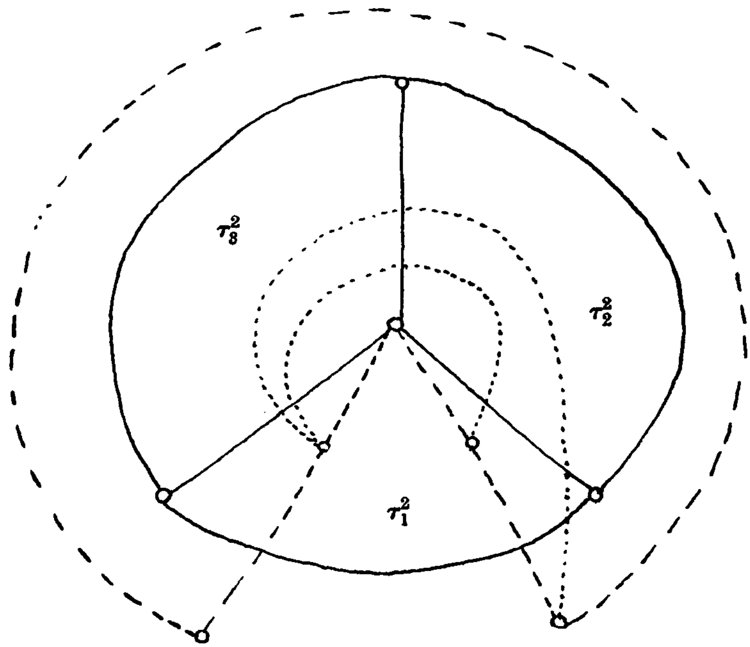

FIG. 5

(*The broken lines show but one* $\bar{\tau}_{ik}^2$, *the dotted lines show how it is to be cut up into three triangles.*)

4. Continuation. Explicit values of $t_1(T_n)$. Examples. Our next task is to prove (23) and to determine the values of a and b. If τ^0 is not a fixed point of an $\eta \in \Gamma_{\mathfrak{T}}$ the complex variable τ is a local uniformizing parameter, and as $\tau_i = \nu_i \bigcirc \tau$ with a certain real matrix ν_i, $a = b = 1$. Thus the multiplicity of such a fixed point is 1.

If $\eta \bigcirc \tau^0 = \tau^0$ for an $\eta \in \Gamma_{\mathfrak{T}}$, and if τ^0 lies in the interior of the complex upper half-plane, we have seen that η is of order $e = 2$ or 3. Therefore $\lambda^{-1} \eta \lambda = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, where ζ is a primitive $2e$th root of unity and λ is a complex two-rowed matrix. We introduce $\tau' = \lambda^{-1} \bigcirc \tau^0$ and find $\lambda^{-1} \eta \lambda \bigcirc \tau' = \zeta^2 . \tau'$. Therefore a uniformizing parameter is τ'^e, small values of τ'^e describing the neighbourhood of τ^0. Now let τ^0 be a fixed point of the substitution ν_i occurring in (11): $\nu_i \bigcirc \tau^0 = \tau^0$. Because of $\eta \bigcirc \tau^0 = \tau^0$ we find

$$\nu_i = U + V\eta \tag{37}$$

with rational diagonal matrices $U = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$, $V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$. Therefore

$$\tau'_i = \lambda^{-1}\nu_i \lambda \circ \tau' = \frac{u + v\zeta}{u + v\zeta^{-1}}\, \tau',$$

and (23) holds with $a = b = 1$. The multiplicity of τ^0 as a fixed point is again 1.

Lastly let τ^0 be a parabolic cusp. This happens only if the algebra Q contains divisors of 0. A local uniformizing paramter is $e^{2\pi i \lambda \circ \tau}$, λ a certain matrix with rational coefficients depending on τ^0 and the group. In this case detailed investigations are necessary, which have been carried out for the type of groups $\Gamma_{\mathfrak{T}}$ which are considered in a former paper [7].

In the same paper the right hand side $f(T_n)$ of (22) has been calculated. The essential point was the fact that, to each class $\epsilon \nu_i \epsilon^{-1}$ with a given ν_i having a fixed point in the complex upper half-plane and all $\epsilon \in \Gamma_{\mathfrak{T}}$, there corresponds exactly one fixed point $P = \Gamma_{\mathfrak{T}} \circ \tau^0$ of $S_{\mathfrak{T}}$. Indeed, if $\nu_i \circ \tau^0 = \tau^0$, $\epsilon \nu_i \epsilon^{-1} \circ (\epsilon \circ \tau^0) = \epsilon \circ \tau^0$.

If now τ^0 is also a fixed point of an $\eta \in \Gamma_{\mathfrak{T}}$, of order $e = 2$ or 3, ν_i and $\nu_i \eta = \eta \nu_i$ (see (37)) would yield the same $P = \Gamma_{\mathfrak{T}} \circ \tau^0$. Therefore we have to count only one substitution of the set $\nu_i \eta^f (f = 0, \ldots, e - 1)$. In the actual calculation we may take all these $\nu_i \eta$ counting each with the multiplicity $1/e$.

There is yet one other point to be observed. If $n = m^2$ with an integral m, one of the ν_i in (11) defining T_n may be $\nu_i = m$ which yields the unity mapping $m \circ \tau = \tau$. For this mapping all points are fixed points, and (22) is not applicable. We now write for (11)

$$T_{m^2} = T_{m^2,1} + T_1, \quad T_{m^2,1} = \sum_{i=1}^{d_{m^2}-1} \Gamma_{\mathfrak{T}} \nu_i, \quad \mathfrak{T}\nu_i \neq \mathfrak{T}_m.$$

T_1 is the unity mapping, and the traces of T_1 as endomorphisms of the Betti groups are

$$t^0(T_1) = t^2(T_1) = 1, \quad t^1(T_1) = g,$$

since the 0th and 2nd Betti groups have rank 1 and the 1st Betti group has rank g = genus of $S_{\mathfrak{T}}$.

The final result is [7]

$$t^1(T_n) = 2\sum_{t|n} t - \sum_{s,f} \prod_{p|q_1} \left(1 - \left\{\frac{(s^2 - 4n)f^{-2}}{p}\right\}\right) \times$$
$$\times \prod_{p|q_2} \left(1 + \left\{\frac{(s^2 - 4n)f^{-2}}{p}\right\}\right) \frac{h((s^2 - 4n)f^{-2})}{e((s^2 - 4n)f^{-2})} + c_n, \quad q_1 \neq 1, \tag{38_1}$$

with

$$c_n = \begin{cases} 0, & \text{for } \sqrt{n} \not\equiv 0 \bmod 1, \\ \frac{1}{6} \prod_{p|q_1} (p-1) \prod_{p|q_2} (p+1), & \text{for } \sqrt{n} \equiv 0 \bmod 1, \end{cases} \quad q_1 \neq 1, \tag{39_1}$$

or

$$t^1(T_n) = 2\sum_{t|n} t - 2^{\kappa+1} \sum_{\substack{t|n \\ t<\sqrt{n}}} t -$$
$$- \sum_{s,f} \prod_{p|q_2} \left(1 + \left\{\frac{(s^2 - 4n)f^{-2}}{p}\right\}\right) \frac{h((s^2 - 4n)f^{-2})}{e((s^2 - 4n)f^{-2})}, \quad q_1 = 1, \tag{38_2}$$

with

$$c_n = \begin{cases} 0, & \text{for } \sqrt{n} \not\equiv 0 \bmod 1, \\ \frac{1}{6} \prod_{p|q_2} (p+1) - 2^{\kappa} |\sqrt{n}|, & \text{for } \sqrt{n} \equiv 0 \bmod , \end{cases} \quad q_2 = 1. \tag{39_2}$$

For the understanding of these formulas the following explanations are necessary : κ is the number of (different) primes dividing q_2. The sums $\sum_{s,f}$ have to be extended over all integers s, f satisfying

$$-2\,|\sqrt{n}| < s < 2\,|\sqrt{n}|,\ 0 < f,\ (s^2 - 4n)f^{-2} \equiv 0 \text{ or } 1 \bmod 4.$$

$h(d)$, $e(d)$ mean the class number and half the number of automorphs of primitive quadratic forms $ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$. Lastly

$$\left\{\frac{d}{p}\right\} = \begin{cases} \left(\frac{d}{p}\right), \text{ the Legendre symbol, if } dp^{-2} \not\equiv 0 \text{ or } 1 \bmod 4, \\ 1, \qquad\qquad\qquad\qquad\qquad \text{if } dp^{-2} \equiv 0 \text{ or } 1 \bmod 4. \end{cases}$$

The proof has only been carried out in the case $q_1 = 1$ which contains the chief difficulty. The generalization does not require new thoughts.

If $g = 0$, the 1st Betti group consists of the element 0 only, and $t^1(T_n) = 0$ for all n. This yields class number relations the most simple of which is obtained by putting $q_1 = q_2 = 1$:

$$\sum_{s,f} \frac{h((s^2 - 4n)f^{-2})}{e((s^2 - 4n)f^{-2})} = 2 \sum_{t > \sqrt{n}} t \quad \begin{cases} 0, & \text{for } \sqrt{n} \not\equiv 0 \bmod 1, \\ \sqrt{n} + \frac{1}{6}, & \text{for } \sqrt{n} \equiv 0 \bmod 1, \end{cases} \tag{40}$$

this is the classical class number relation of the 1st level (Stufe). The genus can also be $g = 0$ if $q_1 \neq 1$, for example in the case $q_1 = 6$, $q_2 = 1$. The class number relations obtained in this way can certainly not be proved by means of modular functions as (40) can. The greatest number of the known class number relations may find a natural explanation in this connection. However, it is doubtful if this would find much interest.

5. The representation of the modular correspondences in the module of differentials of the first kind. In this section we shall study automorphic forms $\phi(\tau)$, satisfying the functional equations

$$\frac{\alpha\delta - \beta\gamma}{(\gamma\tau + \delta)^2} \phi\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \phi(\tau) \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \phi(\tau), \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{\mathfrak{T}}, \tag{41}$$

which are holomorphic throughout the complex upper half-plane. This includes a holomorphic behaviour at the parabolic cusps, if there are any. The holomorphic behaviour at a point $\tau = \tau^0$ is defined as follows: let z be a local uniformizing parameter at τ^0, $z = 0$ for $\tau = \tau^0$, then $\phi(\tau)\ (d\tau/dz)$ is a holomorphic function in some vicinity of $z = 0$. In the case of $\Gamma_{\mathfrak{T}}$ being a subgroup of the modular group the designation of such forms as *cusp-forms* has been widely accepted.

The functions invariant for the substitutions of $\Gamma_{\mathfrak{T}}$ and having no other singularities than poles form a field F of algebraic functions. The Riemann surface of this field is $S_{\mathfrak{T}}$. The expressions $\phi(\tau)\, d\tau$ are the differentials of the first kind of F. It is known that there exist exactly g of them linearly independent.

Let $\phi_1(\tau), \ldots, \phi_g(\tau)$ be a basis of the module of modular forms having the properties mentioned above. Furthermore let $\tau_1^1, \ldots, \tau_{2g}^1$ be a homology basis of dimension 1 of the Riemann surface $S_{\mathfrak{T}}$ (for example a canonical dissection). Now the integrals

$$\int_{\tau_\nu^1} \phi_\mu(\tau)\, d\tau = \omega_{\mu\nu} \tag{42}$$

form the so-called *Riemann matrix* of F. It has g lines and $2g$ columns. We shall make use of the well-known fact that the $2g$ by $2g$ matrix

$$P = \begin{pmatrix} \omega_{\mu\nu} \\ \overline{\omega}_{\mu\nu} \end{pmatrix} \tag{43}$$

(the bar means the complex conjugate) is non-singular.

After these preparations we come to the definition of a new representation $R_1(T_n)$ of the modular correspondences. Employing the abbreviation used in (41) we can see immediately that $\sum_i \phi(\tau) \circ \nu_i$ is again a modular form, if the ν_i are taken from (11) the choice of the ν_i in the respective ideals $\mathfrak{T}_{\nu_i}$ being immaterial. Moreover, as shown by Hecke, these functions are again holomorphic. Thus we get a representation of the T_n which can be written as

$$R_1(T_n). \begin{bmatrix} \phi_1(\tau) \\ \vdots \\ \phi_g(\tau) \end{bmatrix} = \begin{bmatrix} \phi_1(\tau) \\ \vdots \\ \phi_g(\tau) \end{bmatrix} \circ T_n = \sum \begin{bmatrix} \phi_1(\tau) \\ \vdots \\ \phi_g(\tau) \end{bmatrix} \circ \nu_i, \tag{44}$$

or more explicitly, with $R_1(T_n) = (r_{\mu\nu}(T_n))$,

$$\sum_{\nu=1}^{g} r_{\mu\nu}(T_n)\ \phi_\nu(\tau) = \phi_\mu(\tau) \circ T_n = \sum_{i=1}^{dn} \phi_\mu(\tau) \circ \nu_i.$$

There is a close connection between the representations R_1 and R^1. We start from equation (44) and obtain

$$\sum_{\nu=1}^{g} r_{\mu\nu}(T_n) \int_{\tau_\lambda^1} \phi_\nu(\tau)\, d\tau = \sum_{i=1}^{d_n} \int_{\tau_\lambda^1} \frac{\alpha_i \delta_i - \beta_i \gamma_i}{(\gamma_i \tau + \delta_i)^2}\, \phi_\mu\left(\frac{\alpha_i \tau + \beta_i}{\gamma_i \tau + \delta_i}\right) d\tau,$$

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} = \nu_i.$$

The right hand side can be written as

$$\sum_{i=1}^{d_n} \int_{\tau_\lambda^1} \phi_\mu\left(\frac{\alpha_i \tau + \beta_i}{\gamma_i \tau + \delta_i}\right) d\, \frac{\alpha_i \tau + \beta_i}{\gamma_i \tau + \delta_i}$$

which is evidently equal to

$$\int_{T_n(\tau_\lambda^1)} \phi_\mu(\tau)\, d\tau = \sum_{\rho=1}^{2g} \int_{\tau_\rho^1} \phi_\mu(\tau)\, d\tau.\; r_{\rho\lambda}^1\, (T_n)$$

with $(r_{\rho\lambda}^1(T_n)) = R^1(T_n)$. Thus we have proved

$$R_1(T_n)\, (\omega_{\mu\nu}) = (\omega_{\mu\nu})\, R^1(T_n).$$

Here we may substitute for all coefficients their complex conjugates; but we know that $R^1(T_n)$ has rational integral coefficients. Using the abbreviation (43) we then get

$$\begin{pmatrix} R_1(T_n) & 0 \\ 0 & \bar{R}_1(T_n) \end{pmatrix}.\, P = P.\, R^1\, (T_n), \tag{45}$$

which shows that $R^1(T_n)$ and $\begin{pmatrix} R_1 & 0 \\ 0 & \bar{R}_1 \end{pmatrix}$ are equivalent, because P is non-singular. For the trace $t^1(T_n) =$ trace of $R^1(T_n)$ (45) entails

$$t_1(T_n) + \check{t}_1(T_n) = t^1(T_n). \tag{46}$$

In the case of $\Gamma_{\mathfrak{T}}$ being a subgroup of the modular group, $R_1(T_n)$ has real eigenvalues. Therefore $R_1(T_n) \sim \bar{R}_1(T_n)$, and then $R_1(T_n)$ can explicitly be calculated from $R^1(T_n)$, using (46). We shall meet an interesting example later (§ 7).

6. Arithmetics of definite quaternion algebras. The following investigations serve as a preparation for further representations of the modular correspondences which will be discussed in § 7 and § 8. In this section Q is a definite quaternion algebra. The other assumptions made in § 1 on the orders $\mathfrak{T}$ in Q which we are going to examine remain valid.

The chief features of definite quaternion algebras as compared with indefinite ones are : (1) there exists but a finite number of units in each order, (2) the class number of (left or right) ideals is in general $h > 1$.

In a former paper [6] it has been proved that, for any two orders $\mathfrak{T}_1, \mathfrak{T}_2$ of the type described above having the same discriminant, there exist always ideals $\mathfrak{m}$ which are left ideals for $\mathfrak{T}_1$ and right ideals for $\mathfrak{T}_2 : \mathfrak{T}_1 \mathfrak{m} = \mathfrak{m} \mathfrak{T}_2 = \mathfrak{m}$. All ideals belonging in this way to all orders of the same discriminant form a gruppoide G. In the following we shall only deal with ideals of this G without further mentioning it.

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_h$ represent all classes of left ideals for $\mathfrak{T}_1$. Then $\mathfrak{m}_i^{-1} \mathfrak{m}_k (k = 1, \dots, h)$ represent all classes of left ideals for $\mathfrak{T}_i$. Brandt has introduced the following *ideal number matrices* (Anzahlmatrizen [2, 6]) : $P(n) = (p_{ik}(n))$, where $p_{ik}(n)$ is the number of integral ideals of norm n and of the form $\mathfrak{m}_i^{-1} \mathfrak{m}_k \rho$. The sums $\Sigma\, p_{ik}(n)$ are equal to the number d_n of integral left ideals for $\mathfrak{T}_i$ of norm n. We shall now generalize these matrices [8, Ch. IV, p.109]:

Let $r(\rho)$ be an inverse matrix representation of the multiplicative group of all elements $\rho \neq 0$ of Q :

$$r(\rho\sigma) = r(\sigma)\, r(\rho). \tag{47}$$

The elements of $r(\rho)$ are homogeneous polynomials in the coordinates of ρ with respect to some basis of Q. Therefore

$$r(t\rho) = t^{d(r)}\, r(\rho), \tag{48}$$

with a natural number $d(r)$, the *degree* of r. We shall only deal with representations of even degree which have the property that

$$r(-1) = r(1) = 1. \tag{49}$$

Furthermore let $\pm\,\epsilon_1^{(i)}, \dots, \pm\,\epsilon_{e_i}^{(i)}$ be all units of $\mathfrak{T}_i$, e_i being half their number.

If $\mathfrak{l}_{i,\nu} = \mathfrak{m}_i^{-1}\mathfrak{m}_k\,\rho_\nu(\nu = 1, \dots, p_{ik}(n))$ are all integral left ideals for $\mathfrak{T}_i$ of norm n in the kth class, we now define the *generalized ideal number matrix* by

$$P_r(n) = (s_{ik}), \quad s_{ik} = \frac{1}{e_k}\sum_{\mu=1}^{e_k}\sum_{\nu=1}^{p_{ik}(n)} r(\epsilon_\mu^{(k)}\,\rho_\nu).$$

The number of rows of $P_r(n)$ is h times the number of rows of $r(\rho)$. Because of (49), $P_r(n)$ is not changed if some of the $\epsilon_\mu^{(k)}$ are replaced by $-\,\epsilon_\mu^{(k)}$. (This is the reason why representations of even degree are used only.) For $r(\rho) = 1$ we get the former ideal number matrices $P_1(n) = P(n)$.

Our first aim is to prove

$$P_r(m)\,P_r(n) = P_r(mn), \quad \text{for } (m, n) = 1. \tag{50}$$

Let $\mathfrak{l}_{i,\nu} = \mathfrak{m}_i^{-1}\mathfrak{m}_k\,\rho_\nu$ be all integral left ideals for $\mathfrak{T}_i$ of norm mn in the kth class. As it is well known each $\mathfrak{l}_{i,\nu}$ can uniquely be factorized into two integral ideals of norms m, n respectively :

$$\mathfrak{l}_{i,\nu} = \mathfrak{m}_i^{-1}\mathfrak{m}_k\,\rho_\nu = \mathfrak{m}_i^{-1}\mathfrak{m}_j\,\sigma\,\sigma^{-1}\,\mathfrak{m}_j^{-1}\,\mathfrak{m}_k\,\tau\sigma = \mathfrak{l}'_\nu.\,\mathfrak{l}''_\nu. \tag{51}$$

Of course, j, σ and τ depend on ν. Conversely if we take an arbitrary integral left ideal $\mathfrak{m}_i^{-1}\mathfrak{m}_j\,\sigma$ of norm m for $\mathfrak{T}_i$ and an arbitrary left ideal $\sigma^{-1}\,\mathfrak{m}_i^{-1}\,\mathfrak{m}_k\,\tau\sigma$ of norm n for $\sigma^{-1}\,\mathfrak{T}_j\,\sigma$, the product is an integral left ideal of norm mn for $\mathfrak{T}_i$. $\sigma^{-1}\mathfrak{m}_j^{-1}\mathfrak{m}_k\tau\sigma$ is obtained by transformation by σ from an integral left ideal $\mathfrak{m}_j^{-1}\,\mathfrak{m}_k\,\tau$ of norm n for $\mathfrak{T}_j$. The elements σ and τ are determined by ρ_ν only up to a unit $\epsilon_k^{(j)}$ if $\mathfrak{T}_j$ and a unit $\epsilon_\lambda^{(k)}$ of $\mathfrak{T}_k$ as left factors. Therefore we write instead of (51), more clearly:

$$\mathfrak{l}_{i,\nu} = \mathfrak{m}_i^{-1}\,\mathfrak{m}_j\,\epsilon_\kappa^{(j)}\,\sigma.\,\sigma^{-1}\,\mathfrak{m}_j^{-1}\,\mathfrak{m}_k\,\epsilon_\lambda^{(k)}\,\tau\sigma. \tag{52}$$

We now arrange all ideals which have been mentioned in the following schemes: In the first scheme S_1, in the ith line and kth column, we write down all matrices $(1/e_k)r(\epsilon_\lambda^{(k)}\,\rho_\nu,)$ such that S_1 has at the place ik exactly $e_k\,p_{ik}(mn)$ entries. In the second scheme S_2 we write down similarly the matrices $(1/e_j)r(\epsilon_\kappa^{(j)}\sigma)$, and in the third S_3 all matrices $(1/e_k)r(\epsilon_\lambda^{(k)}\tau)$. After the definition of the S_i we define a product $S_2 S_3$ as follows: for each j, all entries at the place ij of S_2 are multiplied with all entries at the place jk of S_3. These products

$$(1/e_k\, e_j)\, r(\epsilon_\kappa^{(j)}\sigma)\, r(\epsilon_\lambda^{(k)}\tau) = (1/e_k\, e_j)\, r(\epsilon_\lambda^{(k)}\tau\epsilon_\kappa^{(j)}\sigma)$$

are arranged at the place ik of a fourth scheme S_4. Since τ represents all integral left ideals $\mathfrak{m}_j^{-1}\mathfrak{m}_k\,\tau$ of norm n of a certain class, and since $\mathfrak{m}_j^{-1}\mathfrak{m}_k\tau\epsilon_\kappa^{(j)}$ is an ideal of the same properties, the $\tau\epsilon_\kappa^{(j)}$ represent all these integrals e_j times each. Now, by virtue of (52), the elements $\tau\epsilon_\kappa^{(j)}\sigma$ represent the left ideals $\mathfrak{m}_i^{-1}\mathfrak{m}_k\,\rho_\nu$ of norm mn of this class, each of them e_j times. Summing up all the entries at each of the places in $S_1, \dots, S_4$, S_1 and S_4 become $P_r(mn)$, S_2 becomes $P_r(m)$, S_3 becomes $P_r(n)$, and the product of S_2 and S_3 becomes the ordinary matrix product.

(50) is analogous to (13). There holds also a formula analogous to (14) for primes p not dividing the discriminant of $\mathfrak{T}$:

$$P_r(p^s)\, P_r(p) = P_r(p^{s+1}) + p^{1+d(r)}\, P_r(p^{s-1}), \tag{53}$$

$d(r)$ being the degree of the representation $r(\rho)$. The proof is based on the same principles as that of (50) and, on the other hand, similar to the proof of (14). Firstly we have, just as in § 2,

$$P_r(p^{s+1}) = P_r^*(p^{s+1}) + p^{d(r)}\, P_r(p^{s-1}),$$

where $P_r^*(p^{s+1})$ is the matrix corresponding to $P_r(p^{s+1})$ formed with primitive ideals only. Secondly the factorization of a primitive ideal $\mathfrak{l}_{i,\nu}$ of norm p^{s+1} into factors of norm p^s and p is unique. However, if $\mathfrak{l}_{i,\nu}$ is divisible by p, $p+1$ such factorisations are possible. Therefore

$$P_r(p^s)\, P_r(p) = P_r^*(p^{s+1}) + (p+1)\, p^{d(r)}\, P_r(p^{s-1}).$$

These two equations yield (53). Eventually, from (53) follows by induction on t:

$$P_r(p^s)\, P_r(p^t) = \sum_{\nu=0}^{\min(s,t)} p^{(1+d(r))\nu}\, P_r(p^{s+t-2\nu}). \tag{54}$$

The trace of $P_1(n) = P(n)$ has been calculated in a former paper [6]. The proof can easily be generalized, which will be done in a later publication.

Here we only give the result

$$t(P_r(n)) = \tfrac{1}{2} \sum_{s,f} \prod_{p|q_1} \left(1 - \left\{\frac{(s^2-4n)f^{-2}}{p}\right\}\right) \times \prod_{p|q_2} \left(1 + \left\{\frac{(s^2-4n)f^{-2}}{p}\right\}\right) p_r(s;n) \frac{h((s^2-4n)f^{-2})}{e((s^2-4n)f^{-2})} + c_n \quad (55)$$

with

$$c_n = \begin{cases} 0, \text{ for } \sqrt{n} \not\equiv 0 \bmod 1, \\ \dfrac{t(r(1))}{12} n^{\frac{1}{2}d(r)} \prod_{p|q_1} (p-1) \prod_{p|q_2} (p+1) \text{ for } \sqrt{n} \equiv 0 \bmod 1. \end{cases} \quad (56)$$

The meaning of the symbols in (55) is the same as in (38), and $p_r(s;n)$ is the trace of $r(\rho)$ for a quaternion ρ having the trace s and norm n. As we shall see soon in a special case, $t(r(\rho))$ is a polynomial of degree $d(r)/2$ in s^2 with coefficients depending on n. Some similarity of (38) and (55) in the case $r(\rho) = 1$ is obvious. It will lead to an interesting application.

It may be remarked that the $P_r(n)$ form a special case of more general matrices introduced by the author. These generate a semi-simple ring. Therefore the knowledge of the trace allows the explicit calculation of the $P_r(n)$ once a basis of the ring has been found.

We are specially interested in *harmonic representations* $r(\rho)$ which are defined as follows: let at first $r_0(\rho) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ($\alpha = \alpha(\rho)$, etc.) be a two-rowed represention in a quadratic splitting field. The coefficients $r_{ik}(\rho)$ of an arbitrary representation $r(\rho)$ are homogeneous polynomials in $\alpha, \beta, \gamma, \delta$. Now $r(\rho)$ is called harmonic if for all i and k the following differential equations hold:

$$\frac{\partial^2 r_{ik}(\rho)}{\partial\alpha\,\partial\delta} - \frac{\partial^2 r_{ik}(\rho)}{\partial\beta\,\partial\gamma} = 0. \quad (57)$$

The reason is that the polynomials $r_{ik}(\rho)$ become spherical harmonics if the coordinates $x_i(\rho)$ of the representation of ρ by the Hamilton quaternions are introduced:

$$\rho = x_0 + \iota_1 x_1 + \iota_2 x_2 + \iota_3 x_3, \quad (\iota_i^2 = -1, \iota_i \iota_k + \iota_k \iota_i = 0).$$

A special harmonic representation is given by the linear transformations which undergo the homogeneous polynomial of degree d in two variables σ_1 and σ_2 when transformed into $\sigma'_1 = \alpha\sigma_1 + \beta\sigma_2$, $\sigma'_2 = \gamma\sigma_1 + \delta\sigma_2$. Its degree is d. We shall now calculate its trace. It is

$$p_r(s; n) = \sum_{r,u} \binom{d-r}{u} \binom{r}{u} \alpha^{d-r-u} \delta^{r-u} \beta^u \gamma^u,$$

the sum extended over all r, s for which the binomial coefficients $\binom{d-r}{u}$, $\binom{r}{u}$ exist. Expressing $\beta\gamma$ by $\alpha\delta$ and n as $\beta\gamma = \alpha\delta - n$, we obtain

$$p_r(s; n) = \sum_{r,u,t} \binom{d-r}{u} \binom{r}{u} \binom{u}{t} (-n)^t \alpha^{d-r-t} \delta^{r-t}.$$

Because the trace is not changed if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is transformed by another matrix, and transformations are possible which give α an arbitrary preassigned value, $p_r(s; n)$ depends on n and s only. Putting $\delta = s - \alpha$ we get

$$p_r(s; n) = \sum_{r,u,t,v} \binom{d-r}{u} \binom{r}{u} \binom{u}{t} \binom{r-t}{v} (-n)^t (-1)^{r-t-v} s^v \alpha^{d-v-2}$$

which must be independent of α. Therefore only the terms with $u = d - 2t$ can be $\neq 0$ and

$$p_r(s; n) = \sum_{r,u,t} (-1)^{d-r} \binom{d-r}{u} \binom{r}{u} \binom{u}{t} \binom{r-t}{d-2t} n^t s^{d-2t}. \quad (58)$$

For example, if $d = 2$, $p_r(s; n) = s^2 - n$.

7. The representation of the modular correspondences in the module of theta functions. The considerations of § 7 are applicable only in the case of $\Gamma_{\mathfrak{T}}$ being a subgroup of the modular group. The assumption on $\Gamma_{\mathfrak{T}}$ made throughout the paper mean in this case that the elements of $\Gamma_{\mathfrak{T}}$ are of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a \equiv b \equiv \frac{c}{q} \equiv d \equiv 0 \bmod 1$, q being a square-free integer. A special case has been treated in [6]. The letter Q will denote a definite quaternion algebra, while the algebra containing the group $\Gamma_{\mathfrak{T}}$ will not occur explicitly.

Having defined $P_r(n)$ for each natural number n we define yet $P_r(0) = 0$ (= the zero matrix), if $r(\rho) \neq 1$, and

$$P_r(0) = (p_{ik}(0)), \quad p_{ik}(0) = \frac{1}{e_k}, \quad \text{if } r(\rho) = 1,$$

e_k being as before the number of units of $\mathfrak{T}_k$. Now we put

$$\vartheta_r(\tau) = \sum_{n=0}^{\infty} P_r(n)\, e^{2\pi i n \tau}. \tag{59}$$

This is a matrix the coefficients of which are certain Fourier series in τ. We are going to show that the functions defined by (59) yield a representation of the T_n similar to that of § 6.

At first we have to mention a basic quality of these functions. As in § 6, the representation $r_0(\rho) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is introduced. Now for each n, let ρ run over all quaternions such that $\mathfrak{m}_i^{-1}\mathfrak{m}_k\, \rho$ is an integral ideal of norm n of its class and let $n = 0, 1, 2, \ldots$. Then the coefficients of $\vartheta_r(\tau)$ are

$$\vartheta_{ik}(\tau) = \sum_{\rho} p_{ik}(\alpha, \beta, \gamma, \delta)\, e^{2\pi i \{n(\mathfrak{m}_k)/n(\mathfrak{m}_i)\}(\alpha\delta - \beta\gamma)\tau},$$

where the $p_{ik}(\alpha, \beta, \gamma, \delta)$ are certain homogeneous polynomials in $\alpha, \beta, \gamma, \delta$, satisfying the differential equations (57), in case $r(\rho)$ is a harmonic representation. On the other hand, the ideal $\mathfrak{m}_i^{-1}\mathfrak{m}_k\, \rho$ is integral if and only if ρ is contained in $\mathfrak{m}_k^{-1}\mathfrak{m}_i$. Therefore if $\mu_1, \ldots, \mu_4$ is a basis of $\mathfrak{m}_k^{-1}\mathfrak{m}_i$, we may write $\rho = \mu_1 r_1 + \ldots + \mu_4 r_4$, and $\alpha, \beta, \gamma, \delta$ are homogeneous linear forms in the r_ν. Now

$$\frac{n(\mathfrak{m}_k)}{n(\mathfrak{m}_i)}(\alpha\delta - \beta\gamma) = \frac{n(\mathfrak{m}_k)}{n(\mathfrak{m}_i)}\, n(\mu_1 r_1 + \ldots + \mu_4 r_4) = F_{ik}(r_1, \ldots r_4)$$

is a definite quadratic form with integral rational coefficients of level (Stufe) $q_1 q_2$. So we may also write

$$\vartheta_{ik}(\tau) = \sum_{r_\nu = -\infty}^{+\infty} p_{ik}(r_1, \ldots, r_4)\, e^{2\pi i F_{ik}(r_1,\ldots,r_4)\tau}.$$

Special attention is due if $r(\rho) = 1$. Then, for each $\rho \neq 0$, $p_{ik}(r_1, \ldots, r_4) = 1/e_k$, and by virtue of the definition above, this equation stays valid for $r_1 = \ldots = r_4 = 0$.

