Lecture 1 : Continuity, Differentiability in several variables

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Motivation and outline

We live in three dimensional space!

Many models necessitate working with variables $x = (x_1, x_2, \cdots, x_n)$.

There is a need to develop continuity, differentiability and integration for functions depending on the variable $\mathbf{x}$, in an Euclidean space $\mathbb{R}^n$.

Let us restrict first to $\mathbb{R}^2$ and develop these concepts.

First of all, recall the notion of distance between any two points, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$:

$$\|\mathbf{x} - \mathbf{y}\| = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{\frac{1}{2}}$$

For any point $\mathbf{x} \in \mathbb{R}^2$, recall that $\|\mathbf{x}\|$ represents the distance from the origin to that point.

Open and Closed sets

$\mathbb{R}^2$ is a vector space with addition and scalar multiplication defined on it.

It is also a metric space with the metric defined by the distance :

$$d(\mathbf{x}, \mathbf{y}) \;=\; \|\mathbf{x} - \mathbf{y}\|.$$

An open ball of radius r, centered at $\mathbf{x}$ is

$$B_r(\mathbf{x}) \;=\; \{\mathbf{y} \mid \|\mathbf{x} - \mathbf{y}\}\| < r\}.$$

An open set Ω : each point $\mathbf{x} \in \Omega$ has an open ball $B_r(\mathbf{x})$, for some sufficiently small $r > 0$, completely contained in Ω.

A closed set is the complement of an open set.

Convergence in $\mathbb{R}^2$

A sequence of vectors $\{\mathbf{x}_k\}$ converges to $\mathbf{z}$ in $\mathbb{R}^2$,

that is $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{z}$,

if for every $\epsilon > 0$, there is an index K such that

$\|\mathbf{x}_k - \mathbf{z}\| < \epsilon, \quad \forall \quad k \geq K.$

A sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{z}$ in $\mathbb{R}^2$, if and only if

each component of $\{\mathbf{x}_k\}$ converges to the corresponding component of $\mathbf{z}$ in $\mathbb{R}$.

Equivalent criteria for continuity at a point

Sequential Continuity

A function f from $\mathbb{R}^2 \to \mathbb{R}^2$ is continuous at a point $\mathbf{x}$ provided that whenever a sequence $\{\mathbf{x_k}\}$ converges to $\{\mathbf{x}\}$,

the image sequence $\{\mathbf{f}(\mathbf{x_k})\}$ converges to $\{\mathbf{f}(\mathbf{x})\}$.

$\epsilon - \delta$ Criterion

For each positive $\epsilon > 0$, there is a positive δ, for a point $\mathbf{x}$ such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\| < \epsilon, \quad \text{if } \|\mathbf{x} - \mathbf{y}\| < \delta.$$

- f is continuous on $\mathbb{R}^2$, if it is continuous at every point.

Pull-back of open set Criteria

For every open set U, the pull back

$$f^{-1}(U) := \{\mathbf{y} \mid f(\mathbf{y}) \in U\}$$

is open in $\mathbb{R}^2$.

Examples, Counter-examples

- A real valued function on $\mathbb{R}^2$:
  $f(\mathbf{x}) \quad = \quad \|x\|$
- A vector valued function on $\mathbb{R}^2$:
  $f(\mathbf{x}) \quad = \quad (x_1 + x_2,\ x_1 - x_2)$
- A real valued discontinuous function on $\mathbb{R}^2$:

$$\begin{aligned} f(\mathbf{x}) \quad &= x_1 \quad \text{if}\, x_2 = 0; \\ &= x_2 \quad \text{if}\, x_1 = 0; \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Partial Derivatives of a real valued function

For a function f from $\mathbb{R}^2$ to $\mathbb{R}$,

the directional derivative in the direction of $\mathbf{z}$, at the point $\mathbf{x}$:

$$D_{\mathbf{z}} f(\mathbf{x}) \; = \; \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{z}) - f(\mathbf{x})}{t}$$

if this limit exists.

The directional derivatives in the direction of the coordinate axes are the partial derivatives

$$D_{\mathbf{x}_i} f(\mathbf{x}), \; i = 1, 2,$$

also denoted by $\frac{\partial f}{\partial x_i}$.

