Lecture 2 : Integration in several variables

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Motivation and outline

The aim of this lecture is to introduce the double integral over rectangles.

After recalling Riemann integrals in one variable briefly, we adapt those concepts to two dimensions, to define the integral over rectangles.

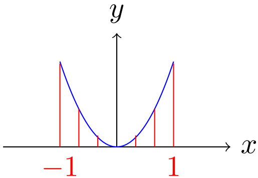
and then extend the definition to more general sets, which have "thin" boundaries.

Then we explore the evaluation of a double integral and examine when the interchange of the order of integration is possible.

Fubini's theorem guarantees when the iterated integrals will be equal.

Recall : One variable Integration

- The area enclosed by the graph of a non-negative function over the region of the interval is $\int_a^b f(t)\, dt$.



The area in the figure on the left is $\int_{-1}^{1} x^2\, dx = 2/3$.

- A partition of the interval $[a, b]$ is a set of points $P = \{a = x_0 \leq x_1 \leq \ldots x_n = b\}$ for some $n \in \mathbb{N}$.
- Define lower and upper sum of f; the lower and upper integrals are $L(f) = \sup\{L(f, P) \colon P$ is a partition of $[a, b]\}$, and $U(f) = \inf\{U(f, P) \colon P$ is a partition of $[a, b]\}$.
- When $L(f) = U(f)$ then f is Darboux integrable and

$$\int_a^b f := L(f) = U(f).$$

Recall : One variable Integration

- Define Riemann sum $S(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$, for some $t_j \in [x_j, x_{j-1}]$
- Define the *norm* of a partition P as $\|P\| = \max_j\{|x_j - x_{j-1}|\}, \quad 1 \leq j \leq n.$
- A function $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* if for some $S \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$, whenever $\|P\| < \delta$. The Riemann integral of f is then S.
- The Riemann integral exists if and only if the Darboux integral exists. Further, the two integrals are equal.
- Let $f : [a, b] \to \mathbb{R}$ be a bounded function which is continuous at all but finitely many points of $[a, b]$. Then f is Riemann integrable on $[a, b]$.

Rectangle in $\mathbb{R}^2$

Any *closed, bounded rectangle R* in $\mathbb{R}^2$:

$$R = [a, b] \times [c, d].$$

Partition of R: A partition P of a rectangle $R = [a, b] \times [c, d]$ is the Cartesian product of a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$.

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b,$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d,$$

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 0, \cdots, n-1.$$

Partitions of rectangles

Note $R = \cup_{i,j} R_{ij}$.

The area of each R_{ij}*:* $\Delta_{ij} := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$.

Diameter of R_{ij} :

$$d(R_{ij}) \ = \ \max\{\|x - y\| \mid x, y \in R_{ij}\}.$$

Mesh of the partition P:
$\|P\| := \ \max\{d(R_{ij}) \mid 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$.

Partition Q is a refinement of partition P, if each subrectangle of Q is contained in a subrectangle of P.

In that case, $\|Q\| \ \leq \ \|P\|$.

Upper and lower sums

Let $f : R \to \mathbb{R}$ be a bounded function where R is a rectangle . Let

$$m(f) = \inf\{f(x,y) \mid (x,y) \in R\} \quad M(f) = \sup\{f(x,y) \mid (x,y) \in R\}.$$

For all $i = 0, 1, \cdots, m-1, \quad j = 0, 1, \cdots, n-1$, let,

$$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\},$$

$$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}.$$

Lower sum: $L(f,P) := \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij},$

Upper sum: $U(f,P) := \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij},$

Refinements of partitions

Note that for any partition P of R

$$m(f)(b-a)(d-c) \leq L(f,P) \leq U(f,P) \leq M(f)(b-a)(d-c).$$

If partition Q is a refinement of partition P, then

$$L(f,P) \ \leq \ L(f,Q), \qquad U(f,Q) \ \leq \ U(f,P).$$

On refinement of the partition P,

- the lower sums of the refinements are increasing and are bounded above.
- the upper sums of the refinements are decreasing and are bounded below.