Both equations for $\vartheta_{ik}(\tau)$ show that these functions are generalized theta functions, introduced by Hecke [10, § 6]. They satisfy the functional equations

$$\vartheta_{ik}(\tau) \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(ad - bc)^{1+\frac{1}{2}d(r)}}{(c\tau + d)^{2+d(r)}}\, \vartheta_{ik}\left(\frac{a\tau + b}{c\tau + d}\right) = \vartheta_{ik}(\tau), \qquad (60)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$, $c \equiv 0 \bmod q_1 q_2$.

Moreover they are holomorphic with the exception of the parabolic cusps. However questions of holomorphy may be left aside until later.

Hecke has shown that, for each modular form $\phi(\tau)$ satisfying (60) of degree $-2 - d(r)$,

$$\phi(\tau) \circ T_n = \sum_{i=1}^{d_n} \phi(\tau) \circ \nu_i = \sum_{i=1}^{d} \frac{(a_i d_i - b_i c_i)^{1+\frac{1}{2}d(r)}}{(c_i\tau + d_i)^{2+d(r)}}\, \phi\left(\frac{a_i\tau + b_i}{c_i\tau + d_i}\right),$$

$$\nu_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \qquad (61)$$

is again a modular form of the same degree (ν_i as in (11)). Also certain properties of holomorphy are preserved by the operator T_n. Here we only want to prove

$$\vartheta_r(\tau) \circ T_n = n^{-\frac{1}{2}d(r)}\, P_r(n)\, \vartheta_r(\tau), \text{ for } (n, q_1 q_2) = 1. \qquad (62)$$

The meaning of (62) is that the modular correspondences can also be represented in the module of (generalized) theta functions. In this way, a connection between modular forms and arithmetics of definite quaternion algebras is established.

The proof of (62) is quite simple. As it is well known, under the assumption that $\Gamma_{\mathfrak{T}}$ is the subgroup of the modular group mentioned in (60), the ν_i constituting T_n in (11) can be chosen as follows :

$$\nu_i = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \, ac = n, \, a > 0, \, 0 \leqslant b < c.$$

Because of (13) and (15) on one side and (50) and (54) on the other, we need prove (62) for $n = p =$ a prime only. For a prime not dividing $q_1 q_2$ we have

$$\vartheta_r(\tau) \circ T_p = \sum_{n=0}^{\infty} P_r(n) \, (p^{1+\frac{1}{2}d(r)} \, e^{2\pi i p n \tau} + p^{-\frac{1}{2}d(r)} \, e^{2\pi i n \tau / p}),$$

where the term $e^{2\pi i n \tau/p}$ is to be cancelled if n is not divisible by p. This is equal to

$$\vartheta_r(\tau) \circ T_p = p^{-\frac{1}{2}d(r)} \sum_{n=1}^{\infty} \Big(P_r(p\,n) + p^{1+d(r)} \, P_r(n/p) \Big) \, e^{2\pi i n \tau},$$

where $P_r(n/p) = 0$ if n is not divisible by p. The right hand side is equal to (62) with $n = p$, because of (50) and (53).

Until the end of § 7, let $r(\rho) = 1$. The series occurring in the coefficients of $\vartheta_r(\tau) = \vartheta(\tau)$ are ordinary theta functions (up to a constant factor $1/e_k$). They are holomorphic throughout the interior of the upper half-plane. The difference of any two theta functions vanishes at $\tau = i\infty$ and is therefore holomorphic even at this point in the sense of § 5. We can easily show that the difference of two theta functions $\vartheta_1(\tau) - \vartheta_2(\tau)$ with the same discriminant of their respective quadratic forms is holomorphic in all parabolic cusps. Indeed $\tau^{-2} (\vartheta_1(-\tau^{-1}) - \vartheta_2(-\tau^{-1}))$ is a difference of theta functions for the inverse quadratic forms, vanishing at $-\tau^{-1} = i\infty$ or $\tau = 0$. Apparently any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the whole modular group applied to this difference yields a difference of certain theta series, vanishing at $c\tau + d = 0$.

As has already been mentioned at the beginning of § 6, the sums over the rows of $P(n)$ yield the same value, whichever row it may be :

$$\sum_{k=1}^{h} p_{ik}(n) = d_n.$$

These equations can also be written as

$$P(n)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = d_n \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix},$$

which shows that d_n is an eigenvalue of $P(n)$. A short calculation shows now

$$M^{-1}P(n)M = \begin{bmatrix} d_n & p(n) \\ 0 & P'(n)\end{bmatrix}, \; M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \\ 1 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \dots & -1 \\ 0 & 1 & \dots & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}, \tag{63}$$

where $p(n)$ is a 1 by $(h-1)$ matrix and $P'(n)$ a $(h-1)$ by $(h-1)$ matrix. The coefficients of the latter are $p_{ik}(n) - p_{1k}(n)$. Therefore

$$\vartheta'(n) = \sum_{n=0}^{\infty} P'(n)\, e^{2\pi i n\tau} \tag{64}$$

is a matrix the coefficients of which are differences of ordinary theta functions and therefore holomorphic modular forms of degree -2. From (62) follows

$$\vartheta'(\tau) \circ T_n = P'(n)\, \vartheta'(\tau). \tag{65}$$

Because of (63), the trace of $P'(n)$ is that of $P(n)$ minus d_n; according to (55):

$$t(P'(n))$$
$$= \frac{1}{2}\sum_{s,f}\prod_{p|q_1}\left(1-\left\{\frac{(s^2-4n)f^{-2}}{p}\right\}\right)\times$$
$$\times\prod_{p|q_2}\left(1-\left\{\frac{(s^2-4n)f^{-2}}{p}\right\}\right)\frac{h((s^2-4n)f^{-2})}{e((s^2-4n)f^{-2})} - \sum_{t/n} t + c_n, \tag{66}$$

with

$$c_n = \begin{cases} 0 & \text{for } \sqrt{n} \not\equiv 0 \bmod 1, \\ \frac{1}{12} \prod_{p|q_1} (p-1) \prod_{p|q_2} (p+1) & \text{for } \sqrt{n} \equiv 0 \bmod 1. \end{cases} \tag{67}$$

Using (40) and comparing (66), (67) with (38_2), (39_2) one finds

$$2t(P'(n)) = t^1(T_n), \tag{68}$$

provided that q_1 is a prime and $q_2 = 1$. Now we have seen in (45) that the representation $R^1(T_n)$ is the sum of the representations $R_1(T_n)$ considered in §5 and the complex conjugate representation $\bar{R}_1(T_n)$. Petersson has shown that all eigenvalues of $R_1(T_n)$ are real and that $R_1(T_n)$ is completely reducible. Therefore $R_1(T_n)$ is real, and (68) is equivalent to

$$t(P'(n)) = t_1(T_n), \quad (n, q_1) = 1. \tag{69}$$

Because $R_1(T_n)$ and $P'(n)$ (for the latter see [6], see § 5) is completely reducible, (69) entails that $P'(n)$ and $R_1(T_n)$ are equivalent representations, and that all cusp-forms of degree -2 can be represented by differences of theta functions if, in the Fourier series, all terms $c_n e^{2\pi i n\tau}$ with $(n, q_1) > 1$ are cancelled. The latter restriction can be seen to be unnecessary however, using an argument of Hecke [10, § 9]. This fact has been conjectured by Hecke in 1936. The result may be considered as a satisfactory answer to the difficult problem of explicitly describing a complete set of linearly independent cusp-forms of degree -2. The Poincaré series are of little practical use since no method is known to determine their Fourier development. The corresponding problem concerning cusp-forms of degrees -2 or of grades $\neq$ primes seems to be far from being solved.

In the case of $q_1 q_2$ being a product of different primes, the connection between $P'(n)$ and $R_1(T_n)$ is more complicated. However, even now certain conclusions are possible [7].

8. Representations of the correspondences connected with differentials of higher degrees. As we have seen in § 5, the expressions

$\phi(\tau)\, d\tau$ are differentials of the field F of automorphic functions, if $\phi(\tau)$ is an automorphic form of degree -2. The study of the integrals (42) connected the representation $R_1(T_n)$ of the T_n in the module of the cusp-forms of degree -2 with the representation $R^1(T_n)$ in the homology group of the underlying Riemann surface. This connection led further to the determination of the trace of $R_1(T_n)$.

Now we wish to investigate the representation $R_f(T_n)$ of the T_n in the module of cusp-forms of degree $-2f$. These are modular forms satisfying (60) with f instead of $1 + \frac{1}{2} d(r)$, which are holomorphic in the interior of the complex upper half-plane and vanish in the parabolic cusps, if there are any. For such modular forms $\phi(\tau)$, the expressions $\phi(\tau)\, d\tau^f$ can be called *differentials of degree f.*

A similar procedure as in § 5 will lead us to certain $(2f-1)$-fold integrals and furthermore to a connection between the representations $R_f(T_n)$ and representations of the T_n in certain cohomology groups.

Incidentally, it would be natural to call such a differential *holomorphic* in $\tau = 0$ if $\phi(\tau) \left(\dfrac{d\tau}{dz}\right)^f$ is a holomorphic function of a local uniformizing parameter z. It should be mentioned, however, that the cusp-forms are generally not holomorphic at the parabolic cusps. On the contrary, $\phi(\tau) \left(\dfrac{d\tau}{dz}\right)^f$ has here in general a pole of order $f-1$. On the other hand, it is known that the T_n transform cusp-forms into cusp-forms, while everywhere holomorphic differentials of degree f are in general not transformed into such differentials.

Let $\phi(\tau)$ be a modular form of degree $-2f$ and holomorphic at least in the interior of the complex upper half-plane. We want to study the integral

$$\Phi(\tau) = \frac{1}{(2f-2)!} \int_{\tau^0}^{\tau} (\tau - \sigma)^{2f-2}\, \phi(\sigma)\, d\sigma, \tag{70}$$

τ being some arbitrarily preassigned point and τ variable. Evidently

$$\frac{(c\,\tau + d)^{2f-2}}{(ad - bc)^{f-1}}\Phi\left(\frac{a\,\tau + b}{c\,\tau + d}\right)$$

$$= \frac{1}{(2f-2)!}\frac{(c\,\tau + d)^{2f-2}}{(ad - bc)^{f-1}} \times$$

$$\times \int\limits_{\tau^0}^{(a\tau^0+b)/(c\tau^0+d)} \left(\frac{a\,\tau + b}{c\,\tau + d} - \sigma\right)^{2f-2} \phi(\sigma)\,d\sigma +$$

$$+ \frac{1}{(2f-2)!}\frac{(c\,\tau + d)^{2f-2}}{(ad - bc)^{f-1}} \int\limits_{(a\tau^0+b)/(c\tau^0+d)}^{(a\tau+b)/(c\tau+d)} \left(\frac{a\,\tau + b}{c\,\tau + d} - \sigma\right)^{2f-2} \phi(\sigma)\,d\sigma.$$

Using the assumption that $\phi(\sigma)$ is a modular form of degree $-2f$, we substitute $\dfrac{a\sigma + b}{c\sigma + d}$ for σ in the second integral and obtain

$$\frac{(c\,\tau + d)^{2f-2}}{(ad - bc)^{f-1}}\Phi\left(\frac{a\,\tau + b}{c\,\tau + d}\right) = \Phi(\tau) \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= c\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \tau\right) + \Phi(\tau), \quad (71)$$

where $c\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \tau\right) = c\,(\alpha\,;\,\tau)$ is a certain polynomial of degree $2f-2$ in τ. The significance of (71) is that $\Phi(\tau)$ behaves similar to a modular form of degree $2f-2$. Conversely, $(2f-1)$-fold differentiation of $\Phi(\tau)$ leads back to $\phi(\tau)$, and (71) contains the fact discovered by G. Bol [1] and further pursued by H. Petersson [14] that the $(2f-1)$-th derivative of a modular form of degree $2f-2$ is a modular form of degree $-2f$.

The polynomials $c(\alpha\,;\,\tau)$ in (71) satisfy a certain functional equation. In order to state it we make the convention that, for any polynomial $p(\tau)$ of degree $2f-2$ in τ and any matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$p(\tau) \circ \alpha = p(\alpha \circ \tau)\,\frac{(c\,\tau + d)^{2f-2}}{(ad - bc)^{f-1}}. \quad (72)$$

In this way, the polynomials of degree $2f-2$ in τ become the elements of a certain representation module $\mathfrak{m}$ of the group of two-rowed matrices. Evidently this representation is just that considered at the end of § 6. From now on we need not mention the argument τ any more. The functional equation of the $c(\alpha)$ in (71) is

$$c(\alpha\beta) = c(\alpha) \circ \beta + c(\beta). \tag{73}$$

For the proof, let

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \beta = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \alpha\beta = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

(71) yields

$$\begin{aligned} \frac{(c''\tau + d'')^{2f-2}}{(a''d'' - b''c'')^{f-1}} \Phi\left(\frac{a''\tau + b''}{c''\tau + d''}\right) &= c(\alpha\beta) + \Phi(\tau) \\ &= c(\alpha) \circ \beta + c(\beta) + \Phi(\tau). \end{aligned}$$

Equation (73) has a meaning quite apart from its origin. A mapping $\alpha \to c(\alpha)$ of $\Gamma_{\mathfrak{T}}$ into the representation module $\mathfrak{m}$ has been called a *cochain*, and a cochain satisfying (73) is a *closed cochain* or a *cocycle*. Special cocycles are the following *coboundaries*

$$c_0(\alpha) = c \circ (\alpha - 1),$$

c being a fixed element in $\mathfrak{m}$. The cocycles form a group C and the coboundaries form a subgroup B. The factor group C/B is the *first cohomology group* of $\Gamma_{\mathfrak{T}}$ in $\mathfrak{m}$, its elements are called *cohomology classes.*

Equation (71) states that, to a modular form of degree $-2f$, there corresponds a cocycle $c(\alpha)$. This cocycle depends yet on the constant τ^0 in (70). A change of τ^0 would add a coboundary to $c(\alpha)$. Therefore, to a $\phi(\tau)$, there corresponds a cohomology class. This correspondence has obviously the following properties: From

$$\begin{cases} \phi_1(\tau) \to c_1(\alpha) \\ \phi_2(\tau) \to c_2(\alpha) \end{cases}$$

follows

$$\begin{cases} \gamma_1\phi_1(\tau) + \gamma_2\phi_2(\tau) \to \gamma_1 c_1(\alpha) + \gamma_2 c_2(\alpha), \\ \qquad \gamma_1 \text{ and } \gamma_2 \text{ constants.} \end{cases}$$

The procedure can be generalized by integrating (70) over not holomorphic modular forms $\phi(\tau)$, however the singularities must be such that no logarithmic terms can occur.

We now study the behaviour of $c(\alpha)$ under the correspondences T_n. Let ν_i be as in (11) and α an arbitrary element of $\Gamma_{\mathfrak{X}}$. Then

$$\nu_i \alpha = \alpha' \nu_j, \quad \alpha' \in \Gamma_{\mathfrak{X}}, \tag{74}$$

with j and α' depending on i and α. The function

$$\begin{aligned} \Psi(\tau) = \Phi(\tau) \circ T_n &= \sum_{i=1}^{d_n} \Phi(\tau) \circ \nu_i \\ &= \sum_{i=1}^{d_n} \frac{(c_i \tau + d_i)^{2f-2}}{(a_i d_i - b_i c_i)^{f-1}} \Phi\left(\frac{a_i \tau + b_i}{c_i \tau + d_i}\right) \end{aligned} \tag{75}$$

satisfies the functional equation (71) with some $c'(\alpha; \tau)$ if $\Phi(\tau)$ does. Indeed, using (74), we obtain

$$\Psi(\tau) \circ \alpha = \sum_{i=1}^{d_n} (\Phi(\tau) \circ \alpha') \circ \nu_j = c'(\alpha; \tau) + \Psi(\tau)$$

with

$$c'(\alpha) = \sum_{i=1}^{d_n} c(\alpha') \circ \nu_j = c(\alpha) \circ T_n. \tag{76}$$

By (76), the T_n are made endomorphisms of the first cohomology group of $\Gamma_{\mathfrak{X}}$ in the module $\mathfrak{m}$. We have only to show that $c'(\alpha)$ is closed if $c(\alpha)$ was closed, and that coboundaries are mapped onto coboundaries. Indeed, if $c(\alpha) = c \circ (\alpha - 1)$,

$$c'(\alpha) = \sum_i c \circ (\alpha' - 1) \nu_j = \sum_i c \circ (\nu_i \alpha - \nu_j) = \left(\sum_i c \circ \nu_i\right) \circ (\alpha - 1).$$

In order to calculate $c'(\alpha \beta)$ we put similarly to (74)

$$\nu_i \alpha \beta = \alpha' \nu_j \beta = \alpha' \beta' \nu_k.$$

Now

$$\begin{aligned} c'(\alpha \beta) &= \sum_i c(\alpha' \beta') \circ \nu_k = \sum_i (c(\alpha') \circ \beta' + c(\beta')) \circ \nu_k \\ &= \sum_i c(\alpha') \beta' \nu_k + \sum_i c(\beta') \nu_k \end{aligned}$$

$$= \left(\sum_i c(\alpha') \circ \nu_j \right) \beta + \sum_i c(\beta') \nu_k = c'(\alpha) \circ \beta + c'(\beta),$$

where one has to bear in mind that j and k are functions of i which assume all values from 1 to d_n.

It is easy to verify that the endomorphisms of the cohomology group defined by (76) are independent of the special choice of the ν_i; the proof may be left to the reader.

We close with some remarks on the algebraic nature of the cocycles.

(1). There exists but a finite number of linearly independent cocycles. For $\Gamma_{\mathfrak{T}}$ has a finite number of generators $\alpha_1, \alpha_2, \ldots$, and because of (73), $c(\alpha)$ is uniquely determined once $c(\alpha_1), \ldots$ are given.

(2). If $c(\alpha)$ is a cocycle and ϵ an element of $\Gamma_{\mathfrak{T}}$,

$$c'(\alpha) = c(\epsilon^{-1} \alpha \epsilon) \circ \epsilon^{-1} = c(\alpha) - c(\epsilon) \circ (\alpha - 1)$$

is also a cocycle.

(3). If $c(\alpha) \sim 0$ for all α in a subgroup Γ' of $\Gamma_{\mathfrak{T}}$ of finite index, then $c(\alpha) \sim 0$ in $\Gamma_{\mathfrak{T}}$. For the proof, let

$$\Gamma_{\mathfrak{T}} = \sum_{i=1}^{n} \Gamma' \beta_i. \tag{77}$$

Without loss of generality we may assume $c(\alpha') = 0$ for all $\alpha' \in \Gamma'$. Now put $c = \sum_{i=1}^{n} c(\beta_i)$. Then for an arbitrary $\alpha \in \Gamma_{\mathfrak{T}}$ we have $\beta_i \alpha = \alpha' \beta_j$, $\alpha' \in \Gamma'$, with j and α' depending on i and α. Furthermore

$$\sum_{i=1}^{n} c(\beta_i \alpha) = c \circ \alpha + n\, c(\alpha) = \sum_{i=1}^{n} c(\alpha' \beta_j)$$

$$= \sum_{i=1}^{n} c(\alpha') \circ \beta_j + \sum_{i=1}^{n} c(\beta_i) = c,$$

therefore

$$c(\alpha) = \frac{1}{n}\, c \circ (\alpha - 1).$$

(4). Let Γ' be a subgroup of finite index n in $\Gamma_{\mathfrak{T}}$ and (77) be the development into left cosets. For an element $\alpha \in \Gamma_{\mathfrak{T}}$, let $\beta_j \alpha = \alpha' \beta_j$, $\alpha' \in \Gamma'$. Then

$$c(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \Big(c(\alpha') \circ \beta_j + c(\beta_i) \circ (1 - \alpha) \Big) \sim \frac{1}{n} \sum_{i=1}^{n} c(\alpha') \circ \beta_j .$$

Therefore $c(\alpha)$ can already be calculated from the values taken in a subgroup of finite index.

PROOF: Because of (73)

$$\frac{1}{n} \sum_{i} (c(\alpha') \circ \beta_j + c(\beta_i) \circ (1 - \alpha))$$

$$= \frac{1}{n} \sum_{i} \Big(c(\alpha') \circ \beta_j + c(\beta_j) - c(\beta_i) \circ \alpha \Big)$$

$$= \frac{1}{n} \sum_{i} (c(\alpha' \beta_j) - c(\beta_i) \circ \alpha) = \frac{1}{n} \sum_{i} (c(\beta_i\, \alpha) - c(\beta_i) \circ \alpha) = c(\alpha)$$

(5). The modular correspondences can also be represented by the higher cohomology classes: for $\alpha_1, \ldots, \alpha_n \in \Gamma_{\mathfrak{T}}$ and $\nu_i\, \alpha_1 \ldots \alpha_n = \alpha'_1 \ldots \alpha'_{n'}\, \nu_j$, $n' = 1, 2, \ldots, n$, with j and $\alpha'_1, \ldots, \alpha'_{n'} \in \Gamma_{\mathfrak{T}}$ depending on i and $\alpha_1, \ldots, \alpha_n$, put

$$c(\alpha_1, \ldots, \alpha_n) \circ T_n = \sum_{i=1}^{d_n} c(\alpha'_1, \ldots, \alpha'_n) \circ \nu_i .$$

Whether there is an application of these representations to modular forms is yet an open question.

Two chief problems remain unsolved: the determination of a form which yields a given cocycle $c(\alpha) = c(\alpha; \tau)$ in (71); the determination of the trace of the representation of T_n by the cohomology classes.

9. Generalizations. The possibility of generalizations becomes evident by merely summarizing some of the contents of this paper in a slightly modified fashion. Let $\mathfrak{r}$ be the group of all matrices $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real coefficients and determinant > 0 and $\mathfrak{L}$ the subgroup of all matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then the right cosets $\alpha\,\mathfrak{L}$ form a variety S of dimension 2. S is mapped on the complex upper half-plane by putting

$$\tau = \alpha \circ i = \frac{a\,i + b}{c\,i + d}, \quad i = \sqrt{-1}.$$

Evidently, τ is the same for all α in the same coset $\alpha\,\mathfrak{L}$. Now $\Gamma = \Gamma_{\mathfrak{T}}$ is a properly discontinuous group of mappings $\epsilon : \alpha\,\mathfrak{L} \rightarrow \epsilon\,\alpha\,\mathfrak{L}$ of S on itself. Therefore the surface $S_{\mathfrak{T}}$, which may now be written S_Γ, is the variety of double cosets $\Gamma\alpha\mathfrak{L}$:

$$S_\Gamma = \{\Gamma\,\alpha\,\mathfrak{L}\}.$$

Let ν be an element of $\mathfrak{T}$ and $\Gamma' = \nu\,\Gamma\,\nu^{-1}$, and assume that Γ and Γ' are both finite extensions of $\Gamma^* = \Gamma \cap \Gamma'$. Then $S_{\Gamma^*} = \{\Gamma^*\,\alpha\,\mathfrak{L}\}$ is a covering variety of both S_Γ and $S_{\Gamma'} = \{\Gamma'\,\alpha\,\mathfrak{L}\}$, consisting of a finite number of sheets. Now the correspondence $C_{\Gamma'/\Gamma}(S_{\Gamma'})$ is defined as in § 1. Furthermore there is a one-to-one mapping $S_{\Gamma'} \longleftrightarrow S_\Gamma$, defined by

$$\Gamma'\,\alpha\,\mathfrak{L} = \nu\,\Gamma\,\nu^{-1}\,\alpha\,\mathfrak{L} \longleftrightarrow \nu.\,\Gamma'\,\alpha\,\mathfrak{L} = \Gamma\,\nu\,\alpha\,\mathfrak{L}.$$

In this way, correspondences of S_Γ to itself are defined.

The same procedure is possible if $\mathfrak{T}$ is a more general topological group, $\mathfrak{L}$ a certain subgroup (e.g. $\mathfrak{L}$ = unity element), and Γ another subgroup which is properly discontinuous in the variety of right cosets $\alpha\,\mathfrak{L}$. For a detailed investigation it is of course necessary to have the elements of Γ defined by some arithmetical law. We may at first try to apply the theorem of Lefschetz as in §3, which is possible if there exist isolated fixed points. Let ν be an element of the property stated above, and let $\alpha\,\mathfrak{L}$ be a fixed point of ν : $\nu\,\alpha\,\mathfrak{L} = \alpha\,\mathfrak{L}$. This entails that $\alpha^{-1}\,\nu\,\alpha \in \mathfrak{L}$. So we see that fixed points can only exist if $\mathfrak{L}$ contains more than one element. Furthermore, $\alpha\,\mathfrak{L}$ is an isolated fixed point, if all elements β of $\mathfrak{T}$ commutable with $\alpha^{-1}\,\nu\,\alpha$ and connected with the unity element are contained in $\mathfrak{L}$. This means that $\mathfrak{L}$ must have some property of maximality.

The conditions for the existence of isolated fixed points are satisfied in the following case. Let F be the matrix of an indefinite quadratic form with rational coefficients, which is in the field of real numbers equivalent to $x_1^2 + \ldots + x_{2n}^2 - x_{2n+1}^2 - \ldots - x_{2m}^2$. $\mathfrak{r}$ is the group of all real matrices X satisfying $X^T F X = F$, Γ the subgroup of all rational integral X, and $\mathfrak{K}$ the normalizer of a maximal compact subgroup of $\mathfrak{r}$. A proof may be published on a later occasion. Whether the variety S_Γ defined in this way is an algebraic variety or not is an open question. However, even if S_Γ were not an algebraic variety, a problem analogous to that of §5 arises: instead of algebraic differentials on S_Γ, which may not exist, one can investigate the behaviour of the harmonic integrals on S_Γ with respect to some appropriate Riemann metrics on S_Γ.

It may even be worthwhile considering the correspondence if no isolated fixed points exist. This is the case when $\mathfrak{r}$ is the group of all elements of norm > 0 of a central simple algebra over the rational field, Γ the group of units of an order $\mathfrak{T}$, and $\mathfrak{K}$ the normalizer of a maximal compact subgroup. The correspondences are now linked with the left ideals for $\mathfrak{T}$ in the same way as in the case of quaternion algebras. But if the algebra is of rank > 4, the fixed points of a correspondence form certain varieties of dimension $\geqslant 1$. So the theorem of Lefschetz is at least not immediately applicable.

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GENERALIZATION OF ONE OF THE HILBERT PROBLEMS

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1. Question of the hypertranscendency of the zeta-function. Functions which do not satisfy any algebraic differential equation are called hypertranscendent.

It is common knowledge that the Riemann zeta function is hypertranscendent. A much more general theorem is proved in A. Ostrowski's studies (*Math. Zeitschrift*, Bd. 8, 1920):

THEOREM (Ostrowski). *If the Dirichlet series*

$$f(s) = \sum a_n e^{-\lambda_n s}, \quad (s = \sigma + it)$$

has a region of absolute convergence, and satisfies the algebraic difference-differential equation

$$\Phi(s, f^{(\nu)}(s + h_{\nu\tau})) = 0,$$

where Φ is a polynomial, and $h_{\nu\tau}$ are real numbers, then among the exponents λ_n there are only a finite number which are linearly independent.

Since the zeta-function, and all the Dirichlet L-series as well, have a region of absolute convergence and their basis of exponents embraces all the logarithms of the prime numbers (except, at the most, a finite number of them), these prime numbers being infinite in number, they are hypertranscendent.

It is interesting to look into the question of the extension of this theorem to functional relationships of a more general nature. Very valuable information on this question is given by L. S. Pontrjagin's

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theorem (published without proof in *Comptes Rendus de L'Academie des Sciences*, Paris, 196 (1933), 1201). We shall formulate a particular case of this theorem (as needed for our purposes).

THEOREM (Pontrjagin). *If the almost periodic functions* $f_i(t)$ *satisfy the system of differential equations*

$$\frac{df_i(t)}{dt} = \Phi_i(f_i(t),\dots, f_r(t)), \quad i = 1, 2,\dots r,$$

where Φ_i *are functions which satisfy the Lipschitz condition in the region of an r-dimensional space containing the closure of a set of points of the type* $(f_1(t), \dots f_r(t))$, $-\infty < t < \infty$, *then there cannot exist among the exponents of the functions an unlimited number which are linearly independent.*

It should be noted that since the almost periodic functions are bounded, the region mentioned in formulating the theorem may always be considered bounded.

The connection of this theorem with the Ostrowski theorem is obvious: if the Dirichlet series $\Sigma a_n e^{-\lambda_n s}$ has a region of absolute convergence, then on any straight line $\sigma = \sigma_1$ in the region of absolute convergence the function represented by the series is almost periodic and the differential equation may be considered a differential equation with regard to t.

Since the exponents of the almost periodic functions $\zeta(\sigma_0 + it)$, $\zeta'(\sigma_0 + it),\dots, \zeta^{(n-1)}(\sigma_0 + it)$, $\sigma_0 > 1$, include the logarithms of prime numbers, these logarithms being linearly independent (this follows from the theorem that an integer can be expressed as a product of prime factors in one way only) and infinite in number, we get the following

THEOREM. *A relation of the type*

$$\frac{d^n \zeta(\sigma_0 + it)}{dt} = \Phi(\zeta(\sigma_0 + it),\dots, \zeta^{(n-1)}(\sigma_0 + it))$$

is impossible on any of the straight lines $\sigma = \sigma_0 > 1$, *where* Φ *is a function satisfying the Lipschitz condition over the region containing the closure of the set of points*

$$(\zeta(\sigma_0 + it), \zeta'(\sigma_0 + it), \ldots, \zeta^{(n-1)} (\sigma_0 + it)).$$

A similar theorem will naturally be true for any $L(s, \chi)$.

It should be noted, finally, that the Pontrjagin theorem is proved by a method quite different from that of A. Ostrowski (its proof involves the theory of continuous groups while the Ostrowski theorem is proved by elementary means).

2. Generalization of one of the Hilbert problems. At the Second Mathematical Congress in Paris, D. Hilbert put forth the assumption that the function

$$\zeta(x, s) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$$

does not satisfy any algebraic differential equation in partial derivatives. The solution of this problem was given by D. D. Morduchai-Boltovskoy (*Izvestiya Warsawskovo Polytechnicheskovo Instituta* 1914, and *Tohoku Math. J.*, 35, 19, 1932) and by A. Ostrowski (*Math. Zeitschrift*, Bd. 8, 1920).

Let χ be a character for the modulus m. Now we introduce the function

$$L(x, s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} x^n.$$

We shall call these functions Dirichlet L_x-series for the modulus m.

A. O. Gelfond suggested that the author prove a theorem which would generalize Hilbert's assumption.

THEOREM. *Let there be (various) Dirichlet L_x-series for the modulus m. The relation*

$$\Phi\left(x, s, \frac{\partial^{p+q} L(x, s, \chi)}{\partial x^p \, \partial s^q}\right) \equiv 0,$$

where Φ is a polynomial, is impossible.

We shall call the system of ordinary Dirichlet series

$$f_1(s) = \sum_{n=1}^{\infty} \frac{a_{1n}}{n^s},\ f_2(s) = \sum_{n=1}^{\infty} \frac{a_{2n}}{n^s},\ \dots,\ f_r(s) = \sum_{n=1}^{\infty} \frac{a_{rn}}{n^s}$$

differentially dependent, if there exists such a polynomial of s, $f_1(s), \dots, f_r(s)$ and their derivatives, that

$$\Phi(s, f_k^{(\nu)}(s)) \equiv 0.$$

The system of ordinary Dirichlet series will be called *difference-differentially dependent*, if there exist such real numbers $h_{k,\nu,\tau}$ and such a polynomial Φ, that

$$\Phi(s, f_k^{(\nu)}(s + h_{k,\nu,\tau})) \equiv 0.$$

We shall base our deductions on a number of lemmas belonging to Ostrowski or obtained according to his scheme (see the cited paper). We shall give proofs only when there is at least some difference from Ostrowski's paper.

