Shapes and Geometry of Surfaces

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Shapes Around Us

A Bewildering Multitude

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Classification

How do we organize all this information?

Formal structure required. What are surfaces? **Answer:** Locally 2-dimensional objects. What does classification mean?

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Want the following:

An exhaustive **list** with no redundant elements.

Given a locally 2-dimensional object want to say which object in the list it 'is'.

Hence we need a way of saying when two surfaces are the 'same'.

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Formal definitions follow.

A surface or **2-manifold** is a topological space that is locally homeomorphic to \mathbb{R}^2 . Any other examples?



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Sure!





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Two surfaces S_1 and S_2 are homeomorphic if there exists a one-to-one onto continuous map $f : S_1 \to S_2$ such that f^{-1} is continuous. Intuitive idea: S_1 can be continuously deformed to S_2 without 'tearing'.

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Continuous deformation:



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Continuous deformation 2:



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Theorem: Any closed *orientable* surface is homeomorphic to a sphere with *n* handles for some non-negative integer *n*.

Orientable? Well-defined *sides*.



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Non-orientable surfaces



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But there's more.

These shapes *are* different at a slightly subtler level. A rabbit is a sphere *topologically* but not *geometrically*.

Geometry = Curvature

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The surfaces of all these solids is homeomorphic to the sphere.

But curvatures at the vertices are different. How to measure curvature χ_v at a vertex v?

$$\chi_{v} = 2\pi - \sum \theta_{i}$$

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Example: The Regular Dodecahedron



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$$\theta_i = \frac{3\pi}{5}, i = 1, 2, 3$$

Therefore $\chi_v = 2\pi - \frac{9\pi}{5} = \frac{\pi}{5}$ for all v .
Therefore $\sum_v \chi_v = 20 \times \frac{\pi}{5} = 4\pi$
Fact: $\sum_v \chi_v = 4\pi$ for all such *cellulations* of the sphere

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The Polyhedral Gauss-Bonnet Theorem: For any *cellulation* of the sphere with *n* handles,

$$\sum_{\mathbf{v}}\chi_{\mathbf{v}}=-2\pi(2n-2).$$

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