Lecture 4 : Line integrals

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Curve and path

Recall that a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$.

Both the curve and path are denoted by the same symbol $\mathbf{c}$.

- In $\mathbb{R}^3$, denote $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$.
- The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous.
- Similarly, $\mathbf{c}$ is a C^1 path, if and only if each component is C^1.

Curve and path

- A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

- A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.

- If we write $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in vector notation, the tangent vector to $\mathbf{c}(t)$ is $\mathbf{c}'(t)$,

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

- If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrized curve.

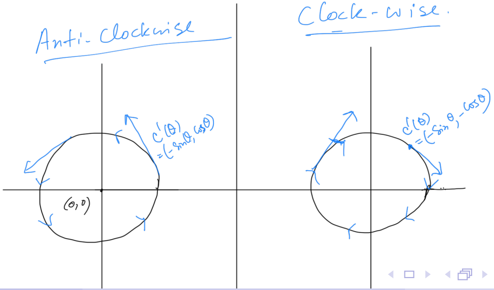
Curves on plane

Let us consider the paths lying in $\mathbb{R}^2$, namely, Planar curves.

For a simple closed planar curve, we get a choice of direction- clockwise or anti-clockwise.

Example . $\gamma(\theta) = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi]$. This is a circle with direction anti-clockwise.

Set $\gamma_1(\theta) = (\cos(\theta), -\sin(\theta))$, $\theta \in [0, 2\pi]$. It is a circle with clockwise direction.



Examples of curves

• Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t)$ where $0 \leq t \leq 1$. This is a simple closed C^1 (actually smooth) curve.

• Let $\mathbf{c}(t) = (t, t^2)$ where $-1 \leq t \leq 5$ is a simple curve but not closed.

• Let $\mathbf{c}(t) = (\sin(2t), \sin t)$ where $-\pi \leq t \leq \pi$. It traces out a figure 8. It is not a simple but a closed C^1 curve.

• Let $\mathbf{c}(t) = (t^3, t)$ where $-1 \leq t \leq 1$ is a part of the graph of the function $y = x^{1/3}$. This is simple but not a closed curve. Though the function $y = x^{\frac{1}{3}}$ is not smooth at origin, but this parametrization is regular!

Different parametrizations of the same path

Example 1:
Let $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$ and

$c_2(t) = (\cos 2t, \sin 2t)$ for $0 \le t \le \pi$.

Then the paths are different as a function but the curves traversed are the same.

Example 2:
Take the straight line segment between $(0, 0, 0)$ and $(1, 0, 0)$.

Here are three different ways of parametrizing it:

$$\{t, 0, 0)\}, \quad \{(t^2, 0, 0)\} \quad \text{and} \quad \{(t^3, 0, 0)\},$$

where $0 \le t \le 1$.

Arc length of a curve

We defined the length of the interval $[a, b]$ as $(b - a)$.

For a line segment connecting two vectors $\mathbf{p}$ and $\mathbf{q}$, the length of the segment is $\|\mathbf{p} - \mathbf{q}\|$.

How to extend this definition to the concept of arclength of a curve?

- For a path $\mathbf{c} : [0, 1] \to \mathbb{R}^n$, let $\Delta t = t_2 - t_1$ be very small and

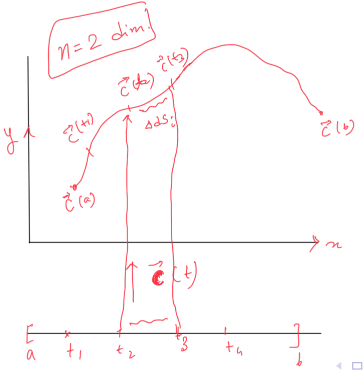
$$\Delta s = \|\mathbf{c}(t_2) - \mathbf{c}(t_1)\| = \|\mathbf{c}'(\widehat{t})(t_2 - t_1)\|$$

for some $\widehat{t} \in [t_1, t_2]$ by mean value theorem.

Arc length of a curve

- Total arc length $\approx \sum_{i=1}^{n} \|\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)\| = \sum_{i=1}^{n} \|\mathbf{c}'(\widehat{t_i}))(t_{i+1} - t_i)\|$.