LEMMA 1. (Ostrowski). *If the Dirichlet series are difference-differentially dependent, s can be excluded from the relation.*

LEMMA 2. (Ostrowski). *If the Dirichlet series have a region of absolute convergence and are difference-differentially dependent, they are formally difference-differentially dependent (by formal dependency is meant that if we carry out the above algebraic operations with the Dirichlet series, we get a Dirichlet series in which all the coefficients equal zero).*

LEMMA 3. (Ostrowski). *If the system of ordinary Dirichlet series*

$$f_1(s) = \sum_{n=1}^{\infty} \frac{a_{1n}}{n^s},\ \dots,\ f_r(s) = \sum_{n=1}^{\infty} \frac{a_{rn}}{n^s}$$

is differentially dependent, then if we leave in these series only terms with prime denominators larger than a certain N, the following homogeneous relation will be true for the series $f^*_k(s)$ obtained:

$$\sum \lambda_{k\nu} f^{*(\nu)}_k(s) \equiv 0,$$

where $\lambda_{k\nu}$ are constants not all equal to zero.

Let the relation with s excluded have the form

$$\Phi(f_k^{(\nu)}(s)) \equiv 0.$$

Now we consider all the $\left.\frac{\partial\Phi}{\partial f_j^{(t)}}\right|_{f_k^{(\nu)}(s)}$. Let the expansion of $\left.\frac{\partial\Phi}{\partial f_j^{(t)}}\right|_{f_k^{(\nu)}(s)}$ into the Dirichlet series begin with $\frac{A_{jt}}{l_{jt}^s}$. Let $\Lambda = \min_{j,t} l_{jt} = l_{j_\mu t_\mu}$, where $j_\mu\, t_\mu$ are indices for which Λ is attained. We express $f_k(s)$ in the form $f_k(s) = F_{km}(s) + g_{km}(s)$, where $g_{km}(s) = \sum_{n=m}^{\infty} \frac{a_{kn}}{n^s}$. It is obvious that if we choose m large enough, and put it in $\frac{\partial\Phi}{\partial f_j^{(t)}}$ instead of $f_k^{(\nu)}(s)$, the expansion of $F_{km}^{(t)}(s)$ again begins with $\frac{A_{jt}}{l_{jt}^s}$. Now we expand $\Phi(f_k^{(\nu)}(s)) = \Phi(F_{km}^{(\nu)}(s) + g_{km}^{(\nu)}(s))$ into Taylor series

$$\begin{aligned} 0 &= \Phi(f_k^{(\nu)}(s)) \\ &= \Phi(F_{km}^{(\nu)}(s)) + \sum_{j,t} \left.\frac{\partial\Phi}{\partial f_j^{(t)}}\right|_{F_{km}^{(\nu)}(s)} g_{jm}^{(t)}(s) + \\ &\qquad + \frac{1}{2}\left(\sum\sum \cdots\right) + \frac{1}{6}\left(\sum\sum\sum \cdots\right) + \cdots \end{aligned}$$

Let us see what term the expansion of the expressions $\frac{1}{2}(\Sigma\,\Sigma\ldots)$, $\frac{1}{6}(\Sigma\,\Sigma\,\Sigma\ldots)$ into Dirichlet series will begin with. The second and third derivatives of Φ may begin even with unity, but the products and degrees of $g_{jm}^{(t)}(s)$ must begin at least with $1/m^{2s}$. Hence, expansions of the expressions $\frac{1}{2}(\Sigma\,\Sigma\ldots)$, $\frac{1}{6}(\Sigma\,\Sigma\,\Sigma\ldots)$ must begin at least with $1/m^{2s}$.

Consider $\sum_{j,t} \left.\frac{\partial\Phi}{\partial f_j^{(t)}}\right|_{F_{km}^{(\nu)}(s)} g_{jm}^{(t)}(s)$. $g_{jm}^{(t)}(s)$ begins with $\frac{a_{jm}(-\log m)^t}{m^s}$ and $\left.\frac{\partial\Phi}{\partial f_j^{(t)}}\right|_{F_{km}^{(\nu)}(s)}$ with $\frac{A_{jt}}{l_{jt}}$. Thus the junior term in $\Sigma_{j,t}$ will be

$$\frac{\sum_{\mu} A_{j_\mu t_\mu}\, a_{j_\mu m}(-\log m)^{t_\mu}}{(\Lambda m)^s}.$$

Two cases are possible:

(1) For all primes p beyond a certain stage,

$$\sum_{\mu} A_{j_\mu t_\mu} a_{j_\mu p}(-\log p)^{t_\mu} = 0;$$

(2) There is an infinite sequence of prime numbers, such that for the p of this sequence,

$$\sum_{\mu} A_{j_\mu t_\mu} a_{j_\mu p}(-\log p)^{t_\mu} \neq 0.$$

For the first case the lemma is proved, since the following relationship is true:

$$\sum_{\mu} A_{j_\mu t_\mu} f_{j_\mu}^{(lp)*}(s) \equiv 0.$$

In the second case, we take $m = p$ and $p > \Lambda$, so that $p^2 > \Lambda p$, and hence the term different from zero,

$$\frac{\Sigma A_{j_\mu t_\mu} a_{j_\mu p}(-\log p)^{t_\mu}}{(\Lambda_p)^s},$$

cannot interfere with the terms originating from the expressions $\frac{1}{2}(\Sigma\,\Sigma\,\ldots)$, $\frac{1}{6}(\Sigma\,\Sigma\,\Sigma\,\ldots)$. Since the left part of the expansion into Taylor series equals zero, this term, in order to be cancelled, must interfere with the terms $\Phi(F_{kp}^{(\nu)}(s))$. But the denominators of $\Phi(F_{kp}^{(\nu)}(s))$ contain only the powers of numbers less than p. Therefore $\Lambda_p = \Pi\, N_i$, where $N_i \leqslant p-1$. This is a contradiction. Hence the case is impossible, and the lemma is proved.

Lemma 4 (Ostrowski). *Let l and q be positive integers; $k_1, \ldots, k_q$ are pairwise unequal real numbers. The equation (with regard to λ)*

$$\sum_{\substack{j=0,\ldots,l\\ i=1,\ldots,q}} c_{ji}\,\lambda^j e^{\lambda k i} = 0$$

(not all c_{ji} are equal to zero) has a finite number of real roots.

Lemma 5 (A. G. Postnikov). *The Dirichlet L-series*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for a given modulus m, are difference-differentially independent.

Assume the contrary. We write

$$L(s+h_\tau, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{h_\tau}} \frac{1}{n^s}$$

(and consider this a separate Dirichlet series in the relationship of Lemma 3).

According to Lemma 3 the relation

$$\sum A_{j_\mu \nu_\mu \tau_\mu} \chi_{j_\mu}(p) \frac{1}{p^{h_{\tau_\mu}}} (-\log p)^{\nu_\mu} = 0$$

should be true for all p beginning with a certain one and with not all $A_{j_\mu \nu_\mu \tau_\mu}$ being equal to zero. The coefficients of $\frac{(-\log)^{\nu_\mu}}{p^{h_{\tau_\mu}}}$ will be certain linear forms of the characters. All these coefficients must be equal to zero. Let a certain coefficient for a certain prime p be not equal to zero. Consider the expression

$$\sum A_{j_\mu \nu_\mu \tau_\mu} \chi_{j_\mu}(p) \frac{1}{p^{h_{\tau_\mu}}} (-\log p)^{\nu_\mu}$$

on a progression for the modulus m containing p. The coefficients will be constant and since according to the Dirichlet theorem there is an infinity of prime numbers in the progression, we come to a contradiction with Lemma 4.

Hence there is at least one non-trivial (not all $A_{j_\mu} = 0$) relation

$$\sum A_{j_\mu} \chi_{j_\mu}(p) = 0$$

fulfilled for all sufficiently large p.

Let the reduced system of residues for the modulus m be $\beta_1 = 1, \ldots \beta_{\phi(m)}$. Since according to the Dirichlet theorem the prime numbers are distributed over all the progressions $mx + l$, $(l, m) = 1$, we have the following system of equations :

$$\sum A_{j_\mu} \chi_{j_\mu}(\beta_i) = 0, \; i = 1, 2, \ldots \phi(m).$$

But it is clear that the determinant

$$\Delta = \begin{vmatrix} \chi_1(\beta_1) & \cdots & \chi_1(\beta_{\phi(m)}) \\ \cdots & \cdots & \cdots \\ \chi_{\phi(m)}(\beta_1) & \cdots & \chi_{\phi(m)}(\beta_{\phi(m)}) \end{vmatrix}$$

(χ_1 is the principal character) differs from zero. Indeed, if it is multiplied by a conjugate transposed, we get, due to the orthogonality of the characters,

$$|\Delta|^2 = \Delta\overline{\Delta^*} = \phi(m)^{\phi(m)} \neq 0.$$

Hence, from the system of equations

$$\sum A_{j_\mu} \chi_{j_\mu}(\beta_i) = 0, \quad i = 1, 2, \ldots \phi(m)$$

we conclude that all $A_{j_\mu} = 0$.

This contradiction proves the lemma.

It should be noted that the statement of the lemma can be referred to the L-series of the fields of algebraic numbers.

Now, making use of the method pointed out by Hilbert, we can prove the above theorem.

PROOF. Suppose there exists the relation

$$\Phi\left(x, s, \frac{\partial^{p+q} L(x, s, \chi)}{\partial x^p \, \partial s^q}\right) \equiv 0.$$

According to a lemma which may be proved in a similar manner to Lemma 1, it may be assumed that the relation contains neither x nor s, i.e. is of the form

$$\Phi\left(\frac{\partial^{p+q} L(x, s, \chi)}{\partial x^p \, \partial s^q}\right) \equiv 0.$$

From the relation

$$x \frac{\partial L(x, s, \chi)}{\partial x} = L(x, s-1, \chi)$$

we then get

$$\left(x\frac{\partial}{\partial x}\right)^{\mu} L(x, s, \chi) = L(x, s-\mu, \chi)$$

(the powers of $x\dfrac{\partial}{\partial x}$ are understood symbolically).

Using these formulas we can successively express the derivatives of $L(x, s, \chi)$ with respect to x through shifted $L(x, s, \chi)$:

$$\frac{\partial^{\mu} L(x, s, \chi)}{\partial x^{\mu}} = \frac{1}{x^{\mu}} L(x, s-\mu, \chi) + c_{\mu,\mu-1} L(x, s-\mu+1, \chi) + \\ + \ldots + c_{\mu,1} L(x, s-1, \chi),$$

where the $c_{\mu k}$ are rational functions of x. In exactly the same manner we can write the reverse relations

$$L(x, s-\mu, \chi) = x^{\mu} \frac{\partial^{\mu} L(x, s, \chi)}{\partial x^{\mu}} + \gamma_{\mu,\mu-1} \frac{\partial^{\mu-1} L(x, s, \chi)}{\partial x^{\mu-1}} + \\ + \ldots + \gamma_{\mu,1} \frac{\partial L(x, s, \chi)}{\partial x}.$$

Differentiating these formulas λ times with respect to s, we get

$$\frac{\partial^{\mu+\lambda} L(x, s, \chi)}{\partial x^{\mu}\, \partial s^{\lambda}} = \frac{1}{x^{\mu}} \frac{\partial^{\lambda} L(x, s-\mu, \chi)}{\partial s^{\lambda}} + c_{\mu,\mu-1} \frac{\partial^{\lambda} L(x, s-\mu+1, \chi)}{\delta s^{\lambda}} + \\ + \ldots + c_{\mu,1} \frac{\partial^{\lambda} L(x, s-1, \chi)}{\partial s^{\lambda}},$$

$$\frac{\partial^{\lambda} L(x, s-\mu, \chi)}{\partial s^{\lambda}} = x^{\mu} \frac{\delta^{\mu+\lambda} L(x, s, \chi)}{\partial x^{\mu}\, \partial s^{\lambda}} + \gamma_{\mu,\mu-1} \frac{\partial^{\mu-1+\lambda} L(x, s, \chi)}{\partial x^{\mu-1}\, \partial s^{\lambda}} + \\ + \ldots + \gamma_{\mu,1} \frac{\partial^{1+\lambda} L(x, s, \chi)}{\partial x\, \partial s^{\lambda}}.$$

These formulas correspond as reverse formulas to each other.

Substituting $\dfrac{\partial^{p+q} L(x, s, \chi)}{\partial x^{p}\, \partial s^{q}}$ in $\Phi\left(\dfrac{\partial^{p+q} L(x, s, \chi)}{\partial x^{p}\, \partial s^{q}}\right)$ according to these formulas and removing the denominators, we get the relation

$$P\left(x, \frac{\partial^{\lambda} L(x, s-\mu, \chi)}{\partial s^{\lambda}}\right) \equiv 0,$$

where P is a polynomial.

The polynomial P must contain at least one $\frac{\partial^\lambda L(x, s-\mu, \chi)}{\partial s^\lambda}$ member, otherwise it would follow from the reverse formulas that all the coefficients in $\Phi\left(\frac{\partial^{p+q} L(x, s, \chi)}{\partial x^p \, \partial x^q}\right)$ are identically equal to zero.

Dividing $P\left(x, \frac{\partial^\lambda L(x, s-\mu, \chi)}{\partial s^\lambda}\right)$ by the greatest possible power of $x-1$, and assuming $x = 1$, we get a difference-differential dependence between the L-series which according to Lemma 5 is impossible. [It is impossible for only the free term $Q(x)|_{x=1}$ to be left with $x = 1$ because it follows from $Q(x)|_{x=1} = 0$ that $Q(x)$ is a multiple of $x-1$, i.e. the expression $P\left(x, \frac{\partial^\lambda L(x, s-\mu, \chi)}{\partial s^\lambda}\right)$ is a multiple of a still higher power of $(x-1)$].

Thus the theorem is proved.

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ON DIRICHET L-SERIES WITH THE CHARACTER MODULUS EQUAL TO THE POWER OF A PRIME NUMBER

By A. G. POSTNIKOV

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1. Analogue of the logarithmic series for the index. It is a well-known fact that the definition and properties of the index are analogous to those of the logarithmic function. We shall now give for the index an analogue of the classical logarithmic series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad (|x| < 1).$$

LEMMA 1. *Let p be a prime > 2, n a natural number, and* $\operatorname{ind}_g x$ *denote the index of x with the base g.*

Let $n \neq \alpha p^f - \nu$, where $(\alpha, p) = 1$, $f = 1, 2, \dots$, $0 \leqslant \nu \leqslant f - 1$. There exists a polynomial with integer coefficients (not depending on g)

$$f(u) = u + a_2 u^2 + \dots + a_{n-1} u^{n-1},$$

where $a_{n-1} \not\equiv 0 (\mathrm{mod}\ p^{n-1})$ and the natural Λ, $(\Lambda p) = 1$ (depending on g) such that

$$\operatorname{ind}_g (1 + pu) \equiv \Lambda(p-1) f(u) \quad \mathrm{mod}\ (p^{n-1}(p-1)).$$

If $n = \alpha p^f - \nu$, $(\alpha, p) = 1$, $0 \leqslant \nu \leqslant f - 1$, then the polynomial $f(u)$ has the degree $n + \nu$ and $a_{n+\nu} \not\equiv 0$ (mod p^{n-1}).

PROOF. It is known that in the multiplicative group of the reduced residue system for the modulus p^n the classes which are comparable with 1 for the modulus p form a cyclic subgroup $\mathfrak{p}$ of order p^{n-1} generated, for instance, by $1 + p$.

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Obviously, all $\operatorname{ind}_g(1+p)$ are multiples of $p-1$ and $\dfrac{\operatorname{ind}_g(1+p)}{p-1}$ is mutually prime with p.

We consider a field of rational p-adic numbers R_p.

The series

$$pu - \frac{(pu)^2}{2} + \frac{(pu)^3}{3} - \dots = \log(1+pu)$$

converges for any p-adic integer u.

Besides

$$\log[(1+pu_1)(1+pu_2)] = \log(1+pu_1) + \log(1+pu_2).$$

If $n \neq \alpha p^f - \nu$, $0 \leqslant \nu \leqslant f-1$, then the polynomial

$$F_n^*(1+pu) = pu - \frac{(pu^2)}{2} + \dots + (-1)^n \frac{(pu)^{n-1}}{n-1}$$

possesses the property

$$F_n^*[(1+pu_1)(1+pu_2)] \equiv F_n^*(1+pu_1) + F_n^*(1+pu_2) \pmod{p^n}.$$

If $n = \alpha p^f - \nu$, $0 \leqslant \nu \leqslant f-1$, then this property can be guaranteed only for the polynomial

$$F^*_{n+\nu-1}(1+pu) = pu - \frac{(pu)^2}{2} + \dots + (-1)^n \frac{(pu)^{n-1}}{n-1} + \\ + \dots + (-1)^{n+\nu+1} \frac{(pu)^{n+\nu}}{n+\nu}.$$

For the polynomial

$$\bar{F}_n(1+pu) = \frac{F_n^*(1+pu)}{p} = u - \frac{pu^2}{2} + \dots + (-1)^n \frac{p^{n-2}u^{n-1}}{n-1}$$

(and respectively for $\bar{F}_{n+\nu+1}(1+pu)$) the following congruence will be fulfilled

$$\bar{F}_n[(1+pu_1)(1+pu_2)] \equiv \bar{F}_n(1+pu_1) + \bar{F}_n(1+pu_2) \pmod{p^{n-1}}.$$

Each coefficient $\bar{F}_n(1+pu)$ (and respectively $\bar{F}_{n+\nu+1}(1+pu)$) can be substituted for the modulus p^{n-1} by a rational integer. Let $k = p^\tau k'$, where $(k', p) = 1$.

Let x_k be the solution of the congruence $k' x_k \equiv 1 \pmod{p^{n-k+\tau}}$. We denote $a_k = (-1)^{k+1} p^{k-1-\tau} x_k$. Then, since $x_k \equiv \frac{1}{k'} \pmod{p^{n-k+\tau}}$, we get

$$(-1)^{k+1} p^{k-1-\tau} x_k \equiv (-1)^{k+1} p^{k-1-\tau} \frac{1}{k'} \pmod{p^{n-1}},$$

that is

$$a_k \equiv (-1)^{k+1} \frac{p^{k-1}}{k} \pmod{p^{n-1}}.$$

Obviously $a_1 = 1$. It should be noted also that $a_{n-1} \not\equiv 0 \pmod{p^{n-1}}$. If $n \leqslant k < n+\nu$, then we assume $a_k = 0$. The coefficient of $u^{n+\nu}$ can also be substituted for the modulus p^{n-1} by an integer, and $a_{n+\nu} \not\equiv 0 \pmod{p^{n-1}}$. We denote

$$f(u) = u + a_2 u^2 + \dots + a_{n-1} u^{n-1},$$

$$f(u) \equiv \bar{F}_n(1+pu) \pmod{p^{n-1}}.$$

If $p \neq 2$, then the coefficients $a_2, a_3, \dots, a_{n-1}$ are sure to be multiples of p because $k - 1 - \tau \geqslant 1$. Hence

$$F_n(1+p) \equiv 1 \pmod{p},$$

$$F_n(1+p) \not\equiv 0 \pmod{p}.$$

Therefore it follows that the congruence

$$\frac{\operatorname{ind}_g (1+p)}{p-1} \equiv \Lambda F_n(1+p) \pmod{p^{n-1}}$$

is solvable. Let Λ be its root. Since $F_n(1+p) \not\equiv 0 \pmod{p}$, and $\frac{\operatorname{ind}_g (1+p)}{p-1} \not\equiv 0$, $(\Lambda, p) = 1$.

When $s = 0, \ldots, p^{n-1}$, we have

$$s\,\frac{\operatorname{ind}_g(1+p)}{p-1} \equiv \Lambda\, s\, F_n(1+p) \pmod{p^{n-1}}$$

or, owing to the multiplicative properties of both sides,

$$\frac{\operatorname{ind}_g((1+p)^s)}{p-1} \equiv \Lambda\, F_n((1+p)^s) \pmod{p^{n-1}}.$$

But $(1+p)^s$ runs through the entire subgroup $\mathfrak{p}$. Therefore, for any u,

$$\frac{\operatorname{ind}_g(1+pu)}{p-1} \equiv \Lambda f(u) \pmod{p^{n-1}}.$$

Hence the result.

Lemma 2. *Let $1 \leqslant a \leqslant p-1$ be the root of the congruence $aa' \equiv 1 \pmod{p^{n-1}}$. Then, for any integer u,*

$$\operatorname{ind}_g(a+pu) \equiv \operatorname{ind}_g a + \Lambda(p-1) f(a'u) \pmod{p^{n-1}(p-1)}.$$

Indeed

$$\operatorname{ind}_g(a+pu) \equiv \operatorname{ind}_g a + \operatorname{ind}_g(1+pa'u) \pmod{p^{n-1}(p-1)}.$$

Applying Lemma 2 we get what was required.

2. Estimation of the sum of characters for a modulus equal to the power of a prime number. According to the lemma proved above, when $D = p^n$ we may consider the sum of characters a trigonometric sum with a polynomial and apply the well-known estimations of I. M. Vinogradov to it.

By C with subscripts we denote positive constants.

Theorem 1. *Let $\chi(k)$ be a character for the modulus p^{n-1} of a power not lower than p^{n-1} (in this case this is a criterion of the character being primitive for the modulus p^n). For $l \geqslant p^2$ and $n \geqslant n_0$,*

$$\left|\sum_{k=N}^{N+l-1} \chi(k)\right| < e^{C_0 n(\log n)^2}\, p^{C_1/n^3 \log n}\, l^{1-C_1/n^3\log n}.$$

PROOF. As we know,

$$\chi(k) = e^{2\pi i m\{\mathrm{ind}_g k/(p-1)p^{n-1}\}}.$$

In view of the requirement for a character $(m, p) = 1$,

$$\left|\sum_{k=N}^{N+l-1} \chi(k)\right| = \sum_{a=1}^{p} \left|\sum_{u=N_a}^{N_a+l_a-1} \chi(a+pu)\right|,$$

where $c/p \leqslant l_a \leqslant l/p + 1$.

$$\left|\sum_{k=N}^{N+l-1} \chi(k)\right| = \sum_{a=1}^{p} \left|\sum_{u=N_a}^{N_a+l_a-1} e^{2\pi i m\Lambda f(a'u)/p^{n-1}}\right|$$

$$= \sum_{a=1}^{p} \left|\sum_{u=1}^{l_a} e^{2\pi i m\{f(a'(u+N_a-1))/p^{n-1}\}}\right|.$$

The major coefficient in $f(a'(u+N_a-1))$, both when $n \neq \alpha p^f - \nu$ and when $n = \alpha p^f - \nu$, since $(m, p) = 1$, $(\Lambda, p) = 1$, $a^{n-1} \not\equiv 0 \pmod{p^{n-1}}$, is of the form $\frac{\tau}{p^{\mu}}$, where $1 + [\log n/\log p] \geqslant \mu \geqslant 1$.

I. M. Vinogradov's estimation of trigonometric sums with polynomials (I. M. Vinogradov: "Upper bound of the modulus of a trigonometric sum", *Selected works*, Moscow, p. 389) may be applied to the internal sum. We apply the estimation for the major coefficient.

We get: If $n \neq \alpha p^f - \nu$,

$$\left|\sum_{u=1}^{l_a} e^{2\pi i m\Lambda f(a'(u+N_a-1))/p^{n-1}}\right| \leqslant (8(n-2))^{\frac{1}{2}(n-2)\log\{12(n-2)(n-1)/\tau\}} \times$$

$$\times\, l_a^{1-\{[3(n-2)^2/\tau]\log[12(n-2)(n-1)/\tau]\}^{-1}},$$

where τ can be determined from the conditions

$$\tau = 1, \text{ when } l_a < p^{\mu} < l_a^{n-2},$$

$$p^{\mu} = l_a^{\tau}, \text{ when } p^{\mu} < l_a.$$

It will suffice to consider the case where $p^n \geqslant l$, because the sum is periodic (with period p^n). The value of τ is smallest when $l_a = p^{n-1}$ and $\mu = 1$, i.e. when $\tau = 1/(n-1)$. Therefore, for $p^{\mu/(n-2)} \leqslant l_a \leqslant p^{n-1}$,

$$\left|\sum_{u=1}^{l_a} e^{2\pi i m \Lambda f(a'(u+N_a-1))/p^{n-1}}\right| \leqslant e^{C'_0 n(\log n)^2}\, l_a^{1-C_1/n^3\log n}.$$

Since $l_a < l/p + 1$ and $\mu/(n-2) \leqslant 1$, we have for $p^2 \leqslant l \leqslant p^n$,

$$\left|\sum_{k=N}^{N+l-1} \chi(k)\right| \leqslant e^{C_0 n(\log n)^2}\, p(l/p)^{1-C_1/n^3\log n}$$

$$= e^{C_0 n(\log n)^2}\, p^{C_1(n^3\log n)-1}\, l_a^{1-C_1(n^3\log n)^{-1}}.$$

If $n = \alpha p^f - \nu$,

$$\left|\sum_{u=1}^{l_a} e^{2\pi i m \Lambda f(a'(u+N_a-1))/p^{n-1}}\right| \leqslant 8(n+\nu-1)^{\frac{1}{2}(n+\nu-1)\log\{12(n+\nu-1)(n+\nu))/\tau\}} \times$$

$$\times\, l_a^{1-\{[3(n+\nu-1)^2/\tau]\log[12(n+\nu-1)(n+\nu-2)/\tau]\}^{-1}},$$

where τ can be determined from the conditions

$$\tau = 1, \quad \text{when} \quad l_a \leqslant p^\mu \leqslant l_a^{n+\nu-1},$$

$$p^\mu = l_a^\tau, \quad \text{when} \quad p^\mu \leqslant l_a.$$

Since $\nu < n$, we find again (with other constants C_0 and C_1) that for $p^2 \leqslant l \leqslant p^n$,

$$\left|\sum_{k=N}^{N+l-1} \chi(k)\right| \leqslant e^{C_0 n(\log n)^2}\, p^{C_1/n^3\log n}\, l^{1-C_1/n^3\log n}.$$

If $l > p^n$, the only "dangerous" place is $p^n < l < p^n + p^2$. Then

$$\left|\sum_{k=N}^{N+l-1} \chi(k)\right| \leqslant p^2 \leqslant e^{C_0 n(\log n)^2}\, p^{C_1/n^3\log n}\, p^{n(1-C_1/n^3\log n)}$$

$$\leqslant e^{C_0 n(\log n)^2}\, p^{C_1/n^3\log n}\, l^{1-C_1/n^3\log n} \quad \text{for } n \geqslant n_0.$$

Thus Theorem 1 is proved.

3. Application to the theory of L-series for a modulus equal to the power a prime number. The estimation of Theorem 1 can be employed in our special case to improve the estimations of the modulus of the L-series on a real straight line in the critical strip. We denote in all cases $D = p^n$. χ denotes the primitive character for the modulus p^n.

THEOREM 2. *If* $n \to \infty$, *and* $n^Q \geqslant \log p \geqslant C_2 n^4 (\log n)^3$, *then if* $2 \geqslant \sigma \geqslant 1 - \dfrac{1}{\log^{Q/(Q+1)} D}$, $|t| \leqslant C_3$,

$$|L(s, \chi)| \ll \log^{Q/(Q+1)} D.$$

PROOF. We denote $D_1 = p^A$ (A is a constant), $\gamma = \dfrac{1}{\log^{Q/(Q+1)} D}$.

$$L(s, \chi) = \sum_{k \leqslant 2D_1 - 1} \frac{\chi(k)}{k^s} + s \int_{2D_1} \frac{\sum_{D_1 \leqslant k \leqslant u} \chi(k)}{u^{s+1}} du - \frac{\sum_{D_1 \leqslant k \leqslant 2D_1 - 1} \chi(k)}{(2D_1)^s}$$

$$|L(s, \chi)| \leqslant \sum_{k \leqslant 2D_1 - 1} \frac{1}{k^{1-\gamma}} + |S| \int_{2D_1} \frac{e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} u^{1 - C_1/n^3 \log n}}{u^{2-\gamma}} du +$$

$$+ \left| \frac{\sum_{D_1 \leqslant k \leqslant 2D_1 - 1} \chi(k)}{(2D_1)^{1-\gamma}} \right|$$

$$\ll \frac{D_1^\gamma}{\gamma} + e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} \int_{2D_1}^{\infty} \frac{du}{u^{1-\gamma + C_1/n^3 \log n}} +$$

$$+ e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} D_1^{\gamma - C_1/n^3 \log n}.$$

It is clear that with $n \geqslant n_0$, $\gamma < \dfrac{1}{2n^3 \log n}$. Therefore

$$|L(s, \chi)| \ll \frac{D_1^\gamma}{\gamma} + e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} \int_{2D_1}^{\infty} \frac{du}{u^{1 + C_1/2n^3 \log n}} +$$

$$+ e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} D_1^{-C_1/2n^3 \log n}$$

$$\ll \frac{D_1^\gamma}{\gamma} + n^3 \log n \, e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} p^{-AC_1/2n^3 \log n} +$$

$$+ e^{C_0 n (\log n)^2} p^{C_1/n^3 \log n} p^{-AC_1/2n^3 \log n}.$$

$$|L(s, \chi)| \ll \frac{D^{A\gamma/n}}{\gamma} + e^{C_4 n(\log n)^2} p^{-(A/2-1)C_1/n^3 \log n}.$$

Since $\log p \geqslant C_2 n^4(\log n)^3$, on taking a sufficiently large A, we get

$$e^{C_4 n(\log n)^2} p^{-(A/2-1)C_1/n^3\log n} = O(1).$$

Since $n^{Q+1} \geqslant \log D$, $n \geqslant \log^{1/(Q+1)} D$,

$$D^{A\gamma/n} \leqslant DA\gamma/\log^{1/(Q+1)} D = D^{A/\log D} = O(1).$$

Consequently

$$|L(s, \chi)| \ll 1/\gamma = \log^{Q/(Q+1)} D.$$

Hence the absence of zeros of the L-series on a section of the real axis in the critical strip can be deduced in the usual manner.

Theorem 3. *Let $D = p^n$, and χ_Q be a primitive (non-real)*† *character for the modulus p^n and $n \geqslant \log p \geqslant C_2 n^4(\log n)^3$, $n \geqslant n_0$. $L(s, \chi)$ has no zeros in the region $|t| \leqslant C_6$, $\sigma > 1 - \dfrac{C_5}{\log^{Q/(Q+1)} D \log\log D}$.*

Proof. Let $\beta + i\gamma$ be a zero of $L(s, \chi)$ with $|\gamma| \leqslant C_6$. We shall consider that it lies in the upper semiplane. We consider the points $s_0 = \sigma_0 + i\gamma$ and $s'_0 = \sigma_0 + 2i\gamma$, where $1 + \dfrac{1}{\log^{10Q/(1Q+1)} D} < \sigma_0 < 2$, while σ_0 will be selected more accurately afterwards.

We circumscribe circles of radius $\tau = \dfrac{1}{\log^{Q/(Q+1)} D}$ around s_0 and s'_0.

Both circles lie in the region $\sigma \geqslant 1 - \dfrac{1}{\log^{Q/(Q+1)} D}$, $|t| \leqslant 2C_6 + 1$.

Employing Theorem 2 we make the estimations.

