Lecture 5 : Surface integrals and Integration theorems

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contents

Recap: Surfaces

Definition

Let E be a path connected subset in $\mathbb{R}^2$ with non-zero area. A parametrized surface is a continuous function $\mathbf{\Phi} : E \to \mathbb{R}^3$.

Examples:

- Graphs of real valued functions of two independent variables.
- A cylinder, A sphere, A cone, Surface of revolution.

For a given $(u, v) \in E$, $\mathbf{\Phi}(u, v)$ can be written as

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

The parametrized surface $\mathbf{\Phi}$ is smooth if the functions x, y, z are C^1 in an open subset of $\mathbb{R}^2$ containing E.

At a point (u, v), the tangent vectors to the surface are $\mathbf{\Phi}_u$ and $\mathbf{\Phi}_v$ and the normal vector $\mathbf{\Phi}_u \times \mathbf{\Phi}_v$ is orthogonal to both the tangent vectors.

Regular surfaces

We call $\mathbf{\Phi}$ a regular of non-singular parametrised surface if $\mathbf{\Phi}$ is C^1 and the normal vector $\mathbf{\Phi}_u \times \mathbf{\Phi}_v \neq 0$ at all points.

Note that the right circular cone is not a regular parametrised surface.

The unit normal $\mathbf{n}$ to the surface at any point $P_0 = \mathbf{\Phi}(u_0, v_0)$ is defined by

$$\mathbf{n}(u_0, v_0) := \frac{\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)}{\|\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)\|}.$$

Example: For the right circular cone, the parametric surface is given by

$$\mathbf{\Phi}(u,v) = (u\cos v, u\sin v, u), \quad (u,v) \in [0,\infty) \times [0,2\pi].$$

In this case we get

$$\mathbf{\Phi}_u(u,v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v(u,v) = -u\sin v\mathbf{i} + u\cos v\mathbf{j},$$

$$\mathbf{\Phi}_u(u,v) \times \mathbf{\Phi}_v(u,v) = (-u\cos v, -u\sin v, u).$$

For any $(u_0, v_0) \in (0,\infty) \times [0,2\pi]$, $\mathbf{n}(u_0,v_0) \neq (0,0,0)$ and the tangent plane

$$(\cos v_0)x + (\sin v_0)y = z.$$

Note that if $(u,v) = (0,0)$, then $\mathbf{n}(0,0) = 0$, so the tangent plane is not defined at the origin. However, it is defined at any other point.

Elementary surface area

Let R be a small rectangle with corners (u, v), $(u + \Delta u, v)$, $(u + \Delta u, v + \Delta v)$ and $(u, v + \Delta v)$, on the $u - v$ plane.

How to compute the "area element", the image of R under $\mathbf{\Phi}$ on S, the geometric surface?

The image is bounded by the four points $\mathbf{\Phi}(u, v)$, $\mathbf{\Phi}(u + \Delta u, v)$, $\mathbf{\Phi}(u + \Delta u, v + \Delta v)$ and $\mathbf{\Phi}(u, v + \Delta v)$.

We computed this, while deriving the formula for the change of variables.

The only difference now is that $\mathbf{\Phi}(R)$ no longer lies in the plane. But this doesn't really change anything.

We must make sure that $\mathbf{\Phi}$ is bijective.

In fact, the inverse function theorem guarantees that if $\mathbf{\Phi}$ is non-singular, then it is bijective in a small enough neighborhood on the surface.

Elementary Surface Area

Let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface.

Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, assuming $\mathbf{\Phi}$ is C^1

$$P := \mathbf{\Phi}(u, v), \quad P_1 := \mathbf{\Phi}(u + h, v) \approx \mathbf{\Phi}(u, v) + h\ \mathbf{\Phi}_u(u, v),$$

$$P_2 := \mathbf{\Phi}(u, v + k) \approx \mathbf{\Phi}(u, v) + k\ \mathbf{\Phi}_v(u, v), \quad Q := \mathbf{\Phi}(u + h, v + k).$$

Area of the parallelogram with sides PP_1 and PP_2

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v)\|\, |h||k|.$$

Surface Area

In view of this approximation, we define on E, a bounded simple domain

$$\text{Area } (\mathbf{\Phi}) := \iint_E \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u,v)\| \, du \, dv.$$

Introduce the following differential notation:

$$dS = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\| \, dudv.$$

Thus Area $(\mathbf{\Phi}) := \iint_E dS$.

