Shapes and Geometry of Surfaces

Mahan Mj,
School of Mathematics,
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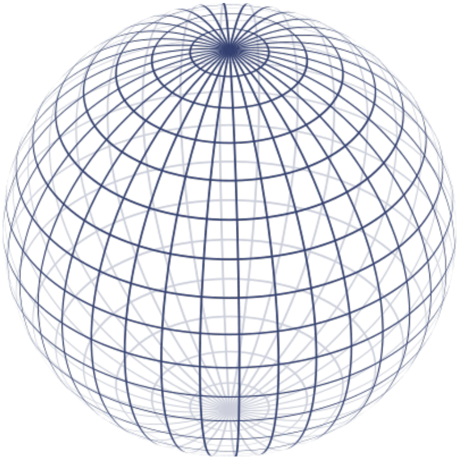
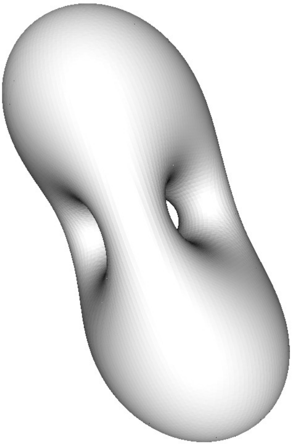
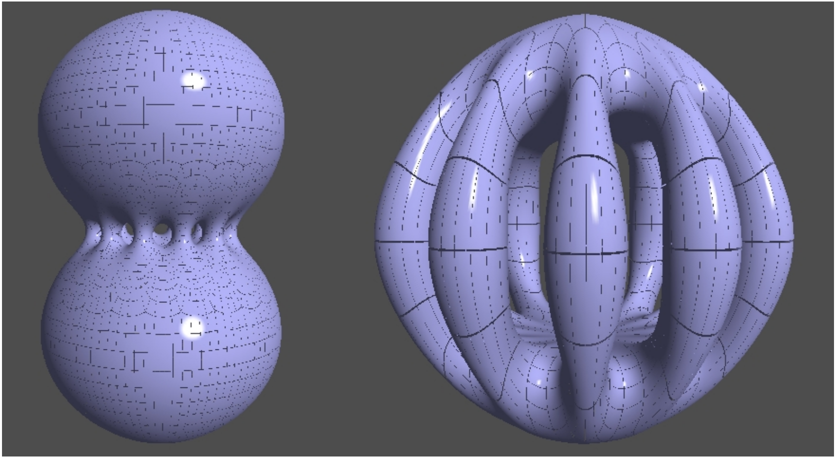
Shapes Around Us

A Bewildering Multitude







Classification

How do we organize all this information?
Formal structure required.
What are surfaces?
Answer: Locally 2-dimensional objects.
What does classification mean?

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An exhaustive **list** with no redundant elements.

Given a locally 2-dimensional object want to say which object in the list it 'is'.

Hence we need a way of saying when two surfaces are the 'same'.

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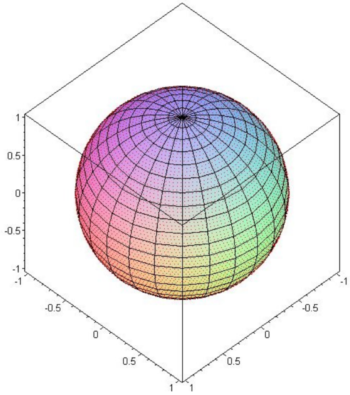
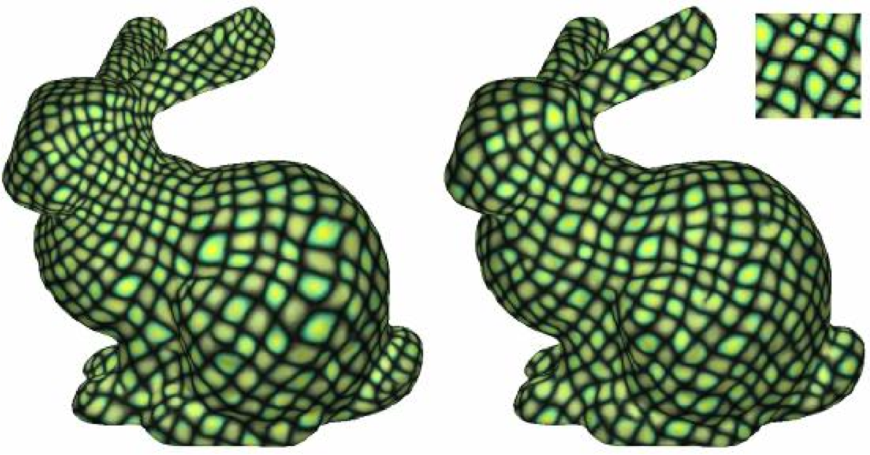

1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1
0.5
0
-0.5
-1



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A surface or **2-manifold** is a topological space that is locally homeomorphic to $\mathbb{R}^2$.