In the circle $|s - s_0| \leqslant \tau$,

$$\left|\frac{L(s, \chi)}{L(s_0, \chi)}\right| < C_7 \log^{Q/(Q+1)} D \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}} < C_8 \log D \frac{1}{\sigma_0 - 1} \leqslant C_8 \log^2 D.$$

† Ed. *J.I.M.S.*

In the circle $|s_0 - s_0'| \leqslant \tau$,

$$\left|\frac{L(s, \chi^2)}{L(s_0, \chi^2)}\right| < C_7 \log^{Q/(Q+1)} D \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_0}} < e^{C_9 \log\log D}.$$

Applying Lemma β, p. 49, Titchmarsh "*The Theory of the Riemann Zeta-Function*", we get :

$$-\operatorname{Re}\frac{L'(\sigma_0 + 2i\gamma, \chi^2)}{L(\sigma_0 + i\gamma, \chi^2)} < C_{10} \log^{Q/(Q+1)} D \log\log D,$$

and if $\beta > \sigma_0 - \dfrac{1}{2\log^{Q/(Q+1)} D}$, then

$$-\operatorname{Re}\frac{L'(\sigma_0 + i\gamma, \chi)}{L(\sigma_0 + i\gamma, \chi)} < C_{10} \log^{Q/(Q+1)} D - \frac{1}{\sigma_0 - \beta}.$$

Let $\beta > \sigma_0 - \dfrac{1}{2\log^{Q/(Q+1)} D}$. We write a well-known inequality

$$-3\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} - 4\operatorname{Re}\frac{L'(\sigma_0 + i\gamma, \chi)}{L(\sigma_0 + i\gamma, \chi)} - \operatorname{Re}\frac{L'(\sigma_0 + 2i\gamma, \chi^2)}{L(\sigma_0 + 2i\gamma, \chi^2)} \geqslant 0.$$

Since, with $\sigma_0 \to 1$, $-\dfrac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} \sim \dfrac{1}{\sigma_0 - 1}$, we have $-\dfrac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} < \dfrac{a}{\sigma_0 - 1}$, where, with a sufficiently small difference $\sigma_0 - 1$, a can be made sufficiently close to 1. We get

$$\frac{3a}{\sigma_0 - 1} + 5C_{10} \log^{Q/(Q+1)} D \log\log D - \frac{4}{\sigma_0 - \beta} \geqslant 0,$$

$$\sigma_0 - \beta \geqslant \frac{1}{3a/4(\sigma_0 - 1) + (5/4) C_{10} \log^{Q/(Q+1)} D \log\log D},$$

$$1 - \beta \geqslant \frac{1}{3a/4(\sigma_0 - 1) + (5/4) C_{10} \log^{Q/(Q+1)} D \log\log D} - (\sigma_0 - 1),$$

$$1 - \beta \geqslant \frac{1 - (3/4)a + (5/4) C_{10}(\sigma_0 - 1) \log^{Q/(Q+1)} D \log\log D}{3a/4(\sigma_0 - 1) + (5/4) C_{10} \log^{Q/(Q+1)} D \log\log D}.$$

We put

$$a = (5/4),\ \sigma_0 = 1 + \frac{1}{40 C_{10} \log^{Q/(Q+1)} D \log\log D}.$$

Then

$$1-\beta \geqslant \frac{1/32}{(155/4)\, C_{10} \log^{Q/(Q+1)} D \log\log D}$$

$$= \frac{1}{1240\, C_{10} \log^{Q/(Q+1)} D \log\log D}.$$

Let $\beta < \sigma_0 - \dfrac{1}{2\log^{Q/(Q+1)} D}$. We again obtain

$$1-\beta \geqslant \frac{1}{2\log^{Q/(Q+1)} D} - \frac{1}{40\, C_{10} \log^{Q/(Q+1)} \log\log D}$$

$$> \frac{C_5}{\log^{Q/(Q+1)} D \log\log D}.$$

Thus the theorem is proved.

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QUADRATIC FORMS OVER INVOLUTORIAL DIVISION ALGEBRAS

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1. Introduction. In 1924 Hasse proved a fundamental theorem concerning quadratic forms over algebraic number fields, namely that two quadratic forms with coefficients in an algebraic number field $\mathscr{K}$ are equivalent if and only if they are so in every completion $\mathscr{K}_{\mathfrak{p}}$ of $\mathscr{K}$ by valuations of $\mathscr{K}$. He also discussed the problem of representing quadratic forms by quadratic forms. These results were later extended, by Witt, to the case where $\mathscr{K}$ is an algebraic function field of one variable over a finite field of constants. Recently Siegel discussed the theory of quadratic forms with coefficients in an involutorial simple algebra over an algebraic number field. It seems therefore of interest to study systematically the arithmetic and analytic theory of quadratic forms over involutorial algebras.

In this paper we extend the results of Hasse and Witt to quadratic forms with coefficients in an involutorial division algebra $\mathscr{D}$ whose centre $\mathscr{K}$ is either an algebraic number field or an algebraic function field of one variable over a finite field of constants and of characteristic $\neq 2$. If $\mathscr{D}$ is a division algebra and the involution leaves the centre $\mathscr{K}$ fixed, then by the results of Albert and Witt, $\mathscr{D}$ is either commutative or is a quaternion division algebra. By extending $\mathscr{K}$ to $\mathscr{K}_{\mathfrak{p}}$, the extended algebra $\mathscr{D}_{\mathfrak{p}}$ splits for almost all $\mathfrak{p}$ and $\mathscr{D}_{\mathfrak{p}}$ is isomorphic to the algebra of two rowed square matrices with elements in $\mathscr{K}_{\mathfrak{p}}$. If the involution in $\mathscr{D}$ does not fix $\mathscr{K}$, and $\mathscr{K}_0$ is the subfield of $\mathscr{K}$ fixed by the involution, we extend $\mathscr{K}_0$ to $\mathscr{K}_{0\mathfrak{p}}$ and then the extended algebra is either simple or semi-simple. In the first case, because of a theorem of Jacobson, $\mathscr{D}_{\mathfrak{p}}$ is isomorphic to the algebra

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of square matrices over $\mathscr{K}_{0\mathfrak{p}}$. In the second case $\mathscr{D}_{\mathfrak{p}}$ is the direct sum of two reciprocal simple algebras. This splitting of $\mathscr{D}_{\mathfrak{p}}$ leads to the study of bilinear forms. In order to deal with all these cases we study in § 3 quadratic and bilinear forms with coefficients in an algebra of appropriate type and whose centre is an arbitrary field of characteristic $\neq 2$. We discuss equivalence of and representability by forms over these algebras. An analogue of Witt's theorem is proved. For later use we study the so-called multiplicative equivalence and multiplicative representation. The theory of skew symmetric forms is discussed but only over a quaternion division algebra whose centre is $\mathscr{K}_{\mathfrak{p}}$. The reason for this restriction is that the analogue of Hasse's theorem seems, in general, to be false.

In a future paper we shall study orthogonal groups and unit groups associated with quadratic forms over these algebras.

2. Notations. $\mathscr{F}$ and $\mathscr{F}_0$ will stand for arbitrary fields of characteristic $\neq 2$. $\mathscr{K}$ and $\mathscr{K}_0$ will stand for an algebraic number field or an algebraic function field of one variable over a finite field of characteristic $\neq 2$. $\mathscr{K}^*$ will denote the group of non-zero elements of $\mathscr{K}$, and $\mathscr{K}^{*2}$ the groups of squares of elements of $\mathscr{K}^*$. For two algebras $\mathscr{A}$ and $\mathscr{B}$ over $\mathscr{K}$, $\mathscr{A} \times \mathscr{B}$ will denote the tensor product algebra. A matrix $D = (d_{kl})$ with $d_{kl} = 0$ if $k \neq l$ will be called a diagonal matrix and denoted by $D = [d_1, \ldots, d_n]$. For an element ξ in an algebra $\mathscr{A}$, $|\xi|$ will denote the reduced norm. If $\mathscr{A}$ is an algebra, $\mathfrak{M}_n(\mathscr{A})$ will denote the algebra of n rowed matrices over $\mathscr{A}$. x, y, z will denote column vectors, E the unit matrix and O the null matrix of order evident from the context. ϵ, η will stand for ± 1.

3. Algebras over arbitrary fields. Let $\mathscr{D}$ be a division algebra of rank m^2 over its centre $\mathscr{F}$ which is an infinite field of characteristic $\neq 2$. Let $\mathscr{D}$ have an involution $a \to \tilde{a}$ so that

$$\tilde{\tilde{a}} = a, \widetilde{a + b} = \tilde{a} + \tilde{b}, \widetilde{ab} = \tilde{b}\,\tilde{a}, \widetilde{\lambda a} = \tilde{\lambda}\,\tilde{a}, \tag{1}$$

where a and b are in $\mathscr{D}$ and $\lambda \in \mathscr{F}$. Since on $\mathscr{F}$ the involution is an automorphism, let $\mathscr{F}_0$ be the fixed field of this automorphism. Then $(\mathscr{F} : \mathscr{F}_0) = 1$ or 2. The involution is said to be of the first or second kind according as $(\mathscr{F} : \mathscr{F}_0) = 1$ or 2. In the second case there exists an element θ in $\mathscr{F}$ such that

$$\tilde{\theta} = - \theta, \tag{2}$$

and $\mathscr{F} = \mathscr{F}_0 (\theta)$.

An element ξ of $\mathscr{D}$ is said to be *symmetric* or *skew* with regard to the involution $\sim$ according as

$$\tilde{\xi} = \xi, \tilde{\xi} = - \xi. \tag{3}$$

In case of involutions of the second kind, there exist $t = m^2$ elements $s_1, \dots , s_t$ in $\mathscr{D}$, which are symmetric and which form a base of $\mathscr{D}$ over $\mathscr{F}$. For every $\omega \in \mathscr{D}$ therefore

$$\omega = \sum_{i=1}^{t} a_i s_i + \theta \sum_{i=1}^{t} b_i s_i, \tag{4}$$

where a_i and b_i are in $\mathscr{F}_0$. In particular, if $\tilde{\omega} = \omega$,

$$\omega = \sum_{i=1}^{t} a_i s_i. \tag{5}$$

Let $M = (a_{kl})$ be an element of $\mathfrak{M}_n(\mathscr{D})$. We can extend the involution in $\mathscr{D}$ to $\mathfrak{M}_n$ by defining

$$\tilde{M} = (\tilde{a}_{lk}). \tag{6}$$

If, on the other hand, $M \to M^*$ is an involution of $\mathfrak{M}_n(D)$ which has on $\mathscr{F}$ the same effect as $\sim$, then because $\mathfrak{M}_n$ is a simple algebra, there exists a C in $\mathfrak{M}_n$ such that

$$M^* = C^{-1} \tilde{M} C, \quad \tilde{C} = \epsilon C. \tag{7}$$

A matrix S in $\mathfrak{M}_n$ is said to be *symmetric* or *skew-symmetric* under(*) if $S^* = S$ or $S^* = - S$ respectively. Using (7), it follows that $T = CS$ satisfies

$$\tilde{T} = \pm T. \tag{8}$$

Let S and T be two n rowed matrices over $\mathscr{D}$ and let $S^* = \epsilon S$ and $T^* = \eta T$. They are said to be *multiplicatively equivalent* if there exists a non-singular V with elements in $\mathscr{D}$ and an element $a \in \mathscr{F}_0$ such that

$$V^* S V = a\, T. \tag{9}$$

This means that $\epsilon = \eta$. Using (7) we get

$$\tilde{V}\,(C\,S)\;V = a(C\;T). \tag{10}$$

Conversely (10) implies (9). If $a = 1$ we say that S and T are *equivalent*, denoted $S \sim T$. Obviously equivalence of matrices as well as multiplicative equivalence are equivalence relations. We can therefore put symmetric and skew matrices into classes of equivalent or multiplicatively equivalent elements.

Let $S = (s_{ij})$ be an n-rowed square matrix over $\mathscr{D}$ and let $\tilde{S} = \epsilon\, S$. Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a column vector of n rows of indeterminates which can take values in $\mathscr{D}$. Then

$$S[x] = \tilde{x}\, S\, x = \sum_{i,j} \tilde{x}_i\, s_{ij}\, x_j$$

is called the *associated form* of the matrix S. If $\epsilon = 1$, the form is said to be a *quadratic form*, otherwise a *skew-symmetric form*. If $q \in \mathscr{D}$, and $\tilde{q} = \epsilon\, q$, and there is a vector x with elements in $\mathscr{D}$ such that $\tilde{x}\, S\, x = q$, we say that S *represents* q. If $q = 0$ and x is not the zero vector we say that the representation is *non-trivial*. If S is also non-singular we say that $S[x]$ is a *zero form*. Clearly equivalent matrices represent the same set of elements of $\mathscr{D}$. If S is non-singular, and $S[x] = 0$ has only the trivial solution $x = 0$, then $S[x]$ is said to be a *definite* form.

In the case of involutions of the first kind Albert has proved that $\mathscr{D}$ has exponent one or two. If $\mathscr{D}$ is non-commutative, then because of the results of Hasse and Witt, it follows that if the centre of $\mathscr{D}$ is $\mathscr{K}$, then $\mathscr{D}$ is a quaternion division algebra. For our purposes we shall study the following situation. If $\mathscr{D}$ is an involutional algebra of the first kind, then $\mathscr{D}$ is either a division algebra or a quaternion algebra *not necessarily a division algebra.* In the latter case $\mathscr{D}$ is isomorphic to the two rowed matrix algebra. Thus in the second case let $\mathscr{D}$ be the cyclic algebra of rank 4 over $\mathscr{F}$ with a basis $1, \omega_1, \omega_2, \omega_3$ satisfying

$$\omega_1^2 = \alpha, \omega_2^2 = \beta, \omega_1 \omega_2 = - \omega_2 \omega_1 = \omega_3, \tag{11}$$

α, β being in $\mathscr{F}$ and 1 is the unit element of $\mathscr{D}$. Every ξ in $\mathscr{D}$ has the form

$$\xi = x_0 + x_1 \omega_1 + x_2 \omega_2 + x_3 \omega_3.$$

$x_i \in \mathscr{F}$ and let $\tilde{\xi}$ denote the conjugate

$$\tilde{\xi} = x_0 - x_1 \omega_1 - x_2 \omega_2 - x_3 \omega_3.$$

Then $|\xi| = \xi \tilde{\xi} = \tilde{\xi} \xi = x_0^2 - \alpha x_1^2 - \beta x_2^2 + \alpha \beta x_3^2$ is the reduced norm of ξ. $\mathscr{D}$ is a division algebra if and only if $|\xi| = 0$ means $\xi = 0$.

$\mathscr{D}$ has the well-known 2-rowed matrix representation

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \omega_1 \to \sqrt{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \omega_2 \to \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}, \ \omega_3 \to \sqrt{\alpha} \begin{pmatrix} 0 & 1 \\ -\beta & 0 \end{pmatrix}. \tag{12}$$

The field $\mathscr{F}(\sqrt{\alpha})$ is obviously a splitting field for $\mathscr{D}$. Every element t in $\mathscr{F}(\sqrt{\alpha})$ is of the form $t = a + b\sqrt{\alpha}$, $a, b \in \mathscr{F}$. Let $\bar{t} = a - b\sqrt{\alpha}$. Every quaternion q has the form

$$q = M(q) = \begin{pmatrix} t_1 & t_2 \\ \beta \bar{t}_2 & \bar{t}_1 \end{pmatrix}. \tag{13}$$

This representation is faithful and absolutely irreducible. Also

$$\tilde{q} = J\, \overline{M(q)}'\, J^{-1},$$

where $J = \begin{pmatrix} -1 & 0 \\ 0 & \beta \end{pmatrix}$, and $|q| = |M(q)|$, the determinant of $M(q)$. Using the above representation, it can be proved that for T in $\mathfrak{M}_n(\mathscr{D})$,

$$|T| = |\tilde{T}|,$$

and that $|T| \in \mathscr{F}$.

3.1. Let $\mathscr{D}$ be an involutorial algebra with $\mathscr{F}$ as centre and satisfying the above conditions if the involution is of the first kind. We now prove

Theorem 1. *Let S be a non-singular n rowed matrix over $\mathscr{D}$, $\tilde{S} = \epsilon S$. There exists then a non-singular V such that*

$$\tilde{V} S V = \mathscr{D} = [\alpha_1, \dots, \alpha_n],$$

where $\alpha_1, \dots, \alpha_n$ are in $\mathscr{D}$ and are not divisors of zero.

Proof. Let $S = (\alpha_{kl})$ so that $\alpha_{kl} = \epsilon \tilde{\alpha}_{lk}$. Put

$$S = \begin{pmatrix} \alpha_{11} & \tilde{q} \\ q & R \end{pmatrix},$$

where q is a column of $n-1$ elements. Suppose $|\alpha_{11}| \neq 0$ so that α_{11} is not a divisor of zero. By the principle of completion of squares we have

$$S = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & S_1 \end{pmatrix} \begin{bmatrix} 1 & \alpha_{11}^{-1}\tilde{q} \\ 0 & E \end{bmatrix}.$$

We can now use induction and apply the theorem to S_1. We have therefore only to prove that, by a non-singular transformation, we can make the first diagonal element not a divisor of zero.

If for some i, $1 < i \leqslant n$, α_{ii} is not a divisor of zero, we may put V to be the matrix which interchanges the first and ith rows of a matrix. Then $S[V]$ will have α_{ii} for first diagonal element.

Suppose α_{ii} are all divisors of zero. Since $|S| \neq 0$, there exists an element α_{it} in the i-th row which is not a divisor of zero. We may assume without loss in generality that α_{12} is not a divisor of zero. We can therefore deal with a 2-rowed matrix T,

$$T = \begin{pmatrix} \alpha & \beta \\ \epsilon\beta & \gamma \end{pmatrix}.$$

where α and γ are divisors of zero and β is not a divisor of zero in $\mathscr{D}$. Let $\mu \in \mathscr{D}$ be chosen so that $\tilde{\mu} + \epsilon\mu$ is not a divisor of zero. This can be done, for if the involution is of the first kind and

$\epsilon = 1$, we put $\mu = 1$, whereas if $\epsilon = -1$, we put $\mu = \omega_1$. If on the other hand the involution is of the second kind, we put $\mu = \theta$ or 1 according as $\epsilon = -1$ or 1. Let t be a variable element of $\mathscr{F}_0$, to be chosen later and put

$$V = \begin{pmatrix} 1 & t\beta^{-1}\mu \\ t\beta^{-1}\mu & 0 \end{pmatrix}.$$

V is obviously non-singular. Let $T[V] = \begin{pmatrix} \alpha_1 & * \\ * & * \end{pmatrix}$. Then

$$\alpha_1 = \alpha + t(\mu + \epsilon \tilde{\mu}) + t^2 \tilde{\mu}\, \tilde{\beta}^{-1}\, \gamma\, \beta^{-1}\, \mu. \tag{14}$$

Since $\mathscr{F}$ is an infinite field, we choose $t \in \mathscr{F}_0$ such that $|\alpha_1| \neq 0$. α_1 is invertible and our theorem is demonstrated.

We remark that Theorem 1 can be extended even to the case where S has rank $r < n$.

Note that Theorem 1 is false if the involution is of the first kind, $\epsilon = -1$ and $\mathscr{D}$ is commutative.

3.2. Before proving the important Theorem 2 we shall prove a lemma concerning binary quadratic forms over quaternion algebras of the first kind.

Lemma 1. *Let $S = [\xi_1, \xi_2]$ be a symmetric non-singular matrix over a quaternion algebra $\mathscr{D}$. Let a, b be elements of $\mathscr{D}$ not both zero. There exists a two rowed matrix P, such that*

$$\begin{cases} S[P] = S, \\ P\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}, \end{cases}$$

with $1 + a'$ not a divisor of zero.

Proof. We use the method of Siegel. Since $\tilde{S} = S$ we see that ξ_1 and ξ_2 are in $\mathscr{F}$. Put

$$P = \begin{pmatrix} \alpha & \beta\,\xi_2 \\ -\beta\,\xi_1 & \alpha \end{pmatrix},$$

where

$$\alpha = \frac{\xi_1\,\xi_2 - t^2}{\xi_1\,\xi_2 + t^2},\ \beta = \frac{2t}{\xi_1\,\xi_2 + t^2} \tag{15}$$

and $t \in \mathscr{F}$ to be suitably chosen. Obviously $S[P] = S$. Now a' is given by $a' = \alpha\, a + \beta\, \xi_2\, b$. Substituting for α and β from (15) we see that since $\mathscr{F}$ is infinite, $t \in \mathscr{F}$ can be so chosen that $|\,\xi_1 \xi_2 + t^2\,| \neq 0$ and $1 + a'$ is invertible.

We are now ready to prove the following analogue of a theorem of Witt.

THEOREM 2. *Let S and T be two non-singular symmetric n-rowed matrices over $\mathscr{D}$. Let also*

$$S \sim \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},\ T \sim \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where S_1 and T_1 are equivalent r-rowed matrices. Then S_2 and T_2 are equivalent if and only if S and T are equivalent.

PROOF. It is enough to prove the 'if' part of the theorem. We may also assume that $S_1 = T_1$. Since by Theorem 1 we may diagonalize the matrices, it is evidently enough to prove the theorem when

$$S \sim \begin{pmatrix} \xi & 0 \\ 0 & S_0 \end{pmatrix},\ T \sim \begin{pmatrix} \xi & 0 \\ 0 & T_0 \end{pmatrix}.$$

Let $P = \begin{pmatrix} p & \tilde{x} \\ y & L \end{pmatrix}$, where x is a column of $n - 1$ elements of $\mathscr{D}$. P is a matrix over $\mathscr{D}$ and satisfies $S[P] = T$. This means that

$$\left.\begin{aligned} \tilde{p}\,\xi\,p + \tilde{y}\,S_0\,y &= \xi, \\ \tilde{p}\,\xi\,\tilde{x} + \tilde{y}\,S_0\,L &= 0, \\ x\,\xi\,\tilde{x} + \tilde{L}\,S_0\,L &= T_0. \end{aligned}\right\} \tag{16}$$

Suppose now that $\mathscr{D}$ is a division algebra. Since $p \in \mathscr{D}$, either $p + 1$ or $p - 1$ is different from zero. Let $u = p + 1 \neq 0$. Put

$$Q = L - y\,u^{-1}\,\tilde{x} \tag{17}$$

Because of conditions (17), it follows that

$$S_0[Q] = T_0,$$

and the theorem is proved in this case.

So let $\mathscr{D}$ be a quaternion algebra with divisors of zero. Let S_0 and T_0 be both in the diagonal form. Suppose p is not a divisor of zero and let μ be the first diagonal element of S_0. Let q be the first element of the column y. Choose the matrix Q in such a way that

$$\begin{pmatrix} \xi & 0 \\ 0 & \mu \end{pmatrix} [Q] = \begin{pmatrix} \xi & 0 \\ 0 & \mu \end{pmatrix},$$

and $Q\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$ with $1 + p'$ not a divisor of zero. This is possible by Lemma 1. Let

$$P_1 = \begin{pmatrix} Q & 0 \\ 0 & E_{n-2} \end{pmatrix} P.$$

Then $S[P_1] = T$ and P_1 has the first diagonal element p' such that $p' + 1$ is not a divisor of zero. We can now proceed in exactly the same way as before.

Suppose now p is a divisor of zero. Since $|P| \neq 0$, at least one element of y satisfies the condition in Lemma 1. We may assume that the first element of y satisfies this condition. We can apply Lemma 1 again and complete the proof.

We remark that Theorems 1 and 2 are still true if $\mathscr{D}$ is a non-commutative division algebra of infinite rank over $\mathscr{F}$. S and T may be symmetric or skew symmetric.

From now on until the end of this section we only assume that $\mathscr{D}$ is an involutorial division algebra with $\mathscr{F}$ as centre, and of finite rank over the centre.

Let S be an n-rowed matrix, non-singular and $\tilde{S} = \epsilon S$. Suppose it is indefinite; that is, there exists a vector $x \neq 0$ such that $S[x] = 0$. Completing x into a non-singular matrix V, we have

$$S[V] = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

It is easy to see that by completing squares we may even choose V in such a way that

$$S[V] = \begin{pmatrix} J & 0 \\ 0 & S_1 \end{pmatrix}, \tag{18}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Doing this again with S_1 we obtain, after finitely many steps, the matrix

$$S \sim \begin{bmatrix} 0 & E_r & 0 \\ \pm E_r & 0 & 0 \\ 0 & 0 & T \end{bmatrix}, \tag{19}$$

where T is a definite matrix, and E_r is the unit matrix of order r. Obviously

$$r \leqslant n/2.$$

From Theorem 2 the integer r is determined uniquely by the equivalence class of S. r is called the *index* of S. Furthermore T is determined uniquely by S. The integer r and the matrix T are invariants for equivalence.

Let S and T be two matrices of n and $m (\leqslant n)$ rows respectively. Also let $\widetilde{S} = \epsilon\, S$, $\widetilde{T} = \epsilon\, T$, and S non-singular. We say that S *represents* T *in* $\mathfrak{D}$ if there exists a matrix C of n rows and m columns and of *rank* m such that

$$S[C] = T.$$

If T is non-singular, the condition on the rank of C is unnecessary. If $T = O$, the null matrix of order m, we see by completion of squares that

$$S \sim \begin{bmatrix} O & E_m & O \\ \pm E_m & O & O \\ O & O & S_1 \end{bmatrix},$$

which shows that $m \leqslant r$. Hence S cannot represent non-trivially the null matrix of order $> r$.

Let $\tilde{S} = \epsilon S$ and S non-singular. Let $S[x]$ be a zero form. We may take S in the diagonal form. Let

$$\sum_{i=1}^{n} \tilde{x}_i a_i x_i = 0 \tag{20}$$

be a non-trivial representation of zero by S. Let $x_1 \neq 0$. Let $t \in \mathscr{D}$, $\tilde{t} = \epsilon t$. Put

$$\begin{cases} y_1 = x_1(1 + \lambda), \\ y_i = x_i(1 - \lambda), \quad i > 1, \end{cases}$$

where $\lambda \in \mathscr{D}$ is to be suitably chosen. Then, because of (20),

$$\sum_{i=1}^{n} \tilde{y}_i a_i y_i = 2(\lambda \tilde{x}_1 a_1 x_1 + \tilde{x}_1 a_1 x_1 \lambda).$$

Since $\tilde{x}_1 a_1 x_1 \neq 0$, put

$$\lambda = (\tilde{x}_1 a_1 x_1)^{-1} t/4.$$

Then $\sum_{i=1}^{n} \tilde{y}_i a_i y_i = t$. This shows that for every t in $\mathscr{D}$, with $\tilde{t} = \epsilon t$, $S[y] = t$ has a solution. We can now prove the following general

THEOREM 3. *Let $\tilde{S} = \epsilon S$, $|S| \neq 0$, and let S have n rows. If S represents the zero matrix of order l non-trivially, then S represents non-trivially every matrix T of order l with $\tilde{T} = \epsilon T$.*

PROOF. Clearly, by Theorem 2,

$$l \leqslant r \leqslant n/2.$$

We shall therefore use induction on l. If $l = 1$, then the considerations above show that the theorem is proved. Let therefore $l > 1$, and the theorem be proved for all matrices of order $l - 1$ instead of l. Let T be a matrix of order l. Then we may take

$$T \sim \begin{pmatrix} t & 0 \\ 0 & T_1 \end{pmatrix},$$

where $t \neq 0$. (We assume that T is not the zero matrix of order l). Let

$$S \sim \begin{bmatrix} \begin{matrix} 0 & 1 \\ \pm 1 & 0 \end{matrix} & O \\ O & S_1 \end{bmatrix}.$$

Then S_1 represents the zero matrix of order $r-1 \geqslant l-1$ non-trivially. Let x be a column of 2 rows such that

$$\begin{bmatrix} 0 & 1 \\ \pm 1 & 0 \end{bmatrix} [x] = t,$$

and let Y be a matrix of $n-1$ rows, and $l-1$ columns, and rank $l-1$, such that

$$S_1[Y_1] = T_1.$$

This exists by the induction hypothesis. Put

$$C = \begin{bmatrix} x & 0 \\ 0 & Y \end{bmatrix}.$$

Then C has rank l, and $S[C] = T$. Our theorem is proved.

It must be noted that the converse of the theorem is false; that is, that S may represent all $T \neq O$ but not $T = O$ of order l. Clearly this can happen only when $l > r$.

We now define the so-called *multiplicative representation*. Let $S = S^{(n)} = \epsilon S$, and $T = T^{(m)} = \epsilon T$, $n \geqslant m$. S is said to represent T multiplicatively if there is a $C = C^{(n,m)}$ with elements in $\mathscr{D}$ and of rank m, and an element $t \neq 0$ in $\mathscr{F}_0$, such that

$$S[C] = t\, T.$$

This means that $t^{-1} S$ represents T. If $|T| \neq 0$, then we find that there exists a T such that

$$S \sim t \begin{bmatrix} T & 0 \\ 0 & T_1 \end{bmatrix}.$$

3.3. We now consider bilinear forms. Let $\mathscr{D}$ be a division algebra with $\mathscr{F}$ as centre. Let $\mathscr{D}^{-1}$ be the reciprocal algebra. Let $\mathscr{A}$ be the semi-simple algebra

$$\mathscr{A} = \mathscr{D} + \mathscr{D}^{-1},$$

constituted by the direct sum of $\mathscr{D}$ and $\mathscr{D}^{-1}$. For any element $a \in \mathscr{D}$ let $\tilde{a}$ denote the unique element into which a goes by the anti-automorphism between $\mathscr{D}$ and $\mathscr{D}^{-1}$. Similarly for $b \in \mathscr{D}^{-1}$, $\tilde{b} \in D^{-1}$. Denote the generic element α in $\mathscr{A}$ by

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

$a \in \mathscr{D}, b \in \mathscr{D}^{-1}$. Denote by $\tilde{\alpha}$ the element

$$\tilde{\alpha} = \begin{bmatrix} \tilde{b} & 0 \\ 0 & \tilde{a} \end{bmatrix}.$$

It is then easy to see that $\alpha \to \tilde{\alpha}$ is an involution of $\mathscr{A}$. α is said to be symmetric or skew according as $\alpha = \tilde{\alpha}$ or $\alpha = -\tilde{\alpha}$. Thus

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & \tilde{a} \end{bmatrix}, \text{ or } \alpha = \begin{bmatrix} a & 0 \\ 0 & -\tilde{a} \end{bmatrix}.$$

Consider now the matrix algebra $\mathfrak{M}_n(\mathscr{A}) = \bar{\mathscr{A}}$. By a suitable choice of base elements we may write any element M of $\bar{\mathscr{A}}$ in the form

$$M = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}.$$

$P \in \mathfrak{M}_n(\mathscr{D})$, $Q \in \mathfrak{M}_n(\mathscr{D}^{-1})$. Let $P = (p_{kl})$, $Q = (q_{kl})$, and define $\tilde{M}$ by

$$\tilde{M} = \begin{bmatrix} \tilde{Q} & O \\ O & \tilde{P} \end{bmatrix},$$

where $\tilde{Q} = (\tilde{q}_{lk})$, $\tilde{P} = (\tilde{p}_{lk})$. Then $M \to \tilde{M}$ is an involution of $\bar{\mathscr{A}}$ which extends the involution in $\mathscr{A}$. We define symmetric and skew elements in the same way as before. If $S \in \bar{\mathscr{A}}$ and $\tilde{S} = \epsilon S$, then S has the form

$$S = \begin{bmatrix} P & O \\ O & \epsilon\tilde{P} \end{bmatrix}.$$

$|S| \neq 0$ means $|P| \neq 0$.

Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$ be a column of $2n$ rows, x and y each having n rows, with x having elements in $\mathscr{D}$ and $y \in \mathscr{D}^{-1}$. We call $S[z]$ the bilinear form of S. Obviously

$$\tilde{z}\, S\, z = \begin{bmatrix} \tilde{x}\, P\, y & 0 \\ 0 & \epsilon\, \tilde{y}\, \tilde{P}\, x \end{bmatrix}.$$

Let $m \leqslant n$ and $\overline{\mathscr{A}}_{mn}$ denote the module over $\mathscr{A}$ of matrices with n rows and m columns. By proper choice of base elements we write any C in $\overline{\mathscr{A}}_{mn}$ as

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

A being a matrix of n rows and m columns over $\mathscr{D}$ and B similarly defined over $\mathscr{D}^{-1}$. Denote by $\tilde{C}$ the matrix

$$\tilde{C} = \begin{bmatrix} \tilde{B} & O \\ O & \tilde{A} \end{bmatrix},$$

then $\tilde{C}$ is an element of the module $\overline{\mathscr{A}}_{nm}$. Also $C \rightarrow \tilde{C}$ is an isomorphic mapping of $\overline{\mathscr{A}}_{mn}$ on $\overline{\mathscr{A}}_{nm}$. If $T \in \overline{\mathscr{A}}_{mm}$ and $\tilde{T} = \epsilon\, T$, we say that S represents T if there is a C in $\overline{\mathscr{A}}_{mn}$ with

$$\tilde{C}\, S\, C = T.$$

If $|\, T\, | = 0$, we insist that A and B, the components of C, have rank m. Obviously if $S = \begin{bmatrix} M & O \\ O & \epsilon \tilde{M} \end{bmatrix}$, $T = \begin{bmatrix} N & O \\ O & \epsilon \tilde{N} \end{bmatrix}$, then

$$\tilde{Q}\, M\, P = N. \tag{21}$$

If $m = n$, we say that S and T are *equivalent*. In case $|\, T\, | \neq 0$, we see from the theory of linear equations in division rings that (21) always has a solution. In particular

THEOREM 4. *Any two bilinear non-singular symmetric forms are equivalent.*

In the case $m = 1$ and $t = 0$ we see that (21) has a solution

$$S\begin{bmatrix} p \\ q \end{bmatrix} = 0,$$

$p \neq 0, q \neq 0$. Completing p and q to non-singular matrices we get

$$S \sim \begin{bmatrix} L & O \\ O & \epsilon \tilde{L} \end{bmatrix},$$

where

$$L = \begin{bmatrix} 0 & e & 0 \\ e & 0 & 0 \\ 0 & 0 & L_1 \end{bmatrix}, \tag{22}$$

and

$$1 = e + \tilde{e}$$

is the decomposition of the unit element of $\mathscr{A}$ into a sum of orthogonal idempotents. This shows that S can represent the null matrix of order m, where

$$m \leqslant n/2.$$

m is the index of S and $n/2$ is the best possible value of m.

The canonical form for a symmetric bilinear matrix is

$$S = \begin{bmatrix} P & O \\ O & \tilde{P} \end{bmatrix},$$

where

$$P = \begin{cases} \begin{bmatrix} O & eE & O \\ eE & O & O \\ O & O & e \end{bmatrix} & n \text{ odd}, \\ \begin{bmatrix} O & eE \\ eE & O \end{bmatrix} & n \text{ even}. \end{cases} \tag{23}$$

Denote by W the matrix

$$W = \begin{bmatrix} eE & 0 \\ 0 & -\tilde{e}E \end{bmatrix},$$

E being the matrix of order n. Then W is in the centre of $\tilde{\mathscr{A}}$, and if S is skew symmetric, then WS is symmetric. This shows that it is enough to study symmetric bilinear forms.

3.4. Let $\mathscr{A}_0$ be the simple algebra of 3.1 and 3.2 or the semi-simple algebra of 3.3. Let $\mathfrak{M}_s(\mathscr{A}_0)$ be the algebra of s-rowed square matrices over $\mathscr{A}_0$. The involution in $\mathscr{A}_0$ can be extended to $\mathscr{A}$ and one can study symmetric, skew symmetric and bilinear forms in $\mathscr{A}$ in exactly the same way, and all the foregoing theorems can be extended easily to $\mathfrak{M}_s(\mathscr{A}_0)$.

4. Symmetric forms over number and function fields. Hereafter $\mathscr{D}$ will be a division algebra of rank m^2 over $\mathscr{K}$. We shall assume that $\mathscr{D}$ is non-commutative.

We consider first the case where $\mathscr{D}$ has an involution of the first kind. $\mathscr{D}$ is thus a non-commutative quaternion division algebra over $\mathscr{K}$. Let it be defined by (11). Let $S[x]$ be a symmetric non-degenerate n-rowed quadratic form over $\mathscr{D}$. Because of Theorem 1 we may take S to be in the diagonal form. Also since for $\xi \in \mathscr{D}$, $\xi + \tilde{\xi} \in \mathscr{K}$, it follows that if $x_i = x_{i0} + x_{i1}\,\omega_1 + \ldots + x_{i3}\,\omega_3$, then

$$S[x] = \sum_{i=1}^{n} a_i(x_{i0}^2 - \alpha x_{i1}^2 - \beta x_{i2}^2 + \alpha\beta x_{i3}^2), \tag{24}$$

which is a quadratic form over $\mathscr{K}$ in the $4n$ variables x_{ij}.

Let $\mathfrak{p}$ be a prime divisor of a valuation of $\mathscr{K}$, and $\mathscr{K}_\mathfrak{p}$ the complete field under this valuation. Put $\mathscr{D}_\mathfrak{p} = \mathscr{D} \times \mathscr{K}_\mathfrak{p}$. Then $\mathscr{D}_\mathfrak{p}$ is a matrix algebra for almost all $\mathfrak{p}$. Let $\mathfrak{p}$ be a finite prime. Then $K_\mathfrak{p}$ is a p-adic field or a field of formal power series. By the results of Hasse-Witt every element in $\mathscr{K}_\mathfrak{p}$ is a norm of an element in $\mathscr{D}_\mathfrak{p}$. Thus

$$S[x] = b,$$

$b \in \mathscr{K}_\mathfrak{p}$ is always solvable if $b \neq 0$, and if $b = 0$, then $n > 1$, provided $\mathscr{D}_\mathfrak{p}$ is a division algebra. We therefore have, if $\mathscr{K}$ is a function field,

THEOREM 5. *A quadratic form $S[x]$ over $\mathscr{D}$ represents every element $b \neq 0$ in $\mathscr{D}$. If in addition $n > 1$, it represents zero also non-trivially.*

In the case of the function field we can solve the equivalence problem also. For let S and T be symmetric and non-singular over

$\mathfrak{D}$. For equivalence, it is necessary that S and T have the same number of rows. We assert that this is sufficient. For let $S = [s_1,\ldots,s_n]$, $T = [t_1,\ldots, t_n]$. There is then x with elements in $\mathfrak{D}$ such that

$$S[x] = t_1.$$

Completing x to a non-singular matrix we have

$$S \sim \begin{bmatrix} t_1 & 0 \\ 0 & S_1 \end{bmatrix}.$$

Put $T = \begin{bmatrix} t_1 & 0 \\ 0 & T_1 \end{bmatrix}$. Then if we use induction, then $S_1 \sim T_1$ and so $S \sim T$. Hence n is the only invariant for equivalence. Thus

THEOREM 6. *If $\mathscr{K}$ is an algebraic function field of one variable over a finite field of constants, then every non-singular symmetric matrix over $\mathfrak{D}$, a quaternion division algebra over $\mathscr{K}$, is equivalent to the n-rowed unit matrix.*

Let now $\mathscr{K}$ be a number field. The above considerations hold good if $\mathfrak{p}$ is a finite prime or an infinite prime where $\mathfrak{D}$ is unramified. Let $\mathfrak{p}_\infty$ be a real infinite prime spot of $\mathscr{K}$ so that $\mathscr{K}_{\mathfrak{p}_\infty}$ is the real field and $\mathfrak{D}_{\mathfrak{p}_\infty}$ the algebra of real quaternions. Let $J_{\mathfrak{p}_\infty}(S)$ denote the number of negative ones among $a_1, a_2, \ldots, a_n$ in (24). Then $J_{\mathfrak{p}_\infty}(S)$ is an invariant for equivalence in $\mathfrak{D}_{\mathfrak{p}_\infty}$. $J_{\mathfrak{p}_\infty}(S)$ is called the signature of S at $\mathfrak{p}_\infty$. Obviously

$$0 \leqslant J_{\mathfrak{p}_\infty}(S) \leqslant n. \tag{25}$$

Let $\mathfrak{p}_{\infty 1}, \ldots \mathfrak{p}_{\infty t}$ be all the real infinite prime spots of $\mathscr{K}$ at which $\mathfrak{D}$ is ramified and $\{J_{\mathfrak{p}_{\infty i}}(S)\}$ the *system of signatures* of S. If for $b \in \mathfrak{D}$, b denotes the conjugate of b in $\mathscr{K}_{\mathfrak{p}_{\infty t}}$, then using Hasse's theory, we get

THEOREM 7. *$S[x]$ represents $b \in \mathscr{K}$ if and only if it does so in every $\mathfrak{D}_{\mathfrak{p}}$. This is equivalent to*

$$\left.\begin{aligned} J_{\mathfrak{p}_{\infty l}}(S) &\leqslant n-1 \text{ if } b_l > 0 \\ &\geqslant 1 \quad\ \text{ if } b_l < 0 \end{aligned}\right\}, \qquad b \neq 0,$$

$$1 \leqslant J_{\mathfrak{p}_{\infty l}}(S) \leqslant n-1, \qquad b = 0,$$

for $l = 1, \ldots, t$.

We can now solve the problem of equivalence. For $\mathfrak{p}$ finite or $\mathscr{D}_{\mathfrak{p}_\infty}$ unramified at $\mathfrak{p}_\infty$, the only invariant for equivalence is n, the numbers of rows. If however $\mathscr{D}_{\mathfrak{p}_\infty}$ is ramified at $\mathfrak{p}_\infty$, then n and $J_{\mathfrak{p}_\infty}(S)$ are invariants. If S and T are symmetric, non-degenerate, and are equivalent over $\mathscr{D}$, then they have the same number n of rows, and

$$J_{\mathfrak{p}_{\infty l}}(S) = J_{\mathfrak{p}_{\infty l}}(T), \quad l = 1, \dots, t. \tag{26}$$

The converse is also true. For let S and T have same number of rows and satisfy (26). It is easy to see that there exists an $a \in \mathscr{K}$ with the properties

$$a_l \begin{cases} > 0 \text{ if } J_{\mathfrak{p}_\infty}(S) = 0, \\ < 0 \quad ,, \quad ,, \quad = n, \\ \text{arbitrary, } 0 < J_{\mathfrak{p}_\infty}(S) < n. \end{cases} \tag{27}$$

By Theorem 7, S and T both represent a. Hence

$$S \sim \begin{bmatrix} a & 0 \\ 0 & S_1 \end{bmatrix}, T \sim \begin{bmatrix} a & 0 \\ 0 & T_1 \end{bmatrix}.$$

But S_1 and T_1 satisfy conditions similar to those of S and T with $n - 1$ instead of n. Using induction we get

THEOREM 8. *Two non-singular symmetric matrices S and T are equivalent in $\mathscr{D}$ if and only if they are so in every $\mathscr{D}_\mathfrak{p}$. An equivalence class of symmetric matrices is completely determined by the invariants n and $\{J_{\mathfrak{p}_{\infty l}}(S)\}, l = 1, \dots, t$. These can be assigned arbitrarily satisfying only* (25).

It is now trivial to extend Theorem 7 to representations of symmetric matrices. For instance S can represent the null matrix of order r only if

$$r \leqslant \min_l \,(J_{\mathfrak{p}_{\infty l}}(S), n - J_{\mathfrak{p}_{\infty l}}(S)). \tag{28}$$

In order to study multiplicative equivalence we proceed thus. We study only the number field case, the other being trivial. At a finite prime spot of $\mathscr{K}$ or at an infinite prime spot of $\mathscr{K}$ where $\mathscr{D}$ is unramified, n is the only invariant for multiplicative equivalence. If $\mathscr{D}$ is ramified at $\mathfrak{p}_\infty$, then clearly S and T are multiplicatively equivalent over $D_{\mathfrak{p}_\infty}$ if and only if

$$J_{\mathfrak{p}_\infty}(S) = J_{\mathfrak{p}_\infty}(T) \text{ or } J_{\mathfrak{p}_\infty}(T) = n - J_{\mathfrak{p}_\infty}(S).$$

Since there exist in $\mathscr{K}$ elements with prescribed signs at the infinite prime spots we have the main theorem for multiplicative equivalence.

THEOREM 9. *A complete system of invariants for multiplicative equivalance are n and $|n - 2\, J_{\mathfrak{p}_{\infty l}}(S)|$, $l = 1, \dots, t$.*

We can now solve also the problem of multiplicative representation.

Let $S^{(n)} = \tilde{S}$ be non-singular symmetric, and $T^{(n)} = \tilde{T}$ be non-singular and $n \geqslant m$. If S represents T multiplicatively, then there exists T_1 such that

$$S \sim t \begin{pmatrix} T & 0 \\ 0 & T_1 \end{pmatrix},$$

for some $t \in \mathscr{K}_0$. In the case of function fields the problem presents no difficulty. In the case when $\mathscr{K}$ is an algebraic number field, we have only to study the infinite prime spots $\mathfrak{p}_{\infty 1}, \dots \mathfrak{p}_{\infty t}$ at which $\mathscr{D}$ is ramified.

Obviously the conditions are

$$0 \leqslant J_{\mathfrak{p}_{\infty l}}(S) - J_{\mathfrak{p}_{\infty l}}(T) \leqslant n - m,$$

or

$$0 \leqslant J_{\mathfrak{p}_{\infty l}}(S) + J_{\mathfrak{p}_{\infty l}}(T) - m \leqslant n - m.$$

The existence of elements in $\mathscr{K}$ with prescribed signs at real infinite prime spots proves

THEOREM 10. *S represents T multiplicatively if and only if it does so at every $\mathscr{D}_\mathfrak{p}$, $\mathfrak{p}$ a prime spot of $\mathscr{K}$.*

Here we assumed that T is non-singular. It is easy to carry over to the case where T is not non-singular.

We now define a form $\tilde{x}\, S\, x$, S symmetric non-singular, to be *definite* if and only if it is definite in the sense of §3 at every infinite prime spot of $\mathscr{K}$. Thus $\tilde{x}\, S\, x = 0$ should imply $x = 0$ in every $\mathscr{D}_{\mathfrak{p}_\infty}$. This means that $\mathscr{K}$ is *totally* real and $\mathscr{D}$ is ramified at all $\mathscr{K}_{\mathfrak{p}_\infty}$. Hence *$\mathscr{D}$ is a totally definite quaternion algebra over a totally real centre.*

$S[x]$ is said to be *totally positive* if it is totally definite and $S[x] > 0$ for every $x \neq 0$ in $\mathscr{D}_{\mathfrak{p}_\infty}$.

4.1. We now consider involutorial algebras of the second kind. The results here have been obtained already by Landherr in case $\mathscr{K}$ is a number field by using the theory of Lie algebras. Our considerations here are simple and apply even to the function field case.

Let $\mathscr{D}$ be an involutorial division algebra of the second kind over $\mathscr{K}$ and let $\mathscr{K}_0$, be the fixed field of the involution. $\mathscr{D}$ may be commutative also. Let θ be chosen as in (2). Put $\theta^2 = \mu \in \mathscr{K}_0$. Let $\mathfrak{p}$ be a prime divisor of a valuation of $\mathscr{K}_0$, and $\mathscr{K}_{0\mathfrak{p}}$ the complete field. Put $\mathscr{D}_\mathfrak{p} = \mathscr{D} \times \mathscr{K}_{0\mathfrak{p}}$ and $\mathscr{K}_\mathfrak{p} = \mathscr{K} \times \mathscr{K}_{0\mathfrak{p}}$. We consider two cases (1) $\mathscr{K}_\mathfrak{p}$ is a field and (2) $\mathscr{K}_\mathfrak{p}$ is a direct sum of two fields.

Consider now the first case, so that $\mathscr{D}_\mathfrak{p}$ is an involutorial simple algebra over $\mathscr{K}_\mathfrak{p}$. It is known, by a theorem of Jacobson, that $\mathscr{D}_\mathfrak{p}$ is isomorphic to the algebra of n by n matrices over $\mathscr{K}_\mathfrak{p}$. Let M be an element in $\mathscr{D}_\mathfrak{p}$, so that $M = (a_{kl})$, $a_{kl} \in \mathscr{K}_\mathfrak{p}$. If one takes the regular representation of $\mathscr{K}_\mathfrak{p}$ over $\mathscr{K}_\mathfrak{p}$, then $a \in \mathscr{K}_{0\mathfrak{p}}$ is represented by a matrix

$$a = \begin{pmatrix} \lambda & \nu \\ \mu\nu & \lambda \end{pmatrix}. \tag{29}$$

Denote by a^* the conjugate of a over $\mathscr{K}_{0\mathfrak{p}}$. Then

$$a^* = \begin{pmatrix} \lambda & -\nu \\ -\mu\nu & \lambda \end{pmatrix},$$

where $\lambda,\ \nu \in \mathscr{K}_{0\mathfrak{p}}$. Let us denote by M^* the matrix (a^*_{lk}). Then $M \rightarrow M^*$ is an involution in $\mathscr{D}_\mathfrak{p}$ over $\mathscr{K}_\mathfrak{p}$. An element M in $\mathscr{D}_\mathfrak{p}$ is said to be symmetric if $M^* = M$. Note that M is a hermitian matrix in the ordinary sense. We shall find a connection between this involution and the one got by extending the involution in $\mathscr{D}$ to $\mathscr{D}_\mathfrak{p}$.

In order to do this we study the regular representation of $\mathscr{D}$ over $\mathscr{K}_0$. Every element α in $\mathscr{D}$ is represented by a matrix of $2m^2$ rows of elements in $\mathscr{K}_0$. We denote this matrix also by α. We use in $\mathscr{D}_\mathfrak{p}$ the same basis over $\mathscr{K}_{0\mathfrak{p}}$ as $\mathscr{D}$ over $\mathscr{K}_0$. Thus α is a matrix of $2m^2$ rows over $\mathscr{K}_{0\mathfrak{p}}$. $\mathscr{D}_\mathfrak{p}$ being isomorphic to $\mathfrak{M}_n(\mathscr{K}_\mathfrak{p})$ we see that by a proper change of basis we may write

$$\alpha = \gamma \begin{bmatrix} \hat{\alpha} & & \\ & \cdots & \\ & & \hat{\alpha} \end{bmatrix} \gamma^{-1}, \tag{30}$$

where γ is a matrix of $2m^2$ rows over $\mathscr{K}_{0\mathfrak{p}}$, and $\hat{\alpha}$ is an m rowed matrix with elements α_{kl} of the form (29). Note that $\hat{\alpha}$ is the matrix representing α in the irreducible representation of $\mathscr{D}$ over $\mathscr{K}_\mathfrak{p}$. Define α^* by

$$\alpha^* = \gamma \begin{bmatrix} \hat{\alpha}^* & & \\ & \cdots & \\ & & \hat{\alpha}^* \end{bmatrix} \gamma^{-1}, \tag{31}$$

$\hat{\alpha}^*$ being defined already. Then $\alpha \to \alpha^*$ is an involution in the matrix of regular representations. Therefore there exists δ in $\mathscr{D}_\mathfrak{p}$ such that

$$\tilde{\alpha} = \delta^{-1}\,\alpha^*\,\delta, \tag{32}$$

where $\delta^* = \pm\,\delta$. We may choose $\delta^* = \delta$ in view of the property of θ. δ being in $\mathscr{D}_\mathfrak{p}$ we have

$$\delta = \gamma \begin{bmatrix} \hat{\delta} & & \\ & \cdots & \\ & & \hat{\delta} \end{bmatrix} \gamma^{-1},$$

and so using (31) and (32) we see that if α is symmetric in $\mathscr{D}$, then

$$(\hat{\delta}\,\hat{\alpha})^* = \hat{\delta}\,\hat{\alpha}. \tag{33}$$

$\hat{\delta}\,\hat{\alpha}$ is a hermitian matrix in the usual sense of the word. We call $\hat{\delta}\,\hat{\alpha}$ the $\mathfrak{p}$ component of α.

The equivalence class of α in $\mathscr{D}_{\mathfrak{p}}$ is determined by the set of elements $\hat{\beta}(\hat{\delta}\,\hat{\alpha})\,\beta$, where $\beta \in \mathscr{D}_{\mathfrak{p}}$ and is not a divisor of zero. If $\mathfrak{p}$ is a finite prime spot, clearly the only invariant for equivalence at $\mathfrak{p}$ is

$$d_{\mathfrak{p}} = \left(\frac{|\alpha|, \mathscr{K}}{\mathfrak{p}}\right),$$

the norm residue symbol.

Suppose now $\mathscr{K}$ is a number field, and $\mathfrak{p}_\infty$ is a real infinite prime spot of $\mathscr{K}_0$ such that $\mathscr{K}_{\mathfrak{p}}$ is the complex number field. The $\mathfrak{p}_\infty$ component of $\alpha = \tilde{\alpha}$ in $\mathscr{D}$ is the m-rowed hermitian matrix $(\hat{\delta}\,\hat{\alpha})$. If $\alpha \neq 0$, the signature of the matrix $\hat{\delta}\,\alpha$ which we denote $J_{\mathfrak{p}_\infty}(\alpha)$ is an invariant for equivalence in $\mathscr{D}_{\mathfrak{p}_\infty}$. We call $J_{\mathfrak{p}_\infty}(\alpha)$ the *signature* of α at $\mathfrak{p}_\infty$. This is the only invariant for equivalence in $\mathscr{D}_{\mathfrak{p}_\infty}$. Also

$$0 \leqslant J_{\mathfrak{p}_\infty}(\alpha) \leqslant m. \tag{34}$$

If $\mathfrak{p}_{\infty_1}, \ldots, \mathfrak{p}_{\infty_t}$ be the finite number of real infinite prime spots of $\mathscr{K}_0$ at which $\mathscr{K}_{\mathfrak{p}_\infty}$ is a field, then α has the *system of signatures* $\{J_{\mathfrak{p}_{\infty i}}(\alpha)\}$. We shall now prove

THEOREM 11. *There exist elements x in $\mathscr{D}$ with preassigned signatures.*

PROOF. It is to be noted at the outset that these signatures satisfy (34). If $m = 1$, then $\alpha \in \mathscr{K}_0$, and the signature is zero or one means that x at that prime spot is positive or negative. The theorem above would in that case be equivalent to showing that elements in $\mathscr{K}_0$ exist with prescribed signs at 'infinity'. The existence of such elements is well known.

Let H be a symmetric element in $\mathscr{D}$, and let $H^{(\mu)}$ denote its $\mathfrak{p}_{\infty\mu}$th component, where $K_{\mathfrak{p}_{\infty\mu}}$ is the complex number field. $H^{(\mu)}$ is an m-rowed hermitian matrix. If $a \in \mathscr{K}_0$, then $a^{(\mu)}\ H^{(\mu)}$ is again hermitian, $a^{(\mu)}$ being the conjugate of a. By (5) every symmetric element H of $\mathscr{D}$ has the form

$$H = a_1 H_1 + \dots + a_l H_l, \quad l = m^2, \tag{35}$$

$H_1, \dots, H_l$ being a base of symmetric elements of $\mathscr{D}$ over $\mathscr{K}$ and $a_1, \dots a_l$ in $\mathscr{K}$. Let f be the degree of $\mathscr{K}_0$ over Γ, the rational number field. Then

$$t \leqslant f,$$

where $\mathfrak{p}_{\infty 1}, \dots, \mathfrak{p}_{\infty t}$ are the real infinite prime spots of $\mathscr{K}_0$ for which $\mathscr{K}_{\mathfrak{p}_\infty}$ is a field.

Let $\omega_1, \dots, \omega_f$ be a base of $\mathscr{K}_0/\Gamma$, so that ω in $\mathscr{K}_0$ can be uniquely written in the form

$$\omega = \sum_{i=1}^{f} d_i\, \omega_i, \quad d_i \in \Gamma.$$

Put a_i in (35) in the above form

$$a_i = \sum_{i=1}^{f} a_{ij}\, \omega_j, \quad i = 1, \dots, l. \tag{36}$$

The $\mathfrak{p}_{\infty\mu}$th component of H is given by

$$H^{(\mu)} = \sum_{i=1}^{l} \sum_{j=1}^{f} a_{ij}\, \omega_j^{(\mu)} H_i^{(\mu)}, \quad \mu = 1, \dots, t, \tag{37}$$

$a_{ij} \in \Gamma$. $H^{(\mu)} = (h_{kl}^{(\mu)})$ is an m rowed hermitian matrix. We shall show that the a_{ij} can be chosen in Γ in such a way that the $H^{(\mu)}$ has signature e_μ, g_μ, where

$$e_\mu + g_\mu = m, \quad 0 \leqslant e_\mu \leqslant m. \tag{38}$$

Now (37) is a system of tm^2 linear equations in the fm^2 variables a_{ij}, so that if the $H^{(1)}, \dots, H^{(t)}$ are arbitrarily given hermitian m-rowed matrices, there is always a real solution if $t < f$. If $t = f$, then if $H^{(1)}, \dots, H^{(l)}$ are not all zero, there is also a real solution since the determinant of the system of equations is not zero. Thus (37) has always a non-trivial real solution, provided $H^{(1)}, \dots, H^{(t)}$ are not all zero.

Let $H^{(\mu)} = \begin{pmatrix} P^{(\mu)} & * \\ * & Q^{(\mu)} \end{pmatrix}$, where $P^{(\mu)}$ is hermitian with e_μ rows, and $Q^{(\mu)}$ hermitian with g_μ rows. Also let

$$P^{(\mu)} > 0, \quad Q^{(\mu)} < 0. \tag{39}$$

Then $H^{(\mu)}$ has precisely the signature e_μ, g_μ. For every $H^{(1)}, \ldots, H^{(t)}$ satisfying (39) there is a real solution of (37). Let W be the space of all solutions of (37) satisfying (39). This space W is an open convex subset of the Euclidean space of $m^2 f$ dimensions formed by a_{ij}'s. Thus there is a set of rational numbers a_{ij} for which the $H^{(\mu)}$ defined by (37) satisfy (39). $H^{(\mu)} \neq 0$. This proves our theorem.

We now consider the second case where $\mathscr{K}_\mathfrak{p}$ is a direct sum of two fields $e\mathscr{K}_\mathfrak{p}$ and $\tilde{e}\mathscr{K}_\mathfrak{p}$. $\mathscr{D}_\mathfrak{p}$ is then a direct sum of two simple algebras $\mathscr{A}_{\mathfrak{p}1}$ and $\mathscr{A}_{\mathfrak{p}2}$ with centres $e\mathscr{K}_{0\mathfrak{p}}$ and $\tilde{e}\mathscr{K}_{0\mathfrak{p}}$ respectively. Also $\mathscr{A}_{\mathfrak{p}1}$ is a matrix algebra of rank $(m/s)^2$ over a division algebra $\mathscr{D}_1$ of rank s^2 over $e\mathscr{K}_\mathfrak{p}$. Similarly $\mathscr{A}_{\mathfrak{p}2}$ over $\mathscr{D}_2$, and it is to be noted that $\mathscr{D}_1$ and $\mathscr{D}_2$ are reciprocal division algebras. By a suitable choice of basis elements of $\mathscr{D}_\mathfrak{p}$, every element α in $\mathscr{D}_\mathfrak{p}$ may be written

$$\alpha = \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

$A \in \mathscr{A}_{\mathfrak{p}_1}$, $B \in \mathscr{A}_{\mathfrak{p}_2}$. Define α^* by

$$\alpha^* = \begin{pmatrix} B^* & O \\ O & A^* \end{pmatrix}, \tag{40}$$

where $B^* = (b^*_{lk})$. $B = (b_{kl})$, $b_{kl} \in \mathscr{D}_2$ and $b^*_{lk} \in \mathscr{D}_1$ which corresponds to b_{kl} by the anti-isomorphism of $\mathscr{D}_1$ and $\mathscr{D}_2$. Similarly for A^*. Then $\alpha \rightarrow \alpha^*$ is an involution in $\mathscr{D}_\mathfrak{p}$ with $\mathscr{K}_{0\mathfrak{p}}$ as fixed field.

By taking the regular representation of $\mathscr{D}$ over $\mathscr{K}$, we again get

$$\alpha = \gamma \begin{bmatrix} \hat{\alpha} & & \\ & \cdots & \\ & & \hat{\alpha} \end{bmatrix} \gamma^{-1},$$

$\hat{\alpha} = \begin{pmatrix} \hat{\alpha}_1 & 0 \\ 0 & \hat{\alpha}_2 \end{pmatrix}$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ being m-rowed square matrices over $\mathscr{K}_{0\mathfrak{p}}$. Proceeding as before we get for a symmetric α in $\mathscr{D}$ the $\mathfrak{p}$ component

$\hat{\delta}\,\hat{\alpha}$ which equals $(\hat{\delta}\,\hat{\alpha})^*$. This is the matrix of a symmetric bilinear form over $\mathscr{D}_{\mathfrak{p}}$. Note that here

$$d_{\mathfrak{p}} = \left(\frac{|\alpha|, \mathscr{K}_0}{\mathfrak{p}}\right) = 1. \tag{41}$$

Let S be a symmetric, non-singular, n-rowed matrix over $\mathscr{D}$. Let S be equivalent to the diagonal matrix $[\alpha_1, \ldots, \alpha_n]$. If $\mathfrak{p}$ is a prime divisor of $\mathscr{K}_0$, we denote by Δ the diagonal matrix whose diagonal elements are δ defined in (32) and later. We call $\hat{\Delta}\,\hat{S}$ the $\mathfrak{p}$ component of S. Also put

$$d_{\mathfrak{p}} = \left(\frac{|S|, \mathscr{K}_0}{\mathfrak{p}}\right).$$

Then $d_{\mathfrak{p}} = 1$, if $\mathscr{K}_{\mathfrak{p}}$ is not a field. If $\mathfrak{p}$ is a finite prime, then n and $d_{\mathfrak{p}}$ are the only invariants for equivalence. In particular, since

$$\prod_{\mathfrak{p}} d_{\mathfrak{p}} = 1, \tag{42}$$

the only invariants for equivalence in $\mathscr{D}$, in case $\mathscr{K}$ is a function field, are n and the $d_{\mathfrak{p}}$'s. The $d_{\mathfrak{p}}$'s have only to satisfy (42). From the properties of the norm symbol it is obvious that $d_{\mathfrak{p}}$'s can be fixed arbitrarily.

We now assume that $\mathscr{K}$ is a number field. The same considerations hold if $\mathfrak{p}$ is a finite prime divisor, or $\mathscr{K}_{\mathfrak{p}_\infty}$ is not a field. If however $\mathscr{K}_{\mathfrak{p}_\infty}$ is a field, then $J_{\mathfrak{p}_\infty}(S)$, defined as the number of negative eigenvalues of the $\mathfrak{p}_\infty$ component of S, is an invariant for equivalence in $\mathscr{D}_{\mathfrak{p}_\infty}$. We call $J_{\mathfrak{p}_\infty}(S)$ the *signature* of S at $\mathfrak{p}_\infty$. Also

$$0 \leqslant J_{\mathfrak{p}_\infty}(S) \leqslant mn. \tag{43}$$

If Δ is the auxiliary matrix at $\mathscr{D}_{\mathfrak{p}_\infty}$, then

$$d_{\mathfrak{p}_\infty} = \left(\frac{|S|, \mathscr{K}_0}{\mathfrak{p}_0}\right) = (-1)^{J_{\mathfrak{p}_\infty}(S)} \left(\frac{|\Delta|, \mathscr{K}_0}{\mathfrak{p}_0}\right). \tag{44}$$

In view of Theorem 10 the only invariants for equivalence are n, $d_{\mathfrak{p}}$ and $\{J_{\mathfrak{p}_{\infty i}}(S)\}$, $i = 1, \ldots, t$, and they can be assigned arbitrarily so long as they satisfy (42) and (43).

The details regarding representation, multiplicative equivalence, and multiplicative representation can all be fixed up easily.

A form, or equivalently a symmetric matrix, is said to be *definite* if at every infinite prime spot of $\mathcal{K}_0$ the $\mathfrak{p}$ component of S is a definite hermitian matrix. It follows therefore that $\mathcal{K}_0$ is *totally real* and that $\mathcal{K}$ must be *totally complex*. Hence θ^2 is a totally negative number of $\mathcal{K}_0$.

5. Skew symmetric forms. In the last section we had studied the problem of equivalence of quadratic forms by assuming that the involution in $\mathfrak{M}_n(\mathcal{D})$ is the one that is extended from $\mathcal{D}$. If we want to study the problem of equivalance under any involution of $\mathfrak{M}_n$, it is necessary — as is evident from (10) — to study not merely quadratic forms but also alternating or skew forms, skew with regard to the involution in $\mathcal{D}$. This presents no difficulty in case $\mathcal{D}$ has an involution of the second kind. For if S and T are two skew symmetric (with regard to the extended involution) n-rowed matrices over $\mathcal{D}$, and

$$\tilde{V}\, S\, V = T,$$

then we put $S_1 = \theta\, S$, $T_1 = \theta\, T$, where θ is defined by (2). Then

$$\tilde{V}\, S_1\, V = T_1,$$

and $\tilde{S}_1 = S_1$, $\tilde{T}_1 = T_1$. Conversely if S_1 and T_1 are equivalent, then so are S and T. Thus in case of involutions of the second kind the problems can be reduced to the case of symmetric matrices.