Upper and lower Integrals

Lower integral:

$L(f) := \sup\{L(f, P) \mid P$ is any partition of $R\}$.

Upper integral:

$U(f) := \inf\{U(f, P) \mid P$ is any partition of $R\}$.

Note $L(f) \leq U(f)$.

f is Darboux integrable over R if

$$L(f) = U(f).$$

The common value is the Darboux integral of f : $\int_R f$.

Definition of Riemann integral

Another way to define the integral is via Riemann Sum of f associated to P:

$$S(f, P) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij}, \quad \Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$$

for arbitrary points $t_{ij} \in R_{ij}$.

f is Riemann integrable if there exists a real number S such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, P) - S| < \epsilon,$$

for every partition P satisfying $\|P\| < \delta$.

S is the value of Riemann integral of f.

Check both integrals are the same!

Equality of both integrals

Theorem (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bounded function. Then f is Darboux integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f, P_\epsilon) - L(f, P_\epsilon)| < \epsilon.$$

Theorem

A bounded function f from R to $\mathbb{R}$ is Riemann integrable over R if and only if it is Darboux integrable over R.

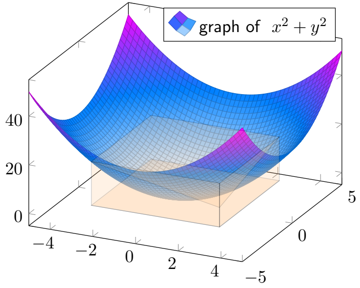
Geometrical meaning of the integral

Take $f(x,y) = x^2 + y^2$, for all $(x,y) \in \mathbb{R}^2$.

Compute the integral of f over the rectangle $[-3,3] \times [-3,3]$.

Define the solid region V in $\mathbb{R}^3$, bounded above by the graph of f over the rectangle $[-3,3] \times [-3,3]$.

$V := \{(x,y,z) \mid (x,y) \in [-3,3] \times [-3,3], \quad 0 \le z \le f(x,y)\}.$



The Riemann sum is indeed an approximation of the volume of the solid V. Thus

$$\int\int_{[-3,3]\times[-3,3]} f(x,y)\,dxdy = \text{Volume} \quad \text{of} \quad V$$

.

Examples

Take f from $R = [-1,1] \times [-1,1]$ to $\mathbb{R}$:

- $f(x,y) = x^2 + y^2$
- A discontinuous function :

$$\begin{aligned} f(x,y) &= x^2 + y^2 \quad \text{if } (x,y) \neq (0,0); \\ &= 1 \quad \text{if } (x,y) = (0,0). \end{aligned}$$

- Another discontinuous function on $R = [0,1] \times [0,1]$ to $\mathbb{R}$:

$$\begin{aligned} f(x,y) &= 0 \quad \text{if } x < y \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

- Dirichlet function :

$$\begin{aligned} f(x,y) &= 1 \quad \text{if both } x \text{ and } y \text{ are rational;} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Discontinuities of an integrable function

For a function to be integrable, it cannot oscillate too wildly.

Oscillation of a bounded function f from R to $\mathbb{R}$ at a point a :

$$\omega_f(a) = \lim_{r\to 0} \sup\{ |f(x) - f(z)| : x, z \in B(a,r)\}$$

Theorem

A bounded function on R is continuous at a if and only if $\omega_f(a) = 0$.

The set of discontinuities of an integrable function has to "small" in some sense.

Sets of zero Jordan Content and Measure

Definition

A set D has Jordan content zero if for every $\epsilon > 0$, there is a finite number of rectangles $\{R_i\}_{i=1}^{m}$ such that

$$D \subset \cup_{i=1}^{m} R_i, \quad \sum_{i=1}^{m} \text{area}(R_i) \; < \; \epsilon.$$

Definition

A set D has measure zero if for every $\epsilon > 0$, there is a sequence of rectangles $\{R_i\}_{i=1}^{\infty}$ such that

$$D \subset \cup_{i=1}^{\infty} R_i, \quad \sum_{i=1}^{\infty} \text{area}(R_i) \; < \; \epsilon.$$

Sets of zero Jordan Content and Measure

- If Jordan content of D is zero, then measure of D is also zero.
- The set $D = \mathbb{Q} \cap [0, 1]$ has measure zero but its Jordan content is not zero.
- If D is compact subset of $\mathbb{R}$ or $\mathbb{R}^2$ and it has measure zero then Jordan Content is also zero.
  Both concepts coincide in this case.
- If $f : R \to \mathbb{R}$ is a continuous function, then the graph of f in $\mathbb{R}^3$

$$S := \{(x, y, f(x, y)) \mid (x, y) \in R\}$$

  is of Jordan content zero.