Example

- Graph of a function: $\mathbf{\Phi}(u,v) = (u, v, f(u,v))$ for $(u,v) \in E$. Then

$$\begin{aligned} \text{Area } (\mathbf{\Phi}) &= \iint_E \|(-f_u, -f_v, 1)\| dudv \\ &= \iint_E \sqrt{1 + f_u^2 + f_v^2} \, dudv \end{aligned}$$

The area vector of an infinitesimal surface element

$\mathbf{\Phi}$ takes the small rectangle R to the parallelogram formed by the vectors $\mathbf{\Phi}_u \Delta u$ and $\mathbf{\Phi}_v \Delta v$.

It follows that the 'area vector' $\Delta \mathbf{S}$ of this parallelogram is

$$\Delta \mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) \Delta u \Delta v.$$

Thus the surface 'area vector' is to be thought of as a vector pointing in the direction of the normal to the surface and in differential notation:

$$d\mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v)\, du\, dv.$$

The magnitude of the surface 'area vector' is given by

$$dS = \|d\mathbf{S}\| = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, du\, dv.$$

If the parametric surface $\mathbf{\Phi}$ is non-singular, we can write

$$d\mathbf{S} = \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit vector normal to the surface.

The magnitude of the area vector

It remains to compute the magnitude dS.

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

Hence,

$$dS = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

where $\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial(x,z)}{\partial(u,v)}$, $\frac{\partial(x,y)}{\partial(u,v)}$ are the determinants of corresponding Jacobian matrix.

The surface area integral

Because of the calculations, the surface area is given by

$$\iint_S dS = \iint_E \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

Integral of any bounded scalar function $f : S \to \mathbb{R}$:

$$\iint_S f dS = \iint_E f(x,y,z)\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

provided the R.H.S double integral exists.

The surface integral of a vector field

Let $\mathbf{F}$ be a bounded vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : E \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$

provided the R.H.S double integral exists. More compactly

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

Orientation of planar curves

By convention, the positive orientation of a simple closed curve on a plane corresponds to the anti-clockwise direction.

Convention: The curve in $\mathbb{R}^2$ is positively oriented if the region bounded by the curve always lies to the left of an observer, walking along the curve in the chosen direction.

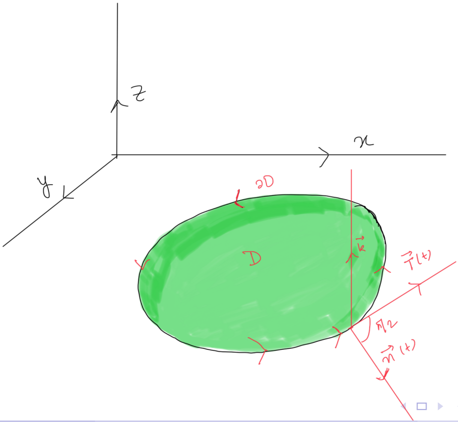
There is a natural notion of positive orientation of the boundary of a region D, given by the vector field $\mathbf{k}$ - the unit normal vector pointing in the direction of the positive z axis.

Orienting the boundary curve

Definition: The positive orientation of a curve C in $\mathbb{R}^2$ is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the normal vector field pointing outward along the curve.



Green's Theorem

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D a simple closed piecewise continuously differentiable curve.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\left(D \cup \partial D\right) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions.*

Then

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

A proof of Green's theorem for regions of special type

Assume that D is of Type 1 or x- simple,

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

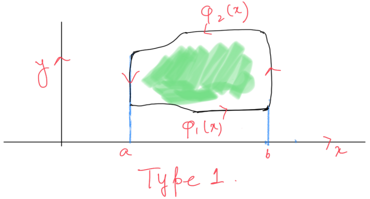
for some continuous functions ϕ_1 and ϕ_2.

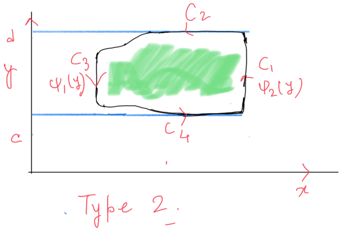
and y simple , Type 2: For two continuous functions ψ_1 and ψ_2,

$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y)\}.$$

The proof follows two main steps:

- Double integrals can be reduced to iterated integrals.
- Then the fundamental theorem of calculus can be applied to the resulting one-variable integrals.