This is *no longer true* if $\mathcal{D}$ is a quaternion division algebra with an involution of the first kind. We shall consider the case of a quaternion division algebra $\mathcal{D}$, whose centre is the completion at a finite prime divisor of $\mathcal{K}$. We denote this by $\mathcal{K}$. If we take the division algebra defined in (11), then an element z is skew if

$$\xi = a_1\, \omega_1 + \dots + a_3\, \omega_3,$$

and then $|\,\xi\,| = -\, a_1^2\, \alpha - a_2^2\, \beta + a_3^2\, \alpha\, \beta$.

Let S and T be two non-singular skew symmetric matrices over $\mathscr{D}$. If they are equivalent, then because of (13), $|S|/|T| \in \mathscr{K}^{*2}$. We shall now prove

THEOREM 12. *The complete system of invariants for equivalence of skew symmetric matrices are n, the number of rows and the coset of $\mathscr{K}^*/\mathscr{K}^{*2}$ to which the reduced norm belongs.*

The proof depends on several lemmas and we prove them below.

LEMMA 2. *If $\tilde{\omega} = -\omega$ and $\tilde{\xi} = -\xi$, then the equation $\tilde{x}\,\omega\, x = \xi$ has a solution x in $\mathscr{D}$ if and only if $|\omega|/|\xi| \in \mathscr{K}^{*2}$.*

PROOF. Let $|\omega| = t^2\,|\xi|$, $t \in \mathscr{K}^*$. By the theorems of Hasse and Witt, there exists a ζ in $\mathscr{D}$ with $|\zeta| = t$. Put $\xi_1 = \tilde{\zeta}\,\xi\,\zeta$. Then $|\xi_1| = |\omega|$, so that it is enough to prove the above lemma in case $|\omega| = |\xi|$. Since ω and ξ are both skew symmetric, there exists $x \in \mathscr{D}$ such that $x^{-1}\omega x = \xi$. This may be written $\tilde{x}\,\omega\, x = p\,\xi$, where $p = |x|$. It is therefore enough to prove the lemma in case $\xi = p\,\omega$ for some $p \in K$.

Consider now the quadratic extension $\mathscr{K}(\omega)$. It is separable over $\mathscr{K}$, since $\mathscr{K}$ has characteristic $\neq 2$. Since $\omega^2 \in \mathscr{K}$, we see that for any λ in $\mathscr{K}(\omega)$, its conjugate over $\mathscr{K}$ is precisely $\tilde{\lambda}$. If $a \in \mathscr{K}$ is norm of λ in $\mathscr{K}(\omega)$, then

$$\tilde{\lambda}\,\omega\,\lambda = a\,\omega.$$

Thus if $p \in \mathscr{K}$ is norm in $\mathscr{K}(\omega)/\mathscr{K}$, our contention is established. Suppose it is not a norm. It is then easy to see that there exists a $\mu \in \mathscr{K}$, which is not norm of an element in $\mathscr{K}(\omega)/\mathscr{K}$, such that

$$\tilde{\lambda}\,\omega\,\lambda = \mu\,\omega,$$

for some λ in $\mathscr{D}$. Now $\mu = p\,\mu/p$. By local classfield theory, μ/p is a norm in $\mathscr{K}(\omega)/\mathscr{K}$ and our assertion in Lemma 2 is proved.

COROLLARY. *A binary form $\tilde{x}\,\xi_1\, x + \tilde{y}\,\xi_2\, y$ represents zero non-trivially if and only if $|\xi_1\,\xi_2| \in \mathscr{K}^{*2}$.*

We now prove

LEMMA 3. *Every quaternary form represents zero non-trivially.*

PROOF. If $S = [\xi_1, \ldots, \xi_4]$ is a quaternary form, then since the skew elements of $\mathscr{D}$ form a vector space of dimension 3 over $\mathscr{K}$, there is a non-trivial relation

$$\sum_i t_i \, \xi_i = 0,$$

$t_i \in \mathscr{K}$. But there exists $x_i \in \mathscr{D}$ such that by Lemma 2, $\tilde{x}_i \, \xi_i x_i = t_i \, \xi_i$. This proves the contention.

LEMMA 4. *Two binary matrices S and T are equivalent if and only if $|S|/|T| \in \mathscr{K}^{*2}$.*

PROOF. If $|S| \in \mathscr{K}^{*2}$, then $|T| \in \mathscr{K}^{*2}$ and then by corollary to Lemma 2, S represents zero non-trivially. Using (18) we have

$$S \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim T.$$

Suppose now that $|S| \notin \mathscr{K}^{*2}$. Let $S \sim \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$, $T \sim \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_4 \end{pmatrix}$. By Lemma 3 there exists a non-trivial relation

$$t_1 \, \xi_1 + \ldots + t_4 \, \xi_4 = 0.$$

t_1 and t_2 cannot both be zero, for then it would mean that T represents zero non-trivially and $|T| \in K^{*2}$. Similarly t_3, t_4 are not both zero. Thus there is a $\xi \in \mathscr{D}$, which both S and T represent. Hence

$$S \sim \begin{pmatrix} \xi & 0 \\ 0 & \xi_5 \end{pmatrix}, \quad T \sim \begin{pmatrix} \xi & 0 \\ 0 & \xi_6 \end{pmatrix}.$$

Thus $|\xi_5|/|\xi_6| \in \mathscr{K}^{*2}$, and Lemma 2 shows that Lemma 4 is true.

LEMMA 5. *A ternary matrix $S = [\xi_1, \xi_2, \xi_3]$ represents zero non-trivially if and only if there exists a $\xi \in \mathscr{D}$ with*

$$\begin{cases} \tilde{\xi} = -\xi, \\ |\xi_1 \, \xi_2 \, \xi_3|/|\xi| \in \mathscr{K}^{*2}, \end{cases}$$

Proof. If $S[x] = 0$ is a non-trivial representation, then at least one of x_1, x_2, x_3, is not zero. Let $x_1 \neq 0$. This means that $[\xi_2, \xi_3]$ represents ξ_1. Thus $[\xi_2, \xi_3]$ is equivalent to $[\xi_1, \xi_4]$ and our contention is proved with ξ_4 instead of ξ.

Let the conditions above be satisfied. Then by Lemma 4, $[\xi_2, \xi_3]$ is equivalent to $[\xi_1, \xi]$. Our Lemma is thereby proved.

In order to prove Theorem 11 we use induction on n. If $n = 1$, then Lemma 2 proves the theorem. Let $n > 1$ and the theorem be proved for $n - 1$ instead of n. In view of Lemma 4 we may take $n > 2$. Take S and T in diagonal forms. Then S and T, because of Lemma 3, represent t, the first diagonal element of T. Thus

$$S \sim \begin{pmatrix} t & 0 \\ 0 & S_1 \end{pmatrix}, \quad T \sim \begin{pmatrix} t & 0 \\ 0 & T_1 \end{pmatrix}.$$

Therefore $|S_1|/|T_1| \in \mathscr{K}^{*2}$. Induction hypothesis applies to S_1 and T_1, and our theorem is established.

The results concerning representations can be easily worked out.

Let us now consider the ordinary algebra of real quaternions, where $\mathscr{K}$ is the real number field. Since $|\omega| > 0$, it follows that for any two skew elements ω, ω' in $\mathscr{D}$, there exists $x \in \mathscr{D}$ with

$$\tilde{x}\, \omega\, x = \omega'.$$

Therefore the canonical form for a skew symmetric matrix is

$$\begin{cases} \begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & i \end{bmatrix}, & \text{if } n \text{ is odd,} \\[2ex] \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}, & \text{if } n \text{ is even,} \end{cases} \tag{45}$$

i being one of the basis elements of $\mathscr{D}$ over $\mathscr{K}$ $\tilde{i} = -i$.

It is very interesting to study the orthogonal group of the matrices in (45). They are generalizations of the symplectic group.

Let $\mathscr{D}$ be a quaternion algebra over the real number field which is not a division algebra. Every element α in $\mathscr{D}$ is a two-rowed matrix

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

a, b, c, d in $\mathscr{K}$. Then $\tilde{\alpha}$ is given by

$$\alpha = J^{-1}\alpha' J, \tag{46}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If α is skew, that is $\tilde{\alpha} = -\alpha$, then $J\alpha = \alpha_1$ is a symmetric two-rowed matrix. α' denotes the transposed matrix.

If α and β are skew symmetric in $\mathscr{D}$, then $J\alpha = \alpha_1$ and $J\beta = \beta_1$ are symmetric ordinary two-rowed matrices. So α and β are equivalent in $\mathscr{D}$ if and only if α_1 and β_1 are equivalent in $\mathfrak{M}_2(K)$. Put now

$$\alpha = -J, \quad \beta = J.$$

Then $|\alpha| = |\beta|$, but they are not equivalent. This shows that Theorem 12 is false in case $\mathscr{D}$ is not a division algebra.

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ON THE COMPACTIFICATION OF THE SIEGEL SPACE

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INTRODUCTION. Let $\mathscr{H}_n$ be the generalized upper half-plane of dimension $\nu = \frac{1}{2}n(n+1)$ and M_n be Siegel's modular group of degree n operating on $\mathscr{H}_n$. Our purpose is to obtain a suitable compactification of the quotient space $\mathscr{V}_n = M_n \backslash \mathscr{H}_n$.

Introducing the notion of V-manifold, which is a generalization of the notion of ordinary complex analytic manifold, I have proved in my former paper [3] that in case $n = 2$, $\mathscr{V}_2$ can be completed to a compact V-manifold $\overline{\mathscr{V}}_2$. $\overline{\mathscr{V}}_2$ has a structure as follows :

$$\overline{\mathscr{V}}_2 = \mathscr{V}_2 \cup \mathscr{W}_1 \cup \mathscr{W}_{12} \cup \mathscr{V}_0,$$

$\mathscr{W}_1$, $\mathscr{W}_{12}$, $\mathscr{V}_0$ denoting the V-manifold of dimension 2, 1, 0 consisting of the points at infinity which correspond to the matrices symbolically written as $\begin{bmatrix} * & * \\ * & \infty \end{bmatrix}$, $\begin{bmatrix} \infty & * \\ * & \infty \end{bmatrix}$, $\begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}$, respectively. Then $\overline{\mathscr{W}}_1 = \mathscr{W}_1 \cup \mathscr{W}_{12} \cup \mathscr{V}_0$ has a structure like a fibre space with base space $\overline{\mathscr{V}}_1 = \mathscr{V}_1 \cup \mathscr{V}_0$, the base space and the fibres being both homeomorphic to $\overline{\mathbf{C}}$ (Riemann sphere). I have also conjectured that $\overline{\mathscr{V}}_2$ may be embeddable in a projective space as an algebraic subvariety. In this respect I was informed recently that W. L. Baily proved this conjecture by a method similar to that of Kodaira†, proving the existence of a positive definite complex line bundle on $\overline{\mathscr{V}}_2$.

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†K. Kodaira: On Kähler varieties of restricted type, *Proc. of Nat. Acad Sci.* 40 (1954). See also W. L. Baily, On the quotient of an analytic manifold by a group of analytic homeomorphisms, *ibid.*

But this procedure of compactification makes use of Minkowski's reduction theory of quadratic forms in its detail, which is particularly simple in case $n = 2$. It becomes very complicated for $n > 2$, mainly because of the fact that the number of conditions for reduced forms is strictly larger than $\nu = \frac{1}{2} n(n+1)$ in case $n > 2$. On the other hand, Siegel [5] gives a method of compactification of the fundamental region $\mathbf{F}_n$ of M_n for general $n \geqslant 2$. But it is difficult to decide whether his method gives rise to a compactification of $\mathscr{V}_n$ as a V-manifold or not.

There is, however, still another possibility of compactification which is much simpler and seems to be more useful for the theory of modular forms. Namely, as I stated above, $\overline{\mathscr{W}}_1$ has a structure like a fibre space. Let us shrink the fibres to the corresponding points on the base space $\overline{\mathscr{V}}_1$. Then from $\overline{\mathscr{V}}_2$ we obtain a compact space $\mathscr{V}_2^* = \mathscr{V}_2 \cup \mathscr{V}_1 \cup \mathscr{V}_0$; the topological structure of $\mathscr{V}_2^*$ is uniquely determined, since the fibres are compact. Though this $\mathscr{V}_2^*$ is no longer a V-manifold, we can define quite easily an analytic structure on $\mathscr{V}_2^*$ in a certain sense.

In the present paper, I shall give a direct construction of the corresponding compactification $\mathscr{V}_n^*$ for general n. In §1 we shall recall the definition of V-manifold and give a general concept of automorphic forms on a V-manifold. Then, considering $\mathscr{V}_n$ as a V-manifold, we define the faisceau $\mathscr{A}_m^{(n)}$ of germs of modular forms of weight m on $\mathscr{V}_n$. In §2 we shall construct the compactified space

$$\mathscr{V}_n^* = \mathscr{V}_n \cup \mathscr{V}_{n-1} \cup \dots \cup \mathscr{V}_0,$$

using some lemmas on Siegel's reduction theory, which are generalizations of those given in [3]. In §3 we shall define the faisceau $\mathscr{A}_m^{(n)*}$ of germs of modular forms of weight m on $\mathscr{V}_n^*$, combining all $\mathscr{A}_m^{(r)}$ $(0 \leqslant r \leqslant n)$. The definition depends essentially on the generalization of the operator Φ, introduced by Siegel [4], to the local modular forms. This can be done by analysing the properties of Fourier coefficients of the local modular forms (Theorem 2) and also by the result of Koecher [2]. Then, in particular, $\mathscr{A}_0^{(n)*}$, which may be called the faisceau of germs of holomorphic functions on $\mathscr{V}_n^*$, defines the analytic structure of $\mathscr{V}_n^*$. I do not here intend to investigate the properties of this analytic space $\mathscr{V}_n^*$ any further. So it is still an open problem whether this $\mathscr{V}_n^*$ can be embedded in a projective space or not.

NOTATIONS. Matrices are denoted by capital roman letters and vectors by small german letters. $Z = X + iY$ denotes always a complex matrix with real part X and imaginary part Y. The components of matrices or vectors are denoted by the corresponding small roman letters; thus y_{kl} denotes the (k, l)–component of Y, and in case Y is symmetric, we put $y_k = y_{kk}$. E and 0 denote respectively the unit matrix and the zero matrix (or vector) of various types. For a symmetric matrix Y we use the notation $Y[Q] = {}^tQYQ$, ${}^tQ$ denoting the transposed of Q. $|A|$ denotes the determinant of a square matrix A.

1. Faisceau of germs of automorphic forms on a V-manifold. We shall first recall the notion of (complex analytic) V-manifold, which is a generalization of the notion of ordinary complex analytic manifold [3]. Let $\mathscr{V}$ be a connected Hausdorff space. We mean by a *local uniformizing system* (abbreviated in the following as l.u.s.) $\{\tilde{U}, G, \phi\}$ for an open set $U \subset \mathscr{V}$, a collection of the following objects:

$\tilde{U}$: a domain in the complex n-space $\mathbf{C}^n$,

G : a finite group of (complex) analytic automorphisms of $\tilde{U}$,

ϕ : a continuous map from $\tilde{U}$ onto U such that $\phi \circ \sigma = \phi$ for all $\sigma \in G$, inducing a homeomorphism from the quotient space $G \backslash \tilde{U}$ onto U.

Let $\{\tilde{U}, G, \phi\}$, $\{\tilde{U}', G', \phi'\}$ be l.u.s. for U, U' respectively, and let $U \subset U'$. By an *injection* λ from $\{\tilde{U}, G, \phi\}$ into $\{\tilde{U}', G', \phi'\}$ we mean a (complex) analytic isomorphism λ from $\tilde{U}$ onto an open subdomain of $\tilde{U}'$, such that for any $\sigma \in G$ there exists $\sigma' \in G'$ satisfying the relation $\lambda \circ \sigma = \sigma' \circ \lambda$, and that $\phi = \phi' \circ \lambda$. Then σ' is uniquely determined by σ, and the correspondence $\sigma \rightarrow \sigma'$ is an isomorphism from G onto a subgroup of G' consisting of those $\sigma' \in G'$ such that $\sigma'(\lambda(\tilde{U})) = \lambda(\tilde{U})$. Clearly for any $\sigma' \in G'$, $\mu = \sigma' \circ \lambda$ becomes also an injection from $\{\tilde{U}, G, \phi\}$ into $\{\tilde{U}', G', \phi'\}$; it can be proved conversely that any injection μ from $\{\tilde{U}, G, \phi\}$ into $\{\tilde{U}', G', \phi'\}$ is given in this form with some $\sigma' \in G'$. In particular, all the injections from $\{\tilde{U}, G, \phi\}$ into itself are given by $\sigma \in G$.

Now $\mathscr{V}$ is called a V-manifold (with a defining family $\mathfrak{F}$), if there exists a family $\mathfrak{F}$ of l.u.s. for open sets in $\mathscr{V}$ satisfying the following conditions.

(I). Let $\{\tilde{U}, G, \phi\}$, $\{\tilde{U}', G', \phi'\} \in \mathfrak{F}$ be l.u.s. for U, U' respectively, and let $U \subset U'$. Then there exists an injection λ from $\{\tilde{U}, G, \phi\}$ into $\{\tilde{U}', G', \phi'\}$. ($\lambda$ is uniquely determined up to $\sigma' \in G'$ in the above sense.)

(II). The open sets U for which there exist l.u.s. $\{\tilde{U}, G, \phi\}$ in $\mathfrak{F}$ form a basis of open sets in $\mathscr{V}$.

Two families $\mathfrak{F}$, $\mathfrak{F}'$ of l.u.s. are said to be equivalent if $\mathfrak{F} \cup \mathfrak{F}'$ satisfies condition (I); equivalent families are regarded as defining one and the same V-manifold structure of $\mathscr{V}$. We shall fix in the following a defining family $\mathfrak{F}$ once for all.

Now let us give the definition of automorphic forms on a V-manifold $\mathscr{V}$. Suppose that for each injection $\lambda : \{\tilde{U}, G, \phi\} \rightarrow \{\tilde{U}', G', \phi'\}$ in condition (I), there is given a function u_λ on $\tilde{U}$ satisfying the following conditions:

(i). u_λ is holomorphic and $\neq 0$ on $\tilde{U}$.

(ii). If λ: $\{\tilde{U}, G, \phi\} \rightarrow \{\tilde{U}', G', \phi'\}$, and λ': $\{\tilde{U}', G', \phi'\} \rightarrow \{\tilde{U}'', G'', \phi''\}$ are injections in (I), we have

$$u_{\lambda' \circ \lambda}(\tilde{p}) = u_{\lambda'}(\lambda(\tilde{p}))\, u_\lambda(\tilde{p}), \text{ for } \tilde{p} \in \tilde{U}, \tag{1}$$

the composed map $\lambda' \circ \lambda$ being an injection: $\{\tilde{U}, G, \phi\} \to \{U'', G'', \phi''\}$.

Then, in particular, we have for $\sigma, \tau \in G$ (which can be considered as injections from $\{\tilde{U}, G, \phi\}$ onto itself)

$$u_{\sigma\circ\tau}(\tilde{p}) = u_\sigma(\tau(\tilde{p}))\, u_\tau(\tilde{p}), \tag{2}$$

and also if $\lambda: \{\tilde{U}, G, \phi\} \to \{\tilde{U}', G', \phi'\}$ is an injection, and $\sigma \in G$ $\sigma' \in G'$ are such that $\lambda \circ \sigma = \sigma' \circ \lambda$, we have

$$u_\lambda(\sigma(\tilde{p})) = u_\lambda(\tilde{p})\, \frac{u_{\sigma'}(\lambda(\tilde{p}))}{u_\sigma(\tilde{p})}. \tag{3}$$

An important example of such a system $\{u_\lambda\}$ is given by (a power of) the Jacobian J_λ.

For $\{\tilde{U}, G, \phi\} \in \mathfrak{F}$, we denote by $A_{\tilde{U}}$ the module of automorphic forms on $\tilde{U}$ with respect to $\{u_\sigma\}$ ($\sigma \in G$), i.e. the module of holomorphic functions $\tilde{f}$ on $\tilde{U}$ satisfying the conditions

$$\tilde{f}(\sigma(\tilde{p})) = \tilde{f}(\tilde{p})\, u_\sigma(\tilde{p})^{-1} \quad \text{for } \sigma \in G. \tag{4}$$

Next, let $\{\tilde{U}, G, \phi\}$, $\{\tilde{U}', G', \phi'\}$ be l.u.s. in $\mathfrak{F}$ for U, U', respectively, $U \subset U'$ and λ be an injection $\{\tilde{U}, G, \phi\} \to \{\tilde{U}', G', \phi'\}$. Then for $\tilde{f}' \in A_{\tilde{U}'}$ we have by (3), (4), $\tilde{f}(\tilde{p}) = \tilde{f}'(\lambda(\tilde{p}))\, u_\lambda(\tilde{p}) \in A_{\tilde{U}}$ and the correspondence

$$\tilde{f}' \to \tilde{f} = (\tilde{f}' \circ \lambda)\, u_\lambda \tag{5}$$

is uniquely determined by $\{\tilde{U}, G, \phi\}$ and $\{\tilde{U}', G', \phi'\}$, independently of the choice of the injection λ. This correspondence defines a canonical homomorphism from $A_{\tilde{U}'}$ into $A_{\tilde{U}}$, which is an isomorphism into, and satisfies the transitivity condition for composed injections by (1). Hence the system of modules $\{A_{\tilde{U}}\}$ ($\{\tilde{U}, G, \phi\} \in \mathfrak{F}$) together with the canonical homorphism (5) defines a faisceau $\mathscr{A}$ on $\mathscr{V}$, which we call the *faisceau of germs of automorphic forms* with respect to $\{u_\lambda\}$ on $\mathscr{V}$.†

† We can also construct an analogue of a complex line bundle on $\mathscr{V}$ corresponding to the system $\{u_\lambda\}$.

Let f be a section of $\mathscr{A}$ on an open set U_1 in $\mathscr{V}$. Then for each $\{\tilde{U}, G, \phi\} \in \mathfrak{F}$ such that $\phi(\tilde{U}) \subset U_1$, there corresponds uniquely an $\tilde{f} \in A_{\tilde{U}}$, and the system of these $\tilde{f} \in A_{\tilde{U}}$ satisfies the following condition. Namely, if λ is an injection: $\{\tilde{U}, G, \phi\} \to \{\tilde{U}', G', \phi'\}$, these being l.u.s. in $\mathfrak{F}$ such that $\phi(\tilde{U}) \subset \phi'(\tilde{U}') \subset U_1$, and if $\tilde{f}, \tilde{f}'$ are the corresponding elements of $A_{\tilde{U}}$, $A_{\tilde{U}'}$, respectively, then $\tilde{f} = (\tilde{f}' \circ \lambda)\, u_\lambda$. Conversely, any system of $\tilde{f} \in A_{\tilde{U}}$ ($\{\tilde{U}, G, \phi\} \in \mathfrak{F}$, $\phi(\tilde{U}) \subset U_1$) satisfying this consistency condition defines a section f of $\mathscr{A}$ on U_1 uniquely. In this sense, we call a section of $\mathscr{A}$ on U_1 a (local) *automorphic form* on U_1 with respect to $\{u_\lambda\}$. In particular, for $u_\lambda = 1$, we have the faisceau $\mathscr{A}_0$ of germs of holomorphic functions on $\mathscr{V}$ and as a section of $\mathscr{A}_0$ a *holomorphic function* on an open subset of $\mathscr{V}$.

Two systems $\{u_\lambda\}$, $\{u'_\lambda\}$ are said to be equivalent, if there is a function $v_{\tilde{U}}$ for each $\{\tilde{U}\ G, \phi\} \in \mathfrak{F}$, which is holomorphic and $\neq 0$ on $\tilde{U}$, and such that

$$u'_\lambda(\tilde{p}) = u_\lambda(p)\, \frac{v_{\tilde{U}'}(\lambda(\tilde{p}))}{v_{\tilde{U}}(\tilde{p})} \quad \text{for } \tilde{p} \in \tilde{U}. \tag{6}$$

The faisceau of germs of automorphic forms corresponding to equivalent systems are mutually isomorphic and so the corresponding automorphic forms on $\mathscr{V}$ are essentially the same.

Now we consider the Siegel space as a special case of the V-manifold. Let $\mathscr{H}_n$ be the generalized upper half-plane of dimension $\nu = \frac{1}{2}\, n(n+1)$, i.e. the space of all complex symmetric matrices $Z = X + iY$ of degree n with the imaginary parts $Y > 0$, and M_n be Siegel's modular group of degree n operating on $\mathscr{H}_n$ as follows:

$$\sigma(Z) = (AZ+B)\,(CZ+D)^{-1} \text{ for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n \text{ and } Z \in \mathscr{H}_n. \tag{7}$$

Then the quotient space $\mathscr{V}_n = M_n \backslash \mathscr{H}_n$ becomes a V-manifold of dimension ν in the following way. Let $p \in \mathscr{V}_n$ and $\tilde{p} = Z^0 \in \mathscr{H}_n$ be such that $\phi_n(Z^0) = p$, ϕ_n denoting the canonical map $\mathscr{H}_n \to \mathscr{V}_n = M_n \backslash \mathscr{H}_n$. Let G be the isotropy subgroup of M_n at Z^0 and $\tilde{U}$ be a connected (open) neighbourhood of Z^0 such that $\sigma(\tilde{U}) = \tilde{U}$ for $\sigma \in G$ and $\sigma(\tilde{U}) \cap \tilde{U} = \phi$ for $\sigma \notin G$. Then it can be shown easily that $\{\tilde{U}, G, \phi_n\}$ becomes a l.u.s. for $U = \phi_n(\tilde{U})$ and that the family $\mathfrak{F}$ of all these $\{\tilde{U}, G, \phi_n\}$ defines a V-manifold structure on $\mathscr{V}_n$.

Next, let $\{\tilde{U}, G, \phi_n\}$, $\{\tilde{U}', G', \phi_n\}$ be l.u.s. in $\mathfrak{F}$ such that $\phi_n(\tilde{U}) \subset \phi_n(\tilde{U}')$. Then every injection $\lambda: \{\tilde{U}, G, \phi_n\} \to \{\tilde{U}', G', \phi_n\}$ is given by some modular transformation. Let m be a fixed even integer. In case λ is given by $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$, we put

$$u_\lambda(Z) = |CZ + D|^{-m}. \tag{8}$$

The system $\{u_\lambda\}$ thus defined satisfies clearly the conditions (i), (ii). The faisceau $\mathscr{A}_m^{(n)}$ corresponding to $\{u_\lambda\}$ is called the *faisceau of germs of modular forms of weight m on* $\mathscr{V}_n$. Then by what we mentioned above it follows that for any section f of $\mathscr{A}_m^{(n)}$ on an open set $U^{(n)}$ in $\mathscr{V}_n$ there exists uniquely a holomorphic function $\tilde{f}$ on $\phi_n^{-1}(U^{(n)})$ satisfying the relation

$$\tilde{f}(\sigma(Z)) = \tilde{f}(Z)\,|CZ + D|^m, \tag{9}$$

for $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_n$, $Z \in \phi_n^{-1}(U^{(n)})$. We denote $\tilde{f} = f \circ \phi_n$ and call $\tilde{f}$ the *local modular form* of weight m on $\phi_n^{-1}(U^{(n)})$ corresponding to f.

2. Construction of the compactified space $\mathscr{V}_n^*$. We shall construct in this section the compactification $\mathscr{V}_n^*$ of $\mathscr{V}_n$ in joining $\mathscr{V}_r$ $(0 \leqslant r \leqslant n-1)$ to $\mathscr{V}_n$ $(\mathscr{V}_0 = \{p_\infty\}$: one point). For that purpose, let us denote by $\mathbf{F}_n$ Siegel's fundamental region of M_n in $\mathscr{H}_n$†:

† See Siegel [4], §2.

namely $\mathbf{F}_n$ is the set of all $Z = X + iY \in \mathscr{H}_n$, $X = (x_{kl})$, $Y = (y_{kl})$ satisfying the following conditions:

(I) abs $|CZ + D| \geqslant 1$,

$\{C, D\}$ running over a complete system of inequivalent coprime symmetric pairs of matrices.

(II) Y is reduced in the sense of Minkowski, namely

$$Y[\mathfrak{g}_k] \geqslant y_k \quad (1 \leqslant k \leqslant n),$$
$$y_{k,k+1} \geqslant 0 \quad (1 \leqslant k \leqslant n-1),$$

$\mathfrak{g}_k = \begin{bmatrix} g \\ \vdots \\ g_n \end{bmatrix}$ running over all integral vectors such that g_i $(k \leqslant i \leqslant n)$ are coprime.

(III) $-\frac{1}{2} \leqslant x_{kl} \leqslant \frac{1}{2}$ $(1 \leqslant k, l \leqslant n)$.

It is known that these infinite number of conditions are not independent, but are equivalent to some finite number of conditions suitably chosen among them. The matrices $Z \in \mathscr{H}_n$ satisfying these conditions are called reduced in the sense of Siegel.

Let $U^{(r)}$ be an open subset of $\mathscr{V}_r$ $(0 \leqslant r \leqslant n)$ and K be a positive number. We denote by $\widetilde{V}^{(n)}(U^{(r)}, K)$ the set of all

$$Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix}$$

such that

(i) $Z \in \mathbf{F}_n$, (hence $Z_1 \in \mathbf{F}_r$)

(ii) $\phi_r(Z_1) \in U^{(r)}$,

(iii) $y_{r+1} > K$.

In case $r = 0$ or $= n$, we drop the condition (ii) or (iii), respectively, which is meaningless in the corresponding case. We put

$$V^{(n)}(U^{(r)}, K) = \phi_n(\widetilde{V}^{(n)}(U^{(r)}, K)).$$

Now we define $\mathscr{V}_n^*$ as a set-theoretical direct sum as follows

$$\mathscr{V}_n^* = \mathscr{V}_n \cup \mathscr{V}_{n-1} \cup \dots \cup \mathscr{V}_0. \tag{10}$$

Let $p \in \mathscr{V}_r$ and $U^{(r)}$ be an open neighbourhood of p in $\mathscr{V}_r$. Then we define a neighbourhood of p in $\mathscr{V}_n{}^*$ as follows

$$V^*(U^{(r)}, K) = \bigcup_{r \leqslant r' \leqslant n} V^{(r')}(U^{(r)}, K). \tag{11}$$

THEOREM 1. *$\mathscr{V}_n{}^*$ becomes a compact Hausdorff space, on taking the $V^*(U^{(r)}, K)$ as a complete system of neighbourhoods of $p \in \mathscr{V}_r$ in $\mathscr{V}_n{}^*$. Here $U^{(r)}$ runs over a complete system of neighbourhoods of p in $\mathscr{V}_r$ and K runs over a sequence divergent to ∞.*

For the proof (as well as for the need in the subsequent section) we give here some lemmas.

LEMMA 1. *Let $\widetilde{U}_0^{(r)}$ be a bounded set in $\mathbf{F}_r$. Then we can find $K > 0$ such that $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \in \mathscr{H}_n$ is reduced in the sense of Siegel if it satisfies the following conditions:*

(*) $Z_1 \in \widetilde{U}_0^{(r)}$,

(**) $y_{r+1} > K$,

(II′) $Y[\mathfrak{g}_k] \geqslant y_k \quad (r+1 \leqslant k \leqslant n)$,

$\qquad y_{k,k+1} \geqslant 0 \quad (r+1 \leqslant k \leqslant n-1)$,

for any integral vector $\mathfrak{g}_k = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$ *with g_i $(k \leqslant i \leqslant n)$ coprime.*

(III′) $-\frac{1}{2} \leqslant x_{kl} \leqslant \frac{1}{2}$ $(1 \leqslant k \leqslant n, r+1 \leqslant l \leqslant n)$.