The limit of these Riemann sums as the length of the subintervals tends to zero, if it exists, is the total arc length.



Arc length of a curve

Definition

A path $\mathbf{c} : [0,1] \to \mathbb{R}^n$ has finite arc length (rectifiable) if there is a number L such that for every ϵ, there is a δ such that

$$|\sum_{i=1}^{n} \|\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)\| - L| < \epsilon$$

for each partition P with $\|P\| < \delta$.

Theorem

If a path $\mathbf{c} : [0,1] \to \mathbb{R}^n$ is C^1, then it is rectifiable and its total arc length is

$$L(\mathbf{c}) \ = \ \int_0^1 \|\mathbf{c}'(t)\| \ dt$$

Examples

- The curve $\mathbf{c}$ is the graph of a smooth function $\phi : [a, b] \to \mathbb{R}$.

Then at a point $(t, \phi(t))$ on the curve, the tangent vector is $(1, \phi'(t))$. The length of the curve is

$$L(\mathbf{c}) \;=\; \int_0^1 \sqrt{1 + \phi'(t)^2}\; dt.$$

- The arc length of the circle $x^2 + y^2 = a^2$.

Parametrize the circle by $\mathbf{c}(t) = (a\cos(t), a\sin(t)),\; t \in [0, 2\pi]$. Then $\mathbf{c}'(t) = (-a\sin(t), a\cos(t))$. The arc length is

$$L \;=\; \int_0^{2\pi} a \; dt \;=\; 2\pi a.$$

Line integrals

Definition

Let a path $\mathbf{c} : [0,1] \to \mathbb{R}^n$ be C^1 and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar field defined in a domain containing the curve $\mathbf{c}$. The line integral of f along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f \; ds \;=\; \int_0^1 f(\mathbf{c}(t)) \; \|\mathbf{c}'(t)\| \; dt$$

Definition

Let a path $\mathbf{c} : [0,1] \to \mathbb{R}^n$ be C^1 and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field defined in a domain containing the curve $\mathbf{c}$. The line integral of F along $\mathbf{c}$ is

$$\int_{\mathbf{c}} F \cdot \; d\mathbf{s} \;=\; \int_0^1 F(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \; dt$$

Reparametrization

Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a non-singular path. ($\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$.)

- Suppose we now make a change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism, from $[\alpha, \beta]$ to $[a, b]$.
- Let $\gamma(u) \;=\; \mathbf{c}(h(u))$ for $u \in [\alpha, \beta]$.
- Assume that the end points are mapped to end points under h.
- Then γ is called a reparametrization of $\mathbf{c}$.
- Because h is a C^1 diffeomorphism, γ is also a C^1curve.

Arc length under reparametrization

- The arc length of a curve is independent of its parametrization.

- The line integral is also independent of the choice of parametrization.

The line integral of a vector field $\mathbf{F}$ along γ is given by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\alpha^\beta \mathbf{F}(\gamma(u)) \cdot \gamma'(u) du = \int_\alpha^\beta \mathbf{F}(\mathbf{c}(h(u)) \cdot \mathbf{c}'(h(u)) h'(u) du,$$

where the last equality follows from the chain rule. Using the fact that $h'(u)du = dt$, we can change variables from u to t to get

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s}.$$

Orientation of Curves

For given two points P and Q on $\mathbb{R}^n$ and a path connecting them, when is the path traversed from P to Q or from Q to P?

Since a path from P to Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.

If the reparametrization $\gamma(\cdot) = \mathbf{c}(h(\cdot))$ preserves the orientation of $\mathbf{c}$, then

$$\int_\gamma \mathbf{F}\mathbf{ds} = \int_\mathbf{c} \mathbf{F}\mathbf{ds}.$$

If the reparamtrization reverses the orientation, then

$$\int_\gamma \mathbf{F}\mathbf{ds} = -\int_\mathbf{c} \mathbf{F}\mathbf{ds}.$$

Work done along a curve

• In Physics, the work done by a particle on which a force $\mathbf{F}$ is applied, is given by $\mathbf{F}.d\mathbf{s}$ where ds is the displacement.

Example : Find the work done by the force field $F = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.

