Group actions and applications

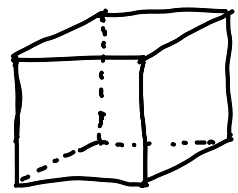
Lecture 1

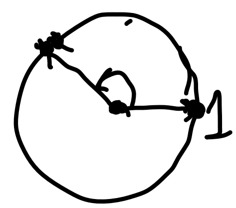
Groups as groups of symmetries: $\mathrm{Perm}(\{1,2,\dots,n\})$

- Permutation groups: S_n $\quad |S_n| = n!$
- Symmetries of regular n-gon: D_n (Dihedral group.)

$$|D_n| = 2n$$

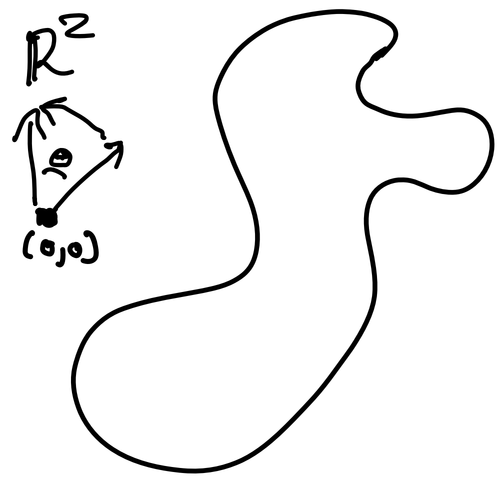
- Group of symmetries of a cube:





$$|G| = 48$$

- Various types of symmetries of $\mathbb{R}^2$



Translations: $\mathbb{R}^2$

Rotations around $(0,0)$: $S^1 \subseteq \mathbb{C}^\times$

Dilations/scalings: $\mathbb{R}^\times$

Linear transformations: $GL_2(\mathbb{R})$

- Symmetries of the field extension $\mathbb{C}/\mathbb{R}$

identity $\qquad \sigma : \mathbb{C} \longrightarrow \mathbb{C}$ such that

& complex conjugation $\sigma(r) = r$

Cayley's Theorem: Every group G is a subgroup of a permutation group for example

$$G \subseteq \mathrm{Perm}(G)$$

<u>Definition of a group:</u> A group is a set G together with a binary operation $*: G \times G \longrightarrow G$ such that:

- $(a*b)*c = a*(b*c)$ for all $a,b,c \in G$ (Associative)
- $\exists\, e \in G$ such that $e*a = a*e = a$ for all $a \in G$ (Existence of identity)
- $\forall\, a \in G\ \exists\, a^{-1} \in G$ such that $a*a^{-1} = a^{-1}*a = e$ (All elements have an inverse.)

A subgroup $H \leq G$ is a subset $H \subseteq G$ which is closed under $*$ and inverse and $e \in H$.

<u>Left cosets</u>: $gH \subseteq G$. They define a partition of G.

The set of all left cosets is denoted by G/H.

<u>Right cosets:</u> $Hg \subseteq G$. They define a partition of G.

The set of all right cosets is denoted by $H\backslash G$.

Under inversion a left coset gets mapped to a right coset and vice versa.

$$(gH)^{-1} = Hg^{-1}$$

$$(Hg)^{-1} = g^{-1}H$$

$$H\backslash G \underset{\text{inversion}}{\overset{\cong}{\longleftrightarrow}} G/H$$

Lagrange's Theorem: If G is a finite group and $H \leq G$, then $|H| \,\big|\, |G|$.

Proof: $|G| = |H| \cdot |G/H| = |H| \cdot |H\backslash G|$ $\qquad (\mathbb{Z}/n\mathbb{Z})^{\times}$

Cor: If G is a finite group then $g^{|G|} = e \ \forall g \in G$.

Cor: (Euler's Theorem) $a^{\phi(n)} \equiv 1 \pmod{n}$ for all $a \in \mathbb{Z}$ co-prime to n.

Group homomorphism:

$$\begin{tikzcd} f: G \arrow[r] \arrow[dr] & H \\ & \mathrm{Im}(f) \arrow[u, phantom, "\leq" rotate=90] \end{tikzcd}$$

such that $f(ab) = f(a)f(b) \ \forall a, b \in G$.

$(\Rightarrow f(e_G) = e_H)$

$f(G) = \mathrm{Im}(f) \leq H$.