φ₂(x)
y
φ₁(x)
a
b
x
Type 1.


C₂
d
y
C₃
ψ₁(y)
C₁
ψ₂(y)
c
C₄
x
Type 2.

The proof of Green's theorem, contd.

Since D is a region of Type 2, it gives

$$\iint_D \frac{\partial F_2}{\partial x} dx dy = \int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x, y) dx dy.$$

Using the Fundamental Theorem of Calculus we get

$$\int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x, y) dx dy = \int_c^d F_2(\psi_2(y), y) - F_2(\psi_1(y), y) dy$$

Note that ∂D can be written as union of four curves C_1, C_2, C_3 and C_4 .

The proof of Green's theorem contd.

On $C_1 = \{(\psi_2(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction upwards. So,

$$\int_{C_1} F_2\, dy = \int_c^d F_2(\psi_2(y), y)\, dy.$$

On $C_3 = \{(\psi_1(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction downwards. So,

$$\int_{C_3} F_2\, dy = -\int_{-C_3} F_2\, dy = -\int_c^d F_2(\psi_1(y), y)\, dy.$$

On $C_2 = \{(x, d) \mid \psi_1(d) \leq x \leq \psi_2(d)\}$ going from right to left and $C_4 = \{(x, c) \mid \psi_1(c) \leq x \leq \psi_2(c)\}$ going from left to right. They are horizontal lines and y is constant along these lines. Thus, for any parametrization of C_2 and C_4, $\frac{dy}{dt} = 0$, and

$$\int_{C_2} F_2\, dy = 0 = \int_{C_4} F_2\, dy.$$

Noting that

$$\int_{\partial D} F_2 dy = \int_{C_1} F_2\, dy + \int_{C_2} F_2\, dy + \int_{C_3} F_2\, dy + \int_{C_4} F_2\, dy,$$

and using previous results, we obtain

$$\int_{\partial D} F_2 dy = \int_c^d F_2(\psi_2(y), y)\, dy - \int_c^d F_2(\psi_1(y), y)\, dy,$$

and thus

$$\iint_D \frac{\partial F_2}{\partial x} dx dy = \int_{\partial D} F_2 dy.$$

As D is also a Type 1 region, similarly we get

$$\iint_D \frac{\partial F_1}{\partial y} dx dy = -\int_{\partial D} F_1 dx.$$

Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \int_{\partial D} F_1 dx + \int_{\partial D} F_2 dy.$$

Orientable surfaces

Definition: A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector, normal to the surface S at P.

At each point of S there are two possible directions for the normal vector to S.

The question is whether the normal vector field can be chosen so as to be continuous.

The orientation of the surface will induce an orientation of its boundary.

Curl and Divergence of a vector field

The del operator ∇ operates on vector fields in two different ways. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

$$\text{curl}\,\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

It is often written as a determinant;

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Divergence of $\mathbf{F} \;=\; \nabla \cdot \mathbf{F}$

Stokes theorem

Theorem

1. *Let S be a bounded piecewise smooth oriented surface with non-empty boundary ∂S.*
2. *Let ∂S, the boundary of S, be a simple, non-singular parametrized curve with the induced orientation.*
3. *Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a C^1 vector field defined on an open set containing S.*

Then

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{ds} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}.$$

In other words,

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{t}\ ds = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ dS,$$

$\mathbf{t}$ unit tangent vector to ∂S and $\mathbf{n}$ unit normal vector to S with orientation.

Gauss's divergence theorem

Theorem (Gauss's Divergence Theorem)

1. *Let W be a closed and bounded region of $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface.*
2. *Suppose ∂W is positively oriented.*
3. *Let $\boldsymbol{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing W.*

Then

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{W} (\text{div } \boldsymbol{F})\ dxdydz.$$

In other words

$$\iint_{\partial W} \boldsymbol{F} \cdot \mathbf{n}\ dS = \iiint_{W} (\text{div } \boldsymbol{F})\ dxdydz$$

The importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

Physical Interpretation of the Gauss Divergence Theorem:

Suppose a solid body W in $\mathbb{R}^3$ is enclosed by a closed geometric surface S, oriented in the direction of the outward normals. Let $\boldsymbol{F}$ be a vector field on D.

The Gauss divergence theorem says that the flux of $\boldsymbol{F}$ across S , $\boldsymbol{F} \cdot \mathbf{n}$, is equal to the triple integral of the divergence of the vector field $\boldsymbol{F}$ over W.

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