Solution: The work done is given by

$$\begin{aligned} W &= \int_{\mathbf{c}} F \cdot ds \\ &= \int_0^{2\pi} F(\mathbf{c}(t)).\mathbf{c}'(t)\ dt \\ &= \int_0^{2\pi} (-\sin t + \cos t) dt \\ &= (\cos t + \sin t)|_0^{2\pi} = 0 \end{aligned}$$

Surfaces : Definition

A curve is a one-dimensional object. Intuitively, this means that it is possible to describe a curve using just one variable or parameter.

In order to describe a surface, we need two parameters.

Definition

Let Ω be a path connected subset in $\mathbb{R}^2$. A parametrized surface is a continuous function $\mathbf{\Phi} : \Omega \to \mathbb{R}^3$.

This definition is the analogue of parametrized curves in one dimension.

Geometric parametrized surfaces

As with curves and paths, we will distinguish between the surface $\mathbf{\Phi}$ and its image.

The image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

For a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$:

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on D.

The parametrized surface $\mathbf{\Phi}$ is smooth if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing Ω.

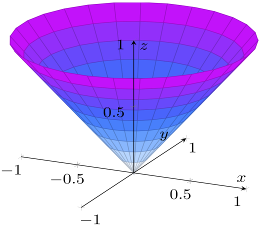
Examples

Example 1: Graphs of real valued functions :
Let $f(x, y)$ be a scalar function and let $z = f(x, y)$, for all $(x, y) \in D$, where D is a path connected region in $\mathbb{R}^2$. Define the parametrized surface $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)), \quad \forall\, (u, v) \in D.$$

Example 2: Sphere of radius a, $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Recall that using spherical coordinates we can represent it using the following parametrization, $\mathbf{\Phi} : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ defined as

$$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v).$$



Example 3: The graph of $z = \sqrt{x^2 + y^2}$ can also be parametrized. We use the idea that at each value of z we get a circle of radius z. We can describe the cone as the parametrized surface $\mathbf{\Phi} : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^3$ as $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u)$.

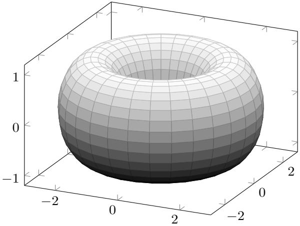
Example 4: Consider the cylinder, $x^2 + y^2 = a^2$. Then this is parametrized surface defined by $\mathbf{\Phi} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ defined as $\mathbf{\Phi}(u, v) = (a \cos u, a \sin u, v)$.

Surfaces of revolution around the z-axis

Example 5: If we have a parametrized curve on the z-y-plane $(0, y(u), z(u))$ which we rotate around z-axis, we can parametrize it as follows:

$$x = y(u)\cos v, \quad y = y(u)\sin v, \quad \text{and} \quad z = z(u).$$

Here $a \leq u \leq b$ if $[a, b]$ is the domain of the curve, and $0 \leq v \leq 2\pi$.



We can parametrize a torus by taking a circle in the y-z plane with center $(0, a, 0)$ of radius b. This is given by the curve $(0, a + b\cos u, b\sin u)$.

Then the parametrization of the torus is then $\mathbf{\Phi}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.

Tangent vectors for a parametrized surface

Let $\mathbf{\Phi}(u,v)$ be a smooth parametrized surface. Fix the variable v, say $v = v_0$, to obtain a curve $\mathbf{c}(u,v_0)$:

$$\mathbf{c}(u) = x(u,v_0)\mathbf{i} + y(u,v_0)\mathbf{j} + z(u,v_0)\mathbf{k}.$$

Its tangent vector is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k}.$$

Define the partial derivative of a vector valued function as

$$\mathbf{\Phi}_u(u_0,v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0,v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing u and varying v, set

$$\mathbf{\Phi}_v(u_0,v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0,v_0) := \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}.$$

The tangent plane

Fix a point on the surface, $P_0 = (x_0, y_0, z_0) := \mathbf{\Phi}(u_0, v_0)$ for some $(u_0, v_0) \in D$.

The two tangent vectors $\mathbf{\Phi}_u(u_0, v_0)$ and $\mathbf{\Phi}_v(u_0, v_0)$ at P_0 define the tangent plane to the surface at P_0.

The normal to this plane at P_0, $\mathbf{n}(u_0, v_0) = \mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

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