Kernel of a group homomorphism

$$\mathrm{Ker}(f) = \{g \in G \mid f(g) = e_H\}.$$

$\boxed{N \trianglelefteq G \text{ iff } N \leq G \ \& \ gN = Ng \ \forall g \in G}$

$N := \mathrm{Ker}(f) \trianglelefteq G$.

$$G/\mathrm{Ker}(f) \cong \mathrm{Im}(f)$$

$\boxed{f(g) = f(h) \text{ iff } gN = hN}$

Conversely if $N \trianglelefteq G$ then $G/N = N\backslash G$ is a group and $N = \mathrm{ker}(\pi)$ where

$$\pi : G \twoheadrightarrow G/N, \quad g \mapsto gN$$

is the natural projection.

Theorem: Let $H \leq G$, $N \trianglelefteq G$ such that H $\boxed{\text{normalizes}}$ N. Then
- $HN = NH \leq G$
- $N \trianglelefteq HN$, $N \cap H \trianglelefteq H$
- $HN/N \cong H/N\cap H$

$\boxed{\begin{array}{c} hN = Nh \\ \text{for all } h \in H \end{array}}$

Theorem: • If $N \trianglelefteq G$ then $\exists$ 1-1 correspondence between subgroups of G/N and subgroups $N \trianglelefteq K \leq G$.
- Normal subgroups correspond to normal K in G. In this case $G/K \cong \dfrac{G/N}{K/N}$.

Group actions:

G group. X set

A (left) action of G on X is:

$$\left\{\begin{array}{l} \text{a map } \cdot : G \times X \longrightarrow X \text{ such that} \\ \qquad e \cdot x = x \quad \forall x \in X \\ \text{and} \quad (ab)\cdot x = a\cdot(b\cdot x) \ \forall a, b \in G, x \in X. \end{array}\right.$$

$\Longleftrightarrow$ $\left\{\begin{array}{l} \text{a group homomorphism} \\ \qquad \alpha : G \longrightarrow \mathrm{Perm}(X) \end{array}\right.$

$$\boxed{\alpha(g)(x) = g\cdot x}$$

$$\boxed{\begin{array}{c} G \\ G^{op} \\ g *^{op} h := h * g \end{array}}$$

A right action of G on X is

$$\left\{\begin{array}{l}\text{a map } \cdot : X \times G \longrightarrow X \text{ such that} \\ \qquad x \cdot e = x \quad \forall\, x \in X \\ \text{and} \quad x\cdot(ab) = (x\cdot a)\cdot b \quad \forall\, a,b \in G,\ x \in X.\end{array}\right.$$

$\Longleftrightarrow$

$$\left\{\begin{array}{l}\text{a group homomorphism} \\ \qquad \alpha : G^{op} \longrightarrow \mathrm{Perm}(X)\end{array}\right.$$

$\Longleftrightarrow$

$\{$ a left action of G^{op} on X

<u>Examples</u>: Groups of symmetries

- Left action of G on itself by left multiplication

$$G \hookrightarrow \mathrm{Perm}(G) \Longrightarrow \text{Cayley's Thm.}$$

- Right action of G on itself by right multiplication.
- Adjoint action or conjugation action:

$$G \times G \longrightarrow G$$

$$\mathrm{ad}(g)(h) = g h g^{-1}$$

$$\mathrm{ad} : G \longrightarrow \mathrm{Perm}(G)$$

$$\mathrm{ad}(g) : G \longrightarrow G$$ Group homomorphism which is a bijection

$$\mathrm{ad} : G \longrightarrow \mathrm{Aut}(G)$$

$$(a * b)^{-1} = b^{-1} * a^{-1} = a^{-1} *^{op} b^{-1}$$

$$G \xrightarrow[\text{inversion}]{\cong} G^{op}$$

$$G^{op} \xrightarrow{\text{inv}} G \longrightarrow \mathrm{Perm}(X)$$

$$\boxed{{}^{g}h} = \boxed{\mathrm{ad}(g).h} \quad \boxed{\cancel{g.h}} = ghg^{-1}$$

So we will use either of these for the adjoint action

This notation can confuse with the group operation.

$${}^{g}h = ghg^{-1}$$

$${}^{g}(h_1h_2) = {}^{g}h_1 \cdot {}^{g}h_2$$