Partial Derivatives need not imply continuity !

Take a function f as follows :

$$\begin{aligned} f(\mathbf{x}) &= 0 \quad \text{if } x_1 \text{ or } x_2 = 0; \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Both the partial derivatives exist at the origin.

But the function is not continuous at the origin and hence not differentiable.

We need another concept of derivative which would imply continuity!

Definition of Total Derivative

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\mathbf{x} \in \mathbb{R}^n$ if there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{z} \to \mathbf{x}} \frac{\|f(\mathbf{z}) - f(\mathbf{x}) - T(\mathbf{x} - \mathbf{z})\|_m}{\|\mathbf{x} - \mathbf{z}\|_n}$$

Such a T, if it exists is unique and is called the total derivative of f at $\mathbf{x}$, denoted by $D_f(\mathbf{x})$.

If $f : \mathbb{R}^n \to \mathbb{R}$, then

$$D_f(\mathbf{x}) = \nabla f(x)$$

the gradient of f at $\mathbf{x}$.

If $f : \mathbb{R} \to \mathbb{R}$, then

$$D_f(\mathbf{x}) = f'(x)$$

the tangent of f at x.

Connection with Partial derivatives

Theorem

$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\mathbf{x} \in \mathbb{R}^n$ with

$$f = (f_1, f_2, \cdots, f_m)$$

if and only if each of the component functions, f_i is differentiable at $\mathbf{x}$, for all $i, 1 \leq i \leq m$.

Further the $m \times n$ derivative matrix is

$$D_f(\mathbf{x}) \quad = \quad [D_{x_i} f_j]_{m \times n} \quad = \quad [\frac{\partial f_j}{\partial x_i}]_{m \times n}$$

For f, a real valued function on $\mathbb{R}^n$, the gradient is given by

$$D_f(\mathbf{x}) \ = \ \nabla f(x) \ = \ (D_{x_1} f, \cdots, D_{x_n} f) \ = \ (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$$

Examples

In the following examples, f is defined on $\mathbb{R}^2$. Discuss the differentiability and write the derivative at different points of $\mathbb{R}^2$.

- $f(\mathbf{x}) \quad = \quad (2x_1 - 3x_2,\ x_1 + x_2) +\ (5, 1)$
- $f(\mathbf{x}) \quad = \quad \|x\|^2$
- $f(\mathbf{x}) \quad = \quad \|x\|$
- $f(\mathbf{x}) \quad = \quad (e^{x_1},\ \log(|x_2|^2 + 1)$
- $f(\mathbf{x}) \quad = \quad (x_1^2 + e^{x_1 x_2},\ x_1 + x_2,\ \ \sin(x_1 x_2))$

Inverse function in one variable

For a differentiable function, the derivative at a point provides a good linear approximation of the function.

A few properties of the derivative are likely to carry over into local properties of the function.

One such property is inversion of the function.

If f is a straight line on (a, b) with a positive slope, then

the inverse function f^{-1} exists from $(f(a), f(b))$ to (a, b).

When can we invert a general nonlinear function?

Inverse function theorem in one variable

Theorem

If $f : (a, b) \to \mathbb{R}$ *is* C^1 *and if* $f'(x_0) \neq 0$, *for some* $x_0 \in (a, b)$, *then*

there exists an open interval I *containing* x_0 *and an open interval* J *containing* $f(x_0)$ *such that*

$$f : I \to J \qquad \text{is one-to-one and onto.}$$

Further, the inverse function is $C^1(J)$.

For $x \in I$, *if* $f(x) = y \in J$, *then*

$$(f^{-1})'(y) \quad = \quad \frac{1}{f'(x)}.$$

Example - one variable

Take the function defined on $\mathbb{R}$:

$$f(x) \quad = \quad x^3 \, + \, x \, + \, \cos(x).$$

Check where inverse function theorem is applicable.

In fact, f is invertible globally!

