Lecture 3 : Change of variable in integration

Mythily Ramaswamy
NASI Senior Scientist, ICTS-TIFR,
Bangalore, India

Summer School
Vigyan Vidhushi Program, TIFR
5-16th July 2021

contents

Motivation and outline

The aim of this lecture is to understand how to change the variables in a double integral.

We first recall the change of variable formula for integrals over real line.

Then we define a change of variables in $\mathbb{R}^2$ and examine how it transforms simple domains.

Then we explore the effect of a linear transformation on the integral.

Finally we indicate the formula when the transformation is a general one.

Recall : Change of variable formula in $\mathbb{R}$

Let $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $g : [a, b] \to \mathbb{R}$ is one-to-one function of class C^1 with $g'(x) \neq 0$ for all $x \in [a, b]$

Then with $y = g(x)$, if g is increasing

$$\int_a^b f(g(x))\ g'(x)\ dx \ = \ \int_{g(a)}^{g(b)} f(y)\ dy,$$

If g is decreasing,

$$\int_a^b f(g(x))\ g'(x)\ dx \ = - \ \int_{g(a)}^{g(b)} f(y)\ dy.$$

One motivation is that the second integral is often easier to compute.

What is a two dimensional analogue?

Change of variables

Let Ω be an open subset of $\mathbb{R}^2$.

A C^1 function $\phi : \Omega \to \mathbb{R}^2$ is a change of variables in $\mathbb{R}^2$ if

- ϕ is one-to -one function on Ω.
- Jacobian $J_\phi(x) = \det D_\phi(x) \neq 0$ for every $x \in \Omega$.

If ϕ is a change of variables in $\mathbb{R}^2$, then by Inverse Function theorem arguments, the image $\phi(\Omega)$ is open and the inverse function $\phi^{-1} : \phi(\Omega) \to \Omega$ is also C^1. Thus ϕ is a diffeomorphism.

On the other hand, every diffeomorphism is a valid change of variables.

Basic questions in change of variables

Let Ω be an open subset of $\mathbb{R}^2$, bounded and simple.

The boundary of Ω has Jordan content zero.

Let $\phi : \Omega \to \mathbb{R}^2$ be a C^1 diffeomorphism.

Let f be a real valued integrable function on $\phi(\Omega)$.

How to relate $\int_{\phi(\Omega)} f(y)dy$ to its integral over Ω?

Often the second integral is easier to compute.

Is $\phi(\Omega)$ also a simple domain?

Diffeomorphism of simple domains

Lemma

Let $\phi : U \to \mathbb{R}^2$ *be a* C^1 *diffeomorphism.*

Let Ω *be a bounded set with* $\overline{\Omega} \subset U$*.*

Let ϕ *restricted to interior of* Ω *be a diffeomorphism. Then*

$$\partial(\phi(\Omega)) \;=\; \phi(\partial(\Omega)).$$

ϕ is Lipshitz if for some $M > 0$, $\|\phi(\mathbf{x}) - \phi(\mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\|$.

Lemma

Let $\phi : U \to \mathbb{R}^2$ *be a Lipschitz function.*

Let D *be a subset of measure zero in* U*.*

Then $\phi(D)$ *has measure zero.*

Diffeomorphism of simple domains

C^1 functions are Lipschitz. Thus a diffeomorphism takes null sets (zero measure sets) to null sets.

Lemma

Let $\phi : U \to \mathbb{R}^2$ be a C^1 diffeomorphism.

Let Ω be a bounded simple set with $\overline{\Omega} \subset U$.

Then $\phi(\Omega)$ is also bounded and simple.

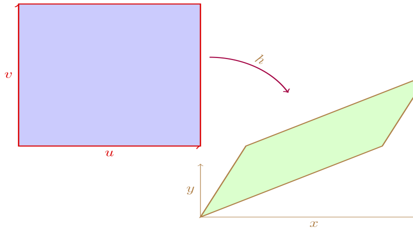
Linear transformations

Take ϕ to be a linear transformation T, from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Then it is represented by a 2×2 matrix. The transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The rectangle $[0,1] \times [0,1]$ in the u-v plane is mapped to a parallelogram in the x-y plane. The sides of the parallelogram are given by (a, c) and (b, d).



How to compute the area of this parallelogram?

Example

Take a simple example of T

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

In this case, it is easy to compute the area of the parallelogram as the sum of two right angled triangles : $2 \times (1/2) = 1$.