PROOF. What we have to show is that, taking K sufficiently large (depending only on $\widetilde{U}_0^{(r)}$), the conditions (*), (**), (II′), (III′) imply the conditions (I), (II), (III).

(I) We can take $\{C, D\}$ in the following form

$$C = \begin{pmatrix} C_0 & 0 \\ 0 & 0 \end{pmatrix}^t \begin{pmatrix} Q & * \end{pmatrix}, D = \begin{pmatrix} D_0 & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} Q & * \end{pmatrix}^{-1}, \tag{12}$$

$\{C_0, D_0\}$ being a coprime symmetric pair of matrices of degree s $(1 \leqslant s \leqslant n)$, $|C_0| \neq 0$ and Q an (n, s)-matrix such that $(Q\ *)$ is unimodular. Then putting

$$S_0 = X[Q] + C_0^{-1} D_0, \quad T_0 = Y[Q],$$

we have

$$\begin{aligned} \text{abs}\,|CZ+D| &= \text{abs}\,|C_0 Z[Q] + D_0| \\ &= \text{abs}\,|C_0|\,\text{abs}\,|S_0 + iT_0|. \end{aligned}$$

As abs $|C_0| \geqslant 1$, abs $|S_0 + iT_0| \geqslant |T_0|$, we have

$$\text{abs}\,|CZ+D| \geqslant |T_0| = |Y[Q]|. \tag{13}$$

Here we can assume furthermore that $Y[Q]$ is reduced in the sense of Minkowski.

Put

$$Q = (\mathfrak{q}_1 \cdots \mathfrak{q}_s), \ \mathfrak{q}_j = \begin{pmatrix} \mathfrak{q}_j^{(1)} \\ \mathfrak{q}_j^{(2)} \end{pmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix}.$$

Then we have

$$\begin{aligned} |Y[Q]| &\geqslant c_s\, Y[\mathfrak{q}_1] \cdots\cdots Y[\mathfrak{q}_s], \\ Y[\mathfrak{q}_j] &= Y_1[\mathfrak{q}_j^{(1)}] + 2\,{}^t\mathfrak{q}_j^{(1)} Y_{12}\, \mathfrak{q}_j^{(2)} + Y_2[\mathfrak{q}_j^{(2)}], \end{aligned} \tag{14}$$

c_s being a constant > 0 depending only on s. Now from (II′) it follows, in particular, that

$$-\frac{y_k}{2} \leqslant y_{kl} \leqslant \frac{y_k}{2} \quad (1 \leqslant k \leqslant r,\ r+1 \leqslant l \leqslant n), \tag{15}$$

namely Y_{12} is in some bounded set and that Y_2 is reduced in the sense of Minkowski. Hence if $\mathfrak{q}_j^{(2)} \neq 0$, we have by (**)

$$Y_2[\mathfrak{q}_j^{(2)}] \geqslant y_{r+1} > K. \tag{16}$$

If $\mathfrak{q}_j^{(2)} = 0$, then $\mathfrak{q}_j^{(1)} \neq 0$ and we have by (*)

$$Y[\mathfrak{q}_j] = Y_1[\mathfrak{q}_j^{(1)}] \geqslant y_1 > \frac{\sqrt{3}}{2}. \tag{17}$$

Hence if $\mathfrak{q}_j^{(2)} \neq 0$ for some j, it follows from (13)—(17) that (*), (**), (II′) imply (I), taking K sufficiently large.

If $\mathfrak{q}_j^{(2)} = 0$, $(1 \leqslant j \leqslant s)$, then $s \leqslant r$ and we may assume that $\{C, D\}$ is of the form

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix},$$

$\{C_1, D_1\}$ being a coprime symmetric pair of matrices of degree r. Then we have by (*)

$$\text{abs}\,|CZ + D| = \text{abs}\,|C_1 Z_1 + D_1| > 1.$$

Hence (1) is again satisfied.

(II) We have only to consider the conditions

$$Y[\mathfrak{g}_k] > y_k \quad (1 < k < r)$$

with

$$\mathfrak{g}_k = \begin{pmatrix} \mathfrak{g}_k^{(1)} \\ \mathfrak{g}_k^{(2)} \end{pmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix},\ \mathfrak{g}_k^{(2)} \neq 0,$$

for all the other conditions in (II) are contained in (*) and (II′). But since we have

$$Y[\mathfrak{g}_k] = Y_1[\mathfrak{g}_k^{(1)}] + 2\,{}^t\mathfrak{g}_k^{(1)}\, Y_{12}[\mathfrak{g}_k^{(2)}] + Y_2[\mathfrak{g}_k^{(2)}],$$

and (Y_1, Y_{12}) is bounded by (*), (II′), the above conditions follow from (*), (**), (II′) by taking K sufficiently large.

(III) is clearly contained in (*) and (III′), q.e.d.

Now let $M_r^{(n)}$ be the subgroup of M_n consisting of all

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ with } \begin{matrix} A = \begin{pmatrix} A_1 & 0 \\ 0 & E \end{pmatrix},\ B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}, \\ C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} D_1 & 0 \\ 0 & E \end{pmatrix}, \end{matrix} \quad \sigma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in M_r, \tag{18}$$

and $\mathfrak{N}$ be the subgroup of M_n consisting of all

$$\sigma = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix}, \text{ with } \begin{matrix} U = \begin{pmatrix} E & U_{12} \\ 0 & U_2 \end{pmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix} : \text{unimodular}, \\ S = \begin{pmatrix} 0 & S_{12} \\ {}^tS_{12} & S_2 \end{pmatrix} \begin{matrix} \}r \\ \}n-r \end{matrix} : \text{symmetric}. \end{matrix} \tag{19}$$

The operations of $\sigma \in M_r^{(n)}$ or $\mathfrak{N}$ on $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \in \mathscr{H}_n$ are given as follows :

$$M_r^{(n)}: Z \to \begin{pmatrix} \sigma_1(Z_1) & (A_1 - \sigma_1(Z_1)C_1)Z_{12} \\ & Z_2 - ((C_1 Z_1 + D_1)^{-1} C_1)[Z_{12}] \end{pmatrix}, \tag{20}$$

$$\mathfrak{N} : Z \to Z[U] + S$$

$$= \begin{pmatrix} Z_1 & Z_{12} U_2 + Z_1 U_{12} + S_{12} \\ & Z_2[U_2] + {}^tU_2 {}^tZ_{12} U_{12} + {}^tU_{12} Z_{12} U_2 + Z_1[U_{12}] + S_2 \end{pmatrix}. \tag{21}$$

We denote by $\mathfrak{G}$ the group composed of $M_r^{(n)}$ and $\mathfrak{N}$. Then it can be seen easily that $\mathfrak{N}$ is a normal subgroup of $\mathfrak{G}$ so that

$$\mathfrak{G} = M_r^{(n)} \mathfrak{N},\ M_r^{(n)} \cap \mathfrak{N} = \{E\}. \tag{22}$$

LEMMA 2. *Let $U^{(r)}$, K be as above and let $\widetilde{W}$ be a bounded set in $\mathscr{H}_r \times \mathbf{C}^{r(n-r)} = \{(Z_1, Z_{12})\}$ such that $(Z_1, Z_{12}) \in \widetilde{W}$ implies $\phi_r(Z_1) \in U^{(r)}$. Then there exists a positive definite symmetric matrix Y_2^0 of degree $n-r$ such that for any $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$ with $(Z_1, Z_{12}) \in \widetilde{W}$, $Z_2 = X_2 + iY_2$, $Y_2 > Y_2^0$, we have $\sigma(Z) \in \widetilde{V}^{(n)}(U^{(r)}, K)$ with some $\sigma \in \mathfrak{G}$.*

PROOF. We may assume that $U^{(r)}$ is relatively compact in $\mathscr{V}_r$ and K is sufficiently large so that we can apply Lemma 1 to $\widetilde{U}_0^{(r)} = \phi_r^{-1}(U^{(r)}) \cap \mathrm{F}_r$. Then the matrices in $\widetilde{V}^{(n)}(U^{(r)}, K)$ are characterized by the conditions (*), (**), (II′), (III′). On the other hand, it follows by the condition on $\widetilde{W}$ that for any $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$ with $(Z_1, Z_{12}) \in \widetilde{W}$, there exists $\sigma_1 \in M_r$ such that $Z'_1 = \sigma_1(Z_1) \in \widetilde{U}_0^{(r)}$. Let σ be the corresponding transformation in $M_r^{(n)}$ and let $Z' = \begin{pmatrix} Z'_1 & Z'_{12} \\ & Z'_2 \end{pmatrix} = \sigma(Z)$. Since $\widetilde{W}$ is bounded, the number of such $\sigma_1 \in M_r$ is also bounded. Hence for any given $Y_2^{0\prime} > 0$, we can find $Y_2^0 > 0$ such that, for the above Z, Z', $Y_2 > Y_2^0$ implies $Y'_2 > Y_2^{0\prime}$. Hence we may assume from the beginning that $(Z_1, Z_{12}) \in \widetilde{W}$ implies $Z_1 \in \widetilde{U}_0^{(r)}$.

Now by Minkowski's reduction theory it is clear that for any $Z \in \mathscr{H}_n$ there exists $\sigma \in \mathfrak{R}$ such that $Z' = \sigma(Z)$ satisfies the conditions (II'), (III'). For $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$ such that $(Z_1, Z_{12}) \in \widetilde{W}$, we have here $Z'_1 = Z_1 \in \widetilde{U}_0^{(r)}$, i.e. (*), and

$$y'_{r+1} = Y\,[\mathfrak{g}_{r+1}],$$

$\mathfrak{g}_{r+1} = \begin{pmatrix} \mathfrak{g}^{(1)} \\ \mathfrak{g}^{(2)} \end{pmatrix}$ being an integral vector with $\mathfrak{g}^{(2)}$ coprime. Hence

$$\begin{aligned} y'_{r+1} &= Y_1\,[\mathfrak{g}^{(1)}] + 2\,{}^t\mathfrak{g}^{(1)}\,Y_{12}\,\mathfrak{g}^{(2)} + Y_2\,[\mathfrak{g}^{(2)}] \\ &= Y_1\,[\mathfrak{g}^{(1)} + Y_1^{-1}\,Y_{12}\,\mathfrak{g}^{(2)}] + (Y_2 - Y_1^{-1}\,[Y_{12}])\,[\mathfrak{g}^{(2)}]. \end{aligned}$$

Since $\mathfrak{g}^{(2)} \neq 0$, we have $y'_{r+1} > K$, i.e. (**), for $Y_2 > Y_2^0$, taking Y_2^0 such that $Y_2^0 > Y_1^{-1}\,[Y_{12}] + KE$ for all $(Z_1, Z_{12}) \in \widetilde{W}$, q.e.d.

By Lemma 1 it follows also that if K is sufficiently large $\widetilde{V}^{(n)}(U^{(r)}, K)$ has the following property : if $Z \in \widetilde{V}^{(n)}(U^{(r)}, K)$ and $\sigma(Z)$ $(\sigma \in M_n)$ is reduced, then $\sigma \in \mathfrak{S}$. On the other hand, by the argument in the first half of the above proof, we can assume that $\widetilde{W}$ has the following properties : if $(Z_1, Z_{12}) \in \widetilde{W}$, then $Z \in \widetilde{U}_0^{(r)}$ and also if $(Z_1, Z_{12}) \in \widetilde{W}$ and $Z'_1 = \sigma_1(Z_1)$ $(\sigma_1 \in M_r)$ is reduced, then $(Z_1, Z_{12}) \in \widetilde{W}$. Hence from the above proof we also obtain the following refinement of Lemma 2.

Lemma 2'. *The assumptions being as in Lemma 2 there exists $Y_2^0 > 0$ of degree $n - r$ such that if $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$, $(Z_1, Z_{12}) \in \widetilde{W}$, $Z_2 = X_2 + iY_2$, $Y_2 > Y_2^0$ and $\sigma(Z)$ $(\sigma \in M_n)$ is reduced, then $\sigma \in \mathfrak{S}$ and $\sigma(Z) \in \widetilde{V}^{(n)}(U^{(r)}, K)$.*

In the following we shall denote by $\widetilde{V}(\widetilde{W}, Y_2^0)$ the set of all $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix} \in \mathscr{H}_n$ such that $(Z_1, Z_{12}) \in \widetilde{W}$, $Z_2 = X_2 + iY_2$, $Y_2 > Y_2^0$.

Now let us prove Theorem 1. To prove that the system of neighbourhoods $\{V^*(U^{(r)}, K)\}$ defines a Hausdorff topology on $\mathscr{V}^*_n$, it will be sufficient to show that for any $V^*(U^{(r)}, K)$ there exists some $V^*(U^{(r)\prime}, K')$ such that any $p' \in V^*(U^{(r)\prime}, K')$ has a neighbourhood in our sense contained in $V^*(U^{(r)}, K)\lambda$; for all the other conditions are quite obvious.

Let $U^{(r)}$ be a relatively compact, open neighbourhood of p in $\mathscr{V}_r$ and let $\tilde{U}^{(r)}$ be a bounded open set in $\mathscr{H}_r$ such that $\tilde{U}_0^{(r)} \subset \tilde{U}^{(r)} \subset \phi_r^{-1}(U^{(r)})$, $\tilde{U}_0^{(r)} = \phi_r^{-1}(U^{(r)}) \cap \mathrm{F}_r$. Let L be an upper bound of y_r for $Z_1 \in \tilde{U}^{(r)}$, y_r denoting the (r, r)-component of the imaginary part of Z_1. Put

$$\tilde{W}_{r,n} = \{(Z_1, Z_{12}); Z_1 \in \tilde{U}^{(r)}, Z_{12} = (x_{kl} + iy_{kl}), \\ -1 < x_{kl} < 1, -L < y_{kl} < L\,(1 \leqslant k \leqslant r, r+1 \leqslant l \leqslant n)\}.$$

Then, $\tilde{W} = \tilde{W}_{r,n}$ satisfying the condition of Lemma 2, we can find $Y_2^0 > 0$ of degree $n - r$ such that $\tilde{V}(\tilde{W}_{r,n}, Y_2^0)$ has the property described in Lemma 2′. Then for $V_{r,n} = \phi_n(\tilde{V}(\tilde{W}_{r,n}, Y_2^0))$ we have

$$\phi_n^{-1}(V_{r,n}) \cap \mathrm{F}_n \subset \tilde{V}^{(n)}(U^{(r)}, K). \tag{23}$$

On the other hand, we have by the definition

$$\tilde{V}^{(n)}(U^{(r)}, K'_{r,n}) \subset \tilde{V}(\tilde{W}_{r,n}, Y_2^0) \tag{24}$$

for some $K'_{r,n}$.

We define similarly $\tilde{W}_{r,r'}$, $V_{r,r'}$, $K'_{r,r'}$ for $r < r' \leqslant n$ and put $V_{r,r} = U^{(r)}$, $K' = \max_{r < r' \leqslant n} K'_{r,r'}$. Then any $p' \in V^*(U^{(r)}, K')$ has a neighbourhood in our sense contained in $V^*(U^{(r)}, K)$. In fact, let $p' \in \mathscr{V}_{r'}$. Then by (24) $p' \in V^{(r')}(U^{(r)}, K') \subset V_{r,r'}$ and $V^*(V_{r,r'}, K)$ is a neighbourhood of p' in our sense. Since by (23), $\tilde{V}^{(r'')}(V_{r,r'}, K) \subset \tilde{V}^{(r'')}(U^{(r)}, K)$ for $r' \leqslant r'' \leqslant n$, we have $V^*(V_{r,r'}, K) \subset V^*(U^{(r)}, K)$. This proves our statement.

To prove the compactness of $\mathscr{V}_n^*$ it will be enough to show that any sequence $\{p_\nu\}$ from $\mathscr{V}_n$ has a cluster point in $\mathscr{V}_n^*$. Let $Z_\nu = (x_{kl}^{(\nu)} + iy_{kl}^{(\nu)})$, $\phi_n(Z_\nu) = p_\nu$, $Z_\nu \in \mathbf{F}_n$. Then, for some r, $0 \leqslant r \leqslant n$, $\{y_r^{(\nu)}\}$ is bounded and $\{y_{r+1}^{(\nu)}\}$ is not bounded; this means in case $r = 0$ that

$\{y_1^{(\nu)}\}$ is not bounded and in case $r = n$ that $\{y_n^{(\nu)}\}$ is bounded. Then, denoting $Z_\nu = \begin{pmatrix} Z_1^{(\nu)} & Z_{12}^{(\nu)} \\ & Z_2^{(\nu)} \end{pmatrix}$, $\{Z_1^{(\nu)}\}$ is bounded and so has a subsequence convergent to some $Z_1^0 \in \mathscr{H}_r$. Then it is clear that $\{p_\nu\}$ has a subsequence convergent in our topology to $p = \phi_r(Z_1^0) \in \mathscr{V}_r$.

It should be noted that our topology of $\mathscr{V}_n^*$ induces on $\mathscr{V}_r$ $(0 \leqslant r \leqslant n)$ the pre-assigned topology, i.e. the topology as a quotient space $M_r\backslash\mathscr{H}_r$, and that $\mathscr{V}_r^* = \mathscr{V}_r \cup \mathscr{V}_{r-1} \cup \ldots \cup \mathscr{V}_0 \, (0 \leqslant r \leqslant n)$ can be considered as a sub-space of V_n^*.

3. Faisceau $\mathscr{A}_m^{(n)*}$ of germs of modular forms of weight m on $\mathscr{V}_n^*$. Let m be a fixed even integer and let $\mathscr{A}_m^{(n)}$ be the faisceau of germs of modular forms of weight m on $\mathscr{V}_n$ defined in §1. We shall define in this section the faisceau $\mathscr{A}_m^{(n)*}$ of germs of modular forms of weight m on $\mathscr{V}_n^*$, combining the faisceau $\mathscr{A}_m^{(r)}$ $(0 \leqslant r \leqslant n)$.

Let U be an open set $\neq \phi$ in $\mathscr{V}_n^*$ and let $U = \bigcup_{r_0 \leqslant r \leqslant n} U^{(r)}$, $U^{(r)} = U \cap \mathscr{V}_r$, $U^{(r_0)} \neq \phi$. Let $r_0 \leqslant r < n$ and $\widetilde{W}$ be a bounded open set in $\mathscr{H}_r \times \mathbf{C}^{r(n-r)}$ such that $(Z_1, Z_{12}) \in \widetilde{W}$ implies $\phi_r(Z_1) \in U^{(r)}$. Then, $\{\phi_r(Z_1);\ (Z_1, Z_{12}) \in \widetilde{W}\}$ being relatively compact in $\mathscr{V}_r$, it follows from Lemma 2 that there exists $Y_2^0 > 0$ of degree $n - r$ such that $Z \in \widetilde{V}(\widetilde{W}, Y_2^0)$, i.e. $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$, $(Z_1, Z_{12}) \in \widetilde{W}$, $Z_2 = X_2 + iY_2$, $Y_2 > Y_2^0$, implies $\phi_n(Z) \in U^{(n)}$. Let f_n be a section of $\mathscr{A}_m^{(n)}$ on $U^{(n)}$ and $\widetilde{f}_n = f_n \circ \phi_n$ be the corresponding local modular form of weight m on $\phi_n^{-1}(U^{(n)})$. Then $\widetilde{f}_n$ is defined on $\widetilde{V}(\widetilde{W}, Y_2^0)$ and since $\widetilde{f}_n$ is invariant under the translation:

$$Z_2 \to Z_2 + S_2,$$

$\widetilde{f}_n$ has a Fourier expansion in $\widetilde{V}(\widetilde{W}, Y_2^0)$ of the following form

$$\widetilde{f}_n(Z) = \sum_{T_2} a_{T_2}(Z_1, Z_{12})\, e^{2\pi i \operatorname{Sp}(T_2 Z_2)}, \tag{25}$$

T_2 running over all half-integral symmetric matrices of degree $n - r$ and $a_{T_2}(Z_1, Z_{12})$ denoting a holomorphic function on $\widetilde{W}$ depending on T_2. This series converges absolutely in $\tilde{V}(\tilde{W}, Y_2^0)$ and uniformly absolutely in any bounded set in $\tilde{V}(\tilde{W}, Y_2^0)$.

Now the open set $\tilde{W}$ can be taken arbitrarily under the conditions that it is bounded and that $(Z_1, Z_{12}) \in \tilde{W}$ implies $\phi_r(Z_1) \in U^{(r)}$. If $\tilde{W}$, $\tilde{W}'$ are two open sets in $\mathscr{H}_r \times \mathbf{C}^{r(n-r)}$ of the described type and if $\sigma_{12}(\tilde{W}) \subset \tilde{W}'$ with some $\sigma \in \mathfrak{G}$, σ_{12} denoting the transformation of the space $\mathscr{H}_r \times \mathbf{C}^{r(n-r)} = \{(Z_1, Z_{12})\}$ induced by $\sigma \in \mathfrak{G}$, then by (20), (21) it follows that for any $Y_2^{0\prime} > 0$ we can find $Y_2^0 > 0$ such that $\sigma(\tilde{V}(\tilde{W}, Y_2^0)) \subset \tilde{V}(\tilde{W}', Y_2^{0\prime})$. Hence by the uniqueness of the Fourier expansions we can extend $a_{T_2}(Z_1, Z_{12})$ to a holomorphic function on $\phi_r^{-1}(U^{(r)}) \times \mathbf{C}^{r(n-r)}$. Then we obtain by (9), (20), (21) the following properties of a_{T_2}:

$$a_{T_2}(Z_1, Z_{12} + S_{12}) = a_{T_2}(Z_1, Z_{12}), \tag{26}$$

$$a_{T_2}(Z_1, Z_{12} + Z_1 U_{12}) = a_{T_2}(Z_1, Z_{12})\, e^{-2\pi i \operatorname{Sp}(T_2(Z_1[U_{12}] + 2\,{}^tU_{12}Z_{12}))} \tag{27}$$

$$a_{T_2[{}^tU_2]}(Z_1, Z_{12}) = a_{T_2}(Z_1, Z_{12}\, U_2), \tag{28}$$

$$a_{T_2}(\sigma_1(Z_1), (A_1 - \sigma_1(Z_1) C_1) Z_{12}) = a_{T_2}(Z_1, Z_{12})\; |C_1 Z_1 + D_1|^m \times \\ \times\, e^{2\pi i \operatorname{Sp}(T_2((C_1Z_1 + D_1)^{-1}C_1)[Z_{12}])}, \tag{29}$$

S_{12}, U_{12} being any integral $(r, n-r)$-matrices, U_2 being any unimodular matrices of degree $n-r$ and $\sigma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in M_r$.

By (26) $a_{T_2}(Z_1, Z_{12})$ can be expressed by Fourier series as follows

$$a_{T_2}(Z_1, Z_{12}) = \sum_{T_{12}} b_{T_{12}, T_2}(Z_1)\, e^{2\pi i \operatorname{Sp}({}^tT_{12}Z_{12})}, \tag{30}$$

T_{12} running over all integral $(r, n-r)$-matrices and $b_{T_{12}, T_2}(Z_1)$ denoting a holomorphic function on $\phi_r^{-1}(U^{(r)})$. By (27), (28) it follows that b_{T_{12}, T_2} satisfies the following properties:

$$b_{T_{12} + 2U_{12}T_2, T_2}(Z_1) = b_{T_{12}, T_2}(Z_1)\, e^{2\pi i \operatorname{Sp}(T_2 Z_1[U_{12}] + {}^tT_{12}Z_1U_{12})}, \tag{31}$$

$$b_{T_{12}{}^tU_2, T_2[{}^tU_2]}(Z_1) = b_{T_{12}, T_2}(Z_1). \tag{32}$$

We call an integral $(r, n-r)$-matrix T_{12} a *rational multiple* of T_2, if there exists a rational $(r, n-r)$-matrix P_{12} such that $T_{12} = P_{12}\, T_2$. We denote by $\{T_2\}$ the set of all integral $(r, n-r)$-matrices which are rational multiples of T_2.

THEOREM 2. *Let $1 \leqslant r < n$. If $T_2 \not\geqslant 0$, we have $a_{T_2}(Z_1, Z_{12}) = 0$ (constant zero). Also if $T_2 \geqslant 0$ and $T_{12} \in \{T_2\}$, we have $b_{T_{12}, T_2}(Z_1) = 0$.*

PROOF. Since (30) is absolutely convergent it follows that for any fixed T_{12}, T_2, the following subseries of (30) is absolutely convergent

$$\sum_{U_{12}} b_{T_{12}+2U_{12}T_2, T_2}(Z_1)\, e^{2\pi i \mathrm{Sp}\, {}^t(T_{12}+2U_{12}T_2)Z_{12}}$$

$$= b_{T_{12}, T_2}(Z_1)\, e^{2\pi i \mathrm{Sp}({}^tT_{12}Z_{12})} \sum_{U_{12}} e^{2\pi i \mathrm{Sp}(T_2 Z_1[U_{12}] + {}^tT_{12}Z_1U_{12} + 2T_2{}^tZ_{12}U_{12})},$$

U_{12} running over all integral $(r, n-r)$-matrices. Hence, if $b_{T_{12}, T_2}(Z_1) \neq 0$, the series

$$e^{2\pi i \mathrm{Sp}({}^tT_{12}Z_{12})} \sum_{U_{12}} e^{2\pi i \mathrm{Sp}(Z_1T_2[{}^tU_{12}] + {}^tU_{12}(Z_1T_{12}+2Z_{12}T_2))} \tag{33}$$

is absolutely convergent.

Let U_{12} be especially of the form

$${}^tU_{12} = (\mathfrak{g}, 0 \ldots 0),$$

$\mathfrak{g}$ being any integral $(n-r)$-vector. Then the series

$$\sum_{\mathfrak{g}} e^{-2\pi(y_1 T_2[\mathfrak{g}] + {}^t\mathfrak{b}\mathfrak{g})}$$

is convergent, ${}^t\mathfrak{b}$ denoting the first row vector of $Y_1 T_{12} + 2Y_{12}\, T_2$. But since $y_1 > 0$, this is possible only if $T_2 \geqslant 0$ and $T_2[\mathfrak{g}] = 0$ implies ${}^t\mathfrak{b}\mathfrak{g} = 0$. This proves already the first half of the theorem. Therefore assume that $T_2 \geqslant 0$. Then $T_2[\mathfrak{g}] = 0$ is equivalent to $T_2\, \mathfrak{g} = 0$. Now if $T_2\, \mathfrak{g} = 0$, we have

$$\begin{aligned} {}^t\mathfrak{b}\mathfrak{g} &= (y_{11}, \cdots, y_{1r})\, T_{12}\, \mathfrak{g} + 2\,(y_{1r+1}, \cdots, y_{1n})\, T_2\, \mathfrak{g} \\ &= (y_{11}, \cdots, y_{1r})\, T_{12}\, \mathfrak{g}. \end{aligned}$$

Since Y_1 can vary in some open set in the space of matrices of degree r, ${}^t\mathfrak{h}\mathfrak{g} = 0$ is equivalent to $T_{12}\,\mathfrak{g} = 0$. Therefore T_{12} must be a rational multiple of T_2. This completes the proof.

Conversely, it is clear that if $T_2 \geqslant 0$ and $T_{12} \in \{T_2\}$ the series (33) is absolutely convergent. Let us denote by $\{\{T_2\}\}$ the set of all $T_{12} = 2U_{12}T_2$, U_{12} being any integral $(r, n-r)$-matrices. Then the factor module $\{T_2\}/\{\{T_2\}\}$ is finite; we denote by $T_{12}^{(\nu)}$ $(1 \leqslant \nu \leqslant \nu(T_2))$ a complete system of representatives of $\{T_2\}$ modulo $\{\{T_2\}\}$. Then we have easily the following

COROLLARY 1. *For a fixed $T_2 \geqslant 0$, the module of all holomorphic functions $a_{T_2}(Z_1, Z_{12})$ on $\phi_r^{-1}(U^{(r)}) \times \mathbf{C}^{r(n-r)}$ satisfying the conditions* (26), (27) *is generated by the series* (33) *corresponding to $T_{12}^{(\nu)}$ $(1 \leqslant \nu \leqslant \nu(T_2))$ over the ring of all holomorphic functions on $\phi_r^{-1}(U^{(r)})$.*

COROLLARY 2. *For $T_2 = 0$, $a_0(Z_1, Z_{12})$ is independent of Z_{12}. Denoting $\tilde{f}_r(Z_1) = a_0(Z_1, Z_{12})$, $\tilde{f}_r$ is a local modular form of weight m on $\phi_r^{-1}(U^{(r)})$.*

The first half of Corollary 2 follows from the fact that in case $T_2 = 0$, $\{0\}$ consists of only the zero matrix. The second half of Corollary 2 follows easily from (29).

In case $r = 0$, a_T are constants satisfying the relation

$$a_{T[{}^tU]} = a_T. \tag{34}$$

Then by the method of Koecher [2, Satz 2] it follows again that $a_T = 0$ for $T \not\geqslant 0$, provided $n \geqslant 2$. But in case $r = 0$, $n = 1$ this does not hold in general. In this case, we *assume* this property, namely we assume that

$$\tilde{f}_1(z) = \sum_{t=0}^{\infty} a_t\, e^{2\pi i t z}, \tag{35}$$

which is of course equivalent to the condition that $\tilde{f}_1$ is bounded in $\tilde{V}^{(1)}(\{p_\infty\}, K)$. We call such a section f_1 on $U^{(1)}$ *bounded*. In case $r = 0$, Corollaries 1, 2 hold trivially.

LEMMA 3. *The Fourier series* (25) *converges uniformly absolutely in* $\tilde{V}(\tilde{W}, Y_2^0 + \epsilon E)$ *for any* $\epsilon > 0$.

PROOF. Since this series converges uniformly absolutely for $(Z_1, Z_{12}) \in \tilde{W}$, $Z_2 = i(Y_2^0 + \frac{1}{2}\epsilon E)$, we have

$$|a_{T_2}(Z_1, Z_{12})|\, e^{-2\pi \mathrm{Sp}(T_2(Y^0_2 + \frac{1}{2}\epsilon E))} \leqq L,$$

with some $L > 0$. Hence for $Z \in \tilde{V}(\tilde{W}, Y_2^0 + \epsilon E)$ we have

$$|a_{T_2}(Z_1, Z_{12})\, e^{2\pi i \mathrm{Sp}(T_2 Z_2)}| \leqq L\, e^{-\pi\epsilon \mathrm{Sp} T_2}.$$

Thus the convergent series

$$L \sum_{T_2 \geqq 0} e^{-\pi\epsilon \mathrm{Sp} T_2} \tag{36}$$

gives a majorant for the absolute value series of (25) in $\tilde{V}(\tilde{W}, Y_2^0 + \epsilon E)$, which proves the uniform convergence of the latter series.

By Theorem 2, Corollary 2, and Lemma 3, we have the following result.

THEOREM 3. *Let* U *be an open set* $\neq \phi$ *in* $\mathscr{V}_n^*$, $U = \bigcup_{r_0 \leqq r \leqq n} U^{(r)}$, $U^{(r)} = U \cap \mathscr{V}_r$, $U^{(r_0)} \neq \phi$. *Let* $r_0 \leqq r < n$. *Then for any section* f_n *of* $\mathscr{A}_m^{(n)}$ *on* $U^{(n)}$, *which is bounded in case* $n = 1$, *there corresponds uniquely a section* f_r *of* $\mathscr{A}_m^{(r)}$ *on* $U^{(r)}$ *as follows. Namely, let* $\tilde{W}$ *be a bounded open set in* $\mathscr{H}_r \times \mathbf{C}^{r(n-r)} = \{(Z_1, Z_{12})\}$ *such that* $(Z_1, Z_{12}) \in \tilde{W}$ *implies* $\phi_r(Z_1) \in U^{(r)}$ *and let* $\tilde{V}(\tilde{W}, Y_2^0)$ *be the set of all* $Z = \begin{pmatrix} Z_1 & Z_{12} \\ & Z_2 \end{pmatrix}$ *with* $(Z_1, Z_{12}) \in \tilde{W}$, $Z_2 = X_2 + iY_2$, $Y_2 > Y_2^0$, Y_2^0 *being a sufficiently large positive definite symmetric matrix of degree* $n - r$ *such that* $Z \in \tilde{V}(\tilde{W}, Y_2^0)$ *implies* $\phi_n(Z) \in U^{(n)}$. *Then* $\tilde{f}_n = f_n \circ \phi_n$ *has a uniformly absolutely convergent Fourier expansion of the form*

$$\tilde{f}_n(Z) = \sum_{T_2 \geqq 0} a_{T_2}(Z_1, Z_{12})\, e^{2\pi i \mathrm{Sp}(T_2 Z_2)} \tag{37}$$

in $\tilde{V}(\tilde{W}, Y_2^0)$, $a_{T_2}(Z_1, Z_{12})$ *being a restriction to* $\tilde{W}$ *of a uniquely determined holomorphic function on* $\phi_r^{-1}(U^{(r)}) \times \mathbf{C}^{r(n-r)}$ *and, in particular,*

$$a_0(Z_1, Z_{12}) = \tilde{f}_r(Z_1) = f_r \circ \phi_r(Z_1). \tag{38}$$

Under the circumstances of Theorem 3, we denote

$$\Phi_r^n(f_n) = f_r \tag{39}$$

and call f_n, f_r *related by the operator* Φ.