Discontinuities of an integrable function

Theorem

If $f : R \to \mathbb{R}$ is a bounded function, then it is integrable if the set of discontinuities of f has Jordan content zero.

Theorem (Lebesgue)

If $f : R \to \mathbb{R}$ is a bounded function, then it is integrable if and only if the set of discontinuities of f has measure zero.

Integral over a bounded subset of $\mathbb{R}^2$

If $f : D \to \mathbb{R}$ is defined on a bounded subset D of $\mathbb{R}^2$, then enclose D by a rectangle R and extend f as f^* to R, by zero outside D.

Define

$$\int_D f \; = \; \int_R f^*$$

if f^* is integrable over R.

We need the boundary of D to be "thin" in some sense.

We will extend the integral to sets which have boundaries of Jordan content zero. These are Simple sets or Jordan domains.

One such example is a domain bounded by graphs of continuous functions.

For such sets, the integral is independent of the extension and hence is well defined.

Example

Take a continuous function f defined on the disk,

$$\mathsf{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Extend the function f by zero to f^* on the rectangle $R = [-2, 2] \times [-2, 2]$.

Is f^* continuous on R?

What are the points of discontinuity?

The points of discontinuity of f^* lie on the boundary of D :

$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ which is of 'content zero'.

Evaluating Integrals

Suppose $f : R \to \mathbb{R}$ is integrable.

How do we compute its double integral?

Geometrically, if f is non-negative then the double integral is the volume of the solid region D between the rectangle R and under the surface $z = f(x, y)$.

Cavalier's method was to compute this volume slice by slice.

That is, first compute area of each slice $A(x) = \int_c^d f(x, y)\, dy$ of the cross section of D perpendicular to the x-axis

Then the volume of $D = \int_a^b A(x)\, dx$.

Can we take the slices with cross section perpendicular to y axis first?

Fubini's Theorem

Theorem

Let $R := [a,b] \times [c,d]$ *and* $f : R \to \mathbb{R}$ *be integrable. Let* I *denote the integral of* f *on* R*.*

1. *If for each* $x \in [a,b]$*, the Riemann integral* $\int_c^d f(x,y)\,dy$ *exists, then the iterated integral* $\int_a^b (\int_c^d f(x,y)\,dy)\,dx$ *exists and is equal to* I*.*
2. *If for each* $y \in [c,d]$*, the Riemann integral* $\int_a^b f(x,y)\,dx$ *exists, then the iterated integral* $\int_c^d (\int_a^b f(x,y)\,dx)\,dy$ *exists and is equal to* I*.*

As a consequence, if f is integrable on R and if both iterated integrals exist in $1.$ and $2.$ in above theorem, then

$$\int_a^b \Big(\int_c^d f(x,y)\,dy \Big)\,dx = I = \int_c^d \Big(\int_a^b f(x,y)\,dx \Big)\,dy.$$

Examples, Counter-examples

- If f is a continuous function on R, then both iterated integrals exist and are equal to $\int_R f(x,y)$.
- Evaluate the integral of $\int_R xe^{xy}$ over $R = [-1,2] \times [0,1]$.
- Take $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$, on $R = [0,1] \times [0,1]$.

Note that $f(x,y) = \frac{\partial}{\partial y}(\frac{y}{x^2+y^2}) = -\frac{\partial}{\partial x}(\frac{x}{x^2+y^2})$.

Show that the iterated integrals exist but are not equal!

- Examine the integral of f over $R = [0,2] \times [0,1]$ where $f(x,y) = \frac{xy(x^2-y^3)}{(x^2+y^3)^3}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

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