This in fact is the determinant of T, which is in fact an orthogonal matrix. (O orthogonal if $OO^t = O^tO = I$)

As orthogonal matrices preserve distances, they preserve the area.

The area of the image parallelogram is equal to the determinant of T, in this case.

Linear transformations

In the case of any general linear transformation also, one can prove that the area of the image of the unit rectangle is equal to $|$determinant of $T|$.

To prove that, use the following Polar decomposition for T.

Lemma

Let T be a non-singular linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then

$$T \; = \; OP$$

for some orthogonal matrix O and a symmetric positive definite matrix P.

As TT^t is symmetric positive definite, for some P, a symmetric positive definite matrix, $TT^t \; = \; P^2$.

Define $O \; = \; TP^{-1}$. Then $T = OP$ with O orthogonal and P symmetric positive definite.

Proof for a general T

$T = OP$ with O orthogonal and P symmetric positive definite.

P can be diagonalized and hence $P = O_1 D (O_1)^t$ for a diagonal matrix D, with diagonal elements a_1, a_2.

Note that

$$|\text{det}(T)| \;=\; |\;\text{det}\;(D)| \;=\; |a_1 a_2|.$$

As the eigenvalues of D are a_1, a_2, D transforms the unit square to the rectangle $[0, a_1] \times [0, a_2]$ (assuming positivity) and the image rectangle area is $|a_1 a_2|$.

By similar arguments, for any rectangle R,

$$\text{area } T(R) \;=\; |\;\text{det}\;(T)|\;(\text{area } R).$$

Geometric meaning of the determinant

Thus the determinant is the factor by which the area of rectangles get multiplied under the linear transformation.

A parallelopiped Π, is the n dimensional analogue of a parallelogram, with adjacent sides $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, linearly independent vectors in $\mathbb{R}^n$, is

$$\Pi \;=\; \{\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_n\mathbf{v}_n,\; 0 \le c_i \le 1\}$$

A rephrasing of our earlier results :

$$\text{volume } \Pi \;=\; |\text{det } T|$$

if T is the $n \times n$ matrix with columns as the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$,

Change of variable formula for linear transformations

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
Let $\Omega \subset \mathbb{R}^2$ be a bounded simple set. Then

$$\mathit{area}(T(\Omega)) \; = \; |\mathit{det}T| \; \mathit{area}(\Omega).$$

Theorem

Let T be a nonsingular linear transformation. Let $\Omega \subset \mathbb{R}^2$ be a bounded simple set. If $f : T(\Omega) \to \mathbb{R}$ is an integrable function, then

(i) $(f \circ T)|\mathit{det}\ T|$ *is also integrable over* Ω *and*

(ii)

$$\int_{T(\Omega)} f(\mathbf{y})\ d\mathbf{y} \; = \; \int_{\Omega} f(T(\mathbf{x}))\ |\mathit{det}\ T|\ d\mathbf{x}.$$

Change of variables in $\mathbb{R}^2$

Let $h : \Omega \to \mathbb{R}^2$ be a diffeomorphism

$$h(u,v) := (h_1(u,v), h_2(u,v)), \quad \forall\, (u,v) \in \Omega.$$

We now want to make a general change of coordinates given by

$$x = h_1(u,v), \quad y = h_2(u,v).$$

Can we compute the area of the image of a rectangle in the u-v plane?

Main idea : local replacement of a nonlinear relation by a linear one.

Recall that a local linear approximation of $h : \mathbb{R}^2 \to \mathbb{R}^2$ at a point $\mathbf{a} \in \Omega$ is by its derivative $D_h(\mathbf{a})$.

$$D_h(\mathbf{a})(\mathbf{z}) \;\approx\; h(\mathbf{a}+\mathbf{z}) \;-\; h(\mathbf{a})$$

in a neighborhood of $\mathbf{a} \in \mathbb{R}^2$.

The area element for a change of coordinates

Let $x = h_1(u, v)$ and $y = h_2(u, v)$.

How does the area of a rectangle in the u-v plane change?

$$\Delta x = h_1(u+\Delta u, v+\Delta v) - h_1(u, v), \quad \Delta y = h_2(u+\Delta u, v+\Delta v) - h_2(u, v),$$

Using the chain rule for functions of two variables we see that

$$\Delta x \sim \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v, \quad \Delta y \sim \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v.$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

The Jacobian

Recall that the matrix

$$J(h) = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

is the derivative matrix at (u, v) for the function $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$, the Jacobian matrix.

In a neighborhood of the point (u_0, v_0), the function h and the function $J(h)$, behave very similarly.

