Group actions and applications

Lecture 4

Direct products and semidirect products.

① Suppose $H, N \trianglelefteq G$. Then $HN \trianglelefteq G$.

② Suppose $H, N \leq G$ and $H \cap N = \{e\}$.

Then $H \times N \longrightarrow HN$ is a bijection of sets.

$(h,n) \longmapsto hn$

Pf: If $h_1 n_1 = h_2 n_2$ $\quad n_1 n_2^{-1} = h_1^{-1} h_2 = e$

$\Rightarrow h_1 = h_2 \,,\; n_1 = n_2$

③ Suppose $H, N \leq G$, H normalizes N as well as N normalizes H.

Also suppose $H \cap N = \{e\}$.

(a) Then elements of H must commute with the elements of N that is

$hn = nh$ for $h \in H$, $n \in N$.

Pf: $[h, n] = \underbrace{h n h^{-1}}\, n^{-1} = e$

(b) Then $HN \cong H \times N$ as groups.

direct product of groups

$(h,n) \longmapsto hn$

(a) $\Rightarrow$ this is a homomorphism

$(h_1, n_1) \longmapsto h_1 n_1$

$(h_2, n_2) \longmapsto h_2 n_2$

$(h_1 h_2, n_1 n_2)$

$\longmapsto h_1 h_2 n_1 n_2$

(4) Suppose $H, N \leq G$, H normalizes N and $H \cap N = \{e\}$.
Then we know that $HN = NH \leq G$.
We also have $N \times H \xrightarrow{\cong} NH$ as <u>sets</u>.

$$(n_1, h_1) \longmapsto n_1 h_1$$
$$(n_2, h_2) \longmapsto n_2 h_2$$

$$\cancel{(n_1 n_2, h_1 h_2)} \mapsto n_1 n_2 h_1 h_2$$

$$? \longmapsto n_1 h_1 n_2 h_2$$
$$\qquad \qquad \qquad \qquad \qquad ||$$
$$(\underbrace{n_1 h_1 n_2 h_1^{-1}}_{\in N}, h_1 h_2) \longmapsto n_1 h_1 n_2 \underbrace{h_1^{-1} h_1}\, h_2$$
$$||$$
$$(n_1\, {}^{h_1}n_2, h_1 h_2)$$
$$||$$
$$(n_1\, ad(h_1)(n_2), h_1 h_2)$$

Semi-direct product.

We had that: H acts on N by conjugation.

$$ad : H \longrightarrow Aut(N)$$

We have an action of H on N by group automorphisms.
(In the setting of (4))

(5) H, N are any groups.
Suppose H acts on N by group automorphisms.

$$\alpha : H \longrightarrow \mathrm{Aut}(N)$$

Then the semidirect product

$N \rtimes H$ is defined as follows

ⓐ As a set $N \rtimes H := N \times H$

ⓑ $(n_1, h_1) \cdot (n_2, h_2) := (n_1 \alpha(h_1)(n_2), h_1 h_2)$

Check that this is a group.

$$N \trianglelefteq N \rtimes H$$

$$N \hookrightarrow N \rtimes H$$

$$n \longmapsto (n, e_H)$$

$$H \hookrightarrow N \rtimes H$$

$$h \longmapsto (e_N, h)$$

$$\alpha : H \longrightarrow \mathrm{Aut}(N) \leq \mathrm{Perm}(N)$$

<u>Exercise</u>: Describe all groups of size 21 up to isomorphism.

<u>Nilpotent groups</u>: If P is a non-trivial p-group we saw that $Z_1(P)$ is also non-trivial. $\circledast$

$$Z\left(\frac{P}{Z(P)}\right) \trianglelefteq \frac{P}{Z_1(P)}$$

$$\|$$

$$\frac{Z_2(P)}{Z_1(P)} \trianglelefteq \frac{P}{Z_1(P)}$$

$$Z_1(P) \trianglelefteq Z_2(P) \trianglelefteq P$$

$Z_2(P)$ is the subgroup of P such that $\frac{Z_2(P)}{Z_1(P)} = Z\left(\frac{P}{Z_1(P)}\right)$

(1-1 Correspondence between subgroups of $\frac{P}{Z_1(P)}$ and intermediate subgroups between $Z_1(P)$ and P.)

$$Z\left(\frac{P}{Z_2(P)}\right) \trianglelefteq \frac{P}{Z_2(P)}$$

$$\| $$

$$\frac{Z_3(P)}{Z_2(P)}$$

$$\{e\} \trianglelefteq Z_1(P) \trianglelefteq Z_2(P) \trianglelefteq Z_3(P) \trianglelefteq \cdots$$

$$Z_n(P) \trianglelefteq P$$

If $Z_n(P) \underset{\neq}{\trianglelefteq} P$

$$\Rightarrow \quad \begin{array}{c} Z_{n+1}(P) \\ \gneq \\ Z_n(P) \end{array}$$

Using $\circledast$ we see that $\exists$ n such that $Z_n(P) = P$.

—x—

$$Z_1(G) = Z(G)$$

Assume $Z_n(G)$ has been defined.

$$\frac{Z_{n+1}(G)}{Z_n(G)} = Z\left(\frac{G}{Z_n(G)}\right) \trianglelefteq \frac{G}{Z_n(G)}$$

Define $Z_{n+1}(G)$ by the above property

$$\{e\} \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq Z_3(G) \trianglelefteq \cdots$$

$$Z_n(G) \trianglelefteq G$$

<u>Def^n:</u> Nilpotent group: If G is a group such that $Z_n(G) = G$ for some n, then we say that G is nilpotent.

We proved above that p-groups (p prime) are always nilpotent.

Thm: A finite group is nilpotent iff all its p-Sylow subgroups for all primes p are normal.

P is a p-Sylow subgroup of G.

$P \trianglelefteq G$.

$\Rightarrow$ P must be the unique p-syl subgp.

If G is nilpotent iff $G \cong P_1 \times \cdots \times P_n$

direct product of sylow subgroups.

Ex: If G has unique p-Syl subgroups for all primes p prove that

$$G \cong P_1 \times \cdots \times P_n$$

product of all Sylow subgroups

$\{e\} \underset{\neq}{\trianglelefteq} H_1 \underset{\neq}{\trianglelefteq} H_2 \underset{\neq}{\trianglelefteq} \cdots \trianglelefteq G$ such that

each $\dfrac{H_{n+1}}{H_n}$ is simple.

Composition series and composition factors