COROLLARY 1. *The assumptions being as in Theorem* 3, *let* $r < r' < n$. *Then*

$$\Phi_r^{r'} \circ \Phi_{r'}^n(f_n) = \Phi_r^n(f_n). \tag{40}$$

PROOF. Put

$$Z = \begin{pmatrix} Z'_1 & Z'_{12} \\ & Z'_2 \end{pmatrix} \begin{matrix} \}r' \\ \}n-r' \end{matrix}$$

and let $\tilde{W}'$ be a bounded open set in $\mathscr{H}_{r'} \times \mathbf{C}^{r'(n-r')} = \{(Z'_1, Z'_{12})\}$ such that $(Z'_1, Z'_{12}) \in \tilde{W}'$ implies $\phi_{r'}(Z'_1) \in U^{(r')}$. We take $\tilde{W}'$ and $Y'^0_2 > 0$ of degree $n - r'$ such that $\tilde{V}(\tilde{W}', Y'^0_2) \subset \tilde{V}(\tilde{W}, Y^0_2)$. Let

$$\tilde{f}_n(Z) = \sum_{T_2' \geqslant 0} a_{T'_2}(Z'_1, Z'_{12})\, e^{2\pi i \mathrm{Sp}(T_2' Z_2')} \tag{41}$$

be the corresponding Fourier expansion of $\tilde{f}_n$ in $\tilde{V}(\tilde{W}', Y'^0_2)$. Then, denoting $\Phi_{r'}^n(f_n) = f_{r'}, \tilde{f}_{r'} = f_{r'} \circ \phi_{r'}$, we have by definition

$$a_0(Z'_1, Z'_{12}) = \tilde{f}_{r'}(Z_1). \tag{42}$$

By (37), (41) we have

$$a_{T_2'}(Z'_1, Z'_{12})\, e^{2\pi i \mathrm{Sp}(T'_2 Z'_2)} = \sum_{T_2 = \begin{pmatrix} * & * \\ * & T'_2 \end{pmatrix}} a_{T_2}(Z_1, Z_{12})\, e^{2\pi i \mathrm{Sp}(T_2 Z_2)}.$$

In particular, for $T_2' = 0$, we have

$$\tilde{f}_{r'}(Z'_1) = \sum_{T_2 = \begin{pmatrix} T''_2 & 0 \\ 0 & 0 \end{pmatrix}} a_{T_2}(Z_1, Z_{12})\, e^{2\pi i \mathrm{Sp}(T_2 Z_2)}.$$

Denoting

$$Z'_1 = \begin{pmatrix} Z_1 & Z''_{12} \\ & Z''_2 \end{pmatrix} \begin{matrix} \}r \\ \}r'-r \end{matrix} ,$$

we can put $a_{T_2}(Z_1, Z_{12}) = a_{T_2''}(Z_1, Z''_{12})$. Then we have

$$\tilde{f}_{r'}(Z'_1) = \sum_{T''_2 \geqslant 0} a_{T_2''}(Z_1, Z''_{12})\, e^{2\pi i \mathrm{Sp}(T''_2 Z''_2)}, \tag{43}$$

with

$$a_0(Z_1, Z''_{12}) = a_0(Z_1, Z_{12}) = f_r(Z_1), \tag{44}$$

for any Z'_1 such that $(Z'_1, Z'_{12}) \in \tilde{W}'$ with some Z'_{12}. Since $\tilde{W}'$ can be taken arbitrarily under the condition $\tilde{V}(\tilde{W}', Y'^0_2) \subset \tilde{V}(\tilde{W}, Y^0_2)$, (43), (44) holds in some $\tilde{V}(W'', Y''^0_2)$. Thus f_r is bounded in case $r' = 1$ and $\Phi^{r'}_r(f_{r'}) = f_r$. q.e.d.

COROLLARY 2. *The notations being as in Theorem* 3, *let* $Z_\nu = \begin{pmatrix} Z_1^{(\nu)} & Z_{12}^{(\nu)} \\ & Z_2^{(\nu)} \end{pmatrix}$ *be a sequence in* $\tilde{V}(\tilde{W}, Y^0_2)$ *such that* $Z_1^{(\nu)} \to Z_1^0$, $Z_2^{(\nu)} = X_2^{(\nu)} + iY_2^{(\nu)}$, $Y_2^{(\nu)} \to \infty$. *Then we have*

$$\lim_{\nu \to \infty} \tilde{f}_n(Z_\nu) = \tilde{f}_r(Z^0_1). \tag{45}$$

PROOF. Since (37) converges uniformly in $\tilde{V}(\tilde{W}, Y^0_2)$, the limit can be calculated term by term. Then (45) follows easily from the fact that $a_{T_2}(Z_1, Z_{12})$ are bounded in $\tilde{W}$ and

$$\lim_{\nu\to\infty} |e^{2\pi i \mathrm{Sp}(T_2 Z_2^{(\nu)})}| = \lim_{\nu\to\infty} e^{-2\pi \mathrm{Sp}(T_2 Y_2^{(\nu)})}$$

$$= \begin{cases} 0 & \text{for } T_2 \geqslant 0,\ T_2 \neq 0, \\ 1 & \text{for } T_2 = 0. \text{ q.e.d.} \end{cases}$$

Now for an open set $U = \bigcup_{r_0 \leqslant r \leqslant n} U^{(r)} \subset \mathscr{V}_n^*$, $U^{(r)} = U \cap \mathscr{V}_r$, $U^{(r_0)} \neq \phi$, let

$$f^* = (f_n, f_{n-1}, \cdots, f_{r_0}) \tag{46}$$

be a set of sections f_r of $\mathscr{A}_m^{(r)}$ on $U^{(r)}$ $(r_0 \leqslant r \leqslant n)$, which are related to each other by the operator Φ. We denote by $A^{(n)}_{m,U}{}^*$ the set of all such f^*. Then $A^{(n)}_{m,U}{}^*$ becomes a module with respect to the addition defined by

$$f^* + g^* = (f_n + g_n, f_{n-1} + g_{n-1}, \cdots, f_{r_0} + g_{r_0}) \tag{47}$$

for $f^* = (f_n, f_{n-1}, \cdots, f_{r_0})$, $g^* = (g_n, g_{n-1}, \cdots, g_{r_0}) \in A^{(n)}_{m,U}{}^*$.

By Theorem 3 and its Corollary 1 it follows that $A^{(n)}_{m,U}{}^*$, is isomorphic to $A^{(n)}_{m,U^{(n)}}$ in case $n \geqslant 2$ or in case $n = r_0 = 1$ and is isormorphic to the submodule of $A^{(1)}_{m,U^{(1)}}$ consisting of all bounded sections of $A^{(1)}_m$ on $U^{(1)}$ in case $n = 1$, $r_0 = 0$.

Moreover we can define the multiplication by

$$f^* g^* = (f_n\, g_n, f_{n-1}\, g_{n-1}, \cdots, f_{r_0}\, g_{r_0}) \tag{48}$$

for $f^* \in A^{(n)}_{m,U}{}^*$, $g^* \in A^{(n)}_{m',U}{}^*$, where the product $f^*g^* \in A^{(n)}_{m+m',U}{}^*$. In particular, $A^{(n)}_{0,U}{}^*$ U forms a ring and all $A^{(n)}_{m,U}{}^*$ become $A^{(n)}_{0,U}{}^*$-modules. Putting

$$f^*(p) = f_r(p) \quad \text{for } p \in U^{(r)}, \tag{49}$$

$f^* = (f_n, f_{n-1}, \cdots, f_{r_0}) \in A^{(n)}_{0,U}{}^*$ can be considered as a function on U, which is continuous by Corollary 2 of Theorem 3. In this sense, we call $f^* \in A^{(n)}_{0,U}{}^*$ a *holomorphic function* on U.

Now the system of modules $\{A^{(n)}_{m,U}{}^*\}$ $(U \subset \mathscr{V}_n{}^*)$ together with the restriction maps define clearly a faisceau on $\mathscr{V}_n{}^*$, which we call the *faisceau of germs of modular forms of weight m on* $\mathscr{V}_n{}^*$ and denote by $\mathscr{A}^{(n)*}_m$. In particular, $\mathscr{A}^{(n)*}_0$ is the faisceau of germs of holomorphic functions on $\mathscr{V}_n{}^*$, which defines the analytic structure of $\mathscr{V}_n{}^*$.

It will not be hard to prove that the analytic space $\mathscr{V}_n{}^*$ thus defined becomes an " espace analytique général " in the sense of H. Cartan [1], i.e. $\mathscr{V}_n{}^*$ is locally isomorphic to a normal analytic subvariety in $\mathbf{C}^k$. Also it will be possible to prove that $\mathscr{A}^{(n)*}_m$ is coherent as a faisceau of $\mathscr{A}^{(n)*}_0$-modules. But we do not enter here into these problems any more.

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GAMMA FACTORS IN FUNCTIONAL EQUATIONS

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In a so-called functional equation for a zeta-function pertaining to an algebraic field or a modular set-up, the Dirichlet series occurring is multiplied by a function $\Delta(s)$ of the complex variable $s = \sigma + i\tau$ which introduces itself each time by some (multiple Euler-) integral of the form

$$\frac{\Delta(s)\ \lambda(x)}{\mu(x)^s} = \int_P e^{-(x,t)}\ R(t)^s d\,\Omega(t), \qquad (1)$$

and which each time, by an appropriate computation, turns out to be a product

$$\prod_{m=1}^{N} \Gamma(p_m s + q_m)$$

in which p_m are positive rational numbers, and q_m are complex numbers. After replacing $\Delta(s)$ by $\Delta(rs)$ for a suitable positive integer r, the numbers p_m may be assumed to be positive integers themselves, and after applying now the classical formula

$$\Gamma(pw) = C_p \exp[D_p w] \prod_{m=1}^{p} \Gamma\left(w + \frac{m-1}{p} \right), \quad w = s + q/p,$$

to each factor, the total product becomes

$$Ae^{as} \prod_{m=1}^{n} \Gamma(s - a_m), \qquad (2)$$

where

$$A > 0,\ a = \textit{real number},\ a_m = \textit{complex number}. \qquad (3)$$

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We are going to give a criterion by which a function $\Delta(s)$ which satisfies a relation (1) is indeed of the form (2). The integral (1) which we will envisage will be rather general, and will very amply include all the particular cases known. But ours will be a pure existence theorem, and none of the particular computations are in any way superseded by it.

We assume that $x = (x_1, \dots, x_k)$, $t = (t_1, \dots, t_k)$ are points in some real Euclidean E_k, $k \geqslant 1$, and (x, t) is the value $x_1 t_1 + \dots + x_k t_k$. The point set P is an arbitrary open subset of $E_k : (t)$, and $\Omega(A)$ is a positive set function which is defined on every Borel set A of P whose closure in E_k is a compact subset of P and which is countably additive on any such compactifiable subset. The function $R(t)$ is a (non-homogeneous) *polynomial* of some (*unprescribed*) degree in $(t_1, \dots, t_k)$, and is positive on P. The functions $\lambda(x)$, $\mu(x)$ are defined real-valued and infinitely differentiable (actually the infinite differentiability would follow from the other assumptions put together) over a neighbourhood G of $E_k : (x)$ no matter how small, and they are positive on G. The complex variable s ranges over some right half-plane

$$\sigma > \sigma_0, \tag{4}$$

and

$$\mu(x) = e^{\log \mu(x) . s}, \quad R(t)^s = e^{\log R(t) . s}$$

for real values of the logarithms, and $\Delta(s)$ is holomorphic in (4). We also assume that for

$$x \textit{ in } G, \quad s \textit{ in } (3), \tag{5}$$

we have

$$\int\limits_P e^{-(x,t)} R(t)^\sigma \, |d\,\Omega\,(t)| < \infty, \tag{6}$$

uniformly for x in G, which implies that the integral (1) is holomorphic in (4), and the over-all assumption is that in (4) the equality (1) holds.

THEOREM. *Such a function* $\Delta(s)$ *is indeed of the form* (2), (3).

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ON RIEMANN'S FUNCTIONAL EQUATION

By S. BOCHNER and K. CHANDRASEKHARAN

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GIVEN THE sequences

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \to \infty, \tag{1}$$

and

$$0 < \mu_1 < \mu_2 < \dots < \mu_n \to \infty, \tag{2}$$

and a number $\delta > 0$, we call the triplet $\{\delta, \lambda_n, \mu_n\}$ a *label*. If s is a complex variable, $s = \sigma + i\tau$, we speak of a solution of the functional equation

$$\pi^{-s/2}\,\Gamma(s/2)\,\phi(s) = \pi^{-(\delta-s)/2}\,\Gamma\{(\delta-s)/2\}\,\psi(\delta-s), \tag{3}$$

pertaining to the label $\{\delta, \lambda_n, \mu_n\}$, if there exists

(i) a Dirichlet series

$$\phi(s) = \sum_1^\infty \frac{a_n}{\lambda_n^s} \tag{4}$$

with arbitrary complex coefficients $\{a_n\}$, not all zero, which converges absolutely for $\sigma > \alpha$, for some finite $\alpha > 0$, and

(ii) a Dirichlet series

$$\psi(s) = \sum_1^\infty \frac{b_n}{\mu_n^s} \tag{5}$$

with arbitrary complex coefficients $\{b_n\}$, not all zero, which converges absolutely for $\sigma > \beta$, for some finite $\beta > 0$, and

(iii) a bounded, closed set S in the s-plane, such that if R is its exterior, then both S and R are symmetric relative to the line

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$$\sigma = \delta/2, -\infty < \tau < +\infty,$$

and there exists in R a holomorphic function $\chi(s)$ which in a right half-plane coincides with

$$\chi_1(s) = \pi^{-s/2}\, \Gamma(s/2)\, \phi(s), \tag{6}$$

and in a left half-plane coincides with

$$\chi_2(s) = \pi^{-(\delta-s)/2}\, \Gamma\{(\delta - s)/2\}\, \psi(\delta - s), \tag{7}$$

and for which

$$\lim_{|\tau|\to\infty} \chi(\sigma + i\tau) = 0, \tag{8}$$

uniformly in every bounded interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $-\infty < \sigma_1 < \sigma_2 < +\infty$. Such a pair of functions $\{\phi(s), \psi(s)\}$ will be called a *solution* of the functional equation (3), pertaining to the label $\{\delta, \lambda_n, \mu_n\}$. Our object is to show that the number of (linearly independent) solutions of equation (3) depends on the *modular density* (a concept which we define presently) of the given sequences λ_n, μ_n.

If $\{\lambda_n\}$ is given as in (1), then the 'lower density' of $\{\lambda_n\}$ is defined by $D_\lambda = \liminf n/\lambda_n$ and the 'upper density' by $D^\lambda = \limsup n/\lambda_n$. If $D_\lambda = D^\lambda = D$, then D is the 'density' of $\{\lambda_n\}$. The *analytic density* of $\{\lambda_n\}$ is the smallest number $D(\lambda)$ such that if one can analytically continue the sum function of the series $\Sigma c_n e^{-\lambda_n s}$, for arbitrary complex c_n, through a gap of width exceeding $D(\lambda)$ into the entire half-plane left of the axis of convergence, then one can so continue it all over the plane. The 'modular density' of $\{\lambda_n\}$ is defined to be $d_\lambda = \max(D(\lambda), D_\lambda)$.

We shall show that, in general, if $\{\mu_n\}$ is of finite modular density, then the number of linearly independent solutions of the functional equation is at most equal to the minimum number of λ_n's that lie in any interval of length greater than d_μ. The functional equation implies the inequality $d_\lambda\, d_\mu \geqslant 1$, and if there exists an integer k such that $d_\lambda, d_\mu < k + 1$, then the equation has at most k linearly independent solutions. Furthermore, if $d_\mu = d_\lambda = 1$, and the coefficients

$\{b_n\}$ are suitably limited in magnitude (e.g. bounded), the functional equation implies that $\lambda_{n+1} - \lambda_n = 1$, for every n, so that there can be at most one solution. On the other hand, if $d_\mu = d_\lambda = 1$ and $\mu_{n+1} - \mu_n = 1$ for $n \geqslant n_0$, then, of necessity, we have $\lambda_n = n$ or $n - 1/2$; $\mu_n = n$ or $n - 1/2$; and $\delta = 1$ or 3; and the only solutions that can occur are those known, namely $\zeta(s)$, $(2^s - 1)\,\zeta(s)$, $(2^{1-s} - 1)\,\zeta(s)$ and $2^{s-1}\,L(s-1)$. Finally, if in a solution we have $\mu_{n+1} - \mu_n = 1$ for $n \geqslant n_0$, and in the sequence $\{b_n\}$ there are at most finitely many different terms, then $\delta = 1$, $\mu_n = n$, $a_n = a_{n+k}$ for a certain positive integer k, $\lambda_n = n/k$, and $b_n = \sum_{q=0}^{k-1} a_n \cos \frac{2\pi qn}{k}$.

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ON THE HECKE OPERATORS FOR MODULAR FORMS OF DEGREE n

By M. KOECHER

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THE FOLLOWING essay is a collection of the most important results of an operator theory of modular forms of degree n. Essentially this theory is a generalization of the Hecke operators, going further than the initial work of H. Maass and M. Sugawara, a generalization which is mainly based on the complex multiplication of Abelian functions of n variables. The main difference between this theory and the classical one of Hecke consists in the fact that we consider an infinite number of equivalent groups instead of the modular group alone (equivalent in so far as none of these groups has any properties which may induce us to prefer it to the others). Those groups appear as unit-groups of skew-symmetric matrices. In relation to them one has to define C-modules of automorphic forms and finally one introduces, as a generalization of the Hecke operators, operators between two of these modules.

As the unit-groups of skew-symmetric matrices are generally not subgroups of the symplectic group of degree n, there are two possibilities for the generalization intended here. Firstly, it is possible to transform all the groups into subgroups of the symplectic group by passing over to conjugated groups. In this case, the so-called symplectic one, the writing and nomenclature is comparatively complicated, but on the other hand, one is able to use all the results of Siegel's theory. Secondly, one can formulate this theory directly by means of the unit-groups, but is then forced to rewrite the results of symplectic geometry etc. a little, which is possible without any difficulty. For the second case we will now put down

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the most important results, which can be easily concluded from the symplectic case [1]. In the cited work there can be found other references to the literature.

A matrix A of m rows and n columns is denoted by $A^{(m,n)}$, and $A^{(n,n)}$ is equivalent to $A(n)$. A' means the transposed matrix of A, and $|A|$ the determinant. A matrix A is called *integral*, if all the elements are rational integral numbers. A non-singular matrix $A^{(n)}$ is called *elementary-divisor-matrix* (ED-matrix) if A is integral, is of diagonal form, and if for the diagonal elements $a_1, a_2, \dots, a_n$, the relation $a_k \mid a_{k+1}$ for $k = 1, 2, \dots, n-1$ is satisfied. It is known that there exist two unimodular matrices U and V for every non-singular integral matrix, so that UAV is ED-matrix. We define for every quadratic matrix H of n rows and columns a matrix H^* of double the number of rows and columns by

$$H^* = \begin{bmatrix} O & H \\ -H' & O \end{bmatrix}.$$

If $Q^{(2n)} = -Q'$ is an integral non-singular matrix, one considers the group of the integral matrices U, which satisfy the equation $U'QU = Q$, i.e. the group of the *units* of Q. As it is possible to find for every skew-symmetric matrix Q a unimodular matrix V so that $V'QV = H^*$ and H is an ED-matrix, we can confine our study of the *unit-groups* to the special matrices of the form $Q = H^*$. The set of the units of H^*, that means the set consisting of all the integral matrices L with $L'H^*L = H^*$, forms the group $M(H)$, which may be looked upon as a generalization of the modular group of degree n, $M(E)$. For $n = 1$ all the $M(H)$ coincide with the classical modular group; for $n > 1$, $M(H_1) = M(H_2)$ is equivalent to $H_1 = hH_2$. Two of these groups are commensurable. The $M(H)$'s have some interesting properties, of which only one shall be mentioned here, namely that they can be generated by a finite number of elements, the order of each of them being a divisor of 12.

The part played by the modular group in the case $n = 1$ is performed by the $M(H)$'s after the generalization of the Hecke operators. Given two ED-matrices H and G, one has to look upon the integral solution R of the equation

$$R'H^*R = G^* \tag{1}$$

as the generalization of the transformation matrices. This equation is solvable by an integral R only if $G = HK$, and K an integral diagonal matrix. If $G = HK$ is valid, we shall write $H \mid G$ (H divides G) and in the following we shall always assume that this relation is satisfied. If R is a solution of (1), then LRN satisfies (1) too, if L and N are respectively chosen out of $M(H)$ and $M(G)$.

Let $P(H)$ be the set of complex matrices $Z^{(n)}$ which satisfy the equation $HZ = Z'H$, and, putting $Z = X + iY$, for which HY is the matrix of a positive-definite quadratic form. Then Z being an element of $P(H)$ implies

$$Z \to (AZ + B)(CZ + D)^{-1} = R(Z),\ R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2}$$

a mapping from $P(G)$ onto $P(H)$, if R satisfies (1). For each function $F(Z)$, defined on $P(H)$, a given rational integral number k and any solution R of (1), R being subdivided in the scheme (Z), we define

$$f/R = f(Z)\underset{(k)}{/}R = |CZ + D|^k f(R(Z)).$$

By reason of (2), f/R is defined on $P(G)$. One easily verifies that

$$f/(RS) = f/R/S,$$

provided that $S'G^*S = F^*$. A function defined on $P(H)$ is called holomorphic, if the function $g(W) = f(H^{-1}W)$, $W = HZ = W' = (w_{kl})$, is holomorphic as a function in the $n(n+1)/2$ variables w_{kl}. Naturally f/R is holomorphic in $P(G)$, if f is holomorphic in $P(H)$.

Now the automorphic functions belonging to $M(H)$ are defined by the following properties :

(1) $f(Z)$ holomorphic on $P(H)$,

(2) $f/L = f$ for every L of $M(H)$.

By these functions a C-module (H, k) is formed. It can be verified that for $n > 1$ these functions are bounded in the fundamental region of $M(H)$ — a fact, which must, however, be postulated for $n = 1$ — and that the module (H, k) possesses a finite basis.

Now we will define the generalization $T(H, G)$ of the Hecke operators. If H and $G = HK$ are two ED-matrices, as before, $f(Z)$ an element of (H, k), we put :

$$f/T(H, G) = \sum_R f/R, \tag{3}$$

the sum being taken over a complete system of inequivalent solutions R of (1). Two solutions of (1) are called equivalent, if they can be transformed into each other by left hand-multiplication with an element of $M(H)$. Because $f/L = f$, $T(H,G)$ does not depend on the selection of the matrices R within a system of equivalent matrices. One proves that the sum (3) is finite, and that $f/T(H,G)$ belongs to (G, k). Therefore we get a mapping

$$(H, k) \xrightarrow{T(H,\, G)} (G, k). \tag{4}$$

Analogously one puts for g of (G, k),

$$g/T^* (G, H) = \sum_R f/R^{-1}, \tag{5}$$

the sum being taken here over a complete system of inequivalent solutions R of (1), inequivalent as regards right-hand multiplication with elements of $M(G)$. One verifies that

$$(G, k) \xrightarrow{T^*(G,\, H)} (H, k). \tag{6}$$

Thus a connection is given by T and T^* between the different modules (H, k).

Now it is the task of the operator theory to examine the relations between the different operators T and T^* for different arguments. The following two important theorems can be deduced :

THEOREM 1. *For H, G, F ED-matrices and $H/G/F$ we have*

$$T(H, G)\ T(G, F) = T(H, F), \quad T^*(F,G)\ T^*(G, H) = T^*(F,H),$$

provided that $|H^{-1} G|$ and $|G^{-1} F|$ are relatively prime.

THEOREM 2. *For H, G_1, G_2 ED-matrices and H/G_j $(j = 1, 2)$ we have*

$$T^*(G_1, H)\ T(H, G_2) = T(G_1, H^{-1} G_1 G_2)\ T^*(H^{-1} G_1 G_2, G_2),$$

provided that $|H^{-1}G_1|$ and $|H^{-1}G_2|$ are relatively prime.

One could think of composing the operators T and T^* in such a way as to get an operator mapping a given module into itself, in order to come to a closer contact with the classical theory. This is indeed possible by defining, for an integral diagonal matrix K,

$$S_H(K) = T(H, HK)\ T^*(HK, H).$$

Then $S_H(K)$ maps the module (H, k) into itself and from Theorems 1 and 2 follows

$$S_H(K_1)S_H(K_2) = S_H(K_1 K_2),$$

provided that K_1, K_2 are integral diagonal matrices with relatively prime determinants. As for $n = 1$, the operators T and T^* are identical, S_H coincides in this case with the square of the corresponding Hecke operator. But this composition of the two operators does not yet seem suitable at this point of the development, one will rather try to find relations between the operators T (respectively T^*) alone.

Analogously to the symplectic case an *inner product* (f, g) between two forms f and g from (H, k) can be defined, which possesses the characteristic properties of a definite Hermitean metric. This definition shall not be given here. But by this product it is possible to prove that for g out of (G, k), and f out of (H, k), the following equation is valid:

$$(f/T(H, G), g) = c\,(H, G).\,(f, g/T^*(G, H)),$$

where $c(H,G)$ is known. One can verify that by forming the direct product of the modules (H, k), it is possible to formulate the results of the operator theory more simply. For this purpose all the ED-matrices H are enumerated in some way, and then the direct product of the (H_j, k)'s is formed. An element of this product can be looked upon as a vector of infinite dimension

$$(f) = (f_1(Z), f_2(Z), \ldots), \quad f_j(Z) \text{ in } (H_j, k).$$

We also write

$$(f) = (f(Z; H)),$$

which is to signify that $f(Z; H)$ from (H, k) is that component of (f) which is numbered by H. For such form-vectors $(f) = (f(Z); H)$ and ED-matrices K, two operators $T(K)$ and $T^*(K)$ are defined as follows:

$$(f)/T(K) = (\delta(HK^{-1}).\, f(Z; HK^{-1})/T(HK^{-1}, H)),$$
$$(f)/T^*(K) = (f(Z; HK)/T^*(HK, H)),$$

with

$$\delta(A) = \begin{cases} 1, \text{ if A is an } ED\text{-matrix}, \\ 0, \text{ otherwise.} \end{cases}$$

Because of (4) and (6), $(f)/T(K)$ and $(f)/T^*(K)$ are again form-vectors. From Theorems 1 and 2 it follows immediately that

$$T(K_1 K_2) = T(K_1)\, T(K_2),\; T^*(K_1 K_2) = T^*(K_1)\, T^*(K_2),$$
$$T(K_1)\, T^*(K_2) = T^*(K_2)\, T(K_1),$$

provided that the K_j's are ED-matrices with relatively prime determinants.

Further we can now formulate some more results concerning the operators $T(K)$ and $T^*(K)$. If the K_j's are two diagonal matrices, only the last component of each of them being a positive integral number different from 1, the following equation can be verified:

$$T(K_1)\, T(K_2) = \sum_D T(D^{-1} K_1 K_2)\, V(D), \tag{8}$$

after norming the $T(H,G)$'s suitably. The sum is to be taken over all the ED-matrices D, for which D/K_j, and $V(D)$ is a simple permutation operator. (8) is the exact analogue of Hecke's theorem.

After norming the operators $T(H,G)$ and $T^*(G, H)$ suitably, one can introduce an inner product for the form-vectors (f) and (g) in a quite natural way. By this definition a subspace of the direct product of the modulus (H, k) becomes a Hilbert space (k). In (k) the operators $T(K)$ and $T^*(K)$ have a bounded norm, and it is possible to norm the $T(H, G)$'s in such a way that there appear only constant additional factors in the above multiplication formulae. Besides this $T(K)$ and $T^*(K)$ are adjoint in (k), so that the following relation exists :

$$((f)/T(K), (g)) = ((f), (g)/T^*(K)).$$

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ON A CERTAIN KIND OF ZETAFUCHSIAN FUNCTIONS

By H. PETERSSON

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IN the theory of the "fonctions zétafuchsiennes" developed by Poincaré and, in other regards by Ritter, some progress may be attained if a non-trivial representation of the given fuchsian group Γ is explicitly known. Such a representation arises from a simple algebraic construction by which there corresponds to each (2, 2)-matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a (n, n)-matrix $D(S) = (d_{jk}(S))$, where $n > 1$ is a preassigned integer, so that $D(S_1 S_2) = D(S_1) D(S_2)$ $|D(S)| = |S|^{\frac{1}{2}n(n-1)}$ holds, when S, S_1, S_2 are any given complex (2, 2)-matrices. The representation of Γ thus defined turns out to be irreducible, as is easily seen in the case of Γ containing parabolic substitutions.

Now, if r denotes a fixed complex number, v a system of multipliers of dimension $-(r-n+1)$ attached to Γ, we consider vectors $f(\tau)$ with components $f_j(\tau)$ $(j = 1, 2, \ldots, n)$, all analytic functions of the complex variable τ in the upper half-plane, so that

$$f(L\tau) = v(L)(\gamma\tau + \delta)^r D(L) f(\tau), \left(L\tau = L(\tau) = \frac{\alpha\tau + \beta}{\gamma\tau + \delta},\ L \in \Gamma \right);$$

here, f is to be treated as a column. Besides, f has to fulfil certain additional conditions of regularity corresponding to those of the scalar automorphic forms.

The general facts which are to be established in the theory of these vectors $f(\tau)$ concern their expansions in power series of the different locally uniformizing variables, their connexion to vectors

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composed by scalar automorphic forms of different dimensions, the effect of a certain differential operator on a scalar automorphic form of dimension $-(r-n+1)$, the Wronskian of a vector $f(\tau)$, the linear independence of its components — all that combined with the transformation of $f(\tau)$ by a real $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of determinant 1. When r is real and $|v|$ is always $= 1$, a scalar product of two vectors $f(\tau)$, $g(\tau)$ belonging to the same "class" $\{\Gamma, -r, v, n\}$ may be defined by a certain integral over the fundamental domain of Γ. In the linear manifold K consisting of the integral vectors of that class vanishing in all the cusps of Γ, the scalar product induces a linear hermitian but not always positive definite metrization.

Whenever $r > n + 1$, $|v| = 1$, Poincaré's series are absolutely convergent, and each vector $f(\tau)$ of a corresponding class—integer or not—admits a representation as a finite linear combination of Poincaré's series. From that it follows that the metrization mentioned above is closed, i.e. a vector of K which is orthogonal to all vectors of K vanishes identically.

In the special case Γ = modular group Γ_0, integral vectors $f(\tau)$ belonging to a real $r > 0$ and to a system of multipliers v of modulus 1 are connected, by means of Mellin's integral, to vectors of Dirichlet's series which represent meromorphic functions with at most a finite number of simple poles, and which satisfy a functional equation of the type of Riemann's $\zeta(s)$. If r is an integer and $v = 1$, then, from the general transformation named above, a theory of operators of Hecke's type arises, leading to results which are, in some regards, but, not throughout, similar to those of the theory of scalar modular forms.

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