They are the same upto the first order terms by Taylor's theorem.

The area of a small rectangle changes by the (absolute value of) determinant of J.

Theorem (Change of Variables Formula)

- *Suppose* U *is an open subset of* $\mathbb{R}^2$,
- $h : U \to \mathbb{R}^2$ *is a diffeomorphism.*
- *Let* Ω *be a bounded simple set such that* $\overline{\Omega} \subset U$.
- *If* $f : h(\Omega) \to \mathbb{R}$ *is integrable on* $h(\Omega)$, *then*

(i) $(f \circ h)|det J_h|$ *is also integrable over* Ω *and*

(ii)

$$\int_{h(\Omega)} f(\mathbf{y})\, d\mathbf{y} \;=\; \int_{\Omega} f(h(u,v))\, |\det J_h|\, du\, dv.$$

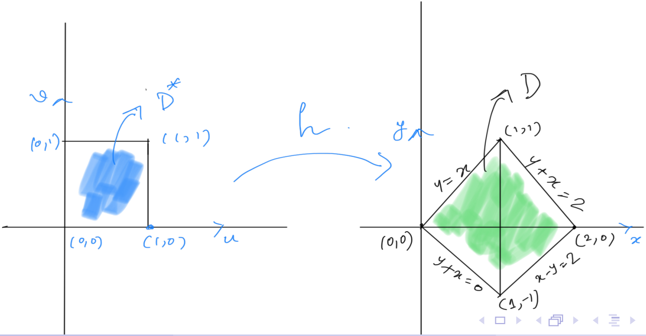
Example Evaluate the integral

$$\iint_D (x^2 - y^2)\,dxdy$$

where D is the square with vertices at $(0,0)$, $(1,-1)$, $(1,1)$ and $(2,0)$.

Solution: Note D is the region in $x-y$ plane bounded by lines $y = x$, $y + x = 0$, $x - y = 2$ and $y + x = 2$.
Put

$$x = u + v, \quad y = u - v,$$



Then the rectangle

$$D^* = \{(u,v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1,\ \ 0 \leq v \leq 1\}$$

in the uv-plane gets mapped to D, in the xy-plane.
Further,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2.$$

$$\int\int_D (x^2 - y^2)\, dxdy = \int\int_{D^*} (4uv) \times 2\, dudv$$

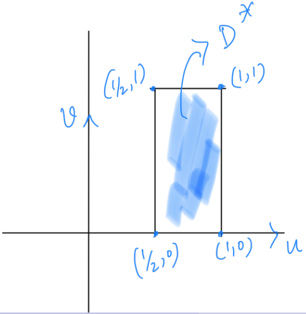
$$= 8 \left(\int_0^1 udu \right) \left(\int_0^1 vdv \right) = 2.$$

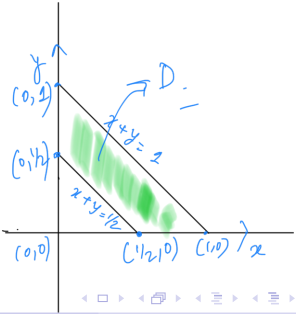
Example

Let D be the region in the first quadrant of the xy-plane bounded by the lines $x+y=\frac{1}{2}$ and $x+y=1$. Find $\iint_D dA$ by transforming it to $\iint_{D^*} dudv$, where $u=x+y, v=\frac{y}{x+y}$.

Solution: Put

$$x=u(1-v),\ \ y=uv.$$





Further,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} = u \neq 0.$$

Hence,

$$\text{Area}(D) = \int\int_D dA = \int\int_{D^*} |u| du dv$$
$$= \left(\int_{\frac{1}{2}}^{1} \frac{u^2}{2} du\right)\left(\int_0^1 dv\right) = \frac{3}{4}.$$

- T.M. Apostol, *Calculus, Volumes 1 and 2*, 2nd ed., Wiley (2007).
- S.R. Ghorpade and B. V. Limaye, *A course in Multivariable Calculus and Analysis*, Springer UTM (2017).
- Patrick M. Fitzpatrick, Advanced Calculus, Pure and Applied Undergraduate Texts - 5, AMS, 2009.
- Moskowitz, Martin; Paliogiannis, Fotios, Functions of several real variables. World Scientific Publishing Co. Pte. Ltd., 2011.
- J.E Marsden, A. J. Tromba, A. Weinstein. *Basic Multivariable Calculus*, South Asian Edition, Springer (2017).