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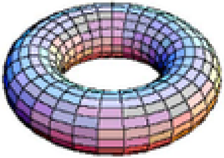
Sure!



Two surfaces S_1 and S_2 are **homeomorphic** if there exists a one-to-one onto continuous map $f : S_1 \to S_2$ such that f^{-1} is continuous.

Intuitive idea: S_1 can be continuously deformed to S_2 without 'tearing'.

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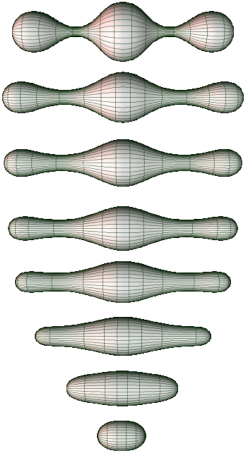
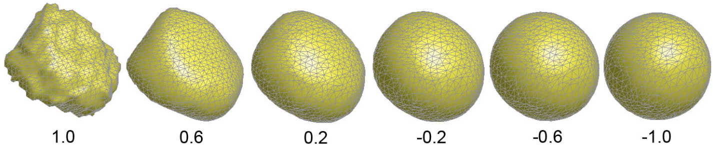
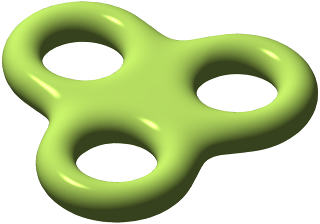
Continuous deformation:

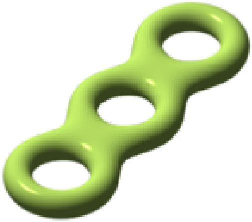


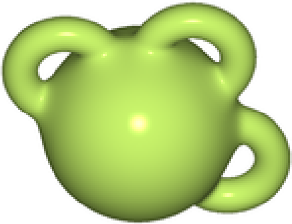
Continuous deformation 2:



CLASSIFY!







Theorem: Any closed *orientable* surface is homeomorphic to a sphere with n handles for some non-negative integer n.

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Well-defined *sides*.

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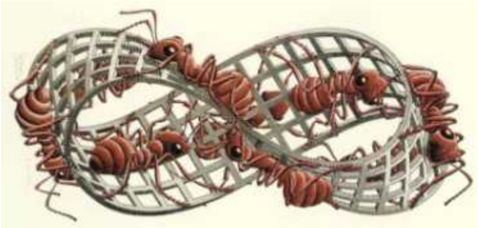
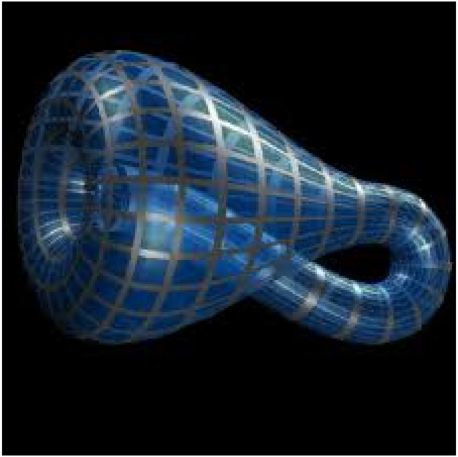
Non-orientable surfaces





Curvature

But there's more.

These shapes *are* different at a slightly subtler level.

A rabbit is a sphere *topologically* but not *geometrically*.

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The surfaces of all these solids is homeomorphic to the sphere.
But curvatures at the vertices are different.
How to measure curvature χ_v at a vertex v?

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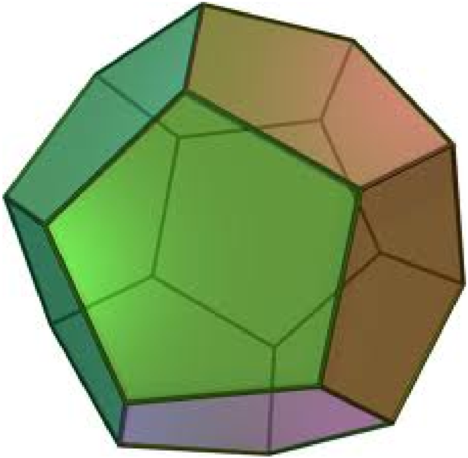
Example: The Regular Dodecahedron



$\theta_i = \frac{3\pi}{5}$, $i = 1, 2, 3$

Therefore $\chi_v = 2\pi - \frac{9\pi}{5} = \frac{\pi}{5}$ for all v.

Therefore $\sum_v \chi_v = 20 \times \frac{\pi}{5} = 4\pi$

Fact: $\sum_v \chi_v = 4\pi$ for all such *cellulations* of the sphere.

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The Polyhedral Gauss-Bonnet Theorem: For any *cellulation* of the sphere with *n* handles,

$$\sum_v \chi_v = -2\pi(2n-2).$$