Inverse function in more variables

Theorem

Suppose that Ω *is an open subset of* $\mathbb{R}^2$ *and* $f : \Omega \to \mathbb{R}^2$ *is* $C^1(\Omega)$*. If for some point* $\mathbf{x}_0 \in \Omega$*, the derivative matrix*

$$D_f(\mathbf{x}_0) \quad \textit{is invertible,}$$

then there is an open set U *containing* $\mathbf{x}_0$ *and an open set* V *containing* $f(\mathbf{x}_0)$ *such that*

$$f : U \to V \quad \textit{is one-to-one and onto.}$$

Further, the inverse function is $C^1(V)$ *and for* $\mathbf{x} \in U$*, if* $f(\mathbf{x}) = \mathbf{y}$*, then*

$$D_{f^{-1}}(\mathbf{y}) \quad = \quad [D_f(\mathbf{x})]^{-1}.$$

Examples - more variables

- Take the function $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$f(\mathbf{x}) \quad = \quad (e^{x_1}\cos(x_2),\ e^{x_1}\sin(x_2))$$

  Check at which points inverse function theorem is applicable.

  Check if it is globally invertible.

- Define the function $f : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2$ by

$$f(r, \theta) \quad = \quad (r\ \cos(\theta),\ r\ \sin(\theta))$$

  Check at which points inverse function theorem is applicable.

Implicit functions

Examples:

- Suppose that x and y implicitly related :

$$4x + 6y - 12 = 0.$$

Then it is easy to write y as a function of x : $y = \frac{12-4x}{6}$.

- Suppose that x and y implicitly related by

$$x^2 + y^2 = 1.$$

Then again y is a function of x : $y = \pm\sqrt{1 - x^2}$.

When can we write y as a function of x, even if we don't know how to solve the equation explicitly ?

Implicit function Theorem

Theorem

Let Ω *be an open subset of* $\mathbb{R}^2$ *and let* $f : \Omega \to \mathbb{R}^2$ *be* $C^1(\Omega)$. *If*

$$f(x_0, y_0) = 0, \quad \text{and if} \quad \frac{\partial f}{\partial y}(x_0, y_0) \neq 0,$$

then there is an open set $B_r(x_0) \times B_p(y_0)$ *containing* $(x_0, y_0) \in \mathbb{R}^2$ *and a* C^1 *function* $g : B_r(x_0) \to \mathbb{R}$ *such that*

$$f(x, g(x)) \; = \; 0, \quad \forall\, x \in B_r(x_0), \qquad g(x_0) = y_0.$$

Further

$$\frac{\partial f}{\partial x}(x, g(x)) \; + \; \frac{\partial f}{\partial y}(x, g(x)) g'(x) \; = \; 0, \; \forall\, x \in (x_0 - r, x_0 + r).$$

Example

Take the function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x, y) \;=\; 4x^2 \;+\; 9y^2 - 1 = 0.$$

Draw the set of solutions.

At which points, can we find g such that $f(x, g(x)) \;=\; 0$?

Find this g explicitly.

Check the formula for the derivative of g.

Implicit Function Theorem - General Case

Theorem

Let Ω *be an open subset of* $\mathbb{R}^n \times \mathbb{R}^m$*.*

Let $F : \Omega \to \mathbb{R}^m$ *be a* C^1 *function such that*

$$F(\mathbf{a},\ \mathbf{b})\ =\ 0$$

for some $(\mathbf{a},\ \mathbf{b}) \in \Omega$*,*

If the derivative matrix $D_y F(\mathbf{a}, \mathbf{b})$ *is nonsingular, then there exist*

- *an open subset* $U_{\mathbf{a}} \times U_{\mathbf{b}}$ *containing* $(\mathbf{a}, \mathbf{b})$
- *and a* C^1 *function* $f : U_{\mathbf{a}} \to \mathbb{R}^m$ *such that*

$$F(\mathbf{x}, f(\mathbf{x}))\ =\ 0 \text{ on } U_{\mathbf{a}} \quad \text{and } f(\mathbf{a})\ =\ \mathbf{b}.$$

Example

Take $F : \mathbb{R}^3 \to \mathbb{R}$ and consider its zero set :

$$F(x, y, z) \ = \ x^2 \ + \ y^2 \ + \ z^2 \ - \ 1 = 0.$$

This represents a surface in $\mathbb{R}^3$.

Can we view this as the graph of some function $z = f(x, y)$?

If so, near which points?

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